Irena Peeva *Editor*

Commutative Algebra

Expository Papers Dedicated to David Eisenbud on the Occasion of his 75th Birthday

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Dedicated to David Eisenbud on the occasion of his 75th birthday.

Biosketch of David Eisenbud

David Eisenbud received his PhD in mathematics in 1970 from the University of Chicago under Professor Saunders MacLane and Professor Chris Robson. He was in the faculty at Brandeis University from 1970 until becoming Professor of Mathematics at UC Berkeley in 1997. Eisenbud has been a visiting professor at Harvard, and in Bonn and Paris.

His mathematical interests range widely over commutative and non-commutative algebra, algebraic geometry, topology, and computer methods.

Eisenbud served as the director of the Mathematical Sciences Research Institute from 1997 to 2007 and 2013 to 2022. He worked for the Simons Foundation between 2009 and 2011, creating the Foundation's grant program in mathematics and the physical sciences. He is currently on the board of directors of the Foundation, and is also a director of Math for America, a foundation devoted to improving mathematics teaching.

Eisenbud has been a member of the Board of Mathematical Sciences and their Applications of the National Research Council, and the U.S. National Committee of the International Mathematical Union.

He currently chairs the editorial board of the *Algebra and Number Theory* journal, which he helped found in 2006. He serves on the board of the *Journal of Software for Algebra and Geometry*, as well as Springer-Verlag's book series Algorithms and Computation in Mathematics and Graduate Texts in Mathematics.

In 2006, Eisenbud was elected a Fellow of the American Academy of Arts and Sciences. He won the 2010 Leroy P. Steele Prize for Mathematical Exposition for his book *Commutative Algebra, with a View toward Algebraic Geometry* and the 2020 Award for Distinguished Public Service, both from the American Mathematical Society.

Eisenbud's interests outside of mathematics include theater, music, and juggling. He loves photography and music, and sings Bach, Brahms, Schubert, and Schumann . . .

Mostly Mathematical Fragments of Autobiography

David Eisenbud

I was born on April 8, 1947, to Leonard Eisenbud, a mathematical physicist (and former student of Eugene Wigner), who was then working at the Oak Ridge National Laboratory, and Ruth-Jean Eisenbud, a psychologist-psychoanalyst (and former student of Robert White) with a large private practice.

The family soon moved to Long Island, where my father worked at Brookhaven National Laboratory, and I developed an early love for the water—I have a photo of my mother lying on the sand at the edge of the waves with me on her back, grinning.

When I was three, we moved to the Swarthmore area and stayed there eight years. My father worked for a research lab, and we lived initially on the edge of the property where the lab had a van de Graaf particle accelerator. I was captivated by the big machine, which my father patiently explained. I attended a public elementary school, and then the tiny progressive "School in Rose Valley." I apparently had such a poor sense of pitch in second grade that I was forbidden to sing with the rest of the class, but my music teacher in Rose Valley rescued me and taught me to hold a tune—a fateful development. Art was an unsolved problem for me too: terminally stuck on what painting to contribute to a frieze about world history, my teacher took pity and suggested I paint "The Dark Ages"—all black.

When I was seven, my parents took me to my first Shakespeare play—Macbeth—preceded by my first lobster dinner, at the original Bookbinders' Restaurant, a classic that is no more. My parents had prepared me for the play as best they could, but my mother told me later that she worried how I would take all the violence. She was relieved when I leaned over during the play and whispered "They forgot one of the murders!" Whether or not I was correct, the experience began a lifelong love of theatre (lobster, too).

We moved back to Long Island when I was 11. There, my father helped found the Stony Brook University physics department, where he worked until he retired, and my mother joined the faculty of the NYU postdoctoral program in psychoanalysis. I had an academically excellent but socially difficult middle school experience at the Friends Academy in Locust Valley, and then the opposite at the public Huntington High School.

When I was about 12, I began announcing, without much apparent cause, that I wanted to be a mathematician. I read Thomas' Calculus at 13, and then began to study mathematical topics proposed by one of my father's colleagues—Mendelson's book on topology, Wilder's book on the foundations of mathematics. Another colleague introduced me to the games of Go and Shogi. My father showed me how to use vectors and algebra to prove simple theorems in plane geometry, which I found very exciting. I entered some science/math fairs with this, with an analogue computation of logarithms, and with a calorimeter. I took folk guitar lessons, and my first serious girlfriend introduced me to the flute—I briefly considered trying to become a professional flutist (luckily, I stuck with math!).

Having exhausted the math and much else in high school I asked, late in my junior year, to leave for college, and was accepted at the University of Chicago, where I entered at 16 (not by any means exceptional there) and left with my PhD at 23. Despite the famed breadth of education at Chicago, I quickly focused on mathematics and music. Fortunately, the music came with some breadth, and I had wonderful mentoring from the well-known musicologist Howard Mayer Brown and from Brown's political scientist partner Roger Strauss, in whose home we practiced. I sang in Brown's small chorus, and played early instruments—recorders, krummhorns, Quantz flute, bass viola da gamba—in Brown's *Collegium Musicum* all seven years I was in Chicago. The group was the first to systematically record the pieces from the Historical Anthology of Music, a collection used by every musicology graduate student, and we gave a formal concert each quarter in the beautiful Bond Chapel. The most memorable concert for me was Schütz' Christmas Oratorio, in which I played the recorder. (I listen to a recording of this piece every December.) Music has remained a passion: after years of serious flute study, I started voice lessons in 1982, focusing on German Lieder, and I still spend many hours a week enjoying this art. The mathematician, pianist, and cellist Arthur Mattuck, my music partner for years, referring to the characteristic subjects of these songs, once wrote that I was "singing songs of puberty in a baritone Schuberty."

Among the math courses I took as an undergraduate, three stand out as exciting and inspiring. They could not have been less alike. The first was taught by Otto Kegel, a postdoc who was a student (and then *Assistent*, in the German sense) in the group of Reinhold Baer, in Frankfurt. Kegel taught a second semester linear algebra course using sesquilinear instead of merely bilinear forms, and with other (too) modern flourishes. In a subject where most objects are called V or W, Kegel was still struggling with the transition from German, where the word for W is pronounced "Vay." These things, combined with Kegel's almost illegible handwriting, made the course extremely hard to follow. Nevertheless, Kegel imbued it with such wonderful excitement that it was a peak experience.

The other two courses were excellent in a more standard way: in one, the famous analyst Antoni Zygmund told a highly polished and perfected version of the story of the Lebesgue integral, and in the other, Felix Browder laid down the basics of the theory of functions of one complex variable. To this day I am amazed by the consequences that flow from a simple hypothesis in that subject. With my inseparable undergraduate friend Joe Neisendorfer, I wrote notes for Browder's course (in pencil, and with plenty of erasures!).

I spent an exciting summer working at the University of Michigan as a counselor for a high-school math program and being tutored by family friend Paul Halmos, with problems from his manuscript *Hilbert Space Problem Book*. This gave me the idea that I wanted to study operator theory. By the end of my third year at Chicago I was taking only graduate math courses, and at the end of the year I officially became a graduate student.

The next summer, my parents treated me to a few months abroad. I chose to go and work with Otto Kegel, who was by this time back in Frankfurt. Kegel proposed a research problem about the order automorphisms of infinite ordered sets. I was possessed by the problem, and could talk of nothing else, no doubt tiresome to those around me! Though I had taken a German class in college (and gotten a very solid D), I unfortunately did not try to speak German, nor did I understand it—with one exception: Saunders MacLane, whom I knew from Chicago, came to visit. Though he spoke German easily, he had such thick American accent that he was easy for me to understand. I was relieved when I was given, as office-mate, a young English mathematician, but Bert Wehrfritz' cockney accent was almost as much of a problem for me as German. Peter Neumann was also a visitor to Frankfurt then, and when I went to England at the end of the summer, he kindly invited me to Oxford and took me to lunch at the High Table. In that hot weather I was living in a youth hostel, and I'll never forget that first time drinking cold hard cider from an ancient silver mug.

Back in Chicago, I was uncertain what direction to study—neither permutation groups nor operator theory were represented on the faculty. Advised by Neisendorfer to choose a thesis advisor first and subject second, I gravitated to Saunders MacLane and—thus—category theory. However, this was not to be my thesis: during MacLane's sabbatical I made friends with J. C. (Chris) Robson, former student of Alfred Goldie in Leeds, who was in Chicago as a postdoc with Israel Herstein. In an intense and exciting (for me) collaboration, Robson and I developed a noncommutative analogue of the theory of Dedekind domains.

At a memorable dinner that spring, the graduate student across the table from me said something implying that the work with Robson would be my thesis. I began to protest. . .when inspiration struck, and I realized how nice it would be to have a thesis done without a "thesis neurosis"! MacLane and Robson were generous, and I was done. Since this was already at the end of the spring term, it made sense to take an extra year, having only the (then) light responsibility of a graduate student and the freedom of a postdoc.

In the spring of 1968, MacLane took me along to a conference on category theory at the Batelle Institute in Seattle. David Buchsbaum, whose thesis had laid the foundation of Abelian categories, was to give a series of lectures on commutative algebra, and MacLane advised me to prepare for these lectures (at the time I knew no commutative algebra at all) and follow them closely. I was strongly drawn to

Buchsbaum for his great warmth and humanity and was also fascinated by his treatment of homological commutative algebra.

I volunteered to write the notes for the lectures and worked with Buchsbaum on them. Things were good in the first lectures—that treatment of the Koszul complex is preserved in my own book on commutative algebra. But in the last lectures, Buchsbaum turned to his thoughts on the resolution of lower-order minors of a generic matrix, then an open problem. I found the lectures muddy, impressionistic, and confusing, and suggested a reorganization. This did not go over well! Buchsbaum and I ultimately agreed to simply leave that material out.

The contact with Buchsbaum was decisive: I decided I would like to go to work near him, at Brandeis University, in Waltham near Boston. The academic job market in 1970, when I got my PhD, was quite different than it is today. This was just at the end of the period, sparked by the U.S. investment in research following Sputnik, when jobs in the sciences and mathematics were plentiful. There was no "Mathjobs," and people applied to few places. I initially applied only to Brandeis, but Nathan Jacobson, who knew of some of my work, wrote to Kaplansky to suggest that I apply to Yale too, and I followed Kap's advice. With offers from both places, I kept to my plan and accepted Brandeis.

I had met Monika Schwabe, a medical student, at the wedding of my cousin Bob to Monika's college friend Karen in the spring of 1966. I was 19 and living in Chicago. Monika was 23, living in NYC, and involved with others. I was interested, but the relationship did not develop. However, a few years later Monika thought that, after all, I had possibilities, and a courtship began. Ultimately Monika braved the disapproval of her medical school and her mother to take a year off to live with me in Chicago. At the end of the year, during a backpacking trip in the high Sierra, we decided to get married, and Monika returned to finish medical school in New York. In the Spring of 1970 I got my PhD, Monika got her MD, and we wed, in quick succession. We packed up and moved to Boston to take up my job at Brandeis, Monika's residency in child psychiatry at the Beth Israel Hospital, and a new life in an apartment in Central Square, Cambridge, a few blocks from the city hall where my parents had been married.

It was not only the job market that was different in 1970. Brandeis had been welcomed as the third member of the former Harvard-MIT colloquium, and the talks rotated, every third week in each place. At least as important for Monika and me as newcomers: there was a large and elaborate colloquium party, often with 30–50 people, nearly every week, at which we met "everyone" in the area. I was only later aware how much this institution, immensely valuable to me as a young mathematician, depended on the non-working spouses—wives, in every case—of senior members of the community. While the mathematician husbands listened to great (or not-so-great) talks, these women prepared and set out great quantities of food and drink, and smilingly welcomed the guests—who were their friends, too. The "job" of Faculty Wife is nearly gone, and largely unlamented, but in this regard, it served the mathematical world well, and certainly not only in Boston.

Mathematically speaking I was quite lonely during the first half-year in Boston. My thesis on noncommutative rings had led to a collaboration with Phillip Griffith (no s) about Artinian rings. Surprising as it now seems, at the time the only people in the Boston area interested in finite dimensional algebras, or indeed in any non-commutative algebra, were Bhama Srinivasan and the great but already elderly Richard Brauer—not a community for me. After a semester I figured out what to do: I went to Buchsbaum and brashly told him that I had a lot of energy that I would like to use to work on a problem with him! He accepted this proposition, and we began an intense collaboration of nearly 10 years, including some of my best work. In my second year at Brandeis Graham Evans arrived for a second postdoc at MIT. I knew Graham and his wife Kaye from graduate school—he graduated a year ahead of me, and we were good friends. Monika and I had admired their early married arrangements, unusual among the students then. During the summer, I joined Graham most days in his office at MIT. We ran a seminar together that included some odd characters. Once, one of the members came to the seminar with a bowl of water and a towel; as the seminar began (I was the speaker), he carefully washed and dried his face, folded his arms on his desk, put his head down, and went to sleep. Graham had a secretary/technical typist to himself that summer, and when we finally produced a manuscript (Basic Elements) she seemed glad to have something to do at last: she drew a cherub, celebrating with a trumpet on the cover page. More importantly, the next academic year we collaborated in solving a famous problem, proving that *Every Algebraic Set in n-space is the Intersection of n Hypersurfaces*. In the end, I think that this is what earned me tenure at Brandeis.

I had another stroke of good luck in my second year at Brandeis: an invitation to a workshop in Oberwolfach. At that time there were (informally!) two kinds of full professors in Germany: those with and those without an annual week reserved in Oberwolfach for them, their groups of students, and their invitees. Baer had such a week, and I had gone along the summer I visited Kegel. Now I was invited to the annual workshop run by Kasch, Rosenberg and Zelinsky—(I later learned that Kasch had noticed a paper I'd written as a graduate student giving a homological proof of a known theorem about when subrings of Noetherian rings are Noetherian.)

I was even given the opportunity to speak, and I explained my newest paper with Buchsbaum, *What Makes a Complex Exact*. Maurice Auslander, my senior colleague at Brandeis, was in the audience, and seemed impressed as well. Ever since, Oberwolfach has seemed a magical place for me, and I have made a point of going back whenever I could—at least 30 times over the intervening 50 years. With perhaps the best mathematical library on the planet, and a perfect setting for walks and afternoon cake, it is a great place to work with others as well as to listen to talks.

After the workshop I was invited to go for a week to Regensburg to visit Juergen Herzog, in the group under Ernst Kunz. I stayed with Juergen, and we became good friends. It was in his household that I first had to try to speak German—a poor showing. I lectured at the University (in English!), and Kunz was extremely kind to me.

Monika and I spent the summer of 1972 traveling. In my mind from that summer are the pleasure of the St. Andrews Mathematical Colloquium in Scotland (Halmos was the principal speaker) and a lecture by Verdier on a very general form of the Riemann-Roch theorem, in Aarhus, Denmark. I knew next to nothing about algebraic geometry, but I dared to approach Verdier afterwards and asked him what the Riemann-Roch theorem was good for. I got no answer—the question left him speechless with disbelief! We then spent the fall in Leeds, England where I visited Chris Robson and Monika worked as a "Registrar" (= Resident) at High Royds psychiatric hospital; she reported that all activity stopped, daily, for afternoon tea, just as in the math department. The earliest notes for Commutative Algebra with a View toward Algebraic Geometry, finally published in 1995, came from lectures I gave there (on Noether normalization).

In our study of free resolutions Buchsbaum and I made many computations by hand, using a method he knew. We hired Ray Zibman, an undergraduate, to program it, and quickly learned that it was NOT an algorithm—without human curation it often looped. At the same time Graham Evans at Urbana hired Mike Stillman to program the computation of free resolutions of homogeneous ideals "up to a given degree" by ordinary linear algebra. Fast forward to 1983, when Mike came to graduate school at Harvard. There he met Dave Bayer and learned about Gröbner bases: soon the program Macaulay was born. Mike was later a postdoc with me, and I felt that I was for many years Macaulay's "uncle," collaborating often with Bayer and Stillman on computations (in recent years, collaborating with Mike and Dan Grayson, I became a member of the Macaulay2 team itself.) Macaulay, Macaulay2, and the computations they enabled have played a major role in my mathematical career. As I said at a Bayer-Stillman 60th Birthday party, Macaulay is the only video game to which I've ever been addicted!

Backing up to 1974, it was time for me to run the tenure gauntlet at Brandeis. Given that I had a powerful advocate in Buchsbaum, one might think that it would be an easy process, and perhaps compared to other tenure processes it was but. . .during it one senior colleague told me outright that he would not vote for me—because I might attract students away from him! Another threatened to vote against me because of an old disagreement with Buchsbaum. These threats could have been fatal, since at that time the Brandeis department operated on unanimity. After the first threat I made a trip to Montpellier, where Buchsbaum was on sabbatical, to tell him of the situation and seek his help—he calmed me down. In the end, neither of the threats was realized, and the vote of the department was positive.

The university still had to grant me tenure, and the Dean proposed to delay a year because of the number of cases pending. I was eager to put it all behind me, and in the end the Dean (whom I didn't yet know) backed down. Dining with Department Chair Jerry Levine a week later, Jerry pointed out the Dean across the room and asked whether I wanted to go and say hello, or perhaps say thank you, but the situation was still so charged for me that I proposed to go and punch him, instead! (I did not do it). These experiences left me highly sympathetic with the bright young researchers who are regularly tortured before promotion.

Tenure gained, Monika and I went for a year to Paris. We traveled on the Queen Mary, and I watched her with pleasure as she drowsed, pregnant with our first child, on the deck. I had a Sloan Fellowship and was a visitor at the IHES; Monika practiced her French as a visitor to the famous Salpêtrière hospital and studied for her psychiatric Board exams, scheduled the same day that Daniel was supposed to be born! Daniel was 3 days late, and Monika, though great with child, took and passed the exam on schedule.

During the first days in Paris, I ran into Harold (Hal) Levine, a colleague from Brandeis. Dining at an old-fashioned restaurant on the left bank of the Seine, he told me a mathematical problem: how could one compute the local degree of a finite map germ? After a few experiments, I had a glimmer of an idea, and over a sleepless night I became sure: the degree would appear as the signature of a natural quadratic form. Hal and I worked this out over the next days.

Arriving for the first time, in the late afternoon, at the Institut des Hautes Etudes Scientifiques, the first person I encountered was Pierre Deligne, only three years my senior but already famous for his proof of the Weil conjectures just a year before. A person of the utmost kindness, Deligne made me feel at home. Though I was in awe and addressed him with "vous," he explained to me that all the French mathematicians "se tutoyent"—that is, use the familiar form of "you"—to one another, because, in the (rather recent) "old days" all the research mathematicians in France had been graduates of one school, the Ecole Normale Superieur. The former students treated each other familiarly (and no doubt lent a hand to each other in careers—the "old boy" network realized on top of Napoleon's system of meritocracy.) Deligne also took me for a wild bicycle ride down paths in the forest nearby—the first time I had done such a thing. I felt that I could ask Deligne any mathematical question, and get an illuminating answer tuned to my state of ignorance.

Of course I told Deligne about the computation of the local degree, the paper with Hal. He immediately asked how we took care of a certain point... that I had not noticed! I stumbled for a while, and finally came up with a plausible fix. Deligne had far more technique than I, and he saw that it could be made rigorous—but I had some learning to do to write the final version of the paper.

I once went with Monika to attend a presentation at the Salpêtrière, and the event left an impression beyond that of any math lecture: one after another, the presenting pathologist would fish a tagged brain from a barrel, and begin slicing with a chef's knife until he came to the fatal lesion, meantime telling the patient's final story ("Entered hospital at 4pm complaining of terrible headache, dead at 6pm... Mais oui!—now you see the cause!")

Mathematically, I had hoped to work with Lucien Szpiro, the most active person in French commutative algebra, but Szpiro ran a seminar listed on the bulletin board as "by invitation only," and when I asked for an invitation...he said, "No!" This rebuff proved a blessing: I fell in with a group around Bernard Teissier, Norbert A'Campo and Monique Lejeune-Jalabert, and began to broaden my interests into singularity theory, initially from Milnor's wonderful book. These became great friends, from whom I found a warm welcome that offset Szpiro's coldness.

When I wasn't going to Teissier's seminar at Paris 7, I would walk in the morning across the Luxembourg gardens to the Metro and take the train to Bures-Sur-Yvette and the IHES. Two seminar experiences stand out from that time:

Renee Thom was still active in that period, and at the first lecture of the year in his seminar he was the speaker. He began by writing down a result on the blackboard and saying that the seminar that semester would be devoted to the consequences of that result, of which the proof had been the subject of the previous semester. Someone in the back of the room raised his hand and proposed a counterexample to the theorem. This was discussed for a few minutes, and the conclusion was: yes, it is a counterexample. Unfazed, Thom continued: "Now we will get to the applications. . . ."

Late in my year in Paris, Daniel Quillen proved that projective modules over polynomial rings are free, solving a famous problem that had been proposed by Jean-Pierre Serre. The proof was, in the end, surprisingly direct, and I was appointed to give an exposition in the main seminar — with Serre himself in the audience. That I was nervous is a gross understatement, and indeed there came a point in the proof when I clutched and couldn't see how to proceed. . .for just a moment. In the end, all was well.

Our son Daniel was born in June. Monika and I returned to Boston soon afterwards. We had bought a small house in a beautiful setting, next to the Charles River at its widest part in Newton, just opposite Brandeis. We could canoe through most of the year—indeed, I took to commuting to Brandeis by canoe—and skate on the ice the rest of the time. Since neither of us could bear the idea of moving out of that spot, we eventually enlarged the house, and our daughter Alina was born. Monika had by this time finished her training (in both Child and Adult psychiatry) and had an active practice in a private office nearby.

During that period I taught a course from Milnor's book on hypersurface singularities and discovered what are now well-known as the matrix factorizations associated with a hypersurface. (This suddenly became my most quoted paper in 2004, when some physicists discovered that matrix factorizations could be used in String Theory.) I also chanced to hear a lecture at MIT by a young postdoc, Joe Harris, which changed my direction again: Harris spoke about the equations of canonical curves (are the quadrics generated by those of rank at most 4? Yes, as Mark Green subsequently proved.) He explained that lots of rank 4 quadrics come from special varieties, called rational normal scrolls, that contain the curves. I recognized the equations of the scrolls as being determinantal, and since Buchsbaum and I had often discussed determinantal ideals, I felt I had something to contribute. We chatted briefly after the lecture. Not much came of the conversation until later, though I did write my first algebraic geometry paper, using scrolls to give the equations of hyperelliptic curves soon afterwards.

During those first 10 years at Brandeis, the work with David Buchsbaum was by no means our only contact. David was deeply committed to Brandeis and to the Brandeis math department, which he had helped to build, and we spoke a great deal about department and university politics. Though I would not have guessed it then, these lessons were the beginning of my interest in such topics, leading much later to my work at MSRI and presidency of the American Mathematical Society. David told me of past struggles on behalf of the department with deans and provosts; of meetings with the President of Brandeis; and of tensions and repercussions within the department itself.

I found all this quite interesting, as a game of chess is interesting. But the first time I was chair of the department, in 1981–1982, it felt very heavy when I had to act myself! Worst of all were the negotiations over salaries. Brandeis math salaries were very low (we thought) compared to what they should have been, and the department's egalitarian culture prevented much forward motion. It seemed strategic to propose a larger increase for a smaller group, hoping to equalize another group in the next round. In my naiveté I found it dismaying that no one was willing not to be in the first group. . .so the plan caused only bad feelings, and never got off the ground. That was the only time in my career when I regularly came home thinking "I need a drink!"

Curiously, that first experience inoculated me against the stress: when I was department chair again in the 90's (and much later director of MSRI) I could more easily act as if the issues were burningly important, and then turn away and be free of the care when I didn't need to be "on." This skill has gotten stronger and stronger, and served me well over the years—though there are still issues that can keep me awake at night.

My second sabbatical was at the Sonderforschungsbereich (forerunner of the Max Planck Institute) headed by Friedrich Hirzebruch at Bonn University. Monika, who was born in Germany, was eager to spend a year nearer her origins and some of her German family with our two children, then 1 and 3 years old, and this helped determine the place. Chance again did its work in my favor: Antonius van de Ven, a well-known Dutch algebraic geometer, was visiting for most of the period, and we fell into a very pleasant collaboration. We would meet in the late morning at the Institute and work together until hunger reached us around 3 or 4; then we would stroll into town for food, and best of all, coffee and cake at one of the many Konditoreien, on which van de Ven was expert.

Van de Ven taught me a great deal about algebraic geometry, as Buchsbaum had about commutative algebra, and changed my direction again. Later in the year, Walter Neumann also spent some time in Bonn, and we began a collaboration that led to a year-long visit by Walter to Brandeis, and our book on knot theory.

It's perhaps worth saying something about my earlier attempts to learn algebraic geometry, as well. When I was a student at Chicago there was no algebraic geometer on the faculty, but I listened to two one-quarter courses that were relevant. In one, Kaplansky lectured from Chevalley's book on algebraic curves. . .except that there were no curves, only fields and valuations. I learned very little. Then the book of Demazure and Gabriel appeared: schemes as functors. MacLane, who liked anything with functors, convinced Swan to give a course on this approach. The high point of the course, reached after a long slog, was to prove: The Grassmannian Exists! Again, I learned nothing that could be called geometry.

When I came to Brandeis I was determined to keep trying. I listened to Paul Monsky's algebraic geometry course first. It was from the Weil foundations, already a little old-fashioned. Big fields and small ones but. . .no geometry that I could discern. Things went better as I listened to Mumford's course from what was to be his book, "Complex Projective Varieties I"—finally, some geometry! But I found I still could not understand any of the frequent algebraic geometry seminars in the area, all cohomology and schemes. Then in 1977 Hartshorne's book appeared. Since my background in commutative algebra was by then strong, I found it relatively easy to read, and over a summer I studied it end-to-end, doing nearly all the problems. That fall, at last, the subject was open to me, though I still had not done anything in it myself. I think of this 10-year effort, and eventual apprenticeship with van de Ven, when people tell me of their troubles in learning this many-sided subject!

Returning from Bonn, I reconnected with Joe Harris, and enjoyed a long and very fruitful collaboration with him around the applications of limit linear series—in particular, the proof that the moduli space of curves of genus $g \geq 24$ is of general type! Joe and I occasionally played Go—he was a much stronger player—and I imagined us growing old together, playing Go in the sun in Harvard square. This was not to be.

In the light of what was to come it is worth mentioning my two longest stays in Berkeley before coming here in earnest in 1997. In 1986 we moved across the country from Boston for a year's sabbatical. We loved living in Berkeley, and it was a particularly productive time for me mathematically. I was a member of what is now called the "complementary program" at MSRI, though I was well-connected to some of the people in the algebra program. At the end of the year Monika and I wondered whether we should try to return—but there seemed no ready way. We happily went back to Boston, and I to Brandeis. Again in 1994 I was a visitor to MSRI, for 7 weeks during a program related to algebraic geometry. I felt at the time that the program badly lacked senior presence. For example, there were no organizers in residence for most of the time I was there, and I was asked, even as a short-term visitor, to run the main seminar. Nevertheless, Berkeley/MSRI was a very attractive place to be (it helped that I house-sat in a wonderful old Berkeley mansion).

Since my contact with MSRI was so slight, it seemed a great stretch when I applied for the position of Director, a dark-horse candidate, 5 years later.

Before getting to that, I want to fill in a few relevant events. The first has to do with my book, *Commutative Algebra with a View Toward Algebraic Geometry*, published in 1995, and now by far my most quoted work. Writing this occupied me off and on for over 20 years: the earliest written material (on Noether normalization) is from a course I gave during my 1972 sabbatical in Leeds, and the ideas in my exposition of the Koszul complex date from my still earlier writing of the notes for Buchsbaum's lectures in 1968. Some of the chapters carry distinctive memories. For example, I can still picture a certain cafe near the Lago Maggiore where I sat for many hours figuring out how to write about Gorenstein rings, after a memorable workshop at the Monte Verità conference center! Springer was happy to publish the book, but the proofreading was a nightmare: for a book with both "Algebra" and "Algebraic" in the title, some typesetter decided that only one was necessary, and changed all occurrences of "Algebraic" with the push of a button. Unfortunately, I wasn't experienced enough to simply say "No!—start again," and instead spent painful hours unsuccessfully trying to catch all the changes and change them back. (As many readers will know, alas, many other slips remained.) Of course there are things I would write differently if I were starting over, but I feel very good about the success the book has had. It won the AMS' Steele Prize for Exposition in 2010. I hope someday to write a short version.

I made a couple of mathematical visits to IMPA, in Rio, and something happened during one of these visits that strongly influenced my future. I was a speaker at a national meeting where Vladimir Arnol'd (Dima to his friends) was giving a series of lectures; I listened with delight. I was lucky enough to stay in the same hotel, and one day at breakfast he mentioned a conjecture that he had made, having to do with the rigidity of algebras filtered by a sequence of ideals with 1-dimensional quotients. I thought the conjecture should be false and produced a counterexample a few days later. Arnol'd was very aware of the stylistic differences between the mathematics in different countries, and I think he was surprised, not so much that there was a counterexample, but that an American should have gotten his hands dirty enough to find it!

Dima and I became good friends, and had some adventures together: for example, during the conference there was a storm, despite which we went swimming together in the sea near the hotel. The waves were big, and the water was very rough. We were separated by a big wave, and when I dragged myself out onto the beach, I looked around. . .and didn't see Dima! I thought "Oh, no! has he drowned?" but he appeared, intact, a few moments later. We didn't go back in. . .. Later I visited Dima and his wife Ella in their flat in Paris. Rather than going 'round the corner to buy the wonderful cheese or croissants, Dima took me on a bike trip to collect berries and wild vegetables on the outskirts of Paris. Visiting Paris, a little later, I was a faithful member of his seminar. Though he sometimes didn't let the lecturer finish a sentence, his explanations were so good—and generally so much more intuitive than the lecturer's—that it was easy for me to forgive him. Another time Dima and Ella visited Monika and me at our vacation cottage in New Hampshire. We all liked to collect mushrooms—but Dima and Ella were far more efficient and far less fussy; they came home with much bigger bags, worms and all.

I'm convinced that Arnol'd's warm letter of recommendation—because I was the American who dared to challenge his conjecture, but also because of the work I had done with Harold Levine on topological degree—was one of the main reasons I was eventually hired at Berkeley and MSRI.

In 1996 I got a letter that changed my life, with the subject line "Retire in Berkeley?" Here's the background: on a visit to Berkeley a year or two before I spent a very pleasant evening over dinner with Bernd Sturmfels and his wife, Hyungsook Kim. I mentioned that Berkeley would be a great place to retire someday (an idea that Marie France Vigneras had once put forward to me). Now Bernd was suggesting that I apply for the job of MSRI Director! Brandeis had been in hard times financially for years, and there had been serious cuts in the mathematics department; I was thoroughly sick of fighting a losing battle to keep the department strong, and the idea of moving to Berkeley was extremely attractive.

However, I wasn't as sure that the administrative job of Director was a good fit for me. I had made only two visits, both times as a peripheral member of programs. And the administrative jobs I'd had—as department chair, as organizer of scientific meetings—were far smaller and simpler than the MSRI Directorship. It seems that the search committee agreed: I was not the first choice for the position, but when the first candidate withdrew, I was apparently the best candidate in a weak field. . .and got the job! Next, I needed a Deputy Director. Hugo Rossi had been Chair at Brandeis when I was first hired, and was now at the University of Utah. A wonderful inspiration led me to phone and offer him the position, and he accepted 24 hours later. Hugo was much a much more experienced administrator than I, and it proved a successful partnership.

There were bad feelings at the time between the Board of Trustees, headed by Elwyn Berlekamp, and the directorate, led by Bill Thurston. A charismatic and immensely brilliant mathematician, Thurston had succeeded in broadening the focus of MSRI in a very positive way, but the tension with the trustees was proving destructive. Fortunately the differences were not very deep, and the rifts were soon mended. Perhaps because of the contrast with Thurston, I got more credit than I deserved.

An immediate problem I faced was a new policy at the NSF: after 15 years of regular renewals, the NSF had decided that in 2000 there should be a "recompetition"—everyone in the world could apply to take the place of MSRI. The NSF was quite aware of the difficulties that MSRI had had under Thurston, and I felt that I might become "the Director who lost MSRI"! Our strongest competitor seemed to be the American Institute of Mathematics (AIM): John Fry, a wealthy businessman, had promised to put his money behind AIM, which was negotiating a partnership with Stanford University—a formidable coupling. Fortunately for MSRI, the AIM/Stanford partnership fell through. Moreover, Berlekamp and others on the Board contributed money to show that MSRI could also get non-government funding. To my great relief, MSRI won the recompetition.

My first two five-year terms at MSRI were intense and full of incident, which will have to be reported elsewhere. Joe Buhler, Michael Singer, Robert Megginson and Julius Zelmanowitz succeeded Hugo as Deputy Directors and I greatly enjoyed working with them. A first serious fundraising project, carried out with Development Director Jim Sotiros, gathered $12 million for a building expansion and renovation that included the grand Simons Auditorium and many other features. Ten years later architect William Glass and celebrated the achievement with a large-format book describing that process and some of the ideas that went into the design. When I retired from the Directorship in 2007, Jim Simons, whom I had recruited to the Board, gave MSRI its first major endowment gift: $5 million outright plus $5 million to match.

In 2007 Robert Bryant became Director of MSRI, and I happily began life as a regular Berkeley professor, but this did not last very long. Shortly before I retired as Director, Jim Simons had inquired about my plans, and soon asked me whether I would come to New York as Director of the Simons Foundation! At the time the Foundation was a very much smaller and less active organization than today: there was a group funding research on Autism, and a group running Math for America—technically a separate foundation. After looking at the situation, I said no.

A couple of years later Jim asked me to come and found a new Division of Mathematics and Physical Sciences (MPS) within the foundation, focusing on fundamental math, physics and computer science. By this time the Foundation had developed further and was about to move into much bigger quarters. My task would be to create a program that could grow over a few years to spend $40 million annually. This was too exciting to pass up! Starting in 2010, I began to spend about half time in NYC, and eventually worked on Foundation business full time. Collaborating with Jim on plans for the MPS program while enjoying the wonderful atmosphere created by Marilyn Simons' management of the Foundation proved a capstone experience.

In 2012–2013 MSRI hosted a year-long program on Commutative Algebra, and already when I joined the Simons Foundation I had decided to return to Berkeley for that program. Jim and Marilyn asked me to stay at the Foundation in New York. Monika and I weighed the possibilities, but we ultimately decided that it would be too disruptive for our family, and I declined. When I stepped down as Director of MPS, Jim and Marilyn asked me to join the Board of the Foundation, and I was delighted to continue in that role.

Robert Bryant's term came to an end in the summer of 2013, and I put myself forward as a candidate to succeed him. I have served as Director for two more terms, but will retire from that job in August of 2022, 25 years after coming to MSRI.

Among the changes at MSRI that I've overseen in these years, several stand out. First, some measurable increases: the number of Academic Sponsor departments has gone from 28 to 110; the annual budget has gone from about $3 million in 1997–1998 to about $12 million in 2019–2020; and the building expansion roughly doubled the floorspace of MSRI, now renamed Chern Hall in honor of the founding director.

With these new resources, the major scientific programs (typically two in each semester) have been significantly enriched. These already had an excellent reputation but, as mentioned above, they didn't always have enough senior participation, and this aspect has improved as we have moved resources from the "Complementary Program" into the main programs and raised endowment and other funds to improve the support of the members. We have emphasized long stays since these are the most productive. We have greatly increased the number of graduate summer schools we offer, now held all over the world, and started new programs emphasizing wide participation from currently under-represented groups, so that (in non-pandemic times) our building is full throughout the year. We have also added programs to serve mathematics in other ways, such as the support of Numberphile, the Mathical Book Prize, the National Math Festival, the prize for mathematical economics given jointly with the Chicago Mercantile Exchange, and the twice-yearly Congressional Briefings in Washington, DC. In these ways MSRI has strengthened both its core missions and its impact on the wider community.

Many people share the credit for these achievements: MSRI has a strong and well-functioning staff, and successive Deputy Directors, and especially Hélène Barcelo, Deputy Director for the last 10 years, have contributed immensely.

Until about 2000, the National Science Foundation was essentially the only financial supporter of MSRI, and it continues to be the most important source. Soon after I came to MSRI, the NSA began to contribute significantly, and continues to do so. Now many private individuals and foundations add their support, amounting to roughly half our budget, a healthy diversification.

I believe that a substantial endowment will be necessary to ensure MSRI's continued ability to serve the community no matter what shifts in federal funding there may be over time. In 2007 we had virtually no endowment. In 2021 the endowment (counting pledges) has reached about $75 million, and our current goal is $100 million.

One sign of the current health of MSRI is the great strength of the field of applicants to succeed me as Director in August 2022. I'm delighted that Tatiana Toro, the Craig McKibben & Sarah Merner Professor of Mathematics at the University of Washington was chosen, and that she has agreed to become MSRI's next Director! The Institute will be in good hands.

As for my own future, I'm looking forward to going back to the life of an ordinary professor at Berkeley, and to being back in the classroom. Despite the scarcity of time to concentrate on mathematics, I've managed to keep up research over these 25 years. From my work in this period, I'm particularly proud of the proof of the Boij-Soederberg conjecture and the analysis of Chow forms (both with Frank Schreyer, continued in work with Daniel Erman); of matrix factorizations for Cohen-Macaulay modules over complete intersections (with Irena Peeva); of the work on residual intersections (with Marc Chardin and Bernd Ulrich); and of the book on intersection theory, "3264 And All That" (with Joe Harris). I've had great pleasure throughout my career in collaborations (you can find the full list of my collaborators in my MathSci record), and I take a special pleasure in those with former students, with many of whom I've kept a close relationship. I look forward particularly to continuing collaborations and to advising of PhD students in the next years.

Berkeley, CA, USA — David Eisenbud
August 2021

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Bernstein-Sato Polynomials in Commutative Algebra

Josep Àlvarez Montaner, Jack Jeffries, and Luis Núñez-Betancourt

Dedicated to Professor David Eisenbud on the occasion of his seventy-fifth birthday.

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1 Introduction

The origin of the theory of D-modules can be found in the works of Kashiwara [70] and Bernstein [9, 10]. The motivation behind Bernstein's approach was to give a solution to a question posed by I. M. Gel'fand [55] at the 1954 edition of the International Congress of Mathematicians regarding the analytic continuation of the complex zeta function. The solution is based on the existence of a polynomial in a

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single variable satisfying a certain functional equation. This polynomial coincides with the b-function developed by Sato in the context of prehomogeneous vector spaces and it is known as the *Bernstein-Sato polynomial*.

The theory of D-modules grew up immensely in the 1970s and 1980s and fundamental results regarding Bernstein-Sato polynomials were obtained by Malgrange [91–93] and Kashiwara [71, 72]. For instance, they proved the rationality of the roots of the Bernstein-Sato polynomial and related the roots to the eigenvalues of the monodromy of the Milnor fiber associated to the singularity. Indeed this link is made through the concept of V-filtrations and the Hilbert-Riemann correspondence.

The theory of D-modules burst into commutative algebra through the seminal work of Lyubeznik [85] where he proved some finiteness properties of local cohomology modules. Nowadays, the theory of D-modules is an essential tool used in the area and has a prominent role. For example, the smallest integer root of the Bernstein-Sato polynomial determines the structure of the localization [143], and thus, using the Čech complex, it is a key ingredient in the computation of local cohomology modules [107–109, 111]. In addition, several results regarding finiteness aspects of local cohomology were obtained via the existence of the Bernstein-Sato polynomial and related techniques [1, 106]. Finally, there are several invariants that measure singularity that are related to the Bernstein-Sato polynomial [36, 38, 51, 102].

In this expository paper we survey several features of the theory of Bernstein-Sato polynomials relating to commutative algebra that have been developed over the last fifteen years or so. For instance, we discuss a version of Bernstein-Sato polynomial associated to ideals was introduced by Budur, Mustaţă, and Saito [36]. We also present a version of the theory for rings of positive characteristic developed by Mustaţă [100] and furthered by Bitoun [14] and Quinlan-Gallego [114]. Finally, we treat a recent extension to certain singular rings [1, 2, 63]. In addition, we discuss relations between the roots of the Bernstein-Sato polynomial and the poles of the complex zeta function [9, 10] and also the relation with multiplier ideals and jumping numbers [36, 38, 51].

In this survey we have extended a few results to greater generality than previously in the literature. For instance, we prove the existence of Bernstein-Sato polynomials of nonprincipal ideals for differentiably admissible algebras in Theorem 5.6. In Proposition 8.2, we show that Walther's proof [143] about generation of the localization as a D-module also holds for nonregular rings. In Theorem 8.6 we observe conditions sufficient for the finiteness of the associated primes of local cohomology in terms of the existence of the Bernstein-Sato polynomial; this covers several cases where this finiteness result is known. We point out that these results are likely expected by the experts and the proofs are along the lines of previous results. They are in this survey to expand the literature on this subject.

We have attempted to collect as many examples as possible. In particular, Sect. 4 is devoted to discuss several examples for classical Bernstein-Sato polynomials. In Sect. 5, we also provide several examples for nonprincipal ideals. In addition, we tried to collect many examples in other sections. We also attempted to present this material in an accessible way for people with no previous experience in the subject.

The theory surrounding the Bernstein-Sato polynomial is vast, and only a portion of it is discussed here. Our most blatant omission is the relation of the roots of Bernstein-Sato polynomials with the eigenvalues of the monodromy of the Milnor fiber [90]. Another crucial aspect of the theory that is not touched upon here is mixed Hodge modules [119]. We also do not discuss the different variants of the Strong Monodromy conjecture which relate the poles of the p-adic Igusa zeta function or the topological zeta function with the roots of the Bernstein-Sato polynomial [48, 68, 105]. We also omitted computational aspects of this subject [13, 107]. We do not discuss in depth several recent results obtained via representation theory [83, 84]. We hope the reader of this survey is inspired to learn more and we enthusiastically recommend the surveys of Budur [31, 33], Granger [57], Saito [122], and Walther [52, 144] for further insight.

2 Preliminaries

2.1 Differential Operators

Definition 2.1 Let $\mathbb{K}$ be a field of characteristic zero, and let A be either

- $A = \mathbb{K}[x_1, \dots, x_d]$, a polynomial ring over $\mathbb{K}$,
- $A = \mathbb{K}[\![x_1, \dots, x_d]\!]$, a power series ring over $\mathbb{K}$, or
- $A = \mathbb{C}\{x_1, \dots, x_d\}$, the ring of convergent power series in a neighborhood of the origin over $\mathbb{C}$.

The ring of differential operators $D_{A|\mathbb{K}}$ is the $\mathbb{K}$-subalgebra of $\text{End}_{\mathbb{K}}(A)$ generated by A and $\partial_1, \dots, \partial_d$, where ∂_i is the derivation $\frac{\partial}{\partial x_i}$.

In the polynomial ring case, $D_{A|\mathbb{K}}$ is the Weyl algebra. We refer the reader to books on this subject [46], [96, Chapter 15] for a basic introduction to this ring and its modules. The Weyl algebra can be described in terms of generators and relations as

$$D_{A|\mathbb{K}} = \frac{\mathbb{K}\langle x_1, \dots, x_d, \partial_1, \dots, \partial_d \rangle}{(\partial_i x_j - x_j \partial_i - \delta_{ij} \mid i, j = 1, \dots, d)},$$

where δ_{ij} is the Kronecker delta. As $D_{A|\mathbb{K}}$ is a subalgebra of $\text{End}_{\mathbb{K}}(A)$, $x_i \in D_{A|\mathbb{K}}$ is the operator of multiplication by x_i. The ring $D_{A|\mathbb{K}}$ has an order filtration

$$D^i_{A|\mathbb{K}} = \bigoplus_{\substack{a_1, \dots, a_d \in \mathbb{N} \\ b_1 + \dots + b_d \le i}} \mathbb{K} \cdot x_1^{a_1} \cdots x_d^{a_d} \partial_1^{b_1} \cdots \partial_d^{b_d}.$$

The associated graded ring of $D_{A|\mathbb{K}}$ with respect to the order filtration is a polynomial ring in $2d$ variables. Many good properties follow from this, for instance, the Weyl algebra is left-Noetherian, is right-Noetherian, and has finite global dimension.

In the generality of Definition 2.1, the associated graded ring of $D_{A|\mathbb{K}}$ with respect to the order filtration is a polynomial ring over A.

Rings of differential operators are defined much more generally as follows.

Definition 2.2 Let $\mathbb{K}$ be a field, and R be a $\mathbb{K}$-algebra.

- $D^0_{R|\mathbb{K}} = \operatorname{Hom}_R(R, R) \subseteq \operatorname{End}_{\mathbb{K}}(R)$.
- Inductively, we define $D^i_{R|\mathbb{K}}$ as

$$\{\delta \in \operatorname{End}_{\mathbb{K}}(R) \mid \delta \circ \mu - \mu \circ \delta \in D^{i-1}_{R|\mathbb{K}} \text{ for all } \mu \in D^0_{R|\mathbb{K}}\}.$$

- $D_{R|\mathbb{K}} = \bigcup_{i\in\mathbb{N}} D^i_{R|\mathbb{K}}$.

We call $D_{R|\mathbb{K}}$ the ring of ($\mathbb{K}$-linear) differential operators on R, and

$$D^0_{R|\mathbb{K}} \subseteq D^1_{R|\mathbb{K}} \subseteq D^2_{R|\mathbb{K}} \subseteq \cdots$$

the order filtration on $D_{R|\mathbb{K}}$.

We refer the interested reader to classic literature on this subject, e.g., [58, §16.8], [16], [104], and [96, Chapter 15]. We now present a few examples of rings of differential operators.

(i) If A is a polynomial ring over a field $\mathbb{K}$, then

$$D^i_{A|\mathbb{K}} = \bigoplus_{a_1+\cdots+a_d\le i} A \cdot \frac{\partial_1^{a_1}}{a_1!} \cdots \frac{\partial_d^{a_d}}{a_d!},$$

where $\dfrac{\partial_i^{a_i}}{a_i!}$ is the $\mathbb{K}$-linear operator given by

$$\frac{\partial_i^{a_i}}{a_i!}(x_1^{b_1}\cdots x_d^{b_d}) = \binom{b_i}{a_i} x_1^{b_1}\cdots x_i^{b_i-a_i}\cdots x_d^{b_d}.$$

Here, we identify an element $a \in A$ with the operator of multiplication by a. In particular, when $\mathbb{K}$ has characteristic zero, this definition agrees with Definition 2.1.

(ii) If R is essentially of finite type over $\mathbb{K}$, and $W \subseteq R$ is multiplicatively closed, then $D^i_{W^{-1}R|\mathbb{K}} = W^{-1}D^i_{R|\mathbb{K}}$. In particular, for $R = \mathbb{K}[x_1, \ldots, x_d]_f$,

$$D^i_{R|\mathbb{K}} = \bigoplus_{a_1+\cdots+a_d \le i} K[x_1, \ldots, x_d]_f \cdot \frac{\partial_1^{a_1}}{a_1!} \cdots \frac{\partial_d^{a_d}}{a_d!}.$$

(iii) If A is a polynomial ring over $\mathbb{K}$, and $R = A/\mathfrak{a}$ for some ideal $\mathfrak{a}$, then

$$D^i_{R|\mathbb{K}} = \frac{\{\delta \in D^i_{A|\mathbb{K}} \mid \delta(\mathfrak{a}) \subseteq \mathfrak{a}\}}{\mathfrak{a}D^i_{A|\mathbb{K}}}.$$

In general, rings of differential operators need not be left-Noetherian or right-Noetherian, nor have finite global dimension [12].

We note that if R is an $\mathbb{N}$-graded $\mathbb{K}$-algebra, then $D_{R|\mathbb{K}}$ admits a compatible $\mathbb{Z}$-grading via $\deg(\delta) = \deg(\delta(f)) - \deg(f)$ for all homogeneous $f \in R$.

Remark 2.3 The ring R is tautologically a left $D_{R|\mathbb{K}}$-module. Every localization of R is a $D_{R|\mathbb{K}}$-module as well. For $\delta \in D_{R|\mathbb{K}}$, and $f \in R$, we define $\delta^{(j),f}$ inductively as $\delta^{(0),f} = \delta$, and $\delta^{(j),f} = \delta^{(j-1),f} \circ f - f \circ \delta^{(j-1),f}$. The action of $D_{R|\mathbb{K}}$ on $W^{-1}R$ is then given by

$$\delta \cdot \frac{r}{f} = \sum_{j=0}^{t} \frac{\delta^{(j),f}(r)}{f^{j+1}}$$

for $\delta \in D^t_{R|\mathbb{K}}$, $r \in R$, $f \in W$.

Definition 2.4 Let $\mathfrak{a} \subseteq R$ be an ideal and $F = f_1, \ldots, f_\ell \in R$ be a set of generators for $\mathfrak{a}$. Let M be any R-module. The Čech complex of M with respect to F is defined by

$$\check{C}^\bullet(F; M): \quad 0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i,j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_\ell} \to 0,$$

where the maps on every summand are localization maps up to a sign. The local cohomology of M with support on $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M) = H^i(\check{C}^\bullet(F; M)).$$

This module is independent of the set of generators of $\mathfrak{a}$.

As a special case, $H^1_{(f)}(R) = \dfrac{R_f}{R}$.

The Čech complex of any left $D_{R|\mathbb{K}}$-module with respect to any sequence of elements is a complex of $D_{R|\mathbb{K}}$-modules, and hence the local cohomology of any $D_{R|\mathbb{K}}$-module with respect to any ideal is a left $D_{R|\mathbb{K}}$-module.

2.2 Differentiably Admissible $\mathbb{K}$-Algebras

In this subsection we introduce what is called now differentiably admissible algebras. To the best of our knowledge, this is the more general class of ring where the existence of the Bernstein-Sato polynomial is known. We follow the extension done for Tate and Dwork-Monsky-Washnitzer $\mathbb{K}$-algebras by Mebkhout and Narváez-Macarro [98], which was extended by the third-named author to differentiably admissible algebras [106]. We assume that $\mathbb{K}$ is a field of characteristic zero.

Definition 2.5 Let A be a Noetherian regular $\mathbb{K}$-algebra of dimension d. We say that A is differentiably admissible if

(i) $\dim(A_{\mathfrak{m}}) = d$ for every maximal ideal $\mathfrak{m} \subseteq A$,
(ii) $A/\mathfrak{m}$ is an algebraic extension of $\mathbb{K}$ for every maximal ideal $\mathfrak{m} \subseteq A$, and
(iii) $\mathrm{Der}_{A|\mathbb{K}}$ is a projective A-module of rank d such that the natural map

$$A_{\mathfrak{m}} \otimes_A \mathrm{Der}_{A|\mathbb{K}} \to \mathrm{Der}_{A_{\mathfrak{m}}|\mathbb{K}}$$

is an isomorphism.

Example 2.6 The following are examples of differentiably admissible algebras:

(i) Polynomial rings over $\mathbb{K}$.
(ii) Power series rings over $\mathbb{K}$.
(iii) The ring of convergent power series in a neighborhood of the origin over $\mathbb{C}$.
(iv) Tate and Dwork-Monsky-Washnitzer $\mathbb{K}$-algebras [98].
(v) The localization of a complete regular rings of mixed characteristic at the uniformizer [86, 106].
(vi) Localization of complete local domains of equal-characteristic zero at certain elements [112].

We note that in the Examples 2.6(i)–(iv), we have that $\mathrm{Der}_{A|\mathbb{K}}$ is free, because there exists $x_1, \ldots, x_d \in R$ and $\partial_1, \ldots, \partial_d \in \mathrm{Der}_{A|\mathbb{K}}$ such that $\partial_i(x_j) = \delta_{i,j}$ [94, Theorem 99].

Theorem 2.7 ([106, Theorem 2.7]) *Let A be a differentiably admissible $\mathbb{K}$-algebra. If there is an element $f \in A$ such that $R = A/fA$ is a regular ring, then R is a differentiably admissible $\mathbb{K}$-algebra.*

Remark 2.8 ([106, Proposition 2.10]) Let A be a differentiably admissible $\mathbb{K}$-algebra. Then,

(i) $D^n_{A|\mathbb{K}} = (\mathrm{Der}_{A|\mathbb{K}} + A)^n$, and
(ii) $D_{A|\mathbb{K}} \cong A\langle \mathrm{Der}_{A|\mathbb{K}} \rangle$.

Theorem 2.9 ([106, Section 2]) *Let A be a differentiably admissible $\mathbb{K}$-algebra. Then,*

(i) *$D_{A|\mathbb{K}}$ is left and right Noetherian;*
(ii) *$\mathrm{gr}_{D^{\bullet}_{A|\mathbb{K}}}(D_{A|\mathbb{K}})$ is a regular ring of pure graded dimension $2d$;*
(iii) *$\mathrm{gl.\,dim}(D_{A|\mathbb{K}}) = d$.*

We recall that for Noetherian rings the left and right global dimension are equal. In fact, this number is also equal to the weak global dimension [116, Theorem 8.27].

Definition 2.10 ([98]) We say that $D_{A|\mathbb{K}}$ is a *ring of differentiable type* if

(i) $D_{A|\mathbb{K}}$ is left and right Noetherian,
(ii) $\mathrm{gr}_{D^{\bullet}_{A|\mathbb{K}}}(D_{A|\mathbb{K}})$ is a regular ring of pure graded dimension $2d$, and
(iii) $\mathrm{gl.\,dim}(D_{A|\mathbb{K}}) = d$.

By Theorem 2.9, the ring of differential operators of any differentiably admissible algebra is a ring of differentiable type.

2.3 Log-Resolutions

Let $A = \mathbb{C}[x_1, \dots, x_d]$ be the polynomial ring over the complex numbers and set $X = \mathbb{C}^d$. A *log-resolution* of an ideal $\mathfrak{a} \subseteq A$ is a proper birational morphism $\pi : X' \to X$ such that X' is smooth, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_\pi)$ for some effective Cartier divisor F_π and $F_\pi + E$ is a simple normal crossing divisor where $E = \mathrm{Exc}(\pi) = \sum_{i=1}^r E_i$ denotes the exceptional divisor. We have a decomposition $F_\pi = F_{exc} + F_{aff}$ into its *exceptional* and *affine* parts which we denote

$$F_\pi := \sum_{i=1}^{r} N_i E_i + \sum_{j=1}^{s} N'_j S_j$$

with N_i, N'_j being nonnegative integers. For a principal ideal $\mathfrak{a} = (f)$ we have that $F_\pi = \pi^* f$ is the total transform divisor and S_j are the irreducible components of the *strict transform* of f. In particular $N'_j = 1$ for all j when f is reduced.

The *relative canonical divisor*

$$K_\pi := \sum_{i=1}^{r} k_i E_i$$

is the effective divisor with exceptional support defined by the Jacobian determinant of the morphism π.

There are many invariants of singularities that are defined using log-resolutions but for now we only focus on *multiplier ideals*. We introduce the basics on these invariants and we refer the interested reader to Lazarsfeld's book [77]. We also want to point out that there is an analytical definition of these ideals that we consider in Sect. 10.

Definition 2.11 The multiplier ideal associated to an ideal $\mathfrak{a} \subseteq A$ and $\lambda \in \mathbb{R}_{\geq 0}$ is defined as

$$\mathcal{J}(\mathfrak{a}^{\lambda}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil) = \{g \in A \mid \operatorname{ord}_{E_i}(\pi^* g) \geq \lfloor \lambda e_i - k_i \rfloor \ \ \forall i\}.$$

An important feature is that $\mathcal{J}(\mathfrak{a}^{\lambda})$ does not depend on the log-resolution $\pi : X' \to X$. Moreover we have $R^i \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil) = 0$ for all $i > 0$.

From its definition we deduce that multiplier ideals satisfy the following properties:

Proposition 2.12 *Let* $\mathfrak{a}, \mathfrak{b} \subseteq A$ *be ideals, and* $\lambda, \lambda' \in \mathbb{R}_{\geq 0}$. *Then,*

(i) *If* $\mathfrak{a} \subseteq \mathfrak{b}$, *then* $\mathcal{J}(\mathfrak{a}^{\lambda}) \subseteq \mathcal{J}(\mathfrak{b}^{\lambda})$.
(ii) *If* $\lambda < \lambda'$, *then* $\mathcal{J}(\mathfrak{a}^{\lambda'}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda})$.
(iii) *There exists* $\epsilon > 0$ *such that* $\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathcal{J}(\mathfrak{a}^{\lambda'})$, *if* $\lambda' \in [\lambda, \lambda + \epsilon)$.

Definition 2.13 We say that λ is a *jumping number* of $\mathfrak{a}$ if

$$\mathcal{J}(\mathfrak{a}^{\lambda}) \neq \mathcal{J}(\mathfrak{a}^{\lambda - \epsilon})$$

for every $\epsilon > 0$.

Notice that jumping numbers have to be rational and we have a nested filtration

$$A \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \cdots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \cdots$$

where the jumping numbers are the λ_i where we have a strict inclusion and $\lambda_1 = \operatorname{lct}(\mathfrak{a})$ is the so-called *log-canonical threshold*. Skoda's theorem states that $\mathcal{J}(\mathfrak{a}^{\lambda}) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda - 1})$ for all $\lambda > \dim A$.

Multiplier ideals can be generalized without much effort to the case where X is a normal $\mathbb{Q}$-Gorenstein variety over a field $\mathbb{K}$ of characteristic zero; one needs to consider $\mathbb{Q}$-divisors. Fix a log-resolution $\pi : X' \to X$ and let K_X be a canonical divisor on X which is $\mathbb{Q}$-Cartier with index m large enough. Pick a canonical divisor $K_{X'}$ in X' such that $\pi_* K_{X'} = K_X$. Then, the relative canonical divisor is

$$K_\pi = K_{X'} - \frac{1}{m} \pi^*(m K_X)$$

and the multiplier ideal of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ is $\mathcal{J}(\mathfrak{a}^{\lambda}) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil)$.

A version of multiplier ideals for normal varieties has been given by de Fernex and Hacon [47]. In this generality we ensure the existence of canonical divisors that are not necessarily $\mathbb{Q}$-Cartier. Then we may find some effective boundary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index m large enough. Then we consider

$$K_\pi = K_{X'} - \frac{1}{m}\pi^*(m(K_X + \Delta))$$

and the multiplier ideal $\mathcal{J}(\mathfrak{a}^\lambda, \Delta) = \pi_*\mathcal{O}_{X'}(\lceil K_\pi - \lambda F\rceil)$ which depends on Δ. This construction allowed de Fernex and Hacon to define the multiplier ideal $\mathcal{J}(\mathfrak{a}^\lambda)$ associated to $\mathfrak{a}$ and λ as the unique maximal element of the set of multiplier ideals $\mathcal{J}(\mathfrak{a}^\lambda, \Delta)$ where Δ varies among all the effective divisors such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A key point proved in [47] is the existence of such a divisor Δ that realizes the multiplier ideal as $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda, \Delta)$.

2.4 *Methods in Prime Characteristic*

In this section we recall definitions and results in prime characteristic that are used in Sect. 6. We focus on Cartier operators, differential operators, and test ideals.

Let R be a ring of prime characteristic p. The Frobenius map $F : R \to R$ is defined by $r \mapsto r^p$. We denote by $F^e_* R$ the R-module that is isomorphic to R as an Abelian group with the sum and the scalar multiplication is given by the e-th iteration of Frobenius. To distinguish the elements of $F^e_* R$ from R we write them as $F^e_* f$. In particular, $r \cdot F^e_* f = F^e_*(r^{p^e} f)$. Throughout this subsection we assume that $F^e_* R$ is a finitely generated R-module: that is, R is *F-finite*.

Definition 2.14 Let R be an F-finite ring.

(i) An additive map $\psi : R \to R$ is a *p^e-linear map* if $\psi(rf) = r^{p^e}\psi(f)$. Let $\mathcal{F}^e_R$ be the set of all the p^e-linear maps.
(ii) An additive map $\phi : R \to R$ is a *p^{-e}-linear map* if $\phi(r^{p^e} f) = r\phi(f)$. Let $\mathcal{C}^e_R$ be the set of all the p^{-e}-linear maps.
(iii) An additive map $\delta : R \to R$ is a *differential operator of level e* if it is R^{p^e}-linear. Let $D^{(e)}_R$ be the set of all differential operator of level e.

Differential operators relate to the Frobenius map in the following important way. This alternative characterization of the ring of differential operators is used in Sect. 6.

Theorem 2.15 ([131, Theorem 2.7], [148, Theorem 1.4.9]) *Let R be a finitely generated algebra over a perfect field $\mathbb{K}$. Then*

$$D_{R|\mathbb{K}} = \bigcup_{e\in\mathbb{N}} D^{(e)}_R = \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R, R).$$

In particular, any operator of degree $\leq p$ *is* R^p*-linear.*

Remark 2.16 Suppose that R is a reduced ring. Then, we may identify $F_*^e R = R^{1/p^e}$. We have that

(i) $\mathcal{F}_R^e \cong \mathrm{Hom}_R(R, F_*^e R)$,
(ii) $\mathcal{C}_R^e \cong \mathrm{Hom}_R(F_*^e R, R)$, and
(iii) $D_R^{(e)} \cong \mathrm{Hom}_R(F_*^e R, F_*^e R)$.

Remark 2.17 Let A be a regular F-finite ring. Then,

$$\mathcal{C}_A^e \otimes_A \mathcal{F}_A^e \cong D_A^{(e)}.$$

This can be reduced to the case of a complete regular local ring. In this case, one can construct explicitly a free basis for $F_*^e A$ as A is a power series over an F-finite field. Then, it follows that $\mathcal{C}_A^e$, $\mathcal{F}_A^e$, and $D_A^{(e)}$ are free A-modules. From this it follows that $\mathcal{C}_A^e \mathfrak{a} = \mathcal{C}_A^e \mathfrak{b}$ if and only $D_A^{(e)} \mathfrak{a} = D_A^{(e)} \mathfrak{b}$ for any two ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$.

We now focus on test ideals. These ideals have been a fundamental tool to study singularities in prime characteristic. They were first introduced by means of tight closure developed by Hochster and Huneke [64–67]. Hara and Yoshida [60] extended the theory to include test ideals of pairs. An approach to test ideals by means of Cartier operators was given by Blickle et al. [21, 22] in the case that A is a regular ring. Test ideals have also been studied in singular rings via Cartier maps [19, 20, 128].

Definition 2.18 Let A be an F-finite regular ring. The test ideal associated to an ideal $\mathfrak{a} \subseteq A$ and $\lambda \in \mathbb{R}_{\geq 0}$ is defined by

$$\tau_A(\mathfrak{a}^\lambda) = \bigcup_{e \in \mathbb{N}} \mathcal{C}_A^e \mathfrak{a}^{\lceil p^e \lambda \rceil}.$$

We note that the chain of ideals $\{\mathcal{C}_A^e I^{\lceil p^e \lambda \rceil}\}$ is increasing [21], and so, $\tau_A(\mathfrak{a}^\lambda) = \mathcal{C}_A^e \mathfrak{a}^{\lceil p^e \lambda \rceil}$ for $e \gg 0$.

We now summarize basic well-known properties of test ideals.

Proposition 2.19 ([21]) *Let* A *be an* F*-finite regular ring,* $\mathfrak{a}, \mathfrak{b} \subseteq A$ *ideals, and* $\lambda, \lambda' \in \mathbb{R}_{>0}$*. Then,*

(i) *If* $\mathfrak{a} \subseteq \mathfrak{b}$, *then* $\tau_A(\mathfrak{a}^\lambda) \subseteq \tau_A(\mathfrak{b}^\lambda)$.
(ii) *If* $\lambda < \lambda'$, *then* $\tau_A(\mathfrak{a}^{\lambda'}) \subseteq \tau_A(\mathfrak{a}^\lambda)$.
(iii) *There exists* $\epsilon > 0$, *such that* $\tau_A(\mathfrak{a}^\lambda) = \tau_A(\mathfrak{a}^{\lambda'})$, *if* $\lambda' \in [\lambda, \lambda + \epsilon)$.

In this way, to every ideal $\mathfrak{a} \subseteq A$ is associated a family of test ideals $\tau_A(\mathfrak{a}^\lambda)$ parameterized by real numbers $\lambda \in \mathbb{R}_{>0}$. Indeed, they form a nested chain of ideals. The real numbers where the test ideals change are called *F-jumping numbers*. To be precise:

Definition 2.20 Let A be an F-finite regular ring and let $\mathfrak{a} \subseteq A$ be an ideal. A real number λ is an *F-jumping number* of $\mathfrak{a}$ if

$$\tau_A(\mathfrak{a}^{\lambda}) \neq \tau_A(\mathfrak{a}^{\lambda-\epsilon})$$

for every $\epsilon > 0$.

3 The Classical Theory for Regular Algebras in Characteristic Zero

3.1 Definition of the Bernstein-Sato Polynomial of an Hypersurface

One basic reason that the ring of differential operators is useful is that we can use its action on the original ring to "undo" multiplication on A: we can bring nonunits in A to units by applying a differential operator. The Bernstein-Sato functional equation yields a strengthened version of this principle. Before we state the general definition, we consider what is perhaps the most basic example.

Example 3.1 Consider the variable $x \in \mathbb{K}[x]$. Differentiation by x not only sends x to 1, but, moreover, decreases powers of x:

$$\partial_x x^{s+1} = (s+1)x^s \qquad \text{for all } s \in \mathbb{N}. \tag{3.1}$$

In this equation, we were able to use one fixed differential operator to turn any power of x into a constant times the next smaller power. Moreover, the constant we obtain is a linear function of the exponent s.

The functional equation arises as a way to obtain a version for Eq. 3.1 for any element in a $\mathbb{K}$-algebra.

Definition 3.2 Let $\mathbb{K}$ a field of characteristic zero and A be a regular $\mathbb{K}$-algebra. A *Bernstein-Sato functional equation* for an element f in A is an equation of the form

$$\delta(s) f^{s+1} = b(s) f^s \qquad \text{for all } s \in \mathbb{N},$$

where $\delta(s) \in D_{A|\mathbb{K}}[s]$ is a polynomial differential operator, and $b(s) \in \mathbb{K}[s]$ is a polynomial. We say that such a functional equation is nonzero if $b(s)$ is nonzero; this implies that $\delta(s)$ is nonzero as well. We may say that $(\delta(s), b(s))$ as above *determine a functional equation for f*.

Theorem 3.3 *Any nonzero element $f \in A$ satisfies a nonzero Bernstein-Sato functional equation. That is, there exist $\delta(s) \in D_{A|\mathbb{K}}[s]$ and $b(s) \in \mathbb{K}[s] \smallsetminus \{0\}$ such that*

$$\delta(s) f^{s+1} = b(s) f^s \qquad \textit{for all } s \in \mathbb{N}.$$

We pause to make an observation. Fix $f \in A$, and suppose that $(\delta_1(s), b_1(s))$ and $(\delta_2(s), b_2(s))$ determine two Bernstein-Sato functional equations for f:

$$\delta_i(s) f^{s+1} = b_i(s) f^s \qquad \text{for all } s \in \mathbb{N} \text{ for } i = 1, 2.$$

Let $c(s) \in \mathbb{K}[s]$ be a polynomial. Then

$$(c(s)\delta_1(s) + \delta_2(s)) f^{s+1} = (c(s)b_1(s) + b_2(s)) f^s \qquad \text{for all } s \in \mathbb{N}.$$

It follows that, for $f \in A$,

$$\{b(s) \in \mathbb{K}[s] \mid \exists \delta(s) \in D_{A|\mathbb{K}}[s] \text{ such that } \delta(s) f^{s+1} = b(s) f^s \text{ for all } s \in \mathbb{N}\}$$

is an ideal of $\mathbb{K}[s]$. By Theorem 3.3, this ideal is nonzero.

Definition 3.4 The *Bernstein-Sato polynomial* of $f \in A$ is the minimal monic generator of the ideal

$$\{b(s) \in \mathbb{K}[s] \mid \exists\, \delta(s) \in D_{A|\mathbb{K}}[s] \text{ such that } \delta(s) f^{s+1} = b(s) f^s \text{ for all } s \in \mathbb{N}\} \subset \mathbb{K}[s].$$

This polynomial is denoted $b_f(s)$.

The polynomial described in Definition 3.4 was originally introduced in independent constructions by Bernstein [9, 10] to establish meromorphic extensions of distributions, and by Sato [125, 126] as the b-function in the theory of prehomogeneous vector spaces.

3.2 The D-Modules $D_{A|\mathbb{K}}[s]\boldsymbol{f^s}$ and $A_f[s]\boldsymbol{f^s}$

For the proof of Theorem 3.3 and for many applications, it is preferable to consider the Bernstein-Sato functional equation as a single equality in a $D_{A|\mathbb{K}}[s]$-module where f^s is replaced by a formal power "$\boldsymbol{f^s}$." We are interested in two such modules that are closely related:

$$D_{A|\mathbb{K}}[s]\boldsymbol{f^s} \subseteq A_f[s]\boldsymbol{f^s}.$$

We give a couple different constructions of each. For much more on these modules, we refer the interested reader to Walther's survey [144].

3.2.1 Direct Construction of $A_f[s]\boldsymbol{f^s}$

Definition 3.5 We define the left $D_{A_f|\mathbb{K}}[s]$-module $A_f[s]\boldsymbol{f^s}$ as follows:

- As an $A_f[s]$-module, $A_f[s]\boldsymbol{f^s}$ is a free cyclic module with generator $\boldsymbol{f^s}$.
- Each partial derivative ∂_i acts by the rule

$$\partial_i(a(s)\boldsymbol{f^s}) = \left(\partial_i(a(s)) + \frac{sa(s)\partial_i(f)}{f}\right)\boldsymbol{f^s}$$

  for $a(s) \in A_f[s]$.

We often consider this as a module over the subring $D_{A|\mathbb{K}}[s] \subseteq D_{A_f|\mathbb{K}}[s]$ by restriction of scalars. To justify that this gives a well-defined $D_{A_f|\mathbb{K}}[s]$-module structure, one checks that $\partial_i(x_i a(s)\boldsymbol{f^s}) = x_i\partial_i(a(s)\boldsymbol{f^s}) + a(s)\boldsymbol{f^s}$.

From the definition, we see that this module is compatible with specialization $s \mapsto n \in \mathbb{Z}$. Namely, for all $n \in \mathbb{Z}$, define the specialization maps

$$\theta_n : A_f[s]\boldsymbol{f^s} \to A_f \quad \text{by} \quad \theta_n(a(s)\boldsymbol{f^s}) = a(n)f^n$$

and

$$\pi_n : D_{A_f|\mathbb{K}}[s] \to D_{A_f|\mathbb{K}} \quad \text{by} \quad \pi_n(\delta(s)) = \delta(n).$$

We then have $\pi_n(\delta(s)) \cdot \theta_n(a(s)\boldsymbol{f^s}) = \theta_n(\delta(s) \cdot a(s)\boldsymbol{f^s})$. This simply follows from the fact that the formula for $\partial_i(a(s)\boldsymbol{f^s})$ in the definition agrees with the power rule for derivations when s is replaced by an integer n and $\boldsymbol{f^s}$ is replaced by f^n.

3.2.2 Local Cohomology Construction of $A_f[s]\boldsymbol{f^s}$

It is also advantageous to consider $A_f[s]\boldsymbol{f^s}$ as a submodule of a local cohomology module.

Consider the local cohomology module $H^1_{(f-t)}(A_f[t])$, where t is an indeterminate over A. As an A_f-module, this is free with basis

$$\left\{\left[\frac{1}{f-t}\right], \left[\frac{1}{(f-t)^2}\right], \left[\frac{1}{(f-t)^3}\right], \dots\right\} : \tag{3.2}$$

indeed, these are linearly independent over A_f, and we can rewrite any element

$$\left[\frac{p(t)}{(f-t)^m}\right] \in H^1_{(f-t)}(A_f[t]), \text{ with } p(t) \in A_f[t]$$

in this form by writing $t = f - (f - t)$, expanding, and collecting powers of $f - t$. By Remark 2.3, $H^1_{(f-t)}(A_f[t])$ is naturally a $D_{A_f[t]|\mathbb{K}}$-module.

Consider the subring $D_{A_f|\mathbb{K}}[-\partial_t t] \subseteq D_{A_f[t]|\mathbb{K}}$. We note that $-\partial_t t$ commutes with every element of $D_{A_f|\mathbb{K}}$ and that $-\partial_t t$ does not satisfy any nontrivial algebraic relation over $D_{A_f|\mathbb{K}}$, so $D_{A_f|\mathbb{K}}[-\partial_t t] \cong D_{A_f|\mathbb{K}}[s]$ for an indeterminate s. We consider $H^1_{(f-t)}(A_f[t])$ as a $D_{A_f|\mathbb{K}}[s]$-module via this isomorphism. Namely,

$$(\delta_m s^m + \cdots + \delta_0) \cdot \left[\frac{a}{(f-t)^n}\right] = (\delta_m(-\partial_t t)^m + \cdots + \delta_0) \cdot \left[\frac{a}{(f-t)^n}\right],$$

where the action on the right is the natural action on the localization.

Lemma 3.6 *The elements*

$$\left\{(-\partial_t t)^n \cdot \left[\frac{1}{f-t}\right] \;\middle|\; n \in \mathbb{N}\right\}$$

are A_f-linearly independent in $H^1_{(f-t)}(A[t]) \subseteq H^1_{(f-t)}(A_f[t])$.

Proof We show by induction on n that the coefficient of $(-\partial_t t)^n \cdot \left[\frac{1}{f-t}\right]$ corresponding to the element $\left[\frac{1}{(f-t)^{n+1}}\right]$ in the A_f-basis (3.2) is nonzero. This is trivial if $n = 0$, and the inductive step follows from the formula

$$-\partial_t t \;\cdot\; \left[\frac{a}{(f-t)^n}\right] = \left[\frac{(n-1)a}{(f-t)^n}\right] + \left[\frac{-nfa}{(f-t)^{n+1}}\right]. \qquad \square$$

Proposition 3.7 *The map*

$$\alpha : A_f[s]\boldsymbol{f}^s \to H^1_{(f-t)}(A_f[t]) \quad \textit{given by} \quad \alpha(a(s)\boldsymbol{f}^s) = a(-\partial_t t) \cdot \left[\frac{1}{f-t}\right]$$

is an injective homomorphism of $D_{A_f|\mathbb{K}}[s]$-modules.

Proof Injectivity of α follows from Lemma 3.6. We just need to check that this map is linear with respect to the action of $D_{A_f|\mathbb{K}}[s]$. We have that α is $A_f[s]$-linear; we just need to check that α commutes with the derivatives ∂_i. We compute that

$$\alpha(\partial_i \boldsymbol{f^s}) = \alpha\left(\frac{s\partial_i(f)}{f}\boldsymbol{f^s}\right) = -\partial_t t\frac{\partial_i(f)}{f}\left[\frac{1}{f-t}\right]$$
$$= -\partial_i(f)\partial_t\left[\frac{1}{f-t}\right] = \partial_i\left[\frac{1}{f-t}\right],$$

where in the penultimate equality we used that

$$t\left[\frac{1}{f-t}\right] = (f-(f-t))\left[\frac{1}{f-t}\right] = f\left[\frac{1}{f-t}\right]. \qquad \square$$

We note that α is not surjective in general.

As $A_f[s]\boldsymbol{f^s}$ is generated by $\boldsymbol{f^s}$ as a $D_{A_f|\mathbb{K}}[s]$-module, Proposition 3.7 yields the following result.

Proposition 3.8 *The $D_{A_f|\mathbb{K}}[s]$-module $A_f[s]\boldsymbol{f^s}$ is isomorphic to the submodule $D_{A_f|\mathbb{K}}[s]\cdot\left[\frac{1}{f-t}\right] \subseteq H^1_{(f-t)}(A_f[t])$, where s acts on the latter by $-\partial_t t$.*

3.2.3 Constructions of the Module $D_{A|\mathbb{K}}[s]\boldsymbol{f^s}$

We now give three constructions of the submodule $D_{A|\mathbb{K}}[s]\boldsymbol{f^s}$ of the module $A_f[s]\boldsymbol{f^s}$. The first is exactly as suggested by the notation.

Definition 3.9 We define $D_{A|\mathbb{K}}[s]\boldsymbol{f^s}$ as the $D_{A|\mathbb{K}}[s]$-submodule of $A_f[s]\boldsymbol{f^s}$ generated by the element $\boldsymbol{f^s}$.

Proposition 3.10 *There is an isomorphism*

$$D_{A|\mathbb{K}}[s]\boldsymbol{f^s} \cong \frac{D_{A|\mathbb{K}}[s]}{\{\delta(s)\in D_{A|\mathbb{K}}[s] \mid \delta(n)f^n = 0 \textit{ for all } n\in\mathbb{N}\}}.$$

Proof We just need to show that the annihilator of $\boldsymbol{f^s}$ in $A_f[s]\boldsymbol{f^s}$ is

$$\{\delta(s)\in D_{A|\mathbb{K}}[s] \mid \delta(n)f^n = 0 \text{ for all } n\in\mathbb{N}\}.$$

We can write $\delta(s)\boldsymbol{f^s}$ as $p(s)\boldsymbol{f^s}$ for some $p(s)\in A_f[s]$. Observe that

$$\begin{aligned} p(s)\boldsymbol{f^s} = 0 &\Leftrightarrow p(s) = 0 \\ &\Leftrightarrow p(n) = 0 \text{ for all } n\in\mathbb{N} \\ &\Leftrightarrow p(n)f^n = 0 \text{ for all } n\in\mathbb{N} \\ &\Leftrightarrow \theta_n(p(s)\boldsymbol{f^s}) = 0 \text{ for all } n\in\mathbb{N}. \end{aligned}$$

Then, $\delta(s)\boldsymbol{f}^s = 0$ if and only if $0 = \theta_n(\delta(s)\boldsymbol{f}^s) = \delta(n)f^n$ for all $n \in \mathbb{N}$, as required. □

Note that this is using characteristic zero in a crucial way: we need that a polynomial that has infinitely many zeroes (or that is identically zero on $\mathbb{N}$) is the zero polynomial.

Remark 3.11 An argument analogous to the above shows that, for $\delta(s) \in D_{A|\mathbb{K}}[s]$, the following are equivalent:

(i) $\delta(s)\boldsymbol{f}^s = 0$ in $A_f[s]\boldsymbol{f}^s$;
(ii) $\delta(n)f^n = 0$ in A for all $n \in \mathbb{N}$;
(iii) $\delta(n)f^n = 0$ in A_f for all $n \in \mathbb{Z}$;
(iv) $\delta(n)f^n = 0$ in A_f for infinitely many $n \in \mathbb{Z}$.

Likewise, by shifting the evaluations, ones sees this is equivalent to:

(v) $\delta(s+t)f^t\boldsymbol{f}^s = 0$ in $A_f[s]\boldsymbol{f}^s$.

Finally, we observe that $D_{A|\mathbb{K}}[s]\boldsymbol{f}^s$ can be constructed via local cohomology as in Sect. 3.2.2. By restricting the isomorphism of Proposition 3.8, we obtain the following result.

Proposition 3.12 *The $D_{A|\mathbb{K}}[s]$-module $D_{A|\mathbb{K}}[s]\boldsymbol{f}^s$ is isomorphic to the submodule $D_{A|\mathbb{K}}[s] \cdot \left[\frac{1}{f-t}\right] \subseteq H^1_{(f-t)}(A[t])$, where s acts on the latter by $-\partial_t t$.*

Proposition 3.13 *The following are equal:*

(i) *The Bernstein-Sato polynomial of f;*
(ii) *The minimal polynomial of the action of s on $\dfrac{D_{A|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{K}}[s]f\boldsymbol{f}^s}$;*
(iii) *The minimal polynomial of the action of $-\partial_t t$ on $[\frac{1}{f-t}]$ in*

$$\frac{D_{A|\mathbb{K}}[-\partial_t t]\cdot[\frac{1}{f-t}]}{D_{A|\mathbb{K}}[-\partial_t t]\cdot f[\frac{1}{f-t}]};$$

(iv) *The monic element of smallest degree in $\mathbb{K}[s] \cap (Ann_{D[s]}(\boldsymbol{f}^s) + D_{A|\mathbb{K}}[s]f)$.*

Proof The equality between the first two follows from the definition. The equality between the second and the third follows from the previous proposition. For the equality between the second and the fourth, we observe that

$$\frac{D_{A|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{K}}[s]f\boldsymbol{f}^s} \cong \operatorname{coker}\left(\frac{D_{A|\mathbb{K}}[s]}{\operatorname{Ann}_{D[s]}(\boldsymbol{f}^s)} \xrightarrow{\cdot f} \frac{D_{A|\mathbb{K}}[s]}{\operatorname{Ann}_{D[s]}(\boldsymbol{f}^s)}\right)$$

$$\cong \frac{D_{A|\mathbb{K}}[s]}{\operatorname{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{A|\mathbb{K}}[s]f}.$$

□

Remark 3.14 For any rational number α, we can consider the $D_{R|\mathbb{K}}$-modules $D_{R|\mathbb{K}} f^{\alpha}$ and $A_f f^{\alpha}$ by specializing $s \mapsto \alpha$ in the $D_{R|\mathbb{K}}[s]$-modules $D_{R|\mathbb{K}}[s]\boldsymbol{f}^s$ and $A_f[s]\boldsymbol{f}^s$. These modules are important in D-module theory, but we do not focus on them in depth here.

We end this subsection with equivalent characterizations on $A_f[s]\boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ for f to have a nonzero functional equation. This lemma plays a role in Corollary 3.21 and Theorem 3.26.

Lemma 3.15 ([2, Proposition 2.18]) *Fix an element $f \in A$. Then, the following are equivalent*:

(i) *There exists a Bernstein–Sato polynomial for f;*
(ii) *$A_f[s]\boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ is generated by $\boldsymbol{f}^s$ as a $D_{A(s)|\mathbb{K}(s)}$-module;*
(iii) *$A_f[s]\boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ is a finitely-generated $D_{A(s)|\mathbb{K}(s)}$-module.*

Proof We first show that (i) implies (ii). For every $m \in \mathbb{Z}$, we have an isomorphism of $D_{A(s)|\mathbb{K}(s)}$-modules

$$\psi_m : A_f \boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s) \to A_f \boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$$

defined by

$$\frac{r(s)h}{f^{\alpha}}\boldsymbol{f}^s \mapsto \frac{r(s-m)h}{f^{\alpha+m}}\boldsymbol{f}^s.$$

Applying these isomorphism to the functional equation, we obtain that $\frac{1}{f^m}\boldsymbol{f}^s \in D_{A(s)|\mathbb{K}(s)}\boldsymbol{f}^s$.

Since (ii) implies (iii) follows from definition, we focus in proving that (iii) implies (i). First we note that (iii) implies that $\frac{1}{f^m}\boldsymbol{f}^s$. generates $A_f\boldsymbol{f}^s \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ for some $m \in \mathbb{N}$. Then, $\frac{1}{f^{m+1}}\boldsymbol{f}^s \in D_{A(s)|\mathbb{K}(s)}\frac{1}{f^m}\boldsymbol{f}^s$. Then, there exists $\delta(s) \in D_{A(s)|\mathbb{K}(s)}$ such that

$$\delta(s)\frac{1}{f^m}\boldsymbol{f}^s = \frac{1}{f^{m+1}}\boldsymbol{f}^s.$$

After clearing denominators and shifting by $-m-1$, we obtain a functional equation. □

3.3 Existence of Bernstein-Sato Polynomials for Polynomial Rings via Filtrations

In this subsection $A = \mathbb{K}[x_1, \ldots, x_d]$ is a polynomial ring over a field, $\mathbb{K}$, of characteristic zero. This was proved in this case by Bernstein [9, 10]. We show the existence of the Bernstein-Sato polynomial using the strategy of Coutinho's book [46].

We define the *Bernstein filtration* of A, $\mathcal{B}^{\bullet}_{A|\mathbb{K}}$ as

$$\mathcal{B}^{i}_{A|\mathbb{K}} = \bigoplus_{a_1+\cdots+a_d+b_1+\cdots+b_d\le i} \mathbb{K}\cdot x_1^{a_1}\cdots x_d^{a_d}\partial_1^{b_1}\cdots\partial_d^{b_d}.$$

We note that

(i) $\dim_{\mathbb{K}} \mathcal{B}^{i}_{A|\mathbb{K}} = \binom{n+i}{i} < \infty$,
(ii) $D_{A|\mathbb{K}} = \bigcup_{i\in\mathbb{N}} \mathcal{B}^{i}_{A|\mathbb{K}}$,
(iii) $\mathcal{B}^{i}_{A|\mathbb{K}}\mathcal{B}^{j}_{A|\mathbb{K}} = \mathcal{B}^{i+j}_{A|\mathbb{K}}$, and
(iv) $[\mathcal{B}^{i}_{A|\mathbb{K}}, \mathcal{B}^{j}_{A|\mathbb{K}}] \subseteq \mathcal{B}^{i+j-2}_{A|\mathbb{K}}$.

We observe that the associated graded ring of the filtration, $\operatorname{gr}(\mathcal{B}^{\bullet}_{A|\mathbb{K}}, D_{A|\mathbb{K}})$, is isomorphic to $\mathbb{K}[x_1,\ldots,x_d,y_1,\ldots,y_d]$.

Given a left, $D_{A|\mathbb{K}}$-module, M, we say that a filtration $\Gamma^{\bullet}$ of $\mathbb{K}$-vector spaces is $\mathcal{B}^{\bullet}_{A|\mathbb{K}}$-compatible if

(i) $\dim_{\mathbb{K}} \Gamma^{i} < \infty$,
(ii) $M = \bigcup_{i\in\mathbb{N}} \Gamma^{i}$, and
(iii) $\mathcal{B}^{i}_{A|\mathbb{K}}\Gamma^{j} \subseteq \Gamma^{i+j}$.

In this manuscript, by a $D_{A|\mathbb{K}}$-module, unless specified, we mean a left $D_{A|\mathbb{K}}$-module.

We observe that $\operatorname{gr}(\Gamma^{\bullet}, M)$ is a graded $\operatorname{gr}(\mathcal{B}^{\bullet}_{A|\mathbb{K}}, D_{A|\mathbb{K}})$-module. Moreover, M is finitely generated as a $D_{A|\mathbb{K}}$-module if and only if there exists a filtration $\Gamma^{\bullet}$ such that $\operatorname{gr}(\Gamma^{\bullet}, M)$ is finitely generated as a $\operatorname{gr}(\mathcal{B}^{\bullet}_{A|\mathbb{K}}, D_{A|\mathbb{K}})$-module. In this case, we say that Γ is a *good filtration for M*.

Proposition 3.16 *Let M be a finitely generated $D_{A|\mathbb{K}}$-module. Let G denote the associated graded ring with respect to the Bernstein filtration. Let $\Gamma^{\bullet}_1$ and $\Gamma^{\bullet}_2$ be two good filtrations for M. Then,*

$$\sqrt{\operatorname{Ann}_G \operatorname{gr}(\Gamma^{\bullet}_1, M)} = \sqrt{\operatorname{Ann}_G \operatorname{gr}(\Gamma^{\bullet}_2, M)}.$$

Thanks to the previous result we are able to define the *dimension* of a finitely generated $D_{A|\mathbb{K}}$-module M as

$$\dim_D(M) = \dim_G\left(\frac{G}{\operatorname{Ann}_G \operatorname{gr}(\Gamma^{\bullet}, M)}\right).$$

Theorem 3.17 (Bernstein's Inequality) *Let M be a finitely generated $D_{A|\mathbb{K}}$-module. Then,*

$$d \leq \dim_D(M) \leq 2d.$$

Definition 3.18 We say that a finitely generated $D_{A|\mathbb{K}}$-module, M, is *holonomic* if either $\dim_D(M) = d$ or $M = 0$.

Theorem 3.19 *Every holonomic $D_{A|\mathbb{K}}$-module has finite length as $D_{A|\mathbb{K}}$-module.*

Proof Let $M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_t \subseteqq M$ be a proper chain of $D_{A|\mathbb{K}}$-submodules. Let $\Gamma^\bullet$ be a good filtration. We note that $\Gamma^i_j = \Gamma^i \cap M_j$ is a good filtration on M_j. In addition, $\overline{\Gamma}^i_j = \phi_j(\Gamma^i_j)$, where $\pi : M_j \to M_j/M_{j-1}$ is the quotient map, is a good filtration for M_j/M_{j-1}. We have the following identity of Hilbert-Samuel multiplicities of graded $\mathrm{gr}(\mathcal{B}^\bullet_{A|\mathbb{K}}, D_{A|\mathbb{K}})$-modules:

$$\mathrm{e}(\mathrm{gr}(\Gamma^\bullet, M)) = \sum_{j=1}^{t} \mathrm{e}(\mathrm{gr}(\overline{\Gamma}^\bullet_j, M_j/M_{j-1})).$$

Since the multiplicities are positive integers, we have that $t \leq \mathrm{e}(\mathrm{gr}(\Gamma^\bullet, M))$, and so, the length of M as a $D_{R|\mathbb{K}}$-module is at most $\mathrm{e}(\mathrm{gr}(\Gamma^\bullet, M))$. □

Theorem 3.20 *Given any nonzero polynomial $f \in A$, $A_f[s]\boldsymbol{f^s} \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ is a holonomic $D_{A(s)|\mathbb{K}(s)}$-module.*

Proof Let $t = \deg(f)$. We set a filtration

$$\Gamma_i = \frac{1}{f^i}\{g \in A(s) \mid \deg(g) \leq (t+1)i\}\boldsymbol{f^s}.$$

We note that Γ_i is a good filtration such that the associated graded of $A_f[s]\boldsymbol{f^s} \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ has dimension d. □

Corollary 3.21 ([10]) *Given any nonzero polynomial $f \in A$, the Bernstein-Sato polynomial of f exists.*

Proof This follows from Proposition 3.15 and Theorems 3.19 and 3.20. □

3.4 Existence of Bernstein-Sato Polynomials for Differentiably Admissible Algebras via Homological Methods

In this subsection we prove the existence of Bernstein-Sato polynomials of differentiably admissible $\mathbb{K}$-algebras (see Sect. 2.2). We assume that $\mathbb{K}$ is a field of characteristic zero.

Definition 3.22 Let A be a differentiably admissible $\mathbb{K}$-algebra. Let $M \neq 0$ be a finitely generated $D_{A|\mathbb{K}}$-module. We define

$$\text{grade}_{D_{A|\mathbb{K}}}(M) = \inf\{j \mid \text{Ext}^j_{D_{A|\mathbb{K}}}(M, D_{A|\mathbb{K}}) \neq 0\}.$$

We note that $\text{grade}_{D_{A|\mathbb{K}}}(M) \leq \text{gl.}\dim(D_{R|\mathbb{K}}) = d$.

Remark 3.23 Given a finitely generated $D_{A|\mathbb{K}}$-module, we can define the filtrations compatible with the order filtration $D^\bullet_{A|\mathbb{K}}$, good filtrations, and dimension as in Sect. 3.3.

Proposition 3.24 ([16, Ch 2., Theorem 7.1]) *Let A be a differentiably admissible $\mathbb{K}$-algebra. Let $M \neq 0$ be a finitely generated $D_{A|\mathbb{K}}$-module. Then,*

$$\dim_{D_{A|\mathbb{K}}}(M) + \text{grade}_D(M) = 2d.$$

In particular,

$$\dim_{D_{A|\mathbb{K}}}(M) \geq d.$$

We stress that the conclusion of the previous result is satisfied for rings of differentiable type [98, 106].

Definition 3.25 Let A be a differentiably admissible $\mathbb{K}$-algebra. Let M be a finitely generated left (right) $D_{A|\mathbb{K}}$-module. We say that M is in the *left (right) Bernstein class* if either $M = 0$ or if $\dim_D(M) = d$.

Let M be a finitely generated $D_{A|\mathbb{K}}$-module. If M is in the Bernstein class of $D_{A|\mathbb{K}}$, then $\text{Ext}^i_{D_{A|\mathbb{K}}}(M, D_{A|\mathbb{K}}) \neq 0$ if and only if $i = d$ [16]. Then, the functor that sends M to $\text{Ext}^d_{D_{A|\mathbb{K}}}(M, D_{A|\mathbb{K}})$ is an exact contravariant functor that interchanges the left Bernstein class and the right Bernstein class. Furthermore, $M \cong \text{Ext}^d_{D_{A|\mathbb{K}}}(\text{Ext}^d_{D_{A|\mathbb{K}}}(M, D_{A|\mathbb{K}}), D_{A|\mathbb{K}})$ for modules in the Bernstein class. Since $D_{R|\mathbb{K}}$ is left and right Noetherian, the modules in the Bernstein class are both Noetherian and Artinian. We conclude that the modules in the Bernstein class have finite length as $D_{A|\mathbb{K}}$-modules [98, Proposition 1.2.5])

This class of Bernstein modules is an analogue of the class of holonomic modules. In particular, it is closed under submodules, quotients, extensions, and localizations [98, Proposition 1.2.7]).

Theorem 3.26 *Let A be a differentiably admissible $\mathbb{K}$-algebra of dimension d. Given any nonzero element $f \in A$, the Bernstein-Sato polynomial of f exists.*

Sketch of proof In this sketch we follow the ideas of Mebkhout and Narváez-Macarro [98] (see also [106]). In particular, we refer the interested reader to their work on the base change $\mathbb{K}$ to $\mathbb{K}(s)$ regarding differentiably admissible algebras

[98, Section 2]. Let $A(s) = A \otimes_{\mathbb{K}} \mathbb{K}(s)$. We observe that $A(s)$ is not always a differentiably admissible $\mathbb{K}(s)$-algebra. Specifically, the residue fields of $A(s)$ might not be always algebraic. However, $D_{A(s)|\mathbb{K}(s)}$ satisfies the conclusions of Theorem 2.9. In particular, the conclusions of Theorem 3.24 hold, and its Bernstein class is well defined. We have that the dimension and global dimension of $D_{A(s)|\mathbb{K}(s)}$ and $D_{A|\mathbb{K}}$ are equal. One can show that $A_f[s]\boldsymbol{f^s} \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ has a $D_{A(s)|\mathbb{K}(s)}$-submodule N is in the Bernstein class of $D_{A(s)|\mathbb{K}(s)}$ such that $N_f = A_f[s]\boldsymbol{f^s} \otimes_{\mathbb{K}[s]} \mathbb{K}(s)$ [98, Proposition 1.2.7 and Proof of Theorem 3.1.1]. Then, there exists $\ell \in \mathbb{N}$ such that $f^\ell \boldsymbol{f^s} \in N$. Since N has finite length as $D_{A(s)|\mathbb{K}(s)}$-module the chain

$$D_{A(s)|\mathbb{K}(s)} f^\ell \boldsymbol{f^s} \supseteq D_{A(s)|\mathbb{K}(s)} f^{\ell+1} \boldsymbol{f^s} \supseteq D_{A(s)|\mathbb{K}(s)} f^{\ell+2} \boldsymbol{f^s} \supseteq \dots$$

stabilizes. Then, there exists $m \in \mathbb{N}$ and a differential operator $\delta(s) \in D_{A(s)|\mathbb{K}(s)}$ such that

$$\delta(s) f^{\ell+m+1} \boldsymbol{f^s} = f^{\ell+m} \boldsymbol{f^s}.$$

After clearing denominators and a shifting, there exists $\widetilde{\delta}(s) \in D_{A|\mathbb{K}}[s]$ such that

$$\widetilde{\delta}(s) f \boldsymbol{f^s} = \boldsymbol{f^s}.$$

□

3.5 First Properties of the Bernstein-Sato Polynomial

A first observation about the Bernstein-Sato polynomial is that $s+1$ is always a factor.

Lemma 3.27 *For $f \in A$, we have $(s+1) \mid b_f(s)$ if and only if f is not a unit.*

Proof If f is a unit, then we can take $f^{-1} f^{s+1} = 1 f^s$ as a functional equation, so $b(s) = 1$ is the Bernstein-Sato polynomial of f.

For the converse, by definition, we have $\delta(s) f \boldsymbol{f^s} = b_f(s) \boldsymbol{f^s}$ in $A_f[s]\boldsymbol{f^s}$. By Remark 3.11, $\delta(n) f^{n+1} = b_f(n) f^n$ in A_f for all $n \in \mathbb{Z}$. In particular, for $n = -1$, we get $\delta(-1)1 = b_f(-1) f^{-1}$. As $\delta(-1) \in D_{A|\mathbb{K}}$, we have $\delta(-1)1 \in A$. Thus, $b_f(-1) = 0$, so $s+1$ divides $b_f(s)$. □

Quite nicely, the factor $(s+1)$ characterizes the regularity of f.

Proposition 3.28 ([27]) *For $f \in A$, we have A/fA is smooth if and only if $b_f(s) = s+1$.*

Definition 3.29 The *reduced Bernstein-Sato polynomial* of a nonunit $f \in A$ is

$$\tilde{b}_f(s) = b_f(s)/(s+1).$$

The analogue of Proposition 3.13 for the reduced Bernstein-Sato polynomial is as follows.

Proposition 3.30 *The following are equal:*

(i) $\tilde{b}_f(s)$,

(ii) *The minimal polynomial of the action of s on* $(s+1)\dfrac{D_{A|\mathbb{K}}[s]\boldsymbol{f^s}}{D_{A|\mathbb{K}}[s]f\boldsymbol{f^s}}$,

(iii) *The monic element of smallest degree in*

$$\mathbb{K}[s] \cap (Ann_{D[s]}\big(\boldsymbol{f^s}) + D_{A|\mathbb{K}}[s](f, \partial_1(f), \dots, \partial_n(f)\big)).$$

Proof Once again, the first two are equivalent by definition.

Given a functional equation $\delta(s) f \boldsymbol{f^s} = (s+1)\tilde{b}(s)\boldsymbol{f^s}$, we have that $\delta(-1) \in D_{A|\mathbb{K}}$ with $\delta(-1)\cdot 1 = 0$. We can write $\delta(s) = (s+1)\delta'(s) + \delta(-1)$ for some $\delta'(s) \in D_{A|\mathbb{K}}[s]$, so $\delta(s) = (s+1)\delta'(s) + \sum_{i=1}^{d} \delta_i \partial_i$ for some $\delta_i \in D_{A|\mathbb{K}}$. Then, using that $\partial_i(f\boldsymbol{f^s}) = (s+1)\partial_i(f)\boldsymbol{f^s}$, we have

$$(s+1)\tilde{b}(s)\boldsymbol{f^s} = (s+1)\delta'(s) f \boldsymbol{f^s} + \sum_{i=1}^{d} \delta_i \partial_i f \boldsymbol{f^s} = (s+1)(\delta'(s) f + \sum_{i=1}^{d} \delta_i \partial_i(f))\boldsymbol{f^s}.$$

Thus, such a functional equation implies that $\tilde{b}(s)\boldsymbol{f^s} \in D_{A|\mathbb{K}}[s](f, \partial_1(f), \dots, \partial_d(f))$. Conversely, if $\tilde{b}(s)\boldsymbol{f^s} \in D_{A|\mathbb{K}}[s](f, \partial_1(f), \dots, \partial_d(f))$, again using that $\partial_i(f\boldsymbol{f^s}) = (s+1)\partial_i(f)\boldsymbol{f^s}$, we can write $(s+1)\tilde{b}(s)\boldsymbol{f^s} \in D_{A|\mathbb{K}}[s]f\boldsymbol{f^s}$. This implies the equivalence of the first and the last characterizations. □

We may also be interested in the characteristic polynomial of the action of s. Traditionally, with the convention of a sign change, the roots of the characteristic polynomial are known as the b-exponents of f.

Definition 3.31 The b-exponents of $f \in A$ are the roots of the characteristic polynomial of the action of $-s$ on $(s+1)\dfrac{D_{A|\mathbb{K}}[s]\boldsymbol{f^s}}{D_{A|\mathbb{K}}[s]f\boldsymbol{f^s}}$.

So far we have considered Bernstein-Sato polynomials over different regular rings A but, a priori, it is not clear how they are related. Our next goal is to address this issue. We start considering $A = \mathbb{K}[x_1, \dots, x_d]$, a polynomial ring over a field $\mathbb{K}$ of characteristic zero and denote by $b_f^{\mathbb{K}[x]}(s)$ the Bernstein-Sato polynomial of $f \in A$. Given any maximal ideal $\mathfrak{m} \subseteq A$ we also consider the Bernstein-Sato polynomial over the localization $A_{\mathfrak{m}}$ that we denote $b_f^{\mathbb{K}[x]_{\mathfrak{m}}}(s)$.

Proposition 3.32 *We have:*

$$b_f^{\mathbb{K}[x]}(s) = lcm\{b_f^{\mathbb{K}[x]_{\mathfrak{m}}}(s) \mid \mathfrak{m} \subseteq A \ \ maximal \ \ ideal\}.$$

Proof Let $b(s) \in \mathbb{K}[s]$ be a polynomial. The module $b(s)\dfrac{D_{A|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{K}}[s]f\boldsymbol{f}^s}$ vanishes if and only if it vanishes locally. The localization at a maximal ideal $\mathfrak{m} \subseteq A$ is

$$b(s)\frac{D_{A_\mathfrak{m}|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A_\mathfrak{m}|\mathbb{K}}[s]f\boldsymbol{f}^s}$$

and the result follows. □

For a polynomial $f \in A$ we may also consider the Bernstein-Sato polynomial $b_f^{\mathbb{K}[\![x]\!]}(s)$ in the formal power series ring $\mathbb{K}[\![x_1, \ldots, x_d]\!]$.

Proposition 3.33 *Let $\mathfrak{m} = (x_1, \ldots, x_d) \subseteq A$ be the homogeneous maximal ideal. We have:*

$$b_f^{\mathbb{K}[x]_\mathfrak{m}}(s) = b_f^{\mathbb{K}[\![x]\!]}(s).$$

Proof $B = \mathbb{K}[\![x_1, \ldots, x_d]\!]$ is faithfully flat over $A_\mathfrak{m} = \mathbb{K}[x_1, \ldots, x_d]_\mathfrak{m}$. Since

$$B \otimes_{A_\mathfrak{m}} b(s)\frac{D_{A_\mathfrak{m}|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A_\mathfrak{m}|\mathbb{K}}[s]f\boldsymbol{f}^s} = b(s)\frac{D_{B|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{B|\mathbb{K}}[s]f\boldsymbol{f}^s}$$

the result follows. □

When $\mathbb{K} = \mathbb{C}$ we may also consider the ring $\mathbb{C}\{x_1 - p_1, \ldots, x_d - p_d\}$ of convergent power series in a neighborhood of a point $p = (p_1, \ldots, p_d) \in \mathbb{C}^d$.

Corollary 3.34 *We have*

(i) $b_f^{\mathbb{C}[x]}(s) = lcm\{b_f^{\mathbb{C}\{x-p\}}(s) \mid p \in \mathbb{C}^d\}$.
(ii) $b_f^{\mathbb{C}\{x-p\}}(s) = b_f^{\mathbb{C}[\![x-p]\!]}(s)$.

Proof Working over $\mathbb{C}$ we have that all the maximal ideals correspond to points so (i) follows from Proposition 3.32. For part (ii) we use the same faithful flatness trick we used in Proposition 3.33 for $\mathbb{C}\{x_1 - p_1, \ldots, x_d - p_d\}$. □

Let $f \in \mathbb{K}[x_1, \ldots, x_d]$ be a polynomial and $\mathbb{L}$ a field containing $\mathbb{K}$. Let $b_f^{\mathbb{K}[x]}(s)$ and $b_f^{\mathbb{L}[x]}(s)$ be the Bernstein-Sato polynomial of f in $\mathbb{K}[x_1, \ldots, x_d]$ and $\mathbb{L}[x_1, \ldots, x_d]$ respectively.

Proposition 3.35 *We have $b_f^{\mathbb{K}[x]}(s) = b_f^{\mathbb{L}[x]}(s)$.*

Proof Notice that $b_f^{\mathbb{L}[x]}(s) \mid b_f^{\mathbb{K}[x]}(s)$ so we have to prove the other divisibility condition. Let $\{e_i\}_{i \in I}$ be a basis of $\mathbb{L}$ as a $\mathbb{K}$-vector space. We have

$$\frac{D_{A|\mathbb{L}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{L}}[s]f\boldsymbol{f}^s} = \mathbb{L} \otimes_\mathbb{K} \frac{D_{A|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{K}}[s]f\boldsymbol{f}^s} = \bigoplus_{i \in I}\left(\frac{D_{A|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{A|\mathbb{K}}[s]f\boldsymbol{f}^s}\right)e_i.$$

Let $b(s) \in \mathbb{L}[s]$ be such that $b(s)\frac{D_{A|\mathbb{L}}[s]f^s}{D_{A|\mathbb{L}}[s]ff^s} = 0$. Then $b(s) = \sum b_i(s)$ with only finitely many nonzero $b_i(s) \in \mathbb{K}[s]$ such that $b_i(s)\frac{D_{A|\mathbb{K}}[s]f^s}{D_{A|\mathbb{K}}[s]ff^s} = 0$. Since $b_f^{\mathbb{K}[x]}(s) \mid b_i(s)$ for all i it follows that $b_f^{\mathbb{K}[x]}(s) \mid b_f^{\mathbb{L}[x]}(s)$. □

Remark 3.36 Let $f \in \mathbb{K}[x_1, \ldots, x_d]$ be a polynomial with an isolated singularity at the origin, where $\mathbb{K}$ is a subfield of $\mathbb{C}$. Then we have $b_f^{\mathbb{K}[x]}(s) = b_f^{\mathbb{K}[\![x]\!]}(s) = b_f^{\mathbb{C}\{x\}}(s)$.

Combining all the results above with the following fundamental result of Kashiwara [71], we conclude that the Bernstein-Sato polynomial of $f \in \mathbb{K}[x_1, \ldots, x_d]$ is a polynomial $b_f(s) \in \mathbb{Q}[s]$.

Theorem 3.37 ([71, 92]) *The Bernstein-Sato polynomial of an element $f \in \mathbb{C}\{x_1, \ldots, x_d\}$, or $f \in \mathbb{K}[x_1, \ldots, x_d]$ for $\mathbb{K} \subseteq \mathbb{C}$, factors completely over $\mathbb{Q}$, and all of its roots are negative rational numbers.*

In Sect. 9 we will provide a refinement of this result given by Lichtin [80].

4 Some Families of Examples

Computing explicit examples of Bernstein-Sato polynomials is a very challenging task. There are general algorithms based on the theory of Gröbner bases over rings of differential operators but they have a very high complexity so only few examples can be effectively computed [78, 107, 110]. In this section we review some of the scarce examples that we may find in the literature. The first systematic method of producing examples can be found in the work of Yano [146] where he considered, among others, the case of isolated quasi-homogeneous singularities (see also [25]). The case of isolated semi-quasi-homogeneous singularities was studied later on by Saito [120] and Briançon et al. [26].

A case that has been extensively studied is that of plane curves, see [29, 43–45, 61, 73, 74, 147]. In particular, a conjecture of Yano regarding the b-exponents of a generic irreducible plane curve among those in the same equisingularity class has been recently proved by Blanco [18] (see also [4, 17, 45]). Finally we want to mention that the case of hyperplane arrangements has been studied by Walther [143] and Saito [124].

We start with some known examples where a Bernstein-Sato functional equation $\delta(s)f^{s+1} = b(s)f^s$ can be given by hand:

(i) Let $f = x_1^2 + \cdots + x_n^2$ be a sum of squares. Then

$$\frac{1}{4}(\partial_1^2 + \cdots + \partial_n^2)f^{s+1} = (s+1)(s+\frac{n}{2})f^s.$$

(ii) Let $f = \det(x_{ij})$ be the determinant of an $n \times n$ generic matrix and set $\partial_{ij} := \frac{d}{dx_{ij}}$. The classic Cayley identity states

$$\det(\partial_{ij}) f^{s+1} = (s+1)(s+2)\cdots(s+n) f^s.$$

There are similar identities for determinants of symmetric and antisymmetric matrices [41].

(iii) Let $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial. Then

$$\frac{1}{\alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}} (\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}) f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{\alpha_i} (s + \frac{k}{\alpha_i}) f^s.$$

We warn the reader that it requires some extra work to prove that the above polynomials are minimal so they are indeed Bernstein-Sato polynomials of the corresponding f.

Let $A = \mathbb{C}\{x_1, \ldots, x_d\}$ and assume that f has an isolated singularity at the origin. In this case, Yano [146] uses the fact that the support of the holonomic $D_{A|\mathbb{C}}$-module $\widetilde{\mathcal{M}} := (s+1)\frac{D_{A|\mathbb{C}}[s] \boldsymbol{f^s}}{D_{A|\mathbb{C}}[s] f \boldsymbol{f^s}}$ is the maximal ideal and thus it is isomorphic to a number of copies of $D_{A|\mathbb{C}}/D_{A|\mathbb{C}}\langle x_1, \ldots, x_d\rangle \cong H_{\mathfrak{m}}^d(A)$. Dualizing this module we get the module of differential d-forms $\Omega^d = D_{A|\mathbb{C}}/\langle \partial_1, \ldots, \partial_d\rangle D_{A|\mathbb{C}}$.

Proposition 4.1 ([146, Theorem 3.3]) *The reduced Bernstein-Sato polynomial $\tilde{b}_f(s)$ of an isolated singularity f is the minimal polynomial of the action of s on either $Hom_{D_{A|\mathbb{C}}}(\widetilde{\mathcal{M}}, H_{\mathfrak{m}}^d(A))$ or $\Omega^n \otimes_{D_{A|\mathbb{C}}} \widetilde{\mathcal{M}}$.*

Then, Yano's method boils down to the following steps:

(i) Compute a free resolution of $\widetilde{\mathcal{M}}$ as a $D_{A|\mathbb{C}}$-module

$$0 \leftarrow \widetilde{\mathcal{M}} \leftarrow (D_{A|\mathbb{C}})^{\beta_0} \leftarrow (D_{A|\mathbb{C}})^{\beta_1} \leftarrow \cdots$$

(ii) Apply the functor $\mathrm{Hom}_{D_{A|\mathbb{C}}}(-, H_{\mathfrak{m}}^d(A))$

$$0 \to \mathrm{Hom}_{D_{A|\mathbb{C}}}(\widetilde{\mathcal{M}}, H_{\mathfrak{m}}^d(A)) \to (H_{\mathfrak{m}}^d(A))^{\beta_0} \to (H_{\mathfrak{m}}^d(A))^{\beta_1} \to \cdots$$

(iii) Compute the matrix representation of the action of s and its minimal polynomial.

Yano could effectively work out some cases depending on the following invariant of the singularity:

$$L(f) := \min\{L \mid \delta(s) = s^L + \delta_1 s^{L-1} + \cdots + \delta_L \in \mathrm{Ann}_{D[s]}(\boldsymbol{f^s}), \mathrm{ord}(\delta_i) \leq i\}.$$

The existence of such a differential operator is given by Kashiwara [71, Theorem 6.3]. More precisely, he could describe step (1) in the cases $L(f) = 1, 2$, and 3 where the case $L(f) = 1$ is equivalent to having a quasi-homogeneous singularity.

4.1 Quasi-Homogeneous Singularities

Let $f = \sum_\alpha a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_d} \in A$ be a quasi-homogeneous isolated singularity of degree N with respect to a weight vector $w := (w_1, \ldots, w_d) \in \mathbb{Q}^d_{>0}$. We have $\chi(f) = Nf$ where

$$\chi = \sum_{i=1}^{d} w_i x_i \partial_i$$

is the Euler operator and $\chi - Ns \in \mathrm{Ann}_{D[s]}(\boldsymbol{f}^s)$. Set $f_i' = \partial_i(f)$ for $i = 1, \ldots, d$. Yano's method is as follows:

(i) We have a free resolution

$$0 \longleftarrow \widetilde{\mathcal{M}} \longleftarrow D_{A|\mathbb{C}} \xleftarrow{(f_1',\ldots,f_d')} (D_{A|\mathbb{C}})^n \longleftarrow 0 \ .$$

(ii) We obtain a presentation $\mathrm{Hom}_{D_{A|\mathbb{C}}}(\widetilde{\mathcal{M}}, H^d_{\mathfrak{m}}(A)) = \{v \in H^d_{\mathfrak{m}}(A) \mid f_i' v = 0 \ \forall i\}$.

(iii) The action of s on $v \in \mathrm{Hom}_{D_{A|\mathbb{C}}}(\widetilde{\mathcal{M}}, H^d_{\mathfrak{m}}(A))$ is the same as the action of $\frac{1}{N}\chi$. Notice that applying χ to a cohomology class $\left[\frac{1}{x_1^{\alpha_1}\cdots x_d^{\alpha_d}}\right]$ is nothing but multiplying by the weight of this class.

Example 4.2 Consider the quasi-homogeneous polynomial $f = x^7 + y^5 \in \mathbb{C}\{x, y\}$ of degree $N = 35$ with respect to the weight $w = (5, 7)$. A basis of the vector space

$$\{v \in H^2_{\mathfrak{m}}(A) \mid x^6 v = 0, y^4 v = 0\}$$

is given by the classes $\left[\frac{1}{x^i y^j}\right]$ with $1 \le i \le 6$ and $1 \le j \le 4$. The action of $\frac{1}{35}\chi = \frac{1}{35}(5x\partial_x + 7y\partial_y)$ on these elements yields

$$\frac{1}{35}\chi\left(\frac{1}{xy}\right) = -\frac{12}{35}\left(\frac{1}{xy}\right), \frac{1}{35}\chi\left(\frac{1}{x^2 y}\right) = -\frac{17}{35}\left(\frac{1}{x^2 y}\right), \ldots,$$
$$\frac{1}{35}\chi\left(\frac{1}{x^6 y^4}\right) = -\frac{58}{35}\left(\frac{1}{x^6 y^4}\right).$$

The matrix representation of the action of $s = \frac{1}{35}\chi$ has a diagonal form with distinct eigenvalues and thus the characteristic and the minimal polynomials coincide.

The negatives of the roots of the reduced Bernstein-Sato polynomial $\tilde{b}_f(s)$, or equivalently, the roots of $\tilde{b}_f(-s)$ are

$$\left\{\frac{12}{35}, \frac{17}{35}, \frac{19}{35}, \frac{22}{35}, \frac{24}{35}, \frac{26}{35}, \frac{27}{35}, \frac{29}{35}, \frac{31}{35}, \frac{32}{35}, \frac{33}{35}, \frac{34}{35}, \frac{36}{35}, \frac{37}{35}, \frac{38}{35}, \frac{39}{35}, \frac{41}{35}, \frac{43}{35}, \frac{44}{35}, \frac{46}{35}, \frac{48}{35}, \frac{51}{35}, \frac{53}{35}, \frac{58}{35}\right\}.$$

Remark 4.3 In general, the diagonal form of the matrix representation of the action of s has repeated eigenvalues so the minimal polynomial only counts them once. Take for example the quasi-homogeneous polynomial $f = x^5 + y^5 \in \mathbb{C}[x, y]$ of degree $N = 5$ with respect to the weight $w = (1, 1)$. The roots of $\tilde{b}_f(-s)$ are

$$\left\{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}\right\}.$$

Theorem 4.4 ([25, 146]) *Let $f \in A$ be a quasi-homogeneous isolated singularity of degree N with respect to a weight vector $w := (w_1, \dots, w_d) \in \mathbb{Q}^d_{>0}$. Then, the Bernstein-Sato polynomial of f is*

$$b_f(s) = (s+1)\prod_{\ell \in W}\left(s + \frac{\ell}{N}\right)$$

where W is the set of weights, without repetition, of the cohomology classes in $\{v \in H^d_{\mathfrak{m}}(A) \mid f'_i v = 0 \;\; \forall i\}$.

Recall from Proposition 4.1 that the reduced Bernstein-Sato polynomial $\tilde{b}_f(s)$ of an isolated singularity f is the minimal polynomial of the action of s on $\Omega^d \otimes_{D_{A|\mathbb{C}}} \widetilde{\mathcal{M}}$. In the quasi-homogeneous case we have

$$\Omega^d \otimes_{D_{A|\mathbb{C}}} \widetilde{\mathcal{M}} \cong A/(f'_1, \dots, f'_d).$$

Notice that the monomial basis of the Milnor algebra is dual, with the convenient shift, of the cohomology classes basis of $\{v \in H^d_{\mathfrak{m}}(A) \mid f'_i v = 0 \;\; \forall i\}$. In this case, the action of s is $-\frac{1}{N}(\chi + \sum_{i=1}^n w_i)$.

4.2 Irreducible Plane Curves

Some of the examples considered by Yano deal with the case of plane curves and his methods were used by Kato to compute the following example which is a continuation of Example 4.2.

Example 4.5 ([73]) The roots of $\tilde{b}_f(-s)$ for $f = x^7 + y^5$ are

$$\Bigg\{\underbrace{\frac{12}{35},\frac{17}{35},\frac{19}{35},\frac{22}{35},\frac{24}{35},\frac{26}{35},\frac{27}{35},\frac{29}{35},\frac{31}{35},\frac{32}{35},\frac{33}{35},\frac{34}{35}}_{\lambda},$$

$$\underbrace{\frac{36}{35},\frac{37}{35},\frac{38}{35},\frac{39}{35},\frac{41}{35},\frac{43}{35},\frac{44}{35},\frac{46}{35},\boxed{\frac{48}{35}},\boxed{\frac{51}{35}},\boxed{\frac{53}{35}},\boxed{\frac{58}{35}}}_{2-\lambda}\Bigg\}.$$

Notice that the roots are symmetric with respect to 1 and we point out that those $\lambda < 1$ are jumping numbers of the multiplier ideals of f (see Sect. 10). Now consider a deformation of the singularity,

$$f_t = x^7 + y^5 - t_{3,3}x^3y^3 - t_{5,2}x^5y^2 - t_{4,3}x^4y^3 - t_{5,3}x^5y^3.$$

Then we have a stratification of the space of parameters where some of the roots of $\tilde{b}_f(-s)$ may change. More precisely, the boxed roots may change to the same root shifted by 1.

$\{t_{3,3} = 0, t_{5,2} = 0, t_{4,3} = 0, t_{5,3} \neq 0\}$. The root $\frac{58}{35}$ changes to $\frac{23}{35}$.
$\{t_{3,3} = 0, t_{5,2} = 0, t_{4,3} \neq 0\}$. The roots $\frac{58}{35}, \frac{53}{35}$ change to $\frac{23}{35}, \frac{18}{35}$.
$\{t_{3,3} = 0, t_{5,2} \neq 0, t_{4,3} = 0\}$. The roots $\frac{58}{35}, \frac{51}{35}$ change to $\frac{23}{35}, \frac{16}{35}$.
$\{t_{3,3} = 0, t_{5,2}t_{4,3} \neq 0\}$. The roots $\frac{58}{35}, \frac{53}{35}, \frac{51}{35}$ change to $\frac{23}{35}, \frac{18}{35}, \frac{16}{35}$.
$\{t_{5,2} \neq 0, 6t_{5,2} + 175t_{3,3}^4 = 0\}$. The roots $\frac{58}{35}, \frac{53}{35}, \frac{48}{35}$ change to $\frac{23}{35}, \frac{18}{35}, \frac{13}{35}$.
$\{t_{5,2} \neq 0, 6t_{5,2} + 175t_{3,3}^4 \neq 0\}$. The roots $\frac{58}{35}, \frac{53}{35}, \frac{51}{35}, \frac{48}{35}$ change to $\frac{23}{35}, \frac{18}{35}, \frac{16}{35}, \frac{13}{35}$.

In this last stratum we have a Zariski open set where the roots are

$$\Bigg\{\frac{12}{35},\boxed{\frac{13}{35}},\boxed{\frac{16}{35}}\frac{17}{35},\boxed{\frac{18}{35}},\frac{19}{35},\frac{22}{35},\boxed{\frac{23}{35}},\frac{24}{35},\frac{26}{35},\frac{27}{35},\frac{29}{35},\frac{31}{35},\frac{32}{35},\frac{33}{35},\frac{34}{35},\frac{36}{35},$$

$$\frac{37}{35},\frac{38}{35},\frac{39}{35},\frac{41}{35},\frac{43}{35},\frac{44}{35},\frac{46}{35}\Bigg\},$$

and thus they are in the interval $[\text{lct}(f), \text{lct}(f)+1)$. We say that these are the generic roots of the Bernstein-Sato polynomial of f_t.

An interesting issue in this example is that, even though they have different Bernstein-Sato polynomials, all the fibres of the deformation f_t have the same Milnor number so they belong to the same equisingularity class. Roughly speaking, all the fibres have the same log-resolution meaning that they have the same combinatorial information, which can be encoded in weighted graphs such as the Enriques diagram [54, §IV.I], [42, S 3.9], the dual graph [42, §4.4], [142, §3.6] or the Eisenbud-Neumann diagrams [53].

From now on let $f \in \mathbb{C}\{x, y\}$ be a defining equation of the germ of an irreducible plane curve. A complete set of numerical invariants for the equisingularity class of f is given by the *characteristic exponents* $(n, \beta_1, \dots, \beta_g)$ where $n \in \mathbb{Z}_{>0}$ is the multiplicity at the origin of f and the integers $n < \beta_1 < \cdots < \beta_g$ can be obtained from the Puiseux parameterization of f. To describe the equisingularity class of f we may also consider its *semigroup* $\Gamma := \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$ that comes from the valuation of $\mathbb{C}\{x, y\}/\langle f \rangle$ given by the Puiseux parametrization of f.

A quasihomogeneous plane curve $f = x^a + y^b$ with $a < b$ and $\gcd(a, b) = 1$ is irreducible with semigroup $\Gamma = \langle a, b \rangle$. Adding higher order terms $x^i y^j$ with $bi + aj > ab$ does not change the equisingularity class but we do not need all the higher order terms. Indeed, every irreducible curve with semigroup $\Gamma = \langle a, b \rangle$ is analytically isomorphic to one of the fibers of the miniversal deformation

$$f = x^a + y^b - \sum t_{i,j}\, x^i y^j,$$

where the sum is taken over the monomials $x^i y^j$ such that $0 \le i \le a - 2$, $0 \le j \le b - 2$ and $bi + aj > ab$. This is the setup considered in Example 4.5.

Cassou-Noguès [44] described the stratification by the Bernstein-Sato polynomial of any irreducible plane curve with a single characteristic exponent using analytic continuation of the complex zeta function.

To construct a miniversal deformation of an irreducible plane curve with g characteristic exponents is much more complicated and one has to use, following Teissier [149], the monomial curve C^Γ associated to the semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$ by the parametrization $u_i = t^{\overline{\beta}_i}$, $i = 1, \dots, g$. Teissier proved the existence of a miniversal semigroup constant deformation of this monomial curve. It turns out that every irreducible plane curve with semigroup Γ is analytically isomorphic to one of the fibres of the miniversal deformation of C^Γ. To give explicit equations in $\mathbb{C}\{x, y\}$ is more complicated and we refer to the work of Blanco [17] for more details. For the convenience of the reader we illustrate an example with two characteristic exponents.

Example 4.6 The semigroup of an irreducible plane curve $f = (x^a + y^b)^c + x^i y^j$ with $bi + aj = d$ is $\Gamma = \langle ac, bc, d \rangle$. All the fibres of the deformation

$$f_t = \left(x^a + y^b + \sum_{bk + a\ell > ab} t_{k,\ell} x^k y^\ell \right)^c + x^i y^j + \sum_{bck + ac\ell + dr > cd} t_{k,\ell} x^k y^\ell \left(x^a + y^b\right)^r$$

have the same semigroup.

The ultimate goal would be to find a stratification by the Bernstein-Sato polynomial of all the irreducible plane curves with a fixed semigroup but this turns out to be a wild problem. However, one may ask about the roots of the Bernstein-Sato polynomial of a generic fibre of a deformation of an irreducible plane curve with a given semigroup. That is, to find the roots in a Zariski open set in the space of parameters of the deformation that we call the generic roots of the Bernstein-Sato polynomial.

Amazingly, Yano [147] conjectured a formula for the generic b-exponents (instead of the generic roots) of any irreducible plane curve. These generic b-exponents can be described in terms of the semigroup Γ but we use a simple interpretation in terms of the numerical data of a log-resolution of f. Let $\pi : X' \to \mathbb{C}^n$ be a log-resolution of an irreducible plane curve with g characteristic exponents. Let F_π be the total transform divisor and K_π the relative canonical divisor. In this case we have g distinguished exceptional divisors, the so-called *rupture divisors* that intersect three or more divisors in the support of F_π. For simplicity we denote them by $E_1, \dots, E_g$ with the corresponding values N_i and k_i in F_π and K_π respectively.

Conjecture 4.7 ([147]) Let $f \in \mathbb{C}\{x, y\}$ be a defining equation of the germ of an irreducible plane curve with semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$. Then, for generic curves in some Γ-constant deformation of f, the b-exponents are

$$\bigcup_{i=1}^{g} \left\{ \lambda_{i,\ell} = \frac{k_i + 1 + \ell}{N_i} \;\middle|\; 0 \leq \ell < N_i,\ \overline{\beta}_i \lambda_{i,\ell} \notin \mathbb{Z}, e_{i-1}\lambda_{i,\ell} \notin \mathbb{Z} \right\}$$

where $e_{i-1} = \gcd(\overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_{i-1})$.

If we consider the irreducible plane curve studied by Kato in Example 4.5 we see that Yano's conjecture holds true.

Example 4.8 The Yano set associated to the semigroup $\Gamma = \langle 5, 7 \rangle$ is

$$\left\{ \lambda_{1,\ell} = \frac{12 + \ell}{35} \;\middle|\; 0 \leq \ell < 35,\ 7\lambda_{1,\ell} \notin \mathbb{Z}, 5\lambda_{1,\ell} \notin \mathbb{Z} \right\}$$

which gives the generic b-exponents given in Example 4.5.

From the stratification given by Cassou-Noguès [44] one gets that Yano's conjecture is true for irreducible plane curves with a single characteristic exponent (see [45]). Almost thirty years later, Artal-Bartolo, Cassou-Noguès, Luengo, and Melle-Hernández [4] proved Yano's conjecture for irreducible plane curves with two characteristic exponents with the extra assumption that the eigenvalues of the monodromy are different. Under the same extra condition, Blanco [17] gave a proof for any number of characteristic exponents. Both papers use the analytic continuation of the complex zeta function. The extra condition on the eigenvalues of the monodromy being different ensures that the characteristic and the minimal polynomial of the action of s on $(s+1)\frac{D_{A|\mathbb{C}}[s]f^s}{D_{A|\mathbb{C}}[s]ff^s}$ are the same.

The shortcomings of the analytic continuation techniques, which deal with the Bernstein-Sato polynomial instead of the b-exponents, can be seen in examples such as the following.

Example 4.9 The Yano sets associated to the semigroup $\Gamma = \langle 10, 15, 36 \rangle$ are

$$\left\{ \lambda_{1,\ell} = \frac{5+\ell}{30} \;\middle|\; 0 \leq \ell < 30,\ 15\lambda_{1,\ell} \notin \mathbb{Z}, 10\lambda_{1,\ell} \notin \mathbb{Z} \right\},$$

and

$$\left\{ \lambda_{2,\ell} = \frac{31+\ell}{180} \;\middle|\; 0 \leq \ell < 180,\ 36\lambda_{2,\ell} \notin \mathbb{Z}, 5\lambda_{2,\ell} \notin \mathbb{Z} \right\}.$$

We have that $\frac{11}{30}, \frac{17}{30}, \frac{23}{30}, \frac{29}{30}$ appear in both sets. Therefore they appear with multiplicity 2 as b-exponents but only once as roots of the Bernstein-Sato polynomial.

Blanco [18] has recently proved Yano's conjecture in its generality. His work uses periods of integrals along vanishing cycles on the Milnor fiber as considered by Malgrange [90, 91] and Varchenko [140, 141]. In particular he extends vastly the results of Lichtin [80] and Loeser [82] on the expansions of these periods of integrals.

4.3 *Hyperplane Arrangements*

Let $f \in \mathbb{C}[x_1, \ldots, x_d]$ be a reduced polynomial defining an arrangement of hyperplanes so $f = f_1 \cdots f_\ell$ decomposes as a product of polynomials f_i of degree one. The Bernstein-Sato polynomial of f has been studied by Walther [143] under the assumptions that the arrangement is:

- *Central:* f is homogeneous so all the hyperplanes contain the origin.
- *Generic:* The intersection of any d hyperplanes is the origin.

The main result of Walther, with the assistance of Saito [124] to compute the multiplicity of -1 as a root, is the following.

Theorem 4.10 ([124, 143]) *The Bernstein-Sato polynomial of a generic central hyperplane arrangement $f \in \mathbb{C}[x_1, \ldots, x_d]$ of degree $\ell \geq d$ is*

$$b_f(s) = (s+1)^{d-1} \prod_{j=0}^{2\ell-d-2} \left(s + \frac{j+d}{\ell} \right).$$

Example 4.11 The homogeneous polynomial $f = x^5 + y^5 \in \mathbb{C}[x, y]$ considered in Remark 4.3 defines an arrangement of five lines through the origin. Walther's formula gives

$$b_f(s) = (s+1)^2 \left(s + \frac{2}{5}\right)\left(s + \frac{3}{5}\right)\left(s + \frac{4}{5}\right)\left(s + \frac{6}{5}\right)\left(s + \frac{7}{5}\right)\left(s + \frac{8}{5}\right).$$

It is an open question to determine the roots of the Bernstein-Sato polynomial of a nongeneric arrangement. In this general setting, Leykin [143] noticed that -1 is the only integer root of $b_f(s)$.

A natural question that arise when dealing with invariants of hyperplane arrangements is whether these invariants are *combinatorial*, meaning that they only depend on the lattice of intersection of the hyperplanes together with the codimensions of these intersections, and it does not depend on the position of the hyperplanes. Unfortunately this is not the case. Walther [145] provides examples of combinatorially equivalent arrangements with different Bernstein-Sato polynomial.

Example 4.12 ([124, 145]) The following nongeneric arrangements have the same intersection lattice

$$f = xyz(x+3z)(x+y+z)(x+2y+3z)(2x+y+z)(2x+3y+z)(2x+3y+4z),$$
$$g = xyz(x+5z)(x+y+z)(x+3y+5z)(2x+y+z)(2x+3y+z)(2x+3y+4z).$$

However the Bernstein-Sato polynomials differ by the root $-\frac{16}{9}$:

$$b_f(s) = (s+1)\prod_{j=2}^{4}\left(s+\frac{j}{3}\right)\prod_{j=3}^{16}\left(s+\frac{j}{9}\right)$$
$$b_g(s) = (s+1)\prod_{j=2}^{4}\left(s+\frac{j}{3}\right)\prod_{j=3}^{15}\left(s+\frac{j}{9}\right).$$

5 The Case of Nonprincipal Ideals and Relative Versions

In this section we study different extensions of Bernstein-Sato polynomials for ideals that are not necessarily principal. Sabbah [118] introduced the notion of Bernstein-Sato ideal $B_F \subseteq \mathbb{K}[s_1, \dots, s_\ell]$ associated to a tuple of elements $F = f_1, \dots, f_\ell$. More recently, Budur et al. [36] defined a Bernstein-Sato polynomial $b_{\mathfrak{a}}(s) \in \mathbb{K}[s]$ associated to an ideal $\mathfrak{a} \subseteq A$ which is independent of the set of generators. The approach to Bernstein-Sato polynomials of nonprincipal ideals has been simplified by Mustaţă [101].

In order to provide a description of the V-filtration of a holonomic D-module, Sabbah introduced a relative version of Bernstein-Sato polynomials that is also considered in the version for nonprincipal ideals [36]. This relative version is also important to describe multiplier ideals (see Sect. 10).

5.1 Bernstein-Sato Polynomial for General Ideals in Differentiably Admissible Algebras

We start studying the Bernstein-Sato polynomial for general ideals using the recent approach given by Mustaţă [101]. In this section we show its existence for general ideals in differentiably admissible algebras in Theorem 5.6.

Definition 5.1 Let $\mathbb{K}$ a field of characteristic zero, A be a regular $\mathbb{K}$-algebra, and $\mathfrak{a} \subseteq A$ be a nonzero ideal. Let $F = f_1, \dots, f_\ell$ be a set of generators for $\mathfrak{a}$, and $g = f_1 y_1 + \cdots + f_\ell y_\ell \in A[y_1, \dots, y_\ell]$. We denote by $b_F(s)$ the monic polynomial in $\mathbb{K}[s]$ of least degree among those polynomials $b(s) \in \mathbb{K}[s]$ such that

$$\delta(s) g^{s+1} = b(s) g^s \qquad \text{for all } \ s \in \mathbb{N},$$

where $\delta(s) \in D_{A[y_1,\dots,y_\ell]|\mathbb{K}}[s]$ is a polynomial differential operator. That is, $b_F(s)$ is the Bernstein-Sato polynomial of g.

Before we discuss properties of this notion of the Bernstein-Sato polynomial, we show that the definition of $b_F(s)$ does not depend on the choice of generators for $\mathfrak{a}$.

Proposition 5.2 ([101, Remark 2.1]) *Let $\mathbb{K}$ a field of characteristic zero, A be a regular $\mathbb{K}$-algebra, and $\mathfrak{a} \subseteq A$ be a nonzero ideal. Let $F = f_1, \dots, f_\ell$ and $G = g_1, \dots, g_m$ be two sets of generators for $\mathfrak{a}$. Then $b_F(s) = b_G(s)$.*

Proof It suffices to show that $b_F(s) = b_G(s) = b_H(s)$, where $H = F \cup G$. This follows from showing that $b_F(s) = b_G(s)$ when $G = F \cup g$ for $g \in \mathfrak{a}$. Let $r_1, \dots, r_\ell$ such that $g = r_1 f_1 + \cdots + r_\ell f_\ell$. We have that

$$\begin{aligned} f_1 y_1 + \cdots + f_\ell y_\ell + g y_{\ell+1} &= f_1 y_1 + \cdots + f_\ell y_\ell + (r_1 f_1 + \cdots + r_\ell f_\ell) y_{\ell+1} \\ & f_1(y_1 + r_1 y_{\ell+1}) + \cdots + f_\ell(y_\ell + r_\ell y_{\ell+1}). \end{aligned}$$

After a change of variables $y_i \mapsto y_i + r_i y_{\ell+1}$, this polynomial becomes f. Since the Bernstein-Sato polynomial does not change by change of variables, we conclude that $b_F(s) = b_G(s)$. □

Given the previous result, we can define the Bernstein-Sato polynomial of a nonprincipal ideal. Notice that $f_1 y_1 + \cdots + f_\ell y_\ell$ is not a unit in $A[y_1, \dots, y_\ell]$ so we may consider its reduced Bernstein-Sato polynomial $\tilde{b}_F(s) = \frac{b_F(s)}{s+1}$.

Definition 5.3 Let $\mathbb{K}$ a field of characteristic zero, A be a regular $\mathbb{K}$-algebra, and $\mathfrak{a} \subseteq A$ be a nonzero ideal. Let $F = f_1 \dots, f_\ell$ be a set of generators for $\mathfrak{a}$. We define the Bernstein-Sato polynomial of $\mathfrak{a}$ as the reduced Bernstein-Sato polynomial of $f_1 y_1 + \cdots + f_\ell y_\ell$. That is

$$b_{\mathfrak{a}}(s) := \tilde{b}_F(s).$$

We point out that the previous definition is not the original given by Budur, Mustaţă, and Saito [36], which we discuss in the next subsection. This approach given by Mustaţă [101] has a couple of differences. First, the existence of Bernstein-Sato polynomials for nonprincipal ideals would follow from the existence of certain Bernstein-Sato polynomials for a single element. This way in particular gives the existence of Bernstein-Sato polynomials for nonprincipal ideals in any differentiably admissible algebras (see Sect. 3.4) such as power series rings over a field of characteristic zero. Second, the treatment given by Mustaţă [101] can be done without using V-filtrations.

We now focus on showing the existence of Bernstein-Sato polynomial for nonprincipal ideals in differentiably admissible algebras. We start recalling a theorem from Matsumura's book [94].

Theorem 5.4 ([94, Theorem 99]) *Let* $(A, \mathfrak{m}, \mathbb{K})$ *be a regular local commutative Notherian ring with unity of dimension* d *containing a field* $\mathbb{K}_0$. *Suppose that* $\mathbb{K}$ *is an algebraic separable extension of* $\mathbb{K}_0$. *Let* $\hat{A}$ *denote the completion of* A *with respect to* $\mathfrak{m}$. *Let* $x_1, \ldots, x_d$ *be a regular system of parameters of* A. *Then,* $\widehat{A} = \mathbb{K}[\![x_1, \ldots, x_d]\!]$ *is the power series ring with coefficients in* $\mathbb{K}$*, and* $\mathrm{Der}_{\hat{A}|\mathbb{K}}$ *is a free* $\widehat{A}$*-module with basis* $\partial_1, \ldots, \partial_d$. *Moreover, the following conditions are equivalent:*

(i) ∂_i *(*$i = 1, \ldots, d$*) maps* A *into* A*, equivalently,* $\partial_i \in Der_{A|\mathbb{K}_0}$*;*
(ii) *there exist derivations* $\delta_1, \ldots, \delta_d \in \mathrm{Der}_{A|\mathbb{K}_0}$ *and elements* $f_1, \ldots, f_d \in A$ *such that* $\delta_i f_j = 1$ *if* $i = j$ *and* 0 *otherwise;*
(iii) *there exist derivations* $\delta_1, \ldots, \delta_d \in \mathrm{Der}_{A|\mathbb{K}_0}$ *and elements* $f_1 \ldots, f_d \in R$ *such that* $\det(\delta_i f_j) \notin \mathfrak{m}$*;*
(iv) $\mathrm{Der}_{A|\mathbb{K}_0}$ *is a free module of rank* d *(with basis* $\delta_1, \ldots, \delta_d$*);*
(v) $\mathrm{rank}(\mathrm{Der}_{A|\mathbb{K}_0}) = d$.

We now show that a power series ring over a differentiably admissible $\mathbb{K}$-algebra is also a differentiably admissible $\mathbb{K}$-algebra. We point out that this fact does not hold for polynomial rings, as the residue field can be a transcendental extension of R. A example of this is $A = \mathbb{K}[\![x]\!]$, where $\mathfrak{n} = (xy - 1) \subseteq A[y]$ is a maximal ideal with residue field $\mathrm{Frac}(A)$.

Proposition 5.5 *Let* A *be a differentiably admissible* $\mathbb{K}$*-algebra of dimension* d. *Then, the power series ring* $A[\![y]\!]$ *is also a differentiably admissible* $\mathbb{K}$*-algebra of dimension* $d + 1$.

Proof Since every regular Noetherian ring is product of regular domains, we assume without loss of generality that A is a domain. Let $\mathfrak{n}$ be a maximal ideal in $A[\![y]\!]$. Then, there exists a maximal ideal $\mathfrak{m} \subseteq A$ such that $\mathfrak{n} = \mathfrak{m}A[\![y]\!] + (y)$. It follows that $\mathfrak{n}$ is generated by a regular sequence of $d + 1$ elements. We conclude that $(A[\![y]\!])_{\mathfrak{n}}$ is a regular ring of dimension $d+1$. We also have that $A[\![y]\!]/\mathfrak{n} \cong A/\mathfrak{m}$ is an algebraic extension of $\mathbb{K}$.

It remains to show that $\mathrm{Der}_{A[\![y]\!]|K}$ is a projective module of rank $d + 1$ and it behaves well with localization. We note that every derivation δ in A can be extended to a derivation $A[\![y]\!]$ by $\delta(\sum_{n=0}^{\infty} f_n y^n) = \sum_{n=0}^{\infty} \delta(f_n) y^n$. Let $M = A[\![y]\!] \otimes_A \mathrm{Der}_{A|K} \oplus A[\![y]\!]\partial_y \subseteq \mathrm{Der}_{A[\![y]\!]|K}$. We note that the natural maps

$$M_{\mathfrak{n}} \to A[\![y]\!]_{\mathfrak{n}} \otimes_A \mathrm{Der}_{A[\![y]\!]|\mathbb{K}} \to \mathrm{Der}_{A[\![y]\!]_{\mathfrak{n}}|\mathbb{K}}$$

are injective. We fix $\mathfrak{n} \subseteq A[\![y]\!]$ a maximal ideal and a maximal ideal $\mathfrak{m} \subseteq R$ such that $\mathfrak{n} = \mathfrak{m}A[\![y]\!] + (y)$. We fix $\delta_1, \ldots, \delta_d \in \mathrm{Der}_{A_{\mathfrak{m}}|\mathbb{K}}$ and elements $f_1, \ldots, f_n \in \mathfrak{m}A_{\mathfrak{m}}$ such that $\delta_i f_j = 1$ if $i = j$ and 0 otherwise. We can do this by Theorem 5.4. Then, $\delta_1, \ldots, \delta_d, \partial_y$ satisfy Theorem 5.4(3). We conclude that $\delta_1, \ldots, \delta_d, \partial_y$ generate $\mathrm{Der}_{A[\![y]\!]_{\mathfrak{n}}|\mathbb{K}}$. Then, the composition of the maps

$$M_{\mathfrak{n}} \to A[\![y]\!]_{\mathfrak{n}} \otimes_A \mathrm{Der}_{A[\![y]\!]|\mathbb{K}} \to \mathrm{Der}_{A[\![y]\!]_{\mathfrak{n}}|\mathbb{K}}$$

is surjective. We conclude that they are isomorphic. Since

$$M_{\mathfrak{m}} = \left(A[\![y]\!]_{\mathfrak{n}} \otimes_{A_{\mathfrak{m}}} (\mathrm{Der}_{A|\mathbb{K}})_{\mathfrak{m}}\right) \oplus A[\![y]\!]_{\mathfrak{n}}\partial_y$$

is free of rank $d + 1$, we have that

$$(M_{\mathfrak{m}})_{\mathfrak{n}} = M_{\mathfrak{n}} \cong \mathrm{Der}_{A[\![y]\!]_{\mathfrak{n}}|\mathbb{K}}$$

is free of rank $d + 1$. □

Theorem 5.6 *Let A be differentiably admissible, and $\mathfrak{a} \subseteq A$. Then, the Bernstein-Sato polynomial of $\mathfrak{a}$ exists.*

Proof Let $f_1, \ldots, f_\ell$ be a set of generators for $\mathfrak{a}$. Let $f = f_1 y_1 + \cdots + f_\ell y_\ell \in A[\![y_1, \ldots, y_\ell]\!]$. There exists $b(s) \in \mathbb{K}[s] \smallsetminus \{0\}$ and $\delta(s) \in A[\![y_1, \ldots, y_\ell]\!][s]$ such that

$$\delta(s) f \boldsymbol{f}^s = b(s) \boldsymbol{f}^s$$

in $A_f[s]\boldsymbol{f}^s$ by Proposition 5.5 and Theorem 3.26. There exist finitely many $\beta \in \mathbb{N}^\ell$, $j \in \mathbb{N}$, $\delta_{\beta,j}[s] \in D_{A|\mathbb{K}}[s]$, and $g_{\beta,j} \in A[\![y_1, \ldots, y_\ell]\!]$ such that

$$\delta(s) = \sum_{\beta,j} g_{\beta,j} \delta_{\beta,j}(s) \frac{\partial^\beta}{\partial y^\beta}$$

because $D_{A[\![y_1,\ldots,y_\ell]\!]|\mathbb{K}}$ is generated by derivations by Remark 2.8, and by the description of $\mathrm{Der}_{A[\![y_1,\ldots,y_\ell]\!]|\mathbb{K}}$ in the proof of Proposition 5.5. Then, there exists $h_{\alpha,\beta,j} \in A$ such that $g_{\beta,j} = \sum_{\alpha \in \mathbb{N}^\ell} h_{\alpha,\beta,j} y^\alpha$. Then,

$$\delta(s) = \sum_{\beta,j} \sum_{\alpha \in \mathbb{N}^\ell} h_{\alpha,\beta,j} \delta_{\beta,j}(s) y^\alpha \frac{\partial^\beta}{\partial y^\beta}.$$

We have that

$$\begin{aligned} b(s)\boldsymbol{f}^s &= \delta(s) f \boldsymbol{f}^s \\ &= \sum_{\beta,j}\sum_{\alpha\in\mathbb{N}^\ell} h_{\alpha,\beta,j}\, y^\alpha \delta_{\beta,j}(s)\frac{\partial^\beta}{\partial y^\beta} f \boldsymbol{f}^s \\ &= \sum_{\beta,j}\sum_{\alpha\in\mathbb{N}^\ell} h_{\alpha,\beta,j}\delta_{\beta,j}(s) y^\alpha \frac{\partial^\beta}{\partial y^\beta} f \boldsymbol{f}^s. \end{aligned}$$

After specializing for $t \in \mathbb{N}$, we have that

$$b(t) f^t = \sum_{\beta,j}\sum_{\alpha\in\mathbb{N}^\ell} h_{\alpha,\beta,j}\delta_{\beta,j}(t) y^\alpha \frac{\partial^\beta}{\partial y^\beta} f^{t+1}.$$

Then,

$$\sum_{\beta,j}\sum_{|\alpha|\neq|\beta|-1} h_{\alpha,\beta,j}\delta_{\beta,j}(t) y^\alpha \frac{\partial^\beta}{\partial y^\beta} f^{t+1} = 0.$$

by comparing the degree in $y_1, \ldots, y_\ell$. Then,

$$\sum_{\beta,j}\sum_{|\alpha|\neq|\beta|-1} h_{\alpha,\beta,j}\delta_{\beta,j}(s) y^\alpha \frac{\partial^\beta}{\partial y^\beta} f \boldsymbol{f}^s = 0.$$

We have that

$$\tilde{\delta}(s) = \sum_{\beta,j}\sum_{|\alpha|=|\beta|-1} h_{\alpha,\beta,j}\delta_{\beta,j}(s) y^\alpha \frac{\partial^\beta}{\partial y^\beta}.$$

satisfies the functional equation and belongs to $D_{A[y_1,\ldots,y_\ell]|\mathbb{K}}[s]$. Then, the Bernstein-Sato polynomial of $\mathfrak{a}$ exists. □

5.2 *Bernstein-Sato Polynomial of General Ideals Revisited*

In this subsection we review the original definition of Bernstein-Sato polynomial of an ideal given by Budur et al. [36]. Indeed they provide two equivalent approaches depending on the ring of differential operators we are working with.

Let $\mathbb{K}$ a field of characteristic zero, A be a regular $\mathbb{K}$-algebra, and let $F = f_1, \ldots, f_\ell$ be a set of generators of an ideal $\mathfrak{a} \subseteq A$. Let $S = \{s_{ij}\}_{1 \le i,j \le \ell}$ be a new set of variables satisfying the following relations:

(i) $s_{ii} = s_i$ for $i = 1, \ldots, \ell$.
(ii) $[s_{ij}, s_{k\ell}] = \delta_{jk} s_{i\ell} - \delta_{i\ell} s_{kj}$,

where δ_{ij} is the Kronecker's delta function. Then we consider the ring $\mathbb{K}\langle S\rangle$ generated by S and $D_{A|\mathbb{K}}\langle S\rangle := D_{A|\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{K}\langle S\rangle$.

In this setting we have the following Bernstein-Sato type functional equation.

Definition 5.7 Let $\mathbb{K}$ be a field of characteristic zero and A a regular $\mathbb{K}$-algebra. A *Bernstein-Sato functional equation* in $D_{A|\mathbb{K}}\langle S\rangle$ for $F = f_1, \ldots, f_\ell$ is an equation of the form

$$\sum_{i=1}^{\ell} \delta_i(S) f_i f_1^{s_1} \cdots f_\ell^{s_\ell} = b(s_1 + \cdots + s_\ell) f_1^{s_1} \cdots f_\ell^{s_\ell}$$

where $\delta_i(S) \in D_{A|\mathbb{K}}\langle S\rangle$ and $b(s) \in \mathbb{K}[s]$.

Definition 5.8 Let $\mathbb{K}$ be a field of characteristic zero and A a regular $\mathbb{K}$-algebra. Let $F = f_1, \ldots, f_\ell$ be a set of generators of an ideal $\mathfrak{a} \subseteq A$. The Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ of $\mathfrak{a}$ is the monic polynomial of smallest degree satisfying a Bernstein-Sato functional equation in $D_{A|\mathbb{K}}\langle S\rangle$.

Budur, Mustaţă, and Saito proved the existence of such Bernstein-Sato polynomial. Moreover, they also proved that it does not depend on the set of generators of the ideal so it is well-defined (see [36, Theorem 2.5]).

After a convenient shifting we can define the Bernstein-Sato polynomial of an algebraic variety.

Theorem 5.9 ([36]) *Let $Z(\mathfrak{a}) \subseteq \mathbb{C}^d$ be the closed variety defined by an ideal $\mathfrak{a} \subseteq A$ and c be the codimension of $Z(\mathfrak{a})$ in $\mathbb{C}^d$. Then*

$$b_{Z(\mathfrak{a})}(s) := b_{\mathfrak{a}}(s - c)$$

depends only on the affine scheme $Z(\mathfrak{a})$ and not on $\mathfrak{a}$.

In this setting we also have that the Bernstein-Sato functional equation in $D_{A|\mathbb{K}}\langle S\rangle$ is an equality in $A_f[s_1, \ldots, s_p]\boldsymbol{f}^s$. The $D_{A|\mathbb{K}}\langle S\rangle$-module structure on this module is given by

$$s_{ij} \cdot a(s_1, \ldots, s_p)\boldsymbol{f}^s := s_i a(s_1, \ldots, s_i - 1, \ldots, s_j + 1, \ldots, s_p)\frac{f_j}{f_i}\boldsymbol{f}^s$$

where $a(s_1, \ldots, s_p) \in A_f[s_1, \ldots, s_p]$. The $D_{A|\mathbb{K}}\langle S\rangle$-submodule generated by $\boldsymbol{f}^{s}$ has a presentation

$$D_{A|\mathbb{K}}\langle S\rangle \boldsymbol{f}^s \cong \frac{D_{A|\mathbb{K}}\langle S\rangle}{\operatorname{Ann}_{D\langle S\rangle}(\boldsymbol{f}^s)},$$

and thus

$$\frac{D_{A|\mathbb{K}}\langle S\rangle \boldsymbol{f}^s}{D_{A|\mathbb{K}}\langle S\rangle (f_1, \ldots, f_p)\boldsymbol{f}^s} \cong \frac{D_{A|\mathbb{K}}\langle S\rangle}{\operatorname{Ann}_{D\langle S\rangle}(\boldsymbol{f}^s) + D_{A|\mathbb{K}}\langle S\rangle(f_1, \ldots, f_p)}.$$

We have an analogue of Proposition 3.13 that is used in order to provide algorithms for the computations of these Bernstein-Sato polynomials [3].

Proposition 5.10 *The Bernstein-Sato polynomial of an ideal $\mathfrak{a} \subseteq A$ generated by $F = f_1, \ldots, f_\ell$ is the monic generator of the ideal*

$$\big(b_{\mathfrak{a}}(s_1 + \cdots + s_p)\big) = \mathbb{K}[s_1 + \cdots + s_p] \cap \big(Ann_{D\langle S\rangle}(\boldsymbol{f}^s) + D_{A|\mathbb{K}}\langle S\rangle(f_1, \ldots, f_p)\big).$$

Budur et al. [36, Section 2.10] gave an equivalent definition of Bernstein-Sato polynomial of $\mathfrak{a}$ using a functional equation in $D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]$ instead of $D_{A|\mathbb{K}}\langle S\rangle$.

Theorem 5.11 ([36]) *Let $\mathbb{K}$ a field of characteristic zero, A be a regular $\mathbb{K}$-algebra, and $\mathfrak{a} \subseteq A$ be a nonzero ideal. Let $F = f_1, \ldots, f_\ell$ be a set of generators for $\mathfrak{a}$. Then, $b_{\mathfrak{a}}(s) \in \mathbb{K}[s]$ is the monic polynomial of least degree, $b(s)$ such that*

$$b(s_1 + \cdots + s_\ell) f_1^{s_1} \cdots f_\ell^{s_\ell} \in \sum_{|\alpha|=1} D_{R|\mathbb{K}}[s_1, \ldots, s_\ell] \cdot \prod_{\alpha_i} \binom{s_i}{-\alpha_i} f_1^{s_1+\alpha_1} \cdots f_\ell^{s_\ell+\alpha_\ell},$$

where $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$, $|\alpha| = \alpha_1 + \cdots + \alpha_\ell$, $\binom{s_i}{m} = \frac{1}{m!}\prod_{j=0}^{m-1}(s_i - j)$.

Mustaţă [101, Theorem 1.1] uses this characterization to show that $b_{\mathfrak{a}}(s)$ coincides with the reduced Bernstein-Sato polynomial of $f_1y_1 + \cdots + f_\ell y_\ell \in A[y_1, \ldots, y_\ell]$.

One may be tempted to consider a general element $\lambda_1 f_1 + \cdots + \lambda_\ell f_\ell \in \mathfrak{a}$ whose log-resolution has the same numerical data as the log-resolution of the ideal $\mathfrak{a}$.

Example 5.12 Let $\mathfrak{a} = (x^4, xy^2, y^3) \subseteq \mathbb{C}[x, y]$ be a monomial ideal and consider a general element of the ideal $g = x^4 + xy^2 + y^3$. The roots of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ are

$$\left\{-\frac{5}{8}, -\frac{2}{3}, -\frac{3}{4}, -\frac{7}{8}, -1, -\frac{9}{8}, -\frac{5}{4}, -\frac{4}{3}, -\frac{11}{8}, -\frac{3}{2}\right\},$$

with -1 being a root with multiplicity 2. Meanwhile, the roots of the reduced Bernstein-Sato polynomial $\tilde{b}_g(s)$ are

$$\left\{-\frac{5}{8}, -\frac{7}{8}, -1, -\frac{9}{8}, -\frac{11}{8}\right\}$$

The exceptional part of the log-resolution divisor F_π in both cases is of the form $3E_1 + 4E_2 + 8E_3$. The roots of $\tilde{b}_g(s)$ are only contributed by the rupture divisor E_3 but this is not the case for $b_{\mathfrak{a}}(s)$.

5.2.1 Monomial Ideals

Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_d]$ be a monomial ideal. Let $P_{\mathfrak{a}} \subseteq \mathbb{R}^d_{\geq 0}$ be the Newton polyhedron associated to $\mathfrak{a}$ which is the convex hull of the semigroup

$$\Gamma_{\mathfrak{a}} = \{a = (a_1, \dots, a_d) \in \mathbb{N}^d \ \mid \ x_1^{a_1} \cdots x_d^{a_d} \in \mathfrak{a}\}.$$

For any face Q of $P_{\mathfrak{a}}$ we define:

(i) M_Q the subsemigroup of $\mathbb{Z}^d$ generated by $a - b$ with $a \in \Gamma_{\mathfrak{a}}$ and $b \in \Gamma_{\mathfrak{a}} \cap Q$.
(ii) $M'_Q := c + M_Q$ for $c \in \Gamma_{\mathfrak{a}} \cap Q$.

M'_Q is a subset of M_Q that is independent of the choice of c. For a face Q of $P_{\mathfrak{a}}$ not contained in a coordinate hyperplane we consider a function $L_Q : \mathbb{R}^d \to \mathbb{R}$ with rational coefficients such that $L_Q = 1$ on Q. Set

$$R_Q = \{L_Q(a) \ \mid \ a \in ((1, \dots, 1) + (M_Q \setminus M'_Q)) \cap V_Q\},$$

where V_Q is the linear subspace generated by Q.

Budur, Mustaţă, and Saito [35] gave a closed formula for the roots of the Bernstein-Sato polynomial of $\mathfrak{a}$ in terms of these sets R_Q.

Theorem 5.13 ([35]) *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_d]$ be a monomial ideal. Let $\rho_{\mathfrak{a}}$ be the set of roots of $b_{\mathfrak{a}}(-s)$. Then*

$$\rho_{\mathfrak{a}} = \bigcup_Q R_Q$$

where the union is over the faces Q of $P_{\mathfrak{a}}$ not contained in coordinate hyperplanes.

5.2.2 Determinantal Varieties

The theory of equivariant D-modules has been successfully used in recent years to study local cohomology modules of determinantal varieties. These techniques have also been used by Lőrincz et al. [84] to determine the Bernstein-Sato polynomial of the ideal of maximal minors of a generic matrix.

Theorem 5.14 ([84]) *Let $X = (x_{ij})$ be a generic $m \times n$ matrix with $m \geq n$. Let $\mathfrak{a}_n \subseteq A = \mathbb{C}[x_{ij}]$ be the ideal generated by the $n \times n$ minors of X. The Bernstein-Sato polynomials of the ideal $\mathfrak{a}_n$ and the corresponding variety are*

$$b_{\mathfrak{a}_n}(s) = \prod_{\ell=m-n+1}^{m} (s+\ell)\,.$$

$$b_{Z(\mathfrak{a}_n)}(s) = \prod_{\ell=0}^{n-1} (s+\ell)\,.$$

They also provided a formula for sub-maximal Pfaffians.

Theorem 5.15 ([84]) *Let $X = (x_{ij})$ be a generic $(2n+1)\times(2n+1)$ skew-symmetric matrix, i.e. $x_{ii} = 0, x_{ij} = -x_{ji}$. Let $\mathfrak{b}_{2n} \subseteq A = \mathbb{C}[x_{ij}]$ be the ideal generated by the $2n \times 2n$ Pfaffians of X. The Bernstein-Sato polynomials of the ideal $\mathfrak{b}_{2n}$ and the corresponding variety are*

$$b_{\mathfrak{b}_{2n}}(s) = \prod_{\ell=0}^{n-1} (s+2\ell+3)\,.$$

$$b_{Z(\mathfrak{b}_{2n})}(s) = \prod_{\ell=0}^{n-1} (s+2\ell)\,.$$

5.3 Bernstein-Sato Ideals

In this subsection we consider the theory of Bernstein-Sato ideals associated to a tuple of elements $F = f_1, \dots, f_\ell$ developed by Sabbah [118].

Definition 5.16 Let $\mathbb{K}$ be a field of characteristic zero and A a regular $\mathbb{K}$-algebra. A *Bernstein-Sato functional equation* for a tuple $F = f_1, \dots, f_\ell$ of elements of A is an equation of the form

$$\delta(s_1,\dots,s_\ell) f_1^{s_1+1} \cdots f_\ell^{s_\ell+1} = b(s_1,\dots,s_\ell) f_1^{s_1} \cdots f_\ell^{s_\ell}$$

where $\delta(s_1,\dots,s_\ell) \in D_{A|\mathbb{K}}[s_1,\dots,s_\ell]$ and $b(s_1,\dots,s_\ell) \in \mathbb{K}[s_1,\dots,s_\ell]$.

All the polynomials $b(s_1,\dots,s_\ell)$ satisfying a Bernstein-Sato functional equation form an ideal $B_F \subseteq \mathbb{K}[s_1,\dots,s_\ell]$ that we refer to as the *Bernstein-Sato ideal*.

Remark 5.17 More generally, given $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell_{\geq 0}$, we may also consider the functional equations

$$\delta(s_1, \ldots, s_\ell) f_1^{s_1+a_1} \cdots f_\ell^{s_\ell+a_\ell} = b(s_1, \ldots, s_\ell) f_1^{s_1} \cdots f_\ell^{s_\ell} \quad \text{for all } s_i \in \mathbb{N},$$

leading to other Bernstein-Sato ideals $B_F^a \subseteq \mathbb{K}[s_1, \ldots, s_\ell]$.

As in the case $\ell = 1$ we first wonder about the existence of such functional equations.

Theorem 5.18 ([118]) *Let $\mathbb{K}$ be a field of characteristic zero, and let A be either $\mathbb{K}[x_1, \ldots, x_d]$ or $\mathbb{C}\{x_1, \ldots, x_d\}$. Any nonzero tuple $F = f_1, \ldots, f_\ell$ of elements of A satisfies a nonzero Bernstein-Sato functional equation and thus $B_F \neq 0$.*

Sabbah [118] proved this result in the local analytic case $A = \mathbb{C}\{x_1, \ldots, x_d\}$. The proof in the polynomial ring case $A = \mathbb{K}[x_1, \ldots, x_d]$ is completely analogous to the one given in Sect. 3.3 for the case $\ell = 1$.

The Bernstein-Sato functional equation is an equality in $A_f[s_1, \ldots, s_\ell]\boldsymbol{f^s}$ where $f = f_1 \cdots f_\ell$ and $\boldsymbol{f^s} := \boldsymbol{f_1^{s_1}} \cdots \boldsymbol{f_\ell^{s_\ell}}$. We also have that the $D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]$-submodule generated by $\boldsymbol{f^s}$ has a presentation

$$D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]\boldsymbol{f^s} \cong \frac{D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]}{\mathrm{Ann}_{D[s_1,\ldots,s_\ell]}(\boldsymbol{f^s})},$$

and, given the fact that

$$\frac{D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]\boldsymbol{f^s}}{D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]f\boldsymbol{f^s}} \cong \frac{D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]}{\mathrm{Ann}_{D[s_1,\ldots,s_\ell]}(\boldsymbol{f^s}) + D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]f}.$$

we get an analogue of Proposition 3.13 that reads as

Proposition 5.19 *The Bernstein-Sato ideal of $F = f_1, \ldots, f_\ell$ is*

$$B_F = \mathbb{K}[s_1, \ldots, s_\ell] \cap (Ann_{D_{A|\mathbb{K}}[s_1,\ldots,s_\ell]}(\boldsymbol{f^s}) + D_{A|\mathbb{K}}[s_1, \ldots, s_\ell]f).$$

Some properties of Bernstein-Sato ideals are the natural extension of those satisfied by Bernstein-Sato polynomials. We start with the ones considered in Sect. 3.5. The analogue of Lemma 3.27 is the following result.

Lemma 5.20 ([30, 95]) *Let $F = f_1, \ldots, f_\ell$ be a tuple where the f_i are pairwise without common factors. Then*

$$B_F \subseteq \Big((s_1 + 1) \cdots (s_\ell + 1)\Big).$$

Equality is achieved if and only if $A/(f_1, \ldots, f_\ell)$ is smooth.

We summarize the relations between the Bernstein-Sato ideals when we change the ring A in the following lemma. For the convenience of the reader we use temporally the same notation as in Sect. 3.5.

Lemma 5.21 ([28]) *We have:*

(i) $B_F^{\mathbb{K}[x]} = \bigcap_{\mathfrak{m} \text{ max ideal}} B_F^{\mathbb{K}[x]_{\mathfrak{m}}}$.
(ii) $B_F^{\mathbb{K}[x]_{\mathfrak{m}}} = B_F^{\mathbb{K}[[x]]}$, *where* $\mathfrak{m}$ *is the homogeneous maximal ideal.*
(iii) $B_F^{\mathbb{C}\{x-p\}} = B_F^{\mathbb{C}[[x-p]]}$, *where* $p \in \mathbb{C}^d$.
(iv) $B_F^{\mathbb{L}[x]} = \mathbb{L} \otimes_{\mathbb{K}} B_F^{\mathbb{K}[x]}$ *where* $\mathbb{L}$ *is a field containing* $\mathbb{K}$.

The first rationality result for Bernstein-Sato ideals is given by Gyoja [59] and Sabbah [118] where they proved the existence of an element of B_F which is a product of polynomials of degree one of the form $a_1 s_1 + \cdots + a_\ell s_\ell + a$, with $a_i \in \mathbb{Q}_{\geq 0}$ and $a \in \mathbb{Q}_{>0}$. This fact prompted Budur [32] to make the following:

Conjecture 5.22 The Bernstein-Sato ideal of a tuple $F = f_1, \ldots, f_\ell$ of elements in $\mathbb{C}\{x_1, \ldots, x_d\}$ is generated by products of polynomials of degree one

$$a_1 s_1 + \cdots + a_\ell s_\ell + a,$$

with $a_i \in \mathbb{Q}_{\geq 0}$ and $a \in \mathbb{Q}_{>0}$

Notice that this would imply that the irreducible components of the zero locus $Z(B_F)$ are linear. The best result so far towards this conjecture is the following.

Theorem 5.23 ([87]) *Every irreducible component of* $Z(B_F)$ *of codimension* 1 *is a hyperplane of type* $a_1 s_1 + \cdots + a_\ell s_\ell + a$, *with* $a_i \in \mathbb{Q}_{\geq 0}$ *and* $a \in \mathbb{Q}_{>0}$. *Every irreducible component of* $Z(B_F)$ *of codimension* > 1 *can be translated by an element of* $\mathbb{Z}^\ell$ *inside a component of codimension* 1.

Recall that the work of Kashiwara and Malgrange relates the roots of the Bernstein-Sato polynomials to the eigenvalues of the monodromy and these eigenvalues are roots of unity by the monodromy theorem. An extension to the case of Bernstein-Sato ideals of Kashiwara and Malgrange result has been given recently by Budur [32] and Budur et al. [39]. There is also an extension of the Monodromy theorem in this setting given by Budur and Wang [40] and Budur et al. [34]. Unfortunately these results are not enough to settle Conjecture 5.22.

The main difference with the classical case is that Bernstein-Sato ideals are not necessarily principally generated. Briançon and Maynadier [30] gave a theoretical proof of this fact for the following example. The explicit computation was given by Balhoul and Oaku [7].

Example 5.24 ([7, 30]) Let $F = z, x^4 + y^4 + zx^2y^2$ be a pair of elements in $\mathbb{C}\{x, y, z\}$. The local Bernstein-Sato ideal is nonprincipal

$$B_F^{\mathbb{C}\{x\}} = \Big((s_1 + 1)(s_2 + 1)^2(2s_2 + 1)(4s_2 + 3)(4s_2 + 5)(s_1 + 2),$$
$$(s_1 + 1)(s_2 + 1)^2(2s_2 + 1)(4s_2 + 3)(4s_2 + 5)(2s_2 + 3)\Big).$$

However, when we consider F in $\mathbb{C}[x, y, z]$ the global Bernstein-Sato ideal is

$$B_F^{\mathbb{C}[x]} = \Big((s_1 + 1)(s_2 + 1)^2(2s_2 + 1)(2s_2 + 3)(4s_2 + 3)(4s_2 + 5)\Big).$$

The following example is also given by Balhoul and Oaku.

Example 5.25 ([7]) Let $F = z, x^5 + y^5 + zx^2y^3$ be a pair of elements in $\mathbb{C}[x, y, z]$. Then the local and the global Bernstein-Sato ideals coincide and are nonprincipal. Specifically, B_F is generated by $(s_1 + 1)(s_2 + 1)^2(5s_2 + 2)(5s_2 + 3)(5s_2 + 4)(5s_2 + 6)(s_1 + 2)(s_1 + 3)(s_1 + 4)(s_1 + 5)$, $(s_1 + 1)(s_2 + 1)^2(5s_2 + 2)(5s_2 + 3)(5s_2 + 4)(5s_2 + 6)(5s_2 + 7)(s_1 + 2)$, and $(s_1 + 1)(s_2 + 1)^2(5s_2 + 2)(5s_2 + 3)(5s_2 + 4)(5s_2 + 6)(5s_2 + 7)(5s_2 + 8)$.

There are interesting examples worked out in several computational articles by Balhoul [6], Balhoul and Oaku [7], Castro-Jiménez and Ucha-Enríquez [139], Andres et al. [3]. However, we cannot find many closed formulas for families of examples. Maynadier [95] studied the case of quasi-homogeneous isolated complete intersection singularities and we highlight the case of hyperplane arrangements.

5.3.1 Hyperplane Arrangements

Let $f \in \mathbb{C}[x_1, \dots, x_d]$ be a reduced polynomial defining an arrangement of hyperplanes. The most natural tuple $F = f_1, \cdots, f_\ell$ associated to f is the one given by its degree one components. The following result is an extension of Walther's work to this setting. It was first obtained by Maisonobe [88] for the case $\ell = d + 1$ and further extended by Bath [8] for $\ell \geq d + 1$. We point out that Bath also provides a formula for other tuples associated to different decompositions of the arrangement f.

Theorem 5.26 ([8, 88]) *Let $f = f_1 \cdots f_\ell \in \mathbb{C}[x_1, \dots, x_d]$, with $\ell \geq d + 1$, be the decomposition of a generic central hyperplane arrangement as a product of linear forms. The Bernstein-Sato ideal of the tuple $F = f_1, \dots, f_\ell$ is*

$$B_F = \left(\prod_{i=1}^{\ell}(s_i + 1) \prod_{j=0}^{2\ell-d-2} (s_1 + \cdots + s_\ell + j + d)\right).$$

5.4 Relative Versions

In this section we discuss a more general version of the Bernstein-Sato polynomials in which the functional equation includes an element of a D-module M [97, 117]. As in the classical case, we consider this functional equation as an equality in a given module that we define next.

Definition 5.27 Let A be a differentiably admissible $\mathbb{K}$-algebra, and M a left $D_{A|\mathbb{K}}$-module. For $f \in A \smallsetminus \{0\}$, we define the left $D_{A_f|\mathbb{K}}[s]$-module $M_f[s]\boldsymbol{f^s}$ as follows:

(i) As an $A_f[s]$-module, $M_f[s]\boldsymbol{f^s}$ is isomorphic to $M_f[s]$.
(ii) Each partial derivative $\partial \in \mathrm{Der}_{A|\mathbb{K}}$ acts by the rule

$$\partial(a(s)v\boldsymbol{f^s}) = \left(a(s)\partial(v) + \frac{sa(s)\partial(f)}{f}\right)\boldsymbol{f^s}$$

for $a(s) \in A_f[s]$.

Alternative descriptions can be given analogously to Sect. 3.2, but we do not need them here.

Theorem 5.28 ([98, Theorem 3.1.1], [117]) *Let A be a differentiably admissible $\mathbb{K}$-algebra, M a left $D_{A|\mathbb{K}}$-module in the Bernstein class, and $f \in A \smallsetminus \{0\}$. For any element $v \in M$ there exists $\delta(s) \in D_{A|\mathbb{K}}[s]$ and $b(s) \in \mathbb{K}[s] \smallsetminus \{0\}$ such that*

$$\delta(s)vf\boldsymbol{f^s} = b(s)v\boldsymbol{f^s}.$$

There are not many explicit examples of Bernstein-Sato polynomials in this generality that we may find in the literature. Torrelli [135, 136] has some results in the case that M is the local cohomology module of a complete intersection or a hypersurface with isolated singularities. Reichelt et al. [115] studied the case of hypergeometric systems.

In the case of M being the ring itself, we find the Bernstein-Sato polynomial of f relative to an element $h \in A$. Of course, when $h = 1$ we recover the classical version.

Corollary 5.29 *Let A be a differentiably admissible $\mathbb{K}$-algebra and $f \in A \smallsetminus \{0\}$. For any element $h \in A$ there exists $\delta(s) \in D_{A|\mathbb{K}}[s]$ and $b(s) \in \mathbb{K}[s] \smallsetminus \{0\}$ such that*

$$\delta(s)hf\boldsymbol{f^s} = b(s)h\boldsymbol{f^s}.$$

Definition 5.30 Let A be a differentiably admissible $\mathbb{K}$-algebra, M a left $D_{A|\mathbb{K}}$-module in the Bernstein class, $f \in A \smallsetminus \{0\}$, and $v \in M$. We define the relative

Bernstein-Sato polynomial $b_{f,v}(s)$ to be the monic polynomial of minimal degree for which there is a nonzero functional equation

$$\delta(s)vf\boldsymbol{f}^s = b_{f,v}(s)v\boldsymbol{f}^s.$$

A basic example shows that $s = -1$ need not always be a root of the relative Bernstein-Sato polynomial $b_{f,g}(s)$.

Example 5.31 Let $A = \mathbb{C}[x]$, and take $f = g = x$. We have a functional equation

$$\partial_x x^{s+1}x = (s+2)x^s x \text{ for all } s,$$

so $s = -1$ is not a root of $b_{x,x}(s)$. It follows from the next proposition that $b_{x,x}(s) = s+2$.

We record a basic property of relative Bernstein-Sato polynomials that may be considered as an analogue to Lemma 3.27.

Lemma 5.32 *Let A be a differentially admissible $\mathbb{K}$-algebra, and $f, g \in A \smallsetminus \{0\}$. If $g \in (f^{n-1}) \smallsetminus (f^n)$, then $s = -n$ is a root of $b_{f,g}(s)$.*

Proof Evaluating the functional equation at $s = -n$, we have

$$\delta(-n)ff^{-n}g = b(-n)f^{-n}g.$$

Since $g/f^{n-1} \in R$, and $g/f^n \notin R$, we must have $b(-n) = 0$. □

We make another related observation.

Lemma 5.33 *Let A be a differentially admissible $\mathbb{K}$-algebra, and $f, g \in A \smallsetminus \{0\}$. Then $b_{f,f^n g}(s) = b_{f,g}(s+n)$ for all n.*

Proof Given a functional equation

$$\delta(s)gf\boldsymbol{f}^s = b_{f,g}(s)g\boldsymbol{f}^s,$$

shifting by n yields

$$\delta(s+n)gf^n f\boldsymbol{f}^s = b_{f,g}(s+n)gf^n\boldsymbol{f}^s,$$

so $b_{f,g}(s+n) \mid b_{f,f^n g}(s)$. Similarly, given a functional equation

$$\delta'(s)gf^n f\boldsymbol{f}^s = b_{f,f^n g}(s)gf^n\boldsymbol{f}^s,$$

we also have

$$\delta'(s-n)gf\boldsymbol{f}^s = b_{f,f^n g}(s-n)g\boldsymbol{f}^s,$$

from which the equality follows. □

This notion of relative Bernstein-Sato polynomials has been extended to the case of nonprincipal ideals by Budur et al. [36] following the approach given in Sect. 5.2.

Theorem 5.34 ([36]) *Let $\mathbb{K}$ a field of characteristic zero, A be a regular finitely generated $\mathbb{K}$-algebra, and $\mathfrak{a} \subseteq A$ be a nonzero ideal. Let $F = f_1, \ldots, f_\ell$ be a set of generators for $\mathfrak{a}$ and consider an element $h \in A$. Then, $b_{\mathfrak{a},h}(s) \in \mathbb{K}[s]$ is the monic polynomial of least degree, $b(s)$ such that*

$$b(s_1+\cdots+s_\ell)hf_1^{s_1}\cdots f_\ell^{s_\ell} \in \sum_{|\alpha|=1} D_{R|\mathbb{K}}[s_1,\ldots,s_\ell]\cdot\prod_{\alpha_i}\binom{s_i}{-\alpha_i}hf_1^{s_1+\alpha_1}\cdots f_\ell^{s_\ell+\alpha_\ell},$$

where $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$, $|\alpha| = \alpha_1 + \cdots + \alpha_\ell$, $\binom{s_i}{m} = \frac{1}{m!}\prod_{j=0}^{m-1}(s_i - j)$.

5.5 *V*-Filtrations

In this subsection, we give a quick overview of the V-filtration and its relationship with the relative versions of Bernstein-Sato polynomials. For further details regarding V-filtrations we refer to Budur's survey on this subject [31].

Definition 5.35 Suppose that $\mathbb{K}$ has characteristic zero. Let A be a regular Noetherian $\mathbb{K}$-algebra. Let $T = t_1, \ldots, t_\ell$ be a sequence of variables, and let $A[t_1, \ldots, t_\ell]$ be a polynomial ring over A. The V-filtration along the ideal (T) on the ring of differential operators $D_{A[T]|\mathbb{K}}$ is the filtration indexed by integers $i \in \mathbb{Z}$ defined by

$$V^i_{(T)}D_{A[T]|\mathbb{K}} = \{\delta \in D_{A[T]|\mathbb{K}} : \delta \bullet (T)^j \subseteq (T)^{j+i} \text{ for all } j \in \mathbb{Z}\},$$

where $(T)^j = A[T]$ for $j \leq 0$.

Remark 5.36 We consider $D_{A[T]|\mathbb{K}}$ as a graded ring where $\deg(t_i) = 1$ and $\deg(\partial_{t_i}) = -1$. Then,

$$V^i_{(T)}D_{A[T]|\mathbb{K}} = \bigoplus_{\substack{a,b\in\mathbb{N}^\ell \\ |a|-|b|\geq i}} D_{A|\mathbb{K}}\cdot t_1^{a_1}\cdots t_\ell^{a_\ell}\partial_{t_1}^{b_1}\cdots\partial_{t_\ell}^{b_\ell}.$$

The V-filtration along the ideal (T) on a $D_{A[T]|\mathbb{K}}$-module M is defined as follows.

Definition 5.37 Suppose that $\mathbb{K}$ has characteristic zero. Let A be a regular Noetherian $\mathbb{K}$-algebra. Let $T = t_1, \ldots, t_\ell$ be a sequence of variables, and let $A[t_1, \ldots, t_\ell]$ be a polynomial ring over A. Let M be a $D_{A[T]|\mathbb{K}}$-module. A *V-filtration on M along the ideal* $(T) = (t_1, \ldots, t_\ell)$ is a decreasing filtration $\{V^\alpha_{(T)}M\}_\alpha$ on M, indexed by $\alpha \in \mathbb{Q}$, satisfying the following conditions.

(i) For all $\alpha \in \mathbb{Q}$, $V^{\alpha}_{(T)}M$ is a Noetherian $V^0_{(T)}D_{A[T]|\mathbb{K}}$-submodule of M.
(ii) The union of the $V^{\alpha}_{(T)}M$, over all $\alpha \in \mathbb{Q}$, is M.
(iii) $V^{\alpha}_{(T)}M = \bigcap_{\gamma<\alpha} V^{\gamma}_{(T)}M$ for all α, and the set J consisting of all $\alpha \in \mathbb{Q}$ for which $V^{\alpha}_{(T)}M \neq \bigcup_{\gamma>\alpha} V^{\gamma}_{(T)}M$ is discrete.
(iv) For all $\alpha \in \mathbb{Q}$ and all $1 \le i \le \ell$,

$$t_i \bullet V^{\alpha}_{(T)}M \subseteq V^{\alpha+1}_{(T)}M \text{ and } \partial_{t_i} \bullet V^{\alpha}_{(T)}M \subseteq V^{\alpha-1}_{(T)}M,$$

i.e., the filtration is compatible with the V-filtration on $D_{A[T]|\mathbb{K}}$.
(v) For all $\alpha \gg 0$, $\sum_{i=1}^{\ell}\left(t_i \bullet V^{\alpha}_{(T)}M\right) = V^{\alpha+1}_{(T)}M$.
(vi) For all $\alpha \in \mathbb{Q}$,

$$\sum_{i=1}^{\ell} \partial_{t_i} t_i - \alpha$$

acts nilpotently on $V^{\alpha}_{(T)}M/(\bigcup_{\gamma>\alpha} V^{\gamma}_{(T)}M)$.

Proposition 5.38 ([31]) *Suppose that $\mathbb{K}$ has characteristic zero. Let A be a regular Noetherian $\mathbb{K}$-algebra. Let $T = t_1, \dots, t_\ell$ be a sequence of variables, and let $A[t_1, \dots, t_\ell]$ be a polynomial ring over A. Let M be a finitely generated $D_{A[T]|\mathbb{K}}$-module. If a V-filtration on M along (T) exists, then it is unique.*

We now define the V-filtration on a $D_{A|\mathbb{K}}$-module M along $F = f_1, \dots, f_\ell \in A$, where M is a $D_{R|\mathbb{K}}$-module. For this, we need the direct image of M under the graph embedding i_F. We recall that this is the local cohomology module $H^{\ell}_{(T-F)}(M[T])$, where $(T - F) = (t_1 - f_1, \dots, t_\ell - f_\ell)$.

Definition 5.39 Suppose that $\mathbb{K}$ has characteristic zero. Let A be a regular Noetherian $\mathbb{K}$-algebra. Given indeterminates $T = t_1, \dots, t_\ell$, and $F = f_1, \dots, f_\ell \in A$, consider the ideal $(T-F)$ of the polynomial ring $A[T]$ generated by $t_1-f_1, \dots, t_\ell-f_\ell$. For a $D_{A|\mathbb{K}}$-module M, let M' denote the $D_{A[T]|\mathbb{K}}$-module $H^{\ell}_{(T-F)}(M[T])$, and identify M with the isomorphic module $0 :_{M'} (T - F) \subseteq M'$. Suppose that M' admits a V-filtration along (T) over $A[T]$. Then the *V-filtration on M along $(T-F)$* is defined, for $\alpha \in \mathbb{Q}$, as

$$V^{\alpha}_{(F)}M := V^{\alpha}_{(T)}M' \cap M = (0 :_{V^{\alpha}_{(T)}M'} (T - F)).$$

We point out that V-filtration over A along F only depends on the ideal $\mathfrak{a} = (F)$ and not on the generators chosen.

We now give a result that guarantees the existence of V-filtrations. We point out that we have not defined regular or quasi-unipotent $D_{A|\mathbb{K}}$-modules. We omit these definitions, but we mention that all principal localizations A_f and all local cohomology modules $H^i_{\mathfrak{a}}(A)$ of the ring A satisfy these properties.

Theorem 5.40 ([72, 93]) *Suppose that* $\mathbb{K}$ *has characteristic zero. Let* $A = \mathbb{K}[x_1, \ldots, x_d]$ *be a polynomial ring and* M *be a quasi-unipotent regular holonomic left* $D_{A|\mathbb{K}}$*-module. Then,* M *has a* V*-filtration along* $F = f_1, \ldots, f_\ell \in A$.

Once we ensure the existence of V-filtrations we have the following characterization in terms of relative Bernstein-Sato polynomials.

Theorem 5.41 ([36, 117]) *Suppose that* $\mathbb{K}$ *has characteristic zero. Let* $A = \mathbb{K}[x_1, \ldots, x_d]$ *be a polynomial ring and* M *be a quasi-unipotent regular holonomic left* $D_{A|\mathbb{K}}$*-module. Then,*

$$V^{\alpha}_{(F)}M = \{v \in M \mid \alpha \leq c \textit{ if } b_{(F),v}(-c) = 0\}.$$

6 Bernstein-Sato Theory in Prime Characteristic

We now discuss Bernstein-Sato theory in positive characteristic. Throughout this section, $\mathbb{K}$ is a perfect field of characteristic $p > 0$, and $A = \mathbb{K}[x_1, \ldots, x_d]$ is a polynomial ring. The main purpose of this section is to discuss the theory developed by Mustaţă [100], Bitoun [14], and Quinlan-Gallego [114].

Before we do so, as motivation, we briefly discuss the notion of the Bernstein-Sato functional equation in positive characteristic. Note that for $b(s) \in \mathbb{K}[s]$, we have $b(s)f^s = c(s)f^s$ for all $s \in \mathbb{N}$ if and only if b and c determine the same function from $\mathbb{F}_p$ to $\mathbb{K}$. This gives a recipe for many unenlightening functional equations: we can take $b(s)$ to be a function identically zero on $\mathbb{F}_p$, e.g., $s^p - s$, and $\delta(s)$ to be some operator that annihilates every power of f, e.g., the zero operator. For this reason, the notion of Bernstein-Sato polynomial in characteristic zero is not as well-suited for consideration in positive characteristic.

Instead, we return to an alternative characterization of the Bernstein-Sato polynomial discussed in Sect. 3.2. As a consequence of Proposition 3.13, for polynomial rings in characteristic zero, we can characterize the roots of the Bernstein-Sato polynomial of f as the eigenvalues of the action of $-\partial_t t$ on $[\frac{1}{f-t}]$ in

$$\frac{D_{A|\mathbb{K}}[-\partial_t t] \cdot [\frac{1}{f-t}]}{D_{A|\mathbb{K}}[-\partial_t t]f \cdot [\frac{1}{f-t}]}.$$

In characteristic $p > 0$, we consider the eigenvalues of a sequence of operators that are closely related to $-\partial_t t$.

Definition 6.1 Consider $D_{A[t]|\mathbb{K}}$ as a graded ring, with grading induced by giving each x_i degree zero, and t degree 1. We set $[D_{A[t]|\mathbb{K}}]_0$ to be the subring of homogeneous elements of degree zero, and $[D_{A[t]|\mathbb{K}}]_{\geq 0}$ to be the subring spanned by elements of nonnegative degree.

We note that $[D_{A[t]|\mathbb{K}}]_{\geq 0}$ is also characterized by the V-filtration as $V^0_{(t)}D_{A[t]|\mathbb{K}}$.

Lemma 6.2 $[D_{A[t]|\mathbb{K}}]_0 = D_{A|\mathbb{K}}[s_0, s_1, \dots]$, *where* $s_e = -\frac{\partial_t^{p^e}}{p^e!} t^{p^e}$. *In this ring, the operators* s_i *commute with one another and elements of* $D_{A|\mathbb{K}}$, *and* $s_i^p = s_i$ *for each* i.

Proof We omit the proof that these elements generate. It is clear that each s_i commutes with elements of $D_{A|\mathbb{K}}$. For an element $f(t) = \sum_j a_j t^j \in A[t]$, with $a_j \in A$, using Lucas' Lemma, we compute

$$s_i f(t) = \sum_j -\binom{j + p^i}{p^i} a_j t^j = \sum_j -([j]_i + 1) a_j t^j,$$

where $[j]_i$ is the ith digit in the base p expansion of j; our convention that the unit digit is the 0th digit. The other claims follow from this computation. □

We can interpret the computation in the previous lemma as saying that the α_i-eigenspace of s_i on $A[t]$ is spanned by the homogeneous elements such that the ith base p digit of the degree is $\alpha_i - 1$. By way of terminology, we say that the $(\alpha_0, \alpha_1, \alpha_2, \dots)$-multieigenspace of $(s_0, s_1, s_2, \dots)$ is the intersection of the α_i-eigenspace of s_i for all i. Then, the $(\alpha_0, \alpha_1, \alpha_2, \dots)$-multieigenspace of $(s_0, s_1, s_2, \dots)$ on $A[t]$ is the collection of homogeneous elements of degree $\sum_i (\alpha_i - 1) p^i$ for a tuple with $\alpha_i = 0$ for $i \gg 0$. This motivates the idea that a "Bernstein-Sato root" in positive characteristic should be determined by a multieigenvalue of the action of $(s_0, s_1, s_2, \dots)$ on $[\frac{1}{f-t}]$ in

$$\frac{[D_{A[t]|\mathbb{K}}]_{\geq 0} \cdot [\frac{1}{f-t}]}{[D_{A[t]|\mathbb{K}}]_{\geq 0} f \cdot [\frac{1}{f-t}]}.$$

Based on this motivation, we give two closely related notions of Bernstein-Sato roots appearing in the literature.

6.1 Bernstein-Sato Roots: p-Adic Version

The first definition of Bernstein-Sato roots that we present follows the treatment of Bitoun [14]. To each element $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \mathbb{F}_p^{\mathbb{N}}$ we associate the p-adic integer $I(\alpha) = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots$.

Theorem 6.3 ([14]) *For any* $f \in A$, *the module*

$$\frac{[D_{A[t]|\mathbb{K}}]_{\geq 0} \cdot [\frac{1}{f-t}]}{[D_{A[t]|\mathbb{K}}]_{\geq 0} f \cdot [\frac{1}{f-t}]}$$

decomposes as a finite direct sum of multieigenspaces of $(s_0, s_1, s_2, \dots)$. *The image of each multieigenvalue under I is negative, rational, and at least negative one. Moreover, the map I induces a bijection between multieigenvalues and the set of negatives of the F-jumping numbers in the interval* $(0, 1]$ *with denominator not divisible by p.*

In this context, we consider the image of the multieigenvaues under the map I as the set of *Bernstein-Sato roots* of f. Moreover, Bitoun constructs a notion of a Bernstein-Sato polynomial as an ideal in a certain ring; however, this yields equivalent information to the set of Bernstein-Sato roots just defined.

Example 6.4 ([14])

(i) Let $f = x_1^2 + \cdots + x_n^2$, with $n \geq 2$, and $p > 2$. Then the set of Bernstein-Sato roots of f is $\{-1\}$. Contrast this with the situation in characteristic zero, where $-n/2$ is also a root.
(ii) Let $f = x^2 + y^3$, and $p > 3$. If $p \equiv 1 \bmod 3$, then the set of Bernstein-Sato roots is $\{-1, -5/6\}$, and if $p \equiv 2 \bmod 3$, then the set of Bernstein-Sato roots is $\{-1\}$.

6.2 Bernstein-Sato Roots: Base p Expansion Version

The second definition of Bernstein-Sato roots that we present is historically the first, following the treatment of Mustaţă. To each element $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_e) \in \mathbb{F}_p^{e+1}$ we associate the real number $E(\boldsymbol{\alpha}) = \frac{1}{p^{e+1}}\alpha_0 + \frac{1}{p^e}\alpha_1 + \cdots + \frac{1}{p}\alpha_e$.

Theorem 6.5 ([100]) *For* $\boldsymbol{\alpha} \in \mathbb{F}_p^{e+1}$, *we have that* $\boldsymbol{\alpha}$ *is a multieigenvalue of*

$$\frac{[D^{(e)}_{A[t]|\mathbb{K}}]_{\geq 0} \cdot [\frac{1}{f-t}]}{[D^{(e)}_{A[t]|\mathbb{K}}]_{\geq 0} f \cdot [\frac{1}{f-t}]}$$

if and only if there is an F-jumping number of f contained in the interval $(E(\boldsymbol{\alpha}), E(\boldsymbol{\alpha}) + 1/p^{e+1}]$.

For each level e, one then obtains a set of *Bernstein-Sato roots*, given as the image of the multieigenvalues under the map E.

Relative versions of the above result, for an element in a unit F-module, were considered by Stadnik [134] and Blickle and Stäbler [23].

6.3 Nonprincipal Case

Both of the approaches above were extended to the nonprincipal case by Quinlan-Gallego [114]. To state these generalizations, for an n-generated ideal $\mathfrak{a} = (f_1, \ldots, f_n)$, we consider the following.

Definition 6.6 Consider $D_{A[t_1,\ldots,t_n]|\mathbb{K}}$ as a graded ring, with grading induced by giving each x_i degree zero, and each t_i degree one. We set $[D_{A[t_1,\ldots,t_n]|\mathbb{K}}]_{\geq 0}$ to be the subring spanned by homogeneous elements of nonnegative degree. We also set

$$s_e = - \sum_{a_1+\cdots+a_n=p^e} \frac{\partial_1^{a_1}}{a_1!} \cdots \frac{\partial_n^{a_n}}{a_n!} t_1^{a_1} \cdots t_n^{a_n}.$$

Theorems 6.3 and 6.5 have analogues in this setting; we state the former here and refer the reader to [114] for the latter.

Theorem 6.7 *Let* $\mathfrak{a} = (f_1, \ldots, f_n)$*, and let*

$$\eta = \left[\frac{1}{(f_1 - t_1) \cdots (f_n - t_n)} \right] \in H^n_{(f_1-t_1,\ldots,f_n-t_n)}(A[t_1, \ldots, t_n]).$$

Then, the module

$$\frac{[D_{A[t_1,\ldots,t_n]|\mathbb{K}}]_{\geq 0} \cdot \eta}{[D_{A[t_1,\ldots,t_n]|\mathbb{K}}]_{\geq 0}\mathfrak{a} \cdot \eta}$$

decomposes as a finite direct sum of multieigenspaces of $(s_0, s_1, s_2, \ldots)$*. The image of each multieigenvalue under the map* I *from Sect. 6.1 is rational and negative. Moreover, there is an equality of cosets in* $\mathbb{Q}/\mathbb{Z}$*:*

$$\{I(\boldsymbol{\alpha}) \mid \alpha \text{ is a multieigenvalue of } (s_0, s_1, s_2, \ldots)\} + \mathbb{Z} =$$

$$\{\text{negatives of } F\text{-jumping numbers of } \mathfrak{a} \text{ with denominator not a multiple of } p\} + \mathbb{Z}.$$

In this setting, we consider the image of the set of multieigenvalues under the map I as the set of *Bernstein-Sato roots* of $\mathfrak{a}$.

Example 6.8 ([113]) Let $\mathfrak{a} = (x^2, y^3)$. Then, for $p = 2$, the set of Bernstein-Sato roots is $\{-4/3, -5/3, -2\}$. For $p = 3$, the set of roots is $\{-3/2, -2\}$. For $p \gg 0$, by [113, Theorem 3.1], the set of roots is $\{-5/6, -7/6, -4/3, -3/2, -5/3, -2\}$.

The connection between Bernstein-Sato roots and F-jumping numbers largely stems from the following proposition, and the fact that $\mathcal{C}^e_A \mathfrak{a} = \mathcal{C}^e_A \mathfrak{b}$ if and only $D^{(e)}_A \mathfrak{a} = D^{(e)}_A \mathfrak{b}$.

Proposition 6.9 ([100, Section 6],[114, Theorem 3.11]) *The multieigenspace corresponding to* $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{e-1})$ *of* $(s_0, s_1, s_2, \ldots, s_{e-1})$ *acting on*

$$\frac{[D^{(e)}_{A[t_1,\ldots,t_n]|\mathbb{K}}]_{\geq 0} \cdot \eta}{[D^{(e)}_{A[t_1,\ldots,t_n]|\mathbb{K}}]_{\geq 0}\mathfrak{a} \cdot \eta}$$

decomposes as the direct sum of the modules

$$\frac{D^{(e)}_A \cdot \mathfrak{a}^{I(\alpha)+sp^e}}{D^{(e)}_A \cdot \mathfrak{a}^{I(\alpha)+sp^e+1}} \qquad s = 0, 1, \ldots, n-1.$$

7 An Extension to Singular Rings

We now consider the notion of Bernstein-Sato polynomial in rings of characteristic zero that may be singular. Throughout this section, $\mathbb{K}$ is a field of characteristic zero, and R is a $\mathbb{K}$-algebra.

As in Sect. 3, the definition is as follows:

Definition 7.1 A *Bernstein-Sato functional equation* for an element f in R is an equation of the form

$$\delta(s) f^{s+1} = b(s) f^s \qquad \text{for all } \ s \in \mathbb{N},$$

where $\delta(s) \in D_{R|\mathbb{K}}[s]$ is a polynomial differential operator, and $b(s) \in \mathbb{K}[s]$ is a polynomial. We say that such a functional equation is nonzero if $b(s)$ is nonzero; this implies that $\delta(s)$ is nonzero as well.

If there exists a nonzero functional equation for f, we say that f admits a Bernstein-Sato polynomial, and the Bernstein-Sato polynomial of f is the minimal monic generator of the ideal

$$\{b(s) \in \mathbb{K}[s] \mid \exists \delta(s) \in D_{R|\mathbb{K}}[s] \text{ such that } \delta(s) f^{s+1} = b(s) f^s \text{ for all } s \in \mathbb{N}\} \subseteq \mathbb{K}[s].$$

We denote this as $b_f(s)$, or as $b^R_f(s)$ if we need to keep track of the ring in which we are considering f as an element.

If every element of R admits a Bernstein-Sato polynomial, we say that R has Bernstein-Sato polynomials.

The set specified above is an ideal of $\mathbb{K}[s]$ for the same reason as in Sect. 3.

The proof of existence of Bernstein-Sato polynomials uses the hypothesis that R is regular crucially in multiple steps; thus, a priori Bernstein-Sato polynomials may or may not exist in singular rings. Before we consider examples, we want to consider the functional equation as a formal equality in a D-module.

Theorem 7.2 ([2]) *There exists a unique (up to isomorphism) $D_{R_f|\mathbb{K}}[s]$-module, $R_f[s]\boldsymbol{f^s}$, that is a free as an $R_f[s]$-module, and that is equipped with maps $\theta_n : R_f[s]\boldsymbol{f^s} \to R_f$, such that $\pi_n(\delta(s)) \cdot \theta_n(a(s)\boldsymbol{f^s}) = \theta_n(\delta(s) \cdot a(s)\boldsymbol{f^s})$ for all $n \in \mathbb{N}$. An element $a(s)\boldsymbol{f^s}$ is zero in $R_f[s]\boldsymbol{f^s}$ if and only if $\theta_n(a(s)\boldsymbol{f^s}) = 0$ for infinitely many (if and only if all) $n \in \mathbb{N}$.*

Remark 7.3 From this theorem, we see that the following are equivalent, as in the regular case:

(i) $\delta(s) f \boldsymbol{f^s} = b(s)\boldsymbol{f^s}$ in $R_f[s]\boldsymbol{f^s}$;
(ii) $\delta(s) f^{s+1} = b(s) f^s$ for all $s \in \mathbb{N}$;
(iii) $\delta(s+t) f^{t+1}\boldsymbol{f^s} = b(s) f^t \boldsymbol{f^s}$ in $R_f[s]\boldsymbol{f^s}$ for some/all $t \in \mathbb{Z}$.

We note also that Proposition 3.13 holds in this setting, by the same argument.

7.1 Nonexistence of Bernstein-Sato Polynomials

In this subsection, we give some examples of rings with elements that do not admit Bernstein-Sato polynomials. This is based on a necessary condition on the roots that utilizes the following definition.

Definition 7.4 A D-ideal of R is an ideal $\mathfrak{a} \subseteq R$ such that $D_{R|\mathbb{K}}(\mathfrak{a}) = \mathfrak{a}$.

As $R \subseteq D_{R|\mathbb{K}}$, we always have $\mathfrak{a} \subseteq D_{R|\mathbb{K}}(\mathfrak{a})$, so the nontrivial condition in the definition above is $D_{R|\mathbb{K}}(I) \subseteq I$. We always have that 0 and R are D-ideals. Sums, intersections, and minimal primary components of D-ideals (when R is Noetherian) are also D-ideals [137, Proposition 4.1]. When R is a polynomial ring, the only D-ideals are 0 and R; in other rings, there may be more. We make a simple observation.

Lemma 7.5 *Let $f \in R$, and let $\mathfrak{a} \subseteq R$ be a D-ideal. Let $\delta(s) f^{s+1} = b(s) f^s$ be a functional equation for f. If $f^{n+1} \in \mathfrak{a}$ and $f^n \notin \mathfrak{a}$, then $b(n) = 0$. In particular, if f admits a Bernstein-Sato polynomial $b_f(s)$, then $b_f(n) = 0$.*

Proof After specializing the functional equation, we have $\delta(n) f^{n+1} = b_f(n) f^n$. Since $\delta(n) f^{n+1} \in \mathfrak{a}$, we must have $b_f(n) f^n \in \mathfrak{a}$, which implies $b_f(n) = 0$. □

From the previous lemma, we obtain the following result.

Proposition 7.6 *Let R be a reduced $\mathbb{N}$-graded $\mathbb{K}$-algebra. If $D_{R|\mathbb{K}}$ lives in nonnegative degrees, then no element $f \in [R]_{>0}$ admits a Bernstein-Sato polynomial.*

Proof Let $\delta(s) f^{s+1} = b(s) f^s$ be a functional equation for f. Suppose $f \in [R]_w \smallsetminus [R]_{w-1}$. Since $D_{R|\mathbb{K}}$ has no elements of negative degree, $[R]_{\geq w(n+1)}$ is a D-ideal for each $n \in \mathbb{N}$, and $f^{n+1} \in [R]_{\geq w(n+1)}$, while $f^n \notin [R]_{\geq w(n+1)}$. Thus, $b(n) = 0$ for all n, so $b(s) \equiv 0$. Thus, f does not admit a Bernstein-Sato polynomial. □

Large classes of rings with no differential operators of negative degree are known. In particular, we have the following.

Theorem 7.7 ([24, Corollary 4.49],[62],[89]) *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let R be a standard-graded normal $\mathbb{K}$-domain with an isolated singularity and that is a Gorenstein ring. If R has differential operators of negative degree, then R has log-terminal and rational singularities.*

In particular, if R is a hypersurface, and R has differential operators of negative degree, then the degree of R is less than the dimension of R.

Mallory recently showed that the hypothesis of log-terminal singularities is not sufficient.

Theorem 7.8 ([89]) *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. There are no differential operators of negative degree on the log-terminal hypersurface $R = \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$.*

Corollary 7.9 *For R as in Theorems 7.7 and 7.8, no element of $[R]_{\geq 1}$ admits a Bernstein-Sato polynomial.*

7.2 Existence of Bernstein-Sato Polynomials

While some rings do not admit Bernstein-Sato polynomials, large classes of singular rings do.

Definition 7.10 Let R, S be two rings. We say that R is a direct summand of S if $R \subseteq S$, and there is an R-module homomorphism $\beta : S \to R$ such that $\beta|_R$ is the identity on R.

A major source of direct summands comes from invariant theory: if G is a linearly reductive group acting on a polynomial ring B, then $R = B^G$ is a direct summand of B. In particular, direct summands of polynomial rings include:

(i) invariants of finite groups (including the simple singularities A_n, D_n, E_n),
(ii) normal toric rings,
(iii) determinantal rings, and
(iv) coordinate rings of Grassmannians.

We note that a ring R may be a direct summand of a polynomial ring in different ways; i.e., as different subrings of polynomial rings. For example, the A_1 singularity $R = \mathbb{C}[a, b, c]/(c^2 - ab)$ embeds as a direct summand of $B = \mathbb{C}[x, y]$ by the maps

$$\phi_1 : R \to B \qquad \phi_1(a) = x^2, \phi_1(b) = y^2, \phi_1(c) = xy, \text{ and}$$

$$\phi_2 : R \to B \qquad \phi_2(a) = x^4, \phi_2(b) = y^4, \phi_2(c) = x^2y^2; \text{ likewise}$$

$$\phi_3 : R \to B[z] \qquad \phi_3(a) = x^2, \phi_3(b) = y^2, \phi_3(c) = xy \text{ splits.}$$

We note also that if R is a direct summand of a polynomial ring, there may be other embeddings of R into a polynomial ring that are not split. E.g., for R and B as above,

$$\phi_4 : R \to B \qquad \phi_4(a) = x, \phi_4(b) = xy^2, \phi_4(c) = xy$$

is injective, but no splitting map $\beta|_R$ exists.

Definition 7.11 ([2, 24]) Let R, S be two rings. We say that R is a differentially extensible direct summand of S if R is a direct summand of S, and for every differential operator $\delta \in D_{R|\mathbb{K}}$, there is some $\tilde{\delta} \in D_{S|\mathbb{K}}$ such that $\tilde{\delta}|_R = \delta$.

This notion is implicit in a number of papers on differential operators, e.g., [69, 79, 99, 127]. Differentially extensible direct summands of polynomial rings include

(i) invariants of finite groups (including the simple singularities A_n, D_n, E_n),
(ii) normal toric rings,
(iii) determinantal rings, and
(iv) coordinate rings of Grassmannians of lines $\mathrm{Gr}(2, n)$.

As with the direct summand property, a ring may be a differentially extensible direct summand of a polynomial ring by some embedding, but fail this property for another embedding into a polynomial ring. For the example considered above, R is a differentially extensible direct summand of B via ϕ_1 and ϕ_3, but not ϕ_2 or ϕ_4.

Theorem 7.12 ([1, 24]) *Let R be a direct summand of a differentiably admissible algebra B over a field $\mathbb{K}$ of characteristic zero. Then every element $f \in R$ admits a Bernstein-Sato polynomial $b_f^R(s)$, and $b_f^R(s) \mid b_f^B(s)$.*

If, in addition, R is a differentially extensible direct summand of B, then $b_f^R(s) = b_f^B(s)$ for all $f \in R$.

Proof Let $\beta : B \to R$ be the splitting map. The key point is that for $\delta \in D_{B|\mathbb{K}}$, the map $\beta \circ \delta|_R$ is a differential operator on R; this is left as an exercise using the inductive definition, or see [133]. Thus, given a functional equation $\forall s \in \mathbb{N}, \delta(s) f^{s+1} = b(s) f^s$ for f in B, we have $\forall s \in \mathbb{N}, \beta \circ \delta(s)|_R\, f^{s+1} = \beta(b(s) f^s) = b(s) f^s$ in R. This implies that f admits a Bernstein-Sato polynomial in R, and that $b_f^R(s) \mid b_f^B(s)$.

If R is a differentially extensible direct summand of B, then for any functional equation $\forall s \in \mathbb{N}, \delta(s) f^{s+1} = b(s) f^s$ for f in R, we can take an extension $\tilde{\delta}(s)$ by extending each s^i-coefficient, and we then have $\forall s \in \mathbb{N}, \tilde{\delta}(s) f^{s+1} = b(s) f^s$ in B. Thus, $b_f^B(s) \mid b_f^R(s)$, so equality holds. □

Note that for direct summands of polynomial rings, all roots of the Bernstein-Sato polynomial are negative and rational, as in the regular case.

We end this section with two examples of Bernstein-Sato polynomials in rings that are not direct summands of polynomial rings.

Example 7.13 ([2]) Let $R = \mathbb{C}[x, y]/(xy)$, and $f = x$. The operator $x\partial_x^2$ is a differential operator on R [138], and it yields a functional equation

$$x\partial_x^2 x^{s+1} = s(s+1)x^s.$$

Thus, $b_f^R(s)$ exists, and divides $s(s+1)$. In fact, we have $b_f^R(s) = s(s+1)$. The ideal (x) is a minimal primary component of (0), hence a D-ideal. By Lemma 7.5, $s = 0$ is a root; $s = -1$ is also a root since x is not a unit.

Example 7.14 ([2]) Let $R = \mathbb{C}[t^2, t^3] \cong \dfrac{\mathbb{C}[x, y]}{(x^3 - y^2)}$ and $f = t^2$. Consider the differential operator of order two

$$\delta = (t\partial_t - 1) \circ \partial_t^2 \circ (t\partial_t - 1)^{-1},$$

where $(t\partial_x t - 1)^{-1}$ is the inverse function of $t\partial_t - 1$ on R. The equation

$$\delta \bullet t^{2(\ell+1)} = (2\ell + 2)(2\ell - 1)t^{2\ell}$$

holds for every $\ell \in \mathbb{N}$. Then, the functional equation

$$\delta \bullet t^2(\boldsymbol{t^2})^s = (2s+2)(2s-1)(\boldsymbol{t^2})^s$$

holds in $R_{t^2}[s](\boldsymbol{t^2})^s$. Thus, $b_{t^2}^R(s)$ divides $(s - \frac{1}{2})(s+1)$.

We now see that the equality holds. We already know that $s = -1$ is a root of $b_{t^2}^R(s)$, because $\frac{1}{t^2} \notin R$. Every differential operator of degree -2 on R can be written as $(t\partial_t - 1) \circ \partial_t^2 \circ \gamma \circ (t\partial_t - 1)^{-1}$ for some $\gamma \in \mathbb{C}[t\partial_t]$ [130, 132]. Since $R_{t^2}[s](\boldsymbol{t^2})^s$ is a graded module we can decompose the functional equation as a sum of homogeneous pieces. Using previous description of such operators, it follows that $s = \frac{1}{2}$ must be a root of $b_{t^2}^R(s)$.

7.3 Differentiable Direct Summands

Definition 7.15 ([1, Definition 3.2]) Let $R \subseteq B$ be an inclusion of $\mathbb{K}$-algebras with R-linear splitting $\beta : B \to R$. Recall that, for $\zeta \in D^n_{B|\mathbb{K}}$, the map $\beta \circ \zeta|_R : R \to R$ is an element of $D^n_{R|\mathbb{K}}$. By abuse of notation, for $\delta \in D_{B|\mathbb{K}}$, we write $\beta \circ \delta|_R$ for the element of $D_{R|\mathbb{K}}$ obtained from δ by applying $\beta \circ -|_R$.

We say that a $D_{R|\mathbb{K}}$-module M is a *differential direct summand* of a $D_{B|\mathbb{K}}$-module N if $M \subseteq N$ and there exists an R-linear splitting $\Theta : N \to M$, called a *differential splitting*, such that

$$\Theta(\delta \bullet v) = (\beta \circ \delta|_R) \bullet v$$

for every $\delta \in D_{B|\mathbb{K}}$ and $v \in M$, where the action on the left-hand side is the $D_{B|\mathbb{K}}$-action, considering v as an element of N, and the action on the right-hand side is the $D_{R|\mathbb{K}}$-action.

A key property for differential direct summands is that one can deduce finite length.

Theorem 7.16 ([1, Proposition 3.4]) *Let $R \subseteq B$ be $\mathbb{K}$-algebras such that R is a direct summand of B. Let M be a $D_{R|\mathbb{K}}$-module and N be a $D_{B|\mathbb{K}}$-module such that M is a differential direct summand of N. Then,*

$$\operatorname{length}_{D_{R|\mathbb{K}}}(M) \leq \operatorname{length}_{D_{B|\mathbb{K}}}(N).$$

In particular, if $\operatorname{length}_{D_{B|\mathbb{K}}}(N)$ *is finite, then* $\operatorname{length}_{D_{R|\mathbb{K}}}(M)$ *is also finite.*

Definition 7.17 ([1, Definition 3.5]) Let $R \subseteq B$ be $\mathbb{K}$-algebras such that R is a direct summand of B. Fix $D_{R|\mathbb{K}}[\underline{s}]$-modules M_1 and M_2 that are differential direct summands of $D_{B|\mathbb{K}}[\underline{s}]$-modules N_1 and N_2, respectively, with differential splittings $\Theta_1 \colon N_1 \to M_1$ and $\Theta_2 \colon N_2 \to M_2$. We call $\phi \colon N_1 \to N_2$ a *morphism of differential direct summands* if $\phi \in \operatorname{Hom}_{D_{B|\mathbb{K}}[\underline{s}]}(N_1, N_2)$, $\phi(M_1) \subseteq M_2$, $\phi|_{M_1} \in \operatorname{Hom}_{D_{R|\mathbb{K}}[\underline{s}]}(M_1, M_2)$, and the following diagram commutes:

$$\begin{array}{ccccc} M_1 & \xrightarrow{\subseteq} & N_1 & \xrightarrow{\Theta_1} & M_1 \\ \downarrow{\scriptstyle \phi|_{M_1}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi|_{M_1}} \\ M_2 & \xrightarrow{\subseteq} & N_2 & \xrightarrow{\Theta_2} & M_2 \end{array}$$

For simplicity of notation, we often write ϕ instead of $\phi|_{M_1}$.

Further, a complex $M_\bullet$ of $D_{R|\mathbb{K}}[\underline{s}]$-modules is called a *differential direct summand* of a complex $N_\bullet$ of $D_{B|\mathbb{K}}[\underline{s}]$-modules if each M_i is a differential direct summand of N_i, and each differential is a morphism of differential direct summands.

Remark 7.18 Let $R \subseteq B$ be $\mathbb{K}$-algebras such that R is a direct summand of B. It is known that the property of being a differential direct summand is preserved under localization at elements of R. In addition, it is preserved under taking kernels and cokernels of morphisms of differential direct summands [1, Proposition 3.6, Lemma 3.7].

We now present several examples of differentiable direct summands built from the previous remark.

Example 7.19 Let $R \subseteq B$ be $\mathbb{K}$-algebras such that R is a direct summand of B

(i) For every $f \in R \setminus \{0\}$, R_f is a differentiable direct summand of B_f.
(ii) For every ideal $\mathfrak{a} \subseteq R$, $H^i_{\mathfrak{a}}(R)$ is a differentiable direct summand of $H^i_{\mathfrak{a}}(B)$.
(iii) For every sequence of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_\ell \subseteq R$, $H^i_{\mathfrak{a}_1} \cdots H^i_{\mathfrak{a}_\ell}(R)$ is a differentiable direct summand of $H^i_{\mathfrak{a}_1} \cdots H^i_{\mathfrak{a}_\ell}(B)$.

We end this subsection showing that $R_f[s]\boldsymbol{f}^s$ is a differentiable direct summand of $R_f[s]\boldsymbol{f}^s$. This gives a more complete approach to prove the existence of the Bernstein-Sato polynomial.

Theorem 7.20 *Let $R \subseteq B$ be $\mathbb{K}$-algebras such that R is a direct summand of B, and $f \in R \smallsetminus \{0\}$. Then, $R_f[s]\boldsymbol{f}^s$ is a differentiable direct summand of $R_f[s]\boldsymbol{f}^s$. In particular, if B is a differentiably admissible $\mathbb{K}$-algebra, then $M^R[\boldsymbol{f}^s] \otimes_{\mathbb{K}} \mathbb{K}(s)$ has finite length as $D_{R(s)|\mathbb{K}(s)}$-module, and so, there exists a functional equation*

$$\delta(s) f \boldsymbol{f}^s = b(s)\boldsymbol{f}^s,$$

where $\delta(s) \in D_{R|\mathbb{K}}$ and $b(s) \in \mathbb{K}[s] \smallsetminus \{0\}$.

8 Local Cohomology

In this section we discuss some properties of local cohomology modules for regular rings that follow from the existence of the Bernstein-Sato polynomial.

Proposition 8.1 *Let $\mathbb{K}$ be a field of characteristic zero, R be a $\mathbb{K}$-algebra, and $f \in R$ be a nonzero element. If R has Bernstein-Sato polynomials, then, R_f is a finitely generated $D_{R|\mathbb{K}}$-module. In particular, if $b_f^R(s)$ has no integral root less than or equal to $-n$, then $R_f = D_{R|\mathbb{K}} \cdot \dfrac{1}{f^{n-1}}$.*

Proof After specializing the functional equation, we have

$$\delta(-t)\frac{1}{f^{t-1}} = b_f^R(-t)\frac{1}{f^t}$$

for all $t \geq n$, with each $b_f^R(-t) \neq 0$. We conclude that each power of f, and hence all of R_f, is in $D_{R|\mathbb{K}} \cdot \frac{1}{f^{n-1}}$. □

In fact, a converse to this theorem is true.

Proposition 8.2 ([143, Proposition 1.3]) *Let $\mathbb{K}$ be a field of characteristic zero, R be a $\mathbb{K}$-algebra, and $f \in R$ have a Bernstein-Sato polynomial. If $-n$ is the smallest integral root of $b_f(s)$, then $\dfrac{1}{f^n} \notin D_{R|\mathbb{K}} \cdot \dfrac{1}{f^{n-1}} \subseteq R_f$.*

We give a proof of this proposition here, since it appears in the literature only in the regular case.

Lemma 8.3 ([71, Proposition 6.2]) *If $-n$ is the smallest integral root of $b_f(s)$, then*

$$(s+n+j)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \cap D_{R|\mathbb{K}}[s]f^j\boldsymbol{f}^s = (s+n+j)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \text{ for all } j > 0.$$

Proof We proceed by induction on j.

Since $b_f(s)$ is the minimal polynomial of the action of s on $\dfrac{D_{R|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s}$ and $-n-j$ is not a root of $b_f(s)$ for $j \geq 1$, the map

$$\frac{D_{R|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s} \xrightarrow{s+n+j} \frac{D_{R|\mathbb{K}}[s]\boldsymbol{f}^s}{D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s}$$

is an isomorphism. Thus,

$$(s+n+j)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \cap D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s = (s+n+j)D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s.$$

In particular, for $j=1$, this covers the base case.

Let $\Sigma : D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \to D_{R|\mathbb{K}}[s]\boldsymbol{f}^s$ be the map given by the rule $\Sigma(\delta(s)\boldsymbol{f}^s) = \delta(s+1)f\boldsymbol{f}^s$. Using the induction hypothesis, for $j \geq 2$ we compute

$$\begin{aligned}(s+n+j)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \cap D_{R|\mathbb{K}}[s]f^j\boldsymbol{f}^s &\subseteq (s+n+j)D_{R|\mathbb{K}}[s]f\boldsymbol{f}^s \cap D_{R|\mathbb{K}}[s]f^j\boldsymbol{f}^s\\ &= \Sigma((s+n+j-1)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s \cap D_{R|\mathbb{K}}[s]f^{j-1}\boldsymbol{f}^s)\\ &= \Sigma((s+n+j-1)D_{R|\mathbb{K}}[s]f^{j-1}\boldsymbol{f}^s)\\ &= (s+n+j)D_{R|\mathbb{K}}[s]f^j\boldsymbol{f}^s.\end{aligned}$$ □

Lemma 8.4 ([71, Proposition 6.2]) *If $-n$ is the smallest integral root of $b_f(s)$, then*

$$Ann_D(f^{-n}) = D_{R|\mathbb{K}} \cap (Ann_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}}[s](s+n)).$$

Proof Let $\delta \in \mathrm{Ann}_D(f^{-n})$. Write $\delta\boldsymbol{f}^s = f^{-m}g(s)\boldsymbol{f}^s$, with $g(s) \in R[s]$. In fact, we can take m to be the order of δ. Then $g(-n)=0$. By Remark 7.3,

$$\delta \cdot f^m\boldsymbol{f}^s = g(s+m)\boldsymbol{f}^s.$$

Set $h(s) = g(s+m)$. We then have that $h(-n-m) = g(-n) = 0$, so $(s+n+m)|h(s)$. Thus, $\delta \cdot f^m\boldsymbol{f}^s \in (s+m+n)D_{R|\mathbb{K}}[s]\boldsymbol{f}^s$, and $\delta \cdot f^m\boldsymbol{f}^s \in D_{R|\mathbb{K}}[s]f^m\boldsymbol{f}^s$ by definition. By the previous lemma, we obtain that $\delta \cdot f^m\boldsymbol{f}^s \in (s+m+n)D_{R|\mathbb{K}}[s]f^m\boldsymbol{f}^s$. We can then write $\delta \cdot f^m\boldsymbol{f}^s = (s+m+n)h'(s)\boldsymbol{f}^s$ for some $h'(s) \in R[s]$. By Remark 7.3, we have that $\delta \cdot \boldsymbol{f}^s = (s+n)h'(s-m)\boldsymbol{f}^s$. Thus, we can write δ as a sum of a multiple of $(s+n)$ and an element in the annihilator of $\boldsymbol{f}^s$. □

Proof of Proposition 8.2 Suppose that $\dfrac{1}{f^n} \in D_{R|\mathbb{K}}\dfrac{1}{f^{n-1}}$. Then we can write $D_{R|\mathbb{K}} = D_{R|\mathbb{K}}f + \mathrm{Ann}_D(\frac{1}{f^n})$. From the previous lemma, we have that

$$\mathrm{Ann}_D(\frac{1}{f^n}) = D_{R|\mathbb{K}} \cap (\mathrm{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}}[s](s+n)).$$

Then,

$$1 \in D_{R|\mathbb{K}} f + \mathrm{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}}[s](s+n).$$

Multiplying by $\frac{b_f(s)}{s+n}$, we get

$$\frac{b_f(s)}{s+n} \in \mathrm{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}} f + D_{R|\mathbb{K}}[s]\, b_f(s).$$

Since $b_f(s) \in D_{R|\mathbb{K}} f + \mathrm{Ann}_{D[s]}(\boldsymbol{f}^s)$, using Remark 7.3 we have

$$\frac{b_f(s)}{s+n} D_{R|\mathbb{K}}[s] \in \mathrm{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}}[s] f,$$

which contradicts that $b_f(s)$ is the minimal polynomial in s contained in

$$\mathrm{Ann}_{D[s]}(\boldsymbol{f}^s) + D_{R|\mathbb{K}}[s] f.$$

□

Remark 8.5 Proposition 8.2 extends to the setting of the $D_{R|\mathbb{K}}$-modules $D_{R|\mathbb{K}} f^\alpha$ for $\alpha \in \mathbb{Q}$ discussed in Remark 3.14. Namely, if $\alpha \in \mathbb{Q}$ is such that $b_f(\alpha) = 0$ and $b_f(\alpha - i) \neq 0$ for all integers $i > 0$, then $f^\alpha \notin D_{R|\mathbb{K}} \cdot f^{\alpha+1}$ in the $D_{R|\mathbb{K}}$-module $R_f f^\alpha$.

It is not true in general that $b_f(\alpha) = 0$ implies $f^\alpha \notin D_{R|\mathbb{K}} \cdot f^{\alpha+1}$, even in the regular case: an example is given by Saito [123]. However, this implication does hold when $R = A$ is a polynomial ring, and f is quasihomogeneous with an isolated singularity [15]. We are not aware of an example where $b_f(n) = 0$ and $f^n \in D_{R|\mathbb{K}} \cdot f^{n+1}$ for an integer n.

We also relate existence of Bernstein-Sato polynomials to finiteness properties of local cohomology.

Theorem 8.6 *Let $\mathbb{K}$ be a field of characteristic zero, R be a $\mathbb{K}$-algebra, and $f \in R$ be a nonzero element. Suppose that R has Bernstein-Sato polynomials and $D_{R|\mathbb{K}}$ is a Noetherian ring. Then, $H^i_{\mathfrak{a}}(R)$ is a finitely generated $D_{R|\mathbb{K}}$-module, and $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(R))$ is finite for every ideal $\mathfrak{a} \subseteq R$.*

Proof Let $F = f_1, \ldots, f_\ell$ be a set of generators for $\mathfrak{a}$. We have that the Čech complex associated to F is a complex of finitely generated $D_{R|\mathbb{K}}$-modules. Since $D_{R|\mathbb{K}}$ is Noetherian, the Čech complex is a complex of Noetherian $D_{R|\mathbb{K}}$-modules. Then, the cohomology of this complex is also a Noetherian $D_{R|\mathbb{K}}$-module.

It suffices to show that a Noetherian $D_{R|\mathbb{K}}$-module, N, has a finite set of associated primes. We build inductively a sequence of $D_{R|\mathbb{K}}$-submodules $N_i \subseteq N$ as follows. We set $N_0 = 0$. Given N_t, we pick a maximal element $\mathfrak{p}_t \in \mathrm{Ass}_R(N/N_t)$. This is possible if and only if $\mathrm{Ass}_R(N/N_t) \neq \varnothing$. We set $\tilde{N}_{t+1} = H^0_{\mathfrak{p}}(N/N_t)$, which is nonzero, and N_{t+1} the preimage of $\tilde{N}_{t+1}$ in N under the quotient map. We have that $\mathrm{Ass}_R(\tilde{N}_{t+1}) = \{\mathfrak{p}\}$, and so, $\mathrm{Ass}_R(N_{t+1}) = \{\mathfrak{p}\} \cup \mathrm{Ass}_R(N_t)$. We note that this sequence cannot be infinite, because N is Noetherian. Then, the sequence stops, and there is a $k \in \mathbb{N}$ such that $N_k = N$. We conclude that $\mathrm{Ass}_R(N) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. □

9 Complex Zeta Functions

The foundational work of Bernstein [9, 10] where he developed the theory of D-modules and proved the existence of Bernstein-Sato polynomials was motivated by a question of I. M. Gel'fand [55] at the 1954 edition of the International Congress of Mathematicians regarding the analytic continuation of the *complex zeta function*. Bernstein's work relates the poles of the complex zeta function to the roots of the Bernstein-Sato polynomials. Previously, Bernstein and S. I. Gel'fand [11] and independently Atiyah [5], gave a different approach to the same question using resolution of singularities.

Throughout this section we consider $A = \mathbb{C}[x_1, \ldots, x_d]$ and the corresponding ring of differential operators $D_{A|\mathbb{C}}$. Given a differential operator $\delta(s) = \sum_\alpha a_\alpha(x, s)\partial^\alpha \in D_{A|\mathbb{C}}[s]$, which is polynomial in s, we denote the *conjugate* and the *adjoint* of $\delta(s)$ as

$$\bar{\delta}(s) := \sum_\alpha a_\alpha(\bar{x}, \bar{s})\overline{\partial}^\alpha, \quad \delta^*(s) := \sum_\alpha (-1)^{|\alpha|}\partial^\alpha a_\alpha(x, s),$$

where we are using the multidegree notation $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\overline{\partial}^\alpha := \overline{\partial}_1^{\alpha_1} \cdots \overline{\partial}_d^{\alpha_d}$ with $\overline{\partial}_i = \frac{d}{d\overline{x_i}}$.

Let $f(x) \in A$ be a non-constant polynomial and let $\varphi(x) \in C_c^\infty(\mathbb{C}^d)$ be a *test function*: an infinitely many times differentiable function with compact support. We define the parametric distribution $f^s : C_c^\infty(\mathbb{C}^d) \longrightarrow \mathbb{C}$ by means of the integral

$$\langle f^s, \varphi\rangle := \int_{\mathbb{C}^d} |f(x)|^{2s}\varphi(x, \bar{x})dxd\bar{x}, \tag{9.1}$$

which is well-defined analytic function for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. We point out that test functions have holomorphic and antiholomorphic part so we use the notation $\varphi = \varphi(x, \bar{x})$. We refer to f^s or $\langle f^s, \varphi\rangle$ as the *complex zeta function* of f.

The approach given by Bernstein in order to solve I. M. Gel'fand's question uses the Bernstein-Sato polynomial and integration by parts as follows:

$$\begin{aligned}\langle f^s, \varphi\rangle &= \int_{\mathbb{C}^d} \varphi(x, \bar{x})|f(x)|^{2s}dxd\bar{x} \\ &= \frac{1}{b_f^2(s)} \int_{\mathbb{C}^d} \varphi(x, \bar{x})\big[\delta(s) \cdot f^{s+1}(x)\big]\big[\bar{\delta}(s) \cdot f^{s+1}(\bar{x})\big]dxd\bar{x}\end{aligned}$$

$$= \frac{1}{b_f^2(s)} \int_{\mathbb{C}^d} \bar{\delta}^* \delta^*(s)\big(\varphi(x,\bar{x})\big)|f(x)|^{2(s+1)} dx d\bar{x}$$

$$= \frac{\langle f^{s+1}, \bar{\delta}^* \delta^*(s)(\varphi)\rangle}{b_f^2(s)}.$$

Thus we get an analytic function whenever $\mathrm{Re}(s) > -1$, except for possible poles at $b_f^{-1}(0)$, and it is equal to $\langle f^s, \varphi\rangle$ in $\mathrm{Re}(s) > 0$. Iterating the process we get

$$\langle f^s, \varphi\rangle = \frac{\langle f^{s+\ell+1}, \bar{\delta}^*\delta^*(s+\ell)\cdots\bar{\delta}^*\delta^*(s)(\varphi)\rangle}{b_f^2(s)\cdots b_f^2(s+\ell)}, \quad \mathrm{Re}(s) > -\ell - 1,$$

In particular we have the following relation between the poles of the complex zeta function and the roots of the Bernstein-Sato polynomial.

Theorem 9.1 *The complex zeta function f^s admits a meromorphic continuation to $\mathbb{C}$ and the set of poles is included in* $\{\lambda - \ell \mid b_f(\lambda) = 0 \quad \textit{and} \quad \ell \in \mathbb{Z}_{\geq 0}\}$.

Both sets are equal for reduced plane curves and isolated quasi-homogeneous singularities by work of Loeser [81].

On the other hand, the approach given by Bernstein and S. I. Gel'fand, and independently Atiyah uses resolution of singularities in order to reduce the problem to the monomial case, which was already solved by Gel'fand and Shilov [56]. Let $\pi : X' \to \mathbb{C}^n$ be a *log-resolution* of $f \in A$ and

$$F_\pi := \sum_{i=1}^{r} N_i E_i + \sum_{j=1}^{s} N'_j S_j \text{ and } K_\pi := \sum_{i=1}^{r} k_i E_i$$

be the total transform and the relative canonical divisors.

The analytic continuation problem is attacked in this case using a change of variables.

$$\langle f^s, \varphi\rangle = \int_{\mathbb{C}^d} |f(x)|^{2s} \varphi(x,\bar{x}) dx d\bar{x} = \int_{X'} |\pi^* f|^{2s}(\pi^*\varphi)|d\pi|^2$$

where $|d\pi|^2 = (\pi^* dx)(\pi^* d\overline{x})$ and $d\pi$ is the Jacobian determinant of π. In order to describe the terms of the last integral we consider a finite affine open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $E \subseteq X'$ such that $\mathrm{Supp}(\varphi) \subseteq \pi(\cup_\alpha U_\alpha)$. Consider a set of local coordinates $z_1, \ldots, z_d$ in a given U_α. Then we have

$$\pi^* f = u_\alpha(z) z_1^{N_{1,\alpha}} \cdots z_d^{N_{d,\alpha}}, \quad |d\pi|^2 = |v_\alpha(z)|^2 |z_1|^{2k_{1,\alpha}} \cdots |z_d|^{2k_{d,\alpha}} dz d\overline{z}$$

where $u_\alpha(z)$ and $v_\alpha(z)$ are units and $N_{i,\alpha}$ may denote both the multiplicities of the exceptional divisors or of the strict transform. Take $\{\eta_\alpha\}$ a partition of unity subordinated to the cover $\{U_\alpha\}_{\alpha\in\Lambda}$. That is, $\eta_\alpha \in C^\infty(\mathbb{C}^d)$, $\sum_\alpha \eta_\alpha \equiv 1$, with only finitely many η_α being nonzero at a point of X' and $\mathrm{Supp}(\eta_\alpha) \subseteq U_\alpha$. Therefore

$$\begin{aligned}\langle f^s, \varphi\rangle &= \int_{X'} |\pi^* f|^{2s}(\pi^*\varphi)(\pi^* dx)(\pi^* d\overline{x}) \\ &= \sum_{\alpha\in\Lambda}\int_{U_\alpha} |z_1|^{2(N_{1,\alpha}s+k_{1,\alpha})}\cdots|z_d|^{2(N_{d,\alpha}s+k_{d,\alpha})}|u_\alpha(z)|^{2s}|v_\alpha(z)|^2\varphi_\alpha(z,\bar{z})dzd\bar{z},\end{aligned}$$

where $\varphi_\alpha := \eta_\alpha\pi^*\varphi$ for each $\alpha \in \Lambda$. Notice that $\pi^{-1}(\mathrm{Supp}(\varphi))$ is a compact set because π is a proper morphism.

Once we reduced the problem to the monomial case, we can use the work of Gel'fand and Shilov [56] on *regularization* to generate a set of candidate poles of f^s.

Theorem 9.2 *The complex zeta function f^s admits a meromorphic continuation to $\mathbb{C}$ and the set of poles is included in*

$$\left\{-\frac{k_i+1+\ell}{N_i} \mid \ell\in\mathbb{Z}_{\geq 0}\right\}\cup\left\{-\frac{\ell+1}{N'_j} \mid \ell\in\mathbb{Z}_{\geq 0}\right\}.$$

The fundamental result of Kashiwara [71] and Malgrange [92] on the rationality of the roots of the Bernstein-Sato mentioned in Theorem 3.37 was refined later on by Lichtin [80]. He provides the same set of candidates for the roots of the Bernstein-Sato polynomial in terms of the numerical data of the log-resolution of f.

Theorem 9.3 ([80]) *Let $f \in A$ be a polynomial. Then, the roots of the Bernstein-Sato polynomial of f are included in the set*

$$\left\{-\frac{k_i+1+\ell}{N_i} \mid \ell\in\mathbb{Z}_{\geq 0}\right\}\cup\left\{-\frac{\ell+1}{N'_j} \mid \ell\in\mathbb{Z}_{\geq 0}\right\}.$$

In particular, the roots of the Bernstein-Sato polynomial of f are negative rational numbers.

This result has recently been extended by Dirks and Mustaţă [49].

We also have a bound for the roots given by Saito [121] in terms of the *log-canonical threshold* of f,

$$\mathrm{lct}(f) := \min_{i,j}\left\{\frac{k_i+1}{N_i}, \frac{1}{N'_j}\right\}.$$

Theorem 9.4 ([121]) *Let $f \in A$ be a polynomial. Then, the roots of the Bernstein-Sato polynomial of f are contained in the interval* $[-d+\mathrm{lct}(f), -\mathrm{lct}(f)]$.

In general the set of candidates that we have for the poles of the complex zeta function or the roots of the Bernstein-Sato polynomial is too big. In order to separate the wheat from the chaff we consider the notion of *contributing divisors*.

Definition 9.5 We say that a divisor E_i or S_j contributes to a pole λ of the complex zeta function f^s or to a root λ of the Bernstein-Sato polynomial of f, if we have $\lambda = -\frac{k_i+1+\ell}{N_i}$ or $\lambda = -\frac{\ell+1}{N'_j}$ for some $\ell \in \mathbb{Z}_{\geq 0}$.

It is an open question to determine the contributing divisors (see [76]). Also we point out that, in general, the divisors contributing to poles are different from the divisors contributing to roots. This is not the case for reduced plane curves and isolated quasi-homogeneous singularities by work of Loeser [81, Theorem 1.9]. In the case of reduced plane curves, Blanco [17] determined the contributing divisors.

Although we have a set of candidate poles of the complex zeta function one has to ensure that a candidate is indeed a pole by checking the corresponding *residue*. This can be quite challenging and was already posed as a question by I. M. Gel'fand [55]. In the case of plane curves we have a complete description given by Blanco [17]. Moreover, it is not straightforward to relate poles of the complex zeta function to roots of the Bernstein-Sato polynomial. We have that a pole $\lambda \in [-d+\mathrm{lct}(f), -\mathrm{lct}(f)]$ such that $\lambda+\ell$ is not a root of $b_f(s)$ for all $\ell \in \mathbb{Z}_{>0}$ is a root of $b_f(s)$ but this is not enough to recover all the roots of the Bernstein-Sato polynomial even if we know all the poles of the complex zeta function.

10 Multiplier Ideals

Let $f \in A = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial. As we mentioned in Sect. 2.3, the family of *multiplier ideals* of f is an important object in birational geometry that is described using a log-resolution of f and comes with a discrete set of rational numbers, the *jumping numbers*, that are also related to the roots of the Bernstein-Sato polynomial.

We start with an analytic approach to multiplier ideals that has its origin in the work of Kohn [75], Nadel [103], and Siu [129]. The idea behind the construction is to measure the singularity of f at a point $p \in Z(f) \subseteq \mathbb{C}^d$ using the convergence of certain integrals.

Definition 10.1 Let $f \in A$ and $p \in Z(f)$. Let $\overline{B}_\epsilon(p)$ be a closed ball of radius ϵ and center p. The multiplier ideal of f at p associated with a rational number $\lambda \in \mathbb{Q}_{>0}$ is

$$\mathcal{J}(f^\lambda)_p = \left\{ g \in A \;\middle|\; \exists\, \epsilon \ll 1 \text{ such that } \int_{\overline{B}_\epsilon(p)} \frac{|g|^2}{|f|^{2\lambda}} dx d\overline{x} < \infty \right\}.$$

More generally we consider $\mathcal{J}(f^\lambda) = \cap_{p \in Z(f)} \mathcal{J}(f^\lambda)_p$.

Similarly to the case of the complex zeta function we may use a log-resolution $\pi : X' \to \mathbb{C}^d$ of f to reduce the above integral to a monomial case where we can easily check its convergence.

$$\int_{\overline{B}_\epsilon(p)} \frac{|g|^2}{|f|^{2\lambda}} dx d\overline{x} = \int_{\pi^{-1}\left(\overline{B}_\epsilon(p)\right)} \frac{|\pi^* g|^2}{|\pi^* f|^{2\lambda}} |d\pi|^2,$$

Consider a finite affine open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $\pi^{-1}\left(\overline{B}_\epsilon(p)\right)$ which is still a compact set since π is proper. We have to check the convergence of the integral at each U_α so let $z_1, \dots, z_d$ be a set of local coordinates in such an open set. Taking local equations for $\pi^* f, \pi^* g$ we get

$$\begin{aligned}&\int_{U_\alpha} \frac{|u(z)\, z_1^{L_{1,\alpha}} \cdots z_d^{L_{d,\alpha}}|^2}{|z_1^{N_{1,\alpha}} \cdots z_d^{N_{d,\alpha}}|^{2\lambda}} |z_1^{k_{1,\alpha}} \cdots z_d^{k_{d,\alpha}}|^2 dz d\overline{z} \\ &= \int_{U_\alpha} |u(z)|\, |z_1|^{2(L_{1,\alpha}+k_{1,\alpha}-\lambda N_{1,\alpha})} \cdots |z_d|^{2(L_{d,\alpha}+k_{d,\alpha}-\lambda N_{d,\alpha})} dz d\overline{z}.\end{aligned}$$

where $u(z)$ is a unit. Using Fubini's theorem we have that the integral converges if and only if

$$L_i + k_i - \lambda N_i > -1, \quad L'_j - \lambda N'_j > -1$$

for all i, j. Here we use that the total transform divisors of f and g are respectively

$$F_\pi := \sum_{i=1}^{r} N_i E_i + \sum_{j=1}^{s} N'_j S_j, \quad G_\pi := \sum_{i=1}^{r} L_i E_i + \sum_{j=1}^{t} L'_j S'_j$$

and the components of the strict transform of g must contain the components of f. Equivalently, we require

$$L_i \geq -\lceil k_i - \lambda N_i \rceil, \quad L'_j \geq \lceil \lambda N'_j \rceil$$

so we are saying that $\pi^* g$ is a section of $\mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil)$. This fact leads to the algebraic geometry definition of multiplier ideals given in Definition 2.11 that we refine to the local case.

Definition 10.2 Let $\pi : X' \to \mathbb{C}^d$ be a *log-resolution* of $f \in A$ and let F_π be the total transform divisor. The multiplier ideal of f at $p \in Z(f)$ associated with a real number $\lambda \in \mathbb{R}_{>0}$ is the stalk at p of

$$\mathcal{J}(f^\lambda) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F_\pi \rceil\right).$$

We omit the reference to the point p if it is clear from the context. Recall that the multiplier ideals form a discrete filtration

$$A \supsetneq \mathcal{J}(f^{\lambda_1}) \supsetneq \mathcal{J}(f^{\lambda_2}) \supsetneq \cdots \supsetneq \mathcal{J}(f^{\lambda_i})) \supsetneq \cdots$$

and the λ_i where we have a strict inclusion of ideals are the *jumping numbers* of f and $\lambda_1 = \mathrm{lct}(f)$ is the log-canonical threshold.

There is a way to describe a set of candidate jumping numbers in a reasonable time. However, contrary to the case of roots of the Bernstein-Sato polynomial, the jumping numbers are not bounded. However they satisfy some periodicity given by the following version of Skoda's theorem, which for principal ideals reads as $\mathcal{J}(f^{\lambda}) = (f) \cdot \mathcal{J}(f^{\lambda-1})$ for all $\lambda \geqslant 1$.

Theorem 10.3 *Let $f \in A$ be a polynomial. Then, the jumping numbers of f are included in the set*

$$\left\{ \frac{k_i + 1 + \ell}{N_i} \mid \ell \in \mathbb{Z}_{\geq 0} \right\} \cup \left\{ \frac{\ell + 1}{N'_j} \mid \ell \in \mathbb{Z}_{\geq 0} \right\}.$$

In particular, the jumping numbers of f form a discrete set of positive rational numbers.

We see that we have the same set of candidates for the roots of the Bernstein-Sato polynomial and the jumping numbers so it is natural to ask how these invariants of singularities are related. The result that we are going to present is due to Ein et al. [51]. A different proof of the same result can be found in the work of Budur and Saito [38] that relies on the theory of V-filtrations.

Theorem 10.4 ([38, 51]) *Let $\lambda \in (0, 1]$ be a jumping number of a polynomial $f \in A$. Then $-\lambda$ is a root of the Bernstein-Sato polynomial $b_f(s)$.*

Proof Let $\lambda \in (0, 1]$ be a jumping number and take $g \in \mathcal{J}(f^{\lambda-\varepsilon}) \smallsetminus \mathcal{J}(f^{\lambda})$ for $\varepsilon > 0$ small enough. Therefore $\frac{|g(x)|^2}{|f(x)|^{2(\lambda-\varepsilon)}}$ is integrable but when we take the limit $\varepsilon \to 0$ we end up with $\frac{|g(x)|^2}{|f(x)|^{2\lambda}}$ that is not integrable.

Consider Bernstein-Sato functional equation $\delta(s) \cdot f^{s+1} = b_f(s) \cdot f^s$ and its application to the analytic continuation of the complex zeta function

$$b_f^2(s) \int_{\mathbb{C}^d} \varphi(x, \bar{x}) |f(x)|^{2s} dx d\bar{x} = \int_{\mathbb{C}^d} \bar{\delta}^* \delta^*(s) \big(\varphi(x, \bar{x})\big) |f(x)|^{2(s+1)} dx d\bar{x}.$$

Notice that $|g(x)|^2 \varphi(x, \bar{x})$ is still a test function so

$$b_f^2(s) \int_{\mathbb{C}^d} |g|^2 \varphi(x, \bar{x}) |f(x)|^{2s} dx d\bar{x} = \int_{\mathbb{C}^d} \bar{\delta}^* \delta^*(s) \big(|g|^2 \varphi(x, \bar{x})\big) |f(x)|^{2(s+1)} dx d\bar{x}.$$

Now we take a test function φ which is zero outside the ball $\overline{B}_\epsilon(p)$ and identically one on a smaller ball $\overline{B}_{\epsilon'}(p) \subseteq \overline{B}_\epsilon(p)$ and thus we get

$$b_f^2(s) \int_{\overline{B}_{\epsilon'}(p)} |g|^2 |f(x)|^{2s} dx d\bar{x} = \int_{\overline{B}_{\epsilon'}(p)} \bar{\delta}^* \delta^*(s)\big(|g|^2\big)|f(x)|^{2(s+1)} dx d\bar{x}.$$

Taking $s = -(\lambda - \varepsilon)$ we get

$$b_f^2(-\lambda+\varepsilon) \int_{\overline{B}_{\epsilon'}(p)} \frac{|g|^2}{|f(x)|^{2(\lambda-\varepsilon)}} dx d\bar{x} = \int_{\overline{B}_{\epsilon'}(p)} \bar{\delta}^* \delta^*(-\lambda+\varepsilon)\big(|g|^2\big)|f(x)|^{2(1-\lambda+\varepsilon)} dx d\bar{x}$$

but the right-hand side is uniformly bounded for all $\varepsilon > 0$. Thus we have

$$b_f^2(-\lambda+\varepsilon) \int_{\overline{B}_{\epsilon'}(p)} \frac{|g|^2}{|f(x)|^{2(\lambda-\varepsilon)}} dx d\bar{x} \leq M < \infty$$

for some positive number M that depends on g. Then, by the monotone convergence theorem we have to have $b_f^2(-\lambda) = 0$. □

So far we have been dealing with the case of an hypersurface $f \in A$ for the sake of clarity but everything works just fine for any ideal $\mathfrak{a} = \langle f_1, \dots, f_m \rangle \subseteq A$. The analytical definition of multiplier ideal at a point $p \in Z(\mathfrak{a})$ associated with a rational number $\lambda \in \mathbb{Q}_{>0}$ is

$$\mathcal{J}(\mathfrak{a}^\lambda)_p = \Big\{ g \in A \;\Big|\; \exists \epsilon \ll 1 \text{ such that } \int_{\overline{B}_\epsilon(p)} \frac{|g|^2}{(|f_1|^2 + \cdots + |f_m|^2)^\lambda} dx d\overline{x} < \infty \Big\}.$$

and $\mathcal{J}(\mathfrak{a}^\lambda) = \cap_{p \in Z(\mathfrak{a})} \mathcal{J}(\mathfrak{a}^\lambda)_p$. One can show that the ideal that we obtain is independent of the set of generators of the ideal $\mathfrak{a}$.

For the algebraic geometry version we consider the stalk at p of the multiplier ideal

$$\mathcal{J}(\mathfrak{a}^\lambda) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F_\pi \rceil\right),$$

given in Definition 2.11. The extension of Theorem 10.4 to this setting was proved by Budur, Mustaţă, and Saito [36] using the theory of V-filtrations.

Theorem 10.5 ([36]) *Let $\lambda \in (lct(\mathfrak{a}), lct(\mathfrak{a}) + 1]$ be a jumping number of $\mathfrak{a} \subseteq A$. Then $-\lambda$ is a root of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$.*

Finally we want to mention that multiplier ideals can be characterized completely in terms of relative Bernstein-Sato polynomials. Namely:

Theorem 10.6 ([36]) *For all ideals $\mathfrak{a} \subseteq A$ and all λ we have the equality*

$$\mathcal{J}(\mathfrak{a}^\lambda) = \{g \in A \mid \gamma > \lambda \textit{ if } b_{\mathfrak{a},g}(-\gamma) = 0\}.$$

This theorem is due to Budur and Saito [38] in the case $\mathfrak{a}$ is principal, and due to Budur et al. [36] as stated. The proofs rely on the theory of mixed Hodge modules. Recent work of Dirks and Mustaţă [49] provides a proof of this result that does not use the theory of mixed Hodge modules.

The analogues of Theorems 10.5 and 10.6 have been shown to hold for certain singular rings.

To illustrate Theorem 10.6, we use this description of multiplier ideals to give a quick proof of Skoda's Theorem in the principal ideal case.

Proposition 10.7 (Skoda's Theorem for Principal Ideals) *For all $f \in A \smallsetminus \{0\}$ and all λ, we have $\mathcal{J}(f^{\lambda+1}) = (f)\mathcal{J}(f^{\lambda})$.*

Proof Let $g \in \mathcal{J}(f^{\lambda})$, so every root of $b_{f,g}(s)$ is less than $-\lambda$. Then, by Lemma 5.33, every root of $b_{f,fg}(s)$ is less than $-\lambda - 1$, and hence $fg \in \mathcal{J}(f^{\lambda+1})$. This shows the containment $\mathcal{J}(f^{\lambda+1}) \supseteq (f)\mathcal{J}(f^{\lambda})$.

Now, if $g \notin (f)$, then $s = -1$ is a root of $b_{f,g}(s)$ by Lemma 5.32. Thus, $\mathcal{J}(f^{\lambda+1}) \subseteq (f)$. In particular, we can write $h \in \mathcal{J}(f^{\lambda+1})$ as $h = fg$ for $g \in A$; since the largest root of $b_{f,g}(s)$ is one greater than the largest root of $b_{f,h}(s)$ by Lemma 5.33, we have that $h \in \mathcal{J}(f^{\lambda})$, and the equality follows. □

Theorem 10.8 ([2]) *Let R be either a ring of invariants of an action of a finite group on a polynomial ring, or an affine normal toric ring. Then, for every ideal $\mathfrak{a} \subseteq R$, we have the log canonical threshold of $\mathfrak{a}$ in R coincides with the smallest root α of $b^R_{\mathfrak{a}}(-s)$, and every jumping number of $\mathfrak{a}$ in $[\alpha, \alpha + 1)$ is a root of $b^R_{\mathfrak{a}}(-s)$. Moreover,*

$$\mathcal{J}_R(\mathfrak{a}^{\lambda}) = \{g \in R \mid \gamma > \lambda \textit{ if } b^R_{\mathfrak{a},g}(-\gamma) = 0\}.$$

The idea behind the proof of this theorem is based on reduction modulo p and a positive characteristic analogue of the notion of differentially extensibility direct summand as in Definition 7.11. We refer the reader to [2] for details.

11 Computations via F-Thresholds

The notion of Bernstein-Sato root in positive characteristic discussed in Sect. 6 is closely related to F-jumping numbers. In this section, we discuss a relationship between the classical Bernstein-Sato polynomial in characteristic zero and similar numerical invariants in characteristic p. This connection was first established by Mustaţă et al. [102], and extended to the singular setting by Àlvarez Montaner et al. [1].

Definition 11.1 ([102]) Let R be a ring of characteristic $p > 0$. Let $\mathfrak{a}$, J be ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$. We set

$$\nu^J_{\mathfrak{a}}(p^e) = \max\{n \in \mathbb{N} \mid \mathfrak{a}^n \not\subseteq J^{[p^e]}\}.$$

We point out that the limit of $\lim_{e\to\infty} \frac{\nu^J_{\mathfrak{a}}(p^e)}{p^e}$ exists [50].

Theorem 11.2 ([1], see also [102]) *Let R be a finitely generated flat $\mathbb{Z}[1/a]$-algebra for some nonzero $a \in \mathbb{Z}$, and $\mathfrak{a} \subseteq \sqrt{J}$ ideals of R. Write R_0 for $R \otimes_{\mathbb{Z}} \mathbb{Q}$, and R_p for R/pR; likewise, write $\mathfrak{a}_0$ for the extension of $\mathfrak{a}$ to R_0, and similarly for $\mathfrak{a}_p$, J_0, J_p, etc. If $\mathfrak{a}_0$ has a Bernstein-Sato polynomial in R_0, then we have*

$$((s+1)b^{R_0}_{\mathfrak{a}_0})(\nu^{J_p}_{\mathfrak{a}_p}(p^e)) \equiv 0 \; mod \; p$$

for all $p \gg 0$.

Sketch of proof First, if $\mathfrak{a} = (f_1, \dots, f_\ell)$, set $g = \sum_i f_i y_i \in R' = R[y_1, \dots, y_\ell]$. Then, one checks easily that for $p \nmid a$, we have $\nu^{J_p}_{\mathfrak{a}_p}(p^e) = \nu^{JR'_p}_{g_p}(p^e)$. Thus, we can reduce to the principal case, where $\mathfrak{a} = (f)$.

Let $\delta(s) f^{s+1} = b_f(s) f^s$ be a functional equation for f in. If we replace a by a nonzero multiple, we can assume that $\delta(s)$ is contained in the image of $D_R[s]$ in $D_{R_0}[s]$ (see [1, Lemma 4.18]) and that $b_f(s) \in \mathbb{Z}[1/a][s]$. Pick n such that $\delta(s) \in D^n_R[s]$ and n is greater than any prime dividing a denominator of a coefficient of $b_f(s)$. Then, for every $p \geq n$, we may take the functional equation modulo p in R_p:

$$\overline{\delta(s)} f^{s+1} = \overline{b_f(s)} f^s.$$

Since $n < p$, we have $\overline{\delta(s)} \in D^{(1)}_{R_p|\mathbb{F}_p}$. In particular, $\overline{\delta(s)}$ is linear over each subring $R^{[p^e]}$, so it stabilizes every ideal expanded from such a subring, namely the Frobenius powers $J^{[p^e]}$ of J. For $s = \nu^{J_p}_{f_p}$, we have $f^s \notin J^{[p^e]}$, and $f^{s+1} \in J^{[p^e]}$, so $\overline{\delta(s)} f^{s+1} \in J^{[p^e]}$; we conclude that $\overline{b_f(s)} = 0$ in $\mathbb{F}_p$, as claimed. □

The previous theorem can be applied to find roots of $b^{R_0}_{\mathfrak{a}_0}(s)$ in $\mathbb{Q}$ when there are sufficiently nice formulas for $\nu^{J_p}_{\mathfrak{a}_p}(p^e)$ for e fixed as p varies.

Proposition 11.3 ([102]) *Let R be a finitely generated flat $\mathbb{Z}[1/a]$-algebra for some nonzero $a \in \mathbb{Z}$, and $\mathfrak{a} \subseteq \sqrt{J}$ ideals of R. Write R_0 for $R \otimes_{\mathbb{Z}} \mathbb{Q}$, and R_p for R/pR; likewise, write $\mathfrak{a}_0$ for the extension of $\mathfrak{a}$ to R_0, and similarly for $\mathfrak{a}_p$, J_0, J_p, etc. Suppose that $\mathfrak{a}_0$ has a Bernstein-Sato polynomial in R_0.*

Let $e > 0$. Suppose that there is an integer N and polynomials $Q_{[i]}$ for each $[i] \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\nu^{J_p}_{\mathfrak{a}_p}(p^e) = Q_{[i]}(p^e)$ for all $p \gg 0$ with $p \in [i]$. Then $Q_{[i]}(0)$ is a root of $b^{R_0}_{\mathfrak{a}_0}(s)$ for each $[i] \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Proof We can consider $b^{R_0}_{\mathfrak{a}_0}(s)$ as a polynomial over $\mathbb{Z}[1/aa']$ for some a'. Fix $[i] \in (\mathbb{Z}/N\mathbb{Z})^\times$. For any $p \in [i]$ with $p \nmid (aa')$, we have

$$(s+1)b^{R_0}_{\mathfrak{a}_0}(Q_{[i]}(0)) \equiv b^{R_0}_{\mathfrak{a}_0}(Q_{[i]}(p^e)) \equiv 0 \mod p,$$

so $p \mid b_{\mathfrak{a}_0}^{R_0}(Q_{[i]}(0))$. As there are infinitely many primes $p \in [i]$, we must have $b_{\mathfrak{a}_0}^{R_0}(Q_{[i]}(0)) = 0$. □

Example 11.4 ([102]) Let $f = x^2 + y^3 \in \mathbb{Z}[x, y]$, and $\mathfrak{m} = (x, y)$. One has

$$\nu_{f_p}^{\mathfrak{m}}(p^e) = \begin{cases} \frac{5}{6}p^e - \frac{5}{6} & \text{if } p \equiv 1 \mod 3 \\ \nu_{f_p}^{\mathfrak{m}}(p^e) = \frac{5}{6}p - \frac{7}{6} & \text{if } p \equiv 2 \mod 3,\ e = 1 \\ \nu_{f_p}^{\mathfrak{m}}(p^e) = \frac{5}{6}p^e - \frac{1}{6}p^{e-1} - 1 & \text{if } p \equiv 2 \mod 3,\ e \geq 2. \end{cases}$$

By the previous proposition, $-5/6$, -1 and $-7/6$ are roots of $b_f(s)$, considering f as an element of $\mathbb{Q}[x, y]$. In fact, $b_f(s) = (s + \frac{5}{6})(s + 1)(s + \frac{7}{6})$.

We note that the method of Proposition 11.3 does not yield any information about the multiplicities of the roots. There are also examples given in [102] of Bernstein-Sato polynomials with roots that cannot be recovered by this method. Nonetheless, we note that this method was successfully employed by Budur et al. [37] to compute the Bernstein-Sato polynomials of monomial ideals.

Remark 11.5 In the case of a regular ring $A = \mathbb{F}_p[x_1, \ldots, x_d]$, and ideals $\mathfrak{a}$, J of A with $\mathfrak{a} \subseteq \sqrt{J}$, the numbers $\nu_{\mathfrak{a}}^J(p^e)$ are closely related to the F-jumping numbers discussed in the introduction. In particular, combining [102, Propositions 1.9 & 2.7] for $\mathfrak{a}$ and e fixed, we have

$$\{\nu_{\mathfrak{a}}^J(p^e) \mid \sqrt{J} \supseteq \mathfrak{a}\} = \{\lceil p^e\lambda \rceil - 1 \mid \lambda \text{ is an } F\text{-jumping number of } \mathfrak{a}\}.$$

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Lower Bounds on Betti Numbers

Adam Boocher and Eloísa Grifo

Dedicated to David Eisenbud on the occasion of his 75th birthday.

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1 Introduction

Consider a polynomial ring over a field k, say $R = k[x_1, \ldots, x_n]$. When studying finitely generated graded modules M over R, there are many important invariants we may consider, with the Betti numbers of M, denoted $\beta_i(M)$, being among some of the richest. The Betti numbers are defined in terms of generators and relations (see Sect. 2), with $\beta_0(M)$ being the number of minimal generators of M, $\beta_1(M)$ the number of minimal relations on these generators, and so on. Despite this simple definition, they encode a great deal of information. For instance, if one knows the Betti numbers[1] of M, one can determine the Hilbert series, dimension, multiplicity, projective dimension, and depth of M. Furthermore, the Betti numbers provide even finer data than this, and can often be used to detect subtle geometric differences (see Example 3.4 for an obligatory example concerning the twisted cubic curve).

There are many questions one can ask about Betti numbers. What sequences arise as the Betti numbers of some module? Must the sequence be unimodal? How small, or how large, can individual Betti numbers be? How large is the sum? Questions like

[1] Really, we mean the graded Betti numbers of M, to be defined in Sect. 3.

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this are but just a few examples of those that have been studied in the past decades, and of the flavor we will discuss in this survey. We will focus on perhaps one of the longest standing open questions in this area, which is due to Buchsbaum–Eisenbud, and independently Horrocks (BEH). Their conjecture proposes a lower bound for each $\beta_i(M)$ depending only on the codimension c of M: that $\beta_i(M) \geqslant \binom{c}{i}$. While the conjecture remains widely open in the general setting, there are some special cases that are known. Moreover, if the conjecture is true, then the total Betti number of M, $\beta(M) := \beta_0(M) + \cdots + \beta_n(M)$, must satisfy $\beta(M) \geqslant 2^c$. Recently, Mark Walker [69] proved this bound on the total Betti number—known as the Total Rank Conjecture—in all cases except when char $k = 2$. Walker also showed that equality holds if and only if M is isomorphic to R modulo a regular sequence—such modules are called complete intersections.

The Betti numbers of modules that are *not* complete intersections are quite interesting. For example, it follows from Walker's result that if our module M is not a complete intersection, then $\beta(M) \geqslant 2^c + 1$, but there is reason to believe that $\beta(M)$ might be much bigger than 2^c. Charalambous, Evans, and Miller [31] asked if in fact we must have $\beta(M) \geqslant 2^c + 2^{c-1}$, and proved that this holds when M is either a graded module small codimension ($c \leqslant 4$), or a multigraded module of finite length (meaning $c = n$) for arbitrary c [29, 30]. More evidence towards this larger bound for Betti numbers has recently been found, including [11, 12].

For example, Erman showed [41] that if M is a graded module of small regularity (in terms of the degrees of the first syzygies), then not only is the BEH Conjecture 4.1 true, but in fact $\beta_i(M) \geqslant \beta_0(M)\binom{c}{i}$. The first author and Wigglesworth [12] then extended Erman's work to say that under the same low regularity hypothesis, $\beta(M) \geqslant \beta_0(M)(2^c + 2^{c-1})$. This stronger bound asserts that on average, each Betti number $\beta_i(M)$ is at least 1.5 times $\beta_0(M)\binom{c}{i}$.

The main goal of this survey is to discuss these lower bounds on Betti numbers and present some of the motivation for these conjectures. We start with a short introduction to free resolutions and Betti numbers, why we care about them, and some of the very rich history surrounding these topics. We also collect some open questions, discuss some possible approaches, and present examples that explain why certain hypothesis are important.

2 What Is a Free Resolution?

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. We will be primarily concerned with **finitely generated graded R-modules** M. One important invariant of such a module is the minimal number of elements needed to generate M. In fact, this number is the first in a sequence of **Betti numbers** that describe how far M

is from being a free module. Indeed, suppose that M is minimally generated by β_0 elements; this means there is a surjection from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M.$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a **free module** of **rank** β_0. Otherwise, it has a nonzero kernel, which will also be finitely generated and can be written as the surjective image of some free module R^{β_1}:

$$\begin{array}{ccccc} R^{\beta_1} & \dashrightarrow & R^{\beta_0} & \xrightarrow{\pi_0} & M. \\ & \searrow & \nearrow & & \\ & & \ker(\pi_0) & & \end{array}$$

Notice that if M is generated by $m_1, \ldots, m_{\beta_0}$, and π_0 is the map sending each canonical basis element e_i in R^{β_0} to m_i, then an element $(r_1, \ldots, r_{\beta_0})^T$ in the kernel of π_0 corresponds precisely to a **relation** among the m_i, meaning that

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[2] of M and the module $\ker \pi_0$ is called the first syzygy module of M.

Continuing this process we can *approximate* M by an exact sequence

$$\cdots \longrightarrow F_p \xrightarrow{\pi_p} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each F_i is free. Such an exact sequence is called a **free resolution of** M.

If at each step we have chosen F_i to have the minimal number of generators, then we say the resolution is **minimal**, and we set $\beta_i(M)$ to be the rank of F_i in any such minimal free resolution. This is well-defined, because it is true that two minimal free resolutions of M are isomorphic as complexes. Furthermore, one has the following,

$$\beta_i(M) = \operatorname{rk} F_i = \operatorname{rk}_k \operatorname{Tor}_i^R(M, k).$$

The i**th syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i, or equivalently the kernel of π_{i-1}. We note that $\Omega_i(M)$ is defined only up to isomorphism.

If at some point in the resolution we obtain an injective map of free modules, then its kernel is trivial, and we obtain a finite free resolution, in this case of length p:

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

[2] Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

If a module M has a finite minimal projective resolution, the length of such a resolution is called the projective dimension of M, and we write it $\operatorname{pdim} M$.

Remark 2.1 We will often implicitly apply the Rank-Nullity Theorem to conclude that

$$\beta_i(M) = \operatorname{rk}\Omega_i(M) + \operatorname{rk}\Omega_{i+1}(M).$$

Example 2.2 If $M = R/(f_1, \dots, f_c)$ where the f_i form a regular sequence, then the minimal free resolution of M is given by the **Koszul complex.** For instance if $c = 4$ then the minimal resolution has the form

$$0 \longrightarrow R^1 \xrightarrow{1} R^4 \xrightarrow{3} R^6 \xrightarrow{3} R^4 \xrightarrow{1} R^1 \xrightarrow{0} M.$$

Note that the numbers over the arrows represent the rank of the corresponding map, which is equal to the rank of the corresponding syzygy module $\Omega_i(M)$. We will discuss this in more detail in Sect. 3.2. We will also see that the ranks occurring in the Koszul complex are conjectured to be the smallest possible for modules of codimension c (see Conjecture 4.2).

Example 2.3 One of the strongest known bounds on ranks of syzygies is the Syzygy Theorem 3.13 which states that except for the last syzygy module, the rank of $\Omega_i(M)$ is always at least i. A typical use of such a result might be as follows. Suppose we had a rank zero module M with Betti numbers $\{1, 7, 8, 8, 7, 1\}$. Then we could calculate the ranks of the syzygy modules by using Remark 2.1 to obtain the ranks labeled in the diagram below:

$$0 \longrightarrow R^1 \xrightarrow{1} R^7 \xrightarrow{6} R^8 \xrightarrow{2} R^8 \xrightarrow{6} R^7 \xrightarrow{1} R^1 \xrightarrow{0} M.$$

We would also obtain from Remark 2.1 that $\operatorname{rk}\Omega_3(M) = 2$, which we will see violates Theorem 3.13. Therefore, such a module does not exist! See also Example 5.17.

Example 2.4 In [36], Dugger discusses almost complete intersection ideals and the tantalizing fact that we currently do not know whether or not there is an ideal I of height 5 with minimal free resolution

$$0 \longrightarrow R^6 \xrightarrow{6} R^{12} \xrightarrow{6} R^{10} \xrightarrow{4} R^9 \xrightarrow{5} R^6 \xrightarrow{1} R^1 \xrightarrow{0} R/I.$$

David Hilbert, interested in studying minimal free resolutions as a way to count invariants, was able to prove that finitely generated modules over a polynomial ring always have finite projective dimension [49].

Theorem 2.5 (Hilbert's Syzygy Theorem, 1890) *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field k. If M is a finitely generated graded R-module, then M has a finite free resolution of length at most n.*

While we are primarily interested in studying polynomial rings over fields, Hilbert's Syzygy Theorem is true more generally for any Noetherian regular ring. In fact, if we focus our study on local rings instead, the condition that every finitely generated module has finite projective dimension characterizes regular local rings [3, 68]. While we will be working over polynomial rings throughout the rest of the paper, we point out that the theory of (infinite) free resolutions over non-regular rings is quite interesting and rich; [59] and [5] are excellent places to start learning about this.

The upshot of Hilbert's Syzygy Theorem is that to each finitely generated R-module M we attach a finite list of Betti numbers $\beta_0(M), \ldots, \beta_n(M)$. Note that while some of these might vanish, M has at most $n+1$ non-zero Betti numbers.

Our main goal in this paper is to discuss the following question:

Question A *If M is a finitely generated graded module over $R = k[x_1, \ldots, x_n]$, where k is a field, can we bound the Betti numbers of M, either from above or below?*

As we will see, there are many results and conjectures relevant to the answer to this question. Feel free to skip the next section if you can't handle the suspense!

3 Why Study Resolutions?

Before getting to the heart of the matter in Sect. 4, we would first like to offer some motivation as to why one might care about Betti numbers at all.

3.1 Betti Numbers Encode Geometry

In a sense, a minimal free resolution of M contains redundant information—after all, the first map $\pi_1 \colon F_1 \to F_0$ is a presentation of M. However, suppose we do not know the maps in the resolution, but just the **numerical data** of the resolution, namely the numbers $\{\beta_i\}$. Surprisingly, this coarse invariant encodes much geometric and algebraic information about M. First of all, the Betti numbers β_i tell us that M has β_0 generators, that there are β_1 relations among those generators, and β_2 relations among those relations, and so on. But the Betti numbers also encode more sophisticated information about M. For instance, since rank is additive across exact sequences, we have

$$\operatorname{rk} M = \beta_0 - \beta_1 + \cdots + (-1)^n \beta_n.$$

Moreover, if we have a graded module M, we can take the resolution of M to be a graded resolution, and if among the β_i generators of $\Omega_i(M)$, exactly β_{ij} of them live in degree j, then the following formula gives the Hilbert series for M:

$$HS(M) = \frac{\sum_{i=0}^{d}(-1)^i \beta_{ij} t^j}{(1-t)^d}. \tag{3.1}$$

We recall that the **Hilbert series** of M is a power series that encodes the k-vector space dimension of each graded piece M_i of M, as follows:

$$HS(M) = \sum_{i=0}^{\infty} \dim_k(M_i) t^i.$$

This is a classical tool that contains important algebraic and geometric information about our module. For example, once we write $HS(M) = p(t)/(1-t)^m$ with $p(1) \neq 0$, we have $\dim(M) = m$ and $p(1)$ is equal to the degree of M. So just by knowing its (graded) Betti numbers, we can then determine the multiplicity (i.e. degree), dimension, projective dimension, Cohen-Macaulayness, and other properties and invariants of a module M.

The following example gives the spirit of these ideas:

Example 3.1 Suppose that $R = k[x, y, z]$ and that $M = R/(xy, xz, yz)$ corresponds to the affine variety defining the union of the three coordinate lines in k^3. This variety has dimension one and degree three. Let us illustrate how the (graded) Betti numbers communicate this. The minimal free resolution for M is

$$0 \longrightarrow R^2 \xrightarrow{\psi = \begin{bmatrix} z & 0 \\ -y & y \\ 0 & -x \end{bmatrix}} R^3 \xrightarrow{\phi = \begin{bmatrix} xy & xz & yz \end{bmatrix}} R \longrightarrow M.$$

From this minimal resolution, we can read the Betti numbers of M:

- $\beta_0 = 1$, since M is a cyclic module;
- $\beta_1 = 3$, and these three quadratic generators live in degree 2;
- $\beta_2 = 2$, and these represent linear (degree 1) syzygies on quadrics (degree 2), and thus live in degree 3 $(= 1 + 2)$.

We can include this *graded* information in our resolution, and write a *graded* free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\psi = \begin{bmatrix} z & 0 \\ -y & y \\ 0 & -x \end{bmatrix}} R(-2)^3 \xrightarrow{\phi = \begin{bmatrix} xy & xz & yz \end{bmatrix}} R \longrightarrow M.$$

The $R(-2)^3$ indicates that we have three generators of degree 2. Formally, the R-module $R(-a)$ is one copy of R whose elements have their degrees shifted by a: the polynomial 1 lives in degree 0 in R and degree a in $R(-a)$, and in general the degree d piece of $R(-a)$ consists of the elements of R of degree $d-a$. With this convention, the map ϕ keeps degrees unchanged—we say it is a degree 0 map: for example, it takes the vector $[1, 1, 1]^T$, which lives in degree 2, to the element $xy + xz + yz$, which is an element of degree 2. When we move on to the next map, ψ, we only need to shift the degree of each generator by 1, but since ψ now lands on $R(-2)^3$, we write $R(-3)^2$.

The graded Betti number $\beta_{ij}(M)$ of M counts the number of copies of $R(-j)$ in homological degree i in our resolution. So we have

$$\beta_{00} = 1, \beta_{12} = 3, \text{and } \beta_{23} = 2.$$

We can collect the graded Betti numbers of M in what is called a *Betti table*:

$$\begin{array}{c|ccc} \beta(M) & 0 & 1 & 2 \\ \hline 0 & \beta_{00} & \beta_{11} & \beta_{22} \\ 1 & \beta_{01} & \beta_{12} & \beta_{23} \\ & & & \end{array}, \qquad \begin{array}{c|ccc} \beta(M) & 0 & 1 & 2 \\ \hline 0 & 1 & - & - \\ 1 & - & 3 & 2 \\ & & & \end{array}.$$

Remark 3.2 To the reader who is seeing Betti tables for the first time, we point out that although we will write resolutions so that the maps go from left to right, and thus the Betti numbers appear from right to left $\{\dots, \beta_2, \beta_1, \beta_0\}$ in a Betti table, the opposite order is used. Furthermore, by convention, the entry corresponding to (i, j) in the Betti table of M is $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$.

Finally, we can use this information to calculate the Hilbert series of M:

$$HS(M) = \frac{1t^0 - 3t^2 + 2t^3}{(1-t)^3} = \frac{1+2t}{(1-t)^1},$$

and since this last fraction is in lowest terms, we see that the dimension of M is 1 (the degree of the denominator) and that the degree of M is equal to $p(1) = 1+2\cdot 1 = 3$. Recall that M corresponded to the union of 3 lines. Notice that in this example, the projective dimension of M is 2, which is equal to the codimension $3-1=2$ of M. Hence, M is Cohen-Macaulay. In summary, we can get lots of information about M from its (graded) Betti numbers.

Example 3.3 (The Hilbert Series Doesn't Determine the Betti Numbers) Let k be a field, $R = k[x, y]$, and consider the two ideals

$$I = (x^2, xy, y^3) \quad \text{and} \quad J = (x^2, xy + y^2).$$

One can check that both R/I and R/J have the same Hilbert series:

$$HS(R/I) = HS(R/J) = 1 + 2t + 1t^2.$$

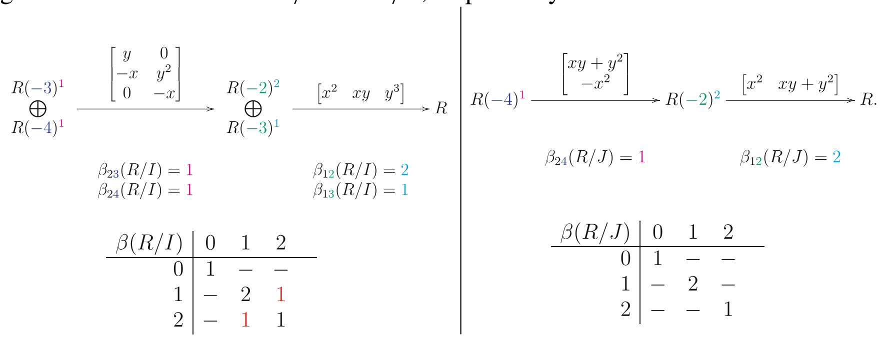
However, these modules have different Betti numbers. We work out the minimal free resolution and Betti numbers for R/I. Since I has two generators of degree 2 and one of degree 3, there are graded Betti numbers β_{12} and β_{13}. Similarly, the two minimal syzygies of R/I correspond to the relations

$$y(x^2) - x(xy) = 0 \text{ which has degree 3, so } \beta_{23} = 1$$

and

$$y^2(xy) - x(y^3) = 0 \text{ which has degree 4, so } \beta_{24} = 1.$$

Continuing this process, we find the following minimal free resolutions and graded Betti numbers for R/I and R/J, respectively:



$\beta(R/I)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

$\beta(R/J)$	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

Finally, if we calculate the Hilbert series from Eq. 3.1, we notice that the calculation is the same for R/I and R/J:

$$HS(R/I) = \frac{1 - 2t^2 - t^3 + t^3 + t^4}{(1-t)^3} = \frac{1 - 2t^2 + t^4}{(1-t)^3} = HS(R/J).$$

The cancellation of the t^3 terms is known as a **consecutive cancellation**, and one can see the two 1s on the diagonal in the Betti table for R/I. For the reader who knows about Gröbner degenerations, I is the initial ideal of J coming from a Lex term-order. Any such degeneration will preserve the Hilbert series, but not necessarily the

Betti numbers. For results concerning the relationship between the Betti numbers of ideals and those of their initial ideals, see [2, 10, 32–34, 61].

Example 3.4 We would be remiss if, in this article dedicated to David Eisenbud on his birthday, we didn't also mention that the connection between graded Betti numbers and geometry is a rich and beautiful story. In his book [37], he paints a story that begins with the following surprising fact from geometry. If X is a set consisting of seven general[3] points in $\mathbb{P}^3$, then the Hilbert series of the coordinate ring for X is completely determined by this data. However, this is not sufficient to determine the Betti numbers of the coordinate ring of X. Indeed, these numbers are either $\{1, 4, 6, 3\}$ or $\{1, 6, 8, 3\}$ depending on whether or not the points lie on a curve of degree 3.

3.2 Resolutions for Ideals with Few Generators

Over a polynomial ring $R = k[x_1, \ldots, x_n]$, calculating a free resolution is tantamount to producing the sets of dependence relations among the generators of a module. In simple cases this is straightforward, as the following example shows:

Example 3.5 Consider the module $M = R/(f)$, where f is a homogeneous polynomial in R. Then

$$0 \longrightarrow R \xrightarrow{[f]} R \longrightarrow M$$

is a minimal free resolution of length 1, since over our polynomial ring R, f is a *regular* element and cannot be killed by multiplication by any nonzero element.

If I is an ideal minimally generated by two polynomials f and g, then the minimal free resolution of R/I has length two. Indeed, if $c = \gcd(f, g)$, then the following is a minimal free resolution:

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} g/c \\ -f/c \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} f & g \end{bmatrix}} R \longrightarrow R/I \ .$$

This example can be summarized by the following result:

Proposition 3.6 *If I is an ideal in a polynomial ring R that is minimally generated by one or two homogeneous polynomials, then the projective dimension of R/I is equal to the minimal number of generators, and the Betti numbers are either* $\{1, 1\}$ *or* $\{1, 2, 1\}$.

[3] This means that no more than 3 lie on a plane and no more than 5 on a conic.

Whatever optimistic generalization of this proposition one might have in mind for ideals with 3 or more generators will certainly fail to be true, as we have the following astonishing results of Burch and Bruns:

Theorem 3.7 (Burch [20]) *For each $N \geqslant 2$, there exists a three-generated ideal I in a polynomial ring $R = k[x_1, \ldots, x_N]$ such that* $\operatorname{pdim}(R/I) = N$.

So we can always find free resolutions of maximal length by simply using 3 generated ideals. In fact, in some sense "every" free resolution is the free resolution of a 3-generated ideal:

Theorem 3.8 (Bruns [15]) *Let $R = k[x_1, \ldots, x_n]$ and*

$$0 \longrightarrow F_n \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

be a minimal free resolution of a finitely generated graded R-module M. Then there exists a 3-generated ideal I in R with minimal free resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_3 \longrightarrow F_2' \longrightarrow R^3 \longrightarrow R \longrightarrow R/I\ .$$

Remark 3.9 Note that the rank of F_2' may be different than that of F_2, but a rank calculation yields that

$$\operatorname{rk} F_2' = 3 - 1 + \operatorname{rk} F_3 - \operatorname{rk} F_4 + \cdots \pm \operatorname{rk} F_n = 2 + \operatorname{rk} F_2 - \operatorname{rk} F_1 + \operatorname{rk} F_0.$$

From this, it follows that β_2 can be arbitrarily large for 3-generated ideals.

Our point in presenting these results is to make plain that free resolutions are complicated—even for ideals with 3 generators! However, if in Example 3.5 we add a further restriction for the ideal $I = (f, g)$ and require that f and g have no common factors (meaning that g is a *regular* element modulo f), then the only relations between f and g are given by the "obvious" relation that $gf - fg = 0$. This fact *does* generalize nicely to any set $\{f_1, \ldots, f_c\}$ of homogeneous polynomials provided f_i is a regular element modulo the previous f_j. Such elements form what is called a **regular sequence**, and the ideal they generate is resolved by the **Koszul complex**. Rather than introducing the topic here, we point the reader to some of the many nice references for learning about the Koszul complex, such as [37, Chapter 17], [16, Section 1.6], or [5, Example 1.1.1].

The most important fact we will need about the Koszul complex is that it is a resolution (of $R/(f_1, \ldots, f_c)$) if and only if the $f_1, \ldots, f_c$ form a regular sequence, and that the Betti numbers (and ranks of syzygy modules) of the Koszul complex are given by binomial coefficients.

Theorem 3.10 *If I is an ideal generated by a regular sequence of c homogeneous polynomials, then*

$$\operatorname{rk} \Omega_i(R/I) = \binom{c-1}{i-1},$$

and therefore

$$\beta_i(R/I) = \binom{c}{i}.$$

Remark 3.11 To the reader not familiar with Koszul complexes, it might be instructive to carefully write out the maps involved to get a feel for how resolutions are constructed. Essentially, the point is that the generating ith syzygies are built from using i generators and the fact that $f_j f_i = f_i f_j$. Alternatively, perhaps the quickest way to define the Koszul complex is just to take the tensor product of the c minimal free resolutions of $R/(f_i)$:

$$0 \longrightarrow R \xrightarrow{f_i} R \longrightarrow 0.$$

Since multiplication by f_i has rank one, if one calculates the ranks in the tensor product inductively, one will see Pascal's Triangle appearing, providing a justification of the claims in Theorem 3.10.

3.3 How Small Can the Ranks of Syzygies Be?

If I is an ideal that is generated by a regular sequence then as we saw in the previous section, the minimal free resolution for R/I is given by the Koszul complex. For instance, if I has height 8, then $\beta_4(R/I)$ will be equal to $\binom{8}{4} = 70$, and the syzygy module $\Omega_4(R/I)$ will have rank $\binom{7}{3} = 35$. We will see in the next section (Conjectures 4.1 and 4.2) that among all ideals of height 8 these numbers are conjectured to be the smallest possible values for β_4 and $\operatorname{rk} \Omega_4$ respectively. In short, these conjectures assert a relationship between the ranks of syzygies and the height (or codimension) of the ideal. Before we present these conjectures, which will occupy the remainder of the paper, we close with an example and theorem that give the sharpest possible bound for ranks of syzygies if one does not refer to codimension.

Example 3.12 (Bruns [15]) Let $R = k[x_1, \dots, x_n]$. There is a finitely generated module M over R with the following resolution:

$$0 \longrightarrow R \longrightarrow R^n \longrightarrow R^{2n-3} \longrightarrow \cdots \longrightarrow R^5 \longrightarrow R^3 \longrightarrow R \longrightarrow M \longrightarrow 0.$$

In other words, the ith Betti number is $2i + 1$ except for the last two Betti numbers. This is the case for an even nicer reason: if one calculates the ranks of each syzygy module (which can be read off as the rank of the ith map π_i in the resolution) one sees that the ranks are:

$$0 \longrightarrow R \xrightarrow{1} R^n \xrightarrow{n-1} R^{2n-3} \xrightarrow{n-2} \cdots \xrightarrow{3} R^5 \xrightarrow{2} R^3 \xrightarrow{1} R \xrightarrow{0} M \longrightarrow 0.$$

In other words, in this example the ith syzygy module has rank equal to i, except for the last one. This bound holds for any module, which is the content of the great Syzygy Theorem.

Theorem 3.13 (Syzygy Theorem, Evans–Griffith [44]) *Let M be a finitely generated module over a polynomial ring R. If $\Omega_i(M)$ is not free, then* $\operatorname{rk}\Omega \geqslant i$. *Hence, if* $\operatorname{pdim} M = p$, *then*

$$\operatorname{rk}\Omega_i(M) \geqslant i, \; \textit{for } i < p.$$

Moreover,

$$\beta_i(M) = \operatorname{rk}\Omega_i(M) + \operatorname{rk}\Omega_{i+1}(M) \geqslant \begin{cases} 2i+1 & \textit{if } i < p-1 \\ p & \textit{if } i = p-1 \\ 1 & \textit{if } i = p \end{cases}$$

where $\Omega_i(M)$ denotes the ith syzygy module of M.

The Syzygy Theorem together with Bruns' example provides a sharp lower bound for $\beta_i(M)$. Without further conditions on M, there is not much more we can say. However, if we add additional hypotheses on M—for instance, requiring M to be Cohen-Macaulay, or of a fixed codimension c—then the bounds above appear to be far from sharp. Indeed, we will discuss a conjecture that states that in fact $\beta_i(M) \geqslant \binom{c}{i}$; when c is large, this conjecture is much stronger than the Syzygy Theorem's bound of $2i+1$. Note that the ideal in Example 3.12 is of codimension 2.

3.4 Other Possible Directions

Before we begin to focus on codimension, we want to say that there are many distinct and interesting alternative questions on bounds for Betti numbers that have been considered. We present some possibilities below.

One could decide to study ideals and then fix the number of generators of I; for example, one could study the sets of Betti numbers of ideals defined by 5 homogeneous polynomials. Theorem 3.8 shows that this approach will not allow for any upper bounds, except in trivial cases.

Refining this idea, one could add a condition on the *degrees* of the generators of these ideals, and for example ask what the maximal Betti numbers for an ideal with 3 quadratic generators might be. This question is tractable, though incredibly difficult. Note that here we are not saying how many variables are in the ring R. For instance, the largest Betti numbers possible for an ideal generated by 3 quadrics is $\{1, 3, 5, 4, 1\}$; note that the projective dimension is 4. More generally, the question of whether there exists an upper bound on the projective dimension of an ideal defined by r forms of degree $d_1, \ldots, d_r$ depending only on r and $d_1, \ldots, d_r$, and not on the number of variables, is known as Stillman's Conjecture, and has been solved by Ananyan and Hochster [1] in general. The question of providing effective upper bounds is much harder, and some of the efforts in this direction can be found in [42]. See [43] for an exposition on some of the followup results that expanded on the ideas initiated by Ananyan and Hochster in their proof of Stillman's conjecture; see also [60] for a survey and [25, 51] for related work on the subject.

We saw in Sects. 2 and 3 that the (graded) Betti number determine the Hilbert series; however, there can be many distinct sets $\{\beta_{ij}(M)\}$ for R-modules M all with the same given Hilbert series. If one fixes a Hilbert series, what are the possible sets $\{\beta_{ij}(M)\}$ for modules M with Hilbert series $h(t)$? The following theorems give a beautiful answer that provides an upper bound for the Betti numbers.

Theorem 3.14 (Bigatti [9], Hulett [50], Pardue [63]) *Let I be a homogeneous ideal in $R = k[x_1, \ldots, x_n]$. Consider the set*

$$\mathcal{H} = \{J \subseteq R \text{ an ideal } \mid HS(R/J) = HS(R/I)\}.$$

There exists an ideal $L \in \mathcal{H}$ with the property that among all ideals in H, the Betti numbers of L are the largest:

$$\beta_{ij}(R/J) \leqslant \beta_{ij}(R/L) \text{ for all } i, j \text{ and for all } J \in \mathcal{H}.$$

The ideal L that achieves the largest Betti numbers in the Theorem can be described explicitly, and goes back 100 years to work of Macaulay [58]; it is the known as the **Lex-segment ideal**. To construct L, we start by going over each degree D and ordering all the monomials in R_D lexicographically. Then we collect the first $\dim_k(J_D)$ monomials in degree D, for all D. Macaulay showed the ideal L generated by all these monomials has the same Hilbert function as our original ideal J; in other words, it is an ideal in $\mathcal{H}$. Bigatti, Hulett, and Pardue then showed that this special ideal has in fact the largest possible Betti numbers with the same Hilbert function as I. Moreover, if we fix a Hilbert polynomial, and consider all the saturated ideals I with that fixed Hilbert polynomial, there is also a particular lex-segment ideal that maximizes the Betti numbers [26].

While we will focus on lower bounds on Betti numbers given by comparing to the case of a complete intersection, there are bounds one may consider. Using Boij-Söderberg theory, Römer showed that the total Betti numbers of R are bounded above by a function of the shifts in the minimal graded free resolution of R, and bounded below by another function of the shifts when R is Cohen-Macaulay [66].

Finally, we remark that while this paper is devoted to the ranks of modules appearing in a minimal resolution—that is, the study of acyclic complexes—there has been much work devoted more generally to complexes, or even more generally to differential modules. For instance, it was conjectured in [7, Conjecture 5.3] that if $F_\bullet$ is any complex over a d-dimensional local ring, and if the homology $H(F)$ has finite length, then $\sum_i \operatorname{rk} F_i \geqslant 2^d$. This was shown in [7] for the case when $d \leqslant 3$, and in [35] in the multigraded setting (for all d). However, the conjecture is false in general. Indeed, in [53], an example is given of a complex of R-modules such that $H(F)$ has length 2 but $\sum_i \operatorname{rk} F_i < 2^d$ for all $d \geqslant 8$. See also [13, 22–24].

In the remainder of the paper we will state several conjectures concerning lower bounds for the $\beta_i(R/I)$ in terms of $c = \operatorname{codim} R/I$. As an appetizer, notice that the Krull altitude theorem asserts that the codimension of R/I must be at most the minimal number of generators, i.e. $\beta_1(R/I) \geqslant c$. Meanwhile, the Auslander-Buchsbaum formula above guarantees that the length of the resolution of R/I is at least the codimension c, which implies that $\beta_c(R/I) \geqslant 1$. With these two classical results giving us information about Betti numbers in terms of codimension, we now proceed to the main conjecture we want to focus on.

4 The Buchsbaum–Eisenbud–Horrocks Conjecture and the Total Rank Conjecture

In the late 1970s, Buchsbaum and Eisenbud [18], and independently Horrocks [47, Problem 24], conjectured that the Koszul complex is the smallest free resolution possible; more precisely, that the Betti numbers of any finitely generated module are at least as large as those of a complete intersection of the same codimension as given in Theorem 3.10:

Conjecture 4.1 (BEH Conjecture) Let $R = k[x_1, \ldots, x_n]$, where k is a field, and M be a nonzero finitely generated graded R-module of codimension c, meaning that $\operatorname{ht} \operatorname{ann}(M) = c$. Then

$$\beta_i(M) \geqslant \binom{c}{i}$$

for all $0 \leqslant i \leqslant \operatorname{pdim}_R M$.

Actually, both Buchsbaum and Eisenbud [18] and Horrocks [47, Problem 24] propose the following stronger conjecture:

Conjecture 4.2 (Stronger BEH Conjecture for the Ranks of the Syzygies) Let $R = k[x_1, \ldots, x_n]$, where k is a field, and M be a nonzero finitely generated graded

R-module of codimension c. Then

$$\operatorname{rk}(\Omega_i(M)) \geqslant \binom{c-1}{i-1}.$$

Originally, Horrocks' problem was stated for finite length modules over a regular local ring, i.e., the case that $\operatorname{codim} M$ was as large as possible, and equal to the dimension of the ring. On the other hand, Buchsbaum and Eisenbud were interested in resolutions of R/I for a general local ring R. They conjectured that the minimal free resolution of R/I possessed the structure of a commutative associative differential graded algebra; they then showed that if this held, and I had grade c, then the corresponding inequalities (which they independently attribute to Jürgen Herzog) on the ranks above would hold:

Theorem 4.3 (Buchsbaum–Eisenbud, Proposition 1.4 in [18]) *If R/I has codimension c and the minimal free resolution of R/I possesses the structure of an associative commutative differential graded algebra, then $\beta_i(R/I) \geqslant \binom{c}{i}$ for all i. Furthermore, the rank of the ith syzygy module is at least $\binom{c-1}{i-1}$.*

For some time it was open whether or not all resolutions could be given such a DGA structure. It turns out that this is not necessarily the case [4, Example 5.2.2], though notably any algebra R/I of projective dimension at most 3 or of projective dimension 4 that is Gorenstein will have such a resolution [18, 55, 56]. See also [8] for more on the $\operatorname{pdim}(R/I) \leqslant 3$ case.

Remark 4.4 Throughout, we will adopt the convention that $\binom{n}{k}$ is zero unless $0 \leqslant k \leqslant n$.

As a motivating example, let R/I be a cyclic module of codimension c.

- The principal ideal theorem guarantees that I must be generated by at least c elements, so $\beta_1(R/I) \geqslant \binom{c}{1}$.
- The Auslander–Buchsbaum formula implies that $\operatorname{pdim}(R/I) \geqslant c$, which implies that $\beta_c(R/I) \geqslant \binom{c}{c}$.
- If I is generated by exactly c elements, then R/I is resolved by the Koszul complex, and then $\beta_i(R/I) = \binom{c}{i}$ for all i.

If I has more than c generators, then I will not be a complete intersection, and in general there is no structural result concerning its minimal free resolution. However, it stands to reason (at least for optimists) that perhaps the Betti numbers can only increase as the number of generators grows and grows.

In the rest of this paper we have two goals. First, we want to survey various generalizations of the BEH Conjecture and give the state of the art for each of these. Second, we want to include a few basic constructions and techniques that could be helpful to those who want to work in this field. For a more thorough treatment, we refer the reader to the book [45] and survey article [28].

We have opted to give a summary of classical results on the BEH Conjecture first, but we want to point out right away that an immediate consequence of the

BEH Conjecture is that if the conjecture is true, then the sum of the Betti numbers will be at least 2^c. This weaker conjecture, known as the Total Rank Conjecture, was proven by Walker in 2018 [69]. Since then, there has been increasing evidence that apart from complete intersections, which are resolved by the Koszul complex, it may be true that in fact the sum of the Betti numbers is always at least $2^c + 2^{c-1}$. In the final section of the survey, we present the case for this stronger conjecture.

4.1 General Purpose Tools

The BEH Conjecture is known in a surprisingly small number of cases. Indeed, as a first challenge, it is open an open question whether $\beta_2(R/I) \geqslant \binom{5}{2}$ whenever I is an ideal of codimension 5. In this section, we present a collection of general purpose tools and use them to show that if $c \leqslant 4$ then the conjecture holds. We also carefully describe how localization can reduce the conjecture to the finite length case, provided we work over arbitrary regular local rings.

Proposition 4.5 (Buchsbaum–Eisenbud, Theorem 2.1 in [17], see also [58]) *Suppose that M is a module of codimension c. Then*

$$\beta_1(M) - \beta_0(M) + 1 \geqslant c.$$

If equality holds, then M is resolved by the Buchsbaum–Rim complex.

Note that this result includes both the Principal Ideal Theorem (when $M = R/I$ and thus $\beta_0(M) = 1$) and the fact that the Koszul complex (a special instance of the Buchsbaum–Rim complex [19]) resolves complete intersections. Below is a version of this result in terms of Betti numbers:

Corollary 4.6 *If M is a module of codimension c, then*

$$\beta_1(M) \geqslant \beta_0(M) + c - 1.$$

If equality holds, then for all $i \geqslant 2$

$$\beta_i(M) = \binom{\beta_0(M) + i - 3}{i - 2}\binom{\beta_1(M)}{\beta_0(M) + i - 1}.$$

As an exercise, the reader can prove that if $\beta_1(M) = \beta_0(M) + c - 1$ then the BEH conjecture holds, by the equality of binomial coefficients above.

Discounting cases when equality holds, this lower bound $\beta_1(M) > \beta_0(M)+c-1$ might not at first glance seem very useful, since it only gives information about $\beta_1(M)$. However, when M is Cohen-Macaulay we can use this result to also gain information about $\beta_{c-1}(M)$ as well by appealing to duality. Indeed, if M is a Cohen-Macaulay module, meaning that the codimension c of M is equal to its projective dimension, then applying $\operatorname{Hom}(-, R)$ to a resolution of M will yield a resolution of $\operatorname{Ext}_R^c(M, R)$, another Cohen-Macaulay module. This yields the following observation:

Proposition 4.7 *If $\{\beta_0, \ldots, \beta_c\}$ is the Betti sequence for a Cohen-Macaulay module, then so is the reverse sequence $\{\beta_c, \ldots, \beta_0\}$.*

As an application of these ideas, let us use these results to prove Conjecture 4.1 for $c \leqslant 4$. We focus on $c = 3$ and $c = 4$, as the smaller cases follow immediately from the principal ideal theorem.

When $\mathbf{c} = \mathbf{3}$, Corollary 4.6 and Proposition 4.7 imply that

$$\{\beta_0, \beta_1, \beta_2, \beta_3\} \geqslant \{\beta_0, 3 + \beta_0 - 1, 3 + \beta_3 - 1, \beta_3\} \geqslant \{1, 3, 3, 1\}$$

where the inequalities are interpreted entry by entry.

Similarly, for $\mathbf{c} = \mathbf{4}$ we obtain

$$\{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4\} \geqslant \{\beta_0, 4 + \beta_0 - 1, \beta_2, 4 + \beta_4 - 1, \beta_4\} \geqslant \{1, 4, \beta_2, 4, 1\}.$$

From here, we can apply the Syzygy Theorem (3.13) and notice that in a minimal free resolution

$$0 \longrightarrow R^{\beta_4} \longrightarrow R^{\beta_3} \xrightarrow{\pi_3} R^{\beta_2} \xrightarrow{\pi_2} R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M,$$

the image of π_3 is equal to $\Omega_3(M)$, and thus the rank of π_3 is at least 3 by the Syzygy Theorem; here we used that $c = 4$, so that $\Omega_3(M)$ is not free.

Similarly, working now on the resolution of the dual $\operatorname{Ext}_R^4(M, R)$, we can see that the rank of π_2 must be at least 3 as well. Hence

$$\beta_2 = \operatorname{rk} \pi_2 + \operatorname{rk} \pi_3 \geqslant 6,$$

as required.

However, if we try the same tricks with $\mathbf{c} = \mathbf{5}$, the best we can get is that

$$\{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\} \geqslant \{1, 5, 7, 7, 5, 1\}.$$

There are, however, other techniques one could use to try and complete this case:

- Suppose M is cyclic, that is, $\beta_0 = 1$. Then one may assume that $\beta_1 > 5$. Indeed, if $\beta_1 = 5$, then M is a complete intersection and the Koszul complex is a resolution. Surprisingly, Conjecture 4.1 is still open even if we assume $c = 5$ and that M is cyclic. More precisely, it is still open whether or not $\beta_2 \geqslant \binom{5}{2} = 10$.
- One could suppose further that $\beta_1 = 6$, so $M = R/I$ is an almost complete intersection. A result of Kunz [54] guarantees that R/I is not Gorenstein, and thus $\beta_5 \geqslant 2$. Using linkage, Dugger [36] was able to show in this case that $\beta_2 \geqslant 9$.

- In general, for cyclic modules, the rank of π_1 will be 1, and thus the Syzygy Theorem implies that

$$\beta_2 = \operatorname{rk}\pi_2 + \operatorname{rk}\pi_3 \geqslant \operatorname{rk}\pi_2 + 3 = (\beta_1 - \operatorname{rk}\pi_1) + 3 = \beta_1 + 2,$$

so whenever $\beta_0 = 1$ and $\beta_1 \geqslant 8$ we will have the BEH bound for β_2.

We close out this section with another general technique and an application. Let M be a graded module and P be a prime ideal in its support. Since localization is exact, any minimal free resolution of M over R will remain exact upon localization at P. Hence, over the local ring R_P, the minimal free resolution of M_P must be a direct summand of this resolution. In other words,

$$\beta_i^{R_P}(M_P) \leqslant \beta_i^R(M).$$

We now give two applications of this idea. The first shows that if we wanted to prove a stronger version of the BEH conjecture, we could restrict to finite length modules.

Conjecture 4.8 (Local BEH Conjecture) Let R be a local ring and M a finitely generated R-module of codimension c. Then for all i,

$$\beta_i(M) \geqslant \binom{c}{i}.$$

Lemma 4.9 *To prove Conjecture 4.8, it suffices to prove it for modules of finite length.*

Proof Let M be an arbitrary module, not necessarily of finite length. Say that M has codimension c, and note that there must be a minimal prime P of M of height c. Then M_P is a finite length module over R_P, and

$$\beta_i^R(M) \geqslant \beta_i^{R_P}(M_P)$$

by our localization argument. Since M_P must then have codimension c, the result follows. □

We apply this idea to the case of monomial ideals and present a short proof that the BEH conjecture holds for monomial ideals. As we will see in Sect. 5, there are in fact stronger bounds that hold in the monomial case.

Theorem 4.10 *Let I be a monomial ideal of height c in a polynomial ring R. Then the BEH conjecture holds and $\beta_i(R/I) \geqslant \binom{c}{i}$.*

Proof Our first step is to reduce to the case that I is squarefree. Indeed, if I is a monomial ideal, then there is a squarefree monomial ideal (perhaps in a larger number of variables) called the *polarization* of I which has the same codimension and Betti numbers as I.

So consider the primary decomposition of a squarefree monomial ideal. It consists entirely of minimal primes that are generated by subsets of the variables, and all must have height at least c. Choose any one you like and call it P. Note that $I_P = P_P$, since P is minimal. Without loss of generality, we can assume $P = (x_1, \ldots, x_r)$ for some $r \geqslant c$. Then upon localizing R/I at P, it is easy to see that

$$(R/I)_P \cong R[x_1, \ldots, x_r]_{(x_1,\ldots,x_r)}/(x_1, \ldots, x_r),$$

whose Betti number are obtained from the Koszul complex on $x_1, \ldots, x_r$. Thus

$$\beta_i(R/I) \geqslant \beta_i(R_P/I_P) = \binom{r}{i} \geqslant \binom{c}{i}.$$

The reader will note that if we choose r as large as possible, then r would be the *big height* of the squarefree monomial ideal I, that is, the largest height of an associated prime. □

Remark 4.11 Notice that it is *not* clear that to prove the original BEH Conjecture (which was stated over a polynomial ring) one can simply study finite length modules. Indeed, this localization argument might require one to work over localizations of polynomial rings, which despite being regular will not be polynomial rings.

Finally, we include another important general result that comes up frequently. As motivation we refer to Example 3.1 with $I = (xy, xz, yz)$. Notice that the element $\ell = x-y-z$ is a regular element on $M = R/I$, for instance by looking at a primary decomposition. If we work over $\overline{R} = R/(\ell) \cong k[y, z]$, then $\overline{M} \cong \overline{R}/(y^2, yz, z^2)$, which is a module of finite length. Standard arguments show that when we go modulo a regular element like this, the homological invariants (including the Betti numbers) do not change. One application of this is the fact that the Betti numbers of Cohen-Macaulay modules are the same as those of finite length modules. We make this sentence precise in the following:

Proposition 4.12 *Let M be a Cohen-Macaulay module of codimension c over the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is any field. There exist a field k' and a finite length module M' over the polynomial ring $R' = k'[y_1, \ldots, y_c]$ such that the Betti numbers of M and M' coincide. Thus the following sets are equal:*

$$\{\beta_i(M) : M \text{ Cohen-Macaulay of codimension } c \text{ over } k[x_1, \ldots, x_n] \text{ for some } k\}$$
$$\|$$
$$\{\beta_i(M) : M \text{ is finite length over } k[x_1, \ldots, x_c] \text{ for some } k\}.$$

Proof Let M be a Cohen-Macaulay module of codimension c over $k[x_1, \ldots, x_n]$. If k is infinite, set $k' = k$. If k is finite, then we may enlarge the field, say to the algebraic closure $k' = \overline{k}$, since flat base change will not affect the Betti numbers of M. Set $\overline{R} = k'[x_1, \ldots, x_n]$ and $\overline{M} = M \otimes_R \overline{R}$, where $\overline{M}$ is regarded as an $\overline{R}$-module. Note that $\beta_i(M) = \beta_i(\overline{M})$. Now, since we are working over an infinite field, there is a sequence of linear forms $\ell_1, \dots, \ell_{n-c} \in \overline{R}$ that is a maximal regular sequence on $\overline{M}$. Now let $R' = \overline{R}/(\ell_1, \dots, \ell_{n-c})$ and set

$$M' = \overline{M} \otimes_{\overline{R}} R'.$$

Then since we have gone modulo a regular sequence, $\beta_i(M') = \beta_i(M)$, and since the ℓ_i were linear forms, R' is isomorphic to a polynomial ring $k'[y_1, \dots, y_c]$. □

4.2 *Other Results*

As we mentioned in the previous section, the BEH conjecture 4.1 remains open for modules of codimension $c \geqslant 5$ except in a small collection of cases. There are, however, some classes of modules for which the BEH Conjecture is known.

A deformation argument was used in [52, Remark 4.14] to show that the conjecture holds for arbitrary c when $M = R/I$ and I is in the linkage class of a complete intersection. Additionally, in [41] it was shown that if the regularity of M is small relative to the degrees of the first syzygies of M, meaning the entries in a presentation matrix for M, then the BEH conjecture holds. This will be discussed more carefully in Sect. 5.

The conjecture holds also when M is multigraded, meaning that M remains graded no matter what weights the generators x_i are given. In fact, there are several proofs of this fact, for example [29, 30, 67], but perhaps the strongest version is the result due to Brun and Römer, [14] which shows that if M is multigraded, then in fact $\beta_i(M) \geqslant \binom{p}{i}$, where p is the projective dimension of M. Since the projective dimension can exceed the codimension, this is a much stronger result. Such a result cannot hold more generally—after all, there are 3-generated ideals I with projective dimension 1000, by Theorem 3.8, and in that case $\beta_1(R/I) = 3 < \binom{1000}{1}$. Nevertheless, it would be interesting to know if there are other classes where $\binom{p}{i}$ is a lower bound for the Betti numbers. We know of at least one other case, when the resolution of R/I is linear, which we present in Theorem 5.14. We will discuss the multigraded case in more detail in Sect. 5, when we discuss stronger bounds on Betti numbers. There is a related conjecture of Herzog, on lower bounds for the Betti numbers of the so-called linear strands. For more information, see [39, 64, 65].

Finally, the BEH conjecture 4.1 also holds for finite length modules of Loewy length 2 over any regular local ring $(R, \mathfrak{m})$, meaning modules M satisfying $\mathfrak{m}^2 M = 0$ [21, 27].

4.3 *The Total Rank Conjecture*

If the Buchsbaum–Eisenbud–Horrocks Rank Conjecture is true, an immediate corollary would be the Total Rank Conjecture, which is obtained by adding the individual inequalities:

Conjecture 4.13 (Total Rank Conjecture) If $M \neq 0$ is a finitely generated graded module over $R = k[x_1, \dots, x_n]$ of codimension c, then

$$\sum_{i=0}^{c} \beta_i(M) \geqslant 2^c.$$

This Conjecture was settled in 2018 by Walker [69], except in the case that k has characteristic 2. In fact, Walker's result also applies to finitely generated modules over an arbitrary local ring R containing a field of odd characteristic. This result truly was a breakthrough in the field.

Even though the Total Rank Conjecture is settled (except in characteristic two), we cannot resist sharing some of the beautiful historical results in this story and compare them with the modern treatment. For example, the odd length case has a simple solution via elementary methods:

Lemma 4.14 *Suppose that M is a finitely generated R-module of (finite) odd length over $R = k[x_1, \dots, x_n]$. Then*

$$\sum_{i=0}^{n} \beta_i(M) \geqslant 2^n.$$

Proof The Hilbert series $h_M(t)$ of M is a polynomial in t, say $h_M(t) = h_0 + h_1 t + \cdots + h_r t^r$. We can also write it as

$$h_M^R(t) = \frac{\sum_{i,j} (-1)^i \beta_{i,j}(M) t^j}{(1-t)^n}.$$

Plugging in $t = -1$, we obtain

$$2^n h_M^R(-1) = \sum_{i,j} (-1)^{i+j} \beta_{i,j}(M),$$

so

$$2^n \left| h_0 - h_1 + \cdots + (-1)^r h_r \right| = \left| \sum_{i,j} (-1)^{i+j} \beta_{i,j}(M) \right| \leqslant \sum_i \beta_i(M).$$

On the other hand, $h_0 + h_1 + \cdots + h_r$ is the rank of M, which we assumed to be odd. Therefore, $h_0 - h_1 + \cdots + (-1)^r h_r$ is also odd, and thus nonzero. In particular,

$$2^n \leqslant \sum_{i=0}^{n} \beta_i(M).$$

□

In other words, for modules of finite odd length, the Total Rank Conjecture holds simply due to constraints on its Hilbert function. In 1993, Avramov and Buchweitz were able to obtain a generalization of this fact in [6]. Their most general bound was that if $d \geqslant 5$ and M is a module of finite length over R, then

$$\sum_{i=0}^{d} \beta_i(M) \geqslant \frac{3}{2}(d-1)^2 + 8.$$

In particular, this shows that when $d = 5$ the lower bound of $32 = 2^5$ in the Total Rank Conjecture does hold. Their results were in fact much finer, depending on the prime factors of the length of M, $\ell(M)$. For instance, they show that

- If $\ell(M)$ is odd, then $\sum \beta_i(M) \geqslant 2^d$, so they recover the above result.
- If $\ell(M)$ is even but not divisible by 6, then $\sum \beta_i(M) \geqslant 3^{d/2} \geqslant 2^{0.79d}$.
- If $\ell(M)$ is divisible by 6 but not by 30, then $\sum \beta_i(M) \geqslant 5^{d/4} \geqslant 2^{0.58d}$.
- If $\ell(M)$ is divisible by 30 but not by 60, then $\sum \beta_i(M) \geqslant 2^{(d+1)/2}$.

If we move forward 25 years, the following is a summary of Walker's results:

Theorem 4.15 (Walker [69, 70]) *Let M be a finitely generated module of codimension c over $k[x_1, \ldots, x_n]$.*

- *If* $\operatorname{char} k \neq 2$, *then* $\sum \beta_i(M) \geqslant 2^c$.
- *If* $\operatorname{char}(k) = 2$, *then* $\sum \beta_i(M) \geqslant 2(\sqrt{3})^{c-1} > 2^{0.79c+0.208}$.

While the Total Rank Conjecture remains open in characteristic 2, for that case Walker [70, Theorem 5] did give the above bound of $2(\sqrt{3})^{d-1}$, which improves the previous bounds by Avramov and Buchweitz [6]. We also remark that the Total Rank Conjecture is related to the Toral Rank Conjecture of Halperin [46]. For a survey on this and related results, see [22–24, 62].

In Table 1, we indicate the current status (as of the writing of this survey) of both the Total Rank Conjecture 4.13 and the BEH Conjecture 4.1. The reader may want to refer to Table 2 at the end of the paper to see what the case for stronger bounds is.

Table 1 Status of the BEH and Total Rank Conjectures for a module of codimension c

	$c \leqslant 4$	$c \geqslant 5$
$\beta_i \geqslant \binom{c}{i}$	Follows from the Syzygy Theorem [44]	Open
$\sum_i \beta_i \geqslant 2^c$	Follows from box above ✓	$c = 5$ [6] all c $\operatorname{char}(k) \neq 2$ [69]

5 Stronger Bounds

We now turn to the question of whether there are larger bounds for Betti numbers and whether or not these bounds are achieved, starting with Walker's result [69].

Theorem 5.1 (Walker, Theorem 1 in [69]) *Suppose that* char $k \neq 2$, *and let* M *be a finitely generated graded* $k[x_1, \ldots, x_n]$*-module of codimension* c. *Then*

$$\sum_{i=0}^{c} \beta_i(M) \geqslant 2^c$$

with equality if and only if M *is not a complete intersection.*

Remark 5.2 The situation where we have a module $M \cong R/I$ with I an ideal generated by a regular sequence is very important, and we will want to distinguish it from any other kind of module; we will abuse notation[4] and say that M is a **complete intersection**. We will say that a module M is not a complete intersection whenever M is not isomorphic to any quotient of R by a regular sequence; in particular, when we refer to modules M that are not a complete intersection, we will include any non-cyclic module.

Notice that this theorem says that the only time that the Betti numbers sum to 2^c is in the case of a complete intersection. Surprisingly, the next smallest value for the sum of the Betti numbers that we know of is $2^c + 2^{c-1}$, which is 50% larger than the bound of 2^c. The next two examples show how to achieve this value. Notice that in one example this stems from the fact that $1 + 3 + 2 = 6$, whereas in the other it is because $1 + 5 + 5 + 1 = 12$.

Example 5.3 Let I be the ideal (x^2, xy, y^2) in $R = k[x, y]$. Then R/I is a finite length module of codimension $c = 2$ with Betti numbers $\{1, 3, 2\}$. Notice that these sum to 6 which is $2^2 + 2^{2-1}$.

By adding new variables (to R and also to I) we can extend this example to any $c \geqslant 2$. Indeed, set $R = k[x, y, z_1, \ldots, z_{c-2}]$, and let $I = (x^2, xy, y^2, z_1^2, z_2^2, \ldots, z_{c-2}^2)$. Then the minimal free resolution of R/I is obtained by tensoring the Koszul complex on $\{z_1, \ldots, z_{c-2}\}$ with the minimal free resolution of $R/(x^2, xy, y^2)$. Thus

$$\beta_i(R/I) = \binom{c-2}{i} + 3\binom{c-2}{i-1} + 2\binom{c-2}{i-2}$$

and we see that $\sum \beta_i(R/I) = 2^c + 2^{c-1}$. We chose to adjoin z_i^2 just so that our generators were all in the same degree, but one could choose these additional

[4] This is an abuse of notation since the expression "complete intersection" typically refers to a ring, not a module.

generators to be of any degree. Note that in all of these examples I is **monomial** and R/I is **of finite length**.

Example 5.4 Consider the ideal

$$G = (x^2, y^2, z^2, xy - yz, yz - xy)$$

in the ring $R = k[x, y, z]$. The height of G is 3 and the Betti numbers of R/I are $\{1, 5, 5, 1\}$. Note that $1 + 5 + 5 + 1 = 2^3 + 2^2$.

Let $c \geqslant 3$. Then as in Example 5.3, we can just add generators in new variables, say

$$I = G + (z_1^2, \ldots, z_{c-3}^2)$$

and after tensoring with a Koszul complex we have that

$$\beta_i(R/I) = \binom{c-3}{i-3} + 5\binom{c-3}{i-2} + 5\binom{c-3}{i-1} + \binom{c-3}{i}.$$

Therefore,

$$\sum \beta_i(R/I) = 2^c + 2^{c-1}.$$

Note that all of the modules R/I in this example are **of finite length**.

Example 5.5 If one repeats Example 5.4 with $R = k[x, y, z, u, v]$ and $J = (xy, yz, zu, uv, vx)$ playing the role of G, then the numerics are exactly the same. R/J has codimension 3 and the Betti numbers are $\{1, 5, 5, 1\}$. The analogous examples obtained by adding new generators will all be monomial but **not of finite colength**. This distinction is important, because we will later see in Corollary 5.8 that there are bounds on the individual Betti numbers for monomial ideal of finite colength that do not hold for monomial ideals more generally, nor for general ideals of finite colength.

The following result in [31] shows that for modules that are not complete intersections, this behavior of Betti numbers adding up to "50% more than 2^c" does hold for $c \leqslant 4$:

Theorem 5.6 (Charalambous–Evans–Miller, Theorem 3 in [31]) *Let M be a finitely generated graded module of height c over a polynomial ring. Suppose M is not a complete intersection. If $c \leqslant 4$, then $\sum \beta_i(M) \geqslant 2^c + 2^{c-1}$.*

In fact, [31] actually provides minimal Betti sequences for each codimension. For example, in codimension $c = 4$ they show that $\{\beta_0, \ldots, \beta_4\}$ must be bigger (entry by entry) than at least one of the following:

$$\begin{matrix} \{1,5,9,7,2\}, & \{1,6,10,6,1\}, & \{2,6,8,6,2\}, & \{1,6,9,6,2\} \\ \{2,7,9,5,1\}, & & & \{2,6,9,6,1\} \end{matrix}.$$

Note that the entries on the bottom row are the reverse of those directly above. The proof of this result uses techniques of linkage and relies on the classification [57] of the possible algebra structures on $\mathrm{Tor}_{\bullet}^{R}(R/I, k)$. This result led the authors to ask the following question:

Question B (Charalambous–Evans–Miller [31]) *If M is finitely generated graded module over $k[x_1, \ldots, x_n]$ of codimension c that is not a complete intersection, is*

$$\sum \beta_i(M) \geqslant (1.5)2^c = 2^c + 2^{c-1}?$$

We will now discuss several instances where we have an affirmative answer to this question. We remark, however, that the techniques—and indeed the underlying reasons—in each instance are completely different! Here are some natural follow-up questions.

Question C *What other modules M of codimension c satisfy $\sum \beta_i(M) = (1.5)2^c$?*

Question D *What are the smallest Betti sequences in a given codimension c, when we range over all finitely generated modules of codimension c over a polynomial ring on any number of variables?*

5.1 The Multigraded Case

Let $R = k[x_1, \ldots, x_n]$ and let M be a finitely generated graded-module over R. We say that M is **multigraded** if it remains graded with respect to any grading of the variables. For example, when I is a monomial ideal, I and R/I are multigraded (each).

Example 5.7 Let $R = k[x, y, z]$. Consider

$$M = \mathrm{Coker} \begin{pmatrix} y & 0 & z \\ -x & z & 0 \\ 0 & -y & -x \end{pmatrix}.$$

The module M is generated by 3 elements; for example, since M is a quotient of R^3, we can take the images of the canonical basis elements e_1, e_2, e_3 for generators of M. Then M has relations

$$ye_1 = xe_2, \quad ze_2 = ye_3, \quad ze_1 = xe_3.$$

Suppose we are given any weights on the variables. Then the module M will be graded as well by setting $\deg e_1 := \deg(x)$, $\deg e_2 := \deg(y)$, $\deg e_3 := \deg(z)$.

In contrast, consider

$$N = \operatorname{Coker}\begin{pmatrix} x & y & z & 0 \\ 0 & x & y & z \end{pmatrix}, \qquad \begin{array}{c|cccc} \beta(N) & 0 & 1 & 2 & 3 \\ \hline 0 & 2 & 4 & - & - \\ 1 & - & - & 4 & 2 \end{array}.$$

Then N is generated by e_1, e_2, and the relations

$$ye_1 + xe_2 = 0, \quad ze_1 + ye_2 = 0$$

imply that

$$\deg(y) + \deg(e_1) = \deg(x) + \deg(e_2)$$

$$\deg(z) + \deg(e_1) = \deg(y) + \deg(e_2)$$

which has no solution for example when $\deg(y) = \deg(x) = 0$ and $\deg(z) = 1$. In other words, N is not multigraded. Notice that N is finite length as an R-module, and thus has codimension 3.

The following theorem gives strong bounds on the individual Betti numbers of modules that are multigraded and of finite-length. For instance they imply that a module of codimension c must have Betti numbers that either exceed $\{1, 4, 5, 2\}$ or $\{2, 5, 4, 1\}$. Noting that the Betti numbers of N in the previous example violate both of these bounds provides yet another reason why it is not multigraded.

Theorem 5.8 (Charalambous–Evans [30]) *Let M be a multigraded module **of finite length** and let $\gamma_i(M)$ denote the rank of the ith syzygy module of M. Then for all i*

$$\gamma_i(M) \geqslant \binom{n-1}{i-1} \text{ and therefore } \beta_i(M) \geqslant \binom{n}{i}.$$

Further if M is not a complete intersection, then at least one of the following holds:

(a) for all i, $\gamma_i(M) \geqslant \binom{n-1}{i-1} + \binom{n-2}{i-2}$, and therefore $\beta_i(M) \geqslant \binom{n}{i} + \binom{n-1}{i-1}$;
(b) for all i, $\gamma_i(M) \geqslant \binom{n-1}{i-1} + \binom{n-2}{i-1}$, and therefore $\beta_i(M) \geqslant \binom{n}{i} + \binom{n-1}{i}$.

Remark 5.9 We want to emphasize that without the assumption that M is multigraded *and* of finite length, Theorem 5.8 is false if $c \geqslant 3$. Indeed, Examples 5.4 (respectively 5.5) give families of modules R/I that are finite length (respectively multigraded) but with Betti numbers that violate the bounds in Theorem 5.8. This is essentially due to the fact that R/I is Gorenstein in both cases. Indeed, since Theorem 5.8 implies that either $\beta_0(M) \geqslant 2$ or $\beta_c(M) \geqslant 2$, any Gorenstein algebra R/I that is not a complete intersection will violate the bounds in Corollary 5.10. In fact, we can use this to deduce the following classical fact: if I is a monomial ideal in a polynomial ring R such that R/I is of finite length and Gorenstein, then R/I is a complete intersection.

Summing the inequalities for the Betti numbers in Theorem 5.8 yields the following result, which is a special case of Question B.

Corollary 5.10 *If M is a **multigraded** module **of finite length** then*

$$\sum \beta_i(M) \geqslant 2^n.$$

Further if M is not a complete intersection, then

$$\sum_{i=0}^{n} \beta_i(M) \geqslant 2^n + 2^{n-1}.$$

We remark that in this case $n = \operatorname{codim} M$.

Notice that one of the examples in Remark 5.9 has Betti numbers $\{1, 5, 5, 1\}$, and although this violates the bounds in Theorem 5.8, they nonetheless add up to $2^3 + 2^2$. Recently, the first author and Seiner were able to show that one can remove the finite length assumption, provided one works with multigraded cyclic modules:

Theorem 5.11 (Boocher–Seiner [11]) *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal of any codimension $c \geqslant 2$. If R/I is not a complete intersection, then*

$$\sum_{i=0}^{c} \beta_i(R/I) \geqslant 2^c + 2^{c-1}.$$

Unlike the proofs in the finite length case, this theorem does not apparently follow from a bound on the individual Betti numbers. Indeed, the argument follows via a degeneration argument that reduces everything to either a Betti sequence $\{1, 3, 2\}$ with $c = 2$ or a Betti sequence $\{1, 5, 5, 1\}$ with $c = 3$. Perhaps it is a coincidence that these Betti numbers sum to $(1.5)2^c$.

Question E *Examples 5.4 and 5.5 are both examples of Gorenstein algebras where the sum of the Betti numbers is equal to $2^c + 2^{c-1}$. What other Gorenstein algebras R/I of codimension c have this sum?*

Question F *In Examples 5.3 and 5.4, we saw two distinct families of Betti numbers whose Betti numbers sum to $2^c + 2^{c-1}$. Are there other examples of Betti numbers that achieve this sum?*

Question G *If M is a **multigraded** $k[x_1, \ldots, x_n]$-module of codimension $c < n$ that is not a complete intersection, then does*

$$\sum \beta_i(M) \geqslant 2^c + 2^{c-1}$$

hold?

As a partial answer to this, we have the following:

Theorem 5.12 (Brun–Romer [14]) *If M is multigraded $k[x_1, \dots, x_n]$-module of projective dimension p, then $\beta_i(M) \geqslant \binom{p}{i}$.*

Since $p \geqslant c$ with equality only in the case that M is Cohen-Macaulay, we see that Question G can be reduced to the Cohen-Macaulay case.

Finally, we cannot resist including the following beautiful result of Charalambous and Evans, which gives a sharp strong bound for monomial ideals of finite colength:

Theorem 5.13 (Charalambous–Evans [30]) *Let $R = k[x_1, \dots, x_n]$ and $M = R/I$, where the ideal I is minimally generated by n pure powers of the variables and one additional generator $m = x_1^{a_1} \cdots x_n^{a_n}$. Suppose that ℓ is the number of nonzero a_i's. Then for all i, we have*

$$\beta_i(M) = \binom{n}{i} + \binom{n-1}{i-1} + \cdots + \binom{n-(\ell-1)}{i-(\ell-1)}.$$

For instance, this says that the Betti numbers of the ideal $I = (x^2, y^2, z^2, w^2, xywz)$ must sum to at least $2^4 + 2^3 + 2^2 + 2 = 30$. Indeed, the Betti numbers are $\{1, 5, 10, 10, 4\}$.

Question H *Can this theorem be extended outside of the case of finite colength monomial ideals? Is there a version for general monomial ideals? Is there a version for multigraded modules? For general ideals?*

5.2 Low Regularity Case

We finish this survey with some of the most recent results on larger lower bounds for Betti numbers. So far we have not paid much attention to the degrees of the syzygies. After all, our bounds are in terms of the Betti numbers β_i, which count the number of generators, but not their degrees, of the ith syzygy module. But since we are working with graded modules, we will now actually look at β_{ij}.

In terms of degrees, the simplest resolutions are those whose matrices all have linear entries. Such resolutions are called linear.

Theorem 5.14 (Herzog–Kühl [48]) *If M is a graded R-module of projective dimension p with a linear resolution, then $\beta_i(M) \geqslant \binom{p}{i}$.*

Remark 5.15 Notice that this is the same bound given by Brun and Römer for multigraded modules in Theorem 5.12. In the same paper, Herzog and Kühl show that apart from this bound, linear resolutions can behave quite wildly.[5] Indeed, they

[5] This is the term used by Herzog and Kühl.

show how to produce squarefree monomial ideals with a linear resolution such that the Betti numbers form a non-unimodal sequence with arbitrarily many extrema.

Linear resolutions have the property that each matrix has entries all of which are linear. This is a particular case of what is called a pure resolution. We say that a module M is pure if it is Cohen-Macaulay and each map has entries all of the same degree. Equivalently, in the free resolution $F_\bullet \to M$, each F_i is generated in a single degree d_i. This sequence of numbers $\{d_0, \dots, d_c\}$ is called the degree sequence of M.

Example 5.16 The module M given in Example 5.7 with Betti table

$\beta(M)$	0	1	2	3
0	2	4	–	–
1	–	–	4	2

is pure with degree sequence $0, 1, 3, 4$. The module R/G in Example 5.4 is an example of a pure module with degree sequences $\{0, 2, 3, 5\}$ and Betti table

$\beta(M)$	0	1	2	3
0	1	–	–	–
1	–	5	5	–
2	–	–	–	1

.

In [48], Herzog and Kühl showed that if M is a pure module with degree sequence $\{d_0, \dots, d_c\}$, then for all $i \geqslant 1$ we have

$$\beta_i(M) = \beta_0(M) \prod_{\substack{1 \leqslant j \leqslant c \\ j \neq i}} \frac{|d_j - d_0|}{|d_i - d_j|}.$$

Quite surprisingly, given any degree sequence $d_0 < d_1 < \cdots < d_p$, there exists a Cohen-Macaulay module M whose resolution is pure with this degree sequence. This was proven in [38, 40] as part of the resolution of the Boij-Söderberg conjectures.

Question I *If M is a pure module of codimension c, is $\beta_i(M) \geqslant \binom{c}{i}$?*

Given the Herzog-Kühl equations, one might expect that this question is numerical in nature, and in a sense it is. However, the following example shows a major obstacle:

Example 5.17 Let M be a pure module with degree sequence $\{0, 2, 3, 7, 8, 10\}$. Such a module has codimension 5 and its Betti table is

$\beta(M)$	0	1	2	3	4	5
0	$\beta_0(M)$	–	–	–	–	–
1	–	$7\beta_0(M)$	$8\beta_0(M)$	–	–	–
2	–	–	–	–	–	–
3	–	–	–	–	–	–
4	–	–	–	$8\beta_0(M)$	$7\beta_0(M)$	–
5	–	–	–	–	–	$\beta_0(M)$

.

Notice that if $\beta_0(M) = 1$, then this would give an example of a module with $\beta_2(M) < \binom{5}{2}$. So part of answering Question I involves showing that $\beta_0(M) \geqslant 2$. One way to prove this is to apply a big hammer—the Total Rank Conjecture, now Walker's Theorem [69]. Using Walker's Theorem, we notice that if $\beta_0(M) = 1$, then the sum of the Betti numbers would be equal to 2^c, but evidently M is not a complete intersection, which contradicts Walker's result. Alternatively, one could note that from the Betti table, the rank of $\Omega_3(M)$ would be $2\beta_0(M)$, which would violate the Syzygy Theorem 3.13 when $\beta_0(M) = 1$.

Extending this sort of argument to general degree sequences will present many challenges. In fact, we need only to turn to the degree sequence $\{0, 1, 2, 3, 5, 7, 8, 9, 10\}$ to see to limits of this argument. A module M possessing a pure resolution with this degree sequence would necessarily be of codimension 8 and would have Betti table

$\beta(M)$	0	1	2	3	4	5	6	7	8
0	$4N$	$25N$	$60N$	$60N$	–	–	–	–	–
1	–	–	–	–	$42N$	–	–	–	–
2	–	–	–	–	–	$60N$	$60N$	$25N$	$4N$

for some positive integer N. Boij-Söderberg Theory guarantees that such a module exists, but N may be large. Note that if $N = 1$ then $\beta_4 < \binom{8}{4}$, and the sum of the Betti numbers would be $340 < 2^8 + 2^7$ which would violate both the BEH Conjecture 4.1 and provide a negative answer to Question B. Notice that the Betti sequence is non-unimodal, regardless of N.

The numerical behavior resulting from the Herzog-Kühl equations is nontrivial to analyze, but is slightly manageable in the case where the last degree d_c is small relative to d_1. Note that d_1 and d_c are essentially degrees of the first syzygies of M and the Castelnuovo-Mumford Regularity. This insight was first noticed by Erman in [41]. Coupling this observation with the full force of the newly proven Boij-Söderberg Theory allowed him to prove the BEH Conjecture for those graded modules whose regularity is low relative to the degrees of the first syzygies.

Theorem 5.18 (Erman [41]) *Let M be a graded R-module of codimension $c \geqslant 3$ generated in degree 0 and let $a \geqslant 2$ be the minimal degree of a first syzygy of M. If* $\operatorname{reg}(M) \leqslant 2a - 2$, *then*

$$\beta_i(M) \geqslant \beta_0(M)\binom{c}{i}.$$

In particular the sum of the Betti numbers is at least $\beta_0(M)2^c$.

To put the regularity bound into perspective, if M is R/I for some ideal I generated by quadrics, then the above theorem would apply to any M with regularity at most 2, which means the Betti table has at most 2 rows. The regularity condition is relaxed enough to include, for example, the coordinate rings of smooth curves embedded by linear systems of high degree, those of toric surfaces, as well as any finite length module whose socle degree is relatively low. In Example 5.17 the two Betti tables do not obey the low regularity bound. In the first, $a = 2$ and $\text{reg}(M) = 5$; in the second, $a = 1$ and $\text{reg}(M) = 2$.

Erman's proof uses general Boij-Söderberg techniques to reduce studying the Betti tables of arbitrary modules to the study of pure modules and then use a degeneration argument to supply the required numerical bound. These techniques were pushed even further in [12], where it is shown that in fact the sum of the Betti numbers is 50% larger:

Theorem 5.19 (Boocher–Wigglesworth [12]) *Let* M *be a graded* R*-module of codimension* $c \geqslant 3$ *generated in degree* 0 *and let* $a \geqslant 2$ *be the minimal degree of a first syzygy of* M. *If* $\text{reg}(M) \leqslant 2a - 2$, *then*

$$\beta(M) \geqslant \beta_0(M)(2^c + 2^{c-1}).$$

If moreover $c \geqslant 9$, *then*

$$\beta_i(M) \geqslant 2\,\beta_0(M)\binom{c}{i}$$

for the first half of the Betti numbers, meaning for $1 \leqslant i \leqslant \lceil c/2 \rceil$.

Essentially, this says that if the regularity is "low", then for $c \geqslant 9$, the *first half* of the Betti numbers are at least *double* the conjectured Buchsbaum–Eisenbud–Horrcks bounds. Then on average the Betti numbers, will be at least 1.5 times the BEH bounds, and thus the sum of all the Betti numbers needs to be at least $1.5(2^c)$. The authors deal with the cases $c \leqslant 8$ separately. Again it seems almost miraculous that the bound of $2^c + 2^{c-1}$ pops up—in this case aided by the fact that the first half of the Betti numbers are twice as large as expected.

Remark 5.20 Notice that if $R = k[x_1, \ldots, x_c]$ with $c \geqslant 2$, then any ideal I generated by $c + 1$ generic quadrics will be an ideal of height c, and $\beta_1(R/I) = c + 1 < 2\binom{c}{1}$. So without some other condition, for example on the regularity, there is no hope of finding a stronger bound for the first Betti number.

Table 2 M is a module of codimension c that is not a complete intersection

<table>
<tr><th></th><th>$c \leqslant 4$</th><th>$c \geqslant 5$</th><th colspan="2">multigraded</th><th>low regularity</th></tr>
<tr><td>$\beta_i \geqslant \binom{c}{i} + \binom{c-1}{i-1}$ for all i
or
$\beta_i \geqslant \binom{c}{i} + \binom{c-1}{i}$ for all i</td><td>False
5.9</td><td>False
5.9</td><td rowspan="3">$c = n$
(CE, 1991)
[30]</td><td>$c < n$
False
5.9</td><td>False
Rk 5.9</td></tr>
<tr><td rowspan="2">$\sum_i \beta_i \geqslant (1.5)\ 2^c$</td><td rowspan="2">(CEM,
1990)
[31]</td><td rowspan="2">Open</td><td>$M \cong R/I$
(BS, ’18 [11])</td><td rowspan="3">True
$\Uparrow$ check $5 \leqslant c \leqslant 8$
True if $c \geqslant 9$
(BW, ’20 [12])</td></tr>
<tr><td>general M
Open</td></tr>
<tr><td>$\beta_i \geqslant 2\binom{c}{i}$ for $i < \frac{c}{2}$</td><td>False
5.20</td><td>False
5.20</td><td>False
5.20</td><td>False
5.20</td></tr>
</table>

As a corollary of Theorem 5.19, for ideals generated by quadrics with $c \geqslant 9$ we have

$$\operatorname{reg}(R/I) < 3, \text{ and } R/I \text{ is not a CI} \implies \beta_1(R/I) \geqslant 2c,\ \beta_2(R/I) \geqslant 2\binom{c}{2}, \ldots$$

In other words, low regularity forces this rather specific bound for the number of generators.

We end with a table summarizing the results concerning these stronger bounds (each). We remind the reader that these entries all concern modules that are not complete intersections.

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The Simplest Minimal Free Resolutions in $\mathbb{P}^1 \times \mathbb{P}^1$

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1 Introduction

In this chapter, we consider bigraded minimal free resolutions in the first nontrivial case. Let $R = \mathbb{K}[s, t; u, v]$ be the bigraded polynomial ring, where $\{s, t\}$ are of degree $(1, 0)$ and $\{u, v\}$ are of degree $(0, 1)$; R is graded by $\mathbb{Z}^2$. For $\mathbf{d} = (d_1, d_2)$, we consider a three dimensional subspace $W = \text{Span}\{f_0, f_1, f_2\} \subseteq R_{\mathbf{d}}$, with the additional constraint that

$$I_W = \langle f_0, f_1, f_2 \rangle \text{ satisfies } \sqrt{I_W} = \langle s, t \rangle \cap \langle u, v \rangle. \tag{1.1}$$

This generic condition arises from the natural geometric condition of being *base-point free*, defined in Sect. 1.2.2 below. We study the minimal free resolution of I_W and we give precise results when $\mathbf{d} = (1, n)$ and $n \geq 3$.

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Example 1.1 For $\mathbf{d} = (1, 1)$ the bigraded Betti numbers of I_W are always

$$0 \leftarrow I_W \leftarrow (-1,-1)^3 \xleftarrow{\partial_1} \begin{matrix}(-1,-3)\\ \oplus \\ (-2,-2)^3 \\ \oplus \\ (-3,-1)\end{matrix} \xleftarrow{\partial_2} \begin{matrix}(-2,-3)^2\\ \oplus \\ (-3,-2)^2\end{matrix} \xleftarrow{\partial_3} (-3,-3) \leftarrow 0$$

The degree (2, 2) syzygies are Koszul. The first syzygies of degree (1, 3) and (3, 1) involve only one set of variables, and arise from the vanishing of a determinant (see Lemma 3.1). The reader is encouraged to work out the remaining differentials.

The $\mathbf{d} = (1, 2)$ case is more complex, and [8] shows that there are two possible bigraded minimal free resolutions for I_W. The resolution type is determined by how $\mathbb{P}(W) \subseteq \mathbb{P}(R_{1,2}) = \mathbb{P}^5$ meets the image $\Sigma_{1,2}$ of the Segre map $\mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\sigma_{1,2}} \mathbb{P}^5$ of factorizable polynomials.

1.1 Motivation from Geometric Modeling

In geometric modeling, it is often useful to approximate a surface in $\mathbb{P}^3$ with a rational surface of low degree. Most commonly the rational surfaces used are $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, and the resulting objects are known as *tensor product surfaces* and *triangular surfaces*. See, for example, [3, 6, 7, 9, 26]. An efficient way to compute the implicit equation is via approximation complexes [20, 21] which use syzygy data as input.

A tensor product surface is mapped to $\mathbb{P}^3$ by a four dimensional subspace $V \subseteq R_{\mathbf{d}}$. For $\mathbf{d} = (1, 2)$, Zube describes the singular locus in [28, 29], and in [14, 16], Elkadi-Galligo-Lê use the geometry of a dual scroll to analyze the image. When $\sqrt{I_V} = \langle s, t\rangle \cap \langle u, v\rangle$, [25] shows that there are exactly six types of free resolution possible, and analyzes the approximation complexes for the distinct resolutions. Degan [10] examines the situation when the subspace has basepoints. For a three dimensional subspace $W \subseteq V$, the ideals I_V and I_W are related via linkage in [25].

1.2 Mathematical Background

We start with a quick review of the cast of principal mathematical players, referring to [12] and [13] for additional details.

1.2.1 Bigraded Betti Numbers

Definition 1.2 For a bihomogeneous ideal $I \subseteq R = \mathbb{K}[s, t; u, v]$ and $\mathbf{d} \in \mathbb{Z}^2$, the *bigraded Betti numbers* are

$$\beta_{i,\mathbf{a}} = \dim_{\mathbb{K}} Tor_i(R/I, \mathbb{K})_{\mathbf{a}}.$$

For all nonnegative integers i and all bidegrees $\mathbf{a}$, $\beta_{i,\mathbf{a}}$ is the number of copies of $R(-\mathbf{a})$ appearing in the ith module of the minimal free resolution of I. Since $\mathbf{a} \in \mathbb{Z}^2$, the $\beta_{i,\mathbf{a}}$ cannot be displayed in the $\mathbb{Z}$-graded Betti table format [13]. Bigraded regularity is studied in [1, 22, 24], and multigraded regularity was introduced in [23].

Example 1.3 [[8], Theorem 7.8] Let $\Sigma_{1,2}$ be the Segre variety of $\mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5 = \mathbb{P}(U)$, where U has basis $\{su^2, suv, sv^2, tu^2, tuv, tv^2\}$. Then $W \cap \Sigma_{1,2}$ is a smooth conic iff I_W has the bigraded Betti numbers as below.

$$0 \leftarrow I_W \leftarrow (-1,-2)^3 \xleftarrow{\partial_1} \begin{array}{c} (-1,-6) \\ \oplus \\ (-2,-4)^3 \\ \oplus \\ (-3,-2) \end{array} \xleftarrow{\partial_2} \begin{array}{c} (-2,-6)^2 \\ \oplus \\ (-3,-4)^2 \end{array} \xleftarrow{\partial_3} (-3,-6) \leftarrow 0$$

For example, $\beta_{1,(2,4)} = 3$ and $\beta_{2,(3,4)} = 2$.

1.2.2 Bigraded Algebra and Line Bundles on $\mathbb{P}^1 \times \mathbb{P}^1$

As noted earlier, the constraint that $\sqrt{I_W}$ is the bihomogeneous maximal ideal in (1.1) arises as a natural geometric condition, and we give a quick synopsis; for additional details, see §V.I of [19].

A line bundle $\mathcal{L}$ on the abstract variety $\mathbb{P}^1 \times \mathbb{P}^1$ is characterized by a choice of $\mathbf{d} \in \mathbb{Z}^2$ and we write $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbf{d})$ for $\mathcal{L}$. Although the global sections

$$H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbf{d})) = R_{\mathbf{d}}$$

are not functions on $\mathbb{P}^1 \times \mathbb{P}^1$, their ratios give well defined functions on $\mathbb{P}^1 \times \mathbb{P}^1$ and so zero sets of sections are defined.

The upshot is that to realize $\mathbb{P}^1 \times \mathbb{P}^1$ as a subvariety of $\mathbb{P}^n$, we choose an $n+1$ dimensional subspace $W \subseteq R_{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{Z}^2_{>(0,0)}$. As long as the $f_i \in W$ do not simultaneously vanish on $\mathbb{P}^1 \times \mathbb{P}^1$, this gives a regular map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^n$. The condition that the f_i do not simultaneously vanish at a point of $\mathbb{P}^1 \times \mathbb{P}^1$ is exactly the condition (1.1); in this situation W is said to be *basepoint free*. For example, if $W = \text{Span}\{su, sv, tu\}$, then $[(0:1), (0:1)] \in \mathbb{P}^1 \times \mathbb{P}^1$ is a basepoint of W.

1.2.3 Koszul Homology and Bigraded Hilbert Series

Koszul homology is defined and discussed in Sect. 2. We will give in Definition 2.8 the precise conditions under which a basepoint free system of polynomials $\mathbf{f} = \{f_0, f_1, f_2\} \subseteq R_{\mathbf{d}}$ is *generic*. In this case, we make a conjecture in Sect. 2 on the bigraded Betti numbers, and show that it is equivalent to a recent conjecture made by Fröberg–Lundqvist in [15] on the bigraded Hilbert series of R/I_W.

1.3 Roadmap of This Chapter

Below is an overview of the sections which make up this chapter.

- In Sect. 2, we study the Koszul homology of I; the first homology encodes the non-Koszul first syzygies. The spectral sequence of the Čech-Koszul double complex has a single d_3 differential, and we explain the connection to local cohomology $H_B^{\bullet}$. We make a conjecture about the first Koszul homology module for *generic* W, and connect it to a conjecture of Fröberg–Lundqvist [15] on the Hilbert series of generic bigraded ideals.
- In Sect. 3, we use tools from commutative algebra such as the Hilbert-Burch theorem to shed additional light on the first syzygies.
- In Sect. 4, we connect the minimal free resolution to the image of the Segre variety $\Sigma_{1,n}$, obtaining canonical syzygies in certain degrees, without any assumptions on genericity. The geometry of $W \cap \Sigma_{1,n}$ plays a key role.
- In Sect. 5 we prove results on higher Segre varieties, in particular about how the geometry of the intersection of W with such varieties influences the free resolution. We close with a number of questions.

2 Koszul Homology $H_1(\mathcal{K}_\bullet(\mathbf{f}, R))$ and the Generic Case

We start this section with an overview of Koszul homology. We then prove in Theorem 2.2 a characterization of the first Koszul homology associated to our ideal I_W under the assumption (1.1). Corollary 2.5 then gives a concrete representation of H_1, that we make explicit in Examples 2.6 and 2.7 for factorizable polynomials. Sect. 2.4 treats the generic case (see Definition 2.8). In this case, we specify some values of the dimensions $(H_1)_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$ and we state Conjecture 2.10 about these dimensions.We prove in Proposition 2.17 that our Conjecture is equivalent to Conjecture 2.14 by Ralf Fröberg and Samuel Lundqvist (Conjecture 8 in [15]).

2.1 Koszul Homology

For notation, we write $\mathfrak{a} = \langle s, t\rangle$, $\mathfrak{b} = \langle u, v\rangle$, $B = \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{m} = R_+ = (s, t, u, v)$

Definition 2.1 For a sequence of polynomials $\mathbf{f} = \{f_0, \dots, f_m\}$ the Koszul complex $\mathcal{K}_\bullet := \mathcal{K}_\bullet(\mathbf{f}, R)$ is the complex

$$\cdots \longrightarrow \Lambda^j(R^{m+1}) \xrightarrow{\delta_j} \Lambda^{j-1}(R^{m+1}) \longrightarrow \cdots$$

where

$$\delta_j(e_{n_1} \wedge \cdots \wedge e_{n_j}) \mapsto \sum_{i=1}^{j} (-1)^i f_{n_i} \cdot (e_{n_1} \wedge \cdots \widehat{e_{n_i}} \cdots \wedge e_{n_j})$$

The ith Koszul homology is $H_i(\mathcal{K}_\bullet)$; Koszul cohomology is $H^i(Hom_R(\mathcal{K}_\bullet, R))$.

The Koszul complex is exact iff $\mathbf{f}$ is a regular sequence. We will focus on the case where $\mathbf{f} = \{f_0, f_1, f_2\}$ is a basepoint free subset of $R_{\mathbf{d}}$. Hence

$$\mathcal{K}_\bullet(\mathbf{f}, R) : 0 \to R(-3\mathbf{d}) \xrightarrow{\delta_3} R(-2\mathbf{d})^3 \xrightarrow{\delta_2} R(-\mathbf{d})^3 \xrightarrow{\delta_1} R \to 0. \qquad (2.1)$$

Let Z_i and B_i be the modules of Koszul i-th cycles and boundaries, graded so that the inclusion maps $Z_i, B_i \subset K_i$ are of degree (0, 0), and let $H_i = Z_i/B_i$ denote the i-th Koszul homology module. Since $\delta_1(p_1, p_2, p_3) = \sum_{i=1}^{3} p_i f_i$,

$$H_0 = \mathrm{coker}(\delta_1) = R/I_W.$$

Since $\sqrt{I_W} = B$, the codimension of I_W is two, so since $\mathbf{f}$ has three generators, $\mathbf{f}$ is not a regular sequence, and thus $H_1 \neq 0$. Our assumption (1.1) that $\mathrm{rad}(I_W) = B$ means that $\mathrm{depth}(I_W) = 2$, and then $H_2 = H_3 = 0$.

From the definition of Koszul homology, the syzygy module $\mathrm{Syz}(\mathbf{f}) := \ker(\delta_1)$. Since $H_1 \neq 0$, the map δ_2 in the Koszul complex (2.1) factors through $\mathrm{Syz}(\mathbf{f})$ as $R(-2\mathbf{d})^3 \xrightarrow{\delta_2} \mathrm{Syz}(\mathbf{f})$ but is not surjective. The module $\mathrm{im}(\delta_2)$ is called the module of Koszul syzygies. Thus, the size of non-Koszul syzygies is measured by H_1.

2.2 Determining $H_1(\mathcal{K}_\bullet(\mathbf{f}, R))$

Since $\mathrm{rad}(I_W) = B$, the modules H_0 and H_1 are supported on B. In particular, we have that $H^i_B(H_1) = 0$ for $i > 0$ and hence, $H^0_B(H_1) = H_1$. This says that the Koszul complex (2.1) is not acyclic globally, but it is acyclic off $V(B)$, i.e. that for every prime $\mathfrak{p} \not\subset B$ the localization $(\mathcal{K}_\bullet(\mathbf{f}, R))_{\mathfrak{p}}$ of (2.1) at $\mathfrak{p}$ is acyclic.

Consider the extended Koszul complex of (2.1)

$$\mathcal{K}_\bullet : 0 \to R(-3\mathbf{d}) \xrightarrow{\delta_3} R(-2\mathbf{d})^3 \xrightarrow{\delta_2} R(-\mathbf{d})^3 \xrightarrow{\delta_1} R \to R/I_W \to 0. \quad (2.2)$$

For the complex (2.2) we have that $H_i = 0$ if $i \neq 1$. The following theorem characterizes H_1.

Theorem 2.2 *There is an isomorphism of bigraded R-modules*

$$H_1 \cong \ker\left(H_B^2(R(-3\mathbf{d})) \xrightarrow{\delta} \left(H_B^2(R(-2\mathbf{d}))\right)^3\right).$$

Proof Consider the Čech-Koszul double complex $\check{\mathcal{C}}_B^\bullet(\mathcal{K}_\bullet)$ that is obtained from (2.2) by applying the Čech functor $\check{\mathcal{C}}_B^\bullet(-)$.

Consider the two spectral sequences that arise from the double complex $\check{\mathcal{C}}_B^\bullet(\mathcal{K}_\bullet)$. We will denote by ${}_hE$ the spectral sequence that arises taking first homology horizontally, this is, computing first the Koszul homology, and by ${}_vE$ the spectral sequence that is obtained by computing first the Čech cohomology. The second page of the spectral sequence of the horizontal filtration is:

$${}_h^2E^{ij} = H_B^i(H_j(\mathcal{K}_\bullet)).$$

Since $H_B^i(H_1) = 0$ for $i > 0$ and $H_B^0(H_1) = H_1$, we have

$${}_h^2E^{ij} = H_B^i(H_j) = \begin{cases} H_1 & \text{for } j = 1 \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$H_B^\bullet(H_\bullet) \Rightarrow H_1.$$

The second spectral sequence has

$${}_v^1E^{ij} = H_B^i(K_j),$$

where K_i is the i-th module from the right in Eq. (2.2). Precisely, we have

$$\begin{aligned} {}_v^1E^{i,-1} &= H_B^i(R/I_W) \\ {}_v^1E^{i,0} &= H_B^i(R) \\ {}_v^1E^{i,1} &= H_B^i(R(-\mathbf{d})^3) \\ {}_v^1E^{i,2} &= H_B^i(R(-2\mathbf{d})^3) \\ {}_v^1E^{i,3} &= H_B^i(R(-3\mathbf{d})) \end{aligned}$$

Therefore, the ${}^1E$ page of the vertical spectral sequence is

$$\begin{array}{cccccc}
0 \longrightarrow \omega_R^*(3\mathbf{d}) & \longrightarrow (\omega_R^*(2\mathbf{d}))^3 & \longrightarrow (\omega_R^*(\mathbf{d}))^3 & \longrightarrow \omega_R^* & \longrightarrow H_B^3(R/_W I) \\
0 \longrightarrow H_B^2(R(-3\mathbf{d})) & \xrightarrow{\delta} (H_B^2(R(-2\mathbf{d})))^3 & \longrightarrow (H_B^2(R(-\mathbf{d})))^3 & \longrightarrow H_B^2(R) & \longrightarrow H_B^2(R/I_W) \\
0 \qquad 0 & 0 & 0 & 0 & H_B^1(R/I_W) \\
0 \qquad 0 & 0 & 0 & 0 & H_B^0(R/I_W)
\end{array}$$

By comparing both spectral sequences, we conclude that

$$H_1 \cong \ker\left(H_B^2(R(-3\mathbf{d})) \xrightarrow{\delta} \left(H_B^2(R(-2\mathbf{d}))\right)^3\right). \qquad \square$$

Corollary 2.3 *The sequence*

$$0 \to R(-3\mathbf{d}) \xrightarrow{\delta_3} R(-2\mathbf{d})^3 \xrightarrow{\delta_2} \mathrm{Syz}(\mathbf{f}) \to H_B^2(R(-3\mathbf{d})) \xrightarrow{\delta} \left(H_B^2(R(-2\mathbf{d}))\right)^3$$

is exact.

Proof From Eq. (2.2), $0 \to R(-3\mathbf{d}) \xrightarrow{\delta_3} R(-2\mathbf{d})^3 \xrightarrow{\delta_2} \mathrm{Syz}(\mathbf{f}) \to H_1 \to 0$ is exact. Theorem 2.2 gives $H_1 \cong \ker\left(H_B^2(R(-3\mathbf{d})) \xrightarrow{\delta} \left(H_B^2(R(-2\mathbf{d}))\right)^3\right)$. The result follows by connecting the two sequences. $\square$

2.3 Understanding $(H_1)_\mathbf{a}$

We have the following consequence of Theorem 2.2.

Corollary 2.4

$$\mathrm{Supp}_{\mathbb{Z}^2}(H_1) \subset -\mathbb{N} \times \mathbb{N} + (3d_1 - 2, 3d_2) \cup \mathbb{N} \times -\mathbb{N} + (3d_1, 3d_2 - 2).$$

Proof A direct computation using the Mayer-Vietoris sequence yields

(1) $H_B^2(R) = H_\mathfrak{a}^2(R) \oplus H_\mathfrak{b}^2(R) = \left(\omega_{R_1}^* \otimes_\mathbb{K} R_2\right) \oplus \left(R_1 \otimes_\mathbb{K} \omega_{R_2}^*\right)$,
(2) $H_B^3(R) = H_\mathfrak{m}^4(R) = \omega_R^*$,
(3) $H_B^i(R) = 0$ for all $i \neq 2, 3$,

where ω_S^* denotes the canonical dualizing module of S. Since we have that the $\mathrm{Supp}_{\mathbb{Z}^2}(H_B^2(R)) = -\mathbb{N} \times \mathbb{N} + (-2, 0) \cup \mathbb{N} \times -\mathbb{N} + (0, -2)$, by shifting we get that

$$\mathrm{Supp}_{\mathbb{Z}^2}(H_1) \subset -\mathbb{N} \times \mathbb{N} + (3d_1 - 2, 3d_2) \cup \mathbb{N} \times -\mathbb{N} + (3d_1, 3d_2 - 2). \qquad \square$$

Corollary 2.5 *Consider the map* $H_B^2(R(-3\mathbf{d})) \xrightarrow{\delta} \left(H_B^2(R(-2\mathbf{d}))\right)^3$. *For every* $\mathbf{a} = (a_1, a_2)$, *we get*

$$(H_1)_{(a_1,a_2)} \cong \ker \left(\begin{pmatrix} R_{(3d_1-a_1-2,-3d_2+a_2)} \\ \oplus \\ R_{(-3d_1+a_1,3d_2-a_2-2)} \end{pmatrix} \xrightarrow{\delta_a} \begin{pmatrix} R_{(2d_1-a_1-2,-2d_2+a_2)} \\ \oplus \\ R_{(-2d_1+a_1,2d_2-a_2-2)} \end{pmatrix}^3 \right)$$

is an isomorphism of $\mathbb{K}$*-modules. Identifying the target with*

$$R^3_{(2d_1-a_1-2,-2d_2+a_2)} \oplus R^3_{(-2d_1+a_1,2d_2-a_2-2)}, \text{ we have}$$

$$\delta_{\mathbf{a}} = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}, \text{ with}$$

$$\begin{aligned} \phi_1 &: R_{(-3d_1+a_1,3d_2-a_2-2)} \to R^3_{(-2d_1+a_1,2d_2-a_2-2)}, \\ \phi_2 &: R_{(3d_1-a_1-2,-3d_2+a_2)} \to R^3_{(2d_1-a_1-2,-2d_2+a_2)}. \end{aligned} \tag{2.3}$$

For $d_1, d_2 \geq 2$, the previous result gives a description of the kernel $(H_1)_{(a_1,a_2)}$:

$[3d_2, +\infty)$	$\ker(\phi_2)$	$R_{(3d_1-a_1-2,-3d_2+a_2)}$	0	0
$3d_2 - 1$	0	0	0	0
$(2d_2 - 2, 3d_2 - 2]$	0	0	0	$R_{(-3d_1+a_1,3d_2-a_2-2)}$
$(-\infty, 2d_2 - 2]$	0	0	0	$\ker(\phi_1)$
	$(-\infty, 2d_1 - 2]$	$(2d_1 - 2, 3d_1 - 2]$	$3d_1 - 1$	$[3d_1, +\infty)$

The next examples illustrate the map $\delta_{\mathbf{a}}$ of Corollary 2.5 in a particular case in which the three polynomials f_i can be factored as two polynomials with bidegrees $(1, 0)$ and $(0, n)$.

Example 2.6 Let $\mathbf{d} = (1, n)$, $f_0 = su^n$, $f_1 = tv^n$, $f_2 = (s + t)(u^n + v^n)$, and $(a_1, a_2) = (3, n)$. Then

$$(H_1)_{(3,n)} = \ker \left((R_1 \otimes_{\mathbb{K}} \omega^*_{R_2})_{(0,-2n+2)} \xrightarrow{\delta_{(3,n)}} \left((R_1 \otimes_{\mathbb{K}} \omega^*_{R_2})_{(1,-n+2)}\right)^3 \right) \tag{2.4}$$

and with the standard identification of the canonical dualizing modules, one has

$$(H_1)_{(3,n)} \cong \ker\left(R_{(0,2n-2)} \longrightarrow \left(R_{(1,n-2)}\right)^3\right)$$

The map $\delta_{(3,n)}$ in Eq. (2.4) is given by multiplication by f_i. Precisely, given $a \geq 0$, $\delta_{(3,n)}$ is as follows

$$\frac{1}{uv}\frac{1}{u^a v^{2n-2-a}} \mapsto \frac{1}{uv}\left(\frac{1}{u^a v^{2n-2-a}} f_0, \frac{1}{u^a v^{2n-2-a}} f_1, \frac{1}{u^a v^{2n-2-a}} f_2\right).$$

Thus, fixing a basis $\mathcal{B}$ for $(R_1 \otimes_{\mathbb{K}} \omega^*_{R_2})_{(0,-2n+2)}$ and also fixing a basis $\mathcal{B}'$ for $(R_1 \otimes_{\mathbb{K}} \omega^*_{R_2})^3_{(1,-n+2)}$, $|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}$ is a $(3 \cdot 2(n-1)) \times (2n-1)$-matrix given by the coefficients $coef_{\mathcal{B}'}((f_0, f_1, f_2) \cdot \mathcal{B}_i)$ of the i-th element of $\mathcal{B}$ multiplied by one of the f_j (j depending on the row), written in the basis $\mathcal{B}'$.

We now exhibit the matrices in Example 2.6 in bidegree $(1, 6)$.

Example 2.7 Set for instance $n = 6$ (so $\mathbf{d} = (1, 6)$), $|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}$ is a $(3 \cdot 10) \times 11$-matrix. One can take $\mathcal{B} = \left\{\frac{1}{uv}\frac{1}{u^6}, \ldots, \frac{1}{uv}\frac{1}{v^6}\right\}$ and
$\mathcal{B}' = \left\{\frac{1}{uv}\frac{s}{u^4}, \ldots, \frac{1}{uv}\frac{s}{v^4}, \frac{1}{uv}\frac{t}{u^4}, \ldots \frac{1}{uv}\frac{t}{v^4}\right\} \times \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
In this case, one has that the 10-tuple, corresponding to the 'upper third' of the first column of $|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}$, induced by multiplying by f_0 is

$$coef_{\mathcal{B}'}(f_0 \cdot \mathcal{B}_1) = coef_{\mathcal{B}'}\left(f_0 \cdot \frac{1}{uv}\frac{1}{u^{10}}\right) = coef_{\mathcal{B}'}\left(\frac{1}{uv}\frac{s}{u^4}\right) = (1, 0, 0, \ldots, 0).$$

And, because of the structure of multiplication on $\omega^*_{R_2}$, it is easy to see that in $(R_1 \otimes_{\mathbb{K}} \omega^*_{R_2})_{(1,-n+2)}$, $f_j \cdot u^{-6}v^{-6} = 0$. Thus, the 6th column of $|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}$ is zero, and the rest are not.

The matrix $|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}$ has the following shape

$$\left(\begin{array}{c|c|c} Id_{5\times 5} & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & Id_{5\times 5} \\ \hline Id_{5\times 5} & 0 & Id_{5\times 5} \\ Id_{5\times 5} & 0 & Id_{5\times 5} \end{array}\right),$$

where $Id_{5\times 5}$ is the 5×5-identity matrix.

Finally, we conclude that $\operatorname{corank}(|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}) = 1$. Since the matrix above induces a morphism from $k^{11} \to k^{30}$, we have that

$$HF_{H_1}(3, n) = \dim(\ker(|\delta_{(3,n)}|_{BB'})) = \operatorname{corank}(|\delta_{(3,n)}|_{\mathcal{B}\mathcal{B}'}) = 1.$$

This in particular says that there is only one non-Koszul syzygy spanning every other non-Koszul syzygy.

Examples 2.6 and 2.7 are explained by Theorem 4.7.

2.4 The Generic Case

We give the definition of *generic* bihomogeneous polynomials f_i of the same bidegree $\mathbf{d} \in \mathbb{Z}^2_{>0}$. Note that our assumption (1.1) implies that no d_i could be equal to 0.

Definition 2.8 Given $\mathbf{d} \in \mathbb{Z}^2_{>0}$ and $\mathbf{f} = \{\mathbf{f_0}, \mathbf{f_1}, \mathbf{f_2}\}$ of bidegree $\mathbf{d}$ satisfying (1.1), we say that $\mathbf{f}$ is generic if the maps ϕ_1 and ϕ_2 in (2.3) in Corollary 2.5 have full rank, for any $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$.

This condition of full rank is not true only under the assumption (1.1). For instance, the factorizable polynomials in Example 2.7 are not generic, but our Conjecture 2.10 below states that the maps have full rank for polynomials with generic coefficients.

If we denote by $n_{\mathbf{d}}(\mathbf{a})$ the difference of dimensions:

$$n_d(\mathbf{a}) := \dim_{\mathbb{K}} \begin{pmatrix} R_{(3d_1-a_1-2,-3d_2+a_2)} \\ \oplus \\ R_{(-3d_1+a_1,3d_2-a_2-2)} \end{pmatrix} - 3\dim_{\mathbb{K}} \begin{pmatrix} R_{(2d_1-a_1-2,-2d_2+a_2)} \\ \oplus \\ R_{(-2d_1+a_1,2d_2-a_2-2)} \end{pmatrix} \in \mathbb{Z}[a,d],$$

we have by Corollary 2.5 that

$$\dim_{\mathbb{K}} (H_1)_{(a_1,a_2)} \geq n_{\mathbf{d}}(\mathbf{a}).$$

For any real number c, denote

$$c_+ = \max(c, 0), \quad c_- = \max(0, -c). \tag{2.5}$$

Note that for any $c \in \mathbb{R}$, $c = c_+ - c_-$ and only one of these two numbers can be positive.

Given $\mathbf{d}, \mathbf{a} \in \mathbb{Z}^2_{\geq 0}$, we set

$$\begin{aligned} \mathrm{dom}_d(\mathbf{a}) := &(0, -3d_1 + a_1 + 1)_+ (0, 3d_2 - a_2 - 1)_+ \\ &+ (0, 3d_1 - a_1 - 1)_+ (0, -3d_2 + a_2 + 1)_+ \end{aligned}$$

$$\begin{aligned} \mathrm{cod}_d(\mathbf{a}) := &(0, -2d_1 + a_1 + 1)_+ (0, 2d_2 - a_2 - 1)_+ \\ &+ (0, 2d_1 - a_1 - 1)_+ (0, -2d_2 + a_2 + 1)_+. \end{aligned}$$

The following Lemma is straigthforward taking into account that a linear map $V_1 \to V_2$ between two $\mathbb{K}$-vector spaces of finite dimension is of maximal rank if and only if the dimension of its kernel of equals $(\dim_{\mathbb{K}}(V_1) - \dim_{\mathbb{K}}(V_2))_+$.

Lemma 2.9 *Let* $\mathbf{f} = \{\mathbf{f_0}, \mathbf{f_1}, \mathbf{f_2}\}$ *of bidegree* $\mathbf{d}$ *satisfying* (1.1)*. Then,* $\mathbf{f}$ *is generic if and only if*

$$n_{\mathbf{d}}(\mathbf{a}) \;=\; (\mathrm{dom}_d(\mathbf{a}) - 3\,\mathrm{cod}_d(\mathbf{a}))_+. \tag{2.6}$$

In this case, for any $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$ *we have the equality:*

$$\dim_{\mathbb{K}} (H_1)_{(a_1,a_2)} = n_{\mathbf{d}}(\mathbf{a}). \tag{2.7}$$

In fact, we conjecture that this is indeed the generic behavior

Conjecture 2.10 *There exists a nonempty open set in the space of coefficients of the polynomials* f_i *where* $\mathbf{f}$ *is generic according to Definition 2.8 and hence* $\dim (H_1)_{\mathbf{a}} = n_{\mathbf{d}}(\mathbf{a})$ *for any* $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$ *by Lemma 2.9.*

Remark 2.11 Note that Corollary 2.5 proves that Conjecture 2.10 is always true for polynomials f_i satisfying assumption (1.1) outside of the range where we have ($a_1 \geq 3d_1$ and $d_2 \leq a_2 \leq 2d_2 - 2$) and ($a_2 \geq 3d_2$ and $d_1 \leq a_1 \leq 2d_1 - 2$).

2.5 The Fröberg-Lindqvist Conjecture on Bigraded Hilbert Series

For any bidegree $\mathbf{a}$, we denote by $\chi_{\mathcal{K}_\bullet}(\mathbf{a})$ the Euler characteristic of the ℓ-strand of the Koszul complex (2.1) and let $S(x, y) = \sum_{\mathbf{a}\in\mathbb{Z}^2_{\geq 0}} \chi_{\mathcal{K}_\bullet}(\mathbf{a}) x^{a_1} y^{a_2}$. Then,

$$S(x, y) \;=\; \frac{(1 - x^{d_1} y^{d_2})^3}{(1 - x)^2(1 - y)^2}. \tag{2.8}$$

We denote by $S(x, y)_+$ the series supported in $\mathbb{Z}^2_{\geq 0}$ with coefficients $\chi_{\mathcal{K}_\bullet}(\ell)_+$. The following lemmas are straightforward.

Lemma 2.12 $S(x, y)_+$ *and* $S(x, y)_-$ *are also rational functions of* x, y.

Lemma 2.13 *Define regions*

$$\begin{aligned}
A_1 &= \quad a_1 < d_1 \text{ or } a_2 < d_2\\
A_2 &= \quad (d_1 \leq a_1 < 2d_1 \text{ and } d_2 \leq a_2) \text{ or } (d_1 \leq a_1 \text{ and } d_2 \leq a_2 < 2d_2)\\
A_3 &= (2d_1 \leq a_1 < 3d_1 \text{ and } 2d_2 \leq a_2) \text{ or } (a_1 < 3d_1 \text{ and } 2d_2 \leq a_2 < 3d_2)\\
A_4 &= \quad 3d_1 \leq a_1 \text{ and } 3d_2 \leq a_2
\end{aligned}$$

For any $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$*, the coefficient* $\chi_{\mathcal{K}_\bullet}(\mathbf{a})$ *equals the following:*

$$\begin{array}{cl} (a_1+1)(a_2+1) & in\ \ A_1 \\ (a_1+1)(a_2+1) - 3(a_1-d_1+1)(a_2-d_2+1) & in\ \ A_2 \\ (a_1+1)(a_2+1) - 3(a_1-d_1+1)(a_2-d_2+1) + 3(a_1-2d_1+1)(a_2-2d_2+1) & in\ \ A_3 \\ 0 & in\ \ A_4 \end{array}$$

The following table shows in which bidegrees the Euler characteristic $\chi_{\mathcal{K}_\bullet}(\mathbf{a})$ is positive, negative or zero.

$[3d_2, +\infty)$	$+$	$+/hiperb/-$	$-$	0	0
$3d_2 - 1$	$+$	$+$	0	0	0
$(2d_2 - 2, 3d_2 - 2]$	$+$	$+$	$+$	0	$-$
$(d_2 - 1, 2d_2 - 2]$	$+$	$+$	$+$	$+$	$+/hiperb/-$
$(0, d_2 - 1]$	$+$	$+$	$+$	$+$	$+$
	$(0, d_1 - 1]$	$(d_1 - 1, 2d_1 - 2]$	$(2d_1 - 2, 3d_1 - 2]$	$3d_1 - 1$	$[3d_1, +\infty)$

Here the notation $+/hiperb/-$ means that there is a hyperbola where $\chi_{\mathcal{K}_\bullet}$ vanishes, separating the positive from the negative values. Here is an equivalent version of the Fröberg and Lundqvist Conjecture 8 from [15]:

Conjecture 2.14 *There exists a nonempty open set in the space of coefficients of the polynomials* f_i *for which* $\dim(R/I_W)_{\mathbf{a}} = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_+$ *for any* $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$.

We now prove that Conjecture 2.10 and 2.14 are indeed equivalent.

Lemma 2.15 *Let* $\mathbf{f} = \{f_0, f_1, f_2\}$ *be bihomogeneous polynomials of bidegree* $\mathbf{d} \in \mathbb{Z}^2_{>0}$ *satisfying* (1.1)*. We have the equality:*

$$\chi_{\mathcal{K}_\bullet}(\mathbf{a}) = \dim(R/I_W)_{\mathbf{a}} - \dim(H_1)_{\mathbf{a}}. \tag{2.9}$$

As we remarked in Sect. 2.1, our assumption (1.1) implies that $H_2 = H_3 = 0$, and so the proof of Lemma 2.15 is immediate.

We compare the conjectural dimension $n_{\mathbf{d}}(\mathbf{a})$ of $H_{1\mathbf{a}}$ with the coefficients of S.

Lemma 2.16 *For any* $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$ *we have the equality:*

$$\dim n_{\mathbf{d}}(\mathbf{a}) = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_-. \tag{2.10}$$

Proof By Lemma 2.13 are sixteen domains of polynomiality of $\chi_{\mathcal{K}_\bullet}$. Consider for instance the case $a_1 > 3d_1 - 1, 2d_2 - 1 < a_2 \leq 3d_2 - 1$. Then, $n_{\mathbf{d}}(\mathbf{a}) = (3d_1 - a_1 - 1)(3d_2 - a_2 - 1)$, while $\chi_{\mathcal{K}_\bullet}(\mathbf{a})_- = (a_1+1)(a_2+1) - 3(-d_1+a_1+1)(-d_2+a_2+1) + 3(-2d_1+a_1+1)(-2d_2+a_2+1)$ and it is a simple computation to check that they coincide. The other cases are similar. □

By Lemma 2.12 the generating series $T(x, y) = \sum_{\mathbf{a}} \dim(H_1)_{\mathbf{a}} x^{a_1} y^{a_2}$ is a rational function. Hence

Proposition 2.17 *Conjectures 2.10 and 2.14 are equivalent.*

Proof Assume $\dim(H_1)_{\mathbf{a}} = n_{\mathbf{d}}(\mathbf{a})$. Using Lemma 2.16, we substitute this value in the statement of Lemma 2.15:

$$\chi_{\mathcal{K}_\bullet}(\mathbf{a}) = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_+ - \chi_{\mathcal{K}_\bullet}(\mathbf{a})_- = \dim(R/I_W)_{\mathbf{a}} - \chi_{\mathcal{K}_\bullet}(\mathbf{a})_-,$$

which says that $\dim(R/I_W)_{\mathbf{a}} = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_+$. The converse is similar. □

Remark 2.18 By Remark 2.11, we have that $\dim(H_1)_{\mathbf{a}} = n_{\mathbf{d}}(\mathbf{a})$ is true for polynomials f_i satisfying assumption (1.1) except in the ranges ($a_1 \geq 3d_1$ and $d_2 \leq a_2 \leq 2d_2 - 2$) and ($a_2 \geq 3d_2$ and $d_1 \leq a_1 \leq 2d_1 - 2$). So, we deduce from the proof of Proposition 2.17 that Conjecture 2.14 is true outside these ranges.

We end this section with an easy corollary.

Corollary 2.19 *If Conjectures 2.10 and 2.14 hold, then for any* $\mathbf{f}$ *regular either* $R/I_{W\mathbf{a}} = 0$ *or* $H_{1\mathbf{a}} = 0$ *for any* $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$.

Proof If the conjectures are valid for any $\mathbf{f}$ regular, we have for any bidegree $\mathbf{a}$ that $\dim(R/I_W)_{\mathbf{a}} = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_+$ and $\dim(H_1)_{\mathbf{a}} = \chi_{\mathcal{K}_\bullet}(\mathbf{a})_-$, and as we remarked after (2.5), at most one of these numbers can be nonzero. □

3 Koszul Homology $H_1(\mathcal{K}_\bullet(\mathbf{f}, R))$ for $\mathbf{d} = (1, n)$

From now on, we specialize our study to bidegrees of the form $\mathbf{d} = (1, n)$, always assuming that W is basepoint free. This case is both the natural sequel to the study of the $(1, 2)$ case studied in [8], as well as a key ingredient for better understanding the general case. It splits the analysis into separate parts, in a way that we make precise below. Theorem 3.7 relates the Betti numbers $\beta_{1,\mathbf{a}}$ with the Koszul homology of H_1 with respect to the sequence $\{s, t, u, v\}$, for any bidegree $\mathbf{d}$.

For degree $(1, n)$, Corollary 2.5 yields the following description of $(H_1)_{(a_1,a_2)}$:

$$(H_1)_{(a_1,a_2)} \cong \ker\left(\begin{array}{c} R_{(1-a_1,-3n+a_2)} \\ \oplus \\ R_{(-3+a_1,3n-a_2-2)} \end{array} \xrightarrow{\delta_{(a_1,a_2)}} \left(\begin{array}{c} R_{(-a_1,-2n+a_2)} \\ \oplus \\ R_{(-2+a_1,2n-a_2-2)} \end{array}\right)^3\right).$$

The table given in Sect. 2.3 reduces to

$[3n, +\infty)$	$R_{(1-a_1,-3n+a_2)}$	0	0
$3n - 1$	0	0	0
$(2n-2, 3n-2]$	0	0	$R_{(-3+a_1,3n-a_2-2)}$
$(-\infty, 2n-2]$	0	0	$\ker(\phi_1)$
	1	2	$[3, +\infty)$

,

with

$$\phi_1 : R_{(-3+a_1,3n-a_2-2)} \to R^3_{(-2+a_1,2n-a_2-2)}.$$

So the region where interesting behavior occurs is in multidegree (a_1, a_2), with

$$\begin{array}{ccc} a_2 \geq 3n & \text{and} & a_1 = 1 \\ & \text{or} & \\ a_1 \geq 3 & \text{and} & a_2 \leq 2n-2 \end{array}$$

Since we need I_W to be nonzero, we have the constraint that $a_1 \geq d_1, a_2 \geq d_2$, so for $\mathbf{d} = (1, n)$, the only region of interest is bidegree (a_1, a_2), with

$$a_1 \geq 3 \quad \text{and} \quad 2n-2 \geq a_2 \geq n,$$

corresponding to $\ker(\phi_1)$ defined in Sect. 2.3. We study $n \geq 3$; $n = 2$ is analyzed in [8].

3.1 Tautological First Syzygies: Degrees (1, ∗) and (2, ∗)

Lemma 3.1 *There is a unique minimal first syzygy on I_W in bidegree $(1, 3n)$.*

Proof Because $\phi_2 : \mathbb{K} \to 0$, $\ker(\phi_2) \simeq \mathbb{K}$, and we can describe the syzygy explicitly as follows (it is of the type appearing in Lemma 6.1 of [8].) Write

$$\begin{aligned} f_0 &= s \cdot p_0 + t \cdot q_0 \\ f_1 &= s \cdot p_1 + t \cdot q_1 \\ f_2 &= s \cdot p_2 + t \cdot q_2, \end{aligned}$$

with the $p_i, q_i \in k[u, v]_n$. Then

$$\det \begin{bmatrix} f_0 & p_0 & q_0 \\ f_1 & p_1 & q_1 \\ f_2 & p_2 & q_2 \end{bmatrix} = 0,$$

so the 2×2 minors in q_i and p_i give a syzygy with entries of bidegree $(0, 2n)$, hence of bidegree $(1, 3n)$ on I_W. It is minimal since any syzygy of lower degree would be of the form $(0, d)$ with $d < 2n$. This would force W to have basepoints; to see this note that a syzygy (s_0, s_1, s_2) of bidegree $(0, d)$ must be in the kernel of the map

$$\theta = \begin{bmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{bmatrix},$$

by splitting out the s and t components. But θ gives a map $\mathcal{O}^3_{\mathbb{P}^1}(-n) \to \mathcal{O}^2_{\mathbb{P}^1}$ with zero cokernel, because the rank of θ drops on the locus of the 2×2 minors of θ; such a point would be a basepoint of W. A Chern class computation shows $\ker(\theta) \simeq \mathcal{O}_{\mathbb{P}^1}(-3n)$; in fact it consists of the 2×2 minors of θ. □

Proposition 3.2 *The syzygy of Lemma 3.1 and the three Koszul syzygies generate a pair of minimal second syzygies of bidegree* $(2, 3n)$. *Furthermore, there is a minimal third syzygy of bidegree* $(3, 3n)$.

Proof Let

$$\begin{aligned} f_0 &= s \cdot p_0 + t \cdot q_0 \\ f_1 &= s \cdot p_1 + t \cdot q_1 \\ f_2 &= s \cdot p_2 + t \cdot q_2, \end{aligned}$$

with the $a_i, b_i \in k[u, v]_n$, and consider the submatrix A of ∂_1 generated by the syzygy of Lemma 3.1 and the three Koszul syzygies:

$$\begin{bmatrix} q_1p_2 - p_1q_2 & t \cdot q_1 + s \cdot p_1 & t \cdot q_2 + s \cdot p_2 & 0 \\ p_0q_2 - q_0p_2 & -(t \cdot q_0 + s \cdot p_0) & 0 & t \cdot q_2 + s \cdot p_2 \\ q_0p_1 - p_0q_1 & 0 & -(t \cdot q_0 + s \cdot p_0) & -(t \cdot q_1 + s \cdot p_1) \end{bmatrix}$$

The columns of the matrix A'

$$\begin{bmatrix} s & t & 0 \\ q_2 & -p_2 & f_2 \\ -q_1 & p_1 & -f_1 \\ q_0 & -p_0 & f_0 \end{bmatrix}$$

are in the kernel of A; the rightmost column is the second Koszul syzygy on I_W. As $[t, -s, 1]$ is in the kernel of A', we see that there is a third syzygy of bidegree $(3, 3n)$. Note that by Theorem 3.4 the second Koszul syzygy is not minimal, but can be represented in terms of the syzygies appearing in Theorem 3.4. □

The results of Sect. 2.3 show that there are no minimal first syzygies in bidegree $(2, *)$ except for the Koszul syzygies in degree $(2, 2n)$. This can be seen explicitly, as follows. First, a syzygy with entries of bidegree $(1, m)$ satisfies

$$(sg_0 + th_0)f_0 + (sg_1 + th_1)f_1 + (sg_2 + th_2)f_2 = 0,$$

with the f_i as in the previous lemma. Note that $\langle p_0, p_1, p_2\rangle$ and $\langle q_0, q_1, q_2\rangle$ are both basepoint free on $\mathbb{P}^1$; for otherwise vanishing of $\{p_0, p_1, p_2, t\}$ would give a basepoint on $\mathbb{P}^1 \times \mathbb{P}^1$ and also for for $\{q_0, q_1, q_2, s\}$. If $(u_0 : v_0) \in \mathbb{P}^1$ is a point where the rank of

$$\begin{bmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{bmatrix}$$

is one, then $s = u_0, t = v_0$ is a basepoint of I_W. Using that $f_i = sp_i + tq_i$, multiplying out and collecting the coefficients of the $\{s^2, st, t^2\}$ terms shows that $[g_0, g_1, g_2, h_0, h_1, h_2]$ is in the kernel of the matrix

$$M = \begin{bmatrix} p_0 & p_1 & p_2 & 0 & 0 & 0 \\ q_0 & q_1 & q_2 & p_0 & p_1 & p_2 \\ 0 & 0 & 0 & q_0 & q_1 & q_2 \end{bmatrix}.$$

The remarks above show that the kernel is free of rank three, with first Chern class $6n$. The matrix K below satisfies these properties and clearly $MK = 0$:

$$\begin{bmatrix} -p_1 & 0 & p_2 \\ p_0 & -p_2 & 0 \\ 0 & p_1 & p_0 \\ -q_1 & 0 & -q_2 \\ q_0 & -q_2 & 0 \\ 0 & q_1 & q_0 \end{bmatrix}.$$

By the Buchsbaum-Eisenbud criterion, $K = \ker(M)$. But K consists of exactly the Koszul syzygies. From this it follows that the lowest possible nonzero multidegree in the degree $(1, 0)$ variables for a non-tautological first syzygy is $(3, m + n) = (2, m) + (1, n)$, with $m \geq 0$. We tackle this next.

3.2 First Syzygies of Degree $(3, *)$

The next theorem gives a complete description of the first syzygies with entries of degree $(2, m)$, hence which are of total degree $(3, m + n)$.

Theorem 3.3 *For the first Betti numbers,*

$$\beta_{1,(3,*)} \in \{1, \ldots, 5\}$$

and all possible values between one and five occur.

Proof Theorems 4.4, 4.7, and 4.8 treat the case where W meets the Segre variety $\Sigma_{1,n}$ in a smooth conic C or 3 noncollinear points Z. In these situations we may choose a basis so $W = \text{Span}\{g_0h_0, g_1h_1, g_2h_2\}$ with g_i degree $(1, 0)$ and h_i degree

$(0, n)$. Theorems 4.7 and 4.8 give explicit resolutions for I_W in these cases.

- When $W \cap \Sigma_{1,n} = C$ there is a single syzygy of bidegree $(3, n)$.
- $W \cap \Sigma_{1,n} = Z$ there are two syzygies of bidegrees $(3, n+\mu)$, $(3, 2n-\mu)$.
- The remaining cases are covered in Theorem 3.4 below.

□

Theorem 3.4 *Suppose* $\{f_0, f_1, f_2\} = (sq_0 + tq_3, sq_1 + tq_4, sq_2 + tq_5)$,*with the* q_i *linearly independent* (*so* $n \geq 5$). *Then there are exactly five minimal first syzygies whose entries are quadratic in* $\{s, t\}$, *obtained from the Hilbert-Burch matrix* ([13], *Theorem 3.2*) N *for the ideal* $Q = \langle q_0, \ldots, q_5 \rangle$. *If the columns of* N *have degrees* $\{b_1, \ldots, b_5\}$, *then the syzygies on* I_W *are of degree* $\{(3, 2n-b_1), \ldots, (3, 2n-b_5)\}$.

Proof A syzygy with entries of bidegree $(2, m)$ satisifies

$$(s^2a_0 + sta_1 + t^2a_2)f_0 + (s^2b_0 + stb_1 + t^2b_2)f_1 + (s^2c_0 + stc_1 + t^2c_2)f_2 = 0,$$

so using that $f_i = sq_i + tq_{i+3}$, multiplying out and collecting the coefficients of the $\{s^3, s^2t, st^2, t^3\}$ terms shows that $[a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2]$ is in the kernel of the matrix

$$M = \begin{bmatrix} q_0 & q_1 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_3 & q_4 & q_5 & q_0 & q_1 & q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3 & q_4 & q_5 & q_0 & q_1 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & q_4 & q_5 \end{bmatrix}.$$

Since W is basepoint free, it follows that as a sheaf, the cokernel of

$$\mathcal{O}^9_{\mathbb{P}^1}(-n) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^1}$$

is zero, hence the kernel of M is a rank five free module with first Chern class $9n$. If N denotes the Hilbert-Burch matrix of Q then N is a 6×5 matrix whose maximal minors are Q. Write n_k^{ij} for the determinant of the submatrix of N obtained by omitting rows i, j and column k (convention-indexing starts with 0), and consider the matrix K

$$\begin{bmatrix} -n_0^{12} & -n_0^{02} & -n_0^{01} & n_0^{15} - n_0^{24} & n_0^{05} - n_0^{23} & n_0^{04} - n_0^{13} & -n_0^{45} & -n_0^{35} & n_0^{34} \\ -n_1^{12} & -n_1^{02} & -n_1^{01} & n_1^{15} - n_1^{24} & n_1^{05} - n_1^{23} & n_1^{04} - n_1^{13} & -n_1^{45} & -n_1^{35} & n_1^{34} \\ -n_2^{12} & -n_2^{02} & -n_2^{01} & n_2^{15} - n_2^{24} & n_2^{05} - n_2^{23} & n_2^{04} - n_2^{13} & -n_2^{45} & -n_2^{35} & n_2^{34} \\ -n_3^{12} & -n_3^{02} & -n_3^{01} & n_3^{15} - n_3^{24} & n_3^{05} - n_3^{23} & n_3^{04} - n_3^{13} & -n_3^{45} & -n_3^{35} & n_3^{34} \\ -n_4^{12} & -n_4^{02} & -n_4^{01} & n_4^{15} - n_4^{24} & n_4^{05} - n_4^{23} & n_4^{04} - n_4^{13} & -n_4^{45} & -n_4^{35} & n_4^{34} \end{bmatrix}.$$

Entries of the ith row of K correspond to combinations of certain 4×4 minors of the submatrix N_i obtained by deleting the ith column of N. A computation shows that $M \cdot K^t = 0$ and therefore

$$0 \longrightarrow \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbb{P}^1}(-2n+b_i) \xrightarrow{K^t} \mathcal{O}_{\mathbb{P}^1}^9(-n) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^1}^4 \longrightarrow 0$$

is exact, by the Buchsbaum-Eisenbud criterion. □

Remark 3.5 If the q_i are not linearly independent, then the basepoint free assumption means they span a space of dimension 5 or 4, or fall under Theorems 4.7, 4.8. When $\dim \mathrm{Span}\{q_0, \ldots, q_5\} \in \{4, 5\}$, the matrix N is 5×4 or 4×3 and the argument of Theorem 3.4 works with appropriate modifications, which we leave to the interested reader.

Corollary 3.6 *The tautological syzygies constructed in Sect. 3.1 and the syzygies of Theorem 3.4 are independent.*

Proof The syzygies constructed in Theorem 3.4 cannot be in the span of the tautological syzygies of Sect. 3.1 because their degree in the $\{u, v\}$ variables is lower than that of the tautological syzygies. On the other hand, the tautological syzygies cannot be in the span of the syzygies of Theorem 3.4, as the tautological syzygies have lower degree in the $\{s, t\}$ variables. □

3.3 Computing Betti Numbers, the General Setting

The Koszul homology of the module H_1 is computed from the complex $\mathcal{M}_\bullet := \mathcal{M}_\bullet((s, t, u, v), H_1)$:

$$\mathcal{M}_\bullet : 0 \to H_1(-4) \xrightarrow{\varphi_4} H_1(-3)^4 \xrightarrow{\varphi_3} H_1(-2)^6 \xrightarrow{\varphi_2} H_1^4 \xrightarrow{\varphi_1} H_1 \to 0. \quad (3.1)$$

The bigraded complex $\mathcal{M}_\bullet$ has the following shape:

$$0 \longrightarrow H_1(-2,-2) \longrightarrow \begin{matrix} H_1(-2,-1)^2 \\ \oplus \\ H_1(-1,-2)^2 \end{matrix} \longrightarrow \begin{matrix} H_1(-2,0) \\ \oplus \\ H_1(-1,-1)^4 \\ \oplus \\ H_1(0,-2) \end{matrix} \longrightarrow \begin{matrix} H_1(0,-1)^2 \\ \oplus \\ H_1(-1,0)^2 \end{matrix} \longrightarrow H_1 \to 0$$

We denote by $H(\mathcal{M})_i$ the i-th homology module.

Theorem 3.7 *For any* $\mathbf{a} \in \mathbb{Z}^2_{\geq 0}$*, we have the equality*

$$\beta_{1,\mathbf{a}} = \dim_{\mathbb{K}}(H(\mathcal{M})_{1,\mathbf{a}}) - \dim_{\mathbb{K}}(H(\mathcal{M})_{2,\mathbf{a}}). \quad (3.2)$$

Proof First, observe that $\beta_{1,\mathbf{a}} \neq 0$ iff $(H_1)_{\mathbf{a}}$ is not spanned by the images of $(H_1)_{\mathbf{a}-e_1}$ and $(H_1)_{\mathbf{a}-e_2}$. Hence $\beta_{1,\mathbf{a}} = \dim_{\mathbb{K}}(H(\mathcal{M})_{0,\mathbf{a}})$. As $\dim_{\mathbb{K}}(H(\mathcal{M})_{0,\mathbf{a}})$ is the alternating sum of $\dim_{\mathbb{K}}(H(\mathcal{M})_{i,\mathbf{a}})$ for $i > 0$, it suffices to show that

$$H(\mathcal{M})_3 = 0 = H(\mathcal{M})_4.$$

Let $K_{i,m} = \mathcal{K}_i((s,t,u,v); R)$ and $K_{i,f} = \mathcal{K}_i(f; R)$ denote the Koszul complex of (s,t,u,v) and the Koszul complex of f on R respectively. We consider the two spectral sequences coming from the double complex $\mathcal{C}_{i,j} = K_{i,m} \otimes_R K_{i,f}$.

$${}^hE^1_{i,j} = H_i(K_{\bullet,m}) \otimes_R K_{i,f}$$

$${}^vE^1_{i,j} = K_{i,m} \otimes_R H_j(K_{\bullet,f})$$

Since $K_{\bullet,m}$ is acyclic, $H_i(K_{\bullet,m}) = 0$ iff $i \neq 0$, and $H_0(K_{\bullet,m}) = \mathbb{K}$. Thus,

$${}^hE^1_{i,j} = \begin{cases} k^{\binom{3}{j}} \text{ if } i = 0 \\ 0 \text{ otherwise.} \end{cases}$$

On the vertical spectral sequence, one has:

$${}^vE^1_{i,j} = \begin{cases} K_{i,m} \otimes_R H_0(K_{\bullet,f}) \text{ if } j = 0 \\ K_{i,m} \otimes_R H_1(K_{\bullet,f}) = \mathcal{M}. \text{ if } j = 1 \\ 0 \text{ otherwise.} \end{cases}$$

Comparing the abutment of both spectral sequences, we have ${}^vE^2_{4,1} = H(\mathcal{M}.)_4 = 0$, ${}^vE^2_{3,1} = H(\mathcal{M}.)_3 = 0$ and ${}^vE^2_{4,0} = H_4(K_{\bullet,m} \otimes_R H_0(K_{\bullet,f})) = 0$. □

Example 3.8 The homologies $H(\mathcal{M})_i$, $i = 1, 2$ in (3.2) might be both nonzero. For instance, let $\mathbf{d} = (1, 6)$. Below we list all nonzero, non-Koszul degree first Betti numbers for generic $\mathbf{f}$:

$$\beta_{1,(3,10)} = 1,\ \beta_{1,(3,11)} = 4,\ \beta_{1,(4,10)} = 3,\ \beta_{1,(6,9)} = 2,\ \beta_{1,(1,18)} = 1. \qquad (3.3)$$

The sum of all these numbers equals 11. On the other side, the sum of the dimensions of all $\dim_{\mathbb{K}}(H(\mathcal{M})_{1,\mathbf{a}})$ equals 18 and the sum of the dimensions of all $\dim_{\mathbb{K}}(H(\mathcal{M})_{2,\mathbf{a}})$ equals 7. Indeed, $11 = 18 - 7$. We plot in Fig. 1 below the bidegrees $\mathbf{a}$ with nonzero $\beta_{1,\mathbf{a}}$ in (3.3), together with the curve $n_{(1,6)}(\mathbf{a}) = 1$.

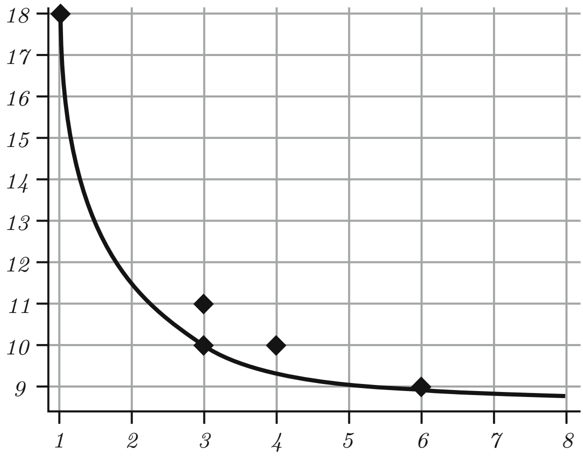

Fig. 1 The generic case with $\mathbf{d} = (1, 6)$

The values of $n_{(1,6)}(a)$ are given below, where the column on the left represents $a_1 = 0$, and the row on the bottom $a_2 = 0$.

0	3	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	2	3	4	5	6	7	8
0	0	0	2	4	6	8	10	12	14	16
0	0	0	3	6	9	12	15	18	21	24
0	0	0	4	8	12	16	20	24	28	32
0	0	0	5	10	15	20	25	30	35	40
0	0	0	6	12	18	24	30	36	42	48
0	0	0	1	5	9	13	17	21	25	29
0	0	0	0	0	0	2	4	6	8	10
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

The values of the Betti numbers in (3.3) can be deduced from Theorem 3.7 and Lemma 2.9. A necessary condition is that $n_{(1,6)}(\mathbf{a}) \geq 1$. For instance, we have that

$$n_{(1,6)}(3, 10) = 1, n_{(1,6)}(3, 11) = 6, \text{ and } n_{(1,6)}(4, 10) = 5,$$

so

$$\beta_{1,(3,11)} = n_{(1,6)}(3, 11) - 2n_{(1,6)}(3, 10) = 6 - 2 = 4.$$

Similarly,

$$\beta_{1,(4,10)} = n_{(1,6)}(4, 10) - 2n_{(1,6)}(3, 10) = 5 - 2 = 3.$$

On the other side,

$$\beta_{1,(6,9)} = n_{2,(6,9)} = 2, \text{ and } \beta_{1,(1,18)} = n_{2,(1,18)} = 1.$$

For a final example, $\beta_{1,(5,10)} = 0$ because

$$n_{2,(5,10)} = 9 = 3n_{2,(3,10)} + 2(n_{2,(4,10)} - 2n_{2,(3,10)}) = 2n_{2,(4,10)} - n_{2,(3,10)},$$

and $\beta_{1,(1,19)} = 0$ because $n_{(1,6)}(1, 19) < 2n_{(1,6)}(1, 18)$.

Example 3.9 Consider the bidegree $\mathbf{d} = (1, 42)$ and $\mathbf{f}$ generic. We list all bidegrees $\mathbf{a}$ with nonzero, non-Koszul first Betti number:

$$(7, 67), (6, 68), (9, 66), (8, 67), (7, 68), (6, 69), (5, 70), (10, 66), (4, 72),$$
$$(12, 65), (3, 75), (3, 76), (17, 64), (18, 64), (33, 63), (1, 126).$$

This is the ordered list of the corresponding Betti numbers: 2, 3, 5, 8, 5, 8, 9, 3, 7, 6, 2, 3, 3, 1, 2, 1. In this example, there are minimal generators of the syzygy module in degrees $\mathbf{a}$, $\mathbf{a} - (1, 0)$ and $\mathbf{a} - (0, 1)$, for $\mathbf{a} = (7, 68)$. We focus on the bidegrees (7, 68), (6, 68), (7, 67), marked with solid diamonds in Fig. 2. Note that we also show a few other bidegrees but we do not display all bidegrees with nonzero, non-Koszul first Betti number in the list above. All these bidegrees must satisfy that $n_{(1,42)}(\mathbf{a}) > 0$. We also plot the curve $n_{(1,42)}(\mathbf{a}) = 1$. Again, the values of the Betti numbers can be deduced from Theorem 3.7 and Lemma 2.9.

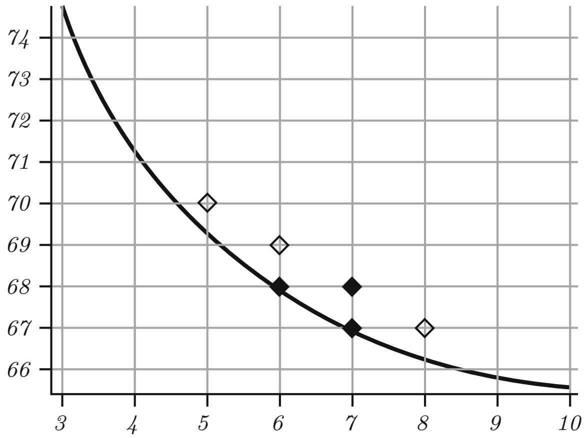

Fig. 2 The generic case with $\mathbf{d} = (1, 42)$

4 Factorization of Sections of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, n)$ and the Segre Variety $\Sigma_{1,n}$

Recall the Segre variety $\Sigma_{r,s}$ is the image of the regular map

$$\mathbb{P}^r \times \mathbb{P}^s \xrightarrow{\sigma_{r,s}} \mathbb{P}^{rs+r+s}$$

given by multiplication

$$(x_0 : \ldots : x_r), (y_0 : \ldots : y_s) \mapsto (x_0 y_0 : \ldots : x_0 y_s : x_1 y_0 : \ldots : x_r y_s).$$

In general one has the following diagram

$$\begin{array}{ccc} \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,i)))\times\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(n-i))) & \longrightarrow & \mathbb{P}^{(2i+1)(n-i)+i+n+1} \\ & \searrow^{\psi_i} & \downarrow \pi \\ & & \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,n))) \end{array} \tag{4.1}$$

The composition of the Segre map

$$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0, n-1))) = \mathbb{P}^3 \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{4n-1},$$

with the projection π onto $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, n)))$ is given with respect to the basis $\{su^n, \cdots, sv^n, tu^n, \ldots, tv^n\}$ for $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, n))$ by

$$\begin{gathered}(a_0 : \cdots : a_3) \times (b_0 : \cdots : b_{n-1}) \mapsto \\ (a_0b_0 : a_0b_1 + a_1b_0 : a_0b_2 + a_1b_1 : \ldots a_0b_{n-1} + a_1b_{n-2} : a_1b_{n-1} : \\ a_2b_0 : a_2b_1 + a_3b_0 : a_2b_2 + a_3b_1 : \ldots : a_2b_{n-1} + a_3b_{n-2} : a_3b_{n-1})\end{gathered}$$

For example, when $n = 2$, the image of ψ_1 is a quartic hypersurface

$$Q = \mathbf{V}(x_2^2x_3^2 - x_1x_2x_3x_4 + x_0x_2x_4^2 + x_1^2x_3x_5 - 2x_0x_2x_3x_5 - x_0x_1x_4x_5 + x_0^2x_5^2).$$

Definition 4.1 The image of the composite map ψ_i is $\Sigma'_{2i+1,n-i}$.

Therefore for $i = 0$ we have $\Sigma_{1,n} = \Sigma'_{1,n}$, but for $i \geq 1$ the variety $\Sigma'_{2i+1,n-i}$ is a linear projection of $\Sigma_{2i+1,n-i}$, with $\operatorname{codim}(\Sigma'_{2i+1,n-i}) = n - i$. In particular $\Sigma'_{2i+1,n-i}$ is a hypersurface in $\Sigma'_{2i+3,n-i-1}$, and

$$\Sigma_{1,n} = \Sigma'_{1,n} \subseteq \Sigma'_{3,n-1} \subseteq \Sigma'_{5,n-2} \subseteq \cdots \subseteq \Sigma'_{2n-3,2} \subseteq \Sigma'_{2n-1,1} \subseteq \mathbb{P}^{2n+1}.$$

If the basepoint free subspace $W \simeq \mathbb{P}^2$ is generic, then $W \cap \Sigma'_{2n-1,1}$ has dimension 1, $W \cap \Sigma'_{2n-3,2}$ is generically finite, and $W \cap \Sigma'_{2i+1,n-i}$ is empty for $i \neq n-1, n-2$.

Lemma 4.2 *If W is a basepoint free three dimensional subspace of $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, n))$, then W is not contained in $\Sigma_{1,n}$.*

Proof If $W \subseteq \Sigma_{1,n}$, then by Theorem 9.22 of [18], the only linear spaces contained in $\Sigma_{n,m}$ are those contained in one of the rulings. So $W \subseteq \Sigma_{1,n}$ would mean that $W = \{a \cdot l, b \cdot l, c \cdot l\}$ with $\{a, b, c\}$ belonging to a fiber. But then l is either in $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0))$ or in $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ and W is not basepoint free. □

As n grows, so do the possibilities for the intersection $W \cap \Sigma'_{2i+1,n-i}$. However, for some special situations governed by geometry, there are resolutions which are independent of n, which we now explore.

4.1 Intersection with $\Sigma_{1,n}$

In the case where $n = 2$, [8] shows that the only way in which W can meet $\Sigma_{1,2}$ in a curve is if the curve is a smooth conic. This phenomenon persists, but we need a bit more machinery.

Theorem (Burau-Zeuge [5]) *If L is a linear space cutting each n-dimensional ruling of $\Sigma_{1,n}$ in at most a single point, then $L \cap \Sigma_{1,n}$ is a rational normal curve.*

Lemma 4.3 *If W meets $\Sigma_{1,n}$ in a curve, then it must be a smooth conic.*

Proof First, suppose W contains a $\mathbb{P}^1$ fiber of $\Sigma_{1,n}$, so that W has basis $\{l_1 s, l_2 s, q\}$ with l_i corresponding to points on the $\mathbb{P}^1$. Then $\mathbf{V}(s, q) \neq \emptyset$ and W is not basepoint free. Next, since W is linear, it cannot meet a $\mathbb{P}^n$ fiber F of $\Sigma_{1,n}$ in more than two noncollinear points, for then it would be contained in F and hence violate Lemma 4.2. If W meets F in two points, since W and F are both linear, $W \cap F$ is a line L, and if F is the fiber over the point l of L, then $W = \{al, bl, c\}$ and since $\mathbf{V}(l, c)$ is nonempty on $\mathbb{P}^1 \times \mathbb{P}^1$, W would have basepoints. In particular, W can meet each $\mathbb{P}^n$ fiber in at most a point, so by the result of Burau-Zeuge, $W \cap \Sigma_{1,n}$ is a rational normal curve. As $W \simeq \mathbb{P}^2$, the curve must be a smooth conic. □

Theorem 4.4 *$W \cap \Sigma_{1,n}$ is a smooth conic iff I_W has a bidegree $(3, n)$ first syzygy.*

Proof Suppose I_W has a minimal first syzygy of bidegree $(3, n)$,

$$a(s, t) \cdot f_0 + b(s, t) \cdot f_1 + c(s, t) \cdot f_2 = 0.$$

If $\langle a(s, t), b(s, t), c(s, t)\rangle \neq \langle s^2, st, t^2\rangle$, then it must be generated by two bidegree $(2, 0)$ quadrics $\{a(s, t), b(s, t)\}$ with no common factor. Changing basis for I_W, the syzygy involves only f_0, f_1, which implies $f_0, f_1 = -b(s, t)g, a(s, t)g$ for some g. This is impossible by degree considerations, so after a change of basis for W, we may assume the f_i satisfy

$$s^2 \cdot f_0 + st \cdot f_1 + t^2 \cdot f_2 = 0$$

Now we switch perspective, and consider $[f_0, f_1, f_2]$ as a syzygy on $[s^2, st, t^2]$. Since the syzygies on the latter space are generated by the columns of

$$\begin{bmatrix} t & 0 \\ -s & -t \\ 0 & s \end{bmatrix},$$

we have

$$\begin{aligned} f_0 &= \quad ta_0 \\ f_1 &= sa_0 + ta_1 \\ f_2 &= \quad sa_1, \end{aligned}$$

with the $a_i \in k[u, v]_n$. In particular, $\{a_0, a_1\}$ are a basepoint free pencil of $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$, and we may parameterize as in the proof of Theorem 4.1 of [8] so that $W \cap \Sigma_{1,n}$ is a smooth conic. On the other hand, if $W \cap \Sigma_{1,n}$ is a smooth conic, the proof follows as in Theorem 4.1 of [8]. □

Example 4.5 If $W = \{su^n, tv^n, sv^n + tu^n\}$, then since $\Sigma_{1,n}$ is given by the 2×2 minors of the matrix

$$\begin{bmatrix} x_0 & \cdots & x_n \\ x_{n+1} & \cdots & x_{2n+1} \end{bmatrix},$$

with $x_i = su^{n-i}v^i$ for $i \in \{0, \ldots, n\}$ and $tu^{n-i}v^i$ for $i \in \{n+1, \ldots, 2n+2\}$, so (dualizing) $W = \mathbf{V}(x_1, \ldots, x_{n-1}, x_{n+2}, \ldots x_{2n+1}, x_n - x_{n+1}) \subseteq \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, n))^\vee)$, and using coordinates $\{x_0, x_n, x_{2n+1}\}$ for W, $W \cap \Sigma_{1,n} = W \cap \mathbf{V}(x_0 x_{2n+1} - x_n^2)$.

Remark 4.6 (Effective Criterion) Notice that given I_W, Theorem 4.4 gives an effective way to understand the geometry of W. Combined with the following result we obtain a complete description of the minimal free resolution of I_W just by computing wether or not I_W has a bidegree $(3, n)$ first syzygy.

4.2 Minimal free Resolutions Determined by the Geometry of $W \cap \Sigma_{1,n}$

We now examine situations where $W \cap \Sigma_{1,n}$ has special geometry.

Theorem 4.7 *$W \cap \Sigma_{1,n}$ is a smooth conic iff I_W has bigraded Betti numbers:*

$$0 \leftarrow I_W \leftarrow (-1,-n)^3 \xleftarrow{\partial_1} \begin{matrix}(-1,-3n)\\ \oplus \\ (-2,-2n)^3 \\ \oplus \\ (-3,-n)\end{matrix} \xleftarrow{\partial_2} \begin{matrix}(-2,-3n)^2 \\ \oplus \\ (-3,-2n)^2\end{matrix} \xleftarrow{\partial_3} (-3,-3n) \leftarrow 0$$

Proof In fact, we will show more, exhibiting the differentials in the minimal free resolution. By Theorem 4.4, we may choose the f_i so that

$$\begin{aligned} f_0 &= \quad ta_0 \\ f_1 &= sa_0 + ta_1 \\ f_2 &= \quad sa_1, \end{aligned}$$

Then the syzygy described by Lemma 3.1 is $(a_1^2, -a_0a_1, a_0^2)$ and we also have the bidegree (2, 0) syzygy $(s^2, -st, t^2)$. Consider the ∂_i below.

$$\partial_1 = \begin{bmatrix} a_1^2 & f_1 & f_2 & 0 & s^2 \\ -a_0a_1 & -f_0 & 0 & f_2 & -st \\ a_0^2 & 0 & -f_0 & -f_1 & t^2 \end{bmatrix}$$

$$\partial_2 = \begin{bmatrix} t & s & 0 & 0 \\ -a_1 & 0 & 0 & -s \\ a_0 & -a_1 & -s & t \\ 0 & a_0 & t & 0 \\ 0 & 0 & a_1 & a_0 \end{bmatrix}$$

$$\partial_3 = \begin{bmatrix} s \\ -t \\ a_0 \\ -a_1 \end{bmatrix}$$

A check shows $\partial_i \partial_{i+1} = 0$, exactness follows by Buchsbaum-Eisenbud [4]. □

Example 1.1 is a consequence of Theorem 4.7, because for $\mathbf{d} = (1, 1)$, W is basepoint free iff it meets $\Sigma_{1,1}$ in a smooth conic. When $W \cap \Sigma_{1,n}$ contains three distinct noncollinear points, the resolution of I_W is also completely determined.

Theorem 4.8 *If $|W \cap \Sigma_{1,n}|$ is finite and contains three noncollinear points, then the bigraded Betti numbers of I_W are*

$$0 \leftarrow I_W \leftarrow (-1,-n)^3 \longleftarrow \begin{array}{c}(-1,-3n)\\ \oplus \\ (-2,-2n)^3 \\ \oplus \\ (-3,-n-\mu) \\ \oplus \\ (-3,-2n+\mu)\end{array} \longleftarrow \begin{array}{c}(-2,-3n)^2 \\ \oplus \\ (-3,-2n)^3\end{array} \longleftarrow (-3,-3n) \leftarrow 0$$

with $0 < \mu \le \lfloor n/2 \rfloor$.

Proof As in the previous theorem, we will describe the differentials in the minimal free resolution. Since $W \cap \Sigma_{1,n}$ contains three noncollinear points, we may choose a basis so that $W = \{l_0 g_0, l_1 g_1, l_2 g_2\}$ with the l_i of bidegree $(1,0)$ and the g_i of bidegree $(0,n)$. If the g_i are not linearly independent, then changing basis we see that there are constants a, b, c, d with

$$W = \langle sg_0, tg_1, (as+bt)\cdot(cg_0+dg_1)\rangle = \langle sg_0, tg_1, acsg_0 + bdtg_1\rangle,$$

so $(bdt^2, acs^2, -st)$ is a bidegree $(2,0)$ syzygy on I_W and Theorem 4.4 applies.

So we may assume $\{g_0, g_1, g_2\}$ are linearly independent; suppose the Hilbert-Burch matrix for $\{g_1, g_2, g_3\}$ has columns of degree a and $n-\mu$. In this case, in addition to the three Koszul syzygies and the syzygy of Lemma 3.1, the Hilbert-Burch syzygies can be lifted: if (b_0, b_1, b_2) is a syzygy of degree a on the g_i, then $(l_1 l_2 b_0, l_0 l_2 b_1, l_0 l_1 b_2)$ is a syzygy of bidegree $(3, n+\mu)$ on I_W, and similarly for the syzygy of degree $n-\mu$. Note that $g_0 = b_1c_2 - c_1b_2, g_1 = c_0b_2 - b_0c_2, g_2 = b_0c_1 - c_0b_1$. A priori, these need not be minimal, but by constructing the remaining differentials and applying the Buchsbaum-Eisenbud criterion, we will see that they are. Changing basis, we may assume W has basis $= \{sg_0, tg_1, (as+bt)g_2\}$, hence the syzygy of Lemma 3.1 takes the form $(-ag_1g_2, -bg_0g_2, g_0g_1)$, and

$$\partial_1 = \begin{bmatrix} -ag_1g_2 & tg_1 & (as+bt)g_2 & 0 & t(as+bt)b_0 & t(as+bt)c_0 \\ -bg_0g_2 & -sg_0 & 0 & (as+bt)g_2 & s(as+bt)b_1 & s(as+bt)c_1 \\ g_0g_1 & 0 & -sg_0 & -tg_1 & stb_2 & stc_2 \end{bmatrix}$$

A check shows that the two matrices below satisfy $\partial_2\partial_1 = 0$ and $\partial_3\partial_2 = 0$:

$$\partial_2 = \begin{bmatrix} t & s & 0 & 0 & 0 \\ ag_2 & -bg_2 & as+bt & 0 & 0 \\ 0 & g_1 & 0 & t & 0 \\ g_0 & 0 & 0 & 0 & s \\ 0 & 0 & c_2 & c_1 & c_0 \\ 0 & 0 & -b_2 & -b_1 & -b_0 \end{bmatrix} \quad \partial_3 = \begin{bmatrix} s \\ -t \\ -g_2 \\ g_1 \\ -g_0 \end{bmatrix}$$

Applying the Buchsbaum-Eisenbud criterion shows the complex is indeed exact. Since $n + \mu \leq 2n - \mu$, then it follows that $\mu \leq \lfloor n/2 \rfloor$. □

Remark 4.9 If $n = 2$ then $|W \cap \Sigma_{1,n}|$ is generically finite and generically contains three noncollinear points. For $n \geq 3$ this is not the case, so this closed condition is very restrictive. Moreover, for $n = 2, 3$, from Theorem 4.8, one has that $\mu = 1$.

5 Higher Segre Varieties

Since $\Sigma'_{2i+1,n-i}$ has codimension $n - i$, unless $i = n - 1$ or $n - 2$, the intersection $W \cap \Sigma'_{2i+1,n-i}$ is generically empty. Theorems 4.7 and 4.8 illustrate the principle that when $W \cap \Sigma'_{2i+1,n-i} \neq \emptyset$ and $i \leq n - 3$, special behavior can occur. The next theorem makes this explicit when $|W \cap \Sigma'_{2i+1,n-i}|$ is finite and contains at least three noncollinear points.

Theorem 5.1 *Suppose W has basis $\{g_0h_0, g_1h_1, g_2h_2\}$ with*

$$g_j \in H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, i)) \text{ and } h_j \in H^0(\mathcal{O}_{\mathbb{P}^1}(n - i)), \text{ with } 0 \leq i \leq n - 1, 3 \leq n. \text{ Then}$$

(1) *$\{h_0, h_1, h_2\}$ and $\{g_0, g_1, g_2\}$ are basepoint free.*
(2) *A syzygy $\{a_0, a_1, a_2\}$ on the h_i lifts to a syzygy (possibly non-minimal)*

$$\{g_1g_2a_0, g_0g_2a_1, g_0g_1a_2\}$$

on I_W, and similarly for a syzygy on the g_i.
(3) *If $\{h_0, h_1, h_2\}$ is a pencil, then there is a bidegree $(3, 2i + n)$ syzygy on I_W.*
(4) *If $\{h_0, h_1, h_2\}$ is not a pencil, then it has a Hilbert-Burch matrix with columns of degrees $\{n - i - \mu, \mu\}$ in the $\{s, t\}$ variables. These give rise to syzygies of type (2) above of bidegree $(3, n + 2i + \mu)$ and $(3, 2n + i - \mu)$.*

Proof For (1), if the g_i or h_i are not basepoint free, then neither is W, and for (2), the result is immediate. For (3), if the h_i are a pencil, then $I_W = \langle g_0h_0, g_1h_1, g_2(ah_0 + bh_1)\rangle$ for constants a, b, and so $(ag_1g_2, bg_0g_2, -g_0g_1)$ is the desired syzygy, and (4) follows by applying (2) to the Hilbert-Burch syzygies. □

The previous theorem deals with the situation where there is a basis for W where all three elements factor in the same way. Even if only one or two elements factor, the minimal free resolutions often behave differently from the generic case. Computations suggest that all nongeneric behavior in the minimal free resolution stems from factorization:

Conjecture 5.2 *If the bigraded minimal free resolution of I_W has nongeneric bigraded Betti numbers, then for some $i \leq n-3$, $W \cap \Sigma'_{2i+1,n-i} \neq \emptyset$.*

5.1 Intersection with $\Sigma'_{3,n-1}$

We now examine the situation where the elements of W factor into components of degree $(1, 1)$ and $(0, n-1)$. The next example shows that the converse to Conjecture 5.2 need not hold.

Example 5.3 Suppose a, b, c are basepoint free elements of bidegree $(1, 1)$, and d, e, f are generic elements of bidegree $(0, 4)$, with $I_W = \langle ad, be, cf \rangle$. A computation shows that the Betti numbers of I_W are as in the diagram below.

$$0 \leftarrow I_W \leftarrow (-1,-5)^3 \longleftarrow \begin{array}{c} (-1,-15) \\ \oplus \\ (-2,-10)^3 \\ \oplus \\ (-3,-9)^5 \\ \oplus \\ (-4,-8)^3 \\ \oplus \\ (-9,-7) \end{array} \longleftarrow \begin{array}{c} (-2,-15)^2 \\ \oplus \\ (-3,-10)^6 \\ \oplus \\ (-4,-9)^6 \\ \oplus \\ (-9,-8)^2 \end{array} \longleftarrow \begin{array}{c} (-3,-15) \\ \oplus \\ (-4,-10)^3 \\ \oplus \\ (-9,-9) \end{array} \leftarrow 0$$

Lemma 3.1 explains the $(1, 15)$ syzygy, and there are three Koszul syzygies. By construction, $W \cap \Sigma'_{3,4}$ consists of three points, and the bigraded Betti numbers for this example agree with the generic case; Theorem 3.4 explains the five $(3, 9)$ syzygies, whereas Theorem 5.1 only accounts for two of them.

Theorem 5.4 *Suppose* $W = \mathrm{Span}\{g_0h_0, g_1h_1, g_2h_2\}$*, with g_i of degree* $(1, 1)$ *and the h_i a pencil of degree* $(0, n-1)$*, so that*

$$I_W = \langle g_0h_0, g_1h_1, g_2(ah_0 + bh_1) \rangle.$$

If $W \cap \Sigma_{1,n}$ is empty and $W \cap \Sigma'_{3,n-1}$ contains three noncollinear points, then there are minimal first syzygies of degrees

$$\{(-1, -3n), (-2, -2n)^3, (-3, -n-2), (-3, -2n+1)^2, (-6, -2n+2)\}$$

Proof The syzygies of degree $(1, 3n)$ and $(2, 2n)$ are tautological. The syzygy of degree $(3, n+2)$ can be explained by Theorem 5.1 (3), but it is more enlightening to treat the syzygies of degree $(3, *)$ as a group. Write $\mathbf{f}$ as in Theorem 3.4:

$$\{sq_0 + tq_1, sq_2 + tq_3, sq_4 + tq_5\}.$$

Using that the h_i are a pencil and expanding, we find that

$$\text{Span}\{q_0, \ldots, q_5\} = \text{Span}\{uh_0, vh_0, uh_1, vh_1\}.$$

Since $\{h_0, h_1\}$ are basepoint free, they are a complete intersection, so the Hilbert-Burch matrix for $\{uh_0, vh_0, uh_1, vh_1\}$ is

$$\begin{bmatrix} v & 0 & p_0 \\ -u & 0 & p_1 \\ 0 & v & p_2 \\ 0 & -u & p_3 \end{bmatrix},$$

with the p_i of degree $n-2$. In particular, the columns have degrees $\{b_1, b_2, b_3\} = \{1, 1, n-2\}$, which by Theorem 3.4 yields syzygies in degrees

$$\{(3, 2n-1), (3, 2n-1), (3, n+2)\}$$

By Corollary 3.6, the seven syzygies constructed so far are independent. We next prove there exists a unique first syzygy of degree $(6, 2n-2)$. Let (s_0, s_1, s_2) be a syzygy on I_W, and rewrite it as below

$$s_0g_0h_0 + s_1g_1h_1 + s_2g_2(ah_0 + bh_1) = 0 = (s_0g_0 + as_2g_2)h_0 + (s_1g_1 + bs_2g_2)h_1. \tag{5.1}$$

So $(s_0g_0 + as_2g_2, s_1g_1 + bs_2g_2)$ is a syzygy on the complete intersection (h_0, h_1), which implies that

$$S = \begin{bmatrix} s_0g_0 + as_2g_2 \\ s_1g_1 + bs_2g_2 \end{bmatrix},$$

is in the image of the Koszul syzygy K on $\{h_0, h_1\}$:

$$K = \begin{bmatrix} h_1 \\ -h_0 \end{bmatrix}.$$

One possibility is $S = 0$, which leads to a syzygy of Theorem 5.1 type 3, of bidegree $(3, n+2)$, which we have accounted for, so we may suppose S is nonzero. This means $S = p \cdot K$ for some polynomial p. Since the h_i are degree $(0, n-1)$, the lowest possible degree for the s_i in the $(0, 1)$ variables is $n-2$. We show that in degree $(a, 2n-2)$, there is a unique minimal syzygy of degree $(6, 2n-2)$ which is not a multiple of the syzygy of degree $(3, n+2)$.

To see this, we write out Eq. (5.1), collecting the coefficients of

$$\{u^{2n-2}, vu^{2n-1}, \ldots, v^{2n-2}\}.$$

Each s_i is of degree $n-2$ in the $(0, 1)$ variables, so there are $3(n-1)$ columns, and we obtain a $2n-1 \times 3n-3$ matrix

$$\mathcal{O}_{\mathbb{P}^1}^{3n-3}(-1) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^1}^{2n-1}.$$

The coefficients in the $(1, 0)$ variables of the s_i (written as polynomials in the $(0, 1)$ variables) correspond to elements of the kernel of this matrix. The nonzero entries of ψ come from the six linear forms in the $(1, 0)$ variables, obtained by writing the $(1, 1)$ components g_i as

$$g_i = a_i u + b_i v, \text{ with } \{a_i, b_i\} \in \mathbb{K}[s, t]_1.$$

Since W is basepoint free and $W \cap \Sigma_{1,n} = \emptyset$, the cokernel of ψ is zero, so $\ker(\psi)$ is free of rank $n-2$, with first Chern class $3-3n$. The key is that the unique syzygy of degree $(3, n+2)$ generates $n-3$ independent syzygies of degree $(3, 2n-2)$, which follows from the computation

$$h^0(\mathcal{O}_{\mathbb{P}^1}((2n-2)-(n+2))) = h^0(\mathcal{O}_{\mathbb{P}^1}(n-4)) = n-3.$$

Since $\ker(\psi)$ has rank $n-2$, this means there is a single additional element in the kernel, which has first Chern class

$$3(n-3) - (3n-3) = -6,$$

yielding a unique first syzygy of degree $(6, 2n-2)$. □

Remark 5.5 If $W \cap \Sigma_{1,n} \neq \emptyset$, some of the g_i will factor. If all three factor, we are in the situation of Theorem 4.8. If only one or two factor, then the first syzygy of degree $(6, 2n-2)$ changes to a syzygy of degree $(4, 2n-2)$ if $|W \cap \Sigma_{1,n}| = 2$, and to a syzygy of degree $(5, 2n-2)$ if $|W \cap \Sigma_{1,n}| = 1$.

Example 5.6 With the hypotheses of Theorem 5.4, computations suggest that the bigraded Betti numbers are

$$
\begin{array}{ccccccc}
 & & (-1,-3n) & & & & \\
 & & \oplus & & & & \\
 & & (-2,-2n)^3 & & (-2,-3n)^2 & & \\
 & & \oplus & & \oplus & & (-3,-3n) \\
0 \leftarrow I_W \leftarrow (-1,-n)^3 & \xleftarrow{\partial_1} & (-3,-n-2) & \xleftarrow{\partial_2} & (-3,-2n)^4 & \xleftarrow{\partial_3} & \oplus \quad \leftarrow 0 \\
 & & \oplus & & \oplus & & (-6,-2n) \\
 & & (-3,-2n+1)^2 & & (-6,-2n+1)^2 & & \\
 & & \oplus & & & & \\
 & & (-6,-2n+2) & & & &
\end{array}
$$

Note that the numbers

$$\beta_{2,\{*,-3n\}} \text{ and } \beta_{3,\{*,-3n\}}$$

are explained by Proposition 3.2; one way to prove the diagram above is the correct Betti table would be to determine explicitly the $(6, 2n-2)$ syzygy, and then write down the differentials and apply the Buchsbaum-Eisenbud criterion for exactness.

5.2 Connections to the Hurwitz Discriminant and Sylvester Map

We consider the case $d = (1, n)$ with $n = 5$ at the particular bidegree $(3, 8)$. The aim is to point out the tip of the iceberg of the relation between non generic rank behavior in the matrix of ϕ_1 from Sect. 2.3 and the different intersections of W with the variety of elementary tensors associated to all the decompositions of $(1, n) = (1, i) + (0, n - i)$.

We set $n = 5$. According to Theorem 3.4, when the f_i are generic there are syzygies in degree $(3, m)$ for $m \geq 2 \cdot 5 - 0 - 1 = 9$ ($\kappa = 0$ for $n = 5$). But in this bidegree, we can read in the table of Sect. 2.3 that the matrix of ϕ_1 has always full rank in the basepoint free case.

We then move one step to bidegree $(3, 8) = (3, 2 \cdot 5 - 2)$ and we try to detect the existence of nontrivial syzygies there. In this bidegree $\phi_1 : R_{(0,5)} \to R^3_{(1,0)}$, so we get a 6×6 matrix M, constructed as follows, according to Example 2.6. Write

$$f_\ell = s\left(\sum_{j=0}^{5} p_{\ell j} u^j v^{5-j}\right) + t\left(\sum_{j=0}^{5} q_{\ell j} u^j v^{5-j}\right), \text{ for } \ell = 0, 1, 2.$$

Then

$$M = \begin{pmatrix} p_{00} & \cdots & p_{05} \\ q_{00} & \cdots & q_{05} \\ \hline p_{10} & \cdots & p_{15} \\ q_{10} & \cdots & q_{15} \\ \hline p_{20} & \cdots & p_{25} \\ q_{20} & \cdots & q_{25} \end{pmatrix}.$$

Let $S_{(3,8)}$ denote the hypersurface $\mathbf{V}(\det(M))$. Consider the 9-dimensional variety $\Sigma'_{7,2} \subset \mathbb{P}^{11}$ of bihomogeneous polynomials factoring as the product of a degree $(1, 3)$ polynomial and a degree $(0, 2)$ polynomial. We expect $\dim(\Sigma'_{7,2} \cap W) = 0$. It would be interesting to relate the geometry of $S_{(3,8)}$ to special features of this intersection (for example, finitely many points but with *special* properties, or a curve). Note that W lives in the Grassmanian of planes in $\mathbb{P}^{11}$ and there is a polynomial, called the *Hurwitz discriminant* by Sturmfels [27], which vanishes whenever the intersection $\Sigma'_{7,2} \cap W$ does not consist of $\deg(\Sigma'_{7,2})$ many points.

5.3 Concluding Remarks

We close with a number of questions:

(1) What happens when there are many "low degree" first syzygies? As shown in [11] and [25], linear first syzygies impose strong constraints.
(2) W is a point of $\mathbb{G}(2, 2n + 1)$. How does the Schubert cell structure impact the free resolution of I_W?
(3) What happens in other bidegrees? For other toric surfaces?
(4) Are there special cases such as in Theorem 4.7, Theorem 4.8, Theorem 5.4 which are of interest to the geometric modeling community?
(5) For computation of toric cohomology, it is sufficient to have a complex with homology supported in B, rather than an exact sequence. This is studied by Berkesch-Erman-Smith in [2], and is a very active area of research.

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Castelnuovo–Mumford Regularity and Powers

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1 Castelnuovo–Mumford Regularity Over General Base Rings

Castelnuovo–Mumford regularity was introduced in the early eighties of the twentieth century by Eisenbud and Goto in [12] and by Ooishi [18] as an algebraic counterpart of the notion of regularity for coherent sheaves on projective spaces discussed by Mumford in [19].

One of the most important features of Castelnuovo–Mumford regularity is that it can be equivalently defined in terms of (and hence it bounds) the vanishing of local cohomology modules, the vanishing of Koszul homology modules and the vanishing of syzygies.

This triple nature of Castelnuovo–Mumford regularity is usually stated for graded rings over base fields, but indeed it holds in general as we will show in this section.

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a $\mathbb{N}$-graded ring with R_0 commutative and Noetherian. We assume that R is standard graded, i.e., it is generated as an R_0-algebra by finitely many elements $x_1, \dots, x_n$ of degree 1. Let $S = R_0[X_1, \dots, X_n]$ with $\mathbb{N}$-graded structure induced by the assignment $\deg X_i = 1$. The R_0-algebra map $S \to R$ sending X_i to x_i induces an S-module structure on R and hence on every R-module.

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Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded R-module. Given $a \in \mathbb{Z}$ we will denote by $M(a)$ the module that it is obtained from M by shifting the degrees by a, i.e. $M(a)_i = M_{i+a}$.

The Castelnuovo–Mumford regularity of M is defined in terms of local cohomology modules $H^i_{Q_R}(M)$ with support on

$$Q_R = R_+ = (x_1, \dots, x_n).$$

For general properties of local cohomology modules we refer the readers to [2, 6, 13]. In our setting the module $H^i_{Q_R}(M)$ is $\mathbb{Z}$-graded and its homogeneous component $H^i_{Q_R}(M)_j$ of degree $j \in \mathbb{Z}$ vanishes for large j. The Castelnuovo–Mumford regularity of M or, simply, the regularity of M is defined as

$$\operatorname{reg}(M) = \max\{i + j : H^i_{Q_R}(M)_j \neq 0\}.$$

We may as well consider M as an S-module by means of the map $S \to R$ and local cohomology supported on

$$Q_S = (X_1, \dots, X_n).$$

Since $H^i_{Q_S}(M) = H^i_{Q_R}(M)$ the resulting regularity is the same.

Here we list some simple properties of regularity that we will freely use.

(1) $\operatorname{reg}(M(-a)) = \operatorname{reg}(M) + a$.
(2) $\operatorname{reg}(S) = 0$ because $H^n_{Q_S}(S) = (X_1 \cdots X_n)^{-1} R_0[X_1^{-1}, \dots, X_n^{-1}]$ and $H^i_{Q_S} = 0$ for all $i \neq n$.
(3) If $0 \to N \to M \to L \to 0$ is a short exact sequence of finitely generated graded R-modules with maps of degree 0 then :

$$\begin{aligned} \operatorname{reg}(N) &\leq \max\{\operatorname{reg}(M), \operatorname{reg}(L) + 1\}, \\ \operatorname{reg}(M) &\leq \max\{\operatorname{reg}(L), \operatorname{reg}(N)\}, \\ \operatorname{reg}(L) &\leq \max\{\operatorname{reg}(M), \operatorname{reg}(N) - 1\}. \end{aligned}$$

A minimal set of generators of M is, by definition, a set of generators that is minimal with respect to inclusion. The number of elements in a minimal set of generators is not uniquely determined, but the set of the degrees of the elements in a minimal set of homogeneous generators of M is uniquely determined because it coincides with the set of $i \in \mathbb{Z}$ such that $[M/Q_R M]_i \neq 0$. So we have a well defined notion of largest degree of a minimal generator of M that we denote by $t_0(M)$, that is,

$$t_0(M) = \max\{i \in \mathbb{Z} : [M/Q_R M]_i \neq 0\}$$

if $M \neq 0$. We use t_0 because $M/Q_R M \simeq \operatorname{Tor}_0^R(M, R_0) = \operatorname{Tor}_0^S(M, R_0)$.

The following result establishes the crucial link between the regularity and the degree of generators of a module. It appears in [18, Thm.2], where it is attributed to Mumford, and it appears also in [2, Thm.16.3.1].

Lemma 1.1 $t_0(M) \le \operatorname{reg}(M)$.

Proof Let $v = t_0(M)$. Then the R_0-module $[M/Q_S M]_v$ is non-zero. Therefore there is a prime ideal P of R_0 such that $[M/Q_S M]_v$ localized at P is non-zero. In other words, the localization M' of M at the multiplicative set $R_0 \setminus P$ is a graded module over $(R_0)_P[X_1, \dots, X_n]$ with $t_0(M') = t_0(M)$. Since $\operatorname{reg}(M') \le \operatorname{reg}(M)$ we may assume right away that R_0 is local with maximal ideal, say, $\mathfrak{m}$. Similarly we may also assume that the residue field of R_0 is infinite. If $M = H^0_{Q_S}(M)$, the assertion is obvious. If $M \ne H^0_{Q_S}(M)$ then set $M' = M/H^0_{Q_S}(M)$. Clearly $t_0(H^0_{Q_S}(M)) \le \operatorname{reg}(M)$ and $\operatorname{reg}(M') \le \operatorname{reg}(M)$. Since $t_0(M) \le \max\{t_0(M'), t_0(H^0_{Q_S}(M))\}$ it is enough to prove the statement for M'. That is to say, we may assume that $\operatorname{grade}(Q_S, M) > 0$. Because the residue field of R_0 is infinite, there exists $L \in S_1 \setminus \mathfrak{m}S_1$ such that L is a non-zero-divisor on M. By a change of coordinates we may assume that $L = X_n$. The short exact sequence

$$0 \to M(-1) \to M \to \overline{M} = M/(X_n)M \to 0$$

implies that $\operatorname{reg}(\overline{M}) \le \operatorname{reg}(M)$ (it is actually equal but we do not need it). As $\overline{M}$ is a finitely generated graded module over $R_0[X_1, \dots, X_{n-1}]$, we may assume, by induction on the number of variables, that it is generated in degree $\le \operatorname{reg}(M)$. But then it follows easily that also M is generated in degree $\le \operatorname{reg}(M)$. □

Next we consider the (graded) Koszul homology $H(Q_R, M) = H(Q_S, M)$ and set:

$$\operatorname{reg}_1(M) = \max\{j - i : H_i(Q_R, M)_j \ne 0\}.$$

In this case, since $H_0(Q_R, M) \cong M/Q_R M$, the assertion

$$t_0(M) \le \operatorname{reg}_1(M)$$

is obvious. Now, let

$$\mathbb{F} : \dots \to F_c \to F_{c-1} \to \dots \to F_0 \to 0$$

be a graded S-free resolution of M, i.e., each F_i is a graded and S-free of finite rank, the maps have degree 0 and $H_i(F) = 0$ for all i with the exception of $H_0(\mathbb{F}) \simeq M$. We say that $\mathbb{F}$ is minimal if a basis of F_0 maps to a minimal set of homogeneous generators of M, a basis of F_1 maps to a minimal set of homogeneous generators of the kernel of $F_0 \to M$ and for $i \ge 2$ a basis of F_i maps to a minimal set of homogeneous generators of the kernel of $F_{i-1} \to F_{i-2}$.

If R_0 is a field then a (finite) minimal S-free resolution always exists and it is unique up to an isomorphism of complexes. For general R_0, it is still true that every module has a minimal free graded resolution but it is, in general, not finite and furthermore it is not unique up to an isomorphism of complexes.

Given a minimal graded S-free resolution $\mathbb{F}$ of M we set:

$$\operatorname{reg}_2(\mathbb{F}) = \max\{t_0(F_i) - i \ : \ i = 0, \ldots, n - \operatorname{grade}(Q_S, M)\}$$

and

$$\operatorname{reg}_3(\mathbb{F}) = \max\{t_0(F_i) - i : i \in \mathbb{N}\}.$$

Obviously we have $t_0(M) \leq \operatorname{reg}_2(\mathbb{F}) \leq \operatorname{reg}_3(\mathbb{F})$. We are ready to establish the following fundamental result:

Theorem 1.2 *With the notation above and for every minimal S-free resolution $\mathbb{F}$ of M, we have:*

$$\operatorname{reg}(M) = \operatorname{reg}_1(M) = \operatorname{reg}_2(\mathbb{F}) = \operatorname{reg}_3(\mathbb{F}).$$

Proof Set $Q = Q_S$ and $g = \operatorname{grade}(Q, M) = \min\{i : H^i_Q(M) \neq 0\}$.

We first prove that $\operatorname{reg}(M) \leq \operatorname{reg}_1(M)$. We prove the statement by decreasing induction on g. Suppose $g = n$. The induced map $H^n_Q(F_0) \to H^n_Q(M)$ is surjective. Hence we have

$$\operatorname{reg}(M) \leq \operatorname{reg}(F_0) = t_0(F_0) = t_0(M) = \max\{j : H_0(Q, M)_j \neq 0\} = \operatorname{reg}_1(M).$$

Now assume that $g < n$ and consider

$$0 \to M_1 \to F_0 \to M \to 0.$$

We have $\operatorname{grade}(Q, M_1) = g + 1$ and

$$\operatorname{reg}(M) \leq \max\{\operatorname{reg}(F_0), \operatorname{reg}(M_1) - 1\}.$$

By induction $\operatorname{reg}(M_1) \leq \operatorname{reg}_1(M_1)$. Since $H_i(Q, M_1) = H_{i+1}(Q, M)$ for $i > 0$ and

$$0 \to H_1(Q, M) \to H_0(Q, M_1) \to H_0(Q, F_0) \to H_0(Q, M) \to 0$$

is an exact sequence, we have

$$\operatorname{reg}_1(M_1) = \max\{j - i : H_i(Q, M_1)_j \neq 0\} = \max\{a, b\}$$

with $a = \max\{j : H_0(Q, M_1)_j \neq 0\}$ and $b = \max\{j - i : H_{i+1}(Q, M)_j \neq 0 \text{ and } i > 0\}$. So $b \leq \mathrm{reg}_1(M) + 1$ and, since $a \leq \max\{t_0(F_0), \max\{j : H_1(Q, M)_j \neq 0\}\}$, we have that $a \leq \mathrm{reg}_1(M) + 1$ as well. Hence

$$\mathrm{reg}_1(M_1) \leq \mathrm{reg}_1(M) + 1$$

and it follows that $\mathrm{reg}(M) \leq \mathrm{reg}_1(M)$.

Secondly we prove that $\mathrm{reg}_1(M) \leq \mathrm{reg}_2(\mathbb{F})$. Since

$$H_i(Q, M) = \mathrm{Tor}_i^S(M, R_0) = H_i(\mathbb{F} \otimes R_0)$$

we have that $H_i(Q, M)$ is a subquotient of $F_i \otimes R_0$ and hence

$$\max\{j : H_i(Q, M)_j \neq 0\} \leq t_0(F_i).$$

Furthermore, $H_i(Q, M) = 0$ if $i > n - g$. Therefore $\mathrm{reg}_1(M) \leq \mathrm{reg}_2(\mathbb{F})$.

That $\mathrm{reg}_2(\mathbb{F}) \leq \mathrm{reg}_3(\mathbb{F})$ is obvious by definition, so it remains to prove that $\mathrm{reg}_3(\mathbb{F}) \leq \mathrm{reg}(M)$. Set $M_0 = M$ and consider the exact sequence

$$0 \to M_{i+1} \to F_i \to M_i \to 0.$$

By the minimality of $\mathbb{F}$ we have $t_0(F_i) = t_0(M_i) \leq \mathrm{reg}(M_i)$. Hence

$$\mathrm{reg}(M_{i+1}) \leq \max\{t_0(F_i), \mathrm{reg}(M_i) + 1\} = \mathrm{reg}(M_i) + 1$$

for all $i \geq 0$. It follows that

$$t_0(F_i) = t_0(M_i) \leq \mathrm{reg}(M_i) \leq \mathrm{reg}(M) + i$$

for every i, that is,

$$t_0(F_i) - i \leq \mathrm{reg}(M),$$

in other words,

$$\mathrm{reg}_3(\mathbb{F}) \leq \mathrm{reg}(M). \qquad \square$$

Remark 1.3 Let $T \to R_0$ be any surjective homomorpism of unitary rings. It extends uniquely to $S' = T[X_1, \ldots, X_n] \to S = R_0[X_1, \ldots, X_n]$. Therefore a finitely generated graded R-module M can be regarded as a finitely generated graded S'-module. Hence the regularity of M can be computed also using a graded minimal free resolution as S'-module.

2 Bigraded Castelnuovo–Mumford Regularity

Assume now $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ is $\mathbb{N}^2$-graded with $R_{(0,0)}$ commutative and Noetherian and that R is generated as an $R_{(0,0)}$-algebra by elements $x_1, \dots, x_n, y_1, \dots, y_m$ with the x_i homogeneous of degree $(1, 0)$ and the y_j homogeneous of degree (0, 1)

We will denote by $R^{(*,0)}$ the subalgebra $\bigoplus_i R_{(i,0)}$ of R and by $Q_{(1,0)}$ the ideal of $R^{(*,0)}$ generated by $R_{(1,0)}$ i.e., by $x_1, \dots, x_n$. Similarly $R^{(0,*)}$ is the subalgebra $\bigoplus_j R_{(0,j)}$ of R and $Q_{(0,1)}$ the ideal of $R^{(0,*)}$ generated by $R_{(0,1)}$ i.e., by $y_1, \dots, y_m$. We have (at least) three ways of getting an $\mathbb{N}$-graded structure out of the $\mathbb{N}^2$-graded structure:

(1) $(1, 0)$-graded structure: the homogeneous component of degree $i \in \mathbb{N}$ is given by $R^{(i,*)} = \bigoplus_j R_{(i,j)}$. The degree 0 part is $R^{(0,*)}$ and the ideal of the homogeneous elements of positive degree is $Q_{(1,0)}R = (x_1, \dots, x_n)$.
(2) (0, 1)-graded structure: the homogeneous component of degree $j \in \mathbb{N}$ is given by $R^{(*,j)} = \bigoplus_i R_{(i,j)}$. The degree 0 part is $R^{(*,0)}$ and the ideal of the homogeneous elements of positive degree is $Q_{(0,1)}R = (y_1, \dots, y_m)$.
(3) total degree: the homogeneous component of degree $u \in \mathbb{N}$ is $\bigoplus_{i+j=u} R_{(i,j)}$. The degree 0 part is $R_{(0,0)}$ and the ideal of the homogeneous elements of positive degree is $(x_1, \dots, x_n, y_1, \dots, y_m)$.

In the same way, any $\mathbb{Z}^2$-graded R-module $M = \bigoplus M_{(i,j)}$ can be turned into a $\mathbb{Z}$-graded module by regrading it with respect to the $(1, 0)$-grading or with respect to the (0, 1)-grading or with respect to the total degree.

We may hence consider the Castelnuovo–Mumford regularity of M with respect to any of these different graded structures. To distinguish them we will denote by $\operatorname{reg}_{(1,0)} M$ the regularity of M with respect to the $(1, 0)$-graded structure and by $\operatorname{reg}_{(0,1)} M$ the regularity of M with respect to the (0, 1)-graded structure.

Given $i, j \in \mathbb{Z}$ we set $M^{(i,*)} = \bigoplus_v M_{(i,v)}$ and $M^{(*,j)} = \bigoplus_v M_{(v,j)}$. Clearly $M = \bigoplus_i M^{(i,*)}$ as a $R^{(0,*)}$-graded module and $M = \bigoplus_j M^{(*,j)}$ as an $R^{(*,0)}$-graded module. Also, it is simple to check that, if M is a finitely generated $\mathbb{Z}^2$-graded module, then $M^{(i,*)}$ is a finitely generated $R^{(0,*)}$-graded module for all $i \in \mathbb{Z}$ and $M^{(*,j)}$ is a finitely generated $R^{(*,0)}$-graded module for all $j \in \mathbb{Z}$.

Let $S = R_{(0,0)}[X_1, \dots, X_n, Y_1, \dots, Y_m]$ with the $\mathbb{N}^2$-graded structure induced by the assignment $\deg X_i = (1, 0)$ and $\deg Y_j = (0, 1)$. We have:

Proposition 2.1 *Let M be a finitely generated $\mathbb{Z}^2$-graded R-module. Let $\mathbb{F}$ be a bigraded S-free minimal resolution of M. Let v_i be the largest integer v such that F_i has a minimal generator in degree $(v, *)$ and w_i be the largest integer w such that F_i has a minimal generator in degree $(*, w)$ Then we have*

$$\max\{\operatorname{reg} M^{(*,j)} : j \in \mathbb{Z}\} = \operatorname{reg}_{(1,0)} M = \max\{v_i - i : i = 0, \dots, n\},$$

$$\max\{\operatorname{reg} M^{(i,*)} : i \in \mathbb{Z}\} = \operatorname{reg}_{(0,1)} M = \max\{w_i - i : i = 0, \dots, m\},$$

where $\operatorname{reg} M^{(*,j)}$ *is the regularity as an* $R^{(*,0)}$*-graded module and* $\operatorname{reg} M^{(i,*)}$ *is the regularity as an* $R^{(0,*)}$*-graded module.*

Proof Set $Q = Q_{(1,0)}$, i.e. Q is the ideal of $R^{(*,0)}$ generated by $R_{(1,0)}$. The $(1,0)$-regularity of M is defined by means of the local cohomology $H^*_{QR}(M)$. We may regard M as an $R^{(*,0)}$-module, so that $H^c_{QR}(M) = H^c_Q(M) = \bigoplus_j H^c_Q(M^{(*,j)})$ for all c. This explains the first equality. For the second equality, by Theorem 1.2 $\operatorname{reg}_{(1,0)} M$ can be computed from any graded minimal free resolution of M as an $R^{(0,1)}[X_1, \dots, X_n]$-module but we have observed in Remark 1.3 that it can be as well computed from any minimal free resolution of M as an S-module. So a minimal bigraded resolution of M as S-modules serves to compute both the $(1,0)$ and the $(0,1)$ regularity. □

3 A Non-standard $\mathbb{Z}^2$-Grading

For later applications we will consider in this section a polynomial ring

$$A = A_0[Y_1, \dots, Y_g]$$

over a ring A_0 with a (non-standard) $\mathbb{Z}^2$-graded structure given by

$$\deg Y_j = (d_j, 1)$$

where $d_1, \dots, d_g \in \mathbb{N}$.

For every $\mathbb{Z}^2$-graded A-module $N = \bigoplus N_{(i,v)}$ and for every $v \in \mathbb{Z}$ we set

$$\rho_N(v) = \sup\{i \in \mathbb{Z} : N_{(i,v)} \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

We will study the behaviour of $\rho_N(v)$ as a function of v. We start with two general facts.

Lemma 3.1 *Given an chain of submodules* $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_p = N$ *of* $\mathbb{Z}^2$*-graded A modules one has* $\rho_N(v) = \max\{\rho_{N_i/N_{i-1}}(v) : i = 1, \dots, p\}$ *for all* v.

Proof The function $\rho_N(v)$ behaves well on short exact sequences with maps of degree 0. Then the statement follows by induction on p using the short exact sequences associated to the chain of submodules. □

Let F be a finitely generated $\mathbb{Z}^2$-graded free A-module with basis $e_1, \dots, e_p$ and let $<$ be a monomial order on F. For every $\mathbb{Z}^2$-graded A-submodule U of F we denote by $\operatorname{in}_<(U)$ the A_0-submodule of F generated by leading monomials (with coefficients!) of the non-zero elements in U. Since U is an A-submodule of F, it turns out that $\operatorname{in}_<(U)$ is an A-submodule of F as well. Furthermore for every monomial $aY^\alpha e_i$ in $\mathrm{in}_<(U)$ there exists an element $u \in U$ such that $\mathrm{in}_<(u) = aY^\alpha e_i$. One has:

Lemma 3.2 $\rho_{F/U}(v) = \rho_{F/\mathrm{in}_<(U)}(v)$ *for all* v.

Proof It is enough to prove that, given (i, v), one has $U_{(i,v)} = F_{(i,v)}$ if and only if $\mathrm{in}_<(U_{(i,v)}) = F_{(i,v)}$ The "only if" implication is obvious. For the "if" implication, we argue by contradiction. Suppose $\mathrm{in}_<(U_{(i,v)}) = F_{(i,v)}$ and $U_{(i,v)} \neq F_{(i,v)}$. Let $Y^\alpha e_i$ be the smallest (with respect to the monomial order) monomial of degree (i, v) which is not in $U_{(i,v)}$. Since $Y^\alpha e_i \in \mathrm{in}_<(U_{(i,v)})$ there exists $u \in U$ such that $\mathrm{in}_<(u) = Y^\alpha e_i$. We may assume that u is homogeneous of degree (i, v). If not, we simply replace u with the homogeneous component of u of degree (i, v) which is in U since U is graded. So we have $u = Y^\alpha e_i + u_1$ where u_1 is a A_0-linear combination of monomials of degree (i, v) that are $< Y^\alpha e_i$. Hence, by assumption, $u_1 \in U_{(i,v)}$. It follows that $Y^\alpha e_i = u - u_1 \in U_{(i,v)}$, a contradiction. □

The fact that A has no elements of degree $(i, 0) \in \mathbb{Z}^2$ with $i \neq 0$ has an important consequence.

Lemma 3.3 *Let N be a $\mathbb{Z}^2$-graded and finitely generated A-module. Then $\rho_N(v)$ is eventually either a linear function of v with leading coefficient in $\{d_1, \ldots, d_g\}$ or $-\infty$.*

Proof First we observe that if n is a generator of N of degree, say, $(\alpha, \beta) \in \mathbb{Z}^2$, then $Y_1^{\alpha_1} \cdots Y_g^{\alpha_g} n$ has degree $(\sum_j \alpha_j d_j + \alpha, \sum_j \alpha_j + \beta)$. Hence $N_{(i,v)}$ is non-zero only if $(i, v) = (\sum_j \alpha_j d_j + \alpha, \sum_j \alpha_j + \beta)$ for some $(\alpha_1, \ldots, \alpha_g) \in \mathbb{N}^g$ and some (α, β) degree of a minimal generator of N. If we set $D = \max\{d_1, \ldots, d_g\}$, then $N_{(i,v)} \neq 0$ implies $\alpha \leq i \leq (v - \beta)D + \alpha$ for some degree (α, β) of a minimal generator of N. As the module N is finitely generated, it follows that $\{i \in \mathbb{Z} : N_{(i,v)} \neq 0\}$ is finite for every $v \in \mathbb{Z}$. To prove that $\rho_N(v)$ is either eventually linear in v or $-\infty$, we present N as F/U where F is a finitely generated A-free bigraded module and U is a bigraded A-submodule of F. Let $<$ be a monomial order on F. Then $\rho_{F/U}(v) = \rho_{F/\mathrm{in}_<(U)}(v)$. Hence we may assume right away that U is generated by monomials (with coefficients). We can consider a bigraded chain of submodules

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_p = N$$

with cyclic quotients $C_i = N_i/N_{i-1}$ annihilated by a monomial prime ideal, i.e., an ideal of the form $pA + J$ where p is a prime ideal of A_0 and J is an ideal generated by a subset of the variables $Y_1, \ldots, Y_g$. It follows that

$$\rho_N(v) = \max\{\rho_{C_i}(v) : i = 1, \ldots, p\}.$$

Since the maximum of finitely many eventually linear functions in one variable is an eventually linear function, it is enough to prove the statement for each C_i. That is, we may assume that, up to a shift $(-w_1, -w_2) \in \mathbb{Z}^2$, the module N has the form A/P with $P = pA + J$ where p is a prime ideal of A_0 and J is generated by a subset of the variables. With $G = \{i : Y_i \notin P\}$, we have

$$\rho_N(v) = \begin{cases} \max\{d_i : i \in G\}(v - w_2) + w_1 & \text{if } G \neq \emptyset \text{ and } v \geq w_2, \\ -\infty & \text{if } G = \emptyset \text{ and } v > w_2. \end{cases}$$

□

4 Regularity and Powers

We return to the notation of Sect. 1. For a finitely generated graded R-module M and a homogeneous ideal I of R we will study the behaviour of $\operatorname{reg}(I^v M)$ as a function of $v \in \mathbb{N}$. For simplicity we will assume throughout that $I^v M \neq 0$ for every v. Let us consider the Rees algebra $\operatorname{Rees}(I)$ of I:

$$\operatorname{Rees}(I) = \bigoplus_{v \in \mathbb{N}} I^v$$

with its natural bigraded structure given by

$$\operatorname{Rees}(I)_{(i,v)} = (I^v)_i.$$

The Rees module of the pair I, M

$$\operatorname{Rees}(I, M) = \bigoplus_{v \in \mathbb{N}} I^v M$$

is clearly a finitely generated $\operatorname{Rees}(I)$-module naturally bigraded by

$$\operatorname{Rees}(I, M)_{(i,v)} = (I^v M)_i.$$

Let $f_1, \ldots, f_g$ be a set of minimal homogeneous generators of I of degrees, say, $d_1, \ldots, d_g \in \mathbb{N}$. We may present $\operatorname{Rees}(I)$ as a quotient of

$$B = R[Y_1, \ldots, Y_g]$$

via the map

$$\psi : B \to \operatorname{Rees}(I), \quad Y_i \to f_i \in I_{d_i} = \operatorname{Rees}(I)_{(d_i,1)}.$$

Actually B is naturally bigraded if we assign bidegree $(i, 0)$ to $x \in R_i$ as an element of B and by set $\deg Y_j = (d_j, 1)$.

Consider the extension $Q_R B$ of Q_R to B and the Koszul homology

$$H(Q_R B, \text{Rees}(I, M)) = H(Q_R, \text{Rees}(I, M)) = \bigoplus_{v \in \mathbb{N}} H(Q_R, I^v M).$$

Since $Q_R H(Q_R, \text{Rees}(I, M)) = 0$ the module $H(Q_R, \text{Rees}(I, M))$ acquires naturally the structure of finitely generated $\mathbb{Z}^2$-graded $B/Q_R B$-module. Here

$$B/Q_R B = R_0[Y_1, \ldots, Y_g]$$

has a bigraded structure defined in Sect. 3. Now for $i = 0, \ldots, n$ we let

$$t_i(M) = \sup\{j : H_i(Q_R, M)_j \neq 0\}.$$

We have:

Theorem 4.1 *Let I be a homogeneous ideal of R minimally generated by homogeneous elements of degree $d_1, \ldots, d_g$ and M be a finitely generated graded R-module. Then there exist $\delta \in \{d_1, \ldots, d_g\}$ and $c \in \mathbb{Z}$ such that*

$$\text{reg}(I^v M) = \delta v + c \textit{ for } v \gg 0.$$

Proof For $i = 0, \ldots, n$ consider the i-th Koszul homology module:

$$H_i = H_i(Q_R, \text{Rees}(I, M)) = \bigoplus_{v \in \mathbb{N}} H_i(Q_R, I^v M).$$

As already observed H_i is a finitely generated $\mathbb{Z}^2$-graded $B/Q_R B$-module. Furthermore $\rho_{H_i}(v) = t_i(I^v M)$. Therefore we may apply Lemma 3.3 and have that either $H_i(Q_R, I^v M) = 0$ for large v or $t_i(I^v M)$ is a linear function of v for large v with leading coefficient in $\{d_1, \ldots, d_g\}$. As $\text{reg}(I^v M) = \max\{t_i(I^v M) - i : i = 0, \ldots, n\}$ we may conclude that $\text{reg}(I^v M)$ is eventually a linear function in v with leading coefficient in $\{d_1, \ldots, d_g\}$. □

Theorem 4.1 has been proved in [11] and [17] when R is a polynomial ring over a field and in [21] for general base rings. Our proof is a modification (and a slight simplification) of the one given in [11]. Here and also in Sect. 2 our work was largely inspired by the papers of Chardin on the subject, in particular by [7–10]. The δ appearing in Theorem 4.1 can be characterized in terms of minimal reductions, see [17, 21] for details. The nature of the others invariants arising from Theorem 4.1, i.e., the constant term c and the least v_0 such that the formula holds for each $v \geq v_0$, have been deeply investigated in [1, 8, 10, 14, 15] and are relatively well understood in small dimension but remain largely unknown in general.

5 Linear Powers

Assume now that the minimal generators of I have all degree d and that the minimal generators of M have all degree d_0. Hence $I^v M$ is generated by elements of degree $vd + d_0$ and therefore $\operatorname{reg}(I^v M) \geq vd + d_0$ for every v.

Definition 5.1 We say that I has linear powers with respect to M if $\operatorname{reg}(I^v M) = vd + d_0$ for every v.

When R_0 is a field, I has linear powers with respect to M if and only if for every v the matrices representing the maps in the minimal S-free resolution of $I^v M$ have entries of degree 1.

We will give a characterization of linear powers in terms of the homological properties of the Rees module $\operatorname{Rees}(I, M)$. Note that, under the current assumptions, $\operatorname{Rees}(I)$ and $\operatorname{Rees}(I, M)$ can be given a compatible and "normalized" $\mathbb{Z}^2$-graded structure:

$$\operatorname{Rees}(I)_{(i,v)} = (I^v)_{vd+i},$$
$$\operatorname{Rees}(I, M)_{(i,v)} = (I^v M)_{vd+d_0+i}.$$

From the presentation point of view, this amounts to set $\deg Y_i = (0, 1)$ so that $B = R[Y_1, \ldots, Y_g]$ is a $\mathbb{Z}^2$-graded R_0-algebra with generators in degree $(1, 0)$, the elements of R_1, and in degree $(0, 1)$, the Y_i's. With the notations introduced in Sect. 2, we have that $\operatorname{Rees}(I, M)^{(*,v)} = (I^v M)(vd+d_0)$. So, applying Proposition 2.1:

$$\operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M) = \max\{\operatorname{reg} \operatorname{Rees}(I, M)^{(*,v)} : v \in \mathbb{N}\} = \max\{\operatorname{reg} I^v M - vd - d_0 : v \in \mathbb{N}\}.$$

Summing up we have:

Theorem 5.2

(1) $\operatorname{reg} I^v M \leq vd + d_0 + \operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M)$ *for all v and the equality holds for at least one v.*

(2) I has linear powers with respect to M if and only if $\operatorname{reg}_{(1,0)} \operatorname{Rees}(I, M) = 0$.

When R is the polynomial ring over a field and $M = R$ Theorem 5.2 part (2) has been proved in [5] extending earlier results of Römer [20].

Theorems 5.2 and 4.1 have been generalized to the case where the single ideal I is replaced with a set of ideals $I_1, \ldots, I_p$ and one looks at the regularity $\operatorname{reg}(I_1^{v_1} \cdots I_p^{v_p} M)$ as a function of $(v_1, \ldots, v_p) \in \mathbb{N}^p$. The main difference is that $\operatorname{reg}(I_1^{v_1} \cdots I_p^{v_p} M)$ is (only) a piecewise linear function unless each ideal I_i is generated in a single degree, see [3, 4, 16] for details.

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The Eisenbud-Green-Harris Conjecture

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1 An Introduction to the Conjecture

A very important problem in Commutative Algebra is the study of the growth of the Hilbert function of an ideal in a given degree *if one knows more than one step of* [its] *history*, cit. Mark Green [28]. A classical theorem, due to Macaulay [36], answers this question by providing an estimate on the Hilbert function in a given degree just by knowing its value in the previous one. This result is very useful, but it is far from being optimal. For instance, there is no way of taking into account any additional information about the ideal. The Eisenbud-Green-Harris, henceforth EGH, Conjecture was first raised in [17, 18], and precisely addresses this matter. By effectively using the additional data that the given ideal contains a regular sequence, it predicts for instance more accurate growth bounds.

We will now introduce some notation and terminology in order to state the EGH Conjecture. Throughout this article, $A = \bigoplus_{d \geqslant 0} A_d$ will denote a standard graded polynomial ring $K[x_1, \dots, x_n]$ over a field K, and $\mathfrak{m} = (x_1, \dots, x_n)$ its homogeneous maximal ideal. We consider A equipped with the lexicographic order $\geqslant$ induced by $x_1 > x_2 > \dots > x_n$. Given polynomials $g_1, \dots, g_s \in A$, we will denote by $\langle g_1, \dots, g_s \rangle$ the K-vector space generated by such elements to distinguish it from the ideal that they generate, which we denote by $(g_1, \dots, g_s)$. We denote the

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Hilbert function of a graded module M and its value in d by $H(M)$ and $H(M; d)$, respectively. On the set of Hilbert functions we consider the partial order given by point-wise inequality. Recall that a K-vector space $V \subseteq A_d$ is called *lex-segment* if there exists a monomial $v \in V$ such that $V = \langle u \in A_d \mid u \text{ monomial}, u \geq v \rangle$.

The classical Macaulay Theorem states that, given any homogeneous ideal I, if one lets $L_d \subseteq A_d$ be the lex-segment of dimension equal to $H(I; d)$, then $\mathrm{Lex}(I) = \bigoplus_{d\geqslant 0} L_d$ is an ideal, that we call *lex-ideal*. In order to take into account that I contains a regular sequence, we will introduce the so-called lex-plus-powers ideals.

Given an integer $0 < r \leqslant n$, we let $\underline{a} = (a_1, \dots, a_r)$ denote an ordered sequence of integers $0 < a_1 \leqslant \dots \leqslant a_r$, and we call it a *degree sequence*. We call the ideal $\mathfrak{a} = (x_1^{a_1}, \dots, x_r^{a_r}) \subseteq A$ the *pure-powers ideal of degree* $\underline{a}$. With any homogeneous ideal $I \subseteq A$ which contains an ideal $\mathbf{f}$ generated by a regular sequence $f_1, \dots, f_r$, of degree $\underline{a} = (a_1, \dots, a_r)$, we associate the K-vector space

$$\mathrm{LPP}^{\underline{a}}(I) = \bigoplus_{d\geqslant 0} \langle L_d + \mathfrak{a}_d \rangle,$$

where $L_d \subseteq A_d$ is the largest, hence unique, lex-segment which satisfies $H(I; d) = \dim_K \langle L_d + \mathfrak{a}_d \rangle$. As Macaulay Theorem proves that $\mathrm{Lex}(I)$ is an ideal, the EGH Conjecture predicts that $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal, which we call *the lex-plus-powers ideal associated with I with respect to the degree sequence* $\underline{a}$.

Conjecture 1.1 (EGH) Let $I \subseteq A$ be a homogeneous ideal that contains a homogeneous ideal $\mathbf{f}$ generated by regular sequence of degree $\underline{a}$. Then $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal.

Observe that the EGH Conjecture is a generalization of Macaulay Theorem, which corresponds to the case $\mathbf{f} = (f_1)$ with respect to any $0 \neq f_1 \in I$ of degree a_1. Just like lexicographic ideals, lex-plus-powers ideals enjoy several properties of extremality. For example, assuming that the EGH Conjecture is true in general, then one can show that the growth of $\mathrm{LPP}^{\underline{a}}(I)$ in each degree is smaller than that of I. That is, $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I)) \leqslant H(\mathfrak{m}I)$, see Lemma 2.14. This immediately translates into an inequality $\beta_{0j}(\mathrm{LPP}^{\underline{a}}(I)) \geqslant \beta_{0j}(I)$ between minimal number of generators in each degree j. We point out that the more refined version of such inequality, i.e.,

$$\beta_{ij}(\mathrm{LPP}^{\underline{a}}(I)) \geqslant \beta_{ij}(I) \qquad \text{for all } i, j,$$

is currently unknown in general, and goes under the name of LPP-Conjecture, see for instance [11, 19, 20, 37, 41, 44].

In the following, it will be useful to have several formulations of the EGH Conjecture, which we will use interchangeably at our convenience.

An equivalent way of approaching the conjecture is degree by degree: given a sequence $\underline{a}$, for a non-negative integer d we say that a homogeneous ideal $I \subseteq A = K[x_1, \dots, x_n]$ *satisfies* $\mathrm{EGH}_{\underline{a}}(d)$ if there exists an $\underline{a}$-lpp ideal J such that $\dim_K(J_d) = \dim_K(I_d)$ and $\dim_K(J_{d+1}) \leqslant \dim_K(I_{d+1})$. We say that I *satisfies*

$\text{EGH}_{\underline{a}}$ if it satisfies $\text{EGH}_{\underline{a}}(d)$ for all non-negative integers d. One can readily verify that Conjecture 1.1 holds true if and only if, for every degree sequence $\underline{a}$, every homogeneous ideal containing a regular sequence of degree $\underline{a}$ satisfies $\text{EGH}_{\underline{a}}$, see [9].

We conclude this introductory section by recalling a weaker version of the EGH Conjecture, raised in [18]. Let $\underline{a} = (a_1, \ldots, a_n)$ be a degree sequence, and D be an integer such that $a_1 \leqslant D \leqslant \sum_{i=1}^{n}(a_i - 1)$. Let b the unique integer such that $\sum_{i=1}^{b}(a_i - 1) \leqslant D < \sum_{i=1}^{b+1}(a_i - 1)$, and set $\delta = \sum_{i=1}^{b+1}(a_i - 1) - D + 1$ if $b < n$, and $\delta = 1$ otherwise.

Conjecture 1.2 (Cayley-Bacharach) Let $\mathbf{f} \subseteq A = K[x_1, \ldots, x_n]$ be an ideal generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_n)$, and $g \notin \mathbf{f}$ be a homogeneous element of degree $D \geqslant a_1$. Let $I = \mathbf{f} + (g)$, and e be the multiplicity of A/I. Then

$$e \leqslant \prod_{i=1}^{n} a_i - \delta \prod_{i=b+1}^{n} a_i .$$

Conjecture 1.2 has been studied by several researchers, from very different points of view; for instance, see [5, 24, 25, 31]. The validity of the EGH Conjecture in the case $r = n$ for almost complete intersections would imply Conjecture 1.2. For an explicit instance of this, see Example 5.6.

This survey paper is structured as follows: in Sect. 2 we treat the case when the given ideal already contains a pure-powers ideal, presenting a new proof of the Clements-Lindström Theorem. Section 3 is very brief, and collects some statements from the theory of linkage, together with a result which yields a reduction to the Artinian case. In Sect. 4 we present proofs of several cases of the conjecture, previously known in the literature. Finally, in Sect. 5 we collect some applications of the techniques and the results illustrated before, together with several examples.

2 Monomial Regular Sequences and the Clements–Lindström Theorem

The goal of this section is to prove the Clements-Lindström Theorem [14], a more general version of the Kruskal-Katona Theorem [33, 34]. The proof presented here relies on recovering a strong hyperplane restriction theorem for strongly-stable-plus-powers and lpp ideals due to Gasharov [21, 22], see also [10, Theorem 2.2]. Our strategy uses the techniques of [7], and is different from the standard one available in the literature [14, 39, 40].

Recall that a monomial ideal $J \subseteq A = K[x_1, \ldots, x_n]$ is called *strongly stable* if for every monomial $u \in J$ and any variable x_i which divides u, one has that

$x_i^{-1}x_j u \in J$ for all $1 \leqslant j \leqslant i$. The ideal J is said to be *$\underline{a}$-strongly-stable-plus-powers*, $\underline{a}$-spp or, simply, spp for short, if there exist a strongly stable ideal S and a pure power ideal $\mathfrak{a}$ of degree $\underline{a}$ such that $J = S + \mathfrak{a}$. Clearly, $\underline{a}$-lpp ideals are $\underline{a}$-spp.

Theorem 2.1 *Let $I \subseteq A$ be a homogeneous ideal that contains a pure-powers ideal $\mathfrak{a}$ of degree $\underline{a}$. Then*

(i) $\mathrm{LPP}^{\underline{a}}(I)$ *is an ideal.*
(ii) If I is $\underline{a}$-spp, then $H(I + (x_n^i)) \geqslant H(\mathrm{LPP}^{\underline{a}}(I) + (x_n^i))$ for all $i > 0$.

We first prove Theorem 2.1 (i) for $n = r = 2$. Since strongly stable ideals in two variables are lex-ideals, $\underline{a}$-spp ideals are automatically $\underline{a}$-lpp in this case.

We start by recalling a few properties of monomial ideals, which are special cases of more general results derived from linkage theory, that we will discuss in Sect. 3.

Let I be a monomial ideal that contains $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$. When we view I as a $K[x_1]$-module, we have a decomposition

$$I = \bigoplus_{i \geqslant 0} x_1^{d_i} K[x_1] \cdot x_2^i; \tag{2.1}$$

observe that, since I is an ideal, one has $d_i \geqslant d_{i+1}$ for all i. Also observe that I is spp if and only if $d_{i+1} + 1 \geqslant d_i$ for all i. Define the link $I^\ell = I^\ell_{\mathfrak{a}}$ of I with respect to the ideal $\mathfrak{a}$ to be the ideal $I^\ell = (\mathfrak{a} :_A I)$. Notice that $I^\ell = (x_1^{a_1-d_0}, x_2^{a_2}) \cap (x_1^{a_1-d_1}, x_2^{a_2-1}) \cap \cdots \cap (x_1^{a_1-d_{a_2-1}}, x_2)$ is an ideal generated by the monomials $x_1^{a_1-d_i}x_2^{a_2-1-i}$, $i = 0, \ldots, a_2 - 1$, and that as a $K[x_1]$-module can be written as

$$I^\ell = \left(\bigoplus_{i=0}^{a_2-1} x_1^{a_1-d_{a_2-1-i}} K[x_1] \cdot x_2^i \right) \oplus \left(\bigoplus_{i \geqslant a_2} K[x_1] \cdot x_2^i \right). \tag{2.2}$$

Remark 2.2 (1) It is immediate from (2.2) that $(I^\ell)^\ell = I$.
(2) The Hilbert function of I^ℓ is determined by that of I. More precisely, if we let $R = A/\mathfrak{a}$ and $s = a_1 + a_2 - 2$, then $H(R; d) = H(R/IR; d) + H(R/I^\ell R; s - d)$.
(3) The link of an $\underline{a}$-lpp ideal is again an $\underline{a}$-lpp ideal. Thus, we may as well prove that I^ℓ is $\underline{a}$-spp if I is $\underline{a}$-spp. To this end, consider the decomposition of I as in (2.1). Given any monomial $x_1^{b_1}x_2^{b_2} \in I^\ell$ with $1 \leqslant b_2 < a_2$, one just needs to show that $x_1^{b_1+1}x_2^{b_2-1} \in I^\ell$. By (2.2), it is enough to verify that $a_1 - d_i + 1 \geqslant a_1 - d_{i+1}$ for all i, which is equivalent to $d_{i+1} + 1 \geqslant d_i$ for all i. Finally, this is true for all i, because I is spp by assumption.

We are now ready to prove the case $n = 2$ of Theorem 2.1 (i).

Proposition 2.3 *Let $\underline{a} = (a_1, a_2)$, and $I \subseteq A = K[x_1, x_2]$ be a homogeneous ideal that contains $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$. Then* $\mathrm{LPP}^{\underline{a}}(I)$ *is an ideal.*

After taking any initial ideal, without loss of generality we may assume that I is monomial. In fact, this operation preserves its Hilbert function, and the initial ideal still contains $\mathfrak{a}$. Next, we give three different proofs of the above proposition.

In the first one, we make use of linkage.

Proof 1 We need to show that the K-vector space $\mathrm{LPP}^{\underline{a}}(I) = \bigoplus_{j\geqslant 0}\langle L_j + \mathfrak{a}_j\rangle$ is indeed an ideal, and we do so by proving that $\mathrm{LPP}^{\underline{a}}(I)$ agrees with an ideal J for all degrees $i \leqslant a_2 - 1$ and it agrees with an ideal J' for all degrees $i \geqslant a_2 - 1$. By Macaulay Theorem, there is a lex-ideal L with the same Hilbert function as I. Consider the $\underline{a}$-lpp ideal $J = L + \mathfrak{a}$. By construction, for all $j = 1, \dots, a_2 - 1$ one has $H(J; j) = H(L; j) = H(I; j)$.

Now we construct the ideal J' as follows. First consider the link $I^{\ell} = (\mathfrak{a} :_A I)$. Since $I^{\ell} \supseteq \mathfrak{a}$, again by Macaulay Theorem there exists a lexicographic ideal L' with the same Hilbert function as I^{ℓ}. Thus, the $\underline{a}$-lpp ideal $L' + \mathfrak{a}$ has the same Hilbert function as I^{ℓ} in degrees $j = 0, \dots, a_1 - 1$. We now let $J' = (L' + \mathfrak{a})^{\ell}$. By Remark 2.2 (2), J' is an lpp ideal and, by Remark 2.2 (3) its Hilbert function in degrees $j \geqslant a_2 - 1$ coincides with that of I. Therefore J' has the desired properties, and the proof is complete. □

In the second proof we use techniques borrowed from [37, Section 3], see also [7, Section 4].

Proof 2 The Hilbert function of a monomial ideal is independent of the base field, thus without loss of generality we may assume that $K = \mathbb{C}$. It suffices to construct an $\underline{a}$-spp ideal with the same Hilbert function as I. Let $\xi_1, \dots, \xi_{a_2}$ the a_2-roots of unity over $\mathbb{C}$, and observe that $x_2^{a_2} - x_1^{a_2} = (x_2 - \xi_1 x_1)(x_2 - \xi_2 x_1)\cdots(x_2 - \xi_{a_2} x_1) \in I$. We consider the distraction $\mathcal{D}$ given by a family of linear forms $\{l_i\}_{i\geqslant 1}$ defined as $l_i = x_2 - \xi_i x_1$, for $i = 1, \dots, a_2$, and $l_i = x_2$ for all $i > a_2$; see [3] for the theory of distractions. Given a decomposition of $I^{(0)} = I = \bigoplus_{i\geqslant 0} I_{[i]} x_2^i$, we let $J^{(0)}$ be the distracted ideal

$$J^{(0)} = J = \bigoplus_{i\geqslant 0} I_{[i]} \prod_{j=1}^{i} l_j = \bigoplus_{i=0}^{a_2} I_{[i]} \prod_{j=1}^{i} l_j \oplus \bigoplus_{i\geqslant a_2} K[x_1] \cdot x_2^i,$$

which shares with I the same Hilbert function, and the same Betti numbers as well. Observe that the last equality is due to the fact that both $x_1^{a_2}$ and $x_2^{a_2} - x_1^{a_2}$ are in J, and therefore $x_2^{a_2} \in J$. We let $I^{(1)}$ be $\mathrm{in}_{>}(J^{(0)})$, where $>$ is any monomial order such that $x_1 > x_2$, and $J^{(1)}$ be the ideal obtained by distracting $I^{(1)}$ with $\mathcal{D}$. We construct in this way a sequence $I^{(0)}, I^{(1)}, \dots, I^{(h)}$ of ideals with the same Hilbert function, each of which contains $\mathfrak{a}$; we finally want to show that this sequence eventually stabilizes at an ideal, we call it L, which is $\underline{a}$-spp. To this end, observe that for all integers $p \geqslant 0$ we have

$$H\left(I^{(h)}_{[0]} \oplus I^{(h)}_{[1]}x_2 \oplus \cdots \oplus I^{(h)}_{[p]}x_2^p\right) = H\left(\mathrm{in}_>(I^{(h)}_{[0]} \oplus I^{(h)}_{[1]}l_1 \oplus \cdots \oplus I^{(h)}_{[p]}\prod_{j=1}^{p} l_j)\right)$$
$$\leqslant H\left(I^{(h+1)}_{[0]} \oplus I^{(h+1)}_{[1]}x_2 \oplus \cdots \oplus I^{(h+1)}_{[p]}x_2^p\right). \tag{2.3}$$

In the above, we consider three modules whose Hilbert functions are computed as homogenous $K[x_1]$-submodules of the graded $K[x_1]$-module $A = K[x_1, x_2]$, where x_2^d has degree d. Notice that the inequality in (2.3) is due to the inclusion of the second module in the third one. Observe that $I^{(0)}_{[0]} \subseteq I^{(1)}_{[0]} \subseteq \ldots$ is an ascending chain of ideals that will eventually stabilize, say at $I^{(h_0)}_{[0]}$. Inductively, assume that for all $i = 0, \ldots, p-1$ the ideals in $I^{(h_{i-1})}_{[i]} \subseteq I^{(h_{i-1}+1)}_{[i]} \subseteq \ldots$ form a chain that stabilizes, say at h_i. The inclusion of the second into the third module of (2.3), for any $h > \max\{h_0, \ldots, h_{p-1}\}$, yields that $I^{(h)}_{[p]} \subseteq I^{(h+1)}_{[p]}$. Thus, for $h \geqslant h_{p-1}$ we have again a chain of ideals which will stabilize, say at h_p. Repeat this process for all $p \leqslant a_2 - 1$, so that for all $h \geqslant h' = \max\{h_1, \ldots, h_{a_2-1}\}$ we have $I^{(h)} = I^{(h+1)}$. Let $L = I^{(h')}$. Keeping in mind how L has been constructed, apply (2.3) to L to obtain, for all $p \geqslant 0$

$$L_{[0]} \oplus L_{[1]}x_2 \oplus \cdots \oplus L_{[p]}x_2^p = \mathrm{in}_>\left(L_{[0]} \oplus L_{[1]}l_1 \oplus \cdots \oplus L_{[p]}\prod_{j=1}^{p} l_j\right)$$
$$= L_{[0]} \oplus L_{[1]}l_1 \oplus \cdots \oplus L_{[p]}\prod_{j=1}^{p} l_j, \tag{2.4}$$

where the second equality can be verified by induction on p, using the first equality and the fact that the least monomial with respect to $>$ in the support of $\prod_{j=1}^{p} l_j$ is x_2^p.

Next, we prove that L is $\underline{a}$-spp. By construction $L \supseteq \mathfrak{a}$, since each $I^{(i)}$ and $J^{(i)}$ does; thus, we have to show that $x_1 L_{[p]} \subseteq L_{[p-1]}$ that for all $0 < p \leqslant a_2 - 1$. Again by induction on p, by (2.4) we have $L_{[0]} \oplus L_{[1]}x_2 = L_{[0]} \oplus L_{[1]}(x_2 - x_1)$, which implies $x_1 L_{[1]} \subseteq L_{[0]}$. Moreover, by induction and again by (2.4), $L_{[0]} \oplus L_{[1]}x_2 \oplus \cdots \oplus L_{[p]}x_2^p = L_{[0]} \oplus L_{[1]}x_2 \oplus \cdots \oplus L_{[p-1]}x_2^{p-1} \oplus L_{[p]}\prod_{j=1}^{p} l_j$. Since $l_j = x_2 - \xi_j x_1$ with $j = 1, \ldots, p$, we have that $\prod_{j=1}^{p} l_j$ has a full support, i.e., its support contains all of the monomials of degree p. In particular it contains $x_1 x_2^{p-1}$. It follows that $x_1 L_{[p]} \subseteq L_{[p-1]}$, as desired. □

The third proof relies on an application of Gotzmann Persistence Theorem [26, 27].

Proof 3 Let $\mathrm{LPP}^{\underline{a}}(I) = L + \mathfrak{a}$, where each L_d is the largest lex-segment such that $\dim_K(L_d + \mathfrak{a}_d) = H(I; d)$. In order to show that $\mathrm{LPP}^{\underline{a}}(I)$ is an ideal we have to show that, for every integer $d \geqslant 0$, we have $H(A/(\mathfrak{m}L + \mathfrak{a}); d + 1) \geqslant H(A/\mathrm{LPP}^{\underline{a}}(I); d + 1)$. For this, without loss of generality we can assume that $(\mathrm{LPP}^{\underline{a}}(I))_j = \mathfrak{a}_j$ for all $j < d$. Let $k = \dim_K \widetilde{L_d}$, where $\widetilde{L_d}$ is the image in $A/\mathfrak{a}$ of the K-vector space $L_d + \mathfrak{a}_d$. If $k = 0$ there is nothing to prove. Let us assume $k > 0$, and study the following three cases separately: $d < a_2 - 1$, $d = a_2 - 1$, and $d \geqslant a_2$. If $d < a_2 - 1$, then $(\mathrm{LPP}^{\underline{a}}(I))_d = L_d$, $(\mathrm{LPP}^{\underline{a}}(I))_{d+1} = L_{d+1}$, and the conclusion follows from Macaulay Theorem.

Now assume $d = a_2 - 1$. If $L_{d+1} = A_{d+1}$, then there is nothing to show, so assume that $L_{d+1} \subsetneq A_{d+1}$. If x_2^{d+1} is a minimal generator of I, then $H(A/(\mathfrak{m}L_d); d+1) \geqslant H(A/\mathfrak{m}I; d+1) \geqslant H(A/I; d+1)+1$. Since $H(A/I; d+1) = H(A/L; d+1) - 1$, it follows that $H(A/\mathfrak{m}L; d+1) \geqslant H(A/L; d+1)$, and therefore $\mathfrak{m}_1 L_d \subseteq L_{d+1}$. A fortiori, we have that $\mathfrak{m}_1(\mathrm{LPP}^{\underline{a}}(I))_d \subseteq (\mathrm{LPP}^{\underline{a}}(I))_{d+1}$, and the proof of this case is complete. If x_2^{d+1} is not a minimal generator of I, then $\dim(A/J) = 0$, where $J = I_{\leqslant d}$. In particular, $H(A/I; j) \leqslant H(A/(x_1^d, x_2^d); d) = d$. By Macaulay Theorem we have that $H(A/I; d+1) \leqslant d$. If equality holds, then I has no minimal generators in degree $d+1$, and thus $H(A/J; d+1) = d$ as well. By Gotzmann Persistence Theorem applied to J, we have that $H(A/H; j) = d$ for all $j \geqslant d$, which contradicts the fact that $\dim(A/J) = 0$.

Finally, if $d \geqslant a_2$, we first observe that once again $k = H(A/I; d) \leqslant d$, and that $H(A/(\mathfrak{m}L_d) + \mathfrak{a}; d + 1) = k - 1$. Since $k = H(A/I; d)$, to conclude the proof it suffices to show that $k > H(A/I; d + 1)$, since the latter is equal to $H(A/\mathrm{LPP}^{\underline{a}}(I); d + 1)$. It follows from Macaulay Theorem $H(A/I; d + 1) \leqslant k = H(A/I; d)$, since we have already observed that $k \leqslant d$. If equality holds, then by Gotzmann Persistence Theorem applied to the ideal $J = I_{\leqslant d}$ we would have that $H(A/J; j) = H(A/J; d) = k > 0$ for all $j \geqslant d$. In particular, this would imply that $\dim(A/J) > 0$, in contrast with the fact that J contains $(x_1^{a_1}, x_2^{a_2})$, and hence it is Artinian. □

Remark 2.4

(1) Observe that Proof 1 can be adapted to any regular sequence of degree $\underline{a} = (a_1, a_2)$ using properties of linkage analogous to those of Remark 2.2, see Theorem 4.1.
(2) It is easy to see that, in Proof 2, we can also keep track of Betti numbers and prove, in characteristic zero, that they cannot decrease when passing to the lex-plus-powers ideal.
(3) In Proof 3 we do not actually use the fact that the regular sequence is monomial. In fact, the same argument can be used to prove that any ideal which contains a regular sequence of degree $\underline{a} = (a_1, a_2)$ satisfies $\mathrm{EGH}_{\underline{a}}$.

We now move our attention from the case $n = 2$ to the general one.

Proposition 2.5 *Under the same assumptions of Theorem 2.1, there exists an $\underline{a}$-spp ideal with the same Hilbert function as that of I.*

Proof We define a total order on the set $\mathcal{S}$ of monomial ideals with the same Hilbert function as I, and which contain the pure-powers ideal $\mathfrak{a} = (x_1^{a_1}, \dots, x_r^{a_r})$. First, given any $J \in \mathcal{S}$, we order the set of its monomials $\{m_i\}$ from lower to higher degrees, and monomials of the same degree lexicographically. Now, given a second ideal $J' \in \mathcal{S}$ and the set of its monomials $\{m'_i\}$, we set $J > J'$ if and only if there exists i such that $m_j = m'_j$ for all $j \leqslant i$ and $m_{i+1} > m'_{i+1}$. Observe that, since J and J' have the same Hilbert function, we are forced to have $\deg m_j = \deg m'_j$ for all j. Let P be the maximal element of $\mathcal{S}$; we claim that P is $\underline{a}$-spp. Assume by contradiction that there exists a monomial $m \in P \setminus \mathfrak{a}$ such that x_i divides m and $x_i^{-1}x_j m \notin P$ for some $j < i$. Write $P = \bigoplus_q P_q \cdot q$, where each $q \in K[x_1, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_n]$ is a monomial, and $P_q \subseteq K[x_j, x_i]$ is an ideal. Notice that each P_q contains $(x_j^{a_j}, x_i^{a_i})K[x_j, x_i]$ since $P \in \mathcal{S}$, and that $P_q \subseteq P_{q'}$ whenever q divides q' since P is an ideal. By Proposition 2.3, for every q there exists an (a_j, a_i)-spp ideal $Q_q \subseteq K[x_j, x_i]$ with the same Hilbert function as P_q.

Let now $Q = \bigoplus_q Q_q \cdot q$, and observe that $Q \in \mathcal{S}$. In fact, Q is clearly spanned by monomials, and it contains $\mathfrak{a}$. Moreover, if q divides q' one gets $H(Q_q) = H(P_q) \leqslant H(P_{q'}) = H(Q_{q'})$. Since Q_q and $Q_{q'}$ are both (a_j, a_i)-spp, it follows that $Q_q \subseteq Q_{q'}$, which in turn that Q is an ideal. Since P is not $\underline{a}$-spp, by our choice of the indices i and j there exists q such that P_q is not (a_j, a_i)-spp. In particular, it follows that $Q > P$, which contradicts maximality of P. □

Remark 2.6 As in the case of two variables, see Remark 2.4 (2), in the proof of Proposition 2.5 one can keep track of how the Betti numbers change in order to prove that, in characteristic zero, the Betti numbers of the $\underline{a}$-spp ideal we obtain cannot decrease. This fact is helpful in order to prove the LPP-Conjecture for ideals that contain pure-powers.

We point out that, in all pre-existing proofs of Clements-Lindström Theorem 2.1 [14, 39, 40], one finds a preliminary reduction step that goes under the name of *compression*. This step consists of assuming that Clements-Lindström Theorem holds in $n-1$ variables in order to construct an $\underline{a}$-spp ideal $J \subseteq A$ in n variables that, for any $i = 1, \dots, n$, has a decomposition $J = \bigoplus_{j \geqslant 0} J_{[j]} x_i^j$, where $J_{[j]}$ is $(a_1, \dots, \widehat{a_i}, \dots, a_r)$-lpp for all j. In our proof, this step corresponds to the reduction provided by Proposition 2.5. Observe that the above ideal J is not necessarily $\underline{a}$-lpp globally in n variables, as the following example shows.

Example 2.7 Let $n \geqslant 4$ and consider the $(2,2)$-spp ideal $I = (x_1^2, x_1x_2, \dots, x_1x_{n-1}, x_2^2, x_2x_3)$ in $A = K[x_1, \dots, x_n]$; then I is compressed, but not $(2,2)$-lpp, since the monomial x_1x_n is missing from its generators.

We introduce some notation and terminology, which will be used henceforth in this section.

Let $A = K[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$, $\underline{a} = (a_1, \dots, a_r)$ be a degree sequence, and $\mathfrak{a} = (x_1^{a_1}, \dots, x_r^{a_r})$ be the corresponding pure-powers ideal. Furthermore, let $\overline{A} = K[x_1, \dots, x_{n-1}]$, and $\overline{\mathfrak{m}} = (x_1, \dots, x_{n-1})\overline{A}$. If $r < n$, we let $\overline{\underline{a}} = \underline{a}$ and $\overline{\mathfrak{a}} = (x_1^{a_1}, \dots, x_r^{a_r})\overline{A}$. Otherwise, if $r = n$, we let $\overline{\underline{a}} = (a_1, \dots, a_{n-1})$ and $\overline{\mathfrak{a}} = (x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}})\overline{A}$.

Given a K-vector space $V \subseteq A_d$ generated by monomials, we say that V is *$\underline{a}$-lpp* if it is the truncation in degree d of an $\underline{a}$-lpp ideal. Similarly, we say that V is *$\underline{a}$-spp* if it is the truncation in degree d of an $\underline{a}$-spp ideal. Observe that a K-vector subspace $V = \bigoplus_{i=0}^{d} V_{[d-i]}x_n^i \subseteq A_d$ containing $\mathfrak{a}_d$ is $\underline{a}$-spp if and only if $V_{[i]}$ is $\overline{\underline{a}}$-spp for all i, and $\overline{\mathfrak{m}}_1 V_{[i]} \subseteq V_{[i+1]}$ for all $i \geqslant \max\{d - a_n + 1, 0\}$; we will refer to the latter property as *stability*. Moreover, if $V \subseteq A_d$ is $\underline{a}$-lpp, respectively $\underline{a}$-spp, then $\mathfrak{m}_1 V + \mathfrak{a}_{d+1}$ is also $\underline{a}$-lpp, respectively $\underline{a}$-spp. Finally, if $V, W \subseteq A_d$ are $\underline{a}$-lpp and $\dim_K(V) \leqslant \dim_K(W)$, then $V \subseteq W$.

Let $L \subseteq \overline{A}_d$ be a lex-segment and $V = L + \overline{\mathfrak{a}}_d$. If $V \neq \overline{A}_d$, there exists the largest monomial $u \in \overline{A}_d \smallsetminus V$ with respect to the lexicographic order. In this case, we let $V^+ = V + \langle u \rangle$; otherwise, we let $V^+ = V = A_d$. Either way, V^+ can be written as $L' + \overline{\mathfrak{a}}_d$, where L' is a lex-segment, and therefore it is $\underline{a}$-lpp.

If $V \neq \overline{\mathfrak{a}}_d$ we may write $V = W \oplus \overline{\mathfrak{a}}_d$, with $W \neq 0$ a vector space minimally generated by monomials $m_1 \geqslant m_2 \geqslant \dots \geqslant m_t$. In this case, we let $V^- = \langle m_1, \dots, m_{t-1} \rangle + \overline{\mathfrak{a}}_d$; otherwise, we set $V^- = V = \overline{\mathfrak{a}}_d$.

The notion of segment we recall next is extracted from [7], and it will be crucial in the proof of Theorem 2.1.

Definition 2.8 Let $V \subseteq A_d$ be a K-vector space, written as $V = \bigoplus_{i=0}^{d} V_{[d-i]}x_n^i$. Then, V is called an *$\underline{a}$-segment*, or simply a *segment*, if it is $\underline{a}$-spp and, for all i,

1. $V_{[i]} \subseteq \overline{A}_i$ is $\overline{\underline{a}}$-lpp, and
2. $V_{[i+j]} \subseteq \overline{\mathfrak{m}}_j (V_{[i]})^+ + \overline{\mathfrak{a}}_{i+j}$ for all $1 \leqslant j \leqslant d - i$.

Note that, if $V \subseteq A_d$ is $\underline{a}$-lpp, then it is an $\underline{a}$-segment.

Remark 2.9 If $V \subseteq A_d$ is a segment, it immediately follows from the definition that $\mathfrak{m}_1 V + \mathfrak{a}_{d+1} \subseteq A_{d+1}$ is also an $\underline{a}$-segment.

Lemma 2.10 *Let V and W be two $\underline{a}$-segments in A_d. Then either $V \subseteq W$, or $W \subseteq V$.*

Proof Write $V = \bigoplus_{i=0}^{d} V_{[d-i]}x_n^i$ and $W = \bigoplus_{i=0}^{d} W_{[d-i]}x_n^i$. If the conclusion is false, since V and W are segments we can find $i \neq j$ such that $V_{[i]} \subsetneq W_{[i]}$ and $V_{[j]} \supsetneq W_{[j]}$; say $j < i$. Since $V_{[j]}$ is lpp, $V_{[j]} \supseteq (W_{[j]})^+$, and therefore $V_{[i]} = V_{[i]} + \overline{\mathfrak{a}}_i \supseteq \overline{\mathfrak{m}}_{i-j} V_{[j]} + \overline{\mathfrak{a}}_i \supseteq \overline{\mathfrak{m}}_{i-j}(W_{[j]})^+ + \overline{\mathfrak{a}}_i \supseteq W_{[i]}$, which is a contradiction. □

Definition 2.11 Let $V \subseteq A_d$ be a K-vector space, written as $V = \bigoplus_{i=0}^{d} V_{[d-i]}x_n^i$. We define the dimension sequence $\underline{\delta}(V) = (\dim_K(V_{[d]}), \dim_K(V_{[d]} \oplus V_{[d-1]}), \dots, \dim_K(V)) \in \mathbb{N}^{d+1}$. On the set of all such sequences, we consider the partial order given by point-wise inequality.

Lemma 2.12 *Let $V \subseteq A_d$ be an $\underline{a}$-spp K-vector space, written as $V = \bigoplus_{i=0}^{d} V_{[d-i]} x_n^i$. Assume that*

1. *$V_{[i]} \subseteq \overline{A}_i$ is $\underline{a}$-lpp for all i, and*
2. *$\underline{\delta}(V)$ is minimal among all dimension sequences of $\underline{a}$-spp K-vector subspaces $W = \bigoplus_{i=0}^{d} W_{[d-i]} x_n^i \subseteq A_d$ such that $\dim_K(W) = \dim_K(V)$ and $W_{[i]}$ is $\underline{a}$-lpp for all i.*

Then, V is a segment.

Proof Assume that V is not a segment; then, there exist $i < j$ such that $V_{[i]} \subsetneq \overline{A}_i$ and $\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j \not\supseteq V_{[j]}$, and choose i and j so that $j - i$ is minimal. Observe that necessarily $i \geqslant \max\{d - a_n + 1, 0\}$, since otherwise $V_{[i]} = \overline{A}_i$. Since $V_{[i]}$ and $V_{[j]}$ are $\underline{a}$-lpp, the fact that $\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j$ does not contain $V_{[j]}$ implies that $\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j$ is properly contained in $V_{[j]}$. In particular, the latter properly contains $\overline{\mathfrak{a}}_j$, and we have that

$$\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j \subseteq (V_{[j]})^-. \tag{2.5}$$

Now, define $W = \bigoplus_{k=0}^{d} W_{[d-k]} x_n^k$, where $W_{[i]} = (V_{[i]})^+$, $W_{[j]} = (V_{[j]})^-$, and $W_{[k]} = V_{[k]}$ for all $k \neq j, i$. We claim that W is an $\underline{a}$-spp vector space.

In fact, let $k \geqslant \max\{d - a_n + 1, 0\}$; by stability, if $k \neq j, j-1, i, i-1$, then $\overline{\mathfrak{m}}_1 W_{[k]} = \overline{\mathfrak{m}}_1 V_{[k]} \subseteq V_{[k+1]} = W_{[k+1]}$; if $k = j$, then $\overline{\mathfrak{m}}_1 W_{[j]} \subseteq \overline{\mathfrak{m}}_1 V_{[j]} \subseteq V_{[j+1]} = W_{[j+1]}$ and if $k = i - 1$, then $\overline{\mathfrak{m}}_1 W_{[i-1]} = \overline{\mathfrak{m}}_1 V_{[i-1]} \subseteq V_{[i]} \subseteq W_{[i]}$.

By costruction, we have that $\overline{\mathfrak{m}}_{j-i} W_{[i]} \subseteq W_{[j]}$, see (2.5); therefore, if $j - i = 1$ we are done, again by stability.

Thus, we may assume that $j - i > 1$ and prove next that $\overline{\mathfrak{m}}_{k-i} W_{[i]} + \overline{\mathfrak{a}}_k = W_{[k]}$ for all $i < k < j$. Since $j - i$ is minimal, we have that $\overline{\mathfrak{m}}_{k-i} W_{[i]} + \overline{\mathfrak{a}}_k = \overline{\mathfrak{m}}_{k-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_k \supseteq V_{[k]} = W_{[k]}$. If the containment were strict, then we would have $\overline{\mathfrak{m}}_{k-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_k \supseteq (V_{[k]})^+$ and, again by minimality, $\overline{\mathfrak{m}}_{j-k}(V_{[k]})^+ + \overline{\mathfrak{a}}_j \supseteq V_{[j]}$; this would in turn imply $\overline{\mathfrak{m}}_{j-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_{j-k}\left(\overline{\mathfrak{m}}_{k-i}(V_{[i]})^+ + \overline{\mathfrak{a}}_k\right) + \overline{\mathfrak{a}}_j \supseteq V_{[j]}$, contradicting our initial assumption on i and j.

The only case left to be shown is now $\overline{\mathfrak{m}}_1 W_{[j-1]} \subseteq W_{[j]}$. By applying what we have proved above for $k = j - 1$, we have that $\overline{\mathfrak{m}}_1 W_{[j-1]} + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_1\left(\overline{\mathfrak{m}}_{j-1-i} W_{[i]}\right) + \overline{\mathfrak{a}}_j = \overline{\mathfrak{m}}_{j-i} W_{[i]} + \overline{\mathfrak{a}}_j \subseteq W_{[j]}$, as desired.

Thus, W is an $\underline{a}$-spp vector space; furthermore, it is clear from definition that each $W_{[i]}$ is an $\underline{a}$-lpp. Finally, observe that $\underline{\delta}(W) < \underline{\delta}(V)$ by construction, which contradicts the minimality of $\underline{\delta}(V)$, and we are done. □

Proposition 2.13 *For every $d \geqslant 0$ and every $D \leqslant \dim_K(A_d)$ there exists a unique segment V with $\dim_K(V) = D$. Moreover, the sequence $\underline{\delta}(V)$ is the minimum of the set of all sequences $\underline{\delta}(W)$ of $\underline{a}$-spp vector spaces $W = \bigoplus_{i=0}^{d} W_{[d-i]} x_n^i \subseteq A_d$ which have dimension D and such that each $W_{[i]}$ is $\underline{a}$-lpp.*

Proof By Lemma 2.12 we have that any vector space with minimal dimension sequence is a segment, and by Lemma 2.10 any two such segments are comparable, hence equal. □

We already mentioned before that, if the EGH Conjecture held in full generality, then $\mathrm{LPP}^{\underline{a}}(I)$ would be the ideal with minimal growth among those containing a regular sequence of degree $\underline{a}$, and with Hilbert function equal to that of I. The proof is easy and we include it here.

Proposition 2.14 *Assume that EGH holds true. Let $I \subseteq A$ be a homogeneous ideal that contains a regular sequence of degree $\underline{a}$. Then $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I)) \leqslant H(\mathfrak{m}I)$.*

Proof Let $d \geqslant 0$ be an integer, and let $\underline{a}' = (a_1, \ldots, a_r)$ be the degree sequence obtained from $\underline{a}$ by considering only the degrees a_i such that $a_i \leqslant d$. Let $J = (I_d)$, and observe that $\mathrm{LPP}^{\underline{a}}(I)_d = \mathrm{LPP}^{\underline{a}'}(I)_d = \mathrm{LPP}^{\underline{a}'}(J)_d$. Moreover, since $J_{d+1} = \mathfrak{m}_1 I_d$, we have $H(\mathrm{LPP}^{\underline{a}'}(J); d+1) = H(J; d+1) = H(\mathfrak{m}I; d+1)$. Since $\mathfrak{m}_1\mathrm{LPP}^{\underline{a}'}(J)_d \subseteq \mathrm{LPP}^{\underline{a}'}(J)_{d+1}$, we finally obtain that $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I); d+1) = H(\mathfrak{m}\mathrm{LPP}^{\underline{a}'}(J); d+1) \leqslant H(\mathfrak{m}I; d+1)$. □

We would like to observe that, even if we do not know that EGH holds in general, we can still get an minimal growth statement in a Clements-Lindström ring $A/\mathfrak{a}$, under milder hypotheses.

Lemma 2.15 (Minimal Growth) *Assume that every homogeneous ideal containing $\mathfrak{a}$ satisfies* $\mathrm{EGH}_{\underline{a}}$. *If $\mathfrak{a} \subseteq I \subseteq A$ is such an ideal, then $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I) + \mathfrak{a}) \leqslant H(\mathfrak{m}I + \mathfrak{a})$.*

Proof Fix an integer $d \geqslant 0$, and let $J = (I_d) + \mathfrak{a}$. Note that both I and J satisfy the EGH, and $\mathrm{LPP}^{\underline{a}}(I)_d = \mathrm{LPP}^{\underline{a}}(J)_d$. Observe that $J_{d+1} = \mathfrak{m}_1 J_d + \mathfrak{a}_{d+1} = \mathfrak{m}_1 I_d + \mathfrak{a}_{d+1}$, and accordingly $H(\mathrm{LPP}^{\underline{a}}(J); d+1) = H(J; d+1) = H(\mathfrak{m}I + \mathfrak{a}; d+1)$. Now, since $(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(J) + \mathfrak{a})_{d+1} = \mathfrak{m}_1(\mathrm{LPP}^{\underline{a}}(J))_d + \mathfrak{a}_{d+1} \subseteq (\mathrm{LPP}^{\underline{a}}(J))_{d+1}$, we may conclude that $H(\mathfrak{m}\mathrm{LPP}^{\underline{a}}(I) + \mathfrak{a}; d+1) \leqslant H(\mathrm{LPP}^{\underline{a}}(J); d+1) = H(\mathfrak{m}I + \mathfrak{a}; d+1)$. □

We are finally in a position to prove the main result of this section. The simple idea underlying the new proof we present here is to demonstrate Clements–Lindström Theorem using Strong Hyperplane Restriction, like Green proved Macaulay Theorem using generic hyperplane section; this also motivates why Part (ii) has been assimilated into the statement.

Proof of Theorem 2.1 By adding sufficiently large powers of the variables $x_{r+1}, \ldots, x_n$, we may assume that $r = n$. After taking any initial ideal, and by Proposition 2.5, we may assume that I is an $\underline{a}$-spp monomial ideal. By induction, we may also assume that both Part (i) and Part (ii) hold true in polynomial rings with less than n variables, since the case $n = 1$ is trivial. In particular, any lpp ideal of $\overline{A}$ has Minimal Growth, see Lemma 2.15.

We write $I = \bigoplus_{i \geqslant 0} I_{[i]} x_n^i$; for all i, we let $J_{[i]} = \mathrm{LPP}^{\underline{a}}(I_{[i]})$, which by induction is an ideal of $\overline{A}$. Next, we prove that

$$J = \bigoplus_{i \geqslant 0} J_{[i]} x_n^i$$

is also an $\underline{a}$-spp ideal. First of all, observe that $I_{[k]} \subseteq I_{[k+1]}$ for all k, since I is an ideal. This implies that $H(J_{[k]}) = H(I_{[k]}) \leqslant H(I_{[k+1]}) = H(J_{[k+1]})$. Since the ideals $J_{[k]}$ and $J_{[k+1]}$ are lpp, it follows that $J_{[k]} \subseteq J_{[k+1]}$, which, in turn, translates into J being an ideal. Since I is $\underline{a}$-spp, for all $i < a_n - 1$ we have $\overline{\mathfrak{m}}_1 I_{[i+1]} \subseteq I_{[i]}$ and $\overline{\mathfrak{a}} \subseteq I_{[i]}$; thus

$$H(J_{[i]}) = H(I_{[i]}) \geqslant H(\overline{\mathfrak{m}}_1 I_{[i+1]} + \overline{\mathfrak{a}}) \geqslant H(\overline{\mathfrak{m}}_1 J_{[i+1]} + \overline{\mathfrak{a}}),$$

where the last inequality follows from Lemma 2.15. This yields that that $\overline{\mathfrak{m}}_1 J_{[i+1]} \subseteq J_{[i]}$ for all $i < a_n - 1$, and J is $\underline{a}$-spp by stability.

Given an $\underline{a}$-spp vector space $V \subseteq A_d$, denote by $\sigma(V)$ the segment contained in A_d which has the same dimension as V. Let $J = \bigoplus_{d \geqslant 0} J_d$ be the homogeneous ideal we constructed above and let

$$\sigma(J) = \bigoplus_{d \geqslant 0} \sigma(J_d).$$

We claim that $\sigma(J)$ is the $\underline{a}$-lpp ideal we are looking for.

First of all we show that it is an ideal. Fix a degree $d \geqslant 0$, and write $J_d = \bigoplus_{i=0}^{d} (J_d)_{[d-i]} x_n^i$, $\sigma(J_d) = \bigoplus_{i=0}^{d} \sigma(J_d)_{[d-i]} x_n^i$; for notational simplicity, in the following we let $\sigma_{[d-i]} = \sigma(J_d)_{[d-i]}$. By stability, we then have

$$\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1} = \begin{cases} \left(\overline{\mathfrak{m}}_1 (J_d)_{[d]} + \overline{\mathfrak{a}}_{d+1}\right) \oplus \left(\bigoplus_{i=0}^{d} (J_d)_{[d-i]} x_n^{i+1}\right), & \text{if } d < a_n - 1, \\ \left(\overline{\mathfrak{m}}_1 (J_d)_{[d]} + \overline{\mathfrak{a}}_{d+1}\right) \oplus \left(\bigoplus_{i=0}^{a_n-2} (J_d)_{[d-i]} x_n^{i+1}\right) \oplus \left(\bigoplus_{i=a_n}^{d} \overline{A}_{d-i} x_n^i\right), & \text{if } d \geqslant a_n - 1, \end{cases}$$

and

$$\mathfrak{m}_1 \sigma(J_d) + \mathfrak{a}_{d+1} = \begin{cases} \left(\overline{\mathfrak{m}}_1 \sigma_{[d]} + \overline{\mathfrak{a}}_{d+1}\right) \oplus \left(\bigoplus_{i=0}^{d} \sigma_{[d-i]} x_n^{i+1}\right), & \text{if } d < a_n - 1, \\ \left(\overline{\mathfrak{m}}_1 \sigma_{[d]} + \overline{\mathfrak{a}}_{d+1}\right) \oplus \left(\bigoplus_{i=0}^{a_n-2} \sigma_{[d-i]} x_n^{i+1}\right) \oplus \left(\bigoplus_{i=a_n}^{d} \overline{A}_{d-i} x_n^i\right), & \text{if } d \geqslant a_n - 1. \end{cases}$$

When $d < a_n - 1$, we set $\sigma_{[a_n-1]} = (J_d)_{[a_n-1]} = 0$. From the above equalities we thus get

$$\begin{aligned}\dim_K(\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1}) - \dim_K(\mathfrak{m}_1\sigma(J_d) + \mathfrak{a}_{d+1}) = \\ = \left(\dim_K(\overline{\mathfrak{m}}_1(J_d)_{[d]} + \overline{\mathfrak{a}}_{d+1}) - \dim_K(\overline{\mathfrak{m}}_1(\sigma_{[d]} + \overline{\mathfrak{a}}_{d+1}))\right) + \\ \left(\dim_K(\sigma_{[a_n-1]}) - \dim_K((J_d)_{[a_n-1]})\right).\end{aligned} \tag{2.6}$$

Since $\sigma(J_d)$ is a segment, $\sigma_{[d]} \subseteq \overline{A}$ is $\overline{\underline{a}}$-lpp and its dimension sequence $\underline{\delta} = \underline{\delta}(\sigma(J_d))$ is minimal for the Proposition 2.13. Moreover, the $\overline{\underline{a}}$-lpp vector space $L_d \subseteq \overline{A}_d$ with the same Hilbert function as $(J_d)_{[d]}$ has Minimal Growth, and $\sigma_{[d]} \subseteq L_d$ by the minimality of $\underline{\delta}$. Therefore,

$$\dim_K(\overline{\mathfrak{m}}_1(J_d)_{[d]} + \overline{\mathfrak{a}}_{d+1}) \geqslant \dim_K(\overline{\mathfrak{m}}_1 L_d + \overline{\mathfrak{a}}_{d+1}) \geqslant \dim_K(\overline{\mathfrak{m}}_1\sigma_{[d]} + \overline{\mathfrak{a}}_{d+1}).$$

Recall that the last entry of the dimension sequence is the dimension of the vector space itself; thus, since $\sigma(J_d)$ and J_d have the same dimension and $\underline{\delta}(J_d) \geqslant \underline{\delta}$ we get $\dim_K(\sigma_{[a_n-1]}) \geqslant \dim_K((J_d)_{[a_n-1]})$. An application of (2.6) now yields

$$\dim_K(\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1}) \geqslant \dim_K(\mathfrak{m}_1\sigma(J_d) + \mathfrak{a}_{d+1}).$$

Since J is an ideal that contains $\mathfrak{a}$, we have that $\mathfrak{m}_1 J_d + \mathfrak{a}_{d+1} \subseteq J_{d+1}$ and, thus,

$$\dim_K(\mathfrak{m}_1\sigma(J_d) + \mathfrak{a}_{d+1}) \leqslant \dim_K(J_{d+1}) = \dim_K(\sigma(J_{d+1})).$$

By Remark 2.9, we know that $\mathfrak{m}_1\sigma(J_d) + \mathfrak{a}_{d+1}$ is a segment, and so is $\sigma(J_{d+1})$ by definition; then, it follows that $\mathfrak{m}_1\sigma(J_d) \subseteq \sigma(J_{d+1})$. We may finally conclude that $\sigma(J)$ is an ideal, which is $\underline{a}$-spp by construction, and has the same Hilbert function as I.

Next, we observe that $\sigma(J)$ satisfies Part (ii) of the theorem, since $H(\sigma(J) + (x_n^i); d)$ is just the i-th entry of $\underline{\delta}(\sigma(J_d))$, $H(J + (x_n^i); d)$ is the i-th entry of $\underline{\delta}(J_d)$, and $\underline{\delta} \leqslant \underline{\delta}(J_d)$.

By construction, $\sigma(J)$ is the ideal with all the required properties, once we have proved the following claim.

Claim $\sigma(J)$ is $\underline{a}$-lpp.

Proof of the Claim By contradiction, there exists a degree d such that $\sigma(J_d)$ is an $\underline{a}$-spp D-dimensional vector space which is not lpp; thus, we may consider a counterexample of degree d and of minimal dimension D for which the operator σ does not return an $\underline{a}$-lpp vector space of dimension D inside A_d; then, if we apply σ to any $(D-1)$-dimensional $\underline{a}$-spp vector space of A_d, we obtain an $\underline{a}$-lpp vector space, but there is an $\underline{a}$-spp vector space of dimension D which is transformed by σ into an $\underline{a}$-spp vector space $V + \langle v\rangle$ which is not lpp. Thus, V is $\underline{a}$-lpp, $V + \langle v\rangle$ is

$\underline{a}$-segment, and we write them as

$$V = \bigoplus_{i=0}^{d} V_{[d-i]} x_n^i, \qquad V + \langle v \rangle = \bigoplus_{i=0}^{d} \widetilde{V}_{[d-i]} x_n^i.$$

Let also w be the monomial such that $V + \langle w \rangle$ is the $\underline{a}$-lpp vector subspace of dimension D of A_d and observe that $w > v$. Write $v = \overline{v} x_n^t$ and $w = \overline{w} x_n^s$, where $\overline{v}, \overline{w}$ are monomials in $\overline{A}$.

Since $V + \langle v \rangle$ is a segment, we have that $t \geqslant s$.

If $t = s$ we immediately get a contradiction, since by construction $\overline{v}$ and $\overline{w}$ would both be the largest monomial of degree $d - t$ which is not contained in $V_{[d-t]}$.

Therefore, we may assume that $t > s$, and $a = \deg(\overline{w}) = d - s > d - t = \deg(\overline{v}) = b$. Observe that $\overline{v} \in \widetilde{V}_{[d-t]}$, and that $d - t < a_n$. Moreover $\overline{\mathfrak{m}}_{a-b} \widetilde{V}_{[d-t]} \subseteq \widetilde{V}_{[d-s]}$ holds by stability applied to $V + \langle v \rangle$. We write $\overline{w} = x_{i_1} \cdots x_{i_a}$ and $\overline{v} = x_{j_1} \cdots x_{j_b}$, with $i_1 \leqslant \ldots \leqslant i_a$ and $j_1 \leqslant \ldots \leqslant j_b$. Since $w > v$ we have two cases, either $\overline{v}$ divides $\overline{w}$, or $x_{i_1} \cdots x_{i_b} > \overline{v}$. In both cases, it is easy to see that $\overline{w} \in \overline{\mathfrak{m}}_{a-b} \widetilde{V}_{[d-t]} \subseteq \widetilde{V}_{[d-s]}$, and thus $w \in V + \langle v \rangle$, which is a contradiction. □

The proofs of Theorem 2.1 (i) previously available in the literature do not include Part (ii), the Strong Hyperplane Section of Gasharov. One advantage of our approach is that, with little additional effort, one can show that the Betti numbers of an $\underline{a}$-spp ideal are at most those of the corresponding $\underline{a}$-lpp ideal; see [8, 42]. Furthermore, combining this fact with Remark 2.6, one recovers the LPP-Conjecture for ideals containing pure-powers ideals in characteristic zero, which is the main result of [37, Section 3]. Note that, in [37], the authors also provide a characteristic-free proof that settles the LPP-Conjecture for ideals that contain pure-powers.

3 Artinian Reduction and Linkage

In this brief section we collect some results which will be useful in what follows. We start with Proposition 10 in [9], which offers in many cases a way to prove the EGH Conjecture in the Artinian case only.

Proposition 3.1 *Let* $\mathbf{f} \subseteq A = K[x_1, \ldots, x_n]$ *be an ideal generated by a regular sequence of degree* $\underline{a}$*, and* ℓ *be a linear* $A/\mathbf{f}$*-regular form. Let also* $\overline{A} = A/(\ell)$*, and* $\bar{\mathbf{f}} = \mathbf{f}\overline{A}$*. If every homogeneous ideal of* $\overline{A}$ *containing* $\bar{\mathbf{f}}$ *satisfies* $\mathrm{EGH}_{\overline{\underline{a}}}$*, then every homogeneous ideal of* A *containing* $\mathbf{f}$ *satisfies* $\mathrm{EGH}_{\underline{a}}$*.*

Proof Let $I \subseteq A$ be a homogeneous ideal that contains $\mathbf{f}$ and for $i \geqslant 0$ we let $I_i = (I :_A \ell^i) + (\ell)$. By assumption, there exist $\overline{\underline{a}}$-lpp ideals $J_i \subseteq K[x_1, \ldots, x_{n-1}]$

with the same Hilbert function as $I_i/(\ell)$. Now, we define

$$J = \bigoplus_{i \geqslant 0} J_i x_n^i,$$

and we claim that J is an ideal with the same Hilbert function as I; since $\mathfrak{a} \subseteq J_0 \subseteq J$, the conclusion will then follow from Theorem 2.1.

By considering the short exact sequences $0 \longrightarrow A/(I :_A \ell^j)(-1) \xrightarrow{\cdot \ell} A/(I :_A \ell^{j-1}) \longrightarrow \overline{A}/I_{j-1}\overline{A} \longrightarrow 0$ for all j, a straightforward computation yields that $H(J) = H(I)$.

What it is left to be shown is that J is an ideal. Let as before $\overline{\mathfrak{m}} = (x_1, \dots, x_{n-1})$; since J_i is an ideal of $\overline{A}$, we have $\overline{\mathfrak{m}} J_i \subseteq J_i$ for all i and, accordingly, $\overline{\mathfrak{m}} J \subseteq J$. The condition $x_n J \subseteq J$ translates into the containments $J_i \subseteq J_{i+1}$ for all $i \geqslant 0$. Since each J_i is an $\overline{\underline{a}}$-lpp ideal, it suffices to show that $H(J_i) \leqslant H(J_{i+1})$, which holds true since $I_i \subseteq I_{i+1}$. □

We now recall some results from the theory of linkage. In Sect. 2 we introduced the following notation: given a homogeneous ideal $I \subseteq A = K[x_1, \dots, x_n]$ containing an ideal $\mathbf{f}$ generated by a regular sequence of degree $\underline{a} = (a_1, \dots, a_n)$, we let $I_{\mathbf{f}}^{\ell} = (\mathbf{f} :_A I)$, and call it the *link of* I *with respect to* $\mathbf{f}$, which is an ideal that contains $\mathbf{f}$. Obviously, the link depends on $\mathbf{f}$; however, when it is clear from the context which $\mathbf{f}$ we consider, we denote $I_{\mathbf{f}}^{\ell}$ simply by I^{ℓ}.

Proposition 3.2 *Let* $\underline{a} = (a_1, \dots, a_n)$ *and* A, I, $\mathbf{f}$ *be as above; let also* $R = A/\mathbf{f}$ *and* $s = \sum_{i=1}^{n}(a_i - 1)$. *Then,*

(i) $(I^{\ell})^{\ell} = I$.
(ii) $H(IR; d) = H(R; d) - H(I^{\ell}R; s - d)$.
(iii) $\operatorname{type}(R/IR) = \mu(I^{\ell}R)$, *i.e., the dimension of the socle of* R/IR *equals the minimal number of generators of its linked ideal.*

In particular, if $I = (\mathbf{f} + (g))$ *is an almost complete intersection, then the ideal* $I^{\ell} = (\mathbf{f} :_A g)$ *defines a Gorenstein ring, and viceversa. Moreover, if* $\deg(g) = D$, *then* $\operatorname{soc}((\mathbf{f} :_A g)R)$ *is concentrated in degree* $s - D$.

Proof Observe that the functor $(-)^{\vee} = \operatorname{Hom}_R(-, R)$ is the Matlis dual, since R is Gorenstein Artinian. The statements that we want to prove are a direct consequence of Matlis duality, see [4, Sections 3.2 and 3.6]. It is well known that a module and its Matlis dual have the same annihilator. In particular, since $(A/I)^{\vee} \cong I^{\ell}/\mathbf{f}$, we obtain that $I = \operatorname{ann}_A(A/I) = \operatorname{ann}_A(I^{\ell}/\mathbf{f}) = (I^{\ell})^{\ell}$, which proves (i). For (ii), recall that in the graded setting one has $((A/I)^{\vee})_d \cong (A/I)_{s-d}$, for all $d \in \mathbb{Z}$. Since $(A/I^{\ell})^{\vee} \cong I/\mathbf{f}$, the claim follows from the graded short exact sequences of K-vector spaces $0 \to (I/\mathbf{f})_d \to (A/\mathbf{f})_d \to (A/I)_d \to 0$. Part (iii) is again a consequence of Matlis duality. □

We conclude this part with an easy lemma.

Lemma 3.3 *Let $\underline{a} = (a_1, \ldots, a_r)$ and $\underline{b} = (b_1, \ldots, b_r)$ be degree sequences satisfying $a_i \leqslant b_i$ for all $i = 1, \ldots, r$. If an ideal I satisfies* $\mathrm{EGH}_{\underline{a}}$*, then it satisfies* $\mathrm{EGH}_{\underline{b}}$.

Proof By assumption, $J = \mathrm{LPP}^{\underline{a}}(I)$ is a $\underline{a}$-lpp ideal with the same Hilbert function as I. By our assumption on the degree sequences, J also contains the pure-powers ideal $(x_1^{b_1}, \ldots, x_r^{b_r})$. Therefore, by Theorem 2.1, $\mathrm{LPP}^{\underline{b}}(J)$ is a $\underline{b}$-lpp ideal with the same Hilbert function as I. □

4 Results on the EGH Conjecture

We collect in the following the most relevant cases when EGH is known to be true. We start with a very recent result, Theorem 4.1, proved by the first two authors in [6, Theorem A], which improves an older result due Maclagan and the first author, [9, Theorem 2]. Indeed, as we show in this section, from Theorem 4.1 one can derive with little effort all of the significant known cases of the EGH Conjecture which take into account only hypotheses on the degree sequence $\underline{a}$ and not on the ideal I. A further generalization can be found in [6], see Theorem 3.6.

Theorem 4.1 *Let $I \subseteq A$ be a homogeneous ideal which contains a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$ and assume that $a_i \geqslant \sum_{j=1}^{i-1}(a_j - 1)$ for all $i \geqslant 3$; then, I satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof For brevity's sake, we present here only the proof of the weaker statement [9, Theorem 2], that is, we will assume that $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ for all $i \geqslant 3$. Observe that, by Proposition 3.1, we may let $r = n$ and work by induction on n. Let $\overline{a} = (a_1, \ldots, a_{n-1})$; by induction, suppose that every ideal of A containing a regular sequence of degree $\overline{a}$ satisfies $\mathrm{EGH}_{\overline{a}}(d)$ for all d.

Clearly, for $d < a_{n-1}$, we have that $\mathrm{EGH}_{\underline{a}}(d)$ is equivalent to $\mathrm{EGH}_{\overline{a}}(d)$. Thus, let $d + 1 \geq a_n$, so that $s - (d + 1) < a_n - 1$; by induction, I^ℓ satisfies $\mathrm{EGH}_{\overline{a}}$ and the previous case yields that I^ℓ satisfies $\mathrm{EGHH}_{\underline{a}}\, s - (d + 1)$ for all $d + 1 \geq a_n$. By Proposition 3.2 (ii), we know that $H(IR; d) = H(R; d) - H(I^\ell R; s - d)$, where $R = A/\mathbf{f}$ and $s = \sum_{i=1}^{n}(a_i - 1)$. It now follows that I satisfies $\mathrm{EGH}_{\underline{a}}(d)$ also for all $d + 1 \geq a_n$, and the proof is complete. □

As we have already observed in Remark 2.4 (1), Theorem 4.1 yields the EGH for $r \leqslant 2$.

One big advantage of Theorem 4.1 is that it can be applied in order to obtain growth bounds for the Hilbert function which are at least as good as the ones given by Macaulay Theorem. This can be done for *any* homogenous ideal, regardless of the degree sequence. The key observation to see this is the following.

Lemma 4.2 *Assume that $|K| = \infty$ and that I contains an ideal* $\mathbf{f}$ *generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$. If $\underline{b} = (b_1, \ldots, b_r)$ is a degree* *sequence such that $b_i \geqslant a_i$ for all i, then I contains an ideal* $\mathbf{g}$ *generated by a regular sequence of degree $\underline{b}$.*

Proof We proceed by induction on $r \geqslant 1$. Let $r = 1$ and observe that $I_{b_1} \neq 0$ since $b_1 \geq a_1$. It follows that there exists a regular element $g_1 \in I$ of degree b_1.

By induction, we have constructed a homogeneous ideal $\mathbf{g}' = (g_1, \ldots, g_{r-1})$, which is unmixed and generated by a regular sequence of degrees $b_1, \ldots, b_{r-1}$. Observe that, since I contains $f_1, \ldots, f_r$, we have that $\operatorname{ht}(I_j A) \geqslant r$ for all $j \geqslant a_r$. In particular, the ideal $\mathbf{g}' + I_{b_r} A$ has height at least r, since $b_r \geqslant a_r$. Thus, by prime avoidance, we find an element $g_r \in I_{b_r}$ which is regular modulo $\mathbf{g}'$ and $\mathbf{g} = (g_1, \ldots, g_r)$ is the ideal we were looking for. □

As another application of the theory of linkage to the EGH Conjecture, we now present a result due to Chong [13], which settles the conjecture for Gorenstein ideals of height three.

Proposition 4.3 *Let I be a homogeneous ideal that contains an ideal* $\mathbf{f}$ *generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_n)$. Assume that $\underline{b} = (b_1, \ldots, b_n)$ is a degree sequence such that $b_i \leqslant a_i$ for all i, and $I^{\ell}_{\mathbf{f}}$ satisfies* $\mathrm{EGH}_{\underline{b}}$*; then I satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof Let $s = \sum_{i=1}^{n}(a_i - 1)$ and $I^{\ell} = I^{\ell}_{\mathbf{f}}$; by hypothesis there exists a $\underline{b}$-lpp ideal J with the same Hilbert function as I^{ℓ} that also contains the pure-powers ideal $\mathfrak{a} = (x_1^{a_1}, \ldots, x_r^{a_r})$, since $a_i \geqslant b_i$ for all i. Consider now $J^{\ell}_{\mathfrak{a}}$; by Proposition 3.2 (ii) for all $d \geqslant 0$ we have

$$
\begin{aligned}
H(I/\mathbf{f}; d) &= H(A/\mathbf{f}; d) - H(I^{\ell}/\mathbf{f}; s-d) \\
&= H(A/\mathring{a}; d) - H(J/\mathring{a}; s-d) = H(J^{\ell}_{\mathring{a}}/\mathring{a}; d).
\end{aligned}
$$

By Theorem 2.1, there exists an $\underline{a}$-lpp ideal with the same Hilbert function as $J^{\ell}_{\mathfrak{a}}$, and we are done. □

Observe that in the above proof we used Theorem 2.1 to transform the monomial ideal $J^{\ell}_{\mathfrak{a}}$ into an $\underline{a}$-lpp ideal. In fact, it can be proved in general that $J^{\ell}_{\mathfrak{a}}$ is already $\underline{a}$-lpp whenever J is $\underline{a}$-lpp, see for instance [45, Theorem 5.7], or [11, Proposition 3.2].

Sequentially bounded licci ideals were first introduced in [13], and are those ideals to which Proposition 4.3 can be applied repeatedly in order to prove the EGH Conjecture. We recall the main definitions here.

Definition 4.4 Let $I \subseteq A = K[x_1, \ldots, x_n]$ be a homogeneous ideal, and set $I_0 = I$. We say that I is *linked to a complete intersection*, or *licci* for short, if there exist ideals $I_j = (I_{j-1})^{\ell}_{\mathbf{f}_j}$ where $\mathbf{f}_1, \ldots, \mathbf{f}_s$ are ideals of the same height as I generated by regular sequences of degrees $\underline{a}_1, \ldots, \underline{a}_s$, such that I_s is generated by a regular sequence of degree $\underline{a}_{s+1}$.

We say that I is *sequentially bounded licci* if the above sequence also satisfies $\underline{a}_1 \geqslant \ldots \geqslant \underline{a}_{s+1}$.

We also recall that I is said to be *minimally licci* if it is licci and, in addition, for each j the regular sequence generating $\mathbf{f}_{j+1}$ can be chosen to be of minimal degree among all the regular sequences contained in I_j. Observe that $\mathbf{f}_j \subseteq I_j$, therefore minimally licci ideals are sequentially bounded licci. It was proved by Watanabe [46] that height three Gorenstein ideals are licci. Later on, Migliore and Nagel show that such ideals are also minimally licci [38]. We see next how these facts together, combined with Proposition 4.3, yield the main result of [13].

Theorem 4.5 *Let $I \subseteq A$ be a sequentially bounded licci ideal, where the first link of I is performed with respect to a regular sequence of degree $\underline{a}$; then I satisfies* $\mathrm{EGH}_{\underline{a}}$.

In particular, if I is a Gorenstein ideal of height 3 containing a regular sequence of degree $\underline{a} = (a_1, a_2, a_3)$, then I satisfies $\mathrm{EGH}_{\underline{a}}$.

Proof We prove the first part only for $n = r$, and we refer the reader to the original paper for the reduction to this case; this is shown in [13, Proposition 10], where the proof runs along the same lines as that of Proposition 3.1.

Since I_s is a complete intersection of degree $\underline{a}_{s+1}$ by assumption, it trivially satisfies $\mathrm{EGH}_{\underline{a}_{s+1}}$; therefore Proposition 4.3 implies that I_{s-1} satisfies $\mathrm{EGH}_{\underline{a}_s}$, and its repeated application to the sequence of linked ideals eventually yields that I satisfies $\mathrm{EGH}_{\underline{a}_1}$, that is $\mathrm{EGH}_{\underline{a}}$. □

Remark 4.6 The height 3 Gorenstein case proved by Chong is also related to a previous result due to Geramita and Kreuzer concerning the Cayley-Bacharach Conjecture in $\mathbb{P}^3$ [24, Corollary 4.4]. In fact, EGH for a height 3 Gorenstein ideal I is equivalent to EGH for its linked ideal I^ℓ, which is an almost complete intersection by Proposition 3.2 (iii). As pointed out in the introduction, EGH for almost complete intersections implies the Cayley-Bacharach Conjecture 1.2.

Next, we present a result due to Francisco [20, Corollary 5.2] which settles $\mathrm{EGH}_{\underline{a}}(D)$ for almost complete intersections $(\mathbf{f} + (g))$ in the first relevant degree, namely $D = \deg(g)$.

Theorem 4.7 *Let $\mathbf{f} \subseteq A$ be an ideal generated by a regular sequence of degree $\underline{a} = (a_1, \ldots, a_r)$, and let $g \notin \mathbf{f}$ be an element of degree $D \geqslant a_1$ such that $I = \mathbf{f} + (g)$. Then, I satisfies* $\mathrm{EGH}_{\underline{a}}(D)$.

Proof We may assume that K is infinite. First, we reduce to the Artinian case by arguing as follows: we choose some $N > D + 1$ and homogeneous elements of degree N such that $f_1, \ldots, f_r, f_{r+1}, \ldots, f_n$ is a full regular sequence of degree $\underline{a}' = (a_1, \ldots, a_r, N, \ldots, N)$. In this way, proving $\mathrm{EGH}_{\underline{a}}(D)$ for I is equivalent to proving $\mathrm{EGH}_{\underline{a}'}(D)$ for $I + (f_{r+1}, \ldots, f_n)$. Thus, for the rest of proof $r = n$ and $A/\mathbf{f}$ is Artinian.

Now, let b be the unique integer such that $\sum_{i=1}^{b}(a_i - 1) \leqslant D < \sum_{i=1}^{b+1}(a_i - 1)$. It is then easy to see that $J = \mathfrak{a} + (h)$, where $h = x_1^{a_1-1} \cdots x_b^{a_b-1} \cdot x_{b+1}^{D-\sum_{i=1}^{b}(a_i-1)}$, is the smallest $\underline{a}$-lpp ideal with $H(J; D) = H(I; D)$.

To conclude the proof, it suffices to show that $H(J; D+1) \leqslant H(I; D+1)$. To this end, let $s = \sum_{i=1}^{n}(a_i - 1)$, and consider the links $I^\ell = I^\ell_{\mathbf{f}} = (\mathbf{f} :_A I)$ and $J^\ell = J^\ell_{\mathfrak{a}} = (\mathfrak{a} :_A J)$. The natural graded short exact sequences

$$0 \to A/I^\ell(-D) \to A/\mathbf{f} \to A/I \to 0 \quad \text{and} \quad 0 \to A/J^\ell(-D) \to A/\mathfrak{a} \to A/J \to 0$$

show that we only have to prove that $H(J^\ell; 1) \geqslant H(I^\ell; 1)$. A direct computation shows that

$$J^\ell = \mathfrak{a} : (h) = (x_1, \ldots, x_b, x_{b+1}^{\sum_{i=1}^{b+1}(a_i-1)-D+1}, x_{b+2}^{a_{b+2}}, \ldots, x_n^{a_n}),$$

that is, $H(J^\ell; 1) = b$.

Suppose, by contradiction, that I^ℓ contains c linear forms, with $c > b$; then, by Prime Avoidance we can find a homogeneous ideal $\mathbf{g} \subseteq I^\ell$ generated by a regular sequence of degree $(1, \ldots, 1, a_{c+1}, \ldots, a_n)$ such that the socle degree of $A/\mathbf{g}$ is $\sum_{i=c+1}^{n}(a_i - 1) < \sum_{i=b+1}^{n}(a_i - 1) \leqslant s - D$. Thus, $H(A/I^\ell; s-D) \leqslant H(A/\mathbf{g}; s-D) = 0$ which is not possible, since the ring A/I^ℓ is Gorenstein of socle degree $s - D$ by Proposition 3.2 (iii). □

Remark 4.8 It is easy to see by means of Lemma 4.2 that the condition $D \geqslant a_1$ in the statement of Theorem 4.7 can always be met.

Observe that, again by Proposition 3.2 (ii), the statement of Theorem 4.7 is equivalent to proving $\mathrm{EGH}_{\underline{a}}(s - D - 1)$ for the ideal $I^\ell = I^\ell_{\mathbf{f}}$. Since the socle of A/I^ℓ is concentrated in degree $s - D$, this is equivalent to controlling the growth of the Hilbert function of a Gorenstein ring from socle degree minus 1 to the socle degree. For other results of this nature, see for instance [43].

The next result we present is due to Abedelfatah, see [1] and [2]; it can be viewed as a generalization of the Clements-Lindström Theorem to ideals that contain a regular sequence generated by products of linear forms. Below we provide the proof of the general version, cf. [2, Theorem 3.4].

Theorem 4.9 *Let* $\mathbf{f} \subseteq A$ *be an ideal generated by a regular sequence of degree* $\underline{a} = (a_1, \ldots, a_r)$*. Assume that* $\mathbf{f} \subseteq P$*, where* P *is an ideal generated by products of linear forms. Then, any ideal* $I \subseteq A$ *that contains* P *satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof By induction we may assume that the claim is true for ideals in polynomial rings with less than n variables, since the base case $n = 1$ is trivial.

Let s be the smallest degree of a minimal generator p of P. Since $s \leqslant a_1$, by Lemma 3.3 it suffices to show that I satisfies $\mathrm{EGH}_{\underline{a}'}$, where $\underline{a}' = (s, a_2, \ldots, a_r)$. Moreover, by Theorem 2.1, it is enough to prove that, for every degree $d \geqslant 0$, there exists a monomial ideal J that contains $(x_1^s, x_2^{a_2}, \ldots, x_r^{a_r})$ such that $H(I; d) = H(J; d)$ and $H(I; d+1) = H(J; d+1)$.

We write $p = \ell_1 \cdots \ell_s$, where ℓ_i are linear forms which we order as follows:

For $k = 1, \ldots, s$, let $I_k^{(0)}$ denote the image ideal of I in $A/(\ell_k)$ and choose ℓ_1 so that $H(I_1^{(0)}; d) = \min_k\{H(I_k^{(0)}; d)\}$.

Inductively, given $\ell_1, \ldots, \ell_j$, for $k = j+1, \ldots, s$ we let $I_k^{(j)}$ denote the image ideal of $(I :_A (\ell_1 \cdots \ell_j))$ in $A/(\ell_k)$ and choose ℓ_{j+1} so that $H(I_{j+1}^{(j)}; d-j) = \min_k\{H(I_k^{(j)}; d-j)\}$.

Now, with some abuse of notation, we let $A_k = A/(\ell_k)$ for $k = 1, \ldots, s$; for notational simplicity, we also set $I_j = I_{j+1}^{(j)}$ for $j = 0, \ldots, s-1$. By construction, we thus have

$$H(I_j; d-j) \leqslant H(I_{j+1}; d-j) \quad \text{for all } j = 0, \ldots, s-1. \tag{4.1}$$

Moreover, for all $j = 1, \ldots, s-1$, the short exact sequences

$$0 \longrightarrow A/(I :_A (\ell_1 \cdots \ell_j))(-1) \longrightarrow A/(I :_A (\ell_1 \cdots \ell_{j-1})) \longrightarrow A_j/I_{j-1} \longrightarrow 0$$

provide that

$$H(A/I; i) = \sum_{j=0}^{s-1} H(A_{j+1}/I_j; i-j), \quad \text{for all } i. \tag{4.2}$$

Let $\tilde{\underline{a}} = (a_2, \ldots, a_n)$ and $\tilde{A} = K[x_2, \ldots, x_n]$. Observe that $A_k \cong \tilde{A}$ for all k, thus, by induction, we can find $\tilde{\underline{a}}$-lpp ideals $J_{[j]}$ in $\tilde{A}$ with the same Hilbert function as I_j, for $j = 0, \ldots, s-1$. Consider now $J = \bigoplus_{j=0}^{s-1} J_{[j]} x_1^j \oplus A x_1^s$, and let J_d denote the degree d component of J. If we show, and we shall do, that $\mathfrak{m}_1 J_d \subseteq J_{d+1}$, that is, J is closed under multiplication from degree d to degree $d+1$, then the proof is complete, since $H(A/J; i) = H(A/I; i)$ for all i by (4.2).

To this end, we clearly have that $(x_2, \ldots, x_n)_1 (J_{[j]})_{d-j} \subseteq (J_{[j]})_{d-j+1}$, since each $J_{[j]}$ is an ideal in $\tilde{A}$. It is left to show that $x_1 J_d \subseteq J_{d+1}$, which translates into $(J_{[j]})_{d-j} \subseteq (J_{[j+1]})_{d-j}$ for all $j = 0, \ldots, s-1$; since such ideals are both $\tilde{\underline{a}}$-lpp, this is yielded by (4.1). □

Corollary 4.10 *The EGH Conjecture is true for monomial ideals.*

Another interesting known case, of different nature, is when the regular sequence that defines $\mathbf{f}$ is a Gröbner basis with respect to some monomial order. In fact, in this situation, the initial forms of the sequence are a regular sequence of monomials.

Proposition 4.11 *Let* $\mathbf{f}$ *be an ideal of* A *generated by a regular sequence* $f_1, \ldots, f_r$ *of degree* $\underline{a}$*, such that* $\{f_1, \ldots, f_r\}$ *is a Gröbner basis with respect to some monomial order* $\succcurlyeq$*. Then, every homogeneous ideal of* A *containing* $\mathbf{f}$ *satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof Let I be a homogeneous ideal that contains $\mathbf{f}$. Let us consider the set $\mathscr{S}$ of all homogeneous ideals of A with the same Hilbert function as I that contain a monomial regular sequence $g_1, \ldots, g_r$ of degree $\underline{a}$. Observe that $\mathscr{S}$ is not empty since, by assumption, the initial ideal of I contains the regular sequence of monomials given by the initial forms of $f_1, \ldots, f_r$, which has degree $\underline{a}$.

Since the monomials $g_1, \ldots, g_r$ are pairwise coprime, we may write $g_i = \prod_{j \in B_i} x_j^{b_{ij}}$, for some subsets $B_i \subseteq \{1, \ldots, n\}$ with $B_i \cap B_{i'} = \emptyset$ if $i \neq i'$, and we let $|g_1, \ldots, g_r| = \sum_{i=1}^{r} |B_i|$ denote the cardinality of the support of $g_1, \ldots, g_r$.

Now, we choose an element J of $\mathscr{S}$ which contains a regular sequence $h_1, \ldots, h_r$ with minimal support and we will show that $|h_1, \ldots, h_r| = r$. In this way we will have that each h_k is the a_k-th power of a variable, which we may assume being equal to $x_k^{a_k}$; the conclusion will then follow by Theorem 2.1.

Clearly $|h_1, \ldots, h_r| \geqslant r$. If we assume by way of contradiction that the inequality were strict, then there would exist $i \in \{1, \ldots, r\}$ and $1 \leqslant j < j' \leqslant n$ such that $x_j x_{j'} \mid h_i$. Consider then the change of coordinates φ defined by

$$x_k \mapsto x_k, \quad \text{for all } k \neq j', \qquad \text{and } x_{j'} \mapsto x_j + x_{j'},$$

let $J' = \mathrm{in}_{\geqslant}(\varphi(J))$, where $\geqslant$ denotes the lexicographic order, and let $h'_k = \mathrm{in}_{\geqslant}(\varphi(h_k)) \in J'$ for $k = 1, \ldots, r$. It is immediate to see that $h'_1, \ldots, h'_r$ is still a monomial regular sequence of degree $\underline{a}$; since J' has the same Hilbert function as I, it belongs to $\mathscr{S}$. However, $h'_k = h_k$ for all $k \neq i$, whereas h'_i has one less variable than h_i in its support. In particular, $|h'_1, \ldots, h'_r| < |h_1, \ldots, h_r|$, which contradicts the minimality of the support of $h_1, \ldots, h_r$, and we are done. □

Clearly, one can generalize the above by using a weight order ω, as long as the given regular sequence form a Gröbner basis with respect to the induced order $\geqslant_\omega$ and the ideal of the initial forms of the sequence satisfies the EGH Conjecture.

Contrary to the "special" case in which the regular sequence $f_1, \ldots, f_r$ is a Gröbner basis, as far as we know the "generic" version of the conjecture is still open. We record this fact as a question.

Question 4.12 Let $\underline{a} = (a_1, \ldots, a_r)$ be a degree sequence. Does there exist a non-empty Zariski open set $U \subseteq \mathbb{P}(A_{a_1}) \times \mathbb{P}(A_{a_2}) \times \cdots \times \mathbb{P}(A_{a_r})$ of general forms of degree $\underline{a}$ such that, for every $[f_1, \ldots, f_r] \in U$, any ideal I containing $\mathbf{f} = (f_1, \ldots, f_r)$ satisfies $\mathrm{EGH}_{\underline{a}}$?

In [32, Proposition 4.2], Herzog and Popescu show that, once a regular sequence of degree $\underline{a} = (2, 2 \ldots, 2)$ is fixed, then any generic ideal generated by quadrics that contains it satisfies $\mathrm{EGH}_{\underline{a}}$. We would like to warn the reader that Question 4.12 addresses a different kind of "genericity". In fact, we are not fixing the regular sequence beforehand, but we are asking whether the EGH Conjecture holds for any ideal containing a general regular sequence.

Remark 4.13

(1) When **f** is a general complete intersection, then the set of monomials of A which do not belong to the monomial complete intersection of the same degree as **f** forms a K-basis of $A/\mathbf{f}$, and this is well-known. This observation could be helpful in giving a positive answer to Question 4.12.
(2) It is currently not known, though, whether or not, after a general change of coordinates $\varphi : A \to A$ the set of monomials of A which do not belong to the monomial complete intersection of the same degree as **f** is a K-basis of $A/\varphi(\mathbf{f})$, when **f** is a complete intersection. A positive answer in this matter would make Question 4.12 even more interesting. In fact, in light of the first part of the remark, it would provide a strategy to attack the EGH Conjecture at once.

There are some other very special cases when EGH is known to hold that can be found in the literature; we complete this section with two of them.

A special case of interest is when I contains a regular sequence of quadrics, and this is the assumption on I in the original statement of the conjecture. In this case, EGH is known to be true in low dimension; for $n \leqslant 4$, it can be proven by a direct application of linkage; see also [12]. The validity of the conjecture for $n = 5$ was first claimed in [44], but a proof was never provided until recently, when Güntürkün and Hochster finally settle the case of five quadrics in [23, Theorem 4.1]. We present an alternative proof of their result which relies on the techniques we used so far.

Theorem 4.14 *$I \subseteq A = K[x_1, \dots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $\underline{a} = (2, 2, 2, 2, 2)$; then, I satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof We may assume that $K = \overline{K}$. By Proposition 3.1 we may assume that $n = 5$, A/I is Artinian and $\mathbf{f} \subseteq I$ is an ideal generated by a regular sequence of five quadrics; notice that the socle degree s of $A/\mathbf{f}$ is $s = 5$.

By Proposition 3.2 (ii) it suffices to show that I satisfies $\mathrm{EGH}_{\underline{a}}(j)$ for $j = 0, 1, 2$; this is clearly true for $j = 0, 1$ and we are left with the case $j = 2$.

If $H(I; 2) = 6$, then we are done by Theorem 4.7. Since the locus of reducible elements in $\mathbb{P}(\mathrm{Sym}^2(A_1))$ has dimension $2n - 2 = 8$, if $H(I; 2) \geqslant 7$ then I must contain a reducible quadric $Q = \ell_1\ell_2$. Proceeding as in the proof of Theorem 4.9, we construct ideals $J_{[0]}$ and $J_{[1]}$ in $\tilde{A} = K[x_2, \dots, x_5]$ such that $J = J_{[0]} \oplus J_{[1]}x_1 \oplus Ax_1^2$ is a monomial vector space which contains $\mathfrak{a} = (x_1^2, \dots, x_5^2)$, $\mathfrak{m}_1 J_2 \subseteq J_3$, $H(A/J; i) = H(A/I; i)$ for all i, and the conclusion follows from an application of Theorem 2.1. □

In [15], Cooper proves some cases of the EGH Conjecture when r is small, including $\underline{a} = (a_1, a_2, a_3)$ with $a_1 = 2, 3$ and $a_2 = a_3$. We present a proof of the case $\underline{a} = (3, a, a)$, which is based on the techniques of [6].

Proposition 4.15 *Let $I \subseteq A = K[x_1, \dots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $\underline{a} = (3, a, a)$. Then, I satisfies* $\mathrm{EGH}_{\underline{a}}$.

Proof We may assume, as accustomed, that K is infinite and, by Proposition 3.1, that $r = n = 3$. Therefore, let $\mathbf{f} = (f_1, f_2, f_3) \subseteq I$ be an ideal generated by a regular sequence of degree $\underline{a}$; since the socle degree of A/I is $2a$, by Proposition 3.2 (ii), we only have to show that I satisfies $\text{EGH}_{\underline{a}}(d)$ for all $d < a$.

Let $\overline{\underline{a}} = (3, a)$, and observe that I satisfies $\text{EGH}_{\overline{\underline{a}}}$ by Theorem 4.1. Thus, since $\text{EGH}_{\underline{a}}(d)$ is equivalent to $\text{EGH}_{\overline{\underline{a}}}(d)$ for all $d < a - 1$, we only have to prove that $\text{EGH}_{\underline{a}}(a-1)$ holds.

Let $Q = (f_1, f_2, u_1, \ldots, u_c) \subseteq I$, where $u_1, \ldots, u_c$ are the pre-images of a K-basis of $(I/\mathbf{f})_{a-1}$. First, assume that $f_3 \notin Q$; thus, Q satisfies $\text{EGH}_{\overline{\underline{a}}}(a-1)$ and, therefore, if $\overline{J}$ denotes the smallest $\overline{\underline{a}}$-lpp ideal such that $H(Q; a-1) = H(\overline{J}; a-1)$, we then have $H(Q; a) \geqslant H(\overline{J}; a)$. Observe that $J = \overline{J} + (x_3^a)$ is an $\underline{a}$-lpp ideal such that $H(J; a-1) = H(\overline{J}; a-1)$ and $H(J; a) = H(\overline{J}; a) + 1$. We then have that

$$H(I; a) \geq H(Q + (f_3); a) = H(Q; a) + 1 \geqslant H(\overline{J}; a) + 1 = H(J; a),$$

and this case is done.

Otherwise, $f_3 \in Q$ and, accordingly, $\text{ht}(Q) = 3$. By Prime Avoidance we may assume that f_1, v_c, f_2 forms a regular sequence of degree $\underline{a}' = (3, a-1, a)$ when $a \neq 3$; when $a = 3$, we may take the sequence v_c, f_1, f_2 of degree $\underline{a}' = (2, 3, 3)$ instead. Either way, I satisfies $\text{EGH}_{\underline{a}'}$ by Theorem 4.1 and, therefore, there exists a $\underline{a}'$-lpp ideal J with the same Hilbert function as I. In particular, since $\underline{a} \geqslant \underline{a}'$, the monomial ideal J also contains $\mathfrak{a} = (x_1^3, x_2^a, x_3^a)$, and we conclude by Theorem 2.1. □

5 Applications and Examples

In this section, we present some applications of the EGH Conjecture, supported by several examples. For our computations, it is convenient to introduce the following integers.

Definition 5.1 Let $\underline{a} = (a_1, \ldots, a_r)$ be a degree sequence, and h, d be non-negative integers with $h \leqslant n$ and $d \geqslant 1$. For $r < i \leqslant n$, we let $a_i = \infty$ and $x_i^{a_i} = 0$. Also, we let

$$\begin{bmatrix} h \\ d \end{bmatrix}_{\underline{a}} = \begin{cases} \dim_K \left(\dfrac{K[x_{n-h+1}, \ldots, x_n]}{(x_i^{a_i} \mid n-h+1 \leqslant i \leqslant n)} \right)_d & \text{if } h \geqslant 1; \\ 0 & \text{if } h = 0. \end{cases}$$

Whenever $\underline{a}$ is clear from the context, we will omit it from the notation.

Remark 5.2 Notice that $\begin{bmatrix} h \\ d \end{bmatrix}_{\underline{a}}$ actually depends on n: for instance $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(2)} = \begin{cases} 0 \text{ if } n = 1; \\ 1 \text{ otherwise.} \end{cases}$

The next definition is based on the Macaulay representation, cf. [4, Section 4.2], but it takes also into account the additional information brought by the degree sequence.

We adopt the standard convention that $\infty - 1 = \infty$ and $a \leqslant \infty$ for all $a \in \mathbb{Z}$.

With the above notation, given an integer $0 < k \leqslant \begin{bmatrix} n \\ d \end{bmatrix}$, we may write

$$k = \begin{bmatrix} k_d \\ d \end{bmatrix} + \begin{bmatrix} k_{d-1} \\ d-1 \end{bmatrix} + \ldots + \begin{bmatrix} k_1 \\ 1 \end{bmatrix},$$

where $k_d \geqslant k_{d-1} \geqslant \ldots \geqslant k_1 \geqslant 0$ and $\#\{t \mid k_t = i\} \leqslant \begin{cases} a_{n-i} - 1 & \text{for } 0 \leqslant i < n; \\ 1 & \text{for } i = n. \end{cases}$

Such an expression is called the $(\underline{a}, n)$-*Macaulay representation of* k *in base* d. As for the classical Macaulay representation, which corresponds to the choice $a_i = \infty$ for all i, the $(\underline{a}, n)$-Macaulay representation of k in base d exists, and it is unique; for instance, see [16, 29, 45].

Finally, given the $\underline{a}$-Macaulay representation of k in base d, we let

$$k_{\underline{a}}^{\langle d \rangle} = \begin{bmatrix} k_d \\ d+1 \end{bmatrix} + \begin{bmatrix} k_{d-1} \\ d \end{bmatrix} + \ldots + \begin{bmatrix} k_1 \\ 2 \end{bmatrix}.$$

Observe that, given any $\underline{a}$-lpp ideal $J \subseteq A$ with $k = H(A/J; d)$, then $H(A/\mathfrak{m}J +$ å$; d+1) = k_{\underline{a}}^{\langle d \rangle}$.

Since $\mathfrak{m}_1 J_d \subseteq J_{d+1}$, the $k_{\underline{a}}^{\langle d \rangle}$ represents the maximal growth in degree $d+1$ of the quotient by an $\underline{a}$-lpp ideal which has Hilbert function equal to k in degree d, as it happens in the classical case.

Next, we present a proof of the following enhanced version of Macaulay Theorem, see for instance [16, 45], which is a direct consequence of Theorem 2.1.

Theorem 5.3 *Let* $\underline{a} = (a_1, \ldots, a_r)$ *be a degree sequence,* $\mathfrak{a}$ *the corresponding pure-powers ideal, and* $R = A/\mathfrak{a}$. *Let* $H : \mathbb{N} \longrightarrow \mathbb{N}$ *be a numerical function; then,* H *is the Hilbert function of* R/I *for some homogeneous ideal* I *of* R *if and only if*

$$H(d+1) \leqslant H(d)_{\underline{a}}^{\langle d \rangle} \quad \text{for all } d \geq 1$$

Proof Let $I \subseteq R$ be a homogeneous ideal and J its lift to A. By Theorem 2.1, $L = \mathrm{LPP}^{\underline{a}}(J)$ is an ideal with the same Hilbert function as I, and from the fact that $\mathfrak{m}_1 J_d \subseteq J_{d+1}$ we get that $H(A/J; d+1) = H(A/L; d+1) \leqslant H(A/L; d)_{\underline{a}}^{\langle d \rangle} = H(A/J; d)_{\underline{a}}^{\langle d \rangle}$.

Conversely, let H be a numerical function that satisfies the growth condition, d be a non-negative integer, and let $V \subseteq A_d$ be an $\underline{a}$-lpp K-vector space such that $\dim_K(A_d/V) = H(d)$. Consider the $\underline{a}$-lpp ideal $J = (V) + \mathfrak{a}$; then $H(d)_{\underline{a}}^{\langle d \rangle}$ coincides with the dimension of $(A/J)_{d+1}$ which, by assumption, is at least $H(d+1)$. By adding appropriate monomials to J_{d+1} if necessary, we can make J into an $\underline{a}$-lpp K-ideal such that $\dim_K((A/J)_{d+1}) = H(d+1)$. Arguing in this way for all d, we obtain a monomial ideal I containing $\mathfrak{a}$, which in fact is an $\underline{a}$-lpp ideal, with Hilbert function H. □

There are implementations of these results in software systems such as Macaulay2, see for instance the one authored by White [47].

Example 5.4 Let $A = K[x_1, x_2, x_3]$, and let $I \subseteq A$ be a homogeneous ideal which contains a regular sequence of degree $\underline{a} = (3, 3, 4)$. Suppose that, regarding its Hilbert function, we only know that $H(A/I; 5) = 5$, and that we would like to estimate $H(A/I; 6)$. Classically, this is achieved by means of Macaulay Theorem, which provides $H(A/I; 6) \leqslant 5$. However, since $\mathrm{EGH}_{\underline{a}}$ holds by Theorem 4.1, we know that $H(I) = H(\mathrm{LPP}^{\underline{a}}(I))$, therefore Theorem 5.3 yields that $H(A/I; 6) \leqslant 5_{\underline{a}}^{\langle 5 \rangle} = 2$.

The following result was observed by Liang [35].

Proposition 5.5 *Let $I \subseteq A = K[x_1, x_2, x_3]$ be an ideal which contains an ideal* $\mathbf{f}$ *generated by a regular sequence of degree (a_1, a_2) and let $\mu(I)$ denote its minimal number of generators; then, $\mu(I) \leqslant a_1 \cdot a_2$.*

Proof Observe that any ideal containing $\mathbf{f}$ satisfies $\mathrm{EGH}_{(a_1,a_2)}$ by Theorem 4.1, therefore by Lemma 2.15 we have $H(I/\mathfrak{m}I) \leqslant H(L/\mathfrak{m}L)$, where $L = \mathrm{LPP}^{(a_1,a_2)}(I)$. Thus, we may as well bound $\mu(L)$. Notice that, if $u = x_1^i x_2^j x_3^k$ is a minimal generator of J, then $0 \leqslant i < a_1$ and $0 \leqslant j < a_2$, since J contains $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2})$. Moreover, if $v = x_1^{i'} x_2^{j'} x_3^{k'}$ is another minimal monomial generator of J, then necessarily $i' \neq i$ or $j' \neq j$. Therefore, there are at most $a_1 \cdot a_2$ possible choices for i and j, as desired. □

Proposition 5.5 can be applied to bound the number of defining equations of curves in $\mathbb{P}^3$. In fact, such a curve is defined by a homogeneous height two ideal $P \subseteq K[x_0, x_1, x_2, x_3]$, which then contains a regular sequence of some degree (a_1, a_2). Pick a general linear form ℓ which is regular modulo P and let $\overline{A} = A/\ell \cong K[x_1, x_2, x_3]$, and $\overline{P} = P\overline{A}$. Then $\mu(\overline{P}) = \mu(P)$, and use Proposition 5.5 on $\overline{P}$, since the latter contains a regular sequence of degree (a_1, a_2).

As we mentioned in the introduction, see Conjecture 1.2, another application of the EGH Conjecture is the Cayley-Bacharach Theorem. Its original formulation states that a cubic $\mathscr{C} \subseteq \mathbb{P}^2$ which contains eight points that lie on the intersection of two cubics, must contain the ninth point as well. Later on, this fact has been extended and generalized in various ways. We illustrate a connection with the EGH in the following example.

Example 5.6 Let $X \subseteq \mathbb{P}^3$ be a complete intersection of degree (3, 3, 3). We show that a cubic hypersurface Y containing at least 22 of the 27 points of X, must contain X.

To see this, let $\mathbf{f} = (f_1, f_2, f_3) \subseteq A = K[x_1, \dots, x_4]$ be an ideal of definition of X. Moreover, let g be a cubic defining Y, let $I = \mathbf{f} + (g)$ and, by way of contradiction, assume that $g \notin \mathbf{f}$. Let $|K| = \infty$; after a general change of coordinates, if necessary, we may write $I^{\text{sat}} = (I :_A x_4^\infty)$ and assume that x_4 is A/I^{sat}-regular.

Clearly, $g \in I^{\text{sat}}$. Next, we claim that we may assume that $g \notin \mathbf{f} + (x_4)$. In fact, if this is not the case, there exists $0 \neq g_1 \in (I :_A x_4) \subseteq I^{\text{sat}}$ of degree at most 2 such that $g = f + g_1 x_4$, for some $f \in \mathbf{f}$. The element g_1 may or may not belong to $\mathbf{f} + (x_4)$. If it does, arguing as above, we obtain that I^{sat} actually contains a linear form ℓ, which is not in $\mathbf{f} + (x_4)$, since x_4 is A/I^{sat}-regular. Either way, we found an element $g_2 \in I^{\text{sat}}$ of degree < 3 which does not belongs to $\mathbf{f} + (x_4)$. Multiplying it by an appropriate power of x_4, we obtain a form g_3 of degree 3 which still belongs to I^{sat}, but does not belong to $\mathbf{f} + (x_4)$. Therefore, we may let $g = g_3$, and our claim is proven.

Henceforth, let $\overline{A} = A/(x_4)$ and denote by $\bar{\mathbf{f}}$, $\overline{I}$, $\overline{g}$ and $\overline{I^{\text{sat}}}$ the images in $\overline{A}$ of $\mathbf{f}$, I, g and I^{sat} respectively; moreover, let $J = \bar{\mathbf{f}} + (\overline{g}) \subseteq \overline{A}$. Then, we immediately have

$$\mathrm{e}(A/I) = \mathrm{e}(A/I^{\text{sat}}) = \mathrm{e}(\overline{A}/\overline{I^{\text{sat}}}) \leqslant \mathrm{e}(\overline{A}/J).$$

By Proposition 4.15, $\mathrm{LPP}^{(3,3,3)}(J)$ is an ideal with the same Hilbert function as J. Moreover, since $\overline{g} \notin \bar{\mathbf{f}}$ by what we have seen above, the ideal $\mathrm{LPP}^{(3,3,3)}(J)$ must contain the monomial $x_1^2 x_2$. In particular,

$$\mathrm{e}(A/I) \leqslant \mathrm{e}(\overline{A}/J) = \mathrm{e}(\overline{A}/\mathrm{LPP}^{(3,3,3)}(J)) \leqslant \mathrm{e}(\overline{A}/(x_1^3, x_1^2 x_2, x_2^3, x_3^3)) = 21.$$

However, our hypothesis guarantees that $\mathrm{e}(A/I) \geqslant 22$, a contradiction.

We conclude the paper by illustrating how the combinatorial Kruskal-Katona Theorem [33, 34], a characterization of all the possible f-vectors of simplicial complexes Δ, is related to the EGH Conjecture for $\underline{a} = (2, 2, \dots, 2)$. For additional details on what follows, see for instance [30, Section 6.4].

Recall that the f-vector $f(\Delta) = (f_0, \dots, f_{r-1})$ of an $(r-1)$-dimensional simplicial complex Δ simply records in its entry f_{i-1} the number of faces of Δ of dimension $i-1$. As it is customary, we set $f_{-1} = 1$. Given positive integers h, d, write its Macaulay representation $h = \binom{h_d}{d} + \binom{h_{d-1}}{d-1} + \dots + \binom{h_1}{1}$, where $h_d \geqslant h_{d-1} \geqslant \dots \geqslant h_1 \geqslant 0$, and set

$$h^{(d)} = \binom{h_d}{d+1} + \binom{h_{d-1}}{d} + \dots + \binom{h_1}{2};$$

the Kruskal-Katona Theorem states that $(f_0, \ldots, f_{r-1})$ is the f-vector of a simplicial complex of dimension $r - 1$ if and only if $f_{d+1} \leqslant f_d^{(d+1)}$ for each $d = 0, \ldots, r - 2$.

Given a simplicial complex Δ, its f-vector $f(\Delta)$ and its Stanley-Reisner ring $K[\Delta]$, we have that $K[\Delta] = K[x_1, \ldots, x_n]/J$, where $n = f_0$ and $J = J_\Delta$ is a square-free monomial ideal. If we let $R = K[x_1, \ldots, x_n]/I$, where $I = J + (x_1^2, \ldots, x_n^2)$, then it is easy to see that $H(R; i) = f_{i-1}$ for all $i \geqslant 0$.

On the other hand, any monomial ideal $I \subseteq A = K[x_1, \ldots, x_n]$ containing $\mathfrak{a} = (x_1^2, \ldots, x_n^2)$, can be written uniquely as $I = J + \mathfrak{a}$, where J is a square-free monomial ideal. If we consider $\Delta = \Delta_J$, then its f-vector $f(\Delta) = (f_0, \ldots, f_{r-1})$, where $f_i = H(A/I; i + 1)$ for all $i \geqslant 0$.

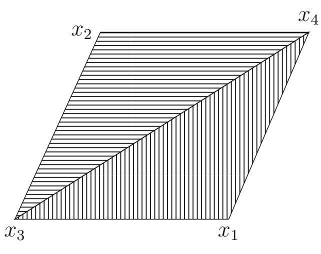
Finally, the crucial observation is that $\begin{bmatrix} k \\ d \end{bmatrix}_{\underline{a}} = \binom{k}{d}$ when if $\underline{a} = (2, 2, \ldots, 2)$.

Therefore, the numerical condition of Theorem 5.3 can be restated as

$$f_d = H(R; d + 1) \leqslant H(R; d)_{\underline{a}}^{\langle d \rangle} = H(R; d)^{(d)} = f_{d-1}^{(d)}, \quad \text{for all } d \geqslant 1,$$

which is precisely the condition of Kruskal-Katona Theorem.

Example 5.7 Let $f = (4, 5, 2)$, and let us construct a simplicial complex Δ such that $f(\Delta) = f$. Consider the numerical function $H : \mathbb{N} \to \mathbb{N}$ defined as $H(0) = 1$, $H(1) = 4$, $H(2) = 5$, $H(3) = 2$, and $H(d) = 0$ for $d > 3$. By means of Theorem 5.3, it can be checked that there exists a $(2, 2, 2, 2)$-lpp ideal I with Hilbert function equal to H, namely, $I = (x_1x_2) + (x_1^2, x_2^2, x_3^2, x_4^2)$. If we let $J = (x_1x_2)$, then $\Delta = \Delta_J$ is the following 2-dimensional simplicial complex



and $f(\Delta) = f$.

Example 5.8 If $f = (4, 5, 3)$, then there is no simplicial complex Δ such that $f(\Delta) = f$, since there is no $(2, 2, 2, 2)$-lpp ideal of $K[x_1, x_2, x_3, x_4]$ with Hilbert function H satisfying $H(2) = 5$ and $H(3) = 3 > H(2)_{(2,2,2,2)}^{\langle 2 \rangle} = 2$.

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Fibers of Rational Maps and Elimination Matrices: An Application Oriented Approach

Laurent Busé and Marc Chardin

Dedicated to David Eisenbud on the occasion of his seventy-fifth birthday.

1 Introduction

In geometric modeling or closely related domains, parameterized curves or surfaces are used intensively. Actually, 2D and 3D geometric objects are often represented by assembling pieces of algebraic rational curves and surfaces that are called rational Bézier curves and surfaces. Typical examples go from the letter fonts stored in a computer to a complex CAD model of mechanical pieces (see e.g. [20, Chapter 3] and [23, 34]). In this context, intersection problems between rational curves and surfaces are central questions to be solved. An important problem is to decide whether a point belongs to a given rational curve or surface. There is a huge literature on this topic with quite different types of techniques (see [34, Chapter 5] and references therein). Among them, the development of algebraic methods to turn the parametric representation of an algebraic curve or surface into an implicit representation received a lot of attention. This so-called implicitization problem is quite useful, because the membership problem we just mentioned can be decided by means of an evaluation operation, which is much simpler. Mathematical tools for solving the implicitization problem go back to the elimination theory as developed by Sylvester, Cayley, Dixon and others (see e.g. [21, 37]). A more modern version, based on resultant theory, can be found in many recent papers and books (see e.g. [17, 24, 29]).

From a practical point of view, the implicitization of a rational curve or surface by means of polynomial equation(s), typically the implicit equation $F(x_0, x_1, x_2) = 0$

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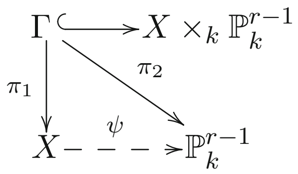
of a rational planar curve, is not enough for solving intersection problems on geometric models. Indeed, as already mentioned, in this field, geometric models are built from pieces of rational curves and surfaces. Therefore, given a point on a parameterized curve or surface, it is necessary to determine its pre-image(s) via the parameterization in order to decide if it belongs to the piece that is used (see e.g. [35]). It is clear that an implicit equation does not allow to do that directly, as it only detects if a point belongs to the entire algebraic curve or surface. In what follows, we will focus on tools and methods from commutative algebra and algebraic geometry that have been developed in order to not only describe implicit equations of the image of a rational map, but also to analyze and determine the fibers of these rational maps, especially the finite fibers.

The setting we will focus on in this survey is the following. Suppose given a rational map $\psi : X \dashrightarrow \mathbb{P}_k^{r-1}$, where X will be typically a (product of) projective space(s) of dimension ≤ 3 ($r = 3, 4$) over a field k, and assume that it is generically finite onto its image. Then, we would like to detect and compute the finite fibers of ψ. The tools that we will present are deeply rooted in elimination theory with a particular focus on elimination matrices. The simplest examples of such matrices are the famous Sylvester matrix, Dixon matrices [21] or Macaulay matrices [29, 31]. The key observation here is to lift the rational map ψ as a projection from its graph Γ to $\mathbb{P}^{r-1}$, turning this way our problem into the study of a projection (elimination) map. More precisely, there is a diagram



and we will describe how the fibers of π_2 can be analyzed by means of elimination matrices.

The paper is organized as follows. In Sect. 2 we set notation and we introduce blowup algebras associated to a rational map ψ, namely the Rees and symmetric algebras, and recall their connection to the graph of ψ. Section 3 is devoted to elimination techniques. We first show how Fitting ideals associated to some graded components of blowup algebras are connected to the image and fibers of a rational map. Then, we explain how the choice of the graded components to be considered is governed by the control of the vanishing of some local cohomology modules. This section ends with an important result showing that for finite fibers the control of this vanishing can be done globally (and not for each point). In Sect. 4 we derive the first consequences of the above-mentioned methods in the case of curve parameterizations, i.e. the case where the source is a projective line. Besides the computation of the fibers of such maps, we also provide an estimate of the Castelnuovo-Mumford regularity of rational curves as a by-product of our approach. Section 5 deals with the case of hypersurface parameterizations without base point, i.e. morphisms from $\mathbb{P}^{n-1}$ to $\mathbb{P}^n$. Here, the emphasis is on the use of non linear equations of the Rees algebra in order to get more compact elimination matrices. Then, the case of surface parameterizations is discussed in Sect. 6, where the presence of finitely many base points is considered. There is also a detailed discussion on the enumeration and determination of positive dimensional fibers of such parameterizations. Finally, the paper ends with Sect. 7 where the challenging case of three-dimensional generically finite and dominant rational maps is treated. This setting is actually motivated by the computation of the Euclidian distance of a point to a parameterized 3D surface, an important problem in geometric modeling. It will also be the occasion to illustrate how to deal with blowup algebras over multigraded rings.

Works of David Eisenbud were very inspiring while exploring this subject and the title of his beautiful book named 'The Geometry of Syzygies" fits perfectly our approach, even though he probably had in mind other strong and fruitful relations between geometry and syzygies while choosing this title.

2 Graph of a Rational Map

Given a rational map, the determination of its image or of the parameters corresponding to one point of its image (the fiber) relies, at least in the algebraic approach that we present, on the choice of compactifications for the source and the target. It turns out in practice that a good choice for compactifying the source could help in speeding up the computations, this choice being adapted to the type of parametrization that is used. For this reason, in this survey we will focus on rational maps of the form

$$\begin{aligned} \psi : X &\dashrightarrow \mathbb{P}^{r-1}_k \\ x &\mapsto (f_1(x) : \cdots : f_r(x)), \end{aligned} \tag{1}$$

where X is a toric variety given in terms of its Cox ring, over a field k. A very important case is when $X = \mathbb{P}^{n-1}_k$ is itself a projective space, but choosing for X a product of projective spaces is also often used.

Recall that a Cox ring is an extension of the usual construction used in the case of a projective space or a product of such spaces [18]. It is given by a grading on a polynomial ring R by an abelian group G (that corresponds to the Picard group of X) and a specific monomial ideal B that defines the empty set in X. The points of X (in the scheme sense, in other words irreducible reduced subvarieties) correspond to G-graded prime ideals and there is a one-to-one correspondence between subschemes of X and G-graded R-ideals that are B-saturated. Coherent G-graded R-modules correspond to coherent sheaves on X and graded pieces of their local cohomology with respect to B correspond to sheaf cohomology of corresponding twists of this coherent sheaf, in a very similar way as in the case of a projective space.

Thus, the rational map ψ, as defined by (1), corresponds to a tuple of homogeneous polynomials $(f_1, \dots, f_r)$ of the same degree $d \in G$. This tuple is uniquely determined, up to multiplication by an element in $k \setminus \{0\}$, assuming that the f_i's have no common factor. To understand image and fibers of ψ, it is no surprise that the graph of ψ plays a very important role. As we will now see, this graph corresponds to the notion of Rees algebra.

2.1 Graph and Rees Algebra

For an ideal I in a ring R, the Rees algebra is defined as the subalgebra $\oplus_{t \geq 0} I^t T^t \subseteq R[T]$. From this description, it is clear that it is a domain if R is a domain. When $I = (f_1, \dots, f_r)$ is a graded ideal in the graded ring R and the f_i's are of the same degree d, the Rees algebra admits a bigrading as follows. Let $S := R[T_1, \dots, T_r]$ with the bigrading (i.e. $G \times \mathbb{Z}$-grading) $\deg(T_i) := (0, 1)$ and $\deg(x) := (\deg(x), 0)$ for $x \in R$, then there is a bigraded onto map

$$S = R[T_1, \dots, T_r] \to \mathcal{R}_I := \oplus_t I^t(td)$$
$$T_i \mapsto f_i \in I(d)_0 = I_d.$$

This grading gives $(\mathcal{R}_I)_{\mu,t} = (I^t)_{\mu+td}$ for $t \geq 0$. Moreover, $\mathcal{R}_I = S/\mathfrak{P}$ for some $G \times \mathbb{Z}$-graded prime ideal $\mathfrak{P}$ whose elements are called the equations of the Rees algebra $\mathcal{R}_I$.

As a key property, the Rees algebra $\mathcal{R}_I$ defines the graph of ψ. To prove it, notice that the defining ideal of the Rees algebra $\mathfrak{P} \subseteq S$ contains the elements $G_{ij} := f_i T_j - f_j T_i$ for any $i < j$. In particular, off $V(I)$, the Rees algebra coincides with the graph of ψ. Thus, the closure Γ of the graph in $X \times \mathbb{P}^{r-1}_k$ is the closure of the variety defined by the G_{ij}'s in $(X \setminus V(I)) \times \mathbb{P}^{r-1}_k$, which is the one defined by $\mathcal{R}_I$, as $\mathcal{R}_I$ is a domain. We notice that another more geometrical way to state this, is that the Koszul relations G_{ij} define a subscheme of $X \times \mathbb{P}^{r-1}_k = \mathrm{Proj}_{G\times\mathbb{Z}}(S)$ that contains the graph of ψ (more precisely the Zariski closure Γ of this graph) as an irreducible component. The following lemma is useful to determine cases where the equations G_{ij} define the Rees algebra (Micali proved that a similar result holds over any commutative ring; see [32, Théorème 1]).

Lemma 2.1 *Let R be a Cohen-Macaulay local domain and $I = (f_1, \dots, f_r)$ be a complete intersection ideal of codimension r. Then the defining ideal $\mathfrak{P}$ of $\mathcal{R}_I$ is generated by the elements $G_{ij} := f_i T_j - f_j T_i$.*

Proof Consider the $(2 \times r)$-matrix

$$M := \begin{bmatrix} f_1 & \cdots & f_r \\ T_1 & \cdots & T_r \end{bmatrix}.$$

Let $J := I_2(M) = (G_{ij})$. The ideals $\mathfrak{P}$ and J coincide off $V(I.S)$, which is of codimension r in S. As $\mathfrak{P}$ is of codimension $r-1$, it follows that J has depth at least $r-1$. Hence the Eagon-Nortcott complex associated to M is a free S-resolution of $J := (G_{ij})$. Thus S/J is Cohen-Macaulay as well, hence unmixed. As J and $\mathfrak{P}$ coincide off $V(I.S)$, it follows that $J = \mathfrak{P}$. □

2.2 Symmetric Algebra

The elements in $\mathfrak{P}$ of T-degree 1, that we can write $\mathfrak{P}_{*,1}$, correspond to linear forms $\sum_i a_i T_i$ with $a_i \in R$ such that $\sum_i a_i f_i = 0$, that is to the first syzygies of the given generators of I, written as linear forms of the T_i's with coefficients in R, in place of a r-tuple of elements in R. The surjection $S/(\mathfrak{P}_{*,1}) \to S/\mathfrak{P}$ is an incarnation of the canonical map $\mathrm{Sym}_R(I) \to \mathcal{R}_I$, whose kernel is the non linear part of $\mathfrak{P}$ (the torsion of the symmetric algebra).

It is important to notice that, locally at a prime $\mathfrak{q} \in \mathrm{Spec}(R)$ where I is a complete intersection, the symmetric and Rees algebras coincide, by Lemma 2.1. This shows that the schemes in $X \times \mathbb{A}^r_k$ defined by $\mathrm{Sym}_R(I)$ and $\mathcal{R}_I$ coincide whenever this holds for any $\mathfrak{q} \in \mathrm{Proj}_G(R/I)$. In other words,

Proposition 2.2 *If* $\mathrm{Proj}_G(R/I)$ *is locally a complete intersection in X, then*

$$\mathcal{R}_I = \mathrm{Sym}_R(I)/H^0_B(\mathrm{Sym}_R(I)).$$

We notice that the case $\mathrm{Proj}_G(R/I) = \emptyset$, i.e. I contains a power of B, is contained in the above proposition.

3 Elimination Matrices and Fibers of Projections

In this section we assume that X is a product of projective spaces and that the subscheme $\Gamma \subset X \times \mathbb{P}^{r-1}_k$ is given by finitely many equations that are homogeneous with respect to the $\mathbb{Z}^s$-grading of the coordinate ring R of X. We denote by $J \subset S = R[T_1, \dots, T_r]$ the ideal generated by these equations, by $\mathcal{S}$ the quotient ring $R[T_1, \dots, T_r]/J$ and by B the irrelevant ideal associated to X. We also set $A := k[T_1, \dots, T_r]$.

3.1 Elimination Ideal

Consider the canonical projection $\pi_2 : X \times \mathbb{P}_k^{r-1} \to \mathbb{P}_k^{r-1}$. It is a classical result that $\pi_2(\Gamma)$ is closed in $\mathbb{P}_k^{r-1}$ and is defined by the elimination ideal

$$\mathfrak{A}(J) := (J : B^\infty) \cap A.$$

An interesting fact is that the elimination ideal can be connected to the graded components of the quotient ring $\mathcal{S}$.

Lemma 3.1 *Assume that $\nu \in \mathbb{N}^s$ is such that $H^0_B(\mathcal{S})_\nu = 0$, then $\mathfrak{A}(J) = \mathrm{ann}_A(\mathcal{S}_\nu)$. Moreover, there exists an integer n_ν such that*

$$\mathfrak{A}(J)^{n_\nu} \subseteq \mathrm{Fitt}^0_A(\mathcal{S}_\nu) \subseteq \mathfrak{A}(J)$$

where $\mathrm{Fitt}^0_A(\mathcal{S}_\nu)$ denotes the initial Fitting ideal of the A-module $\mathcal{S}_\nu$.

Proof $\mathcal{S}' := \mathcal{S}/H^0_B(\mathcal{S})$ is generated over $A/\mathfrak{A}(J) = (\mathcal{S}')_0$ by the variables of R and, as B is contained in the ideal generated by any of these s sets of variables of R (B is the intersection of the ideals B_i generated by the sets of variables), $\mathcal{S}'$ is saturated with respect to B_i. In particular, for any $\nu \in \mathbb{N}^s$ there exists a non zero divisor of degree ν on $\mathcal{S}'$, proving the claimed equality. The inclusions derive from any finite presentation of the A-module $\mathcal{S}_\nu$. □

Now, for any $\nu \in \mathbb{N}^s$ such that $H^0_B(\mathcal{S})_\nu = 0$, denote by $\mathbf{M}_\nu$ a presentation matrix of the A-module $\mathcal{S}_\nu$:

$$A^b \xrightarrow{\mathbf{M}_\nu} A^a \to \mathcal{S}_\nu \to 0.$$

As a consequence of Lemma 3.1, for any point $p \in \mathbb{P}_k^{r-1}$ the corank of $\mathbf{M}_\nu(p)$ (the evaluation of $\mathbf{M}_\nu$ at the point p) is positive if and only if $p \in \pi_2(\Gamma)$. This result can actually be refined as follows.

3.2 Finite Fibers

Let $p \in \mathbb{P}_k^{r-1}$ and denote by $\mathcal{S}_p$ the specialization of $\mathcal{S}$ at p. From the definition of $\mathbf{M}_\nu$ and the stability of Fitting ideals under arbitrary base change (which is not the case for annihilators), for any ν, the corank of $\mathbf{M}_\nu(p)$ is nothing but the Hilbert function of $\mathcal{S}_p$ in degree ν. Thus, if the fiber of p under π_2 is finite then its Hilbert polynomial is a constant which is equal to the degree of this fiber. The next result shows that the degree ν such that the Hilbert function of $\mathcal{S}_p$ reaches its Hilbert polynomial is globally controlled by the vanishing of the local cohomology of $\mathcal{S}$.

Theorem 3.2 *Let $\nu \in \mathbb{N}^s$ be such that $H^0_B(\mathcal{S})_\nu = H^1_B(\mathcal{S})_\nu = 0$ and suppose given $p \in \mathbb{P}^{r-1}_k$ such that the fiber $\pi_2^{-1}(p) \subset X$ is finite (a scheme of dimension zero or empty), then*

$$\operatorname{corank}(\mathbf{M}_\nu(p)) = \deg(\pi_2^{-1}(p)).$$

Sketch of Proof First, Grothendieck-Serre's formula (see e.g. [5, Proposition 4.26]) shows that, for any $\nu \in \mathbb{Z}^s$,

$$\dim_k(\mathcal{S}_p)_\nu = \deg(\pi_2^{-1}(p)) + \sum_{i \geq 0}(-1)^i \dim_k H^i_B(\mathcal{S}_p)_\nu,$$

and by Grothendieck vanishing theorem $H^i_B(\mathcal{S}_p) = 0$ for $i > 1$, as $\pi_2^{-1}(p)$ is of dimension zero or empty (in which case it also holds for $i = 1$).

Let $\mathfrak{p}$ be the graded ideal defining the point p and let $k_p := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be its residue field. Since $H^i_B(\mathcal{S} \otimes_A A_\mathfrak{p})_\nu = H^i_B(\mathcal{S})_\nu \otimes_A A_\mathfrak{p}$, we may replace A by $A_\mathfrak{p}$, S by $S \otimes_A A_\mathfrak{p}$ and $\mathcal{S}$ by $\mathcal{S} \otimes_A A_\mathfrak{p}$ to assume that A is local with residue field k_p.

Next one uses a construction as in [15, Lemma 6.2] to show the existence, for any finitely generated S-module M and A-module N, of two spectral sequences with same abutment and second terms:

$$'_2E^p_q = H^p_B(\operatorname{Tor}^A_q(M, N)) \quad \text{and} \quad ''_2E^p_q = \operatorname{Tor}^A_q(H^p_B(M), N).$$

As the support in S of $\operatorname{Tor}^A_q(M, N)$ sits inside the one of $M \otimes_A N$, one deduces by choosing $M := \mathcal{S}$ and $N := k_p$, first that $\max\{i \mid H^i_B(\mathcal{S}) \neq 0\} = \max\{i \mid H^i_B(\mathcal{S} \otimes_A k_p) \neq 0\} = 1$, and then that $H^1_B(\mathcal{S})_\nu = 0 \Leftrightarrow H^1_B(\mathcal{S} \otimes_A k_p)_\nu = 0$ and $H^0_B(\mathcal{S})_\nu = H^1_B(\mathcal{S})_\nu = 0 \Rightarrow H^0_B(\mathcal{S} \otimes_A k_p)_\nu = 0$. □

4 When the Source Is $\mathbb{P}^1$

In this section we focus on maps whose source is a projective line, which covers the case of rational curves embedded in a projective space of arbitrary dimension. We first describe how fibers of curve parameterizations can be obtained from elimination matrices, following the methods introduced in Sect. 3. Then, in the second part of this section we derive an upper bound for the Castelnuovo-Mumford regularity of rational curves by means of similar elimination techniques. The definition of Castelnuovo-Mumford regularity is also quickly reviewed.

4.1 Matrix Representations

For simplicity, we deviate from general notations and write $R := k[x, y]$. We suppose given a rational map

$$\psi : \mathbb{P}^1_k \to \mathbb{P}^{r-1}_k$$
$$(x : y) \mapsto (f_1 : \cdots : f_r)$$

where the f_i's are homogeneous polynomials in R of degree $d \geq 1$ without common factor. Then, the image of ψ is a curve $\mathcal{C} \subset \mathbb{P}^{r-1}_k$ and ψ is a morphism, in other words it has no base point. A classical intersection theory formula yields the equality

$$\deg(\psi)\deg(\mathcal{C}) = d.$$

In this setting, the equations of the symmetric algebra of $I = (f_1, \ldots, f_r)$ define the graph of ψ, because, as we assume that the f_i's have no common factor, ψ is a morphism. By Hilbert-Burch Theorem, these equations have the following nice structure: these exist non-negative integers $\mu_1, \ldots, \mu_{r-1}$ such that $\sum_{i=1}^{r-1} \mu_i = d$ and R/I has a minimal finite free resolution of the form

$$0 \longrightarrow \sum_{i=1}^{r-1} R(-d-\mu_i) \xrightarrow{\Phi} R(-d)^r \longrightarrow I \longrightarrow 0.$$

The columns of the map Φ yield a basis of the first syzygy module of I. It follows that the forms $L_1, \ldots, L_{r-1}$ defined as

$$\begin{bmatrix} L_1 \\ \vdots \\ L_{r-1} \end{bmatrix} = {}^t\Phi \begin{bmatrix} T_1 \\ \vdots \\ T_r \end{bmatrix}$$

are defining equations of the symmetric algebra $\mathrm{Sym}_R(I)$. Therefore, for any integer ν the multiplication map

$$\oplus_{i=1}^{r-1} R_{\nu-\mu_i}[T_1, \ldots T_r](-1)^{r-1} \xrightarrow{(L_1,\ldots,L_{r-1})} R_\nu[T_1, \ldots T_r]$$

is a presentation of $\mathrm{Sym}_R(I)_\nu$ by free graded A-modules (recall $A = k[T_1, \ldots, T_r]$). We denote by $\mathbf{M}_\nu$ its matrix in some bases and set $B := (x, y)$.

Proposition 4.1 *For all* $\nu \geq \max_{i \neq j}\{\mu_i + \mu_j\}$, $H^0_B(\mathrm{Sym}_R(I))_\nu = H^1_B(\mathrm{Sym}_R(I))_\nu = 0$. *Hence the matrix* $\mathbf{M}_\nu$ *satisfies*

$$\mathrm{corank}(\mathbf{M}_\nu(p)) = \deg(\psi^{-1}(p)), \quad \forall p \in \mathbb{P}^{r-1}_k.$$

Proof As $L_1, \ldots, L_{r-1}$ form a complete intersection in $\mathbb{P}^1_k \times \mathbb{P}^{r-1}_k$, the claimed vanishing of the local cohomology modules follows from the comparison of two spectral sequences associated to the Čech-Koszul double complex of $L_1, \ldots, L_{r-1}$ (see [28, §2.10]). The property on the corank of the matrix $\mathbf{M}_\nu(p)$ then follows from Theorem 3.2. □

Example 4.2 (Plane Curves) In the case $r = 3$, the map ψ defines a curve $\mathcal{C}$ in the projective plane and $\mu_1 + \mu_2 = d$. In suitable bases, the matrix $\mathbf{M}_{d-1}$ is nothing but the Sylvester matrix associated to L_1 and L_2 and we recover here classical results. In particular, its determinant is equal to $F^{\deg(\psi)}$, where F is an implicit equation of $\mathcal{C}$.

Example 4.3 (Twisted Cubic) Consider the following map whose image is the twisted cubic

$$\begin{aligned} \psi : \mathbb{P}^1_k &\to \mathbb{P}^3_k \\ (x : y) &\mapsto (x^3 : x^2y : xy^2 : y^3). \end{aligned}$$

In this case $\mu_1 = \mu_2 = \mu_3 = 1$ and we get the matrix

$$\mathbf{M}_1 = \begin{pmatrix} -T_2 & -T_3 & -T_4 \\ T_1 & T_2 & T_3 \end{pmatrix}.$$

Applying Proposition 4.1, we deduce that the twisted cubic is a smooth curve and ψ is an isomorphism, because $\mathrm{corank}(\mathbf{M}_1(p)) \leq 1$ for any $p \in \mathbb{P}^3_k$.

4.2 Regularity Estimate for Rational Curves

As illustrated above, presentation matrices of the graded components of the symmetric algebra $\mathrm{Sym}_R(I)$ provide an interesting computational tool to manipulate rational curves. Let us recall the notion of Castelnuovo-Mumford regularity, which controls degrees of generators of an ideal, and much more: degrees of all syzygies (as we will see) and degrees of generators of Groebner bases for reverse lexicographic order and general coordinates.

4.2.1 The Four Equivalent Definitions of Castelnuovo-Mumford Regularity

Assume A is a Noetherian ring (hence $S = A[x_1, \ldots, x_n]$ as well) and M is a finitely generated graded S-module. And assume further that $\deg(x_i) = 1$ for all i and $\deg(a) = 0$ for $a \in A$ (standard grading). In what follows, we will denote by $K_\bullet(f; N)$ the Koszul complex associated to a sequence of elements f on a graded module N.

As $K_\bullet(x_1, \ldots, x_n; S)$ is a resolution of the S-module $A = S/(x_1, \ldots, x_n)$, by definition of Tor functors one has first

$$\mathrm{Tor}_i^S(M, A) \simeq H_i(K_\bullet(x_1, \ldots, x_n; M)).$$

By Noetherianity, $\mathrm{Tor}_i^S(M, A)$ is a finitely generated graded A-module (hence not zero in only finitely many degrees). If N is a graded module, we set

$$\mathrm{end}(N) := \sup\{\mu \mid N_\mu \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\},$$

write $S_+ := (x_1, \ldots, x_n)$ and define

$$a^i(M) := \mathrm{end}(H^i_{S_+}(M)) \in \mathbb{Z} \cup \{-\infty\}, \quad b_j(M) := \mathrm{end}(\mathrm{Tor}_j^S(M, R)) \in \mathbb{Z} \cup \{-\infty\}.$$

Recall that, as M is finitely generated, it is generated in degrees at most $b_0(M)$, and this estimate is optimal. We are now ready to introduce regularity (see [16, Proposition 2.4] and [15, §2]).

Theorem 4.4 *Let $M \neq 0$ be a finitely generated graded S-module. Then,*

$$\begin{aligned}\mathrm{reg}(M) := \max_i\{a^i(M) + i\} = \max_j\{b_j(M) - j\} &= \min_{F_\bullet^M}\{\sup_j\{b_0(F_j^M) - j\}\} \\ &= \min_{F_\bullet^M}\{\max_{j \le n}\{b_0(F_j^M) - j\}\} \in \mathbb{Z},\end{aligned}$$

where $F_\bullet^M$ runs over graded free S-resolutions of M.

Notice that $a^i(M) = -\infty$ unless $0 \le i \le n$ and that $b_j(M) = -\infty$ unless $0 \le j \le n$. However, F_j^M could be non zero for any j, even if $F_\bullet^M$ is a minimal resolution over a local ring A, unless A is local regular (in which case $F_j^M = 0$ for $j > n + \dim A$, if $F_\bullet^M$ is minimal).

Remark 4.5

(1) If A is local (or *local: graded with a unique graded maximal ideal) and $F_\bullet^M$ is a minimal graded free S-resolution of M, then

$$\mathrm{reg}(M) = \max_j\{b_0(F_j^M) - j\} = \max_{j \le n}\{b_0(F_j^M) - j\}.$$

(2) If $d := \mathrm{cd}_{S_+}(M) := \max\{i \mid H^i_{S_+}(M) \neq 0\}$ denotes the cohomological dimension of M relatively to S_+, then

$$\mathrm{reg}(M) = \max_{n-d \le j \le n}\{b_j(M) - j\}.$$

For instance, whenever A is a field and M is Cohen-Macaulay of dimension d,

$$\mathrm{reg}(M) = b_{n-d}(M) - n + d.$$

Recall that for a Noetherian local ring $(A, \mathfrak{m})$ and a finitely generated A-module M, $\mathrm{cd}_{\mathfrak{m}}(M) = \dim M := \dim(A/\mathrm{ann}_A(M))$: the cohomological dimension with respect to the maximal ideal is the dimension of the support of the module.

Let us briefly present the geometrical interpretation of the cohomological dimension with respect to S_+. For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, set $k_{\mathfrak{p}} := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \mathrm{Frac}(A/\mathfrak{p})$. The stalk of $\mathcal{F} := \tilde{M}$ at $V(\mathfrak{p})$ is the sheaf defined on $\mathbb{P}^{n-1}_{\mathrm{Spec}(A_{\mathfrak{p}})}$ by $M \otimes_A A_{\mathfrak{p}}$. The fiber of $\mathcal{F}$ at the corresponding point is the sheaf defined on $\mathbb{P}^{n-1}_{k_{\mathfrak{p}}}$ by $M \otimes_A k_{\mathfrak{p}}$, with the usual abuse of notations $\mathbb{P}^{n-1}_{k_{\mathfrak{p}}} := \mathbb{P}^{n-1}_{\mathrm{Spec}(k_{\mathfrak{p}})}$. The following result shows in particular that, in our situation, the cohomological dimension is the maximal dimension of the fibers of the family of sheaves given by $\mathcal{F}$ (plus one, as there is a difference of 1 between the dimension of a graded module and the one of the sheaf on the projective space it represents).

Lemma 4.6 ([15, Proposition 6.3])

(1) For any $\mathfrak{p} \in \mathrm{Spec}(A)$,

$$H^i_{S_+}(M) \otimes_A A_{\mathfrak{p}} \simeq H^i_{S_+}(M \otimes_A A_{\mathfrak{p}}).$$

(2) If $(A, \mathfrak{m}, k)$ *is local, then*

$$d := \max_{\mathfrak{p} \in \mathrm{Spec}(A)} \{\dim(M \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})\} = \max\{i \mid H^i_{S_+}(M \otimes_A k) \neq 0\}$$
$$= \max\{i \mid H^i_{S_+}(M) \neq 0\}, \textit{and}\, H^d_{S_+}(M) \otimes_A k \simeq H^d_{S_+}(M \otimes_A k).$$

Hence, by (1) and (2), $\mathrm{cd}_{S_+}(M)$ is the maximal dimension over the prime ideals $\mathfrak{p}$ of A (equivalently among the maximal ideals) of the modules $M \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. In other words, it is one more than the maximal dimension of support of the fibers of the family of sheaves $\mathcal{F}$ over $\mathrm{Spec}(A)$.

The notion of regularity extends to Cox rings, taking cohomology with respect to B or more generally with respect to any graded S-ideal; we refer the reader to [33] and [5].

4.2.2 The Regularity Estimate for Rational Curves

For the case of rational curves in a projective space, the next result gives another application of the matrices introduced in Sect. 4.1, for Castelnuovo-Mumford regularity estimation. The proof is very much related to the original argument of L'vovsky [30], which itself relies on work of Gruson, Lazarsfeld and Peskine [25]; our presentation has a more ring theoretic flavor.

Theorem 4.7 *Let $\psi : \mathbb{P}^1_k \longrightarrow \mathbb{P}^{r-1}_k$ be a morphism defined by $(f_1, \dots, f_r)$ and $I_{\mathcal{C}}$ be the defining ideal of the image of ψ. If ψ is birational to its image, then $\sum_{i=1}^{r-1} \mu_i = d$ and*

$$\operatorname{reg}(I_{\mathcal{C}}) \leq \max_{i \neq j}\{\mu_i + \mu_j\}.$$

In particular, $\operatorname{reg}(S/I_{\mathcal{C}}) \leq \deg(\mathcal{C}) - \operatorname{codim}(\mathcal{C})$ *if the f_i's are linearly independent (in other words, if the curve $\mathcal{C}$ is not sitting in any hyperplane).*

Proof Let $\rho := \max_{i \neq j}\{\mu_i + \mu_j\}$. From Proposition 4.1, it follows that $H^0_{(x,y)}(\mathcal{S}_I)_\mu = 0$ for $\mu \geq \rho - 1$, in particular $(\mathcal{S}_I)_\mu = (\mathcal{R}_I)_\mu$ for $\mu \geq \rho - 1$. This provides a presentation for the A-module $(\mathcal{R}_I)_{\rho-1}$:

$$\oplus_i A[x,y]_{\rho-1-\mu_i} \xrightarrow{\begin{bmatrix} & \vdots & \\ \cdots & L_{ij} & \cdots \\ & \vdots & \end{bmatrix}} A[x,y]_{\rho-1} \longrightarrow (\mathcal{R}_I)_{\rho-1} \longrightarrow 0$$

where, for j corresponding to $x^a y^b \in A[x,y]_{\rho-1-\mu_i}$ one has $x^a y^b L_i = \sum_{i=1}^{\rho} x^{i-1} y^{\rho-i} L_{ij}(T_1, \dots, T_r)$. We now use the graded presentation of the A-module $(\mathcal{R}_I)_{\rho-1}$ as above

$$A[-1]^N \xrightarrow{\Psi} A^\rho \longrightarrow (\mathcal{R}_I)_{\rho-1} \longrightarrow 0$$

where Ψ is the matrix of the linear forms L_{ij}. The ideal $J := \operatorname{Fitt}^0_A((\mathcal{R}_I)_{\rho-1})$ is an ideal of A generated in degree ρ (by the maximal minors of Ψ). Let $\mathfrak{n} := (T_1, \dots, T_r)$ be the graded irrelevant ideal of A. The conclusion will derive easily from the two following observations: (1) $J = I_{\mathcal{C}} \cap \mathfrak{a}$ with $V(\mathfrak{a})$ supported on the finitely many points of $\mathcal{C}$ where ψ is not locally invertible (and thus an isomorphism), (2) $\operatorname{reg}(J^{sat}) \leq \rho$.

Indeed, $J^{sat} = I_{\mathcal{C}} \cap \mathfrak{a}'$ with $V(\mathfrak{a}) = V(\mathfrak{a}')$, and the exact sequence

$$0 \to I_{\mathcal{C}}/J^{sat} \to S/J^{sat} \to S/I_{\mathcal{C}} \to 0$$

provides an exact sequence

$$H^1_{\mathfrak{n}}(S/J^{sat}) \longrightarrow H^1_{\mathfrak{n}}(S/I_{\mathcal{C}}) \longrightarrow H^2_{\mathfrak{n}}(I_{\mathcal{C}}/J^{sat}) = 0$$

and an isomorphism $H^2_{\mathfrak{n}}(S/I_{\mathcal{C}}) \simeq H^2_{\mathfrak{n}}(S/J^{sat})$, proving that $\operatorname{reg}(I_{\mathcal{C}}) \leq \operatorname{reg}(J^{sat})$.

To prove (1) notice that if $z \in \mathcal{C}$ is such that $\pi := \Gamma \to \mathcal{C}$ is locally an isomorphism at $z = V(\mathfrak{p})$, then $\mathcal{R}_I \otimes_A A_{\mathfrak{p}} \simeq (A/I_{\mathcal{C}})_{\mathfrak{p}}[x]$—in other words, the defining ideal of $\mathcal{R}_I$ over $(A/I_{\mathcal{C}})[x,y]$ contains an equation $fx + gy$ with

$f, g \in A/I_{\mathcal{C}}$ and $g \notin \mathfrak{p}$. Hence, for any $\mu \geq 0$, $(\mathcal{R}_I \otimes_A A_{\mathfrak{p}})_\mu \simeq A_{\mathfrak{p}}/(I_{\mathcal{C}})_{\mathfrak{p}}$ admits a presentation

$$A_{\mathfrak{p}}^t \xrightarrow{\Theta} A_{\mathfrak{p}} \longrightarrow (\mathcal{R}_I \otimes_A A_{\mathfrak{p}})_\mu \longrightarrow 0 \ ,$$

where the entries of Θ are generators of $I_{\mathcal{C}} \otimes_A A_{\mathfrak{p}}$. It follows that $J \otimes_A A_{\mathfrak{p}} = I_{\mathcal{C}} \otimes_A A_{\mathfrak{p}}$, as Fitting ideals are independent of the presentation. This proves (1).

We now prove (2). The Eagon-Northcott complex $E_\bullet$ of the matrix Ψ has the form:

$$\cdots \longrightarrow A[-\rho-2]^{a_3} \longrightarrow A[-\rho-1]^{a_2} \longrightarrow A[-\rho]^{a_1} \xrightarrow{\bigwedge^\rho \Psi} A \longrightarrow 0 \ .$$

We consider the double complex $\mathcal{C}_{\mathfrak{n}}^\bullet(E_\bullet)$ to estimate the regularity of $H_0(E_\bullet) = A/J$, where the notation $\mathcal{C}_{\mathfrak{n}}^\bullet(-)$ stands for the Čech complex with respect to the ideal $\mathfrak{n}$. Setting $H_i := H_i(E_\bullet)$, the two spectral sequences have respective second pages:

$$\begin{array}{cccc}
\cdots & H^0_{\mathfrak{n}}(H_2) & H^0_{\mathfrak{n}}(H_1) & H^0_{\mathfrak{n}}(A/J) \\
\cdots & H^1_{\mathfrak{n}}(H_2) & H^1_{\mathfrak{n}}(H_1) & H^1_{\mathfrak{n}}(A/J) \\
\cdots & H^2_{\mathfrak{n}}(H_2) & H^2_{\mathfrak{n}}(H_1) & H^2_{\mathfrak{n}}(A/J) \\
\cdots & 0 & 0 & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\cdots & H_i(H^n_{\mathfrak{n}}(E_\bullet)) & \cdots
\end{array}$$

As a consequence, $H^2_{\mathfrak{n}}(A/J) \simeq H_{n-2}(H^n_{\mathfrak{n}}(E_\bullet))$ and $H_{n-1}(H^n_{\mathfrak{n}}(E_\bullet))_\mu = 0$ implies that $H^1_{\mathfrak{n}}(A/J)_\mu = 0$. But $H^n_{\mathfrak{n}}(E_p) \simeq H^n_{\mathfrak{n}}(A(-\rho-p+1)^{a_p})$ for $p \geq 1$ vanishes in degrees $> \rho + p - 1 - n$. This shows that $H^1_{\mathfrak{n}}(A/J)_\mu = 0$ for $\mu > \rho - 2$ and $H^2_{\mathfrak{n}}(A/J)_\mu = 0$ for $\mu > \rho - 3$. Hence $\operatorname{reg}(S/J^{sat}) \leq \rho - 1$, as claimed. □

5 Morphisms from $\mathbb{P}^{n-1}_k$ to $\mathbb{P}^n_k$

A morphism has no base point and hence has only finite fibers. In this section, we will consider morphisms associated to hypersurface parameterizations and present some results that partially extend to more general situations, in particular to rational maps whose base locus is of dimension zero (see [8, Section 4]). We keep notations as in Sect. 2; we consider a morphism

$$\psi : \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k$$
$$x = (x_1 : \cdots : x_n) \mapsto (f_1(x) : \cdots : f_{n+1}(x)),$$

and set $R := k[x_1, \ldots, x_n]$, $A := k[T_1, \ldots, T_{n+1}]$.

The original approach of Jouanolou in this situation was to notice first that the Rees algebra is the saturation of the symmetric algebra, i.e. $\mathcal{R}_I = \mathrm{Sym}_R(I)/H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))$ with $\mathfrak{m} = (x_1, \ldots, x_n)$, and second that $\mathrm{Sym}_R(I)$ admits a resolution by the approximation complex of cycles, whose graded components provide free A-resolutions of $\mathrm{Sym}_R(I)_{\mu,*}$. Hence, given $\mu \geq 0$ such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_{\mu,*} = 0$, one gets a (minimal) free A-resolution of the A-module $(\mathcal{R}_I)_{\mu,*}$, whose annihilator is the ideal of the image.

We briefly recall what the approximation complex (of cycles) is. The two Koszul complexes $K_\bullet(f; S)$ and $K_\bullet(T; S)$ where $f := (f_1, \ldots, f_{n+1})$ and $T := (T_1, \ldots, T_{n+1})$, have the same modules $K_p = \bigwedge^p S^{n+1} \simeq S^{\binom{n+1}{p}}$ and differentials $d^f_\bullet$ and $d^T_\bullet$, respectively. Set $Z_p(f; S) := \ker(d^f_p)$. It directly follows from the definitions that $d^f_{p-1} \circ d^T_p + d^T_{p-1} \circ d^f_p = 0$, so that $d^T_p(Z_p(f; S)) \subset Z_{p-1}(f; S)$. The complex $\mathcal{Z}_\bullet := (Z_\bullet(f; S), d^T_\bullet)$ is the $\mathcal{Z}$-complex associated to the f_i's. Notice that $Z_p(f; S) = S \otimes_R Z_p(f; R)$, $Z_0(f; R) = R$, $Z_1(f; R) = \mathrm{Syz}_R(f_1, \ldots, f_{n+1})$, and the map d^T_1 is defined by

$$\begin{aligned} d^T_1 : S \otimes_R \mathrm{Syz}_R(f_1, \ldots, f_{n+1}) &\to S \\ (a_1, \ldots, a_{n+1}) &\mapsto a_1 T_1 + \cdots + a_{n+1} T_{n+1}. \end{aligned}$$

The following result, that holds for any finitely generated ideal I in a commutative ring R, shows the intrinsic nature of the homology of the $\mathcal{Z}$-complex, it is a key point in proving results on its acyclicity (see [27, Section 4]).

Theorem 5.1 *$H_0(\mathcal{Z}_\bullet) \simeq \mathrm{Sym}_R(I)$ and the homology modules $H_i(\mathcal{Z}_\bullet)$ are $\mathrm{Sym}_R(I)$-modules that only depend upon $I \subset R$, up to isomorphism.*

Now, we consider graded pieces:

$$\begin{aligned} \mathcal{Z}^\nu_\bullet \; : \; \cdots \longrightarrow A \otimes_k Z_2(f; R)_{\nu+2d} &\xrightarrow{d^T_2} A \otimes_k Z_1(f; R)_{\nu+d} \\ &\xrightarrow{d^T_1} A \otimes_k Z_0(f; R)_\nu \longrightarrow 0 \end{aligned}$$

where $Z_p(f; R)_{\nu+pd}$ is the part of $Z_p(f; R)$ consisting of elements of the form $\sum a_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ with the $a_{i_1 \cdots i_p}$'s all of the same degree ν. It is proved in [10, Theorem 5.2 and Proposition 5.5] that (we refer the reader to [24, Appendix A] for the notion of determinant of complexes):

Proposition 5.2 *$\mathcal{Z}_\bullet$ is acyclic and for any $\nu \geq (n-1)(d-1)$, $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_{\nu,*} = 0$. Hence, for such an integer ν, $\mathcal{Z}^\nu_\bullet$ is a minimal free A-resolution of $(\mathcal{R}_I)_\nu$ and $\det(\mathcal{Z}^\nu_\bullet) = H^\delta$, where H is the equation of the image and δ the degree of the map ψ onto its image.*

In addition, it can be seen that for these degrees ν, the fibers of ψ can be obtained by means of elimination matrices. More precisely, let $\mathbf{M}_\nu$ be a matrix of the first map of the complex $\mathcal{Z}_\bullet^\nu$, i.e. the map of free A-modules

$$\mathcal{Z}_1^\nu = A \otimes_k Z_1(f; R)_{\nu+d} \xrightarrow{d_1^T} \mathcal{Z}_0^\nu = A \otimes_k R_\nu.$$

Proposition 5.3 *Let $\nu \geq (n-1)(d-1)$ and suppose given $p \in \mathbb{P}_k^n$, then*

$$\operatorname{corank}(\mathbf{M}_\nu(p)) = \deg(\psi^{-1}(p)).$$

Proof As the matrix $\mathbf{M}_\nu$ is a presentation matrix of the A-module $\operatorname{Sym}_R(I)_{\nu,*}$, according to Theorem 3.2 one has to prove that $H^i_{\mathfrak{m}}(\operatorname{Sym}_R(I))_{\nu,*} = 0$ for $i = 0, 1$ and for all $\nu \geq (n-1)(d-1)$. This follows from Proposition 5.2 for $i = 0$ and the case $i = 1$ can be proved in the same vain; we refer the reader to the proof of [1, Proposition 5]. □

A useful structural result to go further, in particular to get more compact elimination matrices, is the following. Recall that $\mathfrak{P}$ denotes the defining ideal of the Rees algebra $\mathcal{R}_I$ in $S = R[T_1, \ldots, T_{n+1}]$. We define $\mathfrak{P}\langle\ell\rangle$ as the ideal generated by equations of the Rees algebra of T-degree at most ℓ. In particular $\operatorname{Sym}_R(I) = S/\mathfrak{P}\langle 1\rangle$ and $\mathcal{R}_I = S/\mathfrak{P}\langle\ell\rangle$ for $\ell \gg 0$.

Proposition 5.4 ([8, Corollary 1]) *For every $\nu \geq \nu_0(I) := \operatorname{reg}(I) - d$, the A-module $(\mathcal{R}_I)_{\nu,*}$ admits a minimal graded free A-resolution of the form*

$$\cdots \to \mathcal{Z}_i^\nu \to \cdots \to \mathcal{Z}_2^\nu \to \mathcal{Z}_1^\nu \oplus_{\ell=2}^n (\mathfrak{P}\langle\ell\rangle/\mathfrak{P}\langle\ell-1\rangle)_\nu \to \mathcal{Z}_0^\nu \qquad (1)$$

with $\mathcal{Z}_i^\nu = A \otimes_k Z_i(f; R)_{\nu+id}$.

Notice that $\mathcal{Z}_1^\nu = (\mathfrak{P}\langle 1\rangle/\mathfrak{P}\langle 0\rangle)_\nu$. In comparison with the admissible degrees in Proposition 5.2, the gain is potentially quite big in terms of the range of degrees it concerns due to the following:

Lemma 5.5 ([8, Corollary 3]) *The threshold degree $\nu_0(I)$ defined in Proposition 5.4 satisfies the inequalities*

$$\left\lfloor \frac{(n-1)(d-1)}{2} \right\rfloor \leq \nu_0(I) \leq (n-1)(d-1).$$

Moreover, the equality on the left holds if the forms $f_1, \ldots, f_{n+1}$ are sufficiently general and k has characteristic 0.

Above the threshold degree $\nu_0(I) := \operatorname{reg}(I) - d = \operatorname{reg}(R/I) + 1 - d$, the following nice property holds:

$$(\mathfrak{P}\langle\ell\rangle/\mathfrak{P}\langle\ell-1\rangle)_\nu \simeq (H_1)_{\nu+\ell d} \otimes_k A[-\ell], \quad \forall \nu \geq \nu_0(I), \quad \forall \ell > 1, \qquad (2)$$

where the brackets stands for degree shifting in the T_i's and H_1 denotes the first homology module of the Koszul complex $K_\bullet(f; R)$. It turns out that the isomorphism (2) is very explicit: given a non-Koszul syzygy $s := (h_1, \ldots, h_{n+1})$ with $\deg(h_i) = \nu + (\ell - 1)d$, which corresponds to an element in $(H_1)_{\nu+\ell d}$, or equivalently to the class of $h_1 T_1 + \cdots + h_{n+1} T_{n+1}$ in $\mathfrak{P}\langle 1\rangle/\mathrm{im}(d_2^T)$, one can write $h_i = \sum_{|\alpha|=\ell-1} c_{i,\alpha} f^\alpha$ as

$$\begin{aligned}\deg(h_i) \geq \nu_0(I) + (\ell-1)d &= \mathrm{reg}(R/I) + 1 + (\ell-2)d \geq \mathrm{reg}(R/I^{\ell-1}) + 1\\ &= \mathrm{end}(R/I^{l-1}) + 1\end{aligned}$$

(see [14, Theorem 1.7.1] for the last above inequality). The map sends s to the element $\sum_i \sum_\alpha c_{i,\alpha} T_i T^\alpha$. In the other direction, one writes (the class) of an element of $(\mathfrak{P}\langle \ell\rangle/\mathfrak{P}\langle \ell - 1\rangle)_\nu$ in the form $\sum_i \sum_\alpha b_{i,\alpha} T_i T^\alpha$ and maps it to $(\sum_\alpha b_{1,\alpha} f^\alpha, \ldots, \sum_\alpha b_{n+1,\alpha} f^\alpha)$. After checking that both maps are well-defined, it is clear that one is the inverse of the other. They are called upgrading and downgrading maps (which refers to the degree in the T_i's of the elements).

One can interpret (2) as a description of the graded strands of $\mathcal{R}_I$, in degrees at least the threshold degree $\nu_0(I)$, purely in terms of syzygies: the non-linear equations are all obtained by upgrading some non-Koszul syzygies, and this is a one-to-one correspondence. For the cases of small dimension, it was already observed previously that incorporating quadratic relations allowed to work in degrees smaller than $(n-1)(d-1)$, hence provides matrix representations of smaller size (but with quadratic, or linear and quadratic entries); see e.g. [6, 13, 19]. In the general case, the following result holds.

Theorem 5.6 *For every $\nu \geq \nu_0(I)$, a matrix $\mathbf{M}_\nu$ of the A-linear map (extracted from* (1)*)*

$$\oplus_{\ell=1}^n (\mathfrak{P}\langle \ell\rangle/\mathfrak{P}\langle \ell-1\rangle)_\nu \longrightarrow R_\nu[T] = A^{\binom{\nu+n-1}{n-1}} \tag{3}$$

is a matrix representation *of the fibers of ψ: for any $p \in \mathbb{P}_k^n$,* $\mathrm{corank}(\mathbf{M}_\nu(p)) = \deg(\psi^{-1}(p))$. *Furthermore, for $\ell > 1$, k-bases of $(\mathfrak{P}\langle \ell\rangle/\mathfrak{P}\langle \ell-1\rangle)_{\nu,\ell}$ and of $(H_1)_{\nu+\ell d}$ are in one-to-one correspondence via the upgrading and downgrading maps described above.*

Proof The algebra $S^{(\nu)} := S/(\mathfrak{P}_{*,1} + \mathfrak{P}_{\geq\nu,*}) = \mathrm{Sym}_R(I)/H^0_\mathfrak{m}(\mathrm{Sym}_R(I))_{\geq\nu,*}$ satisfies the equalities $H^0_\mathfrak{m}(S^{(\nu)})_{\geq\nu,*} = 0$ and $H^i_\mathfrak{m}(S^{(\nu)}) = H^i_\mathfrak{m}(\mathrm{Sym}_R(I))$ for $i > 0$. As $H^i_\mathfrak{m}(\mathrm{Sym}_R(I))_{\geq\nu,*} = 0$ for $i > 0$ by [8, Theorem 1], the conclusion follows from [15, 6.3] since $H^i_\mathfrak{m}(S^{(\nu)})_{\geq\nu,*} = 0$ for all i. □

Remark 5.7 In the particular case of a general map from $\mathbb{P}_k^2$ to $\mathbb{P}_k^3$ ($n = 3$), Theorem 5.6 provides a square matrix of quadrics which is a matrix representation of the fibers: the determinant is the equation of the image and ideals of minors of different sizes yield a filtration of fibers by their degrees; see [8, §5] for more details.

6 When the Source Is of Dimension 2

In this section we consider the case of a rational map $\psi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$ defined by four homogeneous polynomials $f_1, \dots, f_4$ of the same degree $d \geq 1$. Following the discussion in Sect. 2.2, the geometric picture is reflected by the inclusions $\Gamma \subseteq W \subset \mathbb{P}^2_k \times_k \mathbb{P}^3_k$, with $\Gamma = \text{Proj}(\mathcal{R}_I)$ and $W = \text{Proj}(\mathcal{S}_I)$. We are seeking informations on both the image and the fibers of the second canonical projection π from Γ to $\mathbb{P}^3_k$.

Following Proposition 2.2, whenever $\text{Proj}(R/I) \subseteq \mathbb{P}^2_k$ is empty or locally a complete intersection (for instance reduced, as it is of dimension zero) then $\Gamma = W$. Actually, components of W can be easily described: besides Γ, they only differ by linear subspaces of the form $\{p\} \times L_p$ where p is a point where $V(I)$ is not locally a complete intersection. Two cases happen:

- $V(I)$ is locally defined at p by the four equations, and not less, in which case $L_p = \mathbb{P}^3_k$ and therefore $\pi(W) = \mathbb{P}^3_k$,
- $V(I)$ is locally defined at p by three equations, in which case L_p is a hyperplane whose equation is given by the specialization at p of any syzygy that does not specialize to 0, equivalently the specialization of a local relation expressing one f_i in terms of the other three. It can be verified (see [7, Lemma 6]) that this component has multiplicity equal to the difference between the Hilbert-Samuel local multiplicity of the ideal and its colength (these being equal if and only if I is locally a complete intersection).

We keep notations as above, in particular $I = (f_1, \dots, f_4)$ is a graded ideal of R generated in degree d. Recall that $a^i(M) := \text{end}(H^i_{\mathfrak{m}}(M))$ and $a^*(M) := \max_i\{a^i(M)\}$, with $\mathfrak{m} = (x_1, x_2, x_3)$. Finally, in a local ring the notation $\mu(N)$ denotes the minimal number of generators of the module N.

In view of applying Theorem 3.2, a crucial step is to estimate the vanishing degree of local cohomology with respect to $\mathfrak{m}$, which is reflected in the regularity of $\mathcal{S}_I$ as a $\mathbb{Z}$-graded $A[x_1, x_2, x_3]$-module ($\deg(x_i) = 1$).

Proposition 6.1 ([1, Proposition 5]) *Assume that* $\dim(R/I) \leq 1$ *and* $\mu(I_{\mathfrak{p}}) \leq 3$ *for every prime ideal* $\mathfrak{m} \supsetneq \mathfrak{p} \supset I$*, then* $a^*(\mathcal{S}_I) + 1 \leq \nu_0 := 2(d-1) - \text{indeg}(I^{sat})$*, and*

$$\text{reg}(\mathcal{S}_I) \leq \nu_0,$$

unless I is a complete intersection of two forms of degree d.

Notice that if I is a complete intersection of two forms, it defines a surjective map from $\mathbb{P}^2_k$ to $\mathbb{P}^1_k$ that has no finite fiber. We will be interested by the case where ψ is generically finite. To understand and compute fibers, Fitting ideals plays a key role.

6.1 Fitting Ideals Associated to ψ

From the properties of matrix representations of ψ, we see that for all $\nu \geq \nu_0$ the Fitting ideal $\mathrm{Fitt}^0_A((\mathcal{S}_I)_\nu)$ is supported on W and hence provides a scheme structure on W. Following [36] and [22, V.1.3], it is called the *Fitting image* of $\mathcal{S}_I$ by π.

Remark 6.2 Observe that by definition, the ideals $\mathrm{Fitt}^0_A((\mathcal{S}_I)_\nu)$ depend upon the integer ν (they are generated in degree ν) whereas $\mathrm{ann}_A((\mathcal{S}_I)_\nu) = I_Z$, with $Z := \pi(W)$ defined by the elimination ideal, for all $\nu \geq \nu_0$.

We can push further the study of $\mathrm{Fitt}^0_A((\mathcal{S}_I)_\nu)$ by looking at the other Fitting ideals $\mathrm{Fitt}^i_A((\mathcal{S}_I)_\nu)$, $i > 0$, since they provide a natural stratification:

$$\mathrm{Fitt}^0_A((\mathcal{S}_I)_\nu) \subset \mathrm{Fitt}^1_A((\mathcal{S}_I)_\nu) \subset \mathrm{Fitt}^2_A((\mathcal{S}_I)_\nu) \subset \cdots \subset \mathrm{Fitt}^{\binom{\nu+2}{2}}_A((\mathcal{S}_I)_\nu) = A.$$

These Fitting ideals are actually closely related to the geometric properties of the parameterization ψ. For simplicity, the Fitting ideals $\mathrm{Fitt}^i_A((\mathcal{S}_I)_\nu)$ will be denoted $\mathrm{Fitt}^i_\nu(\psi)$. We recall that $\mathrm{Fitt}^i_\nu(\psi) \subset A$ is generated by all the minors of size $\binom{\nu+2}{2} - i$ of any A-presentation matrix of $(\mathcal{S}_I)_\nu$.

Example 6.3 Consider the following parameterization of the sphere

$$\begin{aligned} \psi : \mathbb{P}^2_{\mathbb{C}} &\dashrightarrow \mathbb{P}^3_{\mathbb{C}} \\ (x_1 : x_2 : x_3) &\mapsto (x_1^2 + x_2^2 + x_3^2 : 2x_1x_3 : 2x_1x_2 : x_1^2 - x_2^2 - x_3^2). \end{aligned}$$

It has two base points. Following Theorem 3.2 and Proposition 6.1, matrix representations $\mathbf{M}_\nu$ have the expected properties for all $\nu \geq 2 - 1 = 1$. The computation of $\mathbf{M}_1$ yields

$$\mathbf{M}_1 = \begin{pmatrix} 0 & T_2 & T_3 & -T_1 + T_4 \\ T_2 & 0 & -T_1 - T_4 & T_3 \\ -T_3 & -T_1 - T_4 & 0 & T_2 \end{pmatrix}.$$

A primary decomposition of the 3×3 minors of $\mathbf{M}_1$, i.e. $\mathrm{Fitt}^0_1(\psi)$, gives

$$(T_1^2 - T_2^2 - T_3^2 - T_4^2) \cap (T_3, T_2, T_1^2 + 2T_1T_4 + T_4^2),$$

which corresponds to the implicit equation of the sphere plus one embedded double point $(1 : 0 : 0 : -1)$. Now, a primary decomposition of the 2×2 minors of $\mathbf{M}_1$, i.e. $\mathrm{Fitt}^1_1(\psi)$, is given by

$$(T_3, T_2, T_1 + T_4) \cap (T_4, T_3^2, T_2T_3, T_1T_3, T_2^2, T_1T_2, T_1^2),$$

which corresponds to the same embedded point $(1 : 0 : 0 : -1)$ plus an additional component supported at the origin. Finally, the ideal of 1-minors of $\mathbf{M}_1$, i.e. $\mathrm{Fitt}_1^2(\psi)$, is supported at $V(T_1, \ldots, T_4)$ and hence is empty as a subscheme of $\mathbb{P}_k^3$.

The point $(1 : 0 : 0 : -1)$ is actually a singular point of the parameterization ψ (but not of the sphere itself). Indeed, the line $L = (0 : x_2 : x_3)$ is a $\mathbb{P}^1$ that is mapped to the point $(x_2^2 + x_3^2 : 0 : 0 : -(x_2^2 + x_3^2))$. In particular, the base points of ψ, namely $(0 : 1 : i)$ and $(0 : 1 : -i)$, are lying on this line, and the rest of the points are mapped to $(1 : 0 : 0 : -1)$. Outside L at the source and P at the target, ψ is an isomorphism.

Applying Theorem 3.2 with the estimate of Proposition 6.1 gives the following result that explains the phenomenon just noticed in the example:

Theorem 6.4 *Let $\pi : W \to \mathbb{P}_k^3$ be the second canonical projection of $W \subset \mathbb{P}_k^2 \times \mathbb{P}_k^3$ and $p \in \mathbb{P}_k^3$. If $\dim \pi^{-1}(p) \leq 0$ then, for all $\nu \geq \nu_0 := 2(d-1) - \mathrm{indeg}(I^{sat})$,*

$$p \in V(\mathrm{Fitt}_\nu^i(\psi)) \Leftrightarrow \deg(\pi^{-1}(p)) \geq i + 1. \tag{1}$$

In other words, for any $\nu \geq \nu_0$, the Fitting ideals of a matrix representations stratify the fibers of dimension zero (or empty) by their degrees. One could also remark that the existence of base points improves the value of ν_0 ($\mathrm{indeg}(I^{sat}) = 0$ when $V(I) = \emptyset$); furthermore, it was proved in [7, Proposition 2] that the value of ν_0 is sharp in some sense.

6.2 One Dimensional Fibers

When p is such that the fiber has dimension one, then Theorem 3.2 does not apply and in fact fails without further assumptions. In the special case we are considering, one can nevertheless obtain good estimates for the regularity. One way to see this uses the fact that such a fiber corresponds to a special form of the ideal, namely:

Lemma 6.5 ([1, Lemma 10]) *Assume that the f_i's are linearly independent, the fiber over $p := (p_1 : p_2 : p_3 : p_4) \in \mathbb{P}_k^3$ is of dimension 1, and its unmixed component is defined by $h_p \in R$. Let ℓ_p be a linear form with $\ell_p(p) = 1$ and set $\ell_i(T_1, \ldots, T_4) := T_i - p_i \ell_p(T_1, \ldots, T_4)$. Then, $h_p = \gcd(\ell_1(f), \ldots, \ell_4(f))$ and*

$$I = (\ell_p(f)) + h_p(g_1, \ldots, g_4)$$

with $\ell_i(f) = h_p g_i$ and $\ell_p(g_1, \ldots, g_4) = 0$. In particular

$$(\ell_p(f)) + h_p(g_1, \ldots, g_4)^{sat} \subseteq I^{sat} \subseteq (\ell_p(f)) + (h_p).$$

In [1, Theorem 12] an estimate on the regularity of the specialization of the symmetric algebra and of its Hilbert function (the one of the fiber) is derived, which is at least one less than the one given for fibers of dimension zero. However, this does not provide a very easy way to determine or control the fibers of dimension one.

We now turn to this question using Lemma 6.5 and Jacobian matrices. Let Z be the finite set of points $p \in \mathbb{P}^3_k$ having a one-dimensional fiber. Notice that the unmixed part (i.e. purely one-dimensional part) of a fiber of $\pi : W \to \mathbb{P}^3_k$ or $\pi' : \Gamma \to \mathbb{P}^3_k$ are equal, as the fibers may only differ at points where $V(I)$ is not locally a complete intersection.

Choose a linear form $\ell = \lambda_1 T_1 + \cdots + \lambda_4 T_4$, with $\lambda_i \in k$, not vanishing at any point of Z (i. e. a plane that does not meet Z) with nonzero first coefficient (a general form for instance) and set $f := \lambda_1 f_1 + \cdots + \lambda_4 f_4$. Then for any point $p \in Z$ with a fiber whose unmixed part is defined by $h_p \in R$, there exist $g_{p,1}, g_{p,2}, g_{p,3}$ such that:

$$I = (f) + (h_p)(g_{p,1}, g_{p,2}, g_{p,3}).$$

Examples show that for 4 forms of degree d, one can find birational morphisms with a least $2d - 2$ distinct one-dimensional fibers, hence at least this number of distinct decompositions of this type (private communication of M. Chardin and Hoa Tran Quang). On the other hand, the following result gives an upper bound and a way to determine the one-dimensional fibers:

Theorem 6.6 ([12, 4.4]) *Let $J(f)$ be the Jacobian matrix of the f_i's. Then the ideal $I_3(J(f))$ of maximal minors of this 3×4 matrix is generated by 4 forms of degree $3(d-1)$. If not all zero, the GCD F of these 4 forms is divisible by $\prod_{z \in Z} h_z$, hence:*

$$\sum_{z \in Z} \deg(h_z) \leq \deg F \leq 3(d-1) - \operatorname{indeg}(\operatorname{Syz}(I))$$

(the degree of a homogeneous syzygy $\sum a_i f_i = 0$ is $\deg(a_i)$ for any i).

We notice that if the characteristic is zero, the Jacobian ideal is not zero, and this also holds if the characteristic of k doesn't divide d and $k(X)$ is separable over $k(f)$ (in other words π is generically étale). The simplest proof of this theorem is by choosing the generators in the above form for a given $z \in Z$, then to verify that h_z divides each maximal minor (in fact a little more is true, see [12]) and to conclude using the fact that h_z and $h_{z'}$ have no common factor if $z \neq z'$. The improvement from $3(d-1)$ comes from the fact that maximal minors provide a syzygy of the f_i's via the Euler formula, thus a GCD F of high degree will provide a syzygy of small degree for the f_i's.

Question 6.7 At this moment, we are not aware of an example with $\sum_{z \in Z} \deg(h_z) > 2d - 2$, and wonder if this can hold or not.

Theorem 6.6 can be seen as a particular case of the following result for a rational map ψ from $\mathbb{P}_k^m$ to $\mathbb{P}_k^n$, defined on the complement Ω_ψ of the base locus of ψ. It could be used to detect subvarieties in $\mathbb{P}_k^m$ that are contracted to lower dimensional ones by ψ (see [12, 2.3]):

Proposition 6.8 *Suppose that* V *is a subvariety of* $\mathbb{P}_k^m$ *such that* $V \cap \Omega_\psi \neq \emptyset$ *and let* $r := \dim V - \dim \psi(V)$. *Then* $V \subset V(I_{m-r+2}(J(f)))$, *where* $I_{m-r+2}(J(f))$ *is the ideal generated by the* $(m-r+2)$*-minors of* $J(f)$.

7 When the Base Locus Is of Positive Dimension

In this section, we consider the case of a parameterization whose source is of dimension three. This problem has been recently considered in [2] in order to compute the orthogonal projections of a point in space onto a parameterized algebraic surface. The main idea is to consider the congruence of the normal lines of the surface. Indeed, given a rational surface in $\mathbb{P}_k^3$ parameterized by X, its congruence of normal lines Ψ is a rational map from $X \times \mathbb{P}_k^1$ to $\mathbb{P}_k^3$ and the orthogonal projections of a point $p \in \mathbb{P}_k^3$ on X are in correspondence with the pre-images of p via Ψ. In comparison with the previous cases where the source was of dimension one or two, here the base locus may have a one-dimensional component. In what follows we review the results obtained in [2] with a particular focus on the new techniques that are used to tackle this new difficulty. It will also provide us the opportunity to illustrate how to work with blowup algebras over multigraded rings.

Thus, we consider a homogeneous parameterization

$$\begin{aligned} \Psi : X \times \mathbb{P}_k^1 &\dashrightarrow \mathbb{P}_k^3 \\ \xi \times (\bar{t} : t) &\mapsto (\Psi_1 : \Psi_2 : \Psi_3 : \Psi_4), \end{aligned} \tag{1}$$

where X stands for the spaces $\mathbb{P}_k^2$ or $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ over an algebraically closed field k, and the Ψ_i's are homogeneous polynomials in the coordinate ring of $X \times \mathbb{P}_k^1$. The coordinate ring R_X of X is equal to $k[w, u, v]$ or $k[\bar{u}, u; \bar{v}, v]$, respectively, depending on X. The coordinate ring of $\mathbb{P}^1$ is denoted by $R_1 = k[\bar{t}, t]$ and hence the coordinate ring of $X \times \mathbb{P}_k^1$ is the polynomial ring $R := R_X \otimes_k R_1$. The polynomials $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ are hence multihomogeneous polynomials of the same degree $(\mathbf{d}, e)$, where $\mathbf{d}$ refers to the degree with respect to X, which can be either an integer d if $X = \mathbb{P}_k^2$, or a pair of integers (d_1, d_2) if $X = \mathbb{P}_k^1 \times \mathbb{P}_k^1$.

7.1 The Base Locus

We assume that Ψ is a dominant map. We denote by I the ideal of $R = R_X \otimes R_1$ generated by the defining polynomials of the map Ψ, i.e. $I := (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$.

The irrelevant ideal of $X \times \mathbb{P}^1_k$ is denoted by B; it is equal to the product of ideals $(w, u, v) \cdot (\bar{t}, t)$ if $X = \mathbb{P}^2_k$, or to the product $(\overline{u}, u) \cdot (\overline{v}, v) \cdot (\bar{t}, t)$ if $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$. The notation I^{sat} stands for the saturation of the ideal I with respect to the ideal B, i.e. $I^{sat} = (I : B^\infty)$.

The base locus of Ψ is the subscheme of $X \times \mathbb{P}^1_k$ defined by the ideal I; it is denoted by $\mathcal{B}$. Without loss of generality, $\mathcal{B}$ can be assumed to be of dimension at most one, but the presence of a curve component is a possibility after factoring out the gcd of the Ψ_i's. When $\dim(\mathcal{B}) = 1$ we denote by $\mathcal{C}$ its top unmixed one-dimensional curve component. We will need the following definition.

Definition 7.1 The curve $\mathcal{C} \subset X \times \mathbb{P}^1_k$ has *no section in degree* $< (\boldsymbol{a}, b)$ if for any $\boldsymbol{\alpha} < \boldsymbol{a}$ and $\beta < b$, $H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(\boldsymbol{\alpha}, \beta)) = 0$.

7.2 Fibers

As we already mentioned previously, a proper definition of the fiber of a point under Ψ requires to consider the graph of Ψ and its closure $\Gamma \subset X \times \mathbb{P}^1_k \times \mathbb{P}^3_k$. Thus, the fiber of a point $p \in \mathbb{P}^3_k$ is the subscheme

$$\mathfrak{F}_p := \mathrm{Proj}(\mathrm{Rees}_R(I) \otimes \kappa(p)) \subset X \times \mathbb{P}^1_k, \tag{2}$$

where $\kappa(p)$ denotes the residue field of p. As the equations of the Rees algebra $\mathrm{Rees}_R(I)$ are in general very difficult to get we also consider the corresponding symmetric algebra $\mathrm{Sym}_R(I)$ of the ideal I and hence, as a variation of (2) we introduce the subscheme

$$\mathfrak{L}_p := \mathrm{Proj}(\mathrm{Sym}_R(I) \otimes \kappa(p)) \subset X \times \mathbb{P}^1_k \tag{3}$$

that we call *the linear fiber of* p. We emphasize that the fiber $\mathfrak{F}_p$ is always contained in the linear fiber $\mathfrak{L}_p$ of a point p, and that they coincide if the ideal I is locally a complete intersection at p (see Proposition 2.2).

7.3 The Main Theorem

Theorem 3.2 can be used to analyze finite linear fibers of Ψ following the ideas we introduced in previous sections. In particular, a similar analysis of the regularity of these fibers can be done, but there is an additional difficulty that is appearing if there exists a curve component, $\mathcal{C}$ in the base locus $\mathcal{B}$. In order to describe the multidegrees for which the matrices $\mathbf{M}_{(\boldsymbol{\mu},\nu)}$ (see Theorem 3.2) of the map Ψ yields a representation of its finite fibers, we introduce the following notation.

Notation 7.2 *Let r be a positive integer. For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{Z} \cup \{-\infty\})^r$ we set*

$$\mathbb{E}(\boldsymbol{\alpha}) := \{\boldsymbol{\zeta} \in \mathbb{Z}^r \mid \zeta_i \geq \alpha_i \text{ for all } i = 1, \ldots, r\}.$$

It follows that, for any $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $(\mathbb{Z} \cup \{-\infty\})^r$, $\mathbb{E}(\boldsymbol{\alpha}) \cap \mathbb{E}(\boldsymbol{\beta}) = \mathbb{E}(\boldsymbol{\gamma})$ where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for all $i = 1, \ldots, r$, i.e. $\boldsymbol{\gamma}$ is the maximum of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ component-wise.

Theorem 7.3 ([2, Theorem 8]) *Assume that we are in one of the two following cases:*

(a) *The base locus $\mathcal{B}$ is finite, possibly empty,*
(b) $\dim(\mathcal{B}) = 1$*, $\mathcal{C}$ has no section in degree $< (\mathbf{0}, e)$ and locally at every point $\mathfrak{q} \in \mathrm{Proj}(R) = X \times \mathbb{P}^1_k$, the ideal $I_{\mathfrak{q}}$ is generated by at most three elements.*

Let p be a point in $\mathbb{P}^3_k$ such that $\mathfrak{L}_p$ is finite, then

$$\operatorname{corank} \mathbf{M}_{(\boldsymbol{\mu},\nu)}(p) = \deg(\mathfrak{L}_p)$$

for any degree $(\boldsymbol{\mu}, \nu)$ such that

- *if $X = \mathbb{P}^2_k$, $(\mu, \nu) \in \mathbb{E}(3d - 2, e - 1) \cup \mathbb{E}(2d - 2, 3e - 1)$.*
- *if $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$,*

$$(\boldsymbol{\mu}, \nu) \in \mathbb{E}(3d_1 - 1, 2d_2 - 1, e - 1) \cup \mathbb{E}(2d_1 - 1, 3d_2 - 1, e - 1) \\ \cup \mathbb{E}(2d_1 - 1, 2d_2 - 1, 3e - 1).$$

7.4 Idea of the Proof of the Main Theorem

If the base locus $\mathcal{B}$ of Ψ is composed of finitely many points, then the proof of Theorem 7.3 goes along the same lines as the usual strategy developed in [4, 10, 11]. However, if there exists a curve component in $\mathcal{B}$ then an additional difficulty appears. Indeed, if $\dim(\mathcal{B}) = 0$ then $H^2_B(H_i) = 0$ for all i, where H_j denotes the homology module of the Koszul complex $K_\bullet$ of the Ψ_i's over R and $H^i_B(-)$ the local cohomology modules with respect to the irrelevant B. If $\dim(\mathcal{B}) = 1$ then it is necessary to control the multidegree at which these homology modules vanish. Following [2], we describe the main steps and tools to determine such multidegrees.

Proposition 7.4 ([2, Proposition 10]) *For any integer i, let $\mathcal{R}_i \subseteq \mathbb{Z}^r$ be a subset satisfying*

$$\forall j \in \mathbb{Z} : \mathcal{R}_i \cap \mathrm{Supp}(H^j_B(K_{i+j})) = \emptyset.$$

Then, if $\dim \mathcal{B} \leq 1$ *the following properties hold for any integer* i*:*

- *For all* $\boldsymbol{\mu} \in \mathcal{R}_{i-1}$, $H^1_B(H_i)_{\boldsymbol{\mu}} = 0$.
- *There exists a natural graded map* $\delta_i : H^0_B(H_i) \to H^2_B(H_{i+1})$ *such that* $(\delta_i)_{\boldsymbol{\mu}}$ *is surjective for all* $\boldsymbol{\mu} \in \mathcal{R}_{i-1}$ *and is injective for all* $\boldsymbol{\mu} \in \mathcal{R}_i$.

In particular,

$$H^0_B(H_i)_{\boldsymbol{\mu}} \simeq H^2_B(H_{i+1})_{\boldsymbol{\mu}} \text{ for all } \boldsymbol{\mu} \in \mathcal{R}_{i-1} \cap \mathcal{R}_i.$$

The regions $\mathcal{R}_i$ are obtained by the computation of cohomology of a product of projective spaces that is as follows in our case (recall that the Koszul modules K_{i+j} are direct sums of shifted copies of R):

Lemma 7.5 ([11, §6]) *First,* $H^i_B(R) = 0$ *for all* $i \neq 2, 3, 4$. *In addition, if* $X = \mathbb{P}^2_k$ *then* $R_X = k[w, u, v]$ *and*

$$H^2_B(R) \simeq R_X \otimes \check{R}_1, \quad H^3_B(R) \simeq \check{R}_X \otimes R_1, \quad H^4_B(R) \simeq \check{R}_X \otimes \check{R}_1$$

where $\check{R}_1 = \frac{1}{\bar{t}t} k[\bar{t}^{-1}, t^{-1}]$ *and* $\check{R}_X = \frac{1}{wuv} k[w^{-1}, u^{-1}, v^{-1}]$.

If $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ *then* $R_X = R_2 \otimes R_3$, *where* $R_2 = k[\bar{u}, u]$, $R_3 = k[\bar{v}, v]$, *and*

$$H^2_B(R) \simeq \bigoplus_{\substack{\{i,j,k\}=\{1,2,3\},\\ j<k}} \check{R}_i \otimes R_j \otimes R_k,$$

$$H^3_B(R) \simeq \bigoplus_{\substack{\{i,j,k\}=\{1,2,3\},\\ j<k}} R_i \otimes \check{R}_j \otimes \check{R}_k, \quad H^4_B(R) \simeq \check{R}_1 \otimes \check{R}_2 \otimes \check{R}_3$$

where $\check{R}_2$ *and* $\check{R}_3$ *are defined similarly to* $\check{R}_1$.

For the control of vanishing degrees of $H^2_B(H_i)$, a key ingredient is Serre duality. To be more precise, we have the following lemma that we state in a little more generality, when $\mathbb{P} = \mathbb{P}^{n_1}_k \times \cdots \times \mathbb{P}^{n_r}_k$ is a product of projective spaces.

Lemma 7.6 ([2, Lemma 13]) *Assume that* $\dim(\mathcal{B}) = 1$ *and that the* $s + 1$ *forms* $\Psi_1, \ldots, \Psi_{s+1}$ *are of the same degree* $\boldsymbol{\delta}$. *Let* $\mathcal{C}$ *be the unmixed curve component of* $\mathcal{B}$ *and set* $p := s - \dim \mathbb{P} + 2$ *and* $\boldsymbol{\sigma} := (s+1)\boldsymbol{\delta} - (n_1 + 1, \cdots, n_r + 1)$. *Then, for all* $\boldsymbol{\mu} \in \mathbb{Z}^r$,

$$H^2_B(H_p)_{\boldsymbol{\mu}} \simeq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-\boldsymbol{\mu} + \boldsymbol{\sigma}))^{\vee}.$$

In particular, if $\mathcal{C}$ *has no section in degree* $< \boldsymbol{\mu}_0$, *for some* $\boldsymbol{\mu}_0 \in \mathbb{Z}^r$, *then*

$$H^2_B(H_p)_{\boldsymbol{\mu}} = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{E}((s+1)\boldsymbol{\delta} - (n_1, \cdots, n_r) - \boldsymbol{\mu}_0).$$

Proof As locally at a closed point $x \in \mathbb{P}$, the Ψ_i's contain a regular sequence of length $s-1$, and of length s unless $x \in \mathcal{C}$, by [9, §1-3] there are isomorphisms

$$\widetilde{H_p(\boldsymbol{\sigma})} \simeq \mathrm{Ext}_S^{s-1}\widetilde{(S/I, \omega_S)} \simeq \mathrm{Ext}_S^{s-1}\widetilde{(S/I_{\mathcal{C}}, \omega_S)} \simeq \omega_{\mathcal{C}}$$

from which we deduce that

$$H_B^2(H_p) \simeq \bigoplus_{\boldsymbol{\mu}} H^1(\mathcal{C}, \omega_{\mathcal{C}}(\boldsymbol{\mu} - \boldsymbol{\sigma})). \tag{4}$$

Now, applying Serre's duality Theorem [26, Corollary 7.7] we get

$$H^1(\mathcal{C}, \omega_{\mathcal{C}}(\boldsymbol{\mu} - \boldsymbol{\sigma})) \simeq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-\boldsymbol{\mu} + \boldsymbol{\sigma}))^{\vee},$$

which concludes the proof. □

Lemma 7.7 ([2, Lemma 14]) *In the setting of Lemma 7.6, let $s = \dim \mathbb{P}$ and let I' be an ideal generated by s general linear combinations of the Ψ_i's. If $I^{sat} = I'^{sat}$ then for all $\boldsymbol{\mu} \in \mathbb{Z}^r$ there exists an exact sequence*

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-\boldsymbol{\mu} - \boldsymbol{\delta} + \boldsymbol{\sigma}))^{\vee} \to H_B^2(H_1)_{\boldsymbol{\mu}} \to H_B^2(S/I)_{\boldsymbol{\mu}-\boldsymbol{\delta}} \to 0.$$

In particular, if $\mathcal{C}$ has no section in degree $< \boldsymbol{\mu}_0$, for some $\boldsymbol{\mu}_0 \in \mathbb{Z}^r$, then

$$H_B^2(H_1)_{\boldsymbol{\mu}} = 0, \quad \forall \boldsymbol{\mu} \in \mathbb{E}(s\boldsymbol{\delta} - (n_1, \cdots, n_r) - \boldsymbol{\mu}_0) \cap (\boldsymbol{\delta} + \mathcal{R}_{-2}).$$

Proof We will denote by H_i' the ith homology module of the Koszul complex associated to $I' \subset R$. By [9, Corollary 1.6.13] and [9, Corollary 1.6.21] we have the following graded exact sequence

$$0 \to M \to H_1' \to H_1 \to H_0'(-\boldsymbol{\delta}) \to N \to 0 \tag{5}$$

with the property that the modules M and N are supported on $V(B)$, which implies that $H_B^i(M) = H_B^i(N) = 0$ for $i \geq 1$.

This implies that the sequence

$$H_B^2(H_1') \to H_B^2(H_1) \to H_B^2(H_0')(-\boldsymbol{\delta}) \to 0$$

is exact. By Lemma 7.6, $H_B^2(H_1')_{\boldsymbol{\mu}} \simeq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-\boldsymbol{\mu} - \boldsymbol{\delta} + \boldsymbol{\sigma}))^{\vee}$ and one verifies that the equalities $H_B^2(H_0')(-\boldsymbol{\delta})_{\boldsymbol{\mu}} = H_B^2(S/I)(-\boldsymbol{\delta})_{\boldsymbol{\mu}} = 0$ hold for all $\boldsymbol{\mu} - \boldsymbol{\delta} \in \mathcal{R}_{-2}$. □

From here, the proof of Theorem 7.3 follows by the usual consideration of the Čech-Koszul spectral sequences associated to the approximation complex of cycles of I and comparison between cohomology of Koszul cycles and homologies (see [2, §4.2] for more details).

7.5 Curve with No Section in Negative Degree

To apply Theorem 7.3 it is necessary that the curve component in the base locus, if any, has no section in negative degrees. Therefore, we now discuss when such a property holds.

A reduced irreducible scheme of positive dimension in a projective space has no section in negative degrees, this is due to the fact that the section ring is finite over R and has no non-zero nilpotent element. Over a product of projective spaces it is typically not the case that the section ring is finitely generated, unless the scheme is a product of projective schemes. However, for instance using Veronese-Segre embeddings, one can easily show that it has no section in degrees < 0 (all degrees are strictly negative), which is sufficient for several applications.

An interesting question is anyhow to understand in which multidegrees a scheme could have sections in a product of projective spaces, and a closely related question (equivalent for schemes satisfying S_2) is to determine in what twists the top cohomology of the canonical module is not zero. Another related issue is to understand, for a projective scheme, if it has sections in negative degrees and what is the geometric meaning of these. In this direction, we reproduce a result (together with a proof) showing that symbolic powers of prime ideals determine schemes with no sections in negative degrees (unless it is of dimension zero).

Lemma 7.8 *Let k be a field, $\mathcal{C}$ a geometrically reduced curve in $\mathbb{P}^n_k$ and $t > 0$. Then, the natural map*

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}^{(t)}}(\nu)) \to H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}^{(t-1)}}(\nu))$$

is injective for $\nu < t-1$. In particular $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}^{(t)}}(\nu)) = 0$ for $\nu < 0$ and $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}^{(t)}}) = H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}})$.

Proof This is clear for $t = 1$. Let $t \geq 2$. Write I for the defining ideal of $\mathcal{C}$, $I^{(j)}$ for its j-th symbolic power and $\omega_{A/I^{(j)}} := \mathrm{Ext}^{n-1}_A(A/I^{(j)}, \omega_A)$. For any ν, $(\omega_{A/I^{(j)}})_\nu = H^0(\mathbb{P}^n_k, \omega_{\mathcal{C}^{(j)}}(\nu))$ and setting $-^\vee := \mathrm{Hom}_k(-, k)$,

$$H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathcal{C}^{(j)}}(\nu)) = H^1(\mathbb{P}^n_k, \omega_{\mathcal{C}^{(j)}}(-\nu))^\vee = H^2_{\mathfrak{m}}(A/I^{(j)})^\vee_{-\nu}.$$

Consider the exact sequence

$$0 \longrightarrow \omega_{A/I^{(t-1)}} \longrightarrow \omega_{A/I^{(t)}} \longrightarrow \mathrm{Ext}^{n-1}_A(I^{(t-1)}/I^{(t)}, \omega_A) \xrightarrow{\psi} \mathrm{Ext}^n_A(A/I^{(t-1)}, \omega_A).$$

As $\mathrm{Ext}^n_A(A/I^{(t-1)}, \omega_A)$ has finite length and the other three modules are Cohen-Macaulay of dimension two, it gives rise to an exact sequence,

$$0 \longrightarrow \mathrm{im}(\psi) \longrightarrow H^2_{\mathfrak{m}}(\omega_{A/I^{(t-1)}}) \longrightarrow H^2_{\mathfrak{m}}(\omega_{A/I^{(t)}}) \longrightarrow H^2_{\mathfrak{m}}(\mathrm{Ext}^{n-1}_A(I^{(t-1)}/I^{(t)}, \omega_A)) \longrightarrow 0$$

and it remains to show that $H^2_{\mathfrak{m}}(\mathrm{Ext}^{n-1}_A(I^{(t-1)}/I^{(t)}, \omega_A))_\nu = 0$ for $\nu > 1-t$.

First notice that

$$\mathrm{Ext}_A^{n-1}(I^{(t-1)}/I^{(t)}, \omega_A) \simeq \mathrm{Ext}_A^{n-1}(I^{t-1}/I^t, \omega_A) \simeq \mathrm{Ext}_A^{n-1}(\mathrm{Sym}_A^{t-1}(I/I^2), \omega_A),$$

as I is generically a complete intersection. There is an exact sequence

$$0 \longrightarrow K \longrightarrow I/I^2 \xrightarrow{\delta} A/I[-1]^{n+1} \longrightarrow \Omega_{A/I} \longrightarrow 0$$

where K is supported on the locus where $\mathcal{C}$ is not a complete intersection. Furthermore, locally on the smooth locus of $\mathcal{C}$, δ is split injective. One deduces an exact sequence,

$$0 \longrightarrow K_t \longrightarrow \mathrm{Sym}_{A/I}^{t-1}(I/I^2) \xrightarrow{\delta_t} \mathrm{Sym}_{A/I}^{t-1}(A/I[-1]^{n+1}) = A/I[-t+1]^{\binom{n+t-1}{n}},$$

with K_t supported on the non complete intersection locus of $\mathcal{C}$ and $\mathrm{coker}(\delta_t)$ of dimension two. This in turn gives an exact sequence

$$(\omega_{A/I}[t-1])^{\binom{n+t-1}{n}} \to \mathrm{Ext}_A^{n-1}(\mathrm{Sym}_{A/I}^{t-1}(I/I^2), \omega_A) \to \mathrm{Ext}_A^n(\mathrm{coker}(\delta_t), \omega_A).$$

As $\mathrm{Ext}_A^n(\mathrm{coker}(\delta_t), \omega_A)$ is of dimension at most 1, it follows that the natural map

$$H_{\mathfrak{m}}^2((\omega_{A/I}[t-1]))^{\binom{n+t-1}{n}} \to H_{\mathfrak{m}}^2(\mathrm{Ext}_A^{n-1}(\mathrm{Sym}_{A/I}^{t-1}(I/I^2), \omega_A))$$

is onto. On the other hand, $H_{\mathfrak{m}}^2(\omega_{A/I})_\nu \simeq H_0(\mathbb{P}_k^n, \mathcal{O}_{\mathcal{C}}(-\nu))^\vee = 0$ for $\nu > 0$ as $\mathcal{C}$ is reduced. Therefore $H_{\mathfrak{m}}^2(\mathrm{Ext}_A^{n-1}(\mathrm{Sym}_{A/I}^{t-1}(I/I^2), \omega_A))_\nu = 0$ for $\nu > 1-t$, and the result follows. □

As, by Bertini theorem, the general hyperplane section $Y = X \cap H$ of a (geometrically) reduced scheme X is a (geometrically) reduced scheme of dimension one less and $Y^{(t)} = X^{(t)} \cap H$, the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}_k^n, \mathcal{O}_{X^{(t)}}(\nu-1)) \longrightarrow H^0(\mathbb{P}_k^n, \mathcal{O}_{X^{(t)}}(\nu)) \longrightarrow H^0(\mathbb{P}_k^n, \mathcal{O}_{Y^{(t)}}(\nu))$$

gives by induction on the dimension:

Theorem 7.9 *If X is a geometrically reduced scheme with all irreducible components of positive dimension and $t > 0$, then*

$$H^0(\mathbb{P}_k^n, \mathcal{O}_{X^{(t)}}(\nu)) = 0, \quad \forall \nu < 0,$$

and $H^0(\mathbb{P}_k^n, \mathcal{O}_{X^{(t)}}) = k$ if X is equidimensional and connected in codimension one.

In the case X is irreducible and locally a complete intersection, and k has characteristic zero, the above result follows from the generalization of Kodaira vanishing proved in [3, Theorem 1.4]: $H^\ell(\mathbb{P}_k^n, \mathcal{O}_{X^{(t)}}(\nu)) = 0$ for all $\nu < 0$ and $\ell < \mathrm{codim}(\mathrm{Sing}(X))$.

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Three Takes on Almost Complete Intersection Ideals of Grade 3

Lars Winther Christensen, Oana Veliche, and Jerzy Weyman

To David Eisenbud on the occasion of his 75th birthday

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1 Introduction

Let R be a commutative Noetherian local ring. A celebrated result of Buchsbaum and Eisenbud [10] states that every Gorenstein ideal in R of grade 3 is generated by the $2m \times 2m$ Pfaffians of a $(2m + 1) \times (2m + 1)$ skew symmetric matrix. Later, Avramov [2] and Brown [6] independently proved a similar result for almost complete intersections. Their proofs are based on the fact that an almost complete intersection ideal is linked to a Gorenstein ideal; this means that their descriptions of the resolutions depend on certain choices, so they are not coordinate free.

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In this paper we take three approaches to almost complete intersection ideals of grade 3. They involve different languages, so they can be appreciated by different audiences. However we show how these three approaches intertwine and influence each other.

The first approach uses only commutative and linear algebra. The main theorems about grade 3 almost complete intersection ideals in the local ring R are stated in Sect. 2. They are proved in Sect. 3 and Appendices A–C by specialization from the generic case. In the generic case we use the Buchsbaum–Eisenbud Acyclicity Criterion and a computation with Pfaffians inspired by the Buchsbaum–Eisenbud Structure Theorems, see Remark 3.4, to construct the minimal free resolutions. We emphasize that our description of the resolutions in the generic case does not depend on linkage; this avoids an implicit change of basis present in [6], see Remark 3.8 Under this first and purely algebraic approach all statements are given full proofs; the next two approaches offer interpretations of the same statements.

The second approach, taken in Sect. 4, is to provide canonical equivariant forms of almost complete intersections. The ideals one obtains depend on a skew symmetric matrix and three vectors. This view of almost complete intersections was reached by analyzing the generic ring $\hat{R}_{gen}$ constructed by Weyman [27]. The idea was to look for an open set in $\hat{R}_{gen}$ of points where the corresponding resolution is a resolution of a perfect ideal. This set can be explicitly described as the points where localization of certain complex over $\hat{R}_{gen}$ is split exact. Calculating this "splitting form" of an ideal of grade 3 with four generators led to our form of almost complete intersection. One could use the geometric technique of calculating syzygies to prove the acyclicity of these complexes but they are identical to those from the commutative algebra approach so we do not follow through on that. The advantage of this method is that one can give a geometric interpretation of the zero set of almost complete intersection ideals. Moreover the fact, first noticed in [2], that the skew symmetric matrix associated to an almost complete intersection ideal can be chosen with a 3×3 block of zeros on the diagonal is particularly natural under this approach.

Finally, in Sect. 5, we give a geometric interpretation of both Gorenstein ideals and almost complete intersections of grade 3. It turns out that they are intersections of the so-called big open cell with two Schubert varieties of codimension 3 in the connected component of the orthogonal Grassmannian $\mathrm{OGr}(n, 2n)$ of isotropic subspaces of dimension n in a $2n$-dimensional orthogonal space. It is interesting that in this construction the two Schubert varieties appear together with a regular sequence by which they are linked. This pattern generalizes from the D_n root system to E_6, E_7 and E_8; see Sam and Weyman [24]. We show that the defining ideals are exactly the same as in commutative algebra approach, but we indicate how one could see the graded format of the finite free resolutions just from representation theory viewpoint. Also, the fact about three submaximal Pfaffians forming a regular sequence get a clear geometric interpretation, as one can see geometrically that their zero set has codimension 3.

2 Almost Complete Intersections Following Avramov and Brown

For a grade 3 perfect ideal $\mathfrak{a}$ in a commutative Noetherian local ring $(R, \mathfrak{m}, k)$, the minimal free resolution of the quotient ring $R/\mathfrak{a}$ has the form

$$F \;=\; 0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \,,$$

and we refer to the rank of F_3 as the *type* of $R/\mathfrak{a}$; if R is Cohen–Macaulay, then this is indeed the Cohen–Macaulay type. Throughout the paper we treat quotients of odd and even type separately.

By a result of Buchsbaum and Eisenbud [10] the minimal free resolution F has a structure of a skew commutative differential graded algebra. This structure is not unique, but the induced skew commutative algebra structure on $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$ is unique. It provides for a classification of quotients $R/\mathfrak{a}$ as worked out Weyman [26] and by Avramov, Kustin, and Miller [3].

To state the main theorems about grade 3 almost complete intersection ideals in local rings we introduce some matrix-related notation.

Notation 2.1 Let M be an $m \times n$ matrix with entries in a commutative ring. For subsets

$$I \;=\; \{i_1, \ldots, i_k\} \;\subseteq\; \{1, \ldots, m\} \qquad \text{and} \qquad J \;=\; \{j_1, \ldots, j_l\} \;\subseteq\; \{1, \ldots, n\}$$

with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$ we write $M[i_1 \ldots i_k; j_1 \ldots j_l]$ for the submatrix of M obtained by taking the rows indexed by I and the columns indexed by J. At times, it is more convenient to specify a submatrix in terms of removal of rows and columns: The symbol $M[\overline{i_1 \ldots i_k}; \overline{j_1 \ldots j_l}]$ specifies the submatrix of M obtained by removing the rows indexed by I and the columns indexed by J. These notations can also be combined: For example, $M[\overline{i_1 \ldots i_k}; j_1 \ldots j_l]$ is the submatrix obtained by taking the rows indexed by the complement of I and the columns indexed by J.

For an $n \times n$ skew symmetric matrix T, the Pfaffian of T is written $\mathrm{Pf}(T)$. For a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ the Pfaffian of the submatrix $T[i_1 \ldots i_k; i_1 \ldots i_k]$ is written $\mathrm{Pf}_{i_1 \ldots i_k}(T)$ while the Pfaffian of $T[\overline{i_1 \ldots i_k}; \overline{i_1 \ldots i_k}]$ is written $\mathrm{Pf}_{\overline{i_1 \ldots i_k}}(T)$.

Theorem 2.2 *Let $n \geqslant 5$ be an odd number. Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathfrak{a} \subset R$ a grade 3 almost complete intersection ideal such that $R/\mathfrak{a}$ is of type $n - 3$. There* *exists an $n \times n$ skew symmetric block matrix*

$$U = \left(\begin{array}{c|c} O & B \\ \hline -B^T & A \end{array}\right) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & t_{14} & t_{15} & \dots \\ 0 & 0 & 0 & t_{24} & t_{25} & \dots \\ 0 & 0 & 0 & t_{34} & t_{35} & \dots \\ \hline -t_{14} & -t_{24} & -t_{34} & 0 & t_{45} & \dots \\ -t_{15} & -t_{25} & -t_{35} & -t_{45} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

with entries in $\mathfrak{m}$ such that the minimal free resolution of $R/\mathfrak{a}$,

$$F = 0 \longrightarrow R^{n-3} \xrightarrow{\partial_3} R^n \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R ,$$

has differentials

$$\partial_3 = \left(\frac{B}{A}\right) ,$$

$$\partial_2 = \begin{pmatrix} \mathrm{Pf}(A) & 0 & 0 & -\mathrm{Pf}_{\overline{234}}(U) & \mathrm{Pf}_{\overline{235}}(U) & \cdots & \mathrm{Pf}_{\overline{23n}}(U) \\ 0 & \mathrm{Pf}(A) & 0 & -\mathrm{Pf}_{\overline{134}}(U) & \mathrm{Pf}_{\overline{135}}(U) & \cdots & \mathrm{Pf}_{\overline{13n}}(U) \\ 0 & 0 & \mathrm{Pf}(A) & -\mathrm{Pf}_{\overline{124}}(U) & \mathrm{Pf}_{\overline{125}}(U) & \cdots & \mathrm{Pf}_{\overline{12n}}(U) \\ \mathrm{Pf}_{\overline{1}}(U) & -\mathrm{Pf}_{\overline{2}}(U) & \mathrm{Pf}_{\overline{3}}(U) & -\mathrm{Pf}_{\overline{4}}(U) & \mathrm{Pf}_{\overline{5}}(U) & \cdots & \mathrm{Pf}_{\overline{n}}(U) \end{pmatrix} ,$$

and

$$\partial_1 = \begin{pmatrix} -\mathrm{Pf}_{\overline{1}}(U) & \mathrm{Pf}_{\overline{2}}(U) & -\mathrm{Pf}_{\overline{3}}(U) & \mathrm{Pf}(A) \end{pmatrix} .$$

In particular, $\mathfrak{a}$ is generated by $\mathrm{Pf}_{\overline{1}}(U)$, $\mathrm{Pf}_{\overline{2}}(U)$, $\mathrm{Pf}_{\overline{3}}(U)$, *and* $\mathrm{Pf}(A)$. *Moreover, the multiplicative structure on* $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$ *is of class* $\mathbf{H}(3, 2)$ *if $R/\mathfrak{a}$ is of type* 2 *and otherwise of class* $\mathbf{H}(3, 0)$.

The proof of this theorem is given in 3.7 and the next theorem is proved in 3.12.

Theorem 2.3 *Let $n \geqslant 6$ be an even number. Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathfrak{a} \subset R$ a grade* 3 *almost complete intersection ideal such that $R/\mathfrak{a}$ is of type $n - 3$. There exists an $n \times n$ skew symmetric block matrix U as in Theorem 2.2 such that the minimal free resolution of $R/\mathfrak{a}$,*

$$F = 0 \longrightarrow R^{n-3} \xrightarrow{\partial_3} R^n \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R ,$$

has differentials

$$\partial_3 = \begin{pmatrix} B \\ \hline A \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & -\operatorname{Pf}_{\overline{1234}}(U) & \operatorname{Pf}_{\overline{1235}}(U) & \cdots & -\operatorname{Pf}_{\overline{123n}}(U) \\ \operatorname{Pf}_{\overline{13}}(U) & -\operatorname{Pf}_{\overline{23}}(U) & 0 & \operatorname{Pf}_{\overline{34}}(U) & -\operatorname{Pf}_{\overline{35}}(U) & \cdots & \operatorname{Pf}_{\overline{3n}}(U) \\ -\operatorname{Pf}_{\overline{12}}(U) & 0 & \operatorname{Pf}_{\overline{23}}(U) & -\operatorname{Pf}_{\overline{24}}(U) & \operatorname{Pf}_{\overline{25}}(U) & \cdots & -\operatorname{Pf}_{\overline{2n}}(U) \\ 0 & \operatorname{Pf}_{\overline{12}}(U) & -\operatorname{Pf}_{\overline{13}}(U) & \operatorname{Pf}_{\overline{14}}(U) & -\operatorname{Pf}_{\overline{15}}(U) & \cdots & \operatorname{Pf}_{\overline{1n}}(U) \end{pmatrix},$$

and

$$\partial_1 = \begin{pmatrix} \operatorname{Pf}(U) & \operatorname{Pf}_{\overline{12}}(U) & \operatorname{Pf}_{\overline{13}}(U) & \operatorname{Pf}_{\overline{23}}(U) \end{pmatrix}.$$

In particular, $\mathfrak{a}$ *is generated by* $\operatorname{Pf}(U)$, $\operatorname{Pf}_{\overline{12}}(U)$, $\operatorname{Pf}_{\overline{13}}(U)$, *and* $\operatorname{Pf}_{\overline{23}}(U)$. *Moreover, the multiplicative structure on* $\operatorname{Tor}_*^R(R/\mathfrak{a}, k)$ *is of class* **T**.

The proofs Theorems 2.2 and 2.3 have been deferred to the next section because we obtain them by specialization of statements about generic almost complete intersections.

3 Generic Almost Complete Intersections

In this section and the appendices we deal extensively with relations between Pfaffians of submatrices $T[i_1 \dots i_k; i_1 \dots i_k]$ of a fixed skew symmetric matrix T. It is, therefore, convenient to have the following variation on the notation from 2.1:

$$\operatorname{pf}_T(i_1 \dots i_k) = \operatorname{Pf}_{i_1 \dots i_k}(T) \qquad \text{and} \qquad \operatorname{pf}_T(\overline{i_1 \dots i_k}) = \operatorname{Pf}_{\overline{i_1 \dots i_k}}(T). \qquad (3.0.1)$$

It emphasizes the subset, which changes, over the matrix, which is fixed; for homogeneity we set $\operatorname{pf}_T = \operatorname{Pf}(T)$.

Setup 3.1 Let n be a natural number and $\mathcal{R} = \mathbb{Z}[\tau_{ij} \mid 1 \leqslant i < j \leqslant n]$ the polynomial algebra in indeterminates τ_{ij} over $\mathbb{Z}$. Let $\mathcal{T}$ be the $n \times n$ skew symmetric matrix with entries $\mathcal{T}[i; j] = \tau_{ij} = -\mathcal{T}[j; i]$ for $1 \leqslant i < j \leqslant n$ and zeros on the diagonal. It looks like this:

$$\mathcal{T} = \begin{pmatrix} 0 & \tau_{12} & \tau_{13} & \tau_{14} \dots \\ -\tau_{12} & 0 & \tau_{23} & \tau_{24} \dots \\ -\tau_{13} & -\tau_{23} & 0 & \tau_{34} \dots \\ -\tau_{14} & -\tau_{24} & -\tau_{34} & 0 \ \dots \\ \vdots & \vdots & \vdots & \vdots \ \ddots \end{pmatrix}.$$

Lemma 3.2 *Adopt the setup from 3.1 and denote by $\mathfrak{M}$ the ideal generated by the indeterminates τ_{ij}. Let $\partial\colon \mathcal{R}^{n-3} \longrightarrow \mathcal{R}^n$ be the linear map given by the matrix $\mathcal{T}[1\ldots n; 4\ldots n]$. One has $\partial(\mathcal{R}^{n-3}) \cap \mathfrak{M}^2\mathcal{R}^n = \partial(\mathfrak{M}\mathcal{R}^{n-3})$.*

Proof Let $g_4, \ldots, g_n$ and $f_1, \ldots f_n$ be the standard bases for the free modules $\mathcal{R}^{n-3}$ and $\mathcal{R}^n$. Let $x = \sum_{i=4}^n a_i g_i$ be an element of $\mathcal{R}^{n-3}$; one has

$$\partial(x) = \sum_{i=4}^{n} a_i\Big(\sum_{j=1}^{i-1}\tau_{ji} f_j - \sum_{j=i+1}^{n} \tau_{ij} f_j\Big) = \sum_{j=1}^{n}\Big(\sum_{i=4}^{j-1} -a_i\tau_{ij} + \sum_{i=j+1}^{n} a_i\tau_{ji}\Big) f_j \,.$$

Thus, if $\partial(x)$ is contained in $\mathfrak{M}^2\mathcal{R}^n$, then all the elements $a_i\tau_{ij}$ and $a_i\tau_{ji}$ belong to $\mathfrak{M}^2$, which implies that the every a_i is in $\mathfrak{M}$. Thus, $\partial(\mathcal{R}^{n-3}) \cap \mathfrak{M}^2\mathcal{R}^n$ is contained in $\partial(\mathfrak{M}\mathcal{R}^{n-3})$, and the opposite containment is trivial. □

3.1 Quotients of Even Type

Theorem 3.3 *Let $n \geqslant 5$ be an odd number; consider the ring $\mathcal{R}$ and the $n \times n$ matrix $\mathcal{T}$ from 3.1. The homomorphisms given by the matrices*

$$\partial_3 = \mathcal{T}[1\ldots n; 4\ldots n]\,,$$

$$\partial_2 = \begin{pmatrix} \mathrm{pf}_{\mathcal{T}}(\overline{123}) & 0 & 0 & -\mathrm{pf}_{\mathcal{T}}(\overline{234}) & \mathrm{pf}_{\mathcal{T}}(\overline{235}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{23n}) \\ 0 & \mathrm{pf}_{\mathcal{T}}(\overline{123}) & 0 & -\mathrm{pf}_{\mathcal{T}}(\overline{134}) & \mathrm{pf}_{\mathcal{T}}(\overline{135}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{13n}) \\ 0 & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{123}) & -\mathrm{pf}_{\mathcal{T}}(\overline{124}) & \mathrm{pf}_{\mathcal{T}}(\overline{125}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{12n}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{1}) & -\mathrm{pf}_{\mathcal{T}}(\overline{2}) & \mathrm{pf}_{\mathcal{T}}(\overline{3}) & -\mathrm{pf}_{\mathcal{T}}(\overline{4}) & \mathrm{pf}_{\mathcal{T}}(\overline{5}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{n}) \end{pmatrix},$$

and

$$\partial_1 = \big(-\mathrm{pf}_{\mathcal{T}}(\overline{1}) \quad \mathrm{pf}_{\mathcal{T}}(\overline{2}) \quad -\mathrm{pf}_{\mathcal{T}}(\overline{3}) \quad \mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\,,$$

define an exact sequence

$$\mathcal{F} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}\,.$$

That is, denoting by $\mathfrak{A}_n$ the ideal generated by the entries of ∂_1, the complex $\mathcal{F}$ is a free resolution of $\mathcal{R}/\mathfrak{A}_n$. Moreover, the ideal $\mathfrak{A}_n$ is perfect of grade 3.

The proof of this theorem relies on a series of technical results that we defer to Appendix C. The proof shows how they come together.

Proof It follows from Lemma C.1 that $\mathcal{F}$ is a complex. The expected ranks of the homomorphisms ∂_3, ∂_2, and ∂_1 are $n-3$, 3, and 1. To show that the complex is exact at $\mathcal{R}^{n-3}$, $\mathcal{R}^n$, and $\mathcal{R}^4$ it suffices by the Buchsbaum–Eisenbud Acyclicity Criterion [8] to verify the inequalities

$$\text{grade}_{\mathcal{R}}(I_{n-3}(\partial_3)) \geqslant 3\,, \quad \text{grade}_{\mathcal{R}}(I_3(\partial_2)) \geqslant 2\,, \quad \text{and} \quad \text{grade}_{\mathcal{R}}(I_1(\partial_1)) \geqslant 1\,,$$

where as usual $I_r(\partial)$ denotes the ideal generated by the $r \times r$ minors of ∂. By Lemma C.2 the ideal $I_1(\partial_1) = \mathfrak{A}_n$ has grade at least 3. By Lemma C.3 the radical $\sqrt{\mathfrak{A}_n}$ is contained in $\sqrt{I_{n-3}(\partial_3)}$, so $\text{grade}_{\mathcal{R}}(I_{n-3}(\partial_3)) \geqslant \text{grade}_{\mathcal{R}}(\mathfrak{A}_n) \geqslant 3$ holds. By Proposition C.4, the generators of $I_3(\partial_2)$ are products of generators of the ideals $I_{n-3}(\partial_3)$ and $I_1(\partial_1) = \mathfrak{A}_n$. It follows that the radical $\sqrt{I_3(\partial_2)}$ contains $\sqrt{\mathfrak{A}_n}$, so one also has $\text{grade}_{\mathcal{R}}(I_3(\partial_2)) \geqslant 3$. Thus, $\mathcal{F}$ is a free resolution of $\mathcal{R}/\mathfrak{A}_n$; in particular, the projective dimension of $\mathcal{R}/\mathfrak{A}_n$ is at most 3. As the grade of $\mathfrak{A}_n$ is at least 3, it follows that $\mathfrak{A}_n$ is perfect of grade 3. □

The following commentary also applies to the proof of Theorem 3.9.

Remark 3.4 The proof of Theorem 3.3 is based on establishing containments among radicals to ensure that the rank conditions in the Buchsbaum–Eisenbud Acyclicity Criterion are met; per [9, Theorem 2.1] the conclusion that $\mathcal{F}$ is exact implies that the radicals $\sqrt{I_{n-3}(\partial_3)}$, $\sqrt{I_3(\partial_2)}$, and $\sqrt{I_1(\partial_1)}$ agree.

The inspiration for the pivotal Proposition C.4 came from the same paper, namely from the Buchsbaum–Eisenbud Structure Theorems which, in the guise of [9, Corollary 5.1], say that for $\mathcal{F}$ to be a resolution the equality $I_{n-3}(\partial_3)I_1(\partial_1) = I_3(\partial_2)$ must hold. The vehicle for the proof of Proposition C.4 is a relation between the sub-Pfaffians and general minors of a skew symmetric matrix; it was first discovered by Brill [4] and reproved by us in [11] using Knuth's [17] combinatorial approach to Pfaffians in the same vein as in Appendices A–C.

As noticed in [2] one can replace the upper left 3×3 block in $\mathcal{T}$ with a block of zeros without changing the ideal $\mathfrak{A}_n$.

Lemma 3.5 *Let $n \geqslant 5$ be an odd number and $T = (t_{ij})$ an $n \times n$ skew symmetric matrix with entries in a commutative ring R. Let U be the matrix obtained from T by replacing the upper left 3×3 block by a block of zeros; i.e.*

$$U = \begin{pmatrix} 0 & 0 & 0 & t_{14} & t_{15}\dots \\ 0 & 0 & 0 & t_{24} & t_{25}\dots \\ 0 & 0 & 0 & t_{34} & t_{35}\dots \\ -t_{14} & -t_{24} & -t_{34} & 0 & t_{45}\dots \\ -t_{15} & -t_{25} & -t_{35} & -t_{45} & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix}.$$

There is an equality of ideals in R,

$$\big(\text{Pf}_{\overline{1}}(T)\,,\ \text{Pf}_{\overline{2}}(T)\,,\ \text{Pf}_{\overline{3}}(T)\,,\ \text{Pf}_{\overline{123}}(T)\big) = \big(\text{Pf}_{\overline{1}}(U)\,,\ \text{Pf}_{\overline{2}}(U)\,,\ \text{Pf}_{\overline{3}}(U)\,,\ \text{Pf}_{\overline{123}}(U)\big)\,.$$

Proof Notice that $\mathrm{Pf}_{\overline{12i}}(T) = \mathrm{Pf}_{\overline{12i}}(U)$ holds for $i \in \{3, \dots, n\}$. Lemma A.2 applied with $u_1 \dots u_k = 2 \dots n$ and $\ell = 1$ now yields

$$\begin{aligned}\mathrm{Pf}_{\overline{1}}(T) &= \sum_{i=3}^{n} t_{2i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{12i}}(T)\\ &= t_{23}\,\mathrm{Pf}_{\overline{123}}(U) + \sum_{i=4}^{n} t_{2i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{12i}}(U) \;=\; t_{23}\,\mathrm{Pf}_{\overline{123}}(U) + \mathrm{Pf}_{\overline{1}}(U)\,.\end{aligned}$$

Similarly, one gets

$$\begin{aligned}\mathrm{Pf}_{\overline{2}}(T) &= \sum_{i=3}^{n} t_{1i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{12i}}(T)\\ &= t_{13}\,\mathrm{Pf}_{\overline{123}}(U) + \sum_{i=4}^{n} t_{1i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{12i}}(U) \;=\; t_{13}\,\mathrm{Pf}_{\overline{123}}(U) + \mathrm{Pf}_{\overline{2}}(U)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Pf}_{\overline{3}}(T) &= t_{12}\,\mathrm{Pf}_{\overline{123}}(T) + \sum_{i=4}^{n} t_{1i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{13i}}(T)\\ &= t_{12}\,\mathrm{Pf}_{\overline{123}}(U) + \sum_{i=4}^{n} t_{1i}(-1)^{i-1}\,\mathrm{Pf}_{\overline{13i}}(U) \;=\; t_{12}\,\mathrm{Pf}_{\overline{123}}(U) + \mathrm{Pf}_{\overline{3}}(U)\,.\end{aligned}$$

The asserted equality of ideals is immediate from these three expressions. □

Proposition 3.6 *Let $n \geqslant 5$ be an odd number. Consider the ring $\mathcal{R}$ and the $n \times n$ matrix $\mathcal{T}$ from 3.1 as well as the ideal $\mathfrak{A}_n$ from Theorem 3.3. Let $\mathcal{U}$ be the matrix obtained from $\mathcal{T}$ by replacing the upper left 3×3 block by a block of zeros; i.e.*

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 0 & \tau_{14} & \tau_{15} \dots \\ 0 & 0 & 0 & \tau_{24} & \tau_{25} \dots \\ 0 & 0 & 0 & \tau_{34} & \tau_{35} \dots \\ -\tau_{14} & -\tau_{24} & -\tau_{34} & 0 & \tau_{45} \dots \\ -\tau_{15} & -\tau_{25} & -\tau_{35} & -\tau_{45} & 0 \;\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \;\ddots \end{pmatrix}.$$

The homomorphisms given by the matrices

$$\partial_3 = \mathcal{U}[1 \dots n; 4 \dots n] ,$$

$$\partial_2 = \begin{pmatrix} \mathrm{pf}_{\mathcal{U}}(\overline{123}) & 0 & 0 & -\mathrm{pf}_{\mathcal{U}}(\overline{234}) & \mathrm{pf}_{\mathcal{U}}(\overline{235}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{23n}) \\ 0 & \mathrm{pf}_{\mathcal{U}}(\overline{123}) & 0 & -\mathrm{pf}_{\mathcal{U}}(\overline{134}) & \mathrm{pf}_{\mathcal{U}}(\overline{135}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{13n}) \\ 0 & 0 & \mathrm{pf}_{\mathcal{U}}(\overline{123}) & -\mathrm{pf}_{\mathcal{U}}(\overline{124}) & \mathrm{pf}_{\mathcal{U}}(\overline{125}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{12n}) \\ \mathrm{pf}_{\mathcal{U}}(\overline{1}) & -\mathrm{pf}_{\mathcal{U}}(\overline{2}) & \mathrm{pf}_{\mathcal{U}}(\overline{3}) & -\mathrm{pf}_{\mathcal{U}}(\overline{4}) & \mathrm{pf}_{\mathcal{U}}(\overline{5}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{n}) \end{pmatrix} ,$$

and

$$\partial_1 = \begin{pmatrix} -\mathrm{pf}_{\mathcal{U}}(\overline{1}) & \mathrm{pf}_{\mathcal{U}}(\overline{2}) & -\mathrm{pf}_{\mathcal{U}}(\overline{3}) & \mathrm{pf}_{\mathcal{U}}(\overline{123}) \end{pmatrix} ,$$

define a free resolution $\mathcal{L} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ *of* $\mathcal{R}/\mathfrak{A}_n$*. In particular, the ideal* $\mathfrak{A}_n$ *is generated by* $\mathrm{pf}_{\mathcal{U}}(\overline{1})$, $\mathrm{pf}_{\mathcal{U}}(\overline{2})$, $\mathrm{pf}_{\mathcal{U}}(\overline{3})$, *and* $\mathrm{pf}_{\mathcal{U}}(\overline{123})$.

Proof By Lemma 3.5 one has

$$\big(\mathrm{pf}_{\mathcal{U}}(\overline{1}), \mathrm{pf}_{\mathcal{U}}(\overline{2}), \mathrm{pf}_{\mathcal{U}}(\overline{3}), \mathrm{pf}_{\mathcal{U}}(\overline{123})\big) = \big(\mathrm{pf}_{\mathcal{T}}(\overline{1}), \mathrm{pf}_{\mathcal{T}}(\overline{2}), \mathrm{pf}_{\mathcal{T}}(\overline{3}), \mathrm{pf}_{\mathcal{T}}(\overline{123})\big) = \mathfrak{A}_n .$$

Next we show how to obtain the free resolution $\mathcal{L}$ from the resolution $\mathcal{F}$ from Theorem 3.3. To distinguish the differentials on the resolutions we introduce superscripts $\mathcal{L}$ and $\mathcal{F}$. Consider the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \tau_{23} & -\tau_{13} & \tau_{12} & 1 \end{pmatrix} \quad \text{with inverse} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\tau_{23} & \tau_{13} & -\tau_{12} & 1 \end{pmatrix} .$$

As in the proof of Lemma 3.5 one has

$$\mathrm{pf}_{\mathcal{T}}(\overline{1}) = \tau_{23}\,\mathrm{pf}_{\mathcal{U}}(\overline{123}) + \mathrm{pf}_{\mathcal{U}}(\overline{1}) , \tag{1}$$

$$\mathrm{pf}_{\mathcal{T}}(\overline{2}) = \tau_{13}\,\mathrm{pf}_{\mathcal{U}}(\overline{123}) + \mathrm{pf}_{\mathcal{U}}(\overline{2}) , \tag{2}$$

$$\mathrm{pf}_{\mathcal{T}}(\overline{3}) = \tau_{12}\,\mathrm{pf}_{\mathcal{U}}(\overline{123}) + \mathrm{pf}_{\mathcal{U}}(\overline{3}) , \quad \text{and} \tag{3}$$

$$\mathrm{pf}_{\mathcal{T}}(\overline{123}) = \mathrm{pf}_{\mathcal{U}}(\overline{123}) . \tag{4}$$

These identities show that one has $\partial_1^{\mathcal{L}} = \partial_1^{\mathcal{F}} S$. Thus, the matrices

$$\partial_3^{\mathcal{F}} , \quad S^{-1}\partial_2^{\mathcal{F}} , \quad \text{and} \quad \partial_1^{\mathcal{F}} S$$

determine a free resolution of $\mathcal{R}/\mathfrak{A}_n$. As the submatrices $\partial_3^{\mathcal{L}} = \mathcal{U}[1 \dots n; 4 \dots n]$ and $\partial_3^{\mathcal{F}} = \mathcal{T}[1 \dots n; 4 \dots n]$ agree, it suffices to show that $\partial_2^{\mathcal{L}} = S^{-1}\partial_2^{\mathcal{F}}$ holds.

For indices $1 \leqslant i \leqslant n$ one has

$$\mathrm{pf}_{\mathcal{T}}(\overline{12i}) = \mathrm{pf}_{\mathcal{U}}(\overline{12i}), \quad \mathrm{pf}_{\mathcal{T}}(\overline{13i}) = \mathrm{pf}_{\mathcal{U}}(\overline{13i}), \quad \text{and} \quad \mathrm{pf}_{\mathcal{T}}(\overline{23i}) = \mathrm{pf}_{\mathcal{U}}(\overline{23i}).$$

It follows that the first three rows in the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. We now focus of the fourth rows of $\partial_2^{\mathcal{L}}$ and $S^{-1}\partial_2^{\mathcal{F}}$. The first three entries in the fourth rows agree by the identities (1), (2), and (3). Now fix $j \in \{4, \ldots, n\}$. Another application of Lemma A.2 yields

$$\begin{aligned}\mathrm{pf}_{\mathcal{T}}(\overline{j}) &= \tau_{12}\,\mathrm{pf}_{\mathcal{T}}(\overline{12j}) - \tau_{13}\,\mathrm{pf}_{\mathcal{T}}(\overline{13j}) \\ &\qquad + \sum_{i=4}^{j-1}(-1)^i \tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{1ij}) + \sum_{i=j+1}^{n}(-1)^{i-1}\tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{1ij}).\end{aligned}$$

One has $\mathrm{pf}_{\mathcal{T}}(\overline{1ij}) = \tau_{23}\,\mathrm{pf}_{\mathcal{T}}(\overline{123ij}) + \mathrm{pf}_{\mathcal{U}}(\overline{1ij})$, again by Lemma A.2, and therefore

$$\begin{aligned}&\mathrm{pf}_{\mathcal{T}}(\overline{j}) - \tau_{12}\,\mathrm{pf}_{\mathcal{T}}(\overline{12j}) + \tau_{13}\,\mathrm{pf}_{\mathcal{T}}(\overline{13j}) \\ &\quad = \tau_{23}\Big(\sum_{i=4}^{j-1}(-1)^i \tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{123ij}) + \sum_{i=j+1}^{n}(-1)^{i-1}\tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{123ij})\Big) \\ &\qquad\quad + \sum_{i=1}^{j-1}(-1)^i \tau_{1i}\,\mathrm{pf}_{\mathcal{U}}(\overline{1ij}) + \sum_{i=j+1}^{n}(-1)^{i-1}\tau_{1i}\,\mathrm{pf}_{\mathcal{U}}(\overline{1ij}) \\ &\quad = \tau_{23}\,\mathrm{pf}_{\mathcal{T}}(\overline{23j}) + \mathrm{pf}_{\mathcal{U}}(\overline{j}),\end{aligned}$$

where the last equality follows from two applications of Lemma A.2. This identity shows that the fourth row entries of $\partial_2^{\mathcal{L}}$ and $S^{-1}\partial_2^{\mathcal{F}}$ agree in column j. □

One could also establish Proposition 3.6 as follows: After invoking Lemma 3.5, repeat the proof of Theorem 3.3 noticing at every step that the conclusions remain valid after evaluation at $\tau_{12} = \tau_{13} = \tau_{23} = 0$.

3.7 Proof of Theorem 2.2 An almost complete intersection ideal of grade 3 is by [10, Proposition 5.2] linked to a Gorenstein ideal of grade 3, and Brown [6, Proposition 4.3] uses this to show that there exits a skew symmetric matrix T with entries in $\mathfrak{m}$ such that $\mathfrak{a}$ is generated by the Pfaffians $\mathrm{Pf}_{\overline{1}}(T)$, $\mathrm{Pf}_{\overline{2}}(T)$, $\mathrm{Pf}_{\overline{3}}(T)$, and $\mathrm{Pf}_{\overline{123}}(T)$. Lemma 3.5 shows than one can replace the upper left 3×3 block in T with zeroes and arrive at the asserted block matrix U.

Adopt Setup 3.1. Let $\mathcal{R} \to R$ be given by $\tau_{ij} \mapsto t_{ij}$; it makes R an $\mathcal{R}$-algebra and maps Pfaffians of submatrices of $\mathcal{T}$ to the corresponding Pfaffians of submatrices of U, i.e. $\mathrm{pf}_{\mathcal{T}}(\overline{123})$ maps to $\mathrm{Pf}_{\overline{123}}(U)$ etc. Let $\mathcal{F}$ be the free resolution of $\mathcal{R}/\mathfrak{A}_n$ from Theorem 3.3. As one has $R/\mathfrak{a} = \mathcal{R}/\mathfrak{A}_n \otimes_{\mathcal{R}} R$ and $\mathfrak{a}$ has grade 3 it follows from

Bruns and Vetter [7, Theorem 3.5] that

$$F \;=\; \mathcal{F} \otimes_{\mathcal{R}} R \tag{0}$$

is a free resolution of $R/\mathfrak{a}$ over R, and it is minimal as the differentials are given by matrices with entries in $\mathfrak{m}$.

We now establish parts of a multiplicative structure on F: just enough to determine the multiplicative structure on the k-algebra $\operatorname{Tor}_*^R(R/\mathfrak{a}, k)$. Let $e_1, \ldots, e_4$, $f_1, \ldots, f_n$, and $g_1, \ldots, g_{n-3}$ be the standard bases for the free modules F_1, F_2, and F_3. From the three obvious Koszul relations one gets

$$\begin{aligned}
\partial_2(e_4e_1) &= \operatorname{Pf}_{\overline{123}}(U)e_1 + \operatorname{Pf}_{\overline{1}}(U)e_4 = \partial_2(f_1)\,,\\
\partial_2(e_4e_2) &= \operatorname{Pf}_{\overline{123}}(U)e_2 - \operatorname{Pf}_{\overline{2}}(U)e_4 = \partial_2(f_2)\,, \text{ and}\\
\partial_2(e_4e_3) &= \operatorname{Pf}_{\overline{123}}(U)e_3 + \operatorname{Pf}_{\overline{3}}(U)e_4 = \partial_2(f_3)\,.
\end{aligned}$$

Thus one can set

$$e_4e_1 = f_1\,, \quad e_4e_2 = f_2\,, \quad \text{and} \quad e_4e_3 = f_3\,. \tag{1}$$

These three products in F induce non-trivial products in $\operatorname{Tor}^R_{R/\mathfrak{a}}(*, k)$. Applying Lemma A.2 the same way as in the proof of Lemma 3.5 one gets:

$$\begin{aligned}
\partial_2(e_1e_2) &= -\operatorname{Pf}_{\overline{1}}(U)e_2 - \operatorname{Pf}_{\overline{2}}(U)e_1 = \partial_2\Big(\sum_{i=4}^{n} t_{3i} f_i\Big)\,,\\
\partial_2(e_2e_3) &= \operatorname{Pf}_{\overline{2}}(U)e_3 + \operatorname{Pf}_{\overline{3}}(U)e_2 = \partial_2\Big(\sum_{i=4}^{n} t_{1i} f_i\Big)\,, \text{ and}\\
\partial_2(e_3e_1) &= -\operatorname{Pf}_{\overline{3}}(U)e_1 + \operatorname{Pf}_{\overline{1}}(U)e_3 = \partial_2\Big(\sum_{i=4}^{n} t_{2i} f_i\Big)\,.
\end{aligned}$$

Thus one can set

$$e_1e_2 = \sum_{i=4}^{n} t_{3i} f_i\,, \quad e_2e_3 = \sum_{i=4}^{n} t_{1i} f_i\,, \quad \text{and} \quad e_3e_1 = \sum_{i=4}^{n} t_{2i} f_i\,. \tag{2}$$

The products (2) induce trivial products in $\operatorname{Tor}_*^R(R/\mathfrak{a}, k)$. We have now accounted for all products of elements from F_1, so $R/\mathfrak{a}$ is of class $\mathbf{H}(3, q)$, where q denotes the dimension of the subspace $\operatorname{Tor}_1^R(R/\mathfrak{a}, k) \cdot \operatorname{Tor}_2^R(R/\mathfrak{a}, k)$ of $\operatorname{Tor}_3^R(R/\mathfrak{a}, k)$; see [3, Theorem 2.1].

For $j \in \{1, 2, 3\}$ one has

$$\partial_3(e_4 f_j) \;=\; \mathrm{Pf}_{\overline{123}}(U) f_j - e_4\big(\mathrm{Pf}_{\overline{123}}(U) e_j \pm \mathrm{Pf}_{\overline{j}}(U) e_4\big) \;=\; 0$$

by (1). Since ∂_3 is injective, one has

$$e_4 f_1 \;=\; e_4 f_2 \;=\; e_4 f_3 \;=\; 0\,.$$

For $j \in \{4, \dots, n\}$ one gets

$$\begin{aligned}\partial_3(e_4 f_j) \;&=\; \mathrm{Pf}_{\overline{123}}(U) f_j \\ &\qquad + (-1)^j e_4\big(\mathrm{Pf}_{\overline{23j}}(U) e_1 + \mathrm{Pf}_{\overline{13j}}(U) e_2 + \mathrm{Pf}_{\overline{12j}}(U) e_3\, \mathrm{Pf}_{\overline{j}}(U) e_4\big) \\ &=\; (-1)^j \big(\mathrm{Pf}_{\overline{23j}}(U) f_1 + \mathrm{Pf}_{\overline{13j}}(U) f_2 + \mathrm{Pf}_{\overline{12j}}(U) f_3\big) - \mathrm{Pf}_{\overline{123}}(U) f_j\,,\end{aligned} \tag{3}$$

again by (1). Thus, for $n \geqslant 7$ one has $\partial_3(e_4 f_j) \in \mathfrak{m}^2 F_2$. In this case it follows from Lemma 3.2 and (0) that there is an element $x_j \in \mathfrak{m} F_3$ with $\partial_3(x_j) = \partial_3(e_4 f_j)$, so by injectivity of ∂_3 one has $e_4 f_j = x_j$; in particular this product induces a trivial product in $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$. For $n = 5$ one has $j \in \{4, 5\}$ and (3) specializes to

$$\begin{aligned}\partial_3(e_4 f_4) \;&=\; t_{15} f_1 + t_{25} f_2 + t_{35} f_3 - t_{45} f_4 \;=\; \partial_3(g_2) \quad \text{and} \\ \partial_3(e_4 f_5) \;&=\; -(t_{14} f_1 + t_{24} f_2 + t_{34} f_3 + t_{45} f_5) \;=\; -\partial_3(g_1)\,.\end{aligned}$$

As ∂_3 is injective, this shows that in this case one has

$$e_4 f_4 \;=\; g_2 \qquad \text{and} \qquad e_4 f_5 \;=\; -g_1\,.$$

To prove the assertion about the multiplicative structure on $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$, it suffices to show that there are no further non-zero products in $\mathrm{Tor}_1^R(R/\mathfrak{a}, k) \cdot \mathrm{Tor}_2^R(R/\mathfrak{a}, k)$. To this end it suffices by Lemma 3.2 and (0) to shows that $\partial_3(e_i f_j)$ belongs to $\mathfrak{m}^2 F_2$ for indices $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant n$. For $1 \leqslant i, j \leqslant 3$ one has

$$\partial_3(e_i f_j) \;=\; (-1)^i\, \mathrm{Pf}_{\overline{i}}(U) f_j - e_i\big(\mathrm{Pf}_{\overline{123}}(U) e_j + (-1)^{j-1}\, \mathrm{Pf}_{\overline{j}}(U) e_4\big)\,.$$

This is indeed in $\mathfrak{m}^2 F_2$ as $\mathrm{Pf}_{\overline{i}}(U)$ and $\mathrm{Pf}_{\overline{j}}(U)$ belong to $\mathfrak{m}^2$, and $e_i e_j \in \mathfrak{m} F_2$ by (2). For indices $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant n$ one has

$$\begin{aligned}\partial_3(e_i f_j) \;&=\; (-1)^i\, \mathrm{Pf}_{\overline{i}}(U) f_j \\ &\qquad - e_i (-1)^{j-1}\big(\mathrm{Pf}_{\overline{23j}}(U) e_1 + \mathrm{Pf}_{\overline{13j}}(U) e_2 + \mathrm{Pf}_{\overline{12j}}(U) e_3 + \mathrm{Pf}_{\overline{j}}(U) e_4\big)\,.\end{aligned}$$

As above $\mathrm{Pf}_{\bar{i}}(U)$ and $\mathrm{Pf}_{\bar{j}}(U)$ belong to $\mathfrak{m}^2$, and the products e_ie_1, e_ie_2, and e_ie_3 belong to $\mathfrak{m}F_2$ by (2). □

3.2 Quotients of Odd Type

Remark 3.8 In Brown's work [6], the statements to the effect that all almost complete intersection ideals come from skew symmetric matrices—Propositions 4.2 and 4.3 in *loc. cit.* as cited in our proofs of Theorems 2.2 and 2.3—are separated from the descriptions of the free resolutions: Propositions 3.2 and 3.3 in *loc. cit.* The proofs of all four statements in [6] rely on the fact from [10] that almost complete intersection ideals are linked to Gorenstein ideals, but compare the proofs of [6, Propositions 3.2 and 4.2] for almost complete intersections of odd type: The linking sequence used in the description of the free resolution is different from the one used to associate a skew symmetric matrix; a change of basis argument is thus required to reconcile the two. Our explicit construction of the free resolution in the generic case, Theorems 3.3 and 3.9, allows us to avoid such issues in the proofs of Theorems 2.2 and 2.3.

Theorem 3.9 *Let $n \geqslant 6$ be an even number; consider the ring $\mathcal{R}$ and the $n \times n$ matrix $\mathcal{T}$ from 3.1. The homomorphisms given by the matrices*

$$\partial_3 = \mathcal{T}[1\dots n; 4\dots n]\,,$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{pf}_{\mathcal{T}}(\overline{1234}) & \mathrm{pf}_{\mathcal{T}}(\overline{1235}) & \cdots & -\mathrm{pf}_{\mathcal{T}}(\overline{123n}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{13}) & -\mathrm{pf}_{\mathcal{T}}(\overline{23}) & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{34}) & -\mathrm{pf}_{\mathcal{T}}(\overline{35}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{3n}) \\ -\mathrm{pf}_{\mathcal{T}}(\overline{12}) & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{23}) & -\mathrm{pf}_{\mathcal{T}}(\overline{24}) & \mathrm{pf}_{\mathcal{T}}(\overline{25}) & \cdots & -\mathrm{pf}_{\mathcal{T}}(\overline{2n}) \\ 0 & \mathrm{pf}_{\mathcal{T}}(\overline{12}) & -\mathrm{pf}_{\mathcal{T}}(\overline{13}) & \mathrm{pf}_{\mathcal{T}}(\overline{14}) & -\mathrm{pf}_{\mathcal{T}}(\overline{15}) & \cdots & \mathrm{pf}_{\mathcal{T}}(\overline{1n}) \end{pmatrix},$$

and

$$\partial_1 = \big(\mathrm{pf}_{\mathcal{T}} \quad \mathrm{pf}_{\mathcal{T}}(\overline{12}) \quad \mathrm{pf}_{\mathcal{T}}(\overline{13}) \quad \mathrm{pf}_{\mathcal{T}}(\overline{23})\big)$$

define an exact sequence

$$\mathcal{F} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}\,.$$

That is, denoting by $\mathfrak{A}_n$ the ideal generated by the entries of ∂_1, the complex $\mathcal{F}$ is a free resolution of $\mathcal{R}/\mathfrak{A}_n$. Moreover, the ideal $\mathfrak{A}_n$ is perfect of grade 3.

Proof The proof of Theorem 3.3 applies, one only needs to replace the references to C.1–C.4 with references to C.5–C.8. □

Lemma 3.10 *Let $n \geqslant 6$ be an even number and $T = (t_{ij})$ an $n \times n$ skew symmetric matrix with entries in a commutative ring R. Let U be the matrix obtained from T*

by replacing the upper left 3×3 *block by a block of zeros; see Lemma* 3.5. *There is an equality of ideals in* R

$$\big(\mathrm{Pf}(T)\,,\ \mathrm{Pf}_{\overline{12}}(T)\,,\ \mathrm{Pf}_{\overline{13}}(T)\,,\ \mathrm{Pf}_{\overline{23}}(T)\big) \;=\; \big(\mathrm{Pf}(U)\,,\ \mathrm{Pf}_{\overline{12}}(U)\,,\ \mathrm{Pf}_{\overline{13}}(U)\,,\ \mathrm{Pf}_{\overline{23}}(U)\big)\,.$$

Proof First notice that one has

$$\mathrm{Pf}_{\overline{12}}(T) = \mathrm{Pf}_{\overline{12}}(U)\,, \quad \mathrm{Pf}_{\overline{13}}(T) = \mathrm{Pf}_{\overline{13}}(U)\,, \quad \text{and} \quad \mathrm{Pf}_{\overline{23}}(T) = \mathrm{Pf}_{\overline{23}}(U)\,. \tag{1}$$

Lemma A.2 applied with $u_1 \dots u_k = 1 \dots n$ and $\ell = 1$ yields

$$\begin{aligned} \mathrm{Pf}(T) \;&=\; \sum_{i=2}^{n} t_{1i}(-1)^i \,\mathrm{Pf}_{\overline{1i}}(T) \\ &=\; t_{12}\,\mathrm{Pf}_{\overline{12}}(T) - t_{13}\,\mathrm{Pf}_{\overline{13}}(T) + \sum_{i=4}^{n} t_{1i}(-1)^i\,\mathrm{Pf}_{\overline{1i}}(T)\,. \end{aligned} \tag{2}$$

For $i \geqslant 4$ the same lemma applied with $u_1 \dots u_k = 2 \dots n \setminus i$ and $\ell = 2$ yields

$$\mathrm{Pf}_{\overline{1i}}(T) \;=\; t_{23}\,\mathrm{Pf}_{\overline{123i}}(T) + \mathrm{Pf}_{\overline{1i}}(U)\,. \tag{3}$$

From (2), (3), and further applications of A.2 one now gets

$$\begin{aligned} &\mathrm{Pf}(T) - t_{12}\,\mathrm{Pf}_{\overline{12}}(T) + t_{13}\,\mathrm{Pf}_{\overline{13}}(T) \\ &\qquad =\; t_{23}\sum_{i=4}^{n} t_{1i}(-1)^i\,\mathrm{Pf}_{\overline{123i}}(T) + \sum_{i=4}^{n} t_{1i}(-1)^i\,\mathrm{Pf}_{\overline{1i}}(U) \\ &\qquad =\; t_{23}\,\mathrm{Pf}_{\overline{23}}(T) + \mathrm{Pf}(U)\,. \end{aligned} \tag{4}$$

The asserted equality of ideals is immediate from (1) and (4). □

Proposition 3.11 *Let* $n \geqslant 6$ *be an even number. Consider the ring* $\mathcal{R}$ *and the* $n \times n$ *matrix* $\mathcal{T}$ *from* 3.1 *as well as the ideal* $\mathfrak{A}_n$ *from Theorem* 3.9. *Let* $\mathcal{U}$ *be the matrix obtained from* $\mathcal{T}$ *by replacing the upper left* 3×3 *block by a block of zeros; see Proposition* 3.6. *The homomorphisms given by the matrices*

$$\partial_3 \;=\; \mathcal{U}[1,\dots,n;4,\dots,n]\,,$$

$$\partial_2 \;=\; \begin{pmatrix} 0 & 0 & 0 & -\,\mathrm{pf}_{\mathcal{U}}(\overline{1234}) & \mathrm{pf}_{\mathcal{U}}(\overline{1235}) & \cdots & -\,\mathrm{pf}_{\mathcal{U}}(\overline{123n}) \\ \mathrm{pf}_{\mathcal{U}}(\overline{13}) & -\,\mathrm{pf}_{\mathcal{U}}(\overline{23}) & 0 & \mathrm{pf}_{\mathcal{U}}(\overline{34}) & -\,\mathrm{pf}_{\mathcal{U}}(\overline{35}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{3n}) \\ -\,\mathrm{pf}_{\mathcal{U}}(\overline{12}) & 0 & \mathrm{pf}_{\mathcal{U}}(\overline{23}) & -\,\mathrm{pf}_{\mathcal{U}}(\overline{24}) & \mathrm{pf}_{\mathcal{U}}(\overline{25}) & \cdots & -\,\mathrm{pf}_{\mathcal{U}}(\overline{2n}) \\ 0 & \mathrm{pf}_{\mathcal{U}}(\overline{12}) & -\,\mathrm{pf}_{\mathcal{U}}(\overline{13}) & \mathrm{pf}_{\mathcal{U}}(\overline{14}) & -\,\mathrm{pf}_{\mathcal{U}}(\overline{15}) & \cdots & \mathrm{pf}_{\mathcal{U}}(\overline{1n}) \end{pmatrix}$$

and

$$\partial_1 = \big(\mathrm{pf}_{[}\mathcal{U}] \quad \mathrm{pf}_{\mathcal{U}}(\overline{12}) \quad \mathrm{pf}_{\mathcal{U}}(\overline{13}) \quad \mathrm{pf}_{\mathcal{U}}(\overline{23})\big)$$

define a free resolution $\mathcal{L} = 0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ *of* $\mathcal{R}/\mathfrak{A}_n$. *In particular, the ideal* $\mathfrak{A}_n$ *is generated by* $\mathrm{pf}_{[}\mathcal{U}]$, $\mathrm{pf}_{\mathcal{U}}(\overline{12})$, $\mathrm{pf}_{\mathcal{U}}(\overline{13})$, *and* $\mathrm{pf}_{\mathcal{U}}(\overline{23})$.

Proof By Lemma 3.10 one has

$$\big(\mathrm{pf}_{[}\mathcal{U}],\ \mathrm{pf}_{\mathcal{U}}(\overline{12}),\ \mathrm{pf}_{\mathcal{U}}(\overline{13}),\ \mathrm{pf}_{\mathcal{U}}(\overline{23})\big) = \big(\mathrm{pf}_{\mathcal{T}},\ \mathrm{pf}_{\mathcal{T}}(\overline{12}),\ \mathrm{pf}_{\mathcal{T}}(\overline{13}),\ \mathrm{pf}_{\mathcal{T}}(\overline{23})\big) = \mathfrak{A}_n\,.$$

As in the proof of Proposition 3.6, we proceed to show how the free resolution $\mathcal{L}$ is obtained from the resolution $\mathcal{F}$ from Theorem 3.9. To distinguish the differentials on the resolutions we introduce superscripts $\mathcal{L}$ and $\mathcal{F}$. Consider the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\tau_{12} & 1 & 0 & 0 \\ \tau_{13} & 0 & 1 & 0 \\ -\tau_{23} & 0 & 0 & 1 \end{pmatrix} \quad \text{with inverse} \quad S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_{12} & 1 & 0 & 0 \\ -\tau_{13} & 0 & 1 & 0 \\ \tau_{23} & 0 & 0 & 1 \end{pmatrix}.$$

Notice that one has

$$\mathrm{pf}_{\mathcal{T}}(\overline{12}) = \mathrm{pf}_{\mathcal{U}}(\overline{12}), \quad \mathrm{pf}_{\mathcal{T}}(\overline{13}) = \mathrm{pf}_{\mathcal{U}}(\overline{13}), \quad \text{and} \quad \mathrm{pf}_{\mathcal{T}}(\overline{23}) = \mathrm{pf}_{\mathcal{U}}(\overline{23})\,. \tag{1}$$

As in the proof of Lemma 3.10 one gets

$$\mathrm{pf}_{\mathcal{T}} = \tau_{23}\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) + \mathrm{pf}_{[}\mathcal{U}]\,. \tag{2}$$

These identities yield $\partial_1^{\mathcal{L}} = \partial_1^{\mathcal{F}} S$. Thus, the matrices $\partial_3^{\mathcal{F}}$, $S^{-1}\partial_2^{\mathcal{F}}$, and $\partial_1^{\mathcal{F}} S$ determine a free resolution of $\mathcal{R}/\mathfrak{A}_n$. As the matrices $\partial_3^{\mathcal{L}}$ and $\partial_3^{\mathcal{F}}$ agree, it suffices to show that $\partial_2^{\mathcal{L}} = S^{-1}\partial_2^{\mathcal{F}}$ holds.

By (1) the first three columns of the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. For indices $1 \leqslant i \leqslant n$ one has $\mathrm{pf}_{\mathcal{T}}(\overline{123i}) = \mathrm{pf}_{\mathcal{U}}(\overline{123i})$, so also the top rows in the matrices $\partial_2^{\mathcal{L}}$, $\partial_2^{\mathcal{F}}$, and $S^{-1}\partial_2^{\mathcal{F}}$ agree. Now fix $i \in \{4, \dots, n\}$. The $(4, i)$ entry in the matrix $S^{-1}\partial_2^{\mathcal{F}}$ is

$$\tau_{23}(-1)^{i-1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i}) + (-1)^i\,\mathrm{pf}_{\mathcal{T}}(\overline{1i})\,.$$

To see that this is indeed $(-1)^{i-1}\,\mathrm{pf}_{\mathcal{U}}(\overline{123i})$, the $(4, i)$ entry in $\partial_2^{\mathcal{L}}$, apply Lemma A.2 with $u_1 \dots u_k = 2 \dots n \setminus i$ and $\ell = 2$ to get

$$\mathrm{pf}_{\mathcal{T}}(\overline{1i}) = \tau_{23}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i}) + \mathrm{pf}_{\mathcal{U}}(\overline{1i})\,.$$

Similar applications of Lemma A.2 yield the identities

$$\mathrm{pf}_{\mathcal{T}}(\overline{2i}) \;=\; \tau_{13}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i}) + \mathrm{pf}_{\mathcal{U}}(\overline{2i}) \quad \text{and} \quad \mathrm{pf}_{\mathcal{T}}(\overline{3i}) \;=\; \tau_{12}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i}) + \mathrm{pf}_{\mathcal{U}}(\overline{3i})\,,$$

which show that also the $(3,i)$ and $(2,i)$ entries in the two matrices agree. □

3.12 Proof of Theorem 2.3 An almost complete intersection ideal of grade 3 is by [10, Proposition 5.2] linked to a Gorenstein ideal of grade 3, and Brown [6, Proposition 4.2] uses this to show that there exits a skew symmetric matrix T with entries in $\mathfrak{m}$ such that $\mathfrak{a}$ is generated by the Pfaffians $\mathrm{Pf}(T)$, $\mathrm{Pf}_{\overline{12}}(T)$, $\mathrm{Pf}_{\overline{13}}(T)$, and $\mathrm{Pf}_{\overline{23}}(T)$. Lemma 3.10 shows than one can replace the upper left 3×3 block in T with zeroes and arrive at the asserted block matrix U. Adopt Setup 3.1 and let $\mathcal{F}$ be the free resolution of $\mathcal{R}/\mathfrak{A}_n$ from Theorem 3.9. As in the proof of Theorem 2.2 one sees that

$$F \;=\; \mathcal{F} \otimes_{\mathcal{R}} R \tag{0}$$

is a minimal free resolution of $R/\mathfrak{a}$ over R.

As in the proof of Theorem 2.2 we proceed to determine enough of a multiplicative structure on F to recognize the multiplicative structure on $\mathrm{Tor}^R_*(R/\mathfrak{a}, k)$. Let $e_1, \ldots, e_4$, $f_1, \ldots, f_n$, and $g_1, \ldots, g_{n-3}$ be the standard bases for the free modules F_1, F_2, and F_3. From the three obvious Koszul relations one gets

$$\begin{aligned}
\partial_2(e_2e_3) &= \mathrm{Pf}_{\overline{12}}(U)e_3 - \mathrm{Pf}_{\overline{13}}(U)e_2 \;=\; \partial_2(-f_1)\,,\\
\partial_2(e_3e_4) &= \mathrm{Pf}_{\overline{13}}(U)e_4 - \mathrm{Pf}_{\overline{23}}(U)e_3 \;=\; \partial_2(-f_3)\,, \text{ and}\\
\partial_2(e_4e_2) &= \mathrm{Pf}_{\overline{23}}(U)e_2 - \mathrm{Pf}_{\overline{13}}(U)e_4 \;=\; \partial_2(-f_2)\,.
\end{aligned}$$

Thus one can set

$$e_2e_3 \;=\; -f_1\,, \quad e_3e_4 \;=\; -f_3\,, \quad \text{and} \quad e_4e_2 \;=\; -f_2\,. \tag{1}$$

These three products in F induce non-trivial products in $\mathrm{Tor}^R_*(R/\mathfrak{a}, k)$. Repeated applications of Lemma A.2 yield:

$$\begin{aligned}
\partial_2(e_1e_2) &= \mathrm{Pf}(T)e_2 - \mathrm{Pf}_{\overline{12}}(U)e_1\\
&= -\Big(\sum_{i=4}^{n} t_{3i}(-1)^i\,\mathrm{Pf}_{\overline{123i}}(U)\Big)e_1 + \Big(\sum_{i=4}^{n} t_{3i}(-1)^i\,\mathrm{Pf}_{\overline{3i}}(U)\Big)e_2\\
&= \partial_2\Big(\sum_{i=4}^{n} t_{3i}f_i\Big)\,,
\end{aligned}$$

$$\partial_2(e_1e_3) \;=\; \mathrm{Pf}(T)e_3 - \mathrm{Pf}_{\overline{13}}(U)e_3 \;=\; \partial_2\Big(\sum_{i=4}^{n} t_{2i} f_i\Big)\,, \text{ and}$$

$$\partial_2(e_1e_4) \;=\; \mathrm{Pf}(T)e_4 - \mathrm{Pf}_{\overline{23}}(U)e_1 \;=\; \partial_2\Big(\sum_{i=4}^{n} t_{1i} f_i\Big)\,.$$

Thus one can set

$$e_1e_2 \;=\; \sum_{i=4}^{n} t_{3i} f_i\,, \quad e_1e_3 \;=\; \sum_{i=4}^{n} t_{2i} f_i\,, \quad \text{and} \quad e_1e_4 \;=\; \sum_{i=4}^{n} t_{1i} f_i\,. \tag{2}$$

The products (2) induce trivial products in $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$. We have now accounted for all products of elements from F_1. To prove that $R/\mathfrak{a}$ is of class $\mathbf{T}$ it suffices to show that all products of the form $e_i f_j$ induce the zero product in $\mathrm{Tor}_*^R(R/\mathfrak{a}, k)$; see [3, Theorem 2.1]. To this end, it suffices by Lemma 3.2 and (0) to shows that $\partial_3(e_i f_j)$ belongs to $\mathfrak{m}^2F_2$ for all indices $1 \leqslant i \leqslant 4$ and $1 \leqslant j \leqslant n$. One has

$$\partial_3(e_i f_j) \;=\; \partial_1(e_i) f_j - \partial_2(f_j)e_i\,.$$

For all i one has $\partial_1(e_i) \in \mathfrak{m}^2$, and for $1 \leqslant j \leqslant 3$ also $\partial_2(f_j)$ belongs to $\mathfrak{m}^2$. For $4 \leqslant j \leqslant n$ one has

$$\partial_2(f_j)e_i \;=\; (-1)^{j-1}\big(\mathrm{Pf}_{\overline{123j}}(U)e_1 - \mathrm{Pf}_{\overline{3j}}(U)e_2 + \mathrm{Pf}_{\overline{2j}}(U)e_3 - \mathrm{Pf}_{\overline{1j}}(U)e_4\big)e_i\,.$$

This too is in $\mathfrak{m}^2F_2$ as the coefficients $\mathrm{Pf}_{\overline{1j}}(U)$, $\mathrm{Pf}_{\overline{2j}}(U)$, and $\mathrm{Pf}_{\overline{3j}}(U)$ belong to $\mathfrak{m}^2$ and the product e_1e_i is in $\mathfrak{m}F_2$. □

4 The Equivariant Form of the Format $(1, 4, n, n-3)$

In this section we give an equivariant interpretation of generic four generated perfect ideals of codimension three. These ideals were already considered from a purely algebraic point of view in Sect. 3, and they will be treated as linear sections of Schubert varieties in Sect. 5.

4.1 Quotients of Even Type

Let $n = 2m + 3$ where m is a natural number. Consider a $2m \times 2m$ generic skew symmetric matrix $A = (c_{ij})$ and a $3 \times 2m$ generic matrix $B = (u_{ki})$. Thus we work over a ring

$$\mathcal{A} = \mathrm{Sym}_{\mathbb{Z}}(\bigwedge^2 F \oplus F \otimes G) \cong \mathbb{Z}[c_{ij}, u_{ki}]$$

where $F = \mathbb{Z}^{2m}$ and $G = \mathbb{Z}^3$ are free $\mathbb{Z}$-modules. The ring $\mathcal{A}$ has an obvious bigrading with $|c_{ij}| = (1, 0)$ and $|u_{ki}| = (0, 1)$.

Proposition 4.1 *Let* $\{g_1, \ldots, g_{2m}\}$ *be a basis for* F *and set*

$$C = \sum_{1 \leqslant i < j \leqslant 2m} c_{ij} g_i \wedge g_j \quad \text{and} \quad u_k = \sum_{i=1}^{2m} u_{ki} g_i \ \text{ for } 1 \leqslant k \leqslant 3\,.$$

We denote by C^j *the* jth *exterior power in* $\bigwedge^{2j} F$. *The ideal*

$$\mathcal{I}_n = (C^m, C^{m-1} \wedge u_1 \wedge u_2, C^{m-1} \wedge u_1 \wedge u_3, C^{m-1} \wedge u_2 \wedge u_3)$$

is a grade 3 *almost complete intersection ideal of type* $2m = n - 3$.

Proof The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A,

$$C^j = \sum_{1 \leqslant i_1 < \cdots < i_{2j} \leqslant 2m} \mathrm{Pf}_{i_1 \ldots i_{2j}}(A) \cdot g_{i_1} \wedge \ldots \wedge g_{i_{2j}}\ .$$

Plugging these in, we see that we get the generators of the ideals described in this proposition from the matrix $\mathcal{U}$ in Proposition 3.6 via the substitutions $c_{ij} = \tau_{i+3, j+3}$ and $u_{i,j} = \tau_{i, j+3}$. □

Let us work out the minimal free resolution of the ideal defined above.

Proposition 4.2 *Let* $n = 2m + 3$ *and* $\mathcal{I}_n$ *be the ideal from Proposition 4.1. The minimal graded free resolution of the cyclic* $\mathcal{A}$*-module* $\mathcal{A}/\mathcal{I}_n$ *is*

$$\mathcal{F}_\bullet : 0 \longrightarrow F \otimes (\bigwedge^{2m} F)^{\otimes 2} \otimes \bigwedge^{3} G \otimes \mathcal{A}(-2m+1, -3) \xrightarrow{\partial_3}$$

$$(\bigwedge^{2m} F)^{\otimes 2} \otimes \bigwedge^{2} G \otimes \mathcal{A}(-2m+1, -2) \oplus \bigwedge^{2m} F \otimes \bigwedge^{2m-1} F \otimes \bigwedge^{3} G \otimes \mathcal{A}(-2m$$

$$+2, -3) \xrightarrow{\partial_2} \bigwedge^{2m} F \otimes \mathcal{A}(-m, 0) \oplus \bigwedge^{2m} F \otimes \bigwedge^{2} G \otimes \mathcal{A}(-m+1, -2) \xrightarrow{\partial_1} \mathcal{A}.$$

The differentials ∂_3, ∂_2, ∂_1 are described in the proof below. For every field k the resolution $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ is minimal over

$$\mathcal{A}_k = \mathrm{Sym}_{\mathbb{Z}}(\textstyle\bigwedge^2 \overline{F} \oplus \overline{F} \otimes \overline{G}) \cong k[c_{ij}, u_{ki}],$$

with $\overline{F} = F \otimes_{\mathbb{Z}} k$ and $\overline{G} = G \otimes_{\mathbb{Z}} k$. The ideal $\mathcal{I}_n \otimes_{\mathbb{Z}} k$ is thus perfect of grade three.

Proof Let us first describe the differentials in the complex $\mathcal{F}_\bullet$ in this setting. The last differential ∂_3 is just a $(2m+3) \times 2m$ matrix with the $2m \times 2m$ block given by the matrix A and the $3 \times 2m$ block given by the matrix B. The differential ∂_2 can be expressed in block form as

$$\partial_2 = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

where A_{21} is given by multiplication by the representation $\bigwedge^{2m} F$ occurring in the degree $(m, 0)$ component of $\mathcal{A}$. The matrix A_{22} is given by multiplication by the representation $\bigwedge^{2m-1} F \otimes G$ occurring in the degree $(m-1, 1)$ of $\mathcal{A}$. The matrix A_{11} is given by multiplication by the representation $\bigwedge^{2m} F \otimes \bigwedge^2 G$ occurring in the degree $(m-1, 2)$ component of $\mathcal{A}$ and A_{12} is given by multiplication by the representation $\bigwedge^{2m-1} F \otimes \bigwedge^3 G$ occurring in the degree $(m-2, 3)$ component of $\mathcal{A}$. The relations coming from the second summand are three Koszul relations between the last generator and the three others. The differential ∂_1 is given by the generators of $\mathcal{I}_n$.

The matrices of the differentials are:

$$\partial_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1\,2m} \\ u_{21} & u_{22} & u_{23} & \dots & u_{2\,2m} \\ u_{31} & u_{32} & u_{33} & \dots & u_{3\,2m} \\ 0 & c_{12} & c_{13} & \dots & c_{1\,2m} \\ -c_{12} & 0 & c_{23} & \dots & c_{2\,2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1\,2m} & -c_{2\,2m} & -c_{3\,2m} & \dots & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -x_1 & -x_2 & -x_3 & w_1 & w_2 & \dots & w_{2m} \\ x_4 & 0 & 0 & v_{\{2,3\}\,1} & v_{\{2,3\}\,2} & \dots & v_{\{2,3\}\,2m} \\ 0 & x_4 & 0 & v_{\{1,3\}\,1} & -v_{\{1,3\}\,2} & \dots & -v_{\{1,3\}\,2m} \\ 0 & 0 & x_4 & v_{\{1,2\}\,1} & v_{\{1,2\}\,2} & \dots & v_{\{1,2\}\,2m} \end{pmatrix},$$

and

$$\partial_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

with entries as defined below

$$x_1 = C^m,\ x_2 = C^{m-1} \wedge u_2 \wedge u_3\,,\ x_3 = C^{m-1} \wedge u_1 \wedge u_3\,,\ x_4 = C^{m-1} \wedge u_1 \wedge u_2\,, v_{\{\alpha,\beta\}\,i} = \sum_j u_{\gamma j}\,\mathrm{Pf}_{\overline{ij}}(C)\,, \qquad \text{and} \qquad w_i = \sum_{j,k,l} \pm\Delta^{j,k,l}\,\mathrm{Pf}_{\overline{ijkl}}(C)\,.$$

Here $\Delta^{j,k,l}$ is a 3×3 minor of the $3 \times (2m)$ matrix B on columns j, k, l. Finally γ is the complement of $\{\alpha, \beta\}$ in the set $\{1, 2, 3\}$.

The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A,

$$C^j = \sum_{1 \leqslant i_1 < \cdots < i_{2j} \leqslant 2m} \mathrm{Pf}_{i_1 \ldots i_{2j}}(A) \cdot g_{i_1} \wedge \ldots \wedge g_{i_{2j}}\ .$$

Plugging these in, we see that we get the generators of the ideals described in Proposition 4.1 from the matrix $\mathcal{U}$ from Proposition 3.6 via the substitution $c_{ij} = \tau_{i+3,j+3}$ and $u_{i,j} = \tau_{i,j+3}$. Using this substitution we see our complex is just the complex described in Proposition 3.6. □

Notice that the representation theory dictates what the differentials should be, as each component of ∂_3, ∂_2, ∂_1 is determined by the equivariance property with respect to $\mathrm{GL}(F) \times \mathrm{GL}(G)$ up to a non-zero scalar.

4.2 Quotients of Odd Type

There is a nice analogue in the odd case. Let $n = 2m + 4$ where m is a natural number. Consider a $(2m+1) \times (2m+1)$ generic skew symmetric matrix $A = (c_{ij})$ and a $3 \times (2m + 1)$ generic matrix $B = (u_{ki})$. Thus we work over a ring $\mathcal{A} = \mathrm{Sym}_{\mathbb{Z}}(\bigwedge^2 F \oplus F \otimes G) \cong \mathbb{Z}[c_{ij}, u_{ki}]$ where $F = \mathbb{Z}^{2m+1}$ and $G = \mathbb{Z}^3$ are free $\mathbb{Z}$-modules.

Proposition 4.3 *Let* $\{g_1, \ldots, g_{2m+1}\}$ *be a basis for* F *and set*

$$C = \sum_{1 \leqslant i < j \leqslant 2m+1} c_{ij}\, g_i \wedge g_j \quad \textit{and} \quad u_k = \sum_{i=1}^{2m+1} u_{ki}\, g_i \ \textit{ for } \ 1 \leqslant k \leqslant 3\,.$$

Again we denote by C^j *the* j*-th exterior power of* C *in* $\bigwedge^{2j} F$. *The ideal*

$$\mathcal{I}_n = (C^{m-1} \wedge u_1 \wedge u_2 \wedge u_3, C^m \wedge u_1, C^m \wedge u_2, C^m \wedge u_3)$$

is a grade 3 *almost complete intersection of type* $2m + 1 = n - 3$.

Proof The exterior powers C^j have a natural description in terms of Pfaffians of the matrix A.

$$C^j = \sum_{1 \leqslant i_1 < \cdots < i_{2j} \leqslant 2m} \mathrm{Pf}_{i_1 \ldots i_{2j}}(A) \cdot g_{i_1} \wedge \ldots \wedge g_{i_{2j}} \,.$$

Plugging these in, we see that we get the generators of the ideals described in this proposition from the matrix $\mathcal{U}$ in Proposition 3.11 via the substitution $c_{ij} = \tau_{i+3,j+3}$ and $u_{i,j} = \tau_{i,j+3}$. □

Let us work out the minimal free resolution of the ideal defined above.

Proposition 4.4 *Let $n = 2m + 4$ and $\mathcal{I}_n$ be the ideal from Proposition 4.3. The minimal graded free resolution of the cyclic $\mathcal{A}$-module $\mathcal{A}/\mathcal{I}_n$ is*

$$\mathcal{F}_\bullet : 0 \longrightarrow F \otimes (\bigwedge^{2m+1} F)^{\otimes 2} \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m, -3) \xrightarrow{\partial_3}$$
$$(\bigwedge^{2m+1} F)^{\otimes 2} \otimes \bigwedge^2 G \otimes \mathcal{A}(-2m, -2) \oplus \bigwedge^{2m+1} F \otimes \bigwedge^{2m} F \otimes \bigwedge^3 G \otimes \mathcal{A}(-2m+1, -3)$$
$$\xrightarrow{\partial_2} \bigwedge^{2m+1} F \otimes \bigwedge^3 G \otimes \mathcal{A}(-m+1, -3) \oplus \bigwedge^{2m+1} F \otimes G \otimes \mathcal{A}(-m, -1)) \xrightarrow{\partial_1} \mathcal{A} \,.$$

The differentials ∂_3, ∂_2, ∂_1 are described in the proof below. For every field k the resolution $\mathcal{F}_\bullet \otimes_{\mathbb{Z}} k$ is minimal over

$$\mathcal{A}_k = \mathrm{Sym}_k(\bigwedge^2 \overline{F} \oplus \overline{F} \otimes \overline{G})) \cong k[c_{ij}, u_{ki}] \,,$$

with $\overline{F} = F \otimes_{\mathbb{Z}} k$ and $\overline{G} = G \otimes_{\mathbb{Z}} k$. The ideal $\mathcal{I}_n \otimes_{\mathbb{Z}} k$ is thus perfect of grade three.

Proof Let us describe the differentials of the resolution in this setting. The last differential ∂_3 is just a $(2m + 3) \times 2m$ matrix with the $2m \times 2m$ block given by the matrix A and the $3 \times 2m$ block given by three vectors. The differential ∂_2 can be expressed in block form as

$$\partial_2 = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

where A_{21} is zero. The matrix A_{22} is given by multiplication by the representation $\bigwedge^{2m} F$ occurring in the degree $(m, 0)$ component of $\mathcal{A}$. The matrix A_{11} is given by multiplication by the representation $\bigwedge^{2m+1} F \otimes G$ occurring in the degree $(m, 1)$ component of $\mathcal{A}$ and A_{12} is given by multiplication by the representation $\bigwedge^{2m} F \otimes \bigwedge^2 G$ occurring in the degree $(m - 1, 2)$ component of $\mathcal{A}$. The differential ∂_1 is given by the generators of $\mathcal{I}_n$.

The matrices of differentials are:

$$\partial_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1\,2m+1} \\ u_{21} & u_{22} & u_{23} & \dots & u_{2\,2m+1} \\ u_{31} & u_{32} & u_{33} & \dots & u_{3\,2m+1} \\ 0 & c_{12} & c_{13} & \dots & c_{1\,2m+1} \\ -c_{12} & 0 & c_{23} & \dots & c_{2\,2m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1\,2m+1} & -c_{2\,2m+1} & -c_{3\,2m+1} & \dots & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & w_1 & w_2 & \dots & w_{2m+1} \\ x_3 & x_4 & 0 & v_{11} & v_{12} & \dots & v_{1\,2m+1} \\ -x_2 & 0 & x_4 & v_{21} & v_{22} & \dots & v_{2\,2m+1} \\ 0 & -x_2 & -x_3 & v_{31} & v_{32} & \dots & v_{3\,2m+1} \end{pmatrix},$$

and

$$\partial_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

where

$$x_1 = C^{m-1} \wedge u_1 \wedge u_2 \wedge u_3 , \quad x_2 = C^m \wedge u_1 , \quad x_3 = C^m \wedge u_2 , \quad x_4 = C^m \wedge u_3 ,$$
$$w_i = \pm \operatorname{Pf}_{\bar{i}}(C) , \qquad \text{and} \qquad v_{\gamma i} = \sum_{j,k} \pm \Delta^{j,k}_{\alpha,\beta} \operatorname{Pf}_{\overline{ijk}}(C) .$$

Here $\Delta^{j,k}_{\alpha,\beta}$ is a 2×2 minor of the $3 \times (2m+1)$ matrix B on rows α, β and columns j, k. Finally γ is the complement of $\{\alpha, \beta\}$ in the set $\{1, 2, 3\}$.

In order to prove exactness, notice that the exterior powers C^j have a natural description in terms of Pfaffians of the matrix A.

$$C^j = \sum_{1 \leqslant i_1 < \dots < i_{2j} \leqslant 2m} \operatorname{Pf}_{i_1 \dots i_{2j}}(A) \cdot g_{i_1} \wedge \dots \wedge g_{i_{2j}} .$$

Plugging these in we see that we get the generators of the ideals described in Proposition 4.3 from the matrix $\mathcal{U}$ from Proposition 3.11, with substitutions $c_{ij} = \tau_{i+3,j+3}$ and $u_{i,j} = \tau_{i,j+3}$. Using this substitution we see our complex is just the complex described in Proposition 3.11. □

Notice that the representation theory dictates what the differentials should be, as each component of ∂_3, ∂_2, ∂_1 is determined by the equivariance property with respect to $\mathrm{GL}(F) \times \mathrm{GL}(G)$ up to a non-zero scalar.

5 Schubert Varieties in Orthogonal Grassmannians vs. Almost Complete Intersection and Gorenstein Ideals of Codimension 3

In this section we discuss connections between the ideals described in the previous sections and Schubert varieties in the isotropic Grassmannian of even dimensional orthogonal space. We start with a $2n$-dimensional vector space $\mathbb{W}$ over a field k. We denote by $Q(\cdot,\cdot)$ a non-degenerate quadratic form on $\mathbb{W}$ that admits a hyperbolic basis $\{e_1, e_2, \ldots, e_n, \bar{e}_n, \ldots, \bar{e}_2, \bar{e}_1\}$. We deal with the special orthogonal group $\mathrm{SO}(\mathbb{W})$ of isometries of $\mathbb{W}$ of determinant 1, and its double cover $\mathrm{Spin}(\mathbb{W})$. The maximal torus $T \cong (k^*)^n$ is contained in $\mathrm{SO}(\mathbb{W})$ as the diagonal matrices $\underline{t}$ acting on $\mathbb{W}$ as follows

$$\underline{t}(e_i) = t_i e_i \quad \text{and} \quad \underline{t}(\bar{e}_i) = t_i^{-1}\bar{e}_i \quad \text{for} \quad 1 \leqslant i \leqslant n\,.$$

We consider the lattice of integral weights for T, which is a free $\mathbb{Z}$-module with coordinate basis $\{\epsilon_1, \ldots, \epsilon_n\}$. We identify ϵ_i with the weight of e_i under this action; the weight of $\bar{e}_i$ is $-\epsilon_i$.

There is an associated root system of type D_n with roots

$$\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant n\}\,.$$

Simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. If $R(D_n)$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}^n$ generated by roots, then the *fundamental weights* ω_i in the dual $\mathbb{Z}$-module $(\mathbb{Z}^n)^*$ are generators of the dual lattice Λ, called the *weight lattice*, defined by $\omega_i(\alpha_j) = \delta_{i,j}$. We see that

$$\begin{aligned}
\omega_i &= \epsilon_1 + \cdots + \epsilon_i \quad \text{for } 1 \leqslant i \leqslant n-2\,,\\
\omega_{n-1} &= \tfrac{1}{2}\textstyle\sum_{i=1}^{n} \epsilon_i\,, \text{ and}\\
\omega_n &= \tfrac{1}{2}\textstyle\sum_{i=1}^{n-1} \epsilon_i - \tfrac{1}{2}\epsilon_n\,.
\end{aligned}$$

5.1 The Action of the Weyl Group

The Weyl group $\mathrm{W}(D_n)$ acts on Λ by linear maps that permute the roots. It is a subgroup of index 2 in a hyperoctahedral group. $\mathrm{W}(D_n)$ is generated by simple reflections $s_1, s_2, \ldots, s_n$. For $1 \leqslant i \leqslant n-1$ the reflection s_i simply permutes ϵ_i and ϵ_{i+1}, and s_n acts as follows: $s_n(\epsilon_i) = \epsilon_i$ for $1 \leqslant i \leqslant n-2$, $s_n(\epsilon_{n-1}) = -\epsilon_n$, and $s_n(\epsilon_n) = -\epsilon_{n-1}$. It contains the permutation group $\mathrm{W}(A_{n-1})$ on n elements generated by simple reflections $s_1, \ldots, s_{n-2}, s_n$.

Over the field of complex numbers, one can classify representations of the group $\mathrm{SO}(\mathbb{W})$ and its double cover $\mathrm{Spin}(\mathbb{W})$. First, the category of representations of the Spin group is semi-simple, so every representation is a direct sum of irreducible ones. The irreducible representations are so-called highest weight representations $V(\lambda)$, where $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$ is an integral linear combination of fundamental weights with non-negative coefficients λ_i. The representations of $\mathrm{Spin}(\mathbb{W})$ are direct sums of only those irreducibles $V(\lambda)$ for which λ written in terms of ϵ_i's belongs to Λ. Over other fields, and over $\mathbb{Z}$, one can define appropriate analogues of highest weight representations.

We are interested in two particular representations: the half-spinor representations $V(\omega_{n-1})$ and $V(\omega_n)$. They are closely connected, as we will show, to the space of skew symmetric matrices. To that end we recall some generalities about homogeneous spaces.

Let us work over an algebraically closed field k. Let G be a reductive algebraic group and let $P_i \subset G$ be a parabolic subgroup stabilizing a fundamental weight $\omega_i \in \Lambda$. It is well-known that there is a canonical embedding of G/P_i into $\mathbb{P}(V(\omega_i))$. To describe this embedding, consider the Weyl group W, which naturally acts on Λ, and in it the stabilizer W_{ω_i} of the i^{th} fundamental weight. For each $w \in W/W_{\omega_i}$, let $\dot{w} \in W$ be the unique minimal length representative. There is a cell decomposition

$$G/P_i = \bigsqcup_{w \in W/W_{\omega_i}} B\dot{w}P_i$$

called the Bruhat decomposition, where B is the Borel subgroup contained in P_i. The embedding $G/P_i \hookrightarrow \mathbb{P}(V(\omega_i))$ is given by $b\dot{w} \mapsto [b\dot{w}\omega_i]$. In fact, we know that $G/P = \overline{G \cdot v_{\omega_i}}$, where v_{ω_i} is the highest weight vector in $V(\omega_i)$.

The cardinality of W/W_{ω_i} is the same as the cardinality of the orbit $W \cdot \omega_i$. Now, if the fundamental weight ω_i is *minuscule*, then this number coincides with the dimension $\dim_k V(\omega_i)$ of the fundamental representation. This implies that the Bruhat graph of the Bruhat interval in the Coxeter group (W, S) corresponding to the minimal length representatives of the elements in W/W_{ω_i} coincides with the *crystal graph* associated to the representation $V(\omega_i)$.

Throughout the rest of this section we are interested in the case of a root system of type D_n and the parabolic subgroup P_{n-1}, the homogeneous space $\mathrm{Spin}(2n)/P_{n-1}$ is one of the two connected components of the isotropic Grassmannian $\mathrm{OGr}(n, 2n)$.

It is well-known, see for example Laksmibai and Raghavan [18, Section 3.3], that the homogeneous coordinate ring of the connected component of $\mathrm{OGr}(n, 2n)$, considered as a projective subvariety of the projective space $\mathbb{P}(V(\omega_{n-1}))$, has a decomposition

$$k[\mathrm{OGr}(n, 2n)] = \bigoplus_{d\geqslant 0} V(d\omega_{n-1}) .$$

into irreducible representations of Spin($\mathbb{W}$), so each graded component of this ring is irreducible. The half-spinor representation $V(\omega_{n-1})$ is a representation of dimension 2^{n-1} whose weights with respect to the Cartan subalgebra are $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ with an even number of minuses. It has a twin representation $V(\omega_n)$ of dimension 2^{n-1} whose weights with respect to the Cartan subalgebra are $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ with an odd number of minuses. Both half-spinor representations are constructed from the Clifford algebra of the quadratic form Q.

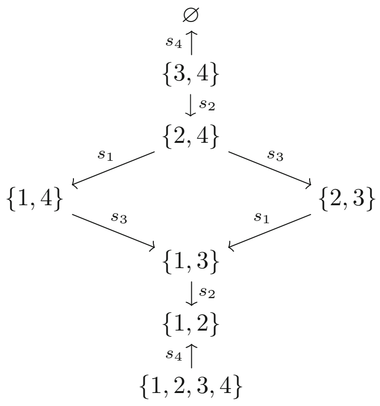
It is also known—Kostant's Theorem, see Garfinkle's dissertation [14]—that as a factor of $\mathrm{Sym}_k(V(\omega_{n-1}))$ the coordinate ring $k[\mathrm{OGr}(n, 2n)]$ is generated by quadratic equations. The generators of $k[\mathrm{OGr}(n, 2n)]$ are the spinor coordinates; they can be indexed by the cosets $W(D_n)/W(A_{n-1})$. We denote by q_w the spinor coordinate corresponding to $w \in W(D_n)/W(A_{n-1})$; the Schubert varieties in $\mathrm{OGr}(n, 2n)$ are also indexed by $W(D_n)/W(A_{n-1})$. There is a natural partial order on these coordinates, which in the case of Schubert varieties corresponds to the inclusion order. This partially ordered set has two combinatorial interpretations; it is a set of 2^{n-1} elements.

The first interpretation of $W(D_n)/W(A_{n-1})$ is as the set $\mathcal{PE}_n$ of even cardinality subsets of $\{1, 2, \ldots, n\}$. The Weyl group $W(D_n)$ acts on this set as follows. For $1 \leqslant i \leqslant n-1$ the simple reflection s_i acts by switching the numbers i and $i+1$. This means the subset is fixed by s_i if it contains both or none of the numbers i and $i+1$. The reflection s_n acts non-trivially only on subsets either containing or not intersecting the subset $\{n-1, n\}$. It either adds the numbers $n-1$ and n or takes them away. For a subset $I \in \mathcal{PE}_n$ let $\ell(I)$ denote the length of a minimal representative of the corresponding coset in $W(D_n)/W(A_{n-1})$. For a reflection s_i such that $s_i(I) \neq I$ one can prove that $\ell(I) = \ell(s_i(I)) \pm 1$. The partial order is generated by comparing I and $s_i(I)$ according to the length. In the case at hand, there is a concrete description: The induced partial order $\mathcal{PE}_n$ compares subsets of a given cardinality as usual by setting

$$\{i_1, \ldots, i_r\} \leqslant \{i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_r\}$$

for $1 \leqslant i_1 < i_2 < \cdots < i_r$ and $i_j + 1 < i_{j+1}$. The partial order is generated by these inequalities together with the inequalities $\{i_1, \ldots, i_r\} \leqslant \{i_1, \ldots, i_r, n-1, n\}$ for $1 \leqslant i_1 < \cdots < i_r < n-1$; this includes $\varnothing \leqslant \{n-1, n\}$.

Example 5.1 The induced partial order on $\mathcal{PE}_4$ is illustrated below where the arrows are directed such that $s_i(I) \leqslant I$ holds.



In the second interpretation one views $W(D_n)/W(A_{n-1})$ as a $W(D_n)$-orbit of the weight—thought of as assigning an integer to each node of the Dynkin diagram D_n—under the natural action of $W(D_n)$ on these weights. The action of the simple reflection s_i on a weight

$$a_1\ a_2\ \cdots\ a_{n-3}\ a_{n-2} \begin{array}{c} a_{n-1} \\ \\ a_n \end{array}$$

changes a_i to $-a_i$ and adds a_i to the value at all neighboring nodes. The partial order is generated by setting $s_i(w) > w$ if and only if $s_i(w) \neq w$ and the node $w(i)$ is positive.

Example 5.2 The bijection between the set $\mathcal{PE}_4$ and the $W(D_4)$-orbit of the weight

$$w = 0\ 0 \begin{array}{c} 0 \\ \\ 1 \end{array}$$

is as follows

$$\varnothing \leftrightarrow w = 0\ 0 \begin{array}{c} 0 \\ \\ 1 \end{array} \qquad \{3,4\} \leftrightarrow s_4(w) = 0\ 1 \begin{array}{c} 0 \\ \\ -1 \end{array}$$

$$\{2,4\} \leftrightarrow s_2s_4(w) = 1\ -1 \begin{array}{c} 1 \\ \\ 0 \end{array} \qquad \{1,4\} \leftrightarrow s_1s_2s_4(w) = -1\ 0 \begin{array}{c} 1 \\ \\ 0 \end{array}$$

$$\{2,3\} \leftrightarrow s_3s_2s_4(w) = 1\ 0 \begin{array}{c} -1 \\ \\ 0 \end{array} \qquad \{1,3\} \leftrightarrow \begin{array}{c} s_3s_1s_2s_4(w) \\ s_1s_3s_2s_4(w) \end{array} = -1\ 1 \begin{array}{c} -1 \\ \\ 0 \end{array}$$

$$\{1,2\} \leftrightarrow s_2s_3s_1s_2s_4(w) = 0\ -1 \begin{array}{c} 0 \\ \\ 1 \end{array} \qquad \{1,2,3,4\} \leftrightarrow s_4s_2s_3s_1s_2s_4(w) = 0\ 0 \begin{array}{c} 0 \\ \\ -1 \end{array}$$

Notice that this bijection commutes with the Weyl group action and preserves the associated partial order.

5.2 Schubert Varieties

It is known, see for example [18, Section 3.3], that the defining ideal in $k[\mathrm{OGr}(n, 2n)]$ of every Schubert variety Ω_w is generated by spinor coordinates q_v for $v \nleq w$ in the associated partial order. In our case this translates as follows. Consider the big open cell Y in $\mathrm{OGr}(n, 2n)$ consisting of points with Plücker coordinate $p_{\mathrm{id}} \neq 0$. Recall the hyperbolic basis $\{e_1, \dots, e_n, \bar{e}_n, \dots, \bar{e}_1\}$ of $\mathbb{W}$. To every subspace $V \in \mathrm{OGr}(n, 2n)$ and every basis $\{v_1, \dots, v_n\}$ of V we associate an $n \times 2n$ matrix M whose i^{th} row consists of the coordinates of the vector v_i written in the basis $\{e_1, \dots, e_n, \bar{e}_n, \dots, \bar{e}_1\}$. The big open cell Y in $\mathrm{OGr}(n, 2n)$ discussed above consists of subspaces V such that for every basis $\{v_1, \dots, v_n\}$ of V the corresponding matrix M has a minor corresponding to columns $e_1, \dots, e_n$ not equal to zero. The set Y is an affine space of dimension $\binom{n}{2}$ as for $V \in Y$ we can find a unique basis of V such that the corresponding matrix has a form

$$M = \left(\begin{array}{ccccc|ccccc} 0 & 0 & \cdots & 0 & 1 & 0 & x_{1\,2} & \cdots & x_{1\,n-1} & x_{1\,n} \\ 0 & 0 & \cdots & 1 & 0 & -x_{1\,2} & 0 & \cdots & x_{2\,n-1} & x_{2\,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & -x_{1\,n-1} & -x_{2\,n-1} & \cdots & 0 & x_{n-1\,n} \\ 1 & 0 & \dots & 0 & 0 & -x_{1\,n} & -x_{2\,n} & \dots & -x_{n-1\,n} & 0 \end{array}\right).$$

We refer to the skew symmetric $n \times n$ block as X. The restrictions to Y of the spinor coordinates correspond to sub-Pfaffians of X of all possible sizes; see for example Manivel [19]. More precisely, the weights of the half-spinor representation correspond to the subsets I of the set $\{1, \dots, n\}$ of even cardinality. For a given I, the corresponding weight is a vector $w_I = (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with n coordinates and an even number of minuses occurring in the positions determined by I. The corresponding spinor coordinate q_I restricts to Y as the Pfaffian of the skew symmetric matrix obtained by picking from a generic $n \times n$ skew symmetric matrix the rows and columns determined by I. Thus, the quadratic equations generating the defining ideal of the homogeneous coordinate ring $k[\mathrm{OGr}(n, 2n)]$ are just the quadratic equations in Pfaffians of all sizes of a generic skew symmetric matrix.

There are more facts that are known about Schubert varieties, the reference for this is [18, Chapter 7]. The half-spinor representation $V(\omega_{n-1})$ is an example of so-called *minuscule* representation. This means all its weight vectors are in one $W(D_n)$-orbit. This implies that the defining ideals of Schubert varieties and their unions behave in the optimal way described below. For each cofinal subset U of the partially ordered set $\mathcal{PE}_n$ we consider the ideal J_U in $k[Y]$, the coordinate ring of Y, generated by the spinor coordinates from that subset. This set of ideals forms a distributive lattice L_1 with the join and meet operations given by $+$ and $\cap$. On the other hand we can form a lattice L_2 of the cofinal subsets in $\mathcal{PE}_n$ with operations of join and meet given by $\cup$ and $\cap$. The first part of the next statement follows from of [18, Section 7.2]; the assertion about J_U being radical follows from Brion and Kumar [5, Corollary 2.3.3].

Proposition 5.3 *The lattices L_1 and L_2 are isomorphic. Moreover, the ideal J_U is the defining ideal of the union of the Schubert varieties it defines set-theoretically. Thus all ideals J_U are radical.*

Notice also that if we change the half-spinor representation $V(\omega_{n-1})$ to the other one, i.e. $V(\omega_n)$ then the lattice of Schubert varieties will change to the poset $\mathcal{PO}_n$ of odd sized subsets of $\{1, \dots, n\}$. The action of the Weyl group $W(D_n)$ and the poset structure are similar to those on $\mathcal{PE}_n$. We refer the reader to [18, Section 7.2].

To give an interpretation of the Schubert varieties of codimension 3 in concrete terms, we adopt the notation from the notes by Coskun [12, Lecture 5]. Let $\mathrm{OGr}(n, 2n)$ be one of the two connected components of the orthogonal Grassmannian of n-dimensional isotropic subspaces in a $2n$-dimensional vector space $\mathbb{W}$. As above denote by Q a non-degenerate quadratic form on $\mathbb{W}$ that admits a hyperbolic basis. Fix an isotropic flag

$$F_\bullet : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F_n^\perp \subset F_{n-1}^\perp \subset \cdots \subset F_1^\perp \subset \mathbb{W}\,.$$

Here F_n is isotropic and $F_i^\perp$ denotes the orthogonal complement of F_i. The Schubert varieties in $\mathrm{OGr}(n, 2n)$ are parameterized by sequences λ,

$$n - 1 \geqslant \lambda_1 > \lambda_2 > \cdots > \lambda_s \geqslant 0\,,$$

of strictly decreasing integers where s has the same parity as n; notice that $s \leqslant n$ holds. The sequence λ determines a unique sequence $\tilde{\lambda}$ of strictly decreasing integers

$$n - 1 \geqslant \tilde{\lambda}_{s+1} > \tilde{\lambda}_{s+2} > \cdots > \tilde{\lambda}_n \geqslant 0$$

satisfying the condition that there is no i, j such that $\lambda_i + \tilde{\lambda}_j = n - 1$. In other words, we obtain $\tilde{\lambda}$ by removing from the sequence $n-1, n-2, \dots, 0$ the numbers $n - 1 - \lambda_1, \dots, n - 1 - \lambda_s$.

The Schubert variety $\Omega_\lambda = \Omega_\lambda(F_\bullet)$ is defined as the closure of the locus

$$\Omega_\lambda^{(0)}(F_\bullet) = \left\{ V \in \mathrm{OGr}(n, 2n) \;\middle|\; \begin{array}{l} \dim_k(V \cap F_{n-\lambda_i}) = i, \text{ for } 1 \leqslant i \leqslant s \\ \dim_k(V \cap F_{\tilde{\lambda}_j}^\perp) = j, \text{ for } s < j \leqslant n \end{array} \right\}.$$

The codimension of Ω_λ is $|\lambda| = \sum_i \lambda_i$. The cells $\Omega_\lambda^{(0)}(F_\bullet)$ are exactly the orbits of the Borel subgroup B of the spin group $\mathrm{Spin}(2n)$ acting on the connected component of $\mathrm{OGr}(n, 2n)$.

Remark 5.4 In order to connect with the previous description, let us indicate how the partitions λ translate to the subsets $\mathcal{PE}_n$ and $\mathcal{PO}_n$. The partition $(\lambda_1, \dots, \lambda_s)$ such that

$$n - 1 \geqslant \lambda_1 > \ldots > \lambda_s \geqslant 0$$

corresponds to the set $w(\lambda)$ of s minuses in places $n-\lambda_1, n-\lambda_2, \ldots, n-\lambda_s$. This set is either in $\mathcal{PE}_n$ or $\mathcal{PO}_n$ depending on parity of n.

The variety $\Omega_\lambda(F_\bullet)$ in Coskun's notation is then is then equal to the variety $\Omega_{w(\lambda)}$ in the notation of this section.

It is well-known—see for example the works [20–23] by Mehta, Ramanan, Ramanathan, and Srinivas—that the Schubert varieties are defined over $\mathbb{Z}$ and are normal and arithmetically Cohen-Macaulay and so are the affine varieties $Y_\lambda = \Omega_\lambda \cap Y$. Our goal in this section is to explicitly describe the varieties Y_λ of codimension 3 in the affine space Y, i.e. the subvarieties Y_λ such that $|\lambda| = 3$.

5.3 *Spinor Coordinates*

Finally we describe the bijection between the spinor coordinates and the Pfaffians of the matrix X.

For n even and $I \in \mathcal{PE}_n$ the corresponding spinor coordinate q_I is a square root of the minor of the matrix M with columns corresponding to e_i with $i \in I$ and $\bar{e}_I$ with $i \notin I$. This is the Pfaffian of the submatrix of X obtained by removing the rows and columns with indices $n+1-i$ for $i \in I$. The first spinor coordinate corresponds to the subset $\emptyset$, and it is the Pfaffian of X.

For n odd we consider the elements $I \in \mathcal{PO}_n$. The corresponding spinor coordinate q_I is the Pfaffian of the submatrix of X obtained by removing the rows and columns with indices $n+1-i$ for $i \in I$. The first spinor coordinate corresponds to the subset $\{n\}$, and it is the Pfaffian of the submatrix obtained from X by removing the first row and column, i.e. the matrix $X[\bar{1}; \bar{1}]$ in the notation from 2.1.

5.4 *The Case of Even n*

Let us intersect our Schubert varieties with the open cell Y.

For $n = 2$ there are evidently no Schubert varieties of codimension 3, but for $n \geqslant 4$ there are precisely two of them, namely $\Omega_{(3,0)}$ and $\Omega_{(2,1)}$. For $n = 4$ it is easy to see they are both complete intersections, one given by $x_{12} = x_{13} = x_{14} = 0$ and the other by $x_{12} = x_{13} = x_{23} = 0$. We now assume $n \geqslant 6$.

We start with the intersection $Y_{(3,0)}$. The rank conditions related to the flags $F_j^\perp$ are easily seen to be empty because our subspace is isotropic. The condition $\dim_k(V \cap F_{n-3}) \geqslant 1$ means exactly that the rank of the submatrix $X[1\ldots n; 4\ldots n]$ has to be less than $n-3$. This is the matrix of the third differential of the almost complete intersection ideal of format $(1, 4, n, n-3)$ described in Theorem 3.3. Note that the other condition $\dim_k(V \cap F_n) \geqslant 2$ holds as the matrix X has to be singular and therefore of rank at most $n-2$. So the defining ideal of the Schubert variety $Y_{(3,0)}$ is almost complete intersection of odd type.

We turn to the intersection $Y_{(2,1)}$. Again the rank conditions related to the flags $F_j^{\perp}$ are empty because our subspace is isotropic, so we get the conditions $\dim_k(V \cap F_{n-2}) \geqslant 1$ and $\dim_k(V \cap F_{n-1}) \geqslant 2$. The first condition is now redundant. The second condition means that the rank of the submatrix of $X[1 \dots n; 2 \dots n]$ has to be at most $n-3$. This means that the submaximal Pfaffians $\mathrm{Pf}_{\overline{1i}}(X)$ of the matrix $X[\bar{1}; \bar{1}]$ vanish for $2 \leqslant i \leqslant n$. It follows that the rank of this matrix is at most $n-4$, so the rank of $X[1 \dots n; 2 \dots n]$ is at most $n-3$. We conclude that $Y_{(2,1)}$ is the subvariety given by vanishing of Pfaffians $\mathrm{Pf}_{\overline{1i}}(X)$; in other words, the defining ideal is a generic Gorenstein ideal of codimension 3.

5.5 The Case of Odd n

For $n \leqslant 3$ there are evidently no Schubert varieties of codimension 3, but for $n \geqslant 5$ there are precisely two of them, namely $\Omega_{(3)}$ and $\Omega_{(2,1,0)}$. Let us intersect them with the open cell Y.

We start with the intersection $Y_{(3)}$. The rank conditions related to the flags $F_j^{\perp}$ are easily seen to be empty because our subspace is isotropic. The condition $\dim_k(V \cap F_{n-3}) \geqslant 1$ means that the rank of the submatrix $X[1 \dots n; 4 \dots n]$ has to be less than $n-3$. But this is the matrix of third differential of the almost complete intersection ideal of format $(1, 4, n, n-3)$ described in Theorem 3.9. So the defining ideal of the Schubert variety $Y_{(3)}$ is an almost complete intersection of even type.

We turn to the intersection $Y_{(2,1,0)}$. Again the rank conditions related to the flags $F_j^{\perp}$ are empty because our subspace is isotropic. This leaves us with the conditions $\dim_k(V \cap F_{n-2}) \geqslant 1$ and $\dim_k(V \cap F_{n-1}) \geqslant 2$. This just means that submaximal Pfaffians of the matrix X are zero, so we get a generic Gorenstein ideal of codimension 3.

5.6 Minimal Free Resolutions

Let us look at the minimal free resolutions of the coordinate rings of the codimension 3 varieties Y_λ from the point of view of Schubert varieties. The defining equations of Schubert varieties have a general description in terms of ideals J_U; we recall its meaning in our case. The Schubert varieties of codimension 3 correspond to elements

$$w' = s_n s_{n-2} s_{n-1} \qquad \text{and} \qquad w'' = s_{n-3} s_{n-2} s_{n-1} ,$$

as these are the only two elements of length 3 in $W(D_n)/W(A_{n-1})$. The generators of the corresponding ideals are:

$$(q_{\mathrm{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}, q_{s_{n-3}s_{n-2}s_{n-1}}, \ldots, q_{s_{n-i}\ldots s_{n-3}s_{n-2}s_{n-1}}, \ldots, q_{s_1\ldots s_{n-3}s_{n-2}s_{n-1}}) ,$$

where there are n generators in total, and

$$(q_{\mathrm{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}, q_{s_n s_{n-2}s_{n-1}}) .$$

Identifying these generators with the corresponding Pfaffians, we see that for n odd the first ideal generated by the submaximal Pfaffians of X. The second ideal gives the almost complete intersection ideal described in Theorem 3.3. For even n, the first ideal gives the Pfaffian of X and submaximal Pfaffians of $X[\bar{1}; \bar{1}]$; in this case the first generator is redundant. The four generator ideal gives the almost complete intersection ideal described in Theorem 3.9.

There is one more statement one can make which plays an important role. It is be proved in terms of commutative algebra Lemmas C.2 and C.6. Here we give a geometric reasoning proving the statement.

Proposition 5.5 *The ideal generated by the first three spinor coordinates,*

$$(q_{\mathrm{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}}) ,$$

in the partial order on the homogeneous coordinate ring $k[\mathrm{OGr}(n, 2n)]$ *of the orthogonal Grassmannian is generated by a regular sequence. Therefore, these coordinates restricted to the open cell* Y *also generate an ideal generated by a regular sequence in* $k[Y]$*. Moreover, these elements generate a radical ideal.*

Proof Let B be a Borel subgroup of the group $\mathrm{Spin}(2n)$. The almost complete intersection ideal $(q_{\mathrm{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$ in $k[\mathrm{OGr}(n, 2n)]$ is B-equivariant. This means that its vanishing locus is a union of Schubert cells. It follows that this vanishing set consists of the closure of the union of two Schubert varieties of codimension 3. It is therefore an ideal of depth three generated by three elements. Such ideal is then generated by a regular sequence, as the ring $k[\mathrm{OGr}(n, 2n)]$ is Cohen-Macaulay. The ideal is radical by of [5, Corollary 2.3.3]. The result for $k[Y]$ follows by localization. □

This result means we have an occurrence of the situation described by Ulrich [25]. The ideal $(q_{\mathrm{id}}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$ is the intersection $I_{w'} \cap I_{w''}$, and thus the ideals $I_{w'}$ and $I_{w''}$ are linked via the regular sequence $(q_{id}, q_{s_{n-1}}, q_{s_{n-2}s_{n-1}})$. This is exactly the procedure described in [6]. By this argument, we can describe the format of the resolutions of our almost complete intersections.

Set $R = k[Y]$ and $n = 2m + 2$ for some natural number m. The resolution of the Gorenstein ideal $I_{w'}$ has format

$$0 \longrightarrow R(-2m-1) \longrightarrow R^{2m+1}(-m-1) \longrightarrow R^{2m+1}(-m) \longrightarrow R ;$$

we link by a regular sequence of elements of degrees m, m, and $m+1$. Looking at the mapping cone

$$\begin{array}{ccccccccc}
0 \longrightarrow & R(-2m-1) & \longrightarrow & R^{2m+1}(-m-1) & \longrightarrow & R^{2m+1}(-m) & \longrightarrow & R \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & R(-3m-1) & \longrightarrow & R^2(-2m-1) \oplus R(-2m) & \longrightarrow & R(-m-1) \oplus R^2(-m) & \longrightarrow & R
\end{array}$$

we deduce that the other ideal has a resolution with the format

$$0 \longrightarrow R^{2m-1}(-2m-1) \longrightarrow R^{2m+2}(-2m) \longrightarrow R(-m-1) \oplus R^3(-m) \longrightarrow R$$

which is exactly the format of the resolution from Sect. 3.

Let us do this calculation for odd $n = 2m+3$. The resolution of Gorenstein ideal of codimension 3 has format

$$0 \longrightarrow R(-2m-3) \longrightarrow R^{2m+3}(-m-2) \longrightarrow R^{2m+3}(-m-1) \longrightarrow R$$

and we link by a regular sequence with degrees $m+1, m+1, m+1$. Looking at the mapping cone

$$\begin{array}{ccccccccc}
0 \longrightarrow & R(-2m-3) & \longrightarrow & R^{2m+3}(-m-2) & \longrightarrow & R^{2m+3}(-m-1) & \longrightarrow & R \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & R(-3m-3) & \longrightarrow & R^3(-2m-2) & \longrightarrow & R^3(-m-1) & \longrightarrow & R
\end{array}$$

we deduce that the other ideal has a resolution with the format

$$0 \longrightarrow R^{2m}(-2m-2) \longrightarrow R^{2m+3}(-2m-1) \longrightarrow R(-m) \oplus R^3(-m-1) \longrightarrow R$$

which is exactly the format of the resolution from Sect. 3.

Next we interpret the matrices of the free resolutions of almost complete intersections in terms of spinor coordinates. Before we start, let us comment on the defining ideals of the coordinate rings $k[\mathrm{OGr}(n, 2n)]$ thought of as factors of the symmetric algebra on the half-spinor representation. By Kostant's Theorem [14] these ideals are defined by quadratic equations, therefore, they are generated by the kernel of the map

$$S_2(V(\omega_{n-1})) \longrightarrow V(2\omega_{n-1}) .$$

One has the following formula, see Adams [1, p. 25],

$$S_2(V(\omega_{n-1}) \;=\; V(2\omega_{n-1}) \oplus \bigoplus_{i\geqslant 1} V(\omega_{n-4i})\,.$$

We use the notation from Sect. 4 for the differentials in our complexes.

Let us start with the case of odd $n = 2m+3$. The generators of our ideal in terms of Plücker coordinates are

$$\begin{aligned} x_1 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},2m+1,2m+2,2m+3} \\ x_2 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},2m+1,\overline{2m+2},\overline{2m+3}} \\ x_3 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},\overline{2m+1},2m+2,\overline{2m+3}} \\ x_4 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},\overline{2m+1},\overline{2m+2},2m+3}\,. \end{aligned}$$

The entries of the second differential ∂_2 are as follows. The element w_i is the Plücker coordinate with $2m+2$ bars, the only number without bar is $2m+1-i$. The element $v_{\{\alpha,\beta\}\,i}$ is a Plücker coordinate with $2m$ bars. The numbers without bars are $2m+1-i$, $2m+\alpha$, $2m+\beta$.

The entries of the matrix ∂_3 are also Plücker coordinates. The element $u_{\alpha i}$ is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+4-\alpha$. The element c_{ij} is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+1-j$.

The gradings of the basis vectors in the modules of the complex are: The basis element in $F_0 = R$ has weight (0^{2m+3}). In the following we use 1^m to denote $(1, 1, \ldots, 1)$ with m coordinates. The basis elements in $F_1 = R \oplus R^3$ have weights

$$((\tfrac{1}{2})^{2m}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), ((\tfrac{1}{2})^{2m}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), ((\tfrac{1}{2})^{2m}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}), ((\tfrac{1}{2})^{2m}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})\,.$$

The basis elements in $F_2 = R^{2m} \oplus R^3$ have weights

$$\begin{gathered} (1^{2m-1}, 0, 0^3), \quad (1^{2m-2}, 0, 1, 0^3), \ldots, (0, 1^{2m-1}, 0^3), \\ (1^{2m}, 0, 0, -1), \quad (1^{2m}, 0, -1, 0), \quad (1^{2m}, -1, 0, 0)\,. \end{gathered}$$

Finally the basis vectors in $F_3 = R^{2m}$ have weights $(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, (-\frac{1}{2})^3)$.

The composite $\partial_1\partial_2$ is easily explained. It is a $1 \times (2m+3)$ matrix. Its entry in the ith row is the Plücker coordinate with the weight with $2m-1$ entries of -1's and 4 zeros in positions $2m+1-i$, $2m+1$, $2m+2$, $2m+3$. This entry is zero because it corresponds to the extremal weight vector in the representation $V(\omega_{2m-1})$ which occurs in the 2nd symmetric power of $V(\omega_{2m+2})$.

The composite $\partial_2\partial_3$ is a $4 \times 2m$ matrix with the weights in the first row being $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 0, 0, 0)$, where -1 appears in positions $1, \ldots, 2m$; the entries in the second row are $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 0, 1, 1)$, where -1 appears in positions $1, \ldots, 2m$; the entries in the third row are

$(0, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, 1)$, where -1 appears in positions $1, \ldots, 2m$, finally the entries in the fourth row are $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 1, 0)$, where -1 appears in positions $1, \ldots, 2m$.

The interpretation of the identity $\partial_2\partial_3 = 0$ from this point of view requires further analysis.

Similarly we treat the even case $n = 2m + 4$. The generators of the ideal $I_{w''}$ in terms of Plücker coordinates are

$$\begin{aligned} x_1 &= p_{\bar{1},\bar{2},\ldots,\overline{2m+4}} \\ x_2 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},\overline{2m+1},\overline{2m+2},2m+3,2m+4} \\ x_3 &= p_{\bar{1},\bar{2},\ldots\overline{2m},\overline{2m+1},2m+2,\overline{2m+3},2m+4} \\ x_4 &= p_{\bar{1},\bar{2},\ldots,\overline{2m},\overline{2m+1},2m+2,2m+3,\overline{2m+4}} \,. \end{aligned}$$

The entries of the second differential ∂_2 are as follows. The element w_i is the Plücker coordinate with $2m$ bars, the only numbers without bar are $2m + 2 - i$ and $2m + 2, 2m + 3, 2m + 4$. The element $v_{\alpha i}$ is a Plücker coordinate with $2m + 2$ bars. The numbers without bars are $2m + 2 - i, 2m + 1 + \alpha$.

The entries of the matrix ∂_3 are also Plücker coordinates. The element $u_{\alpha i}$ is a Plücker coordinate with two bars, at numbers $2m+1-i$ and $2m+4-\alpha$. The element c_{ij} is a Plücker coordinate with two bars, at numbers $2m + 1 - i$ and $2m + 1 - j$.

The gradings of the basis vectors in the modules of the complex are: The basis element in $F_0 = R$ has weight (0^{2m+4}). The basis elements in $F_1 = R \oplus R^3$ have weights

$$((\tfrac{1}{2})^{2m+4}),\ \ ((\tfrac{1}{2})^{2m+1}, \tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}),\ \ ((\tfrac{1}{2})^{2m+1}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}),\ \ ((\tfrac{1}{2})^{2m+1}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}) \,.$$

The basis elements in $F_2 = R^{2m} \oplus R^3$ have weights

$$\begin{gathered} (1^{2m}, 0, 0^3), (1^{2m-1}, 0, 1, 0^3), \ldots, (0, 1^{2m}, 0^3), \\ (1^{2m+1}, 0, 0, -1), \quad (1^{2m+1}, 0, -1, 0), \quad (1^{2m+1}, -1, 0, 0) \,. \end{gathered}$$

Finally the basis vectors in $F_3 = R^{2m}$ have weights $(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, (-\frac{1}{2})^3)$.

The composite $\partial_1\partial_2$ is easily explained. It is a $1 \times (2m + 4)$ matrix. Its entry in the ith row is the Plücker coordinate with the weight with $2m$ entries of -1's and 4 zeros in positions $2m + 2 - i, 2m + 2, 2m + 3, 2m + 4$. This entry is zero because it corresponds to the extremal weight vector in the representation $V(\omega_{2m})$ which occurs in the 2nd symmetric power of $V(\omega_{2m+3})$.

The composition $\partial_2\partial_3$ is a $4 \times (2m + 1)$ matrix with the weights in the first row being $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 1, 1)$, where -1 appears in positions $1, \ldots, 2m + 1$; the entries in the second row are $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, 0)$, where -1 appears in positions $1, \ldots, 2m + 1$; the third row entries are $(0, 0, \ldots, 0, -1, 0, \ldots, 0, 0, 1, 0)$, where -1 appears in positions $1, \ldots, 2m + 1$;

finally the entries in the fourth row are $(0, 0, \dots, 0, -1, 0, \dots, 0, 0, 0, 1)$, where -1 appears in positions $1, \dots, 2m + 1$.

The interpretation of the identity $\partial_2\partial_3 = 0$ from this point of view requires further analysis.

Appendix

A Pfaffian Identities Following Knuth

For the benefit of the reader, we quote from [11] a short introduction to Knuth's [17] combinatorial approach to Pfaffians.

Let $T = (t_{ij})$ be an $n \times n$ skew symmetric matrix with entries in a commutative ring. Assume that T has zeros on the diagonal; this is, of course, automatic if the characteristic of the ring is not 2. Set $\mathcal{P}[ij] = t_{ij}$ for $i, j \in \{1, \dots, n\}$ and extend $\mathcal{P}$ to a function on words in letters from $\{1, \dots, n\}$ as follows:

$$\mathcal{P}[i_1 \dots i_m] = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \sum \operatorname{sgn}\left(\begin{smallmatrix} i_1 \dots i_{2k} \\ j_1 \dots j_{2k} \end{smallmatrix}\right) \mathcal{P}[j_1 j_2] \cdots \mathcal{P}[j_{2k-1} j_{2k}] & \text{if } m = 2k \text{ is even} \end{cases}$$

where the sum is over all partitions of $\{i_1, \dots, i_{2k}\}$ in k subsets of cardinality 2. The order of elements in each subset is irrelevant as the difference in sign $\mathcal{P}[jj'] = -\mathcal{P}[j'j]$ is offset by a change of sign of the permutation; see [17, Section 0]. The value of $\mathcal{P}$ on the empty word is by convention 1, and the value of $\mathcal{P}$ on a word with a repeated letter is 0. The latter is a convention in characteristic 2 and otherwise automatic.

The function $\mathcal{P}$ computes the Pfaffians of submatrices of T. Indeed, for a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with elements $i_1 < \dots < i_k$ one has

$$\operatorname{pf}_T(i_1 \dots i_k) = \mathcal{P}[i_1 \dots i_k],$$

in the notation introduced in 2.1 and (3.0.1).

A.1 Overlapping Pfaffians Let α, β, and γ be disjoint words in letters from $\{1, \dots, n\}$. For b a letter in β, the formula [17, (5.0)] reads

$$\begin{aligned} \mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha\gamma] = &\sum_{i \in \beta} \operatorname{sgn}\left(\begin{smallmatrix} \beta \\ bi(\beta \setminus bi) \end{smallmatrix}\right) \mathcal{P}[\alpha\beta \setminus bi]\,\mathcal{P}[\alpha\gamma bi] \\ &+ \sum_{j \in \gamma} \operatorname{sgn}\left(\begin{smallmatrix} \beta \\ b(\beta \setminus b) \end{smallmatrix}\right) \operatorname{sgn}\left(\begin{smallmatrix} \gamma \\ j(\gamma \setminus j) \end{smallmatrix}\right) \mathcal{P}[\alpha j\beta \setminus b]\,\mathcal{P}[\alpha b\gamma \setminus j]\,. \end{aligned} \tag{A.1.1}$$

We record a number of special cases of this formula.

For $\beta = b$ the formula (A.1.1) reduces to

$$\mathcal{P}[\alpha b]\,\mathcal{P}[\alpha\gamma] \;=\; \sum_{j\in\gamma} \operatorname{sgn}\left(\begin{smallmatrix}\gamma\\ j(\gamma\setminus j)\end{smallmatrix}\right) \mathcal{P}[\alpha j]\,\mathcal{P}[\alpha b\gamma \setminus j]\,. \tag{A.1.2}$$

For $\gamma = c$ the formula (A.1.1) reduces to

$$\begin{aligned}\mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha c] \;=\; &\sum_{i\in\beta} \operatorname{sgn}\left(\begin{smallmatrix}\beta\\ bi(\beta\setminus bi)\end{smallmatrix}\right) \mathcal{P}[\alpha\beta \setminus bi]\,\mathcal{P}[\alpha cbi]\\ &+ \operatorname{sgn}\left(\begin{smallmatrix}\beta\\ b(\beta\setminus b)\end{smallmatrix}\right) \mathcal{P}[\alpha c\beta \setminus b]\,\mathcal{P}[\alpha b]\,.\end{aligned} \tag{A.1.3}$$

With γ empty the formula (A.1.1) reduces to

$$\mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha] \;=\; \sum_{i\in\beta} \operatorname{sgn}\left(\begin{smallmatrix}\beta\\ bi(\beta\setminus bi)\end{smallmatrix}\right) \mathcal{P}[\alpha\beta \setminus bi]\,\mathcal{P}[\alpha bi]\,. \tag{A.1.4}$$

With α and γ empty the formula (A.1.1) reduces to

$$\mathcal{P}[\beta] \;=\; \sum_{i\in\beta} \operatorname{sgn}\left(\begin{smallmatrix}\beta\\ bi(\beta\setminus bi)\end{smallmatrix}\right) \mathcal{P}[\beta \setminus bi]\,\mathcal{P}[bi]\,. \tag{A.1.5}$$

In this first appendix we derive some consequences of (A.1.1) that facilitate the computations in Appendices B and C. The first lemma is just the classic Laplacian expansion of the Pfaffian of a skew symmetric submatrix of T.

Lemma A.2 *For integers $1 \leqslant u_1 < \cdots < u_k \leqslant n$ and every integer ℓ with $1 \leqslant \ell \leqslant k$ one has*

$$\begin{aligned}(-1)^{\ell-1}\operatorname{pf}_T(u_1\ldots u_k) \;=\; &\sum_{i=1}^{\ell-1}(-1)^i t_{u_iu_\ell}\operatorname{pf}_T(u_1\ldots u_k \setminus u_iu_\ell)\\ &+ \sum_{i=\ell+1}^{k}(-1)^i t_{u_\ell u_i}\operatorname{pf}_T(u_1\ldots u_k \setminus u_iu_\ell)\,.\end{aligned}$$

Proof With $\beta = u_1\ldots u_k$ and $b = u_\ell$ the formula (A.1.5) yields

$$\begin{aligned}&\mathcal{P}[\beta]\\ &= \sum_{i=1}^{\ell-1}(-1)^{i+\ell}\mathcal{P}[\beta \setminus u_\ell u_i]\,\mathcal{P}[u_\ell u_i] + \sum_{i=\ell+1}^{k}(-1)^{i+\ell-1}\mathcal{P}[\beta \setminus u_\ell u_i]\,\mathcal{P}[u_\ell u_i]\\ &= \sum_{i=1}^{\ell-1}(-1)^{i+\ell-1}\mathcal{P}[\beta \setminus u_\ell u_i]\,\mathcal{P}[u_i u_\ell] + \sum_{i=\ell+1}^{k}(-1)^{i+\ell-1}\mathcal{P}[\beta \setminus u_\ell u_i]\,\mathcal{P}[u_\ell u_i]\,.\end{aligned}$$ □

Lemma A.3 *For integers* $1 \leqslant u_1 < \cdots < u_k \leqslant n$ *and for every integer* ℓ *with* $1 \leqslant \ell \leqslant k$ *one has*

$$\sum_{i=1}^{\ell-1}(-1)^i t_{u_i u_\ell} \operatorname{pf}_T (u_1 \dots u_{i-1} u_{i+1} \dots u_k) = \sum_{i=\ell+1}^{n} (-1)^i t_{u_\ell u_i} \operatorname{pf}_T (u_1 \dots u_{i-1} u_{i+1} \dots u_k) .$$

Proof First assume that $\ell \geqslant 2$ holds. With $\alpha = u_\ell$, $b = u_1$, and $\gamma = u_2 \dots u_k \setminus u_\ell$ the equation (A.1.2) yields

$$\mathcal{P}[u_\ell u_1]\,\mathcal{P}[u_\ell \gamma] = \sum_{j=2}^{\ell-1}(-1)^j\,\mathcal{P}[u_\ell u_j]\,\mathcal{P}[u_\ell u_1 \gamma \setminus u_j] + \sum_{j=\ell+1}^{k} (-1)^{j-1}\,\mathcal{P}[u_\ell u_j]\,\mathcal{P}[u_\ell u_1 \gamma \setminus u_j] ,$$

which after reordering and multiplication by a sign becomes

$$\mathcal{P}[u_1 u_\ell]\,\mathcal{P}[u_2 \dots u_k] = \sum_{j=2}^{\ell-1}(-1)^j\,\mathcal{P}[u_j u_\ell]\,\mathcal{P}[u_1 \dots u_k \setminus u_j] + \sum_{j=\ell+1}^{n} (-1)^{j-1}\,\mathcal{P}[u_\ell u_j]\,\mathcal{P}[u_1 \dots u_k \setminus u_j] ,$$

and that can be rewritten as

$$\sum_{j=1}^{\ell-1}(-1)^j\,\mathcal{P}[u_j u_\ell]\,\mathcal{P}[u_1 \dots u_k \setminus u_j] = \sum_{j=\ell+1}^{k} (-1)^j\,\mathcal{P}[u_\ell u_j]\,\mathcal{P}[u_1 \dots u_k \setminus u_j] .$$

Next assume that $\ell = 1$ holds. With $\alpha = u_1$, $b = u_2$, and $\gamma = u_3 \dots u_k$ Eq. (A.1.2) yields

$$\mathcal{P}[u_1 u_2]\,\mathcal{P}[u_1 \gamma] = \sum_{j=3}^{k}(-1)^{j-1}\,\mathcal{P}[u_1 u_j]\,\mathcal{P}[u_1 u_2 \gamma \setminus u_j]$$

which can be rewritten as

$$\sum_{j=2}^{k}(-1)^j\,\mathcal{P}[u_1 u_j]\,\mathcal{P}[u_1 \dots u_k \setminus u_j] = 0 .$$

□

Lemma A.4 *For integers* $1 \leqslant u_1 < \cdots < u_k \leqslant n$ *and every integer* ℓ *with* $1 \leqslant \ell \leqslant k$ *one has,*

$$(-1)^{\ell-1}\,\mathrm{pf}_T\,\mathrm{pf}_T(\overline{u_1\ldots u_k}) \;=\; \sum_{i=1}^{k}(-1)^i\,\mathrm{pf}_T(\overline{u_iu_\ell})\,\mathrm{pf}_T(\overline{u_1\ldots u_k\setminus u_iu_\ell})\,.$$

Proof With $\alpha = 1\ldots n \setminus u_1\ldots u_k$, $\beta = u_1\ldots u_k$, and $b = u_\ell$ the formula (A.1.4) yields

$$\begin{aligned}\mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha] \;=\; &\sum_{i=1}^{\ell-1}(-1)^{i+\ell}\,\mathcal{P}[\alpha\beta\setminus u_\ell u_i]\,\mathcal{P}[\alpha u_\ell u_i]\\ &+\sum_{i=\ell+1}^{k}(-1)^{i+\ell-1}\,\mathcal{P}[\alpha\beta\setminus u_\ell u_i]\,\mathcal{P}[\alpha u_\ell u_i]\,,\end{aligned}$$

which after reordering and multiplying by a sign becomes

$$\begin{aligned}&\mathcal{P}[1\ldots n]\,\mathcal{P}[1\ldots n\setminus u_1\ldots u_k]\\ &\qquad= \sum_{i=1}^{k}(-1)^{i+\ell-1}\,\mathcal{P}[1\ldots n\setminus u_\ell u_i]\,\mathcal{P}[1\ldots n\setminus(u_1\ldots u_k\setminus u_\ell u_i)]\,.\end{aligned}$$ □

Lemma A.5 *For integers* $1 \leqslant u_1 < \cdots < u_k \leqslant n$ *and every integer* ℓ *with* $1 \leqslant \ell \leqslant k-1$ *one has*

$$\sum_{i=1}^{\ell-1}(-1)^i\,\mathrm{pf}_T(\overline{u_1\ldots u_k\setminus u_i})\,\mathrm{pf}_T(\overline{u_iu_\ell}) \;=\; \sum_{i=\ell+1}^{k}(-1)^i\,\mathrm{pf}_T(\overline{u_1\ldots u_k\setminus u_i})\,\mathrm{pf}_T(\overline{u_\ell u_i})\,.$$

Proof With $\alpha = 1\ldots n\setminus u_1\ldots u_k$, $b = u_k$, and $\gamma = u_1\ldots u_{k-1}\setminus u_\ell$ formula (A.1.2) yields

$$\begin{aligned}\mathcal{P}[\alpha u_k]\,\mathcal{P}[\alpha\gamma] \;=\; &\sum_{j=1}^{\ell-1}(-1)^{j-1}\,\mathcal{P}[\alpha u_j]\,\mathcal{P}[\alpha u_k\gamma\setminus u_j]\\ &+\sum_{j=\ell+1}^{k-1}(-1)^{j}\,\mathcal{P}[\alpha u_j]\,\mathcal{P}[\alpha u_k\gamma\setminus u_j]\,,\end{aligned}$$

which after reordering and multiplication by a sign can be rewritten as

$$(-1)^{k-1}\,\mathcal{P}[1\ldots n\setminus u_1\ldots u_{k-1}]\,\mathcal{P}[1\ldots n\setminus u_\ell u_k]$$
$$= \sum_{j=1}^{\ell-1}(-1)^{j-1}\,\mathcal{P}[1\ldots n\setminus(u_1\ldots u_k\setminus u_j)]\,\mathcal{P}[1\ldots n\setminus u_j u_\ell]$$
$$+\sum_{j=\ell+1}^{k-1}(-1)^{j}\,\mathcal{P}[1\ldots n\setminus(u_1\ldots u_k\setminus u_j)]\,\mathcal{P}[1\ldots n\setminus u_\ell u_j]\,.$$

This can also be written

$$\sum_{j=1}^{\ell-1}(-1)^{j}\,\mathcal{P}[1\ldots n\setminus(u_1\ldots u_k\setminus u_j)]\,\mathcal{P}[1\ldots n\setminus u_j u_\ell]$$
$$= \sum_{j=\ell+1}^{k}(-1)^{j}\,\mathcal{P}[1\ldots n\setminus(u_1\ldots u_k\setminus u_j)]\,\mathcal{P}[1\ldots n\setminus u_\ell u_j]\,.$$ □

Lemma A.6 *For integers* $1\leqslant u_1<\cdots<u_k\leqslant n$ *one has*

$$\sum_{i=1}^{k}(-1)^{i}\,\mathrm{pf}_T(\overline{u_i})\,\mathrm{pf}_T(\overline{u_1\ldots u_{i-1}u_{i+1}\ldots u_k}) \;=\; 0\,.$$

Proof With $\alpha=1\ldots n\setminus u_1\ldots u_k$, $b=u_1$, and $\gamma=u_2\ldots u_k$ Eq. (A.1.2) yields

$$\mathcal{P}[\alpha u_1]\,\mathcal{P}[\alpha\gamma] \;=\; \sum_{j=2}^{k}(-1)^{j}\,\mathcal{P}[\alpha u_j]\,\mathcal{P}[\alpha u_1\gamma\setminus u_j]\,,$$

which after reordering and multiplication by a sign becomes

$$\sum_{j=1}^{k}(-1)^{j}\,\mathcal{P}[1\ldots n\setminus u_1\ldots u_{j-1}u_{j+1}\ldots u_k]\,\mathcal{P}[1\ldots n\setminus u_j] \;=\; 0\,.$$ □

Lemma A.7 *For integers* $1\leqslant u<v<w<x<y<z\leqslant n$ *one has*

$$\mathrm{pf}_T(\overline{y})\,\mathrm{pf}_T(\overline{uvwxz})-\mathrm{pf}_T(\overline{z})\,\mathrm{pf}_T(\overline{uvwxy})$$
$$=\;\mathrm{pf}_T(\overline{uyz})\,\mathrm{pf}_T(\overline{vwx})-\mathrm{pf}_T(\overline{vyz})\,\mathrm{pf}_T(\overline{uwx})$$
$$+\,\mathrm{pf}_T(\overline{wyz})\,\mathrm{pf}_T(\overline{uvx})-\mathrm{pf}_T(\overline{xyz})\,\mathrm{pf}_T(\overline{uvw})\,.$$

Proof With $\alpha=1\ldots n\setminus uvwxyz$, $\beta=uvwxy$, $b=y$, and $c=z$ Eq. (A.1.3) yields

$$\mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha z] - \mathcal{P}[\alpha z\beta \setminus y]\,\mathcal{P}[\alpha y] \;=\; \sum_{i\in\beta} \operatorname{sgn}\left(\begin{smallmatrix}\beta\\ yi\,(\beta\setminus yi)\end{smallmatrix}\right)\mathcal{P}[\alpha\beta \setminus yi]\,\mathcal{P}[\alpha zyi]\,,$$

which expands into

$$\begin{aligned}\mathcal{P}[\alpha uvwxy]\,\mathcal{P}[\alpha z] - \mathcal{P}[\alpha zuvwx]\,\mathcal{P}[\alpha y] = {} & \mathcal{P}[\alpha vwx]\,\mathcal{P}[\alpha zyu]\\ & - \mathcal{P}[\alpha uwx]\,\mathcal{P}[\alpha zyv]\\ & + \mathcal{P}[\alpha uvx]\,\mathcal{P}[\alpha zyw]\\ & - \mathcal{P}[\alpha uvw]\,\mathcal{P}[\alpha zyx]\,.\end{aligned}$$

After reordering and multiplication by $(-1)^{u+v+w+x+y+z}$ it becomes

$$\begin{aligned}&-\mathcal{P}[1\ldots n \setminus z]\,\mathcal{P}[1\ldots n \setminus uvwxy] + \mathcal{P}[1\ldots n \setminus y]\,\mathcal{P}[1\ldots n \setminus uvwxz]\\ &\quad= \mathcal{P}[1\ldots n \setminus uyz]\,\mathcal{P}[1\ldots n \setminus vwx]\\ &\qquad - \mathcal{P}[1\ldots n \setminus vyz]\,\mathcal{P}[1\ldots n \setminus uwx]\\ &\qquad + \mathcal{P}[1\ldots n \setminus wyz]\,\mathcal{P}[1\ldots n \setminus uvx]\\ &\qquad + \mathcal{P}[1\ldots n \setminus xyz]\,\mathcal{P}[1\ldots n \setminus uvw]\,.\end{aligned}$$

□

Lemma A.8 *For integers* $1 \leqslant u < v < w < x < y < z \leqslant n$ *one has*

$$\begin{aligned}&\operatorname{pf}_T(\overline{xy})\operatorname{pf}_T(\overline{uvwz}) - \operatorname{pf}_T(\overline{xz})\operatorname{pf}_T(\overline{uvwy}) + \operatorname{pf}_T(\overline{yz})\operatorname{pf}_T(\overline{uvwx})\\ &\quad= \operatorname{pf}_T(\overline{uv})\operatorname{pf}_T(\overline{wxyz}) - \operatorname{pf}_T(\overline{uw})\operatorname{pf}_T(\overline{vxyz}) + \operatorname{pf}_T(\overline{vw})\operatorname{pf}_T(\overline{uxyz})\,.\end{aligned}$$

Proof With $\alpha = 1\ldots n \setminus uvwxyz$, $\beta = uvwx$, $b = x$, and $\gamma = yz$ Eq. (A.1.1) yields

$$\begin{aligned}&\mathcal{P}[\alpha uvwx]\,\mathcal{P}[\alpha yz]\\ &\quad= -\mathcal{P}[\alpha vw]\,\mathcal{P}[\alpha yzxu] + \mathcal{P}[\alpha uw]\,\mathcal{P}[\alpha yzxv]\\ &\qquad - \mathcal{P}[\alpha uv]\,\mathcal{P}[\alpha yzxw] - \mathcal{P}[\alpha yuvw]\,\mathcal{P}[\alpha xz] + \mathcal{P}[\alpha zuvw]\,\mathcal{P}[\alpha xy]\\ &\quad= \mathcal{P}[\alpha vw]\,\mathcal{P}[\alpha uxyz] - \mathcal{P}[\alpha uw]\,\mathcal{P}[\alpha vxyz] + \mathcal{P}[\alpha uv]\,\mathcal{P}[\alpha wxyz]\\ &\qquad + \mathcal{P}[\alpha uvwy]\,\mathcal{P}[\alpha xz] - \mathcal{P}[\alpha uvwz]\,\mathcal{P}[\alpha xy]\,,\end{aligned}$$

which after reordering and multiplication by a sign becomes

$$\begin{aligned}\mathcal{P}[1\ldots n\setminus yz]\,\mathcal{P}[1\ldots n\setminus uvwx]-\mathcal{P}[1\ldots n\setminus xz]\,\mathcal{P}[1\ldots n\setminus uvwy]\\ +\mathcal{P}[1\ldots n\setminus xy]\,\mathcal{P}[1\ldots n\setminus uvwz]\\ = \mathcal{P}[1\ldots n\setminus vw]\,\mathcal{P}[1\ldots n\setminus uxyz]-\mathcal{P}[1\ldots n\setminus uw]\,\mathcal{P}[1\ldots n\setminus vxyz]\\ +\mathcal{P}[1\ldots n\setminus uv]\,\mathcal{P}[1\ldots n\setminus wxyz]\,.\end{aligned}$$ □

Lemma A.9 *For integers* $1\leqslant u<x<y\leqslant n$ *and* $1\leqslant v<w<x$ *one has*

$$\begin{aligned}\mathrm{pf}_T(\overline{uxy})\,\mathrm{pf}_T(\overline{uvw})-\mathrm{pf}_T(\overline{u})\,\mathrm{pf}_T(\overline{uvwxy})\\ = \mathrm{pf}_T(\overline{uvx})\,\mathrm{pf}_T(\overline{uwy})-\mathrm{pf}_T(\overline{uwx})\,\mathrm{pf}_T(\overline{uvy})\,.\end{aligned}$$

Proof With $\alpha=1\ldots n\setminus uvwxy$, $\beta=vwxy$, and $b=x$ Eq. (A.1.4) yields

$$\mathcal{P}[\alpha\beta]\,\mathcal{P}[\alpha]\;=\;\sum_{i\in\beta}\operatorname{sgn}\left(\begin{smallmatrix}\beta\\ xi(\beta\setminus xi)\end{smallmatrix}\right)\mathcal{P}[\alpha\beta\setminus xi]\,\mathcal{P}[\alpha xi]\,,$$

which expands into

$$\mathcal{P}[\alpha vwxy]\,\mathcal{P}[\alpha]\;=\;\mathcal{P}[\alpha wy]\,\mathcal{P}[\alpha xv]-\mathcal{P}[\alpha vy]\,\mathcal{P}[\alpha xw]+\mathcal{P}[\alpha vw]\,\mathcal{P}[\alpha xy]\,.$$

After reordering and multiplication by $(-1)^{v+w+x+y}$ this expression becomes

$$\begin{aligned}\mathcal{P}[1\ldots n\setminus u]\,\mathcal{P}[\alpha]\;=\;-\mathcal{P}[1\ldots n\setminus uvx]\,\mathcal{P}[1\ldots n\setminus uwx]\\ +\mathcal{P}[1\ldots n\setminus uwx]\,\mathcal{P}[1\ldots n\setminus uvy]\\ +\mathcal{P}[1\ldots n\setminus uxy]\,\mathcal{P}[1\ldots n\setminus uvw]\,.\end{aligned}$$ □

B Minors via Pfaffians Following Brill

The formula in the next theorem was first discovered by Brill [4]; the theorem stated here is [11, Theorem 2.1].

Theorem B.1 *Let* T *be an* $n\times n$ *skew symmetric matrix. Let* $\{i_1,\ldots,i_m\}$ *and* $\{j_1,\ldots,j_m\}$ *be subsets of* $\{1,\ldots,n\}$ *with* $i_1<\cdots<i_m$ *and* $j_1<\cdots<j_m$*,*

and set $\rho = i_1 \dots i_m$ *and* $\sigma = j_1 \dots j_m$. *The following equality holds:*

$$\det(T[i_1 \dots i_m;\, j_1 \dots j_m]) = (-1)^{\lfloor \frac{m}{2} \rfloor} \sum_{0 \leqslant k \leqslant \lfloor \frac{m}{2} \rfloor} (-1)^k \sum_{\substack{|\omega|=2k \\ \omega \subseteq \rho}} \operatorname{sgn}\left(\begin{smallmatrix} \rho \\ \omega(\rho\setminus\omega) \end{smallmatrix}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho \setminus \omega)\sigma]\,.$$

Notice that only subwords ω of ρ that contain $\rho \cap \sigma$ contribute to the sum above.

The two lemmas proved below are applied in Appendix C to calculate the maximal minors of the matrices ∂_3 from Theorems 3.3 and 3.9.

Lemma B.2 *Let* $n \geqslant 5$ *be an odd number. For integers* $1 \leqslant r_1 < r_2 < r_3 \leqslant n$ *one has*

$$\det\bigl(T[\overline{r_1 r_2 r_3};\, \overline{123}]\bigr) = \begin{cases} \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123}) & \text{if } r_2 \leqslant 3 \\ \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123}) - \operatorname{pf}_T(\overline{123r_2r_3})\operatorname{pf}_T(\overline{r_1}) & \text{if } r_1 \leqslant 3 < r_2 \\ \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123}) - \operatorname{pf}_T(\overline{23r_1r_2r_3})\operatorname{pf}_T(\overline{1}) & \\ \quad + \operatorname{pf}_T(\overline{13r_1r_2r_3})\operatorname{pf}_T(\overline{2}) - \operatorname{pf}_T(\overline{12r_1r_2r_3})\operatorname{pf}_T(\overline{3}) & \text{if } 3 < r_1\,. \end{cases}$$

Proof Consider the words

$$\rho = 1 \dots n \setminus r_1 r_2 r_3 \qquad \text{and} \qquad \sigma = 4 \dots n$$

of length $n-3$. One has $\rho \cap \sigma = \sigma \setminus r_1r_2r_3$, and Theorem B.1 yields

$$\det(T[\overline{r_1r_2r_3};\, \overline{123}]) = (-1)^{\frac{n-3}{2}} \sum_{k=\left\lceil \frac{|\sigma \setminus r_1r_2r_3|}{2} \right\rceil}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma\setminus r_1r_2r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}\left(\begin{smallmatrix} \rho \\ \omega(\rho\setminus\omega) \end{smallmatrix}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho \setminus \omega)\sigma]\,. \tag{1}$$

If $r_2 \leqslant 3$ holds, then one has $|\rho \cap \sigma| = n-3$ or $|\rho \cap \sigma| = n-4$. In either case the shortest word ω contributing to the sum (1) has length $n-3$. Thus, $\omega = \rho$ is the only contributing word and one gets

$$\det(T[\overline{r_1r_2r_3};\, \overline{123}]) = \mathcal{P}[\rho]\,\mathcal{P}[\sigma] = \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123})\,.$$

If $r_1 \leqslant 3 < r_2$ holds, then one has $|\rho \cap \sigma| = n-5$. As $n-5$ is even, the shortest subwords ω of

$$\rho = (123 \setminus r_1)(\sigma \setminus r_2r_3)$$

that contribute to the sum (1) have length $n-5$, so $\omega = \sigma \setminus r_2r_3$ is the only one. Now one has

$$\begin{aligned}
&\det(T[\overline{r_1r_2r_3};\overline{123}]) \\
&\quad = (-1)^{\frac{n-3}{2}} \sum_{k=\frac{n-5}{2}}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma\setminus r_2r_3\subseteq\omega\subseteq\rho}} \operatorname{sgn}\left(\overset{\rho}{\omega(\rho\setminus\omega)}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma] \\
&\quad = (-1)^{\frac{n-3}{2}}\Big((-1)^{\frac{n-5}{2}} \operatorname{sgn}\left(\overset{\rho}{\sigma\setminus r_2r_3(123\setminus r_1)}\right) \mathcal{P}[\sigma\setminus r_2r_3]\,\mathcal{P}[(123\setminus r_1)\sigma] \\
&\qquad\qquad + (-1)^{\frac{n-3}{2}}\,\mathcal{P}[\rho]\,\mathcal{P}[\sigma]\Big) \\
&\quad = -\operatorname{sgn}\left(\overset{\rho}{\sigma\setminus r_2r_3(123\setminus r_1)}\right) \mathcal{P}[\sigma\setminus r_2r_3]\,\mathcal{P}[(123\setminus r_1)\sigma] + \mathcal{P}[\rho]\,\mathcal{P}[\sigma] \\
&\quad = -\mathcal{P}[\sigma\setminus r_2r_3]\,\mathcal{P}[(123\setminus r_1)\sigma] + \mathcal{P}[\rho]\,\mathcal{P}[\sigma] \\
&\quad = -\operatorname{pf}_T(\overline{123r_2r_3})\operatorname{pf}_T(\overline{r_1}) + \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123})\,.
\end{aligned}$$

If $3 < r_1$ holds, then one has $|\rho\cap\sigma| = n-6$. As $n-6$ is odd, the shortest subwords ω of

$$\rho = 123(\sigma\setminus r_1r_2r_3)$$

that contribute to the sum (1) have length $n-5$. Now one has

$$\begin{aligned}
&\det(T[\overline{r_1r_2r_3};\overline{123}]) \\
&\quad = (-1)^{\frac{n-3}{2}} \sum_{k=\frac{n-5}{2}}^{\frac{n-3}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma\setminus r_1r_2r_3\subseteq\omega\subseteq\rho}} \operatorname{sgn}\left(\overset{\rho}{\omega(\rho\setminus\omega)}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma] \\
&\quad = (-1)^{\frac{n-3}{2}}\Big((-1)^{\frac{n-5}{2}} \sum_{\substack{|\omega|=n-5 \\ \sigma\setminus r_1r_2r_3\subseteq\omega\subseteq\rho}} \operatorname{sgn}\left(\overset{\rho}{\omega(\rho\setminus\omega)}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma] \\
&\qquad\qquad + (-1)^{\frac{n-3}{2}}\,\mathcal{P}[\rho]\,\mathcal{P}[\sigma]\Big) \\
&\quad = -\Big(\operatorname{sgn}\left(\overset{\rho}{1(\sigma\setminus r_1r_2r_3)23}\right)\mathcal{P}[1\sigma\setminus r_1r_2r_3]\,\mathcal{P}[23\sigma] \\
&\qquad + \operatorname{sgn}\left(\overset{\rho}{2(\sigma\setminus r_1r_2r_3)13}\right)\mathcal{P}[2\sigma\setminus r_1r_2r_3]\,\mathcal{P}[13\sigma] \\
&\qquad\qquad + \operatorname{sgn}\left(\overset{\rho}{3(\sigma\setminus r_1r_2r_3)12}\right)\mathcal{P}[3\sigma\setminus r_1r_2r_3]\,\mathcal{P}[12\sigma]\Big) + \mathcal{P}[\rho]\,\mathcal{P}[\sigma] \\
&\quad = \mathcal{P}[1\sigma\setminus r_1r_2r_3]\,\mathcal{P}[23\sigma] - \mathcal{P}[2\sigma\setminus r_1r_2r_3]\,\mathcal{P}[13\sigma] + \mathcal{P}[3\sigma\setminus r_1r_2r_3]\,\mathcal{P}[12\sigma] \\
&\qquad\qquad + \mathcal{P}[\rho]\,\mathcal{P}[\sigma]
\end{aligned}$$

$$= -\operatorname{pf}_T(\overline{23r_1r_2r_3})\operatorname{pf}_T(\overline{1}) + \operatorname{pf}_T(\overline{13r_1r_2r_3})\operatorname{pf}_T(\overline{2}) \\ - \operatorname{pf}_T(\overline{12r_1r_2r_3})\operatorname{pf}_T(\overline{3}) + \operatorname{pf}_T(\overline{r_1r_2r_3})\operatorname{pf}_T(\overline{123})\,. \qquad \square$$

Lemma B.3 *Let $n \geqslant 6$ be an even number. For integers $1 \leqslant r_1 < r_2 < r_3 \leqslant n$ one has*

$$\det\big(T[\overline{r_1r_2r_3};\, \overline{123}]\big)$$
$$= \begin{cases} 0 & \textit{if } r_3 = 3 \\ \operatorname{pf}_T(\overline{123r_3})\operatorname{pf}_T(\overline{r_1r_2}) & \textit{if } r_2 \leqslant 3 < r_3 \\ \operatorname{pf}_T(\overline{12r_2r_3})\operatorname{pf}_T(\overline{13}) - \operatorname{pf}_T(\overline{13r_2r_3})\operatorname{pf}_T(\overline{12}) & \textit{if } 1 = r_1 \leqslant 3 < r_2 \\ \operatorname{pf}_T(\overline{12r_2r_3})\operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{23r_2r_3})\operatorname{pf}_T(\overline{12}) & \textit{if } 2 = r_1 \leqslant 3 < r_2 \\ \operatorname{pf}_T(\overline{13r_2r_3})\operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{23r_2r_3})\operatorname{pf}_T(\overline{13}) & \textit{if } 3 = r_1 < r_2 \\ \operatorname{pf}_T(\overline{1r_1r_2r_3})\operatorname{pf}_T(\overline{23}) - \operatorname{pf}_T(\overline{2r_1r_2r_3})\operatorname{pf}_T(\overline{13}) & \\ \quad + \operatorname{pf}_T(\overline{3r_1r_2r_3})\operatorname{pf}_T(\overline{12}) - \operatorname{pf}_T(\overline{123r_1r_2r_3})\operatorname{pf}_{[}\,T] & \textit{if} 3 < r_1\,. \end{cases}$$

Proof Consider the words

$$\rho \;=\; 1\ldots n \setminus r_1r_2r_3 \quad \text{and} \quad \sigma \;=\; 4\ldots n$$

of length $n-3$. One has $\rho \cap \sigma = \sigma \setminus r_1r_2r_3$ and Theorem B.1 yields

$$\det(T[\overline{r_1r_2r_3};\, \overline{123}])$$
$$= (-1)^{\frac{n-4}{2}} \sum_{k=\left\lceil \frac{|\sigma\setminus r_1r_2r_3|}{2}\right\rceil}^{\frac{n-4}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma\setminus r_1r_2r_3 \subseteq \omega \subseteq \rho}} \operatorname{sgn}\left(\begin{smallmatrix} \rho \\ \omega(\rho\setminus\omega)\end{smallmatrix}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma]\,. \tag{1}$$

If $r_3 = 3$ holds, then one has $|\rho \cap \sigma| = |\sigma| = n-3$, so the sum (1) is empty. i.e.

$$\det(T[\overline{123};\, \overline{123}]) \;=\; 0\,.$$

If $r_2 \leqslant 3 < r_3$ hold, then one has $|\rho\cap\sigma| = n-4$, so the shortest subwords ω of

$$\rho \;=\; (123 \setminus r_1r_2)(\sigma \setminus r_3)$$

contributing to the sum (1) have length $n - 4$. Thus, $\omega = \sigma \setminus r_3$ is the only contributing word, and one gets

$$\det(T[\overline{r_1 r_2 r_3};\, \overline{123}]) \;=\; \mathcal{P}[\sigma \setminus r_3]\, \mathcal{P}[(123 \setminus r_1 r_2)\sigma] = \mathrm{pf}_T(\overline{123 r_3})\, \mathrm{pf}_T(\overline{r_1 r_2})\,.$$

If $r_1 \leqslant 3 < r_2$ holds, then one has $|\rho \cap \sigma| = n - 5$, which is odd. Therefore, the shortest subwords ω of

$$\rho \;=\; (123 \setminus r_1)(\sigma \setminus r_2 r_3)$$

contributing to the sum (1) have length $n - 4$. Hence, one gets

$$\begin{aligned}&\det(T[\overline{r_1 r_2 r_3};\, \overline{123}])\\ &\quad= \sum_{r \in 123 \setminus r_1} \mathrm{sgn}\left(\overset{\rho}{{}_{(r\sigma \setminus r_2 r_3)(123 \setminus r r_1)}}\right) \mathcal{P}[r\sigma \setminus r_2 r_3]\, \mathcal{P}[(123 \setminus r_1 r)\sigma]\,.\end{aligned} \tag{2}$$

For $r_1 = 1$ this specializes to

$$\begin{aligned}\det(T[\overline{1 r_2 r_3};\, \overline{123}]) \;&=\; \sum_{r \in 23} \mathrm{sgn}\left(\overset{\rho}{{}_{(r\sigma \setminus r_2 r_3)(23 \setminus r)}}\right) \mathcal{P}[r\sigma \setminus r_2 r_3]\, \mathcal{P}[(23 \setminus r)\sigma]\\ &=\; -\,\mathrm{pf}_T(\overline{13 r_2 r_3})\, \mathrm{pf}_T(\overline{12}) + \mathrm{pf}_T(\overline{12 r_2 r_3})\, \mathrm{pf}_T(\overline{13})\,.\end{aligned}$$

The specialization of (2) with $r_1 = 2$ is

$$\begin{aligned}\det(T[\overline{2 r_2 r_3};\, \overline{123}]) \;&=\; \sum_{r \in 13} \mathrm{sgn}\left(\overset{\rho}{{}_{(r\sigma \setminus r_2 r_3)(13 \setminus r)}}\right) \mathcal{P}[r\sigma \setminus r_2 r_3]\, \mathcal{P}[(13 \setminus r)\sigma]\\ &=\; -\,\mathrm{pf}_T(\overline{23 r_2 r_3})\, \mathrm{pf}_T(\overline{12}) + \mathrm{pf}_T(\overline{12 r_2 r_3})\, \mathrm{pf}_T(\overline{23})\,.\end{aligned}$$

The specialization of (2) with $r_1 = 3$ is

$$\begin{aligned}\det(T[\overline{3 r_2 r_3};\, \overline{123}]) \;&=\; \sum_{r \in 12} \mathrm{sgn}\left(\overset{\rho}{{}_{(r\sigma \setminus r_2 r_3)(12 \setminus r)}}\right) \mathcal{P}[r\sigma \setminus r_2 r_3]\, \mathcal{P}[(12 \setminus r)\sigma]\\ &=\; -\,\mathrm{pf}_T(\overline{23 r_2 r_3})\, \mathrm{pf}_T(\overline{13}) + \mathrm{pf}_T(\overline{13 r_2 r_3})\, \mathrm{pf}_T(\overline{23})\,.\end{aligned}$$

If $3 < r_1$ holds, then one has $|\rho \cap \sigma| = n - 6$, which is even. Therefore, the shortest subwords ω of

$$\rho \;=\; 123(\sigma \setminus r_1 r_2 r_3)$$

that contribute to the sum (1) have length $n-6$, which means that $\omega = \sigma \setminus r_1r_2r_3$ is the only one. Thus one has

$$\begin{aligned}
&\det(T[\overline{r_1r_2r_3}, \overline{123}]) \\
&= (-1)^{\frac{n-4}{2}} \sum_{k=\frac{n-6}{2}}^{\frac{n-4}{2}} (-1)^k \sum_{\substack{|\omega|=2k \\ \sigma\setminus r_1r_2r_3\subseteq\omega\subseteq\rho}} \operatorname{sgn}\left(\begin{smallmatrix} \rho \\ \omega(\rho\setminus\omega)\end{smallmatrix}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma] \\
&= -\mathcal{P}[\sigma\setminus r_1r_2r_3]\,\mathcal{P}[123\sigma] + \sum_{\substack{|\omega|=n-4 \\ \sigma\setminus r_1r_2r_3\subseteq\omega\subseteq\rho}} \operatorname{sgn}\left(\begin{smallmatrix} \rho \\ \omega(\rho\setminus\omega)\end{smallmatrix}\right) \mathcal{P}[\omega]\,\mathcal{P}[(\rho\setminus\omega)\sigma] \\
&= -\mathcal{P}[\sigma\setminus r_1r_2r_3]\,\mathcal{P}[123\sigma] + \mathcal{P}[23\sigma\setminus r_1r_2r_3]\,\mathcal{P}[1\sigma] \\
&\qquad - \mathcal{P}[13\sigma\setminus r_1r_2r_3]\,\mathcal{P}[2\sigma] + \mathcal{P}[12\sigma\setminus r_1r_2r_3]\,\mathcal{P}[3\sigma] \\
&= -\operatorname{pf}_T(\overline{123r_1r_2r_3})\operatorname{pf}_[\, T] + \operatorname{pf}_T(\overline{1r_1r_2r_3})\operatorname{pf}_T(\overline{23}) \\
&\qquad - \operatorname{pf}_T(\overline{2r_1r_2r_3})\operatorname{pf}_T(\overline{13}) + \operatorname{pf}_T(\overline{3r_1r_2r_3})\operatorname{pf}_T(\overline{12})\,. \qquad \square
\end{aligned}$$

C Generic Almost Complete Intersections: The Proofs

In this final appendix we provide the detailed computations that underpin the theorems in Sect. 3.

C.1 Quotients of Even Type

Lemma C.1 *Let $n \geqslant 5$ be an odd number and adopt the setup from 3.3. The sequence $0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R} \longrightarrow 0$ is a complex.*

Proof The product $\partial_1\partial_2$ is a $1\times n$ matrix; the first three entries are evidently 0. For $i \in \{4,\dots,n\}$ the i^{th} entry is

$$\pm\big(-\operatorname{pf}_{\mathcal{T}}(\overline{1})\operatorname{pf}_{\mathcal{T}}(\overline{23i}) + \operatorname{pf}_{\mathcal{T}}(\overline{2})\operatorname{pf}_{\mathcal{T}}(\overline{13i}) - \operatorname{pf}_{\mathcal{T}}(\overline{3})\operatorname{pf}_{\mathcal{T}}(\overline{12i}) + \operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{i})\big)\,,$$

which is zero by Lemma A.6 applied with $u_1 \dots u_k = 123i$.

The product $\partial_2\partial_3$ is a $4\times(n-3)$ matrix. Let $i \in \{4,\dots,n\}$; the entry in position $(1, i-3)$ is

$$\tau_{1i}\operatorname{pf}_{\mathcal{T}}(\overline{123}) + \sum_{j=4}^{i-1}(-1)^{j-1}\tau_{ji}\operatorname{pf}_{\mathcal{T}}(\overline{23j}) - \sum_{j=i+1}^{n}(-1)^{j-1}\tau_{ij}\operatorname{pf}_{\mathcal{T}}(\overline{23j})\,.$$

Applied with $u_1 \dots u_k = 14 \dots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. Similarly, Lemma A.3 applied with $u_1 \dots u_k = 24 \dots n$ and $u_\ell = i$ shows that the entry in position $(2, i-3)$ is zero, and an application with $u_1 \dots u_k = 3 \dots n$ and $u_\ell = i$ shows that the entry in position $(3, i-3)$ is zero. The entry in position $(4, i-3)$ is

$$\sum_{j=1}^{i-1}(-1)^{j-1}\tau_{ji}\,\mathrm{pf}_{\mathcal{T}}(\overline{j}) - \sum_{j=i+1}^{n}(-1)^{j-1}\tau_{ij}\,\mathrm{pf}_{\mathcal{T}}(\overline{j})\,.$$

Applied with $u_1 \dots u_k = 1 \dots n$ and $u_\ell = i$, Lemma A.3 shows that also this quantity is zero. □

Józefiak and Pragacz [16] calculate the grade of ideals generated by Pfaffians; we combine this with a classic result of Eagon and Northcott [13] to obtain the next lemma and Lemma C.6, which deals with the case of even n.

Lemma C.2 *Let $n \geqslant 5$ be an odd number and adopt the setup from 3.3. The Pfaffians* $\mathrm{pf}_{\mathcal{T}}(\overline{1})$, $\mathrm{pf}_{\mathcal{T}}(\overline{2})$, *and* $\mathrm{pf}_{\mathcal{T}}(\overline{123})$ *form a regular sequence in* $\mathcal{R}$.

Proof The $(n-3)\times(n-3)$ Pfaffians of the matrix $\mathcal{T}[3\dots n; 3\dots n]$ generate by [16, Corollary 2.5] an ideal of grade 3 in the subring $\mathcal{R}' = \mathbb{Z}[\tau_{ij} \mid 3 \leqslant i < j \leqslant n]$ of $\mathcal{R}$; they are the Pfaffians $\mathrm{pf}_{\mathcal{T}}(\overline{12i})$ for $3 \leqslant i \leqslant n$. As $\mathrm{pf}_{\mathcal{T}}(\overline{123})$ is a regular element in the domain $\mathcal{R}'$, the Pfaffians $\mathrm{pf}_{\mathcal{T}}(\overline{12i})$ for $4 \leqslant i \leqslant n$ generate an ideal of grade 2 in $\mathcal{S}' = \mathcal{R}'/\,\mathrm{pf}_{\mathcal{T}}(\overline{123})$. In $\mathcal{S} = \mathcal{R}/\,\mathrm{pf}_{\mathcal{T}}(\overline{123})$ one has,

$$\mathrm{pf}_{\mathcal{T}}(\overline{1}) = \sum_{i=4}^{n}(-1)^{i-1}\tau_{2i}\,\mathrm{pf}_{\mathcal{T}}(\overline{12i}) \quad \text{and} \quad \mathrm{pf}_{\mathcal{T}}(\overline{2}) = \sum_{i=4}^{n}(-1)^{i-1}\tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{12i})\,.$$

Indeed, the first equality follows from Lemma A.2 applied with $u_1 \dots u_k = 2 \dots n$ and $\ell = 1$; the same lemma applied with $u_1 \dots u_k = 13 \dots n$ and $\ell = 1$ yields the second equality. Now it follows from [13, Lemma 6] that $\mathrm{pf}_{\mathcal{T}}(\overline{1})$ and $\mathrm{pf}_{\mathcal{T}}(\overline{2})$ form a regular sequence in $\mathcal{B}$. □

Lemma C.3 *Let $n \geqslant 5$ be an odd number and adopt the setup from 3.3. The ideal generated by the $(n-3)\times(n-3)$ minors of the matrix ∂_3 contains the elements*

$$(\mathrm{pf}_{\mathcal{T}}(\overline{1}))^2\,, \quad (\mathrm{pf}_{\mathcal{T}}(\overline{2}))^2\,, \quad (\mathrm{pf}_{\mathcal{T}}(\overline{3}))^2\,, \quad \textit{and} \quad (\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2\,.$$

Proof One has $(\mathrm{pf}_{\mathcal{T}}(\overline{1}))^2 = \det(\mathcal{T}[2\dots n; 2\dots n])$ and expansion of this determinant along the first two columns, see Horn and Johnson [15, 0.8.9], yields:

$$\begin{aligned}\det(\mathcal{T}[2\dots n; 2\dots n]) &= \sum_{2\leqslant i<j\leqslant n} \pm\det(\mathcal{T}[ij; 23])\det(\mathcal{T}[\overline{1ij}; \overline{123}])\\ &= \sum_{2\leqslant i<j\leqslant n} \pm\det(\mathcal{T}[ij; 23])\det(\partial_3[\overline{1ij}; 1\dots n-3])\,.\end{aligned}$$

Similarly, one gets

$$\begin{aligned}(\mathrm{pf}_{\mathcal{T}}(\overline{2}))^2 &= \det(\mathcal{T}[13\dots n;\,13\dots n]) \\ &= \sum_{3\leqslant j\leqslant n} \pm\det(\mathcal{T}[1j;\,13])\det(\partial_3[\overline{12j};\,1\dots n-3]) \\ &\quad + \sum_{3\leqslant i<j\leqslant n} \pm\det(\mathcal{T}[ij;\,13])\det(\partial_3[\overline{1ij};\,1\dots n-3])\end{aligned}$$

and

$$\begin{aligned}(\mathrm{pf}_{\mathcal{T}}(\overline{3}))^2 &= \det(\mathcal{T}[124\dots n;\,124\dots n]) \\ &= \sum_{\substack{1\leqslant i<j\leqslant n\\ i\neq 3\neq j}} \pm\det(\mathcal{T}[ij;\,12])\det(\partial_3[\overline{1ij};\,1\dots n-3])\,.\end{aligned}$$

Finally, one trivially has

$$(\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2 = \det(\mathcal{T}[\overline{123};\,\overline{123}]) = \det(\partial_3[\overline{123};\,1\dots n-3])\,.$$

□

Proposition C.4 *Let $n \geqslant 5$ be an odd number and adopt the setup from 3.3. For integers $1 \leqslant r_1 < r_2 < r_3 \leqslant n$ and $1 \leqslant s_1 < s_2 < s_3 \leqslant 4$ one has*

$$\det(\partial_3[\overline{r_1r_2r_3};\,1\dots n-3])\det(\partial_1[1;\,\overline{s_1s_2s_3}]) = \pm\det(\partial_2[s_1s_2s_3;\,r_1r_2r_3])\,.$$

Proof First notice that one has $\det(\partial_3[\overline{r_1r_2r_3};\,1\dots n-3]) = \det(\mathcal{T}[\overline{r_1r_2r_3};\,\overline{123}])$. With the notation

$$\begin{aligned}\mathrm{LHS} &= \det(\mathcal{T}[\overline{r_1r_2r_3};\,\overline{123}])\det(\partial_1[1;\,\overline{s_1s_2s_3}]) \quad \text{and} \\ \mathrm{RHS} &= \det(\partial_2[s_1s_2s_3;\,r_1r_2r_3])\end{aligned}$$

the goal is to prove that $\mathrm{LHS} = \pm\mathrm{RHS}$ holds. Set

$$\rho = 1\dots n \setminus r_1r_2r_3 \qquad \text{and} \qquad \{s\} = \overline{\{s_1, s_2, s_3\}}\,.$$

The possible values of s_3 are 3 and 4, and we treat these cases separately.

Case I. Assuming that $s_3 = 3$ holds one has $s = 4$ and, therefore,

$$\det(\partial_1[1;\,\overline{123}]) = \mathrm{pf}_{\mathcal{T}}(\overline{123})\,. \tag{1}$$

Because the first three columns of the matrix ∂_2 are special, our argument depends on the size of the intersection $\{1,2,3\} \cap \{r_1, r_2, r_3\}$. We therefore consider four subcases determined by the (in)equalities

$$r_3 = 3\,, \qquad r_2 \leqslant 3 < r_3\,, \qquad r_1 \leqslant 3 < r_2\,, \qquad \text{and} \qquad r_1 < 3\,. \tag{2}$$

Subcase I.a. If $r_3 = 3$ holds, then (1) and Lemma B.2 yield

$$\text{LHS} \;=\; (\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,,$$

and evidently one has $\text{RHS} = (\mathrm{pf}_{\mathcal{T}}(\overline{123}))^3$.

Subcase I.b. If $r_2 \leqslant 3 < r_3$ hold, then (1) and Lemma B.2 yield

$$\text{LHS} \;=\; \mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})(\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2\,.$$

Expanding the determinant along the first column one has

$$\begin{aligned}
\pm\text{RHS} \;&=\; \det\begin{pmatrix} \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) \\ 0 & \delta_{3r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) \end{pmatrix} \\
&=\; \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\delta_{2r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\delta_{3r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big) \\
&\qquad + \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\delta_{3r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) \\
&=\; (\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2 \\
&\qquad \cdot\big(\delta_{1r_1}\delta_{2r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \delta_{1r_1}\delta_{3r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) + \delta_{2r_1}\delta_{3r_2}\,\mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\big)\,.
\end{aligned}$$

For all three choices of $r_1 < r_2$ in $\{1,2,3\}$ one gets $\text{RHS} = \pm(\mathrm{pf}_{\mathcal{T}}(\overline{123}))^2$ $\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})$ as desired.

Subcase I.c. If $r_1 \leqslant 3 < r_2$ hold, then (1) and Lemma B.2 yield

$$\text{LHS} \;=\; \big(\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_1})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,.$$

In view of Lemma A.9 this can be rewritten as

$$\begin{aligned}
\text{LHS} \;=\; &\delta_{1r_1}\big(\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) \\
&+ \delta_{2r_1}\big(\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) \\
&\quad + \delta_{3r_1}\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,.
\end{aligned}$$

Expansion of the determinant along the first column yields the matching expression

$$
\begin{aligned}
\pm\text{RHS} &= \det\begin{pmatrix} \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \delta_{3r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) \end{pmatrix} \\
&= \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\big) \\
&\qquad - \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\big) \\
&\qquad\qquad + \delta_{3r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big)\,.
\end{aligned}
$$

Subcase I.d. If $3 < r_1$ holds, then (1) and Lemma B.2 yield

$$
\begin{aligned}
\text{LHS} &= \big(\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{1}) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\,.
\end{aligned}
$$

Expansion of the determinant along the first column yields the second equality in the computation below. The third equality follows from Lemma A.9 while the fifth follows from Lemmas A.6 and A.7. Finally, the last equality follows from another application of Lemma A.6.

$$
\begin{aligned}
\pm\text{RHS} &= \det\begin{pmatrix} \mathrm{pf}_{\mathcal{T}}(\overline{23r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{13r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{12r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) \end{pmatrix} \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\big) \\
&\qquad\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{12r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{1}) - \mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{1r_2r_3})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2r_3})\big) \\
&\qquad\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{12r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2r_3})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{1}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) + \mathrm{pf}_{\mathcal{T}}(\overline{12r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{1r_2r_3}) \\
&\qquad\qquad\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2r_3}) + \mathrm{pf}_{\mathcal{T}}(\overline{12r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2r_3})\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{pf}_{\mathcal{T}}(\overline{123r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{r_1}) \\
&\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_2}) \\
&\qquad\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\big) \\
&= -\operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123}) - \operatorname{pf}_{\mathcal{T}}(\overline{123r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_1}) \\
&\qquad\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_2}) - \operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{r_3})\big) \\
&= -\operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123}) - \operatorname{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{1}) \\
&\qquad\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{2}) - \operatorname{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{3})\big)\,.
\end{aligned}
$$

Thus LHS $= \pm$RHS holds, also in this subcase.

Case II. Assuming now that $s_3 = 4$ holds, one has $s \in \{1, 2, 3\}$ and hence

$$
\det(\partial_1[1; \overline{s_1s_2s_3}]) = (-1)^s \operatorname{pf}_{\mathcal{T}}(\overline{s})\,. \tag{3}
$$

As in Case I the argument is broken into subcases following the (in)equalities (2).

Subcase II.a. If $r_3 = 3$ holds, then (3) and Lemma B.2 yield

$$
\text{LHS} = \pm(\operatorname{pf}_{\mathcal{T}}(\overline{123}))^2 \operatorname{pf}_{\mathcal{T}}(\overline{s})\,,
$$

and evidently one has RHS $= \pm(\operatorname{pf}_{\mathcal{T}}(\overline{123}))^2 \operatorname{pf}_{\mathcal{T}}(\overline{s})$.

Subcase II.b. If $r_2 \leqslant 3 < r_3$ hold, then (3) and Lemma B.2 again yield

$$
\text{LHS} = \pm\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{s})\,.
$$

This has to be compared to

$$
\text{RHS} = \pm\det\begin{pmatrix}
\delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & 0 & \operatorname{pf}_{\mathcal{T}}(\overline{23r_3}) \\
\delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{13r_3}) \\
0 & \delta_{3r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{12r_3}) \\
(-1)^{r_1-1}\operatorname{pf}_{\mathcal{T}}(\overline{r_1}) & (-1)^{r_2-1}\operatorname{pf}_{\mathcal{T}}(\overline{r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{r_3})
\end{pmatrix}[s_1s_24;\,123]\,.
$$

Notice that the zeros in the matrix stand for $\delta_{3r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123})$ and $\delta_{1r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123})$; the determinant is thus symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. By this symmetry it is sufficient to treat the choice $\{s_1, s_2\} = \{1, 2\}$. In this case one has $s = 3$ and, therefore,

$$
\text{LHS} = \pm\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{3})\,. \tag{4}
$$

Expansion of the determinant along the first column yields

$$\begin{aligned}\pm\text{RHS} &= \det\begin{pmatrix} \delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & 0 & \operatorname{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{13r_3}) \\ (-1)^{r_1-1}\operatorname{pf}_{\mathcal{T}}(\overline{r_1}) & (-1)^{r_2-1}\operatorname{pf}_{\mathcal{T}}(\overline{r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} \\ &= \delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{r_3}) + (-1)^{r_2}\operatorname{pf}_{\mathcal{T}}(\overline{13r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_2})\big) \\ &\qquad + (-1)^{r_2}\delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{23r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_2}) \\ &\qquad\qquad + (-1)^{r_1}\delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{23r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_1})\,. \end{aligned} \tag{5}$$

For $\{r_1, r_2\} = \{1, 2\}$ one has LHS $= \pm\operatorname{pf}_{\mathcal{T}}(\overline{12r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\bar{3})$. In the next computation, which shows that this agrees with ±RHS, the last equality follows from Lemma A.6.

$$\begin{aligned}\pm\text{RHS} &= \operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{r_3}) \\ &\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{13r_3})\operatorname{pf}_{\mathcal{T}}(\bar{2})\big) - \operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{23r_3})\operatorname{pf}_{\mathcal{T}}(\bar{1})\big) \\ &= \operatorname{pf}_{\mathcal{T}}(\overline{123})\big(\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\overline{r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{23r_3})\operatorname{pf}_{\mathcal{T}}(\bar{1}) + \operatorname{pf}_{\mathcal{T}}(\overline{13r_3})\operatorname{pf}_{\mathcal{T}}(\bar{2})\big) \\ &= \operatorname{pf}_{\mathcal{T}}(\overline{123})(-\operatorname{pf}_{\mathcal{T}}(\bar{3})\operatorname{pf}_{\mathcal{T}}(\overline{12r_3}))\,. \end{aligned}$$

For $\{r_1, r_2\} = \{1, 3\}$ one has LHS $= \pm\operatorname{pf}_{\mathcal{T}}(\overline{13r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\bar{3})$, see (4), and (5) specializes to the same expression. Similarly, for $\{r_1, r_2\} = \{2, 3\}$ one has LHS $= \pm\operatorname{pf}_{\mathcal{T}}(\overline{23r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123})\operatorname{pf}_{\mathcal{T}}(\bar{3})$ and (5) specializes to the same expression.

Subcase II.c. If $r_1 \leqslant 3 < r_2$ hold, then (3) and Lemma B.2 yield

$$\text{LHS} = \pm\big(\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123}) - \operatorname{pf}_{\mathcal{T}}(\overline{123r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_1})\big)\operatorname{pf}_{\mathcal{T}}(\bar{s})\,.$$

This has to be compared to

$$\text{RHS} = \pm\det\begin{pmatrix} \delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{23r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{13r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{13r_3}) \\ \delta_{3r_1}\operatorname{pf}_{\mathcal{T}}(\overline{123}) & \operatorname{pf}_{\mathcal{T}}(\overline{12r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{12r_3}) \\ (-1)^{r_1-1}\operatorname{pf}_{\mathcal{T}}(\overline{r_1}) & \operatorname{pf}_{\mathcal{T}}(\overline{r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix}[s_1s_24;\,123]\,.$$

This determinant is symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. It suffices to treat the case $\{s_1, s_2\} = \{1, 2\}$, where one has $s = 3$ and, therefore,

$$\text{LHS} = \pm\big(\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123}) - \operatorname{pf}_{\mathcal{T}}(\overline{123r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{r_1})\big)\operatorname{pf}_{\mathcal{T}}(\bar{3})\,. \tag{6}$$

Expanding the determinant along the first column one gets

$$
\begin{aligned}
\pm\mathrm{RHS} &= \det\begin{pmatrix} \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3}) \\ \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) \\ (-1)^{r_1-1}\,\mathrm{pf}_{\mathcal{T}}(\overline{r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_3}) \end{pmatrix} \\
&= \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big) \\
&\quad - \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big) \\
&\quad + (-1)^{r_1-1}\,\mathrm{pf}_{\mathcal{T}}(\overline{r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big)\,.
\end{aligned}
$$

For $r_1 = 1$ this expression specializes to

$$
\begin{aligned}
\pm\mathrm{RHS} &= \mathrm{pf}_{\mathcal{T}}(\overline{123})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{1})\big) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{1}) - \mathrm{pf}_{\mathcal{T}}(\overline{123})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{12r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3})) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})(\mathrm{pf}_{\mathcal{T}}(\overline{12r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2})\big) \\
&= \big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_2}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{3}) \\
&= \big(\mathrm{pf}_{\mathcal{T}}(\overline{1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{1})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\,,
\end{aligned}
$$

where the third equality follows from Lemma A.6 and the last equality holds by Lemma A.9. This matches (6).

For $r_1 = 2$ a parallel computation using the same lemmas yields

$$
\mathrm{RHS} = \pm\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{2r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\,,
$$

which again matches (6).

For $r_1 = 3$ the RHS expression specializes to

$$
\begin{aligned}
\pm\mathrm{RHS} &= \mathrm{pf}_{\mathcal{T}}(\overline{3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big) \\
&= \big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{3r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\,,
\end{aligned}
$$

where the second equality holds by Lemma A.9. This matches (6).

Subcase II.d. If $3 < r_1$ holds, then (3) and Lemma B.2 yield

$$
\begin{aligned}
\mathrm{LHS} &= \pm\big(\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{1}) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{s})\,.
\end{aligned}
$$

This has to be compared to

$$\text{RHS} = \pm\det\begin{pmatrix}\mathrm{pf}_{\mathcal{T}}(\overline{23r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\\ \mathrm{pf}_{\mathcal{T}}(\overline{13r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\\ \mathrm{pf}_{\mathcal{T}}(\overline{12r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{12r_3})\\ \mathrm{pf}_{\mathcal{T}}(\overline{r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\end{pmatrix}[s_1s_24;\,123]\,.$$

This determinant is symmetric in the three possible choices of $\{s_1, s_2\} \subset \{1, 2, 3\}$. It is sufficient to treat the case $\{s_1, s_2\} = \{1, 2\}$, where one has $s = 3$ and, therefore,

$$\begin{aligned}\text{LHS} = \pm\big(&\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\bar{1})\\ &+ \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\bar{2}) - \mathrm{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\bar{3})\big)\,\mathrm{pf}_{\mathcal{T}}(\bar{3})\,.\end{aligned} \tag{7}$$

Expansion along the third row yields

$$\begin{aligned}\pm\text{RHS} &= \det\begin{pmatrix}\mathrm{pf}_{\mathcal{T}}(\overline{23r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\\ \mathrm{pf}_{\mathcal{T}}(\overline{13r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{13r_3})\\ \mathrm{pf}_{\mathcal{T}}(\overline{r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\end{pmatrix}\\
&= \mathrm{pf}_{\mathcal{T}}(\overline{r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2})\big)\\
&\quad - \mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\big)\\
&\quad + \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_2}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_1})\big)\\
&= \mathrm{pf}_{\mathcal{T}}(\overline{r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\bar{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{3r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\\
&\quad - \mathrm{pf}_{\mathcal{T}}(\overline{r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\bar{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\\
&\quad + \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\bar{3}) - \mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\big)\\
&= \big(\mathrm{pf}_{\mathcal{T}}(\overline{r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{123r_2r_3})\\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_3}) + \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_2})\big)\,\mathrm{pf}_{\mathcal{T}}(\bar{3})\\
&\quad - \big(\mathrm{pf}_{\mathcal{T}}(\overline{r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2r_3})\\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_3}) + \mathrm{pf}_{\mathcal{T}}(\overline{r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_2})\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\\
&= \big(\mathrm{pf}_{\mathcal{T}}(\bar{1})\,\mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2r_3}) - \mathrm{pf}_{\mathcal{T}}(\bar{2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2r_3})\\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\bar{3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12r_1r_2r_3})\big)\,\mathrm{pf}_{\mathcal{T}}(\bar{3}) - \mathrm{pf}_{\mathcal{T}}(\bar{3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{123})\end{aligned}$$

where the last two equalities follow from Lemmas A.6 and A.9. □

C.2 Quotients of Odd Type

The proofs of C.5–C.8 below are, if anything, slightly simpler than the proofs of C.1–C.4.

Lemma C.5 *Let* $n \geqslant 6$ *be an even number and adopt the setup from* 3.9. *The sequence* $0 \longrightarrow \mathcal{R}^{n-3} \xrightarrow{\partial_3} \mathcal{R}^n \xrightarrow{\partial_2} \mathcal{R}^4 \xrightarrow{\partial_1} \mathcal{R}$ *is a complex.*

Proof The product $\partial_1\partial_2$ is a $1 \times n$ matrix; the first three entries are evidently 0. For $i \in \{4, \ldots, n\}$ the i^{th} entry is

$$\pm\big(\mathrm{pf}_{\mathcal{T}}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i}) - \mathrm{pf}_{\mathcal{T}}(\overline{12})\,\mathrm{pf}_{\mathcal{T}}(\overline{3i}) + \mathrm{pf}_{\mathcal{T}}(\overline{13})\,\mathrm{pf}_{\mathcal{T}}(\overline{2i}) - \mathrm{pf}_{\mathcal{T}}(\overline{23})\,\mathrm{pf}_{\mathcal{T}}(\overline{1i})\big)\,,$$

which is zero by Lemma A.4 applied with $u_1 \ldots u_k = 123i$ and $u_\ell = i$.

The product $\partial_2\partial_3$ is a $4 \times (n-3)$ matrix. Let $i \in \{4, \ldots, n\}$; the entry in position $(1, i-3)$ is

$$\sum_{j=4}^{i-1}(-1)^{j-1}\tau_{ji}\,\mathrm{pf}_{\mathcal{T}}(\overline{123j}) - \sum_{j=i+1}^{n}(-1)^{j-1}\tau_{ij}\,\mathrm{pf}_{\mathcal{T}}(\overline{123j})\,.$$

Applied with $u_1 \ldots u_k = 4 \ldots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. The entry in position $(2, i-3)$ is

$$\tau_{1i}\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \tau_{2i}\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) + \sum_{j=4}^{i-1}(-1)^{j}\tau_{ji}\,\mathrm{pf}_{\mathcal{T}}(\overline{3j}) - \sum_{j=i+1}^{n}(-1)^{j}\tau_{ij}\,\mathrm{pf}_{\mathcal{T}}(\overline{3j})\,.$$

Applied with $u_1 \ldots u_k = 124 \ldots n$ and $u_\ell = i$, Lemma A.3 shows that this quantity is zero. Similarly, Lemma A.3 applied with $u_1 \ldots u_k = 134 \ldots n$ and $u_\ell = i$ shows that the entry in position $(3, i-3)$ is zero, and an application with $u_1 \ldots u_k = 2 \ldots n$ and $u_\ell = i$ shows that the entry in position $(4, i-3)$ is zero. □

Lemma C.6 *Let* $n \geqslant 6$ *be an even number and adopt the setup from* 3.9. *The Pfaffians* $\mathrm{pf}_{\mathcal{T}}(\overline{12})$, $\mathrm{pf}_{\mathcal{T}}(\overline{13})$, *and* $\mathrm{pf}_{\mathcal{T}}(\overline{23})$ *form a regular sequence in* $\mathcal{R}$.

Proof The $(n-4) \times (n-4)$ Pfaffians of the matrix $\mathcal{T}[4 \ldots n; 4 \ldots n]$ generate by [16, Corollary 2.5] an ideal of grade 3 in the subring $\mathcal{R}' = \mathbb{Z}[\tau_{ij} \mid 4 \leqslant i < j \leqslant n]$ of $\mathcal{R}$; they are the Pfaffians $\mathrm{pf}_{\mathcal{T}}(\overline{123i})$ for $4 \leqslant i \leqslant n$. Applied with $u_1 \ldots u_k = 3 \ldots n$ and $\ell = 1$, Lemma A.2 yields

$$\mathrm{pf}_{\mathcal{T}}(\overline{12}) = \sum_{i=4}^{n}(-1)^{i}\tau_{3i}\,\mathrm{pf}_{\mathcal{T}}(\overline{123i})\,.$$

Similarly, applied with $u_1 \dots u_k = 24 \dots n$ and $\ell = 2$ the same lemma yields

$$\mathrm{pf}_{\mathcal{T}}(\overline{13}) = \sum_{i=4}^{n} (-1)^i \tau_{2i}\, \mathrm{pf}_{\mathcal{T}}(\overline{123i}) .$$

Finally, with $u_1 \dots u_k = 14 \dots n$ and $\ell = 1$ one gets

$$\mathrm{pf}_{\mathcal{T}}(\overline{23}) = \sum_{i=4}^{n} (-1)^i \tau_{1i}\, \mathrm{pf}_{\mathcal{T}}(\overline{123i}) .$$

Now it follows from [13, Lemma 6] that $\mathrm{pf}_{\mathcal{T}}(\overline{12})$, $\mathrm{pf}_{\mathcal{T}}(\overline{13})$, and $\mathrm{pf}_{\mathcal{T}}(\overline{23})$ form a regular sequence in $\mathcal{R}$. □

Lemma C.7 *Let* $n \geqslant 6$ *be an even number and adopt the setup from 3.9. The ideal generated by the* $(n-3) \times (n-3)$ *minors of the matrix* ∂_3 *contains the elements*

$$(\mathrm{pf}_{\mathcal{T}})^2, \quad (\mathrm{pf}_{\mathcal{T}}(\overline{12}))^2 , \quad (\mathrm{pf}_{\mathcal{T}}(\overline{13}))^2 , \quad \textit{and} \quad (\mathrm{pf}_{\mathcal{T}}(\overline{23}))^2 .$$

Proof One has $(\mathrm{pf}_{\mathcal{T}})^2 = \det(\mathcal{T})$ and expansion of this determinant along the first three columns, see [15, 0.8.9], yields:

$$\begin{aligned}\det(\mathcal{T}) &= \sum_{1 \leqslant i<j<k \leqslant n} \pm \det(\mathcal{T}[ijk;\, 123]) \det(\mathcal{T}[\overline{ijk};\, \overline{123}]) \\ &= \sum_{1 \leqslant i<j<k \leqslant n} \pm \det(\mathcal{T}[ijk;\, 123]) \det(\partial_3[\overline{ijk};\, 1 \dots n-3]) .\end{aligned}$$

Similarly, expanding along the first column one gets

$$(\mathrm{pf}_{\mathcal{T}}(\overline{12}))^2 = \det(\mathcal{T}[3 \dots n;\, 3 \dots n]) = \sum_{i=3}^{n} \pm \mathcal{T}[i;\, 3] \det(\partial_3[\overline{12i};\, 1 \dots n-3]) ,$$

$$(\mathrm{pf}_{\mathcal{T}}(\overline{13}))^2 = \det(\mathcal{T}[24 \dots n;\, 24 \dots n]) = \sum_{\substack{2 \leqslant i \leqslant n \\ i \neq 3}} \pm \mathcal{T}[i;\, 2] \det(\partial_3[\overline{13i};\, 1 \dots n-3]) ,$$

and

$$(\mathrm{pf}_{\mathcal{T}}(\overline{23}))^2 = \det(\mathcal{T}[14 \dots n;\, 14 \dots n]) = \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq 2,3}} \pm \mathcal{T}[i;\, 1] \det(\partial_3[\overline{23i};\, 1 \dots n-3]) . \quad \square$$

Proposition C.8 *Let* $n \geqslant 6$ *be an even number and adopt the setup from* 3.9. *For integers* $1 \leqslant r_1 < r_2 < r_3 \leqslant n$ *and* $1 \leqslant s_1 < s_2 < s_3 \leqslant 4$ *one has*

$$\det(\partial_3[\overline{r_1 r_2 r_3};\, 1 \dots n-3]) \det(\partial_1[1;\, \overline{s_1 s_2 s_3}]) \;=\; \pm \det(\partial_2[s_1 s_2 s_3;\, r_1 r_2 r_3])\,.$$

Proof Notice that $\det(\partial_3[\overline{r_1 r_2 r_3};\, 1 \dots n-3]) = \det(\mathcal{T}[\overline{r_1 r_2 r_3};\, \overline{123}])$ holds and set

$$\begin{aligned} \text{LHS} &= \det(\mathcal{T}[\overline{r_1 r_2 r_3};\, \overline{123}]) \det(\partial_1[1;\, \overline{s_1 s_2 s_3}]) \quad \text{and} \\ \text{RHS} &= \det(\partial_2[s_1 s_2 s_3;\, r_1 r_2 r_3])\,. \end{aligned}$$

The goal is now to prove that $\text{LHS} = \pm\text{RHS}$ holds. Set

$$\rho \;=\; 1 \dots n \setminus r_1 r_2 r_3 \qquad \text{and} \qquad \{s\} \;=\; \overline{\{s_1, s_2, s_3\}}\,.$$

The possible values of s_1 are 1 and 2, and we treat these cases separately.

Case I. Assuming that $s_1 = 1$ holds, one has $s \in \{2, 3, 4\}$. By symmetry it suffices to treat the case $s = 4$. In this case one has

$$\det(\partial_1[1;\, \overline{s_1 s_2 s_3}]) \;=\; \mathrm{pf}_{\mathcal{T}}(\overline{23})\,. \tag{1}$$

Because the first three columns of the matrix ∂_2 are special, our argument depends on the size of the intersection $\{1, 2, 3\} \cap \{r_1, r_2, r_3\}$. We therefore consider four subcases determined by the (in)equalities

$$r_3 = 3\,, \qquad r_2 \leqslant 3 < r_3\,, \qquad r_1 \leqslant 3 < r_2\,, \qquad \text{and} \qquad r_1 < 3\,. \tag{2}$$

Subcase I.a. If $r_3 = 3$ holds, then Lemma B.3 yields $\text{LHS} = 0$, and ∂_2 has a zero row, so $\text{RHS} = 0$ holds as well.

Subcase I.b. If $r_2 \leqslant 3 < r_3$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} \;=\; \mathrm{pf}_{\mathcal{T}}(\overline{123 r_3})\, \mathrm{pf}_{\mathcal{T}}(\overline{r_1 r_2})\, \mathrm{pf}_{\mathcal{T}}(\overline{23})\,.$$

Expansion of the determinant along the first row yields

$$\begin{aligned} \pm\text{RHS} &= \det \begin{pmatrix} 0 & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{123 r_3}) \\ \delta_{1 r_1} \mathrm{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2 r_1} \mathrm{pf}_{\mathcal{T}}(\overline{23}) & -\delta_{2 r_2} \mathrm{pf}_{\mathcal{T}}(\overline{23}) & -\mathrm{pf}_{\mathcal{T}}(\overline{3 r_3}) \\ -\delta_{1 r_1} \mathrm{pf}_{\mathcal{T}}(\overline{12}) & \delta_{3 r_2} \mathrm{pf}_{\mathcal{T}}(\overline{23}) & \mathrm{pf}_{\mathcal{T}}(\overline{2 r_3}) \end{pmatrix} \\ &= \mathrm{pf}_{\mathcal{T}}(\overline{123 r_3})\, \mathrm{pf}_{\mathcal{T}}(\overline{23}) \big(\delta_{1 r_1} \big(\delta_{3 r_2} \mathrm{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2 r_2} \mathrm{pf}_{\mathcal{T}}(\overline{12})\big) - \delta_{2 r_1} \delta_{3 r_2} \mathrm{pf}_{\mathcal{T}}(\overline{23})\big) \\ &= \pm\, \mathrm{pf}_{\mathcal{T}}(\overline{123 r_3})\, \mathrm{pf}_{\mathcal{T}}(\overline{23})\, \mathrm{pf}_{\mathcal{T}}(\overline{r_1 r_2})\,. \end{aligned}$$

Subcase I.c. If $r_1 \leqslant 3 < r_2$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} = \mathrm{pf}_{\mathcal{T}}(\overline{23}) \cdot \begin{cases} \mathrm{pf}_{\mathcal{T}}(\overline{12r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 1 \\ \mathrm{pf}_{\mathcal{T}}(\overline{12r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 2 \\ \mathrm{pf}_{\mathcal{T}}(\overline{13r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) - \mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) & \text{if } r_1 = 3\,. \end{cases}$$

This has to be compared to

$$\begin{aligned} &\pm\,\text{RHS} \\ &= \det\begin{pmatrix} 0 & \mathrm{pf}_{\mathcal{T}}(\overline{123r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{123r_3}) \\ \delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) & -\mathrm{pf}_{\mathcal{T}}(\overline{3r_2}) & -\mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) + \delta_{3r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) & \mathrm{pf}_{\mathcal{T}}(\overline{2r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{2r_3}) \end{pmatrix} \\ &= -\delta_{1r_1}\big(\mathrm{pf}_{\mathcal{T}}(\overline{13})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2})\big) \\ &\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{12})\big(-\mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) + \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2})\big)\big) \\ &\quad + \delta_{2r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2})\big) \\ &\qquad\quad + \delta_{3r_1}\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\big(-\mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) + \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2})\big)\,. \end{aligned}$$

Lemma A.5 applied with $u_1 \dots u_k = 123r_2r_3$ and $\ell = 2$ and $\ell = 3$ yields

$$\begin{aligned} &-\mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) + \mathrm{pf}_{\mathcal{T}}(\overline{12r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) \\ &\qquad = \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_3}) \end{aligned}$$

and

$$\begin{aligned} &-\mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) + \mathrm{pf}_{\mathcal{T}}(\overline{13r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) \\ &\qquad = \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_3})\,. \end{aligned}$$

The first of these identities immediately yields LHS $= \pm$RHS in case $r_1 = 2$, and for $r_1 = 3$ the second identity yields the same conclusion. In case $r_1 = 1$ one applies both identities to see that LHS $= \pm$RHS holds.

Subcase I.d. If $3 < r_1$ holds, then (1) and Lemma B.3 yield

$$\begin{aligned} \text{LHS} = \big(&\mathrm{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) - \mathrm{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) \\ &+ \mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}\big)\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\,. \end{aligned}$$

Expansion of the determinant along the third row yields the second equality in the computation below. The third equality follows from three applications of Lemma A.5 with $u_1 \dots u_k = 123r_2r_3/123r_1r_3/123r_2r_2$ and $\ell = 3$. The fifth follows from Lemma A.5 applied with $u_1 \dots u_k = 23r_1r_2r_3$ and $\ell = 1$ and Lemma A.4 applied with $u_1 \dots u_k = 123r_1r_2r_3$ and $\ell = 2$.

$$
\begin{aligned}
\pm\text{RHS} &= \det\begin{pmatrix} \mathrm{pf}_{\mathcal{T}}(\overline{123r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{123r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{123r_3}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{3r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{3r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) \\ \mathrm{pf}_{\mathcal{T}}(\overline{2r_1}) & \mathrm{pf}_{\mathcal{T}}(\overline{2r_2}) & \mathrm{pf}_{\mathcal{T}}(\overline{2r_3}) \end{pmatrix} \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{2r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{2r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_3}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_1})\big) \\
&\qquad\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{2r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{123r_1})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_2}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{3r_1})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{2r_1})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{2r_2})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\big) \\
&\qquad\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{2r_3})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{13})\big(\mathrm{pf}_{\mathcal{T}}(\overline{23r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_1}) \\
&\qquad\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2}) + \mathrm{pf}_{\mathcal{T}}(\overline{23r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_3})\big) \\
&\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{23})\big(\mathrm{pf}_{\mathcal{T}}(\overline{13r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_1}) \\
&\qquad\qquad - \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_2}) + \mathrm{pf}_{\mathcal{T}}(\overline{13r_1r_2})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_3})\big) \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{13})\,\mathrm{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) - \mathrm{pf}_{\mathcal{T}}(\overline{23})\big(\mathrm{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) \\
&\qquad + \mathrm{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{23}) - \mathrm{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\,\mathrm{pf}_{\mathcal{T}}\big)\,.
\end{aligned}
$$

Up to a sign, this is LHS.

Case II. Assuming that $s_1 = 2$ holds one has $s = 1$ and, therefore,

$$\det(\partial_1[1;\overline{234}]) = \mathrm{pf}_{\mathcal{T}}\,. \tag{3}$$

As in Case I the argument is broken into subcases following the (in)equalities (2).

Subcase II.a. If $r_3 = 3$, then (3) and Lemma B.3 yield LHS $= 0$, and one has

$$
\begin{aligned}
\text{RHS} &= \det\begin{pmatrix} \mathrm{pf}_{\mathcal{T}}(\overline{13}) & -\mathrm{pf}_{\mathcal{T}}(\overline{23}) & 0 \\ -\mathrm{pf}_{\mathcal{T}}(\overline{12}) & 0 & \mathrm{pf}_{\mathcal{T}}(\overline{23}) \\ 0 & \mathrm{pf}_{\mathcal{T}}(\overline{12}) & -\mathrm{pf}_{\mathcal{T}}(\overline{13}) \end{pmatrix} \\
&= \mathrm{pf}_{\mathcal{T}}(\overline{13})\,\mathrm{pf}_{\mathcal{T}}(\overline{23})\,\mathrm{pf}_{\mathcal{T}}(\overline{12}) - \mathrm{pf}_{\mathcal{T}}(\overline{23})\,\mathrm{pf}_{\mathcal{T}}(\overline{12})\,\mathrm{pf}_{\mathcal{T}}(\overline{13}) = 0\,.
\end{aligned}
$$

Subcase II.b. If $r_2 \leqslant 3 < r_3$ hold, then (1) and Lemma B.3 yield

$$\text{LHS} = \mathrm{pf}_{\mathcal{T}}(\overline{123r_3})\,\mathrm{pf}_{\mathcal{T}}(\overline{r_1r_2})\,\mathrm{pf}_{\mathcal{T}}\,.$$

In the computation below, the last equality follows from Lemma A.4 applied with $u_1 \dots u_k = 123r_3$ and $\ell = 4$; it shows that LHS and RHS agree up to a sign.

$$\begin{aligned}
&\pm \text{RHS} \\
&= \det\begin{pmatrix} \delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{23}) & -\delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{23}) & \operatorname{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{12}) & \delta_{3r_2}\operatorname{pf}_{\mathcal{T}}(\overline{23}) & -\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) \\ \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{12}) & \delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{12}) - \delta_{3r_2}\operatorname{pf}_{\mathcal{T}}(\overline{13}) & \operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) \end{pmatrix} \\
&= \delta_{1r_1}\delta_{2r_2}\operatorname{pf}_{\mathcal{T}}(\overline{12})\big(\operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big) \\
&\quad + \delta_{1r_1}\delta_{3r_2}\operatorname{pf}_{\mathcal{T}}(\overline{13})\big(\operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big) \\
&\quad - \delta_{2r_1}\delta_{3r_2}\operatorname{pf}_{\mathcal{T}}(\overline{23})\big(\operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big) \\
&= \pm\operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2})\big(\operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big) \\
&= \operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2})\operatorname{pf}_{\mathcal{T}}\operatorname{pf}_{\mathcal{T}}(\overline{123r_3})\,.
\end{aligned}$$

Subcase II.c. If $r_1 \leqslant 3 < r_2$ hold, then (3) and Lemma B.3 yield

$$\text{LHS} = \operatorname{pf}_{\mathcal{T}} \cdot \begin{cases} \operatorname{pf}_{\mathcal{T}}(\overline{12r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{13}) - \operatorname{pf}_{\mathcal{T}}(\overline{13r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 1 \\ \operatorname{pf}_{\mathcal{T}}(\overline{12r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{23}) - \operatorname{pf}_{\mathcal{T}}(\overline{23r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{12}) & \text{if } r_1 = 2 \\ \operatorname{pf}_{\mathcal{T}}(\overline{13r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{23}) - \operatorname{pf}_{\mathcal{T}}(\overline{23r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{13}) & \text{if } r_1 = 3\,. \end{cases}$$

This has to be compared to

$$\begin{aligned}
\pm\text{RHS} &= \det\begin{pmatrix} \delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{13}) - \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{23}) & \operatorname{pf}_{\mathcal{T}}(\overline{3r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{3r_3}) \\ -\delta_{1r_1}\operatorname{pf}_{\mathcal{T}}(\overline{12}) + \delta_{3r_1}\operatorname{pf}_{\mathcal{T}}(\overline{23}) & -\operatorname{pf}_{\mathcal{T}}(\overline{2r_2}) & -\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) \\ \delta_{2r_1}\operatorname{pf}_{\mathcal{T}}(\overline{12}) - \delta_{3r_1}\operatorname{pf}_{\mathcal{T}}(\overline{13}) & \operatorname{pf}_{\mathcal{T}}(\overline{1r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) \end{pmatrix} \\
&= \delta_{1r_1}\big(\operatorname{pf}_{\mathcal{T}}(\overline{13})\big(-\operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3})\big) \\
&\qquad\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{12})\big(\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big)\big) \\
&\quad - \delta_{2r_1}\big(\operatorname{pf}_{\mathcal{T}}(\overline{23})\big(-\operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3})\big) \\
&\qquad\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{12})\big(-\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big)\big) \\
&\quad - \delta_{3r_1}\big(\operatorname{pf}_{\mathcal{T}}(\overline{23})\big(\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big) \\
&\qquad\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{12})\big(-\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\big)\big)\,.
\end{aligned}$$

For $r_1 = 1$ it follows from two applications of Lemma A.4, namely with $u_1 \dots u_k = 12r_2r_3/13r_2r_3$ and $\ell = 1$, that LHS and RHS agree up to a sign. For $r_1 = 2$ one gets the same conclusion by applying Lemma A.4 with $u_1 \dots u_k = 12r_2r_3/23r_2r_3$ and

$\ell = 1$. For $r_1 = 3$ one gets the desired conclusion from Lemma A.4 applied with $u_1 \dots u_k = 13r_2r_3/23r_2r_3$ and $\ell = 1$.

Subcase II.d. If $3 < r_1$ holds, then (3) and Lemma B.3 yield

$$\begin{aligned}\text{LHS} \;=\; &\big(\operatorname{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{23}) - \operatorname{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{13})\\ &\quad + \operatorname{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{12}) - \operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\operatorname{pf}_{\mathcal{T}}\big)\operatorname{pf}_{\mathcal{T}}\,.\end{aligned}$$

Expansion of the determinant along the first column yields the second equality in the computation below. The third equality follows from three applications of Lemma A.4. The fifth follows from two applications of Lemma A.4 with $u_1 \dots u_k = 123r_1r_2r_3/123r_1$ and $\ell = 4$. The last equality follows from Lemma A.8.

$$\begin{aligned}
&\pm\,\text{RHS}\\
&= \det\begin{pmatrix}\operatorname{pf}_{\mathcal{T}}(\overline{3r_1}) & \operatorname{pf}_{\mathcal{T}}(\overline{3r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\\ \operatorname{pf}_{\mathcal{T}}(\overline{2r_1}) & \operatorname{pf}_{\mathcal{T}}(\overline{2r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{2r_3})\\ \operatorname{pf}_{\mathcal{T}}(\overline{1r_1}) & \operatorname{pf}_{\mathcal{T}}(\overline{1r_2}) & \operatorname{pf}_{\mathcal{T}}(\overline{1r_3})\end{pmatrix}\\
&= \operatorname{pf}_{\mathcal{T}}(\overline{3r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\big)\\
&\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{2r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{1r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\operatorname{pf}_{\mathcal{T}}(\overline{1r_2})\big)\\
&\qquad\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{1r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{3r_2})\operatorname{pf}_{\mathcal{T}}(\overline{2r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{3r_3})\operatorname{pf}_{\mathcal{T}}(\overline{2r_2})\big)\\
&= \operatorname{pf}_{\mathcal{T}}(\overline{3r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{12r_2r_3})\operatorname{pf}_{\mathcal{T}} - \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{r_2r_3})\big)\\
&\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{2r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{13r_2r_3})\operatorname{pf}_{\mathcal{T}} - \operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{r_2r_3})\big)\\
&\qquad\qquad + \operatorname{pf}_{\mathcal{T}}(\overline{1r_1})\big(\operatorname{pf}_{\mathcal{T}}(\overline{23r_2r_3})\operatorname{pf}_{\mathcal{T}} - \operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{r_2r_3})\big)\\
&= \big(\operatorname{pf}_{\mathcal{T}}(\overline{1r_1})\operatorname{pf}_{\mathcal{T}}(\overline{23r_2r_3}) - \operatorname{pf}_{\mathcal{T}}(\overline{2r_1})\operatorname{pf}_{\mathcal{T}}(\overline{13r_2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{3r_1})\operatorname{pf}_{\mathcal{T}}(\overline{12r_2r_3})\big)\operatorname{pf}_{\mathcal{T}}\\
&\qquad + \big(-\operatorname{pf}_{\mathcal{T}}(\overline{1r_1})\operatorname{pf}_{\mathcal{T}}(\overline{23}) + \operatorname{pf}_{\mathcal{T}}(\overline{2r_1})\operatorname{pf}_{\mathcal{T}}(\overline{13}) - \operatorname{pf}_{\mathcal{T}}(\overline{3r_1})\operatorname{pf}_{\mathcal{T}}(\overline{12})\big)\operatorname{pf}_{\mathcal{T}}(\overline{r_2r_3})\\
&= \big(\operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\operatorname{pf}_{\mathcal{T}} - \operatorname{pf}_{\mathcal{T}}(\overline{r_1r_2})\operatorname{pf}_{\mathcal{T}}(\overline{123r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{r_1r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123r_2})\\
&\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{r_2r_3})\operatorname{pf}_{\mathcal{T}}(\overline{123r_1})\big)\operatorname{pf}_{\mathcal{T}}\\
&= \big(\operatorname{pf}_{\mathcal{T}}(\overline{123r_1r_2r_3})\operatorname{pf}_{\mathcal{T}} - \operatorname{pf}_{\mathcal{T}}(\overline{12})\operatorname{pf}_{\mathcal{T}}(\overline{3r_1r_2r_3}) + \operatorname{pf}_{\mathcal{T}}(\overline{13})\operatorname{pf}_{\mathcal{T}}(\overline{2r_1r_2r_3})\\
&\qquad - \operatorname{pf}_{\mathcal{T}}(\overline{23})\operatorname{pf}_{\mathcal{T}}(\overline{1r_1r_2r_3})\big)\operatorname{pf}_{\mathcal{T}}\,.
\end{aligned}$$

Up to a sign, this is LHS. □

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Stickelberger and the Eigenvalue Theorem

David A. Cox

To David Eisenbud on the occasion of his 75th birthday.

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1 Introduction

The Eigenvalue Theorem is a standard result in computational algebraic geometry. Given a field F and polynomials $f_1, \dots, f_s \in F[x_1, \dots, x_n]$, it is well known that the system

$$f_1 = \cdots = f_s = 0 \tag{1.1}$$

has finitely many solutions over the algebraic closure $\overline{F}$ of F if and only if

$$A = F[x_1, \dots, x_n]/\langle f_1, \dots, f_s\rangle$$

has finite dimension over F (see, for example, Theorem 6 of [7, Ch. 5, §3]).

A polynomial $f \in F[x_1, \dots, x_n]$ gives a multiplication map

$$m_f : A \longrightarrow A.$$

A basic version of the Eigenvalue Theorem goes as follows:

Theorem 1.1 (Eigenvalue Theorem) *When* $\dim_F A < \infty$, *the eigenvalues of* m_f *are the values of* f *at the finitely many solutions of* (1.1) *over* $\overline{F}$.

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For $\overline{A} = A \otimes_F \overline{F}$, we have a canonical isomorphism of F-algebras

$$\overline{A} = \prod_{a \in \mathbf{V}_{\overline{F}}(f_1,\dots,f_s)} \overline{A}_a,$$

where $\overline{A}_a$ is the localization of $\overline{A}$ at the maximal ideal corresponding to a. Following [12, Thm. 3.3], we get a more precise version of the Eigenvalue Theorem:

Theorem 1.2 (Stickelberger's Theorem) *For every $a \in \mathbf{V}_{\overline{F}}(f_1, \dots, f_s)$, we have $m_f(\overline{A}_a) \subseteq \overline{A}_a$, and the restriction of m_f to $\overline{A}_a$ has only one eigenvalue $f(a)$.*

This result easily implies Theorem 1.1 and enables us to compute the characteristic polynomial of m_f. Namely, the multiplicity of a as a solution of (1.1) is

$$\mu(a) = \dim_{\overline{F}} \overline{A}_a,$$

and then Theorem 1.2 tells us that the characteristic polynomial of m_f is

$$\det(m_f - xI) = \prod_{a \in \mathbf{V}_{\overline{F}}(f_1,\dots,f_s)} (f(a) - x)^{\mu(a)}. \tag{1.2}$$

Furthermore, since the trace of a matrix can be read off from its characteristic polynomial, (1.2) gives the formula

$$\mathrm{Tr}(m_f) = \sum_{a \in \mathbf{V}_{\overline{F}}(f_1,\dots,f_s)} \mu(a) f(a). \tag{1.3}$$

This trace formula will play an important role in what follows.

The name "Stickelberger's Theorem" in Theorem 1.2 is from [12]. Versions of Theorems 1.1 and 1.2 also named "Stickelberger's Theorem" can be found in the papers [11, 26, 32], and [23] has a "Stickelberger's Theorem" for positive-dimensional solution sets. A "Stickelberger's Theorem" that focuses on (1.2) and (1.3) can be found in [2]. A common feature of these papers is that no reference to Stickelberger is given! An exception is [11], which refers to the wrong paper of Stickelberger.

There is an actual theorem of Ludwig Stickelberger lurking in the background, in the paper *Über eine neue Eigenschaft der Diskriminanten algebraischer Zahlkörper* [28] that appeared in the proceedings of the first International Congress of Mathematicians, held in Zürich in 1897. This paper includes Theorems I–XIII, most dealing with traces and properties of the discriminant of a number field.

In [28], Stickelberger fixes a number field Ω of degree n and discriminant D. Here are two of the theorems from [28]:

Theorem 1.3 (Theorems VII and XIII of [28]) *If a prime p does not divide D, then the Legendre symbol $\left(\frac{D}{p}\right)$ satisfies*

$$\left(\frac{D}{p}\right) = (-1)^{n-m},$$

where $p\mathcal{O} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ is the prime factorization in the ring $\mathcal{O}$ of algebraic integers of Ω.

This result is well known in number theory. See, for example, [5] and [15]. But for our purposes, Stickelberger's most interesting theorem in [28] involves the trace function of $\mathcal{O}$ modulo an ideal $\mathfrak{a}$ containing a prime p. This is the map

$$\mathrm{Tr}_{\mathfrak{a}} : \mathcal{O} \longrightarrow \mathbb{F}_p$$

where multiplication by $\alpha \in \mathcal{O}$ gives a $\mathbb{F}_p$-linear map $m_\alpha : \mathcal{O}/\mathfrak{a} \to \mathcal{O}/\mathfrak{a}$ with trace

$$\mathrm{Tr}_{\mathfrak{a}}(\alpha) = \mathrm{Tr}(m_\alpha) \in \mathbb{F}_p.$$

When $\mathfrak{a} = p\mathcal{O}$, we write $\mathrm{Tr}_p(\alpha)$ instead of $\mathrm{Tr}_{p\mathcal{O}}(\alpha)$. Here is Stickelberger's theorem:

Theorem 1.4 (Theorem III of [28]) *Let p be prime with factorization $p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are distinct primes. Then for any $\alpha \in \mathcal{O}$, we have*

$$\mathrm{Tr}_p(\alpha) = \sum_{i=1}^{m} e_i \mathrm{Tr}_{\mathfrak{p}_i}(\alpha).$$

Given the similarity to the trace formula (1.3), it becomes clear why Stickelberger's paper is relevant to the Eigenvalue Theorem. The link was made explicit in 1988 when Günter Scheja and Uwe Storch published *Lehrbuch der Algebra* [24]. However, even though Scheja and Storch invoke Stickelberger's name, they do not refer to his 1897 paper [28].

In what follows, we will say more about Stickelberger and his mathematics in Sect. 2 and explore the history of the Eigenvalue Theorem in Sect. 3. Section 4 will describe how Stickelberger and the Eigenvalue Theorem came together in 1988 under the influence of Scheja and Storch, and Sect. 5 will explain the unexpected role played by real solutions. We end with some final remarks in Sect. 6

2 Ludwig Stickelberger

Ludwig Stickelberger was a Swiss mathematician born in 1850 in the canton of Schaffhausen and died in 1936 in Basel. He got his PhD from Berlin in 1874 under the direction of Ernst Kummer and Karl Weierstrass. After spending a few years at the forerunner of ETH in Zürich, Stickelberger went to the University of Freiburg in 1879. He retired in 1919 but remained in Freiburg as an "Honorarprofessor" until 1924, when he returned to Switzerland.

Stickelberger's mathematical work is described in a 1937 article [14] written by his Freiburg colleague Lothar Heffter. Stickelberger's mathematical output was modest: besides his dissertation, he published 12 papers during his lifetime, four jointly written with Frobenius. One unpublished manuscript from 1915 appeared posthumously in 1936. Heffter gives a brief description of each paper in [14].

His papers cover a range of topics, including quadratic forms, real orthogonal transformations, differential equations, algebraic geometry, group theory, elliptic functions, and algebraic number theory. Heffter comments that

> … he definitely adopted Gauss' point of view "Pauca sed matura" [few but mature]. He recognized and filled essential gaps in fundamental theories, often having the last word with the keystone of a development that gives the theory its final, simplest form.

Stickelberger's best known result, published in 1890 in *Mathematische Annalen* [29], concerns an element θ in the group ring $\mathbb{Q}[G]$, where G is the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$ of the cyclotomic extension $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_m)$. This gives the ideal

$$I = (\theta\mathbb{Z}[G]) \cap \mathbb{Z}[G] \subseteq \mathbb{Z}[G].$$

It is customary to call θ the *Stickelberger element* and I the *Stickelberger ideal*. Here is his theorem:

Theorem 2.1 (Stickelberger's Theorem [29]) *The Stickelberger ideal I annihilates the class group of $\mathbb{Q}(\zeta_m)$.*

If you search MathSciNet for reviews that mention "Stickelberger" anywhere, the vast majority involve the Stickelberger element, the Stickelberger ideal, and their generalizations. When mathematicians say "Stickelberger's Theorem", they are usually referring to Theorem 2.1. This is probably what led the authors of [11] to cite [29] as the source for their version of the Eigenvalue Theorem.

In 1897, Stickelberger published the paper [28] discussed in the Introduction. The main focus here is on properties of the discriminant D of a number field Ω. Besides proving Theorem 1.3, Stickelberger's results also imply that $D \equiv 0, 1 \bmod 4$. This standard fact appears in many textbooks on algebraic number theory (see, for example, Exercise 7 on p. 15 of [20], where the congruence is called *Stickelberger's discriminant relation*).

There are several ways to define D; the one most relevant to us uses a $\mathbb{Z}$-basis $\beta_1, \ldots, \beta_n$ of the ring $\mathcal{O}$ of algebraic integers of Ω. The trace function $\mathrm{Tr} : \Omega \to \mathbb{Q}$ maps $\mathcal{O}$ to $\mathbb{Z}$. Then the discriminant of Ω is defined to be

$$D = \det(\mathrm{Tr}(\beta_i \beta_j)) \in \mathbb{Z}.$$

Given this definition, it is not surprising that Stickelberger begins [28] with some properties of traces. He quickly gets to the trace formula given in Theorem 1.4, which we propose calling the *Stickelberger Trace Formula* to distinguish it from the more famous Stickelberger Theorem 2.1.

In Sect. 4, we will explain carefully how the Stickelberger Trace Formula relates to the Eigenvalue Theorem. But first, we need to learn more about the evolution of the Eigenvalue Theorem.

3 The Eigenvalue Theorem

A key feature of the Eigenvalue Theorem is that the quotient algebra $A = F[x_1, \ldots, x_n]/\langle f_1, \ldots, f_s\rangle$ is finite dimensional over F when $f_1 = \cdots = f_s = 0$ has finitely many solutions over $\overline{F}$. This was known by the end of the 1970s and is what allows us to use linear algebra to find solutions. But getting from here to the Eigenvalue Theorems 1.1 and 1.2 involved several independent discoveries, each with its own point of view. In what follows, I will mention some but not all of the relevant papers.

We begin in 1981 with Daniel Lazard's paper *Résolutions des systèmes d'équations algébriques* [18], which gives an algorithm to solve a zero-dimensional system. To relate his approach to ours, observe that setting $x = 0$ in (1.2) gives the formula

$$\det(m_f) = \prod_{a \in \mathbf{V}_{\overline{F}}(f_1,\ldots,f_s)} f(a)^{\mu(a)}. \tag{3.1}$$

For new variables $U_0, \ldots, U_n$, let $L = U_0 + U_1x_1 + \cdots + U_nx_n$. Given a point $a = (a_1, \ldots, a_n) \in \overline{F}^n$, applying L to a gives

$$L(a) = U_0 + U_1a_1 + \cdots + U_na_n,$$

from which we can recover a. Thus, if we could somehow set $f = L$ in (3.1), we would get

$$\det(m_L) = \prod_{a \in \mathbf{V}_{\overline{F}}(f_1,\ldots,f_s)} L(a)^{\mu(a)}, \tag{3.2}$$

which would give the solutions and their multiplicities.

In [18], Lazard describes an algorithm for computing a projective version of the right-hand side of (3.2). He replaces A with

$$A_U = F[U_0, \ldots, U_n, x_0, \ldots, x_n]/\langle F_1, \ldots, F_s\rangle,$$

where $F_i(x_0, \ldots, x_n)$ is the homogenization of $f_i(x_1, \ldots, x_n)$ and L becomes $L = U_0x_0 + \cdots + U_nx_n$. The ring A_U is graded with respect to $x_0, \ldots, x_n$, and multiplication by L between graded pieces of A_U appears explicitly in §4 of [18].

The product in (3.2) is an example of a U-resultant, and (very large) determinantal formulas for such resultants were known by the early twentieth century. Lazard's paper is important because of its efficient algorithm for computing this product. For us, the key feature of [18] is the use of a multiplication map on a quotient algebra.

The next advance came in 1988 with the paper *An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations* by Winfried Auzinger and Hans Stetter [1]. For a system of n equations in $x_1, \ldots, x_n$, their initial is goal is to compute the right-hand side of (3.1) when $f = b_0 + b_1x_1 + \cdots + b_nx_n$. Coming from a background in numerical analysis, they begin with the classical theory of resultants and describe an approach that works "in the general case (without degeneracies)".

In §5 of [1], Auzinger and Stetter construct matrices $B^{(k)}$, $k = 1, \ldots, n$, whose eigenvalues are the kth coordinates of the solutions, together with simultaneous eigenvectors. They also explain how these eigenvectors enable one to find the solutions. Eigenvalues and eigenvectors finally take center stage!

For us, §6 of [1] is the most interesting, for here, $B^{(k)}$ is interpreted as the matrix of the linear map $x_k : A \to A$ given by multiplication by x_k. Then comes a key observation: while the treatment so far assumes that there are no degeneracies, one can avoid this assumption by simply *defining* $B^{(k)}$ to be the matrix of multiplication by x_k on A. Everything still works and we finally have the Eigenvalue Theorem!

A more complete treatment of this circle of ideas appears in the *Central Theorem* (Theorem 2.27) in Stetter's 2004 book *Numerical Polynomial Algebra* [27]. You can also read about this in *Using Algebraic Geometry* [8], where §2.4 discusses the Eigenvalue Theorem and the role of eigenvectors, and §3.6 makes the link to resultants when there are no degeneracies. We should also mention the 1992 paper *Solutions of systems of algebraic equations and linear maps on residue class rings* [31] by Yokoyama, Noro and Takeshima that draws on ideas of Lazard, Auzinger and Stetter, together with papers of Kobayashi.

In the Historical and Bibliographical Notes to Chapter 2 of [27], Stetter writes

> The fundamental relation between the eigenelements of multiplication in the quotient ring and the zeros of the ideal must have been known to algebraists of the late 19th and early 20th centuries, in the language of the time.... There are quotations of a theorem of Stickelberger from the 1920s, which is equivalent to Theorem 2.27, but its relevance remained concealed.

Sorting out what was known 100 years ago is not an easy task. The only name mentioned by Stetter is our friend Stickelberger, though as we have seen, the date is 1897, not the 1920s.

It is now time to turn to Stickelberger, even though the above discussion omits some important papers from the early 1990s that are relevant to the ideas behind the Eigenvalue Theorem. We will consider this work in §5 when we study real solutions of a polynomial system.

4 Scheja and Storch 1988

In 1988, Günter Scheja and Uwe Storch published the two-volume algebra text *Lehrbuch der Algebra*. In Volume 2, §94 deals with trace forms (Spurformen) and is where Stickelberger enters the picture:

> **Beispiel 7** (Die Sätze von Stickelberger)

(see [24, p. 795]). But before giving the theorems, they observe that

> In some cases, the fine structure of the trace form of a finite free algebra can be described with the help of simple features of the algebra itself.

They begin with a “simple lemma” that goes as follows. Let A be a finite-dimensional F-algebra with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. The localizations $A_{\mathfrak{m}_i}$ have residue fields $L_i \simeq A/\mathfrak{m}_i$ and satisfy

$$A \simeq \prod_{i=1}^{r} A_{\mathfrak{m}_i}.$$

For each i, define λ_i by the equation

$$\dim_F A_{\mathfrak{m}_i} = \lambda_i [L_i : F] \tag{4.1}$$

Note also that $\alpha \in A$ gives F-linear multiplication maps $m_\alpha : A \to A$ and $m_\alpha : L_i \to L_i$.

Theorem 4.1 (Lemma 94.6 in [24]) *Assume that L_i is a separable extension of F for $1 \leq i \leq r$. Then for $\alpha \in A$, the multiplication maps m_α defined above satisfy*

$$\mathrm{Tr}_A(m_\alpha) = \sum_{i=1}^{r} \lambda_i \mathrm{Tr}_{L_i}(m_\alpha).$$

Proof Since $A \simeq A_{\mathfrak{m}_1} \times \cdots \times A_{\mathfrak{m}_r}$, we can reduce to the case where A is local with maximal ideal $\mathfrak{m}$ and residue field $L = A/\mathfrak{m}$. Then (4.1) can be written

$$\dim_F A = \lambda [L : F].$$

In the F-algebra A, the separable hull $A_{\mathrm{sep}} \subseteq A$ consists of all elements of A whose minimal polynomial over F is separable. Since $F \subseteq L$ is separable by hypothesis, the composition

$$A_{\mathrm{sep}} \hookrightarrow A \longrightarrow A/\mathfrak{m} = L$$

is an isomorphism of fields by Corollary 91.14 of [24].

Thus A is a vector space over A_{sep} of dimension λ. A basis $\{\beta_1, \dots, \beta_\lambda\}$ of A over A_{sep} gives a direct sum

$$A = A_{\mathrm{sep}}\beta_1 \oplus \cdots \oplus A_{\mathrm{sep}}\beta_\lambda.$$

To compute the trace of $m_\alpha : A \to A$ over F, first assume $\alpha \in A_{\mathrm{sep}}$. Then m_α is compatible with the direct sum decomposition, which easily implies

$$\mathrm{Tr}_A(m_\alpha) = \lambda \mathrm{Tr}_{A_{\mathrm{sep}}}(m_\alpha).$$

Via the isomorphism $A_{\mathrm{sep}} \simeq L$, this becomes

$$\mathrm{Tr}_A(m_\alpha) = \lambda \mathrm{Tr}_L(m_\alpha). \tag{4.2}$$

Now suppose $\alpha \in A$ is arbitrary. Since $A_{\mathrm{sep}} \simeq L = A/\mathfrak{m}$, there is $\alpha' \in A_{\mathrm{sep}}$ such that $\alpha = \alpha' + \beta$ with $\beta \in \mathfrak{m}$. Note that $\mathfrak{m}$ is nilpotent since A is finite-dimensional over F. Then $m_\alpha = m_{\alpha'} + m_\beta$ implies

$$\begin{aligned}\mathrm{Tr}_A(m_\alpha) &= \mathrm{Tr}_A(m_{\alpha'}) + \mathrm{Tr}_A(m_\beta)\\ &= \lambda \mathrm{Tr}_L(m_{\alpha'}) + 0\\ &= \lambda \mathrm{Tr}_L(m_{\alpha'}) + \lambda \mathrm{Tr}_L(m_\beta) = \lambda \mathrm{Tr}_L(m_\alpha),\end{aligned}$$

where the second line follows from (4.2) with α' and the fact that m_β is nilpotent, and the third line follows since m_β is the zero map on L. We conclude that (4.2) holds for all $\alpha \in A$, and the theorem follows. □

A key feature of the proof is that everything becomes clear once we understand the structure of A, i.e., the "simple features of the algebra itself".

Scheja and Storch then use Theorem 4.1 to prove various results of Stickelberger, including the Stickelberger Trace Formula given in Theorem 1.4. For this reason, it makes sense to call Theorem 4.1 the *Stickelberger Trace Formula* in honor of Stickelberger's contribution.

Here is an easy consequence of Theorem 4.1.

Corollary 4.2 *With the notation and assumptions of Theorems 1.1 and 1.2, we have*

$$\mathrm{Tr}(m_f) = \sum_{a \in \mathbf{V}_{\overline{F}}(f_1,\dots,f_s)} \mu(a) f(a).$$

Proof Maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of $\overline{A} = A \otimes_F \overline{F}$ correspond to solutions $a_1, \dots, a_r$ in $\mathbf{V}_{\overline{F}}(f_1, \dots, f_s)$. In the notation of Sect. 1, the localization $\overline{A}_{a_i}$ has residue field $L_i \simeq \overline{F}$ via the map $\overline{A}_{a_i} \to \overline{F}$ defined by $f \mapsto f(a_i)$. It follows that $\mathrm{Tr}_{L_i}(m_f) = f(a_i)$. Furthermore, for $\overline{A}_{a_i}$, the λ_i in (4.1) is multiplicity $\mu(a_i)$. Then the desired formula for $\mathrm{Tr}(m_f)$ is an immediate consequence of Theorem 4.1. □

Corollary 4.2 is the trace formula (1.3) from the Introduction. There, we deduced (1.3) from the version of "Stickelberger's Theorem" given in [12]. We now see how this follows from Theorem 4.1, which is Scheja and Strorch's version of the actual Stickelberger Trace Formula from 1897.

This is nice, but where are the eigenvalues? After all, our main concern is the relation between Stickelberger and the Eigenvalue Theorem. Fortunately, the trace formula given in Corollary 4.2 is powerful enough to determine the eigenvalues of m_f when F has characteristic zero. Here is the precise result:

Proposition 4.3 *With the notation and assumptions of Theorems 1.1 and 1.2, the following are equivalent when* $\mathrm{char}(F) = 0$:

(1) *For every* $f \in F[x_1, \ldots, x_n]$,

$$\mathrm{Tr}(m_f) = \sum_{a \in \mathbf{V}_{\overline{F}}(f_1,\ldots,f_s)} \mu(a) f(a).$$

(2) *For every* $f \in F[x_1, \ldots, x_n]$,

$$\det(m_f - xI) = \prod_{a \in \mathbf{V}_{\overline{F}}(f_1,\ldots,f_s)} (f(a) - x)^{\mu(a)}.$$

Proof We proved (2) ⇒ (1) in the discussion leading up to (1.3). As for (1) ⇒ (2), let M be the diagonal matrix whose diagonal entries are $f(a)$ repeated $\mu(a)$ times, for each $a \in \mathbf{V}_{\overline{F}}(f_1, \ldots, f_s)$. Then for any integer $\ell \geq 0$, we have

$$\mathrm{Tr}(M^\ell) = \sum_{a \in \mathbf{V}_{\overline{F}}(f_1,\ldots,f_s)} \mu(a) f(a)^\ell = \mathrm{Tr}(m_{f^\ell}) = \mathrm{Tr}((m_f)^\ell),$$

where the second equality uses (1) with f^ℓ and the third equality follows from $m_{fg} = m_f \circ m_g$. Thus M^ℓ and $(m_f)^\ell$ have the same trace for all $\ell \geq 0$.

It has been known since 1840 that in characteristic zero, the characteristic polynomial of a matrix is determined by the traces of its powers (a formula for the coefficients in terms of the traces is given in [19]).[1] Thus the previous paragraph implies that M and m_f have the same characteristic polynomial, and (2) follows immediately. □

We now have a direct path from Stickelberger to the characteristic zero version of the Eigenvalue Theorem. Our final task is to explore how Stickelberger's name

[1] If $\mathrm{char}(F) = p > 0$, then $A = \lambda I_p$ has $\mathrm{Tr}(A^\ell) = 0$ for all $\ell \geq 0$, independent of λ, while $\det(A - xI) = (\lambda - x)^p = \lambda^p - x^p$.

began to appear in the literature following the 1988 publication of Scheja and Storch's book [24].

In Theorem 1.2, we stated "Stickelberger's Theorem" from the 1999 book chapter [12] by Gonzalez-Vega, Roullier and Roy. Their Corollary 3.6 states the formulas for the trace, determinant and characteristic polynomial of m_f given in (1.3), (3.1) and (1.2) respectively. Not surprisingly, there is no reference to Stickelberger. Nor is there a reference to Scheja and Storch!

However, there is a reference to the 1995 paper [13] by Gonzalez-Vega and Trujillo, which includes the following result (reproduced verbatim):

Theorem 1 (Stickelberger Theorem) Let $\mathbb{K} \subset \mathbb{F}$ be a field extension with $\mathbb{F}$ algebraically closed, $h \in \mathbb{K}[\underline{x}]$ and J be a zero dimensional ideal in $\mathbb{K}[\underline{x}]$. If $\mathcal{V}_{\mathbb{F}}(J) = \{\Delta_1, \ldots, \Delta_s\}$ are the zeros in $\mathbb{F}^n$ of J then there exists a basis of $\mathbb{F}[\underline{x}]/J$ such that the matrix of M_h, with respect to this basis, has the following block structure:

$$\begin{pmatrix} \mathbb{H}_1 & 0 & \ldots & 0 \\ 0 & \mathbb{H}_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \mathbb{H}_s \end{pmatrix} \qquad \text{where} \qquad \mathbb{H}_i = \begin{pmatrix} h(\Delta_i) & \star & \ldots & \star \\ 0 & h(\Delta_i) & \ldots & \star \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & h(\Delta_i) \end{pmatrix}$$

The dimension of the i-th submatrix is equal to the multiplicity of Δ_i as a zero of the ideal J.

As far as I know, this is the first explicit mention of "Stickelberger's Theorem" in the literature. As usual, Stickelberger does not appear in the references to [13], and there is also no reference to Scheja and Storch. To see why, we look to Trujillo's 1997 PhD thesis [30]. She states a version of Theorem 1 and says:

> The version presented here was introduced by L. Stickelberger ([SS88]) in 1930

So we have a direct link between "Stickelberger's Theorem" and Scheja and Storch, though the date 1930 is not correct.

Trujillo also notes that this result was rediscovered independently in 1991 by Pedersen, Roy and Szpirglas (see [22]) and by Becker and Wörmann (see [4]). The references to [12] and [13] cite these authors. Hence we need to examine [22] and [4]. These papers deal with solutions over $\mathbb{R}$, which leads to our next topic.

5 Counting Real Solutions

Given a finite-dimensional F-algebra A, multiplication by $\alpha \in A$ gives a F-linear map $m_\alpha : A \to A$ as usual. In §94 of [24], Scheja and Storch define the *trace form* to be the symmetric bilinear form T_A on A defined by

$$T_A(\alpha, \beta) = \mathrm{Tr}(m_{\alpha\beta}) \in F.$$

When $F = \mathbb{R}$, the *type* of T_A is (p, q), where $p = \#$ positive eigenvalues and $q = \#$ negative eigenvalues, and the *signature* is $\sigma(T_A) = p - q$. Scheja and Storch apply the Stickelberger Trace Formula (Theorem 4.1) to determine the type of T_A:

Theorem 5.1 (Theorem 94.7 of [24]) *If A is a finite-dimensional $\mathbb{R}$-algebra, then the trace form T_A has type $(r_1 + r_2, r_2)$, where r_1 (resp. r_2) is the number maximal ideals $\mathfrak{m} \subseteq A$ with quotient $A/\mathfrak{m} \simeq \mathbb{R}$ (resp. $\mathbb{C}$).*

Proof For maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of A with quotients $L_1, \ldots, L_r$, Theorem 4.1 implies that

$$T_A = \sum_{i=1}^{r} \lambda_i T_{L_i}. \tag{5.1}$$

Using the bases $\{1\}$ of $\mathbb{R} \subseteq \mathbb{R}$ and $\{1, \sqrt{-1}\}$ of $\mathbb{R} \subseteq \mathbb{C}$, one easily computes that

$$\text{matrix of } T_{L_i} = \begin{cases} (1) & L_i = \mathbb{R} \quad \text{(happens } r_1 \text{ times)} \\ \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} & L_i = \mathbb{C} \quad \text{(happens } r_2 \text{ times).} \end{cases}$$

Since $\lambda_i > 0$ for all i, (5.1) implies that T_A is represented by a diagonal matrix with $r_1 + r_2$ positive entries, r_2 negative entries, and possibly many zero entries. The theorem follows. □

Over $\mathbb{R}$, there is a bijective correspondence between symmetric bilinear forms and quadratic forms. Thus one can speak of the type and signature of a quadratic form. In what follows, the quadratic form associated to T_A will be denoted Q_A, so

$$Q_A(\alpha) = T_A(\alpha, \alpha) = \mathrm{Tr}(m_{\alpha^2}).$$

An immediate consequence of Theorem 5.1 is the following wonderful result about real solutions of a zero-dimensional polynomial system over $\mathbb{R}$.

Corollary 5.2 *Assume $\langle f_1, \ldots, f_s \rangle \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is a zero-dimensional ideal. Set $A = \mathbb{R}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_s \rangle$ and let*

$$S = \{a \in \mathbb{R}^n \mid f_1(a) = \cdots = f_s(a) = 0\}$$

be the set of real solutions of $f_1 = \cdots = f_s = 0$. Then the quadratic form Q_A has signature

$$\sigma(Q_A) = \#S = \textit{the number of real solutions}.$$

Proof The maximal ideals $\mathfrak{m}$ of A come in two flavors: the r_1 maximal ideals with $A/\mathfrak{m} \simeq \mathbb{R}$ correspond to real solutions, hence elements of S, and the r_2 maximal ideals with $A/\mathfrak{m} \simeq \mathbb{C}$ correspond to complex-conjugate pairs of nonreal solutions. Thus

$$\#S = r_1 = (r_1 + r_2) - r_2 = \sigma(Q_A),$$

where the last equality follows since Q_A has type $(r_1 + r_2, r_2)$ by Theorem 5.1. □

There is a long history of using quadratic forms to study the number of real solutions, going back to the work of Jacobi, Hermite and Sylvester in the nineteenth century. In 1936, Krein and Naimark wrote a nice survey of these developments. An English translation of their paper was published in 1981 as [16].

Historically, real positive solutions were preferred (in one variable, negative solutions where called *false roots* by Cardan). More generally, given $h \in \mathbb{R}[x_1, \ldots, x_n]$, one can ask for solutions $a \in \mathbb{R}^n$ of $f_1 = \cdots = f_s = 0$ that satisfy $h(a) > 0$ or $h(a) < 0$. An easy adaptation of the proofs of Theorem 5.1 and Corollary 5.2 leads to the following result:

Theorem 5.3 *Assume $\langle f_1, \ldots, f_s\rangle \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is a zero-dimensional ideal. Set $A = \mathbb{R}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_s\rangle$ and let*

$$S = \{a \in \mathbb{R}^n \mid f_1(a) = \cdots = f_s(a) = 0\}$$

be the set of real solutions of $f_1 = \cdots = f_s = 0$. If $h \in \mathbb{R}[x_1, \ldots, x_n]$, then the quadratic form $Q_{A,h}$ defined by

$$Q_{A,h}(\alpha) = \mathrm{Tr}(m_{\alpha^2 h})$$

has signature

$$\sigma(Q_{A,h}) = \#\{a \in S \mid h(a) > 0\} - \#\{a \in S \mid h(a) < 0\}.$$

Proof As in the proof of Theorem 5.1, the Stickelberger Trace Formula from Theorem 4.1 easily implies

$$Q_{A,h} = \sum_{i=1}^{r} \lambda_i Q_{L_i,h}.$$

The r_1 indices with $L_i \simeq \mathbb{R}$ correspond to elements $a \in S$, and the isomorphism is given by evaluation at a. Thus we can rewrite the above sum as

$$Q_{A,h} = \sum_{a\in S} \lambda_i h(a)\, Q_{\mathbb{R}} + \sum_{L_i \simeq \mathbb{C}} \lambda_i Q_{L_i,h}.$$

The first sum is a quadratic form of signature

$$\#\{a \in S \mid h(a) > 0\} - \#\{a \in S \mid h(a) < 0\}.$$

Hence it suffices to show that $Q_{L_i,h}$ has signature zero when $L_i \simeq \mathbb{C}$. Such an isomorphism (there are two) is given by evaluation at one of the corresponding pair of complex-conjugate roots of the system. Call this root b and set $h(b) = u + iv$. We leave it as an exercise for the reader to show that for the basis $\{1, \sqrt{-1}\}$, the corresponding bilinear form is represented by the symmetric matrix

$$\begin{pmatrix} 2u & -2v \\ -2v & -2u \end{pmatrix},$$

which has eigenvalues $\pm 2|h(b)|$. Thus $Q_{L_i,h}$ has signature zero, and we are done. □

This path from Stickelberger to Corollary 5.2 and Theorem 5.3 is lovely but not what happened historically. Instead, Paul Pedersen [21] and Eberhard Becker [3] discovered these results independently in 1991, with no knowledge at the time of §94 of Scheja and Storch. In 1993, Pedersen joined forces with Marie-Françoise Roy and Aviva Szpirglas to write [22], where the authors comment that

> The structure theory for finite dimensional algebras which we shall present was first developed by Stickelberger (see [SS 88]).

While they never say “Stickelberger’s Theorem”, this is the first instance I could find of Stickelberger. Naturally, there is no reference to a paper of his, though the citation to Scheja and Storch is clear. A year later, in 1994, Becker and Thorsten Wörmann published [4], which includes [22] in its references. Thus the link to Stickelberger via Scheja and Storch was established in the literature by 1993.

6 Conclusion

We have seen how Stickelberger’s 1897 paper influenced Scheja and Storch in 1988. His name and the link to Scheja and Storch appeared in papers on real solutions starting in 1993, and in 1995, we finally see the label *Stickelberger’s Theorem* applied to the Eigenvalue Theorem, with the name becoming standard in the late 1990s. But in the process, the link to Stickelberger’s actual work got lost. The purpose of this paper is to reestablish the connection and get a better sense of Stickelberger’s contribution.

One thing to notice in the papers from the 1990s is the emphasis on *structure*. In 1993, Pedersen, Roy and Szpirglas [22] use a structure theory for finite dimensional algebras “first developed by Stickelberger”, and in 1995, Gonzalez-Vega and Trujillo [13] state a “Stickelberger Theorem” that describes the structure of multiplication matrices. This is not what Stickelberger did; rather, in [28], he proved a trace formula using the known factorization $p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$.

The emphasis on structure is really due to Scheja and Storch in their version of Stickelberger's Trace Formula in Lemma 94.6 in [24]: the "fine structure of the trace form" is a consequence of "simple features of the algebra itself". This is borne out by their proof of the lemma. However, their treatment is abstract and non-constructive, while the papers from the 1990s are interested in algorithms. In these papers, the goal is not to *describe* the structure but rather to *compute* the structure. This is a significant advance beyond what Stickelberger, Scheja and Storch did.

A striking feature of this story is wide range of mathematics involved:

- Abstract algebra: Günter Scheja and Uwe Storch.
- Algebraic number theory: Ludwig Stickelberger.
- Computer algebra: Daniel Lazard and Paul Pedersen.
- Numerical analysis: Winfried Auzinger and Hans Stetter.
- Real algebraic geometry: Eberhard Becker, Marie-Françoise Roy, Aviva Szpirglas and Thorsten Wörmann.

Of course, the names mentioned here are involved in other areas of research; what the list represents is the perspectives they brought to the story of Stickelberger and the Eigenvalue Theorem.

I draw two lessons from this diversity. First, polynomial systems have a wide interest that touches on many areas of mathematics, and second, abstract algebra provides a powerful language that enables us to understand the structure of what is going on. As an algebraic geometer, I find this to be deeply satisfying.

A final comment is that the linear maps m_f commute since $m_f \circ m_g = m_{fg}$. This point of view features in the version of the Eigenvalue Theorem, valid over an arbitrary field, that appears in [17, Theorem 2.4.3]. See also [17, Section 6.2.A].

As noted at the beginning of Sect. 3, my account of Stickelberger and the Eigenvalue Theorem omits many fine papers. I apologize for any omissions or inaccuracies on my part.

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I also have a story to tell. In the late 1970s, I learned about the Eisenbud-Levine Theorem [10], which computes the topological degree of a C^∞ map germ as the signature of a certain algebraically defined quadratic form (see [9] for a lovely exposition). The topological degree is usually defined using singular cohomology. I was working on étale cohomology at the time, and my paper [6] studied a question raised by David of whether étale cohomology can be used to define the topological degree (it can't). In reading the paper of Eisenbud and Levine, I learned about a splendid 1975 paper of Scheja and Storch [25] on Spurfunktionen (trace functions). In their algebra book [24], written thirteen years later, §94 is entitled *Die Spurformen* (Trace Forms). This is where they make the connection between Stickelberger and the ideas behind the Eigenvalue Theorem. So dedicating this paper to David is wonderfully appropriate.

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Multiplicities and Mixed Multiplicities of Filtrations

Steven Dale Cutkosky and Hema Srinivasan

In this article we survey recent progress in the theory of multiplicities and mixed multiplicities of filtrations. This theory is introduced and developed in the papers [8–11] and [12]. In this article, we discuss results, overviews of proofs and context of these papers.

Multiplicities and mixed multiplicities exist for filtrations of m-primary ideals, and many of the foundational results of the theory of multiplicities and mixed multiplicities of m-primary ideals in a local ring are true for filtrations of m-primary ideals; especially, inequalities generalize to filtrations. The theorems of Rees [21] and Teissier [27], Rees and Sharp [23] and Katz [14] characterizing equality of multiplicity and of mixed multiplicities do not extend to arbitrary filtrations, but they are true for divisorial and bounded filtrations.

Mixed multiplicities of filtrations of m-primary ideals and their basic properties are derived by Cutkosky, Sarkar and Srinivasan in the paper [11] and overviewed in Sect. 5 of this article. The characterization of equality of multiplicity for divisorial and bounded filtrations of m-primary ideals is given in [8] and [9] and discussed in Sects. 3 and 4 of this article. The Minkowski inequalities for filtrations of m-primary ideals are proven in [11] and presented in Sect. 6 of this paper. The characterization of equality in the Minkowski inequality for divisorial and bounded filtrations of m-primary ideals is given in [9] and discussed in Sects. 6 and 7 of this article.

All local rings will be Noetherian. The maximal ideal of a local ring R will be denoted by m_R. The length of an R-module N will be written as $\ell(N) = \ell_R(N)$. The round up $\lceil x \rceil$ of a real number x is the smallest integer n such that $x \leq n$.

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1 Filtrations

Let R be a local ring. An m_R-filtration is a family of ideals $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$,

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

such that I_n is m_R-primary for $n > 0$ and $I_i I_j \subset I_{i+j}$ for all i, j.

The filtration $\mathcal{I}$ is said to be Noetherian if the ring $\oplus_{n\geq 0} I_n$ is a finitely generated R-algebra.

We give a few important examples of m_R-filtrations.

Example 1.1 Let I be an m_R-primary ideal and $\mathcal{I} = \{I^n\}$. This is the m_R-filtration of the powers of a fixed m_R-primary ideal.

Example 1.2 Let $R \subset S$ be local rings such that S dominates R ($m_S \cap R = m_R$) and $\mathcal{I} = \{m_S^n \cap R\}$.

Example 1.3 Let R be a local domain, μ a valuation with value group $\mathbb{Z}$ and valuation ring $\mathcal{O}_\mu$ with maximal ideal m_μ such that $R \subset \mathcal{O}_\mu$ and μ dominates R ($m_\mu \cap R = m_R$). We define valuation ideals

$$I(\mu)_n = \{f \in R \mid \mu(f) \geq n\},$$

and an associated m_R-filtration $\mathcal{I}(\mu) = \{I(\mu)_n\}$.

Example 1.4 Let R be an excellent local domain and $\varphi : X \to \mathrm{Spec}(R)$ be the normalization of the blowup of an m_R-primary ideal with prime exceptional divisors $E_1, \ldots, E_r$. Let ν_{E_i} be the valuation with valuation ring $\mathcal{O}_{X,E_i}$. The valuations of this type are the m_R-valuations. They are the Rees valuations of the ideal blown up ([22, Statement (G)]).

For $a_1, \ldots, a_r \in \mathbb{N}$ and $D = a_1E_1 + \cdots + a_rE_r$ (a Weil divisor), let

$$\begin{aligned} I(mD) &= \Gamma(X, \mathcal{O}_X(-ma_1E_1 - \cdots - ma_rE_r)) \cap R \\ &= I(\nu_{E_1})_{ma_1} \cap \cdots \cap I(\nu_{E_r})_{ma_r}. \end{aligned} \tag{1}$$

The m_R-filtration $\mathcal{I}(D) = \{I(mD)\}$ is called a divisorial m_R-filtration.

The last line of Eq. (1) gives a definition of the filtration $\mathcal{I}(D)$ which is independent of X.

A choice of a pair $\varphi : X \to \mathrm{Spec}(R)$ and Weil divisor $D = a_1E_1 + \cdots + a_rE_r$ on X giving the filtration of (1) will be called a representation of the divisorial m_R-filtration $\{I(\nu_{E_1})_{ma_1} \cap \cdots \cap I(\nu_{E_r})_{ma_r}\}$. We will abuse notation and call this a representation of $\mathcal{I}(D)$.

Example 1.1 is always Noetherian. However, Examples 1.2, 1.3 and 1.4 are often not Noetherian, even in regular local rings. An example is given in [3].

In a two-dimensional normal local ring R, the condition that the filtration of valuation ideals of R is Noetherian for all m_R-valuations dominating R is the condition (N) of Muhly and Sakuma [18]. It is proven in [4] that a complete normal local ring of dimension two satisfies condition (N) if and only if its divisor class group is a torsion group.

Let R be a local ring and $\mathcal{I} = \{I_n\}$ be an m_R-filtration. The integral closure $\overline{\sum_{n\geq 0} I_n t^n}$ of the R-algebra $\sum_{n\geq 0} I_n t^n$ in the polynomial ring $R[t]$ is the ring

$$\overline{\sum_{n\geq 0} I_n t^n} = \sum_{n\geq 0} J_n t^n$$

where $\{J_n\}$ is the m_R-filtration

$$J_n = \{f \in R \mid f^r \in \overline{I_{nr}} \text{ for some } r > 0\}.$$

Here $\overline{I}$ denotes the integral closure of the ideal I. This is proven in [10, Lemma 5.5]. In the classical case that $\mathcal{I} = \{I^n\}$ is the filtration of the powers of a fixed m_R-primary ideal, we have that $J_n = \overline{I^n}$ for all n.

If $\mathcal{I}(D)$ is a divisorial m_R-filtration, then ([10, Lemma 5.7]) $\sum_{n\geq 0} I(nD)t^n$ is integrally closed; that is,

$$\overline{\sum_{n\geq 0} I(nD)t^n} = \sum_{n\geq 0} I(nD)t^n.$$

Definition 1.5 An m_R-filtration $\mathcal{I} = \{I_n\}$ is said to be bounded if there exists a divisorial m_R-filtration $\mathcal{I}(D)$ on R such that we have equality of R-algebras

$$\overline{\sum_{n\geq 0} I_n t^n} = \sum_{n\geq 0} I(nD)t^n.$$

The category of bounded m_R-filtrations contains the classical m_R-filtrations $\mathcal{I} = \{I^n\}$ of the powers of a fixed m_R-primary ideals ([10, Remark 5.6]).

2 Multiplicity of m_R-Primary Ideals and of m_R-Filtrations

Let R be a local ring of dimension d and I be an m_R-primary ideal. For all $m \gg 0$,

$$\ell(R/I^m) = \frac{e(I)}{d!}m^d + \text{ lower order terms}$$

is a polynomial of degree m with rational coefficients, which is called the Hilbert Samuel polynomial. The multiplicity of I is defined to be $e(I)$, which is always a positive integer. A good exposition of multiplicity is given in [24, Chapter 11]. We see that the multiplicity is the limit

$$e(I) = \lim_{m\to\infty} \frac{\ell(R/I^m)}{m^d/d!}.$$

This last formulation allows us to define multiplicities of filtrations.

Definition 2.1 Suppose that $\mathcal{I} = \{I_n\}$ is an m_R-filtration on a local ring R. Then the multiplicity of $\mathcal{I}$ is

$$e(\mathcal{I}) = \limsup_{m\to\infty} \frac{\ell(R/I_m)}{m^d/d!}.$$

Since I_1 is an m_R-primary ideal, there exists $c > 0$ such that $m_R^{cn} \subset I_1$ so that $m_R^{cn} \subset I_n$ for all n, thus $\ell(R/I_n) \le \ell(R/m_R^{cn})$ for all n and thus the sequence $\frac{\ell(R/I_n)}{n^d}$ is bounded, and so the limsup of the sequence exists.

Rings for which all multiplicities exist as a limit have a simple characterization.

Theorem 2.2 ([5, Theorem 1.1] and [6, Theorem 4.2]) *Suppose that R is a local ring of dimension d, and $N(\hat{R})$ is the nilradical of the m_R-adic completion $\hat{R}$ of R. Then the limit*

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} \tag{1}$$

exists for any m_R-filtration $\mathcal{I} = \{I_n\}$ if and only if $\dim N(\hat{R}) < d$.

The problem of existence of such limits (1) has been considered by Ein, Lazarsfeld and Smith [13] and Mustaţă [19]. When the ring R is a domain and is essentially of finite type over an algebraically closed field k with $R/m_R = k$, Lazarsfeld and Mustaţă [17] showed that the limit exists for all m_R-filtrations. Cutkosky [6] proved it in the complete generality stated above in Theorem 2.2. Lazarsfeld and Mustaţă use in [17] the method of counting asymptotic vector space dimensions of graded families using "Okounkov bodies". This method, which is reminiscent of the geometric methods used by Minkowski in number theory, was developed by Okounkov [20], Kaveh and Khovanskii [15] and Lazarsfeld and Mustaţă [17]. We also use this wonderful method. The fact that $\dim N(R) = d$ implies there exists a filtration without a limit was observed by Dao and Smirnov.

As can be seen from this theorem, one must impose the condition that the dimension of the nilradical of the completion $\hat{R}$ of R is less than the dimension of R to ensure the existence of limits. The nilradical $N(R)$ of a d-dimensional local ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

We have that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced. In particular, the condition $\dim N(\hat{R}) < d$ holds if R is analytically unramified; that is, $\hat{R}$ is reduced. Thus $\dim N(\hat{R}) < d$ if R is excellent and reduced.

Example 2.3 The multiplicity of an m_R-filtration can be any nonnegative real number. Here is a very simple example showing that any positive real number can be realized as a limit. Let k be a field and $R = k[[x]]$ be a power series ring. Given $\lambda \in \mathbb{R}_{>0}$ let $I_n = (x^{\lceil n\lambda \rceil})$ and $\mathcal{I} = \{I_n\}$ where $\lceil \lambda \rceil$ is the round up of λ. Then $e(\mathcal{I}) = \lambda$.

Examples of m_R-filtrations $\mathcal{I}(\mu)$ where μ is an m_R-valuation on a normal local domain such that the multiplicity $e(\mathcal{I}(\mu))$ is irrational are given in [11] and [8].

3 Rees's Theorem

The following theorem by Rees [21] characterizes when the Rees algebras of two m_R-primary ideals have the some integral closure. An exposition is given in [24, Theorem 11.3.1].

Theorem 3.1 (Rees [21]) *Suppose that R is a formally equidimensional local ring and $I' \subset I$ are m_R-primary ideals. Then the following are equivalent*

1) $e(I') = e(I)$
2) *There is equality of integral closures* $\overline{\sum_{n\geq 0}(I')^n t^n} = \overline{\sum_{n\geq 0} I^n t^n}$
3) $\overline{I'} = \overline{I}$.

The statements 2) and 3) are equivalent on any local ring R and 2) implies 1) is true on any local ring R. The original statement of Rees is that 1) is equivalent to 3). We have added the equivalent condition 2) since this condition is the generalization of 3) to filtrations.

This theorem suggests the following question for filtrations.

Question 3.2 Suppose that $\mathcal{I}' \subset \mathcal{I}$ are m_R-filtrations. Are the conditions

1) $e(\mathcal{I}') = e(\mathcal{I})$
2) There is equality of integral closures $\overline{\sum_{n\geq 0} I'_n t^n} = \overline{\sum_{n\geq 0} I_n t^n}$

equivalent?

Writing $\mathcal{I}' = \{I'_n\}$ and $\mathcal{I} = \{I_n\}$, the condition $\mathcal{I}' \subset \mathcal{I}$ means that $I'_n \subset I_n$ for all n.

It is shown in [11, Theorem 6.9] and [8, Appendix] that 2) $\Rightarrow$ 1) is true for arbitrary m_R-filtrations (if $\dim N(\hat{R}) < d$). However 1) $\Rightarrow$ 2) is false for arbitrary m_R-filtrations, as shown in the following simple example from [11], which appears after the statement of [11, Theorem 6.9].

Example 3.3 Let R be a regular local ring and $\mathcal{I}' = \{m_R^{n+1}\} \subset \mathcal{I} = \{m_R^n\}$. Then $e(\mathcal{I}') = e(\mathcal{I})$ but $\overline{\sum_{n\geq 0} I'_n t^n} \neq \overline{\sum_{n\geq 0} I_n t^n}$.

We prove in [10] that 3.2 does have a positive answer for bounded m_R-filtrations (Definition 1.5).

Theorem 3.4 *(Rees's theorem for bounded filtrations)([10, Theorem 13.1] and [10, Theorem 14.4]) Suppose that R is an analytically irreducible or excellent local domain and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded m_R-filtrations such that $\mathcal{I}(1) \subset \mathcal{I}(2)$. Then the following are equivalent*

1) $e(\mathcal{I}') = e(\mathcal{I})$
2) *There is equality of integral closures* $\overline{\sum_{n\geq 0} I'_n t^n} = \overline{\sum_{n\geq 0} I_n t^n}$.

4 Outline of the Proof of Rees's Theorem for Filtrations

In this section, we suppose that R is a d-dimensional normal excellent local domain.

4.1 Multiplicities of Filtrations

We summarize [10, Section 6] and [10, Section 7]. We use the method of counting asymptotic vector space dimensions of graded families by computing volumes of convex bodies associated to appropriate semigroups introduced in [17, 20] and [15]. Let ν be a valuation of the quotient field K of R which dominates R and has value group isomorphic to $\mathbb{Z}^d$. Further suppose that $\nu(f) \in \mathbb{N}^d$ for $0 \neq f \in R$. Then we can associate to an m_R-filtration $\mathcal{I} = \{I_n\}$ a semigroup $\Gamma(\mathcal{I}) \subset \mathbb{N}^{d+1}$ defined by $\Gamma(\mathcal{I}) = \{(\nu(f), n) \mid f \in I_n\}$. Let $\Delta(\mathcal{I})$ be the intersection of the closure of the real cone generated by $\Gamma(\mathcal{I})$ with $\mathbb{R}^d \times \{1\}$. Similarly, we define $\Delta(R)$ to be the subset of $\mathbb{R}^d$ constructed from $\Gamma(R)$ by replacing I_n with R for all n.

For $c \in \mathbb{R}_{>0}$, let

$$H_c^- = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d \leq c\}.$$

Suppose that μ is an m_R-filtration. In [10, Section 6] and [10, Section 7] we construct a valuation ν as above such that $\nu = (\mu, -)$ (ν is composite with μ) and so that there is a constant $c > 0$ such that

$$\Delta(\mathcal{I}) \setminus (\Delta(\mathcal{I}) \cap H_c^-) = \Delta(R) \setminus (\Delta(R) \cap H_c^-). \tag{1}$$

Then $\Delta(\mathcal{I}) \cap H_c^-$ and $\Delta(R) \cap H_c^-$ are compact convex sets and by equation (34) of [10],

$$\frac{e_R(\mathcal{I})}{d!} = \delta[\mathrm{Vol}(\Delta(R) \cap H_c^-) - \mathrm{Vol}(\Delta(\mathcal{I}) \cap H_c^-)] \tag{2}$$

where $\delta = [\mathcal{O}_\nu/m_\nu : R/m_R]$.

4.2 The Integral Closure of a Filtration $\mathcal{I}$ and the Convex Sets $\Delta(\mathcal{I})$

Suppose that $\mathcal{I}' \subset \mathcal{I}$ are m_R-filtrations. Then we have $\Delta(\mathcal{I}') \subset \Delta(\mathcal{I})$, so we have $e_R(\mathcal{I}) = e_R(\mathcal{I}')$ if and only if $\Delta(\mathcal{I}') = \Delta(\mathcal{I})$.

If $\mathcal{I}'$ is a Noetherian m_R-filtration, and $\mathcal{I}$ is an m_R-filtration such that $\mathcal{I}' \subset \mathcal{I}$, then we have that $e_R(\mathcal{I}') = e_R(\mathcal{I})$ if and only if $\Delta(\mathcal{I}') = \Delta(\mathcal{I})$ which holds if and only if $\overline{\sum_{m\geq 0} I_m t^m} = \overline{\sum_{m\geq 0} I'_m t^m}$. This is proven as follows. By taking suitable Veronese subalgebras, we reduce to the case where $\mathcal{I}$ and $\mathcal{I}'$ are the filtrations of powers of fixed m_R-primary ideals I and I', so that this follows from Rees's Theorem [21] (Theorem 3.1) for normal excellent local domains.

For arbitrary m_R-filtrations $\mathcal{I}' \subset \mathcal{I}$ such that $\overline{\sum I_m t^m} = \overline{\sum_{m\geq 0} I'_m t^m}$ we have that $e_R(\mathcal{I}') = e_R(\mathcal{I})$ (and $\Delta(\mathcal{I}') = \Delta(\mathcal{I})$), as shown in [11, Theorem 6.9] and [8, Appendix]. However, as we showed in Example 3.3, there exists a non-Noetherian m_R-filtration $\mathcal{I}'$ and a Noetherian m_R-filtration $\mathcal{I}$ such that $\mathcal{I}' \subset \mathcal{I}$, $e_R(\mathcal{I}') = e_R(\mathcal{I})$, so that $\Delta(\mathcal{I}') = \Delta(\mathcal{I})$, but $\sum_{m\geq 0} I_m t^m$ is not a subset of $\overline{\sum_{m\geq 0} I'_m t^m}$.

4.3 The Invariant $\gamma_\mu(\mathcal{I})$

The invariant $\gamma_\mu(\mathcal{I})$ is introduced in [8] and [10, Subsection 5.1]. Suppose that $\mathcal{I} = \{I_n\}$ is an m_R-filtration and μ is an m_R-valuation. Define

$$\tau_{\mu,m}(\mathcal{I}) = \mu(I_m) = \min\{\mu(f) \mid f \in I_m\} \in \mathbb{Z}_{>0},$$

and define

$$\gamma_\mu(\mathcal{I}) = \inf_m \frac{\tau_{\mu,m}(\mathcal{I})}{m}.$$

Let $\mathcal{I}(D)$ be a divisorial m_R-filtration on R. Let $\varphi : X \to \mathrm{Spec}(R)$ be the blow up of an m_R-primary ideal with prime exceptional divisors $E_1, \ldots, E_r$ such that X is normal with a representation $D = a_1E_1 + \cdots + a_rE_r$ with $a_1, \ldots, a_r \in \mathbb{N}$. Let ν_{E_i} be the m_R-valuation with valuation ring $\mathcal{O}_{X,E_i}$. Let $\gamma_{E_i}(D) = \gamma_{\nu_{E_i}}(\mathcal{I}(D))$. We have that $\gamma_{E_i}(D) > 0$ for all i, but γ_{E_i} can be an irrational number ([8, Theorem 15.2]). We have the important inequality

$$\gamma_{E_i}(D) \geq a_i \tag{3}$$

for all i. Let $\lceil x \rceil$ be the round up of a real number x (the greatest integer in x). If $b_1E_1 + \ldots + b_rE_r$ with $b_1, \ldots, b_r \in \mathbb{R}$ is a real divisor, then define the integral divisor

$$\lceil b_1E_1 + \cdots + b_rE_r \rceil = \lceil b_1 \rceil E_1 + \cdots + \lceil b_r \rceil E_r.$$

If F is an (integral) Weil divisor on X then $\mathcal{O}_X(F)$ denotes the associated reflexive rank 1 sheaf on X (c.f. [7, Section 3.2]).

For all $m \in \mathbb{N}$ we have that

$$\begin{aligned}\Gamma(X, \mathcal{O}_X(-\lceil m\gamma_{E_1}(D)E_1 + \cdots + m\gamma_{E_r}(D)E_r \rceil) \\ = \Gamma(X, \mathcal{O}_X(-ma_1E_1 - \cdots - ma_rE_r)) = I(mD).\end{aligned} \tag{4}$$

Here ma_i is the prescribed order of vanishing of elements of $I(mD)$ along E_i while $m\gamma_{E_i}(D)$ is asymptotically the actual order of vanishing along E_i.

We have the following fundamental statement.

$$\text{If } \mathcal{I}' \subset \mathcal{I} \text{ and } e_R(\mathcal{I}') = e_R(\mathcal{I}) \text{ then } \gamma_\mu(\mathcal{I}') = \gamma_\mu(\mathcal{I}) \text{ for all } m_R\text{-valuations } \mu. \tag{5}$$

This is proven in [10, Theorem 7.3] by taking the valuation ν used to compute Δ to be composite with μ, so $\nu(f) = (\mu(f), -) \in \mathbb{N}^d$ for $f \in R$. The condition $e_R(\mathcal{I}') = e_R(\mathcal{I})$ implies $\Delta(\mathcal{I}') = \Delta(\mathcal{I})$ and $\gamma_\mu(\mathcal{I}')$, $\gamma_\mu(\mathcal{I})$ are the smallest points of the projections of $\Delta(\mathcal{I}')$, respectively $\Delta(\mathcal{I})$ onto the first coordinate of $\mathbb{R}^d$.

4.4 Rees's Theorem for Divisorial m_R-Filtrations

We now indicate the proof in [10, Section 7] that if $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$ are divisorial m_R-filtrations such that $e(\mathcal{I}(D_2)) = e(\mathcal{I}(D_1))$, then $\mathcal{I}(D_2) = \mathcal{I}(D_1)$.

Let $X \to \text{Spec}(R)$ be a representation of D_1 and D_2, and write $D_1 = \sum a_iE_i$ and $D_2 = \sum b_iE_i$ as Weil divisors on X.

By Theorem (5), $\gamma_{E_i}(D_1) = \gamma_{E_i}(D_2)$ for $1 \leq i \leq r$. Thus $I(mD_1) = I(mD_2)$ for all $m \in \mathbb{N}$ by (4).

5 Mixed Multiplicities of m_R-Primary Ideals and of m_R-Filtrations

Mixed multiplicities of m_R-primary ideals where defined by Battacharya [1], Rees [21], Risler and Teissier [25]. A good exposition of this subject is given in [24, Chapter 17]. Let R be a d-dimensional local ring and let $I_1, \ldots, I_r$ be m_R-primary ideals in a local ring R. Then for all $n_1, \ldots, n_r \in \mathbb{N}$ with $n_1 + \cdots + n_r \gg 0$, $\ell(R/I_1^{n_1} \cdots I_r^{n_r})$ is a polynomial of total degree d with rational coefficients. The mixed multiplicities $e(I_1^{[d_1]}, \ldots, I_r^{[d_r]})$ of the ideals $I_1, \ldots, I_r$ are defined by the writing

$$\ell(R/I_1^{n_1} \cdots I_r^{n_r}) = H(n_1, \ldots, n_r) + \text{ terms of lower total degree}$$

for $n_1, \ldots, n_r \gg 0$ where $H(n_1, \ldots, n_r)$ is a homogeneous polynomial of total degree d written as

$$H(n_1, \ldots, n_r) = \sum_{d_1+\cdots+d_r=d} \frac{e(I_1^{[d_1]}, \ldots, I_r^{[d_r]})}{d_1! \cdots d_r!} n_1^{d_1} \cdots n_r^{d_r}.$$

The mixed multiplicities $e(I_1^{[d_1]}, \ldots, I_r^{[d_r]})$ are always positive integers.

We see that the polynomial $H(n_1, \ldots, n_r)$ is equal to the function

$$\lim_{m\to\infty} \frac{\ell(R/I_1^{mn_1} \cdots I_r^{mn_r})}{m^d} = H(n_1, \ldots, n_r)$$

for all $n_1, \ldots, n_r \in \mathbb{N}$, so that this limit computes the mixed multiplicities. This formula is generalized to arbitrary m_R-filtrations in our paper [11].

Theorem 5.1 ([11, Theorem 6.6]) *Suppose that R is a d-dimensional local ring such that $\dim N(\hat{R}) < d$ and $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are m_R-filtrations. Then there exists a homogeneous polynomial $H(x_1, \ldots, x_r) \in \mathbb{R}[x_1, \ldots, x_r]$ of degree d such that for all $n_1, \ldots, n_r \in \mathbb{N}$,*

$$\lim_{m\to\infty} \frac{\ell(R/I(1)_{mn_1} \cdots I(r)_{nm_r})}{m^d} = H(n_1, \ldots, n_r).$$

Now define the mixed multiplicities $e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]})$ of the m_R-filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ from the coefficients of the polynomial $H(n_1, \ldots, n_r)$ of Theorem 5.1 by the writing

$$H(n_1, \ldots, n_r) = \sum_{d_1+\cdots+d_r=d} \frac{e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]})}{d_1! \cdots d_r!} n_1^{d_1} \cdots n_r^{d_r}.$$

The mixed multiplicities $e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]})$ of m_R-filtrations are always nonnegative real numbers ([12, Proposition 1.3]).

Theorem 5.1 extends to define mixed multiplicities of finitely generated modules with respect to m_R-filtrations. With the assumptions of Theorem 5.1, [11, Theorem 6.6] shows that if M is a finitely generated R-module, then the function $\lim_{m\to\infty} \frac{\lambda(M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)}{m^d}$ is a homogeneous polynomial of degree d for $n_1, \ldots, n_r \in \mathbb{N}$. We define the mixed multiplicities of M with respect to the m_R-filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ by the expansion

$$\begin{aligned} P(n_1, \ldots, n_r) &:= \lim_{m\to\infty} \frac{\ell(M/I(1)_{mn_1}\cdots I(r)_{mn_r}M)}{m^d} \\ &= \sum_{d_1+\cdots+d_r=d} \frac{1}{d_1!\cdots d_r!} e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) n_1^{d_1} \cdots n_r^{d_r}. \end{aligned} \tag{1}$$

The following Associativity Formula for m_R-filtrations generalizes the classical theorem for m_R-primary ideals (c.f. [24, Theorem 17.4.8]).

Theorem 5.2 (Associativity Formula [11, Theorem 6.8]) *Suppose that R is a Noetherian local ring of dimension d with* $\dim N(\hat{R}) < d$. *Suppose that M is a finitely generated R-module and $\mathcal{I}(1) = \{I(1)_i\}, \ldots, \mathcal{I}(r) = \{I(r)_i\}$ are m_R-filtrations. Let P be a minimal prime of R. Then* $\dim N(\widehat{R/P}) < d$. *For any $d_1, \ldots, d_r \in \mathbb{N}$ with $d_1 + \cdots + d_r = d$,*

$$\begin{aligned} e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) = \sum \ell_{R_P}(M_P) e_{R/P}((\mathcal{I}(1)R/P)^{[d_1]}, \ldots, \\ (\mathcal{I}(r)R/P)^{[d_r]}; R/P) \end{aligned}$$

where the sum is over the minimal primes of R such that $\dim R/P = d$ *and $\mathcal{I}(j)R/P = \{I(j)_i R/P\}$.*

The first step in the construction of mixed multiplicities for m_R-filtrations is to construct them for Noetherian m_R-filtrations. In this case the associated multigraded Hilbert function is a quasi polynomial, whose highest degree terms are constant, rational numbers, as we show in [11, Proposition 3.5]. We next restrict in [11, Section 4] to the case that R is analytically irreducible. Using methods of volumes of Newton-Okounkov bodies adapted to our situation, we show in [11, Proposition 4.3] and [11, Corollary 4.4] that the coefficients of the polynomials $P_a(n_1, \ldots, n_r)$ of (1) for successive Noetherian approximations $\mathcal{I}_a(1), \ldots, \mathcal{I}_a(r)$ of $\mathcal{I}(1), \ldots, \mathcal{I}(r)$, all have a limit as $a \to \infty$. We then define $G(x_1, \ldots, x_n)$ to be the real polynomial with these limit coefficients, and show in [11, Theorem 4.5] that for $n_1, \ldots, n_r \in \mathbb{Z}_+$, $G(n_1, \ldots, n_r)$ is the function $P(n_1, \ldots, n_r)$ of (1) for the filtrations $\mathcal{I}(1), \ldots, \mathcal{I}(r)$. In [11, Section 5], we reduce Theorem 5.1 to the case where R is analytically irreducible so that Theorem 5.1 follows.

6 The Minkowski Inequalities

The Minkowski inequalities for mixed multiplicities of m_R-primary ideals in local rings were proven by Teissier [25], [26] and Rees and Sharp [23]. An exposition is in [24, Section 17.7]. We generalize these inequalities to show that they are also true for m_R-filtrations in our paper [11].

Theorem 6.1 (Minkowski Inequalities for Filtrations [11, Theorem 6.3]) *Suppose that R is a d-dimensional local ring with* $\dim N(\hat{R}) < d$, *and $\mathcal{I}(1) = \{I(1)_j\}$ and $\mathcal{I}(2) = \{I(2)_j\}$ are m_R-filtrations. Let $e_i = e_R(\mathcal{I}(1)^{[d-i]}, \mathcal{I}(2)^{[i]})$ for $0 \le i \le d$. Then*

1) $e_i^2 \le e_{i-1}e_{i+1}$ *for* $1 \le i \le d-1$
2) $e_i e_{d-i} \le e_0 e_d$ *for* $0 \le i \le d$
3) $e_i^d \le e_0^{d-i} e_d^i$ *for* $0 \le i \le d$
4) $e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} \le e_0^{\frac{1}{d}} + e_d^{\frac{1}{d}}$, *where* $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_j I(2)_j\}$.

We write out the last inequality without abbreviation as

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} \le e_R(\mathcal{I}(1))^{\frac{1}{d}} + e_R(\mathcal{I}(2))^{\frac{1}{d}} \tag{1}$$

where $\mathcal{I}(1)\mathcal{I}(2) = \{I(1)_m I(2)_m\}$. This equation is often called "The Minkowski Inequality".

The inequality (1) was proven by Mustaţă [19] for regular local rings with algebraically closed residue field.

There is a beautiful characterization of when equality holds in the Minkowski inequality (1) by Teissier [27] (for Cohen-Macaulay normal two-dimensional complex analytic R), Rees and Sharp [23] (in dimension 2) and Katz [14] (in complete generality).

Theorem 6.2 (Teissier [27], Rees and Sharp [23], Katz [14]) *Suppose that R is a d-dimensional formally equidimensional local ring and $I(1)$, $I(2)$ are m_R-primary ideals. Then the following are equivalent*

1) *The Minkowski equality*

$$e_R(I(1)I(2))^{\frac{1}{d}} = e(I(1))^{\frac{1}{d}} + e(I(2))^{\frac{1}{d}}$$

holds

2) *There exist positive integers a and b such that*

$$\overline{\sum_{n\ge 0} I(1)^{an}t^n} = \overline{\sum_{n\ge 0} I(2)^{bn}t^n}$$

3) *There exist positive integers a and b such that* $\overline{I(1)^a} = \overline{I(2)^b}$.

The conditions 2) and 3) are equivalent on any local ring and 2) implies 1) is true on any local ring. The original statement of Teissier, Rees and Sharp and Katz is that 1) is equivalent to 3).

The Teissier, Rees and Sharp, Katz Theorem leads to the question of whether the following conditions are equivalent for m_R-filtrations $\mathcal{I}(1)$ and $\mathcal{I}(2)$.

Question 6.3 Suppose that $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are m_R-filtrations. Are the conditions

1) The Minkowski equality

$$e_R(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2)^{\frac{1}{d}}$$

holds

2) There exist positive integers a and b such that

$$\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n}$$

equivalent?

If $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are m_R-filtrations on a local ring R such that $\dim N(\hat{R}) < d$ and condition 2) holds then the Minkowski equality 1) holds, but the converse statement, that the Minkowski equality 1) implies condition 2) is not true for m_R-filtrations, even in a regular local ring, as follows from Example 3.3.

In our paper [10], it is shown that 1) and 2) are equivalent for bounded m_R-filtrations (Definition 1.5) on an analytically irreducible or excellent local domain, giving a complete generalization of the Teissier, Rees and Sharp, Katz Theorem for bounded m_R-filtrations.

Theorem 6.4 ([10, Theorem 13.2] and [10, Theorem 14.5]) *Suppose that R is a d-dimensional analytically irreducible or excellent local domain and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded m_R-filtrations. Then the following are equivalent*

1) *The Minkowski equality*

$$e(\mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d}} = e(\mathcal{I}(1))^{\frac{1}{d}} + e(\mathcal{I}(2))^{\frac{1}{d}}$$

holds

2) *There exist positive integers a, b such that there is equality of integral closures*

$$\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n}$$

in $R[t]$.

7 An Overview of the Proof of the Characterization of the Minkowski Equality

In this section, we give an overview of the proof of Theorem 6.4.

We suppose that R is a d-dimensional normal excellent local domain. The proof of Theorem 6.4 reduces to this case.

We mention some positivity results about multiplicities and mixed multiplicities of m_R-filtrations that we will need. Suppose that $\mathcal{I}(1), \ldots, \mathcal{I}(r)$ are m_R-filtrations on an analytically irreducible local domain such that $e(\mathcal{I}(j)) > 0$ for all j. It is shown in [12, Theorem 1.4] that then $e(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) > 0$ for all the mixed multiplicities. Suppose that $\mathcal{I}(D)$ is a divisorial m_R-filtration. Then $e(\mathcal{I}(D)) > 0$ by [8, Proposition 2.1].

Theorem 6.4 follows from the following theorem.

Theorem 7.1 ([10, Theorem 11.1], [10, Theorem 11.4]) *Suppose that R is a d-dimensional normal excellent local domain. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be divisorial m_R-filtrations. Let $X \to Spec(R)$ be a representation of $D_1 = \sum \alpha_i E_i$ and $D_2 = \sum_{i=1}^{r} \beta_i E_i$. Then $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ satisfy the Minkowski equality if and only if*

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{\gamma_{E_j}(D_2)}{\gamma_{E_j}(D_1)} \text{ for } 1 \leq i, j \leq r. \tag{1}$$

When this condition holds, we have that

$$\frac{\gamma_{E_i}(D_2)}{\gamma_{E_i}(D_1)} = \frac{e(\mathcal{I}(D_2))^{\frac{1}{d}}}{e(\mathcal{I}(D_1))^{\frac{1}{d}}} \in \mathbb{Q} \tag{2}$$

for $1 \leq i \leq r$.

By Theorem 7.1, when the Minkowski equality holds, we can write

$$\frac{e(\mathcal{I}(D_2))^{\frac{1}{d}}}{e(\mathcal{I}(D_1))^{\frac{1}{d}}} = \frac{a}{b} \text{ with } a, b \in \mathbb{Z}_{>0},$$

so that by (4),

$$\begin{aligned} I(amD_1) &= \Gamma(X, \mathcal{O}_X(-\lceil \textstyle\sum_{i=1}^{r} ma\gamma_{E_i}(D_1)E_i \rceil) \\ &= \Gamma(X, \mathcal{O}_X(-\lceil \textstyle\sum_{i=1}^{r} mb\gamma_{E_i}(D_2)E_i \rceil) = I(bmD_2) \end{aligned}$$

for all $m \in \mathbb{N}$. Thus $\sum_{n \geq 0} I(amD_1)t^n = \sum_{n \geq 0} I(bmD_2)t^n$.

The equivalence of (1) with the Minkowski Equality is proven in [10, Theorem 11.1] and the rationality of the real number of (2) is proven in [10, Theorem 11.4].

We will now give an outline of the proof that the Minkowski equality implies (1). Let

$$f(n_1, n_2) = \lim_{m\to\infty} \frac{\ell(R/I(mnD_1)I(mn_2D_2))}{m^d}.$$

Since the Minkowski equality holds, also using the Minkowski inequalities in Theorem 6.1, we calculate that

$$f(n_1, n_2) = \frac{1}{d!}(e_0^{\frac{1}{d}} n_1 + e_d^{\frac{1}{d}} n_2)^d \tag{3}$$

where $e_0 = e(\mathcal{I}(D_1))$ and $e_d = e(\mathcal{I}(D_2))$.

Let μ be an m_R-valuation and let $\nu : R \to \mathbb{N}^d$ be a valuation of the quotient field of R which dominates R such that $\nu = (\mu, -)$ (ν is composite with μ) as in Sect. 4. For $n_1, n_2 \in \mathbb{N}$, let $\Gamma(n_1, n_2)$ be the semigroup

$$\Gamma(n_1, n_2) = \{(\nu(f), m) \mid f \in I(mn_1D_1)I(mn_2D_2)\} \subset \mathbb{N}^{d+1}.$$

Let $\Delta(n_1, n_2)$ be the intersection in $\mathbb{R}^{d+1}$ of the closure (in the Euclidean topology) of the real cone generated by $\Gamma(n_1, n_2)$ with $\mathbb{R}^d \times \{1\}$. Let $\Gamma(R) = \Gamma(0, 0)$ and $\Delta(R) = \Delta(0, 0)$. We will denote the volume of a compact convex subset Δ of $\mathbb{R}^d$ by $\mathrm{Vol}(\Delta)$.

By the methods of Sect. 4, we obtain the following theorem.

Theorem 7.2 ([10, Section 8]) *There exists $\varphi \in \mathbb{R}_{>0}$ such that letting*

$$H^-_{\Phi,n_1,n_2} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d \le \varphi e_0^{\frac{1}{d}} n + \varphi e_d^{\frac{1}{d}} n_2\},$$

$$\Delta_\Phi(n_1, n_2) = \Delta(n_1, n_2) \cap H^-_{\Phi,n_1,n_2}$$

and

$$\widetilde{\Delta}_\Phi(n_1, n_2) = \Delta(R) \cap H^-_{\Phi,n_1,n_2}$$

we have that

$$f(n_1, n_2) = \delta[\mathrm{Vol}(\widetilde{\Delta}_\Phi(n_1, n_2)) - \mathrm{Vol}(\Delta_\Phi(n_1, n_2))] \tag{4}$$

for all $n_1, n_2 \in \mathbb{N}$, where $\delta = [\mathcal{O}_\nu/m_\nu : R/m_R]$.

We have that $\Delta(R)$ is a closed convex cone with vertex at the origin since $\nu(1) = 0$ ([10, Lemma 7.1]). Thus

$$\mathrm{Vol}(\widetilde{\Delta}_\Phi(n_1, n_2) = (n_1 e_0^{\frac{1}{d}} + n_2 e_d^{\frac{1}{d}})^d \varphi^d \mathrm{Vol}(\Delta(R) \cap \{(x_1, \ldots, x_d) \mid x_1 + \cdots + x_d \le 1\}). \tag{5}$$

Define

$$h(n_1, n_2) = \text{Vol}(\Delta_\Phi(n_1, n_2)).$$

We have that

$$\begin{aligned} h(n_1, n_2) &= \text{Vol}(\widetilde{\Delta}_\Phi(n_1, n_2)) - \tfrac{f(n_1,n_2)}{\delta d!} \\ &= \lambda(e_0^{\frac{1}{d}} n_1 + n_2 e_d^{\frac{1}{d}})^d + n_2 e_d^{\frac{1}{d}} \end{aligned}$$

for some $\lambda \in \mathbb{R}_{>0}$ by (4), (3) and (5). Let

$$g(n_1, n_2) = \text{Vol}(n_1 \Delta_\Phi(1, 0) + n_2 \Delta_\varphi(0, 1)),$$

where $n_1, n_2 \in \mathbb{R}_{\geq 0}$ and $n_1 \Delta_\Phi(1, 0) + n_2 \Delta_\varphi(0, 1)$ is the Minkowski sum

$$n_1 \Delta_\Phi(1, 0) + n_2 \Delta_\varphi(0, 1) = \{n_1(a_1, b_1) + n_2(a_2, b_2) \mid (a_1, b_1) \in \Delta_\Phi(1, 0) \text{ and } (a_2, b_2) \in \Delta_\Phi(0, 1)\}.$$

The function $g(n_1, n_2)$ is a homogeneous real polynomial of degree d ([2, Section 29, page 42]). We have

$$n_1 \Delta_\Phi(1, 0) + n_2 \Delta_\Phi(0, 1) \subset \Delta_\Phi(n_1, n_2)$$

for all $n_1, n_2 \in \mathbb{N}$, from which we deduce that

$$g(n_1, n_2) \leq h(n_1, n_2)$$

for all $n_1, n_2 \in \mathbb{R}$. We have

$$g(1, 0) = h(1, 0) > 0 \text{ and } g(0, 1) = h(0, 1) > 0$$

so for $0 < t < 1$,

$$\begin{aligned} h(1-t, t)^{\frac{1}{d}} &= (1-t)h(1, 0)^{\frac{1}{d}} + t h(0, 1)^{\frac{1}{d}} \\ &= (1-t)g(1, 0)^{\frac{1}{d}} + t g(0, 1)^{\frac{1}{d}} \\ &\leq g(1-t, t)^{\frac{1}{d}} \\ &\leq h(1-t, t)^{\frac{1}{d}} \end{aligned}$$

where the equality of the first line follows from our expression for $h(n_1, n_2)$ in (3) and the inequality between the second and third lines is the Brunn-Minkowski inequality of convex geometry ([2, Page 94], [16]). Thus we have equality in the Brunn-Minkowski inequality which implies that $\Delta_\Phi(1, 0)$ and $\Delta_\Phi(0, 1)$ are homothetic; that is, there exists a map $T(\vec{x}) = c\vec{x} + \vec{\gamma}$ of $\mathbb{R}^d$ (with $c > 0$) such that $T(\Delta_\Phi(1,0)) = \Delta_\Phi(0,1)$. We have just enough information about these sets to calculate that

$$c = \frac{e_d^{\frac{1}{d}}}{e_0^{\frac{1}{d}}}$$

and $\vec{\gamma} = 0$. Thus

$$e_d^{\frac{1}{d}} \Delta_\Phi(1,0) = e_0^{\frac{1}{d}} \Delta_\Phi(0,1).$$

Now recalling that $\nu = (\mu, -)$, where μ is an m_R-valuation, we see that $\gamma_\mu(D_1)$ is the smallest point of the projection of $\Delta(1,0)$ onto the first real axis and $\gamma_\mu(D_2)$ is the smallest point of the projection of $\Delta(0,1)$ onto the first real axis. Taking $\mu = \nu_{E_j}$, we obtain that

$$\frac{\gamma_{E_j}(D_1)}{e_0^{\frac{1}{d}}} = \frac{\gamma_{E_j}(D_2)}{e_d^{\frac{1}{d}}}$$

for $1 \le j \le r$, from which we obtain the statement of (1) and the first part of the statement of (2) of Theorem 7.1.

8 Examples

The above concepts and results are analyzed in [9] and [10]. An example of a blowup $\varphi : X \to \mathrm{Spec}(R)$ of an m_R-primary ideal in a normal and excellent three dimensional local ring R which is a resolution of singularities is constructed in [9]. The map φ has two prime exceptional divisors E_1 and E_2. The function

$$f(n_1, n_2) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1E_1 + mn_2E_2))}{m^3}$$

is computed in [9] and is reproduced here.

Theorem 8.1 ([9, Theorem 1.4]) *For* $n_1, n_2 \in \mathbb{N}$,

$$f(n_1,n_2) = \begin{cases} 33n_1^3 & \text{if } n_2 < n_1 \\ 78n_1^3 - 81n_1^2n_2 + 27n_1n_2^2 + 9n_2^3 & \text{if } n_1 \le n_2 < n_1\left(3 - \frac{\sqrt{3}}{3}\right) \\ \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right)n_2^3 & \text{if } n_1\left(3 - \frac{\sqrt{3}}{3}\right) < n_2. \end{cases}$$

Thus $f(n_1, n_2)$ is not a polynomial, but it is "piecewise a polynomial"; that is, $\mathbb{R}^2_{\geq 0}$ consists of three triangular regions determined by lines through the origin such that $f(n_1, n_2)$ is a polynomial function within each of these three regions. The line separating the second and third regions has irrational slope, and the function $f(n_1, n_2)$ has an irrational coefficient in the third region. The middle region is the ample cone and is also the Nef cone.

We compute the functions γ_{E_1} and γ_{E_2} in [9, Theorem 4.1], as summarized in the following theorem. Observe that γ_{E_1} is an irrational number in the third region.

Theorem 8.2 ([9, Theorem 4.1]) *Let $D = n_1E_1 + n_2E_2$ with $n_1, n_2 \in \mathbb{N}$, an effective exceptional divisor on X.*

1) *Suppose that $n_2 < n_1$. Then $\gamma_{E_1}(D) = n_1$ and $\gamma_{E_2}(D) = n_1$.*
2) *Suppose that $n_1 \leq n_2 < n_1\left(3 - \frac{\sqrt{3}}{3}\right)$. Then $\gamma_{E_1}(D) = n_1$ and $\gamma_{E_2}(D) = n_2$.*
3) *Suppose that $n_1\left(3 - \frac{\sqrt{3}}{3}\right) < n_2$. Then $\gamma_{E_1}(D) = \frac{3}{9-\sqrt{3}}n_2$ and $\gamma_{E_2}(D) = n_2$.*

In all three cases, $-\gamma_{E_1}(D)E_1 - \gamma_{E_2}(D)E_2$ is nef on X.

The divisors for which Minkowski's inequality holds has a simple classification, as is shown in [10].

Corollary 8.3 ([10, Corollary 1.3]) *Suppose that D_1 and D_2 are effective integral exceptional divisors on X. If D_1 and D_2 are in the first region of Theorem 8.2, then Minkowski's equality holds between them. If D_1 and D_2 are in the second region, then Minkowski's equality holds between them if and only if D_2 is a rational multiple of D_1. If D_1 and D_2 are in the third region, then Minkowski's equality holds between them. Minkowski's equality cannot hold between D_1 and D_2 in different regions.*

Theorem 8.2 allows us to compute the mixed multiplicities of any two divisors $D_1 = a_1E_1 + a_2E_2$ and $D_2 = b_1E_1 + b_2E_2$ by interpreting mixed multiplicities as anti positive intersection multiplicities (as explained in [8] and [9]).

In particular, we can compare $f(n_1, n_2)$ with the polynomial

$$P(n_1, n_2) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1E_1)I(mn_2E_2))}{m^d}$$

which computes the mixed multiplicities of the m_R-filtrations $\mathcal{I}(E_1)$ and $\mathcal{I}(E_2)$. We have that

$$\begin{aligned} P(n_1, n_2) &= \tfrac{1}{3!}e(\mathcal{I}(E_1)^{[3]})n_1^3 + \tfrac{1}{2!}e(\mathcal{I}(E_1)^{[2]}, \mathcal{I}(E_2)^{[1]})n_1^2n_2 \\ &\quad + \tfrac{1}{2!}e(\mathcal{I}(E_1)^{[1]}, \mathcal{I}(E_2)^{[2]})n_1n_2^2 + \tfrac{1}{3!}e(\mathcal{I}(E_2)^{[3]})n_2^3 \\ &= 33n_1^3 + (\tfrac{891}{26} + \tfrac{99}{26}\sqrt{3})n_1^2n_2 + (\tfrac{12042}{338} - \tfrac{27}{338}\sqrt{3})n_1n_2^2 \\ &\quad + \left(\tfrac{2007}{169} - \tfrac{9\sqrt{3}}{338}\right)n_2^3. \end{aligned}$$

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Stanley-Reisner Rings

Ralf Fröberg

1 Introduction

Sometimes a breakthrough in mathematics is made when somebody realizes that results in one branch of mathematics can be applied to problems in another branch. The rings now called Stanley-Reisner rings or face rings is a good example. To each simplicial complex a factor ring of a polynomial ring is defined. These rings were independently defined by Mel Hochster and Richard Stanley to be able to use commutative algebra on combinatorial problems. Hochster wrote an influential paper, [86]. He also gave the problem to characterize Cohen-Macaulay face rings to his student Reisner, who solved this problem in his thesis, see [108]. From Reisner's result it followed that face rings of spheres were Cohen-Macaulay, and this was the missing piece for Stanley to be able to prove "The upper bound conjecture for spheres", [115]. Now there are hundreds of papers on Stanley-Reisner rings, and this article is a try to describe the algebraic side of the story.

2 Simplicial Complexes and Stanley-Reisner Rings

2.1 Stanley-Reisner Rings

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set. A simplicial complex Δ on V is a set of subsets of V such that $v_i \in V$ for all i and closed under subsets. The elements F of V are called faces, and $\dim(F) = |F| - 1$. A maximal face is called a facet. The dimension of Δ is the largest dimension of a facet. The complex is pure if all facets

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have the same dimension. In the sequel we just use $V = \{1, \ldots, n\} = [n]$ as the finite set.

Definition 1 For a finite simplicial complex Δ with vertices $\{1, \ldots, n\}$, and a field k, the Stanley-Reisner ring (or face ring) is $k[x_1, \ldots, x_n]/I = k[\Delta]$, where I is generated by all squarefree monomials $x_{i_1} \cdots x_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is NOT a face of Δ.

In this way we get a 1-1 correspondance between squarefree monomial ideals and simplicial complexes.

One much studied source for simplicial complexes is finite posets (partially ordered sets). We suppose that the poset has a max and a min. The faces are the chains in the poset when the max and the min are removed. This is the order complex of the poset. If all maximal chains have the same length, the poset is called pure (or graded).

2.2 Edge Rings and Clutters

There is another much studied construction of squarefree monomial ideals, edge ideal of a graph. Let G be a graph with vertex set $V = \{x_i\}, i \in [n]$ and edge set E. Here the ideal is $I(G) = (x_i x_j; \{i, j\} \in E)$, the edge ideal of G, and $K[x_1, \ldots, x_n]/I(G)$ is called the edge ring. This was introduced by Villarreal in [133] and followed by [114].

Although edge ideals were not introduced as Stanley-Reisner ideals, they could be.

Definition 2 A set of vertices $\{i_1, \ldots, i_t\}$ in a graph is independent if there is no edge between vertices in the set. Since a subset of an independent set is independent, the independent sets constitute a simplicial complex, the independent complex of the graph.

Definition 3 A clique of a graph is a complete subgraph. Since a subset of a clique is a clique, the set of all cliques constitute a simplicial complex, the clique complex.

The edge ring of G is the Stanley-Reisner ring of the independence complex of G, or, which is the same, the clique complex of the complementary graph $\bar{G}$. These complexes have only quadratic relations. A complex with only quadratic relations is called a flag complex. It is determined by its 1-skeleton.

Edge rings has been generalized to clutters. A clutter (or a simple hypergraph), is a family of subsets,"edges", of the vertices of size at least two, in which none of the sets contains another. Thus "edges" can consist of more than two vertices. There is a 1-1 correspondance between clutters and simplicial complexes.

2.3 Facet Rings

A facet ideal, introduced by Faridi in [46] of a simplicial complex is generated by $\{x_{i_1} \cdots x_{i_t}\}$ for all facets $\{i_1, \ldots, i_t\}$. Facet ideals can of course also be viewed as edge ideals of clutters.

Stanley-Reisner rings, edge rings of clutters, and facet rings, being different ways to look at simplicial complexes, give us possibilities to use commutative algebra on combinatorial problems.

3 Hilbert Series and Hochster's Formula

3.1 Hilbert Series

We continue by describing some close connections between a simplicial complex and its Stanley-Reisner ring.

Definition 4 Let Δ be a simplicial complex of dimension d. The f-vector of Δ is $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$, where f_i is the number of faces of dimension i in Δ (or the number of faces of cardinality $i+1$). ($f_{-1} = 1$ counting the empty set.)

A Stanley-Reisner ring is standard graded, $k[\Delta] = R = \oplus_{i\geq 0} R_i$, $R_0 = k$, R generated by R_1, $R_i R_j = R_{i+j}$. It is even multigraded, graded over $\mathbb{N}^n$, so if $\alpha, \beta \in \mathbb{N}^n$, then $R_\alpha R_\beta \in R_{\alpha+\beta}$..

Definition 5 The Hilbert series of a graded k-algebra $R = \oplus_{i=0}^{\infty} \dim_k R_i$ is $R(Z) = \sum_{i\geq 0} \dim_k(R_i) Z^i$. If R is $\mathbb{N}^n$-graded, the multigraded Hilbert series is $R(Z_1, \ldots, Z_n) = \sum_{(i_1 \ldots, i_n)\in\mathbb{N}^n} \dim_k R_{(i_1,\ldots,i_n)} Z_1^{i_1} \cdots Z_n^{i_n}$. Thus $\dim_k R_{(i_1,\ldots,i_k)}$ is 0 or 1 depending on wether the image of $x_1^{i_1} \cdots x_n^{i_n}$ is 0 or not.

The following is easy.

Lemma 6 *Let R be a face ring of embedding dimension n and let $\bar{R} = R/(x_1^2, \ldots, x_n^2)$. Then $R(Z) = \bar{R}(Z)/(1-Z))$.*

Proposition 7 *Let Δ be a simplicial complex with f-vector $(f_{-1}, \ldots, f_d)$. Then*

$$k(\Delta)(Z) = f_{-1} + f_0 Z/(1-Z) + \cdots + f_d Z^{d+1}/(1-Z)^{d+1}.$$

In particular, $\dim(k[\Delta]) = \dim(\Delta) + 1$.

Proof $\bar{R}(Z) = f_{-1} + f_0 Z + \cdots + f_d Z^{d+1}$, so the first statement follows from the lemma. The dimension of the ring equals the order of the pole $Z = 1$.

For future use we also define the h-vector of a simplicial complex.

Definition 8 If the Hilbert series of $k[\Delta]$ is $(\sum_{i=0}^{d+1} h_i z^i)/(1-Z)^{d+1}$, then $(h_0, \dots, h_{d+1})$ is the h-vector of Δ.

The h-vector contains the same information as the f-vector.

3.2 Homology

To get a closer connection between combinatorics of simplicial complexes and algebra, we need to define homology of simplicial complexes.

Let Δ be a simplicial complex on $[n]$. For a field k, let $C_q(\Delta)$ be the k-vector space with all ordered q-faces $[i_1, \dots, i_q]$ as basis, and $d_q\colon C_q(\Delta) \to C_{q-1}(\Delta)$ defined by

$$d_q([i_1, \dots, i_q]) = \sum_{j=1}^{q} (-1)^{j-1}[i_1, \dots, i_{j-1}, \hat{i_j}, i_{j+1}, \dots, i_q],$$

where $\hat{i_j}$ means omit i_j. The homology of

$$0 \to C_{\dim(\Delta)} \to \cdots \to C_0(\Delta) \to C_{-1}(\Delta) = k \to 0,$$

i.e., $\tilde{H}_q(\Delta; k) = \ker d_q / \operatorname{im} d_{q+1}$, is the reduced simplicial homology of Δ.

In general $\tilde{h}_0 = \dim_k(\tilde{H}_0)$ counts the number of connected components minus 1, and $\tilde{h}_i = \dim_k(\tilde{H}_i)$ counts the number of "i-dimensional holes" in Δ if $i > 0$, if Δ is interpreted geometrically.

3.3 Hochster's Formula

Let $R = k[x_1, \dots, x_n]/I = S/I$, I generated by monomials. Then R is $\mathbb{N}^n$-graded. The Koszul complex K_R respects this ordering, and the homology $H_*(K_R) = \operatorname{Tor}_*^S(R, k)$ is $\mathbb{N}^n$-graded. (If you are unfamiliar with the Koszul complex, see e.g. [106, Chapter I:14].)

It is not hard (see e.g. [58, Lemma]) to see that the homology of a face ring can be non-zero only in square-free degrees, i.e., in degrees $(i_1, \dots, i_n)$ with $i_j \le 1$ for all j. For $J \subseteq \{1, 2, \dots, n\}$, let $\delta(J) = (d_1, d_2, \dots, d_n)$, where $d_i = 1$ if $i \in J$ and $d_i = 0$ otherwise (so $d_i = 1$ on the support of J). Let $\dim_k \operatorname{Tor}_i^S(R, k)_{\delta(J)} = \beta_{i,\delta(J)}$, $\sum_{|\delta(J)|=j} \beta_{i,\delta(J)} = \beta_{i,j}$, and $\sum_{\delta(J)} \beta_{i,\delta(J)} = \beta_i$, the Betti numbers.

Lemma 9 ([86, Theorem 5.1]) *Let $K_{R(J)}$ be the part of K_R which is of degree $\delta(J)$. Then*

$$H_{i,\delta(J)} = H_i(K_{R(J)}) \simeq \tilde{H}_{|J|-i-1}(\Delta_J),$$

where $|J|$ denotes the number of elements in J and Δ_J denotes the induced simplicial complex on J.

Proof Let Δ_J^* denote the dual of Δ_J. We will define a map of k-spaces $f : K_{R(J)} \to \Delta_J^*$ such that $f((K_{R(J)})_i = \Delta_J^{|J|-i-1}$. Suppose that $d_1 < d_2 < \cdots < d_r$ and $e_1 < e_2 < \cdots < e_s$, where $\{d_, \ldots, d_r\} \cup \{e_1, \ldots, e_s\} = J$, then $f(x_{d_1} \cdots x_{d_r} T_{e_1} \cdots_{e_r}) = \pm\{d_1, \ldots, d_r\}^*$. With an appropriate choice of signs this is an isomorphism of complexes. Thus $\tilde{H}^{|J|-i-1}(\Delta_J) = H_i(K_{R(J)})$. Since $\tilde{H}^{|J|-i-1}(\Delta_J) = \tilde{H}_{|J|-i-1}(\Delta_J)$ over a field, we get the result.

Example 10 Let Δ be C_5, a cycle with 5 vertices. The facets are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, and $\{5, 1\}$. The minimal nonfaces are $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, and $\{3, 5\}$, so the Stanley-Reisner ring is $k[\Delta] = k[x_1, \ldots, x_5]/(x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5) = S/I$. The f-vector is $(1, 5, 5)$, so the Hilbert series is $1+5Z/(1-Z)+5Z^2/(1-Z)^2 = (1+3Z+Z^2)/(1-Z)^2$. The dimension of the ring is 2, since the complex has dimension 1, so the h-vector is $(1, 3, 1)$. The induced complexes with 1 or 4 vertices have no homology. Also the complexes with 2 or 3 consecutive vertices have no homology. There are 5 subsets with two vertices, namely $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$ and $\{3, 5\}$ with homology $\tilde{H}_0$ one-dimensional. These give $\beta_{1,2} = 5$ (corresponding to the minimal generators of the ideal). Likewise there are 5 subsets with 3 vertices, e.g. $\{1, 2, 4\}$, with $\tilde{H}_0$ one-dimensional. These give $\beta_{2,3} = 5$. Finally the whole complex has $\tilde{H}_1$ one-dimensional, which gives $\beta_{3,5} = 1$.

Corollary 11 ([86, Corollary 5.3]) *Let R be a face ring of Krull dimension d and embedding dimension n. Then R is a Cohen-Macaulay ring if and only if for $i = 0, 1, \ldots, d-2$ we have $H_i(\Delta_J) = 0$ for all i and J with $|J| = n-d+i+2$.*

Proof We always have $\operatorname{depth}(R) \leq \dim(R)$ and the ring is CM (Cohen-Macaulay) if and only if we have equality. Now $\operatorname{depth}(R) = \max\{g; H_{n-g} \neq 0\}$. We have $H_{n-d+1}(K_R) = 0$ if and only if $H_{n-d+1}(K_{R(J)}) = 0$ for all J with $|J| > n-d+i+1$. This is equivalent by Hochster's lemma to $\tilde{H}_{|J|-n+d-2}(\Delta_J) = 0$ for all J with $|J| > n-d+1$, i.e. to $\tilde{H}_i(\Delta_J) = 0$ for all i and all J with $|J| = n-d+i+2$.

Example 12 (Continuation) Since for all i, $\tilde{H}_i(\Delta) = 0$ for all subsets with $5+i$ vertices, the ring $k[\Delta]$ is CM.

We will soon give a more efficient criterion for a face ring to be CM, but first we will draw a conclusion.

Corollary 13 *If $k[\Delta]$ is CM, then all facets of Δ have the same dimension.*

Proof We observe that if $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime in $k[\Delta]$, then $[n] \setminus \{i_1, \ldots, i_k\}$ is a facet of Δ and vice versa. If $k[\Delta]$ is CM all minimal primes have the same dimension, which is equivalent to that all facets have the same dimension.

4 Reisner's Criteria

This is one of the most cited results on Stanley-Reisner rings, and the reason for the name. First some preliminaries.

Definition 14 Let F be a face in the simplicial complex Δ. The link of F is $lk_\Delta(F) = \{G \in \Delta, G \cap F = \emptyset, G \cup F \in \Delta\}$.

Here is the more efficient criterion for a simplicial complex to be CM.

Theorem 15 ([108]) *Let $R = k[\Delta]$ be the face ring of Δ. Then the following are equivalent*

(i) *R is CM.*
(ii) *$\tilde{H}_i(\Delta) = 0$ if $i < \dim \Delta$ and $\tilde{H}_i(lk_\Delta(\sigma)) = 0$ if $i < \dim lk_\Delta(\sigma)$ for all simplices $\sigma \in \Delta$.*
(iii) *Δ has CM links of vertices and $\tilde{H}_i(\Delta) = 0$ if $i < \dim \Delta$.*

There is a much cited corollary to Reisner's theorem, namely that CMness depends on the field. His example is the minimal triangulation of the projective plane. Then the Stanley-Reisner ring is CM if and only if $\mathrm{char}(k) \neq 2$, see [108].

There is also an explicit formula for the local cohomology of a Stanley-Reisner ring $k[\Delta]$ with respect to the graded maximal ideal M in [86]. Expressed in the multigraded Hilbert series it looks like this.

Theorem 16 $H^i_M(k[\Delta])(Z_1, \ldots, Z_n) = \sum_{F\in\Delta} \dim_k \tilde{H}_{i-|F|-1}(lk_\Delta F) \prod_{j\in F} Z_j^{-1}/(1 - Z_j^{-1})$

Some early papers on Cohen-Macaulay complexes are [20, 117–119], and [13]. The last two papers treat posets. In this case the Cohen-Macaulay property is equivalent to that all intervals have no homology except in the highest dimension. It is shown that certain classes of posets (such as locally semimodular) are CM, and that certain operations preserve CMness.

There is another way to study Cohen-Macaulay complexes, avoiding homological algebra. It uses the fact that $k[\Delta]$ is CM if and only if there is a linear system of parameters $f_1, \ldots, f_d$ such that $k[\Delta]$ is free over $k[f_1, \ldots, f_d]$, see [60] and [92].

5 Gorenstein, Buchsbaum Rings, and Serre's Condition S_r

5.1 Gorenstein Rings

To give a characterization of Gorenstein complexes, we have to introduce some new terminology.

Definition 17 A cone point of Δ is a vertex that is part of all facets of Δ. The core of a simplicial complex is the complex which consists of those faces that are disjoint to the core of Δ.

Thus the core of Δ is a simplicial complex on those vertices which do not occur in a minimal system of generators of the ideal of nonfaces. If R is the face ring of core(Δ) and the cone points are $i_1, \ldots, i_k$, then the face ring of Δ is $R[x_{i_1}, \ldots, x_{i_k}]$.

Theorem 18 ([116, 117]) *The face ring of* Δ *is Gorenstein if and only if for all* $F \in \mathrm{core}(\Delta)$ *we have*

(i) $\dim_k \tilde{H}_i(lk_{\mathrm{core}\Delta}(F)) = 1$ *if* $i = \dim lk_{\mathrm{core}\Delta}(F)$
(ii) $\dim_k \tilde{H}_i(lk_{\mathrm{core}\Delta}(F)) = 0$ *if* $i < \dim lk_{\mathrm{core}\Delta}(F)$

If furthermore $\Delta = \mathrm{core}(\Delta)$*,* $k[\Delta]$ *is called Gorenstein**.

Example 19 With this theorem it is easy to check that if $\Delta = C_5$, $k[\Delta]$ is Gorenstein.

5.2 Buchsbaum Rings

We also mention the characterization of Buchsbaum complexes, i.e., complexes with Buchsbaum Stanley-Reisner ring, see [120], and a characterization of Stanley-Reisner rings satisfying S_r, see [122].

If R is CM, then $l_R(R/q) = e_q(R)$ for all parameter ideals q, where l_R denotes length and e_q multiplicity with respect to q.

Definition 20 R is a Buchsbaum ring if $l_R(R/q) - e_q(R)$ is independent of the parameter ideal q.

Theorem 21 ([120]) *A complex* Δ *is Buchsbaum over the field k if it is pure, and if for all* $F \in \Delta$*,* $F \neq \emptyset$*, and for all* $i < \dim(lk_\Delta(F))$ *we have* $\tilde{H}_i(lk_\Delta(F; k)) = 0$.

5.3 Rings Satisfying S_r

Definition 22 A ring R satisfies Serre's condition S_r if for all $P \in \mathrm{Ass}(R)$ we have $\mathrm{depth}(R) \geq \min\{r, \dim R_P\}$.

Theorem 23 *[122] let* $k[\Delta]$ *have dimension* $d - 1$*. Then* $k[\Delta]$ *satisfies* S_r *if and only if for all* $F \in \Delta$ *with* $|F| \leq d-i-2$ *and all* i*,* $-1 \leq i \leq r-2$ $\tilde{H}_i(lk_\Delta F; k) = 0$.

There is a survey article on this, see [107].

5.4 (Locally) Complete Intersection

A Stanley-Reisner ring is a complete intersection if and only if the generators of the ideal are pairwise relatively prime. A graded ring R is locally a complete intersection if R_P is a complete intersection for all primes P. For Stanley-Reisner rings $k[\Delta]$ this means that $k[lk_\Delta\{v\}]$ is a complete intersection for each v.

Theorem 24 ([126, Theorem 1]) *Let* Δ *be connected with* $\dim(\Delta) \geq 2$ *(resp.* $\dim(\Delta) = 1$*). If* $k[\Delta]$ *is a locally a complete intersection, then it is a complete intersection (resp. an* n*-gon for* $n \geq 3$ *or an* n*-pointed path for some* $n \geq 2$*).*

The converse is trivially true.

Definition 25 If Δ is a simplicial complex and $i \leq \dim \Delta$. the i'th skeleton, $\Delta^{(i)}$, of Δ is the subcomplex of all faces of dimension $\leq i$ of Δ. The pure i'th skeleton, $\Delta^{[i]}$, of Δ is the subcomplex of Δ whose facets are the faces of dimension i in Δ.

Now we can formulate the following. One can find the result in e.g. [56, Theorem 8].

Theorem 26 *The depth of* $k[\Delta]$ *is* $\max\{i; k[\Delta^{(i)}]$ *is CM.*

6 Shellability

The concept of shellability apparently first occurred in [29].

6.1 Pure Shellability

Definition 27 A pure simplicial complex Δ of dimension d is called pure shellable if the facets can be ordered $F_1, F_2, \ldots, F_m$ such that the subcomplex generated by $F_1, \ldots, F_j$ intersects F_{j+1} in a pure complex of dimension $d-1$ for $j = 1, 2, \ldots, m-1$.

The following was proved in [50], also see [119].

Theorem 28 *If* Δ *is pure shellable, then* Δ *is CM.*

In some vague sense it seems that "most" Cohen-Macaulay complexes are pure shellable. Since shellability is often rather easy to prove, it has been the most common way to show CM-ness by showing that the complex is pure shellable. Shellable posets (i.e. the order complex is shellable) are treated in [19]. Generalizing earlier results, it is shown that a class of posets, called admissible, containing upper semimodular and supersolvable are shellable. Furthermore that some operations on poset, such as taking intervals, direct product, and barycentric subdivision preserve shellability.

There are some other concepts close to shellability.

Definition 29 A simplicial complex Δ is homotopy CM if the homotopy groups $\pi_i(lk_\Delta F)$ are trivial for all $F \in \Delta$ and $i < \dim(lk_\Delta F)$.

Definition 30 A simplicial complex Δ is constructible if it satisfies the following recursive conditions

(1) A simplex is constructible.
(2) If Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$ are constructible and $\dim(\Delta_1) = \dim(\Delta_2) = \dim(\Delta_1 \cap \Delta_2) + 1$, then $\Delta_1 \cup \Delta_2$ is constructible.

It is shown in [19, Appendix] that Pure shellable $\Rightarrow$ Constructible $\Rightarrow$ Homotopy CM $\Rightarrow$ CM over $\mathbb{Z}$. If a complex is CM over $\mathbb{Z}$, it is CM over all fields.

Some variants of shellability, CL-shellability and EL-shellability are studied in [22].

6.2 Non-pure Shellability

Shellability was greatly generalized to cover the nonpure case in two papers, [23] and [24]. Most properties of pure shellability were generalized. Furthermore doubly indexed f- and h-vectors were defined. The usual h-vector has positive values in the Cohen-Macaulay case, since one can factor out with a linear regular sequence (at least after extending the field which doesn't change the Hilbert series) to get an Artinian ring with Hilbert series $\sum h_i Z^i$, so the h_i's are dimensions. In the shellable case they show that the doubly indexed h-vector is positive. These two papers have become the natural sources to refer to with respect to shellability.

Definition 31 A simplicial complex Δ is called (nonpure) shellable if the facets can be ordered $F_1, F_2, \ldots, F_m$ such that the subcomplex generated by $F_1, \ldots, F_j$ intersects F_{j+1} in a pure complex of dimension $\dim F_{j+1} - 1$ for $j = 1, 2, \ldots, m - 1$. In the sequel we mean non-pure shellable when we write shellable.

Here is an example from [23].

Theorem 32 ([23]) *If Δ is shellable, both all skeletons and all pure skeletons of Δ and links to all faces are shellable.*

Dress has an alternative criterion for shellability in [39].

Definition 33 Let M be a module over a commutative ring R and let $(0) \subset M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a filtration. The filtration of M is called clean if for all i we have $M_i/M_{i-1} = R/P_i$, with P_i a minimal prime over $\mathrm{Ann}(M)$. A module is clean if it has a clean filtration.

Theorem 34 *A simplicial complex Δ is nonpure shellable if and only if $k[\Delta]$ is clean.*

The theory on clean face rings has continued and been extended in e.g. [87] and [82].

There are many papers on shellability for different classes of complexes, and shellability is still used very actively. Almost all articles refer to [23].

In the sequel we mean nonpure shellable if we write shellable.

7 Eagon-Reiner's Theorem

Although Alexander duality had been used before, Eagon and Reiner discovered the full potential of it. They found that the homologies of $k[\Delta]$ in Hochster's theorem could be expressed in the homologies of links of faces of the Alexander dual Δ^*. Eagon-Reiner's article has been one of the most influential since Reisner's result.

Definition 35 If Δ is a simplicial complex, its Alexander dual is $\Delta^* = \{[n]\setminus\sigma, \sigma \notin \Delta\}$.

The following is easy to prove.

Theorem 36 *If $R = k[x_1, \ldots, x_n]/I$ and I is minimally generated by $m_1, \ldots, m_k$, where $m_i = x_{i_1 1} \cdots x_{i_1 k_1}$, then the Alexander dual has face ring $R = k[x_1, \ldots, x_n]/I^*$, where $I^* = \cap_{i=1}^{k}(x_{i_1 1}, \ldots, x_{i_1 k_1})$.*

The following gives the essence of Alexander duality.

Theorem 37 *Let δ be a multidegree and $k[\Delta] = S/I(\Delta)$. We have*

$$\sum \dim_k \operatorname{Tor}_i^S(k[\Delta], k)_\delta Z^\delta = \sum_{F\in\Delta^*} \dim_k \tilde{H}_{i-2}(lk_{\Delta^*} F; k) Z^{[n]\setminus\delta}$$

Definition 38 A graded algebra $k[x_1, \ldots, x_n]/I = S/I$ is said to have a t-linear resolution if $(\operatorname{Tor}_i^S(S/I, k)_j = 0$ for $j \neq t + i - 1$. This means that all minimal generators of I have degree t, and that the matrices describing all higher syzygies only contain linear elements. A resolution is linear, if it is t-linear for some t.

The following is the most used corollary.

Corollary 39 ([42, Proposition 1]) *Let Δ be a simplicial complex with n vertices. Then $k[\Delta]$ has a t-linear resolution if and only if the face ring of Δ^* is CM of dimension $n - t$, and vice versa.*

Example 40 If $\Delta = C_5$, we have shown that $k[\Delta]$ is CM of dimension 2. Thus $k[\Delta^*] = k[x_1, \ldots, x_5]/(x_2x_4x_5, x_2x_3x_5, x_1x_3x_5, x_1x_3x_4, x_1x_2x_4)$ has a 3-linear resolution.

The following are easy consequences:

Corollary 41

1. $(\Delta \setminus v)^* = lk_{\Delta^*} v$ *and vice versa.*
2. $\tilde{H}_{i-1}(\Delta^*; k) = \tilde{H}_{n-i-1}(\Delta; k)$
3. $f_i(\Delta^*) = \binom{n}{i-1} - f_{n-i-1}(\Delta)$

Terai has an article, [122], which contains several new results and new proofs of known results on Alexander duality. Among other things, the following is proved.

Theorem 42 ([122]) *The projective dimension of* $k[\Delta]$ *equals the regularity of* $k[\Delta^*]$.

The regularity of $k[\Delta] = S/I(\Delta)$ is $\max\{j; (\mathrm{Tor}_i^S(R, k))_{i+j} \neq 0$.

Example 43 If $\Delta = C_5$, then $k[\Delta]$ has projective dimension 2 which equals the regularity of $k[\Delta^*]$.

Another consequence of Alexander duality is shown in [123], namely that the second Betti number of a face ring does not depend on the characteristic of the field.

There is a more elementary proof of Alexander duality in [21].

8 Polarization

Let I be a monomial ideal, not necessarily squarefree. Several authors have described a method to get a squarefree monomial ideal with most properties common with I.

Theorem 44 *Let* I *be a monomial ideal in* $k[x_1, \ldots, x_n] = S$. *There exists an* N *and a squarefree ideal* $I' \subset [y_1, \ldots, y_N]$ *and a linear regular sequence* $f_1, \ldots, f_N$ *in* $S[y_1, \ldots, y_N]/I'$ *such that* $S[y_1, \ldots, y_N]/(I' + (f_1, \ldots, f_N))$ *is isomorphic to* S/I.

Proof Let $I = (M_1, \ldots, M_r) = (x_1^{i_1} M_1', \ldots, x_1^{i_s} M_s', M_{s+1}, \ldots, M_r)$ be a monomial ideal where neither M_i', $i = 1, \ldots, s$ nor M_i, $i = s+1, \ldots, r$ are divisible by x_1, and let $I' = (x_{11} x_1^{i_1 - 1} M_1', \ldots, x_{11} x_1^{i_s - 1} M_s', M_{s+1}, \ldots, M_r) \subset S/I[x_{11}]$. Then $S[x_{11}]/(I' + (x_{11} - x_1)) \simeq S/I$ and $x_{11} - x_1$ is a nonzero divisor. Continuing like this with all variables, we get the result.

The construction to get a squarefree ideal is called polarization. Most homological invariants are preserved, such as being CM or Gorenstein. Also Betti numbers are preserved. There are results about general monomial ideals from their polarization in [47]. In particular primary decomposition is studied.

Example 45 The monomial ideal $(x_1, x_2, x_3)^2$ has polarization $(x_{11}x_{12}, x_{11}x_{21}, x_{11}x_{31}, x_{21}x_{31}, x_{21}x_{22}, x_{31}x_{32})$

9 Resolutions, Betti Numbers, Regularity

9.1 Resolutions

By polarization it follows that it is just as hard to find resolutions of all monomial rings as it is for all squarefree. We will first mention some methods which applies for all, not only squarefree, monomial rings. The simplest is the Taylor resolution, [121]. Let $J = (m_1, \dots, m_r)$ be a monomial ideal in $S = k[x_1, \dots, x_n]$. The Taylor complex is

$$0 \longrightarrow V_r \longrightarrow V_{r-1} \longrightarrow \cdots \longrightarrow V_0 \longrightarrow S/J,$$

where $V_k = \oplus_{I \subset [r], |I|=k} S e_I$. We have $d(e_I) = \sum_{j=1}^{k} (-1)^{j-1} M_I / M_{I \setminus i_j} e_{I \setminus ij}$ if $I = \{i_1, \dots, i_k\}, i_1 < \cdots < i_k$. Here $M_I = \operatorname{lcm}\{m_i; i \in I\}$. This is a resolution, but seldom minimal. It is described closer in e.g [106] and [74]. Methods giving smaller, but still not minimal, resolutions are described in [93]. Even smaller resolutions, but still not minimal in general are described in [15] and generalized in [17]. Other methods to get resolutions are described in [61] and [88]. There are surveys on resolutions of monomial ideals in [106] and [96]. In [6] a resolution of a squarefree ideal in the exterior algebra is constructed from a resolution of the ideal in the symmetric algebra.

Next we will mention resolutions which apply only for some class of ideals. We need some definitions.

Definition 46 Let I be a monomial ideal. For a monomial M we set $\operatorname{supp}(M) = \{i; x_i | M\}$ and $m(M) = \max\{i; i \in \operatorname{supp}(M)\}$. I is called stable if for all minimal generators M and all $i \leq m(M)$ we have $x_i(M/x_{m(M)}) \in I$. I is called strongly stable if for all $M \in I$ and $x_j | M$, we have $x_i M / x_j$ whenever $i < j$.

Definition 47 A monomial ideal is Borel-fixed if it is invariant under the Borel subgroup of $\mathrm{GL}(n.k)$ consisting of upper triangular invertible matrices.

Definition 48 A monomial ideal I is called a lexsegment ideal if for each degree d, if $m \in I$, $\deg(m) = d$, then all other monomials of degree d which are larger than m in the lexicographic ordering, also belong to I.

We have that Lexsegment ideals $\Rightarrow$ Strongly stable ideals $\Rightarrow$ Borel fixed ideals. If $\operatorname{char}(k) = 0$ Borel fixed ideals are strongly stable. Let $g(I)$ be the image of I for a general linear combination. Then the ideal of leading monomials, $\operatorname{gin}(I)$, is strongly stable, [16, 59].

Definition 49 A monomial ideal I is called squarefree stable if for all minimal generators M and all $i < m(M)$, $i \notin \operatorname{supp}(M)$, we have $x_i(M/x_{m(M)}) \in I$.

Stable ideals were defined in [43], and squarefree stable in [8].

A resolution for stable ideals is constructed in [43]. A corresponding resolution for squarefree stable ideals is studied in [8] and further generalized to so called weakly stable ideals in [9]. More similarities of squarefree stable ideals to stable ideals are discussed in [10], where the corresponding ring to Stanley-Reisner rings in the exterior algebra is studied.

Another method to get resolutions for some classes of ideals is to use iterated mapping cones. They are used in [84] for certain ideals with linear quotients (defined below), and in [33] for some ideal constructed from lexicographic ideals by cutting some powers.

Sometimes one can split the monomial ideal in two simpler, $I = K \cup J$, and get

$$\beta_{i,j}(I) = \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K),$$

[52]. This has been used in several articles, e.g. [66] for edge ideals and generalized to clutters in [68]. There is a survey in [67].

9.2 *Linear and Pure Resolutions*

Stanley-Reisner rings with 2-linear resolution has been characterized in [56]. Then all generators of the ideal is of degree 2, and the complex Δ is determined by its 1-skeleton, a graph $G(\Delta)$.

Definition 50 A graph is chordal if every cycle of length at least 4 has a chord.

Theorem 51 ([56]) *$k[\Delta]$ has a 2-linear resolution if and only if the complement graph $\overline{G(\Delta)}$ to $G(\Delta)$ is chordal.*

Example 52 Let Δ be the octahedron with facets $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{1, 4, 5\}$, $\{1, 5, 2\}$, $\{6, 2, 3\}$, $\{6, 3, 4\}$, $\{6, 4, 5\}$, $\{6, 5, 2\}$. Then $\overline{G(\Delta)}$ has facets $\{1, 6\}$, $\{2, 4\}$, $\{3, 5\}$ and clearly is chordal, thus $k[\Delta]$ has a 2-linear resolution.

The result of the theorem has been partially extended to rings with pure resolution in two papers by Bruns and Hibi, [31] and [32]. (A graded algebra $k[x_1, \dots, x_n]/I = S/I$ has a pure resolution if for each i $(\mathrm{Tor}_i^S(S/I, k))_j \neq 0$ for at most one j.) Note that, in the Cohen-Macaulay case, the Betti numbers are determined by shifts in a pure resolution, see [81].

Let α and β be two monomials in $k[x_1, \dots, x_n]$. In [7] ideals $I(\alpha, \beta)$ generated by all monomials $\{m; \alpha \le m \le \beta\}$ in lexicographic order, are treated, and it is determined when they have linear resolution.

Definition 53 The a-invariant of a graded algebra R of dimension d is defined in terms of local cohomology with respect to the graded maximal ideal $\mathbf{m}$, $a(R) = \max\{n \in \mathbb{Z}; (H^d_{\mathbf{m}})_n \neq 0\}$.

Let $A = k[\Delta]$ be a d-dimensional Buchsbaum Stanley–Reisner ring of embedding dimension n. The following was proved by Hibi, [85].

Theorem 54 *A has t-linear resolution if and only if the following two conditions are satisfied:*

(i) $\tilde{H}_i(\Delta; k) = 0$ *for all* $i \neq t-2$.
(ii) $a(k[lk_\Delta\{i\}]) \leq t-d$ *for all* $i = 1, \dots, n$

This was improved by Terai and Yoshida, [125], who showed that (i) could be replaced by $\tilde{H}_{t-1}(\Delta; k) = 0$.

A Ferrer's graph is a bipartite graph G on $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ such that if $(x_r, y_s) \in G$, then $(x_i, y_j) \in G$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. In [37] it is shown that a bipartite graph without isolated vertices has a 2-linear resolution if and only if it is a Ferrer's graph.

In [9] it is shown that weakly stable ideals (and thus squarefree stable ideals) generated by monomials of the same degree have linear resolutions.

It is not possible to get a purely combinatorial description of complexes with 3-linear resolution. The example of Reisner, the minimal triangulation of the projective plane, has a ring with 3-linear resolution exactly when $\mathrm{char}(k) \neq 2$. Nevertheless partial generalizations to t-linear resolutions has been given, see e.g. [36, 44, 66, 137], and [101].

9.3 Betti Numbers and Regularity

Recursive methods to determine Betti numbers of different classes of Stanley-Reisner rings are discussed in [66]. Bounds for Betti numbers for general Stanley-Reisner rings and for those generated in degree 2 are determined in [104].

Definition 55 For a graded algebra $R = S/I$, $S = k[x_1, \dots, x_n]$, the regularity is

$$\max\{j; (\mathrm{Tor}_i^S(R, K))_{i+j}\} \neq 0.$$

Bounds on the regularity of edge rings are given in e.g. [69] ([70]) and [18]. There is a survey in [63].

9.4 Infinite Resolutions

If $R = k[x_1, \dots, x_n]/I = S/I$ is a graded algebra the minimal R-resolution of k is infinite (except when $I = 0$). Also for these resolutions one can, by polarization, restrict to squarefree monomial ideals. The series $P_k^R(t) = \sum_{i \geq 0} \dim_k \mathrm{Tor}_i^R(k, k)t^i$ is often called the Poincaré series of R. Serre [111] asked if the Poincaré series always is a rational function. This was disproved by Anick [4] (also [5]). But if I is generated by monomials, the series is rational [12]. If I is generated by monomials of degree 2, the resolution is linear (S/I is a Koszul algebra) [57].

10 Linear Quotients

Definition 56 A monomial ideal $I = (M_1, \ldots, M_k)$ has linear quotients if $(M_1, \ldots, M_i) : M_{i+1}$ (in some order) is generated by a subset of the variables for all i.

The main reference for ideals with linear quotients is the paper [84] by Herzog and Takayama. They give a lot of examples of monomial rings with linear quotients, among those stable ideals, squarefree stable ideals, and matroidal ideals.

Definition 57 A monomial ideal is polymatroidal if for each pair of minimal generators $u = \prod_{j=1}^{n} x_j^{a_j}$ and $v = \prod_{j=1}^{n} x_j^{b_j}$ with $a_i > b_i$ there exists a k, with $a_k < b_k$, such that $x_k u / x_i$ is another minimal generator.

Definition 58 A polymatroidal monomial ideal is called matroidal if it is squarefree.

Polymatroidal ideals have linear quotients, see [35]. Polymatroidal ideals that are CM are classified in [76].

The following is easy to see.

Theorem 59 *The squarefree monomial $I(\Delta)$ has linear quotients if and only if Δ^* is nonpure shellable.*

In case all minimal generators of $I(\Delta)$ have the same degree, Δ^* is pure shellable, so $k[\Delta^*]$ is CM. Thus $I(\Delta)$ has a linear resolution according to Eagon-Reiner.

The theory of ideals with linear quotients has continued in e.g. [113].

11 Componentwise Linear Ideals and Sequentially CM Complexes

Componentwise linear ideals were introduced by Herzog and Hibi in [73]. For a graded ideal I, let $I_{(d)}$ be the ideal generated by the elements of degree d in I.

Definition 60 I is componentwise linear if $I_{(d)}$ has a linear resolution for all d.

Examples of componentwise linear ideals are stable ideals, see [73].

For a squarefree monomial ideal I, let $I_{[d]}$ be generated by the squarefree monomials of degree d in I.

Definition 61 I is squarefree componentwise linear if $I_{[d]}$ has a linear resolution for all d.

Theorem 62 ([73]) *A squarefree monomial that is squarefree componentwise linear is componentwise linear.*

From the long Tor-sequence coming from

$$0 \longrightarrow (x_1, \ldots, x_n)I \longrightarrow I \longrightarrow I/(x_1, \ldots, x_n)I \longrightarrow 0$$

it follows that if I has a linear resolution, then also $(x_1, \ldots, x_n)I$ has a linear resolution. (Note that $I/(x_1, \ldots, x_n)I$ is a direct sum of the field, placed in the degree of the generators of I.) Thus

Theorem 63 *If I has a linear resolution, then I is componentwise linear.*

Also Betti numbers of componentwise linear ideals are treated in [73]. They show:

Theorem 64 *If I is a componentwise linear ideal, then $\beta_{i,i+j}(I) = \beta_i(I_{(j)}) - \beta_i(\mathbf{m}I_{(j-1)})$*

There are more results on componentwise linear ideals in e.g. [54, 77], and [1].

Theorem 65 *If I is a squarefree componentwise linear ideal, then $\beta_{i,i+j}(I) = \beta_i(I_{[j]}) - \beta_i(\mathbf{m}I_{[j-1]})$*

Stanley gave the following definition, and showed that a shellable complex has a sequentially CM ring.

Definition 66 ([116, Chapter III:2]) A graded module M over a polynomial ring is sequentially Cohen-Macaulay if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay for all i, and the dimensions increase:

$$\dim M_1/M_0 < \dim M_2/M_1 < \cdots < \dim M_r/M_{r-1}.$$

The following theorem extends the Eagon-Reiner theorem.

Theorem 67 ([73, 83]) *$k[\Delta]$ is componentwise linear if and only if the Alexander dual $k[\Delta^*]$ is sequentially CM.*

Duval, [41], shows the following.

Theorem 68 ([41, Theorem 3.3]) *$k[\Delta]$ is sequentially CM if and only if all pure skeletons of Δ are CM.*

Using this one can show

Theorem 69 *A shellable complex is sequentially CM.*

There is an alternative characterization of sequentially CM complexes.

Definition 70 Let $\Delta_{\langle m \rangle}$ be the subcomplex of Δ generated by all facets of dimension $\geq m$.

Theorem 71 *Δ is sequentially CM if and only if $\tilde{H}_r((lk_\Delta F)_{\langle m\rangle}) = 0$ for all $F \in \Delta$ and all $r < \dim(lk_\Delta F)$.*

Björner et al. [25] contains a lot on sequentially CM complexes and posets.

12 Powers and Symbolic Powers of Stanley-Reisner Ideals

In [79] the authors study monomial ideals I such that I^k has a linear resolution for all k. If $I = (M_1, \ldots, M_r) \subset S = k[x_1, \ldots, x_n]$, then $R(I) = \oplus_j I^j t^j = S[M_1 t, \ldots, M_r t] \subset S[t]$ is the Rees algebra of I. Let $f : S[y_1, \ldots, y_r] \to R(I)$, where $f(x_i) = x_i$, $f(y_j) = M_j t$. If $\deg(x_i) = (1, 0), \deg(y_j) = (0, 1)$ we get a bigraded resolution of the kernel. They give a short proof of the fact that if the regularity with respect of the x-variables is 0, then all powers of I have linear resolution. (This was earlier proved in [110].) Their main result is that if I is generated in degree 2, then I has a linear resolution if and only if S/I has linear quotients and if and only if all powers of I have linear resolution.

In [132] it is proved that all symbolic powers of a squarefree monomial ideal are CM if and only if the complex is a matroid.

Let $I \subset S = k[x_1, \ldots, x_n]$ be an ideal.

Definition 72 The mth symbolic power of I is $I^{(m)} = S \cap (\cap_{P \in \mathrm{Ass} I} I^m S_P)$. In case I is radical, e.g. a Stanley-Reisner ideal, $I^{(m)}$ consist of the elements that vanish to order m on $V(I)$. The symbolic Reesalgebra is $\oplus_m I^{(m)}$.

The symbolic Reesalgebra is not Noetherian in general, but it is for squarefree monomial ideals, [94]. For all monomial ideals it is normal and CM, see [78]. Minimal generators for the Rees algebra for some edge rings are determined in [40].

In many articles the ordinary and symbolic powers and integral closure of powers of Stanley-Reisner ideals are compared, e.g. [26, 65, 127], and [51].

Investigations on when symbolic powers are CM has been done e.g. in [98, 99, 109], and [124].

Associated primes to powers and symbolic powers are studied in e.g. [53, 78], and [64].

Finite length cohomology in [62].

13 Shifting

Algebraic shifting was introduced by Kalai [89].

Definition 73 Let Δ be a complex on $[n]$. Δ is called shifted if for all r, s, $1 \le r < s \le n$ and all faces in $F \in \Delta$ such that $r \in F, s \notin F$ we have $F \setminus \{r\} \cup \{s\} \in \Delta$.

A shifting operation on a complex Δ gives a complex Shift(Δ) such that

(1) Shift(Δ) is shifted
(2) If Δ is shifted, then Shift(Δ) $= \Delta$
(3) Δ and Shift(Δ) have the same **f**-vector
(4) If Γ is a subcomplex of Δ, then Shift(Γ) $\subset$ Shift(Δ)

Shifting a complex has become a common mean to get an easier complex preserving f-vector. Different kinds of shifting operations and connections to generic initial ideals and lexsegment ideals is described in [72]. Shifting has been used in e.g. [1, 11, 103], and [105].

14 Edge Ideals, Path Ideals, Facet Ideals

14.1 Edge Ideals of Graphs and Clutters

Much work has been done to classify CM graphs. Since this seems too ambitious, one has restricted to e.g. trees [133], bipartite graphs, [75] and [131], or chordal graphs, [80] and [55].

The presentation of the Rees algebra of an edge ideal is determined in [135]. In [45] it is shown that bipartite CM edge rings are shellable. In [136] all unmixed bipartite edge rings are classified. A subset C of the vertices is called a vertex cover if every edge is incident to C. The graph is unmixed if all minimal vertex covers have the same size.

There is a description of a package for calculating on edge ideals in Macaulay 2 in [129]. There is a survey on edge rings in [128], and [14] is a survey on regularity for edge ideals and their powers.

Chordal graphs were generalized to clutters in [137], where it is shown that they are shellable. In [44] another generalization of chordal graphs is discussed.

There is a survey in [102]. In [91] multipartite uniform (all parts have the same size) CM edge ideals are classified.

14.2 Facet Ideals

A special kind of facet ideals, those for simplicial trees, is studied in [48, 49], and [139]. A facet F is called a leaf if it is the only facet, or if for there is a facet $G \neq F$ such that $F \cap H \subset F \cap G$ for each facet $H \neq F$. A facet ideal is called a simplicial tree if each nonempty subcomplex has a leaf.

14.3 Vertex Decomposability

Definition 74 A simplicial complex Δ is vertex decomposable if either it is a simplex or it has a vertex v (shedding vertex) such that

(1) Both $\Delta \setminus v$ and $lk_\Delta v$ are vertex decomposable
(2) No face of $lk_\Delta v$ is a facet of $\Delta \setminus v$

Vertex decomposable complexes are shellable, and thus sequentially CM, see e.g [138]. It is shown that chordal rings are vertex decomposable.

In [38] many earlier results are reproved or extended with easy proofs. They show e.g. that Ferrer's graphs, cycles with a triangle attached to an edge, and graphs with whiskers (a new edge) attached to each vertex are vertex decomposable and thus shellable and sequentially CM. Further that graphs with complement a tree are pure and shellable, thus CM. The homology of C_5-free vertex decomposable are studied in [90]. In [130] it is proved that bipartite sequentially graphs are vertex decomposable. In [95] it is shown that for well-covered graphs CM implies vertex decomposable. A graph is well-covered if it is unmixed without isolated vertices and with $2\text{ht}(I(G) = |V|$.

Further in [100] the concept vertex splittable is introduced, and it is shown that Δ is vertex decomposable if and only if the Alexander dual is vertex splittable. This is used to get a Betti splitting and to show that the dual has linear quotients. There are more on vertex decomposable ideals in [38].

14.4 Path Ideals

The original edge ideals were generalized by Conca and de Negri in [34] to path ideals and path algebras. Given an integer t, and a directed graph, the ideal is generated by $\{x_{i_1} \cdots x_{i_t}\}$ for all paths $(i_1, \dots, i_t)$ in the graph. (Thus, edge ideals correspond to $t = 2$.) They showed, among other things, that the Rees algebra of a rooted tree is normal and CM. The study has continued in e.g. [71] and [27, 28]. Later path ideals of cycles and lines have been investigated, see [2, 112], and [3].

15 Books

There are several books containing material on Stanley-Reisner rings. We mention books by Stanley, Bruns-Herzog, Villarreal, and Miller-Sturmfels. In [116, Chapter 3] **f**- and **h**-vectors, Gorenstein complexes, canonical modules of Stanley-Reisner rings, Buchsbaum complexes, and doubly CM complexes are treated. A complex Δ is doubly CM if it is CM and for each vertex v, $\Delta \setminus v = \{F \in \Delta; v \notin F\}$ is CM. In [116, Chapter 4] shellable complexes and sequentially CM rings as well as flag complexes are treated. In [30, Chapter 5] one finds results on local cohomology and canonical module of Stanley-Reisner rings, as well as Gorenstein complexes and Betti numbers. [134, Chapter 6 and 7] deals with Stanley-Reisner rings and edge ideals. In [97] there are chapters on Borel-fixed ideals, cellular resolutions, and Alexander duality.

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Symbolic Rees Algebras

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1 Introduction

Symbolic powers arise from the theory of primary decomposition. It is often surprising to the novice algebraist that the powers of an ideal can acquire associated primes that were not associated to the ideal itself. In that sense, the symbolic powers of I are more natural.

Definition 1.1 Let R be a Noetherian ring and I an ideal in R with no embedded primes. The n-**th symbolic power** of I is the ideal

$$I^{(n)} \quad := \bigcap_{P \in \mathrm{Ass}(R/I)} I^n R_p \cap R.$$

This is the ideal obtained by intersecting the components in a primary decomposition of I corresponding to the associated primes of I, which by assumption are all minimal. When I does have embedded primes, there are two possible definitions of symbolic power to chose from: either taking P to range over the associated primes of I, or over the minimal primes of I. To avoid this, we will focus on the case of ideals with no embedded primes. Note that the symbolic powers of a prime ideal are

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already very interesting, and thus our assumption that I has no embedded primes is fairly mild.

When I is a radical ideal in $R = k[x_1, \ldots, x_d]$, where k is a perfect field, $I^{(n)}$ coincides with the set of polynomials that vanish to order n on the variety defined by I [32, 92, 118]. In general, we always have $I^{(1)} = I$, by definition, and it is easy to show that $I^n \subseteq I^{(n)}$ always holds. However, given an ideal I and some $n > 1$, determining whether the equality $I^n = I^{(n)}$ holds can be a very difficult question. This stems from the fact that computing primary decompositions is a difficult problem; as Decker, Greuel, and Pfister write in [24], "providing efficient algorithms for primary decomposition of an ideal [...] is [...] still one of the big challenges for computational algebra and computational algebraic geometry". In fact, even if one restricts to monomial ideals, the problem of finding a primary decomposition is NP-complete [68]. This is one of the reasons why many innocent sounding questions one could ask about symbolic powers remain open.

Nevertheless, there exist sufficiently efficient methods for computation of symbolic power ideals using computer algebra systems such as *Macaulay2* [57]. Some of these methods are used in the Macaulay2 package *SymbolicPowers*; we refer to [25] for an account of the functionality offered by this package.

Symbolic powers are ubiquitous throughout commutative algebra, with connections to virtually all topics in the field. For a more general survey on symbolic powers, we direct the reader to [21]. In this survey, we focus on symbolic Rees algebras.

2 Symbolic Rees Algebras

The symbolic powers of I form a graded family of ideals, meaning that $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$ for all a and b. Thanks to this simple property, we can package together all the symbolic powers of I to form a graded ring. This is the so called symbolic Rees algebra of I, which contains much information about I and its symbolic powers, and the main character in this survey.

Definition 2.1 (Symbolic Rees Algebra) Let R be a Noetherian ring and I an ideal in R. The **symbolic Rees algebra** of I, also known as the **symbolic blow-up ring** of I, is the graded ring

$$\mathcal{R}_s(I) := R[It, I^{(2)}t^{(2)}, \ldots] = \bigoplus_{n \geqslant 0} I^{(n)}t^n \subseteq R[t].$$

The indeterminate t of degree one is helpful in keeping track of the degree of elements in the symbolic Rees algebra. It helps distinguish an element $f \in I^{(n)}$, which we write ft^n, from the element $f \in I$, which we write ft.

This construction is akin to that of the Rees algebra of I, which is the graded ring

$$\mathcal{R}(I) := R[It, I^2t^2, \ldots] = \bigoplus_{n \geqslant 0} I^n t^n \subseteq R[t].$$

The study of Rees algebras is very rich and presents its own challenges (see [112] for an overview), and yet the symbolic Rees algebra of I is often much more complicated than the ordinary Rees algebra. While the study of symbolic Rees algebras is certainly inspired by Rees algebras, there is a crucial difference: $\mathcal{R}(I)$ is a finitely generated R-algebra, while $\mathcal{R}_s(I)$ may fail to be an algebra-finite extension of R. Indeed, the Rees algebra of I is generated over R in degree 1, by a (finite) generating set of I, that is, $\mathcal{R}(I)$ is a standard graded Noetherian ring, i.e., generated as an R-algebra by elements of degree 1. In contrast, the symbolic Rees algebra may require infinitely many generators. As we will see, the symbolic Rees algebra of I is a finitely generated R-algebra if and only if $\mathcal{R}_s(I)$ is a Noetherian ring. Even if $\mathcal{R}_s(I)$ is Noetherian, it may be generated in different degrees; we introduce the generation type in Sect. 3.2 to quantify this. The symbolic Rees algebra of I is generated in degree 1 precisely if $I^n = I^{(n)}$ for all $n \in \mathbb{N}$, in which case $\mathcal{R}(I)$ and $\mathcal{R}_s(I)$ coincide. Sufficient criteria for this equality are presented in [67] and [100].

2.1 A Brief History

Although symbolic Rees algebras appear implicitly in the 1950s in work of Rees, Zariski, Nagata, and others surveyed below, this class of algebras did not acquire a name until several decades later. To our knowledge, the terminology "symbolic Rees algebra" appears for the first time in Huneke's paper [72] in 1982, while the monograph [112] by Vasconcelos proposes the alternative terminology "symbolic blowup algebra".

The first example of an ideal whose symbolic Rees algebra is not finitely generated appears in Rees' counterexample to Zariski's Formulation of Hilbert's 14th Problem (Question 2.2).

Question 2.2 (Hilbert's 14th Problem) Let k be a field. For all $n \geqslant 1$, and all subfields K of $k(x_1, \ldots, x_n)$, is $K \cap k[x_1, \ldots, x_n]$ finitely generated over k?

An important special case that provided the original motivation for this question concerns the ring of invariants of a linear action of a group of matrices on a polynomial ring over a field. For $R = k[x_1, \ldots, x_n]$, a polynomial ring with coefficients in a field k equipped with a linear action of a group $G \subseteq \mathrm{GL}_n(k)$, one studies the subring of G-invariant polynomials

$$R^G = \{f \in R \mid g \cdot f = f \text{ for all } g \in G\}.$$

A fundamental question in invariant theory is whether R^G is finitely generated as a k-algebra. For finite groups, an affirmative answer is due to E. Noether [93]. The finite generation of R^G is the particular case of Question 2.2 where K is the subfield of elements of the fraction field of R fixed by G.

We point the reader to the surveys [39, 90] for more on Hilbert's 14th problem, and we will instead focus on the connections between symbolic Rees algebras and this famous question. The foundation of this connection was laid by Zariski in the early 1950s in [119] by interpreting the rings $K \cap k[x_1, \ldots, x_n]$ as rings of rational functions on a nonsingular projective variety X with poles restricted to a specified divisor D. Such varieties X can be obtained geometrically by the procedure of blowing up, and D is usually taken to be the exceptional divisor of the blow up X.

Zariski [95] formulated a more general version of Question 2.2, by taking any integrally closed domain that is finitely generated over k in place of $R = k[x_1, \ldots, x_d]$. The first counterexample to Zariski's version of Question 2.2 was given by Rees [95], and this is where the connection with symbolic Rees algebras first appears. The crux of Rees' proof, while not written in the language of symbolic Rees algebras, consists of showing that if P is a height 1 prime ideal in the affine cone over an elliptic curve with infinite order in the divisor class group, then its symbolic Rees algebra is not finitely generated. We give a numerical example to illustrate the principles used by Rees.

Example 2.3 (Rees) Consider the elliptic curve C cut out by the equation $x^3 - y^2z - 2z^3$ in the projective plane $\mathbb{P}^2_{\mathbb{Q}}$. The point $p = (3, 5, 1)$ is a rational point on this curve which has infinite order with respect to the group law on C [80, Example 2.4.6(3)]. Consider the coordinate ring $R = \mathbb{Q}[x, y, z]/(x^3 - y^2z - 2z^3)$ of C and the ideal $P = (x - 3z, y - 5z)$ of R which defines p. Then [95] yields that $\mathcal{R}_S(P)$ is not a finitely generated $\mathbb{Q}$-algebra.

By contrast, consider the point $q = (2, 3, 1)$ on the elliptic curve with coordinate ring $S = \mathbb{Q}[x, y, z]/(x^3 - y^2z + z^3)$. The point q has order six with respect to the group law of this curve [80, Example 2.4.2], and examining the ideal $Q = (x - 2z, y - 3z)$ defining this point with Macaulay2 [57] yields

$$Q^{(6)} = (12x^2 - 6xy + y^2 - 6xz - 6yz + 9z^2),$$

which is a principal ideal. Moreover, $Q^{(6n)} = (Q^{(6)})^n$ for all $n \geqslant 0$, which as we will see in Proposition 3.1 implies that $\mathcal{R}_S(Q)$ is a finitely generated $\mathbb{Q}$-algebra.

In the late 1950s, Nagata found the first example of an ideal I in a polynomial ring whose symbolic Rees algebra is not finitely generated, giving a counterexample to Hilbert's 14th Problem [91]. In fact, he constructed a ring of invariants which is not a finitely generated algebra. The ideal constructed by Nagata defines a set of 16 points in the projective plane, and hence is not a prime ideal like in the example provided by Rees. Nagata's method is to relate the structure of $\mathcal{R}_s(I)$ to an interpolation problem in the projective plane, namely, that for each $m \geqslant 1$, there does not exist a curve of degree $4m$ having multiplicity at least m at each of 16 general points of the projective plane.

In the 1980s, Roberts constructed new examples of symbolic Rees algebras that are not finitely generated based on Nagata's examples. His work shows that $\mathcal{R}_s(I)$ may fail to be finitely generated even when I is a prime ideal in a regular ring [98], thus answering a question of Cowsik in the negative [18]. Roberts' examples are prime ideals in a polynomial ring over a field of characteristic 0, and later Kurano [79] showed that if we consider the same examples as in [98] in prime characteristic p, their symbolic Rees algebras are in fact Noetherian. Roberts' examples [98], while prime, are not analytically irreducible, meaning that these prime ideals do not stay prime after passing to the completion; he later improved this by providing an example that was in fact analytically irreducible [99].

Still, as shown below, finite generation has powerful consequences for some symbolic Rees algebras, and thus it is natural to ask when this occurs. Huneke gave a general criterion for a symbolic Rees algebra of a height 2 prime ideal in a three-dimensional regular local ring to be finitely generated [72, 74], which we will discuss in more detail in Sect. 3.

In the early 1980s, Cowsik showed that if I determines a curve in $\mathbb{A}^n_k$, where k is an infinite field, and the symbolic Rees algebra of I is finitely generated, then I is a set-theoretic complete intersection [18]. This has been exploited to show that certain curves are indeed set-theoretic complete intersections in [33]. A modern generalization of Cowsik's result states that, if the symbolic Rees algebra of an ideal I is finitely generated, the arithmetic rank of I, that is, the least number of generators of an ideal whose radical agrees with the radical of I, is bounded above by the polynomial order of growth for the number of generators of $I^{(n)}$ as a function of n; see [27, Proposition 2.3].

One interesting case is that of space monomial curves, which are Zariski closures of images of maps of the form $\mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t^a, t^b, t^c)$. We abbreviate this by referring to a monomial curve as (t^a, t^b, t^c). The defining ideals of space monomial curves were known to be set theoretic complete intersections since 1970 [59], and thus one could hope that in fact their symbolic Rees algebras are always finitely generated. This is, however, false: Goto, Nishida, and Watanabe [54] found the first counterexamples, a family of choices of (a, b, c) which give infinitely generated symbolic Rees algebras over a field in characteristic 0. We record this interesting family of examples below.

Example 2.4 (Goto–Nishida–Watanabe) Let P be the defining ideal in the power series ring $k[\![x, y, z]\!]$ over a field k of the space monomial curve

$$x = t^{7n-3},\ y = t^{(5n-2)n},\ z = t^{8n-3} \quad \text{where } n \geqslant 4 \text{ and } n \not\equiv 0 \pmod 3.$$

For example, when $n = 4$, our curve is parametrized by $x = t^{25}$, $y = t^{72}$, $z = t^{29}$, and

$$P = (y^3 - x^4z^4, x^{11} - yz^7, x^7y^2 - z^{11}).$$

Then $\mathcal{R}_s(P)$ is a non-Cohen-Macaulay Noetherian ring if char $k > 0$, and $\mathcal{R}_s(P)$ is not a Noetherian ring if char $k = 0$.

Other examples where $\mathcal{R}_s(P)$ is not Cohen-Macaulay with P the defining ideal of (t^a, t^b, t^c) in prime characteristic were already known by [51].

Recently, Sannai and Tanaka [109] showed that there are primes ideals with non-finitely generated symbolic Rees algebras over *any* field:

Theorem 2.5 (Sannai–Tanaka, 2019 [109]) *Let k be a field. There exists a prime ideal* $\mathfrak{p}$ *in* $k[x_1, \ldots, x_{12}]$ *such that* $\mathcal{R}_s(P)$ *is not Noetherian.*

The late 1980s and early 1990s saw a program to classify when these symbolic Rees algebras are (or are not) finitely generated [86, 103, 108]. Most notably, Cutkosky gave criteria which say, for example, that over any field, $\mathcal{R}_s(P)$ is finitely generated whenever $(a + b + c)^2 > abc$. Cutkosky's work [19, Lemma 7] also uncovered a deep connection to a different geometric problem: that $\mathcal{R}_s(P)$, where P defines (t^a, t^b, t^c), is Noetherian if and only if a certain space—the blow-up at a general point of the weighted projective space $\mathbb{P}(a, b, c)$—is a Mori dream space—meaning its Cox ring is Noetherian. Using this connection, González Anaya, González, and Karu [40–42, 49, 50] have more recently found several large families of examples in characteristic 0 that in particular recover the original family of examples of Goto–Nishida–Watanabe of Non-noetherian $\mathcal{R}_s(P)$; in fact, they give a complete characterization of when $\mathcal{R}_s(P)$ is (non)Noetherian for large families of curves of type (t^a, t^b, t^c). The smallest of their examples to date are the curves (t^7, t^{15}, t^{26}) and (t^{12}, t^{13}, t^{17}), each of these examples being smallest in a different manner. For more examples of this kind, see also [58]. The original family of Non-noetherian examples in [54] has also been generalized via different methods in [76].

Finally, the story of symbolic Rees algebras of space monomial curves has deeper connections to Hilbert's 14th Problem: Kurano and Matsuoka showed that whenever the symbolic Rees algebra of the defining ideal P of (t^a, t^b, t^c) is not Noetherian, then in fact $\mathcal{R}_s(P)$ is a counterexample to Hilbert's 14th Problem [77].

Even when the symbolic Rees algebra $\mathcal{R}_s(P)$ of the curve (t^a, t^b, t^c) in indeed Noetherian, it may still be generated in various degrees. As a corollary of a result of Huneke's [73, Corollary 2.5], we know $P^{(n)} = P^n$ for all $n \geqslant 1$, or equivalently $\mathcal{R}_s(P)$ is generated in degree 1, exactly when P is a complete intersection. In the language of Sect. 3.2, we say that $\mathcal{R}_s(P)$ has generation type 1. Herzog and Ulrich characterized when the symbolic Rees algebra $\mathcal{R}_s(P)$ is generated in degree up to 2, or has generation type 2, and showed that this implies that P is self-linked [70]. The cases when $\mathcal{R}_s(P)$ has generation type 3 [53] and 4 [96, 97] have also been completely characterized; these characterizations are all in terms of the Hilbert-Burch matrix of P.

With the subject of finite generation presenting such a difficult problem, the literature on other ring-theoretic properties of $\mathcal{R}_s(I)$ is not as vast. Watanabe [117] asked whether $\mathcal{R}_s(I)$ must be Cohen-Macaulay whenever it is Noetherian, where I is a divisorial ideal in a strongly F-regular ring R. Watanabe constructed an example [117, Example 4.4] of a divisorial ideal I in an F-rational ring whose Rees algebra is Noetherian but not Cohen-Macaulay. When R is strongly F-regular, Singh showed that the answer to Watanabe's question is affirmative provided that a certain auxiliary ring is finitely generated over R [106]. The construction of this auxiliary ring is an iterated symbolic Rees algebra.

In positive characteristic, the symbolic Rees algebra of the canonical module $\omega = \omega_R$ of a local, normal, complete ring R plays an important role in studying Frobenius actions on the injective hull of the residue field. A significant construction in this context is the anticanonical cover $\bigoplus_{n\geqslant 0} \operatorname{Hom}_R(\omega^{(n)}, R)$. The number of generators for this ring as an algebra over R, if finite, bounds the Frobenius complexity of R as shown by Enescu and Yao [37].

The research and literature surrounding symbolic Rees algebras is abundant and growing at a steady rate. While we cannot do complete justice to this topic by presenting an exhaustive review, we expand in some directions which are closest to our interests in the following sections.

3 Criteria for Noetherianity

In this section we discuss criteria for finite generation, and equivalently Noetherianity, of symbolic Rees algebras and structural invariants of finitely generated symbolic Rees algebras.

3.1 Noetherianity

The most comprehensive criterion, described below in Proposition 3.1 (4) $\Leftrightarrow$ (1), states that, under mild hypotheses, finite generation of a symbolic Rees algebra $\mathcal{R}_s(I)$ is equivalent to the fact that there exists a Veronese subalgebra $\bigoplus_{n\geqslant 0} I^{(kn)}t^{kn}$ isomorphic to the (ordinary) Rees algebra $\mathcal{R}(I^{(k)}) = \bigoplus_{n\geqslant 0}(I^{(k)})^n t^n$. An equivalent assertion is that a Veronese subalgebra of $\mathcal{R}_s(I)$ admits a standard grading.

The various parts of the following criterion appear in different places in the literature: the equivalence of (1) and (3) is developed in [95] and (4) appears in work of Schenzel [104, Theorem 1.3]. We include a proof since this result is central to our discussion.

Proposition 3.1 (Standard Graded Subalgebra Criterion) *Let R be a Noetherian ring and I an ideal in R. The following are equivalent:*

(1) $\mathcal{R}_s(I)$ is a finitely generated R-algebra.
(2) $\mathcal{R}_s(I)$ is a Noetherian ring.
(3) There exists d such that for all $n \geqslant 1$,

$$I^{(n)} = \sum_{a_1+2a_2+\cdots+da_d=n} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}.$$

Furthermore, when these equivalent conditions hold, then

(4) There exists k such that $I^{(kn)} = \left(I^{(k)}\right)^n$ *for all* $n \geqslant 1$.

Conditions (1)–(4) are equivalent whenever R is an excellent ring.

Proof The fact that (1) implies (2) is a consequence of Hilbert's Basis Theorem. Moreover, since $\mathcal{R}_s(I)$ is an $\mathbb{N}$-graded algebra and $\mathcal{R}_s(I)_0 = R$ is a Noetherian ring, the equivalence between (1) and (2) is a general fact about graded R-algebras; see for example [8, Proposition 1.5.4] for a proof. Statement (3) says that $\mathcal{R}_s(I)$ is generated in degree up to d as an R-algebra, and thus is equivalent to (1).

To show that (3) implies (4), we follow [95, Lemma 2], where in fact a stronger statement is proved. We will show that k can in fact be taken to be $k = d \cdot d!$.

First, suppose that $n \geqslant k$. For each choice of $a_1 + 2a_2 + \cdots + da_d = n \geqslant d \cdot d!$, we must have $ia_i \geqslant d!$ for some i, by the pigeonhole principle. Moreover, $q := \frac{d!}{i}$ is an integer, so

$$I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d} = \left(I^{(i)}\right)^q I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(i)}\right)^{a_i-q}\cdots\left(I^{(d)}\right)^{a_d}$$
$$\subseteq I^{(d!)}I^{(n-d!)}.$$

In particular, $I^{(n)} \subseteq I^{(d!)}I^{(n-d!)}$ for all $n \geqslant d \cdot d!$, but since $I^{(d!)}I^{(n-d!)} \subseteq I^{(n)}$ holds because symbolic powers form a graded family, in fact we have shown that $I^{(n)} = I^{(d!)}I^{(n-d!)}$.

Now consider any $n \geqslant 1$. Since $nk \geqslant k = d \cdot d!$, then

$$I^{(kn)} = I^{(d!)}I^{(kn-d!)} = \left(I^{(d!)}\right)^2 I^{(kn-2d!)} = \cdots = \left(I^{(d!)}\right)^d I^{(kn-d\cdot d!)}$$
$$\subseteq I^{(d\cdot d!)}I^{(kn-d\cdot d!)},$$

so that

$$I^{(kn)} = I^{(d\cdot d!)}I^{(kn-d\cdot d!)} = I^{(k)}I^{(k(n-1))}.$$

By induction, the statement follows.

On the other hand, if (4) holds, then the algebra

$$A := \bigoplus_{n\geqslant 0} I^{(kn)}t^{kn} = \bigoplus_{n\geqslant 0}\left(I^{(k)}t^k\right)^n \subseteq \mathcal{R}_s(I) \subseteq R[t]$$

is finitely generated. The fact that (4) implies the remaining equivalent statements will follow once we show that $\mathcal{R}_s(I)$ is a finitely generated algebra over A. To do that, we follow the argument in [104, (2.2)].

Let B denote the integral closure of A inside $R[t]$. Recall[1] that B is the subring of $R[t]$ given as follows:

$$B = \left\{ f \in R[t] : f^d + a_{d-1} f^{d-1} + \cdots + a_1 f + a_0 = 0 \text{ for some } f_i \in A \right\}.$$

We claim that $\mathcal{R}_s(I) = \bigoplus I^{(n)} t^n \subseteq B$. To show that, consider $u \in I^{(i)} t^i$. Then

$$u^k \in \left(I^{(i)}\right)^k t^{ik} \subseteq I^{(ki)} t^{ki} = \left(I^{(k)} t^k\right)^i,$$

so that u is a root of $T^k - u^k$. Since $u^k \in A$, u is integral over A, which implies that $u \in B$. Since $\mathcal{R}_s(I)$ is generated by such elements, we conclude that $\mathcal{R}_s(I) \subseteq B$. Moreover, B is a finitely generated module over A, by [105, Remark 12.3.11 or Theorem 9.2.2]. Therefore, $\mathcal{R}_s(I)$ must be finitely generated over A by the Artin-Tate theorem [1]. □

For Proposition 3.1 (4) $\Rightarrow$ (3), the condition we need is that the integral closure of a finitely generated R-algebra B in a finite extension is a finitely generated algebra over B; rings with this property are called Nagata rings (see [82, Chapter 13]). This holds whenever R is excellent or analytically unramified, and in particular every polynomial or power series ring over a field has this property.

Remark 3.2 The proof of Proposition 3.1 shows that when the symbolic Rees algebra is Noetherian and generated in degree up to d, then for $k = d \cdot d!$, we do have $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. In fact, it is shown in [95, Lemma 2] that if the symbolic Rees algebra is generated in degrees $a_1, \ldots, a_s$, and r is the least common multiple of $a_1, \ldots, a_s$, then we can take $k = sr$.

Under mild assumptions, part (3) of Proposition 3.1 above might be rewritten, as follows:

Lemma 3.3 *Let R be an excellent ring, and I an ideal in R. Suppose that k is such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. Then there exists $A \geqslant 1$ such that for all $n \geqslant 1$, if $n = qk + r$, with $0 \leqslant r < k$, then*

$$I^{(n)} = \sum_{a=0}^{A} \left(I^{(k)}\right)^{q-a} I^{(ak+r)}.$$

Proof As before, note that the R-algebra

$$B := \bigoplus_{n \geqslant 0} I^{(kn)} t^{kn} = \bigoplus_{n \geqslant 0} \left(I^{(k)} t^k\right)^n \subseteq R[t]$$

is finitely generated, and that $\mathcal{R}_s(I)$ is finitely generated over B.

[1] The book [105] is a comprehensive reference on the subject of integral closure.

Suppose that $\mathcal{R}_s(I)$ is generated over B in degrees $a_1, \ldots, a_d$. Then

$$\bigoplus_{n \geqslant 0} I^{(n)} t^n = \mathcal{R}_s(I) = I^{(a_1)} t^{a_1} B \oplus \cdots \oplus I^{(a_d)} t^{a_d} B = \bigoplus_{m \geqslant 1} I^{(a_i)} \left(I^{(k)}\right)^m t^{a_i + km}.$$

Finally, the theorem follows once we collect the pieces in degree n. □

While very useful, the criteria in Proposition 3.1 often prove challenging to apply because they require checking infinitely many equalities of ideals. The next results of Huneke [74, Theorems 3.1 and 3.25] present ideal-theoretical criteria for the symbolic Rees ring $\mathcal{R}_s(P)$ of a height two prime P of a three-dimensional regular ring R to be Noetherian, which are relatively simple to apply. These criteria suffice to establish that every affine space curve of degree three as well as every monomial space curve of degree four have Noetherian symbolic Rees algebras.

Proposition 3.4 (Multiplicity Criterion) *Let R be a regular local ring with* $\dim(R) = 3$ *and infinite residue field and let P be a height two prime ideal of R. The following are equivalent:*

(1) $\mathcal{R}_s(P)$ is a finitely generated R-algebra.
(2) There exist $k, l \geqslant 1$, $f \in P^{(k)}$, $g \in P^{(l)}$ and $x \notin P$ such that

$$\lambda(R/(f, g, x)) = kl\lambda\left(R/(P + (x)\right).$$

(3) There exist $f.g \in P$ such that $\sqrt{(f, g)} = P$ and the leading forms f^*, g^* of f, g in the associated graded ring of PR_P form a regular sequence.

It is possible to extend this criterion to reduced ideals of height two that are not necessarily prime. For example, by [61, Proposition 3.5], if an ideal I defines a set of s points in $\mathbb{P}^2$ and if there exist $m \in \mathbb{N}$ and $f, g \in I^{(m)}$ such that f, g form a regular sequence and $\deg(f)\deg(g) = m^2 s$, then $\mathcal{R}_s(I)$ is a Noetherian ring. This criterion can be applied to show that any set of $s \leqslant 8$ points in $\mathbb{P}^2$ gives rise to a Noetherian symbolic Rees algebra. However, the converse implication is no longer valid, as shown in [94].

It turns out that the analytic spread $\ell(I)$ of I, which is defined to be the Krull dimension of the special fiber ring of I, $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, plays an important role in the study of symbolic Rees algebras. Its contribution is due to work of McAdam on asymptotic primes of I. For the rest of this section, we assume R is an excellent domain, although a weaker condition, locally quasi-unmixed, would suffice. Brodmann shows in [12] that the following set, known as the set of *asymptotic primes* of I, is finite:

$$A^*(I) := \bigcup_{n \geqslant 0} \mathrm{Ass}(I^n).$$

McAdam [84, Theorem 3] (see also [85, Proposition 4.1]) shows that $P \in A^*(I)$ if and only if $\ell(IR_P) = \dim(R_P)$. Setting

$$J = \bigcap_{P \in A^*(I) \setminus \mathrm{Min}(I)} P$$

yields another description of the symbolic powers of I as saturations: $I^{(n)} = I^n : J^\infty$. We are now ready to state another criterion for finite generation of the symbolic Rees algebra. This appears for primary ideals in work of Katz and Ratliff [78, Theorem A], and in the form presented here in [15, Theorem 2.6].

Proposition 3.5 (Analytic Spread Criterion) *Let $(R, \mathfrak{m})$ be an excellent domain. Then $\mathcal{R}_s(I)$ is a finitely generated R-algebra if and only if for all $P \in V(J)$ we have that*

$$\ell\left((I : J^\infty)R_P\right) < \dim(R_P).$$

In a similar vein, Goto, Herrmann, Nishida, and Villamayor give a sufficient criterion for the symbolic Rees algebra to be Noetherian in terms of an equimultiplicity condition of some symbolic power. Their result in [47, Theorem 3.3] states that if $\ell(I^{(n)}) = \mathrm{ht}(I^{(n)})$ for some natural number n and ideal I in an unmixed local ring, then $\mathcal{R}_s(I)$ is Noetherian.

A large class of ideals with finitely generated symbolic Rees algebra is the class of monomial ideals. While none of the above criteria apply to show this, finite generation of the respective symbolic Rees algebras follows from Gordan's lemma, which says that the set of all lattice points in a rational cone is a finitely generated affine semigroup. This approach is taken by Herzog, Hibi, and Trung in [62, Proposition 1.4]. For squarefree monomial ideals, finite generation of the symbolic Rees algebra was previously shown in work of Lyubeznik; see [81, Proposition 1].

3.2 Generation Type and Standard Veronese Degree

In this section, we explore the maximum degree of elements required to generate the symbolic Rees ring as an R-algebra and the minimum degree of a standard graded Veronese subalgebra.

Following Bahiano [2], we define

Definition 3.6 The *generation type* of a symbolic Rees algebra $\mathcal{R}_s(I)$ is the value

$$\mathrm{gt}(R_s(I)) := \inf\{d \mid \mathcal{R}_s(I) = R[It, I^{(2)}t^2, \ldots, I^{(d)}t^d]\}.$$

Note that $\mathrm{gt}(R_s(I)) \in \mathbb{N} \cup \{\infty\}$ and $\mathrm{gt}(R_s(I)) \in \mathbb{N}$ if and only if $\mathcal{R}_s(I)$ is a Noetherian ring. A challenging problem is to determine or bound this invariant for interesting classes of ideals.

Problem 3.7 Find effective bounds on $\mathrm{gt}(R_s(I))$, when finite, in terms of invariants of I.

This problem has been studied predominantly in combinatorial contexts, when I is a monomial ideal [2, 62, 83]; it has also been studied for the ideal defining a space monomial curve [53, 96, 97], and in some cases of more general monomial curves, for example in [26]. For a monomial ideal I, finding a minimal set of algebra generators for $\mathcal{R}_s(I)$ translates into finding a Hilbert basis for an appropriate convex polyhedron [83, Corollary 3.2]. This is a computationally intensive problem, which can nevertheless be approached with the aid of specialized software [11, 111].

When I is a monomial ideal and $I^{(n)} = \overline{I^n}$ for each $n \in \mathbb{N}$, i.e. when all the symbolic powers are the integral closures of the corresponding ordinary powers, then [36, Corollary 3.11] yields $\mathrm{gt}(R_s(I)) \leqslant \dim(R) - 1$. When I is the edge ideal of a simple graph, [2] yields $\mathrm{gt}(R_s(I)) \leqslant (\dim(R) - 1)(\dim(R) - \mathrm{ht}(I))$. However, for arbitrary monomial ideals the best known bound seems to be given by [62, Theorem 5.6]:

$$\mathrm{gt}(\mathcal{R}_s(I)) \leqslant \frac{(\dim(R) + 1)^{(\dim(R)+3)/2}}{2^{\dim(R)}}.$$

Proposition 3.1 reveals the importance of standard graded Veronese subalgebras of the symbolic Rees algebra. We introduce a new invariant that captures the least degree where they occur.

Definition 3.8 The *standard Veronese degree* of an ideal I is the value

$$\mathrm{svd}(I) := \inf\{k \mid (I^{(k)})^n = I^{(kn)} \text{ for all } n \in \mathbb{N}\}.$$

As before, $\mathrm{svd}(I) < \infty$ is equivalent to $\mathcal{R}_s(I)$ being a Noetherian ring by Proposition 3.1, and the proof of this proposition yields the upper bound $\mathrm{svd}(I) \leqslant \mathrm{gt}(\mathcal{R}_s(I)) \cdot \mathrm{gt}(\mathcal{R}_s(I))!$. Remark 3.2 yields a sharper upper bound. For particular families of ideals, specific upper bounds can be found in the literature, for example for some space monomial curve families [86] and for ideals defining Fermat-type point configurations [94] (cf. Example 4.6).

We explore these invariants for a specific family of monomial ideals below, with an eye towards evaluating the optimality of these bound.

Example 3.9 Let n and $h \leqslant n - 1$ be positive integers and let $I_{n,h}$ denote the following monomial ideal in the polynomial ring $R_n = k[x_1, \ldots, x_n]$ with coefficients in a field k

$$I_{n,h} := \bigcap_{1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant n} (x_{i_1}, x_{i_2}, \cdots, x_{i_h}).$$

This family of ideals is known as *monomial star configurations*. Then the following hold:

$$\mathcal{R}_s(I) = R_n[x_{i_1}x_{i_2}\cdots x_{i_{n-h+m}}t^m, 1 \leqslant m \leqslant h, i_1 < i_2 < \cdots < i_{n-c+m} \leqslant n], \tag{3.1}$$
$$\mathrm{gt}(\mathcal{R}_s(I_{n,h})) = h, \quad \text{and} \quad \mathrm{svd}(I_{n,h}) \text{ is divisible by } \mathrm{lcm}(1, 2, \ldots, h).$$

Proof The description of the symbolic Rees algebra in (3.1) is established using different notation in [62, Proposition 4.6].

It follows that the generation type of this algebra is h, provided that the unique algebra generator of degree h, $\prod_{i=1}^{n} x_i \in I^{(h)}$, listed in (3.1) cannot be decomposed as a product of squarefree monomials $m_1, m_2, \ldots, m_s$ with $s > 1$, $m_i = x_{i,1}x_{i,2}\cdots x_{i,n-h+a_i} \in I^{(a_i)}$. This would yield $a_1 + \cdots + a_s = h$ and because the degrees of these monomials are $\deg(m_i) = n - h + a_i$ we obtain the following impossible inequality

$$\deg(m_1)+\cdots+\deg(m_s) \geqslant a_1+\cdots+a_s+s(n-h) = h+s(n-h) > n = \deg \prod_{i=1}^{n} x_i.$$

Continuing to a discussion of the standard Veronese degree, let us first observe that the lowest degree of a nonzero element of $I_{n,h}^{(m)}$ is $\alpha(I_{n,h}^{(m)}) = m + (n-h)\lceil \frac{m}{h} \rceil$. A simple calculation now verifies that when $\frac{m}{h}$ is not an integer, then $\alpha(I_{n,h}^{(mk)}) < k\alpha(I_{n,h}^{(m)}) = \alpha((I_{n,h}^{(m)})^k)$ whenever $k > h$. This restricts the possible values for $r := \mathrm{svd}(I_{n,h})$ to multiples of the height h. However, further restrictions on r are imposed by consideration of the manner in which our family of ideals contracts with respect to the inclusions $R_{n-i} \subset R_n$. Specifically, for all h, n, m there are identities

$$\begin{aligned}
I_{n,h}^{(m)} \cap R_{n-i} &= \bigcap_{1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant n} (x_{i_1}, x_{i_2}, \cdots, x_{i_h})^m \cap R_{n-i} \\
&= \bigcap_{j=h-i}^{h} \bigcap_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n-i} (x_{i_1}, x_{i_2}, \cdots, x_{i_j})^m \\
&= I_{n-i,h-i}^{(m)} \cap I_{n-i,h-i+1}^{(m)} \cap \cdots \cap I_{n-i,h}^{(m)} \\
&= I_{n-i,h-i}^{(m)},
\end{aligned}$$

where we make the convention that $I_{n,u} = R_n$ whenever $u < 0$. Similarly one deduces

$$I_{n,h}^{m} \cap R_{n-i} = (I_{n,h} \cap R_{n-i})^m = I_{n-i,h-i}^{m}.$$

If $I^{(rm)} = (I^{(r)})^m$ for all $m \in \mathbb{N}$ then for $0 \leqslant i \leqslant h-1$ we deduce the identities

$$I_{n,h}^{(rm)} \cap R_{n-i} = (I_{n,h}^{(r)})^m \cap R_{n-i}, \text{ i.e.,}$$
$$I_{n-i,h-i}^{(rm)} = (I_{n-i,h-i}^{(r)})^m.$$

By the previous reasoning, we see that r must be divisible by all integers $1 \leqslant h-i \leqslant h$, thus $\operatorname{lcm}(1, 2, \ldots, h)$ divides r. □

We conjecture that for the family of ideals in Proposition 3.9 there is in fact an equality $\operatorname{svd}(I_{n,h}) = \operatorname{lcm}(1, 2, \ldots, h)$. This prompts the following question:

Question 3.10 Can the bound in Remark 3.2 be improved for all monomial ideals I to

$$\operatorname{svd}(I) \leqslant \text{ the lcm of the degrees of any set of algebra generators for } \mathcal{R}_s(I)?$$

At this time we are unaware of any ideals that satisfy $\operatorname{svd}(I) < \operatorname{gt}(I)$. Hence we ask:

Question 3.11 Does the inequality $\operatorname{gt}(I) \leqslant \operatorname{svd}(I)$ hold for every ideal I?

4 Applications to Containment Problems and Asymptotic Invariants

4.1 The Containment Problem

Containments of the form $I^n \subseteq I^{(n)}$ are a direct consequence of Definition 1.1, which further implies that $I^b \subseteq I^{(a)}$ if and only if $b \geqslant a$. Containments of the converse type $I^{(a)} \subseteq I^b$ are a lot more interesting. Together these form the basis for comparison of the ordinary and symbolic ideal topologies, which has been pioneered by Schenzel [102] and later Swanson [110]. This line of inquiry is nowadays known as the containment problem:

Question 4.1 (Containment Problem) Let R be a ring and let I be an ideal of R without embedded primes. For which pairs a, b does the containment $I^{(a)} \subseteq I^b$ hold?

If for each value of b there is a pair a, b answering the above question, then the families $\{I^{(n)}\}_n$ and $\{I^n\}_n$ are cofinal, and induce equivalent topologies. In [110], Swanson shows that the equivalence of ordinary and symbolic ideal topologies is linear, that is, if $\{I^{(n)}\}_n$ and $\{I^n\}_n$ are cofinal then there is a constant c, possibly depending on I, such that $I^{(cn)} \subseteq I^n$ for all $n \geqslant 1$. When the ambient ring is regular, this constant can be expressed explicitly, in terms of the *big height* of I, the largest height of on associated prime of I. In fact, in this case the constant c can even be taken uniformly, depending only on R, as shown by the following important results [34, 60, 87].

Theorem 4.2 (Ein–Lazarsfeld–Smith, Hochster–Huneke, Ma–Schwede) *Let R be a regular ring and I an ideal in R. If h is the big height of I, then $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$. In particular, if $d = \dim(R)$, then $I^{((d-1)n)} \subseteq I^n$ for $n \geqslant 1$.*

If we remove the regular assumption, and ask that R be a complete normal local domain, it is still an open problem in general to determine whether there exists a uniform constant c, depending only on R, such that $P^{(cn)} \subseteq P^n$ for all $n \geqslant 1$ and all primes P. When P is a prime ideal in a complete normal local domain, the P-symbolic and P-adic topologies are equivalent [101]. More generally, if R is an excellent Noetherian domain, the P-symbolic and P-adic topologies are equivalent for every prime P if and only if going down holds between R and its integral closure [66]. Some of the recent progress on this problem in [63–65, 114–116] is also described in some detail in [21].

More surprisingly, the containment problem is not settled even in the regular case. In fact, the containments provided by Theorem 4.2 are not necessarily best possible. In fact, examining the proof of the above theorem in [60] one sees that in the case of positive characteristic, $\mathrm{char}(R) = p$, it relies on containments of the form $I^{(hq)} \subseteq I^{[q]}$, where $q = p^e$ for $e \in \mathbb{N}$ and $I^{[q]}$ denotes the qth Frobenius power of I. The stronger containment $I^{(hq-q+1)} \subseteq I^{[q]}$ follows in this context using localization and the pigeonhole principle as explained in [60, p.351]. This yields the following improved containments:

Proposition 4.3 *Let R be a regular ring, I an ideal of R, and h the big height of I. If $\mathrm{char}(R) = p > 0$, the containments $I^{(hq-h+1)} \subseteq I^q$ hold for $q = p^e$ and for each integer $e \geqslant 1$.*

This leads to the question of whether similar improvements can be carried over to arbitrary characteristic and arbitrary exponents. Harbourne proposed this as a conjecture in [4, 61] for homogeneous ideals, which we write here for radical ideals.

Conjecture 4.4 (Harbourne) Let I be a radical homogeneous ideal in a polynomial ring, and let h be the big height of I. Then the containments $I^{(hn-h+1)} \subseteq I^n$ hold for all $n \geqslant 1$.

Remark 4.5 To compare Conjecture 4.4 to Theorem 4.2, it is instructive to note that Theorem 4.2 implies that $I^{(n)} \subseteq I^{\lfloor \frac{n}{h} \rfloor}$ for $n \geqslant 1$, while Harbourne's Conjecture 4.4 asks if $I^{(n)} \subseteq I^{\lceil \frac{n}{h} \rceil}$ for all $n \geqslant 1$.

There are various cases where Conjecture 4.4 is known to hold: if I is a monomial ideal [4, Example 8.4.5] or more generally if I defines an F-pure ring [43], if I corresponds to a general set of points in $\mathbb{P}^2$ [9] or $\mathbb{P}^3$ [31], and if I defines a matroid configuration [46], that is, a union of codimension c intersections of hypersurfaces such that any subset of at most $c + 1$ of the equations of these hypersurfaces forms a regular sequence. Moreover, versions of Conjecture 4.4 hold in some singular settings as well [52]. Despite that, asking that Conjecture 4.4 holds for any radical ideal in a regular ring turns out to be too general.

Example 4.6 (Dumnicki–Szemberg–Tutaj-Gasińska [29], Harbourne–Seceleanu [69]) Let $n \geqslant 3$ be an integer and let k be a field of with $\mathrm{char}(k) \neq 2$ that contains n distinct roots of unity. Let $R = k[x, y, z]$, and consider the ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$$

defining $n^2 + 3$ points in $\mathbb{P}^2_k$, namely the 3 coordinates points together with the n^2 points defined by the complete intersection $(x^n - y^n, y^n - z^n)$. For this ideal $h = 2$ but $I^{(3)} \not\subseteq I^2$, thus Conjecture 4.4 is not satisfied for $n = 2$.

Example 4.6 is particularly interesting because it shows that Conjecture 4.4 can fail even for ideals with Noetherian symbolic Rees algebra. Indeed, in [94] it is shown that the symbolic Rees algebras of the ideals in Example 4.6 are finitely generated. On the other hand, the family of space monomial curves in Example 2.4, which have (big) height 2, satisfy $I^{(4)} \subseteq I^3$ by [56, Example 4.7]. This is a stronger containment than the one proposed by Conjecture 4.4, and yet these ideals have Non-noetherian symbolic Rees algebras.

In [69], Harbourne and Seceleanu show that the containment $I^{(hn-h+1)} \subseteq I^n$ can fail for arbitrarily high values of n that grow with the dimension of R, if R is a polynomial ring of characteristic $p > 0$. However, in characteristic 0 all the known counterexamples to Conjecture 4.4 found to date [5, 14, 28, 29, 75, 88] are for the value $n = 2$. There are moreover no prime counterexamples to Harbourne's Conjecture 4.4. We emphasize this by asking:

Question 4.7 (Harbourne Conjecture for Primes) If P is a prime ideal in a regular ring and $\mathrm{ht}(P) = h$, then do the containments $P^{(hn-h+1)} \subseteq P^n$ hold for all $n \geqslant 1$?

For example, in characteristic other than 3, it is know that all space monomial curves (t^a, t^b, t^c) satisfy this containment for $n = 2$ [56, Theorem 4.1], and also for $n \gg 0$ [44, Corollary 4.3]. There are also no known counterexamples to the following asymptotic version of Harbourne's conjecture formulated in [56].

Conjecture 4.8 (Stable Harbourne Conjecture) Let R be a regular ring and I a radical ideal of R with big height h. Then there exists $N > 0$ such that the containment $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \geqslant N$.

This stable version of Harbourne's Conjecture does hold for various classes of ideals in equicharacteristic rings, including examples with Non-noetherian symbolic Rees algebra. In Sect. 4.3, we will discuss ideals with expected resurgence, and all of these satisfy the Stable Harbourne Conjecture.

4.2 Noetherian Symbolic Rees Algebras and the Containment Problem

We now consider the implications of having a Noetherian symbolic Rees algebra on the Containment Problem 4.1. The first easy implication is that I satisfies a version of Harbourne's Conjecture with the big height replaced by the generation type.

Lemma 4.9 *Let R be a Noetherian ring and I an ideal in R with $\operatorname{gt}(I) = d$. Then for all $n \geqslant 1$,*

$$I^{(dn-d+1)} \subseteq I^n.$$

In particular, I satisfies Harbourne's Conjecture whenever $\operatorname{gt}(I) \leqslant \operatorname{bight}(I)$.

Proof Fix $n \geqslant 1$. By Lemma 3.1, it is enough to show that for all choices of $a_1, \ldots, a_n \geqslant 0$ such that $a_1 + 2a_2 + 3a_3 + \cdots + da_d = dn - d + 1$,

$$I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d} \subseteq I^{dn-d+1}.$$

To see this holds, note that $\left(I^{(i)}\right)^{a_i} \subseteq I^{a_i}$ for each i, so that

$$I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d} \subseteq I^{a_1+a_2+\cdots+a_d}.$$

For each such choice of $a_1, \ldots, a_d$,

$$d\,(a_1 + \cdots + a_d) \geqslant a_1 + 2a_2 + \cdots + da_d = dn - d + 1,$$

so that

$$a_1 + \cdots + a_d \geqslant \frac{d(n-1)+1}{d}.$$

Since $a_1 + \cdots + a_d$ is an integer, we conclude that

$$a_1 + \cdots + a_d \geqslant (n-1) + 1 = n.$$

□

Moreover, if $\mathcal{R}_s(I)$ is Noetherian, it suffices to check the containments for $n \leqslant \operatorname{gt}(I)$ in Conjecture 4.4 to conclude Harbourne's Conjecture holds for I:

Lemma 4.10 *Let R be a Noetherian ring and I an ideal in R such that $\operatorname{gt}(I) = d$. If h is an integer such that*

$$I^{(i)} \subseteq I^{\left\lceil \frac{i}{h} \right\rceil}$$

for all $i \leqslant d$*, then for all* $n \geqslant 1$*,*

$$I^{(hn-h+1)} \subseteq I^n.$$

Proof Given $n \geqslant 1$,

$$I^{(hn-h+1)} = \sum_{a_1+2a_2+\cdots+da_d=hn-h+1} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}.$$

It is enough to show that for all $a_1, \ldots, a_d \geqslant 0$ such that $a_1 + 2a_2 + \cdots + da_d = hn - h + 1$, the ideal

$$J := I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}$$

is contained in I^n. By assumption, $I^{(i)} \subseteq I^{\lceil \frac{i}{h} \rceil}$ for each i. Therefore, $J \subseteq I^N$, where

$$N \geqslant \sum_{i=1}^{d} a_i \left\lceil \frac{i}{h} \right\rceil \geqslant \sum_{i=1}^{d} \frac{ia_i}{h} = \frac{hn-h+1}{h}.$$

Since N is an integer, we must have

$$N \geqslant \left\lceil \frac{hn-h+1}{h} \right\rceil = n.$$

□

For example, as a consequence of Lemma 4.10 and [56, Theorem 4.4], Harbourne's Conjecture holds for space monomial curves of generation type up to 6.

In a similar vein, one may ask if the Stable Harbourne Conjecture holds when $\mathcal{R}_s(I)$ is Noetherian. Here is some evidence in that direction (cf. [55, Theorem 5.28]).

Theorem 4.11 *Let* I *be a radical ideal of big height* h *in a regular ring* R *containing a field. If* $\operatorname{svd}(I)$ *divides* h*, then* $I^{(hn-h+1)} \subseteq I^n$ *for all* $n \gg 0$*.*

Proof First, notice there is nothing to show in the case when $h = 1$, so we assume $h \geqslant 2$. By Lemma 3.3, there exists an integer $A \geqslant 1$ such that for all $n \geqslant 1$,

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)}.$$

In prime characteristic p, consider e such that $q := p^e \geqslant A + 1$. Whenever $n \geqslant q$, we have $hn - h + 1 \geqslant hq - h + 1 \geqslant hA + 1$, and thus

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-q} \left(I^{(h)}\right)^{q-1-a} I^{(ha+1)}$$
$$\subseteq I^{(h(n-q))} I^{(hq-h+1)}.$$

Now as we have mentioned above, $I^{(hq-h+1)} \subseteq I^q$ and $I^{(h(n-q))} \subseteq I^{n-q}$ by [60]; the latter is true more generally, but the first statement requires specifically that we are in characteristic p and $q = p^e$. Combining these two containments with the line above, we conclude that

$$I^{(hn-h+1)} \subseteq I^{(hq-h+1)} I^{(h(n-q))} \subseteq I^q I^{n-q} = I^n.$$

To prove the statement in equicharacteristic 0, we need [71, Theorem 1.2], which says that there exists $N > 0$ such that $\left(I^{(2)}\right)^n \subseteq I^{n+1}$ for all $n \geqslant N$. Fix such N, and let $n \geqslant N + A + 1$. Then

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{N} \left(I^{(h)}\right)^{n-1-N-a} I^{(ha+1)}.$$

By [34, 60], $I^{(ha+1)} \subseteq I^a$. Moreover, $\left(I^{(h)}\right)^{n-1-N-a} \subseteq I^{n-1-N-a}$ since $I^{(h)} \subseteq I$. By choice of N, $\left(I^{(h)}\right)^N \subseteq \left(I^{(2)}\right)^N \subseteq I^{N+1}$. Therefore,

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{N} \left(I^{(h)}\right)^{n-1-N-a} I^{(ha+1)} \subseteq \sum_{a=0}^{A} I^{N+1} I^{n-1-N-a} I^a = I^n.$$

□

So if the big height of an ideal I is divisible by its standard Veronese degree, then I satisfies the stable Harbourne Conjecture 4.8. The ideals in Example 3.9, for example, do not have this property although they satisfy Conjecture 4.8. Thus we ask:

Question 4.12 Which ideals satisfy the condition that $\operatorname{bight}(I)$ is divisible by $\operatorname{svd}(I)$?

In prime characteristic, there are other cases where the Noetherianity of $\mathcal{R}_s(I)$ implies the Stable Harbourne Conjecture 4.8; for example, see [55, Theorem 5.19 and Theorem 5.23].

4.3 Asymptotic Invariants

One way to study symbolic powers and the containment problem is through the development of asymptotic invariants. This is an idea pioneered by Bocci and Harbourne in [9, 10] with the definition of the resurgence of an ideal, and extended in [48] with the definition of the asymptotic resurgence. We present these invariants and their relationship to the symbolic Rees algebra below.

Definition 4.13 The *resurgence* of an ideal I and the *asymptotic resurgence* are given, respectively, by

$$\rho(I) = \sup\left\{\frac{a}{b} \mid I^{(a)} \not\subseteq I^b\right\} \text{ and } \widehat{\rho}(I) = \sup\left\{\frac{a}{b} \mid I^{(at)} \not\subseteq I^{bt} \text{ for } t \gg 0\right\}.$$

The importance of (asymptotic) resurgence to containment problems lies in the fact that, by definition, if a, b are positive integers with $a > \rho(I)b$, then $I^{(a)} \subseteq I^b$.

If I is an ideal of a regular ring and has big height h, Theorem 4.2 implies that $1 \leqslant \rho(I) \leqslant h$ and since the definitions yield $\widehat{\rho}(I) \leqslant \rho(I)$ we deduce that $1 \leqslant \widehat{\rho}(I) \leqslant h$ as well. If we have equality of ordinary and symbolic powers $I^{(n)} = I^n$ for all $n \geqslant 1$, then $\widehat{\rho}(I) = \rho(I) = 1$. However, the resurgence attaining its lowest possible value of 1 does not guarantee equality of the ordinary and symbolic powers.

Example 4.14 (DiPasquale–Drabkin [20]) The ideal $I = (abc, aef, cde, bdf)$ of the polynomial ring $R = k[a, b, c, d, e, f]$ satisfies $\rho(I) = 1$ and $I^{(n)} = I^n + (abcdef)I^{n-2}$ for $n \geqslant 2$. In particular, the ordinary and symbolic powers do not coincide for any $n \geqslant 2$.

At the other end of the spectrum, whether the (asymptotic) resurgence attains its largest possible value equal to the big height has implications on the stable Harbourne Conjecture 4.8. First, it follows easily from the definition that Conjecture 4.8 holds for ideals with $\rho(I) < \text{bight}(I)$ (see [56, Remark 2.7]); moreover, it is sufficient to show that $\widehat{\rho}(I) < \text{bight}(I)$.

Theorem 4.15 (Grifo–Huneke–Mukundan [44, Proposition 2.11]) *Let I be a radical ideal in either a regular local ring containing a field, or a quasi-homogeneous radical ideal in a polynomial ring over a field. If $\widehat{\rho}(I) < \text{bight}(I)$, then the containment $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$.*

Ideals satisfying $\rho(I) < \text{bight}(I)$ have been termed *ideals with expected resurgence* in [44]. Classes of ideals with expected resurgence include: those defining general points in $\mathbb{P}^n$ [6, Theorem 4.2], locally complete intersection ideals I a polynomial ring that are minimally generated by forms of degree lower than $\text{bight}(I)$ [44, Theorem 3.1], ideals I of a local or standard graded regular ring $(R, \mathfrak{m}, k)$ which contains a field so that R/I is Gorenstein, $I^{(n)} = I^n : \mathfrak{m}^\infty$ and either k has positive characteristic or the symbolic Rees algebra of I is Noetherian [45].

Because of these considerations it becomes important to develop methods for determining the (asymptotic) resurgence of an ideal. In order to do this, it is helpful to investigate another asymptotic invariant.

Definition 4.16 Let I be an ideal of a graded ring and denote by $\alpha(I)$ the smallest degree of a nonzero homogeneous from in I. The *Waldschmidt constant* of I is the value

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n} = \inf_{n} \frac{\alpha(I^{(n)})}{n}.$$

For homogeneous ideals of a polynomial ring, the following inequalities discovered by Bocci and Harbourne often hold the key to computing resurgence.

Theorem 4.17 (Bocci–Harbourne [9, Theorem 1.2.1]) *Let I be a homogeneous ideal of a polynomial ring. Then there is an inequality*

$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leqslant \rho(I).$$

If, in addition, I defines a zero-dimensional subscheme, then

$$\rho(I) \leqslant \operatorname{reg}(I)/\widehat{\alpha}(I),$$

where $\operatorname{reg}(I)$ *denotes the Castelnuovo-Mumford regularity of I.*

However, resurgences and Waldschmidt constants remain elusive invariants. In the case of ideals having Noetherian symbolic Rees algebras, however, one can get a better handle on these invariants by expressing them in terms of finitely many symbolic power of ideals.

Theorem 4.18 (Drabkin–Guerrieri [23, Theorem 3.6], DiPasquale–Drabkin [20, Proposition 2.2, Corollary 3.6]) *Suppose I is an ideal of a polynomial ring which has Noetherian symbolic Rees algebra. Then the Waldschmidt constant, asymptotic resurgence, and the resurgence of I are rational numbers and can be computed as follows*

$$\widehat{\alpha}(I) = \min_{n\leqslant \operatorname{gt}(I)} \frac{\alpha(I^{(n)})}{n},$$

$$\widehat{\rho}(I) = \max_{1\leqslant i\leqslant r, 1\leqslant j\leqslant \operatorname{gt}(I)} \left\{ \frac{j\nu_i(I)}{\nu_i(I^{(j)})} \right\}, \text{ and}$$

$$\rho(I) = \begin{cases} \max_{(a,b)\in \text{finite set}} \left\{ \frac{a}{b} \mid I^{(a)} \not\subseteq I^b \right\} & \text{if } \rho(I) \neq \widehat{\rho}(I) \\ \widehat{\rho}(I) & \text{otherwise}, \end{cases}$$

where $\nu_1, \dots, \nu_r$ denote the distinct Rees valuations[2] *of I and the finite set in the last displayed equation is given explicitly in [20, Proposition 2.2].*

Remark 4.19 If I is an ideal of a polynomial ring which has Noetherian symbolic Rees algebra, then in fact the Waldschmidt constant is determined by a single symbolic power corresponding to the standard Veronese degree. Indeed, Definition 4.16 and Proposition 3.1(3) yield $\widehat{\alpha}(I) = \alpha(I^{(\operatorname{svd}(I))})/\operatorname{svd}(I)$.

Ideals with irrational values of the Waldschmidt constant are expected to abound. Indeed, Nagata's conjecture [91] would imply that the Waldschmidt constant of a radical ideal I defining s general points in $\mathbb{P}^2$ is $\widehat{\alpha}(I) = \sqrt{s}$, often producing an irrational value. However, no examples of ideals with confirmed irrational Waldschmidt constant, resurgence, or asymptotic resurgence have been constructed yet. Thus we propose the following task.

Problem 4.20 Provide examples of ideals with irrational Waldschmidt constant, resurgence, or asymptotic resurgence.

The lower bound $\widehat{\alpha}(I) \geqslant \alpha(I)/(d-1)$ holds for homogeneous ideals in a d dimensional polynomial ring, and it follows easily from the containments in Theorem 4.2. The details can be found in [61], but the lower bound itself—although phrased in a different language—appears in work of Waldschmidt [113] and Skoda [107]. Improvements on this lower bound have been proposed by Chudnovsky [16] and Demailly [22] in relation to the difficult question of finding the least degree of a homogeneous polynomial vanishing at a given set of points in projective space to a prescribed order. The validity of the bounds suggested by Chudnovsky and Demailly follows if one can establish containments of the symbolic power ideals deeper within the ordinary powers than provided by Theorem 4.2. We make these containments precise in Question 4.21 below, while also abstracting the bounds suggested by Chudnovsky and Demailly to the more general setting of homogeneous radical ideals in Question 4.22.

Question 4.21 Let I be either a radical ideal of big height h in a regular local ring $(R, \mathfrak{m})$, or a homogeneous radical ideal of big height h in a polynomial ring R with maximal homogeneous ideal $\mathfrak{m}$. Do the following containments

$$I^{(rh)} \subseteq \mathfrak{m}^{r(h-1)} I^r \tag{4.1}$$

$$I^{(r(h+m-1))} \subseteq \mathfrak{m}^{r(h-1)} \left(I^{(m)}\right)^r \tag{4.2}$$

hold for all $m, r \geqslant 1$?

Note that (4.1) is the particular case of (4.2) with $m = 1$. These first appeared as a question for ideals of points in [61, Question 4.2.3], and the more general version for

[2] For details on Rees valuations and their applications the reader is invited to consult [105, §10.1].

radical ideals of big height h appeared in [13, Conjecture 2.9]. Both containments are satisfied for squarefree monomial ideals by [13, Corollary 4.3], where in fact a stronger statement was shown [13, Theorem 4.2]. Similar containment for the defining ideal of a general set of points in $\mathbb{P}^2$ were investigated in [3].

The validity of the containments in Eqs. (4.1) and (4.2) of Question 4.21 would imply the bounds for Waldschmidt constants of homogeneous radical ideals given in (4.3) and (4.4) respectively of the following question.

Question 4.22 (Chudnovsky and Demailly Type Bounds on the Waldschmidt Constant) Let I be a homogeneous radical ideal of big height h in a polynomial ring R. Do the following inequalities

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I)+h-1}{h} \text{ and thus } \widehat{\alpha}(I) \geqslant \frac{\alpha(I)+h-1}{h} \tag{4.3}$$

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I^{(m)})+h-1}{m+h-1} \text{ and thus } \widehat{\alpha}(I) \geqslant \frac{\alpha(I^{(m)})+h-1}{h} \tag{4.4}$$

hold for all $n, m \geqslant 1$?

An affirmative answer to Question 4.22 (4.1) has been given for ideals defining general points in $\mathbb{P}^2$ in [61], for ideals defining general sets of projective points of sufficiently large cardinality in [30], for very general sets of points in arbitrary projective spaces in [38], and for ideals defining sufficiently many general sets of points in projective space in [6], where Question 4.21 (4.1) is shown to hold for $r \gg 0$. Question 4.22 (4.4) is answered in the affirmative for general points in $\mathbb{P}^2$ by Esnault and Viehweg [35] and for very general sets of projective points of sufficiently large cardinality in arbitrary projective spaces by work of Malara, Szemberg and Szpond [89], extended by Chang and Jow [17]. Recently, an affirmative answer to Question 4.22 (4.4) has also been established for sufficiently large general sets of points in arbitrary projective spaces by Bisui, Grifo, Hà and Nguyễn in [7], where an affirmative answer to Question 4.21 (4.2) is also provided in the same context for infinitely many values of r, although not for all r or even $r \gg 0$. Outside of the context of points, the answer to all of these questions is also affirmative for generic determinantal ideals and the defining ideals of star configurations in any codimension [7].

However, both of the above questions remain open in the form stated here, and are open even for ideals having Noetherian symbolic Rees algebra.

Remark 4.23 Suppose that I has a finitely generated symbolic Rees algebra. Since the limit in the definition of $\widehat{\alpha}(I)$ exists, we obtain

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(\mathrm{svd}(I)n)})}{\mathrm{svd}(I)n} = \frac{\alpha(I^{(\mathrm{svd}(I))})}{\mathrm{svd}(I)}.$$

Alternatively, since $\widehat{\alpha}(I)$ is also given as an infimum, one can compute $\widehat{\alpha}(I)$ by taking

$$\widehat{\alpha}(I) = \min\left\{\alpha(I), \frac{\alpha(I^{(2)})}{2}, \ldots, \frac{\alpha(I^{\mathrm{gt}(I)})}{\mathrm{gt}(I)}\right\}.$$

As a consequence, the containments (4.1) and (4.2) of Question 4.22 can be reduced to checking only those instances with $r \leqslant \mathrm{gt}(I)$ or, alternatively, only the case $r = \mathrm{svd}(I)$. Similarly, the inequalities (4.3) and (4.4) of Question 4.22 reduce to checking

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I)+h-1}{h} \text{ and } \frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I^{(m)})+h-1}{m+h-1} \text{ for } 1 \leqslant n \leqslant \mathrm{gt}(I) \text{ and } m \geqslant 1$$

or, equivalently,

$$\frac{\alpha(I^{(\mathrm{svd}(I))})}{\mathrm{svd}(I)} \geqslant \frac{\alpha(I)+h-1}{h} \text{ and } \frac{\alpha(I^{(\mathrm{svd}(I))})}{\mathrm{svd}(I)} \geqslant \frac{\alpha(I^{(m)})+h-1}{m+h-1} \text{ for } m \geqslant 1.$$

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The Alexander–Hirschowitz Theorem and Related Problems

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Dedicated to David Eisenbud, on the occasion of his 75th birthday.

1 Introduction: The Alexander–Hirschowitz Theorem

The *polynomial interpolation problem* originates from the simple fact that a polynomial in one variable over $\mathbb{C}$ is completely determined by its zeros. In fact, given $r \leq d$ distinct points $x_1, \ldots, x_r$ on the affine line $\mathbb{A}^1_{\mathbb{C}}$ and positive integers $m_1, \ldots, m_r$ such that $m_1 + \ldots + m_r = d + 1$, a polynomial $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ of degree d is uniquely determined by the following $(d+1)$ vanishing conditions on its derivatives, namely $f^{(j)}(x_i) = 0$ for all $i = 1, \ldots, r$ and $j = 0, \ldots, m_i - 1$. Equivalently, the matrix arising from these vanishing conditions, which determines the parameters $a_0, \ldots, a_d$, has *maximal rank*. A natural question, that has been studied for a long time, is: *what happens in higher dimension, meaning for polynomials in several variables?*

The problem is much more difficult for several variables, even when the multiplicities $m_1, \ldots, m_r$ are all equal and the ambient space is a projective space over the complex numbers. The aim of this paper is to explore a fundamental result due to Alexander and Hirschowitz, obtained in a series of papers [1–4, 35] (and simplification to its proof given by Chandler in [11, 12]), which shows that, if $m_1 = \cdots = m_r = 2$ and the points are chosen to be *general* points in a projective space, then the same phenomenon happens for homogeneous polynomials in several variables, except for a few identified exceptional cases.

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Before proceeding, it may be useful to clarify our intention with this survey. There are already a few surveys discussing part of this topic in the literature. For instance, the surveys by C. Ciliberto [14] and J. Harris [33] introduce (Hermite) interpolation problems and related results, and include brief discussions of some of the geometric ideas behind the Alexander–Hirschowitz theorem. A survey by R. A. Lorentz [36] discusses these topics from a more Numerical Analysis perspective. A survey by M. C. Brambilla and G. Ottaviani [7] discusses the history and presents many details of the core arguments needed in the proof of the Alexander–Hirschowitz theorem.

These existing surveys assume advanced knowledge and tools from Algebraic Geometry, and are written in languages that may be more familiar to an algebraic geometer (cf. [7]) or an analyst (cf. [36]). Some of the stated facts from these surveys may not appear so obvious for a young reader who is not specifically well trained in algebraic geometry; for instance, the use of curvilinear subschemes and the semi-continuity of Hilbert function. Furthermore, the recent large body of work on symbolic powers of ideals in commutative algebra has drawn our attention and convinced us that it is a good time to reintroduce the Alexander–Hirschowitz theorem to commutative algebraists.

For these reasons, and partly due to a personal interest, our survey is intended for an audience consisting of young commutative algebraists. We aim to present a self-contained proof of the Alexander-Hirschowitz theorem and, particularly, to provide all details that may not be easy to see for commutative algebraists who are new to this research area. We will follow an approach similar to the one of [7]. However, our style of presentation reflects our choices in using algebraic notions and techniques. At the same time, we still identify and appreciate the fundamental geometric ideas at the core of the proof.

We should mention that, to the best of our knowledge, there is no survey or paper with a completely self-contained proof of the Alexander–Hirschowitz theorem. While [7] does include many details of the core argument, its emphasis is geared towards techniques that historically have been used to approach the Interpolation Problem, and the tight connections of this problem with secant varieties. We have also discovered in the literature a few computational inaccuracies and incorrect statements; while they are minor, yet a rechecking was required. Additionally, we shall include all necessary tools in a few appendices; there we state basic results on symbolic powers of ideals, secant varieties, Hilbert functions, generic points, curvilinear schemes and the semi-continuity of Hilbert function.

The proof we present in this survey incorporates all up-to-date simplifications of the arguments in the original proof of the Alexander–Hirschowitz theorem, including, for instance, the work done by K. Chandler [11, 12], and Brambilla and Ottaviani [7] (regarding the case of cubics).

We shall now give a number of important notations and terminology needed to state the Alexander–Hirschowitz theorem. Fix a positive integer n and let $R = \mathbb{C}[x_0, \dots, x_n] = \mathbb{C}[\mathbb{P}^n]$ be the homogeneous coordinate ring of $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. For a zero-dimensional subscheme $X \subseteq \mathbb{P}^n$, let $I_X \subseteq R$ denote its defining ideal. It is a basic fact the Hilbert function H_{R/I_X} of R/I_X is bounded above by its *multiplicity* $e(R/I_X)$ and the Hilbert function of R (see Corollary C.5). Particularly, $H_{R/I_X}(d) \le \min\left\{e(R/I_X), \binom{n+d}{d}\right\}$ for all $d \in \mathbb{N}$. We say that a zero-dimensional subscheme X in $\mathbb{P}^n$ *has maximal Hilbert function in degree* d, or simply *is* $\mathrm{AH}_n(d)$, if

$$H_{R/I_X}(d) = \min\left\{e(R/I_X), \binom{n+d}{d}\right\}.$$

This is equivalent to what is often referred to as *imposing independent conditions on degree d hypersurfaces in* $\mathbb{P}^n$. This latter name however could be slightly misleading because the given property is equivalent to the linear system of equations associated to the points having maximal rank; it is not equivalent to the stronger property that these equations are linearly independent. Thus, we choose to use the notation $\mathrm{AH}_n(d)$, which has essentially been used already in [7, 11].

Another basic fact about Hilbert function of zero-dimensional subschemes in $\mathbb{P}^n$, see Propositions C.3 and C.4, is that $H_{R/I_X}(d) = e(R/I_X)$ for all $d \gg 0$. Thus, we say that X *is multiplicity* d*-independent* if

$$H_{R/I_X}(d) = e(R/I_X).$$

In the known literature, this property is commonly referred to as being simply d-*independent*. We add the word "multiplicity" to the terminology to emphasize the fact that the Hilbert function of R/I_X at degree d equals its multiplicity, in this case, and to avoid the potential confusion between the similar-sounding properties of *imposing independent conditions in degree d* and *being d-independent*.

Let $Y = \{P_1, \ldots, P_r\}$ be a set of distinct points in $\mathbb{P}^n$ and suppose that the defining ideal of P_i is $\mathfrak{p}_i \subseteq R$ for all $i = 1, \ldots, r$. Then, the defining ideal of Y is $I_Y = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. A celebrated theorem of Zariski and Nagata (Theorem B.5) implies that the *symbolic square* $I_Y^{(2)} = \mathfrak{p}_1^2 \cap \cdots \cap \mathfrak{p}_r^2$ consists of all homogeneous polynomials in R passing through each point of Y at least twice. Let X be the zero-dimensional subscheme in $\mathbb{P}^n$ defined by $I_Y^{(2)}$. We call X the set of r *double points supported on* Y, and write $X = 2Y = \{2P_1, \ldots, 2P_r\}$ for simplicity of notation. $X = 2Y$ is called a *general* set of r double points if Y is a general set of r simple points (see Definition D.2 for the precise definition of general sets of simple points).

We are ready to state the main theorem surveyed in this paper.

Theorem 1.1 (Alexander-Hirschowitz) *Let n, d be positive integers. Let X be a general set of r double points in $\mathbb{P}^n_{\mathbb{C}}$. Then, X is $AH_n(d)$ with the following exceptions:*

(1) $d = 2$ and $2 \le r \le n$;
(2) $d = 3$, $n = 4$ and $r = 7$; and
(3) $d = 4$, $2 \le n \le 4$ and $r = \binom{n+2}{2} - 1$.

Our proof of Theorem 1.1 follows an outline similar to the one of [7]. Theorem 1.1 is proved by double-induction, on n and d. For sporadic small values of n

and d the inductive hypotheses are not satisfied. Some of these sporadic cases are indeed the exceptions appearing in the statement, but the other ones are not and they are checked to be $\mathrm{AH}_n(d)$ on an *ad hoc* basis. In general, for the inductive step, two fundamental ingredients of the proof are the so-called *méthode d'Horace différentielle* and the use of zero-dimensional schemes of prescribed length and with support on a set of points. Their refined and delicate use is at the core of the simplifications of the original proof.

We now outline the structure of this survey. In Sect. 2, we present the proof of Theorem 1.1 for $d \geq 4$ and $n \geq 2$, when induction works. This is the most technical section of the paper. We will summarize the main ideas behind the core inductive argument before giving the details of this inductive step in Theorem 2.9. Theorem 2.9 is then employed to prove Theorem 1.1 as well as other results in the survey. In Sect. 3, we discuss the exceptional cases, leaving out some details when $n = 2$ and when $d = 3$ until later in Sects. 4 and 5. In Sect. 4, we give the proof of Theorem 1.1 when $n = 2$, i.e., for points on the projective plane. We have chosen to write a proof which employs Theorem 2.9 to provide the reader with another illustration of the use of this core inductive argument. In Sect. 5, we conclude the proof of the Alexander–Hirschowitz theorem by examining the case when $d = 3$, i.e., for cubics. The paper continues with a list of open problems and questions in Sect. 6.

As mentioned, we end the paper with a number of short appendices to complement the previous sections. In Appendix A, we briefly illustrate the connection between the (homogeneous, Hermite) double interpolation problem and computing the dimension of certain secant varieties as well as determining the Waring rank of forms. In Appendix B, we recall the definition of symbolic powers and the statement of a fundamental theorem of Zariski and Nagata drawing the connection between symbolic powers of ideals of points and the interpolation problems. Since this paper is largely about the Hilbert function of zero-dimensional subschemes in $\mathbb{P}^n$, we have included an appendix about Hilbert functions and, especially, the *lower semi-continuity* property of Hilbert function; see Appendices C and D. The proof of Theorem 1.1 uses a number of known facts about *Hilbert schemes of points* and *curvilinear subschemes*, which may not be obvious for an algebraist (they were not obvious for us), so we include an appendix about Hilbert schemes and curvilinear subschemes; see Appendix E.

Finally, for sake of clarity, we have chosen to work over $\mathbb{C}$, however, a large number of results would still be valid over any perfect field (and using divided powers rather than the usual derivatives, in case the characteristic of the field is positive).

2 The General Case ($d \geq 4$ and $n \geq 3$)

In this section, we discuss the core inductive argument for the proof of Theorem 1.1. It is known, see Proposition C.4, that if X is a set of r double points in $\mathbb{P}^n$ then $e(R/I_X) = r(n+1)$. Thus, a set X of r double points in $\mathbb{P}^n$ is $\mathrm{AH}_n(d)$ if and only if

$$H_{R/I_X}(d) = \min\left\{\binom{n+d}{n}, r(n+1)\right\}.$$

The following observations allow us to *specialize*, i.e., to deduce the statement of Theorem 1.1 by constructing a specific set X of r double points in $\mathbb{P}^n$ which is $\mathrm{AH}_n(d)$, and to consider at most two values of r.

Remark 2.1 Fix $n, d \in \mathbb{Z}_+$.

- (Corollary D.4) If there exists *one* collection of r double points in $\mathbb{P}^n$ that is $\mathrm{AH}_n(d)$, then any general set of r double points in $\mathbb{P}^n$ is $\mathrm{AH}_n(d)$.
- (Corollary D.5) To prove that any set of r general double points in $\mathbb{P}^n$ is $\mathrm{AH}_n(d)$, it suffices to verify the statement for the following two (possibly coinciding) values of r:

$$\left\lfloor \frac{1}{n+1}\binom{n+d}{d} \right\rfloor \le r \le \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil.$$

A key ingredient for the inductive argument of Theorem 1.1 is the so-called Castelnuovo's Inequality which we now recall.

Lemma 2.2 (Castelnuovo's Inequality) *Let R be a polynomial ring. Let I be a homogeneous ideal and let ℓ be a linear form in R. Set $\widetilde{I} = I : \ell$, $\overline{R} = R/(\ell)$, and $\overline{I} = I\overline{R}$. Then,*

$$H_{R/I}(d) \ge H_{R/\widetilde{I}}(d-1) + H_{\overline{R}/(\overline{I})^{sat}}(d). \tag{2.1}$$

Additionally, the equality holds for every d if and only if $\overline{I}$ is saturated in $\overline{R}$.

Proof From the standard exact sequence

$$0 \longrightarrow R/I : \ell(-1) \xrightarrow{\cdot\ell} R/I \longrightarrow R/(I,\ell) \longrightarrow 0,$$

and the fact that $\overline{I} \subseteq (\overline{I})^{\mathrm{sat}}$, one obtains

$$H_{R/I}(d) = H_{R/\widetilde{I}}(d-1) + H_{\overline{R}/\overline{I}}(d) \ge H_{R/\widetilde{I}}(d-1) + H_{\overline{R}/(\overline{I})^{\mathrm{sat}}}(d).$$

It is also clear that the equality holds for every d if and only if $H_{\overline{R}/\overline{I}}(d) = H_{\overline{R}/(\overline{I})^{\mathrm{sat}}}(d)$ for all d, which is the case if and only if $\overline{I} = (\overline{I})^{\mathrm{sat}}$. □

An intuitive natural approach to Theorem 1.1 is to apply Casteluovo's Inequality to obtain a proof by induction on $n \ge 1$. Indeed, Terracini already employed this method to study the case of $n = 3$ by partly reducing to the case of $n = 2$. We shall capture the modern version of Terracini's argument.

Theorem 2.3 (Terracini's Inductive Argument) *Fix integers $r \geq q \geq 1$ and $d \in \mathbb{Z}_+$ satisfying either*

$$r(n+1) - \binom{d+n-1}{n} \leq qn \leq \binom{d+n-1}{n-1} \quad or$$
$$\binom{d+n-1}{n-1} \leq qn \leq r(n+1) - \binom{d+n-1}{n}.$$

Let L be a hyperplane in $\mathbb{P}^n$. If

(1) a set of q general double points in $L \simeq \mathbb{P}^{n-1}$ is $AH_{n-1}(d)$, and
(2) the union of a set of $r-q$ general double points in $\mathbb{P}^n$ and a set of q general simple points in L is $AH_n(d-1)$,

then a set of r general double points in $\mathbb{P}^n$ is $AH_n(d)$.

Proof Without loss of generality, we may assume that $x_n = 0$ is the equation of L. Let $R = \mathbb{C}[x_0, \ldots, x_n]$, and let $\overline{R} = \mathbb{C}[x_0, \ldots, x_{n-1}] \simeq R/(x_n)$. Let Y_1 be a set of q general simple points in $L \simeq \mathbb{P}^{n-1}$, with defining ideal $\overline{I_{Y_1}} \subseteq \overline{R}$. If we consider Y_1 as a set of points in $\mathbb{P}^n$, then its defining ideal is $I_{Y_1} := (\overline{I_{Y_1}}, x_n)R$. Let Y_2 be a set of $r-q$ general simple points in $\mathbb{P}^n - L$ with defining ideal $I_{Y_2} \subseteq R$. Let $I = I_{Y_1}^{(2)} \cap I_{Y_2}^{(2)}$ be the defining ideal of $2Y = 2Y_1 \cup 2Y_2$, and $\overline{I} = I\overline{R}$. By Remark 2.1 and Corollary C.5, it suffices to show that $H_{R/I} \geq \min\left\{\binom{n+d}{n}, r(n+1)\right\}$.

Since $Y_1 \subseteq L$ and none of the points in Y_2 lies on L, we have

$$\widetilde{I} := I : x_n = (I_{Y_1}^{(2)} : x_n) \cap (I_{Y_2}^{(2)} : x_n) = I_{Y_1} \cap I_{Y_2}^{(2)},$$

which is the defining ideal of the union of a set of q general simple points in H and $r-q$ general double points in $\mathbb{P}^n$. Next we show that $(\overline{I})^{sat} = \overline{I_{Y_1}}^{(2)}$. First, observe that $\mathrm{ht}(\overline{I}) = \dim(\overline{R}) - 1$, so $(\overline{I})^{\mathrm{sat}}$ is the intersection of the minimal components of $\overline{I}$. These minimal components are the images in $\overline{R}$ of the minimal components of (I, x_n). Now, the primes containing $(I, x_n) = (I_{Y_1}^{(2)} \cap I_{Y_2}^{(2)}, x_n)$ are precisely the primes containing x_n and either I_{Y_1} or I_{Y_2}. Since for any $\mathfrak{p} \in \mathrm{Min}(I_{Y_2})$ we have $x_n \notin \mathfrak{p}$, then $(\mathfrak{p}, x_n) = (x_0, \ldots, x_n)$ is the maximal ideal of R and, thus, it is not a minimal prime of (I, x_n) (which has height n). On the other hand, for any $\mathfrak{p} \in \mathrm{Min}(I_{Y_1})$ we have $\mathrm{ht}(\mathfrak{p}) = n$ and $x_n \in \mathfrak{p}$, so $\mathfrak{p} \in \mathrm{Min}(I, x_n)$. It follows that the minimal primes $\mathfrak{p}$ of (I, x_n) are precisely the minimal primes of I_{Y_1} and when we localize at any of them we get $(I, x_n)_\mathfrak{p} = (I_{Y_1}^{(2)}, x_n)_\mathfrak{p}$. It follows that $(I, x_n)^{sat} = (I_{Y_1}^{(2)}, x_n)^{sat}$ and, by taking images in $\overline{R}$, we derive that $(\overline{I})^{sat} = \overline{I_{Y_1}}^{(2)}$.

Now, by assumptions (1) and (2), we have

$$H_{R/\widetilde{I}}(d-1) = \min\left\{\binom{n+d-1}{n}, q + (n+1)(r-q)\right\}$$

and

$$H_{\overline{R}/\overline{I}^{\text{sat}}}(d) = \min\left\{\binom{(n-1)+d}{n-1}, qn\right\}.$$

These inequalities together with Lemma 2.2 yield

$$\begin{aligned} H_{R/I}(d) &\geq H_{R/\widetilde{I}}(d-1) + H_{\overline{R}/\overline{I}^{\text{sat}}}(d) \\ &= \min\left\{\binom{n+d-1}{n}, q+(n+1)(r-q)\right\} + \min\left\{\binom{n-1+d}{n-1}, qn\right\}. \end{aligned}$$

Now, if $r(n+1) - \binom{d+n-1}{n} \leq qn \leq \binom{d+n-1}{n-1}$, then $\min\left\{\binom{n+d}{n}, r(n+1)\right\} = r(n+1)$, and

$$\begin{aligned} &\min\left\{\binom{n+d-1}{n}, q+(n+1)(r-q)\right\} + \min\left\{\binom{n-1+d}{n-1}, qn\right\} \\ &\quad = q+(n+1)(r-q)+qn = r(n+1). \end{aligned}$$

Thus,

$$H_{R/I}(d) \geq r(n+1) = \min\left\{\binom{n+d}{n}, r(n+1)\right\}.$$

Similarly, if $\binom{d+n-1}{n-1} \leq qn \leq r(n+1) - \binom{n-1+d}{n}$ holds, then

$$\begin{aligned} H_{R/I}(d) &\geq \min\left\{\binom{n+d-1}{n}, q+(n+1)(r-q)\right\} + \min\left\{\binom{n-1+d}{n-1}, qn\right\} \\ &= \binom{n+d-1}{n} + \binom{n-1+d}{n-1} \\ &= \binom{n+d}{n} \\ &= \min\left\{\binom{n+d}{n}, r(n+1)\right\}. \end{aligned}$$

This concludes the proof. □

Assumption (1) in Theorem 2.3 is usually provided by the inductive hypothesis. Assumption (2) is more delicate, because we have a mix of double points and simple points—Proposition C.13 provides the tool to handle this situation. What prevents one from using Theorem 2.3 to prove Theorem 1.1 is the fact that there may not be an integer q satisfying both of the numerical assumptions of Theorem 2.3. For instance, to prove the case where $n = 3$ and $d = 6$, by Remark 2.1, we need to prove that a set of $r = 21$ general double points satisfies $\text{AH}_3(6)$. To apply Theorem 2.3, we need to find $q \in \mathbb{Z}$ with $84 - 56 \leq 3q \leq 28$, i.e. $q = 28/3$. Thus, Theorem 2.3 is not applicable. There are in fact infinitely many choices of n and d for which we run into the same problem, i.e., when we cannot apply Theorem 2.3 directly.

The *méthode d'Horace différentielle* of [2] is designed to overcome this difficulty. For a subscheme $X \subseteq \mathbb{P}^n$ and a hyperplane L defined by a linear form ℓ, we use $\widetilde{X}$ to denote the *residue* of X with respect to L; that is, the subscheme of $\mathbb{P}^n$ defined by the ideal $I_X : \ell$. The underlying ideas of the *méthode d'Horace différentielle* are:

Step 1. Fix a hyperplane $L \simeq \mathbb{P}^{n-1}$ in $\mathbb{P}^n$. For a suitable choice of q and ϵ, choose a general collection 2Ψ of $r - q - \epsilon$ double points not in L, a general collection 2Λ of q double points in L, and a general collection 2Γ of ϵ double points in L.

Step 2. By induction on the dimension, the sets $2\Lambda \cup 2\Gamma_{|L}$ and $\Psi \cup 2\Lambda \cup 2\Gamma_{|L}$ have maximal Hilbert function in degree $(d-1)$ in $L \simeq \mathbb{P}^{n-1}$. One shows that to prove the theorem it suffices to prove that 2Γ is multiplicity $[I_{2\Psi \cup 2\Lambda}]_d$-independent (see Definition 2.5 below).

Step 3. The last statement in Step 2 is proved using deformation. For $\mathbf{t} = (t_1, \ldots, t_\epsilon) \in K^\epsilon$ we take a flat family of general points $\Gamma_\mathbf{t}$ lying on a family of hyperplanes $\{L_{t_1}, \ldots, L_{t_\epsilon}\}$ having Γ as a limit when $\mathbf{t} \longrightarrow 0$, and the problem reduces to showing that $2\Gamma_\mathbf{t}$ is multiplicity $[I_{2\Psi \cup 2\Lambda}]_d$-independent for some $\mathbf{t}$.

Step 4. To establish this latter fact, the existence of $\mathbf{t}$ in Step 3, we argue by contradiction and another deformation argument reduces the problem to understanding the Hilbert function of schemes of the form $2\Psi \cup 2\Lambda \cup \Theta_\mathbf{t}$, for a suitable curvilinear subscheme $\Theta_\mathbf{t}$ supported on $\Gamma_\mathbf{t}$ and contained in $2\Gamma_\mathbf{t}$ (see Appendix E for basic facts about curvilinear schemes). Since $\Gamma_\mathbf{t}$ is a family of curvilinear schemes, the family has a limit which can be used in the process. Finally, arguments employing the semi-continuity of the Hilbert function, the Castelnuovo inequality (2.1) and the material developed in Step 2 allows us to arrive at the desired conclusion.

The deformation argument in Step 4 of the *méthode d'Horace différentielle* is possible by the use of curvilinear subschemes and, particularly, Lemma 2.7, which we shall now introduce.

Definition 2.4 Let V be a $\mathbb{C}$-vector space of homogeneous polynomials of the same degree in $R = \mathbb{C}[x_0, \ldots, x_n]$ and let $I \subseteq R$ be a homogeneous ideal. Let $I \cap V$ denote the $\mathbb{C}$-vector space of forms (necessarily of the same degree) belonging to both I and V.

Recall that a zero-dimensional subscheme $X \subseteq \mathbb{P}^n$ is *multiplicity d-independent* if $H_{R/I_X}(d) = e(R/I_X)$.

Definition 2.5 Let $X \subseteq \mathbb{P}^n$ be a zero-dimensional subscheme and let V a $\mathbb{C}$-vector space of homogeneous polynomials of the same degree in R.

(1) The *Hilbert function of X (or I_X) with respect to V* is defined to be

$$h_{\mathbb{P}^n}(X, V) = \dim_\mathbb{C} V - \dim_\mathbb{C}(I_X \cap V).$$

(2) We say that X (or I_X) is *multiplicity V-independent* if

$$h_{\mathbb{P}^n}(X, V) = e(R/I_X).$$

This definition generalizes multiplicity d-independence in the sense that X is multiplicity R_d-independent if and only if X is multiplicity d-independent. We now prove a couple of basic facts.

Lemma 2.6 *Let $X \subseteq \mathbb{P}^n$ be a zero-dimensional subscheme and let V a $\mathbb{C}$-vector space of homogeneous polynomials of the same degree in R. Then*

(1) $h_{\mathbb{P}^n}(X, V) \le \min\{H_{R/I_X}(d), \dim_{\mathbb{C}} V\}$;
(2) if X is multiplicity V-independent then X is multiplicity d-independent;
(3) if X is multiplicity d-independent then so is Y, for any zero-dimensional subscheme Y of X.

Proof (1) Let $[I_X]_d := I_X \cap R_d$, thus we have $(I_X \cap V) = [I_X]_d \cap V$ and $H_{I_X}(d) = \dim_{\mathbb{C}}[I_X]_d$. By definition $h_{\mathbb{P}^n}(X, V) = \dim_{\mathbb{C}} V - \dim_{\mathbb{C}}(I_X \cap V) \le \dim_{\mathbb{C}} V$, so we only need to prove $h_{\mathbb{P}^n}(X, V) \le H_{R/I_X}(d)$. From the short exact sequence of vector spaces

$$0 \longrightarrow I_X \cap V \longrightarrow [I_X]_d \oplus V \longrightarrow [I_X]_d + V \longrightarrow 0,$$

and the additivity of dimension of vector spaces we obtain that $h_{\mathbb{P}^n}(X, V) = \dim_{\mathbb{C}}([I_X]_d + V) - H_{I_X}(d)$, which is at most $H_{R/I_X}(d)$ because $[I_X]_d + V \subseteq R_d$.

(2) By (1) and the fact that $H_{R/I_X}(d) \le e(R/I_X)$ for every d (see Proposition C.3) we have $h_{\mathbb{P}^n}(X, V) \le H_{R/I_X}(d) \le e(R/I_X)$. So if X is multiplicity V-independent then $h_{\mathbb{P}^n}(X, V) = e(R/I_X) = H_{R/I_X}(d)$ and, particularly, X is also multiplicity d-independent.

(3) Since R/I_X and R/I_Y are one-dimensional Cohen-Macaulay modules, the short exact sequence

$$0 \longrightarrow I_Y/I_X \longrightarrow R/I_X \longrightarrow R/I_Y \longrightarrow 0$$

implies that I_Y/I_X is a one-dimensional Cohen-Macaulay module and $e(I_Y/I_X) = e(R/I_X) - e(R/I_Y)$. Now, by the above short exact sequence and Proposition C.3, we obtain

$$H_{R/I_Y}(d) = e(R/I_X) - H_{I_Y/I_X}(d) \ge e(R/I_X) - [e(R/I_X) - e(R/I_Y)] = e(R/I_Y).$$

Since $H_{R/I_Y}(d) \le e(R/I_Y)$, by Proposition C.3, we conclude that $H_{R/I_Y}(d) = e(R/I_Y)$. □

Let Z be a set of finitely many simple points, we shall now prove that to check whether a scheme X contained in $2Z$ is V-independent it suffices to consider curvilinear subschemes of X. This reduction and the fact that curvilinear schemes form a dense open subset of the Hilbert scheme (see Proposition E.7) play an important role in the proof of Theorem 2.9.

Lemma 2.7 (Curvilinear Lemma) *Let $X \subseteq \mathbb{P}^n$ be a zero-dimensional scheme contained in a finite union of double points and let V be a $\mathbb{C}$-vector space of homogeneous polynomials of degree d in R. Then X is multiplicity V-independent if and only if every curvilinear subscheme of X is multiplicity V-independent.*

Proof Suppose first that X is multiplicity V-independent, then X is multiplicity d-independent by Lemma 2.6(2) and $h_{\mathbb{P}^n}(X, V) = e(R/I_X) = H_{R/I_X}(d)$. Particularly, V contains all homogeneous polynomials of degree d that are not in I_X. Let $Y \subseteq X$ be any zero-dimensional subscheme, then clearly V contains all homogeneous polynomials of degree d that are not in I_Y, and Y is multiplicity d-independent by Lemma 2.6(3). Therefore,

$$h_{\mathbb{P}^n}(X, V) - h_{\mathbb{P}^n}(Y, V) = \dim_{\mathbb{C}}(I_Y \cap V) - \dim_{\mathbb{C}}(I_X \cap V) = H_{R/I_X}(d) - H_{R/I_Y}(d) = e(R/I_X) - e(R/I_Y).$$

Since by assumption $h_{\mathbb{P}^n}(X, V) = e(R/I_X)$, then $h_{\mathbb{P}^n}(Y, V) = e(R/I_Y)$, and so Y is multiplicity V-independent.

Suppose now that every curvilinear subscheme of X is multiplicity V-independent. We shall use induction on the number r of points in the support of X and $e(R/I_X)$ to show that $h_{\mathbb{P}^n}(X, V) = e(R/I_X)$.

CASE 1: X is supported at a single point $P \in \mathbb{P}^n$. If $e(R/I_X) = 1$ then $X = \{P\}$ and the statement is trivial. If $e(R/I_X) = 2$ then, locally at P, $X \cong \operatorname{Spec}(T)$ where T is a local $\mathbb{C}$-algebra of vector space dimension 2 over $\mathbb{C}$. This implies that the maximal ideal $\mathfrak{m}$ of T is of vector space dimension 1 over $\mathbb{C}$ and $\mathfrak{m}^2 = 0$. It follows that $T \cong \mathbb{C}[t]/(t^2)$. As a consequence (see Lemma E.2), X is a curvilinear scheme. Therefore, X is multiplicity V-independent by the hypotheses.

Assume that $e(R/I_X) > 2$. Let $Y \subseteq X$ be any subscheme with $e(R/I_Y) = e(R/I_X) - 1$. Clearly, $h_{\mathbb{P}^n}(Y, V) \leq h_{\mathbb{P}^n}(X, V)$. Observe that any curvilinear subscheme of X restricts to a curvilinear subscheme of Y. Thus, by the induction hypothesis, we conclude that Y is multiplicity V-independent. That is,

$$h_{\mathbb{P}^n}(Y, V) = e(R/I_Y) = e(R/I_X) - 1.$$

Particularly, this implies that $h_{\mathbb{P}^n}(X, V) \leq e(R/I_X) = h_{\mathbb{P}^n}(Y, V) + 1$. Thus, to show that X is multiplicity V-independent it suffices to construct a subscheme Y of X such that $e(R/I_Y) = e(R/I_X) - 1$ and $h_{\mathbb{P}^n}(X, V) = h_{\mathbb{P}^n}(Y, V) + 1$ (equivalently, $h_{\mathbb{P}^n}(X, V) > h_{\mathbb{P}^n}(Y, V)$).

To this end, let $\zeta \subseteq X$ be a subscheme of multiplicity 2. As shown above ζ is a curvilinear subscheme of X. Thus, ζ is multiplicity V-independent, i.e., $h_{\mathbb{P}^n}(\zeta, V) = 2$. On the other hand, $h_{\mathbb{P}^n}(P, V) \leq H_{R/I_P}(d) = 1$, by Lemma 2.6(1). Therefore, there exists a homogeneous polynomial f in V that vanishes at P but not on ζ. Set $Z = \mathbb{V}(f)$ be the zero locus of f, and define $Y = X \cap Z$. Since

X is contained in $2P$, by imposing the condition that $f = 0$ on Y, we have $e(R/I_Y) = e(R/I_X) - 1$. Furthermore, f vanishes on Y but not on X, and so $h_{\mathbb{P}^n}(X, V) > h_{\mathbb{P}^n}(Y, V)$. The assertion follows in this case.

CASE 2: X is supported at r points $P_1, \ldots, P_r$ for $r \geq 2$. By induction on r, we may assume that the statement is true for schemes supported at $r - 1$ points.

Let $I := I_X = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be an irredundant primary decomposition of $I = I_X$ and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for every i. Let $\mathfrak{q} := \mathfrak{q}_r$, let Q be its associated scheme and let $Z \subseteq X$ be the scheme defined by $I_Z := \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{r-1}$. By assumption, every curvilinear scheme contained in X is V-independent, and then so is every curvilinear scheme contained in Z. Since Z is supported at $r-1$ points, by inductive hypothesis we have

$$h_{\mathbb{P}^n}(Z, V) = e(R/I_Z).$$

Claim 1 X is V-independent if one proves that Q is $V \cap [I_Z]_d$-independent.

Proof of Claim 1 To prove that X is V-independent we compute

$$\begin{aligned} h_{\mathbb{P}^n}(X, V) &= \dim_{\mathbb{C}} V - H_{I_Z \cap \mathfrak{q} \cap V}(d) \\ &= [\dim_{\mathbb{C}} V - H_{I_Z \cap V}(d)] + [H_{I_Z \cap V}(d) - H_{I_Z \cap \mathfrak{q} \cap V}(d)] \\ &= h_{\mathbb{P}^n}(Z, V) + [H_{I_Z \cap V}(d) - H_{I_Z \cap \mathfrak{q} \cap V}(d)] \\ &= e(R/I_Z) + [H_{I_Z \cap V}(d) - H_{I_Z \cap q \cap V}(d)]. \end{aligned}$$

If Q is $V \cap [I_Z]_d$-independent, then $H_{I_Z \cap V}(d) - H_{I_Z \cap \mathfrak{q} \cap V}(d) = e(R/\mathfrak{q})$, and thus $h_{\mathbb{P}^n}(X, V) = e(R/I_Z) + e(R/\mathfrak{q}) = e(R/I)$.

Claim 2 It suffices to prove that $Z \cup Q'$ is V-independent for any curvilinear scheme $Q' \subseteq Q$.

Proof of Claim 2 Let $\mathfrak{q}' \supseteq \mathfrak{q}$ be the defining ideal of $Q' \subseteq Q$. By Claim 1 it suffices to prove that Q is $V \cap [I_Z]_d$-independent. By the base case of induction, it suffices to prove that Q' is $V \cap [I_Z]_d$-independent for any curvilinear scheme $Q' \subseteq Q$. We compute $h_{\mathbb{P}^n}(Q', V \cap [I_Z]_d)$:

$$\begin{aligned} h_{\mathbb{P}^n}(Q', V \cap [I_Z]_d) &= [\dim_{\mathbb{C}} V - H_{I_Z \cap \mathfrak{q}' \cap V}(d)] - [\dim_{\mathbb{C}} V - H_{I_Z \cap V}(d)] \\ &= h_{\mathbb{P}^n}(Z \cup Q', V) - h_{\mathbb{P}^n}(Z, V) \\ &= h_{\mathbb{P}^n}(Z \cup Q', V) - e(R/I_Z). \end{aligned}$$

Therefore, if $Z \cup Q'$ is V-independent, then $h_{\mathbb{P}^n}(Z \cup Q', V) = e(R/I_Z \cap \mathfrak{q}')$ and, by the above computation, $h_{\mathbb{P}^n}(Q', V \cap [I_Z]_d) = e(R/I_Z \cap \mathfrak{q}') - e(R/I_Z) = e(R/\mathfrak{q}')$, proving that Q' is $V \cap [I_Z]_d$-independent. This establishes Claim 2.

Claim 3 It suffices to show that for any curvilinear scheme $Z' \subseteq Z$ one has Z' is $V \cap [\mathfrak{q}']_d$-independent.

Proof of Claim 3 Recall that by Claim 2 it suffices to prove $Z \cup Q'$ is V-independent. We observe that

$$\begin{aligned} h_{\mathbb{P}^n}(Z \cup Q', V) &= \dim_{\mathbb{C}} V - H_{I_Z \cap \mathfrak{q}' \cap V}(d) \\ &= [\dim_{\mathbb{C}} V - H_{\mathfrak{q}' \cap V}(d)] + [H_{\mathfrak{q}' \cap V}(d) - H_{I_Z \cap \mathfrak{q}' \cap V}(d)] \\ &= h_{\mathbb{P}^n}(Q', V) + h_{\mathbb{P}^n}(Z, V \cap [Q']_d). \end{aligned}$$

Since $Q' \subseteq Q \subseteq X$ is curvilinear, Q' is V-independent by the assumption, i.e., $h_{\mathbb{P}^n}(Q', V) = e(R/\mathfrak{q}')$. Thus, it suffices to prove that $h_{\mathbb{P}^n}(Z, V \cap [Q']_d) = e(R/I_Z)$, because then, by the above computation, $h_{\mathbb{P}^n}(Z \cup Q', X) = e(R/\mathfrak{q}') + e(R/I_Z) = e(R/I_Z \cap \mathfrak{q}')$, exhibiting that $Z \cup Q'$ is V-independent.

Since Z is supported at $r - 1$ points, by the inductive hypothesis, to prove that Z is $V \cap [Q']_d$-independent it suffices to show that Z' is $V \cap [Q']_d$-independent for any curvilinear subscheme $Z' \supseteq Z$. This proves Claim 3.

We conclude the proof of Lemma 2.7 by showing that Z' is $V \cap [\mathfrak{q}']_d$-independent. First, we compute $h_{\mathbb{P}^n}(Z, V \cap [\mathfrak{q}']_d)$:

$$\begin{aligned} h_{\mathbb{P}^n}(Z, V \cap [\mathfrak{q}']_d) &= H_{\mathfrak{q}' \cap V}(d) - H_{I_{Z'} \cap \mathfrak{q}' \cap V}(d) \\ &= [\dim_{\mathbb{C}} V - H_{I_{Z'} \cap \mathfrak{q}' \cap V}(d)] - [\dim_{\mathbb{C}} V - H_{\mathfrak{q}' \cap V}(d)] \\ &= h_{\mathbb{P}^n}(Z' \cup Q', V) - h_{\mathbb{P}^n}(Q', V). \end{aligned}$$

Since $Q' \subseteq Q \subseteq X$ is curvilinear, $h_{\mathbb{P}^n}(Q', V) = e(R/\mathfrak{q}')$ by the assumption. Since Q' and Z' are curvilinear and have disjoint support (because $\mathrm{Ass}(R/\mathfrak{q}') = \{\mathfrak{p}_r\}$ and $\mathrm{Ass}(R/I_{Z'}) \subseteq \mathrm{Ass}(R/I_Z) = \{\mathfrak{p}_1, \dots, p_{r-1}\}$), we have that $Z' \cup Q'$ is locally curvilinear at points of the support, and so it is curvilinear. Since $Z' \cup Q' \subseteq Z \cup Q = X$, $h_{\mathbb{P}^n}(Z' \cup Q', V) = e(R/I_{Z'} \cap \mathfrak{q}')$ by the assumption. Therefore,

$$h_{\mathbb{P}^n}(Z', V \cap [\mathfrak{q}']_d) = e(R/I_{Z'} \cap \mathfrak{q}') - e(R/\mathfrak{q}') = e(R/I_{Z'}),$$

and Lemma 2.7 is established. □

Recall that, by Remark 2.1, to prove Theorem 1.1 for values of n, d not in the list of exceptional cases, it suffices prove that a general set of r double points has $\mathrm{AH}_n(d)$ for

$$\left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor \le r \le \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil.$$

When r takes one of these two (sometimes coinciding) values, we let q and ϵ be the quotient and remainder of the division of $r(n+1) - \binom{n+d-1}{n}$ by n. For ease of references, we now provide the values of q and ϵ for a few special choices of n and d.

We start with the case where $d = 4$.

n	Value of r	$\Delta := r(n+1) - \binom{n+d-1}{n}$	Value of q	Value of ϵ	Value of $r - q - \epsilon$
$n = 2$	$r = 5$	$\Delta = 5$	$q = 2$	$\epsilon = 1$	$r - q - \epsilon = 2$
$n = 3$	$r = 8$	$\Delta = 12$	$q = 4$	$\epsilon = 0$	$r - q - \epsilon = 4$
	$r = 9$	$\Delta = 16$	$q = 5$	$\epsilon = 1$	$r - q - \epsilon = 3$
$n = 4$	$r = 14$	$\Delta = 35$	$q = 8$	$\epsilon = 3$	$r - q - \epsilon = 3$
$n = 5$	$r = 21$	$\Delta = 70$	$q = 14$	$\epsilon = 0$	$r - q - \epsilon = 7$
$n = 6$	$r = 30$	$\Delta = 126$	$q = 21$	$\epsilon = 0$	$r - q - \epsilon = 9$
$n = 7$	$r = 41$	$\Delta = 208$	$q = 29$	$\epsilon = 5$	$r - q - \epsilon = 7$
	$r = 42$	$\Delta = 216$	$q = 30$	$\epsilon = 6$	$r - q - \epsilon = 6$
$n = 8$	$r = 55$	$\Delta = 330$	$q = 41$	$\epsilon = 2$	$r - q - \epsilon = 12$
$n = 9$	$r = 71$	$\Delta = 490$	$q = 54$	$\epsilon = 4$	$r - q - \epsilon = 13$
	$r = 72$	$\Delta = 500$	$q = 55$	$\epsilon = 5$	$r - q - \epsilon = 12$

For $d = 5$ and we get the following table.

n	Value of r	$\Delta := r(n+1) - \binom{n+d-1}{n}$	Value of q	Value of ϵ	Value of $r - q - \epsilon$
$n = 2$	$r = 7$	$\Delta = 6$	$q = 3$	$\epsilon = 0$	$r - q - \epsilon = 4$
$n = 3$	$r = 14$	$\Delta = 21$	$q = 7$	$\epsilon = 0$	$r - q - \epsilon = 7$
$n = 4$	$r = 25$	$\Delta = 55$	$q = 13$	$\epsilon = 3$	$r - q - \epsilon = 9$
	$r = 26$	$\Delta = 60$	$q = 15$	$\epsilon = 0$	$r - q - \epsilon = 11$
$n = 5$	$r = 42$	$\Delta = 126$	$q = 25$	$\epsilon = 1$	$r - q - \epsilon = 16$
$n = 6$	$r = 66$	$\Delta = 252$	$q = 42$	$\epsilon = 0$	$r - q - \epsilon = 24$
$n = 7$	$r = 99$	$\Delta = 462$	$q = 66$	$\epsilon = 0$	$r - q - \epsilon = 33$

We now prove a few basic numeric facts that will be employed later.

Lemma 2.8 *For fixed integers* $n \geq 2, d \geq 4$ *and* $0 \leq r \leq \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$, *let* $q \in \mathbb{Z}$ *and* $0 \leq \epsilon < n$ *be such that* $nq + \epsilon = r(n+1) - \binom{n+d-1}{n}$. *Then,*

(1) $n\epsilon + q \leq \binom{n+d-2}{n-1}$,
(2) $\binom{n+d-2}{n} \leq (r - q - \epsilon)(n+1)$,
(3) $r - q - \epsilon \geq n + 1$, *for* $d = 4$ *and* $n \geq 8$.
(4) $q \geq \epsilon$.

Proof (1) We prove the equivalent statement that $n(n\epsilon + q) \leq n\binom{n+d-2}{n-1}$. Clearly, $nq \leq r(n+1) - \binom{n+d-1}{n}$. Since $r \leq \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$, we have $(n+1)r \leq \binom{n+d}{n} + n$, and so

$$n^2\epsilon + nq \le n^2(n-1) + \binom{n+d}{n} + n - \binom{n+d-1}{n} = n^2(n-1) + \binom{n+d-1}{n-1} + n. \tag{2.2}$$

The right-hand side is at most $n\binom{n+d-2}{n-1}$ except when $d = 4$ and $3 \le n \le 5$. In these three cases however the inequality still holds, as one can check directly with given values of q and ϵ in the above tables.

(2) Since $(r-q-\epsilon)(n+1) = r(n+1) - (nq+\epsilon) - (n\epsilon + q)$, we have

$$(r-q-\epsilon)(n+1) = \binom{n+d-1}{n} - (n\epsilon+q) \ge \binom{n+d-1}{n} - \binom{n+d-2}{n-1} = \binom{n+d-2}{n},$$

where the middle inequality follows from (1).

(3) We prove the equivalent statement that $(r-q-\epsilon)(n+1) \ge (n+1)^2$ for $d = 4$ and $n \ge 8$. By the computation in (2), $(r-q-\epsilon)(n+1) \ge (n+1)^2$ holds if and only if $\binom{n+3}{n} - (n\epsilon+q) \ge (n+1)^2$. This holds if and only if

$$(n+1)\left(\frac{(n+3)(n+2)}{6} - (n+1)\right) \ge n\epsilon + q \iff \binom{n+1}{3} \ge n\epsilon + q.$$

By Eq. (2.2), $n\epsilon + q \le \frac{1}{n}\left(n^2(n-1) + \binom{n+3}{4} + n\right)$. The right-hand side is at most $\binom{n+1}{3}$ if and only if $n^3 - 10n^2 + 3n - 10 \ge 0$. This inequality holds for all $n \ge 10$. For the cases $n = 8, 9$ the inequality is easily checked using the above table.

(4) Assume by contradiction that $q < \epsilon$. Then, $r(n+1) - \binom{n+d-1}{n} = nq + \epsilon < (n+1)\epsilon \le (n+1)(n-1)$. From the definition of r, one also sees that $r(n+1) > \binom{n+d}{n} - (n+1)$, and so $r(n+1) \ge \binom{n+d}{n} - n$. As a consequence, we have $\binom{n+d}{n} - n \le r(n+1) < (n+1)(n-1) + \binom{n+d-1}{n}$. Thus, $\binom{n+d-1}{n-1} < (n+1)(n-1) + n$. Since $(n+1)(n-1) + n - 1 = (n+2)(n-1)$, this leads to $\binom{n+d-1}{n-1} \le (n+2)(n-1)$.

It is well-known that $\binom{n+d-1}{n-1}$ increases as d increases, so the left-hand side is at least $\binom{n+3}{n-1} = \binom{n+3}{4}$. In particular, $\frac{(n+3)(n+2)(n+1)n}{24} \le (n+2)(n-1)$. Therefore, $f(n) \le 0$, where $f(n) := (n+3)(n+1)n - 24(n-1)$.

On the other hand, it is easily seen that $f(n)$ is increasing for $n \ge 2$ and $f(2) = 6 > 0$, thus $f(n) > 0$ for every $n \ge 2$, yielding a contradiction. □

We are ready to present the core inductive argument for Theorem 1.1. The proof follows the four steps we outlined when we illustrated the *méthode d'Horace différentielle*.

Theorem 2.9 *For fixed $n \geq 2, d \geq 4$ and $\left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor \leq r \leq \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$, let $q \in \mathbb{Z}$ and $0 \leq \epsilon < n$ be such that $nq + \epsilon = r(n+1) - \binom{n+d-1}{n}$. Suppose that*

(i) q general double points are $AH_{n-1}(d)$,
(ii) $r - q$ general double points are $AH_n(d-1)$,
(iii) $r - q - \epsilon$ general double points are $AH_n(d-2)$.

Then, r general double points are $AH_n(d)$.

Proof By Remark 2.1, it suffices to construct a set of r double points in $\mathbb{P}^n$ which is $\mathrm{AH}_n(d)$. This set of r double points arises in the form $2\Psi \cup 2\Lambda \cup 2\Gamma_{\mathbf{t}}$, for some family of parameters $\mathbf{t}$, where the sets Ψ, Λ and $\Gamma_{\mathbf{t}}$ are constructed as in the outlined steps. To understand the construction better, we shall use 21 double points in $\mathbb{P}^3$ and degree 6 as our running example; in this particular situation, $r = 21$, $d = 6$, $q = 9$ and $\epsilon = 1$.

Step 1. We first fix a hyperplane $L \simeq \mathbb{P}^{n-1}$ in $\mathbb{P}^n$, with defining equation $\ell = 0$. We take a set of $q + \epsilon$ general points in L, let $\Gamma = \{\gamma_1, \ldots, \gamma_\epsilon\}$ be a subset of ϵ of these points, and let Λ be the set consisting of the remaining q points. Finally, we take a set Ψ of $r - q - \epsilon$ general points in $\mathbb{P}^n$ outside of L. (In our running example, Γ consists of a single point in L, Λ of 9 general points in L, and Ψ of 11 general points outside of L.)

Step 2. By (ii), we have

$$H_{R/(I_\Psi^{(2)} \cap I_\Gamma^{(2)})}(d-1) = \min\left\{(n+1)(r-q), \binom{n+d-1}{n}\right\} = (n+1)(r-q),$$

where the rightmost equality holds because Lemma 2.8(4) yields $\binom{n+d-1}{n} = (n+1)(r-q) - \epsilon + q \geq (n+1)(r-q)$. Now, if we consider $\Gamma|_L$ instead of Γ, then the linear system associated to $[I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)}]_{d-1}$ is obtained by removing ϵ equations from the linear system of equations defined by $[I_\Psi^{(2)} \cap I_\Gamma^{(2)}]_{d-1}$ (more precisely, the ones corresponding to setting the partial derivatives with respect to ℓ equal to 0). One then obtains

$$H_{R/(I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)})}(d-1) = \min\left\{(n+1)(r-q) - \epsilon, \binom{n+d-1}{n}\right\} = (n+1)(r-q) - \epsilon$$

(= 47 for the running example), and then $H_{I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)}}(d-1) = \binom{n+d-1}{n} - H_{R/(I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)})}(d-1) = q$ (= 9 in the running example).

Claim 1. $H_{R/(I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)} \cap I_\Lambda)}(d-1) = e(R/(I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)} \cap I_\Lambda)) = \binom{n+d-1}{d-1}$.

(that is, $H_{R/(I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)} \cap I_\Lambda)}(5) = \binom{3+6-1}{3} = 56$ for the running example.)

Notice that Claim 1 implies that $2\Psi \cup 2\Gamma_{|L} \cup \Lambda$ is multiplicity $(d-1)$-independent. The rightmost equality in the claim holds because $\binom{n+d-1}{d-1} = \binom{n+d-1}{d-1} - \epsilon + q = e(R/(I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda))$. To conclude the proof it then suffices to show that $H_{R/(I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda)}(d-1) = \binom{n+d-1}{d-1} = H_R(d-1)$, i.e. $[I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda]_{d-1} = 0$.

If we restate the paragraph before the Claim in terms of linear algebra, we see that the solution set of the linear system defined by $[2\Psi \cup 2\Gamma_{|L}]_{d-1}$ is a q-dimensional vector space. Now, for each simple general point in $\mathbb{P}^n$ that we are adding to $2\Psi \cup 2\Gamma_{|L}$, we are adding a general linear equation to this system, so we are reducing the dimension of the solution set by 1. Thus, if we add q general simple points in $\mathbb{P}^n$ to $2\Psi \cup 2\Gamma_{|L}$, then the corresponding ideal contains no forms of degree $d-1$. It follows that if we add q points to $2\Psi \cup 2\Gamma_{|L}$, and these q additional points lie on L, then the defining equation ℓ of L divides the equation of any hypersurface of degree $d-1$ passing through $2\Psi \cup 2\Gamma_{|L}$ and these q points. In particular, any form $F \in [I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda]_{d-1}$ is divisible by ℓ, and we can write $F = F_1\ell$.

Since $\ell \in I_\Lambda$ (because $\Lambda \subseteq L$) and ℓ is regular on R/I_Ψ (because none of the points of Ψ lies on L), we have that F_1 is a degree $(d-2)$ form in

$$(I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda) : \ell = I_\Psi^{(2)} \cap (I_{\Gamma_{|L}}^{(2)} : \ell) \subseteq I_\Psi^{(2)}.$$

However, by (iii), we know that $H_{I_\Psi^{(2)}}(d-2) = \max\left\{0, \binom{n+d-2}{n} - (r-q-\epsilon)(n+1)\right\} = 0$. Therefore, $F_1 = 0$, and so $F = 0$. Hence, $[I_\Psi^{(2)} \cap I_{\Gamma_{|L}}^{(2)} \cap I_\Lambda]_{d-1} = 0$, and Claim 1 is proved.

To prove the theorem we need to prove the following equality

$$H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap I_\Gamma^{(2)})}(d) = \min\left\{(n+1)r, \binom{n+d}{n}\right\}.$$

We now proceed by considering two different cases depending on which of the two possible values the right-hand side may take. Since, by assumption, $\left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor \le r \le \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$, it can be easily seen that

- $\min\left\{(n+1)r, \binom{n+d}{n}\right\} = (n+1)r$ holds precisely if $r = \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor$,
- $\min\left\{(n+1)r, \binom{n+d}{n}\right\} = \binom{n+d}{n} > r(n+1)$ holds if $r = \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil > \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor$.

The running example of 21 double points in $\mathbb{P}^3$ falls in the first possibility.

<u>CASE 1:</u> $r = \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor$. In this case, $r(n+1) \leq \binom{n+d}{n}$, $nq + \epsilon \leq \binom{n+d-1}{n-1}$, and by the above observation we need to show that

$$H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap I_\Gamma^{(2)})}(d) = (n+1)r.$$

Claim 2. $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})}(d) = e(R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})) = (n+1)(r-\epsilon)$.

Castelnuovo's inequality (2.1) gives $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})}(d) \geq H_{R/(I_\Lambda \cap I_\Psi^{(2)})}(d-1) + H_{\overline{R}/I^{(2)}_{\Lambda|_L}}(d)$. By Claim 1 and Lemma C.12(1), $H_{R/(I_\Psi^{(2)} \cap I_\Lambda)}(d-1) = (n+1)(r-q-\epsilon)+q$. By assumption (ii) and the inequality $nq \leq \binom{n+d-1}{n-1}$, we have

$$H_{\overline{R}/I^{(2)}_{\Lambda|_L}}(d) = \min\left\{nq, \binom{n-1+d}{n-1}\right\} = nq = e(\overline{R}/I^{(2)}_{\Lambda|_L}).$$

Thus, $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})}(d) \geq (n+1)(r-q-\epsilon)+q+nq = (n+1)(r-\epsilon)$. Since the other inequality always holds by Corollary C.5, Claim 2 is proved.

To finish this case it suffices to prove that $I_\Gamma^{(2)}$ is multiplicity $[I_\Lambda^{(2)} \cap I_\Psi^{(2)}]_d$-independent, because then

$$\begin{aligned} H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap I_\Gamma^{(2)})}(d) &= H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})}(d) + e(R/I_\Gamma^{(2)}) = (n+1)(r-\epsilon) + (n+1)\epsilon \\ &= (n+1)r. \end{aligned}$$

Instead of proving this statement directly, we will use deformation to consider a family of general points $\Gamma_{\mathbf{t}}$ having Γ as a limit (as we shall explain in the upcoming Step 3).

For now, we observe the following fact. As before, if we add ϵ general points of L to $2\Lambda|_L$ then we are adding ϵ general equations to the linear system determined by $[I^{(2)}_{\Lambda|_L}]_d$, and so

$$H_{\overline{R}/(I^{(2)}_{\Lambda|_L} \cap I_\Gamma)}(d) = \min\left\{nq+\epsilon, \binom{n-1+d}{n-1}\right\} = nq+\epsilon. \tag{2.3}$$

(In our running example, $H_{\overline{R}/I^{(2)}_{\Lambda|_L} \cap I_\Gamma}(d) = (3)(9)+1 = 28$.)

Step 3. For $\mathbf{t} = (t_1, \ldots, t_\epsilon) \in K^\epsilon$, consider a flat family of general points $\Gamma_{\mathbf{t}} = \{\gamma_{1,t_1}, \ldots, \gamma_{\epsilon,t_\epsilon}\}$ in $\mathbb{P}^n$ and a family of hyperplanes $\{L_{t_1}, \ldots, L_{t_\epsilon}\}$ such that

(1) the point γ_{i,t_i} lies in L_{t_i}, for all $i = 1, \ldots, \epsilon$,
(2) $\gamma_{i,t_i} \notin L$ for any $t_i \neq 0$ and any $i = 1, \ldots, \epsilon$,
(3) $L_0 = L$ and $\gamma_{i,0} = \gamma_i \in L$ for any $i = 1, \ldots, \epsilon$.

(For the running example, we have a family of general points $\Gamma_t = \{\gamma_t\} \subseteq \mathbb{P}^3$ and a family of hyperplanes L_t, for $t \in K$.)

Step 4. To prove $I_\Gamma^{(2)}$ is multiplicity $[I_\Lambda^{(2)} \cap I_\Psi^{(2)}]_d$-independent, by Corollary C.5 and Theorem D.9 it suffices to prove that there exists $\mathbf{t} = (t_1, \dots, t_\epsilon) \in K^\epsilon$ such that $I_{\Gamma_\mathbf{t}}^{(2)}$ is multiplicity $[I_\Lambda^{(2)} \cap I_\Psi^{(2)}]_d$-independent, because then

$$(n+1)r \geq H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap I_\Gamma^{(2)})}(d) \geq H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap I_{\Gamma_\mathbf{t}}^{(2)})}(d) = (n+1)r.$$

Suppose, by contradiction, that such a $\mathbf{t}$ does not exist. Then, by Lemma 2.7, for each $\mathbf{t} = (t_1, \dots, t_\epsilon)$, there exist curvilinear ideals J_{i,t_i} such that $I_{\gamma_{i,t_i}}^{(2)} \subseteq J_{i,t_i} \subseteq I_{\gamma_{i,t_i}}$ and, by letting $J_\mathbf{t} = \bigcap_{i=1}^{\epsilon} J_{i,t_i}$, we then have

$$H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_\mathbf{t})}(d) < H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)})}(d) + e(R/J_\mathbf{t}) = (n+1)(r-\epsilon) + e(R/J_\mathbf{t}). \tag{2.4}$$

(For the running example, there is a single point γ_t, so I_{γ_t} is a linear prime and $J_\mathbf{t}$ is a curvilinear ideal J_t with $I_{\gamma_t}^{(2)} \subseteq J_t \subseteq I_{\gamma_t}$ and $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_t)}(6) < (3+1)(20) + e(R/J_t) = 80 + e(R/J_t)$.)

Since $J_\mathbf{t}$ is a curvilinear ideal, by Proposition E.7, for every $i = 1, \dots, \epsilon$ the family $\{J_{i,t_i}\}$ has a limit $J_{i,0}$. Let $J_0 = \bigcap_{i=1}^{\epsilon} J_{i,0}$.

Let $A := \{i \mid \ell \notin J_{i,0}\}$, $B := \{i \mid \ell \in J_{i,0}\}$ and $A' := \{i \in A \mid \ell \in \sqrt{J_{i,0}}\}$. We set $a = |A|$, $a' = |A'|$, and $b := |B|$. For each $\mathbf{t} \in K^\epsilon$, set $J_\mathbf{t}^A = \bigcap_{i \in A} J_{i,t_i}$, and $J_\mathbf{t}^B = \bigcap_{i \in A} J_{i,t_i}$, in particular $J_\mathbf{t} = J_\mathbf{t}^A \cap J_\mathbf{t}^B$. We also set $I_\Gamma^{A'} = \bigcap_{i \in A'} I_{\gamma_i}$.

By the semi-continuity of Hilbert function and (2.4), there exists an open neighborhood U of 0 such that for any $\mathbf{t} \in U$, we have the equalities $e(R/J_\mathbf{t}^B) = e(R/J_0^B)$, $e(R/J_\mathbf{t}^A) = e(R/J_0^A)$ (so $e(R/J_\mathbf{t}) = e(R/J_0)$), and

$$H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0^A \cap J_\mathbf{t}^B)}(d) = H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_\mathbf{t}^A \cap J_\mathbf{t}^B)}(d) < (n+1)(r-\epsilon) + e(R/J_\mathbf{t}), \tag{2.5}$$

(for the running example, we have $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0^A \cap J_\mathbf{t}^B)}(d) < 82$), and

$$H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_\mathbf{t}^B)}(d-1) = H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_0^B)}(d-1).$$

We want to show $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0^A \cap J_\mathbf{t}^B)}(d) \geq (n+1)(r-\epsilon) + e(R/J_\mathbf{t})$, which would then contradict (2.5). For any $\mathbf{t} \in U$, the Castelnuovo inequality gives

$$H_{R/I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0^A \cap J_\mathbf{t}^B}(d) \geq H_{R/I_\Lambda \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_\mathbf{t}^B}(d-1) + H_{\overline{R}/(I_{\Lambda|_L}^{(2)} \cap I_{\Gamma|_L}^{A'})}(d),$$

where $\overline{R} \cong R/(\ell)$ and $I_{\Gamma|_L}^{A'}$ is the defining ideal of $\{\gamma_i \mid i \in A'\}$ in $\overline{R}$.

We examine the first summand which, by the choice of U, equals $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_0^B)}(d-1)$ for every $\mathbf{t} \in U$. By Claim 1, the ideal $I_\Lambda \cap I_\Psi^{(2)} \cap I_{\Gamma|_L}^{(2)}$

is multiplicity $(d-1)$-independent, so –by Lemma 2.6– the larger ideal $I_\Lambda \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_0^B$ is multiplicity $(d-1)$-independent too. Thus,

$$\begin{aligned} H_{R/(I_\Lambda \cap I_\Psi^{(2)} \cap (J_0^A:\ell)\cap J_0^B)}(d-1) &= e(R/(I_\Lambda \cap I_\Psi^{(2)} \cap (J_0^A : \ell) \cap J_0^B)) \\ &= e(R/(I_\Lambda \cap I_\Psi^{(2)})) + e(R/(J_0^A : \ell) \cap J_0^B) \\ &= q + (n+1)(r-q-\epsilon) + e(R/(J_0^A : \ell) \cap J_0^B) \\ &= q + (n+1)(r-q-\epsilon) + e(R/J_0) - a', \end{aligned} \tag{2.6}$$

where a' is the cardinality of $\{i \in A \mid \ell \text{ is not regular on } R/J_{i,0}\}$. The last equality holds because $e(R/J_0) = e(R/J_0 : \ell) + e(R/(J_0, \ell)) = e(R/J_0^A : \ell) + e(R/(J_0, \ell)) = e(R/(J_0^A : \ell)) + e(R/J_0^B) + a'$.

Now, the inclusion $2\Lambda_{|L} \cup \{\gamma^i \mid i \in F\} \subseteq 2\Lambda_{|L} \cup \Gamma$ and (2.3) allow the use of Lemma C.12(1) to deduce that

$$H_{\overline{R}/(I_{\Lambda|_L^{(2)}} \cap I^{A'}_{\Gamma|_L})}(d) \geq e\left(\overline{R}/\left(I_{\Lambda|_L^{(2)}} \cap I^{A'}_{\Gamma|_L}\right)\right) = nq + a'.$$

Putting all these together, we obtain that, for any $\mathbf{t} \in U$,

$$\begin{aligned} H_{R/I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0^A \cap J_{\mathbf{t}}^B}(d) &\geq H_{R/I_\Lambda \cap I_\Psi^{(2)} \cap (J_0^A:\ell)\cap J_{\mathbf{t}}^B}(d-1) + H_{\overline{R}/I^{(2)}_{\Lambda|_L} \cap I_\Gamma^{A'}}(d) \\ &\geq q + (n+1)(r-q-\epsilon) + e(R/J_0) - a' + (nq + a') \\ &= (n+1)(r-\epsilon) + e(R/J_0) = (n+1)(r-\epsilon) + e(R/J_{\mathbf{t}}). \end{aligned}$$

(For the running example, this implies one of the following inequalities $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_0)}(6) \geq 80 + e(R/J_t)$ or $H_{R/(I_\Lambda^{(2)} \cap I_\Psi^{(2)} \cap J_t)}(6) \geq 80 + e(R/J_t)$.) This is a contradiction to (2.4), and we are done.

<u>Case 2:</u> $r > \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor$. In this case, $r = \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$, and we have

$$r(n+1) > \binom{n+d}{n} \text{ and } nq + \epsilon > \binom{n+d-1}{n-1}.$$

First, one considers the case where $nq \geq \binom{n+d-1}{n-1}$. Then by (i), we have

$$H_{\overline{R}/I^{(2)}_{\Lambda|_L}}(d) = \min\left\{nq, \binom{n+d-1}{n-1}\right\} = \binom{n+d-1}{n-1}.$$

On the other hand, by (ii), we have

$$H_{R/(I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d-1)=\min\left\{(n+1)(r-q),\binom{n+d-1}{n}\right\}.$$

For any set Δ of $q-\epsilon$ general points in $\mathbb{P}^n$ one has $H_{R/(I_\Delta\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d-1)=\binom{n+d-1}{n}=e(R/(I_\Delta\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)}))$. As in the proof of Claim 1, this yields that for any subset $\Lambda'\subseteq\Lambda$ consisting of $q-\epsilon$ points one has $H_{R/(I_{\Gamma'}\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d-1)=\binom{n+d-1}{n}$. By Lemma C.12(2), one obtains $H_{R/(I_\Lambda\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d-1)=\binom{n+d-1}{n}$. Thus, by the Castelnuovo inequality, we get

$$\begin{aligned}H_{R/(I_\Lambda^{(2)}\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d)&\geq H_{R/(I_\Lambda\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d-1)+H_{\overline{R}/I^{(2)}_{\Lambda|_L}}(d)\\&=\binom{n+d-1}{n}+\binom{n+d-1}{n-1}=\binom{n+d}{n}.\end{aligned}$$

Hence, the desired equality holds and $2\Lambda\cup 2\Psi\cup 2\Gamma$ satisfies $\mathrm{AH}_{n,d}$.

We may now assume that $0<\nu:=\binom{n+d-1}{n-1}-nq<\epsilon$, and let $\Gamma'=\{\gamma_1,\ldots,\gamma_\nu\}\subseteq\Gamma$. By a similar argument (or similar to the proof of Claim 1), it can be shown that

$$H_{\overline{R}/(I^{(2)}_{\Lambda|_L}\cap I_{\Gamma'|L})}(d)=\binom{n+d-1}{n-1}=nq+\nu.$$

Thus, by Lemma C.12(2), one has

$$H_{\overline{R}/(I^{(2)}_{\Lambda|_L}\cap I_{\Gamma|L})}(d)=\min\left\{nq+\epsilon,\binom{n+d-1}{n-1}\right\}=\binom{n+d-1}{n-1}.$$

To show that $2\Lambda\cup 2\Psi\cup 2\Gamma$ satisfies $\mathrm{AH}_{n,d}$ we need to prove

$$H_{R/(I_\Lambda^{(2)}\cap I_\Gamma^{(2)}\cap I_\Psi^{(2)})}(d)=\binom{n+d}{n}.$$

Let $\mathbf{t}$ and $\Gamma_{\mathbf{t}}$ be defined as in Case 1. By the semi-continuity of the Hilbert function, there exists a neighborhood U of 0 such that for $\mathbf{t}\in U$ we have

$$H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap I_\Gamma)}(d)=H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap I_{\Gamma_{\mathbf{t}}})}(d).$$

Claim 3. To finish the proof it suffices to find an ideal $K\supseteq I^{(2)}_{\Gamma_{\mathbf{t}}}$ such that K is multiplicity $[I_\Lambda^{(2)}\cap I_\Psi^{(2)}]_d$-independent, and $e(R/K)=n\epsilon+\nu$.

Assume such an ideal K exists. The first assumption gives $H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap K)}(d) = H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)})}(d) + e(R/K)$. Also, by Claim 2, $H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)})}(d) = (n+1)(r-\epsilon)$. Finally, $e(R/K)$ is precisely the amount needed to ensure that $H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap K)}(d) = \binom{n+d}{n}$, because one has

$$\begin{aligned} H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap K)}(d) &= (n+1)(r-\epsilon) + e(R/K) &&= (n+1)(r-\epsilon) + n\epsilon + \nu \\ &= (n+1)r - \epsilon + \nu &&= (n+1)r - (nq+\epsilon) + (nq+\nu) \\ &= \tbinom{n+d-1}{n} + \tbinom{n+d-1}{n-1} &&= \tbinom{n+d}{n}. \end{aligned}$$

Now,

$$H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap I_\Gamma)}(d) = H_{R/(I_\Lambda^{(2)}\cap I_\Psi^{(2)}\cap I_{\Gamma_{\mathbf{t}}})}(d) = \binom{n+d}{n},$$

where the rightmost equality follows from Lemma C.12(2). This proves Claim 3.

Finally, it is easily seen that $K := I_{\Gamma'}^{(2)} \cap I_{(\Gamma-\Gamma')|_L}$ satisfies the desired properties. This concludes the proof of the theorem. □

Modulo the exceptional cases, which are considered in the following sections, we now give a complete proof of the Alexander–Hirschowitz theorem.

Theorem 2.10 (Alexander–Hirschowitz) *For every $n \geq 1$ and $d \geq 1$, a set X of r general double points in $\mathbb{P}^n_{\mathbb{C}}$ is $AH_n(d)$, with the following exceptions:*

(1) $d = 2$ and $2 \leq r \leq n$;
(2) $d = 3$, $n = 4$ and $r = 7$;
(3) $d = 4$, $2 \leq n \leq 4$ and $r = \binom{n+2}{2} - 1$.

Proof By Remarks C.9 and C.10, we may assume that $r \geq 2$ and $d \geq 2$. The statement for $n = 1$ is proved in Proposition C.11. The case where $n = 2$ is treated in Sect. 4. Thus, we may also assume that $n \geq 3$. The exceptional cases are discussed in Sects. 3, 4 and 5. Furthermore, it will be shown that for fixed d and n, the given value of r is the only exceptional case of r general double points not being $\mathrm{AH}_n(d)$. Finally, for n and d not in the list of exceptional cases, by Lemma 2.1, we only need to consider values of r such that

$$\left\lfloor \frac{1}{n+1}\binom{n+d}{d} \right\rfloor \leq r \leq \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil.$$

Our argument proceeds by considering small values of d and then using induction together with Theorem 2.9. The statement for $d = 2$ is proved in Lemma 3.1. The statement for $d = 3$ is examined in Sect. 5. Therefore, we may assume now that $n \geq 3$ and $d \geq 4$.

We will use induction on n to prove the assertion for $d = 4$. Note that the statement for $d = 4$ and $3 \leq n \leq 4$ is proved in Lemma 3.2. On the other hand, if the statement has been shown for $5 \le n \le 7$, then Theorem 2.9 applies to prove the desired assertion for all $n \ge 8$ too. This is because condition (i) holds by the induction hypothesis on n, condition (ii) holds as shown in Sect. 5, and condition (iii) holds because, for $n \ge 8$, by Lemma 2.8(3) we have $r - q - \epsilon \ge n + 1$, and thus $\mathrm{AH}_{r-q-\epsilon}(2)$ holds as shown in Lemma 3.1. It remains to consider $d = 4$ and $5 \le n \le 7$. We shall leave this case until later in the proof.

In general, for $d \ge 5$, the proof proceeds by a double induction on d and n. Observe that if the statement has been proved for $d = 5, 6$ and $n = 3, 4$, then Theorem 2.9 applies to prove the statement for all $d \ge 5$ and $n \ge 3$. Therefore, we only need to establish the desired assertion for $d = 5, 6$ and $n = 3, 4$.

We conclude the proof by analyzing the needed cases, i.e. when $d = 4$ and $5 \le n \le 7$, or when $d = 5, 6$ and $n = 3, 4$. Most cases are also proved by applying Theorem 2.9.

Case 1: $d = 4, n = 5$. In this case, we need to consider $r = 21$ general double points in $\mathbb{P}^5$, $q = 14$ and $\epsilon = 0$. Direct Macaulay 2 [26] computation can be used to verify that the assertion holds.

Case 2: $d = 4, n = 6$. In this case, we need to consider $r = 30$ general double points in $\mathbb{P}^6$, $q = 21$ and $\epsilon = 0$. Theorem 2.9 applies because 21 general double points are $\mathrm{AH}_5(4)$ by Case 1, and 9 general double points are $\mathrm{AH}_6(3)$ (as shown in Sect. 5) and $\mathrm{AH}_6(2)$ (by Lemma 3.1).

Case 3: $d = 4, n = 7$. In this case, we need to consider $r = 41$ or 42 general double points in $\mathbb{P}^7$, $q = 29$ or 30 and $\epsilon = 5$ or 6. Direct Macaulay 2 [26] computation shows that 41 and 42 general double points in $\mathbb{P}^7$ are indeed $\mathrm{AH}_7(4)$.

Case 4: $d = 5, n = 3$. In this case, we need to consider $r = 14$ general double points in $\mathbb{P}^3$, $q = 7$ and $\epsilon = 0$. Theorem 2.9 applies because 7 general double points are $\mathrm{AH}_2(5)$ (by Theorem 4.1), $\mathrm{AH}_3(4)$ (by Lemma 3.2), and $\mathrm{AH}_3(3)$ (as shown in Sect. 5).

Case 5: $d = 5, n = 4$. In this case, we need to consider $r = 25$ or 26 general double points in $\mathbb{P}^4$, $q = 13$ or $q = 15$, and $\epsilon = 3$ or 0. For $r = 25, q = 13$ and $\epsilon = 3$, Theorem 2.9 applies because 13 general double points are $\mathrm{AH}_3(5)$ by Case 4, 12 general double points are $\mathrm{AH}_4(4)$ (by Lemma 3.2), and 9 general double points are $\mathrm{AH}_4(3)$ (as shown in Sect. 5). For $r = 26, q = 15$ and $\epsilon = 0$, Theorem 2.9 applies because 15 general double points are $\mathrm{AH}_3(5)$ by Case 4, 11 general double points are $\mathrm{AH}_4(4)$ (by Lemma 3.2) and $\mathrm{AH}_4(3)$ (as shown in Sect. 5).

Case 6: $d = 6, n = 3$. In this case, we need to consider $r = 21$ general double points in $\mathbb{P}^3$, $q = 9$ and $\epsilon = 1$. Theorem 2.9 applies because 9 general double points are $\mathrm{AH}_2(6)$ (by Theorem 4.1), 12 general double points are $\mathrm{AH}_3(5)$ by Case 4, and 11 general double points are $\mathrm{AH}_3(4)$ (by Lemma 3.2).

Case 7: $d = 6, n = 4$. In this case, we need to consider $r = 42$ general double points in $\mathbb{P}^4$, $q = 21$ and $\epsilon = 0$. Theorem 2.9 applies because 21 general double points are $\mathrm{AH}_3(6)$ by Case 6, and 21 general double points are $\mathrm{AH}_4(5)$ by Case 5 and $\mathrm{AH}_4(4)$ (by Lemma 3.2). □

3 The Exceptional Cases

In this section, we consider the exceptional cases listed in Theorem 1.1 and show that they are indeed the only exceptional cases for given n and d. We begin by considering the case where $d = 2$.

Lemma 3.1 *A set of $r \geq 1$ general double points in $\mathbb{P}^n$ is not $AH_n(2)$ if and only if $2 \leq r \leq n$.*

Proof A single double point is $\mathrm{AH}_n(2)$ (e.g. by Remark C.9), so we may assume $r \geq 2$. First we prove that a set of $r \geq n+1$ general double points in $\mathbb{P}^n$ is $\mathrm{AH}_n(2)$. Let $Y = \{P_1, \ldots, P_r\}$ denote a set of $r \geq n+1$ general points in $\mathbb{P}^n$ and let $X = 2Y$. It is easily seen that X is $\mathrm{AH}_n(2)$ if and only if I_X contains no quadrics.

By Lemma C.12, it suffices to show that I_X contains no quadrics when $r = n+1$. When $r = n+1$, by a change of variables, we can assume that P_i is the i-th coordinate point, for $i = 1, \ldots, n+1$. That is, $P_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, where the value 1 appears at the i-th position. In this case, I_Y is the squarefree monomial ideal

$$I_Y = \mathfrak{p}_0 \cap \ldots \cap \mathfrak{p}_n = (x_i x_j \mid 0 \leq i < j \leq n)$$

where $\mathfrak{p}_i = (x_j \mid 0 \leq j \leq n,\ j \neq i)$ for every $i = 0, \ldots, n$. It is well-known that $I_X = I_Y^{(2)} = (x_i x_j x_h \mid 0 \leq i < j < h \leq n)$ (e.g. [21, Cor. 3.8], or [37, Cor. 4.15(a)]). Thus, I_X indeed contains no quadrics.

To conclude the proof we need to show that any set X of $2 \leq r \leq n$ general double points in $\mathbb{P}^n$ is not $\mathrm{AH}_n(2)$. Since $r \leq n$, we may assume that P_i is the i-th coordinate point for $1 \leq i \leq r$. We first claim that I_X contains precisely $\binom{n-r+2}{2}$ linearly independent quadrics. Indeed, again, let $\mathfrak{p}_i$ be the defining ideal of P_i, for $i = 1, \ldots, r$. It is easy to see that $(x_r, x_{r+1}, \ldots, x_n) \subseteq \mathfrak{p}_i$ for all $i = 1, \ldots, r$. Thus, $(x_r, \ldots, x_n)^2 \subseteq \bigcap_{i=1}^r \mathfrak{p}_i^2 = I_X$. By modularity law, it follows that

$$I_X = \left(I_{r-1,r}\right)^{(2)} + (x_r, \ldots, x_n)^2$$

where $\left(I_{r-1,r}\right)^{(2)} = \bigcap_{0 \leq j_1 < j_2 < \ldots, < j_{r-1} \leq r-1} (x_{j_1}, \ldots, x_{j_{r-1}})^2$ (this is called the second symbolic power of the star configuration of codimension $r-1$ in the variables $x_0, \ldots, x_{r-1}$). It is known that $I_{r-1,r}$ is generated by all squarefree quadrics in $x_0, \ldots, x_{r-1}$ (e.g. [41, Thm 2.3]), and $\left(I_{r-1,r}\right)^{(2)}$ is generated in degree 3 and higher (see e.g. [21, Cor. 3.8]). It follows that the quadrics in I_X are precisely the $\binom{n-r+2}{2}$ quadrics contained in $(x_{r+1}, \ldots, x_n)^2$, proving the claim.

Now, our claim on $I_X(2)$ implies that X is $\mathrm{AH}_n(2)$ only if

$$\binom{n-r+2}{2} = \binom{n+2}{2} - H_{R/I_X}(2) = \max\left\{0, \binom{n+2}{2} - r(n+1)\right\}.$$

Since $\binom{n-r+2}{2} > 0$, this is only possible if $\binom{n-r+2}{2} = \binom{n+2}{2} - r(n+1)$, which implies $r^2 - r = 0$, and thus gives a contradiction. Therefore, X is not $\mathrm{AH}_n(2)$. □

We continue with the cases $d = 4$ and $2 \leq n \leq 4$.

Lemma 3.2 *Suppose that* $2 \leq n \leq 4$. *Then, a set of* r *general double points in* $\mathbb{P}^n$ *is not* $AH_n(4)$ *if and only if* $r = \binom{n+2}{2} - 1$.

Proof Let $Y = \{P_1, \ldots, P_r\}$ be a set of r general points in $\mathbb{P}^n$ and let $X = 2Y$. We shall first show that for $r = \binom{n+2}{2} - 1$, X is not $\mathrm{AH}_n(4)$. Indeed, since $r < \binom{n+2}{2}$, I_Y contains a nonzero quadric, say Q. Then, Q^2 is a nonzero quartic in $I_Y^2 \subseteq I_Y^{(2)} = I_X$. This implies that $H_{R/I_X}(4) \leq \binom{n+4}{4} - 1$. It is easy to check that for $2 \leq n \leq 4$, $\binom{n+4}{4} - 1 < \left[\binom{n+2}{2} - 1\right](n+1) = r(n+1)$. Therefore, X is not $\mathrm{AH}_n(4)$.

We shall now show that $r = \binom{n+2}{2} - 1$ is indeed the only exceptional case. The statement for $n = 2$ is proved in Theorem 4.1. Suppose that $3 \leq n \leq 4$.

For $n = 3$, by Corollary D.5, it suffices to prove that a set of 8 general double points and a set of 10 general double points in $\mathbb{P}^3$ are both $\mathrm{AH}_3(4)$. Similarly, for $n = 4$, it suffices to establish $\mathrm{AH}_4(4)$ property for a set of 13 general double points and a set of 15 general double points in $\mathbb{P}^4$.

<u>$n = 3$ and $r = 8$.</u> Observe that $\left\lfloor \frac{1}{3+1}\binom{4+3}{3} \right\rfloor = 8 = r$, so Theorem 2.9 applies if its hypotheses are satisfied. In this case, we have $q = 4$ and $\epsilon = 0$. Thus, condition (i) holds because 4 general double points in $\mathbb{P}^2$ are $\mathrm{AH}_2(4)$ (by Theorem 4.1), and condition (iii) holds because 4 general double points are $\mathrm{AH}_3(2)$ (by Lemma 3.1). To prove that condition (ii) holds, we need to show that 4 general double points are $\mathrm{AH}_3(3)$. This follows from Sect. 5.

We can also prove this statement directly by considering the 4 coordinate points in $\mathbb{P}^3$. Let I be the defining ideal of these coordinate points. Then, $I = (x_i x_j \mid 0 \leq i < j \leq 3)$, and it can be checked that $I^{(2)}$ is minimally generated by the four squarefree monomials of degree 3. In particular, $H_{R/I^{(2)}}(3) = 16$ which is the expected dimension, so condition (ii) of Theorem 2.9 holds.

In the remaining 3 cases, i.e. when $n = 3$ and $r = 10$, or when $n = 4$ and $r = 13$ or 15 we cannot apply Theorem 2.9 because r is not one of the two possible values needed to apply the theorem. We will instead use Theorem 2.3.

<u>$n = 3$ and $r = 10$.</u> We shall apply Theorem 2.3 for $q = 6$. Clearly, 6 general double points is $\mathrm{AH}_2(4)$ (by Theorem 4.1). Thus, it remains to show that the union of 4 general double points and 6 general simple points on a hyperplane is $\mathrm{AH}_3(3)$.

Let Y_1 be the set of the four coordinate points in $\mathbb{P}^3$. As shown above, we have

$$H_{R/I_{Y_1}^{(2)}}(2) = 10 \qquad \text{and} \qquad H_{R/I_{Y_1}^{(2)}}(3) = 16.$$

Let L be a hyperplane not containing any point of Y_1. By taking $I = I_{Y_1}^{(2)}$, Proposition C.13(2) holds for any u satisfying

$$H_{R/I}(3) + u \leq H_{R/I}(2) + \binom{3+3-1}{3-1},$$

i.e., whenever $16 + u \leq 10 + 10$, i.e., $u \leq 4$. Therefore, if we let Y_0 be a set of $u = 4$ general points on L, then $I_{Y_1}^{(2)} \cap I_{Y_0}$ does not contain any cubic. Now, let Y_2 be obtained by adding two points to Y_0, then $I_{Y_1}^{(2)} \cap I_{Y_2} \subseteq I_{Y_1}^{(2)} \cap I_{Y_0}$ contains no cubics. That is, $2Y_1 \cup Y_2$ is $\mathrm{AH}_3(3)$.

$\underline{n = 4 \text{ and } r = 13.}$ We shall apply Theorem 2.3 for $q = 8$. So one may take Y_1 to be the set of the 5 coordinate points of $\mathbb{P}^4$ and L to be a hyperplane not containing any of these points. Then I_{Y_1} is again generated by all squarefree monomials of degree 2 in R, and $I_{Y_1}^{(2)}$ by the squarefree monomials of degree 3. It follows that Y_1 is $\mathrm{AH}_4(3)$, and in particular $H_{R/I_{Y_1}^{(2)}}(3) = 25$. Then inequality (2) of Proposition C.13 then becomes $25 + q \leq 15 + 20$, so if we add 10 general simple points in L to $2Y_1$ we obtain a scheme containing no cubics.

In particular, if we take Y_2 to be a set of $q = 8$ general points on L, then assumption (2) of Theorem 2.3 is satisfied, so $Y_1 \cup Y_2$ is a set of 13 points in $\mathbb{P}^4$ which is $\mathrm{AH}_4(4)$. By Lemma D.4 any set of 13 general points is $\mathrm{AH}_4(4)$.

$\underline{n = 4 \text{ and } r = 15.}$ We shall apply Theorem 2.3 for $q = 10$. Clearly, a set of $q = 10$ general double points is $\mathrm{AH}_3(4)$ as shown above. Thus, it suffices to show that the union of 5 general double points and 10 general simple points in a hyperplane is $\mathrm{AH}_4(3)$. This follows by the same argument of the previous case. □

We conclude this section with the case where $d = 3$ and $n = 4$.

Lemma 3.3 *A set of r general double points in $\mathbb{P}^4$ is $\mathrm{AH}_4(3)$ if and only if $r \neq 7$.*

Proof We first prove that a set of 7 general double points in $\mathbb{P}^4$ is not $\mathrm{AH}_4(3)$. Let $Y = \{P_1, \ldots, P_7\} \subseteq \mathbb{P}^4$ be a set of 7 general points, a simple computation shows that $2Y$ is $\mathrm{AH}_4(3)$ if and only if $I_X^{(2)}$ contains no non-zero cubic.

By a result of Castelnuovo (e.g. [18, Thm 1]), given any set of $t + 3$ points in general position in $\mathbb{P}^t$, there exists a unique rational normal curve C_t passing through all of them, whose equations are given by the 2×2 minors of a *1-generic* matrix. In particular, there is a (unique) rational normal curve C_4 passing through our 7 points in $\mathbb{P}^4$, whose equation, in an appropriate coordinate system, is

$$I := I_2\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

One can check directly that $I^{(2)}$ contains (precisely) one cubic, namely

$$x_2^3 - 2x_1x_2x_3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4.$$

Thus, a set of 7 general double points in $\mathbb{P}^4$ is not $\mathrm{AH}_4(3)$.

Alternatively, it is also known that $I = I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$ and it can be seen that $f = \det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$ is singular at all points of C_4.

By Corollary D.5, to conclude it suffices to show that sets of $r = 6$ and $r = 8$ general double points in $\mathbb{P}^4$ are $\mathrm{AH}_4(3)$. As Theorem 2.9 could only be applied if $r = 7$, then we invoke Theorem 2.3 in both cases. First, observe that by Lemma 3.1, sets of 5 general double points in $\mathbb{P}^4$ are $\mathrm{AH}_4(2)$. If $r = 6$, to apply Theorem 2.3 we need q with $15 \leq 4q \leq 18$, thus $q = 4$. Then, assumption (1) holds for the reasons stated in the proof of Lemma 3.2 (the case where $n = 3$ and $r = 8$), and (2) holds because $r - q = 2$ general double points are $\mathrm{AH}_4(2)$ (because 5 double coordinate points are, and because of Lemma C.12(1)) and by Proposition C.13 (we need to add $u = q = 4$ general points to the two double points).

The case $r = 8$ is proved similarly. In this case, one may take q satisfying $15 \leq 4q \leq 25$. If we take $q = 4$ then, as above, assumption (1) of Theorem 2.3 is satisfied. For assumption (2), we need to prove there exists no quadric through a set Z of 4 general double points and 4 general simple points. However it is easily seen that the only quadric through 4 general double points in $\mathbb{P}^4$ is the square of the hyperplane containing them. Since the remaining 4 simple points are general, we may take them outside this hyperplane, so there is no quadric in I_Z.

An application of Theorem 2.3 now finishes the proof. □

We end this section by noting that the case of cubics, i.e., when $d = 3$, for an arbitrary value of n is much more subtle. Section 5 is devoted to handle this case.

4 The Case of $\mathbb{P}^2$ ($n = 2$)

This section focuses on the double points in $\mathbb{P}^2$. Particularly, we shall identify all exceptional cases when $n = 2$. While one could prove this case with more elementary arguments, we have chosen to employ Theorem 2.9 to provide the reader with a further illustration of its application.

Theorem 4.1 *Let X be any set of r general points in $\mathbb{P}^2$. Then $2X$ is $AH_2(d)$ for every $d \geq 1$, except for the exceptional cases of $r = 2$ and $d = 2$, and $r = 5$ and $d = 4$.*

Proof Let $R = \mathbb{C}[x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^2$. We shall consider different cases based on the values of d.
Case 1: $d = 1$. It suffices to prove the assertion for $r = 1$ since the degree of a double point in $\mathbb{P}^3$ is $3 = H_R(1)$. This case follows from Remark C.9.

Case 2: $d = 2$. The assertion is true for $r = 1$ by Remark C.9. The case where $r = 2$ is an exceptional case by Lemma 3.1. Suppose that $r \geq 3$. Since the degree of 3 double points in $\mathbb{P}^2$ is 9, which is bigger than $6 = H_R(2)$, then $2X$ is $AH_2(2)$ if and only if there exists no conic in $\mathbb{P}^2$ it suffices to prove that there is no conic in $\mathbb{P}^2$ with r double points. Clearly, it suffices to prove it when $r = 3$. By Bézout theorem, the equation of every conic with 3 double points is divisible by the equations of the three lines connecting 2 of these points—this gives a contradiction.
Case 3: $d = 3$. The statement is true for $r = 1$, again by Remark C.9. When $r = 2$ we need to show that $H_{R/I_X^{(2)}}(3) = 4$. Observe that by Bézout theorem, a cubic with 2 double points must contain the line connecting these points. That is, this cubic factors as a line and a conic going through these 2 points. Since the Hilbert function of 2 general points in $\mathbb{P}^2$ is $1, 2, 2, \ldots$, it follows that the space of conic going through these 2 points has dimension 4. Particularly, the space of cubic with 2 double points has dimension 4. Thus, the assertion is true for $r = 2$.

Observe further that by Bézout theorem, a cubic with 3 double points must contain 3 lines connecting 2 of these points, and so there is a unique such cubic, which is the union of the 3 lines. It follows that $H_{I_X^{(2)}}(3) = 10 - 1 = 9 = e(R/I_X^{(2)})$, therefore, the assertion is true for $r = 3$.

Suppose that $r \geq 4$. Since the degree of 4 double points is $12 > 10 = H_R(3)$, it suffices to show that there is no cubic containing 4 double points. By Bézout theorem again, if such a cubic existed then it would contain the 6 lines connecting any 2 of these 4 points, a contradiction.
Case 4: $d \geq 4$. Recall that, from Theorem 2.9, a set of r general double points in $\mathbb{P}^2$ with $\left\lfloor \frac{1}{3}\binom{d+2}{2} \right\rfloor \leq r \leq \left\lceil \frac{1}{3}\binom{d+2}{2} \right\rceil$ is $\mathrm{AH}_2(d)$ if

(1) q general double point in $\mathbb{P}^1$ is $\mathrm{AH}_1(d)$ (which holds by Proposition C.11),
(2) $r - q$ general double points in $\mathbb{P}^2$ are $\mathrm{AH}_2(d-1)$, and
(3) $r - q - \epsilon$ general double points in $\mathbb{P}^2$ are $\mathrm{AH}_2(d-2)$,

where $q \in \mathbb{N}_0$ and $0 \leq \epsilon \leq 1$ are such that $2q + \epsilon = 3r - \binom{d+1}{2}$.

When $d = 4$, Remark 2.1 says there are no exceptions if the case $r = \frac{1}{3}\binom{4+2}{2} = 5$ is not an exceptional case. However, it is an exceptional case (and in fact in this case $q = 2, \epsilon = 1$, so condition (3) of Theorem 2.9 is not satisfied—because it is the exceptional case of 2 double points in degree 2). By Lemma C.12, we need to show that the cases $r = 4, 6$ are not exceptional cases.

When $r = 4$, the first numerical condition in Theorem 2.3 is $2 \leq 2q \leq 5$, so $1 \leq q \leq 2$. Taking $q = 1$, assumption (1) of Theorem 2.3 is satisfied by Proposition C.11. On the other hand, a set of $r - q = 3$ general double points in $\mathbb{P}^2$ is $AH_2(3)$ by the above and there is precisely one cubic passing through all the three points twice. So there is no cubic passing through them twice and passing through an additional general simple point (which we can take to be outside the cubic). Therefore, assumption (2) is satisfied too, and this case follows by Theorem 2.3.

When $r = 6$ the proof is very similar. The second numerical condition in in Theorem 2.3 is $5 \leq 2q \leq 8$, so $3 \leq q \leq 4$. We take $q = 3$ so again we have

$r - q = 3$, and then assumptions (1) and (2) of Theorem 2.3 are satisfied as above, thus proving that $r = 6$ is not an exceptional case, and concluding the case $d = 4$.

For $d = 5$, by Remark 2.1 it suffices to prove that a set of $r = 7$ general double points is $AH_2(5)$. In this case Theorem 2.9 applies, because the induction hypotheses (1)–(3) are satisfied with the only possible exception of (2) when $r - q = 5$ (as it reduces to the exceptional case of 5 double points in degree 4), i.e. $q = 2$. Since $2q + \epsilon = 3r - 15$, then $\epsilon = 2$, which is a contradiction.

For $d = 6$, by Remark 2.1 we need to prove that sets of $r = 9, 10$ general double points in $\mathbb{P}^2$ are $AH_2(6)$. The induction hypotheses (1)–(3) of Theorem 2.9 are satisfied except possibly assumption (3) when $r - q - \epsilon = 5$ (in this case (3) reduces to the exceptional case of 5 double points in degree 4). Since $2q + \epsilon = 3r - 21$, we get $q = 2r - 16$ and $\epsilon = 11 - r$. Since $r \leq 10$ and $0 \leq \epsilon \leq 1$, we must have $r = 10$, $\epsilon = 1$ and $q = 4$. This particularly shows that Theorem 2.9 applies when $r = 9$, so the case $r = 9$ and $d = 6$ is not an exceptional case. As a consequence, there is a unique sextic containing 9 general double points (since the degree of 9 double points is 27). On the other hand, the Hilbert function of 9 general points is $1, 3, 6, 9, 9, \ldots$, and so there is only one cubic passing through 9 general points. Thus, the unique sextic with 9 general double points is the double cubic passing through these 9 general points. As the remaining point is general, we can take it outside the sextic, resulting in no sextic passing through 10 general double points.

Since there are no exceptional cases in degrees 5 and 6, by Theorem 2.9, we conclude that there is no exceptional cases in any degree $d \geq 5$, finishing the proof. □

5 The Case of Cubics ($d = 3$)

In this section, we consider the case of cubics for any value of n. The main result in this section extends Lemma 3.3 and completes the case where $d = 3$.

Theorem 5.1 *Suppose that $n \geq 2$. A set of r general double points in $\mathbb{P}^n$ is not $AH_n(3)$ if and only if $n = 4$ and $r = 7$.*

Proof The case where $n = 2$ was already proved in Sect. 4. The case of $n = 4$ has been discussed in Lemma 3.3. For $n \geq 3$ and $n \neq 4$ we proceed by considering two possibilities depending on the congruence of n modulo 3.

CASE 1: $n \equiv 0, 1 \pmod 3$. In these cases $(n+2)(n+3)$ is a multiple of 6. Thus, $\frac{1}{n+1}\binom{n+3}{3} = \frac{(n+2)(n+3)}{6} \in \mathbb{Z}$ and by Remark 2.1, it suffices to show that a set of $r = \frac{(n+2)(n+3)}{6}$ general double points in $\mathbb{P}^n$ is $\mathrm{AH}_n(3)$.

We shall use induction on n to show that the ideal of r general double points in $\mathbb{P}^n$ contains no cubics. The first base case, when $n \equiv 0 \pmod 3$, is $n = 3$. By Remark 2.1, the assertion amounts to showing that a set X of 5 general double points in $\mathbb{P}^3$ is $\mathrm{AH}_3(3)$, ie., its defining ideal contains no cubics. Without loss of generality we may write $X = Y \cup \{Q\}$ where $Q = [1 : 1 : 1 : 1]$, $Y = \{P_0, P_1, P_2, P_3\}$ and $P_i = [e_i] = [0 : \dots : 1 : 0 \dots : 0]$ for $0 \le i \le 3$. Then Y is a star configuration of 4 points, and a basis of $[I_Y^{(2)}]_3$ is $\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3\}$ (see, e.g., [21, Cor. 3.8]). So any cubic f in $I_X^{(2)} \subseteq I_Y^{(2)}$ is a linear combination of these basis elements. It is easily seen that imposing that the partial derivatives $(\partial/\partial x_i)f(Q) = 0$ forces $f = 0$.

The other base case, when $n \equiv 1 \pmod 3$, is $n = 7$ and $r = 15$. This can be computed directly (and verified via Macaulay 2 [26] computations).

Suppose now that $n \ge 6$ and $n \ne 7$. The inductive hypothesis applies to $n_1 = n-3$. So we let r_1 be the integer obtained by replacing n by $n_1 = n-3$ in the formula for r, i.e. $r_1 = \frac{n(n-1)}{6}$. Let L be a codimension 3 linear subspace in $\mathbb{P}^n$, after possibly a change of variables we may assume the defining ideal of L is $\mathfrak{p}_L = (x_{n-2}, x_{n-1}, x_n)$. Let X be a set of r_1 general double points in L together with $r - r_1 = n+1$ general double points outside of L. By the semi-continuity of Hilbert function (Remark 2.1), it is enough to show that I_X contains no cubics. Consider a point Q in the support of X that lies in L, and let $\mathfrak{q}$ be its defining ideal. Clearly, $\mathfrak{q} \supseteq \mathfrak{p}_L$. Thus, we can write $\mathfrak{q} = \overline{\mathfrak{q}} + \mathfrak{p}_L$, where $\overline{\mathfrak{q}}$ is a linear prime in $R_1 = \mathbb{C}[x_0, \dots, x_{n-3}] \simeq R/P_L$. It follows from [28, Theorem 3.4] that

$$\mathfrak{q}^{(2)} = \overline{\mathfrak{q}}^{(2)} + \overline{\mathfrak{q}} \cdot \mathfrak{p}_L + \mathfrak{p}_L^{(2)}.$$

Particularly, it implies that $\mathfrak{q}^{(2)} + \mathfrak{p}_L = \overline{\mathfrak{q}}^{(2)} + \mathfrak{p}_L$ is the defining ideal of the double point $2Q$ in L. Thus, by letting $\overline{X}$ be the set of r_1 general double points of X in L, considered as a subscheme of $L \simeq \mathbb{P}^{n-3}$, we obtain

$$I_X + \mathfrak{p}_L \subseteq I_{\overline{X}} + \mathfrak{p}_L.$$

Moreover, by the induction hypothesis applied to $\overline{X} \subseteq L \simeq \mathbb{P}^{n-3}$, we have $\left[I_{\overline{X}}\right]_3 = (0)$. Therefore, $I_X + \mathfrak{p}_L/\mathfrak{p}_L$ contains no cubics. Hence, by considering the exact sequence

$$0 \longrightarrow I_X \cap \mathfrak{p}_L \longrightarrow I_X \longrightarrow I_X + \mathfrak{p}_L/\mathfrak{p}_L \longrightarrow 0,$$

to prove that I_X contains no cubics, it remains to show that $I_X \cap \mathfrak{p}_L$ contains no cubics. This is the content of Claim 5.1.1 below. □

Claim 5.1.1 Suppose that $n \ge 3$ and $n \ne 4$. Let L be a codimension 3 linear subspace of $\mathbb{P}^n$ and let X be the union of $r_1 = \frac{n(n-1)}{6}$ general double points in L and $n+1$ general double points outside of L. Then, $I_X \cap \mathfrak{p}_L$ contains no cubics.

Proof of Claim 5.1.1 We use also induction on n to prove the assertion. The base case $n = 3$ holds because, by the above, I_X contains no cubics. The other base case $n = 7$ can be verified directly, or by Macaulay 2 [26] computations. Assume that $n \ge 6$. For the inductive step, let M be a codimension 3 linear subspace of $\mathbb{P}^n$ such that $L \cap M$ has codimension 6 in $\mathbb{P}^n$ (any general codimension 3 linear subspace would work). Let $\mathfrak{p}_M$ be the defining ideal of M. We specialize to the following situation:

- $r_2 := \frac{(n-3)(n-3-1)}{6} = \frac{(n-3)(n-4)}{6}$ of the points of X in L are general double points in $L \cap M$;
- the $r_1 - r_2 = n - 2$ remaining points of X in L lie outside M;
- $n - 2$ of the $n + 1$ points of X lying outside of L are general double points in M;
- and the last 3 points of X outside of L are general double points outside $L \cup M$.

By the semi-continuity of Hilbert function, it suffices to show that $I_X \cap \mathfrak{p}_L$ contains no cubics in this particular case. From the short exact sequence

$$0 \longrightarrow I_X \cap \mathfrak{p}_L \cap \mathfrak{p}_M \longrightarrow I_X \cap \mathfrak{p}_L \longrightarrow (I_X \cap \mathfrak{p}_L) + \mathfrak{p}_M/\mathfrak{p}_M \longrightarrow 0,$$

it suffices to prove the other two terms of this exact sequence contain no cubics. As before, observe that

$$(I_X \cap \mathfrak{p}_L) + \mathfrak{p}_M \subseteq (I_{\overline{X}} \cap \mathfrak{p}_{\overline{L}}) + \mathfrak{p}_M,$$

where $\overline{X}$ denotes the set of points of X lying in $M \simeq \mathbb{P}^{n-3}$, and $\overline{L}$ denotes the codimension 3 subspace $L \cap M$ of $M \simeq \mathbb{P}^{n-3}$. As above, it can be seen that, in M, $\overline{X}$ is the union of r_2 general double points lying in $\overline{L}$ and $n - 2$ general double points outside of $\overline{L}$. Thus, by the induction hypothesis, the ideal $(I_{\overline{X}} \cap \mathfrak{p}_{\overline{L}}) + \mathfrak{p}_M/\mathfrak{p}_M$ of $R/\mathfrak{p}_M \simeq \mathbb{C}[y_0, \dots, y_{n-3}]$ contains no cubics. Hence, it remains to show that $I_X \cap \mathfrak{p}_L \cap \mathfrak{p}_M$ contains no cubics. This follows from Claim 5.1.2 below. □

Claim 5.1.2 Suppose that $n \geq 3$ and $n \neq 4$. Let L, M be two general codimension 3 linear subspaces of $\mathbb{P}^n$. Let $X \subseteq \mathbb{P}^n$ be the union of $r_2 = \frac{(n-3)(n-4)}{6}$ general double points in $L \cap M$, $n-2$ general double points in $L \setminus M$, $n-2$ general double points in $M \setminus L$, and 3 general double points outside of $L \cup M$. Then, $I_X \cap \mathfrak{p}_L \cap \mathfrak{p}_M$ contains no cubics.

Proof of Claim 5.1.2 Let Z be the set of double points obtained by removing the r_2 double points in $L \cap M$ from X. Clearly, $I_Z \supseteq I_X$. We shall prove the stronger statement that $I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M$ contains no cubics. The statement for $n = 3, 5, 6,$ and 7 can be verified by direct computations (e.g. using Macaulay 2 [26]). We shall use induction to prove the statement for $n \geq 8$.

Let N be another general codimension 3 linear subspace of $\mathbb{P}^n$ and let $\mathfrak{p}_N$ be its defining ideal. We specialize the points as follow: we take $n-5$ of the $n-2$ double points of Z lying in L to be in $L \cap N$, we take $n - 5$ of the $n - 2$ double points of Z lying in M to be in $M \cap N$, and we take the 3 general double points of Z outside of $L \cup M$ to be in $N \simeq \mathbb{P}^{n-3}$. By the semi-continuity of Hilbert function, it suffices to show that for this special configuration of Z, the ideal I_Z contains no cubics.

Consider the following short exact sequence

$$0 \longrightarrow I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M \cap \mathfrak{p}_N \longrightarrow I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M \longrightarrow I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M + \mathfrak{p}_N/\mathfrak{p}_N \longrightarrow 0.$$

By an argument similar to the proof of Claim 5.1.1, we have $I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M + \mathfrak{p}_N \subseteq I_{\overline{Z}} \cap \mathfrak{p}_{\overline{L}} \cap \mathfrak{p}_{\overline{M}} + \mathfrak{p}_N$, where $\overline{\bullet}$ represents the restrictions to $N \simeq \mathbb{P}^{n-3}$. The induction hypothesis applies to $\overline{Z}$, so $I_{\overline{Z}} \cap \mathfrak{p}_{\overline{L}} \cap \mathfrak{p}_{\overline{M}} + \mathfrak{p}_N/\mathfrak{p}_N$ contains no cubics. Therefore, to establish the desired statement, it remains to show that $I_Z \cap \mathfrak{p}_L \cap \mathfrak{p}_M \cap \mathfrak{p}_N$ contains no cubics. This follows from Claim 5.1.3 below, noting that $n - 5 \geq 3$. □

Claim 5.1.3 Suppose that $n \geq 5$. Let L, M, and N be general codimension 3 linear subspaces of $\mathbb{P}^n$. Let $X \subseteq \mathbb{P}^n$ be the union of 3 general double points in $L \setminus (M \cup N)$, 3 general double points in $M \setminus (L \cup N)$, and 3 general double points in $N \setminus (L \cup M)$. Then $I_X \cap \mathfrak{p}_L \cap \mathfrak{p}_M \cap \mathfrak{p}_N$ contains no cubics.

Proof of Claim 5.1.3 Direct computations (e.g. via Macaulay 2 [26]) verify the statement for $n = 5$ and $n = 6$. (Notice that in [7, Prop. 5.2] it is incorrectly stated that when $n = 6$ the ideal $\mathfrak{p}_L \cap \mathfrak{p}_M \cap \mathfrak{p}_N$ contains no quadrics.) Assume that $n \geq 7$. Without loss of generality, we may assume that $\mathfrak{p}_L = (x_0, x_1, x_2)$ and $\mathfrak{p}_M = (x_3, x_4, x_5)$, so $\mathfrak{p}_L \cap \mathfrak{p}_M = \mathfrak{p}_L\mathfrak{p}_M$; in particular, $\mathfrak{p}_L \cap \mathfrak{p}_M$ is minimally generated by 9 quadrics, so $H_{R/\mathfrak{p}_L \cap \mathfrak{p}_M}(2) = H_R(2) - 9$

Let $\kappa := \mathfrak{p}_L \cap \mathfrak{p}_M \cap \mathfrak{p}_N$, so we need to show that $I_X \cap \kappa$ contains no cubics.

We shall first show that κ contains no quadrics. Indeed, if $n \geq 8$ then we may assume that $\mathfrak{p}_N = (x_6, x_7, x_8)$. In this case, $\kappa = \mathfrak{p}_L\mathfrak{p}_M\mathfrak{p}_N$ is generated in degree 3. On the other hand, if $n = 7$ then we may assume that $\mathfrak{p}_N = (x_6, x_7, x_0 - x_3)$. Now, consider the short exact sequence

$$0 \longrightarrow R/\kappa \longrightarrow R/\mathfrak{p}_L \cap \mathfrak{p}_M \oplus R/\mathfrak{p}_N \longrightarrow R/(\mathfrak{p}_L \cap \mathfrak{p}_M) + \mathfrak{p}_N \longrightarrow 0.$$

Since $R/(\mathfrak{p}_L \cap \mathfrak{p}_M) + \mathfrak{p}_N = R/\mathfrak{p}_L\mathfrak{p}_M + \mathfrak{p}_N = R/(x_0, x_1, x_2)(x_3, x_4, x_5), x_0 - x_3, x_6, x_7)$ is isomorphic to $B := \mathbb{C}[x_1, \dots, x_5]/(x_1, x_2, x_3)(x_3, x_4, x_5)$, then we have

$$\begin{aligned} H_{R/\kappa}(2) &= H_{R/\mathfrak{p}_L \cap \mathfrak{p}_M}(2) + H_{R/\mathfrak{p}_N}(2) - H_B(2) = (H_R(2) - 9) + 15 - H_B(2) \\ &= 27 + 15 - H_B(2). \end{aligned}$$

Since B contains all the quadrics in $\mathbb{C}[x_1, \dots, x_5]$ except for the 9 generators of the ideal $(x_1, x_2, x_3)(x_3, x_4, x_5)$, then $H_B(2) = 15 - 9 = 6$. Therefore, $H_{R/\kappa}(2) = 42 - 6 = 36 = H_R(2)$, showing that $[\kappa]_2 = 0$.

Now, by the above short exact sequence, since $\dim R \geq 5$ one has $\operatorname{depth} R/\kappa \geq 2$. Let h be a general linear form in R and let H be the hyperplane in $\mathbb{P}^n$ defined by h; since $\operatorname{depth} R/\kappa \geq 2$, we may assume h is regular on R/κ. Let $\overline{R} = R/(h)$ and $\overline{\kappa}$ be the image of κ in $\overline{R}$. From the standard short exact sequence

$$0 \longrightarrow R/\kappa \longrightarrow R/\kappa \longrightarrow \overline{R}/\overline{\kappa} \longrightarrow 0$$

one obtains that $\operatorname{depth} \overline{R}/\overline{\kappa} \geq 1$, i.e. $\overline{\kappa}$ is saturated in $\overline{R}$.

We now specialize the configuration so that all 9 double points of X are on the hyperplane $H \simeq \mathbb{P}^{n-1}$ and let $I = I_X \cap \kappa$ for simplicity of notation. Consider the short exact sequence

$$0 \longrightarrow (I : h)(-1) \longrightarrow I \longrightarrow (I, h)/(h) \longrightarrow 0.$$

Since the points in X are lying on H, we have $I : h = \kappa : h = \kappa$. Thus, this sequence can be rewritten as

$$0 \longrightarrow \kappa(-1) \longrightarrow I \longrightarrow (I, h)/(h) \longrightarrow 0.$$

As we have shown, κ has no quadrics, so $\kappa(-1)$ has no cubics. Hence, to show that I contains no cubics, it remains to show that the image $\overline{I}$ of I in $\overline{R}$ has no cubics. This is indeed true by induction on n, since $\overline{I} \subseteq (\overline{I})^{\text{sat}}$ and $(\overline{I})^{\text{sat}}$ is the defining ideal of X in $H \simeq \mathbb{P}^{n-1}$. □

CASE 2: $n \equiv 2 \pmod 3$. In this case, $\frac{\binom{n+3}{3}}{n+1} = \frac{n^2+5n+6}{6} = \frac{(n+1)(n+4)}{6} + \frac{1}{3}$, and since $n \equiv 2 \pmod 3$, we know $\frac{(n+1)(n+4)}{6}$ is an integer. So, we let $r_0 = \frac{(n+2)(n+3)}{6} - \frac{1}{3} = \frac{(n+1)(n+4)}{6}$ and set $\delta = \binom{n+3}{3} - r_0 = \frac{n+1}{3}$. By Remark 2.1, to prove the desired statement, it suffices to show that sets of $r = r_0$ and $r_0 + 1$ general double points are $\text{AH}_n(3)$. To this end, it is enough to show that a scheme $X \subseteq \mathbb{P}^n$ consisting of r_0 general double points and a general subscheme η supported at another general point with degree δ is $\text{AH}_n(3)$. Indeed, it is easy to see that X has multiplicity exactly $\binom{n+3}{3}$. Thus, by a proof similar to the one of Lemma C.12, it can be shown that if X is $\text{AH}_n(3)$ then so is a set of r_0 general double points in $\mathbb{P}^n$. On the other hand, a set of $r_0 + 1$ general double points contains X as a subscheme, so its Hilbert function in degree d is at least that of X, which is $\binom{n+3}{3}$, i.e. it is already maximal. Particularly, a set of $r_0 + 1$ general double points also has maximal Hilbert function in degree 3.

As in Case 1, we shall use induction on $n \geq 2$ to show that X is $\text{AH}_n(3)$. The case $n = 2$ is proved in Theorem 4.1. The induction step proceeds along the same lines as Case 1. The only difference is at Claim 5.1.1, which shall be replaced by the following

Claim 5.1.4 Suppose that $n \geq 2$. Let L be a general codimension 3 linear subspace in $\mathbb{P}^n$. Let $X \subseteq \mathbb{P}^n$ the union of $r_1' = \frac{(n-2)(n+1)}{6}$ general double points in L, $(n + 1)$ general double points outside of L, and a general subscheme η supported at a point $Q \in L$ and of multiplicity δ such that $\eta \cap L$ has multiplicity $\delta - 1 = \frac{n-2}{3}$. Then, $I_X \cap \mathfrak{p}_L$ contains no cubics.

Proof of Claim 5.1.4 One proceeds by induction exactly as in the proof of Claim 5.1.1. □

The proof of Theorem 5.1 is now completed.

6 Open Problems

In this section we discuss a few open problems. Let us state clearly that there are many other interesting questions outside the ones that we include here. For instance, as indicated by Appendix A below, the polynomial interpolation is closely connected to secant varieties and Waring rank. Thus, there are many other problems and questions that are of interest to researchers working in these areas or studying, for example, containment problems for ordinary and symbolic powers of ideals, other interpolation problems and invariants associated to symbolic powers of ideals.

However, to keep this section aligned with the other sections, we restrict ourselves to problems and questions *related to the Alexander–Hirschowitz theorem*. It is implicit that this small set of problems and conjectures is far from being comprehensive, and it should be considered as a sample –aimed at young researchers– of the many problems in this active area of research.

We begin by observing that Theorem 1.1 describes the Hilbert function of $I_Y^{(2)}$ for every set Y of *general* points in $\mathbb{P}^n$ with a finite list of exceptions (the Hilbert functions in these cases can be worked out individually). A starting point is asking for a characterization of the Hilbert function of $I_Y^{(2)}$ for any set of points Y in $\mathbb{P}^n$.

To state this general problem, for $n \geq 1$ and $r \geq 1$, let $\mathcal{H}_n(r)$ be the set of all Hilbert functions $H_{R/I_Y^{(2)}}$ where Y is a set of r points in $\mathbb{P}^n$.

Problem 6.1 Characterize the numerical functions which are Hilbert functions of $I_Y^{(2)}$ for some set Y of points in $\mathbb{P}^n$, i.e. for every $n \geq 1$ characterize all elements in

$$\mathcal{H}_n := \bigcup_{r\geq 1} \mathcal{H}_n(r) = \left\{ H_{R/I_Y^{(2)}} \mid Y \text{ is a set of points in } \mathbb{P}^n \right\}.$$

In this generality, so far this has been a very challenging problem, see, for instance, the surveys of Gimigliano [25] and Harbourne [29]. Since Problem 6.1 is easy for points in $\mathbb{P}^1$ (see Proposition C.11), and, to the best of our knowledge, it is still open in $\mathbb{P}^2$ (see [24] and [23] for some recent work in this direction), then one might attempt to tackle this first nontrivial case:

Problem 6.2 Characterize the numerical functions which are Hilbert functions of $I_Y^{(2)}$ for some set Y of points in $\mathbb{P}^2$, i.e. characterize all elements in

$$\mathcal{H}_2 := \left\{ H_{R/I_Y^{(2)}} \mid Y \text{ is a set of points in } \mathbb{P}^2 \right\}.$$

In investigating a family of Hilbert functions, it is natural to determine the existence of "minimal" and "maximal" elements. In fact, we can define a partial order on $\mathcal{H}_n(r)$ by setting

$$H_{R/I_Y^{(2)}} \leq H_{R/I_Z^{(2)}} \qquad \text{if} \qquad H_{R/I_Y^{(2)}}(d) \leq H_{R/I_Z^{(2)}}(d) \text{ for every } d \geq 1.$$

Notice that every $H \in \mathcal{H}_n(r)$ satisfies

$$H(d) \leq \min\left\{\binom{n+d}{d}, r(n+1)\right\}$$

and, by Theorem 1.1, equality holds for any general set of points (with a few exceptions). Therefore, Theorem 1.1 in particular proves the existence of maximal elements in $\mathcal{H}_n(r)$ (with a few exceptions), and numerically characterizes what these maximal Hilbert functions are. It is a natural problem to determine the potential existence and characterization of *minimal* elements of $\mathcal{H}_n(r)$.

Problem 6.3 Fix $n, r \geq 1$.

(a) Prove the existence of a minimal element in $\mathcal{H}_n(r)$.
(b) Determine the minimal element in $\mathcal{H}_n(r)$.

A partial answer to Problem 6.3 was given for double points in $\mathbb{P}^2$ in [22, 24], where the problem is solved when $r = \binom{t}{2}$ or $r \leq 11$. E

Another natural approach in examining the Hilbert function of double points is to specify that the points are *lying on a given subscheme*, e.g. on a rational normal curve or a conic.

Problem 6.4 For $n \geq 1$, let C_n be the rational normal curve in $\mathbb{P}^n$. For any $r \geq 1$, determine the Hilbert function of $R/I_Y^{(2)}$ where Y is a set of r general points on C_n.

If the rational normal curve C_n is replaced by a conic then Problem 6.4 has a satisfactory answer, given by Geramita, Harbourne and Migliore [23].

Another problem along the lines of the Alexander–Hirschowitz theorem is to determine the Hilbert functions of sets of general double points in *multiprojective spaces*. In general, however, points in multiprojective spaces are harder to understand than points in projective spaces. (e.g., a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ does not need to be Cohen–Macaulay.) Much work has been put forward to understand, in general, numerical invariants and properties of points in the first nontrivial case of a multiprojective space, i.e., $\mathbb{P}^1 \times \mathbb{P}^1$, (see, e.g., [27]).

While the Hilbert function for a general set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ is known (see [42]), that for an *arbitrary* set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ is not yet completely classified.

Problem 6.5 Let $R = \mathbb{C}[x_0, \ldots, x_3]$ and fix any $r \geq 1$. Determine the possible Hilbert functions of $R/I_Y^{(2)}$ where Y is any set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$.

We observe, in passing, that similarly to how the Alexander-Hirschowitz theorem is closely related to the study of secant varieties of Veronese embeddings of $\mathbb{P}^n$, Problem 6.5 is intimately connected to the study of secant varieties of Segre-Veronese varieties (cf. [8]).

In general, understanding the symbolic square $I_Y^{(2)}$ of a set Y of simple points is far from being a completed task. Since for certain questions I_Y^2 is more understood than $I_Y^{(2)}$, a possible approach is to compare $I_Y^{(2)}$ and I_Y^2, or simply to consider the module $I_Y^{(2)}/I_Y^2$.

For instance, Galetto, Geramita, Shin and Van Tuyl [21] defined a first possible measure aimed at quantifying the gap between the m-th symbolic power of an ideal and the m-th ordinary power. They dubbed this measure the *m-th symbolic defect of an ideal J*, and they defined it to be

$$\operatorname{sdef}(J, m) := \mu(J^{(m)}/J^m).$$

(Here, $\mu(M)$ denotes the minimal number of generators of a finitely generated R-module M.) The problem of determining symbolic defects of an ideal is open, even for the defining ideal of a general set of points.

Problem 6.6 Compute $\operatorname{sdef}(I_Y, 2)$ for any set Y of general simple points in $\mathbb{P}^n$.

Problem 6.6 seems to be open even in $\mathbb{P}^2$.

Problem 6.7 Compute $\operatorname{sdef}(I_Y, 2)$ for any set Y of general simple points in $\mathbb{P}^2$.

A first partial result towards Problem 6.7 is [21, Thm 6.3], where the authors determined the second symbolic defect when $|Y| \leq 9$ and $|Y| \neq 6$. These are precisely the set of points whose second symbolic defect is either 0 or 1. They also proved that if $|Y| = 6$ of $|Y| > 10$, then $\operatorname{sdef}(I_Y, 2) > 1$, however, the precise value is not known.

Inspired by studies on symbolic defects of an ideal, we can consider a similar invariant defined by examining the Hilbert function instead of the minimum number of generators. Particularly, for $m \in \mathbb{N}$, define the *m-th symbolic HF-defect* of an ideal J to be the Hilbert function of $J^{(m)}/J^m$, i.e.

$$\operatorname{sHFdef}(J, m) := H_{J^{(m)}/J^m}.$$

Problem 6.8 Compute $\operatorname{sHFdef}(I_Y, 2)$ for any set Y of general points in $\mathbb{P}^n$. Equivalently, compute the Hilbert function H_{R/I_Y^2} for any set Y of general points in $\mathbb{P}^n$.

The equivalence of the statements given in Problem 6.8 follows because $H_{I_Y^{(2)}/I_Y^2} = H_{R/I_Y^2} - H_{R/I_Y^{(2)}}$, and by Theorem 1.1 we already know $H_{R/I_Y^{(2)}}$.

Most of the above problems are aimed at understanding symbolic squares of ideals of points; however, the most natural, important and challenging question raised by Theorem 1.1 is to prove an analogue of Theorem 1.1 for any symbolic power of any ideal defining a set of general points in $\mathbb{P}^n$.

Problem 6.9 Let $n \geq 1$ and $R = \mathbb{C}[x_0, \ldots, x_n]$. For every fixed $m \geq 3$, determine the Hilbert function of $R/I_Y^{(m)}$ for a set Y of general points in $\mathbb{P}^n$.

Problem 6.9 is one of the main open problems in interpolation theory. Even the case where $m = 3$ is still wide open.

Problem 6.10 Let $n \geq 1$ and $R = \mathbb{C}[x_0, \dots, x_n]$. Determine the Hilbert function of $R/I_Y^{(3)}$ for a set Y of general points in $\mathbb{P}^n$.

As we have seen in Theorem 1.1, one expects to have a finite list of exceptional cases, for which the general statement does not hold. A starting point toward Problem 6.10 is to determine a similar list of exceptional cases for triple general points.

Problem 6.11 Let $n \geq 1$ and $R = \mathbb{C}[x_0, \dots, x_n]$. Determine all the potential exceptional cases for Problem 6.10, i.e., find a finite list $\mathcal{L}$ such that if $3Y$, for a general set of points $Y \subseteq \mathbb{P}^n$, is not $\mathrm{AH}_n(d)$ then $Y \in \mathcal{L}$.

A well-known conjecture, often referred to as the *SHGH Conjecture*, raised (and refined) over the years by Segre, Harbourne, Gimigliano and Hirschowitz, provides the first step toward a solution to Problem 6.11 by predicting what these exceptional cases are expected to be. We shall state a special case of this conjecture, namely, the uniform points in $\mathbb{P}^2$. See, for instance, [9] for a more general statement and details on the SHGH Conjecture.

An irreducible homogeneous polynomial $F \in R = \mathbb{C}[x, y, z]$ is said to be *exceptional* for a set $Y = \{P_1, \dots, P_r\}$ of points in $\mathbb{P}^2$ if

$$\deg(F)^2 - \sum_{i=1}^{r} n_i^2 = -3\deg(F) + \sum_{i=1}^{r} n_i = -1,$$

where n_i is the highest vanishing order of F at P_i, for $i = 1, \dots, r$, i.e. $n_i = \max\{t \in \mathbb{N}_0 \mid F \in \mathfrak{p}_i^t\}$ (and $\mathfrak{p}_i$ is the defining ideal of P_i)

Conjecture 6.12 (SHGH Conjecture) Let Y be a general set of points in $\mathbb{P}^2$ and let $m \in \mathbb{N}$. Then, mY is not $\mathrm{AH}_n(d)$ if and only if there exists an irreducible homogeneous polynomial $F \in R$ that is exceptional for Y such that F^s, for some $s > 1$, divides every homogeneous polynomial of degree d in $I_Y^{(m)}$.

The ultimate goal naturally would be to determine the Hilbert function of every non-uniform symbolic power of any set Y of general points in $\mathbb{P}^n$ (i.e., the Hilbert function of $\mathfrak{p}_1^{m_1} \cap \ldots \cap \mathfrak{p}_r^{m_r}$, where $Y = \{P_1, \dots, P_r\}$ is a set of general points in $\mathbb{P}^n$ and $\mathfrak{p}_i$ is the defining ideal of P_i for every i).

Harbourne [31] showed that this problem would be solved if one is able to determine $\alpha(\mathfrak{p}_1^{m_1} \cap \ldots \cap \mathfrak{p}_r^{m_r})$ for every choice of the multiplicities $m_i \in \mathbb{Z}_+$. Here, for any homogeneous ideal J,

$$\alpha(J) := \min\{d \geq 0 \mid [J]_d \neq 0\},$$

is the *initial degree* of J. Hence, the problem of determining the initial degree of symbolic powers of ideals of points would solve the ultimate problem on interpolation. However, as one may expect, determining α is usually very challenging, even in the uniform case and even for points in $\mathbb{P}^2$. For instance, the following celebrated conjecture of Nagata, which arose from his work on Hilbert's 14-th problem [39], remains open.

Conjecture 6.13 (Nagata's Conjecture) For any set Y of $r \geq 10$ general points in $\mathbb{P}^2$, and any $m \geq 2$ one has

$$\alpha(I_Y^{(m)}) > m\sqrt{r}.$$

Conjecture 6.13 was proved by Nagata when r is a perfect square. A large body of literature is dedicated to this conjecture (cf. [30] and references therein and thereafter). Connections have also been made with other problems, for instance symplectic packing problems (see, e.g., [5, Section 5]). Nevertheless, the conjecture still seems out of reach at the moment. The interested reader will find in the literature many variations and different viewpoints on Nagata's Conjecture, e.g., in [10].

Given the difficulty in establishing the bound predicted by Nagata's conjecture, it is natural to ask for weaker bounds. In this direction, we mention that for an arbitrary set of points Y in $\mathbb{P}^n$, a weaker bound for $\alpha(I_Y^{(m)})$ was formulated by G. V. Chudnovsky in [13].

Conjecture 6.14 (Chudnovsky) Let Y be an arbitrary set of points in $\mathbb{P}^n$. For every $m \geq 1$, we have

$$\frac{\alpha(I_Y^{(m)})}{m} \geq \frac{\alpha(I_Y) + n - 1}{n}.$$

Chudnovsky's conjecture has been established for

- points in $\mathbb{P}^2$ (see [13, 32]),
- *general* points in $\mathbb{P}^3$ (see [16]),
- points on a quadric (see [20]),
- *very general* points in $\mathbb{P}^n$ (in [17] for large number of points, and in [20] for any number of points),
- large number of general points in $\mathbb{P}^n$ (see [6]).

Very recently, in a personal communication with the authors, R. Lazarsfeld suggested a geometric intuitive evidence for why one may expect the existence of counterexamples to Conjecture 6.14. Thus, instead of trying to prove Conjecture 6.14, one may look for counterexamples. It should be noted that partial results stated earlier suggest that a potential counterexample should be a special configuration and have high singularity outside of the given set of points.

A Appendix: Secant Varieties and the Waring Problem

In this section, we briefly describe the connection between the polynomial interpolation problem, particularly the Alexander–Hirschowitz theorem, and studies on secant varieties and the big Waring problem for forms.

Throughout this section, let V be a vector space of dimension $(n+1)$ over $\mathbb{C}$. Then $\mathbb{P}^n$ can also be viewed as $\mathbb{P}(V)$, the projective space of lines going through the origin in V. For $f \in V \setminus \{0\}$, let $[f]$ denote the line spanned by f in V and, at the same time, the corresponding point in $\mathbb{P}(V)$.

Let S be the symmetric algebra of V. Then S is naturally a graded algebra, given by $S = \bigoplus_{d \geq 0} S^d V$, where the d-th symmetric tensor $S^d V$ is a $\mathbb{C}$-vector space of dimension $\binom{n+d}{n}$. Note that the dual $R = S^*$ is the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ identified as the coordinate ring of $\mathbb{P}(V)$.

Definition A.1 Let V be a $(n+1)$-dimensional vector space over $\mathbb{C}$.

(1) The *d-th Veronese embedding* of $\mathbb{P}(V)$ is the map $\nu_d : \mathbb{P}(V) \to \mathbb{P}(S^d V)$, given by

$$[v] \mapsto [v^d] = [\underbrace{v \otimes \cdots \otimes v}_{d \text{ times}}].$$

Equivalently, ν_d is the map $\mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$, defined by

$$[a_0 : \cdots : a_n] \mapsto [a_0^d : a_0^{d-1} a_1 : \cdots : a_n^d],$$

where the coordinates on the right are given by all monomials of degree d in the a_i's.

(2) The *d-th Veronese variety* of $\mathbb{P}(V)$, denoted by V_d^n, is defined to be the image $\nu_d(\mathbb{P}(V))$.

Lemma A.2 *Let $f \in V \setminus \{0\}$. The tangent space $T_{[f^d]}(V_d^n)$ of V_d^n at the point $[f^d]$ is spanned by*

$$\{[f^{d-1} g] \in \mathbb{P}(S^d V) \bigm| g \in V\}.$$

Proof Let $g \in V \setminus \{0\}$. The line ℓ passing through $[f] \in \mathbb{P}(V)$, whose tangent vector at $[f]$ is given by $[g]$, is parameterized by $\epsilon \mapsto [f + \epsilon g]$. The image of this line via the Veronese embedding ν_d is given by $\epsilon \mapsto [(f + \epsilon g)^d]$. As f^d corresponds to the value $\epsilon = 0$, then the tangent vector of $\nu_d(\ell)$ at $\nu_d([f])$ is

$$\Big[\frac{d}{d\epsilon}\Big|_{\epsilon=0} (f + \epsilon g)^d\Big] = [d f^{d-1} g] = [f^{d-1} g].$$

The statement then follows. ☐

Lemma A.3 *Let* $f \in V \setminus \{0\}$.

(1) There is a one-to-one correspondence between hyperplanes in $\mathbb{P}(S^d V)$ *containing* $[f^d]$ *and hypersurfaces of degree* d *in* $\mathbb{P}(V)$ *containing* $[f]$.
(2) There is a one-to-one correspondence between hyperplanes in $\mathbb{P}^(S^d V)$ *containing* $T_{[f^d]}(V_d^n)$ *and hypersurfaces of degree* d *in* $\mathbb{P}(V)$ *singular at* $[f]$.

Proof (1) Let $z_0, \ldots, z_N$, where $N = \binom{n+d}{d} - 1$ be the homogeneous coordinates of $\mathbb{P}(S^d V)$. The equation for a hyperplane H in $\mathbb{P}(S^d V)$ has the form

$$a_0 z_0 + \cdots + a_N z_N = 0.$$

By replacing z_i with the corresponding monomial of degree d in the x_i's, this equation gives a degree d equation that describes a degree d hypersurface in $\mathbb{P}(V)$. Clearly, this hypersurface is the preimage $\nu_d^{-1}(H)$ of H. Furthermore, since $\nu_d^{-1}([f^d]) = [f]$, if H passes through $[f^d]$ then $\nu_d^{-1}(H)$ contains $[f]$.

(2) Let $\{e_0, \ldots, e_n\}$ be a basis of V whose dual basis in R is $\{x_0, \ldots, x_n\}$. By a linear change of variables, we may assume that $f = e_0$. That is, $[f] = [1 : 0 : \cdots : 0] \in \mathbb{P}_K^n$. Then, the defining ideal of $[f]$ is $(x_1, \ldots, x_n)$.

It follows from Lemma A.2 that $T_{[f^d]}(V_d^n)$ is spanned by

$$\{[e_0^d], [e_0^{d-1} e_1], \ldots, [e_0^{d-1} e_n]\}.$$

As before, the equation for a hyperplane H in $\mathbb{P}(S^d V)$ has the form $a_0 z_0 + \cdots + a_N z_N = 0$. By using lexicographic order, we may assume that $z_0, \ldots, z_n$ are variables corresponding to monomials $x_0^d, x_0^{d-1} x_1, \ldots, x_0^{d-1} x_n$. Then, H contains $T_{[f^d]}(V_d^n)$ if and only if $a_0 = \cdots = a_n = 0$. It follows that the equation for $\nu_d^{-1}(H)$ is a linear combination of monomials of degree d not in the set $\{x_0^d, x_0^{d-1} x_1, \ldots, x_0^{d-1} x_n\}$. Particularly, these monomials have degree at least 2 in the variables $x_1, \ldots, x_n$. Hence, $\nu_d^{-1}(H)$ is singular at $[f]$. □

We obtain an immediate corollary.

Corollary A.4 *Let* $X = \{[f_1], \ldots, [f_k]\} \subseteq \mathbb{P}(V)$ *be a set of points. Let* $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *be the defining ideal of these points. Then, there is a bijection between the vector space of degree* d *elements in* $\bigcap_{i=1}^k \mathfrak{m}_i^2$ *and the vector space of hyperplanes in* $\mathbb{P}(S^d V)$ *containing the linear span of* $T_{[f_1^d]}(V_d^n), \ldots, T_{[f_k^d]}(V_d^n)$.

Proof The conclusion follows from Lemma A.3, noticing that $\mathfrak{m}_i^2$ is the ideal of polynomials in R singular at $[f_i] \in \mathbb{P}(V)$ for all $i = 1, \ldots, k$. □

Recall that if I is any (homogeneous) ideal in R, then $[I]_d = I \cap R_d$ is the vector space of all homogeneous equations of degree d in I.

Corollary A.5 *Let $X = \{[f_1], \ldots, [f_k]\} \subseteq \mathbb{P}(V)$ be a set of points. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the defining ideals of the points in X. Then,*

$$\dim_{\mathbb{C}} \Big[\bigcap_{i=1}^{k} \mathfrak{m}_i^2\Big]_d = N - \dim\langle T_{[f_1^d]}(V_d^n), \ldots, T_{[f_k^d]}(V_d^n)\rangle.$$

Proof The conclusion is an immediate consequence of Corollary A.4 and basic linear algebra facts. □

Definition A.6 Let X be a projective variety. For any nonnegative integer r, the *r-secant variety* of X, denoted by $\sigma_r(X)$, is defined to be

$$\sigma_r(X) = \overline{\bigcup_{P_1,\ldots,P_r \in X} \langle P_1, \ldots, P_r\rangle}^{\text{Zariski closure}}.$$

Note that $\sigma_r(V_d^n)$ is an irreducible variety for all r.

Remark A.7 Let $X \subseteq \mathbb{P}^N$ be a projective scheme of dimension n. Then,

$$\dim \sigma_r(X) \leq \min\{rn + r - 1, N\} = \min\{(n+1)r - 1, N\}.$$

When the equality holds we say that $\sigma_r(X)$ has *expected dimension*.

Lemma A.8 (First Terracini's Lemma) *Let $Y \subseteq \mathbb{P}^n$ be a projective scheme. Let $p_1, \ldots, p_r$ be general points in Y. Let $z \in \langle p_1, \ldots, p_r\rangle$ be a general point in the linear span of $p_1, \ldots, p_r$. Then,*

$$T_z(\sigma_r(Y)) = \langle T_{p_1}(Y), \ldots, T_{p_r}(Y)\rangle.$$

Proof Let $Y(\tau) = Y(\tau_1, \ldots, \tau_n)$ be a local parametrization of Y. Let $Y_j(\tau)$ represent the partial derivative with respect to τ_j, for $j = 1, \ldots, n$. Suppose that p_i corresponds to $\tau^i = (\tau_1^i, \ldots, \tau_n^i)$ in this local parametrization.

By definition, $T_{p_i}(Y)$ is spanned by the tangent vectors $Y(\tau^i) + \epsilon Y_j(\tau^i)$, for $j = 1, \ldots, n$. Thus, $\langle T_{p_1}(Y), \ldots, T_{p_r}(Y)\rangle$ is the affine span of $\{Y(\tau^i), Y_j(\tau^i) \mid i = 1, \ldots, r, j = 1, \ldots, n\}$.

On the other hand, a general point z in $\sigma_r(Y)$ is parametrized by $Y(\tau^k) + \sum_{i=1}^{r-1} \gamma_i Y(\tau_i)$. By considering partial derivatives at z, it can be seen that $T_z(\sigma_r(Y))$ is also the affine span of $\{Y(\tau^i), Y_j(\tau^i) \mid i = 1, \ldots, r, j = 1, \ldots, n\}$. The lemma is proved. □

The following theorem establishes the equivalence between being $\mathrm{AH}_n(d)$ for double points and having expected dimension for secant varieties.

Theorem A.9 *A set of r general double points in $\mathbb{P}^n$ is $AH_n(d)$ if and only if $\sigma_r(V_d^n)$ has expected dimension.*

Proof Let $X = \{q_1, \dots, q_r\}$ be a set of r general simple points in $\mathbb{P}^n$, and let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be their defining ideals. Set $p_i = \nu_d(q_i)$ for $i = 1, \dots, r$. By Corollary A.5, we have

$$\dim_{\mathbb{C}} \big[\bigcap_{i=1}^{r} \mathfrak{m}_i^2\big]_d = N - \dim\langle T_{p_1}(V_d^n), \dots, T_{p_r}(V_d^n)\rangle.$$

It follows that

$$\begin{aligned} h_{\mathbb{P}^n}(2X, d) &= \binom{n+d}{d} - \dim_{\mathbb{C}} \big[\bigcap_{i=1}^{r} \mathfrak{m}_i^2\big]_d \\ &= \dim\langle T_{p_1}(V_d^n), \dots, T_{p_r}(V_d^n)\rangle + 1. \end{aligned}$$

By the genericity assumption of the points and the fact that $\sigma_r(V_d^n)$ is an irreducible variety, Lemma A.8 now gives

$$h_{\mathbb{P}^n}(2X, d) = \dim \sigma_r(V_d^n) + 1.$$

The conclusion now follows, noting that $h_{\mathbb{P}^n}(2X, d) \le \min\{(n+1)r, \binom{n+d}{n}\}$ and $\dim \sigma_r(V_d^n) \le \min\{(n+1)r - 1, \binom{n+d}{n} - 1\}$. □

Via its connection to secant varieties of Veronese varieties, the interpolation problem is also closely related to the big Waring problem for forms.

Definition A.10 Let $F \in R$ be a homogeneous polynomial of degree d. The *Waring rank* of F, denoted by $\mathrm{rk}(F)$, is defined to be the *minimum* s such that

$$F = \ell_1^d + \cdots + \ell_s^d$$

for some linear forms $\ell_1, \dots, \ell_s \in R_1$.

The Waring problems for forms ask for bounds or precise values for the Waring rank of homogeneous polynomials.

Definition A.11 Let n, d be positive integers and $R = \mathbb{C}[x_0, \dots, x_n]$.

(1) Set $G(n, d) := \min\{s \in \mathbb{N} \mid \mathrm{rk}(F) \le s \text{ for a general element } F \in R_d\}$.
(2) Set $g(n, d) := \min\{s \in \mathbb{N} \mid \mathrm{rk}(F) \le s \text{ for any element } F \in R_d\}$.

The *Big Waring Problem* and *Little Waring Problem*, respectively, are to determine $G(n, d)$ and $g(n, d)$. It is easy to see that $g(n, d) = \max\{\mathrm{rk}(F) \mid F \in R_d\}$. The connection between $G(n, d)$ and secant varieties comes from the following result.

Lemma A.12 $G(n, d) = \min\{r \mid \sigma_r(V_d^n) = \mathbb{P}(S^d V)\}$.

Proof Fix a basis $\{e_0, \dots, e_n\}$ of V whose dual basis in R is $\{x_0, \dots, x_n\}$. Let $\theta : R \to S$ be the natural isomorphism defined by $x_i \mapsto e_i$. Consider a form $F \in R_d$.

By definition, $\mathrm{rk}(F) \le r$ if and only if there exist linear forms $\ell_1, \ldots, \ell_r \in R_1$ such that

$$F = \ell_1^d + \cdots + \ell_r^d \tag{A.1}$$

Let $h_i' = \theta(\ell_i)$ for $i = 1, \ldots, r$. Then, (A.1) holds if and only if $\theta(F) = h_1^d + \cdots + h_r^d$. By scalar scaling if necessary, this is the case if and only if $[\theta(F)] = \langle [h_1^d], \ldots, [h_r^d] \rangle$.

By the definition of $\sigma_r(V_d^n)$ (being the Zariski closure of the union of secant linear subspaces), it then follows that $\sigma_r(V_d^n) = \mathbb{P}(S^d V)$ if and only if $\bigcup_{P_1,\ldots,P_r \in V_d^n} \langle P_1, \ldots, P_r \rangle$ contains a general point of $\mathbb{P}(S^d V)$. Hence, $\mathrm{rk}(F) \le r$ for a general element $F \in R_d$ if and only if $\sigma_r(V_d^n) = \mathbb{P}(S^d V)$. □

B Appendix: Symbolic Powers

Considering the considerable literature on symbolic powers of ideals, we have included in this appendix only a minimal amount of definitions and results. We refer the interested reader to the recent, comprehensive survey on the subject [15].

Definition B.1 Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and let I be an ideal with no embedded associated primes. For every $m \in \mathbb{Z}_+$, the *m-th symbolic power of I* is the R-ideal

$$I^{(m)} = \bigcap_{p \in \mathrm{Ass}(R/I)} \left(I^m R_p \cap R \right).$$

Additionally, one sets $I^{(0)} = R$.

For every integer $m \ge 2$ and ideal I, one has $I^m \subseteq I^{(m)}$ and in general this is a strict inclusion. A notable exception is when I is a complete intersection, in which case, $I^m = I^{(m)}$ for every $m \ge 1$.

The following result gives a way to compute symbolic powers of ideals of points.

Proposition B.2 *Let $X = \{P_1, \ldots, P_r\}$ be a set of simple points in $\mathbb{P}^n$, i.e. its defining ideal $I_X = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ in $R = \mathbb{C}[x_0, \ldots, x_n]$ is a radical ideal. Then for every $m \ge 1$*

$$I_X^{(m)} = \mathfrak{p}_1^m \cap \cdots \cap \mathfrak{p}_r^m.$$

Definition B.3 Let P be a point in $\mathbb{P}^n$ with defining ideal $\mathfrak{p}$. For $m \ge 1$, we write mP for the subscheme of $\mathbb{P}^n$ with defining ideal $\mathfrak{p}^m$. One often calls mP a *fat point* subscheme of $\mathbb{P}^n$.

If $P_1, \ldots, P_r$ are points in $\mathbb{P}^n$ with defining ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, then the defining ideal of the fat point scheme $X = m_1P_1 + m_2P_2 + \ldots + m_rP_r$ is $I_X := \mathfrak{p}_1^{m_1} \cap \ldots \cap \mathfrak{p}_r^{m_r}$.

Example B.4 (The defining ideal of 3 non-collinear double points in $\mathbb{P}^2$) Let $X = \{[1:0:0], [0:1:0], [0:0:1]\} \subseteq \mathbb{P}^2$ be the three coordinate points in $\mathbb{P}^2$, and let $I_X = (x_1, x_2) \cap (x_0, x_2) \cap (x_0, x_1) = (x_0x_1, x_0x_2, x_1x_2) \subseteq \mathbb{C}[x_0, x_1, x_2]$ be its defining ideal. The defining ideal of mX is

$$I_X^{(m)} = (x_1, x_2)^m \cap (x_0, x_2)^m \cap (x_0, x_1)^m.$$

For instance, $2X$ is defined by $I_X^{(2)} = (x_1, x_2)^2 \cap (x_0, x_2)^2 \cap (x_0, x_1)^2$, and by computing this intersection one obtains

$$I_X^{(2)} = (x_0x_1x_2) + I_X^2.$$

In particular, $I_X^2 \neq I_X^{(2)}$. (In fact, more generally, $x_0^t x_1^t x_2^t \in I_X^{(2t)}$ for all $t \in \mathbb{Z}_+$.)

An important theorem proved by Zariski [43] and Nagata [40] (and generalized by Eisenbud and Hochster [19]) provides a first illustration of the geometric relevance of symbolic powers of ideals: they consist of all hypersurfaces vanishing with order at least m on the variety defined by I.

Theorem B.5 (Zariski-Nagata) *Let I be a radical ideal in $R = \mathbb{C}[x_0, \ldots, x_n]$ and let $s \geq \mathbb{N}$. Then,*

$$\begin{aligned} I^{(s)} &= \bigcap_{\mathfrak{m} \in \mathrm{Max}(R),\, I \subseteq \mathfrak{m}} \mathfrak{m}^s. \\ &= \{f \in R \mid \text{ all partial derivatives of } f \text{ of order } \leq s-1 \text{ lie in } I\}. \end{aligned}$$

Example B.6 Let X, I_X and R be as in Example B.4. We have claimed that $x_0x_1x_2 \in I_X^{(2)}$. One can check this easily using Zariski–Nagata theorem. Each of the partial derivatives of $x_0x_1x_2$ with respect to one of the variables is a minimal generator of I_X. Thus, all partial derivatives of order at most 1 of $x_0x_1x_2$ lie in I_X, and so by Zariski–Nagata theorem, $x_0x_1x_2 \in I_X^{(2)}$.

More geometrically, $V(x_0x_1x_2)$ is the union of the three lines $V(x_0) \cup V(x_1) \cup V(x_2)$. Each of the points at the intersection of two of the three lines are singular points. Since these three intersections are the points of X, it follows by Zariski–Nagata theorem that $x_0x_1x_2 \in I_X^{(2)}$.

C Appendix: Hilbert Function

In this section, we record some basic properties of Hilbert functions, especially relative to sets of points. We start with a simple lemma potentially allowing the use of Linear Algebra to investigate interpolation problems.

Lemma C.1 *Let $P \in \mathbb{P}^n$ be a point with defining ideal $\mathfrak{p} \subseteq R = \mathbb{C}[x_0, \ldots, x_n]$. Then $[\mathfrak{p}^m]_d$ consists of all solutions of a homogeneous linear system of $\binom{n+m-1}{n}$ equations in $\binom{n+d}{n}$ variables. In particular, the rank of this linear system is $H_{R/\mathfrak{p}^m}(d)$.*

Proof Let F be a generic homogeneous equation of degree d in $n+1$ variables, i.e.

$$F = \sum_{M \in T_d} c_M M \in \mathbb{C}[\{c_M\}, x_0, \ldots, x_n]$$

where T_d consists of all the $\binom{n+d}{d}$ monomials of degree d in R.

By Zariski–Nagata's theorem, the equation F vanishes with multiplicity m at a point $P \in \mathbb{P}^n \iff$ all the $(m-1)$-order (divided power) derivatives of F vanish at P.

Now, consider any $(m-1)$-th partial order derivative of F with respect to the x_i's and substitute the coordinates of P in for the variables. We obtain a linear combination of the coefficients c_M's; this linear combination is zero if and only if that partial derivative of F vanish at P. Therefore, there is a bijective correspondence between the solutions to the system of these $\binom{n+m-1}{n}$ linear equations in the unknowns c_M's and all hypersurfaces of degree d passing through P at least m times. This proves the statement. □

The Hilbert function of a graded ring counts the number of linearly independent forms in a given degree.

Definition C.2 Let $R = \mathbb{C}[x_0, \ldots, x_n]$, and let $M = \bigoplus_{i \geq 0} M_i$ be a graded R-module (e.g., $M = R/I$ where I is a homogeneous ideal). Then, M_i is a $\mathbb{C}$-vector space for every $i \geq 0$. The *Hilbert function of* M is the function $H_M : \mathbb{Z}_{\geq 0} \to \mathbb{N}$, given by

$$H_M(d) := \dim_{\mathbb{C}} M_d.$$

For any $a \in \mathbb{Z}$, $M(a)$ is defined as the graded R-module whose degree j component is $[M(a)]_j = M_{a+j}$.

In general, for d large, the Hilbert function of M agrees with a polynomial of degree $\dim(M) - 1$, which is called the *Hilbert polynomial* of M. Its normalized leading coefficient is an integer $e(M)$ called the *multiplicity of* M. When M is one-dimensional and Cohen-Macaulay, $H_M(d)$ is non-decreasing and eventually equals the multiplicity of M. We recapture this property in the following proposition.

Proposition C.3 *Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and let M be a Cohen-Macaulay graded R-module with $\dim(M) = 1$. Then $H_M(d-1) \leq H_M(d)$ for all $d \in \mathbb{N}$, and $H_M(d) = e(M)$ for $d \gg 0$. In particular, $H_M(d) \leq e(M)$ for every $d \in \mathbb{N}$.*

Proof Since $\dim(M) = 1$, then the Hilbert polynomial of M is just the constant function $e(M)$, so $H_M(d) = e(M)$ for $d \gg 0$.

Since M is Cohen-Macaulay there exists a linear form $x \in R$ that is regular on M. Let $\overline{R} = R/(x)$ be the Artinian reduction of R, and let $\overline{M} = M/(x)M$. The short exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{\cdot x} M \longrightarrow \overline{M} \longrightarrow 0$$

and the additivity of Hilbert function under short exact sequence yield $H_{\overline{M}}(d) = H_M(d) - H_M(d-1)$. As $H_{\overline{M}}(d)$ is of course non-negative for every $d \in \mathbb{N}$, then $H_M(d-1) \le H_M(d)$ for all $d \in \mathbb{N}$. □

To complement the previous result, one can use the so-called Associativity Formula for $e(R/I)$ to prove the following statement computing the multiplicity of any set of fat points in $\mathbb{P}^n$.

Proposition C.4 *Let $Y = \{P_1, \ldots, P_r\}$ be a set of points in $\mathbb{P}^n$, and let $X = \{m_1 P_1, \ldots, m_r P_r\}$. Then*

$$e(R/I_X) = \sum_{i=1}^{r} \binom{n+m_i-1}{n}.$$

By counting equations and variables, one immediately obtains an upper bound for the Hilbert function of any ideal associated to (possibly fat) points.

Corollary C.5 *Let $X = \{m_1 P_1, \ldots, m_1 P_r\}$ be a set of points in $\mathbb{P}^n$ with multiplicities $m_1, \ldots, m_r$. Then*

$$H_{R/I_X}(d) \le \min\left\{\binom{d+n}{n}, \sum_{i=1}^{r} \binom{n+m_i-1}{n}\right\},$$

or, equivalently, $H_{I_X}(d) \ge \max\left\{0, \binom{d+n}{n} - \sum_{i=1}^{r} \binom{n+m_i-1}{n}\right\}$.

Proof By Propositions C.3 and C.4 we have $H_{R/I_X}(d) \le e(R/I_X) = \sum_{i=1}^{r} \binom{n+m_i-1}{n}$. One also has $H_{R/I_X}(d) \le H_R(d) = \binom{d+n}{n}$. □

When the equality in Corollary C.5 is achieved, we obtain the definition of $\mathrm{AH}_n(d)$, or maximal Hilbert function in degree d. In other papers, this property is often referred to as X *imposes independent conditions on degree d hypersurfaces in* $\mathbb{P}^n$.

Definition C.6 Let $X = \{m_1 P_1, \ldots, m_r P_r\} \subseteq \mathbb{P}^n$ be a set of r points with multiplicities $m_1, m_2, \ldots, m_r$. We say that X is $\mathrm{AH}_n(d)$ (or has *maximal Hilbert function in degree d*), if

$$H_{R/I_X}(d) = \min\left\{\binom{d+n}{n}, \sum_{i=1}^{r} \binom{n+m_i-1}{n}\right\}.$$

The number $\min\left\{\binom{d+n}{n}, \sum_{i=1}^{r}\binom{n+m_i-1}{n}\right\}$ is called *the expected codimension in degree d* (for r general points in $\mathbb{P}^n$ with multiplicities $m_1, \ldots, m_r$).

The easiest situation in Definition C.6 is when $m_1 = \cdots = m_r = 1$, i.e., the points in X are all simple points.

Theorem C.7 *A set X of r general simple points in $\mathbb{P}^n$ is $AH_n(d)$ for every $d \geq 1$.*

Proof The statement follows from a simple observation, via Lemma C.1, that the condition $H_{R/I_X}(d) = \min\{\binom{n+d}{d}, r\}$ is an open condition. □

Example C.8 Let $X = \{P_1, 2P_2, 4P_3\} \subseteq \mathbb{P}^3$, then $e(R/I_X) = 25$, and X is $\mathrm{AH}_3(d)$ if and only if

$$H_{R/I_X}(d) = \min\left\{\binom{d+3}{3}, 1+4+20\right\}$$

i.e., if its Hilbert function is $H_{R/I_X} = (1, 4, 10, 20, 25, 25, 25, \ldots)$.

Other simple situations, where we can quickly prove that the property $\mathrm{AH}_n(d)$ holds, are when X is supported at a single point, i.e. $r = 1$, or when $d = 1$, or $n = 1$.

Remark C.9 A single double point $X = \{2P\}$ in $\mathbb{P}^n$ is $\mathrm{AH}_n(d)$ for all n and d.

Proof After a change of coordinates we may assume that $I_P = (x_1, \ldots, x_n)$, so $I_X = I_P^{(2)} = I_P^2$. If $d = 1$ then $[I_P^{(2)}]_1 = 0$, so $H_{R/I_P^{(2)}}(1) = n+1 = \min\{n+1, n+1\}$. If $d \geq 2$ then $[R/I_P^{(2)}]_d = \langle x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n\rangle$. Thus,

$$H_{R/I_P^{(2)}}(d) = n+1 = \min\left\{\binom{n+d}{d}, n+1\right\},$$

and the statement follows. □

Remark C.10 Any set $2Y$ of r double points (not necessarily general) in $\mathbb{P}^n$ is $\mathrm{AH}_n(1)$.

Proof For any $r \geq 1$, one needs to show that

$$H_{I_Y^{(2)}}(1) = \max\{0, \binom{n+1}{n} - r(n+1)\} = 0.$$

Let $P \in Y$ be any point, then $[I_Y^{(2)}]_1 \subseteq [I_P^{(2)}]_1 = [I_P^2]_1 = (0)$. Thus, $I_Y^{(2)}$ contains no linear forms. □

Proposition C.11 *Let Y be a set of r distinct simple points in $\mathbb{P}^1$. Then mY is $AH_1(d)$ for every $d, m \in \mathbb{Z}_+$.*

Proof Notice that the defining ideal of any point in $\mathbb{P}^1$ is just a principal prime ideal generated by a linear form. Thus, if $Y = \{P_1, \ldots, P_r\}$ then I_Y is a principal ideal generated by a form of degree r. It follows that $I_Y^{(m)} = I_Y^m \subseteq R = \mathbb{C}[x_0, x_1]$ is a principal ideal of degree rm, so $I_Y^{(m)} \cong R(-rm)$. Thus, $H_{I_Y^{(m)}}(d)$ is

$$\max\left\{0, \binom{d+1-rm}{1}\right\} = \max\{0, d+1-rm\} = \max\left\{0, \binom{d+1}{d} - rm\right\}$$

Therefore, mY is $\mathrm{AH}_1(d)$. □

The following lemma allows us to restrict attention to only a finite number of values of r in proving the Alexander-Hirschowitz theorem.

Lemma C.12 *Let X be a set of r points in $\mathbb{P}^n$, with multiplicities $m_1, \ldots, m_r$, which is $AH_n(d)$.*

(1) If $H_{R/I_X}(d) = \binom{d+n}{n}$, then X' is also $AH_n(d)$ for any larger set $X' \supseteq X$ consisting of $r' \geq r$ points of X with multiplicities $m'_1 \geq m_1, \ldots, m'_{r'} \geq m_{r'}$.
(2) If $H_{R/I_X}(d) = \sum_{i=1}^{r} \binom{n+m_i-1}{n} = e(R/I_X)$, then X' is also $AH_n(d)$ for any subset $X' \subseteq X$ consisting of $r' \leq r$ points of X with multiplicities $m'_1 \leq m_1, \ldots, m'_{r'} \leq m_{r'}$.

Proof

(1) Since $X \subseteq X'$ then $I_{X'} \subseteq I_X$. By assumption $[I_X]_d = 0$, thus also $[I_{X'}]_d = 0$, which implies that $H_{R/I_{X'}}(d) = \binom{d+n}{n}$.
(2) Since X' is a subscheme of X and X is multiplicity d-independent, the assertion is a direct consequence of Lemma 2.6.

□

We conclude this appendix by stating a numerical characterization of when it is possible to find simple points to be added to a given scheme in order to change its Hilbert function by a prescribed value.

Proposition C.13 ([11, Lemma 3]) *Let I be a saturated homogeneous ideal in $R = \mathbb{C}[x_0, \ldots, x_n]$, and let ℓ be a linear form that is regular on R/I. TFAE:*

(1) There exists a set Y_0 of u points in $V(\ell)$ such that

$$H_{R/(I \cap I_{Y_0})}(t) = H_{R/I}(t) + u.$$

(2) $H_{R/I}(t) + u \leq H_{R/I}(t-1) + \binom{n+t-1}{t}$.

D Appendix: Semi-continuity of the Hilbert Function and Reduction to Special Configurations

The starting point of the proof of Theorem 1.1 is the observation that to establish the $\mathrm{AH}_n(d)$ property for a general set of double points, in non-exceptional cases, we only need to exhibit a specific collection of double points with the $\mathrm{AH}_n(d)$ property. This is because Hilbert functions have the so-called *lower semi-continuity* property. This is the content of this appendix.

We begin by defining *generic* points and the *specialization* of points. Throughout this appendix, we shall fix a pair of positive intgers n and r. Recall that $R = \mathbb{C}[x_0, \ldots, x_n]$ is the homogeneous coordinate ring of $\mathbb{P}^n$. Let $\underline{z} = \{z_{ij} \mid 1 \le i \le r, 0 \le j \le n\}$ be a collection of $r(n+1)$ indeterminates, and let $\mathbb{C}(\underline{z})$ be the purely transcendental field extension of $\mathbb{C}$ be adjoining the variables in $\underline{z}$. Let $S = \mathbb{C}(\underline{z})[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_{\mathbb{C}(\underline{z})}$.

Set-up D.1

(1) By the *generic set of r points*, we mean the set $Z = \{Q_1, \ldots, Q_r\}$, where $Q_i = [z_{i0} : \cdots : z_{in}]$, for $i = 1, \ldots, r$, are points with the generic coordinates in $\mathbb{P}^n_{\mathbb{C}(\underline{z})}$. Let $I_Z \subseteq S$ denote the defining ideal of Z.

(2) Let $\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}^{r(n+1)}_{\mathbb{C}}$ be such that for each $i = 1, \ldots, r$, $\lambda_{ij} \neq 0$ for some j. Define the set $Z(\underline{\lambda}) = \{Q_1(\underline{\lambda}), \ldots, Q_r(\underline{\lambda})\}$ of points in $\mathbb{P}^n$, with $Q_i(\underline{\lambda}) = [\lambda_{i0} : \cdots : \lambda_{in}]$. Let $I_{\underline{\lambda}} \subseteq R$ be the defining ideal of $Z(\underline{\lambda})$.

We call $Z(\underline{\lambda})$ the *specialization of the generic points at* $\underline{\lambda}$, and call $I_{\underline{\lambda}}$ the *specialization of the ideal I_Z at $\underline{\lambda}$.*

To define precisely the notions of general points and very general points one often employs Chow varieties. However, one can also use dense Zariski-open subsets of $A^{r(n+1)}$ (see, e.g., [20, Lemma 2.3]) for these purposes, and this is the point of view we take.

Definition D.2 One says that a property $\mathcal{P}$

- *holds for a general set of r points* of $\mathbb{P}^n_{\mathbb{C}}$ if there is a dense Zariski-open subset $U \subseteq \mathbb{A}^{r(n+1)}_{\mathbb{C}}$ such that $\mathcal{P}$ holds for $Z(\underline{\lambda})$ for all $\underline{\lambda} \in U$;
- *holds for a very general set of r points* of $\mathbb{P}^n_{\mathbb{C}}$ if $\mathcal{P}$ holds for $Z(\underline{\lambda})$ for all $\underline{\lambda} \in U$ where U is an intersection of countably many dense Zariski-open subsets of $A^{r(n+1)}_{\mathbb{C}}$.

The lower semi-continuity of Hilbert functions that we shall use is stated in the following theorem.

Theorem D.3 (Lower-Semi-Continuity of the Hilbert Function) *Assume Set-up D.1. Then, for any $m, d \in \mathbb{N}$, we have*

$$H_{I_Z^{(m)}}(d) \le H_{I_Y^{(m)}}(d)$$

for any set Y *of* r *points in* $\mathbb{P}^n$*. Moreover, for fixed* $m \geq 1$ *and* $d \geq 1$*, the equality* $H_{I_Z^{(m)}}(d) = H_{I_Y^{(m)}}(d)$ *holds for a general set of points* $Y \subseteq \mathbb{P}^n$*.*

Proof Note that every set Y of r points in $\mathbb{P}^n$ can be viewed as a specialization $Z(\underline{\lambda})$ of the generic set of r points. The proof is similar to the proof of [20, Thm 2.4]. For every $s \geq 1$, set

$$W_s := \{\underline{\lambda} \in \mathbb{A}_{\mathbb{C}}^{r(n+1)} \mid H_{I_{\underline{\lambda}}^{(m)}}(d) \geq s\}.$$

We claim that W_s is a Zariski-closed subset of $\mathbb{A}_{\mathbb{C}}^{r(n+1)}$ for any $s \geq 1$.

To see it, let $f = \sum_{|\alpha|=d} C_\alpha \underline{x}^\alpha \in R[C_\alpha]$ be a generic homogeneous polynomial of degree d, where $\underline{x}^\alpha$ are the monomials of degree d in R. Let $\partial_{\underline{\beta}} \underline{x}^\alpha$ denote the partial derivative of $\underline{x}^\alpha$ with respect to $\underline{\beta}$.

Now, let $\mathbb{D}_{m,d}$ be the matrix with columns indexed by all monomials in R_d, rows indexed by all partial derivatives $\underline{\beta}$ with $|\underline{\beta}| \leq m-1$, and whose rows are

$$\begin{bmatrix} \partial_{\underline{\beta}} x_0^d & \dots & \partial_{\underline{\beta}} \underline{z_i}^{\underline{\alpha}} & \dots & \partial_{\underline{\beta}} x_n^d \end{bmatrix}.$$

Let $[\mathbb{B}_{m,d}]_{\underline{\lambda}}$ be the r by 1 block matrix

$$\mathbb{B}_{m,d} = \begin{bmatrix} \mathbb{D}_{m,d}(P_1) \\ \mathbb{D}_{m,d}(P_2) \\ \vdots \\ \mathbb{D}_{m,d}(P_k) \end{bmatrix}$$

where $\mathbb{D}_{m,d}(P_1)$ is the specialization of the matrix $\mathbb{D}_{m,d}$ at the point P_i, i.e. we replace $x_0, \dots, x_n$ by $\lambda_{i,0}, \dots, \lambda_{i,n}$, respectively.

Then the forms $f = \sum_{|\alpha|=d} C_\alpha \underline{x}^\alpha$ of degree d in $I_{\underline{\lambda}}^{(m)}$ are in a bijective correspondence with the non-trivial solutions to the system of equations (in the variables C_α)

$$[\mathbb{B}_{m,d}]_{\underline{\lambda}} \cdot \begin{bmatrix} C_{(d,\dots,0)} & \dots & C_{\underline{\alpha}} & \dots & C_{(0,\dots,d)} \end{bmatrix}^T = \underline{0}.$$

It follows that $\underline{\lambda} \in W_s$ if and only if the null-space of this linear system has dimension at least s, which is holds if and only if the number $r\binom{m+n}{m-1}$ of rows of $[\mathbb{B}_{m,d}]_{\underline{\lambda}}$ is less than $\binom{d+n}{n} - (s-1)$ or $r\binom{m+n}{m-1} \geq \binom{d+n}{n} - (s-1)$ and $\mathrm{rk}[\mathbb{B}_{m,d}]_{\underline{\lambda}} < \binom{d+n}{n} - (s-1)$. In either case we have a closed condition in $\mathbb{A}^{r(n+1)}$. This proves the claim.

To prove the inequality in the statement we prove that when one takes $s_0 := H_{I_Z^{(m)}}(d)$, then W_{s_0} also contains a dense Zariski-open subset, thus showing that W_{s_0} is the entire space.

Indeed, let $f_1, \dots, f_{s_0}$ be linearly independent forms of degree d in $I_Z^{(m)}$. We may assume that each $f_i \in \mathbb{C}(\underline{z})[x_0, \dots, x_n]$. Let M be the matrix whose i-th row consists of the coefficients of each monomial $\underline{x}^\alpha$ in f_i. By assumption M has maximal rank, i.e. s_0, so at least one of the minors of size s_0 of M does not vanish. It follows that there exists a dense Zariski-open subset $\widetilde{U}_t$ of specializations $\underline{z} \longmapsto \underline{\lambda}$ ensuring that the specialization does not make this minor vanish, thus for any $\underline{\lambda} \in \widetilde{\widetilde{U}}$ we have that $(f_1)_{\underline{z}\mapsto\underline{\lambda}}, (f_2)_{\underline{z}\mapsto\underline{\lambda}}, \dots, (f_{s_0})_{\underline{z}\mapsto\underline{\lambda}}$ are s_0 linearly independently forms of degree d in $I_{\underline{\lambda}}^{(m)}$. This concludes the proof of the first part.

The equality now follows from this last paragraph, as it is shown in there that for any $\underline{\lambda} \in \widetilde{U}_d$ one has that $s_0 := H_{I_Z^{(m)}}(d) = H_{I_{\underline{\lambda}}^{(m)}}(d)$. □

We obtain the following immediate consequences of Theorem D.3.

Corollary D.4 *Fix positive integers n, r, d and m. TFAE:*

(1) There exists a set Y of r points in $\mathbb{P}^n$ such that mY is $AH_n(d)$.
(2) For any set Y of r general points in $\mathbb{P}^n$, mY is $AH_n(d)$.

Corollary D.5 *Fix $n, d \in \mathbb{Z}_+$. Then every set Y of r general double points in $\mathbb{P}^n$ is $AH_n(d)$ if and only if there exist sets of r double points in $\mathbb{P}^n$ which are $AH_n(d)$ for $\left\lfloor \frac{1}{n+1}\binom{d+n}{n} \right\rfloor \leq r \leq \left\lceil \frac{1}{n+1}\binom{d+n}{n} \right\rceil$.*

Similarly, if a set of r_0 general points is not $AH_n(d)$, then any set of $r \neq r_0$ general double points in $\mathbb{P}^n$ is $AH_n(d)$ if and only if there exist sets of $r_0 - 1$ and $r_0 + 1$ double points in $\mathbb{P}^n$ that are $AH_n(d)$.

Proof The desired statements are direct consequences of Corollary D.4 and Lemma C.12. □

In the last part of this section we prove a semi-continuity results in the more general setting of flat families of projective schemes.

Definition D.6 Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *f-flat at $x \in X$* if the stalk $\mathcal{F}_x$, seen as an $\mathcal{O}_{Y,f(x)}$-module, is flat. We say that $\mathcal{F}$ is *f-flat* if it is f-flat at every point in X.

Definition D.7 A *family of (closed) projective schemes $f : X \to Y$* is a morphism f of (locally) Noetherian schemes which factors through a closed embedding $X \subseteq \mathbb{P}^r \times Y = \mathbb{P}$, for some r. The family is *flat* if $\mathcal{O}_X$ if f-flat.

Let $\mathfrak{p}$ be a point in Y. Let $\mathbb{C}(\mathfrak{p})$ be the residue field of the local ring $\mathcal{O}_{Y,\mathfrak{p}}$. Let $X_\mathfrak{p} = X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,\mathfrak{p}})$ and let $\mathbb{P}_\mathfrak{p} = \mathbb{P} \times_Y \operatorname{Spec}(\mathcal{O}_{Y,\mathfrak{p}})$. For example, if $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Proj}(R/I)$, where $R = A[x_0, \dots, x_r]$ and $I \subset R$ is a homogeneous ideal, then $X_\mathfrak{p} = \operatorname{Proj}((R/I) \otimes_A \mathbb{C}(\mathfrak{p}))$ and $\mathbb{P}_\mathfrak{p} = \operatorname{Proj}(R \otimes_A \mathbb{C}(\mathfrak{p}))$. Note that, in general, the defining ideal of $X_\mathfrak{p}$ in $\mathbb{P}_\mathfrak{p}$ may not be the same as $I \otimes_A \mathbb{C}(\mathfrak{p})$; rather, it is the image of the canonical map $(I \otimes_A \mathbb{C}(\mathfrak{p}) \to R \otimes_A \mathbb{C}(\mathfrak{p}))$.

The following result is well-known; see, for example, [34, Theorem III.12.8].

Theorem D.8 *Let $f : X \to Y$ be a family of projective schemes and let $\mathcal{F}$ be a coherent sheaf over X which is also f-flat. Then, for each $i \geq 0$, the function $Y \to \mathbb{Z}$ defined by*

$$\mathfrak{p} \mapsto \dim_{\mathbb{C}(\mathfrak{p})}(H^i(X_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}}))$$

is upper semicontinuous on Y.

Theorem D.9 *Let $f : X \to Y$ be a flat family of projective schemes. Then, for any degree $d \geq 0$, the function $Y \to \mathbb{Z}$ defined by*

$$\mathfrak{p} \mapsto h_{\mathbb{P}_{\mathfrak{p}}}(X_{\mathfrak{p}}, d)$$

is lower semicontinuous on Y.

Proof Let $\mathcal{I}$ be its ideal sheaf of the embedding $X \subseteq \mathbb{P}$. Let $\mathfrak{p} \in Y$ be any point and let $A = \mathcal{O}_{Y,\mathfrak{p}}$. We have a short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0$. By tensoring with $\mathbb{C}(\mathfrak{p})$, we obtain the following short exact sequence

$$0 \to \mathcal{I} \otimes_A \mathbb{C}(\mathfrak{p}) \to \mathcal{O}_{\mathbb{P}_{\mathfrak{p}}} \to \mathcal{O}_{X_{\mathfrak{p}}} \to 0.$$

Particularly, this shows that $\mathcal{I} \otimes_A \mathbb{C}(\mathfrak{p})$ is the ideal sheaf of the embedding $X_{\mathfrak{p}} \subseteq \mathbb{P}_{\mathfrak{p}}$. Set $\mathcal{I}_{\mathfrak{p}} = \mathcal{I} \otimes_A \mathbb{C}(\mathfrak{p})$. We then have

$$h_{\mathbb{P}_{\mathfrak{p}}}(X_{\mathfrak{p}}, n) = h^0(\mathcal{O}_{\mathbb{P}_{\mathfrak{p}}}(n)) - h^0(\mathcal{I}_{\mathfrak{p}}(n)).$$

Observe that $\mathcal{O}_X$ is f-flat, and so $\mathcal{I}$ is also f-flat. Therefore, by Theorem D.8, the function $\mathfrak{p} \mapsto h^0(\mathcal{I}_{\mathfrak{p}}(n))$ is an upper semicontinuous function on Y. The conclusion now follows, since $h^0(\mathcal{O}_{\mathbb{P}_{\mathfrak{p}}}(n))$ is constant on Y. □

E Appendix: Hilbert Schemes of Points and Curvilinear Subschemes

We end the paper with our last appendix giving basic definitions and properties of curvilinear subschemes that allow the deformation argument in the *méthode d'Horace différentielle* to work.

Definition E.1 A finite zero-dimensional scheme Z is said to be *curvilinear* if Z locally can be embedded in a smooth curve. That is, for every point P in Z, the dimention $T_P(Z)$ of the tangent space is at most 1.

Lemma E.2 *Let Z be a zero-dimensional scheme supported at one point P. Then Z is curvilinear if and only if $Z \simeq \operatorname{Spec} \mathbb{C}[t]/(t^l)$, where l is the degree of Z.*

Proof Without loss of generality, assume that $(x_1, \ldots, x_n)$ are local parameters at P. Let C be a smooth curve to which Z can be embedded in. Clearly, $P \in C$. Let $I_C = (f_1, \ldots, f_s)$ be the defining ideal of C in $\mathcal{O}_P = \mathbb{C}[x_1, \ldots, x_n]$ (particularly, $s \geq n - 1$). Since C is smooth at P, the Jacobian matrix of C at P has rank $n - 1$. Thus, by a change of variables and a re-indexing, if necessary, we may further assume that $f_i = x_i + g_i$, for $i = 1, \ldots, n - 1$, and $g_1, \ldots, g_{n-1} \in \mathcal{O}_P$.

Let I_Z be the defining ideal of Z in $\mathcal{O}_P$. Since Z can be embedded in C, we have $I_C \subseteq I_Z$. Therefore, locally at P, $\mathcal{O}_Z$ is a quotient ring of $\mathbb{C}[x_n]$. It follows that, locally at P, $\mathcal{O}_Z \cong \mathbb{C}[x_n]/(x_n^l)$ for some l.

The converse is clear by the same arguments. Observe further that localizing at P (a minimal prime in $\mathcal{O}_Z$) does not change the multiplicity of $\mathcal{O}_Z$, or equivalently, the degree of Z. Hence, $\deg(Z) = l$. □

Corollary E.3 *Let Z be a curvilinear subscheme of a double point. Then the degree of Z is either 1 or 2.*

Proof By Lemma E.2, we have $Z \cong \operatorname{Spec} \mathbb{C}[t]/(t^l)$. Since Z is contained in a double point, we must have l is equal to 1 or 2. Hence, $\deg(Z)$ is either 1 or 2. □

Remark E.4 Let Z be a zero-dimensional scheme with irreducible components $Z_1, \ldots, Z_r$. Then, Z is curvilinear if and only if $Z_1, \ldots, Z_r$ are curvilinear.

The next lemma gives another way of seeing curvilinear schemes.

Lemma E.5 *Let Z be a zero-dimension scheme supported at one point P. Then, Z is curvilinear if and only if, locally at P, the $\mathcal{O}_Z$ is generated by one element, that is, $\mathcal{O}_Z = \mathbb{C}[f]$ for some $f \in \mathcal{O}_Z$.*

Proof By Lemma E.2, if Z is curvilinear then, clearly, $\mathcal{O}_Z$ is generated by one element. Suppose, conversely, that $\mathcal{O}_Z = \mathbb{C}[f]$ for some $f \in \mathcal{O}_Z$. Since Z is zero-dimensional, we must have $f^l = 0$ for some l. By taking the smallest such l, we then have $\mathcal{O}_Z \cong \mathbb{C}[t]/(t^l)$, and so Z is curvilinear by Lemma E.2. □

The main result about curvilinear subschemes that we shall use is that they form an open dense subset in the Hilbert scheme of zero-dimensional subscheme of a given degree in $\mathbb{P}^n$. Particularly, this allows us to take the limit of a family of curvilinear subschemes. For this, we shall need the following lemma.

Lemma E.6 *Let A be a Noetherian ring, let B be a free A-algebra of rank n. Then the set*

$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \text{ the } K(\mathfrak{p})\text{-algebra } B \otimes_A K(\mathfrak{p}) \text{ is generated by one element}\}$$

is an open subset U of Spec$A$. *Here, $K K(\mathfrak{p})$ is the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ at $\mathfrak{p}$.*

Proof Let $U := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid$ the $K(\mathfrak{p})$-algebra $B \otimes_A K(\mathfrak{p})$ is generated by one element$\}$. Clearly, if $\mathfrak{p} \in U$ and $\mathfrak{q} \subseteq \mathfrak{p}$ then $\mathfrak{q} \in U$. Thus, by Nagata's topological criterion (e.g. [38, Thm 24.2]), to prove that U is open it suffices to show that if $\mathfrak{p} \in U$, then there exists a non-empty open subset of $V(\mathfrak{p})$ contained in U.

Write $B = A[T_1, \dots, T_r]/J$, since $\mathfrak{p} \in U$ then (after possibly relabelling) we may assume that there exist $a_1, \dots, a_{r-1} \in A \setminus \mathfrak{p}$ and $g_1, \dots, g_{r-1} \in A[T_1, \dots, T_r]$ with $T_i \notin \operatorname{supp}(g_i)$ for any $i = 1, \dots, r-1$ such that

$$(a_1 T_1 + g_1, \dots, a_{r-1} T_{r-1} + g_{r-1}) \subseteq J.$$

Clearly, for any $\mathfrak{q} \in V(\mathfrak{p}) \setminus [V(a_1) \cup V(a_2) \cup \dots \cup V(a_r)]$ we have $\mathfrak{q} \in U$; this concludes the proof. □

We are now ready to state and prove the density result of curvilinear subschemes.

Proposition E.7 *Let* $\mathbf{H}_l$ *denote the Hilbert scheme of zero-dimensional subscheme of degree* l *in* $\mathbb{P}^n$ *and let* $\mathbf{H}_l^{curv}$ *denote the subset of* $\mathbf{H}_l$ *consisting of curvilinear subschemes of degree* l *in* $\mathbb{P}^n$. *Then* $\mathbf{H}_l^{curv}$ *is an open dense subset of* $\mathbf{H}_l$.

Proof The statement follows from Lemmas E.5 and E.6. □

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Depth Functions and Symbolic Depth Functions of Homogeneous Ideals

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1 Introduction

Let $\Bbbk$ be a field and let S be a standard graded $\Bbbk$-algebra. For a homogeneous ideal $Q \subseteq S$, we call the functions depth S/Q^t and depth $S/Q^{(t)}$, for $t \geq 1$, the *depth function* and the *symbolic depth function* of Q, respectively.

Depth is an important cohomological invariant (cf. [1, 4, 30]). For instance, we can compute the projective dimension via depth by the Auslander and Buchsbaum formula:

$$\text{pd}\ S/Q = \dim S - \text{depth}\ S/Q.$$

However, our understanding of the depth function and the symbolic depth function of ideals has been quite limited. This is partly because there are no effective methods to compute and/or to compare the depth of powers and symbolic powers of an arbitrary ideal. The aim of paper is to present recent studies, which have led to satisfactory solutions to the problem of classifying depth functions and symbolic depth functions of homogeneous ideals in polynomial rings.

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It is a classical result of Brodmann [3] that the depth function of an ideal in a Noetherian ring is asymptotically a constant function. From a few initially known examples, the depth function of an ideal appeared to be a non-increasing function. As more examples surfaced, it became a surprising fact that the depth function may otherwise exhibit wild behaviors; see [2, 10, 12, 21]. Herzog and Hibi [12] conjectured that the eventual behavior, as shown by Brodmann's result, is the only condition for the depth function of homogeneous ideals in polynomial rings. In Sect. 2, we survey results in our recent joint work with H.D. Nguyen and T.N. Trung [7], in which we establish the conjecture of Herzog and Hibi in its full generality.

Symbolic depth functions are much less understood compared to depth functions. For instance, there is no similar result to that of Brodmann for the eventual behavior of symbolic depth functions of ideals. This is because the symbolic Rees algebra $R_s(Q) := \bigoplus_{t \ge 0} Q^{(t)}$, which governs the behavior of symbolic powers of Q, is not always finitely generated. If $R_s(Q)$ is finitely generated then depth $S/Q^{(t)}$ is an asymptotically periodic function. In Sect. 3, we survey recent results of the second author and H.D. Nguyen in [24, 25], which shows that any positive and asymptotically periodic numerical function is the symbolic depth function of a homogeneous ideal in a polynomial ring.

In both Sects. 2 and 3, we shall thoroughly explain the ideas and techniques which have led to the surveyed results in [7] and [24, 25]. We believe that *they may provide effective tools for the study of other numerical invariants*, such as the projective dimension, the Castelnuovo-Mumford regularity and the number of associated primes of powers and symbolic powers of homogeneous ideals.

We end the paper with Sect. 4, where we discuss a number of open questions on depth functions and symbolic depth functions, and related problems on the projective dimension of powers and symbolic powers of homogeneous ideals. For unexplained notions and terminology we refer the readers to [4].

2 Ordinary Depth Functions

One of the main motivations for the study of depth functions of ideals is the following classical result of Brodmann [3].

Theorem 2.1 [3, Theorem (2)] *Let S be a Noetherian ring and let $Q \subseteq S$ be an ideal. Then,* depth S/Q^t *is asymptotically a constant function, i.e.,* depth $S/Q^t =$ depth S/Q^{t+1} *for all $t \gg 0$.*

The first systematic study on depth functions of homogeneous ideals was carried out by Herzog and Hibi [12]. In their work, Herzog and Hibi observed that Theorem 2.1 is only a special case of a more general phenomenon, which we shall now describe. Note that a graded S-algebra R is said to be *standard graded* if it is generated over S by homogeneous elements of degree one. For a graded module E, we denote by E_t its degree t component.

Theorem 2.2 [12, Theorem 1.1] *Let R be a finitely generated standard graded S-algebra. Let E be a finitely generated graded R-module. Then,* $\operatorname{depth} E_t = \operatorname{depth} E_{t+1}$ *for all $t \gg 0$.*

For an ideal $Q \subseteq S$, let $R(Q) := \bigoplus_{t\geq 0} Q^t$ be the *Rees algebra* of Q. Then, $R(Q)$ is a finitely generated standard graded S-algebra. Thus, Theorem 2.2 applies to imply Theorem 2.1.

Thanks to Theorem 2.1, to investigate all possible depth functions, we only need to focus on *convergent* non-negative *numerical* functions, i.e. functions $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ with the property that $f(t) = f(n+1)$ for $t \gg 0$.

Herzog and Hibi [12, Theorem 4.1] showed that any non-decreasing convergent non-negative numerical function is the depth function of a monomial ideal. This result was astonishing because, as mentioned, depth functions of homogeneous ideals initially tend to be non-increasing. Also in [12], Herzog and Hibi exhibited monomial ideals whose depth functions display unusual behaviors. These results seemed to suggest that the beginning of the depth function of a homogeneous deal in a polynomial ring can be *arbitrarily* wild. In fact, at the end of [12], Herzog and Hibi made the following conjecture.

Conjecture 2.3 (Herzog-Hibi) Let f be any convergent non-negative numerical function. There exists a homogeneous ideal Q in a polynomial ring S such that f is the depth function of Q, i.e., $f(t) = \operatorname{depth} S/Q^t$ for all $t \geq 1$.

In a later work, Bandari, Herzog and Hibi [2] showed that the depth function can have an arbitrary number of local maxima and local minima, which provided a strong evidence for Conjecture 2.3. Until then, the constructions of particular depth functions were all more or less ad hoc.

There was a more general attempt by the authors, together with T.N. Trung, in [10, Theorem 4.9] to show that any non-increasing convergent non-negative numerical function is the depth function of a monomial ideal. An important new ingredient in [10] is the following result on the depth function of sums of ideals.

Let A and B be polynomial rings over a field $\Bbbk$ with disjoint sets of variables. Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals. By abuse of notations, we shall also use I and J to denote their extensions in the tensor product $R := A \otimes_k B$.

Proposition 2.4 [10, Corollary 3.6(i)] *Assume that* $\operatorname{depth} I^{i-1}/I^i \geq \operatorname{depth} I^i/I^{i+1}$ *for $i \leq t-1$. Then,*

$$\operatorname{depth} R/(I+J)^t = \min_{i+j=n-1} \{\operatorname{depth} I^i/I^{i+1} + \operatorname{depth} J^j/J^{i+1}\}.$$

The proof of [10, Theorem 4.9] contained an error, discovered by Matsuda, Suzuki and Tsuchiya in [21]. It only gave the desired conclusion for a large class of non-increasing convergent non-negative numerical functions, as established in [21, Theorem 2.1]. However, a modification of this approach has finally led to a complete characterization of depth functions of monomial ideals, which confirms Conjecture 2.3.

Theorem 2.5 [7, Theorem 4.1] *Let $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be any convergent non-negative numerical function and let $\Bbbk$ be any field. There exists a monomial ideal Q in a polynomial ring S over $\Bbbk$ such that f is the depth function of Q.*

The key idea in [7] is that depth functions are *additive*, i.e., the sum of two depth functions is again a depth function. This makes use of the following result of Hoa and Tam [15].

Lemma 2.6 [15, Lemmas 1.1 and 2.2] *Let A and B be polynomial rings over $\Bbbk$ with disjoint sets of variables. Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals, which are also seen as their extensions in $R = A \otimes_{\Bbbk} B$. Then,*

(i) $I \cap J = IJ$, *and*
(ii) $\operatorname{depth} R/IJ = \operatorname{depth} A/I + \operatorname{depth} B/J + 1$.

By setting $S = R/(x - y)$ and $Q = (IJ, x - y)/(x - y)$, where x and y are arbitrary variables in A and B, respectively, Lemma 2.6 gives rise to the following result.

Proposition 2.7 [7, Proposition 2.3] *Let $I \subseteq A$ and $J \subseteq B$ be homogeneous ideals as in Lemma 2.6. There exists a homogeneous ideal Q in a polynomial ring S such that for all $t \geq 1$,*

$$\operatorname{depth} S/Q^t = \operatorname{depth} A/I^t + \operatorname{depth} B/J^t.$$

Moreover, if I and J are monomial ideals then Q can be chosen to be a monomial ideal.

We will use the additivity of depth functions to construct a monomial ideal whose depth function is any given convergent non-negative numerical function. The construction is based on the following simple arithmetic observation.

To ease on notations, we shall identify a numerical function $f : \mathbb{N} \to \mathbb{Z}$ with the sequence of its values $f(1), f(2), \ldots$. Let f be a convergent non-negative numerical function which is not the constant function $0, 0, \ldots$. Then f can be written as a sum of numerical functions of the following two types:

Type I: $0, \ldots, 0, 1, 1, \ldots$
Type II: $0, \ldots, 0, 1, 0, 0, \ldots$.

Note that if f is the constant function $0, 0, \ldots$ then f is the depth function of the maximal homogeneous ideal in any polynomial ring over $\Bbbk$.

By Proposition 2.7 and the above observation, to prove Theorem 2.5 we only need to construct monomial ideals that admit any function of both Types I and II as their depth functions. Functions of Type I are non-decreasing convergent functions, and so they are the depth functions of monomial ideals, as constructed in [12, Theorem 4.1]. For functions of Type II, we shall make use of monomial ideals whose depth functions are of the form $1, \dots, 1, 0, 0, \dots$, which exist as shown by [10, Example 4.10] and [21, Proposition 1.5].

As before, let A and B be polynomial rings over $\Bbbk$ with disjoint sets of variables, and let $R = A \otimes_{\Bbbk} B$. Let $I \subseteq A$ and $J \subseteq B$ be monomial ideals with depth functions $0, \dots, 0, 1, 1, \dots$ and $1, \dots, 1, 0, 0, \dots$, where the first 1 of the former function and the last 1 of the later function are at the same position. By Proposition 2.7, the function depth $R/((IJ)^t, x - y)$ is of the form $1, \dots, 1, 2, 1, 1, \dots$ for some variables x, y. If we can find variables x' and y' such that $x' - y'$ is a non-zerodivisor in $R/((IJ)^t, x - y)$ for all $t \geq 1$, then

$$\operatorname{depth} R/((IJ)^t, x - y, x' - y') = \operatorname{depth} R/((IJ)^t, x - y) - 1$$

is of the form $0, \dots, 0, 1, 0, 0, \dots$, i.e., of Type II. Clearly, we can identify $S = R/(x - y, x' - y')$ with a polynomial ring and $(IJ, x - y, x' - y')/(x - y, x' - y')$ with a monomial ideal in S. To find such variables x' and y' we need to know the associated primes of the ideal $((IJ)^t, x - y)$ for all $t \geq 1$. This is given in the next proposition.

For an ideal Q, denote the set of the associated primes and the set of the minimal associated primes of Q by $\operatorname{Ass}(Q)$ and $\operatorname{Min}(Q)$, respectively.

Proposition 2.8 [7, Proposition 3.2] *Let $I \subseteq A$ and $J \subseteq B$ be proper monomial ideals in polynomial rings. Let x and y be variables in A and B, respectively. Then,* $\operatorname{Ass}(IJ, x - y)$ *is given by*

$$\{(\mathfrak{p}, x - y) \mid \mathfrak{p} \in \operatorname{Ass}(I)\} \cup \{(\mathfrak{q}, x - y) \mid \mathfrak{q} \in \operatorname{Ass}(J)\} \cup \left(\bigcup_{\substack{\mathfrak{p} \in \operatorname{Ass}(I), x \in \mathfrak{p} \\ \mathfrak{q} \in \operatorname{Ass}(J), y \in \mathfrak{q}}} \operatorname{Min}(\mathfrak{p} + \mathfrak{q}) \right).$$

Using Proposition 2.8 one can give sufficient conditions for the existence of variables x', y' such that $x' - y'$ is a non-zerodivisor in $R/((IJ)^t, x - y)$ for all $t \geq 1$ [7, Proposition 3.5]. It turns out that the monomial ideals I and J, as exhibited in [12, Theorem 4.1] and [10, Example 4.10], satisfy these conditions. This completes the construction of monomial ideals with depth functions of Type II and, therefore, the proof of Theorem 2.5.

The following concrete example illustrates the construction of monomial ideals with depth functions of Type II.

Example 2.9 Let $A = \Bbbk[x, y, z]$ and $I = (x^{d+2}, x^{d+1}y, xy^{d+1}, y^{d+2}, x^d y^2 z)$, for some $d \geq 2$. By [12, Theorem 4.1] we have

$$\operatorname{depth} A/I^t = \begin{cases} 0 & \text{if } t \leq d - 1, \\ 1 & \text{if } t \geq d. \end{cases}$$

Let $B = \Bbbk[w, u, v]$. Let J be the integral closure of the ideal $(w^{3d+3}, wu^{3d+1}v, u^{3d+2}v)^3$ or $J = (w^{d+1}, wu^{d-1}v, u^d v)$. By [10, Example 4.10] we have

$$\operatorname{depth} B/J^t = \begin{cases} 1 & \text{if } t \leq d, \\ 0 & \text{if } t \geq d+1. \end{cases}$$

Let $R = \Bbbk[x, y, z, w, u, v]$. By Proposition 2.7, we have

$$\operatorname{depth} R/((IJ)^t, y-u) = \begin{cases} 1 & \text{if } t \neq d, \\ 2 & \text{if } t = d. \end{cases}$$

Using Proposition 2.8, it is easy to check that $z - v$ is a non-zerodivisor modulo $((IJ)^t, y - u)$ for all $t > 0$. Therefore,

$$\operatorname{depth} R/((IJ)^t, y-u, z-v) = \begin{cases} 0 & \text{if } t \neq d, \\ 1 & \text{if } t = d. \end{cases}$$

If we set $S = \Bbbk[x, w, u, v]$ and $Q = (x^{d+2}, x^{d+1}u, xu^{d+1}, u^{d+2}, x^d u^2 v)J$, which is obtained from IJ by setting $y = u$ and $z = v$, then

$$\operatorname{depth} S/Q^t = \operatorname{depth} R/((IJ)^t, y-u, z-v).$$

Hence, the depth function of Q is of Type II.

Theorem 2.5 also settles affirmatively a long standing question of Ratliff in [28, (8.9)], that has remained open since 1983.

Question 2.10 (Ratliff) Given a finite set Γ of positive integer, do there exist a Noetherian ring S, an ideal Q and a prime ideal $P \supseteq Q$ in S such that P is an associated prime of Q^t if and only if $t \in \Gamma$?

Specifically, the following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.11 *Let Γ be a set of positive integers which either is finite or contains all sufficiently large integers. There exists a monomial ideal Q in a polynomial ring S, with maximal homogeneous ideal $\mathfrak{m}$, such that $\mathfrak{m} \in \operatorname{Ass}(Q^t)$ if and only if $t \in \Gamma$.*

Corollary 2.11, furthermore, gives a monomial ideal as a counterexample to the following question, which was also due to Ratliff [28, (8.4)]. This question was answered negatively by Huckaba [16, Example 1.1], in which the given ideal was not a monomial ideal.

Question 2.12 (Ratliff) Let Q be an arbitrary ideal in a Noetherian ring S. Let $P \supseteq Q$ be a prime ideal such that $P \in \operatorname{Ass}(Q^m)$ for some $m \geq 1$ and $P \in \operatorname{Ass}(Q^t)$ for all $t \gg 0$. Is $P \in \operatorname{Ass}(Q^t)$ for all $t \geq m$?

3 Symbolic Depth Functions

Let Q be an ideal in a Noetherian ring S. For $t \geq 0$, the t-th *symbolic power* of Q is the ideal

$$Q^{(t)} := \bigcap_{\mathfrak{p} \in \mathrm{Min}(Q)} (Q^t_{\mathfrak{p}} \cap S).$$

In other words, $Q^{(t)}$ is the intersection of the primary components of the minimal associated primes of Q^t. We remark here that there is another variant of symbolic powers, in which $\mathrm{Min}(Q)$ is replaced by $\mathrm{Ass}(Q)$, that has also been much investigated. If Q is a radical ideal in a polynomial ring then these definitions agree. Symbolic powers of homogeneous ideals are much harder to study compared to their ordinary powers. This is seen from, for example, the fact that the generators of $Q^{(t)}$ in general cannot be derived merely from the generators of Q.

Inspired by Theorem 2.1, one may incline to ask if the symbolic depth function is also necessarily a convergent numerical function; that is, if $\operatorname{depth} S/I^{(t)} = \operatorname{depth} S/I^{(t+1)}$ for all $t \gg 0$. Theorem 2.2 does not apply in this case because the *symbolic Rees algebra*

$$R_s(Q) := \bigoplus_{t \geq 0} Q^{(t)}$$

is not always a standard graded S-algebra; it needs not even be finitely generated (see, for instance, [6, 17, 29]). An application of Theorem 2.2, when the symbolic Rees algebra of Q is finitely generated, gives us the following result.

Proposition 3.1 *Let Q be a homogeneous ideal in a polynomial ring S. Assume that $R_s(Q)$ is a finitely generated S-algebra. Then,* $\operatorname{depth} S/Q^{(t)}$ *is an asymptotically periodic function, i.e., it is periodic for $t \gg 0$.*

By [13, Theorem 3.2], $R_s(Q)$ is finitely generated if Q is a monomial ideal. Therefore, the symbolic depth functions of monomial ideals are asymptotically periodic. For several classes of squarefree monomial ideals, it is known that their symbolic depth functions are actually convergent functions (cf. [5, 14, 19, 32]). It was an open question whether the symbolic depth function of any monomial ideal is convergent [14, p. 308].

Remark 3.2 It is an easy observation that if $\operatorname{depth} S/Q^{(t)} = 0$ for some $t > 0$ then $\operatorname{depth} S/Q^{(t)} = 0$ for all $t > 0$. Therefore, we shall only consider *positive* symbolic depth functions.

In this section, we survey is a recent result of the second author and H.D. Nguyen [24], which shows that any asymptotically periodic positive numerical function is the symbolic depth function of a homogeneous ideal. In particular, there are plenty monomial ideals whose symbolic depth functions are not necessarily convergent.

Theorem 3.3 [24, Theorem 6.1] *Let $\mathbb{k}$ be a field and let $\phi : \mathbb{N} \to \mathbb{N}$ be an asymptotically periodic positive numerical function. Then, there exist a polynomial ring S over a purely transcendental extension of $\mathbb{k}$ and a homogeneous ideal $Q \subseteq S$ which admits ϕ as its symbolic depth function, i.e.,*

$$\operatorname{depth} S/Q^{(t)} = \phi(t) \text{ for all } t \geq 1.$$

The proof of Theorem 3.3 is inspired by that of Theorem 2.5. The key idea is to construct any asymptotically periodic positive numerical function from basic symbolic depth functions by using closed operations within the class of symbolic depth functions.

Once again, let A and B be polynomial rings over $\mathbb{k}$ with disjoint sets of variables, and let $R = A \otimes_{\mathbb{k}} B$. Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals. It follows from Lemma 2.6(i) that

$$(IJ)^{(t)} = I^{(t)} \cap J^{(t)} = I^{(t)} J^{(t)}.$$

This, together with Lemma 2.6(ii), implies that

$$\operatorname{depth} R/(IJ)^{(t)} = \operatorname{depth} A/I^{(t)} + \operatorname{depth} B/J^{(t)} + 1.$$

As in the study of depth function, at this point, we need to find a Bertini-type theorem to get the additivity property of symbolic depth functions. That is, for a given polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$ and a homogeneous ideal $K \subseteq R$, we need to find a linear form $f \in R$ such that for all $t \geq 1$, f is a non-zerodivisor of $K^{(t)}$ and if we set $S = R/(f)$ and $Q = (K, f)/(f)$, then

$$S/Q^{(t)} \simeq R/(K^{(t)}, f).$$

The first difficulty in finding such a result is that f has to be the same for all symbolic powers $K^{(t)}$, which form an infinite families of ideals.

The method employed in [24] to address this issue is using *generic hyperplane section*. Let $u = \{u_1, \ldots, u_n\}$ be a collection of indeterminates and let $R(u) = R \otimes_{\mathbb{k}} \mathbb{k}(u)$, where $\mathbb{k}(u) = \mathbb{k}(u_1, \ldots, u_n)$ is a purely transcendental extension of $\mathbb{k}$. Set

$$f_u := u_1 x_1 + \cdots + u_n x_n.$$

We call f_u a *generic linear form*. The associated primes of the ideal $(K^{(t)}, f_u)$ were already studied in a more general setting in [14]. Using results from [14], the following Bertini-type theorem was given in [24].

Proposition 3.4 [24, Proposition 5.3] *Let R be a polynomial ring over $\mathbb{k}$ and let $K \subseteq R$ be an ideal with* $\operatorname{depth} R/K^{(t)} \geq 2$ *for some*

$t \geq 1$. Let $S = R(u)/(f_u)$ and $Q = (K, f_u)/(f_u)$. Then f_u is a regular element on $K^{(t)}R(u)$ and

$$S/Q^{(t)} \simeq R(u)/(K^{(t)}, f_u).$$

Proposition 3.4 has the following consequences on symbolic depth functions.

Corollary 3.5 *Let $\phi(t)$ be a symbolic depth function over a field $\Bbbk$ such that $\phi(t) \geq 2$ for all $t \geq 1$. Then $\phi(t) - 1$ is also a symbolic depth function over a purely transcendental extension of $\Bbbk$.*

Corollary 3.6 *Let $\phi(t)$ and $\psi(t)$ be symbolic depth functions over a field $\Bbbk$. Then $\phi(t) + \psi(t) - 1$ is a symbolic depth function over a purely transcendental extension of $\Bbbk$.*

Corollaries 3.5 and 3.6 particularly show that the operations

$$\overline{\phi}(t) := \phi(t) - 1,$$
$$(\phi * \psi)(t) := \phi(t) + \psi(t) - 1$$

are closed in the set of symbolic depth functions with values ≥ 2 and the set of all symbolic depth functions, respectively.

It is not hard to see that any asymptotically periodic positive numerical function is obtained from finitely many functions of the following types by using the operations $\overline{\phi}$ with $\phi(t) \geq 2$ for all $t \geq 1$ and $\phi * \psi$:

Type A: $1, \dots, 1, 2, 2, \dots$, which is a monotone function converging to 2,
Type B: $1, \dots, 1, 2, 1, 1, \dots$, which has the value 2 at only one position,
Type C: $1, 1, 1, \dots$ or $1, \dots, 1, 2, 1, \dots, 1, 1, \dots, 1, 2, 1, \dots, 1, \dots$, which is a periodic function with a period of the form $1, \dots, 1, 2, 1, \dots, 1$, where 2 can be at any position.

The proof of Theorem 3.3 now reduces to showing that all functions of types A, B and C are symbolic depth functions of homogeneous ideals. In fact, any function of types A, B or C is the symbolic depth function of a monomial ideal. This is the most difficult part of the arguments in [24].

By focusing on monomial ideals, whose symbolic powers are then also monomial ideals, one can invoke a formula of Takayama [31], which relates local cohomology modules of a monomial ideal with the reduced homology groups of certain simplicial complexes. Since depth can be characterized by the vanishing of the local cohomology modules, the study of symbolic depth functions can be reduced to the investigation of combinatorial properties of monomial ideals.

To be more precise, let $R = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over $\Bbbk$ and let $\mathfrak{m}$ be its maximal homogeneous ideal. Let $K \subseteq R$ be a monomial ideal. Note that

$$\operatorname{depth} R/K = \min\{i \mid H^i_{\mathfrak{m}}(R/K) \neq 0\}.$$

Since R/K has a $\mathbb{N}^n$-graded structure, the local cohomology modules $H^i_{\mathfrak{m}}(R/K)$ also have a $\mathbb{Z}^n$-graded structure. For $\mathbf{a} \in \mathbb{Z}^n$, let $H^i_{\mathfrak{m}}(R/K)_{\mathbf{a}}$ denote the degree $\mathbf{a}$ component of $H^i_{\mathfrak{m}}(R/K)$. Takayama [31] gave a formula to relate the dimension and vanishing of $H^i_{\mathfrak{m}}(R/K)_{\mathbf{a}}$ to that of the reduced homology groups of certain simplicial complex $\Delta_{\mathbf{a}}(K)$, which depends on the primary component of K. The simplicial complex $\Delta_{\mathbf{a}}(K)$ is a subcomplex of the Stanley-Reisner simplicial complex $\Delta(K)$ of the squarefree monomial ideal $\sqrt{K}$. Particularly, the facets of $\Delta_{\mathbf{a}}(K)$ are facets of $\Delta(K)$ if $\mathbf{a} \in \mathbb{N}^n$. (See [22] for more details and a different interpretation of Takayama's formula.)

A consequence of Takayama's formula is the following criterion for depth $R/K \geq 2$.

Proposition 3.7 (cf. [24, Proposition 1.4]) *Let K be an unmixed ideal in R. Then* depth $R/K \geq 2$ *if and only if $\Delta_{\mathbf{a}}(K)$ is connected for all $\mathbf{a} \in \mathbb{N}^n$.*

On the other hand, if K has a minimal prime M such that $\dim R/M = 2$, then depth $R/K \leq 2$ (see, e.g., [4, Proposition 1.2.13]). Since K and $K^{(t)}$ share the same minimal primes, in this case, we also have $1 \leq$ depth $R/K^{(t)} \leq 2$ for all $t \geq 1$. Hence, the symbolic depth function of K is a 1–2 functions. Note that $\Delta(K) = \Delta(K^{(t)})$ for all $t \geq 1$. If we choose K such that $\Delta(K)$ has only one disconnected subcomplex Δ' whose facets are facets of $\Delta(K)$, then we only need to check when $\Delta_{\mathbf{a}}(K^{(t)}) = \Delta'$ for all $\mathbf{a} \in \mathbb{N}^n$ in order to know when depth $R/K^{(t)} = 1$. An instance when this observation applies is given in the following proposition, in which for a monomial ideal with 1–2 symbolic depth function we can test which symbolic powers has depth exactly 2.

Proposition 3.8 [24, Proposition 3.7] *Let $R = \Bbbk[x, y, z, u, v]$ be a polynomial ring. Let M, P, Q be primary monomial ideals of R such that*

$$\sqrt{M} = (x, y, z),$$

$$\sqrt{P} = (x, y),$$

$$\sqrt{Q} = (z).$$

Let $K = M \cap (P, u) \cap (Q, v)$. Then depth $R/K^{(t)} \leq 2$ *and* depth $R/K^{(t)} = 2$ *if and only if $M^t \subseteq P^t + Q^t$ for all $t > 1$.*

For the ideal K in Proposition 3.8, we have $\Delta(K) = \langle \{u, v\}, \{z, v\}, \{x, y, u\} \rangle$, which consists of two disjoint facets $\{z, v\}$ and $\{x, y, u\}$ that are connected by the facet $\{u, v\}$. Therefore, $\Delta' = \langle \{z, v\}, \{x, y, u\} \rangle$ is the only disconnected subcomplex of $\Delta(K)$ whose facets are facets of $\Delta(K)$.

Proposition 3.8 allows us to construct monomial ideals admitting any given function of Types A, B, C as symbolic depth functions and, thus, completes the proof of Theorem 3.3. This construction is illustrated in the following examples.

Example 3.9 [24, Lemma 4.2] Let $m \geq 2$ be a fixed integer and let $R = \Bbbk[x, y, z, u, v]$. Consider the ideal

$$K = (x^{2m-2}, y^m, z^{2m})^2 \cap (x^{2m-1}, y^{2m-1}, u) \cap (z, v).$$

Then, the symbolic depth function of K is of Type A, i.e.,

$$\operatorname{depth} R/K^{(t)} = \begin{cases} 1 \text{ if } t \leq m-1, \\ 2 \text{ if } t \geq m. \end{cases}$$

Example 3.10 [24, Lemma 4.3] Let $m \geq 1$ be a fixed integer and let $R = \Bbbk[x, y, z, u, v]$. Consider the ideal

$$K = (x^{2m}, y^{2m}, xy^{m-1}z, z^{2m})^2 \cap (x^m, y^m, u) \cap (z^{2m+2}, v).$$

Then, the symbolic depth function of K is of Type B, i.e.,

$$\operatorname{depth} R/K^{(t)} = \begin{cases} 2 \text{ if } t = m, \\ 1 \text{ if } t \neq m. \end{cases}$$

For functions of Type C, we first note that the existence of the symbolic depth function $1, 1, 1, \ldots$ is trivial, for example, with $R = \Bbbk[x, y]$ and $K = (x)$. The construction of other symbolic depth functions of Type C is much more subtle because these functions are periodic. (For instance, the construction depends on the period of the given function.) The existence of ideals with symbolic depth functions of Type C is summarized in the following result.

Theorem 3.11 [24, Theorem 4.4] *Let $m \geq 2$ and $0 \leq d < m$ be integers. There exists a monomial ideal K in $R = \Bbbk[x, y, z, u, v]$ such that*

$$\operatorname{depth} R/K^{(t)} = \begin{cases} 2 & \text{if } t \equiv d \text{ modulo } m, \\ 1 & \text{otherwise}. \end{cases}$$

4 Open Questions

In this section, we discuss open problems and questions related to depth functions and symbolic depth functions that we would like to see answered.

The following question arises naturally from the relationship between depth and projective dimension:

> "*Which numerical functions describe the projective dimension of powers and symbolic powers of homogeneous ideals in polynomial rings?*"

This question seems to be very difficult. We could not give the answer even in the following basic situation.

Question 4.1 Let Q be a homogeneous ideal in a polynomial ring S. Suppose that $\text{pd } Q = 1$ and $\text{pd } Q^t = 1$ for all $t \gg 0$. Is it true that $\text{pd } Q^t = 1$ for all $t \geq 1$?

It follows from the Auslander-Buchsbaum formula and Brodmann's result that the projective dimension of powers of an ideal is a convergent function. Inspired by Theorem 2.5 and Question 4.1, we raise the following question.

Question 4.2 Let $g : \mathbb{N} \to \mathbb{Z}$ be a convergent function such that $g(t) \geq 2$ for all $t \geq 1$. Does there exist a monomial ideal Q in a polynomial ring S such that $g(t) = \text{pd } Q^t$ for all $t \geq 1$?

As a consequence of Theorem 2.5, we can give partial answer to Question 4.2.

Corollary 4.3 *Let $g : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be any convergent numerical function. There exists a monomial ideal Q and a number c such that* $\text{pd } Q^t = g(t) + c$ *for all* $t \geq 1$.

The constant c in Corollary 4.3 can be computed as follows. Let $m = \max_{t \geq 1} g(t)$. Then $f(t) = m - g(t)$ is a convergent numerical function. Let n be the number of variables of a polynomial ring S which contains a homogeneous ideal Q such that $\text{depth } S/Q^t = f(t)$ for all $t \geq 1$. Then $\text{pd } Q^t = g(t) + c$ for $c = n - m - 1$. To this end, it is of interest to have an answer to the following question.

Question 4.4 What is the smallest number of variables of a polynomial ring which contains a homogeneous ideal with a given depth function $f(t)$?

Note that the proof of Theorem 2.5 uses a large number of variables compared to the values of $f(t)$.

In making use of Corollaries 3.5 and 3.6, the ideals constructed in Theorem 3.3 are non-monomial ideals in polynomial rings over purely transcendental extensions of the given field $\Bbbk$. Using the theory of specialization [20, 26, 27], we can construct such ideals in polynomial rings over any uncountable field. This is because the Bertini-type result, Proposition 3.4, holds without having to go to purely transcendental extensions of the ground field; see [24, Proposition 5.8]. This leads us to the following question.

Question 4.5 Given a field $\Bbbk$ and an asymptotically periodic positive numerical function $\phi(t)$, do there exist a polynomial ring S over $\Bbbk$ and a *monomial ideal* $Q \subset S$ such that $\text{depth } S/Q^{(t)} = \phi(t)$ for all $t \geq 1$?

The analogous question for the depth function of homogeneous ideals has a positive answer by Theorem 2.5.

Again, due to the relationship between depth and projective dimension and inspired by Theorem 3.3, we raise the following question on projective dimension of symbolic powers.

Question 4.6 Let $g : \mathbb{N} \to \mathbb{N}$ be an asymptotically periodic function such that $g(t) \geq 2$ for all $t \geq 1$. Does there exist a monomial ideal Q in a polynomial ring S such that $g(t) = \text{pd } Q^{(t)}$ for all $t \geq 1$?

As a consequence of Theorem 3.3, we obtain a partial answer to Question 4.6.

Corollary 4.7 *Let $\phi : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be an asymptotically periodic numerical function. Let $\Bbbk$ be a field and let $m = \max_{t\geq 1} \phi(t)$. Then, there exist a positive integer c, a polynomial ring S in $m + c + 2$ variables over a purely transcendental extension of $\Bbbk$, and a homogeneous ideal Q in S such that* $\text{pd } Q^{(t)} = \phi(t) + c$ *for all* $t \geq 1$.

Similarly to Corollary 4.3, the constant c in Corollary 4.7 is determined by the number of variables of a polynomial ring S which contains a homogeneous ideal Q with the given symbolic depth function.

The proof of Theorem 3.3 uses a large number of variables. However, all constructed examples of symbolic depth functions of types A, B, C (except $1, 1, 1, \dots$) are ideals of height 2 in polynomial rings in 5 variables. It is naturally of interest to consider the following question.

Question 4.8 Let $\phi(t)$ be an asymptotically periodic positive numerical function and $m = \max_{t\geq 1} \phi(t)$. Does there exist a polynomial ring S in $m + 3$ variables that contains a height 2 homogeneous ideal Q such that $\text{depth } S/Q^{(t)} = \phi(t)$ for all $t \geq 1$?

Theorem 3.3 classifies a large class of symbolic depth functions. It remains an open problem to determine if Theorem 3.3 indeed covers all symbolic depth functions.

Question 4.9 Does there exist a homogeneous ideal whose symbolic depth function is not asymptotically periodic?

According to Proposition 3.1, if such an ideal existed, its symbolic Rees algebra would have to be non-Noetherian. To find non-Noetherian symbolic Rees algebras is a difficult problem that is related to Hilbert's fourteenth problem; see, for instance, [29]. To the best of our knowledge, there are only examples of non-Noetherian symbolic Rees algebras for one-dimensional ideals (cf. [6, 17, 29]). In this case, we have $\text{depth } S/I^{(t)} = 1$ for all $t \geq 1$, whence the symbolic depth function is a constant function.

It was shown in [23, 24] that the symbolic depth function of a squarefree monomial ideal Q is *almost* non-increasing, in the sense that $\text{depth } S/Q^{(s)} \leq \text{depth } S/Q^{(t)}$ for $s \gg t$. There are examples of ideals generated by squarefree monomials of degrees ≥ 3 whose symbolic depth functions need not be monotone [24].

Question 4.10 Is the symbolic depth function of the edge ideal of a graph a non-increasing function?

The analogous question for the depth function of the edge ideal of a graph is also an open question (cf. [11, 12]). Note that the depth function of a squarefree monomial ideal in general needs not be non-increasing; see [8, 18].

Beside powers and symbolic powers of an ideal, the integral closures of powers have been extensively investigated. It is also a classical result of Brodmann [3] that for an ideal Q in a Noetherian ring S, depth $S/\overline{Q^t}$ is asymptotically a constant function, i.e., the function depth $S/\overline{Q^t}$ is a convergent numerical function.

Question 4.11 For which convergent numerical function $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ does there exist a homogeneous ideal Q in a polynomial ring S such that depth $S/\overline{Q^t} = f(t)$ for all $t \geq 1$?

The monomial generators of $\overline{Q^t}$ can be derived from that of Q by combinatorial means; see, for instance, [9]. This fact was used in [24] to examine the depth of integrally closed symbolic powers of monomial ideals.

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Algebraic Geometry, Commutative Algebra and Combinatorics: Interactions and Open Problems

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Dedicated to David Eisenbud, on the occasion of his 75th birthday.

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1 Introduction

We survey three topics of recent research interest, in each case starting with a question, conjecture or result of David Eisenbud. A theme that will recur throughout this survey is that of the longstanding, still open and interesting question of what singularities a plane curve of given degree can have. Indeed, the numerical question of what vectors $(d, t_2, t_3, \dots, t_d)$ arise for reduced plane curves is an open problem (here d is the degree of the curve, and t_k is the number of points of multiplicity exactly k). This is open even when restricted to curves all of whose components are lines (i.e., line arrangements). In fact, just classifying complex line arrangements with $t_2 = 0$ is an open problem (see Problem 2.14).

This survey is divided into three parts, corresponding to work on semi-effectivity, on the containment problem of symbolic powers of ideals of fat points in their ordinary powers, and on splitting types of rank 2 bundles on rational curves.

The discussion on semi-effectivity starts with a question of Eisenbud and M. Velasco about computability of semi-effectivity of divisors on the blow up X of $\mathbb{P}^n$ at a finite set of points. This leads to the SHGH Conjecture in the case of points in $\mathbb{P}^2$. For points in $\mathbb{P}^n$ more generally, it leads to a consideration of Waldschmidt constants of ideals of fat point subschemes of $\mathbb{P}^n$, and to conjectures of G.V. Chudnovsky and J.-P. Demailly, and then to questions about semi-effectivity index, bounded

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negativity and H-constants, instances of which are of interest even for curves given by line arrangements.

The discussion on the containment problem evolves out of a conjecture of Eisenbud and B. Mazur. This leads to questions of C. Huneke, one of which (Question 3.3) is now known to have a negative answer, thanks to examples coming from line arrangements. All known examples showing that Question 3.3 has a negative answer and giving counterexamples to a more general conjecture of the author subsuming Question 3.3 (viz., Conjecture 3.4) are very constrained. This led to a conjecture of E. Grifo (see Conjecture 3.6) which proposes that Conjecture 3.4 holds asymptotically. Grifo's conjecture is related to how large the value of asymptotic quantities known as resurgences can be. Other open problems relate to the relationship between the minimal values of various versions of the resurgence, some of which involve questions relating to integral closure of ideals.

The discussion on splitting types begins with a result of Eisenbud and A. Van de Ven on splitting types of certain rational space curves. The discussion then turns to the work of M.-G. Ascenzi on splitting types for rational plane curves which more recently has led to the concept of an Ascenzi curve (a concept of interest in computational applications; see [31, 61, 99, 100]). This work is related to a conjecture on polynomial interpolation known as the SHGH Conjecture. Splitting types also come up in a different way in work on unexpected curves, a concept which was partly motivated by wanting to understand the SHGH Conjecture better. Unexpected curves arise in association to special finite point sets $Z \subset \mathbb{P}^2$; the splitting type in this context is defined in terms of the line arrangement comprising the lines dual to the points of Z.

Throughout this survey, $\mathbb{K}$ will denote an algebraically closed field. We will indicate when we need $\mathbb{K}$ to be the complex numbers, but there are times when it will be of interest to allow $\mathbb{K}$ to be an arbitrary algebraically closed field, even possibly of positive characteristic.

2 Semi-effectivity

Around 2009, a problem of Eisenbud and M. Velasco was circulating by email. The context was that $\pi : X \to \mathbb{P}^n$ was the blow up of distinct points $p_1, \ldots, p_r \in \mathbb{P}^n$. Thus the divisor class group of X is the free abelian group on the class $[H]$ of a hyperplane H and the classes $[E_1], \ldots, [E_r]$ where E_i is the blow up of p_i. To simplify notation, when referring to a divisor class $F = d[H] - m_1[E_1] - \cdots - m_r[E_r]$, we will dispense with the brackets and just write $F = dH - m_1E_1 - \cdots - m_rE_r$ (thereby relying on the discernment of the reader to distinguish between divisor classes and divisors in those rare cases where it might matter).

We will say F is *semi-effective* if sF is the class of an effective divisor for some $s > 0$. The following problem is still open.

Problem 2.1 (Eisenbud-Velasco Problem) Given integers $m_i > 0$ and a divisor class $F = dH - m_1E_1 - \cdots - m_rE_r$, give a finite procedure to determine whether $|sF|$ is nonempty for some $s > 0$ (i.e., whether F is semi-effective).

2.1 Waldschmidt Constants

Problem 2.1 relates directly to Waldschmidt constants, a concept introduced in [110] but which was gaining renewed interest in 2009. Some background is needed in order to define Waldschmidt constants. Given points $p_1, \ldots, p_r \in \mathbb{P}^n$ and positive integers m_i, a subscheme $Z \subset \mathbb{P}^n$ is defined by the ideal $I(Z) = \cap_i I(p_i)^{m_i} \subset \mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \ldots, x_n]$, where $I(p_i)$ is the ideal generated by all homogeneous polynomials (i.e., forms) that vanish at p_i. The scheme Z is called a *fat point* subscheme and is denoted $Z = m_1p_1 + \cdots + m_rp_r$. The ideal $I(Z)$ is homogeneous; the $\mathbb{K}$-vector space span of the homogeneous elements of $I(Z)$ of degree t is denoted $[I(Z)]_t$. The connection to F in Problem 2.1 is that $\dim|F| + 1 = h^0(X, \mathcal{O}_X(F)) = \dim H^0(X, \mathcal{O}_X(F))$, and there is a natural $\mathbb{K}$-vector space isomorphism $H(X, \mathcal{O}_X(sF)) \cong [I(sZ)]_{sd}$ [70, Proposition 4.1.1].

To make use of this connection, given a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$ and a rational number $t > 0$, it will be convenient to define the $\mathbb{Q}$-class $F(Z)_t = tH - m_1E_1 - \cdots - m_rE_r$. We will say $F(Z)_t$ is *semi-effective* if mt is an integer for some integer $m > 0$ for which $|m(F(Z)_t)| = |F(mZ)_{mt}| \neq \varnothing$ (i.e., for which $H^0(X, \mathcal{O}_X(F(mZ)_{mt})) = [I(mZ)]_{mt} \neq (0)$).

A quantity of interest with respect to $I(Z)$ is the least integer $t \geq 0$ such that $(0) \subsetneq [I(Z)]_t$. This least t is denoted $\alpha(I(Z))$; thus $\alpha(I(Z))$ is the least degree t such that there is a form $F \neq 0$ of degree t in $I(Z)$. The Waldschmidt constant $\widehat{\alpha}(I(Z))$ for $I(Z)$ is an asymptotic version of $\alpha(I(Z))$:

$$\widehat{\alpha}(I(Z)) = \inf\left\{\frac{\alpha(I(mZ))}{m} : m \geq 1\right\}.$$

This infimum is actually a limit [19, Lemma 2.3.1] (also see [69, Example 1.3.4]), so

$$\widehat{\alpha}(I(Z)) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m}.$$

On the one hand, if $d > \widehat{\alpha}(I(Z))$, then for some $s > 0$ we have $[I(sZ)]_{sd} \neq (0)$, so $|sF| \neq \varnothing$ and hence F is semi-effective. On the other, if $d < \widehat{\alpha}(I(Z))$, then for all $s > 0$ we have $[I(sZ)]_{sd} = (0)$, so F is not semi-effective. However, it is unclear how to tell whether or not F is semi-effective when $d = \widehat{\alpha}(I(Z))$, so computing $\widehat{\alpha}(I(Z))$ is not a complete solution when $\widehat{\alpha}(I(Z))$ turns out to be an integer. There is also the issue of how does one compute $\widehat{\alpha}(I(Z))$.

Although integer values of $\widehat{\alpha}(I(Z))$ occur, in most cases $\widehat{\alpha}(I(Z))$ is in fact not an integer. It is believed that examples of Z exist for which $\widehat{\alpha}(I(Z))$ is not even rational, but no examples for which an irrational value can be proved are currently known.

Problem 2.2 Confirm that fat point schemes Z occur for which $\widehat{\alpha}(I(Z))$ is not rational.

When $Z = p_1 + \cdots + p_r \subset \mathbb{P}^2$ with the points p_i being generic and $r > 9$ not a square, it is (in different words) a still open conjecture of M. Nagata that $\widehat{\alpha}(I(Z)) = \sqrt{r}$ [90]. At the same time that he made the conjecture, Nagata showed that $F(Z)_{\sqrt{r}}$ is not semi-effective when $r = s^2$ is a square bigger than 9. In this case we have $\widehat{\alpha}(I(Z)) = \sqrt{r} = s$.

A complete solution to Problem 2.1 can be given in the special case that $n = 2$ when the points p_i are generic, assuming the SHGH Conjecture [60, 71, 80, 101]. Given any divisor class F on X when $n = 2$, without assuming any conjectures, there is a routine [72] based on quadratic Cremona transformations which results in either a nef divisor G with $G \cdot F < 0$ (and hence F is not semi-effective) or a reduction to the case that F is a nonnegative integer linear combination of the classes $H, H-E_1, 2H-E_1-E_2, 3H-E_1-E_2-E_3, \dots, 3H-E_1-\cdots-E_r$. In the latter case, one version of the SHGH Conjecture is that $h^0(X, \mathcal{O}_X(F)) = \max(0, (F^2 - F \cdot K_X)/2 + 1)$, where $-K_X = 3H - E_1 - \cdots - E_r$. It is not hard to see that a nonnegative integer linear combination F of the classes H, $H-E_1, 2H-E_1-E_2$, $3H-E_1-E_2-E_3, \dots, 3H-E_1-\cdots-E_r$ satisfies $((mF)^2 - mF \cdot K_X)/2 + 1 > 0$ for some $m > 0$ if and only if either $F^2 > 0$, or $F^2 = 0$ and $-F \cdot K_X \geq 0$.

Thus a complete solution to Problem 2.1 in the case of $n = 2$ and generic points $p_i \in \mathbb{P}^2$ reduces to determining when a nonnegative integer linear combination F of the classes $H, H-E_1, 2H-E_1-E_2, 3H-E_1-E_2-E_3, \dots, 3H-E_1-\cdots-E_r$ is semi-effective. If either $F^2 > 0$, or $F^2 = 0$ and $-F \cdot K_X \geq 0$, then F is semi-effective; the converse is true if the SHGH Conjecture is true.

Much less can be said in general about when $|F|$ is nonempty for $F = dH - m_1E_1 - \cdots - m_rE_r$ in the case of general fat points in $\mathbb{P}^n$ for $n > 2$. Conjectures have been given for $n = 3$ ([86, 87]). For $n > 3$ conjectures are lacking, but work has been done (as a starting point see, for example, [16, 21, 83, 92] and the references therein).

2.2 Computing and Bounding Waldschmidt Constants

Given a fat point subscheme $Z \subset \mathbb{P}^n$, no algorithm is known for computing $\widehat{\alpha}(I(Z))$ in general, but bounds are known which allow one, in principle, to compute $\widehat{\alpha}(I(Z))$ arbitrarily accurately, and these bounds have given rise to additional open problems.

One bound is obvious from the definition, namely

$$\widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(mZ))}{m}$$

holds for all $m \geq 1$. Since $\widehat{\alpha}(I(Z))$ is a limit, this bound approaches $\widehat{\alpha}(I(Z))$ from above as m grows.

A fundamental lower bound due to M. Waldschmidt and H. Skoda [102, 110] is

$$\frac{\alpha(I(Z))}{n} \leq \widehat{\alpha}(I(Z)).$$

While the original proof was quite hard, this is an easy consequence of the containment result $I(mnZ) \subseteq I(Z)^m$ of [51, 81]. (The original statement was in terms of symbolic powers, so $I(Z)^{(s)} \subseteq I(Z)^m$ for all $s \geq nm$, but in the context of an ideal $I(Z)$ of fat points, we can take $I(sZ)$ as the definition of the sth symbolic power $I(Z)^{(s)}$ of $I(Z)$.) Indeed, $I(mnZ) \subseteq I(Z)^m$ implies $\alpha(I(mnZ)) \geq \alpha(I(Z)^m) = m\alpha(I(Z))$, so $\alpha(I(mnZ))/(mn) \geq \alpha(I(Z))/n$, and taking limits gives the result.

A refinement of the Waldschmidt-Skoda lower bound [111, Lemme 7.5.2] is

$$\frac{\alpha(I(mZ))}{m+n-1} \leq \widehat{\alpha}(I(Z)).$$

This follows (see [74]) by a similar argument from the containment $I(s(m+n-1)Z) \subseteq I(mZ)^s$ [51, 81].

Given the points p_i, $\alpha(Z)$ is in principle computable for every Z, and since both $\frac{\alpha(I(mZ))}{m+n-1}$ and $\frac{\alpha(I(mZ))}{m}$ converge to $\widehat{\alpha}(I(Z))$ as $m \to \infty$, we see that we can in principle determine by computation semi-effectivity of $F(Z)_t$ whenever $t \neq \widehat{\alpha}(I(Z))$; viz., $F(Z)_t$ is semi-effective if $t > \widehat{\alpha}(I(Z))$ and $F(Z)_t$ is not semi-effective if $t < \widehat{\alpha}(I(Z))$. If $t \neq \widehat{\alpha}(I(Z))$, one can confirm this by computing $\frac{\alpha(I(mZ))}{m+n-1}$ and $\frac{\alpha(I(mZ))}{m}$ for sufficiently large m, but when $t = \widehat{\alpha}(I(Z))$, one will never be sure whether $t = \widehat{\alpha}(I(Z))$ or whether $t \neq \widehat{\alpha}(I(Z))$ and one just has not yet checked for large enough values of m.

A further refinement of the Waldschmidt-Skoda lower bound was conjectured by Chudnovsky [26], which he proved for $n = 2$ but which remains open in general:

Conjecture 2.3 (Chudnovsky) Let $Z \subset \mathbb{P}^n$ be a fat point subscheme. Then

$$\frac{\alpha(I(Z)) + n - 1}{n} \leq \widehat{\alpha}(I(Z)).$$

Demailly [36, p. 101] extended this to the following still open conjecture:

Conjecture 2.4 (Demailly) Let $Z \subset \mathbb{P}^n$ be a fat point subscheme. Then

$$\frac{\alpha(I(mZ)) + n - 1}{m + n - 1} \leq \widehat{\alpha}(I(Z)).$$

Both, Conjectures 2.3 and 2.4, would follow if certain containments were true. This will be discussed in more detail in §3. See [6, 13, 17], [14, 23, 24, 30, 44, 50, 54, 59, 67, 88, 92, 108] for selected articles exhibiting partial progress on these conjectures and on work on computing Waldschmidt constants of ideals.

2.3 Index of Semi-effectivity

Semi-effectivity raises additional questions of current interest; see, for example, [11]. Let $Z \subset \mathbb{P}^n$ be a fat point scheme and d an integer such that $F(Z)_d$ is semi-effective. Call the least integer $m > 0$ such that $|m(F(Z)_d)| = |F(mZ)_{md}| \neq \varnothing$ the *semi-effectivity index* of $F(Z)_d$.

When $n = 2$, one can show the semi-effectivity index can be arbitrarily large.

Example 2.5 Let $s > 3$ be an integer and let $r = s^2$ and $t = s+1$. Let $p_1, \ldots, p_r$ be general points in $\mathbb{P}^2$ and let $Z = p_1 + \cdots + p_r$. Then it is known that $|F(mZ)_{mt}| \neq \varnothing$ if and only if $(F(mZ)_{mt})^2 - F(mZ)_{mt} \cdot K_X \geq 0$ [28, 55, 96]; i.e., if and only if $m^2(2s+1) + m(3s+3-s^2) \geq 0$. When $s \geq 4$, the least $m \geq 1$ for which this occurs is $m = \lfloor (s/2) - 1 \rfloor$. Thus the semi-effectivity index for $F(Z)_t$ is approximately $s/2$.

Example 2.6 Assuming the SHGH Conjecture, one can even find $F(Z)_t$ with arbitrarily large semi-effectivity index for any fixed $r > 9$ [27].

Example 2.7 In Examples 2.5 and 2.6 we have $(F(Z)_t)^2 > 0$. For an example of $F(Z)_t$ with semi-effectivity index 2 such that $2(F(Z)_t)$ is the class of a reduced irreducible curve of negative intersection (see [11, Example 3.2]), take the image C of a general map of $\mathbb{P}^1 \to \mathbb{P}^2$ with image of degree $2d$ with $d \geq 3$. Then C has $\binom{2d-1}{2}$ nodes. Take Z to be the set of nodes with each nodal point taken with multiplicity 1. Then $F(Z)_d$ is not the class of an effective divisor because $2(F(Z)_d)$ is the class of the proper transform C' of C, and C' has negative self-intersection. For $d = 3$ we get a sextic with 10 nodes; see [82, p. 26, p. 51] for a specific such curve (see [11, Example 3.1]).

The following problem seems to be open:

Problem 2.8 Determine whether there exists a fat point subscheme $Z \subset \mathbb{P}^2$ and an integer t such that $F(Z)_t$ has semi-effectivity index $m > 2$ where $m(F(Z)_t)$ is the class of a reduced irreducible curve of negative self-intersection.

Line arrangements give examples $F(Z)_t$ with various values of the semi-effectivity index m and various negativity conditions but in these cases $m(F(Z)_t)$ is the class of a reducible curve.

Example 2.9 Let $C \subset \mathbb{P}^2$ be the union of $s > 2$ general lines for s even, so $s = 2a$. Let $Z = p_1 + \cdots + p_r$, $r = \binom{s}{2}$, be the singular points of C. Note that each of the s lines goes through $s - 1$ of the points, and each point p_i is on exactly 2 of the lines. Let C' be the proper transform of C on the blow up $X \to \mathbb{P}^2$ of the points p_i. Let $L_1, \ldots, L_s$ be the s lines and let L'_j be the proper transform of L_j, so $C' = L'_1 + \cdots + L'_s$. Then $C' \cdot L'_j = s - 2(s-1) = 2 - s < 0$ and $2(F(Z)_a) = [C']$. For $i > 0$, $i(F(Z)_a) \cdot L' < 0$ so if $i(F(Z)_a)$ is the class of an effective divisor, then each of L'_j must be a component of $i(F(Z)_a)$, hence $i(F(Z)_a) - [C'] = (i-2)(F(Z)_a)$ is the class of an effective divisor, so $i \geq 2$. Thus $F(Z)_a$ has semi-effective index 2. Note that the intersection matrix $(L'_i \cdot L'_j)$ of C' is diagonal (since the components L'_j are disjoint) and thus $(L'_i \cdot L'_j)$ is negative definite (since $(L'_j)^2 = 2 - s < 0$).

Example 2.10 ([11, Example 3.4]) Now assume $\mathbb{F} \subset \mathbb{K}$ is a finite field with q elements. Then $\mathbb{P}^2_{\mathbb{F}}$ has $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Let p be any of the points. Let Z be all of the points but p, so $|Z| = q^2 + q$. There are q^2 lines that do not contain p. Together they give a curve C of degree q^2 whose singular locus is Z, and the multiplicity of C at each point of Z is q. Thus $q(F(Z)_q)$ is the class of the proper transform C' of C with respect to the blow up $X \to \mathbb{P}^2$ of the points of Z. If L' is the proper transform of any of the q^2 lines comprising the components of C, then $C' \cdot L' = (L')^2 = 1 - (q+1) = -q$ so, arguing as in Example 2.9, we see $F(Z)_q$ has semi-effectivity index q and C' has a negative definite intersection matrix.

Examples of complex line arrangements giving rise to classes $F(Z)_t$ with semi-effectivity index $m > 2$ (in particular $m = 3$) such that $m(F(Z)_t)$ is the class of a reduced curve with negative definite intersection matrix are especially interesting.

Example 2.11 ([11, Example 3.3]) Consider the curve C defined in $\mathbb{P}^2$ over the complex numbers by $(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$ for $n = 3s$ for $s \geq 1$. Then C consists of $3n$ lines, with n^2 triple points and 3 points of multiplicity n (i.e., $t_3 = 12$ but $t_2 = 0$ if $n = 3$, and $t_3 = n^2$ and $t_n = 3$, but $t_2 = 0$ if $n > 3$). Take Z to be the singular points where the triple points are taken with multiplicity 1 and the 3 points of multiplicity n are taken with multiplicity s. Take X to be $\mathbb{P}^2$ blown up at the points of Z. Then $3(F(Z)_s)$ is the class of the proper transform C' of C; C' consists of the disjoint union of the proper transforms of the components of C (each of which is a line, and hence goes through $n + 1$ of the points of Z and thus the proper transform of each line has self-intersection $-n$). Thus $i(F(Z)_s)$ is not the class of an effective divisor for $0 < i < 3$, so the semi-effectivity index of $F(Z)_s$ is 3. The intersection matrix of C' is negative definite since it is the disjoint union of the proper transforms of the lines, which have negative self-intersection.

Two more related examples can be given, but in neither case is the intersection matrix of $m(F(Z)_t)$ negative definite and in the second case $m(F(Z)_t)$ is not reduced. They come from examples of complex line arrangements having no simple crossings (i.e., any point where two lines cross is also on at least one additional line, so $t_2 = 0$).

Example 2.12 There is a curve C due to F. Klein [8, 9, 85] consisting of 21 lines whose singular locus consists of 21 quadruple points and 28 triple points (i.e., $t_3 = 28$ and $t_4 = 21$; in particular, there are no double points so $t_2 = 0$). Each of the 21 lines goes through 4 of the triple points and 4 of the quadruple points. Let Z be the 49 singular points and let $X \to \mathbb{P}^2$ be the blow up of the points of Z. Let $E_{3,i}$, $1 \leq i \leq 28$, be the exceptional curves of the 28 triple points, and let $E_{4,i}$, $1 \leq i \leq 21$, be the exceptional curves of the 21 quadruple points. If L' is the proper transform of any of the 21 lines, then $i(F(Z)_7) \cdot L' = -i$, so if $i(F(Z)_7)$ is the class of an effective divisor, then L' is a component of the effective divisor. Since $3(F(Z)_7)$ is the class of $C' + E_{4,1} + \cdots + E_{4,21}$, where C' is the proper transform of C and thus consists of the proper transforms of the 21 lines, we see that $F(Z)_7$ has semi-effectivity index 3. The intersection matrix is not negative definite since $C' + 4(E_{4,1} + \cdots + E_{4,21})$ has positive self-intersection.

Example 2.13 There a curve C due to A. Wiman [8, 9, 112] consisting of 45 lines whose singular locus consists of 36 quintuple points, 45 quadruple points and 120 triple points (i.e., $t_3 = 120$, $t_4 = 45$ and $t_5 = 36$; there are no double points so $t_2 = 0$). Each of the 45 lines goes through 8 of the triple points, 4 of the quadruple points and 4 of the quintuple points. Let Z be the 201 singular points and let $X \to \mathbb{P}^2$ be the blow up of the points of Z. Let $E_{3,i}$, $1 \leq i \leq 120$, be the exceptional curves of the 120 triple points, let $E_{4,i}$, $1 \leq i \leq 45$, be the exceptional curves of the 45 quadruple points and let $E_{5,i}$, $1 \leq i \leq 36$, be the exceptional curves of the 36 quadruple points. If L' is the proper transform of any of the 45 lines, then $i(F(Z)_{15}) \cdot L' = -i$, so if $i(F(Z)_{15})$ is the class of an effective divisor, then L' is a component of the effective divisor. Since $3(F(Z)_{15})$ is the class of $C' + E_{4,1} + \cdots + E_{4,45} + 2(E_{5,1} + \cdots + E_{5,36})$, where C' is the proper transform of C and thus consists of the proper transforms of the 45 lines, we see that $F(Z)_{15}$ has semi-effectivity index 3, but $3(F(Z)_{15})$ is not the class of a reduced divisor. The intersection matrix is not negative definite since $C' + 4(E_{4,1} + \cdots + E_{4,45}) + 5(E_{5,1} + \cdots + E_{5,36})$ has positive self-intersection.

The preceding three examples come from complex line arrangements having no simple crossings (i.e., $t_2 = 0$). It is interesting to ask what other such line arrangements there are. There is of course the trivial arrangement, consisting of 3 or more lines through a point. It is well known that $t_2 > 0$ for any nontrivial arrangement of lines over the reals [8]. Over the complex numbers, the only nontrivial examples currently known are those of Examples 2.11, 2.12 and 2.13; no examples have been found since the example of Klein in 1879 and of Wiman in 1896. This leads to an open problem relevant also to some of the topics that will be discussed later.

Problem 2.14 Classify all reduced curves in the complex projective plane which are unions of 2 or more lines but which have no singular points of multiplicity 2 (i.e., such that $t_2 = 0$).

Examples $Z \subset \mathbb{P}^2$ where $F(Z)_t$ can have arbitrarily large semi-effectivity index m even though $m(F(Z)_t)$ has negative definite intersection matrix can be given but in these cases some of the points of Z are infinitely near and $m(F(Z)_t)$ is the class of a non-reduced curve [11, Example 3.5].

Note that the surfaces X on which these negative definite examples live are not all the same. These examples obtain larger and larger semi-effectivity indices by blowing up more and more points of $\mathbb{P}^2$. This raises the following question, which is open regardless of the characteristic of $\mathbb{K}$:

Question 2.15 Can one blow up a fixed number of points of $\mathbb{P}^2$, possibly infinitely near, to obtain a surface X having a sequence $F_1, F_2, F_3, \ldots$ of classes of increasing semi-effective index m_i such that each $m_i F_i$ has negative definite intersection matrix?

If we require only that $F_i^2 < 0$ and not that the intersection matrix of $m_i F_i$ be negative definite, then Question 2.15 has an affirmative answer if the SHGH Conjecture is true (see [11, Example 1.2])

2.4 Bounded Negativity Conjecture

If Question 2.15 were to have an affirmative answer, then we would have a sequence of classes $m_i F_i$ of effective divisors C_i with $0 > (m_1 F_1)^2 > (m_2 F_2)^2 > (m_3 F_3)^2 > \ldots$.

Such an example would be interesting in terms of semi-effectivity indices, but it would be even more interesting if the curves C_i were all reduced, or reduced and irreducible. A conjecture relevant to this is known as the Bounded Negativity Conjecture. This is a still open folklore conjecture. It is not known who first proposed it, but it has an oral tradition that goes back at least to F. Enriques [10]. We state two versions:

Conjecture 2.16 (Bounded Negativity Conjecture) Let X be any smooth complex projective rational surface. Then there is a bound B_X such that for every reduced curve $C \subset X$ we have $C^2 \geq B_X$.

A second equivalent [10, Proposition 5.1] version is:

Conjecture 2.17 (Bounded Negativity Conjecture) Let X be any smooth complex projective rational surface. Then there is a bound b_X such that for every reduced irreducible curve $C \subset X$ we have $C^2 \geq b_X$.

Remark 2.18 Examples of nonrational surfaces in positive characteristics have been known for some time for which the bounds B_X and b_X do not exist; see [79, Exercise

V.1.10]. It is now known that such failure of bounded negativity is a very general positive characteristic phenomenon: in positive characteristic, after blowing up an appropriate finite set of points, every smooth projective surface has unbounded negativity [25]. Previous to [25], the main fact known was that if X is a smooth projective surface in any characteristic such that $-K_X$ is semi-effective, then it follows by adjunction that bounded negativity holds for X. Indeed, given a reduced, irreducible curve $C \subset X$, we then have $C^2 = 2g_C - 2 - C \cdot K_X$, so $C^2 \geq -2$ unless C is a component of $-mK_X$, where m is the index of semi-effectivity of $-K_X$, but $-C \cdot mK_X$ is bounded below since $|-mK_X|$ has only finitely many fixed components.

Remark 2.19 If X is obtained by blowing up r distinct points of $\mathbb{P}^2$, and if b_X is a lower bound for C^2 for reduced irreducible curves C, then we can take $B_X = rb_X$ [10, Proposition 5.1]. Here we note that if the points are generic, the SHGH Conjecture implies that $b_X = -1$, hence $B_X = -r$ is optimal (take C to be the union of the exceptional curves of the points blown up). If X is obtained by blowing up $r \leq 9$ distinct points of $\mathbb{P}^2$, using $C^2 + C \cdot K_X = 2g_C - 2$ and the fact that $-K_X$ is the class of an effective divisor, it follows that we can take $b_X = -8$. (We get $C^2 \geq -2$ unless C is a component of $-K_X$, in which case we have $C^2 \geq -8$, where $C^2 = -8$ comes from taking C to be the proper transform of the line through 9 collinear points.)

The concept of H-constants was introduced to explore the Bounded Negativity Conjectures (see for example [8, 48, 93, 103]). Given a reduced singular curve $C \subset \mathbb{P}^2$, let $S = \{p_1, \ldots, p_r\}$ be the set of singular points of C, let m_i be the multiplicity of p_i and let d be the degree of C.

Define

$$H(C) = \frac{d^2 - \sum m_i^2}{r} = \frac{(C')^2}{r},$$

where X is the surface obtained by blowing up the points p_i and $C' \subset X$ is the proper transform of C.

Example 2.20 If C is the image of $\mathbb{P}^1$ in $\mathbb{P}^2$ under a general map of degree $d > 2$ (and hence as discussed above has $\binom{d-1}{2}$ singular points, all nodes), then

$$H(C) = \frac{d^2 - 4\binom{d-1}{2}}{\binom{d-1}{2}} = -2 + \frac{6d - 4}{d^2 - 3d + 2}.$$

The results of [25] give examples in positive characteristic of irreducible reduced singular curves $C \subset \mathbb{P}^2$ with $H(C) \leq -2$, but no examples are currently known in characteristic 0.

Question 2.21 Does there exist an irreducible reduced singular complex curve $C \subset \mathbb{P}^2$ with $H(C) \leq -2$?

Remark 2.22 If the answer is negative, then for the blow up X of $\mathbb{P}^2$ at r distinct points we would have the bounds $b_X = -2r$ and $B_X = -2r^2$.

At the extreme opposite to irreducible curves we can consider $H(C)$ for totally reducible curves C; i.e., reduced curves C which are unions of lines. Given a curve C consisting of $d \geq 2$ lines, so for $k > 1$, t_k is the number of points which are on exactly k lines. Then the number of singular points of C is $\sum_{k\geq 2} t_k$ and an easy counting argument gives $\binom{d}{2} = \sum_{k\geq 2} \binom{t_k}{2}$. Using this we obtain

$$H(C) = \frac{d - \sum_{k\geq 2} kt_k}{\sum_{k\geq 2} t_k}.$$

Example 2.23 ([8]) If the lines of which C is composed are defined over the reals, then one can show $H(C) > -3$. If we denote by C_r the union of the lines determined by the edges of a regular r-gon together with the lines of bilateral symmetry of the r-gon, then

$$\lim_{r\to\infty} H(C_r) = -3.$$

If the lines of which C is composed are defined over the complex numbers, then one can show $H(C) > -4$. For such a C, the most negative value for $H(C)$ currently known is for the arrangement of 45 lines discussed in Example 2.13, which gives $H(C) = \frac{-225}{67}$ or about -3.358. Part of what makes $H(C)$ as negative as it is, is the fact that $t_2 = 0$. Thus Problem 2.14 is relevant here. If there were other line arrangements with $t_2 = 0$ than those currently known, they might give more negative values for the H-constant than Wiman's curve gives. (The curves in Example 2.11 have $H > -3$ but approach -3 in the limit, while the curve in Example 2.12 has $H = -3$.)

Remark 2.24 If one considers reduced but not irreducible complex curves which are not unions of lines it is possible to get values of H-constants less than the $H(C) = \frac{-225}{67}$ given by the Wiman line arrangement, but they are rare. For example, [94] gives an example with $H(C) = -1173/347 \approx -3.38$, while [95] gives examples of curves C with ordinary singularities where $H(C)$ is arbitrarily close to but bigger than $-25/7 \approx -3.571$ (using an approach similar to that used later by [25]). The most negative values currently known have $H(C)$ arbitrarily close to but bigger than -4 (see [12, 97, 98]). No examples are yet known with $H(C) \leq -4$.

This raises the following question:

Question 2.25 Does there exist a reduced singular curve $C \subset \mathbb{P}^2$ over the complex numbers with $H(C) \leq -4$?

If the answer is negative, then for the blow up X of $\mathbb{P}^2$ at r distinct points over the complex numbers we would have the bound $B_X = -4r$.

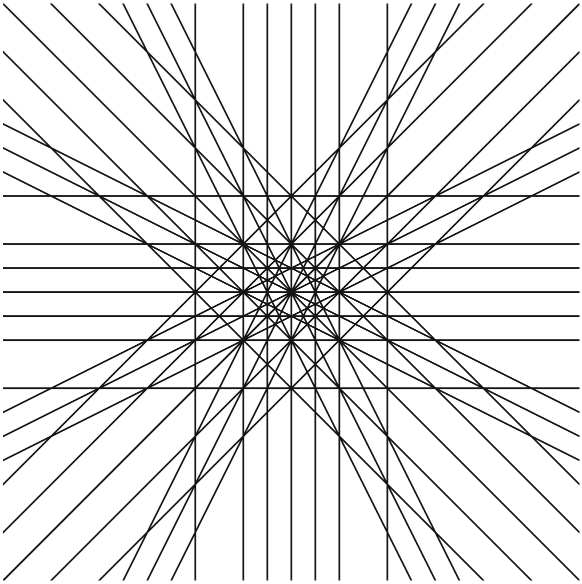

Fig. 1 Grünbaum's rational simplicial configuration $\mathcal{A}(37, 3)$ of 37 lines (only 36 are shown; the 37th is the line at infinity)

Here is another open problem:

Problem 2.26 Determine the infimum of $H(C)$ when C is a union of lines defined over the rationals.

Example 2.27 The most negative value of $H(C)$ currently known when C is a union of lines defined over the rationals is $H(C) = -503/181 \approx -2.779$, where C consists of 37 lines with $t_2 = 72, t_3 = 72, t_4 = 24, t_6 = 10$ and $t_8 = 3$. This arrangement (shown in Fig. 1) is denoted $\mathcal{A}(37, 3)$ in Grünbaum's list of real simplicial arrangements of lines [65]. (A real arrangement of lines being *simplicial* means that the lines give a simplicial decomposition of the real projective plane. Simplicial arrangements are hard to find. Only three infinite families and 42 sporadic examples are currently known, and none of the sporadic examples have more than 37 lines. The infinite families are: the arrangements where all but one of the lines go through a single point; the arrangements described above corresponding to the sides and lines of symmetry of regular n-gons for odd n; and the arrangements corresponding to the sides and lines of symmetry of regular n-gons for even n with the addition of the line at infinity. Only 37 of the 42 known sporadic cases are given in [65]. Four more were found in 2011 [32] and another in 2020 [33].)

Remark 2.28 If $\mathbb{K}$ is a finite field of q elements, then $\mathbb{P}^2$ has $q^2 + q + 1$ lines and $q^2 + q + 1$ points. If we take C to be the union of all of the $q^2 + q + 1$ lines, then the singular set of C consists of all of the $q^2 + q + 1$ points, and each point has multiplicity $q + 1$, so we get $H(C) = -q$. Thus over an algebraically closed field $\mathbb{K}$ of positive characteristic p, there are line arrangements C where $H(C)$ can be arbitrarily negative.

3 Containment Problems

The Containment Problem concerns studying containments of symbolic powers of an ideal in their ordinary powers. For a prime ideal P in a Noetherian commutative ring R, the mth symbolic power $P^{(m)}$ is just the P-primary component of the primary decomposition of P^m. For a fat point subscheme $Z \subset \mathbb{P}^n$ we have $R = \mathbb{K}[\mathbb{P}^n]$, a polynomial ring, and we recall the mth symbolic power $(I(Z))^{(m)}$ can be defined most easily as $(I(Z))^{(m)} = I(mZ)$.

Related to Wiles proof of the Fermat Conjecture, Eisenbud and Mazur proposed the following still open conjecture ([52]; see [35] for discussion):

Conjecture 3.1 (Eisenbud and Mazur) Let $\mathbb{K}$ be the field of complex numbers and let $P \subset R = \mathbb{K}[[x_0, \dots, x_n]]$ be a prime ideal. Then $P^{(2)} \subseteq MP$, where $M = (x_0, \dots, x_n)$.

3.1 Related Containment Problems

A natural thing to do is to replace M by P, $\mathbb{K}$ by any field, and R by $\mathbb{K}[x_0, \dots, x_n]$. Examples show that $P^{(2)} \subseteq P^2$ can fail, so one needs to make $P^{(2)}$ a little smaller to have a reasonable problem; how much smaller depends on the height. This line of thinking is suggestive of a version of a still open problem proposed around 2000 by Huneke (see [35, Question 2.21]):

Question 3.2 (Huneke) Does there exist a prime ideal $P \subset \mathbb{K}[x_0, \dots, x_n]$ of height 2 such that the containment $P^{(3)} \subseteq P^2$ fails?

About the same time Huneke also asked:

Question 3.3 (Huneke) Does there exist a finite set of points $Z = p_1 + \cdots + p_r \subset \mathbb{P}^2$ such that the containment $I(3Z) \subseteq I(Z)^2$ fails?

Note by [51, 81] that $P^{(4)} \subseteq P^2$ always holds for height 2 primes (so it is compelling to ask if $P^{(3)} \subseteq P^2$ also holds), and that we always have $I(mZ) \subseteq I(Z)^r$ when $m \geq rn$ for a fat point subscheme $Z \subset \mathbb{P}^n$. The bound $m \geq rn$ is optimal in the sense that if $c < n$, then there are Z and m and r with $m \geq rc$ for which $I(mZ) \subseteq I(Z)^r$ fails [19].

However, there are other senses in which the bound $m \geq rn$ for ensuring $I(mZ) \subseteq I(Z)^r$ might not be optimal. For example, perhaps there is a c depending only on n such that $m \geq rn - c$ implies $I(mZ) \subseteq I(Z)^r$. It is easy to find examples of Z for which $I((nr - n)Z) \subseteq I(Z)^r$ fails, so the author proposed the following conjecture [7]:

Conjecture 3.4 Let $Z \subset \mathbb{P}^n$ be a fat point subscheme. Then $I((nr - n + 1)Z) \subseteq I(Z)^r$ holds for all $r > 0$.

The first failure found is for $n = r = 2$ and thus also answers Question 3.3 negatively. In this case Z is the set of singular points of the curve $(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$ in the complex projective plane [49]. In fact, the reduced scheme Z of singular points for any of the known nontrivial examples of complex curves given by unions of lines with $t_2 = 0$ (see the discussion after Problem 2.14) give failures of $I(3Z) \subseteq I(Z)^2$. (This is another reason why Problem 2.14 is of interest.)

Given a failure of containment, one can construct more using flat extensions [2]. Failures of $I(3Z) \subseteq I(Z)^2$ for a fat point subscheme $Z \subset \mathbb{P}^2$ occur also in positive characteristics [18], over the reals [34] and even over the rationals [46]. Some failures of containment are related to both line arrangements and hyperplane arrangements arising from complex reflection groups; see [43]. Failures of $I((nr - n + 1)Z) \subseteq I(Z)^r$ for various r and n occur in positive characteristics [78] but no examples over the complex numbers are known when $n > 2$ or $r > 2$. The next simplest open case is $n = 2$ and $r = 3$. Thus we have the following open problem:

Problem 3.5 Let $Z \subset \mathbb{P}^2$ be a fat point subscheme over the complex numbers. Must $I(5Z) \subseteq I(Z)^3$ hold?

In fact, regardless of $\mathbb{K}$, in all cases checked $I((nr - n + 1)Z) \subseteq I(Z)^r$ holds for $r \gg 0$. This suggests the following conjecture (it is a version of a conjecture of Grifo [63]):

Conjecture 3.6 (Grifo) Let $Z \subset \mathbb{P}^n$ be a fat point subscheme. Then $I((nr - n + 1)Z) \subseteq I(Z)^r$ holds for all $r \gg 0$.

Note that the difference between $I(mZ) \subseteq I(Z)^r$ with $m = nr$ versus $m = nr - n + 1$ is that in the second case m is smaller, and thus $I(mZ)$ is bigger. Another approach to the optimality of $I(nrZ) \subseteq I(Z)^r$ is instead of replacing $I(nrZ)$ by something bigger, one might replace $I(Z)^r$ by something smaller. This is one of the approaches taken in [74], which proposed the following still open conjecture (and which has a flavor similar to Conjecture 3.1):

Conjecture 3.7 (Harbourne and Huneke) Let $Z \subset \mathbb{P}^n$ be a fat point subscheme and let $M = (x_0, \ldots, x_n)$, where $\mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \ldots, x_n]$. Then $I(nrZ) \subseteq M^{(n-1)r} I(Z)^r$ holds for all $r > 0$.

If this conjecture is true, then so is Conjecture 2.3, using the same argument as used near the beginning of §2.2 to prove the Waldschmidt-Skoda bound; see [74], which also asked:

Question 3.8 (Harbourne and Huneke) Let $Z \subset \mathbb{P}^n$ be a fat point subscheme and let $M = (x_0, \ldots, x_n)$, where $\mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \ldots, x_n]$. Does $I(r(m + n - 1)Z) \subseteq M^{(n-1)r} I(mZ)^r$ hold for all $r > 0$?

In the same way, if this conjecture is true, then so is Conjecture 2.4 (see [74]).

To study containments $I(mZ) \subseteq I(Z)^r$, the notion of the *resurgence* was introduced by [19]. Given a fat point subscheme $Z \subset \mathbb{P}^n$, define the resurgence $\rho(I(Z))$ as

$$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq I(Z)^r\right\}.$$

Subsequently an asymptotic version was introduced [66]:

$$\widehat{\rho}(I(Z)) = \sup\left\{\frac{m}{r} : I(mtZ) \not\subseteq I(Z)^{rt}, t \gg 0\right\}.$$

(A second version of asymptotic resurgence was also introduced in [66], but by [15] it is the same as this one.)

Although values of resurgences have been computed in some special cases (see, for example, [19, 23, 37, 46, 66, 84]), it is typically quite hard to compute $\rho(I(Z))$. However, by [19, 66] we have

$$1 \le \frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} \le \widehat{\rho}(I(Z)) \le \rho(I(Z)) \le \min\left\{n, \frac{\operatorname{reg}(I(Z))}{\widehat{\alpha}(I(Z))}\right\},$$

where $\operatorname{reg}(I(Z))$ is the Castelnuovo-Mumford regularity.

Examples are known with $\widehat{\rho}(I(Z)) < \rho(I(Z))$ ([46]; see also [41, Corollary 4.14] and [42]) and with $\widehat{\rho}(I(Z)) = 1$ (see [20] and [41, Corollary 4.16]), but no examples are known with $1 = \widehat{\rho}(I(Z)) < \rho(I(Z))$ (cf. [41, Corollary 4.17]) or with either $\widehat{\rho}(I(Z)) = n$ or $\rho(I(Z)) = n$ for $n > 1$. This raises the following open questions (see [75]):

Question 3.9 Does there exist a fat point subscheme $Z \subset \mathbb{P}^n$ with $1 = \widehat{\rho}(I(Z)) < \rho(I(Z))$?

Question 3.10 Does there exist a fat point subscheme $Z \subset \mathbb{P}^n$ with $\widehat{\rho}(I(Z)) = n$ or $\rho(I(Z)) = n$ for $n > 1$?

In relation to Question 3.10, if $\widehat{\rho}(I(Z)) < n$ (and hence also if $\rho(I(Z)) < n$), then Grifo's Conjecture holds for Z (see [63, Remark 2.7] and [64, Proposition 2.11]).

There is some evidence that the answer to Question 3.9 is also negative. In particular, using a version of the Briançon-Skoda Theorem, one can show that $\widehat{\rho}(I(Z)) = 1$ implies $\rho(I(Z)) = 1$ for fat point subschemes $Z \subset \mathbb{P}^2$ [75, Corollary 2.9].

The proof uses ideas involving integral closure (thereby bringing in the Briançon-Skoda Theorem as a tool). In fact, one can show [41] that

$$\widehat{\rho}(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq \overline{I(Z)^r}\right\},$$

where $\overline{I(Z)^r}$ is the integral closure of $I(Z)^r$.

This suggests defining an integral closure version of the resurgence, namely

$$\rho_{int}(I(Z)) = \sup\left\{\frac{m}{r} : \overline{I(Z)^m} \not\subseteq I(Z)^r\right\}.$$

One can then show [75, Corollary 2.10] that $\rho(I(Z)) = 1$ if and only if $\rho_{int}(I(Z)) = \widehat{\rho}(I(Z)) = 1$. But for a fat point subscheme $Z \subset \mathbb{P}^2$ one can show [75, Corollary 2.8] (as a consequence of a version of the Briançon-Skoda Theorem) that $\rho_{int}(I(Z)) = 1$, and hence we obtain for $\mathbb{P}^2$ that $\rho(I(Z)) = 1$ if and only if $\widehat{\rho}(I(Z)) = 1$.

This thus raises the following question [75]:

Question 3.11 Does there exist a fat point subscheme $Z \subset \mathbb{P}^n$ for $n > 2$ with $\rho_{int}(I(Z)) > 1$?

We close this section with another open problem.

Problem 3.12 Let Z be the set of 49 singular points of Klein's curve of degree 21 which is a union of lines with $t_2 = 0$ (see Example 2.12). Compute $\widehat{\rho}(I(Z))$.

For Z as in Problem 3.12, computing $\widehat{\rho}(I(Z))$ is equivalent to computing $\widehat{\alpha}(I(Z))$ (see [9, Theorems 1.1, 1.4]).

4 Splitting Types

A locally free sheaf on a smooth rational curve C splits as a sum of line bundles. Thus a rank 2 bundle V will be isomorphic to $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ for some a and b (the couple (a, b) with $a \leq b$ is called the splitting type of V). This raises the question of what values of a and b can occur, based on some knowledge of V and C.

This was answered by Eisenbud and Van de Van [53] in the case that V is the normal bundle of C where C is a smooth rational curve in $\mathbb{P}^3$ of degree $d \geq 4$ (and $\mathbb{K}$ has characteristic 0). The answer they found was that the splitting type (a, b) satisfied $a + b = 4d - 2$ with $d + 3 \leq a \leq 2d - 1$ and that every such splitting type occurs for some C.

4.1 Ascenzi Curves and the SHGH Conjecture

Ascenzi [4] (a 1985 PhD student of Eisenbud at Brandeis) studied a related problem. Given a reduced irreducible rational curve $C \subset \mathbb{P}^2$ of degree d and the normalization morphism $\pi : C' \to C \subset \mathbb{P}^2$ (so C' is smooth), Ascenzi studied the pullback $\pi^*(T)$ of the tangent bundle T of $\mathbb{P}^2$. For expositional efficiency, it is convenient to express her results in terms of the splitting type (a, b) of $\pi^*(T(-1))$ (where $T(-1) = T \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$). One can show that $a + b = d$ and $0 \leq a, b$ and we may assume $a \le b$. In this situation we will call (a, b) the splitting type of C and also of C'. For the special case that $d = 1$ we have $(a, b) = (0, 1)$. For $d > 1$ we always have $a > 0$.

There are significant applications where what is wanted is to know whether a rational curve $C \subset \mathbb{P}^2$ of degree d_C is *balanced*, meaning the splitting type (a, b) satisfies $0 \le b - a \le 1$. Ascenzi [4] showed a general plane rational curve is balanced. Ascenzi also showed that each pair (a, b) satisfying $a + b = d > 1$ and $0 < a \le d/2$ arises as the splitting type for some C of degree $d_C = d$.

The type is heavily affected by the multiplicities of the singular points of C. For example, if C has a point of multiplicity m, then Ascenzi showed that

$$\min(m, d_C - m) \le a \le \min(d_C - m, d_C/2).$$

Note that we get tighter bounds on a the larger m is. In particular, if m is small, then we get only $m \le a \le d_C/2$, but if m is big enough we get $d_C - m \le a \le d_C - m$ and hence $a = d_C - m$. Thus there is a point where m is big enough to determine a. Indeed, if $2m + 1 = d_C$, then $a = m$ while if $2m + 1 > d$, then $a = d_C - m$. Putting these together we get that if $2m + 1 \ge d_C$, then Ascenzi's bounds imply $a = \min(m, d_C - m)$ and hence $b = \max(m, d_C - m)$.

Thus it makes sense to focus on the maximum multiplicity, hence let m_C be the multiplicity of the point of maximum multiplicity of C. We now make a definition: if $2m_C + 1 \ge d_C$, the curve C is said to be *Ascenzi* (see [61, Definition 1.1]).

If C is Ascenzi, it follows it is balanced if $2m_C \le d_C + 1$, otherwise it is unbalanced. Thus we know which Ascenzi curves are balanced and that a general plane rational curve of degree d is balanced. Note that a general plane rational curve C has $m_C = 2$, and so cannot be Ascenzi if $d_C > 5$. Moreover, we have the following fact.

Proposition 4.1 *The double points of a general plane rational curve C are not themselves general if $d_C \ge 6$.*

Proof The curve C has $\binom{d-1}{2}$ double points. Let $d = d_C$. If the points were general, we could fix $\binom{d-1}{2} - 1$ of the double points and choose different points p and p' for the last one. We thus get two curves, C and C', where both are singular at the $\binom{d-1}{2} - 1$ fixed points but C is singular at p and C' is singular at p'. We get $C \cdot C' \ge 4\binom{d-1}{2} - 4$ and this is more than d^2 when $d > 6$, which implies $C = C'$, a contradiction. Thus the double points cannot be general if $d > 6$. So now say $d = 6$. It is easy to see that there is a unique cubic through 9 general points and it is known that there is a unique sextic through 9 general points of multiplicity 2 (see [72] or [73]). Indeed the sextic is the cubic doubled. In any case, the 10th double point of a sextic with 10 cannot be general. □

This suggests it might be of interest to invert the problem. Instead of studying the singularities of general plane rational curves, we can study plane rational curves whose singular points are general. The splitting type of such curves turns out to be relevant to determining the Betti numbers of ideals $I(Z)$ of fat point subschemes $Z \subset \mathbb{P}^2$ when the points of Z are general.

Let $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^2$ be a fat point subscheme. The minimal free graded resolution of $I(Z) \subset R = \mathbb{K}[\mathbb{P}^2]$ is of the form

$$0 \to F_1 \to F_0 \to I(Z) \to 0,$$

where $F_0 = \oplus_i G_i R$ and $F_1 = \oplus_j S_j R$ are graded R-modules, with $\{G_i\}$ being a minimal set of homogeneous generators for $I(Z)$ and $\{S_j\}$ being a minimal set of homogeneous syzygies. Finding the Betti numbers of $I(Z)$ amounts to finding how many G_i and S_j there are of each degree. Let γ_t be the number of elements G_i there are of degree t, and let σ_t be the number of elements S_j there are of degree t. The Hilbert function of $I(Z)$ is defined to be $H_{I(Z)}(t) = \dim[I(Z)]_t$. Likewise, the Hilbert function of F_i is $H_{F_i}(t) = \dim[F_i]_t$. If we know $H_{I(Z)}$, then finding γ_t amounts to finding the dimension of the cokernel of the multiplication map $\mu_{Z,t}$: $[I(Z)]_t \otimes [R]_1 \to [I(Z)]_{t+1}$ for each $t \geq 0$. Knowing γ_t allows us to compute the Hilbert function of F_0 as $H_{F_0}(t) = \sum_{s \leq t} \gamma_s \binom{t-s+2}{2}$. Then we have $\sigma_t = H_{F_0}(t) - H_{I(Z)}(t)$.

If the points p_i are general, we can, assuming the SHGH Conjecture, obtain $H_{I(Z)}$. Computing the dimension of the cokernel of $\mu_{Z,t}$ in any degree $t > \alpha(Z)$, reduces (assuming the SHGH Conjecture) to the case of $\mu_{iZ',it+1}$, where $F(Z')_t = C$ is a smooth rational curve of self-intersection -1 with $t = \alpha(Z')$ and $1 \leq i \leq t$ (see [62]).

When the points p_i are general, given t it is known exactly which multiplicities m_i give rise to a fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$ such that $F(Z)_t$ is the class of a smooth rational curve C of self-intersection -1 with $t = \alpha(Z)$; they can be enumerated recursively for each t (see [91]). For such a Z, this raises the following problem:

Problem 4.2 ([62, Problem 3.2]) Let the points p_i be general and let $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^2$ be a fat point subscheme such that $F(Z)_t = C$ is the class of a smooth rational curve of self-intersection -1 with $t = \alpha(Z)$. Find the minimal number of homogeneous generators for $I(iZ)$ in degree $s = it + 2$ for $1 \leq i \leq t$.

A still open conjectural answer has been proposed:

Conjecture 4.3 ([62, Conjecture 3.4]) Let the points p_i be general and let $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^2$ be a fat point subscheme such that $F(Z)_t = C$ is the class of a smooth rational curve of self-intersection -1 with $t = \alpha(Z)$. The minimal number of homogeneous generators for $I(iZ)$ in degree $s = it + 2$ for $1 \leq i \leq t$ is $\binom{i-a}{2} + \binom{i-b}{2}$, where (a, b) is the splitting type of C.

This conjecture holds when C is Ascenzi by [62, Theorem 3.3], but when C is not Ascenzi one faces two hurdles: the first is that the conjecture is still mostly open in those cases, and the second is that we do not know the splitting type in those cases.

For the blow up X of $\mathbb{P}^2$ at $r < 9$ generic points, there are only finitely many classes of smooth rational curves C with $C^2 = -1$, and each is Ascenzi, but for $r = 9$ generic points there are infinitely many such C, of which only finitely many are Ascenzi; see [61]. Conjecture 1.7 of [61] proposes a formula for their splitting types (there is not even a conjecture when $r > 9$):

Conjecture 4.4 Let X be the blow up of $\mathbb{P}^2$ at 9 general points. Let H be the pullback to X of the class of a line. Let C be a smooth rational curve on X with $C^2 = -1$, and assume C is the proper transform of a plane curve $\overline{C}$. Let (a, b) be the splitting type of C (so $a+b = C \cdot H$ is the degree of $\overline{C}$). If there is a divisor class A such that $2A = C + K_X + H$, then $b_C - a_C = 2$; otherwise, $0 \leq b_C - a_C \leq 1$.

Recent work of Ascenzi [5] developing a recursive procedure for determining splitting types for non-Ascenzi curves suggests progress on this conjecture is within reach.

4.2 Unexpected Curves

Splitting types associated to line arrangements [38, 56, 89] also arise as an important feature in research on *unexpected curves* [29, 39, 40, 57]. This burgeoning area has taken off in several directions [45, 47, 58, 76, 77, 104–106, 109]. Here we will focus on the original setting [29] of plane curves of degree $d = m + 1$ having a general point p of multiplicity m and containing a reduced subscheme $Z' = p_1 + \cdots + p_r \subset \mathbb{P}^2$ of points, where mp fails to impose independent conditions on $[I(Z')]_d$.

One motivation for the concept of unexpected curves is the SHGH Conjecture. This conjecture proposes an answer to the question of when a fat point subscheme $Z = m_1 p_1 + \cdots + m_r p_r \subset \mathbb{P}^2$ with $[I(Z)]_d \neq (0)$ fails to impose independent conditions on the vector space $[R]_d = [\mathbb{K}[\mathbb{P}^2]]_d$ of forms of degree d. A form F which vanishes at a point p_i to order m_i must satisfy $\binom{m_i+1}{2}$ linear conditions; if $\mathbb{K}$ has characteristic 0, these conditions are that all partial derivatives of order up to $m_i - 1$ must vanish at p_i. Thus $[I(Z)]_d \subset [R]_d$ is the solution set of $\sum_i \binom{m_i+1}{2}$ linear conditions, hence $\dim[I(Z)]_d \geq \max\left(0, \dim[R]_d - \sum_i \binom{m_i+1}{2}\right) = \max\left(0, \binom{d+2}{2} - \sum_i \binom{m_i+1}{2}\right)$. These conditions are independent for $d \gg 0$ (indeed for $d + 1 \geq \sum_i m_i$), but can fail to be independent in general even in cases when $[I(Z)]_d \neq (0)$, and it is of interest to understand when this can happen.

It is hoped that insight into this problem can be gained by making it broader. One way to do that is to consider vector subspaces $V \subset [R]_d$ and fat point subschemes Z such that $[I(Z)]_d \cap V \neq (0)$ yet Z fails to impose impose independent conditions on V. The paper [29] that introduced this approach focused on the case where $V = [I(Z')]_d$ for $d = m + 1$ and $Z' = p_1 + \cdots + p_r \subset \mathbb{P}^2$, with $Z = mp$ for a general point p, hence $[I(Z')]_d \cap V = [I(mp + Z')]_d$.

We say that Z' has *unexpected curves* in degree $d = m + 1$ if for a general point p we have

$$\dim[I(mp + Z')]_d > \max\left(0, \dim[I(Z')]_d - \binom{m+1}{2}\right).$$

One can also define unexpected hypersurfaces in $\mathbb{P}^n$ more generally. The reason for starting with $n = 2$, where $Z' \subset \mathbb{P}^2$ is a finite reduced scheme of points, $V = [I(Z')]_d$, $Z = mp$ and $d = m + 1$, is that it was an example of this kind that first arose and prompted the definition of the concept of unexpectedness, and tools were at hand for trying to understand examples of this kind better.

One of the principal tools is that of the splitting type. Given $Z' = p_1 + \cdots + p_r \subset \mathbb{P}^2$, let L_i be the line dual to the point p_i, and let $C_{Z'}$ be the curve $C_{Z'} = L_1 \cup \cdots \cup L_r$. The curve $C_{Z'}$ is defined by a single reduced form F of degree r. Let L be a general line. Assume $\mathbb{K}$ has characteristic 0 and take $\mathbb{K}[\mathbb{P}^2]$ to be $\mathbb{K}[x, y, z]$. We have the sheaf map $\mathcal{O}^3_{\mathbb{P}^2} \xrightarrow{\nabla F} \mathcal{O}_{\mathbb{P}^2}(d-1)$ given by the matrix $\nabla F = [F_x, F_y, F_z]$ of partials of F. This gives the exact sheaf sequence

$$0 \to \mathcal{D} \to \mathcal{O}^3_{\mathbb{P}^2} \xrightarrow{\nabla F} \mathcal{O}_{\mathbb{P}^2}(d-1)$$

where $\mathcal{D}$ is a rank 2 bundle, called the derivation bundle. By Grothendieck's splitting lemma, the restriction $\mathcal{D}_L$ of $\mathcal{D}$ to L is thus isomorphic to $\mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b)$ for some integers $0 \le a \le b$. We refer to (a, b) as the splitting type of $C_{Z'}$, denoted $(a_{Z'}, b_{Z'})$. (See [29, Appendix] for the definition when $\mathbb{K}$ has positive characteristic.)

Example 4.5 Let $Z' = p_1 + p_2$. We may, up to choice of coordinates, assume $C_{Z'}$ is defined by $F = xy$, so $\nabla F = [y, x, 0]$, hence the global sections of $\mathcal{D}(t)$ consist of expressions of the form $(Gx, -Gy, A)$ where G is a form of degree $t - 1$ and A is a form of degree t, so $\mathcal{D}$ already splits as $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$, thus $(a, b) = (0, 1)$.

Let $(a, b) = (a_{Z'}, b_{Z'})$ be the splitting type of $C_{Z'}$. The relevance to unexpected curves is [29, Lemma 3.3], which says that

$$\dim[I(jp + Z')]_{j+1} = \max(0, j - a + 1) + \max(0, j - b + 1). \tag{4.1}$$

Note that (a, b) always satisfies $a+b+1 = |Z'|$, where $|Z'|$ is the number of points of Z'.

One of the main results of [29] is:

Theorem 4.6 ([29, Theorem 1.2]) *Let $Z' \subset \mathbb{P}^2$ be a finite set of points. Let $(a, b) = (a_{Z'}, b_{Z'})$ be the splitting type of $C_{Z'}$. Then Z' admits an unexpected curve of some degree if and only if $2a + 2 < |Z'| = a + b + 1$ but no subset of $a + 2$ (or more) of the points is collinear. Moreover, if Z' admits an unexpected curve of some degree, the degrees j for which Z' has an unexpected curve are precisely $a < j \le |Z'| - a - 2 = b - 1$.*

We thus have:

Corollary 4.7 *Let $Z' \subset \mathbb{P}^2$ be a finite set of points. Let $(a, b) = (a_{Z'}, b_{Z'})$ be the splitting type of $C_{Z'}$. If Z' admits an unexpected curve of degree $m + 1$, then $b - a > 1$ and $2m - b - a - 1 = 2m - |Z'| < b - a - 4$.*

Proof We have $b - a > 1$ since $2a + 2 < |Z'| = a + b + 1$. We also have $m + 1 = j \leq |Z'| - a - 2 = b - 1$, so $2m + 2 \leq 2b - 2 = (|Z'| - a - 1) + b - 2$ so $2m + 1 - |Z'| \leq b - a - 4$ hence $2m - b - a - 1 = 2m - |Z'| < b - a - 4$. ◻

Example 4.8 If Z' has an unexpected curve and has splitting type (a, b), then $a \geq 2$ with $a = 2$ if and only if $\mathbb{K}$ has characteristic 2. Indeed, there cannot be unexpected curves of degree less than 3 for any Z' (and hence by Theorem 4.6 we must have $a + 1 \geq 3$). To see this, note from the definition that an unexpected curve of degree $d = m + 1$ must have $m > 0$, but a general point of multiplicity 1 always imposes independent conditions on $[I(Z')]_d$, so in fact we need $m > 1$. Thus the first possibility is $m = 2$, and indeed there is a Z' with a cubic unexpected curve; it occurs for a unique Z' [3] and it is irreducible. In this case Z' consists of the 7 points of the Fano plane and $m = 2$ (so $\mathbb{K}$ has characteristic 2; see [39] for a proof that there is no unexpected cubic in characteristic 0) and $C_{Z'}$ has splitting type $(a, b) = (2, 4)$. If q is a power of a prime, there is in a similar way an unexpected curve of degree $q + 1$ having a general point of multiplicity q with Z' consisting of the $q^2 + q + 1$ points of $\mathbb{P}^2$ defined over a field $\mathbb{F} \subset \mathbb{K}$ of q elements; see [22, Theorem 13.6].

Example 4.9 There is in characteristic 0 a unique Z' having an unexpected quartic [39]. In this case Z' is the set of 9 points obtained by projectivizing the B_3 root system; see [38, 76]. (We will refer to the line arrangement dual to Z' also as B_3; it is shown in Fig. 2.) In this case $C_{Z'}$ has splitting type $(a, b) = (3, 5)$ and we get an irreducible unexpected quartic.

Example 4.10 Assuming $\mathbb{K}$ has characteristic 0, an unexpected quintic must be irreducible. Suppose Z' is a point set having a reducible unexpected quintic, so $C_{Z'}$ has degree $d = m + 1 = 5$ and a general point of multiplicity $m = 4$. Since $C_{Z'}$ is not irreducible, by [29] there is an unexpected quartic $C_{Z''}$ for a subset $Z'' \subseteq Z'$ with either $Z'' = Z'$ or $|Z''| = |Z'| - 1$, but $|Z''| = 9$ by Example 4.9 so $|Z'|$ is either 9 or 10. Since the splitting type (a, b) for Z' satisfies $a + b + 1 = |Z'|$, we have $9 \leq a + b + 1 \leq 10$, and we have $a \geq 3$ by Example 4.8. By Corollary 4.7 we have $b - a > 1$ which together with $9 \leq a + b + 1 \leq 10$ and $a \geq 3$ means (a, b) is either (3, 5) or (3, 6). But by Corollary 4.7 we have $2m - b - a - 1 < b - a - 4$,

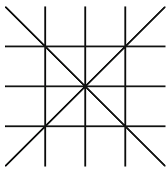

Fig. 2 The real (in fact rational) supersolvable configuration B_3 of 9 lines (only 8 are shown; the 9th is the line at infinity)

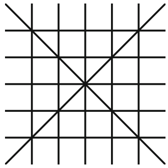

Fig. 3 The real (and rational) supersolvable configuration $\mathcal{A}_8$ of 13 lines (only 12 are shown; the 13th is the line at infinity) from [29, Proposition 6.15]

which for $(a, b) = (3, 5)$ and $m = 4$ is $-1 < -2$, so $(3, 5)$ is ruled out. This leaves $(a, b) = (3, 6)$ and $|Z'| = 10$. But then by Theorem 4.6 we'd have a 10 point set Z' with an unexpected quartic, contradicting Example 4.9. Thus an unexpected quintic must be irreducible. In fact, if Z' is a point set having an irreducible unexpected quintic, then $11 \le |Z'| \le 12$ and examples occur with both 11 and 12 points [40]. In contrast to the case of unexpected quartics, it is not known exactly which point sets Z' give rise to an unexpected quintic.

Example 4.11 Assuming $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, let Z' be the 13 point set corresponding to the line arrangement $\mathcal{A}_8$ shown in Fig. 3. The equation of $C_{Z'}$ is $xyz(x^2 - y^2)(x^2 - z^2)(x^2 - 4z^2)(y^2 - z^2)(y^2 - 4z^2)$. By [29, Proposition 6.15] Z' has a unique irreducible unexpected curve of degree 6.

Example 4.12 Assuming $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, the point set Z' dual to the line arrangement defined by the linear factors of $(x^s - y^s)(x^s - z^s)(y^s - z^s)$ has splitting type $(a, b) = (s + 1, 2s - 2)$. By [29, Proposition 6.12], it has an unexpected curve of degree $a + 1$ exactly when $s \ge 5$, and the curve is unique and irreducible.

Remark 4.13 We noted above that the curves $(x^s - y^s)(x^s - z^s)(y^s - z^s)$ have $t_2 = 0$ when $s \ge 3$. In fact, the other known nontrivial examples of complex curves given by unions of lines with $t_2 = 0$ (i.e., Klein's curve and Wiman's; see the discussion preceding Problem 2.14) also give Z' having unexpected curves (see [29]). This is yet another reason why Problem 2.14 is of interest!

Examples 4.9, 4.10, 4.11, 4.12 show over the complex numbers that there are irreducible unexpected curves of all degrees $d \ge 4$. The examples shown in Figs. 2 and 3 are the first two of an obvious infinite family of line arrangements, each of which has two interesting features, namely it is supersolvable and dual to point a set Z having an unexpected curve.

A line arrangement $\mathcal{L}$ of two or more lines is said to be *supersolvable* if there is a point q where two or more lines cross such that for every other point q' where two or more of the lines cross, then $L_{qq'} \in \mathcal{L}$; i.e., the line $L_{qq'}$ through the points q and q' is a line of the arrangement. Such a point q is called a *modular* point. We write $d_{\mathcal{L}}$ for the number $|\mathcal{L}|$ of lines in $\mathcal{L}$, and we write $m_{\mathcal{L}}$ for the maximum number of coincident lines in $\mathcal{L}$. When $\mathcal{L}$ is supersolvable, every point on $m_{\mathcal{L}}$ lines of $\mathcal{L}$ is a modular point (see [107, Lemma 2.1] or [68, Lemma 2]). For example, the B_3 configuration shown in Fig. 2 has three modular points (all three have multiplicity 4; one is the point at the center of the figure while the other two are at infinity, where the horizontal and vertical lines converge). The $\mathcal{A}_8$ configuration shown in Fig. 3 has two modular points (both have multiplicity 6 and are where the horizontal and vertical lines converge at infinity).

By [39, Theorem 3.7], if Z' is the set of points dual to a supersolvable line arrangement $\mathcal{L}$, then Z' has an unexpected curve if and only if $d_{\mathcal{L}} > 2m_{\mathcal{L}}$, and in this case the splitting type of $\mathcal{L}$ is $(a, b) = (m_{\mathcal{L}} - 1, d_{\mathcal{L}} - m_{\mathcal{L}})$. Thus, when $\mathcal{L}$ is supersolvable and $d_{\mathcal{L}} > 2m_{\mathcal{L}}$, Z' has an unexpected curve of each degree d in the range $m_{\mathcal{L}} \le d \le d_{\mathcal{L}} - m_{\mathcal{L}} - 1$.

It is an open problem to classify line arrangements $\mathcal{L}$ whose dual points Z' have unexpected curves. Even doing this for supersolvable line arrangements is open. (However, see [1, 68] for a partial classification of complex supersolvable line arrangements.)

A less ambitious open problem is as follows:

Problem 4.14 For each $a \ge 3$, what range of values of b occur such that (a, b) is the splitting type for a complex line arrangement $\mathcal{L}$ whose dual points Z' have an unexpected curve? Said differently, for each degree d, what range of values of $|Z'|$ occur such that Z' has an unexpected curve of degree d?

Getting good bounds on b given a seems challenging. There is however the following fact:

Theorem 4.15 *Let $\mathcal{L}$ be a complex supersolvable line arrangement of splitting type (a, b) such that the dual points Z' have an unexpected curve of degree d.*

(a) Then $a + 2 \le b \le 2a - 1$.
(b) Moreover,

$$\frac{3d}{2} + 3 \le |Z'| \le 3(d - 1).$$

Proof

(a) We have $a+1 \le d \le b-1$, so $a+2 \le b$. We also have $d_{\mathcal{L}} > 2m_{\mathcal{L}}$ [39], so by the results of [68], all of the modular points of $\mathcal{L}$ have the same multiplicity. Thus we also have $d_{\mathcal{L}} \le 3m_{\mathcal{L}}-3$ by [1]. By [39] we have $(a, b) = (m_{\mathcal{L}}-1, d_{\mathcal{L}}-m_{\mathcal{L}})$. From $d_{\mathcal{L}} \le 3m_{\mathcal{L}} - 3$ we now get $b = d_{\mathcal{L}} - m_{\mathcal{L}} \le 2m_{\mathcal{L}} - 3 = 2a - 1$.
(b) We have $a + 1 \le d \le b - 1$ by Theorem 4.6, and we have $b \le 2a - 1$ by (a), so $|Z'| = a + b + 1 \le d + b \le d + 2a - 1 \le d + 2(d - 1) - 1 = 3(d - 1)$. From $d \le b - 1 \le 2a - 2$ we get $(d + 2)/2 \le a$ so we have $(3d/2) + 3 = (d + 2)/2 + d + 2 \le a + b + 1 = |Z'|$. □

Remark 4.16 For the B_3 line arrangement we have $a = 3, b = 5$, $|Z'| = 9$ and $d = 4$. In this case we get $5 = a + 2 \le b \le 2a - 1 = 5$ and $9 = \frac{3d}{2} + 3 \le |Z'| \le 3(d - 1) = 9$.

For the case of a complex unexpected quintic, so $d = 5$, by [40] we have $11 \le |Z'| \le 12$ and both extremes occur. Theorem 4.15 gives $10.5 \le |Z'| \le 12$ for Z' dual to complex supersolvable line arrangements.

For the $\mathcal{A}_8$ line arrangement we have $a = 5$, $b = 7$, $|Z'| = 13$ and $d = 6$. In this case we get $7 = a+2 \leq b \leq 2a-1 = 9$ and $12 = \frac{3d}{2}+3 \leq |Z'| \leq 3(d-1) = 15$.

Remark 4.17 Note that $a + 1 \leq d \leq b - 1$ and $a + 2 \leq b$ hold for any line arrangement $\mathcal{L}$ whose dual points Z' have an unexpected curve. In order to extend Theorem 4.15 to complex line arrangements generally, it is clear from the proof that we need an upper bound on b in terms of a. Suppose that the general singular point p of an unexpected curve was in general regular (i.e., had distinct tangents). Let C_p be the unexpected curve of degree $a + 1$ whose general point of multiplicity a is p (by Eq. (4.1), C_p is unique). Choose a point p' close to p. Then by the structural results of [29], C_p and $C_{p'}$ would not in general have any components in common, and the branches of the singularities at p and p' would have $a(a-1)$ intersections points near p. Thus by Bezout's Theorem the intersection of C_p with $C_{p'}$ would give $(a+1)^2 \geq a(a-1) + |Z'| = a(a-1) + (a+b+1)$, or $2a \geq b$. Mimicking the proof of Theorem 4.15(b) now gives $(3d+5)/2 \leq |Z'| \leq 3d-2$.

Problem 4.18 Let $\mathcal{L}$ be a complex line arrangement and Z' the points dual to $\mathcal{L}$. Assume $\mathcal{L}$ has splitting type (a, b) and that Z' has an unexpected curve C of degree d. Is it true that that C in general has only regular singularities? Is it true that $a+2 \leq b \leq 2a$ and $(3d+5)/2 \leq |Z'| \leq 3d-2$?

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Maximal Cohen-Macaulay Complexes and Their Uses: A Partial Survey

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To David Eisenbud on his 75th birthday.

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1 Introduction

Big Cohen-Macaulay modules over (commutative, noetherian) local rings were introduced by Hochster around 50 years ago and their relevance to local algebra is established beyond doubt. Indeed, they play a prominent role in Hochster's lecture notes [21], where he describes a number of homological conjectures that can be proved using big Cohen-Macaulay modules, and their finitely generated counterparts, the maximal Cohen-Macaulay modules; the latter are sometimes called, as in *loc. cit.*, "small" Cohen-Macaulay modules see also. Hochster [20, 21] proved that big Cohen-Macaulay modules exist when the local ring contains a field;

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and conjectured that even rings of mixed-characteristic possess such modules. This conjecture was proved by André [1], see also Bhatt, thereby settling a number of the homological conjectures. In fact, by results of Hochster and Huneke [23, 24], and André [1] there exist even big Cohen-Macaulay *algebras* over any local ring. The reader will find a survey of these developments in [25, 35].

In this work we introduce three versions of the Cohen-Macaulay property that apply also to complexes of modules, discuss various constructions that give rise to them, and present some consequences that follow from their existence. In fact such complexes have come up earlier, in the work of Roberts [41, 42], recalled in Sect. 4.3, and in recent work of Bhatt [4], though only in passing. What we found is that results that were proved using big Cohen-Macaulay modules can often be proved using one of their complex variants. This assertion is backed up the material presented in Sects. 3 and 5. Moreover, as will be apparent in the discussion in Sect. 4, the complex versions are easier to construct, and with better finiteness properties. It thus seems worthwhile to shine an independent light on them. Let us begin by defining them.

We say that a complex M over a local ring R with maximal ideal $\mathfrak{m}$ has *maximal depth* if $\operatorname{depth}_R M = \dim R$, where depth is as in Sect. 2.4; we ask also that $\mathrm{H}(M)$ be bounded and the canonical map $\mathrm{H}_0(M) \to \mathrm{H}_0(k \otimes^{\mathrm{L}}_R M)$ be non-zero. Any complex that satisfies the last condition has depth at most $\dim R$, whence the name "maximal depth". An R-module has maximal depth precisely when it is big Cohen-Macaulay. The depth of a complex can be computed in terms of its local cohomology modules, $\mathrm{H}^i_{\mathfrak{m}}(M)$, with support on $\mathfrak{m}$. Thus $\operatorname{depth}_R M = \dim R$ means that $\mathrm{H}^i_{\mathfrak{m}}(M)$ is zero for $i < \dim R$, and nonzero for $i = \dim R$. A complex of maximal depth is *big Cohen-Macaulay* if $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i > \dim R$ as well. When in addition the R-module $\mathrm{H}(M)$ is finitely generated, M is *maximal Cohen-Macaulay* (MCM). Thus an MCM module is what we know it to be. These notions are discussed in detail in Sect. 3 and 4.

When R is an excellent local domain with residue field of positive characteristic, R^+, its integral closure in an algebraic closure of its field of fractions, is big Cohen-Macaulay. This was proved by Hochster and Huneke [23], see also Huneke and Lyubeznik [26], when R itself contains a field of positive characteristic. When R has mixed characteristic this is a recent result of Bhatt [4]. Thus for such rings there is a canonical construction of a big Cohen-Macaulay module, even an *algebra*. See also the work of André [2] and Gabber [14] concerning functorial construction of big Cohen-Macaulay algebras; see also [37, Appendix A]. On the other hand, R^+ is never big Cohen-Macaulay when R contains the rationals and is a normal domain of Krull dimension at least 3, by a stadard trace argument. As far as we know, in this context there are no such "simple" models of big Cohen-Macaulay modules, let alone algebras. See however Schoutens' work [44].

When R is essentially of finite type containing a field of characteristic zero, the derived push-forward of the structure sheaf of a resolution of singularities of Spec R is an MCM complex [41]. What is more, this complex is equivalent to a graded-commutative differential graded algebra; see 4.3. This is noteworthy because when such a ring R is also a normal domain of dimension ≥3 it cannot have any MCM algebras, by the same trace argument as for R^+. For a local ring R with a dualizing complex there are concrete constructions of MCM complexes; see Corollaries 4.6 and 4.9 and the paragraph below. However we do not know any that are also differential graded algebras. In [5] Bhatt gives examples of complete local rings, containing a field of positive characteristic, that do not have any MCM algebras.

As to applications, in Sect. 3 we prove the New Intersection Theorem and its improved version using complexes of maximal depth, extending the ideas from [27] where they are proved using big Cohen-Macaulay modules. It follows from the work of Hochster [22] and Dutta [9] that the Improved New Intersection Theorem is equivalent to the Canonical Element Theorem. In Sect. 4 we use results from *loc. cit.* to prove that for local rings with dualizing complexes the Canonical Element Theorem implies the existence of MCM complexes. An interesting point emerges: replacing "module" with "complex" puts the existence of big Cohen-Macaulay modules on par with the rest of the homological conjectures.

In Sect. 5 we paraphrase Boutot's proof of his theorem on rational singularities to highlight the role of MCM complexes. We also give a new proof of a subadditivity property for multiplier ideals. On the other hand, there are applications of MCM modules that do require working with modules; see 4.15. Nevertheless, it is clear to us that big Cohen-Macaulay complexes and MCM complexes have their uses, hence this survey.

2 Local Cohomology and Derived Completions

In this section, we recall basic definitions and results on local cohomology and derived completions. Throughout R will be a commutative noetherian ring. By an *R-complex* we mean a complex of R-modules; the grading will be upper or lower, depending on the context. In case of ambiguity, we indicate the grading; for example, given an R-complex M, the supremum of $\mathrm{H}(M)$ depends on whether the grading is upper or lower. So we write $\sup \mathrm{H}_*(M)$ for the largest integer i such that $\mathrm{H}_i(M) \neq 0$, and $\sup \mathrm{H}^*(M)$ for the corresponding integer for the upper grading.

We write $\mathrm{D}(R)$ for the (full) derived category of R viewed as a triangulated category with translation Σ, the usual suspension functor on complexes. We take [11, 33] as basic references, augmented by Avramov and Foxby [3] and Roberts [41], except that we use the term "semi-injective" in place of "q-injective" as in [33], and "DG-injective", as in [3]. Similarly for the projective and flat analogs.

2.1 *Derived I-torsion*

Let I an ideal in R. The *I-power torsion* subcomplex of an R-complex M is

$$\varGamma_I M := \{m \in M \mid I^n m = 0 \text{ for some } n \geq 0\}.$$

By $m \in M$ we mean that m is in M_i for some i. The corresponding derived functor is denoted $\mathrm{R}\varGamma_I(M)$; thus $\mathrm{R}\varGamma_I(M) = \varGamma_I J$ where $M \xrightarrow{\sim} J$ is any semi-injective resolution of M. In fact, one can compute these derived functors from any complex of injective R-modules quasi-isomorphic to M; see [33, §3.5]. By construction there is a natural morphism $\mathrm{R}\varGamma_I(M) \to M$ in the $\mathrm{D}(R)$. The R-modules

$$\mathrm{H}^i_I(M) := \mathrm{H}^i(\mathrm{R}\varGamma_I(M)) \quad \text{for } i \in \mathbb{Z}$$

are the local cohomology modules of M, supported on I. Evidently, these modules are I-power torsion. Conversely, when the R-module $\mathrm{H}(M)$ is I-power torsion, the natural map $\mathrm{R}\varGamma_I(M) \to M$ is an isomorphism in $\mathrm{D}(R)$; see [11, Proposition 6.12], or [33, Corollary 3.2.1].

In what follows we will use the fact that the class of I-power torsion complexes form a localizing subcategory of $\mathrm{D}(R)$; see [11, §6], or [33, §3.5]. This has the consequence that these complexes are stable under various constructions. For example, this class of complexes is closed under $L \otimes^{\mathrm{L}}_R (-)$ for any L in $\mathrm{D}(R)$. Thus, for any R-complexes L and M the natural map

$$\mathrm{R}\varGamma_I(L \otimes^{\mathrm{L}}_R M) \longrightarrow L \otimes^{\mathrm{L}}_R \mathrm{R}\varGamma_I(M) \tag{2.1}$$

is a quasi-isomorphism.

2.2 Derived *I*-completion

The I-adic completion of an R-complex M with respect to the ideal I, denoted $\varLambda^I M$, is

$$\varLambda^I M := \lim_{n \geqslant 0} M/I^n M\,.$$

This complex is thus the limit of the system

$$\cdots \longrightarrow M/I^{n+1}M \longrightarrow M/I^n M \longrightarrow \cdots \longrightarrow M/IM\,.$$

The canonical surjections $M \to M/I^n M$ induce an R-linear map $M \to \varLambda^I M$. If this is an isomorphism we say that M is *I-adically complete*, or just *I-complete*, though we reserve this name mainly for modules. The *left-derived completion* with respect to I of an R-complex M is the R-complex

$$\mathrm{L}\varLambda^I(M) := \varLambda^I P \quad \text{where } P \simeq M \text{ is a semi-projective resolution.}$$

This complex is well-defined in $\mathrm{D}(R)$, and there is a natural morphism

$$M \longrightarrow \mathrm{L}\Lambda^I(M)\,.$$

We say M is *derived I-complete* if this map is a quasi-isomorphism; equivalently if each $\mathrm{H}_i(X)$ is derived I-complete; see [11, Proposition 6.15], or [46, Tag091N]

The derived I-complete modules from a colocalizing subcategory of $\mathrm{D}(R)$, and this means that for N in $\mathrm{D}(R)$ the natural map

$$\mathrm{L}\Lambda^I(\mathrm{RHom}_R(N, M)) \longrightarrow \mathrm{RHom}_R(N, \mathrm{L}\Lambda^I(M))$$

is a quasi-isomorphism. In particular, when F is a perfect complex, we have an isomorphim in $\mathrm{D}(R)$

$$F \otimes^{\mathrm{L}}_R \mathrm{L}\Lambda^I(M) \simeq \mathrm{L}\Lambda^I(F \otimes^{\mathrm{L}}_R M). \tag{2.2}$$

These isomorphisms will be useful in what follows. It is a fundamental fact, proved by Greenlees and May [16], see also [11, Proposition 4.3] or [33, §4], that derived local cohomology and derived completions are adjoint functors:

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_I(M), N) \simeq \mathrm{RHom}_R(M, \mathrm{L}\Lambda^I(N))\,. \tag{2.3}$$

One can take this as a starting point for defining derived completions, which works better in the non-noetherian settings; see [46]. This adjunction implies that the natural maps are quasi-isomorphisms:

$$\mathrm{L}\Lambda^I(\mathrm{R}\Gamma_I(M)) \xrightarrow{\simeq} \mathrm{L}\Lambda^I(M) \quad \text{and} \quad \mathrm{R}\Gamma_I(M) \xrightarrow{\simeq} \mathrm{R}\Gamma_I(\mathrm{L}\Lambda^I(M))\,. \tag{2.4}$$

The result below, due to A.-M. Simon [45, 1.4], is a version of Nakayama's Lemma for cohomology of complete modules. It is clear from the proof that we only need X to be derived I-complete; see [46, Tag09b9].

Lemma 2.1 *For any R-complex X consisting of I-complete modules, and integer i, if $I\,\mathrm{H}_i(X) = \mathrm{H}_i(X)$, then $\mathrm{H}_i(X) = 0$.*

Proof The point is that Z_i, the module of cycle in degree i, is a closed submodule of the I-complete module X_i, and hence is also I-complete. Moreover $\mathrm{H}_i(X)$ is the cokernel of the map $X_{i+1} \to Z_i$, and a map between I-complete modules is zero if and only if its I-adic completion is zero. This translates to the desired result. □

2.3 Koszul Complexes

Given a sequence of elements $\underline{r} := r_1, \ldots, r_n$ in the ring R, and an R-complex M, we write $K(\underline{r}; M)$ for the Koszul complex on $\underline{r}$ with coefficients in M, namely

$$K(\underline{r}; M) := K(\underline{r}; R) \otimes_R M\,.$$

Its homology is denoted $\mathrm{H}_*(\underline{r}; M)$. For a single element $r \in R$, the complex $K(r; M)$ can be constructed as the mapping cone of the homothety map $M \xrightarrow{r} M$. In particular, one has an exact sequence

$$0 \longrightarrow M \longrightarrow K(r; M) \longrightarrow \Sigma M \longrightarrow 0 \tag{2.5}$$

of R-complexes. The Koszul complex on a sequence can thus be constructed as an iterated mapping cone. From Lemma 2.1 one gets the result below. Recall that $\sup \mathrm{H}_*(-)$ denotes the supremum, in lower grading.

Lemma 2.2 *Let R be a noetherian ring and X a derived I-complete R-complex. For any sequence $\underline{r} := r_1, \ldots, r_n$ in I one has*

$$\sup \mathrm{H}_*(\underline{r}; X) \geq \sup \mathrm{H}_*(X)\,.$$

Proof When X is derived I-complete so is $K(r; X)$ for any $r \in I$. It thus suffices to verify the desired claim for $n = 1$. Replacing X by $\Lambda^I P$, where P is a semi-projective resolution of X, we can assume X, and hence also $K(r; X)$, consists of I-complete modules. The desired inequality is then immediate from the standard long exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_i(X) \xrightarrow{r} \mathrm{H}_i(X) \longrightarrow \mathrm{H}_i(r, X) \longrightarrow \mathrm{H}_{i-1}(X) \longrightarrow \cdots$$

arising from the mapping cone sequence (2.5) and Lemma 2.1. □

To wrap up this section we recall the notion of depth for complexes.

2.4 Depth

The *I-depth* of an R-complex M is

$$\operatorname{depth}_R(I, M) := \inf\{i \mid \mathrm{H}^i_I(M) \neq 0\}.$$

In particular, $\operatorname{depth}_R(I, M) = \infty$ if $\mathrm{H}_I(M) = 0$. When the ring R is local, with maximal ideal $\mathfrak{m}$, the *depth* of M refers to the $\mathfrak{m}$-depth of M.

Depth can also be computed using Ext and Koszul homology:

$$\operatorname{depth}_R(I, M) = \inf\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\},$$

and if a sequence $\underline{r} := r_1, \ldots, r_n$ generates I, then

$$\operatorname{depth}_R(I, M) = n - \sup\{i \mid \mathrm{H}_i(\underline{r}; M) \neq 0\}$$

This last equality can be expressed in terms of Koszul cohomology. All these results are from [13], though special cases (for example, when M is an R-module) had been known for much longer.

Remark 2.3 Let R be a commutative ring, I an ideal in R, and M an R-complex. Set $s = \sup \mathrm{H}_*(M)$.

(1) $\operatorname{depth}_R(I, M) \geq -s$ and equality holds if $\varGamma_I(\mathrm{H}_s(M)) \neq 0$.
(2) When R is local and F is a finite free complex, one has

$$\operatorname{depth}_R(F \otimes_R M) = \operatorname{depth}_R M - \operatorname{proj\,dim}_R F$$

For part (1) see [13, 2.7]. When F is the resolution of a module and $M = R$, part (2) is nothing but the equality of Auslander and Buchsbaum. For a proof in the general case see, for example, [13, Theorem 2.4].

3 Complexes of Maximal Depth and the Intersection Theorems

In this section we introduce a notion of "maximal depth" for complexes over local rings. The gist of the results presented here is that their existence implies the Improved New Intersection Theorem, and hence a whole slew of "homological conjectures", most of which have been recently settled by André [1].

A *module* of maximal depth is nothing but a big Cohen-Macaulay module and Hochster proved, already in [21], that their existence implies the homological conjectures mentioned above. On the other hand, the Canonical Element Conjecture, now theorem, implies that R has a *complex* of maximal depth, even one with finitely generated homology. This will be one of the outcomes of the discussion in the next section; see Remark 4.13. No such conclusion can be drawn about big Cohen-Macaulay modules.

3.1 Complexes of Maximal Depth

Throughout $(R, \mathfrak{m}, k)$ will be a local ring, with maximal ideal $\mathfrak{m}$ and residue field k. We say that an R-complex M has *maximal depth* if the following conditions hold:

(1) $\mathrm{H}(M)$ is bounded;
(2) $\mathrm{H}_0(M) \to \mathrm{H}_0(k \otimes^{\mathrm{L}}_R M)$ is nonzero; and
(3) $\operatorname{depth}_R M = \dim R$.

The nomenclature is based on that fact that $\operatorname{depth}_R M \le \dim R$ for any complex M that satisfies condition (2) above. This inequality follows from Lemma 3.1 applied with $F := K$, the Koszul complex on a system of parameters for R. Condition (3) can be restated as

$$\mathrm{H}^i_{\mathfrak{m}}(M) = 0 \quad \text{for } i < \dim R \quad \text{and} \quad \mathrm{H}^{\dim R}_{\mathfrak{m}}(M) \neq 0\,. \tag{3.1}$$

Clearly when M is a module it has maximal depth precisely when it is big Cohen-Macaulay; condition (2) says that $M \neq \mathfrak{m}M$. Note also that if a complex M has maximal depth then so does $M \oplus \Sigma^{-n} N$ for any R-module N and integer $n \ge \dim R$.

Lemma 3.1 *Let M be an R-complex with the natural map $\mathrm{H}_0(M) \to \mathrm{H}_0(k \otimes^{\mathrm{L}}_R M)$ nonzero. For any R-complex F with $\mathrm{H}_i(F) = 0$ for $i < 0$, if $\mathrm{H}_0(F) \otimes_R k$ is nonzero, then so is $\mathrm{H}_0(F \otimes^{\mathrm{L}}_R M)$.*

Proof We can assume M is semi-projective, so the functor $- \otimes^{\mathrm{L}}_R M$ is represented by $- \otimes_R M$. By hypothesis there exists a cycle, say z, in M_0 whose image in $k \otimes_R M = M/\mathfrak{m}M$ is not a boundary. Consider the morphism $R \to M$ of R-complexes, where $r \mapsto rz$. Its composition $R \to M \to k \otimes_R M$ factors through the canonical surjection $R \to k$, yielding the commutative square

$$\begin{array}{ccc} R & \longrightarrow & M \\ \downarrow & & \downarrow \\ k & \rightleftarrows & k \otimes_R M. \end{array}$$

The dotted arrow is a left-inverse in $\mathrm{D}(R)$ of the induced $k \to k \otimes_R M$. It exists because $k \to \mathrm{H}(k \otimes_R M)$ is nonzero, by the choice of z, and the complex $k \otimes_R M$ is quasi-isomorphic to $\mathrm{H}(k \otimes_R M)$ in $\mathrm{D}(k)$, and hence in $\mathrm{D}(R)$. Applying $F \otimes^{\mathrm{L}}_R -$ to the diagram above yields the commutative square in $\mathrm{D}(R)$ on the left:

$$\begin{array}{ccc} F & \longrightarrow & F \otimes^{\mathrm{L}}_R M \\ \downarrow & & \downarrow \\ F \otimes^{\mathrm{L}}_R k & \rightleftarrows & F \otimes^{\mathrm{L}}_R (k \otimes_R M) \end{array} \qquad \begin{array}{ccc} \mathrm{H}_0(F) & \longrightarrow & \mathrm{H}_0(F \otimes^{\mathrm{L}}_R M) \\ \downarrow & & \downarrow \\ \mathrm{H}_0(F \otimes^{\mathrm{L}}_R k) & \rightleftarrows & \mathrm{H}_0(F \otimes^{\mathrm{L}}_R (k \otimes_R M)) \end{array}$$

The commutative square on the right is obtained by applying $\mathrm{H}_0(-)$ to the one on the left. In this square, the hypotheses on F imply that the vertical map on the left is nonzero, so hence is its composition with the horizontal arrow. The commutativity of the square then yields that $\mathrm{H}_0(F \otimes^{\mathrm{L}}_R M)$ is nonzero. □

The following result is due to Hochster and Huneke for rings containing a field, and due to André in the mixed characteristic case.

Theorem 3.2 (André [1], Hochster and Huneke [23, 24]) *Each noetherian local ring possesses a big Cohen-Macaulay algebra.* □

As has been said before, the existence of big Cohen-Macaulay algebras, and hence big Cohen-Macaulay modules, implies many of the homological conjectures. In particular, it can be used to give a quick proof of the New Intersection Theorem, first proved in full generality by P. Roberts [43] using intersection theory; see also [40]. Here is a proof that uses only the existence of *complexes* of maximal depth; the point being that they are easier to construct than big Cohen-Macaulay modules. Our argument is modeled on that of [27, Theorem 2.5], which uses big Cohen-Macaulay modules.

Theorem 3.3 *Let R be a local ring. Any finite free R-complex*

$$F := 0 \to F_n \to \cdots \to F_0 \to 0$$

with $\mathrm{H}_0(F) \neq 0$ *and* $\mathrm{length}_R \mathrm{H}_i(F)$ *finite for each i satisfies* $n \geq \dim R$.

Proof Let M be an R-complex of maximal depth. As $\mathrm{H}(F)$ is of finite length, the R-module $\mathrm{H}(F \otimes_R M)$ is $\mathfrak{m}$-power torsion, so 2.3(1) yields the second equality:

$$\begin{aligned} \operatorname{proj\,dim}_R F &= \operatorname{depth}_R M - \operatorname{depth}_R (F \otimes_R M) \\ &= \operatorname{depth}_R M + \sup \mathrm{H}_*(F \otimes_R M) \\ &\geq \operatorname{depth}_R M \\ &= \dim R \end{aligned}$$

The first one is by 2.3(2). The inequality is by Lemma 3.1, noting that $\mathrm{H}_0(F) \otimes_R k$ is nonzero by Nakayama's lemma. □

One can deduce also the Improved New Intersection Theorem 3.6 from the existence of complexes of maximal depth, but the proof takes some more preparation.

Lemma 3.4 *Let R be a local ring and M an R-complex. If M has maximal depth, then so does $\mathrm{L}\Lambda^I(M)$ for any ideal $I \subset R$.*

Proof Condition (1) for maximal depth holds because $\mathrm{H}(M)$ bounded implies $\mathrm{H}(\mathrm{L}\Lambda^I(M))$ is bounded; this follows, for example, from (2.3) and the observation $\mathrm{R}\Gamma_I(R)$ has finite projective dimension. As to the other conditions, the main point is that for any R-complex X such that $\mathrm{H}(X)$ is I-power torsion, the canonical map

$M \to \mathrm{L}\Lambda^I(M)$ in $\mathrm{D}(R)$ induces a quasi-isomorphism

$$X \otimes^{\mathrm{L}}_R M \xrightarrow{\simeq} X \otimes^{\mathrm{L}}_R \mathrm{L}\Lambda^I(M).$$

This can be deduced from (2.2) and (2.4). In particular, taking $X = \mathrm{R}\Gamma_{\mathfrak{m}}(R)$, where $\mathfrak{m}$ is the maximal ideal of R, yields

$$\mathrm{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{L}\Lambda^I(M)),$$

so that $\operatorname{depth}_R M = \operatorname{depth}_R \mathrm{L}\Lambda^I(M)$. Moreover, taking $X = k$ gives the isomorphism in the following commutative diagram in $\mathrm{D}(R)$:

$$\begin{array}{ccc} M & \longrightarrow & k \otimes^{\mathrm{L}}_R M \\ \downarrow & & \downarrow \simeq \\ \mathrm{L}\Lambda^I(M) & \longrightarrow & k \otimes^{\mathrm{L}}_R \mathrm{L}\Lambda^I(M) \end{array}$$

that is induced by the morphism $M \to \mathrm{L}\Lambda^I(M)$. Since M has maximal depth, the map in the top row is nonzero when we apply $\mathrm{H}_0(-)$, and so the same holds for the map in the bottom row. Thus $\mathrm{L}\Lambda^I(M)$ has maximal depth. □

Lemma 3.5 *Let $(R, \mathfrak{m}, k)$ be a local ring and M a derived $\mathfrak{m}$-complete R-complex of maximal depth. Set $d := \dim R$. The following statements hold:*

(1) *For any system of parameters $r_1, \dots, r_d$ for R, one has*

$$\operatorname{depth}_R(K(r_1, \dots, r_n; M)) = n \quad \textit{for each } 1 \le n \le d.$$

In other words, the depth of M with respect to the ideal $(r_1, \dots, r_n)$ is n.

(2) *For any $\mathfrak{p} \in \operatorname{Spec} R$ one has*

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \ge \dim R_{\mathfrak{p}},$$

and equality holds when the map $\mathrm{H}_0(M) \to \mathrm{H}_0(k(\mathfrak{p}) \otimes^{\mathrm{L}}_R M)$ is nonzero, in which case the $R_{\mathfrak{p}}$-complex $M_{\mathfrak{p}}$ has maximal depth.

Proof

(1) Set $\underline{r} = r_1, \dots, r_d$. The hypothesis that M has maximal depth and the depth sensitivity of the Koszul complex $K(\underline{r}; R)$ yield $\mathrm{H}_i(\underline{r}; M) = 0$ for $i \ge 1$. One has an isomorphism of R-complexes

$$K(\underline{r}; M) \cong K(r_{n+1}, \dots, r_d; K(r_1, \dots, r_n; M)).$$

Since M is derived complete with respect to $\mathfrak{m}$, it follows from Lemma 2.2, applied to the sequence $r_{n+1}, \dots, r_d$ and $X := K(r_1, \dots, r_n; M)$, that

$$\mathrm{H}_i(K(r_1, \dots, r_n; M)) = 0 \quad \text{for } i \geq 1.$$

On the other hand, since the natural map $\mathrm{H}_0(M) \to \mathrm{H}_0(k \otimes^{\mathrm{L}}_R M)$ is nonzero, Lemma 3.1 applied with $F = K(r_1, \dots, r_n; R)$, yields

$$\mathrm{H}_0(K(r_1, \dots, r_n; M)) \neq 0\,.$$

Thus the depth sensitivity of $K(r_1, \dots, r_n; M)$ yields the equality in (1).

(2) Set $h :=$ height $\mathfrak{p}$ and choose a system of parameters $\underline{r} := r_1, \dots, r_d$ for R such that the elements $r_1, \dots, r_h$ are in $\mathfrak{p}$. One has

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \operatorname{depth}_R(K(r_1, \dots, r_h), M) \geq h\,.$$

where the first inequality is clear and the second one holds by (1). The natural map $M \to k(\mathfrak{p}) \otimes^{\mathrm{L}}_R M$ factors through $M_\mathfrak{p}$, so under the additional hypothesis Lemma 3.1 implies $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leq h$. We conclude that $M_\mathfrak{p}$ has maximal depth. □

Given the preceding result, we argue as in the proof of [27, Theorem 3.1] to deduce the Improved New Intersection Theorem:

Theorem 3.6 *Let R be a noetherian local ring and $F := 0 \to F_n \to \cdots \to F_0 \to 0$ a finite free R-complex with* $\mathrm{H}_0(F) \neq 0$ *and* length $\mathrm{H}_i(F)$ *finite for each $i \geq 1$. If an ideal I annihilates a minimal generator of* $\mathrm{H}_0(F)$*, then* $n \geq \dim R - \dim(R/I)$.

Proof Let M be an R-complex of maximal depth. By Lemma 3.4, we can assume M is derived $\mathfrak{m}$-complete, so Lemma 3.5 applies. Set $s := \sup \mathrm{H}_*(F \otimes_R M)$ and note that $s \geq 0$, by Lemma 3.1.

Fix $\mathfrak{p}$ in $\operatorname{Ass}_R \mathrm{H}_s(F \otimes_R M)$, so that $\operatorname{depth}_{R_\mathfrak{p}} \mathrm{H}_s(F \otimes_R M)_\mathfrak{p} = 0$. The choice of $\mathfrak{p}$ implies that $\mathrm{H}(F \otimes_R M)_\mathfrak{p}$ is nonzero, and hence $\mathrm{H}(F)_\mathfrak{p}$ and $\mathrm{H}(M)_\mathfrak{p}$ are nonzero as well. Therefore one gets

$$\begin{aligned} \operatorname{proj\,dim}_{R_\mathfrak{p}} F_\mathfrak{p} &= \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} (F \otimes_R M)_\mathfrak{p} \\ &= \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + s \\ &\geq \dim R_\mathfrak{p} + s \end{aligned} \tag{3.2}$$

The equalities are by 2.3 and the inequality is by Lemma 3.5(2).

Suppose $s \geq 1$. We claim that $\mathfrak{p} = \mathfrak{m}$, the maximal ideal of R, so (3.2) yields

$$\operatorname{proj\,dim}_R F \geq \dim R\,,$$

which implies the desired inequality.

Indeed if $\mathfrak{p} \neq \mathfrak{m}$, then since $\operatorname{length}_R \mathrm{H}_i(F)$ is finite for $i \geq 1$, one gets that $F_\mathfrak{p} \simeq \mathrm{H}_0(F)_\mathfrak{p}$, which justifies the equality below:

$$\operatorname{depth} R_\mathfrak{p} \geq \operatorname{proj\,dim}_{R_\mathfrak{p}} \mathrm{H}_0(F)_\mathfrak{p} = \operatorname{proj\,dim}_{R_\mathfrak{p}} F_\mathfrak{p} \geq \dim R_\mathfrak{p} + s$$

The first inequality is a consequence of the Auslander-Buchsbaum equality 2.3(2), the second one is from (3.2). We have arrived at a contradiction for $s \geq 1$.

It remains to consider the case $s = 0$. Set $X := F \otimes_R M$. Since $\mathrm{H}_0(F)$ is finitely generated, Nakayama's Lemma and Lemma 3.1 imply that each minimal generator of $\mathrm{H}_0(F)$ gives a nonzero element in $\mathrm{H}_0(X)$. One of these is thus annihilated by I, by the hypotheses. Said otherwise, $\varGamma_I \mathrm{H}_0(F) \neq 0$. Since $\sup \mathrm{H}_*(X) = 0$, this implies $\operatorname{depth}_R(I, X) = 0$, by Remark 2.3, and hence one gets the equality below

$$\operatorname{depth}_R X \leq \operatorname{depth}_R(I, X) + \dim(R/I) = \dim(R/I)$$

The inequality can be verified by arguing as in the proof of [27, Proposition 5.5(4)]: Let $\underline{a} := a_1, \dots, a_l$ be a set of generators for the ideal I, and let $\underline{b} := b_1, \dots, b_n$ be elements in R whose residue classes in R/I form a system of parameters. Since M is derived $\mathfrak{m}$-complete, so is X and hence also $K(\underline{a}; X)$. Then Lemma 2.2 applied to the sequence $\underline{b}$ and complex $K(\underline{a}; X)$ yields

$$\sup \mathrm{H}_*(\underline{a}, \underline{b}; X) \geq \sup \mathrm{H}_*(\underline{a}; X)\,;$$

this gives the desired inequality. Finally it remains to invoke the Auslander-Buchsbaum equality once again to get

$$\operatorname{proj\,dim}_R F = \operatorname{depth}_R M - \operatorname{depth}_R X \geq \dim R - \dim(R/I)\,.$$

This completes the proof. □

4 MCM Complexes

In this section we introduce two strengthenings of the notion of complexes of maximal depth, and discuss various constructions that yield such complexes. As before let $(R, \mathfrak{m}, k)$ be a local ring, of Krull dimension d.

4.1 Big Cohen-Macaulay Complexes

We say that an R-complex M is *big Cohen-Macaulay* if the following conditions hold:

(1) $\mathrm{H}(M)$ is bounded;
(2) $\mathrm{H}^0(M) \to \mathrm{H}^0(k \otimes^{\mathrm{L}}_R M)$ is nonzero.
(3) $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i \neq \dim R$;

If in addition $\mathrm{H}(M)$ is finitely generated, M is *maximal Cohen-Macaulay*; usually abbreviated to MCM. Condition (2) implies in particular that $\mathrm{H}^0(k \otimes^{\mathrm{L}}_R M)$ is nonzero, and from this it follows that $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$ for some i. Thus condition (3) implies $\operatorname{depth}_R M = \dim R$; in particular, a big Cohen-Macaulay complex has maximal depth, in the sense of 3.1 and $\mathrm{H}^{\dim R}_{\mathfrak{m}}(M) \neq 0$. However (3) is more restrictive, as the following observation shows.

Lemma 4.1 *If M is an MCM R-complex, then $\mathrm{H}^i(M) = 0$ for $i \notin [0, \dim R]$; moreover, $\mathrm{H}^0(M) \neq 0$.*

Proof The last part of the statement is immediate from condition (2).

Set $d = \dim R$. Let K be the Koszul complex on a system of parameters for R. Then one has isomorphisms

$$K \otimes_R M \simeq K \otimes^{\mathrm{L}}_R \mathrm{R}\Gamma_{\mathfrak{m}}(M) \simeq K \otimes^{\mathrm{L}}_R \Sigma^{-d}\, \mathrm{H}^d_{\mathfrak{m}}(M)$$

where the first one is from (2.1), since $K \otimes_R M$ is $\mathfrak{m}$-power torsion, and the second isomorphism holds by the defining property (3) of a big Cohen-Macaulay complex. Hence

$$\inf \mathrm{H}^*(K \otimes_R M) \geq 0 \quad \text{and} \quad \sup \mathrm{H}^*(K \otimes_R M) \leq d\,.$$

By our hypotheses, the R-module $\mathrm{H}^i(M)$ is finitely generated for each i, and since K is a Koszul complex on d elements, a standard argument leads to the desired vanishing of $\mathrm{H}^i(M)$. □

Any nonzero MCM R-module is MCM when viewed as complex. However, even over Cohen-Macaulay rings, which are not fields, there are MCM complexes that are not modules; see the discussion in (3.1). In the rest of this section we discuss various ways MCM complexes can arise, or can be expected to arise. It turns out that often condition (2) is the one that is hardest to verify. Here is one case when this poses no problem; see 4.3 for an application. The main case of interest is where A is a dg (=differential graded) R-algebra.

Lemma 4.2 *Let A be an R-complex with a unital (but not necessarily associative) multiplication rule such that the Leibniz rule holds and $i := \inf \mathrm{H}_*(A)$ is finite. If $\mathrm{H}_i(A)$ is finitely generated, then the identity element of A is nonzero in $\mathrm{H}_0(A \otimes^{\mathrm{L}}_R k)$.*

Proof One has $\mathrm{H}_i(A \otimes^{\mathrm{L}}_R k) \cong \mathrm{H}_i(A) \otimes_R k$ and the latter module is nonzero, by Nakayama's lemma and the finite generation hypothesis. We have $A \otimes^{\mathrm{L}}_R k = A \otimes_R T$ where T is a Tate resolution of k; see [48]. So $A \otimes_R T$ is also a (possibly non-associative) dg algebra. Thus if the identity element were trivial in $\mathrm{H}(A \otimes_R T)$, then $\mathrm{H}(A \otimes_R T) = 0$ holds, contradicting $\mathrm{H}_i(A \otimes^{\mathrm{L}}_R k) \neq 0$. □

The MCM property for complexes has a simple interpretation in terms of their duals with respect to dualizing complexes.

4.2 Dualizing Complexes

Let D be a dualizing complex for R, normalized[1] so D^i is nonzero only in the range $[0, d]$, where $d := \dim R$ and always with nonzero cohomology in degree 0. Thus D is an R-complex with $\mathrm{H}(D)$ finitely generated, and $\mathrm{R}\Gamma_{\mathfrak{m}}(D) \simeq \Sigma^{-d} E$, where E is the injective hull of k; see [41, Chapter 2, §3] see also Chapter V. For any R-complex M set

$$M^{\dagger} := \mathrm{RHom}_R(M, D)\,.$$

One version of the local duality theorem is that the functor $M \mapsto M^{\dagger}$ is a contravariant equivalence when restricted to $\mathrm{D^b}(\operatorname{mod} R)$, the bounded derived category of finitely generated R-modules; see [41, Chapter 2, Theorem 3.5]. For M in this subcategory, this gives the last of following quasi-isomorphisms:

$$\begin{aligned}
\mathrm{RHom}_R(M^{\dagger}, E) &= \mathrm{RHom}_R(\mathrm{RHom}_R(M, D), E) \\
&\simeq \mathrm{RHom}_R(\mathrm{RHom}_R(M, D), \Sigma^d\, \mathrm{R}\Gamma_{\mathfrak{m}}(D)) \\
&\simeq \Sigma^d\, \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{RHom}_R(\mathrm{RHom}_R(M, D), D)) \\
&\simeq \Sigma^d\, \mathrm{R}\Gamma_{\mathfrak{m}}(M)
\end{aligned}$$

The rest are standard. Passing to cohomology yields the usual local duality:

$$\mathrm{Hom}_R(\mathrm{H}^i(M^{\dagger}), E) \cong \mathrm{H}^{d-i}_{\mathfrak{m}}(M) \quad \text{for each } i. \tag{4.1}$$

When R is $\mathfrak{m}$-adically complete, one can apply Matlis duality to express $\mathrm{H}^i(M^{\dagger})$ as a dual of $\mathrm{H}^{d-i}_{\mathfrak{m}}(M)$.

We also need to introduce a class of maps that will play an important role in the sequel: For any R-module N let ζ^i_N denote the composition of maps

[1] In [18, 41], a dualizing complex is normalized to be nonzero in $[-d, 0]$.

$$\operatorname{Ext}^i_R(k,N) \xrightarrow{\cong} \operatorname{Ext}^i_R(k, \mathrm{R}\Gamma_{\mathfrak{m}}(N)) \longrightarrow \operatorname{Ext}^i_R(R, \mathrm{R}\Gamma_{\mathfrak{m}}(N)) \cong \mathrm{H}^i_{\mathfrak{m}}(N) \qquad (4.2)$$

where the one in the middle is induced by the surjection $R \to k$. We will be particular interested in ζ^d_N. If this map is nonzero, then $\dim_R N = \dim R$, but the converse does not hold.

Proposition 4.3 *With D as above and M an R-complex with $\mathrm{H}(M)$ finitely generated, set $N := \mathrm{H}^0(M^\dagger)$. Then M is MCM if and only if $M^\dagger \simeq N$ and the map ζ^d_N is nonzero, for $d = \dim R$.*

Proof Given the hypothesis on the local cohomology on M, it follows that $\mathrm{H}^i(M^\dagger)$ is nonzero for $i \neq 0$ and hence $M^\dagger \simeq N$. Moreover, this quasi-isomorphism yields

$$\mathrm{R}\Gamma_{\mathfrak{m}}(N) \simeq \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{RHom}_R(M,D)) \simeq \mathrm{RHom}_R(M, \mathrm{R}\Gamma_{\mathfrak{m}}(D)) \simeq \Sigma^{-d} \operatorname{Hom}_R(M,E)\,.$$

Therefore the map (4.2) is induced by (to be precise, the degree 0 component of the map in cohomology induced by) the map

$$\mathrm{RHom}_R(k, \operatorname{Hom}_R(M,E)) \longrightarrow \operatorname{Hom}_R(M,E)$$

By adjunction, the map above is

$$\mathrm{RHom}_R(k \otimes^{\mathrm{L}}_R M, E) \longrightarrow \operatorname{Hom}_R(M,E)$$

That is to say, (4.2) is the Matlis dual of the map $\mathrm{H}^0(M) \to \mathrm{H}^0(k \otimes^{\mathrm{L}}_R M)$. This justifies the claims.

Clearly, these steps are reversible: if N is a finitely generated R-module such that the map (4.2) is nonzero, the R-complex $\mathrm{RHom}_R(N,D)$ is MCM. □

Here then is a way (and the only way) to construct MCM complexes when R has a dualizing complex: Take a finitely generated R-module N for which ζ^d_N is nonzero; then the complex $\mathrm{RHom}_R(N,D)$ is MCM. It thus becomes important to understand the class of finitely generated R-modules for which the map ζ^d_N is nonzero.

To that end let F be a minimal free resolution of k, and set

$$\Omega := \operatorname{Coker}(F_{d+1} \to F_d)\,;$$

this is the dth syzygy module of k. Since minimal free resolutions are isomorphic as complexes, this Ω is independent of the choice of resolution, up to an isomorphism. The canonical surjection $F \to F_{\geqslant d}$ gives a morphism in $\mathrm{D}(R)$:

$$\varepsilon\colon k \longrightarrow \Sigma^d \Omega\,. \qquad (4.3)$$

We view it as an element in $\operatorname{Ext}^d_R(k,\Omega)$. The map ζ^d_Ω below is from (4.2).

Lemma 4.4 *One has $\zeta^d_\Omega(\varepsilon) = 0$ if and only if $\zeta^d_\Omega = 0$ if and only if $\zeta^d_N = 0$ for all R-modules N.*

Proof Fix an R-module N. Any map f in $\operatorname{Hom}_R(\Omega, N)$ induces a map

$$f_*\colon \operatorname{Ext}^d_R(k,\Omega) \longrightarrow \operatorname{Ext}^d_R(k,N)\,.$$

Let F be a resolution of k as above, defining Ω. Any map $k \to \Sigma^d N$ in $\mathrm{D}(R)$ is represented by a morphism of complexes $F \to \Sigma^d N$, and hence factors through the surjection $F \to F_{\geqslant d}$, that is to say, the morphism ε. We deduce that any element of $\operatorname{Ext}^d_R(k,N)$ is of the form $f_*(\varepsilon)$, for some f in $\operatorname{Hom}_R(\Omega,N)$.

In particular, $\operatorname{Ext}^d_R(k,\Omega)$ is generated by ε as a left module over $\operatorname{End}_R(\Omega)$. This observation, and the linearity of the ζ^d_Ω with respect to $\operatorname{End}_R(\Omega)$, yields $\zeta^d_\Omega = 0$ if and only if $\zeta^d_\Omega(\varepsilon) = 0$. Also each f in $\operatorname{Hom}_R(\Omega,N)$ induces a commutative square

$$\begin{array}{ccc}
\operatorname{Ext}^d_R(k,\Omega) & \xrightarrow{\zeta^d_\Omega} & \mathrm{H}^d_{\mathfrak{m}}(\Omega) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^d_{\mathfrak{m}}(f)} \\
\operatorname{Ext}^d_R(k,N) & \xrightarrow{\zeta^d_N} & \mathrm{H}^d_{\mathfrak{m}}(N)
\end{array}$$

Thus if $\zeta^d_\Omega = 0$ we deduce that $\zeta^d_N(f_*\varepsilon) = 0$. By varying f we conclude from the discussion above that $\zeta^d_N = 0$. □

We should record the following result immediately. It is one formulation of the Canonical Element Theorem; see [22, (3.15)]. The “canonical element” in question is $\zeta^d_\Omega(\varepsilon)$; see Lemma 4.4.

Theorem 4.5 *For any noetherian local ring R, one has $\zeta^d_\Omega \neq 0$.* □

Here then is first construction of an MCM R-complex.

Corollary 4.6 *If R has a dualizing complex the R-complex $\Omega^\dagger$ is MCM.* □

Remark 4.7 Suppose R has a dualizing complex. Given Proposition 4.3 and Lemma 4.4 it follows that $\Omega^\dagger$ is MCM if and only if there exists *some* R-complex M that is MCM. Therefore, the Canonical Element Theorem, in all its various formulations [22], is equivalent to the statement that R has an MCM R-complex!

We now describe another way to construct an MCM complex. Let D be a dualizing complex for R and set $\omega_R := \mathrm{H}^0(D)$; this is the *canonical module* of R.

Lemma 4.8 *One has $\zeta^d_\Omega \neq 0$ if and only if $\zeta^d_{\omega_R} \neq 0$.*

Proof We write ω for ω_R. Given Lemma 4.4 we have to verify that if $\zeta^d_\Omega \neq 0$, then $\zeta^d_\omega \neq 0$. Let E be an injective hull of k, the residue field of R. Since this is a faithful injective, there exists a map $\alpha\colon \mathrm{H}^d_{\mathfrak{m}}(\Omega) \to E$ such that $\alpha \circ \zeta^d_\Omega \neq 0$.

It follows from local duality 4.1, applied to $M = \Omega$, that α is induced by a morphism $f: \Omega \to D$; equivalently, an R-linear map $f: \Omega \to \omega$. This gives the following commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ext}^d_R(k,\Omega) & \xrightarrow{\zeta^d_\Omega} & \mathrm{H}^d_{\mathfrak{m}}(\Omega) & & \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle \mathrm{H}^d_{\mathfrak{m}}(f)}\downarrow & \searrow^{\alpha} & \\
\operatorname{Ext}^d_R(k,N) & \xrightarrow[\zeta^d_\omega]{} & \mathrm{H}^d_{\mathfrak{m}}(\omega) & \longrightarrow & E
\end{array}$$

Since $\alpha \circ \zeta^d_\Omega \neq 0$ we conclude that $\zeta^d_\omega \neq 0$, as desired. □

Corollary 4.9 *If R has a dualizing complex, the R-complex ${\omega_R}^\dagger$ is MCM.* □

The preceding result prompts a natural question.

Question 4.10 When is the dualizing complex itself an MCM complex?

Let R be a local ring with a dualizing complex D, normalized as in 4.2. The local cohomology of D has the right properties, so, by Proposition 4.3, the R-complex D is MCM precisely when ζ^d_R is nonzero. Easy examples involving non-domains show that this is not always the case; Dutta [10] asked: *Is ζ^d_R nonzero whenever R is a complete normal domain?* Recently, Ma, Singh, and Walther [38] constructed counterexamples.

On the other hand, when R is *quasi-Gorenstein*, that is to say, when ω_R is free, it follows from Corollary 4.9 that D is MCM.

Here is a broader question, also of interest, concerning the maps ζ^i_N: It is easy to check that this is nonzero when $i = \operatorname{depth}_R N$. What conditions on N ensure that this is the only i for which it is true? By taking direct sums of modules of differing depths, we obtain modules N with ζ^i_N nonzero for more than a single i.

Example 4.11 When $(R, \mathfrak{m}, k)$ is a regular local ring and N is a finitely generated R-module, then N is Buchsbaum if and only if ζ^i_N is surjective for each $i < \dim_R N$. So any non-CM Buchsbaum R-module would give an example.

Remark 4.12 Let $F \simeq k$ be a free resolution of k and $\underline{r} := r_1, \dots, r_n$ elements such that $(\underline{r})$ is primary to the maximal ideal. The canonical surjection $R/(\underline{r}) \to k$ lifts to a morphism of complexes $K(\underline{r}; R) \to F$. Applying $\operatorname{Hom}_R(-, N)$ induces maps

$$\operatorname{Ext}^i_R(k, N) \longrightarrow \mathrm{H}^i(\underline{r}; N)$$

It is easy to verify that ζ^i_N factors through this map. What is more, if $\underline{s}$ is another sequence of elements such that $\underline{r} \in (\underline{s})$, then the map above factors as

$$\operatorname{Ext}^i_R(k, N) \longrightarrow \mathrm{H}^i(\underline{s}; N) \longrightarrow \mathrm{H}^i(\underline{r}; N)$$

Thus if any of maps above are zero, so is ζ^i_N.

We would like to record a few more observations about MCM complexes.

Remark 4.13 Let $(R, \mathfrak{m}, k)$ be an arbitrary noetherian local ring. Then its $\mathfrak{m}$-adic-completion, $\widehat{R}$, has a dualizing complex, and hence an MCM $\widehat{R}$-complex, as discussed above. Since any MCM $\widehat{R}$-complex is a big Cohen-Macaulay complex over R, we conclude that R has a big Cohen-Macaulay complex, and, in particular, a complex of maximal depth.

Remark 4.14 Assume R has a dualizing complex and that M is an MCM R-complex. It is easy to check using Proposition 4.3 that $M_{\mathfrak{p}}$ is an MCM $R_{\mathfrak{p}}$-complex for $\mathfrak{p}$ in Spec R, as long as condition (2) defining MCM complexes holds at $\mathfrak{p}$. For example, if A is dg R-algebra that is MCM as an R-complex, then since $A_{\mathfrak{p}}$ is a dg $R_{\mathfrak{p}}$-algebra, Lemma 4.2 implies that it is an MCM $R_{\mathfrak{p}}$-complex.

Remark 4.15 While MCM complexes have their uses, as the discussion in Sect. 3 makes clear, they are not always a good substitute for MCM *modules*. Indeed, in [21, §3] Hochster proves if every local ring has an MCM module, then the Serre positivity conjecture on multiplicities is a consequence of the vanishing conjecture; see also [25, §4]. Hochster's arguments cannot be carried out with MCM complexes in place of modules. The basic problem is this: Given a finite free complex F, over a local ring R, with homology of finite length, if M is an MCM R-module, then $\mathrm{H}(F \otimes_R M)$ is concentrated in at most one degree; this need not be the case when M is an MCM complex. Indeed this is clear from Iversen's Amplitude inequality [28], which is a reformulation of the New Intersection Theorem, and reads:

$$\operatorname{amp}(F \otimes^{\mathrm{L}}_R X) \geq \operatorname{amp}(X)$$

where F is any finite free complex with $\mathrm{H}(F) \neq 0$ and X is an R-complex with $\mathrm{H}(X)$ bounded. Here $\operatorname{amp}(X) := \sup \mathrm{H}_*(X) - \inf \mathrm{H}_*(X)$, the *amplitude* of X. By the way, the Amplitude Inequality holds even when $\mathrm{H}(X)$ is unbounded [13].

4.3 Via Resolution of Singularities

The constructions of MCM complexes described above are independent of the characteristic of the ring, but proving that they are MCM is a non-trivial task, for it depends on knowing that one has MCM complexes to begin with; see Remark 4.7. Next, we describe a complex that arises from a completely different source that one can prove is MCM independently. The drawback is that it is restricted to algebras essentially of finite type and containing the rationals. We first record a well-known observation about proper maps.

Lemma 4.16 *Let R be any commutative noetherian ring and $\pi : X \to \operatorname{Spec}(R)$ a proper map from a noetherian scheme X. Viewed as an object in $\mathrm{D}(R)$ the complex $\mathrm{R}\pi_*\mathcal{O}_X$ is equivalent to a dg algebra with cohomology graded-commutative and* *finitely generated. When R contains a field of characteristic zero, the dg algebra itself can be chosen to be graded-commutative.*

Proof By Grothendieck [17, Theorem 3.2.1], since $\mathcal{O}_X$ is coherent and π is proper, $\mathrm{R}\pi_*\mathcal{O}_X$ is coherent and hence its cohomology is finitely generated. Next, we explain why this complex is equivalent, in $\mathrm{D}(R)$, to a dg algebra. The idea is that $\mathcal{O}_X$ is a ring object in $\mathrm{D}(X)$ and there is a natural morphism

$$\mathrm{R}\pi_*\mathcal{F} \otimes^{\mathrm{L}}_R \mathrm{R}\pi_*\mathcal{G} \longrightarrow \mathrm{R}\pi_*(\mathcal{F} \otimes^{\mathrm{L}}_X \mathcal{G})$$

so $\mathrm{R}\pi_*\mathcal{O}_X$ is ring object in $\mathrm{D}(R)$. One can realize this concretely as follows.

Let $\{U_i\}_{i=1}^n$ be an affine cover of X. Then the Čech complex computing $\mathrm{R}\pi_*\mathcal{O}_X$ is equivalent to the total complex associated to the co-simplicial commutative ring

$$0 \longrightarrow \prod_i \Gamma(X, U_i) \rightrightarrows \prod_{i,j} \Gamma(X, U_i \cap U_j) \Rrightarrow \prod_{i,j,k} \Gamma(X, U_i \cap U_j \cap U_k)$$

It remains to point out that the Alexander-Whitney map makes the normalization of a co-simplicial ring a dg algebra, with graded-commutative cohomology. Moreover, since R contains a field of characteristic zero, it is even quasi-isomorphic to a graded-commutative dg algebra. □

The statement of the next result, which is due to Roberts [41], invokes the resolution of singularities in characteristic zero, established by Hironaka. The proof uses Grothendieck duality for projective maps [18] and the theorem of Grauert and Riemenschneider [15] on the vanishing of cohomology. Given these, the calculation that is needed is standard; see the proof of [19, Proposition 2.2] due to Hartshorne and Ogus. It will be clear from the proof that the result extends to any context where one has sufficient vanishing of cohomology; see [41, Theorem 3.3].

Proposition 4.17 *Let $(R, \mathfrak{m}, k)$ be an excellent noetherian local ring containing a field of characteristic zero, and admitting a dualizing complex. Let $\pi : X \to \mathrm{Spec}(R)$ be a resolution of singularities. The R-complex $\mathrm{R}\pi_*\mathcal{O}_X$ is MCM and equivalent to a graded-commutative dg algebra.*

Proof Given Lemmas 4.16 and 4.2 it remains to verify that $\mathrm{H}^j_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X) = 0$ for $j \neq d$, where $d := \dim R$. Let D be a dualizing complex for R and $\pi^! D = \omega_X$, the dualizing sheaf for X. Since the R-complex $\mathrm{R}\pi_*\mathcal{O}_X$ has finitely generated cohomology, local duality 4.1 yields the first isomorphism below

$$\begin{aligned} \mathrm{H}^j_{\mathfrak{m}}(\mathrm{R}\pi_* O_X) &\cong \mathrm{Ext}^{d-j}_R(\mathrm{R}\pi_*\mathcal{O}_X, D)^{\vee} \\ &\cong \mathrm{Ext}^{d-j}_X(\mathcal{O}_X, \pi^! D)^{\vee} \\ &= \mathrm{H}^{d-j}(X, \omega_X)^{\vee} \end{aligned}$$

The second isomorphism is by coherent Grothendieck duality [18]. It remains to invoke the Grauert-Riemenschneider vanishing theorem [15]—see Murayama [39, Theorems A&B] for the version that applies in the present generality—to deduce that the last module in the display is 0 for all $j \neq d$. □

Here is a natural question, growing out of Proposition 4.17. A positive answer might have a bearing on the theory of multiplier ideals; see Theorem 5.3.

Question 4.18 When R contains a field of positive characteristic, or is of mixed characteristic, does it have an MCM R-complex that is also a dg algebra? What about a graded-commutative dg algebra?

5 Applications to Birational Geometry

In this section we prove two celebrated results in birational geometry using MCM complexes constructed via Proposition 4.17. The first one generalizes Boutot's theorem on rational singularities [8]; the argument is only a slight reworking of Boutot's proof, emphasizing the role of the derived push-forward as an MCM complex. Related circles of ideas can be found in the work of Bhatt, Kollár, Kovács, and Ma [6, 30, 31, 34].

Theorem 5.1 *Let $\rho\colon Z \to \operatorname{Spec} R$ be a map of excellent schemes containing a field of characteristic zero, admitting dualizing complexes, and such that $R \to \mathrm{R}\rho_*\mathcal{O}_Z$ splits in $\mathrm{D}(R)$. If Z has rational singularities, then so does R.*

Proof We may assume $(R, \mathfrak{m})$ is local. Note that the condition implies $R \to \rho_*\mathcal{O}_Z$ is injective so in particular R is reduced (as Z is reduced). Take $\pi : X \to \operatorname{Spec} R$ to be a resolution of singularities. Then there is a (reduced) subscheme of $X \times_{\operatorname{Spec} R} Z$ that is birational over Z for each irreducible component of Z. Let Y be a resolution of singularities of that subscheme. Thus there is a commutative diagram:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\ \rho\ } & \operatorname{Spec} R. \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{R}\rho_*\mathcal{O}_Z \\ \big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ \mathrm{R}\pi_*\mathcal{O}_X & \longrightarrow & \mathrm{R}\rho_*\,\mathrm{R}\sigma_*\mathcal{O}_Y\,. \end{array}$$

The right vertical map is an isomorphism since Z has rational singularities. Now since $R \to \mathrm{R}\rho_*\mathcal{O}_Z$ splits in $D(R)$, chasing the diagram shows that $R \to \mathrm{R}\pi_*\mathcal{O}_X$ splits in $\mathrm{D}(R)$. In particular, we know that the induced map

$$\mathrm{H}^i_{\mathfrak{m}}(R) \hookrightarrow \mathrm{H}^i_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)$$

is split-injective for all i. Because $\mathrm{R}\pi_*\mathcal{O}_X$ is a MCM complex, by Proposition 4.17, it follows that $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ for $i < d$, that is to say, R is Cohen-Macaulay. Finally, the Matlis dual of the injection above yields a surjective map $\pi_*\omega_X \twoheadrightarrow \omega_R$. Therefore $\pi_*\omega_X \cong \omega_R$ since $X \to \operatorname{Spec} R$ is birational.

Putting these together yields $\omega_R^\bullet \cong \mathrm{R}\pi_*\omega_X^\bullet$, where $\omega_R^\bullet$ and $\omega_X^\bullet$ are the normalized dualizing complex of R and X respectively. Applying $\mathrm{RHom}_R(-, \omega_R^\bullet)$ and using Grothendieck duality yields $R \cong \mathrm{R}\pi_*\mathcal{O}_X$. Thus R has rational singularities. □

Here is an application.

Corollary 5.2 *If (R, Δ) is KLT, then R has rational singularities.*

Proof Let $\pi : Y \to X = \operatorname{Spec} R$ be a log resolution of (R, Δ). Since (R, Δ) is KLT, we know that $\lceil K_Y - \pi^*(K_X + \Delta)\rceil$ is effective and exceptional, thus

$$R = \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil) = \mathrm{R}\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil)\,,$$

where the second equality follows from relative Kawamata-Viehweg vanishing [32, Theorem 9.4.1]; see [39, Theorems A&B] for the general version. Then the composition of maps

$$R \to \mathrm{R}\pi_*\mathcal{O}_Y \to \mathrm{R}\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil) \cong R\,,$$

is an isomorphism, that is to say, the map $R \to \mathrm{R}\pi_*\mathcal{O}_Y$ splits in $\mathrm{D}(R)$. Theorem 5.1 then implies R has rational singularities. □

Our second application is a new proof of the subadditivity property of multiplier ideals [32]. The first proof in the generality below is due to Jonsson and Mustaţă [29, Theorem A.2]. Our idea of using the MCM property of $\mathrm{R}\pi_*\mathcal{O}_X$ to prove this comes from the analogous methods in positive and mixed characteristic [36, 47].

Theorem 5.3 *Let $(A, \mathfrak{m})$ be an excellent noetherian regular local ring containing a field of characteristic zero. Given ideals $\mathfrak{a}, \mathfrak{b}$ in A and numbers $s, t \in \mathbb{Q}_{\geq 0}$, one has $J(A, \mathfrak{a}^s\mathfrak{b}^t) \subseteq J(A, \mathfrak{a}^s)J(A, \mathfrak{b}^t)$.*

Proof We first claim that we may assume that $\mathfrak{a}, \mathfrak{b}$ are both principal ideals. This type of reduction is standard for multiplier ideals [32, Proposition 9.2.26], but we do not know a reference for the case of *mixed* multiplier ideals $J(A, \mathfrak{a}^s\mathfrak{b}^t)$. However the argument is the same and we now sketch it. Indeed, fix general elements $f_1, \ldots, f_k$

in $\mathfrak{a}$ and $g_1, \dots, g_l$ in $\mathfrak{b}$ for $k > s$ and $l > t$ and set

$$D_1 = \frac{1}{k} \sum \mathrm{Div}(f_i) = \frac{1}{k} \mathrm{Div}(\prod (f_i))$$
$$D_2 = \frac{1}{l} \sum \mathrm{Div}(g_i) = \frac{1}{l} \mathrm{Div}(\prod (g_i)) .$$

For a log resolution $\pi : X \to \mathrm{Spec} A$ of $(A, \mathfrak{a}, \mathfrak{b})$ with $\mathcal{O}_X(-F) = \mathfrak{a} \cdot \mathcal{O}_X$ and $\mathcal{O}_X(-G) = \mathfrak{b} \cdot \mathcal{O}_X$, we have that

$$\pi_*^{-1} \mathrm{Div}(f_i) + F_{\mathrm{exc}} = \pi^* \mathrm{Div}(f_i) \quad \text{and} \quad \pi_*^{-1} \mathrm{Div}(g_i) + G_{\mathrm{exc}} = \pi^* \mathrm{Div}(g_i).$$

where π_*^{-1} denotes the strict transform and F_{exc} and G_{exc} are the π-exceptional parts of F and G. Since the f_i and g_i are generic, the associated divisors and their strict transforms are reduced. A straightforward computation then shows that

$$\lfloor sF \rfloor = \left\lfloor \frac{s}{k} \sum \pi^* \mathrm{Div}(f_i) \right\rfloor \quad \text{and} \quad \lfloor tG \rfloor = \left\lfloor \frac{t}{l} \sum \pi^* \mathrm{Div}(g_i) \right\rfloor .$$

Thus $J(A, \mathfrak{a}^s \mathfrak{b}^t) = J(A, (\prod f_i)^{s/k} (\prod g_i)^{t/l})$, and likewise $J(A, \mathfrak{a}^s) = J(A, (\prod f_i)^{s/k})$ and $J(A, \mathfrak{b}^t) = J(A, (\prod g_i)^{t/l})$. Therefore we may assume that $\mathfrak{a}$ and $\mathfrak{b}$ are principal.

Now we assume $\mathfrak{a} = (f)$ and $\mathfrak{b} = (g)$. Let R be the normalization of $A[f^{1/d_s}, g^{1/d_t}]$ where d_s and d_t are the denominators of s and t; thus f^s, g^t are elements in R. Let $\pi : X \to \mathrm{Spec}\, R$ be a resolution of singularities. Thus $X \to \mathrm{Spec}\, A$ is a regular alteration; we write π also for this map.

In what follows, to simplify notation, we write E for $\mathrm{H}^d_{\mathfrak{m}}(A)$. Given an element $r \in R$ let 0^r_E be the kernel of the composite map

$$E = \mathrm{H}^d_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(R) \xrightarrow{r} \mathrm{H}^d_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_* \mathcal{O}_X)$$

Now suppose that a power r^m of r lives in A (for instance $r = f^s$ or $r = g^t$). Then by Blickle et al. [7, Theorem 8.1] we have that $\mathrm{Tr}(J(\omega_R, r)) = J(A, (r^m)^{1/m})$. By local duality it is easy to see that

$$J(A, (r^m)^{1/m}) = \mathrm{ann}_A\, 0^r_E .$$

In particular, $J(A, f^s) = \mathrm{ann}_A\, 0^{f^s}_E$ and $J(A, g^t) = \mathrm{ann}_A\, 0^{g^t}_E$.

We next claim that the following inclusion holds:

$$\{\eta \in E \mid J(A, f^s) \cdot \eta \subseteq 0^{g^t}_E\} \subseteq 0^{f^s g^t}_E . \tag{5.1}$$

Indeed, suppose $J(A, f^s)\eta \subseteq 0_E^{g^t}$, then $J(A, f^s) \cdot g^t\eta = 0$ in $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)$. Note that $g^t\eta$ makes sense in $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)$ as the latter is a module over R. Thus

$$g^t\eta \in \operatorname{ann}_{\mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)} J(A, f^s) \cong \operatorname{Hom}_A(A/J(A, f^s), \mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X))\,.$$

Next, because $\mathrm{R}\pi_*\mathcal{O}_X$ is MCM, by Proposition 4.17, one gets the equality below

$$h^{-i}(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E) \cong \mathrm{H}^{d-i}_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X) = 0$$

for all $i \geq 1$. Thus we conclude that $g^t\eta$ is in the module

$$\begin{aligned}
\operatorname{Hom}_A(\frac{A}{J(A, f^s)}, \mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)) &\cong h^0\left(\operatorname{RHom}_A(\frac{A}{J(A, f^s)}, \mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E)\right) \\
&\cong h^0\left(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(\frac{A}{J(A, f^s)}, E)\right) \\
&\cong h^0\left(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A \operatorname{ann}_E J(A, f^s)\right)\,.
\end{aligned}$$

The second isomorphism follows from [12, Proposition 1.1 (4)], noting that A is regular thus every bounded complex is isomorphic to a bounded complex of flat modules in $\mathrm{D}(A)$, and the third isomorphism follows from the fact that E is an injective A-module.

Consider the following composite map; again, the second multiplication by f^s map makes sense since we can view $\mathrm{R}\pi_*\mathcal{O}_X$ as a complex over R and not merely over A:

$$E \to h^0\left(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E\right) \xrightarrow{\cdot f^s} h^0\left(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E\right)\,.$$

Its kernel is $\operatorname{ann}_E J(A, f^s)$, by Matlis duality. Thus the composition of the natural induced maps

$$\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A \operatorname{ann}_E J(A, f^s) \to \mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E \xrightarrow{\cdot f^s} \mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E$$

is zero in h^0. In particular, since $g^t\eta$ is in h^0 of the source of this composite map, we deduce that, viewed as an element in target, namely in

$$h^0\left(\mathrm{R}\pi_*\mathcal{O}_X \otimes^{\mathrm{L}}_A E\right) \cong \mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)$$

it is killed by f^s. Therefore $f^s g^t\eta = 0$ in $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{R}\pi_*\mathcal{O}_X)$ and hence $\eta \in 0_E^{f^s g^t}$.

This justifies (5.1).

Finally, for any $z \in \operatorname{ann}_E J(A, f^s)J(A, g^t)$, we have $J(A, f^s)z \subseteq 0_E^{g^t}$ and thus $z \in 0_E^{f^s g^t}$ by (5.1). Therefore

$$\operatorname{ann}_E J(A, f^s)J(A, g^t) \subseteq 0_E^{f^s g^t}$$

and hence by Matlis duality $J(A, f^s g^t) \subseteq J(A, f^s)J(A, g^t)$. □

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Subadditivity of Syzygies of Ideals and Related Problems

Jason McCullough

1 Introduction

Graded free resolutions are highly useful vehicles for computing invariants of ideals and modules. Even when restricting to finite minimal graded free resolutions of graded ideals and modules over a polynomial ring, there are questions regarding the structure of such resolutions that we do not yet understand. The aim of this survey paper is to collect known results on the degrees of syzygies of graded ideals and pose some open questions.

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ denote a standard graded polynomial ring over $\mathbb{K}$. If M is a finitely generated, graded S-module, let $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(M, k)_j$ denote the graded Betti numbers of M, and let $\bar{t}_i(M) = \sup\{j \mid \beta_{i,j}(M) \neq 0\}$ denote the ith maximal graded shift of M. Thus $\bar{t}_i(M)$ denotes the maximal degree of an element in a minimal generating set of the ith syzygies of M. The shifts $\bar{t}_i(M)$ are primarily of interest due to their connection with another invariant, the regularity $\operatorname{reg}(M)$ of M; indeed one can take $\operatorname{reg}(M) = \max\{\bar{t}_i(M) - i\}$ as a definition of the regularity of M. The underlying question considered in this paper is the following:

Question 1.1 Which sequences of integers $(\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_m)$ can be realized as $(\bar{t}_0(S/I), \bar{t}_1(S/I), \ldots, \bar{t}_m(S/I))$ for some graded ideal $I \subseteq S$?

Note that the more general question in which S/I is replaced by an arbitrary graded module M is not so interesting. One of the main results of Boij-Söderberg theory, proved by Eisenbud, Fløystad, and Weyman [23] in characteristic 0 and Eisenbud and Schreyer [26] in all characteristics, shows that every strictly increasing sequence $\bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_c$ of integers with $c \leq n$ can be realized as the shifts of some

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graded, Cohen-Macaulay, pure S-module of codimension c. On the other hand, one sees immediately that the same statement is not true for cyclic modules S/I. Indeed, the sequence $(0, 1, 3, 4)$ cannot be realized as the maximal graded shifts of any cyclic module S/I. If it could, then I would be generated by linear forms (since $\bar{t}_1(S/I) = 1$) and yet would have minimal quadratic syzygies, which is impossible; any such S/I would be resolved by a Koszul complex with linear differential maps. This degree sequence $(0, 1, 3, 4)$ can be realized by the resolution of Coker(M), where M is a generic 2×4 matrix. (See Example 2.1.) This explains restricting our attention to ideals and cyclic modules.

After a section to collect notation and background, there are four main sections to this paper, each dealing with refinements to Question 1.1. In Sect. 3, we consider effective bounds on $\operatorname{reg}(S/I)$ in terms of some initial segment of the maximal shifts. Thought of another way, we address the question of how much of the resolution of an ideal must be computed to get a reasonable bound on its regularity. In Sect. 4, we consider the subadditivity property of maximal graded shifts; that is, when $\bar{t}_a(S/I) + \bar{t}_b(S/I) \geq \bar{t}_{a+b}(S/I)$ for all $a, b \geq 1$. It is not hard to find examples where this property fails, but for specific classes of ideals, subadditivity has been proved or conjectured. In Sect. 5, we consider bounds on maximal graded shifts for arbitrary ideals. In Sect. 6, we focus specifically on ideals generated by quadrics with linear resolutions for a fixed number of steps. We examine geometric and combinatorial conditions which guarantee resolutions of this form. In the final Sect. 7, we collect some open questions and pose some new problems that we hope will inspire future work in the area.

2 Background

In this section we fix notation used throughout this paper. Let $\mathbb{K}$ denote a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ denote a polynomial ring over $\mathbb{K}$. We assume throughout that S is standard graded, i.e. $\deg(x_i) = 1$ for $1 \leq i \leq n$. We write S_i for the $\mathbb{K}$-vector space spanned by all degree i homogeneous polynomials so that $S = \bigoplus_{i \geq 0} S_i$ as $\mathbb{K}$-vector spaces. We write $S(-j)$ for a rank one free module with generator in degree j so that $S(-j)_i = S_{i-j}$. We consider the resolutions of graded ideals $I = (f_1, \ldots, f_m)$ and graded modules M of S. Note that I is graded if it has a set of homogeneous generators. We write $\mathbf{F}_\bullet$ for the minimal graded free resolution of M so that $F_i = \bigoplus_j S(-j)^{\beta_{ij}(M)}$. The numbers $\beta_{ij}(M)$ are the graded Betti numbers of M and can alternatively be computed as $\beta_{ij} = \dim_k \operatorname{Tor}_i^S(M, k)_j$. (We drop the module M from the notation when it is clear from context.) In particular, the graded Betti numbers are isomorphism invariants of M. It is convenient to keep track of the graded Betti numbers in a matrix called the Betti table; by convention, we place $\beta_{i,i+j}(M)$ in position (i, j). One of the advantages of the Betti table is that it allows us to quickly read off the projective dimension and regularity of the module being resolved. More precisely, we may define $\operatorname{reg}(M) = \max\{j - i \mid \beta_{ij}(M) \neq 0\}$ and

$\mathrm{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0\}$. Thus $\mathrm{reg}(M)$ is the index of the last nonzero row in the Betti table of M and $\mathrm{pd}(M)$ is the index of the last nonzero column.

If we want to study the structure of minimal, graded free resolutions more closely, we can consider the maximal and minimal graded shifts of M. For each $i \geq 0$, set

$$\overline{t}_i(M) = \sup\{j \mid \beta_{ij}(M) \neq 0\}$$

and

$$\underline{t}_i(M) = \inf\{j \mid \beta_{ij}(M) \neq 0\}.$$

In other words, $\overline{t}_i(M)$ ($\underline{t}_i(M)$) denotes the maximal (resp. minimal) degree of an element in a minimal generating set of the ith syzygy module of M. When $M = S/I$ is a cyclic module, $\overline{t}_1(S/I)$ denotes the maximal degree of a minimal generator of I. A module M is called pure if $\overline{t}_i(M) = \underline{t}_i(M)$ for all $0 \leq i \leq \mathrm{pd}(M)$.

Example 2.1 Let $S = \mathbb{K}[x_1, \ldots, x_8]$ and let $M = \mathrm{Coker}(\mathbf{A})$, where

$$\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{pmatrix}.$$

is a generic matrix. M is resolved by a Buchsbaum-Rim complex [21, Appendix A2.6] with the following Betti table.

	0	1	2	3
0:	2	4	-	-
1:	-	-	4	2

In particular, M is a pure module with $\underline{t}_0(M) = \overline{t}_0(M) = 0$, $\underline{t}_1(M) = \overline{t}_1(M) = 1$, $\underline{t}_2(M) = \overline{t}_2(M) = 3$, and $\underline{t}_3(M) = \overline{t}_3(M) = 4$. This is the example mentioned in the introduction.

In a minimal graded free resolution, it is clear that the minimal graded shifts are strictly increasing, that is $\underline{t}_{i-1}(S/I) < \underline{t}_i(S/I)$ for all $1 \leq i \leq \mathrm{pd}(S/I)$. The maximal graded shifts are strictly increasing up to $\mathrm{ht}(I)$.

Proposition 2.2 ([50, Proposition 2.2], [4, Lemma 5.1]) *Let I be a standard graded ideal in a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. For all $1 \leq i \leq \mathrm{ht}(I)$, one has*

$$\overline{t}_{i-1}(S/I) < \overline{t}_i(S/I).$$

To see that $\overline{t}_{i-1}(S/I) \geq \overline{t}_i(S/I)$ is possible for $i > \mathrm{ht}(I)$, see Example 3.11. For the remainder of the paper, we primarily focus on the maximal graded shifts $\overline{t}_i(S/I)$ for graded cyclic modules S/I.

3 Effective Bounds on Regularity

In this section we consider bounds on the regularity of ideals in terms of some initial segment of the maximal graded shifts. Fix a graded ideal I and write $d(I)$ for the maximal degree of an element in a minimal generating set of I. We recall that $\operatorname{reg}(S/I) = \max\{\bar{t}_i(S/I) - i\}$ and so in particular $\operatorname{reg}(S/I) \geq d(I) - 1 = \bar{t}_1(S/I) - 1$. A natural question is to what extent $\operatorname{reg}(S/I)$ can exceed $d(I)$. Without referencing the number of variables, no such bound is possible—an ideal I generated by a complete intersection of n quadrics has $d(I) = 2$ and $\operatorname{reg}(S/I) = n$. If we fix the number of variables to be n, then there is a well-known doubly exponential upper bound on regularity, due to Bayer and Mumford in characteristic 0, and later Caviglia and Sbarra in all characteristics.

Theorem 3.1 ([7, Proposition 3.8], [14, Corollary 2.7]) *Let I be a graded ideal in $S = \mathbb{K}[x_1, \dots, x_n]$. Then*

$$\operatorname{reg}(I) \leq (2d(I))^{2^{n-2}}.$$

Recall that $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. For recent improvements to the above bound, see [15, Corollary 2.3]. Examples based on a construction of Mayr and Meyer [48] due to Bayer and Mumford [7] show that the above bound is close to best possible. (See also [8] and [43].) Thus even by referencing the number of variables, the best bound on the regularity of a cyclic module in terms of the first maximal graded shift is doubly exponential. One could hope that by taking more of the resolution into account, one might be able to formulate a tighter bound on the regularity of ideals. The construction of Ullery in the next subsection shows that, if we do not reference the length of the resolution or number of variables, this is not possible. However, if we do take into account the length of the resolution, one can achieve at least a polynomial bound on regularity in terms of the first half of the maximal graded shifts.

Theorem 3.2 ([49, Theorem 4.7]) *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a homogenous ideal. Set $h = \lceil \frac{n}{2} \rceil$. Then*

$$\operatorname{reg}(S/I) \leq \sum_{i=1}^{h} \bar{t}_i(S/I) + \frac{\prod_{i=1}^{h} \bar{t}_i(S/I)}{(h-1)!}.$$

The proof of the previous result follows from a careful analysis of the Boij-Söderberg decomposition of the Betti table of S/I. A similar idea also proves [49, Theorem 4.4] and hints that stronger statements might be true; see Conjecture 7.5. Note that this result requires no hypotheses on the ideal I.

Using different techniques, the author proved that there is a linear bound on $\operatorname{reg}(S/I)$ in terms of the first $\operatorname{pd}(S/I) - \operatorname{codim}(I)$ maximal graded shifts in the resolution.

Theorem 3.3 ([50, Corollary 3.7]) *Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a graded ideal with $p = \mathrm{pd}(S/I)$ and $c = codim(I)$. Then*

$$\mathrm{reg}(S/I) \leq \max_{1 \leq i \leq p-c} \{\bar{t}_i(S/I) + (p-i)\bar{t}_1(S/I)\} + p.$$

Note that again there are no assumptions on the ideal I. In the Cohen-Macaulay case, the result follows from a result of Eisenbud, Huneke, and Ulrich; see Theorem 5.1. The above result follows by reverse induction on p. Ideals in the next subsection show that the above result cannot be substantially improved; there are quadratic ideals of codimension c with linear resolutions for arbitrarily many steps but whose last $c+1$ steps have arbitrarily large degree. See Example 4.1.

One could also hope for better bounds on regularity under some hypotheses on the ideal I. If S/I is Cohen-Macaulay, we have the following natural bound given by Huneke, Migliore, Nagel, and Ulrich.

Theorem 3.4 ([39, Remark 3.1]) *Let $S = \mathbb{K}[x_1, ..., x_n]$ be a polynomial ring over a field $\mathbb{K}$, and let I be a graded ideal in S of height c such that S/I is Cohen-Macaulay. Then $\mathrm{reg}(S/I) \leq c(d(I) - 1)$, with equality if and only if S/I is a complete intersection generated by c forms of degree $d(I)$. In particular,*

$$\mathrm{reg}(S/I) \leq n(d(I) - 1)$$

for all ideals I with S/I Cohen-Macaulay.

On the other hand, if I is merely prime with fixed $d(I)$, even the first syzygies can have arbitrarily large degree.

Theorem 3.5 ([13, Theorem 6.2]) *Fix a positive integer $s \geq 9$ and field $\mathbb{K}$. There exists a nondegenerate prime ideal P in a polynomial ring S over $\mathbb{K}$ with $d(P) = 6$ and $\bar{t}_1(P) = s$.*

In the following two subsections, we show some of the limits on these sort of regularity bounds by showing to what extent the maximal graded shifts of ideals and cyclic modules mimic those of arbitrary graded modules. The prime ideals in the previous theorem are derived from the following construction of Ullery, which we recast as idealizations.

3.1 Ullery's Designer Ideals via Idealizations

We first recall the idealization construction. Fix a ring R and an R-module M. The idealization (sometimes called the Nagata idealization or trivial extension) of R by M is the ring denoted $R \ltimes M$, which is $R \times M$ as an abelian group with multiplication given by $(r, x) \cdot (s, y) = (rs, ry + sx)$ for $r, s \in R$ and $x, y \in M$. Idealizations are commonly used for constructing Gorenstein rings, when M is the canonical module of R, but here we will be interested in the situation that $R = S$ is a polynomial ring and M is a finitely generated graded S-module. The following is an algebraic description of certain designer ideals of Ullery described in [61].

Fix an increasing sequence of integers $2 = d_1 < d_2 < d_3 < \cdots < d_n$ and set $S = \mathbb{K}[x_1, \ldots, x_n]$. As previously mentioned, there is a pure, Cohen-Macaulay module M with maximal (and minimal) graded shifts $\bar{t}_0(M) = 1$ and $\bar{t}_i(M) = d_i$ for $1 \leq i \leq n$. Denote by $\mathbf{A}$ the first differential in the minimal free resolution of M so that $M = \operatorname{Coker}(\mathbf{A})$. Note that by our choice of shifts, $\mathbf{A}$ is a matrix of linear forms. We then consider the standard graded ring $S \ltimes M$. If M has m minimal generators, then we can represent $S \ltimes M$ as a homogeneous quotient of the standard graded polynomial ring $T = S[y_1, \ldots, y_m] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. To be precise, let $\mathbf{y}$ denote the row matrix $(y_1, \ldots, y_m)$. Then $S \ltimes M \cong T/I$, where $I = (y_1, \ldots, y_m)^2 + (\mathbf{y} \cdot \mathbf{A})$. If we write $I' = (y_1, \ldots, y_m)^2$ and $I'' = (\mathbf{y} \cdot \mathbf{A})$, then we have a graded short exact sequence of T-modules:

$$0 \to \frac{I}{I'} \to \frac{T}{I'} \to \frac{T}{I} \to 0.$$

The middle term T/I' has free resolution $\mathbf{E}_\bullet$ with the structure of an Eagon-Northcott complex. In particular, it is a linear free resolution after the first step of length m. The first term in the short exact sequence has a free resolution of the form $\mathbf{K}_\bullet(\mathbf{x}; T) \otimes_S \mathbf{F}_\bullet$, where $\mathbf{K}_\bullet(\mathbf{x}; T)$ denotes the Koszul complex on T with respect to $x_1, \ldots, x_n$, and $\mathbf{F}_\bullet$ is the minimal free resolution of $\operatorname{Syz}_1(M)$. In particular, much of the structure of the free resolution of M is passed to the resolution of I/I'. Let $\varphi_\bullet : \mathbf{K}_\bullet(\mathbf{x}; T) \otimes_S \mathbf{F}_\bullet \to \mathbf{E}_\bullet$ be a map of complexes lifting the inclusion $I/I' \to T/I'$. It follows from standard homological arguments that $\operatorname{Cone}(\varphi_\bullet)$ is a T-free resolution of T/I. By analyzing the structure of this resolution one can show that it is in fact minimal. For details we refer the reader to [61].

As a consequence of the preceding discussion, we have the following special case of a result of Ullery:

Theorem 3.6 ([61, Theorem 1.3]) *Let M be a graded $S = \mathbb{K}[x_1, \ldots, x_n]$-module minimally generated in degree 0 by m elements with strictly increasing maximal shifts $d_i := \bar{t}_i(M)$. Let I be the defining ideal of $S \ltimes M$ in the polynomial ring $T = S[y_1, \ldots, y_m]$ as above. Then*

$$\bar{t}_i(T/I) = \begin{cases} d_i + 1 & \text{for } 1 \leq i \leq n \\ d_n + i - n + 1 & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

In particular, for any strictly increasing sequence of integers $2 \leq d_1 < d_2 < \cdots < d_n$, there exists an ideal I in a polynomial ring T with $\bar{t}_i(T/I) = d_i$ for $1 \leq i \leq n$.

Thus the maximal shifts of a graded ideal can realize any increasing sequence of integers (beginning with $d_1 \geq 2$) as an initial segment at the expense of a long linear tail of the corresponding resolution. We illustrate this with an example.

Example 3.7 Fix integers $p, r \geq 1$. We show how to construct an ideal generated by quadrics which has linear syzygies for p steps and a $(p+1)$th syzygy of degree $p+r+3$. Let $S = \mathbb{K}[x_1, \ldots, x_{p+2}]$ and let $M = \mathrm{Ext}_S^{p+2}(S/(x_1, \ldots, x_{p+2})^{r+1}, S)(-p-r-2)$. Then M is a pure, Cohen-Macaulay S-module with maximal shifts $(0, 1, 2, \ldots, p, p+1, p+r+2)$. As M has $m = \binom{p+r+1}{r}$ minimal generators, we set $T = S[y_1, \ldots, y_m]$ and $I = (y_1, \ldots, y_m)^2 + (\mathbf{y} \cdot \mathbf{A})$, where $\mathbf{A}$ is the linear presentation matrix of M. Then $S \ltimes M(-1) \cong T/I$, and $\bar{t}_i(T/I) = i+1$ for $1 \leq i \leq p+1$ and $\bar{t}_{p+2}(T/I) = p+r+3$.

When $p = 1$ and $r = 3$, the module M has Betti table:

	0	1	2	3
0 :	10	24	15	-
1 :	-	-	-	-
2 :	-	-	-	-
3 :	-	-	-	1

while T/I has Betti table:

	0	1	2	3	4	5	6	$\cdots$	11	12	13
0 :	1	-	-	-	-	-	-		-	-	-
1 :	-	79	585	2220	5403	9150	11178	$\cdots$	174	15	-
2 :	-	-	-	-	-	-	-		-	-	-
3 :	-	-	-	-	-	-	-		-	-	-
4 :	-	-	-	1	10	45	120	$\cdots$	45	10	1

It is easy to see the copy of $\mathbf{K}_\bullet(y_1, \ldots, y_{10}; T)$ in the 4-linear strand.

Remark 3.8 When $M = J$ is an ideal, the construction of the resolution of T/J can also be found in [51], where Peeva and the author constructed counterexamples to the Eisenbud-Goto Conjecture by way of Rees-like algebras. The Rees-like algebra of J is $S[Jt, t^2] \subseteq S[t]$. As $S[Jt, t^2]/(t^2) \cong S \ltimes J$, the graded Betti table of the defining ideal of $S[Jt, t^2]$ is the same as that of I above. (Although a different grading is used there to make $S[It, t^2]$ a positively graded ring.)

We will return to the study of quadratic ideals with linear syzygies in Sect. 6.

3.2 Graded Bourbaki Ideals

We saw in Sect. 3.1 that we could construct ideals whose resolutions shared many properties with a given module. Bourbaki ideals give another way to construct ideal analogues of modules while preserving much of the structure of the free resolution. While Bourbaki ideals exist in a much wider context, we limit our attention to graded Bourbaki ideals over polynomial rings and refer the interested reader to [11, Chapter VII, §4.9, Theorem 6] for the more general result.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and let M be a finitely generated, torsionfree S-module. A *Bourbaki sequence* for M is a short exact sequence of the form

$$0 \to F \to M \to I \to 0,$$

where F is a finitely generated free S-module and I is an ideal of S. Bourbaki sequences always exist and in the graded setting we can be a bit more precise.

Theorem 3.9 ([36, Theorem 1.2]) *Let $\mathbb{K}$ be an infinite field and let $S = \mathbb{K}[x_1, \ldots, x_n]$. Let M be a finitely generated, graded, torsionfree S-module generated in degree 0 with $\operatorname{rank}(M) = r$. Then there is a graded Bourbaki sequence of the form:*

$$0 \to S^{r-1} \to M \to I(-m) \to 0,$$

where $m \in \mathbb{Z}$ and I is a graded height two ideal of S.

As a result, we have the following corollary

Corollary 3.10 *Let M be a finitely generated, graded, torsionfree S-module generated in degree 0. Then there exists an integer m and a height two graded ideal I such that*

$$\bar{t}_{i+1}(S/I) = \bar{t}_i(I) = \bar{t}_i(M) + m,$$

for all $i \geq 0$. In particular, for any strictly increasing sequence of integers $d_1 < d_2 < \cdots < d_n$, there exists a graded height two ideal I and an integer m (depending on $d_1, d_2, \ldots, d_n$) such that $\bar{t}_i(S/I) = d_i + m$.

In other words, we can construct ideals with any pattern of maximal shifts up to an added constant. We can also use Bourbaki ideals to construct ideals whose maximal graded shifts are not strictly increasing.

Example 3.11 Let $S = \mathbb{K}[x_1, x_2, x_3]$ and set $M = S/(x, y, z)(+1) \oplus S/(x^3, y^3)(+3)$ so that $\operatorname{Syz}_1(M)$ is torsionfree and has the following Betti table.

	0	1	2
0 :	5	3	1
1 :	-	-	-
2 :	-	1	-

The coresponding graded, height two Bourbaki ideal $I \subseteq T$ associated to M has following Betti table.

	0	1	2	3
0 :	1	-	-	-
1 :	-	-	-	-
2 :	-	-	-	-
3 :	-	4	3	1
4 :	-	-	-	-
5 :	-	-	1	-

Note that $\bar{t}_2(T/I) = 7$, while $\bar{t}_3(T/I) = 6$.

4 Subadditivity of Syzygies

Again let $S = \mathbb{K}[x_1, \ldots, x_n]$, and fix a graded S-ideal I. Then I is said to satisfy the *subadditivity* condition if

$$\bar{t}_a(S/I) + \bar{t}_b(S/I) \geq \bar{t}_{a+b}(S/I)$$

for all integers $a, b \geq 1$ with $a + b \leq \text{pd}(S/I)$. This is a natural condition from the perspective of the Koszul homology algebra. Write $H_i(\mathbf{x}, S/I)$ to denote the ith Koszul homology of S/I with respect to $x_1, \ldots, x_n$. Since $\beta_{i,j}(S/I) = \dim_k H_i(\mathbf{x}, S/I)_j$, we can interpret the Betti table of S/I as a bigraded decomposition of the Koszul homology algebra $H_*(\mathbf{x}; S/I)$, with the obvious multiplicative structure coming from the Koszul complex. In particular, if $\bar{t}_{a+b}(S/I) > \bar{t}_a(S/I) + \bar{t}_b(S/I)$, then there is a generator of the Koszul homology algebra in homological degree $a + b$.

If I is generated by a homogeneous regular sequence $f_1, \ldots, f_c$, then it follows easily from the structure of the Koszul complex $\mathbb{K}(f_1, \ldots, f_c; S)$ that S/I satisfies the subadditivity condition [52, Proposition 4.1]. On the other hand, the subadditivity condition fails in general, even for Cohen-Macaulay quotients S/I.

Example 4.1 This example is a modification of [25, Example 4.4]. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, x_2, x_3, x_4]$. Consider the ideals $C = (x_1^4, x_2^4, x_3^4, x_4^4)$ and $I = C + (x_1 + x_2 + x_3 + x_4)^4$. As $\ell = x_1 + x_2 + x_3 + x_4$ is a strong Lefschetz element for S/C [34, Theorem 3.35], it follows that the h-vector of S/I is (1, 4, 10, 20, 30, 36, 34, 20). Let $L = C : I$. Using the Lefshchetz property, we see that L has no generators in degree ≤ 4, except for those of C. As $\text{reg}(S/C) = 12$, the map $(S/C)_5 \to (S/C)_9$ via multiplication by ℓ^4 is surjective. Thus there is a $\dim_{\mathbb{K}}(S/C)_5 - \dim_{\mathbb{K}}(S/C)_9 = 40 - 20 = 20$ dimensional kernel to this map corresponding to 20 generators of L in degree 5. Consider the graded short exact sequence

$$0 \to (S/L)(-4) \to S/C \to S/I \to 0.$$

As S/C is resolved by a Koszul complex, $\bar{t}_1(S/C) = 4$. The degree 5 generators of L force $\bar{t}_1(S/L(-4)) = \bar{t}_1(S/L) + 4 \geq 9$. (Actually we get equality here but the inequality is all we need.) It follows from the long exact sequence of Tor that $\bar{t}_2(S/I) \geq 9$ while $\bar{t}_1(S/I) = 4$. Thus the subadditivity property fails for S/I. The full sequence of maximal graded shifts of S/I is $(0, 4, 9, 10, 11)$.

To construct examples where the subadditivity property fails later in the resolution, we can replicate copies of the ideal I in new sets of variables, four at a time. Their ideal sum is resolved by the tensor product of the copies of the resolution of S/I. For example, taking 3 copies of S/I and tensoring the corresponding resolutions, we get the following sequence of maximal graded shifts: $(0, 4, \underline{9}, 13, \underline{18}, 22, \underline{27}, 28, 29, 30, 31, 32, 33)$. The subadditivity property fails at each of the underlined positions ($9 \not\leq 4+4$, $18 \not\leq 4+13$, $27 \not\leq 13+13$).

Since the subadditivity condition holds for complete intersections but fails for Cohen-Macaulay ideals, It is natural to ask if it holds for Gorenstein ideals. Some positive results are given by Srinivasan and El Khoury in [27]. However, Gorenstein ideals failing the subadditivity condition were constructed by Seceleanu and the author in [52]. More precisely, they proved the following:

Theorem 4.2 [52, Theorem 4.3] *Let $\mathbb{K}$ be an infinite field and $m \geq 2$ an integer. Then there exists a quadratic, Artinian, Gorenstein ideal I in a polynomial ring S over $\mathbb{K}$ such that I has first syzygies in degree $m+2$. In particular, the subadditivity property fails for S/I.*

The ideals in this construction also come from idealizations but of a different sort. The key is to construct a quadratic Artinian $\mathbb{K}$-algebra A whose defining ideal J has arbitrarily large first syzygies and has the superlevel property. A standard graded $\mathbb{K}$-algebra R is called *superlevel* if its canonical module ω_R is linearly presented over R. In this case, it is sufficient to check that the last differential in the resolution of the defining ideal of A is linear. It follows from a result of Mastroeni, Schenck, and Stillman [47, Theorem 3.5] that the idealization $G = A \ltimes \omega_A(-\operatorname{reg}(A) - 1)$ is Gorenstein, Artininian, and quadratic, and while we do not know the full structure of the defining ideal, minimal syzygies of J induce minimal syzygies of the defining ideal of G.

Nevertheless, there are notable classes of ideals where the subadditivity property is expected or even conjectured:

(1) Monomial ideals,
(2) Koszul algebras,
(3) Toric ideals.

If $I \subseteq S$ is a monomial ideal, the Taylor resolution has maximal graded shifts satisfying the subadditivity property, but this resolution is not minimal in general. When one trims this down to a minimal free resolution, it is not clear that the subadditivity property is preserved, although it is expected and partial results are known. The most general result is the following theorem of Herzog and Srinivasan:

Theorem 4.3 ([37, Corollary 4]) *Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$\bar{t}_1(S/I) + \bar{t}_a(S/I) \geq \bar{t}_{a+1}(S/I)$$

for all integers $0 \leq a < \mathrm{pd}(S/I)$.

Note that the monomial ideal hypothesis is necessary as we have previously seen this inequality fails for arbitrary (even Gorenstein) ideals.

Specific classes of monomials have been shown to satisfy the full subadditivity condition, including facet ideals of simplicial forests [28], edge ideals of certain graphs and hypergraphs [9], and monomial ideals with DGA resolutions [41]; see also [1]. The general case remains open.

The subadditivity property of Koszul algebras was studied by Avramov, Conca, and Iyengar [3, 4], where they explicitly conjecture the subadditivity property and extended work of Backelin [5], Kempf [42] and others. Recall that a standard graded ring $R = S/I$ is called *Koszul* if $R/R_+ \cong K$ has a linear free resolution over R, where R_+ denotes the homogeneous maximal ideal; equivalently, $\bar{t}_i^R(K) = i$ for all $i \geq 0$. While subadditivity is still open in general for Koszul algebras, many slightly weaker results on the maximal graded shifts are known.

If I is generated by quadratic monomials, Conca [16] observed that the following inequalities follow from the Taylor resolution of $R = S/I$:

(1) $\bar{t}_i(R) \leq 2i$ for all $i \geq 0$.
(2) If $\bar{t}_i(R) < 2i$ for some i, then $\bar{t}_{i+1}(R) < 2(i+1)$.
(3) $\bar{t}_i(R) < 2i$ if $i > \dim(S) - \dim(R)$.

Therefore, these same properties hold by a deformation argument whenever I has a quadratic Gröbner basis and it is natural to ask if these properties hold for arbitrary Koszul algebras. Kempf [42, Lemma 4] (and also Backelin [6]) proved that (1) above holds for all Koszul algebras. Items (2) and (3) were proved by Avramov, Conca, and Iyengar [3, Main Theorem]. In a later paper, under mild hypotheses, they proved the following improvements.

Theorem 4.4 ([4, Theorem 5.2]) *Suppose $R = S/I$ is a Koszul $\mathbb{K}$-algebra with* $\mathrm{Char}(\mathbb{K}) = 0$. *Let $m = \min\{i \in \mathbb{Z} \,|\, \bar{t}_i(R) \geq \bar{t}_{i+1}(R)\}$. Then*

(1) *If* $\max\{a, b\} \leq m$, *then*

$$\bar{t}_{a+b}(R) \leq \max\{\bar{t}_a(R) + \bar{t}_b(R), \bar{t}_{a-1}(R) + \bar{t}_{b-1}(R) + 3\}.$$

(2) *In particular, if* $\max\{a, b\} \leq m$ *then*

$$\bar{t}_{a+1}(R) \leq \bar{t}_a(R) + 2$$

and

$$\bar{t}_{a+b}(R) \leq \bar{t}_a(R) + \bar{t}_b(R) + 1.$$

Moreover, we may drop the condition $\max\{a, b\} \leq m$ *when R is Cohen-Macaulay, since in this case* $\operatorname{ht}(I) = m = \operatorname{pd}_S(R)$.

Minimal free resolutions of toric ideals have similar combinatorial descriptions to those of monomial ideals; for example, see [55, Section 67]. It seems natural to study the subadditivity property for toric ideals. This problem seems wide open.

5 General Syzygy Bounds

As noted in the previous section, the subadditivity condition fails for arbitrary ideals; however, there are several slightly weaker bounds on syzygy degrees that hold with greater generality. In their paper [25], Eisenbud, Huneke, and Ulrich studied the regularity of Tor modules and obtained consequences for ideals $I \subset S$ such that $\dim(S/I) \leq 1$. Note that such results instantly extend to ideals I such that S/I has Cohen-Macaulay defect at most 1; one extends to an infinite base field and then kills a general sequence of linear forms to reduce to this case. In particular, the following two weak convexity results hold when S/I is Cohen-Macaulay.

Theorem 5.1 ([25, Corollary 4.1]) *Suppose* $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ *is a graded ideal such that* $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. *Set* $p = \operatorname{pd}_S(S/I)$. *Then for any* $0 \leq i \leq p$,

$$\bar{t}_p(S/I) \leq \bar{t}_{p-i}(S/I) + \bar{t}_i(S/I).$$

Theorem 5.2 ([25, Corollary 4.2(a)]) *Suppose* $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ *is a graded ideal such that* $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. *Set* $p = \operatorname{pd}_S(S/I)$. *Suppose that* $f_1, \dots, f_c$ *is a homogeneous regular sequence in I, where* $d_i = \deg(f_i)$. *Then*

$$\bar{t}_p(S/I) \leq \bar{t}_{p-c}(S/I) + \sum_{i=1}^{c} d_i.$$

Both of these results follow from a more general result on the regularity of Tor that requires the hypothesis that $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$; however, it is natural to ask if either of the above results holds without the assumption that $\dim(S/I) - \operatorname{depth}(S/i) \leq 1$; see Conjectures 7.5 and 7.6. While this remains open, slightly weaker statements do hold without assumptions on the ideal I. The author used similar techniques as those used for Theorem 3.2 to show that

$$\bar{t}_p(S/I) \leq \max_{1\leq i\leq p-1} \left\{\bar{t}_i(S/I) + \bar{t}_{p-i}(S/I)\right\},$$

where $p = \mathrm{pd}(S/I)$ [49, Theorem 4.4]. Shortly thereafter, Herzog and Srinivasan proved the following stronger statement:

Theorem 5.3 ([37, Corollary 3]) *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ be a graded ideal and set $p = \mathrm{pd}(S/I)$. Then*

$$\bar{t}_p(S/I) \leq \bar{t}_1(S/I) + \bar{t}_{p-1}(S/I).$$

This result follows from a more general statement [37, Proposition 2], which considers the dual complex of the minimal free resolution of I. Similar techniques yields the stronger statement in Theorem 4.3 for monomial ideals; see Sect. 7.2 for potential stronger statements.

6 Quadratic Ideals and Linear Syzygies

Historically, there has been significant interest in conditions on nondegenerate projective varieties that force the resolutions of the vanishing ideals to be as simple as possible. There are many classical theorems guaranteeing that a variety X is defined by quadrics $q_1, \dots, q_t$ either ideal theoretically ($I_X = (q_1, \dots, q_t)$), set theoretically ($I_X = \sqrt{(q_1, \dots, q_t)}$), or scheme-theoretically ($I_X = (q_1, \dots, q_t)^{\mathrm{sat}}$). See [31, 53, 58, 59]. Green and Lazarsfeld [29] wrote that "one expects that theorems on generation by quadrics will extend to—and be clarified by—analogous statements for higher syzygies." In this section we consider the stronger condition that I_X is generated by quadrics ideal theoretically and has linear resolution for $p-1$ steps. In many geometric situations, it is natural to assume that the corresponding variety is projectively normal, i.e. that S/I_X is a normal ring. Following Green and Lazarsfeld [30], we define property N_p, sometimes called the Green-Lazarsfeld property, as follows. An ideal $I \subseteq S = \mathbb{K}[x_1, \dots, x_n]$ satisfies property N_p for some integer $p \geq 0$ if S/I is normal and $\beta^S_{i,j}(I) = 0$ for $j \neq i + 2$ and $0 \leq i < p$. A projective variety X (with fixed embedding) satisfies property N_p if its homogeneous vanishing ideal I_X does. Note that properties N_0 and N_1 are what Mumford termed "normal generation" and "normal presentation" respectively in [53]. Assuming X is projectively normal and nondegenerate, the ideal I_X satisfies property N_p if and only if $\bar{t}_i(S/I_X) = i + 1$ for $1 \leq i \leq p - 1$.

We first consider geometric or combinatorial conditions that ensure an ideal satisfies property N_p. The first example of this type is the following result of Green.

Theorem 6.1 ([31, Theorem 4.a.1]) *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth projective curve of degree d and genus g. If $d \geq 2g + 1 + p$, then I_X satisfies property N_p.*

This theorem was recovered by Green and Lazarsfeld as a result of the following theorem on points in projective space:

Theorem 6.2 ([29, Theorem 1]) *Suppose that $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ consists of $2n + 1 - p$ points in linear general position (no $n + 1$ lying on a hyperplane). Then I_X satisfies property N_p.*

Specifically regarding curves with their canonical embedding, Green's Conjecture predicts that the N_p property is connected with the Clifford index.

Conjecture 6.3 ([31, Conjecture 5.1]) Let $X \subset \mathbb{P}^g_{\mathbb{C}}$ be a smooth curve in its canonical embedding. Then the Clifford index $\mathrm{Cliff}(X)$ is equal to the least integer p for which property N_p fails for I_X.

See [22, Section 9A] or [46, Section 1.8] for a precise definition of the Clifford index. Note that the hyperelliptic case is simple as X is then a rational normal curve with a linear free resolution. Green and Lazarsfeld [31, Appendix] showed that if $\mathrm{Cliff}(X) = p$, then property N_p fails, so the content of the theorem is the reverse implication. Voisin [62, 63] showed that Green's Conjecture holds for general curves. More recently, a shorter proof of the general case, which also applies in characteristic $p \gg 0$ was given by Aprodu, Farkas, Papadima, Raicu, and Weyman [2] via the theory of Koszul Modules, while Green's Conjecture can fail in small characteristics [60].

For related statements regarding higher dimensional varieties, we refer the reader to the survey [20] by Ein and Lazarsfeld.

Of particular interest are the resolutions of Segre and Veronese varieties. We restrict our discussion to the case when $\mathrm{char}(\mathbb{K}) = 0$, as the resolutions can change in certain small characteristics; see [35]. In the case of Veronese embedding, Green proved the following via a Koszul vanishing argument.

Theorem 6.4 ([32, Theorem 2.2]) *Let $V_{d,r}$ denote the defining ideal of the image of $\mathbb{P}^r$ in $\mathbb{P}^{\binom{r+d}{d}-1}$ under the dth Veronese embedding. Then $V_{d,r}$ satisfies property N_d.*

Ottoviani and Paoletti [54] later conjectured that if $d \geq 3$, then property N_{3d-3} should hold while showing that N_{3d-2} failed. When $d = 2$, the ideals $V_{2,r}$ are generated by the 2×2 minors of a symmetric $(r + 1) \times (r + 1)$ matrix, whose resolutions are described by Józefiak, Pragacz, and Weyman [40] via representation theoretic techniques. It follows that $V_{2,r}$ satisfies property N_5 and fails N_6 for $r \geq 3$, while $V_{2,2}$ has a linear free resolution. (i.e. satisfies property N_p for all p.) See Sect. 7 for more on this problem, and see the paper [12] by Bruce, Erman, Goldstein, and Yang for an interesting computational approach.

The situation for Segre embeddings is better understood as resolutions, again in characteristic 0, are given by Lascoux [45] and Pragacz and Weyman [56]. Such ideals are generated by the 2×2 minors of a generic matrix. The next result follows from their construction.

Theorem 6.5 *Let I denote the defining ideal of the Segre embedding $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \to \mathbb{P}^{ab-1}$ with $a, b \geq 3$. Then I satisfies property N_3 and fails property N_4.*

When $a = 2$ or $b = 2$, it is well known that the resolution of I is linear. For a more detailed treatment of representation theoretic techniques for computing free resolutions, we refer the reader to the book [65] of Weyman. For summaries of related statements on the N_p property, see [57] and [46].

Especially in combinatorial settings it may not be natural to assume normality; we can also generalize the situation to arbitrary degree. Following [24], we say that an ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ satisfies property $N_{d,p}$ if $\beta^S_{i,j}(I) = 0$ for $j \neq d + i$ and $0 \leq i < p$. Thus I satisfies property N_p if and only if S/I is normal and I satisfies property $N_{2,p}$.

When I is a square-free monomial ideal, we can identify it with its Stanley-Reisner complex Δ_I whose faces correspond to the monomials not in I. In the specific case when I is generated in degree two, we can identify I with a graph whose vertices correspond to the variables and whose edges $\{i, j\}$ correspond to monomial generators $x_i x_j$ of I. We write $I(G)$ for the edge ideal of the graph G and I_Δ for the square-free monomial ideal corresponding to the simplicial complex Δ. We write $I^\vee = I_{\Delta^\vee}$ for the ideal corresponding to the Alexander dual of the squarefree monomial ideal I.

The following result shows that property $N_{2,p}$ for an edge ideal of a graph can be detected combinatorially.

Theorem 6.6 ([24, Theorem 2.1],[66, Corollary 3.7]) *Let G be a graph and let $p \geq 2$. Then the following are equivalent:*

(1) *$I(G)$ satisfies property $N_{2,p}$.*
(2) *$S/I(G)^\vee$ satisfies Serre's property S_p.*
(3) *G has no induced k-cycle for $4 \leq k \leq p + 2$.*

The equivalence of items (1) and (2) is a result of Terai and Yanagawa [66]; the equivalence of (1) and (3) is a result of Eisenbud, Green, Hukek, and Popescu [24] and holds when $p = 1$.

Assuming one has such an edge ideal, Dao, Huneke, and Schweig [19] proved the following logarithmic bound on the regularity.

Theorem 6.7 ([19, Theorem 4.1]) *Let G be a graph on n vertices such that $I(G)$ satisfies property $N_{2,p}$ for some integer $p \geq 2$. Then*

$$\operatorname{reg}(S/(I(G)) \leq \log_{\frac{p+3}{2}} \left(\frac{n-1}{p}\right) + 2.$$

For some time it was an open question as to whether there was a bound, independent of the number of variables, on the regularity of quadratic monomial ideals satisfying property $N_{2,p}$ [19, p. 8]. The following theorem of Constantinescu, Kahle, and Varbaro, improving earlier work [17], shows that this is not the case.

Theorem 6.8 ([18, Corollary 6.12]) *Suppose* $\mathrm{Char}(\mathbb{K}) = 0$ *and fix positive integers* p *and* r. *Then there exists a quadratic square-free monomial ideal* $I \subseteq S = \mathbb{K}[x_1, \ldots, x_{N(p,r)}]$ *satisfying property* $N_{2,p}$ *with*

$$\mathrm{reg}(S/I) = r.$$

These ideals are constructed via an interesting connection between the regularity of certain edge ideals and the virtual projective dimension of hyperbolic Coxeter groups. It is worth noting though that the construction requires a large number of variables. Also, unlike Ullery's construction in Sect. 3.1, the jump in syzygy degrees cannot happen all at once by Theorem 4.3.

Finally, we recall that Avramov, Conca, and Iyengar proved that Koszul ideals satisfying property $N_{2,p}$ also satisfy a regularity bound.

Theorem 6.9 ([4, Theorem 6.1]) *Let* $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ *be a graded ideal such that* S/I *is Koszul and satisfies property* $N_{2,p}$ *for some* $p \geq 1$. *Then*

$$\mathrm{reg}(S/I) \leq 2\left\lfloor \frac{n}{p+1} \right\rfloor + 1.$$

We consider potential stronger regularity bounds in the next section.

7 Questions and Conjectures

We end by collecting a number of open questions and conjectures related to the subadditivity type problems.

7.1 Subadditivity of Syzygies

First, we recall the main open cases for the subadditivity condition.

Conjecture 7.1 ([4, Conjecture 5.5]) Let S/I be a Koszul algebra. Then S/I satisfies the subadditivity condition.

While it is not explicitly conjectured in print, the results in [28, 37] strongly indicate that we should expect the subadditivity condition to hold for all monomial ideals. We make this conjecture here.

Conjecture 7.2 Let $M \subseteq S$ be a monomial ideal. Then S/M satisfies the subadditivity condition.

Given the combinatorial nature of resolutions of toric ideals, it seems natural to expect that they also satisfy the subadditivity condition. The author knows of no counterexamples to the following conjecture.

Conjecture 7.3 Let $I \subseteq S$ be a toric ideal (meaning prime and generated by binomials). Then S/I satisfies the subadditivity condition.

As both monomial ideals and toric ideals have $\mathbb{Z}^m$-gradings for some integer m, we could strengthen both of the previous conjectures to ask about subadditivity of the multi-graded Betti numbers. It would also be interesting to find other classes of ideals where the subadditivity condition holds. We state this formally as a problem.

Open-ended Problem 7.4 *Find other classes of ideals that satisfy the subadditivity condition.*

7.2 Weak Convexity of Syzygies

The results of Eisenbud, Huneke, and Ulrich [25] hold for ideals $I \subseteq S$ with Cohen-Macaulay defect at most 1; that is, $\dim(S/I) - \operatorname{depth}(S/I) \leq 1$. The author previously conjectured that several of these results hold in greater generality. We record these here.

Conjecture 7.5 ([49, Question 5.1]) Let $I \subseteq S$ be a graded ideal and let $p = \operatorname{pd}(S/I)$. Then for any integer $0 \leq i \leq p$,

$$\bar{t}_p(S/I) \leq \bar{t}_i(S/I) + \bar{t}_{p-i}(S/I).$$

This appears to be open even when $p = 4$ and $i = 2$. Note that Theorem 5.3 shows the conjecture holds in the case $i = 1$.

Conjecture 7.6 ([50, Conjecture 1.4]) Let $I \subseteq S$ be a graded ideal and suppose $f_1, \ldots, f_c \in I$ is a homogeneous regular sequence. Set $d_i = \deg(f_i)$ for $1 = 1, \ldots, c$ and $p = \operatorname{pd}(S/I)$. Then

$$\bar{t}_p(S/I) \leq \bar{t}_{p-c}(S/I) + \sum_{i=1}^{c} d_i.$$

7.3 Syzygy Bounds on Regularity

We know that the subadditivity condition fails in general for Cohen-Macaulay ideals; see Example 4.1. More precisely, there are Cohen-Macaulay ideals generated in fixed degree and with arbitrarily large first syzygies. What is not clear is whether resolutions of Cohen-Macaulay ideals can exhibit more extreme behavior beyond the first two steps.

Question 7.7 Let $I \subseteq S$ be a graded ideal such that S/I is Cohen-Macaulay. Fix an integer $0 \leq i \leq \mathrm{pd}(S/I)$. Does the following inequality hold:

$$\bar{t}_i(S/I) \leq \max\{i \cdot \bar{t}_1(S/I), \frac{i}{2} \cdot \bar{t}_2(S/I)\}?$$

Ullery's designer ideals show that the Cohen-Macaulay hypothesis cannot be removed from the previous question.

7.4 Syzygies of Quadratic Ideals

Recall that property N_d holds for the dth Veronese embedding of $\mathbb{P}^n_{\mathbb{K}}$ and property N_{3d-2} fails. Ottoviani and Paoletti have conjectured that this is sharp.

Conjecture 7.8 ([54]) For integers $n \geq 2$ and $d \geq 3$ and a field $\mathbb{K}$ of characteristic 0, the defining ideal of the dth Veronese embedding of $\mathbb{P}^n_{\mathbb{K}}$ satisfies N_p if and only if $p \leq 3d - 3$.

When $d = 2$ and $n \geq 3$, it follows from work of Józefiak, Pragacz, and Weyman [40] that property N_5 holds and N_6 fails. When $n = 1$ (and when $d = n = 2$), the corresponding resolution in linear, i.e. property N_p holds for all p. When $n = 2$, the conjecture holds by work of Birkenhake [10] and Green [31]. When $d = 3$, the conjecture holds by work of Vu [64]. All other cases are open.

Finally, we recall the following question of Constantinescu, Kahle, and Varbaro regarding the regularity of linearly presented quadratic ideals. In the more restrictive Koszul setting, this question was previously posed by Conca [16, Question 2.8].

Question 7.9 ([18, Question 1.1]) Does there exist a family of quadratically generated, linearly presented, graded ideals $I_n \subseteq \mathbb{K}[x_1, \ldots, x_n]$ such that

$$\lim_{n\to\infty} \frac{\mathrm{reg}(I_n)}{n} > 0?$$

We may replace n with $\mathrm{pd}(I_n)$ as one can mod out by a regular sequence of linear forms to reduce to the above question. One expects that no such families of ideals exist, so we pose the following stronger question:

Question 7.10 If $I_n \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a family of quadratic, homogeneous, linearly presented (i.e. satisfying property $N_{2,2}$) ideals, is

$$\limsup_{n\to\infty} \frac{\mathrm{reg}(I_n)}{\sqrt{n}} < \infty?$$

Clearly a positive answer to Question 7.10 gives a negative answer to Question 7.9. Let's calibrate on some examples.

(1) Let $V_{d,r}$ denote the defining ideal of the dth Veronese embedding of $\mathbb{P}^r$ in $\mathbb{P}^{\binom{r+1}{d}-1}$ in characteristic 0. This ideal satisfies property N_d by Theorem 6.4. One checks that $\operatorname{reg}(V_{d,r}) = r - \lceil \frac{r+1}{d} \rceil + 2$, while $\operatorname{codim}(V_{d,r}) = \binom{r+d}{d} - r - 1$. Setting $d = 2$ and taking an Artinian reduction (so that $n = \binom{r+2}{2} - r - 1$), we see that the above lim sup is a limit with value $\sqrt{2}$ as n approaches ∞. Note also that this example shows that the stronger question asking if families of $N_{2,p}$ ideals satisfy

$$\limsup_{n\to\infty} \frac{\operatorname{reg}(I_n)}{\sqrt[p]{n}} < \infty$$

fails for $p = 3$, since $V_{2,r}$ satisfies property N_3; see above.

(2) If $G_{1,r}$ is the defining ideal of the Grassmannian of lines in $\mathbb{P}^r$, it is known that $\operatorname{reg}(G_{1,r}) = r - 3$ [44, Theorem 5.3], while $G_{1,r}$ is Cohen-Macaulay with $\operatorname{codim}(G_{1,r}) = \binom{r-2}{2}$ [38, Corollary 3.2]. Again taking an Artinian reduction and letting r tend to ∞ we get a limit of $\sqrt{2}$.

(3) If $I_n \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a quadratic monomial ideal satisfying property $N_{2,2}$, then by Theorem 6.7, $\operatorname{reg}(I_n) \le \log_{\frac{5}{2}} \left(\frac{n-1}{2}\right) + 2$. It follows that the above lim sup is 0 for all such families of monomial ideals.

(4) Finally, we can construct quadratic, linearly presented ideals I_r with regularity r via the idealization construction in Sect. 3.1. However, the projective dimension of such ideals is $\frac{r^2+3r}{2} + 4$, which is quadratic in r, meaning the above lim sup is always finite.

See also [4, Section 6] and [33] for relevant examples and computations.

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Applications of Liaison

J. Migliore and U. Nagel

Dedicated to David Eisenbud on the occasion of his 75th birthday

1 Introduction

Liaison theory has a long and rich history, with several periods of pronounced activity in the last century and a half. Many important questions have been answered, and important questions still remain. We refer the reader to [59, 61] and [65] for detailed treatments of liaison theory, and the authors of this paper hope to update [61] in the coming years to account for the progress made in the intervening years since its original publication, of which there is quite a bit.

What is less chronicled, though, are the many areas in which liaison techniques have been applied. In this paper we have selected a handful of examples of such applications. We begin, in Sect. 2, by giving enough of a background to make the subsequent sections readable, and then we describe just a few of the many directions in which these tools have led to interesting, and perhaps surprising, contributions. The table of contents lists the topics that we will cover, and will not be repeated here.

One of the breakthroughs in liaison theory came in 1983 by Lazarsfeld and Rao [56], and in a sense it was intended as a warning that a classical idea for applying liaison was more limited than was previously known. They say: *Classically, linkage was seen as a method for producing interesting examples of space curves starting from simpler ones. . . . A priori, one could hope—as some of the classical geometers apparently did—that techniques of liaison could be used to study space curves inductively, by linking a given curve to a (possibly very special) curve of lower*

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degree or genus. Believing that at least for general curves such an approach is fundamentally flawed, Harris suggested that a general curve should in various senses be minimal in its liaison class. Our results may be seen, then, as giving additional support (if any is needed) to the philosophy that there is no easy way to get one's hands on a "general" curve.

In a sense, the book of Martin-Deschamps and Perrin [59] showed that this result of Lazarsfeld and Rao was not restricted to general space curves, but in fact was just part of a beautiful and much larger picture for space curves. The paper [4], appearing at about the same time, showed that it was not even restricted to space curves.

The applications in this paper illustrate the fact that nevertheless, the classical ideas were not so far off. In many situations liaison *can* be used to study general objects, and in any case it can be used to produce examples of varieties or ideals with very nice properties, or to produce interesting results of other kinds.

For many applications, it is essential to use a more general concept of liaison. Classically and in the references above, complete intersections were used to link subschemes. However, in [81] it is already discussed that one could use, more generally, arithmetically Gorenstein subschemes to link. Several decades went by before a systematic investigation of Gorenstein liaison was initiated in [52]. It led to a flurry of new results whose power we illustrate in some of the following sections.

We end the paper with a short list of open questions from liaison theory, hoping that they supplement the descriptions of known applications as a motivation for further study in liaison theory, and that their eventual resolution will in turn lead to new applications. We also include a long list of references for the interested reader.

2 Background

Let $R = k[x_0, \dots, x_n]$, where k is at least an infinite field. In different parts of the paper we make different assumptions about k.

Definition 2.1 Let $C_1, C_2, X \subset \mathbb{P}^n$ be subschemes of the same dimension, with X arithmetically Gorenstein. Assume that $I_X \subset I_{C_1} \cap I_{C_2}$ and that $I_X : I_{C_1} = I_{C_2}$ and $I_X : I_{C_2} = I_{C_1}$. Then C_1 and C_2 are said to be *(directly) algebraically G-linked by* X, and we say that C_2 is *residual* to C_1 in X. We write $C_1 \overset{X}{\sim} C_2$. If X is a complete intersections, we say that C_1 and C_2 are *(directly) algebraically CI-linked*.

Suppose two subschemes C_1 and C_2 of $\mathbb{P}^n$ are directly G-linked by an arithmetically Gorenstein subscheme X, and assume that the last twist in the minimal free resolution of X is $-t$. Then it was shown in [81] that there is a short exact sequence of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{I}_{C_1} \to \omega_{C_2}(n+1-t) \to 0. \tag{2.1}$$

Of course this gives a short exact sequence on global sections, since $\mathcal{I}_X$ has vanishing first cohomology. If furthermore C_1 and C_2 are arithmetically Cohen-Macaulay, and if we know a minimal free resolution for I_X and one for I_{C_1}, then a mapping cone gives a free resolution for the canonical module of C_2, and the dual of this resolution is a free resolution for I_{C_2}. (Something more general holds, but we will not need it here.) This construction for the free resolution of I_{C_2} is called the *mapping cone construction*.

In some sense, the theories of CI-linkage and G-linkage have moved in different directions, although in codimension two in projective space they coincide. Two important properties that these two kinds of linkage have in common are the invariance of the deficiency module under (even) liaison, and the formula for the Hilbert function of the residual scheme in the arithmetically Cohen-Macaulay case.

We will see in a moment that the invariance of the deficiency modules implies that the property of being arithmetically Cohen-Macaulay is preserved under liaison. Accepting this for now, we first mention the Hilbert function formula. Suppose V and V' are arithmetically Cohen-Macaulay schemes of codimension c directly linked by an arithmetically Gorenstein scheme X. It turns out that linkage is also preserved under general hyperplane sections, including Artinian reductions, so we can assume that J and J' are Artinian ideals directly linked by an Artinian Gorenstein ideal I in a polynomial ring R.

Let $\underline{c} = (1, c, c_2, \dots, c_{s-1}, c_s)$ be the Hilbert function of R/I (i.e. the h-vector of X). Note that $\underline{c}$ is symmetric. Let $\underline{a} = (1, a_1, \dots, a_t)$ be the Hilbert function of R/J and let $\underline{a}' = (1, a'_1, \dots, a'_u)$ be the Hilbert function of R/J'. Note that $a_1 \leq c$, with equality if and only if V is non-degenerate (and similarly for V'). By Davis et al. [20, Theorem 3] (see also [61, Corollary 5.2.19]) we have the following result.

Theorem 2.2 *Under the above assumptions and notation, the h-vector of V_2 is given by*

$$a'_i = c_{s-i} - a_{s-i}$$

for $i \geq 0$.

If C is a subscheme of $\mathbb{P}^n$ of dimension r, then for $1 \leq i \leq r$ we will denote by $M^i(C)$ the *i-th deficiency module*, i.e. the graded R-module

$$M^i(C) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathcal{I}_C(t)).$$

Recall that C is arithmetically Cohen-Macaulay (ACM) if and only if $M^i(C) = 0$ for all $1 \leq i \leq r$.

It was shown by Hartshorne and by Rao (cf. for instance [84]) that in any codimension (assuming dimension $r \geq 1$), up to shift $M^i(C)$ is an invariant of the even liaison class of C. There is also a result relating the modules under an odd number of links, involving dual modules. We omit this here, but note that it follows from this that the property of being arithmetically Cohen-Macaulay is thus an invariant of a liaison class.

In fact, the whole configuration of modules is invariant up to shift. However, except in one case (see below), they do not uniquely determine the even liaison class, and in codimension ≥ 3 we know very little about what invariant(s) uniquely determine an even liaison class.

In [9, Proposition 1.4], it was shown that in fact there is a left-most shift for this configuration of modules within an even liaison class:

Proposition 2.3 *Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$ ($1 \leq r \leq n-2$). Then there exists $X \in \mathcal{L}$ such that for all $V \in \mathcal{L}$ and for all $1 \leq i \leq r$, we have*

$$M^i(V) \cong M^i(X)(-d) \text{ for some } d \geq 0.$$

Note that it is the same value of d for each of the modules. This motivates the following partition of a non-ACM even liaison class according to the shift of the modules:

Definition 2.4 Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$. Then $\mathcal{L}^0$ is the set of subschemes whose associated modules attain the leftmost possible shift, and $\mathcal{L}^h$ is the set of subschemes whose associated modules are shifted h places to the right of the leftmost shift.

We now consider curves in $\mathbb{P}^3$. As a special case, if C is a curve in $\mathbb{P}^3$, we set

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathcal{I}_C(t)).$$

This is the *Hartshorne-Rao module* of C. It serves (at least) two purposes in this paper:

- It is invariant for the even liaison class of C (Hartshorne-Rao), and in fact up to shift it determines the even liaison class (Rao—see Theorem 2.5);
- It measures the failure of C to be ACM In particular,

$$C \text{ is ACM if and only if } M(C) = 0.$$

For codimension two subschemes of a smooth arithmetically Gorenstein variety (in particular, codimension two subschemes of $\mathbb{P}^n$), we have necessary and sufficient conditions for two subschemes to be in the same even liaison class (cf. [74, 77, 84]). However, for the purposes of this paper we focus on the necessary and sufficient condition found by Rao for curves in $\mathbb{P}^3$.

Theorem 2.5 (Rao [83])

(i) *Let* C, C' *be curves in* $\mathbb{P}^3$ *whose homogeneous ideals are unmixed (i.e. the curves are locally Cohen-Macaulay and equidimensional). Then* C *and* C' *are in the same even liaison class if and only if* $M(C) \cong M(C')(\delta)$ *for some* $\delta \in \mathbb{Z}$.

(ii) *Let* M *be a graded module of finite length over* $k[x_0, x_1, x_2, x_3]$. *Then there exists a curve* $C \subset \mathbb{P}^3$ *and a positive integer* δ *such that* $M \cong M(C)(-\delta)$.

Rao has also solved this question for locally Cohen-Macaulay, codimension two subschemes of projective space [84] in terms of stable equivalence classes of vector bundles. We omit the details here, except to remark that both Nagel [74] and Nollet [77] extended the result to codimension two subschemes that are equidimensional but not necessarily locally Cohen-Macaulay.

We also remark that among curves in $\mathbb{P}^3$ (and indeed, among codimension two subschemes of $\mathbb{P}^n$), the ACM subschemes *form* an even liaison class. In higher codimension, it is an interesting question to determine which ACM subschemes are *licci* (the CI-liaison class of a complete intersection), or *glicci* (the G-liaison class of a complete intersection). We do not deal with this question in this paper except to mention an important open question in the last section.

Next we recall the construction of Liaison Addition, which was discovered and first proved by Phil Schwartau in his Ph.D. thesis in 1982 [86]. We note that while Schwartau never published his thesis, the result has been generalized in the literature [9, 31, 46]. Below we state the version proved in his thesis, very slightly revised to agree with our terminology.

Theorem 2.6 (Schwartau, [86] Theorem 50) *Let* C, C' *be codimension two subschemes of* $\mathbb{P}^n$. *Let* $F \in I_C$ *and* $F' \in I_{C'}$ *be homogeneous polynomials such that* (F, F') *forms a regular sequence, defining a complete intersection* Y. *Assume that* $\deg F = d, \ \deg F' = d'$. *Let* $I = F' \cdot I_C + F \cdot I_{C'}$. *Then*

(i) I *is a saturated ideal.*

(ii) *As sets,* I *defines* $C \cup C' \cup Y$. *This is also true as schemes if pairwise* C, C' *and* Y *have no common component.*

(iii) *Let* X *be the scheme defined by the saturated ideal* I. *Then for* $1 \le i \le n-2$ *we have*

$$M^i(X) = M^i(C)(-d') \oplus M^i(C')(-d).$$

(iv) *In particular, if* C *and* C' *are ACM then so is* X.

(v) *The Hilbert function of* X *satisfies*

$$h_X(t) = h_Y(t) + h_C(t-d') + h_{C'}(t-d).$$

The ideal I *(or the subscheme* X*) is called the* liaison addition *of* C *and* C'.

The construction of Basic Double Linkage was introduced by Lazarsfeld and Rao [56] in 1982 in the context of curves in $\mathbb{P}^3$. We first state it in the context of codimension two subschemes of $\mathbb{P}^n$.

Theorem 2.7 (Lazarsfeld-Rao [56]) *Let $C \subset \mathbb{P}^n$ be a codimension 2 subscheme. Let $F \in I_C$ and let A be a form such that (F, A) is a regular sequence. Let Y be the complete intersection subscheme defined by (F, A).*

Consider the ideal $J = A \cdot I_C + (F)$. Then:

(i) *$J = A \cdot I_C + (F)$ is a saturated ideal, defining a scheme X.*
(ii) *If I_C is unmixed then X is CI-linked to C in two steps.*
(iii) *In particular:*
   - *if C is ACM then so is X.*
   - *If $C \in \mathcal{L}^h$ and $\deg A = a$ then $X \in \mathcal{L}^{h+a}$.*
(iv) *If C and Y have no common component then $X = C \cup Y$ as schemes.*

The ideal J (resp. subscheme X) is called a Basic Double Link *of I_C (resp. of C).*

Notice that this version of basic double linkage can be viewed as a special case of Liaison Addition, by taking C' to be the empty set, with ideal R. This theorem has been generalized, and we next give a more general version. This version was discovered by the two authors with Kleppe, Miró-Roig and Peterson [52].

Theorem 2.8 ([52] Lemma 4.8, Proposition 5.10) *Let $S \subset \mathbb{P}^n$ be a generically Gorenstein, ACM subscheme. Let $C \subset S$ be an equidimensional subscheme of codimension 1 in S, and let $A \in R$ be a homogeneous element of degree d such that $I_S : A = I_S$. Let*

$$J = A \cdot I_C + I_S.$$

(i) *J is unmixed (in particular saturated). Let Y be the scheme defined by J, so we have $J = I_Y$.*
(ii) $\deg(Y) = d \cdot \deg(S) + \deg(C)$.
(iii) *Y is ACM if and only if C is ACM.*
(iv) *Let C_A be the subscheme of S cut out by A. We have $I_{C_A} = I_S + (A)$. As sets, $Y = C \cup C_A$, and if A does not vanish on any component of C then this equality is also true as schemes.*
(v) *C and Y are evenly G-linked in two steps. When S is a complete intersection, C and Y are evenly CI-linked in two steps.*

The ideal J (resp. the subscheme Y) is called a Basic Double G-link *of I_C (resp. of C).*

Remark 2.9 We want to highlight the second half of item (v) of the above theorem. When S is a complete intersection, then Y is not only G-linked in two steps, but in fact CI-linked in two steps. In this case we call Y a *Basic Double CI-link* of C.

Example 2.10 As a first application, basic double linkage can be used in a simple way to show that powers of complete intersections are saturated and Cohen-Macaulay, and with a little more work, to find their graded Betti numbers. This was carried out in [36]. The main tool is [36, Lemma 1.4], which applies Theorem 2.8 above and says the following.

Let $F_1, \dots, F_r$ be a regular sequence in $R = k[x_0 \dots, x_n]$ with $\deg F_i = d_i$. Set $I = \langle F_1, \dots, F_r \rangle$ and $J = \langle F_2, \dots, F_r \rangle$. Then for each positive integer s,

$$I^s = J^s + F_1 \cdot I^{s-1}.$$

Furthermore, we have the following short exact sequence

$$0 \to J^s(-d_1) \to I^{s-1}(-d_1) \oplus J^s \to J^s + f_1 \cdot I^{s-1} = I^s \to 0.$$

Using this sequence and induction, it is not hard to see that R/I^s is Cohen-Macaulay (so in particular I^s is saturated), and using a mapping cone and an inductive argument, with a bit of work one gets the graded Betti numbers.

The original significance of Theorem 2.8 is that it showed how Gorenstein liaison can be thought of as a theory about divisors. Indeed, Corollary 5.14 of [52] gives that if S as above satisfies property G_1 and C is a divisor, then a divisor in the linear system $|C + tH|$ (where H is the hyperplane section class and $t \in \mathbb{Z}$) can be obtained from C in two Gorenstein links. The analogous statement for complete intersection liaison was already known (for instance [59]). Hartshorne has further explored the consequences of this point of view (e.g. [45]). See also [62].

As we saw above, in codimension two it is fairly well understood what determines an even liaison class, while in higher codimension it is wide open. Another natural question is whether the even liaison classes (say with fixed codimension) have a common structure. This question has a long history [4, 9, 11, 44, 46, 56, 57, 59, 74, 77]. The following was proposed in general in [9], although as we will see, the only positive result is in codimension two.

Definition 2.11 ([9] Definition 1.8) Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$. Then $\mathcal{L}$ has the *Lazarsfeld-Rao (LR)-Property* if the following conditions hold.

(a) If $V_1, V_2 \in \mathcal{L}^0$ then there is a deformation from one to the other through subschemes all in $\mathcal{L}^0$.
(b) Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ ($h \geq 1$), there exists a sequence of subschemes $V_0, V_1, \dots, V_t$ such that for all i, $1 \leq i \leq t$, V_i is a basic double link of V_{i-1} and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.

Remark 2.12 This definition was motivated by the paper [56] of Lazarsfeld and Rao, who proved that this structure holds for a "general" curve in $\mathbb{P}^3$. Their motivation was not so much to prove a general structure theorem as it was to prove a conjecture of Harris that a "general" curve is the smallest in its (even) liaison class.

The first broad case not covered by Lazarsfeld and Rao [56] where this structure was proven was for arithmetically Buchsbaum curves in $\mathbb{P}^3$ ([8]; see also

Remark 2.14). Much more generally, this structure (and more) was shown to hold for even liaison classes of unmixed curves in $\mathbb{P}^3$ by Martin-Deschamps and Perrin [59], and at about the same time for locally Cohen-Macaulay, equidimensional codimension two subschemes of $\mathbb{P}^n$ by Ballico, Bolondi, and Migliore [4]. We quote the latter result since we will refer to it.

Theorem 2.13 ([4] Theorem 2.4) *Every even liaison class of codimension two, locally Cohen-Macaulay, equidimensional subschemes of $\mathbb{P}^n$ has the Lazarsfeld-Rao property.*

Earlier, a very special case (but the first for dimension ≥ 2) was proven in [9]. It was generalized to unmixed codimension two subschemes by Nollet [77] and (separately) by Nagel [74].

It is known [46, 57] that a G-liaison class in codimension ≥ 3 does not have such a structure. It is an open question whether it holds for CI-liaison in codimension ≥ 3.

Remark 2.14 As already noted, for a curve C, $M(C)$ is a graded module over the polynomial ring, i.e. multiplication by a linear form L induces a homomorphism from any component $M(C)_t$ to the next. However, it sometimes happens that this multiplication is trivial for all L and all t. In this case, C is said to be a(n) *(arithmetically) Buchsbaum* curve.

Ignoring the shift, any finite sequence $(d_1, \dots, d_s)$ of non-negative integers (say $d_1, d_s \neq 0$) is the sequence of dimensions (up to shift) of the components of many possible graded modules of finite length, of which one, say M, is the one with trivial multiplication by linear forms. Ballico and Bolondi [3] have studied how these structures fit together, although we will not go into this here. By Rao's theorem (Theorem 2.5), there is a curve C so that $M(C)$ is some shift of M. By Basic Double Linkage, all rightward shifts of this module also exist for some curves in $\mathbb{P}^3$. Of course if $(d_1, \dots, d_s) \neq (e_1, \dots, e_t)$ then the modules, and hence the corresponding even liaison classes, are distinct. Thus each of these tuples represents a unique *Buchsbaum even liaison class*. A good deal of work has gone into studying Buchsbaum even liaison classes and Buchsbaum curves, some of which will be described here in passing. We also refer to work of Amasaki (e.g. [1]) on this subject.

Buchsbaum curves have provided a setting in which progress on several interesting questions was made, and liaison tools have played an important role. They arise in several ways in this paper.

Another liaison tool that has been very useful in the literature, in the construction of arithmetically Gorenstein subschemes of projective (and graded Artinian Gorenstein algebras) with desired properties, is often referred to as *sums of linked ideals*. We briefly describe the background. Although this method has been in existence for a long time, our exposition here is mostly from [32] and [66], and we will describe these applications in Sect. 3.

It is well known that the sum of the ideals of two geometrically linked, arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ is arithmetically Gorenstein of height one greater, whether they are CI-linked [81] or G-linked (cf. [61]). Harima

[38, Lemma 3.1], has computed the Hilbert function of the Gorenstein ideals so obtained in the case of CI-linkage under a special numerical assumption. Here we would like to record this result in a more general way, more in line with our needs.

Lemma 2.15 *Let $V_1 \overset{X}{\sim} V_2$, where X is arithmetically Gorenstein, V_1 and V_2 are arithmetically Cohen-Macaulay of codimension c with saturated ideals I_{V_1} and I_{V_2}, and the link is geometric (meaning that V_1 and V_2 have no common components). Then $I_{V_1} + I_{V_2}$ is the saturated ideal of an arithmetically Gorenstein scheme Y of codimension $c + 1$. The Hilbert functions are related by Theorem 2.2. Then the sequence $d_i = (a_i + a_i' - c_i)$ is the* first difference *of the h-vector of Y.*

Example 2.16 A twisted cubic curve V_1 in $\mathbb{P}^3$ is linked to a line V_2 by the complete intersection of two quadrics. The intersection of these curves is the arithmetically Gorenstein zeroscheme Y consisting of two points. This is reflected in the following diagram of h-vectors:

$$\begin{array}{rl} X: & 1\ 2\ \ 1 \\ V_1: & 1\ 2 \\ V_2: & 1 \\ \Delta Y: & 1\ 0\ {-1} \end{array}$$

adding the second and third rows and subtracting the first to obtain the fourth, and so the h-vector of Y is (1,1), obtained by "integrating" the vector $(1, 0 - 1)$. The notation ΔY serves as a reminder that the row is really the first difference of the h-vector of Y.

It is well known that the h-vector of an arithmetically Gorenstein subscheme of projective space is symmetric, and a lot of work has been done to try to describe the symmetric sequences $(1, h_1, \ldots, h_e = 1)$ that arise as the h-vector of an arithmetically Gorenstein scheme (which from now on we will call *Gorenstein sequences*). Note that the h-vector of an arithmetically Cohen-Macaulay (e.g. arithmetically Gorenstein) scheme is the Hilbert function of its Artinian reduction. Some of this work involves restricting to certain classes of arithmetically Gorenstein schemes, for instance reduced ones. It is also interesting to understand how "non-unimodal" a Gorenstein sequence can be, and many papers have explored this.

An important special case of a Gorenstein sequence is a so-called *SI-sequence*:

Definition 2.17 A symmetric sequence of integers $\underline{h} = (1, h_1, \ldots, h_1, 1)$ is an *SI-sequence* if

(i) $\underline{h}$ is an O-sequence (i.e. it satisfies Macaulay's growth condition, and hence is the Hilbert function of some Artinian algebra);
(ii) the positive part of the first difference, $(1, h_1 - 1, h_2 - h_2, \ldots)$ is again an O-sequence. In this case we say that $\underline{h}$ is a *differentiable O-sequence*.

Definition 2.18 An Artinian graded algebra R/I is said to have the *strong Lefschetz property (SLP)* if there exists a linear form ℓ such that for all i and all d the homomorphism $\times\ell^d : [R/I]_i \to [R/I]_{i+d}$ has maximal rank. It has the *weak Lefschetz property (WLP)* if the above holds just for the case $d = 1$.

Remark 2.19 Notice that SI-sequences are always unimodal. They are important for (at least) two reasons:

- When $h_1 = 3$, they are *exactly* the set of Hilbert functions of Artinian Gorenstein algebras [87, 92].
- In any codimension they are exactly the set of Hilbert functions of Artinian Gorenstein algebras with the *Weak Lefschetz Property* [38].

Interestingly, in codimension 3 it is not known if all Artinian Gorenstein algebras have the Weak Lefschetz Property, in spite of the two bullet points above.

3 Stick Figures, Zeuthen's Problem and Configurations of Linear Subvarieties

The question of when an irreducible flat family of subschemes of projective space contains an element that is a union of linear varieties is a very classical one. In this case we say that any element of the family *specializes* to the union of linear varieties. Throughout this section we assume that our union of linear varieties is equidimensional.

The most desirable kind of union of linear varieties is one for which the singularities are as "nice" as possible. We will see that different terminology has been used in different situations. For curves, these are universally called *stick figures*," i.e. configurations of lines where at most two lines meet in a point. In higher dimension, we also want the components to meet "nicely," depending on the situation.

- In [10] the authors defined a *good linear configuration* of codimension two in $\mathbb{P}^n$ to be a union of codimension two linear varieties such that the intersection of any three has dimension at most $n-4$. This was mentioned also in [32, Remark 2.5].
- In [32] the authors defined a *good linear configuration* of codimension three in $\mathbb{P}^n$ "in the obvious way" without specifying it, but it is clear that it was meant that the intersection of any three of the codimension three linear components has dimension at most $n - 5$.
- In [66] the authors defined a *generalized stick figure* to be a union of linear varieties of any dimension d such that the intersection of any three components has dimension at most $d - 2$ (where the empty set is taken to have dimension -1).

The latter clearly includes all the previous special cases, so in this section from now on we will use the term "generalized stick figure," except for the case of curves where we simply use "stick figure."

3.1 *Stick Figure Curves in* $\mathbb{P}^3$

Returning to families, the most celebrated such problem is the *Zeuthen problem*. Indeed, quoting [43], "at the suggestion of H.G. Zeuthen, the Royal Danish Academy of Arts and Sciences proposed in 1901 a prize problem with a gold medal [78] p. 29:

> *To determine if every family of space curves—after the customary division—contains limit curves which are composed of lines. In the case of a negative answer there should also be an investigation of conditions for the existence of such limit curves, and possible restrictions on the results which have been found using such limit curves.*"

The first result in this direction was given by Gaeta [25], who showed that any arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ specializes to a stick figure.

The Zeuthen problem was solved in full by Hartshorne [43] (his solution appeared in 1997), where he gives a careful discussion of the history of the problem, partial results, and the significance of a possible positive answer. He points out that "(F)rom the earliest work on this problem, it has been understood that a "curve composed of lines" should be taken to mean a stick figure, ... and that "limits" should preserve the arithmetic genus, so that we are dealing with flat families."

In this subsection we mention a partial result on space curves predating Hartshorne's complete solution and generalizing Gaeta's work. In the next subsection we will give some results on arithmetically Gorenstein subschemes of higher codimension. These results used different ideas from liaison.

In the spirit of Zeuthen's problem, we first give a result for space curves. We refer to Remark 2.14 for the definition of Buchsbaum curves and Buchsbaum even liaison classes.

Theorem 3.1 ([10] Proposition 3.4) *Every Buchsbaum curve in* $\mathbb{P}^3$ *specializes to a stick figure.*

The proof is based on the following simple idea. Let C be a stick figure in $\mathbb{P}^3$ and assume that C lies on a surface S consisting of a union of planes, no three containing the same line. Assume further that no component of C lies in the singular locus of S. Let H be a general plane. Then the union of C and $S \cap H$ is again a stick figure. Notice that $C \cup (S \cap H)$ is a basic double link of C.

Let $\mathcal{L}$ be an even liaison class of Buchsbaum curves (i.e. an even liaison class all of whose elements are Buchsbaum with the same Hartshorne-Rao module). If we can show that $\mathcal{L}$ contains a minimal element $C_0 \in \mathcal{L}^0$ that is a stick figure, then the above idea, combined with the Lazarsfeld-Rao property (Theorem 2.13), shows that every $C \in \mathcal{L}$ (and hence every Buchsbaum curve in $\mathbb{P}^3$, since $\mathcal{L}$ is an arbitrary Buchsbaum liaison class in $\mathbb{P}^3$) specializes to a stick figure. (Actually, it is a little more subtle than this. The construction also needs the observation that the sequence of basic double links described in part (b) of Definition 2.11 can be chosen to be strictly increasing, in a sense that we omit here. See [61, Example 6.4.12] and [9, Corollary 5.3].)

So it remains to show that $\mathcal{L}^0$ contains a stick figure. This is done in [10] using Liaison Addition (Theorem 2.6) and induction (see in particular [10, Lemma 3.3]). One first notes that a set of two skew lines is a stick figure, and its Hartshorne-Rao module is one-dimensional (occurring in degree 0). One then builds up any finite length module with trivial R-multiplication using Liaison Addition, using surfaces of suitably chosen degree (but as efficiently as possible). The fact that if the choices are as efficient as possible then the stick figure so constructed is minimal in its even liaison class is a consequence of the description of Buchsbaum curves in [6], [7] and [30].

Example 3.2 Let us construct a Buchsbaum curve C that is a stick figure, and such that $M(C)$ is a module whose dimensions are $(1, 0, 2)$ (meaning 1-dimensional in some degree, 0 in the next degree, and 2-dimensional in the next). Let C_1 and C_2 be two sets of two skew lines, chosen generally. Let $F_i \in I_{C_i}$ $(i = 1, 2)$ be surfaces of degree 2, each a union of planes. Then by Theorem 2.6, $F_2 \cdot I_{C_1} + F_1 \cdot I_{C_2}$ is the saturated ideal of a stick figure Y of degree 8, with $\dim M(Y) = 2$, and the only non-zero component is 2-dimensional occurring in degree 2. (The fact that it is a stick figure of degree 8 is from the geometric interpretation of Liaison Addition and is left to the reader.)

Now let C_3 again be a sufficiently general choice of two skew lines, and let F_3 be a union of *four* planes containing C_3, chosen so that one plane contains one component of C_3, one contains the other component, and the other two are chosen generally. Then $F_1F_2 \cdot I_{C_3} + F_3 \cdot I_Y$ defines a stick figure, C. Its module $M(C)$ is the direct sum of a shift of $M(Y)$ and a shift of $M(C_3)$. What are these shifts? $M(Y)$ is 2-dimensional in degree 2, and it gets shifted to the right by $\deg(F_3) = 4$ to degree 6. $M(C_3)$ is 1-dimensional in degree 0 and gets shifted by $\deg(F_1)+\deg(F_2) = 4$ to degree 4. Thus $M(C)$ has the desired dimensions, with the 1-dimensional component coming in degree 4. The fact that this is the minimal shift among modules in the even liaison class of C follows from [30, Corollary 3.10]. See also [7] for relevant facts about the even liaison class of a Buchsbaum curve.

3.2 Arithmetically Gorenstein Generalized Stick Figures of Codimension Three

It is known from work of Stanley [87] and Buchsbaum and Eisenbud [14] exactly what Hilbert functions can occur for codimension three arithmetically Gorenstein subschemes, and in fact from their work we also know what sets of graded Betti numbers can occur. (From now on we will refer to *Betti diagrams* as a way of collecting the graded Betti numbers for a given graded module, in what is now a standard way. For Gorenstein algebras we will refer to *Gorenstein Betti diagrams*.) Concerning the Hilbert functions, the corresponding Gorenstein sequences are the SI-sequences (Remark 2.19). Diesel [21] described an algorithm to find all possible

Betti diagrams given the SI-sequence. She also showed that the Gorenstein algebras for such a Hilbert function form an irreducible family.

Although the possible Betti diagrams for a given Hilbert function were known, it was not known "how nice" the arithmetically Gorenstein subschemes are for any such Betti diagram. In particular, does each such irreducible family contain a reduced set of points in the case of arithmetically Gorenstein zero-dimensional schemes in $\mathbb{P}^3$, a stick figure in the case of curves in $\mathbb{P}^4$, or a generalized stick figure in the case of codimension three subschemes in $\mathbb{P}^n$? This was shown in the affirmative in [32], and in fact not only for each family but indeed for each possible Betti diagram.

Theorem 3.3 ([32] Theorem 2.1, Corollary 2.4, Remark 2.5) *For any Gorenstein Betti diagram for codimension three subschemes of* $\mathbb{P}^n$*, there is a arithmetically Gorenstein generalized stick figure having that Betti diagram.*

The idea of the proof is as follows. We begin with a possible Gorenstein Betti diagram. From this diagram one finds the Betti diagram of a suitable arithmetically Cohen-Macaulay codimension two subscheme V_1, a suitable complete intersection X containing it, and the Betti diagram of the residual scheme V_2 (using the mapping cone construction mentioned above), so that the sum of the linked ideals is arithmetically Gorenstein and has the desired Gorenstein Betti diagram. This is done using a certain mapping cone, building off of the resolutions for the linked curves. (So far this is all numerical.) Then we use Gaeta's result mentioned above (generalized to $\mathbb{P}^n$) to arrange that V_1, X and V_2 are all generalized stick figures. This gives that $I_{V_1} + I_{V_2}$ defines an arithmetically Gorenstein generalized stick figure of codimension three with the desired Betti diagram.

With this result for codimension three Gorenstein subschemes, it is natural to wonder what we can say about higher codimension.

3.3 Arithmetically Gorenstein (Generalized) Stick Figures of Any Codimension

Recall from Remark 2.19 that in codimension three, the h-vector of a (codimension three) arithmetically Gorenstein subscheme is always an SI-sequence. We also noted that in higher codimension, the SI-sequences are exactly the Hilbert functions of Artinian Gorenstein algebras with the Weak Lefschetz Property. In this setting, though, the Artinian Gorenstein algebras with the same Hilbert function do not in general form an irreducible family.

Remark 3.4 It has also been asked (and we conjecture) whether a general Artinian reduction of a reduced, arithmetically Gorenstein subscheme of projective space necessarily has the Weak Lefschetz Property in characteristic zero [67, Question 3.8]. Of course since Artinian Gorenstein algebras exist with non-unimodal Hilbert function, the extension of such an ideal in a larger polynomial ring defines a non-reduced arithmetically Gorenstein subscheme whose general Artinian reducton does not have the Weak Lefschetz Property. Also, Mats Boij has shown us an example of a reduced, arithmetically Gorenstein set of points in projective space such that a *special* Artinian reduction fails to have the Weak Lefschetz property. Thus the assumptions *general* and *reduced* are important in this question. In any case, the problem of classifying all possible Hilbert functions of Artinian Gorenstein algebras is probably intractable, and the problem of classifying the Hilbert functions of reduced, arithmetically Gorenstein subschemes of projective space is still open. Thus the fact that we at least do know precisely the Hilbert functions of Artinian Gorenstein algebras with the Weak Lefschetz Property is a very welcome result.

In view of Remark 3.4, it is natural to wonder whether every SI-sequence also occurs as the h-vector of a reduced, arithmetically Gorenstein subscheme, and then it is worth asking if it also occurs for a generalized stick figure. In this subsection we are also interested in the question of whether there is a set of maximal graded Betti numbers among arithmetically Gorenstein subschemes with the given Hilbert function which have general Artinian reduction with the Weak Lefschetz Property.

The story starts with the paper [28] of A.V. Geramita, T. Harima and Y.S. Shin. In that paper they used CI-liaison to construct certain Artinian Gorenstein algebras with the Weak Lefschetz Property (and as such, whose Hilbert function is an SI-sequence). They were not interested in generalized stick figures or reduced arithmetically Gorenstein algebras, and most importantly they did not produce an example for every possible SI-sequence. They did, however, prove the extremality of the graded Betti numbers for the class of algebras that they constructed. The paper [66] goes beyond these results (and in fact Remark 10.2 of [66] shows that CI-links are not enough to produce all SI-sequences).

In fact, [66] was innovative in the application of liaison in two ways. First, it was one of the first papers that applied G-liaison (rather than CI-liaison) to construct interesting objects. Second, and more surprisingly, the approach was in some sense the reverse of the usual one: instead of starting with a scheme and producing a Gorenstein scheme containing it to produce a desired link, the approach was to first produce a totally reducible (i.e. union of linear varieties) arithmetically Gorenstein scheme and find *within it* a suitable arithmetically Cohen-Macaulay subscheme, and then perform the desired G-link. The scheme with the desired Hilbert function is then obtained as a sum of G-linked ideals.

The first main result of [66] is the following. We will return to this paper when we discuss simplicial polytopes.

Theorem 3.5 ([66] Theorem 1.1) *Let* $\underline{h} = (1, c, h_2, \ldots, h_{s-2}, c, 1)$ *be an SI-sequence and let* K *be an arbitrary field containing sufficiently many elements. Then for every integer* $d \geq 0$ *there is a reduced arithmetically Gorenstein union of linear varieties,* $G \subset \mathbb{P}^{c+d}$*, of dimension* d*, whose general Artinian reduction has the Weak Lefschetz Property, and whose* h*-vector is* $\underline{h}$*.*

Remark 3.6

1. Of course the assumption that K has sufficiently many elements depends on the choice of $\underline{h}$.
2. The theorem does not, unfortunately, guarantee that the arithmetically Gorenstein scheme produced is a generalized stick figure, only that it is reduced. However, the "large" Gorenstein scheme referred to before the statement of the theorem *is* a generalized stick figure, and this is what guarantees that when it is used to produced a sum of linked ideals, the result will again be reduced. The authors conjecture that the schemes are, in fact, again generalized stick figures [66, Remark 6.5].

As mentioned, the arithmetically Gorenstein union of linear varieties produced in Theorem 3.5 also has an extremality property. We remark that the schemes produced in [28] also have such a property.

Theorem 3.7 *Fix an SI-sequence $\underline{h}$. The scheme produced in Theorem 3.5 with h-vector $\underline{h}$ has maximal graded Betti numbers among arithmetically Gorenstein subschemes of $\mathbb{P}^n$ whose general Artinian reductions have the Weak Lefschetz Property and Hilbert function $\underline{h}$.*

In a way analogous to the approach of [28], the idea is to arrange that the linked varieties are not only arithmetically Cohen-Macaulay, but in fact have extremal Betti numbers for their (prescribed) Hilbert functions.

4 The Singular Locus of a Hyperplane Arrangement

In this section we will assume that k has characteristic zero.

If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{P}^n$, it is defined by a product, F, of linear forms such that none is a scalar multiple of another. Let

$$J = \langle F_{x_0}, F_{x_1}, \dots, F_{x_n} \rangle$$

be the Jacobian ideal, generated by the partial derivatives of F. Note that J is not necessarily saturated, and that the saturation J^{sat} is not necessarily unmixed. It does, however, have height two. Consider a primary decomposition of J,

$$J = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s \cap \dots$$

where $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are all the primary components of height 2. For each such primary ideal, let $\mathfrak{p}_i$ be the associated prime. Then set

$$J^{top} = \bigcap_{1 \le i \le s} q_i \quad \text{and} \quad \sqrt{J} = \bigcap_{1 \le i \le s} p_i .$$

Both J^{top} and $\sqrt{J}$ are unmixed ideals of height 2. We denote by X^{top} the scheme defined by J^{top} and by X^{red} the scheme defined by $\sqrt{J}$.

Remark 4.1 Although the saturation of J can still have embedded components, it is interesting to study the unmixed singular locus of $\mathcal{A}$, and this could refer to either X^{top} (which is not necessarily reduced) or to X^{red}. The results described here are a contribution to this.

As a first step to studying the schemes X^{top} and X^{red}, one asks if they are necessarily ACM. If they are not, are there conditions that guarantee that they are ACM? And in terms of the invariants $M^i(C)$ (which we saw in Sect. 2 is a measure of the failure to be ACM), how far can these schemes be from being ACM? In this section we address all of these questions.

We first give a version of Liaison Addition (Theorem 2.6) in the language of arrangements, that we will use in the rest of this section.

Theorem 4.2 (Arrangement Version of Liaison Addition in $\mathbb{P}^3$) *Let $\mathcal{A}_1 = \bigcup_{i=1}^{s_1} H_i$ and $\mathcal{A}_2 = \bigcup_{i=1}^{s_2} H_i'$ be plane arrangements in $\mathbb{P}^n$ with corresponding schemes X_i^{top}, X_i^{red} ($i = 1, 2$).*

(∗) *Assume that no plane of $\mathcal{A}_1$ contains a component of X_2^{red} and vice versa.*

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, with schemes X^{top} and X^{red}. Then for each $1 \le i \le n-2$,

$$M^i(X^{top}) \cong M^i(X_1^{top})(-s_2) \oplus M^i(X_2^{top})(-s_1).$$

Similarly,

$$M^i(X^{red}) \cong M^i(X_1^{red})(-s_2) \oplus M^i(X_2^{red})(-s_1).$$

In particular, if X_1^{top} and X_2^{top} (resp. X_1^{red} and X_2^{red}) are ACM then also X^{top} (resp. X^{red}) is ACM.

We also give a version of Basic Double Linkage (Theorem 2.7) in the language of arrangements.

Theorem 4.3 (Arrangement Version of Basic Double Linkage in $\mathbb{P}^n$) *Let $\mathcal{A}$ be an arbitrary hyperplane arrangement in $\mathbb{P}^n$ with corresponding schemes X^{top} and X^{red}. Let H be a plane* not containing any component of X^{red}. *Let $\mathcal{A}' = \mathcal{A} \cup H$, with corresponding schemes Y^{top} and Y^{red}. Then X^{top} and Y^{top} are linked in two steps, as are X^{red} and Y^{red}. In particular:*

(i) *We have isomorphisms*

$$M^i(Y^{top}) \cong M^i(X^{top})(-1) \quad \textit{and} \quad M^i(Y^{red}) \cong M^i(X^{red})(-1)$$

for $1 \le i \le \dim X = \dim Y$.

(ii) *X^{top} (resp. X^{red}) is ACM if and only if Y^{top} (resp. Y^{red}) is ACM.*

A very special case of Theorem 4.3 is the following.

Corollary 4.4 ([26]) *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$ and assume that no three hyperplanes contain the same codimension two linear subvariety. Then $X^{top} = X^{red}$ is ACM.*

Proof One simply notes that the intersection of two planes is ACM, and then applies Theorem 4.3 successively for the remaining planes. □

Remark 4.5 The schemes described in Corollary 4.4 are called *codimension two star configurations*. Theorem 2.8 can be used to extend this result. Assume that the intersection of any j of the hyperplanes of $\mathcal{A}$ is either empty or of codimension j. Then for any $1 \le c \le \min(s, n)$ let $V_c(\mathcal{A})$ be the union of the codimension c linear varieties defined by the intersections of these hyperplanes, taken c at a time. In [26] these were called *codimension c star configurations*. Then it was shown in [26] (among other results) that $V_c(\mathcal{A})$ is also ACM. The machinery of Theorem 2.8 can also be used to give Hilbert functions and Betti numbers of $V_c(\mathcal{A})$, which we omit here.

Our first main result is obtained by combining Liaison Addition (Theorem 4.2) and Basic Double Linkage (Theorem 4.3). It says that under a condition on the hyperplanes, X^{top} and X^{red} are both ACM.

Theorem 4.6 ([69]) *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$. Assume that*

$$(*)\ \begin{cases} \text{no linear factor of } F \text{ is in the associated prime of any two \underline{non-reduced} components} \\ \text{of } J^{top}. \end{cases}$$

Then both R/J^{top} and $R/\sqrt{J}$ are Cohen-Macaulay (i.e. X^{top} and X^{red} are ACM). If $(*)$ fails then both X^{top} and X^{red} may *fail to be ACM.*

Remark 4.7 In 1983, Hiroaki Terao [90] conjectured that the freeness of a hyperplane arrangement is a combinatorial property, i.e. whether it is determined from its intersection lattice. The above result does not address freeness. Nevertheless, notice that $(*)$ is a combinatorial property of the intersection lattice of $\mathcal{A}$. Thus one can also ask whether the property that X^{top} (resp. X^{red}) is Cohen-Macaulay is a combinatorial property of the intersection lattice of $\mathcal{A}$. Theorem 4.6 says that the answer is yes provided that condition $(*)$ holds, so the issue is whether it is true without condition $(*)$.

We now focus on arrangements in $\mathbb{P}^3$. As indicated in Theorem 4.6, if condition $(*)$ fails then it is possible for either X^{top} or X^{red} or both to fail to be ACM. The first example, where X^{top} fails, was given in [73, Example 4.5] for hyperplane arrangements in $\mathbb{P}^3$, and in their example X^{red} is ACM. Experimenting with subarrangements of their example, we were able to find examples for X^{red} to fail to be ACM while X^{top} is ACM, and examples where both fail to be ACM. In all cases, the dimension of the Hartshorne-Rao module was 1. As such, these curves are automatically Buchsbaum.

This leads to the question of by how much the Cohen-Macaulay property can fail to hold, and one answer is provided by another application of Liaison Addition:

Theorem 4.8 *Let $r \geq 1$ be a positive integer. Then:*

(i) *There exists a positive integer N and an arrangement $\mathcal{A}$ in $\mathbb{P}^3$, such that*

$$\dim M(X^{top})_i = \begin{cases} r & \text{if } i = N; \\ 0 & \text{if } i \neq N \end{cases}$$

(ii) *The same result holds for X^{red}, although the value of N is not necessarily the same.*
(iii) *For each $h \geq 1$ we can replace N by $N + h$ and obtain the same result.*
(iv) *The curves obtained above are all in the same even liaison class.*

The proof involves making sufficiently general "copies" of the curves described before the statement of the theorem, and applying Liaison Addition $r - 1$ times, keeping careful track of the degrees. Then part (iii) is obtained by basic double linkage using general linear forms (i.e. adding general hyperplanes to $\mathcal{A}$). The conclusion (iv) is a direct application of Rao's theorem (Theorem 2.5). Notice that the curves described here are again automatically Buchsbaum, having only one non-zero component in the Hartshorne-Rao module.

Remark 4.9 It is natural to ask which other even liaison classes arise among the schemes X^{top} or X^{red} for arrangements in $\mathbb{P}^3$. We have found several examples to show that the curves in Theorem 4.8 are not the only non-ACM examples. However, a classification remains out of reach. We would be very interested to know, for example, whether any arithmetically Buchsbaum curve whose Hartshorne-Rao module is non-zero in more than one degree can arise in this way.

5 The Eisenbud-Green-Harris Conjecture and Cayley-Bacharach

It is hard to imagine a more appropriate topic, in a paper on applications of liaison theory, than a description of the paper "An application of liaison theory to the Eisenbud-Green-Harris conjecture" [16] by Ernest Chong.

The classical Cayley-Bacharach theorem (see [22, Exercise 21.24]) says the following. Let Z be a reduced complete intersection of two plane cubics in $\mathbb{P}^2$. Let $P \in Z$ be any point and let $Y = Z \backslash P$. If F is any cubic vanishing at the eight points of Y, then F must also vanish at P. Another way to say this is that the Hilbert function of Z and the Hilbert function of Y agree in degrees ≤ 3. More precisely, the Hilbert function of Z is $(1, 3, 6, 8, 9, 9, \dots)$ and the Hilbert function of *any* eight points of Z must be $(1, 3, 6, 8, 8, \dots)$.

Remark 5.1 It is beyond the scope of this paper to discuss all the different directions beyond this result that have been studied, but we make a few comments.

The classical Cayley-Bacharach theorem has led to the notion of the *Cayley-Bacharach property* for a set of points, defined as follows. It is not hard to show that given *any* reduced subset of degree d in any projective space $\mathbb{P}^n$ (not only $\mathbb{P}^2$) with Hilbert function

$$(1, n+1, h_2, \dots, h_{t-1}, h_t = d, d, \dots),$$

with $h_{t-1} < d$, there must be at least one subset of $d-1$ points with the truncated Hilbert function

$$(1, n+1, h_2, \dots, h_{t-1}, h_t - 1 = d-1, d-1, \dots).$$

(See for instance [29].) The set Z has the Cayley-Bacharach property if this is true for *every* choice of $d-1$ points. A standard fact, which follows easily from liaison (specifically from Theorem 2.2) is that any arithmetically Gorenstein set of points in any projective space has the Cayley-Bacharach property. One generalization has been the notion of the *uniform position property*, which has been important in the study of the genus of space curves (see for instance [40]). A set of points Z in $\mathbb{P}^n$ has the uniform position property if, for any fixed cardinality p, all subsets of p points have the same Hilbert function (which must be the truncation at level p of the Hilbert function of Z).

The Cayley-Bacharach property has been studied in many papers, for example [15] and [53] (both of which use liaison as a tool).

The fact that the Cayley-Bacharach property is closely related to liaison has been studied for many years (see for instance [20]), but in 1996 David Eisenbud, Mark Green and Joe Harris wrote the beautiful paper [23], starting with historical versions of this result and developing the theory until they arrived at several versions of what is now called the Eisenbud-Green-Harris (EGH) conjecture. We refer the reader to [23] for all of the beautiful intricacies and interrelations between the different ideas, and here we will focus on the version addressed by Chong.

Let $S = k[x_1, \dots, x_n]$, where k is an infinite field. The version of the conjecture quoted by Chong is the following.

Conjecture 5.2 (Eisenbud-Green-Harris Conjecture [23]) Let $2 \le e_1 \le \cdots \le e_n$ be integers. If $I \subsetneq S$ is a homogeneous ideal that minimally contains an $(e_1, \dots, e_n)$-regular sequence of forms, then there exists a homogeneous ideal $J \subsetneq S$ containing $x_1^{e_1}, \dots, x_n^{e_n}$, such that I and J have the same Hilbert function.

Some special cases of this conjecture have been proven (we refer to [16] for a partial list). Chong's idea is to prove it for a new special case using a result of the current authors in [68]. His main theorem proves it for height three Gorenstein ideals:

Theorem 5.3 ([16] Theorem 1) *Let $2 \leq e_1 \leq e_2 \leq e_3$ be integers. If $I \subsetneq k[x_1, x_2, x_3]$ is a homogeneous Gorenstein ideal that minimally contains an (e_1, e_2, e_3)-regular sequence of forms, then there exists a monomial ideal J in $k[x_1, x_2, x_3]$ containing $x_1^{e_1}, x_2^{e_2}, x_3^{e_3}$, such that I and J have the same Hilbert function.*

As noted, Chong's idea was to use liaison to prove this result. First, it follows immediately from Theorem 2.2 that if I_1 and I_2 are two Cohen-Macaulay ideals (of any codimension) with the same Hilbert function, and if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are complete intersections in I_1 and I_2, respectively, of the same type, then the linked ideals $\mathfrak{c}_1 : I_1$ and $\mathfrak{c}_2 : I_2$ have the same Hilbert function.

The key ingredient of Chong's proof is the notion of minimal linkage, which we now describe. Given an ideal I, there is certainly an initial degree d_1 in which I is non-zero (i.e. the initial degree of I). Then there is a smallest degree $d_2 \geq d_1$ such that I contains a regular sequence of type (d_1, d_2). Continuing in this way, there is a smallest c-tuple (where c is the codimension of I) $(d_1, d_2, \dots, d_c)$ for which I contains a regular sequence of those degrees. Certainly such a regular sequence can be used to perform a link of $I := I_1$, obtaining a residual ideal I_2.

Next, one can apply the same construction to the residual I_2. The resulting tuple is lexicographically smaller than or equal to the original one. One sequentially applies this construction (in [60] it was called the *minimal link procedure*) until one of two things happens. Either the smallest tuple is no smaller than the one just before, or else one arrives at a complete intersection ideal.

In the former case, if R/I is not Cohen-Macaulay one can hope that the final ideal is minimal in its even liaison class, in the sense of Definition 2.11. It was shown in [68] that among curves in $\mathbb{P}^3$, in some even liaison classes this is true, and in others it is not true. In the latter case, i is said to be *licci* (i.e. in the linkage class of a complete intersection) in particular, and hence R/I is Cohen-Macaulay. We now focus on the Cohen-Macaulay situation.

In codimension two, Gaeta proved that this procedure always leads to a complete intersection. In codimension 3, an example was given in [27] of a licci ideal that could not be minimally linked to a complete intersection, but no proof was given. This was remedied in [47], where it was even shown that for the Hilbert function (1, 3, 6, 8, 7, 6, 2) there exist three ideals I_1, I_2, I_3 such that all three quotients have this Hilbert function, but one is not licci, one is licci but cannot be linked to a complete intersection by minimal links, and one is licci and can be linked to a complete intersection by minimal links. The latter two even have the same graded Betti numbers.

This all goes to show that it is interesting to study what properties of an ideal give that it can be minimally linked to a complete intersection. (The example just mentioned shows that the Hilbert function and the graded Betti numbers are not enough, in general.) One such property is that of being Gorenstein of codimension three.

J. Watanabe showed in [91] that any such ideal is licci, but he did not consider minimal links. Watanabe's result is extended to minimal links:

Theorem 5.4 ([68] Theorem 6.3) *If $I \subset k[x_1, x_2, \ldots, x_n]$ ($n \geq 3$) is a homogeneous Gorenstein ideal then I can be minimally linked to a complete intersection.*

We now return to Chong's nice idea to use this result to prove Theorem 5.3. He first weakens the notion of minimal links down to a complete intersection.

Definition 5.5 Let I be a licci ideal of height r. Suppose there exists a sequence of CI-links

$$I = I_0 \overset{J_1}{\sim} I_1 \overset{J_2}{\sim} \cdots \overset{J_s}{\sim} I_s$$

where I_s is a complete intersection. Say the type of J_i is $\mathbf{a}^{(i)} \in \mathbb{Z}_+^r$, and assume

$$\mathbf{a}^{(1)} \geq \cdots \geq \mathbf{a}^{(s)}$$

in the lexicographic order. Then I is said to be a *sequentially bounded* licci ideal. If, furthermore, J_1 is a minimal link, we say that I is a sequentially bounded licci ideal *that admits a minimal first link.*

Chong then proves the following important theorem.

Theorem 5.6 *Let $2 \leq e_1 \leq \cdots \leq e_n$ be integers. If $I \subsetneq S = k[x_1, \ldots, x_n]$ is a sequentially bounded licci ideal that admits a minimal first link and minimally contains an $(e_1, \ldots, e_n)$-regular sequence of forms, then there exists a monomial ideal $J \subsetneq S$ containing $x_1^{e_1}, \ldots, x_n^{e_n}$ such that I and J have the same Hilbert function.*

The proof is a bit technical, but essentially the existence of the specified sequence of links starting with I down to a complete intersection allows one to construct a numerically equivalent sequence of links using monomial ideals.

Once this is established, Theorem 5.3 follows immediately from Theorem 5.4.

6 The Genus of Space Curves

A very classical problem, going back well over a century, is to classify the smooth curves in $\mathbb{P}^3$ (also called *space curves*). In particular, one can ask which pairs (d, g) occur for a smooth space curves, and what role is played by the least degree of a surface containing the curve. Furthermore, what Hilbert functions can arise for such curves, or for their general hyperplane sections? This can also be extended from smooth curves to locally Cohen-Macaulay, equidimensional curves (see for example [42]), but we consider here only the smooth case.

As of about 1980, two outstanding references for much of what was known at the time were Hartshorne's book [41] (Chapter IV, section 6) and Harris's Montreal Notes [39]. And of course one can extend all this to curves in $\mathbb{P}^n$, where open questions remain. It is very far beyond the scope of this paper to describe this rich history, but two results in fact are connected (and use) liaison, and we will describe these. The first was written by J. Harris [40] and appeared in 1980, and the second was written by R. Maggioni and A. Ragusa and appeared in 1988.

The first complete answer to the question of which pairs (d, g) occur for smooth, non-degenerate space curves is due to Gruson and Peskine [35], and we omit the slightly technical statement. A weaker question is to ask for a *bound* on the genus of a smooth curve of degree d, and this was settled many years ago:

$$g(C) \le \begin{cases} (\frac{d}{2}-1)^2, & d \text{ even} \\ (\frac{d-1}{2})\left(\frac{d-3}{2}\right), & d \text{ odd} \end{cases}$$

with equality if and only if C lies on a quadric surface and is either a complete intersection (in the even case) or residual to a line (in the odd case) in a suitable complete intersection with a quadric. See [40] or [44]. The latter attributes this result to [6].

Among curves not lying on a quadric but lying on a cubic surface there is a new bound:

$$g(C) \le \begin{cases} \frac{d^2}{6}-\frac{d}{2}+1, & d \equiv 0 \pmod 3 \\ \frac{d^2}{6}-\frac{d}{2}+\frac{1}{3}, & d \equiv 1, 2 \pmod 3 \end{cases}$$

and Harris notes that these three values represent (respectively) (i) the genus of the complete intersection of a cubic and a surface of degree $\frac{d}{3}$, (ii) the genus of the residual to a conic in a complete intersection of a cubic and a surface of degree $\frac{d+2}{3}$, or (iii) the genus of the residual to a line inside a complete intersection of a cubic and a surface of degree $\frac{d+1}{3}$ (see [40]).

Notice that in some sense what matters is not so much that the extremal curves in the latter case fail to lie on a quadric surface as that they *do* lie on an *irreducible* cubic surface. So, for instance, if $d = 10$ then such a curve may lie on a quadric surface, but then (by Bezout's theorem) it has no hope of lying on an irreducible cubic surface. If $d = 5$, on the other hand, then the argument is a little bit more subtle: Bezout's theorem does not rule out the possibility that C lie on both a quadric surface and an irreducible cubic surface, but the residual must be a line, which is arithmetically Cohen-Macaulay, so C also must be arithmetically Cohen-Macaulay and we must have that C has genus 2, which is extremal using either formula.

Harris's idea was to extend this to higher degree surfaces. He produces a formula (which we will not repeat here, but which is analogous to the two formulas above) which gives an upper bound for the genus of a smooth curve lying on an irreducible surface of degree k. His argument breaks into two cases, namely $d > k(k-1)$ and $d \leq k(k-1)$, as one might guess from the preceding paragraph. Furthermore, he shows that the extremal curves are always residual, in a suitable complete intersection, to a plane curve of degree $n = \lceil \frac{d-1}{k} \rceil + 1$.

Central to Harris's argument was an approach using hyperplane sections, and studying the Hilbert function (or more precisely the h-vector) of the corresponding points in $H = \mathbb{P}^2$. It was in this paper that he introduced the crucial notion of points being in *uniform position*, meaning that all subsets of m points (for any m) have the same Hilbert function. One also says that the points have the *Uniform Position Property*. Harris showed that the general hyperplane section of a reduced, irreducible curve $C \subset \mathbb{P}^3$ has this property. Furthermore, he shows that the h-vector of the general hyperplane section of C is of (what has come to be called) *decreasing type*. This means that the beginning of the h-vector agrees with the polynomial ring, i.e. `(1,2,3,4,...)`, then is possibly flat at the highest point, and after that is *strictly decreasing*. So for example `(1,2,3,4,5,5,5,3,2)` is of decreasing type, while `(1,2,3,4,5,5,5,3,2,2)` is not.

A more general bound for the genus (now for curves in $\mathbb{P}^r$) was given in [39], by Eisenbud and Harris. In this book the authors ask what may be the Hilbert function of the general hyperplane section of a reduced, irreducible curve in $\mathbb{P}^r$. Furthermore, what may be the Hilbert function of a set of points in $\mathbb{P}^{r-1}$ with the Uniform Position Property? And are the two answers the same?

This question was the launching point for the second liaison-related result that we describe here, namely the paper [58] of R. Maggioni and A. Ragusa. In this paper the authors show that when $r = 3$, the answers are indeed the same and the possible h-vectors are exactly those of decreasing type. Part of this of course was done by Harris, and the task remaining for the authors was to show that given an h-vector of decreasing type, there exists a smooth curve (in fact an arithmetically Cohen-Macaulay smooth curve) whose general (in fact arbitrary) hyperplane section has the given h-vector.

As mentioned, the proof uses liaison. One starts with an h-vector $\underline{h}$ of decreasing type. From $\underline{h}$ one can read the least degree, a_1, of a minimal generator of the ideal I_C of any arithmetically Cohen-Macaulay curve C with this h-vector. One can also read the degree, a_2, where such curve would have its second minimal generator. In general it is not necessarily true that I_C contains a regular sequence of type (a_1, a_2), but it is true if C is irreducible, and it is true for the general hyperplane section of C. One then formally produces the "residual" h-vector to $\underline{h}$, by a complete intersection of type (a_1, a_2), using Theorem 2.2. Call this sequence $\underline{h}'$.

Next, the authors construct a reduced, ACM union of lines, C', in $\mathbb{P}^3$ whose h-vector is $\underline{h}'$. They show that C' lies on a smooth surface, S, of degree a_1. They do this with a variation of Bertini's theorem. Finally, they look at the general residual to C' in a complete intersection of S and a surface of degree a_2; that is, they look at the linear system $|a_2 H - C'|$ on S and show that the general element is smooth. This general element is the desired smooth arithmetically Cohen-Macaulay curve C.

Remark 6.1 In the above argument we have ignored the issue of why we need decreasing type. As the authors remark, in this case it can never happen that C (with h-vector $\underline{h}$) lies in a complete intersection of type (a_1, a_2).

7 Liaison and Graded Betti Numbers

Liaison theory has been used in a number of contexts in order to achieve information on minimal free resolutions. We will highlight a few instances.

It is useful to recall a module version of the exact sequence of sheaves (2.1). Let I, J be ideals of a polynomial ring $R = k[x_0, \ldots, x_n]$ that are directly linked by a Gorenstein ideal $\mathfrak{c} \subset R$, that is, $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$. Then there is a short exact sequence (see, e.g., [74, Lemma 3.5])

$$0 \to \mathfrak{c} \hookrightarrow I \to \omega_{R/J}(-t) \to 0, \tag{7.1}$$

where t is the integer such that $\omega_{R/\mathfrak{c}} \cong R/\mathfrak{c}(t)$ and $\omega_{R/J} \cong \operatorname{Ext}^c_R(M, R)(-t)$ is the canonical module of R/J with $c = n + 1 - \dim R/J$, the codimension of J. If R/J is Cohen-Macaulay then the Betti numbers of R/J and those of its canonical module determine each other. Explicitly, if

$$0 \to F_c \to \cdots \to F_1 \to R \to R/J \tag{7.2}$$

is a graded minimal free resolution of R/J over R, then dualizing with respect to R gives a graded minimal free resolution of $\operatorname{Ext}^c_R(M, R)$,

$$0 \to R \to F_1^* \to \cdots \to F_c^* \to \operatorname{Ext}^c_R(M, R) \to 0.$$

Consider an ideal I of R that is minimally generated by s homogeneous polynomials of degrees $d_1, \ldots, d_s$. This is not enough information to determine the Hilbert function of R/I. However, if the generators of I are sufficiently general polynomials with the specified degrees, then the Hilbert function of R/I depends only on the integers $n, d_1, \ldots, d_s$. In fact, Fröberg's conjecture (see [24]) predicts this Hilbert function. This conjecture is known in a number of cases, but open in general.

The graded Betti numbers of such ideals are much less understood. In fact, one can show that the graded Betti numbers of an ideal I generated by sufficiently general forms are determined by n and the generator degrees $d_1, \ldots, d_s$. However, even in interesting special cases there is not even a conjecture that gives a precise description of the minimal graded free resolution of I. Note that the ideal I is resolved by a Koszul complex if $s \leq n+1$. Thus the first interesting case is $s = n+2$, that is, I is an almost complete intersection. Although even in this case the graded Betti numbers are not known in general, many partial results have been established by the first author and Miro-Roig in [63] (see also [64]). Most notably, if $n = 2$

and so $s = 4$ the Betti numbers have been determined in [63, Theorems 4.2]. In any case, the authors obtain very good upper bounds on the Betti numbers. The basic strategy is to use induction on the number of variables.

Consider a complete intersection $\mathfrak{c}$ and an almost complete intersection $I = (\mathfrak{c}, f)$. Then I is linked by c to a Gorenstein ideal $J = \mathfrak{c} : I = \mathfrak{c} : f$. Using duality, the exact sequence (7.2) becomes

$$0 \to R/J(-\deg f) \to R/\mathfrak{c} \to R/I \to 0. \tag{7.3}$$

Now assume that f and the generators of $\mathfrak{c}$ are sufficiently general. Then $R/\mathfrak{c}$ has the Strong Lefschetz Property, and thus so does $R/(\mathfrak{c} : f) = R/J$ because f is generic. Moreover, one gets $[R/J]_j = [R/\mathfrak{c}]_j$ if $j \leq \frac{1}{2}(d_0 + \cdots + d_n - n - 1 - \deg f)$, where $d_0, \ldots, d_n$ are the degrees of the generators of $\mathfrak{c}$. Let $\ell \in R$ be a generic linear form. The last equality implies $[R/(J, \ell)]_j = [R/(\mathfrak{c}, \ell)]_j$ if $j \leq \frac{1}{2}(d_0 + \cdots + d_n - n - 1 - \deg f)$. Using that R/J has the Weak Lefschetz Property, one knows the Hilbert function of $R/(J, \ell)$. Moreover, $R/(\mathfrak{c}, \ell)$ is isomorphic to a quotient of $R/\ell R \cong k[x_0, \ldots, x_{n-1}]$ by an almost complete intersection whose generators have degrees $d_0, \ldots, d_n$. Thus, by induction on n we have information on the graded Betti numbers of $R/(\mathfrak{c}, \ell)$, and so on the graded Betti numbers of $R/(J, \ell)$ in low degrees. Invoking [66, Proposition 8.7], this gives upper bounds on the graded Betti numbers of R/J in low degrees. Since the resolution of R/J is self-dual as R/J is Gorenstein, one obtains upper bounds on all graded Betti numbers of R/J. Finally, using Sequence (7.3), this gives information on the resolution of R/I. The base case for the induction is $n = 2$, where R/J is a Gorenstein ideal of codimension three. Thus, its minimal free resolution is known by work of Diesel [21].

It turns out that the obtained bounds are optimal in several situations, once cancellations in the mapping cone procedure are taken into account. In general, it is a difficult problem to establish if a cancellation occurs or not. Since I is a generic complete intersection, one may hope that its minimal free resolution has few if any ghost terms, that is, free summands that appear in consecutive homological degrees. Note that ghost terms cannot be entirely avoided. For example, if I has two generators of degree 5 and one generator of degree 10, there is a Koszul syzygy of degree 10 producing a ghost summand $R(-10)$. A natural conjecture, due to Iarrobino [48], predicted that these Koszul syzygies are the only source of ghost terms. However, this is too optimistic. Consider, for example, a generic almost complete intersection I in three variables with generator degrees 4, 4, 4, 8. Its minimal free resolution has the form (see [63, Example 4.3]):

$$0 \to \begin{array}{c} R(-10) \\ \oplus \\ R(-11)^2 \end{array} \to \begin{array}{c} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{array} \to \begin{array}{c} R(-4)^3 \\ \oplus \\ R(-8) \end{array} \to I \to 0.$$

It has a ghost term $R(-10)$, which is not a consequence of a Koszul syzygy. The presence of ghost terms makes it challenging to predict the minimal free resolution when the number of variables is large.

More recently, liaison theory has been used with regards to a conjecture of Mustaţă [72] on the minimal free resolution of a general set of points X on an irreducible subvariety $S \subset \mathbb{P}^n$. Essentially, the conjecture posits that the top part of the Betti diagram of R/I_X is the Betti diagram of R/I_S and the bottom part has only two rows with no ghost terms. Although it is not true in general, this conjecture has motivated several investigations.

Assume X is a general set of points on a surface S of $\mathbb{P}^3$. In this case, Mustaţă's conjecture has been established if S is a smooth quadric [33], a smooth cubic (see [70] and [71]) or a general quartic surface [5]. The last two results use liaison theory in order to prove the conjecture by induction on the number of points on X. Once the result is shown for a certain number of points this set is linked by a suitable Gorenstein set of points to a larger set of points. Sequence (7.1) is used to guarantee that the new set of points satisfies the conjecture as well. Establishing the existence of suitable Gorenstein sets of points is rather subtle. Thus, in [5] the conjecture is first shown for sets of points on a carefully constructed quartic surface. Semicontinuity implies then the desired result on a general quartic surface.

As indicated in Theorem 3.5 above, a different set of tools from liaison theory has been used in order to construct reduced Gorenstein schemes with prescribed properties. In fact, the methods also provide information on their graded Betti numbers.

A key is to use geometric linkage. Suppose ideals I and J are geometrically linked, that is, I and J do not have associated prime ideals in common and $I \cap J$ is a Gorenstein ideal of codimension, say, c. Then Sequence (7.1) implies

$$\omega_{R/J}(-t) \cong I/I \cap J \cong (I+J)/J.$$

It follows that $I + J$ is a Gorenstein ideal of codimension $c + 1$ that fits into an exact sequence

$$0 \to \omega_{R/J}(-t) \to R/J \to R/(I+J) \to 0. \tag{7.4}$$

As pointed out above, if R/J is Cohen-Macaulay, then a resolution of R/J determines a resolution of its canonical module $\omega_{R/J}$. Using Sequence (7.4), one obtains upper bounds on the graded Betti numbers of $I + J$. If the Castelnuovo-Munford regularity of J is large enough compared to the regularity of $I \cap J$, then these bounds are sharp by Migliore and Nagel [66, Corollary 8.2]. This is an important ingredient of the following result.

Theorem 7.1 ([66, Theorem 8.13]) *Let A be a graded Gorenstein k-algebra whose Artinian reduction has the Weak Lefschetz Property. Then for any integers i, j, there is an upper bound on* $\dim_k[\mathrm{Tor}_i^R(A,k)]_j$ *depending on the Hilbert function of A.*

Moreover, given any Hilbert function of a graded Gorenstein k-algebra of positive dimension whose Artinian reduction has the Weak Lefschetz Property, there is a reduced *Gorenstein algebra with these properties such that the above bounds are equalities for every i and j, provided the field k has sufficiently many elements,*

The Gorenstein algebras proving sharpness of the bounds are constructed using sums of geometrically G-linked ideals. The bounds are a consequence of [66, Proposition 8.7] that compares the graded Betti numbers of an Artinian Gorenstein algebra with those of $A/\ell A$, where $\ell \in [A]_1$ is sufficiently general. Since $A/\ell A$ is Cohen-Macaulay its graded Betti numbers are bounded above by whose of R/L where L is a lexicographic ideal such that R/L has the same Hilbert function as A. The graded Betti numbers of a lexicographic ideal were explicitly computed in [75, Proposition 3.8].

The above result has consequences for the theory of simplicial polytopes. Consider a d-dimensional simplicial polytope P. Let $\Delta(P)$ its boundary complex. The Stanley-Reisner ring $k[P] = R/I_{\Delta(P)}$ is a Gorenstein ring of dimension d. The so-called g-theorem classifies face vectors of simplicial polytopes, equivalently, Hilbert functions of $k[P]$. In particular, Stanley showed in [88] that a general Artinian reduction of $k[P]$ has the Weak Lefschetz Property if k has characteristic zero.

Theorem 7.2 ([66, Theorem 9.5]) *Suppose k has characteristic zero. If P is a d-dimensional simplicial polytope then there is an upper bound for any graded Betti number of the Stanley-Reisner ring $k[P]$ that depends only on the face vector of P.*

Moreover, for every face vector of a simplicial polytope, there is a simplicial polytope with the given face vector such that the above bounds are all simultaneously sharp.

As pointed out in [75], any empty simplex of a simplicial polytope P corresponds to a minimal generator of the monomial ideal $I_{\Delta(P)}$. Hence, Theorem 7.2 implies a conjecture by Kalai, Kleinschmidt and Lee on the number of empty simplices of a simplicial polytope (see [75, Theorem 2.3]). Additional work is needed to establish the following result:

Theorem 7.3 ([75, Corollary 4.16]) *Let P be a d-dimensional simplicial polytope with n vertices, which is not a simplex. Then P has at most $\binom{g+k}{g-1} + \binom{g+k-1}{g-1}$ empty simplices of dimension $\leq k$, where $g = n - d - 1$.*

8 Gröbner Bases and Rees Algebras

Suppose $I \subset R$ is a homogeneous ideal with a generating set G whose initial monomials (with respect to some monomial ordering on R) generate a monomial ideal I'. Using Buchberger's classical criterion in order to decide whether G is a Gröbner basis of I is often not feasible. In interesting cases, liaison theory offers an alternate approach. In fact, if I and I' can be linked to complete intersections of the same type and both chains of links have the same pattern, then this implies that both, R/I and R/I', are Cohen-Macaulay ideals with the same Hilbert function. Hence the inclusion $I' \subset I$ must be an equality, and G is indeed a Gröbner basis of I.

In order to implement this basic idea it is often enough to leave out every other ideal in a chain of direct links by using suitable generalizations of basic double links as discussed in Theorems 2.7 and 2.8. Here we give a more algebraic version.

Definition 8.1

(i) Let $\mathfrak{a} \subset I \subset R$ be homogeneous ideals such that $\operatorname{codim} \mathfrak{a} + 1 = \operatorname{codim} I$ and $R/\mathfrak{a}$ is Cohen-Macaulay. If $f \in R$ is homogeneous with $\mathfrak{a} : f = \mathfrak{a}$, then the ideal $fI + \mathfrak{a}$ is called a *basic double link* of degree $\deg f$ on $\mathfrak{a}$.
(ii) Let $\mathfrak{a}, I, J$ be unmixed homogeneous ideals of R such that $\mathfrak{a} \subset I \cap J$, $\operatorname{codim} \mathfrak{a} + 1 = \operatorname{codim} I = \operatorname{codim} J$ and $R/\mathfrak{a}$ is Cohen-Macaulay. If there is an isomorphism of graded R-modules $J/\mathfrak{a} \cong (I/\mathfrak{a})(-t)$, then it is said that J is obtained from I by an *elementary biliaison* of height t on $\mathfrak{a}$.

The above names are motivated by the following result.

Theorem 8.2

(a) Suppose $J = fI + \mathfrak{a}$ is a basic double link of I height $t = \deg f$.

(i) If I is a perfect ideal, then so is J. Moreover, their Hilbert functions are related by

$$h_{R/J}(j) = h_{R/I}(j-t) + h_{R/\mathfrak{a}}(j) - h_{R/\mathfrak{a}}(j-t) \quad \textit{for all } j \in \mathbb{Z}.$$

In particular, I and J have the same codimension.

(ii) If I is unmixed and $R/\mathfrak{a}$ is generically Gorenstein then J is unmixed and Gorenstein linked to I in two steps.

(b) Suppose J is obtained from I by an elementary biliaison of height t.

(i) The Hilbert functions are related by

$$h_{R/J}(j) = h_{R/I}(j-t) + h_{R/\mathfrak{a}}(j) - h_{R/\mathfrak{a}}(j-t) \quad \textit{for all } j \in \mathbb{Z}.$$

(ii) If I and J are unmixed and $R/\mathfrak{a}$ is generically Gorenstein then I and J are Gorenstein linked to I in two steps.

Proof

(a) Claim (i) is part of [52, Lemma 4.8] and (ii) is shown in [52, Proposition 5.10]
(b) The first assertion is an immediate consequence of the definition. The second assertion is shown in [45].

□

The concepts of basic double links and elementary biliaison are closely related.

Remark 8.3

(i) If $fI + \mathfrak{a}$ is a basic double link of I, then there is a graded isomorphism $(I/\mathfrak{a})(-\deg f) \cong J/\mathfrak{a}$. Thus, basic double linkage is a special case of elementary biliaison.

(ii) If J is obtained from I by an elementary biliaison of height t, then there are homogeneous polynomials $f, g \in R$ with $\deg f = t + \deg g$, $\mathfrak{a} : f = \mathfrak{a} = \mathfrak{a} : g$ and $fI + \mathfrak{a} = gJ + \mathfrak{a}$. Thus, I and J are related via two basic double links, and, by Theorem 8.2(a), J can be obtained (not optimally) from I by four Gorenstein links.

The above result implies a sufficient condition for a set of polynomials to be a Gröbner basis (with respect to a given term order) for the ideal that they generate. We denote by $\text{in}(I)$ the initial ideal of I with respect to the chosen term order.

Lemma 8.4 ([34, Lemma 1.12]) *Fix a monomial order on R. Consider an ideal J that is obtained from I by an elementary biliaison of height t on $\mathfrak{a}$. If the initial ideals* $\text{in}(I)$ *and* $\text{in}(\mathfrak{a})$ *are perfect and there is a monomial ideal $J' \subset J$ that is obtained from* $\text{in}(I)$ *by an elementary biliaison of height t on* $\text{in}(\mathfrak{a})$*, then $J' = \text{in}(J)$.*

Following [34], we illustrate the use of this Gröbner basis criterion in a simple well-known case.

Theorem 8.5 *Let $X = (x_{i,j})$ be an $m \times n$ matrix with $m \leq n$ whose entries are distinct variables. Then the set of maximal minors of X forms a Gröbner basis of the ideal $I_m(X)$, generated by the maximal minors of X.*

Sketch of Proof Fix a monomial order such that the product of the variables on the main diagonal of any maximal minor is its initial monomial, and denote by J' the ideal generated by these monomials. We want to show that $\text{in}(I_m(X)) = J'$.

We use induction on $|X| = mn$. If $m = 1$, then $J' = I_1(X)$ is generated by variables.

Let $m \geq 2$. If $n = m$, then $I_m(X)$ is a principal ideal. Let $n \geq m + 1$. Denote by Z the $m \times (n-1)$ matrix obtained from X by deleting the last column, and let Y be the $(m-1) \times (n-1)$ matrix obtained from Z by deleting the last row. The induction hypothesis implies that the maximal minors of Y and of Z are Gröbner bases of $I_{m-1}(Y)$ and $I_m(Z)$. It follows by inspection that

$$J' = \text{in}(I_{m-1}(Y)) + x_{m,n}\,\text{in}(I_m(Z)).$$

Since $x_{m,n}$ does not appear in Y, we get $\text{in}(I_{m-1}(Y)) : x_{m,n} = \text{in}(I_{m-1}(Y))$. Moreover, we have $\operatorname{codim} I_{m-1}(Y) = m - n + 1 = 1 + \operatorname{codim} I_m(Z)$. Hence J' is a basic double link of $\text{in}(I_{m-1}(Y))$ of height one on $\text{in}(I_m(Z))$. The proof of [52, Theorem 3.6] shows that $I_m(X)$ is obtained from $I_{m-1}(Y)$ by an elementary biliaison of height one on $I_m(Z)$. Hence Lemma 8.4 gives $J' = \text{in}\, I_m(X)$. □

Even if a linkage pattern of an ideal is not known, variations of the above aproach can be productive. We illustrate the idea by explicitly determining equations of some blow-up algebras. If I is an ideal of R, then its *Rees algebra* is the ring $R[It] = \bigoplus_{j \geq 0} I^j t^j \subset R[t]$, where t is a new variable. The *special fiber ring* of $I \subset R$ is the algebra

$$\mathcal{F}(I) = \bigoplus_{j \geq 0} I^j/\mathfrak{m}I^j \cong R[It] \otimes_R R/\mathfrak{m},$$

where $\mathfrak{m} = (x_0, \dots, x_n)$ is the unique maximal homogeneous ideal of R. Both rings are finitely generated k-algebras, and so they are quotients of polynomial rings by suitable ideals. One often refers to generators of these ideals as equations of the Rees algebra and the special fiber ring, respectively. Determining these equations is typically a challenging problem. We describe a solution for some important classes of monomial ideals.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ be a vector such that $0 \leq \mu_1 \leq \cdots \leq \mu_n < \lambda_n$ and $\mu_i \geq i - 1$ for $i = 1, \dots, n$. Set $m = \lambda_1$. Following [18], define a *generalized Ferrers ideal* $I_{\lambda-\mu}$ as

$$I_{\lambda-\mu} := (x_i y_j \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i) \subset k[x_1, \dots, x_n, y_1, \dots, y_m].$$

It is isomorphic to a Ferrers ideal as considered in [17]. Substituting $y_j \mapsto x_j$ gives the *specialized Ferrers ideal*

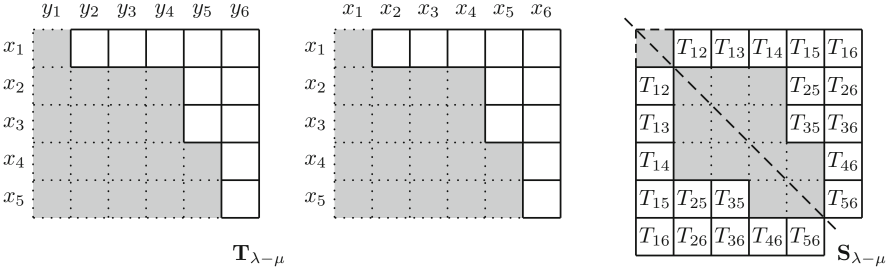
$$\overline{I}_{\lambda-\mu} := (x_i x_j \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i) \subset K[x_1, \dots, x_{\max\{n,m\}}].$$

Note that any squarefree strongly stable monomial ideal corresponds to a unique ideal $\overline{I}_{\lambda-\mu}$ with $\mu = (1, 2, \dots, n)$.

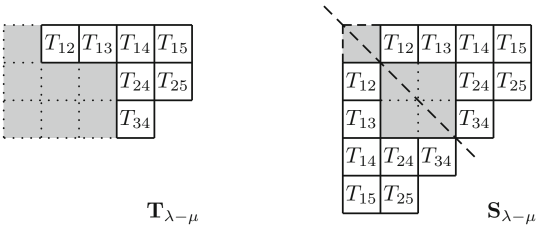
The above ideals can be visualized using a suitable tableau. Form a skew shape $\mathbf{T}_{\lambda-\mu}$, obtained from the Ferrers diagram $\mathbf{T}_\lambda$ by removing the leftmost μ_i boxes in row i. Then the generators of $I_{\lambda-\mu}$ and $\overline{I}_{\lambda-\mu}$ correspond to the boxes of the $\mathbf{T}_{\lambda-\mu}$, where the rows are labelled by $x_1, \dots, x_n$ and the columns by $y_1, \dots, y_m$ and $x_1, \dots, x_m$, respectively. We may also label a box in position (i, j) of $\mathbf{T}_{\lambda-\mu}$ by a variable $T_{i,j}$. Thus, it corresponds to a polynomial ring

$$k[\mathbf{T}_{\lambda-\mu}] := K[T_{ij} \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i].$$

The *symmetrized tableau* $\mathbf{S}_{\lambda-\mu}$ is obtained by reflecting $\mathbf{T}_{\lambda-\mu}$ along the main diagonal. It may have holes along the main diagonal. For example, if $\lambda = (6, 6, 6, 6, 6)$ and $\mu = (1, 4, 4, 5, 5)$, one gets



Observe that in general neither $\mathbf{T}_{\lambda-\mu}$ nor $\mathbf{S}_{\lambda-\mu}$ is a ladder. Denote by $I_2(\mathbf{T}_{\lambda-\mu})$ and $I_2(\mathbf{S}_{\lambda-\mu})$ the ideals in $K[\mathbf{T}_{\lambda-\mu}]$ generated by the determinants of 2×2 submatrices of $\mathbf{T}_{\lambda-\mu}$ and $\mathbf{S}_{\lambda-\mu}$, respectively. For instance, if $\lambda = (5, 5, 4)$ and $\mu = (1, 3, 3)$ we obtain



and so

$$I_2(\mathbf{T}_{\lambda-\mu}) = (T_{14}T_{25} - T_{15}T_{24})$$

and

$$I_2(\mathbf{S}_{\lambda-\mu}) = (T_{14}T_{25} - T_{15}T_{24}, T_{12}T_{34} - T_{13}T_{24}).$$

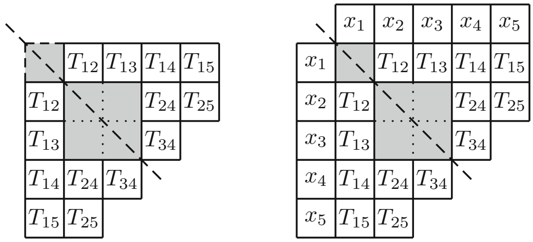
We need one further construction. Given vectors $\lambda, \mu \in \mathbb{Z}^n$ as above, set

$$\lambda' = (\lambda_1 + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1) \in \mathbb{Z}^{n+1}$$

and

$$\mu' = (1, \mu_1 + 1, \mu_2 + 1, \mu_n + 1) \in \mathbb{Z}^{n+1}.$$

Augment the tableau $\mathbf{S}_{\lambda-\mu}$ with a new top row and a new leftmost column. Leave the new northwest corner empty and fill the new top row with the variables $x_1, \ldots, x_m$ from left to right and the leftmost column with $x_1, \ldots, x_m$ from top to bottom. Up to the names of the variables, the augmented tableau is the same as $\mathbf{S}_{\lambda'-\mu'}$.



$\mathbf{S}_{\lambda-\mu}$ The augmented tableau $\mathbf{S}_{\lambda'-\mu'}$

Theorem 8.6 ([19, Theorem 4.2 and Corollary 4.6]) *The special fiber ring and the Rees algebra of $\overline{I}_{\lambda-\mu}$ are determinantal rings.*

More precisely, there are graded isomorphisms

$$\mathcal{F}(\overline{I}_{\lambda-\mu}) \cong k[\mathbf{T}_{\lambda-\mu}]/I_2(\mathbf{S}_{\lambda-\mu})$$

and, if $\mu_1 \leq n$,

$$R[\overline{I}_{\lambda-\mu}t] \cong \mathcal{F}(\overline{I}_{\lambda'-\mu'}) \cong k[\mathbf{T}_{\lambda'-\mu'}]/I_2(\mathbf{S}_{\lambda'-\mu'}).$$

By Nagel et al. [19, Remark 4.5], the above result also gives a description of the special fiber ring and the Rees algebra of a generalized Ferrers ideal $I_{\lambda-\mu}$ as established first in [17]. In particular, one has $\mathcal{F}(I_{\lambda-\mu}) \cong K[\mathbf{T}_{\lambda-\mu}]/I_2(\mathbf{T}_{\lambda-\mu})$.

As explained in [19, Remark 4.5], the assumption $\mu_1 \leq n$ for the second isomorphism is harmless. Its proof is similar to that of the first isomorphism. The latter is shown as follows. By Nagel et al. [19, Theorem 2.4] the 2-minors of $\mathbf{T}_{\lambda-\mu}$ and $\mathbf{S}_{\lambda-\mu}$ form a Gröbner basis of $I_2(\mathbf{T}_{\lambda-\mu})$ and $I_2(\mathbf{S}_{\lambda-\mu})$, respectively. Their initial ideals can be obtained from ideals generated by variables via sequences of basic double links, which, in particular, allows one to determine the codimension of these ideals (see [19, Theorem 3.3]). Consider now the algebra epimorphism

$$\pi : k[\mathbf{T}_{\lambda-\mu}] \twoheadrightarrow k[x_i x_j \mid x_i x_j \in \overline{I}_{\lambda-\mu}] \cong \mathcal{F}(\overline{I}_{\lambda-\mu}),$$

induced by $\pi(T_{ij}) = x_i x_j$. Since π maps all 2-minors of $\mathbf{S}_{\lambda-\mu}$ to zero we get $I_2(\mathbf{S}_{\lambda-\mu}) \subset \ker \pi$. Both ideals are prime ideals (see [19, Proposition 3.5]). Thus, the desired equality follows if the two ideals have the same codimension. This is indeed true as a comparison of the codimension of $I_2(\mathbf{S}_{\lambda-\mu})$ and $\dim \mathcal{F}(\overline{I}_{\lambda-\mu})$ reveals.

9 Vertex Decomposability

The use of liaison-theoretic methods to study simplicial complexes has been pioneered in [76]. The starting point is a well-known bijection between squarefree monomial ideals and simplicial complexes.

Recall that a *simplicial complex* Δ on n vertices is a collection of subsets of $[n] = \{1, \ldots, n\}$ that is closed under inclusion. The elements of Δ are called the *faces* of Δ. The dimension of a face F is $|F| - 1$. The *Stanley-Reisner ideal* of Δ is $I_\Delta = (\prod_{i \in F} x_i \mid F \subseteq [n],\ F \notin \Delta) \subset R = k[x_1, \ldots, x_n]$, and the corresponding *Stanley-Reisner ring* is $k[\Delta] = R/I_\Delta$. Note that the dimensions of Δ and $k[\Delta]$ determine each other because $\dim \Delta = \dim k[\Delta] - 1$. We say that Δ has an algebraic property such as Cohen-Macaulayness if $K[\Delta]$ has this property. For more details on simplicial complexes, Stanley-Reisner rings and their algebraic properties we refer to the books of Bruns-Herzog [13] and Stanley [89].

Following [76, Definition 2.2], a squarefree monomial I is said to be *squarefree glicci* if I can be linked in an even number of steps to a complete intersection I' generated by variables such that every other ideal in the chain linking I to I' is a squarefree monomial ideal. In other words the simplicial complex Δ corresponding to I can be "linked" to a simplex in an even number of steps, where every other step corresponds to a simplicial complex.

Example 9.1 Denote by Δ the simplicial complex on [4] consisting of 4 vertices. Its Stanley-Reisner ideal is $I_\Delta = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$. It is squarefree glicci because

$$I_\Delta = x_4 \cdot (x_1, x_2, x_3) + (x_1x_2, x_1x_3, x_2x_3)$$

implies that I_Δ is a basic double link of (x_1, x_2, x_3).

Provan and Billera introduced in [82] an important property of a simplicial complex. To state it recall that, given a vertex j of a simplicial complex Δ, the *link* of j is

$$\mathrm{lk}_\Delta(j) = \{G \in \Delta \mid \{j\} \cup G \in \Delta, \{j\} \cap G = \emptyset\},$$

and the *deletion* with respect to j is

$$\Delta_{-j} = \{G \in \Delta \mid \{j\} \cap G = \emptyset\}.$$

A pure simplicial complex Δ is said to be *vertex decomposable* if Δ is a simplex or equal to $\{\emptyset\}$, or there exists a vertex j such that $\mathrm{lk}_\Delta j$ and Δ_{-j} are both pure and vertex-decomposable and $\dim \Delta = \dim \Delta_{-j} = \dim \mathrm{lk}_\Delta j + 1$.

Every vertex decomposable simplicial complex is shellable, and so Cohen-Macaulay. Thus, the following concept, introduced in [76, Definition 3.1], is less restrictive. A pure simplicial simplex $\Delta \neq \emptyset$ on $[n]$ is said to be *weakly vertex* *decomposable* if there is some $j \in [n]$ such that Δ is a cone over the weakly vertex-decomposable deletion Δ_{-j} or there is some $j \in [n]$ such that $\mathrm{lk}_\Delta(j)$ is weakly vertex decomposable and Δ_{-j} is Cohen-Macaulay of the same dimension as Δ.

We now relate these combinatorial concepts via liaison theory. For a simplicial complex Δ on $[n]$, consider any vertex $j \in [n]$. Then the cone over the link $\mathrm{lk}_\Delta(j)$ with apex j considered as complex on $[n]$ has as Stanley-Reisner ideal $J_{\mathrm{lk}_\Delta(j)} = I_\Delta : x_j$. Denote by $J_{\Delta_{-j}} \subset R$ the extension ideal of the Stanley-Reisner ideal of Δ_{-j} considered as a complex on $[n] \setminus \{j\}$. Note that x_j does not divide any of the minimal generators of $J_{\Delta_{-j}}$, thus $J_{\Delta_{-j}} : x_j = J_{\Delta_{-j}}$. Furthermore, it follows that

$$I_\Delta = x_j J_{\mathrm{lk}_\Delta(j)} + J_{\Delta_{-j}}. \tag{9.1}$$

Comparing with Definition 8.1 and Theorem 8.2, this equation implies that Δ is a basic double link of the cone over its link $\mathrm{lk}_\Delta(j)$ and Gorenstein linked to it in two steps if Δ is pure and if the deletion Δ_{-j} is Cohen-Macaulay and has the same dimension as Δ when both are considered as complexes on $[n]$. These observations lead to the following result.

Theorem 9.2 ([76, Theorem 3.3]) *If Δ is a weakly vertex decomposable simplicial complex, then Δ is squarefree glicci. In particular, Δ is Cohen-Macaulay.*

This result applies to a number of well-studied classes of simplicial complexes. In fact, it is known that any pure shifted complex, any matroid complex, any Gorenstein complex and any 2-Cohen-Macaulay complex (see [2] for the definition) is weakly vertex decomposable.

It has been observed in [76] that in general both of the properties considered in the above theorem depend on the characteristic of the ground field.

Example 9.3 ([76, Example 5.5])

(i) Consider a tringulation Δ of the real projective plane $\mathbb{P}^2$ with six vertices. Using the notation from [13, p. 236], its Stanley-Reisner ideal in $k[x_1, \ldots, x_6]$ is

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

If $\operatorname{char} k \neq 2$ this is a 2-dimensional Cohen-Macaulay complex, whereas Δ is *not* Cohen-Macaulay if $\operatorname{char} k = 2$.

(ii) Let $R = k[x_1, \ldots, x_7]$ and denote by $\mathfrak{a}$ the extension ideal of I_Δ in R. Set $J = (x_1, \ldots, x_4) \subset R$. Consider the squarefree monomial ideal

$$I = x_7 J + \mathfrak{a}.$$

Since $R/\mathfrak{a}$ is Cohen-Macaulay if and only if $\operatorname{char} k \neq 2$, I is a basic double link of the complete intersection J if $\operatorname{char} k \neq 2$. It follows that in this case I is squarefree glicci and that the induced simplicial complex Δ' is weakly vertex-decomposable. However, if $\operatorname{char} k = 2$ then Δ' is not Cohen-Macaulay and so neither (squarefree) glicci nor weakly vertex decomposable.

Example 9.3(i) also gives rise to a challenging problem. One of the main open questions in liaison theory is whether every Cohen-Macaulay ideal is glicci. In view of the above dependence of the Cohen-Macaulayness of $k[\Delta]$ on the characteristic, the following problem was proposed in [76, Problem 5.3]:

Problem 9.4 Decide whether the Stanley-Reisner ideal of the above triangulation of $\mathbb{P}^2_{\mathbb{R}}$ is glicci if $\operatorname{char} k \neq 2$.

Recently, in [50] Klein and Rajchgot established a vast generalization of Theorem 9.2. To discuss it, it is useful to rewrite Eq. (9.1) as

$$I_\Delta = J_{\mathrm{lk}_\Delta(j)} \cap (x_j, J_{\Delta_{-j}}). \tag{9.2}$$

Note that $(x_j, J_{\Delta_{-j}}) = (x_j, I_\Delta)$ is the Stanley-Reisner ideal of the deletion Δ_{-j} when it is considered as a simplicial complex on $[n]$. We also observed that $J_{\mathrm{lk}_\Delta(j)} = I_\Delta : x_j$. Knutson, Miller, Yong introduced in [49] geometric vertex decomposition as an analog of the decomposition in Eq. (9.2) for an ideal $I \subset R = k[x_1, \ldots, x_n]$ that is not necessarily homogeneous. We need some notation.

Let y be any variable of R. Any nonzero polynomial $f \in R$ can be uniquely written as $f = y^d q + r$ with polynomials $q, r \in R$ and $d \in \mathbb{N}_0$ such that no monomial in $q \neq 0$ is divisible by y, no monomial in r is divisible by y^d if $d > 0$ and $r = 0$ if $d = 0$. Set $\mathrm{in}_y\, f = y^d q$ and define the initial ideal of I with respect to y as

$$\mathrm{in}_y\, I = (\mathrm{in}_y\, f \mid f \in I) \subset R.$$

A monomial order $<$ on R is said to be *y-compatible* if $\mathrm{in}_< f = \mathrm{in}_<(\mathrm{in}_f\, f)$ for every $f \neq 0$ in R. Consider now a Gröbner basis G of an ideal $I \subset R$ with respect to a y-compatible order. Write each element of G as above, that is,

$$G = \{y^{d_i} q_i + r_i \mid 1 \leq i \leq s\},$$

and so in particular $\mathrm{in}_y(y^{d_i} q_i + r_i) = y^{d_i} q_i$. Define the following ideals of R:

$$\mathfrak{b}_{y,I} = (q_i \mid 1 \leq i \leq s) \quad \text{and} \quad \mathfrak{a}_{y,I} = (q_i \mid d_i = 0).$$

Note that these definitions do not depend on the choice of Gröbner basis G because one has by Knutson et al. [49, Theorem 2.1],

$$\mathfrak{b}_{y,I} = \bigcup_{i \geq 1} (\mathrm{in}_y\, I : y^i) \quad \text{and} \quad (y, \mathfrak{a}_{y,I}) = (y, \mathrm{in}_y\, I).$$

Definition 9.5 ([49]) If

$$\operatorname{in}_y I = \mathfrak{b}_{y,I} \cap (y, \mathfrak{a}_{y,I}) \tag{9.3}$$

then this is called a *geometric vertex decomposition of I with respect to y*.

Comparing with Eq. (9.2) and the discussion below it, it follows that Eq. (9.2) is a geometric vertex decomposition of I_Δ with respect to x_j. Using the definition of a vertex decomposable simplicial complex as a role model, one says (see [50, Definition 2.6]) that an unmixed ideal I of R is *geometrically vertex decomposable* if (i) $I = R$ or I is generated by variables, or (ii) for some variable y of R, $\operatorname{in}_y I = \mathfrak{b}_{y,I} \cap (y, \mathfrak{a}_{y,I})$ is a geometric vertex decomposition and the contractions of $\mathfrak{b}_{y,I}$ and $\mathfrak{a}_{y,I}$ to $k[x_1, \ldots, \hat{y}, \ldots, x_n]$ are geometrically vertex decomposable. By induction, it follows that every geometrically vertex decomposable ideal is radical.

Similarly to weakly vertex decomposable simplicial complexes, there are also *weakly geometrically vertex decomposable ideals*, see [50, Definition 4.6]. Analogously to Theorem 9.2, one has:

Theorem 9.6 ([50, Corollary 4.8]) *Any weakly geometrically vertex decomposable ideal $I \subset R$ is both radical and glicci. In particular, R/I is Cohen-Macaulay.*

This result applies, for example, to Schubert determinantal ideals and homogeneous ideals of lower bound Cluster algebras [50, Propositions 5.2 and 5.3]. The key observation for establishing Theorem 9.6 is that a geometric vertex decomposition often gives rise to an elementary biliaison (see Definition 8.1(ii)).

Theorem 9.7 ([50, Theorem 4.1]) *Suppose an unmixed ideal $I \subset R$ has a geometric vertex decomposition with respect to some variable y of R such that neither $\mathfrak{b}_{y,I} = \mathfrak{a}_{y,I}$ nor $\mathfrak{b}_{y,I} = R$. If $I, \mathfrak{a}_{y,I}$ and $I, \mathfrak{a}_{y,I}$ are homogeneous then there is a graded isomorphism $I/\mathfrak{a}_{y,I} \cong (\mathfrak{b}_{y,I}/\mathfrak{a}_{y,I})(-1)$.*

Remarkably, some form of converse to this result is true as well.

Theorem 9.8 ([50, Theorem 6.1]) *Fix a y-compatible monomial order and consider ideals $I, \mathfrak{b}, \mathfrak{a}$ with $\mathfrak{a} \subset I \cap \mathfrak{b}$. Suppose that y^2 does not divide any term of any element of the reduced Gröbner basis of I and that no term of any element of the reduced Gröbner basis of $\mathfrak{a}$ is divisible by y. If there is an isomorphism of $R/\mathfrak{a}$ modules $I/\mathfrak{a} \to \mathfrak{b}/\mathfrak{a}$ induced by multiplication with $\frac{f}{g}$ with $\frac{\operatorname{in}_y f}{g} = y$ then $\operatorname{in}_y I = \mathfrak{b} \cap (y, \mathfrak{a})$ is a geometric vertex decomposition.*

Combining these results with Lemma 8.4, allows one to determine Gröbner bases of further classes of ideals (see [50, Corollary 4.13] and [51]).

10 Unprojections

In 1983 Kustin and Miller introduced a construction of Gorenstein ideals in local Gorenstein rings, starting from smaller such ideals. More precisely, given Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ with grades g and $g-1$, respectively, in a Gorenstein local ring R, in [54] they construct a new Gorenstein ideal I of grade g in a larger Gorenstein ring $R[v]$. Here v is a new indeterminate. In [55] they give an interpretation for their construction via liaison theory. The Kustin-Miller construction has been used to produce many interesting classes of Gorenstein ideals. In birational geometry it is known as *unprojection* (see, e.g., [12, 79, 80]). Following [37], we discuss a modification of the Kustin-Miller construction in the case of graded rings within the framework of Gorenstein liaison theory.

Let R be a graded Gorenstein k-algebra. Let $\mathfrak{a}$ and $\mathfrak{b} \subset \mathfrak{a}$ be homogeneous Gorenstein ideals in R of codimension g and $g-1$, respectively. The embedding $\mathfrak{b} \hookrightarrow \mathfrak{a}$ induces the following commutative diagram, where the rows are minimal free resolutions of $R/\mathfrak{b}$ and $R/\mathfrak{a}$, respectively:

$$\begin{array}{ccccccccccc} 0 & & \longrightarrow & B_{g-1} = R(-u) & \xrightarrow{b_{g-1}} & \dots & \longrightarrow & B_1 & \xrightarrow{b_1} & R & \longrightarrow 0 \\ \downarrow & & & \downarrow \alpha_{g-1} & & & & \downarrow \alpha_1 & & \downarrow \alpha_0 = \mathrm{id} & \\ 0 \longrightarrow & A_g = R(-v) & \xrightarrow{a_g} & A_{g-1} & \xrightarrow{a_{g-1}} & \dots & \longrightarrow & A_1 & \xrightarrow{a_1} & R & \longrightarrow 0 \end{array} \tag{10.1}$$

Fixing bases for all the free modules, we identify the maps with their coordinate matrices. As above, we denote by ω_M the canonical module of a graded R-module M. It is isomorphic to the k-dual of the local cohomology module $H_{\mathfrak{m}}^{\dim M}(M)$.

Theorem 10.1 ([37, Theorem 3.1]) *Assume $d = u - v \geq 0$. Let $y \in \mathfrak{a}$ be a homogeneous element such that $\mathfrak{b} : y = \mathfrak{b}$. The embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces an R-module homomorphism $\omega_{R/\mathfrak{a}} \to \omega_{R/(\mathfrak{b},y)}$ that is multiplication by some homogeneous element $\omega \in R$. Its degree is $d + \deg y$.*

Assume there is a homogeneous element $f \in R$ of degree d such that $\mathfrak{b} : (\omega + fy) = \mathfrak{b}$. Consider the ideal I obtained from $\mathfrak{a}$ by the two Gorenstein links

$$\mathfrak{a} \sim_{(\mathfrak{b},y)} \sim_{(\mathfrak{b},\omega+fy)} I,$$

that is, $I = (\mathfrak{b}, \omega + fy) : [(\mathfrak{b}, y) : \mathfrak{a}]$. Then I is a Gorenstein ideal with the same codimension as $\mathfrak{a}$. It can be written as

$$I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*) = (\mathfrak{b}, \alpha_{g-1}^* + (-1)^g f a_g^*),$$

where α_{g-1}^* and a_g^* are interpreted as row vectors and "+" indicates their component-wise sum whose entries, together with generators of $\mathfrak{b}$, generate I.

Observe that a sufficiently general choice of the element f always gives a desired element $\omega + fy$ in Theorem 10.1, at least if the field k is infinite.

We illustrate the result by a simple example.

Example 10.2 Consider the complete intersections $\mathfrak{a} = (x, y, z)$ and $\mathfrak{b} = (x^2 - z^2, y^2 - z^2)$ in the polynomial ring $k[x, y, z]$, where k is a field of characteristic zero. Linking $\mathfrak{a}$ by $\mathfrak{b} + (z^2)$, we get as residual $J = \mathfrak{b} + (z^2, xyz)$. Choosing $f = 5z$, we link J by $\mathfrak{b} + (xyz + fz^2)$ to

$$I = \mathfrak{b} + (xf + yz, yf + xz, zf + xy) = (x^2 - z^2, y^2 - z^2, xz, yz, xy + 5z^2).$$

Notice that for the second link we cannot take $f = z$ because $xyz + z^3$ is a zero divisor modulo $\mathfrak{b}$.

Given a minimal free resolution of $\mathfrak{b}$, it is easy to determine minimal free resolutions of the ideals $(\mathfrak{b}, y)$ and $(\mathfrak{b}, \omega + fy)$ that are used for the links in Theorem 10.1. Combined with the mapping cone procedure applied twice to sequences as in (7.1), one obtains a free resolution of I. However, this resolution is not minimal if $g \geq 3$. In fact, by identifying the construction in Theorem 10.1 as an elementary biliaison one gets a smaller free resolution.

Theorem 10.3 ([37, Theorem 4.1]) *Adopt the notation and assumptions of Theorem 10.1. Then there is a short exact sequence of graded R-modules*

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Moreover, the ideal I has a graded free resolution of the form

$$0 \to B_{g-1}(-d) \to \begin{array}{c} A_{g-1}(-d) \\ \oplus \\ B_{g-2}(-d) \end{array} \to \begin{array}{c} B_{g-2} \\ \oplus \\ A_{g-2}(-d) \\ \oplus \\ B_{g-3}(-d) \end{array} \to \ldots \to \begin{array}{c} B_2 \\ \oplus \\ A_2(-d) \\ \oplus \\ B_1(-d) \end{array} \to \begin{array}{c} B_1 \\ \oplus \\ A_1(-d) \end{array} \to I \to 0.$$

Notice that the maps in the constructed free resolution of I are described in the proof of the statement.

Corollary 10.4 ([37, Proposition 4.3]) *The homogeneous Gorenstein ideal $I = (\mathfrak{b}, \alpha^*_{g-1} + (-1)^g f a^*_g)$ in Theorem 10.1 is obtained from $\mathfrak{a}$ by an elementary biliaison on $\mathfrak{b}$.*

Proof The short exact sequence in Theorem 10.3 gives a graded isomorphism $\mathfrak{a}/\mathfrak{b}(-d) \cong I/\mathfrak{b}$. Since $\mathfrak{b}$ is Gorenstein the claim follows directly from the definition of an elementary biliaison. □

The free resolution constructed in Theorem 10.3 is often minimal. In fact, if the polynomial f is not a unit and each map α_i in Diagram (10.1) is minimal whenever $1 \leq i \leq g - 1$, that is, $\text{Im}\alpha_i \subset \mathfrak{m}A_i$, then the resolution of I described in Theorem 10.3 is a graded minimal free resolution of I (see [37, Corollary 4.2]).

We illustrate the versatility of the above construction by some examples. Even if one starts with complete intersections the resulting Gorenstein ideal is more complicated.

Example 10.5 ([37, Example 51]) Let $R = k[x_1, \dots, x_n$ be a polynomial ring. For an integer g with $2 \le g \le n$, consider two ideals that are generated by regular sequences

$$\mathfrak{b} = (x_1^{m_1}, x_2^{m_2}, \cdots, x_{g-1}^{m_{g-1}}) \subset (x_1^{n_1}, x_2^{n_2}, \cdots, x_g^{n_g}) = \mathfrak{a}.$$

If $d := \sum_{i=1}^{g-1} m_i - \sum_{i=1}^{g} n_i \ge 0$ then, for a sufficiently general polynomial $f \in R$ of degree d,

$$I = (x_1^{m_1}, \cdots, x_{g-1}^{m_{g-1}}, fx_1^{n_1}, \cdots, fx_{g-1}^{n_{g-1}}, fx_g^{n_g} + \prod_{j=1}^{g-1} x_j^{m_j - n_j})$$

is a Gorenstein ideal. Moreover, if $m_j > n_j$ for each $j = 1, \dots, g-1$, then the resolution in Theorem 10.3 is a minimal free resolution of I.

The next example shows that every Artinian Gorenstein ideal whose Castelnuovo-Mumford regularity is three can be obtained by one elementary biliaison from a complete intersection.

Example 10.6 Consider an ideal $I \subset R = k[x_1, \dots, x_n]$ such that R/I is a graded compressed Gorenstein algebra with h-vector $(1, n, 1)$. According to Sally [85, Theorem 1.1], after a suitable change of coordinates any such ideal is of the form

$$I = (x_i x_j \mid 1 \le i < j \le n) + (x_1^2 - c_1 x_n^2, \dots, x_{n-1}^2 - c_{n-1} x_n^2),$$

where $c_1, \dots, c_{n-1} \in k$ are suitable units. It can be obtained by an elementary biliaison as in Theorem 10.1 from $\mathfrak{a} = (x_1, \dots, x_n)$ on $\mathfrak{b}R$, where $\mathfrak{b}$ is a Sally ideal in $n-1$ variables, namely

$$\mathfrak{b} = (x_i x_j \mid 1 \le i < j \le n-1) + (x_1^2 - \frac{c_1}{c_{n-1}} x_{n-1}^2, \dots, x_{n-2}^2 - \frac{c_{n-2}}{c_{n-1}} x_{n-1}^2).$$

More precisely, there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b}, x_n)} (\mathfrak{b}, x_n, x_{n-1}^2) \sim_{(\mathfrak{b}, x_{n-1}^2 - c_{n-1} x_n^2)} I.$$

Note that $(\mathfrak{b}, x_n, x_{n-1}^2) = (x_1, \dots, x_{n-1})^2 + (x_n)$.

We now consider some codimension four Gorenstein ideals with 9 generators and 16 syzygies. Such Gorenstein ideals are investigated in depth from the point of view of unprojections in [12].

Example 10.7 Let $R = k[a, b, c, d, e, f, x, y, z]$ be a polynomial ring in 9 variables over a field k. Consider a generic 3×3 symmetric matrix A and a generic skew-symmetric matrix B:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

For $\lambda \neq 0$ in k, define a 6×6 skew-symmetric matrix $N = \begin{bmatrix} B & A \\ -A & \lambda B \end{bmatrix}$. The ideal $\mathfrak{a}$ generated by the 4×4 Pfaffians of N is a homogeneous Gorenstein ideal of grade 4:

$$\mathfrak{a} = (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2, cd - be + \lambda xz, ce - bf + \lambda yz, e^2 - df + \lambda z^2, cx - by + az, ex - dy + bz, fx - ey + cz).$$

It is the defining ideal of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$ and a typical case of a Tom unprojection (see [12, 80]). In particular, $\mathfrak{a}$ is equal to the ideal generated by the 2×2 minors of a 3×3 generic matrix $A + \sqrt{-\lambda} B$. Hence, the Gulliksen and Negård complex gives its minimal free resolution:

$$0 \longrightarrow R(-6) \xrightarrow{\alpha_4} R^9(-4) \xrightarrow{\alpha_3} R^{16}(-3) \xrightarrow{\alpha_2} R^9(-2) \xrightarrow{\alpha_1} \mathfrak{a} \longrightarrow 0.$$

In order to perform the construction of Theorem 10.1, we choose the first three listed generators of $\mathfrak{a}$ to define a complete intersection

$$\mathfrak{b} = (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2)$$

inside $\mathfrak{a}$. Then we link as follows:

$$\mathfrak{a} \sim_{(\mathfrak{b}, cd - be + \lambda xz)} (\mathfrak{b}, cd - be + \lambda xz, ax) \sim_{(\mathfrak{b}, ax + (cd - be + \lambda xz))} I.$$

Explicitly, the resulting ideal I is

$$I = (e^2 - df - cx + by + az + \lambda z^2, ce - bf + ay + \lambda yz, cd - be + ax + \lambda xz, c^2 - af + \lambda y^2, bc - ae + \lambda xy, ac + \lambda fx - \lambda ey + \lambda cz, b^2 - ad + \lambda x^2, ab + \lambda ex - \lambda dy + \lambda bz, a^2 + \lambda cx - \lambda by + \lambda az).$$

It has the same Betti table as $\mathfrak{a}$. In fact, I is generated by the 4×4 Pfaffians of the matrix

$$M = \begin{bmatrix} 0 & x & y & a & b & c \\ -x & 0 & \frac{1}{\lambda}a + z & b & d & e \\ -y & -\frac{1}{\lambda}a - z & 0 & c & e & f \\ -a & -b & -c & 0 & \lambda x & \lambda y \\ -b & -d & -e & -\lambda x & 0 & a + \lambda z \\ -c & -e & -f & -\lambda y & -a - \lambda z & 0 \end{bmatrix}.$$

Using the description of the minimal free resolution in Theorem 10.3, one can compare the Castelnuovo-Mumford regularities of the ideals involved in Theorem 10.1. In fact, one has (see the proof of [37, Corollary 4.4])

$$\operatorname{reg} I - \operatorname{reg} \mathfrak{a} = 2d.$$

In particular, we get $\operatorname{reg} I \geq \operatorname{reg} \mathfrak{a}$, which is also expressed by saying that I has been obtained from $\mathfrak{a}$ by an *ascending* elementary biliaison. The last equation also leads to an explicit example of a Gorenstein ideal that cannot be obtained using the construction of Theorem 10.1 with a strictly ascending biliaison.

Example 10.8 ([37, Example 5.5]) Let I be a generic Artinian Gorenstein ideal in $R = k[x_1, \ldots, x_5]$ with h-vector $(1, 5, 5, 1)$, where k is an infinite field. It has the least possible Betti numbers. Its graded minimal free resolution has the form

$$0 \to R(-8) \to R^{10}(-6) \to R^{16}(-5) \to R^{16}(-3) \to R^{10}(-2) \to I \to 0. \tag{10.2}$$

This is the key to showing that there are no Gorenstein ideals $\mathfrak{a}$ and $\mathfrak{b}$ to produce I using a biliaison as in Theorem 10.1 that is strictly ascending, i.e., $d > 0$ or, equivalently, $\mathfrak{a}$ has smaller regularity than I.

11 Open Questions

We end with a short list of open questions from liaison theory. Besides being important in and of themselves from a theoretical perspective, it is to be hoped that their resolution will lead to further examples of beautiful and unexpected applications.

1. It is well-known that if a homogeneous ideal I is glicci then it is Cohen-Macaulay. What about the converse: is every Cohen-Macaulay ideal glicci? The first result in this direction is still arguably the cleanest in that it is a direct generalization of Gaeta's theorem [52, Theorem 3.6]: if I is the ideal of maximal

minors of a homogeneous $t \times (t + c)$ matrix, and if I has the expected height $c + 1$, then I is glicci. As mentioned on page 557, this converse is one of the main open questions in liaison theory and was first proposed in [52], page 18. See Problem 9.4 above for a particular example. One can also ask, more generally, whether for curves in $\mathbb{P}^n$, $n \geq 4$, the Hartshorne-Rao module determines the even Gorenstein liaison class.

2. We have seen several applications of the LR property (see Definition 2.11 and Theorem 2.13), and as noted above, this property is only known to hold in codimension two. It was studied in the context of Gorenstein liaison in higher codimension in [46], and the general conclusion was that there is no hope of getting an analogous result in that setting. However, it seems to us to be quite reasonable to hope that for CI-liaison in higher codimension, the analogous property *does* hold. And since it had so many applications in codimension two, one can furthermore expect many consequences in higher codimension.
3. We have seen above that questions about the genus of curves in $\mathbb{P}^n$, and about possible Hilbert functions of sets of points in uniform position, have used liaison theory to make advances. One kind of measure of uniformity is given by the Cayley-Bacharach property, and we saw above that Chong used liaison to say something also here. It seems almost certain that Gorenstein liaison will open still further doors for us in this direction. Is there in fact an approach via Gorenstein liaison?
4. We saw above in Sects. 3 and 7 that liaison theory has been used to produce a broad family of arithmetically Gorenstein unions of linear varieties in any codimension, with important properties. Predominant among these are the fact that the general Artinian reduction has the WLP, and the fact that the graded Betti numbers are maximal in a precise sense. In the paper [67] is a discussion of how this relates to the so-called g-conjecture and, perhaps, an even stronger result as a consequence of a positive answer to the following open question: Does the general Artinian reduction of an arithmetically Gorenstein set of points have the WLP? SLP?
5. In several papers (see, e.g., [44, 45]) Hartshorne has studied aspects of the following open question: Can every Gorenstein ideal be produced by an ascending elementary biliaison from another Gorenstein ideal? This is interesting in its own right, but it would also give further applications along the lines of unprojection, as described in Sect. 10.

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Survey on Regularity of Symbolic Powers of an Edge Ideal

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Dedicated to Professor David Eisenbud on the occasion of his 75th birthday

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1 Introduction

Let G be a simple graph on n vertices. Denote I the edge ideal of G, i.e. squarefree monomial ideal of $S = k[x_1, \dots, x_n]$ generated by $x_i x_j$ where $\{i, j\}$ is an edge of G. By a celebrated result of Cutcosky, Herzog, and Trung [8] and Kodiyalam [29] there exist natural numbers q and $b(G)$ depending only on G such that $\operatorname{reg}(I^s) = 2s + b(G)$ for all $s \geq q$. The smallest such natural number q is called the regularity stabilization index of I, denoted by $\operatorname{rstab}(G)$. For general graphs, studying the behavior of the sequence $\{\operatorname{reg}(I^n) \mid n \geq 1\}$ is a very challenging problem. In particular, there is no description for $\operatorname{rstab}(G)$ or $b(G)$ in terms of combinatorial data of G. For a comprehensive list of problems and known results for the regularity of ordinary powers of edge ideals, we refer to [9] and its references. In this note, we complement the discussion in [9] by focusing on the problem of computing the regularity of symbolic powers of I and its relation to the regularity of ordinary powers. Recently, explicit computation of the regularity of symbolic powers of edge ideals has been carried out for certain classes of edge ideals verifying the following conjecture of the first author.

Conjecture A Let I be the edge ideal of a simple graph G. Then, for all $s \geq 1$,

$$\operatorname{reg}(I^{(s)}) = \operatorname{reg}(I^s).$$

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Table 1 Classes of graphs with known regularity of ordinary/symbolic powers

Graphs	Ordinary powers	Symbolic powers
Forests	Beyarslan, Ha, T. Trung [6]	$I^{(s)} = I^s$
Unicyclic graphs	Alilooee et. al. [1]	Fakhari [14]
Co-chordal graphs	Herzog, Hibi, Zheng [24]	?
Very well-covered graphs	Jayanthan, Selvaraja [28]	?
Gap-free and criket-free graphs	Barnejee [2]	?
Gap-free and diamond-free graphs	Erey [13]	?
Chordal graphs	?	Fakhari [15]
Cameron-Walker graphs	Barnejee, Beyarslan, Ha [4]	Fakhari [16]
Graphs with $\alpha(G) = 2$	Minh, Vu [34]	Hoa, T. Trung [26]
Some bicyclic graphs	Gu [21]; Cid-Ruiz et. al. [10]	?

We would like to note that, for general graphs, it is not known whether $\operatorname{reg}(I^{(s)})$ is asymptotically a linear function. On the other hand, by the result of [23], $\operatorname{reg}(I^{(s)})$ is asymptotically quasi-linear, and thus is bounded above by $2s+c$ for some constant c depending only on G. Recall that, for a radical homogeneous ideal J of S, the s-th symbolic powers of J, denoted by $J^{(s)}$ consists of polynomials vanishing at zeros of J of order at least s. The difficulty in understanding the regularity of $I^{(s)}$ partly comes from the fact that explicit description of the minimal generators of $I^{(s)}$ is only available in very few classes, e.g. perfect graphs [40], unicyclic graphs [20].

We start our discussion by tabulating classes of graphs where the regularity of ordinary/symbolic powers are known in the following.

A first natural problem arises

Problem 1.1 Verify Conjecture A for each class of graphs in Table 1.

It is noted that Conjecture A is also true for some the other class of graphs (see [27]). We further note that except the case of $\dim \Delta(G) = 1$, where $\Delta(G)$ is the independence complex of G, the computation of $\operatorname{reg} I^s$ and $\operatorname{reg} I^{(s)}$ is based on analyzing the regularity of certain colon ideals. In this survey, we focus on the use of degree complexes. We expect that exploiting this technique further would allow us to expand the table to include more classes of graphs and to fill in the gap for the symbolic powers. The fact that using degree complexes to study the regularity of symbolic powers are very potential comes from the remark that a facet of the degree complex $\Delta_{\mathbf{a}}(I^{(s)})$ is also a facet of $\Delta(G)$ (see [26, 33]). We then propose the following.

Problem 1.2 Using degree complexes to compute the regularity of symbolic powers of edge ideals of graphs listed in Table 1.

To accomplish that, a related interesting problem is

Problem 1.3 Verify that extremal exponents of $I^{(s)}$ (see Definition 2.8) satisfies $|\mathbf{a}| \le 2s - 2$ for each $s \ge 1$.

On the other extreme, even when the regularity of powers of I is unknown, by studying degree complexes of the ordinary/symbolic powers of edge ideals, together with Nam, Phong, and Thuy, in [31], we prove Conjecture A for $s = 2, 3$. From the proof, we note the following rigidity of regularity of intermediate ideals between I^s and $I^{(s)}$, for $s = 2, 3$.

Theorem 1.4 *Let I be the edge ideal of a simple graph G. For $s = 2, 3$, let $J = I^s + (f_1, \dots, f_t)$ where f_i are minimal monomial generators of $I^{(s)}$, then*

$$\operatorname{reg}(J) = \operatorname{reg}(I^s) = \operatorname{reg}(I^{(s)}).$$

For simplicity of notation, for two monomial ideals $I \subset J$, we define $\operatorname{Inter}(I, J)$ the set of monomial ideals L such that $L = I + (f_1, \dots, f_t)$ where f_i are among minimal monomial generators of J. We call this set intermediate ideals between I and J. Theorem 1.4 says that all intermediate ideals in $\operatorname{Inter}(I^s, I^{(s)})$ have the same regularity for $s = 2, 3$.

In recent work, we complete Lu's work [30] to compute the regularity of powers of squarefree monomial ideals of dimension two and show that the rigidity of regularity holds for intermediate ideals for all powers.

Theorem 1.5 *Let Δ be an one-dimensional simplicial complex. Let $I = I_\Delta$ be the Stanley-Reisner ideal of Δ. Then for all $s \geq 1$ and all intermediate ideal J in* $\operatorname{Inter}(I^s, I^{(s)})$*, we have*

$$\operatorname{reg}(J) = \operatorname{reg}(I^s) = \operatorname{reg}(I^{(s)}).$$

We propose the following:

Conjecture B Let I be the edge ideal of a simple graph G. For all $s \geq 1$, let J be an intermediate ideal in $\operatorname{Inter}(I^s, I^{(s)})$. Then

$$\operatorname{reg}(J) = \operatorname{reg}(I^s) = \operatorname{reg}(I^{(s)}).$$

Problem 1.6 Establish Conjecture B for classes of graphs listed in Table 1.

For general graphs, computing the regularity of powers is very challenging, bounding $\operatorname{rstab}(G)$ and $b(G)$ is of particular interest. In this note, we discuss recent development on the following Conjecture of Alilooe-Barnejee-Bayerslan-Ha [3] and its analog for symbolic powers [17].

Conjecture C Let I be the edge ideal of a simple graph G. Then

(1) $\operatorname{reg}(I^s) \leq \operatorname{reg}(I) + 2s - 2$.
(2) $\operatorname{reg}(I^{(s)}) \leq \operatorname{reg}(I) + 2s - 2$.

Note that Conjecture C implies all known upper bounds for the regularity of powers of edge ideals, e.g. [19, 22]. In [31], we establish Conjecture C (1) for $s = 2, 3$ and obtain the same bound for symbolic powers for $s = 2, 3, 4$.

Furthermore, using the results on mixed sums and fiber products, we note that if Conjecture A/Conjecture C holds for I and J, it also holds for their mixed sum and fiber product.

Now we explain the organization of the survey. In Sect. 2, we recall some notation and basic facts about the symbolic powers of a squarefree monomial ideal, the degree complexes, and Castelnuovo-Mumford regularity. In Sect. 3, we prove the rigidity property for second and third powers. In Sect. 4, we discuss results on intermediate ideals for two-dimensional squarefree monomial ideals. In Sect. 5 we discuss bounds on the regularity of ordinary/symbolic powers of edge ideals. Finally, in Sect. 6, we note the results on mixed sums and fiber products of edge ideals.

2 Castelnuovo-Mumford Regularity, Symbolic Powers and Degree Complexes

In this section, we recall some definitions and properties concerning Castelnuovo-Mumford regularity, the symbolic powers of a squarefree monomial ideal, and the degree complexes of a monomial ideal. The interested reader is referred to [5, 11, 12, 39] for more details. The material in this section follows closely [31, Section 2].

2.1 Graph Theory

Throughout this paper, G denotes a finite simple graph over the vertex set $V(G) = [n] = \{1, 2, \dots, n\}$ and the edge set $E(G)$. For a vertex $x \in V(G)$, let the neighbour of x be the subset $N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$, and set $N_G[x] = N_G(x) \cup \{x\}$. For a subset U of the vertices set $V(G)$, $N_G(U)$ and $N_G[U]$ are defined by $N_G(U) = \cup_{u\in U} N_G(u)$ and $N_G[U] = \cup_{u\in U} N_G[u]$. If G is fixed, we shall use $N(U)$ or $N[U]$ for short.

An independent set in G is a set of vertices no two of which are adjacent to each other. An independent set of maximum size will be referred to as a maximum independent set of G, and the independence number of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set in G.

A subgraph H is called an induced subgraph of G if for any vertices $u, v \in V(H) \subseteq V(G)$ then $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(G)$. For a subset U of the vertices set $V(G)$, we shall denote by $G[U]$ the induced subgraph of G on U, and denote by $G - U$ the induced subgraph of G on $V(G) \setminus U$.

An induced matching is a subset of the edges that do not share any vertices and it is an induced subgraph. The induced matching number of G, denoted by $\mu(G)$, is the largest size of an induced matching in G.

A m-cycle in G is a sequence of m distinct vertices $1, \dots, m \in V(G)$ such that $\{1, 2\}, \dots, \{m-1, m\}, \{m, 1\}$ are edges of G. We shall also use $C = 12\dots m$ to denote the m-cycle whose sequence of vertices is $1, \dots, m$.

An anticycle over $[n]$ is the complement of a n-cycle for $n \geq 4$. It is clear that the independence number of an anticycle is always two.

2.2 Simplicial Complex

Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$ that is a collection of subsets of $[n]$ closed under taking subsets. We put $\dim F = |F| - 1$, where $|F|$ is the cardinality of F. The dimension of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. It is clear that Δ can be uniquely determined by the set of its maximal elements under inclusion, called by facets, which is denoted by $\mathfrak{F}(\Delta)$.

A simplicial complex Δ is called a cone over $x \in [n]$ if $x \in B$ for any $B \in \mathfrak{F}(\Delta)$. If Δ is a cone, then it is acyclic (i.e., has vanishing reduced homology).

For a face $F \in \Delta$, the link of F and the star of F in Δ are the subsimplicial complexes of Δ defined by

$$\mathrm{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\} \text{ and } \mathrm{st}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta\}.$$

2.3 Stanley-Reisner Correspondence

Let $S = K[x_1, \dots, x_n]$. We now recall the Stanley-Reisner correspondence which corresponds a squarefree monomial ideal of S and a simplicial complex Δ on $[n]$. For each subset F of $[n]$, let $x_F = \prod_{i \in F} x_i$ be a squarefree monomial in S.

Definition 2.1 For a squarefree monomial ideal I, the Stanley-Reisner complex of I is defined by

$$\Delta(I) = \{F \subset [n] \mid x_F \notin I\}.$$

For a simplicial complex Δ, the Stanley-Reisner ideal of Δ is defined by

$$I_\Delta = (x_F \mid F \notin \Delta).$$

The Stanley-Reisner ring of Δ is the quotient by the Stanley-Reisner ideal, $K[\Delta] = S/I_\Delta$.

2.4 Castelnuovo-Mumford Regularity

Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the maximal homogeneous ideal of $S = K[x_1, \dots, x_n]$ a polynomial ring over a field K. For a finitely generated graded S-module L, let

$$a_i(L) = \begin{cases} \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(L)_j \neq 0\} & \text{if } H^i_{\mathfrak{m}}(L) \neq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

where $H^i_{\mathfrak{m}}(L)$ denotes the i-th local cohomology module of L with respect to $\mathfrak{m}$. Then, the Castelnuovo-Mumford regularity (or regularity for short) of L is defined to be

$$\operatorname{reg}(L) = \max\{a_i(L) + i \mid i = 0, \dots, \dim L\}.$$

The regularity of L can also be defined via the minimal graded free resolution. Assume that the minimal graded free resolution of L is

$$0 \longleftarrow L \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_p \longleftarrow 0.$$

Let $t_i(L)$ be the maximal degree of graded generators of F_i. Then,

$$\operatorname{reg}(L) = \max\{t_i(L) - i \mid i = 0, \dots, p\}.$$

From the minimal graded free resolution of S/J, we obtain $\operatorname{reg}(J) = \operatorname{reg}(S/J) + 1$ for a non-zero and proper homogeneous ideal J of S.

2.5 Symbolic Powers

Let I be a non-zero and proper homogeneous ideal of S. Let $\{P_1, \dots, P_r\}$ be the set of the minimal prime ideals of I. Given a positive integer s, the s-th symbolic power of I is defined by

$$I^{(s)} = \bigcap_{i=1}^{r} I^s S_{P_i} \cap S.$$

For a monomial f in S, we denote $\frac{\partial^*(f)}{\partial^*(x^{\mathbf{a}})}$ the $*$-partial derivative of f with respect to $x^{\mathbf{a}}$, which is derivative without coefficients. In general, $\partial f / \partial x^{\mathbf{a}} = c\partial^*(f)/\partial^*(x^{\mathbf{a}})$ for some constant c. We define

$$I^{[s]} = (f \in S \mid \frac{\partial^* f}{\partial^* x^{\mathbf{a}}} \in I, \text{ for all } x^{\mathbf{a}} \text{ with } |\mathbf{a}| \leq s - 1),$$

the s-th $*$-differential power of I. When I is a squarefree monomial ideal, the symbolic powers of I is equal to the $*$-differential powers of I.

Lemma 2.2 *Let I be a squarefree monomial ideal. Then $I^{(s)} = I^{[s]}$.*

For a monomial f in S, we denote $\operatorname{Supp}(f)$, the support of f, the set of all indices $i \in [n]$ such that $x_i | f$. For an exponent $\mathbf{a} \in \mathbb{Z}^n$, we denote $\operatorname{Supp}(\mathbf{a}) = \{i \in [n] \mid a_i \neq 0\}$, the support of $\mathbf{a}$. For any subset $V \subset [n]$, we denote

$$I_V = (f \mid f \text{ is a monomial which belongs to } I \text{ and } \operatorname{Supp}(f) \subseteq V)$$

be the restriction of I on V. We have

Corollary 2.3 *Let I be a squarefree monomial ideal and f be a monomial in S. Denote $V = \operatorname{Supp}(f)$. Then, $f \in I^{(s)}$ if and only if $f \in I_V^{(s)}$.*

As a consequence of Corollary 2.3, we deduce a generalization of [20, Corollary 4.5] for squarefree monomial ideals.

Corollary 2.4 *Let I be a squarefree monomial ideal in S. Let $V \subseteq [n]$, and I_V be the restriction of I to V. Then for all $s \geq 1$,*

$$\operatorname{reg}(I_V^{(s)}) \leq \operatorname{reg}(I^{(s)}).$$

Proof By Corollary 2.3, $I_V^{(s)}$ is the restriction of $I^{(s)}$ to V. Let $\{t, \ldots, n\} = [n] \setminus V$. Then, $I_V^{(s)} + (x_t, \ldots, x_n) = I^{(s)} + (x_t, \ldots, x_n)$. The conclusion follows from Lemma 2.11 and the fact that $x_t, \ldots, x_n$ is a regular sequence with respect to $S/I_V^{(s)}$. □

2.6 Edge Ideals and Their Symbolic Powers

Let G be a simple graph over the vertex set $V(G) = [n] = \{1, 2, \ldots, n\}$. The edge ideal of G is defined to be

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq S.$$

For simplicity, we often write $i \in G$ (resp. $ij \in G$) instead of $i \in V(G)$ (resp. $\{i, j\} \in E(G)$).

It is noted that the Krull dimension $\dim(S/I) = \alpha(G)$.

A clique of size t in G is an induced subgraph of G which is a complete graph over t-vertices. We also called a clique of size 3 a triangle.

Let $J_1(G)$ be the ideal generated by all squarefree monomials $x_i x_j x_r$ where $\{i, j, r\}$ forms a triangle in G. Let $J_2(G)$ be the ideal generated by all squarefree monomials $x_i x_j x_r x_s$ where $\{i, j, r, s\}$ forms a clique of size 4 in G and all squarefree monomials x_C where C is a 5-cycle of G.

We have the following expansion formula of the second and third symbolic powers of an edge ideal. Note that the first formula is [40, Corollary 3.12]. And, the second formula appeared in [31, Theorem 2.7].

Theorem 2.5 *Let I be the edge ideal of a simple graph G. Then*

$$I^{(2)} = I^2 + J_1(G).$$

Theorem 2.6 *Let I be the edge ideal of a simple graph G. Then*

$$I^{(3)} = I^3 + I J_1(G) + J_2(G).$$

2.7 Degree Complexes

For a monomial ideal I in S, Takayama in [41] found a combinatorial formula for $\dim_K H^i_{\mathfrak{m}}(S/I)_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$ in terms of certain simplicial complexes which are called degree complexes. For every $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we set $G_{\mathbf{a}} = \{i \mid a_i < 0\}$ and write $x^{\mathbf{a}} = \Pi_{j=1}^n x_j^{a_j}$. Thus, $G_{\mathbf{a}} = \emptyset$ whenever $\mathbf{a} \in \mathbb{N}^n$. The degree complex $\Delta_{\mathbf{a}}(I)$ is the simplicial complex whose faces are $F \setminus G_{\mathbf{a}}$, where $G_{\mathbf{a}} \subseteq F \subseteq [n]$, so that for every minimal generator $x^{\mathbf{b}}$ of I there exists an index $i \notin F$ with $a_i < b_i$. It is noted that $\Delta_{\mathbf{a}}(I)$ may be either the empty set or $\{\emptyset\}$ and its vertex set may be a proper subset of $[n]$. Moreover, the degree complexes can be computed by using tools from linear programming (see [32]). The next lemma is useful to compute the regularity of a monomial ideal in terms of its degree complexes.

Lemma 2.7 *Let I be a monomial ideal in S. Then*

$$\operatorname{reg}(S/I) = \max\{|\mathbf{a}| + i \mid \mathbf{a} \in \mathbb{N}^n, i \geq 0, \widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0 \text{ for some } F \in \Delta_{\mathbf{a}}(I) \text{ with } F \cap \operatorname{Supp} \mathbf{a} = \emptyset\}.$$

In particular, if $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ then $\operatorname{reg}(K[\Delta]) = \max\{i \mid i \geq 0, \widetilde{H}_{i-1}(\operatorname{lk}_\Delta F; K) \neq 0$ *for some* $F \in \Delta\}$.

Remark Let I be a monomial ideal in S and a vector $\mathbf{a} \in \mathbb{N}^n$. In the proof of Theorem 1 in [41], he showed that if there exists $j \in [n]$ such that $a_j \geq \rho_j = \max\{\deg_{x_j}(u) \mid u$ is a minimal monomial generator of $I\}$ then $\Delta_{\mathbf{a}}(I)$ is either a cone over $\{j\}$ or the void complex. Thus, we only consider some vectors $\mathbf{a}$ which belongs to the finite set

$$\Gamma(I) = \{\mathbf{a} \in \mathbb{N}^n \mid a_j < \rho_j \text{ for all } j = 1, \dots, n\}.$$

Definition 2.8 Let I be a monomial ideal in S. A pair $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ is called *an extremal exponent of the ideal* I, if $\operatorname{reg}(S/I) = |\mathbf{a}| + i$ as in Lemma 2.7.

It is clear that if $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ is an extremal exponent of I then $x^{\mathbf{a}} \notin I$ and $\Delta_{\mathbf{a}}(I)$ is not a cone over t with $t \in \operatorname{Supp} \mathbf{a}$. From the definition, it is easy to see the following:

Lemma 2.9 *Let* I, J *be proper monomial ideals of* S. *Let* $(\mathbf{a}, i)$ *be an extremal exponent of* I. *If* $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$, *then* $\operatorname{reg}(I) \le \operatorname{reg}(J)$. *In particular, if* $J \subseteq I$ *and* $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$ *for all exponent* $\mathbf{a} \in \mathbb{N}^n$ *such that* $x^{\mathbf{a}} \notin I$ *then* $\operatorname{reg}(I) \le \operatorname{reg}(J)$.

Lemma 2.10 *Let* I *be a monomial ideal in* S *and* $\mathbf{a} \in \mathbb{N}^n$. *Then*

$$I_{\Delta_{\mathbf{a}}(I)} = \sqrt{I : x^{\mathbf{a}}}.$$

In particular, $x^{\mathbf{a}} \in I$ *if and only if* $\Delta_{\mathbf{a}}(I)$ *is the void complex.*

We first deduce the following inequality on the regularity of restriction of a monomial ideal.

Lemma 2.11 *Let* I *be a monomial ideal and* x_j *is a variable. Then*

$$\operatorname{reg}(I, x_j) \le \operatorname{reg} I.$$

Proof Let $(\mathbf{a}, i)$ be an extremal exponent of (I, x_j). Then $x_j \nmid x^{\mathbf{a}}$ (i.e. $j \notin \operatorname{Supp}(\mathbf{a})$). It is noted that

$$\sqrt{(I, x_j) : x^{\mathbf{a}}} = \sqrt{I : x^{\mathbf{a}}} + (x_j).$$

In other words, $\Delta_{\mathbf{a}}(I, x_j)$ is the restriction of $\Delta_{\mathbf{a}}(I)$ to $[n] \setminus \{j\}$. Let F be a face of $\Delta_{\mathbf{a}}(I)$ such that $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I,x_j)} F) \neq 0$. Denote $\Delta = \operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F$, $\Gamma = \operatorname{lk}_{\Delta_{\mathbf{a}}(I,x_j)} F$. We have

$$\Delta = \Gamma \cup \operatorname{st}_{\Delta}\{j\} \text{ and } \operatorname{lk}_{\Delta}\{j\} = \Gamma \cap \operatorname{st}_{\Delta}\{j\}.$$

The conclusion follows by looking at the Mayer-Vietoris sequence as follows

$$\cdots \longrightarrow \widetilde{H}_{i-1}(\operatorname{lk}_{\Delta}\{j\}) \longrightarrow \widetilde{H}_{i-1}(\Gamma) \oplus \widetilde{H}_i(\operatorname{st}_{\Delta}\{j\}) \longrightarrow \widetilde{H}_{i-1}(\Delta) \longrightarrow \cdots$$

It is noted that $\operatorname{st}_{\Delta}\{j\} = \operatorname{lk}_{\Delta}\{j\} * \{j\}$ is a cone over j. Therefore, $\widetilde{H}_r(\operatorname{st}_{\Delta}\{j\}) = 0$ for all r. From this sequence, if the middle term is non-zero then either the term on the left or the term on the right is non-zero. This completes our lemma. □

Remark As in the proof of [6] (also see [35]), we may use the upper Koszul complexes to deduce inequalities on Betti numbers.

The following lemma is essential to using the induction method in studying the regularity of a monomial ideal [31, Lemma 2.14].

Lemma 2.12 *Let I be a monomial ideal and the pair $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ be its extremal exponent. If x is a variable that appears in $\sqrt{I : x^{\mathbf{a}}}$ and $x \notin \operatorname{Supp} \mathbf{a}$, then*

$$\operatorname{reg}(I) = \operatorname{reg}(I, x).$$

Since we shall deal with radical ideals of the colon ideals of monomial ideals, the following simple observation will be useful later on.

Lemma 2.13 *Let I be a monomial ideal in S generated by the monomials $f_1, \dots, f_s$ and $\mathbf{a} \in \mathbb{N}^n$. Then $\sqrt{I : x^{\mathbf{a}}}$ is generated by $\sqrt{f_1/\gcd(f_1, x^{\mathbf{a}})}, \dots, \sqrt{f_s/\gcd(f_s, x^{\mathbf{a}})}$, where $\sqrt{x^{\mathbf{b}}} = \prod_{i \in \operatorname{Supp} \mathbf{b}} x_i$ for each $\mathbf{b} \in \mathbb{N}^n$.*

3 Intermediate Ideals for Second and Third Powers

Let G be a simple graph with vertex set $[n]$ and I its edge ideal. In this section we prove that $\operatorname{reg}(J) = \operatorname{reg}(I^s)$ for all monomial ideal $J \in \operatorname{Inter}(I^s, I^{(s)})$ for $s = 2, 3$ by using the same method in [31] with some more detailed statements.

Firstly, for $s = 2$, we have the following analog of [31, Lemma 3.1].

Lemma 3.1 *Let $J \subseteq L$ be intermediate monomial ideals in $\operatorname{Inter}(I^2, I^{(2)})$. Let $\mathbf{a} \in \mathbb{N}^n$ be such that $x^{\mathbf{a}} \notin L$. Then*

(1) $\sqrt{L : x^{\mathbf{a}}} = \sqrt{J : x^{\mathbf{a}}}$.
(2) $\operatorname{reg}(L) \le \operatorname{reg}(J)$.

Proof Part (2) follows from Part (1) and Lemma 2.9. Thus, we shall now prove Part (1) which follows the same line as the proof of [31, Lemma 3.1].

Since $L \supseteq J \supseteq I^2$, we have $\sqrt{L : x^{\mathbf{a}}} \supseteq \sqrt{J : x^{\mathbf{a}}} \supseteq \sqrt{I^2 : x^{\mathbf{a}}}$. By Lemma 2.13, it suffices to prove that $f = \sqrt{g/\gcd(g, x^{\mathbf{a}})} \in \sqrt{J : x^{\mathbf{a}}}$ for a minimal monomial generator g of L. By Theorem 2.5, we may assume that $g = x_1x_2x_3$ where 123 is a triangle in G. Since $x^{\mathbf{a}} \notin L$, $f \neq 1$. There are two cases:
Case 1: $\deg f \ge 2$. This implies that $f \in I \subseteq \sqrt{J : x^{\mathbf{a}}}$.
Case 2: $\deg f = 1$. We may assume that $f = x_1$. Thus $x_2x_3 \mid x^{\mathbf{a}}$. Since $x_1^2x_2x_3 \in I^2$, $f \in \sqrt{J : x^{\mathbf{a}}}$ as required. □

Consequently, we have:

Theorem 3.2 *Let $J \in \operatorname{Inter}(I^2, I^{(2)})$ be an intermediate ideal lying between I^2 and $I^{(2)}$. Then*

$$\operatorname{reg}(J) = \operatorname{reg}(I^2) = \operatorname{reg}(I^{(2)}).$$

Proof By Lemma 3.1, we have

$$\operatorname{reg}(I^{(2)}) \le \operatorname{reg}(J) \le \operatorname{reg}(I^2).$$

The conclusion follows from [31, Theorem 1.1] as $\operatorname{reg}(I^{(2)}) = \operatorname{reg}(I^2)$. $\square$

For $s = 3$, we have the following analog of [31, Lemma 4.1].

Lemma 3.3 *Let $J \subseteq L$ be intermediate monomial ideals lying between I^3 and $I^{(3)}$. Let $\mathbf{a} \in \mathbb{N}^n$ be such that $x^{\mathbf{a}} \notin L$. Assume that $\sqrt{L : x^{\mathbf{a}}} \neq \sqrt{J : x^{\mathbf{a}}}$. Let f be a minimal squarefree monomial generator $\sqrt{L : x^{\mathbf{a}}}$ such that $f \notin \sqrt{J : x^{\mathbf{a}}}$. Then, we have:*

(1) There exists a triangle 123 in G such that $x_1x_2x_3 \mid x^{\mathbf{a}}$ and $\deg(f) = 1$ with $f \nmid x^{\mathbf{a}}$.
(2) $\operatorname{reg}(L) \leq \operatorname{reg}(J)$.

Proof The proof is the same as the proof of [31, Lemma 4.1] and [31, Theorem 4.3]. We include the argument here for completeness. By Theorem 2.6 and Lemma 2.13, there are three cases as follows.
Case 1: There exists a clique of size 4, $C = 1234$, of G such that

$$f = \sqrt{x_1x_2x_3x_4/\gcd(x_1x_2x_3x_4, x^{\mathbf{a}})}.$$

If $\deg f \geq 2$ then $\operatorname{Supp}(f)$ must contain at least two vertices i, j among $\operatorname{Supp}(C)$. In particular, $f \in I \subseteq \sqrt{I^3 : x^{\mathbf{a}}}$, which is a contradiction. If $\deg f = 1$, say $f = x_1$. This implies that $x_2x_3x_4 \mid x^{\mathbf{a}}$. But, $f^3x^{\mathbf{a}} \in I^3$ by $x_1^3(x_2x_3x_4) = (x_1x_2)(x_1x_3)(x_1x_4) \in I^3$, which is a contradiction.
Case 2: There exists a 5-cycle, $C = 12345$, of G such that

$$f = \sqrt{x_1x_2x_3x_4x_5/\gcd(x_1x_2x_3x_4x_5, x^{\mathbf{a}})}.$$

Since $f \notin \sqrt{I^3 : x^{\mathbf{a}}}$, $f \notin I$. Furthermore, $f \mid x_1x_2x_3x_4x_5$, we have three subcases.

Subcase 2.1: $\deg f = 3$. We may assume that $f = x_1x_3x_5$ then $x_2x_4 \mid x^{\mathbf{a}}$. In this case $f^2x_2x_4 \in I^3$, which is a contradiction.

Subcase 2.2: $\deg f = 2$. We may assume that $f = x_1x_3$ then $x_2x_4x_5 \mid x^{\mathbf{a}}$. In this case $f^2x_2x_4x_5 \in I^3$, which is a contradiction.

Subcase 2.3: $\deg f = 1$. We may assume that $f = x_1$ then $x_2x_3x_4x_5 \mid x^{\mathbf{a}}$. In this case $f^2x_2x_3x_4x_5 \in I^3$, which is a contradiction.
Case 3: There exists an edge uv and a triangle 123 in G such that

$$f = \sqrt{x_ux_vx_1x_2x_3/\gcd(x_ux_vx_1x_2x_3, x^{\mathbf{a}})},$$

note that u, v might belong to $\{1, 2, 3\}$. In particular, $\operatorname{Supp}(f) \subseteq \{u, v\} \cup \{1, 2, 3\}$. Since $f \notin \sqrt{I^3 : x^{\mathbf{a}}}$, $f \notin I$. In particular, $|\operatorname{Supp}(f) \cap \{u, v\}| \leq 1$ and $|\operatorname{Supp}(f) \cap \{1, 2, 3\}| \leq 1$. There are two subcases.

Subcase 3.1: $\deg f \geq 2$. Since f is squarefree, $|\operatorname{Supp}(f) \cap \{u, v\}| = 1$ and $|\operatorname{Supp}(f) \cap \{1, 2, 3\}| = 1$. We may assume that $f = x_1x_u$. In particular $x_vx_2x_3$ is a divisor of $x^{\mathbf{a}}$. In this case, $f^2x^{\mathbf{a}} \in I^3$ by $x_u^2x_1^2x_vx_2x_3 = x_u(x_ux_v)(x_1x_2)(x_1x_3) \in I^3$, which is a contradiction.

Subcase 3.2: $\deg f = 1$. We first prove that $\mathrm{Supp}(f) \notin \{1, 2, 3\}$. Assume by contradiction that $\mathrm{Supp}(f) \in \{1, 2, 3\}$. We may assume that $f = x_1$.

(1) If $x_1 \in \{x_u, x_v\}$. Assume $x_1 = x_u$. It implies that $x_v x_2 x_3$ is a divisor of $x^{\mathbf{a}}$. By $(x_1x_v)(x_1x_2)(x_1x_3) \in I^3$, we have $x_1 \in \sqrt{I^3 : x^{\mathbf{a}}}$.
(2) If $x_1 \notin \{x_u, x_v\}$. Then $x_u x_v x_2 x_3$ is a divisor of $x^{\mathbf{a}}$. Since $x_1^2(x_u x_v x_2 x_3) = (x_u x_v)(x_1x_2)(x_1x_3) \in I^3$, $x_1 \in \sqrt{I^3 : x^{\mathbf{a}}}$.

This is a contradiction. Thus $\mathrm{Supp}(f) \notin \{1, 2, 3\}$. Therefore, $x_1x_2x_3 \mid x^{\mathbf{a}}$. Furthermore, $\mathrm{Supp}(f) \in \{u, v\}$. We may assume that $f = x_u$. If $u \in \mathrm{Supp}(\mathbf{a})$, then $x_u^2 \mid x_u x_v x_1 x_2 x_3$. Thus $u \in \{1, 2, 3\}$, which is a contradiction. It implies the statement of part (1).

Let $(\mathbf{a}, i)$ be an extremal exponent of L. By Lemma 2.9, we may assume that $\Delta_{\mathbf{a}}(L) \neq \Delta_{\mathbf{a}}(J)$. By part (1), there exists a variable x_u such that $x_u \in \sqrt{L : x^{\mathbf{a}}}$, and $u \notin \mathrm{Supp}\,\mathbf{a}$. By Lemma 2.12, $\mathrm{reg}\, L = \mathrm{reg}(L, x_u)$. Let I_0, J_1, L_1, I_1 be the restriction of $I^3, J, L, I^{(3)}$ to $W = [n] \setminus \{u\}$. Then $I_0 \subseteq J_1 \subseteq L_1 \subseteq I_1$. In fact, $I_0 = I_W^3$; $I_1 = I_W^{(3)}$ and $J_1 \subseteq L_1$ are intermediate monomial ideals lying between I_W^3 and $I_W^{(3)}$. By induction, $\mathrm{reg}\, L_1 \le \mathrm{reg}\, J_1$. Since x_u is a regular element with respect to $S/L_1, S/J_1$, this implies that $\mathrm{reg}\, L = \mathrm{reg}(L, x_u) = \mathrm{reg}(L_1, x_u) = \mathrm{reg}\, L_1 \le \mathrm{reg}\, J_1 = \mathrm{reg}(J_1, x) \le \mathrm{reg}\, J$. This is our statement. □

Consequently, we have:

Theorem 3.4 *Let $J \in \mathrm{Inter}(I^3, I^{(3)})$ be an intermediate ideal lying between I^3 and $I^{(3)}$. Then*

$$\mathrm{reg}(J) = \mathrm{reg}(I^3) = \mathrm{reg}(I^{(3)}).$$

Proof By Lemma 3.3, we have

$$\mathrm{reg}(I^{(3)}) \le \mathrm{reg}(J) \le \mathrm{reg}(I^3).$$

The conclusion follows from [31, Theorem 1.1] as $\mathrm{reg}(I^{(3)}) = \mathrm{reg}(I^3)$. □

4 Intermediate Ideals for Edge Ideals of Small Dimensions

In this section, we provide examples where we can verify Conjecture B. The first class where we can show the rigidity for all powers is the class of one-dimensional simplicial complexes.

Theorem 4.1 ([34, Theorem 1.1]) *Let Δ be an one-dimensional simplicial complex. For any $s \ge 1$, let $J \in \mathrm{Inter}(I^s, I^{(s)})$ be an intermediate ideal lying between I^s and $I^{(s)}$. Then*

$$\mathrm{reg}(J) = \mathrm{reg}(I^s) = \mathrm{reg}(I^{(s)}).$$

There are two main steps in proving Theorem 4.1. The first one is bounding the degrees of extremal exponents of the regular powers, the second one is the reduction of the computation of the regularity of intermediate ideals to that of the regular powers. The arguments can be extended to established Conjecture B for some small two-dimensional simplicial complexes. In this survey, we give two simple cases where we can reduce the computation of the intermediate ideals to the regular powers.

Theorem 4.2 *Let G be a simple graph with* $\dim \Delta(G) = 1$ *and* $\deg_G(i) \geq n-3$ *for all $i \in V(G) = [n]$ and I its edge ideal. Let $s \geq 2$ be an integer and J an intermediate ideal lying between I^s and $I^{(s)}$. Then,*

$$\operatorname{reg}(J) \leq \operatorname{reg}(I^s).$$

Before proving Theorem 4.2, we introduce two lemmas (first appeared in [34]) that will be useful later on.

Lemma 4.3 *Let $J \in \operatorname{Inter}(I^s, I^{(s)})$ be an intermediate ideal lying between I^s and $I^{(s)}$. Let $\mathbf{a} \in \mathbb{N}^n$ be an exponent such that $x^{\mathbf{a}} \notin J$. Assume that $f \in \sqrt{J : x^{\mathbf{a}}}$ and that $f \notin I$. Let F be a facet of $\Delta(I)$ that contains* $\operatorname{Supp}(f)$. *Then $\sum_{i \notin F} a_i \geq s$.*

Proof Assume by contradiction that $\sum_{i \notin F} a_i \leq s-1$. Let $\mathbf{b} \in \mathbb{N}^n$ be an exponent such that $b_i = 0$ for all $i \in F$, $b_i = a_i$ for all $i \notin F$. Then $|\mathbf{b}| \leq s-1$. Furthermore,

$$\operatorname{Supp}\left(\frac{\partial^*(f^u x^{\mathbf{a}})}{\partial^*(x^{\mathbf{b}})}\right) \subseteq F$$

implying that $f^u x^{\mathbf{a}} \notin I^{(s)}$ for all $u \geq 1$ by Lemma 2.2, which is a contradiction to the fact that $f \in \sqrt{J : x^{\mathbf{a}}} \subseteq \sqrt{I^{(s)} : x^{\mathbf{a}}}$. □

Lemma 4.4 *Let $f \in S$ be a monomial and $\mathbf{a} \in \mathbb{N}^n$ be an exponent such that $\sum_{i \in N(\operatorname{Supp}(f))} a_i \geq s$. Then $f \in \sqrt{I^s : x^{\mathbf{a}}}$.*

Proof We have $f^{|\mathbf{a}|} x^{\mathbf{a}}$ is divisible for

$$\prod_{i \in N(\operatorname{Supp}(f)),\ j_i \in \operatorname{Supp}(f) \text{ such that } i j_i \in G} (x_i x_{j_i})^{a_i} \in I^s.$$

Thus $f \in \sqrt{I^s : x^{\mathbf{a}}}$, as required. □

Proof of Theorem 4.2 Let $x^{\mathbf{a}} \notin J$ be a monomial. Assume that $\sqrt{J : x^{\mathbf{a}}} \neq \sqrt{I^s : x^{\mathbf{a}}}$. Let g be a minimal generator of $\sqrt{J : x^{\mathbf{a}}}$ such that $g \notin \sqrt{I^s : x^{\mathbf{a}}}$. Since $\alpha(G) = 2$ and $I \subseteq \sqrt{I^s : x^a}$, we deduce that $g \notin I$ and thus $\deg(g) \leq 2$.

If $\deg(g) = 2$, then $\operatorname{Supp}(g)$ is a facet of $\Delta(G)$. By Lemma 4.3, $\sum_{i \notin \operatorname{Supp}(g)} a_i \geq s$. Furthermore, $\alpha(G) = 2$ implies that $N(\operatorname{Supp}(g)) = [n] \backslash \operatorname{Supp}(g)$. By Lemma 4.4, $g \in \sqrt{I^s : x^{\mathbf{a}}}$, which is a contradiction.

If $\deg(g) = 1$, write $g = x_1$. By $\deg_G(1) \geq n - 3$, there are at most two non-neighbors of 1. There are two cases. If x_1 has less than two non-neighbors, then by Lemma 4.3, $\sum_{i \notin N_G[1]} \geq s$. By Lemma 4.4, it follows that $x_1 \in \sqrt{I^s : x^{\mathbf{a}}}$ which is a contradiction. Thus, we may assume that there are two non-neighbors of 1 called 2, 3. Since $\alpha(G) = 2$, $x_2x_3 \in I$. By Lemma 4.3, $a_2 + a_4 + \cdots + a_n \geq s$ and $a_3 + a_4 + \cdots + a_n \geq s$. By Lemma 4.4, $x_1^{|\mathbf{a}|} x^{\mathbf{a}} \in I^t$ where $t = \min(a_2, a_3) + a_4 + \cdots + a_n \geq s$, which is a contradiction.

Therefore, $\sqrt{J : x^{\mathbf{a}}} = \sqrt{I^s : x^{\mathbf{a}}}$. By Lemma 2.10, $\Delta_{\mathbf{a}}(I^s) = \Delta_{\mathbf{a}}(J)$ for any $x^{\mathbf{a}} \notin J$. Using Lemma 2.9, we have $\operatorname{reg}(J) \leq \operatorname{reg}(I^s)$. □

Corollary 4.5 *Let G be an anticycle over $[n]$ $(n \geq 4)$ and I its edge ideal. Let $s \geq 2$ be an integer and J be an intermediate ideal lying between I^s and $I^{(s)}$. Then,*

$$\operatorname{reg}(J) = \operatorname{reg}(I^s) = \operatorname{reg}(I^{(s)}).$$

Proof If $n = 4$ then I is a complete intersection. Then, $I^s = I^{(s)} = J$. In this case, we have

$$\operatorname{reg} J = \operatorname{reg} I^s = 2s + 1$$

for all $s \geq 1$.

If $n \geq 5$ then G is a gap-free and criket-free (the definition as in [2]). So, $\operatorname{reg}(I^s) = 2s$ for any $s \geq 2$ by Banerjee [2, Theorem 6.17]. Moreover, G satisfies the condition of Theorem 4.2, we deduce that $\operatorname{reg} J \leq \operatorname{reg} I^s = 2s$. Since $\operatorname{reg} J \geq 2s$, we have the required conclusion. □

We now give an example of two-dimensional simplicial complexes.

Theorem 4.6 *Let $n = 6$. Assume that*

$$G = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \{1, 6\}, \{3, 6\}$$

(see the Fig. 1) and I its edge ideal. For each $s \geq 2$, let $J \in \operatorname{Inter}(I^s, I^{(s)})$ be an intermediate ideal. Then, $\operatorname{reg} J = 2s$.

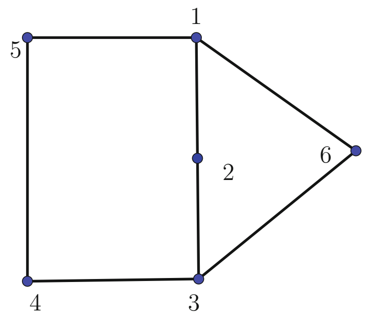

Fig. 1 The graph G

Proof It suffices to show that reg $J \le 2s$. Let $(\mathbf{a}, i) \in \mathbb{N}^6 \times \mathbb{N}$ be an extremal exponent of the ideal J. Note that

$$\Delta(I) = \{\{2,5,6\},\{2,4,6\},\{1,3\},\{1,4\},\{3,5\}\}.$$

We shall reduce the computation of the regularity of J to the regularity of regular power I^s. To do so, we shall show that $\Delta_{\mathbf{a}}(J) = \Delta_{\mathbf{a}}(I^s)$. Write $\mathbf{a} = (a_1, a_2, \dots, a_6)$. Assume by contradiction that $\Delta_{\mathbf{a}}(J) \neq \Delta_{\mathbf{a}}(I^s)$. In other words, there exists $f \in \sqrt{J : x^{\mathbf{a}}}$ but $f \notin \sqrt{I^s : x^{\mathbf{a}}}$.

Since $J \subseteq I^{(s)}$, if $f \in \sqrt{J : x^{\mathbf{a}}}$, then $f^u \cdot x^{\mathbf{a}} \in I^{(s)}$ for some u large enough. Since $I \subseteq \sqrt{I^s : x^{\mathbf{a}}}$, $f \notin I$. In particular Supp(f) is an independent set.

By Lemmas 4.3, 4.4, with argument similar to that of the proof of Theorem 4.2, we may assume that $\deg(f) = 1$.

By symmetry, x_1 plays the same role as x_3, x_4 plays the same role as x_5, there are four cases:

Case 1: $f = x_1$. By Lemma 4.3, $a_2 + a_5 + a_6 + \min(a_3, a_4) \ge s$. As $2, 5, 6 \in N_G(1)$, by Lemma 4.4, $x_1^{|\mathbf{a}|} x^{\mathbf{a}}$ belongs to $I^t \subseteq I^s$ where $t = a_2 + a_5 + a_6 + \min(a_3, a_4)$, which is a contradiction.

Case 2: $f = x_2$. By Lemma 4.3, $a_1 + a_3 + \min(a_4, a_5) \ge s$. Similarly, this implies $x_2 \in \sqrt{I^s : x^{\mathbf{a}}}$.

Case 3: $f = x_6$. By Lemma 4.3, $a_1 + a_3 + \min(a_4, a_5) \ge s$. Similarly, this implies $x_6 \in \sqrt{I^s : x^{\mathbf{a}}}$.

Case 4: $f = x_4$. By Lemma 4.3, $a_3 + a_5 + a_1 \ge s$ and $a_2 + a_3 + a_5 + a_6 \ge s$. If $a_3 + a_5 \ge s$, then $x_4 \in \sqrt{I^s : x^{\mathbf{a}}}$. Assume that $a_3 + a_5 = u < s$. Then $a_1 \ge s - u$ and $a_2 + a_6 \ge s - u$. This implies that $x_1^{a_1} x_2^{a_2} x_6^{a_6} \in I^{s-u}$. Thus $x_4 \in \sqrt{I^s : x^{\mathbf{a}}}$.

By Lemma 2.9, reg $J \le$ reg I^s. Since G is gap-free and cricket-free, by Banerjee [2, Theorem 6.17], $\operatorname{reg}(I^s) = 2s$ for any $s \ge 2$. Thus reg $J = 2s$, which completes our argument. □

Remark In general, even for the case $\dim \Delta(G) = 1$, it is not true that $\Delta_{\mathbf{a}}(J) = \Delta_{\mathbf{a}}(I^s)$ for all $x^{\mathbf{a}} \notin J$. We refer to [34] for more details.

5 Bounds on Regularity of Powers/Symbolic Powers

In this section, we discuss bounds on the regularity of powers/symbolic powers of edge ideals, paying attention to the recent development on the following Conjecture of Alilooe-Barnejee-Bayerslan-Ha [3] and its analog for symbolic powers:

Conjecture C Let I be the edge ideal of a simple graph G. Then, for all $s \ge 1$,

(1) $\operatorname{reg}(I^s) \le \operatorname{reg}(I) + 2s - 2$;
(2) $\operatorname{reg}(I^{(s)}) \le \operatorname{reg}(I) + 2s - 2$.

In [7], Barnejee and Nevo establish Conjecture C for bipartite graphs, and Part (1) for $s = 2$ for arbitrary graphs.

In [17], Fakhari study bounds on the regularity of symbolic powers of edge ideals, and prove the following

Theorem 5.1 *Let I be the edge ideal of a simple graph G. Then, for any $s \geq 2$,*

$$\operatorname{reg}(I^{(s)}) \leq \max\{\operatorname{reg}(I^{(s)} + I^{s-1}), \operatorname{reg}(I) + 2s - 2\}.$$

Together with the result of Barnejee and Nevo, he deduced that Conjecture C (2) for holds for $s = 2, 3$ for arbitrary graphs. This work also suggests that studying regularity of regular powers helps in understanding regularity of symbolic powers.

In [31], by combining [31, Theorem 1.1] and Theorem 5.1 we prove that Conjecture C (1) holds for $s = 2, 3$ and Conjecture C (2) holds for $s = 2, 3, 4$.

Theorem 5.2 *Let I be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I^3) \leq 4 + \operatorname{reg}(I).$$

Theorem 5.3 *Let I be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I^{(4)}) \leq 6 + \operatorname{reg}(I).$$

For co-chordal graphs, we can proceed a bit further and extend the previous work of Fakhari.

Theorem 5.4 *Let G be a co-chordal graph and I its edge ideal. Then* $\operatorname{reg} I^{(5)} = 10$.

Proof By Theorem 5.1, it suffices to proof that $\operatorname{reg} J \leq 10$ where $J = I^{(5)} + I^4$. By the result of Sullivant [40, Theorem 3.10], as co-chordal graphs are perfect,

$$J = I^4 + I(J_1 J_1 + J_3) + J_1 J_2,$$

where J_i is generated by cliques of sizes $i + 2$. We shall prove by induction on the number of variables that $\operatorname{reg} J \leq 8$. Let $(\mathbf{a}, i)$ be an extremal exponent of J. By Lemma 2.9 and the fact that $\operatorname{reg} I^4 = 8$, we may assume that $\Delta_{\mathbf{a}}(J) \neq \Delta_{\mathbf{a}}(I^4)$. Let $f \in \sqrt{J : x^{\mathbf{a}}}$ be such that $f \notin \sqrt{I^4 : x^{\mathbf{a}}}$. Since $I \subseteq \sqrt{I^4 : x^{\mathbf{a}}}$, $f \notin I$. By Lemma 2.13, with argument similar to that of Lemma 3.3, we deduce that $\deg f = 1$ and that $f \nmid x^{\mathbf{a}}$. By Lemma 2.12, $\operatorname{reg} J = \operatorname{reg}(J, f) \leq 8$ by induction. □

Remark Modifying [31, Example 4.2], we have an example of a co-chordal graph and a monomial $x^{\mathbf{a}} \notin I^{(s)}$ such that

$$\sqrt{I^{(s)} : x^{\mathbf{a}}} \neq \sqrt{I^s : x^{\mathbf{a}}} + (\text{variables}).$$

Indeed, let

$I = (x_1x_2, x_2x_3, x_3x_1, x_1x_4, x_4x_5, x_2x_6, x_6x_7, x_1x_6, x_2x_4, x_4x_6)$ and $x^{\mathbf{a}} = x_1x_2x_3x_4x_6$, then x_5x_7 is a minimal generator of $\sqrt{I^{(4)} : x^{\mathbf{a}}}$ but does not belong to $\sqrt{I^4 : x^{\mathbf{a}}}$. Thus, verifying Conjecture A for co-chordal graphs requires more work.

Besides, there are a few other bounds given for the regularity of ordinary powers of edge ideals. One expects that these bounds hold for symbolic powers. We single out the bound given by Herzog and Hibi [22] to propose the following.

Problem 5.5 Let I be the edge ideal of a simple graph G. Then, for any $s \geq 1$,

$$\operatorname{reg}(I^{(s)}) \leq 2s + c,$$

where c is the dimension of the stable complex of G.

As in [22], to settle Problem 5.5 it suffices to establish Conjecture A for very-well covered graphs.

6 Mixed Sum and Fiber Product

We shall provide the result on mixed sums and fiber products. From this, we shall reduce Conjectures A and C to the case G is a connected graph.

Theorem 6.1 *Let $P = I + J$ be the mixed sum of two edge ideals I and J. The following statements hold*

(1) If Conjecture A holds for I and J then so is P.
(2) If Conjecture C holds for I and J then so is P.

Proof By Nguyen and Vu [36, Theorem 1.1] and Ha et al. [25, Theorem 1.1], we have

$$\operatorname{reg} P^s = \max\{\operatorname{reg} I^i + \operatorname{reg} J^{s-i} + 1, \operatorname{reg} I^i + \operatorname{reg} J^{s-i+1}\}$$
$$\operatorname{reg} P^{(s)} = \max\{\operatorname{reg} I^{(i)} + \operatorname{reg} J^{(s-i)} + 1, \operatorname{reg} I^{(i)} + \operatorname{reg} J^{(s-i+1)}\}.$$

The conclusions follow. □

A similar result holds for fiber products. The regularity of symbolic powers of fiber products has been computed for squarefree monomial ideals by O'Rourke [38] and for arbitrary radical ideals by Fakhari and Nguyen [18]. Precisely, we have

Lemma 6.2 *Let $F = I + J + \mathfrak{m}\mathfrak{n}$ be the fiber product of squarefree monomial ideals I and J. Then*

$$\operatorname{reg}(F^{(s)}) = \max_{i \in [1,s]} \{2s, \operatorname{reg}(I^{(i)}) + s - i, \operatorname{reg}(J^{(i)}) + s - i\}.$$

Proof The main idea in the work of [18] is that we have

$$F^{(s)} = (I, \mathfrak{n})^{(s)} \cap (J, \mathfrak{m})^{(s)}.$$

We give a simple argument for equality in the case of squarefree monomial ideals as below. Since $F = (I, \mathfrak{n}) \cap (J, \mathfrak{m})$, the left hand side is contained in the right hand side.

Conversely, let $f \in (I, \mathfrak{n})^{(s)} \cap (J, \mathfrak{m})^{(s)}$ be a monomial. Then for any $x^{\mathbf{a}}$ we have $\partial f/\partial x^{\mathbf{a}} \in (I, \mathfrak{n}) \cap (J, \mathfrak{m}) = F$. Thus $f \in F^{(s)}$ as required.

Thus, we see that

$$\operatorname{reg}(F^{(s)}) = \max\{\operatorname{reg}((I, \mathfrak{n})^{(s)}), \operatorname{reg}((J, \mathfrak{m})^{(s)}), \operatorname{reg}((I, \mathfrak{n})^{(s)} + (J, \mathfrak{m})^{(s)}) + 1\}.$$

Note that

$$(I, \mathfrak{n})^{(s)} + (J, \mathfrak{m})^{(s)} = \mathfrak{m}^s + \mathfrak{n}^s,$$

which has regularity $2s - 1$. The conclusion follows from [25, Theorem 1.1]. □

Theorem 6.3 *Let $F = I + J + \mathfrak{m}\mathfrak{n}$ be the fiber product of two edge ideals I and J. The following statements hold*

(1) If Conjecture A holds for I and J then so is F.
(2) If Conjecture C holds for I and J then so is F.

Proof Follows from Theorem 6.2 and [37, Theorem 5.1]. □

Remark It is interesting if a similar statement holds for Conjecture B. It is not clear to us to draw any conclusions from what we knew about mixed sums and fiber products.

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Applications of Differential Graded Algebra Techniques in Commutative Algebra

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To David Eisenbud on the occasion of his 75th birthday

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1 Introduction

Throughout this paper, the term "ring" is short for "commutative noetherian ring with identity."

In algebraic topology, it is incredibly useful to know that the singular cohomology of a manifold has a natural algebra structure. Similarly, in commutative algebra the fact that certain Ext and Tor modules carry algebra structures is a powerful tool. Both of these notions arise by considering differential graded (DG) algebra structures on certain chain complexes. In short, a DG algebra is a chain complex that is also a graded commutative ring, where the differential and multiplication are compatible; see Sect. 2 for definitions and background material.

Avramov, Buchsbaum, Eisenbud, Foxby, Halperin, Kustin, and others pioneered the use of DG algebra techniques in homological commutative algebra. The idea is to prove results about rings by broadening one's context to include vast generalizations. A deep, rich sample of the theory and applications can be found in Avramov's lecture notes [14].

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The current paper is a modest follow-up to *op. cit.*, documenting a few applications that have appeared in the twenty-some years since *op. cit.* appeared. To be clear, we focus on applications: results whose statements make no reference to DG algebras but whose proofs use them extensively. Furthermore, this survey is by no means comprehensive. We focus on small number of some of our favorite applications, limited by constraints of time and space.

Most of the sections below begin by describing an application with little reference to DG algebras. This is followed by a certain amount of DG background material, but generally only enough to give a taste for the material. The sections conclude with an indication of how the DG technology helps to obtain the application.

As we noted above, the work of David Eisenbud is foundational in this area, especially the paper [39] with Buchsbaum; see 2.6. Those of us working in this area owe him a huge debt of gratitude for this and other seminal work in the field.

2 Growth of Bass and Betti Numbers

In this section, let $(R, \mathfrak{m}, k)$ be a local ring with $d := \operatorname{depth} R$. The *embedding codepth* of R, denoted $c := \operatorname{ecodepth} R$, is defined to be $e - d$, where $e := \operatorname{edim} R$ is the minimal number of generators of $\mathfrak{m}$. Cohen's Structure Theorem states that the $\mathfrak{m}$-adic completion $\widehat{R}$ admits a minimal Cohen presentation, i.e., there is a complete regular local ring $(P, \mathfrak{p}, k)$ and an ideal $I \subseteq \mathfrak{p}^2$ such that $\widehat{R} \cong P/I$. Note that the projective dimension $\operatorname{pd}_P(\widehat{R})$, i.e., the length of the minimal free resolution of $\widehat{R}$ over P, is equal to ecodepth R.

Fundamental invariants of a finitely generated R-module M are the Bass and Betti numbers. These numerically encode structural information about the module M, e.g., the minimal number of generators and relations and higher degree versions of these. A hot topic of research in commutative algebra is the growth of these sequences. In this section, we describe some recent progress by Avramov on this subject including how he uses DG techniques to get information about these invariants. Along the way, we also present foundational material about the DG context.

Bass Numbers, Betti Numbers, and a Question of Huneke

Let M be a finitely generated R-module. For each integer i, the ith *Bass number* and the ith *Betti number* of M are defined to be, respectively

$$\mu_R^i(M) := \operatorname{rank}_k (\operatorname{Ext}[R]ikM)$$

$$\beta_i^R(M) := \operatorname{rank}_k (\operatorname{Ext}[R]iMk) = \operatorname{rank}_k \left(\operatorname{Tor}_i^R(M, k)\right).$$

The *Bass series* and the *Poincaré series* of M are the formal power series

$$I_R^M(t) := \sum_{i\in\mathbb{Z}} \mu_R^i(M)t^i \qquad P_M^R(t) := \sum_{i\in\mathbb{Z}} \beta_i^R(M)t^i. \tag{2.0.1}$$

In case that $M = R$, the Bass numbers and Bass series are denoted μ_R^i and $I_R(t)$.

In this section we are concerned with the following unpublished question of Huneke; see [107]. This question is motivated in part by the fact that R is Gorenstein if and only if its Bass numbers are eventually 0.

Question 2.1 Let R be a Cohen-Macaulay local ring. If $\{\mu_R^i\}$ is bounded, must R be Gorenstein? If $\{\mu_R^i\}$ is bounded above by a polynomial in i, must R be Gorenstein? If R is not Gorenstein, must $\{\mu_R^i\}$ grow exponentially?

Very little progress has been made on this question. Christensen, Striuli, and Veliche [45] conduct a careful analysis of several special cases of this and other related questions. Other progress comes from Jorgensen and Leuschke [75] and Borna, Sather-Wagstaff, and Yassemi [33, 107].

In this section, we focus on work of Avramov [16] on this question for non-Gorenstein rings R with $c = \operatorname{ecodepth}(R) \leq 3$ which includes the following result. The proof relies heavily on DG techniques as we explain in the next subsections. This is a true application of DG tools, as the statement makes no mention of DG algebras, though they are used extensively in the proof.

Theorem 2.2 ([16, Theorem 4.1]) *If $c \leq 3$ and R is not Gorenstein, then there is a real number $\gamma_R > 1$ such that for all $i \geq 1$*

$$\mu_R^{d+i} \geq \gamma_R \mu_R^{d+i-1} \tag{2.2.1}$$

with two exceptions for $i = 2$: If $I = (wx, wy)$ or $I = (wx, wy, z)$, where $x, y \in P$ is a regular sequence, $w \in P$, and $z \in \mathfrak{p}^2$ is a $P/(wx, wy)$-regular element, then $\mu_R^{d+2} = \mu_R^{d+1} = 2$. If R is Cohen-Macaulay, the inequality (2.2.1) holds for all i.

DG Algebra Resolutions and DG Modules

Let S be a ring. A *associative, commutative differential graded S-algebra* (*DG S-algebra* for short) is a chain complex $A = \cdots \to A_2 \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \to 0$ such that $A^\natural = \bigoplus_{i\geq 0} A_i$ has the structure of a graded commutative S-algebra

- for all $a, b \in A$ the equality $ab = (-1)^{|a||b|}ba$ holds, and $a^2 = 0$ if the homological degree $|a|$ is odd
  that satisfies the *Leibniz rule*
- for all $a, b \in A$ we have $\partial^A(ab) = \partial^A(a)b + (-1)^{|a|}a\partial^A b$, i.e., the assignment $a \otimes b \mapsto ab$ describes a chain map $A \otimes_S A \to A$.

The DG S-algebra A is called *homologically degreewise noetherian* if $\mathrm{H}_0(A)$ is noetherian, and each $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated.

Examples of homologically degreewise noetherian DG S-algebras include S itself, considered as a complex concentrated in degree 0, and the Koszul complex $K^S(\underline{\mathbf{x}})$ over S on a sequence $\underline{\mathbf{x}} = x_1, \ldots, x_n$ in S with the exterior algebra structure.

A *morphism* of DG S-algebras is a chain map $f: A \to B$ such that for all $a, a' \in A$ we have $f(aa') = f(a)f(a')$ and $f(1) = 1$. A *quasiisomorphism* of DG algebras is a morphism that is a quasiisomorphism, i.e., such that the induced map on homology is an isomorphism in each degree. A *DG algebra resolution of an S-algebra T* is a quasiisomorphism $F \xrightarrow{\simeq} T$ of DG S-algebras such that each F_i is free over S. Several examples of DG algebra resolutions are given below starting with 2.4.

In case that $(S, \mathfrak{n})$ is a local ring, a DG S-algebra A is called *local* if it is homologically degreewise noetherian and $\mathrm{H}_0(A)$ is a local S-algebra. In this case, setting $\mathfrak{n}_0$ to be the preimage of $\mathfrak{m}_{\mathrm{H}_0(A)}$ in A_0, we let $\mathfrak{m}_A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{n}_0 \to 0$ be the *augmentation ideal* of A. By definition it is a subcomplex of A. Moreover, it is a *DG ideal* of A meaning that it absorbs multiplication by elements of A. In this situation, we say that $(A, \mathfrak{m}_A, A/\mathfrak{m}_A)$ is a local DG S-algebra. As an example, if $\underline{\mathbf{x}} \in \mathfrak{n}$ is a sequence of n elements, then $K = K^S(\underline{\mathbf{x}})$ is a local DG S-algebra with the augmentation ideal $\mathfrak{m}_K = 0 \to S \to \cdots \to S^n \to \mathfrak{n} \to 0$ and $K/\mathfrak{m}_K \cong S/\mathfrak{n}$.

2.3 A construction of Tate [113] (see also Avramov [14, Proposition 2.2.8]) guarantees the existence of a DG algebra resolution F of $\widehat{R}$ over P, where each F_i is finitely generated and free over P and $F_i = 0$ for all $i > \mathrm{pd}_P(\widehat{R})$.

Examples of DG algebra resolutions include the following.

2.4 If I is generated by a P-regular sequence (that is, if R is a formal complete intersection), then the Koszul complex $K^P(I)$ on a minimal generating sequence for I is a DG algebra resolution of $\widehat{R}$ over P.

2.5 If $\mathrm{pd}_P(\widehat{R}) = 2$, then it follows from the Hilbert-Burch Theorem [51, Theorem 20.15] that there is an element $f \in P$ and a matrix Y of size $n \times (n-1)$ such that the minimal P-free resolution of $\widehat{R}$ can be chosen with the form

$$0 \to P^{\oplus n-1} \xrightarrow{Y} P^{\oplus n} \xrightarrow{X} P \to \widehat{R} \to 0$$

with $X = f\left(\det(Y_1), \ldots, (-1)^{j-1}\det(Y_j), \ldots, (-1)^{n-1}\det(Y_n)\right)$, where Y_j is the minor obtained from Y by deleting the j-th row. Herzog [67] describes a DG algebra structure on this resolution, as follows. Let $\{a_1, \ldots, a_n\}$ be a basis for $P^{\oplus n}$ and $\{b_1, \ldots, b_{n-1}\}$ be a basis for $P^{\oplus n-1}$, and set

$$(a_i)^2 = 0$$

$$a_i \cdot a_j = -a_j \cdot a_i = \sum_{t=1}^{n-1} (-1)^{i+j+t+1} \det(Y_{ij,t}) f b_t \qquad \text{for } i < j$$

where $Y_{ij,t}$ denotes the minor obtained from Y by deleting rows i, j and column t.

2.6 Assume $\mathrm{pd}_P(\widehat{R}) = 3$. Buchsbaum and Eisenbud [39] show that the minimal free resolution of $\widehat{R}$ over P has the structure of a DG algebra, though the explicit structure of the resolution is not given.

Let n denote the minimal number of generators for I, and assume that R is Gorenstein (that is, I is a Gorenstein ideal). Then it is shown in *op. cit.* that the minimal free resolution of $\widehat{R}$ over P is of the form

$$0 \to P \xrightarrow{Z} P^{\oplus n} \xrightarrow{Y} P^{\oplus n} \xrightarrow{X} P \to \widehat{R} \to 0 \tag{2.6.1}$$

for some $n \times n$ alternating matrix Y with entries in $\mathfrak{p}$ and

$$X = \left(\mathrm{pf}(Y_1), \ldots, (-1)^{j-1}\,\mathrm{pf}(Y_j), \ldots, (-1)^{n-1}\,\mathrm{pf}(Y_n)\right)$$

where Y_i denotes the matrix obtained from Y by deleting row i and column i. (Here, pf is the Pfaffian; see [38] for details.) Also, $Z = \mathrm{Hom}[P]XP$.

An explicit DG algebra structure on (2.6.1) is given by Avramov [9] as follows. Let $\{a_1, \ldots, a_n\}$ be a basis for $P^{\oplus n}$ in degree 1, let $\{b_1, \ldots, b_n\}$ be a basis for $P^{\oplus n}$ in degree 2, and let $\{c\}$ be a basis for P in degree 3. Define

$$(a_i)^2 = 0 \qquad\qquad a_i \cdot b_j = b_j \cdot a_i = \delta_{ij} c$$

$$a_i \cdot a_j = -a_j \cdot a_i = \sum_{t=1}^{n} (-1)^{i+j+t} \rho_{ijt}\,\mathrm{pf}(Y_{ijt}) b_t \qquad \text{for } i < j$$

where Y_{ijt} is the matrix obtained from Y by deleting rows i, j, t and columns i, j, t, and δ_{ij} is the Kronecker delta, and

$$\rho_{ijt} = \begin{cases} -1 & i < t < j \\ 1 & \text{otherwise.} \end{cases}$$

There are many examples of DG algebra resolutions in the monomial situation. We summarize a few here very briefly and point the reader to references for more details. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and let I be a monomial ideal in S, i.e., an ideal generated by monomials. A general DG algebra resolution of S/I is given by Taylor [115] and Bayer, Peeva, and Sturmfels [32], but it is not minimal in general. In the following cases the minimal free resolution of S/I over S has a DG algebra structure: stable ideals (see Eliahou and Kervaire [52] or Peeva [100]), matroidal ideals (see Sköldberg [111]), and ideals of the form $I = fJ$, where J is a monomial ideal in S and f is the least common multiple of the generators of J (see Katthän [78]).

2.7 It is important to note for 2.3 that the minimal free resolution of $\widehat{R}$ over P may not admit a DG algebra structure in general. Examples for this are given by Khinich (as documented in [8]), Avramov [10], and Katthän [78]. The example of Katthän is generic and disproves a claim by Bayer, Peeva, and Sturmfels [32].

2.8 In contrast to 2.7, if R satisfies one of the following conditions, then the minimal free resolution of $\widehat{R}$ over P admits a DG algebra structure:

(a) $c \leq 3$, by 2.5–2.6;
(b) $c = 4$ and R is Gorenstein, by Kustin and Miller [79, 82];
(c) $c = 4$, R is Cohen-Macaulay, almost complete intersection, and $1/2 \in R$, by Kustin [81];
(d) R is complete intersection, by 2.4;
(e) R is one link from a complete intersection, by Avramov, Kustin, and Miller [31];
(f) R is two links from a complete intersection and is Gorenstein, by Kustin and Miller [83].

The translation to DG algebras uses the following DG analogue of modules.

2.9 Let B be a DG S-algebra. A *DG B-module* is an S-complex M such that $M^\natural = \bigoplus_i M_i$ is a graded $A^\natural$-module satisfying the Leibniz rule. For the case of S considered as a DG S-algebra, the DG S-modules are just the S-complexes. A DG B-module M is *homologically bounded* if $\mathrm{H}_i(M) = 0$ for all $|i| \gg 0$; it is *homologically finite* if $\bigoplus_i \mathrm{H}_i(M)$ is a finitely generated $\mathrm{H}_0(B)$-module.

Let M be a DG B-module. The *trivial extension* $B \ltimes M$ is the DG algebra with the underlying complex $B \oplus M$ equipped with the product that is given as follows:

$$(b, m)(b', m') := (bb', bm' + (-1)^{|m||b'|} b'm).$$

Avramov's Machine

Some of our favorite applications of DG techniques use the following tool which Kustin [80] calls *Avramov's machine*.

2.10 Let $\mathbf{x}$ be a minimal generating sequence for $\mathfrak{m}$, and let $\mathbf{y}$ be a minimal generating sequence for the maximal ideal $\mathfrak{p}$. Since P is a regular local ring, the Koszul complex $K^P(\mathbf{y})$ is a minimal free resolution of k over P. Since $K^P(\mathbf{y}) \simeq k$, we obtain the following diagram of DG algebra quasiisomorphisms:

$$K^R(\mathbf{x}) \xrightarrow{\simeq} K^{\widehat{R}}(\mathbf{x}\widehat{R}) \xleftarrow{\cong} K^P(\mathbf{y}) \otimes_P \widehat{R} \xleftarrow{\simeq} K^P(\mathbf{y}) \otimes_P F \xrightarrow{\simeq} k \otimes_P F =: A. \tag{2.10.1}$$

The assumptions on F in 2.3 imply that A is a finite-dimensional DG k-algebra. It follows that $\mathrm{Tor}^P(\widehat{R}, k)$ inherits the structure of a finite-dimensional DG k-algebra; this is the *Tor algebra*. If F is minimal, e.g., in any of the cases from 2.8, the algebra A has zero differential, so $A \cong \mathrm{Tor}^P(\widehat{R}, k)$.

Rationality of Poincaré series, which we discuss next, is an important application of Avramov's machine.

2.11 Consider the notation from 2.10. In this paragraph, assume that one of the conditions (a), (b), (e), or (f) in 2.8 holds. Using the fact that the minimal free resolution of $\widehat{R}$ over P has a DG algebra structure, Avramov, Kustin, and Miller [31] give a factorization $P \xrightarrow{\varphi} Q \xrightarrow{\psi} \widehat{R}$ of the canonical map $P \to \widehat{R}$ such that φ is complete intersection and ψ is Golod (see, e.g., [12] for the definition). Then they invoke a result of Levin [85] to conclude the following;

(∗) the Poincaré series of every finitely generated R-module is rational with common denominator.

In case (d) of 2.8 where R is a complete intersection, conclusion (∗) was proved for $P_k^R(t)$ by Tate [113] and, in general, by Gulliksen [64] and Avramov [13]. In case (c) of 2.8, conclusion (∗) was proved by Kustin and Palmer [84].

Growth Rates in Embedding Codepth at Most 3

With these tools in hand, the proof of Theorem 2.2 proceeds in the following steps. First, consider the following structure result for the Tor algebra.

2.12 Assume that $c \leq 3$. Using the notation from 2.10, we know that A is a finite-dimensional DG algebra with zero differential. In this case, by Avramov et al. [31] and Weyman [124], the ring R belongs to one of the following classes

Class	c	A	B	C	D
$\mathbf{C}(c)$	≤ 3	B	$\bigwedge_k \Sigma k^c$		
$\mathbf{S}$	2	$B \ltimes W$	k		
$\mathbf{T}$	3	$B \ltimes W$	$C \ltimes \Sigma(C/C_{\geq 2})$	$\bigwedge_k \Sigma k^2$	
$\mathbf{B}$	3	$B \ltimes W$	$C \ltimes \Sigma C_+$	$\bigwedge_k \Sigma k^2$	
$\mathbf{G}(r)$	3	$B \ltimes W$	$C \ltimes \mathrm{Hom}[k]C\Sigma^3 k$	$k \ltimes \Sigma k^r$	
$\mathbf{H}(p,q)$	3	$B \ltimes W$	$C \otimes_k D$	$k \ltimes (\Sigma k^p \oplus \Sigma^2 k^q)$	$k \ltimes \Sigma k$

where W is a finitely generated positively graded k-vector space with $B_+ W = 0$ and $\ltimes$ designates the trivial extension from 2.9. The ring R is in class $\mathbf{S}$ (that is, A is of the form $k \ltimes W$) if and only if R is Golod; see [61]. If R is in class $\mathbf{C}(c)$, then R is a complete intersection.

The next step in the proof of Theorem 2.2 is to connect the Poincaré and Bass series of R to analogous series for A:

$$I_R(t) = t^e \cdot I_A(t) \qquad\qquad P_k^R(t) = (1+t)^e \cdot P_k^A(t)$$

where $I_A(t)$ and $P_k^A(t)$ are the Bass series and the Poncaré series for A which are defined in the DG setting as in (2.0.1). These equalities are based on work in [9, 23].

The third step in the proof of Theorem 2.2 is to analyze the Poincaré and Bass series of A to draw the following conclusions about the corresponding series for R; the proof then concludes from an analysis of the coefficients in the displayed series.

Theorem 2.13 ([16, Theorem 2.1]) *Use the notation from 2.12. Assume that $c \leq 3$ and set $l := \mathrm{rank}_k A_1 - 1$, $n := \mathrm{rank}_k A_3$, $p := \mathrm{rank}_k (A_1)^2$, $q := \mathrm{rank}_k (A_1 \cdot A_2)$, and*

$r := \operatorname{rank}_k(\delta_2)$, where $\delta_2 \colon A_2 \to \operatorname{Hom}[k]A_1A_3$ is defined by $\delta_2(a_2)(a_1) := a_2a_1$ for all $a_1 \in A_1$ and $a_2 \in A_2$. Then the following equalities hold for the Poincaré series and Bass series of R:

$$P_k^R(t) = \frac{(1+t)^{e-1}}{g(t)} \qquad \textit{and} \qquad I_R^R(t) = t^d \cdot \frac{f(t)}{g(t)}$$

where $f(t), g(t) \in \mathbb{Z}[t]$ are described as follows, where $p + q \geq 1$:

Class	$g(t)$	$f(t)$
$\mathbf{C}(c)$	$(1-t)^c(1+t)^{c-1}$	$(1-t)^c(1+t)^{c-1}$
$\mathbf{S}$	$1-t-lt^2$	$1+t-t^2$
$\mathbf{T}$	$1-t-lt^2-(n-3)t^3-t^5$	$n+lt-2t^2-t^3+t^4$
$\mathbf{B}$	$1-t-lt^2-(n-1)t^3+t^4$	$n+(l-2)t-t^2+t^4$
$\mathbf{G}(r)$	$1-t-lt^2-nt^3+t^4$	$n+(l-r)t-(r-1)t^2-t^3+t^4$
$\mathbf{H}(0,0)$	$1-t-lt^2-nt^3$	$n+lt+t^2-t^3$
$\mathbf{H}(p,q)$	$1-t-lt^2-(n-p)t^3+qt^4$	$n+(l-q)t-pt^2-t^3+t^4$

We end this section with the discussion of some properties of the class $\mathbf{G}(r)$ including recent counterexamples to a conjecture of Avramov [16]. Consider the notation from 2.12. Let n denote the minimal number of generators for I, and assume that $c = 3$. In case that R is a Gorenstein ring which is not complete intersection, it is known from work of J. Watanabe [123] that $n \geq 5$ and n is odd. Also, in this case, R belongs to the class $\mathbf{G}(2i+1)$ for some $i \geq 2$ by Avramov [16]. In particular, R belongs to the class $\mathbf{G}(n)$.

Conversely, Avramov *op. cit.* conjectured that if R is in the class $\mathbf{G}(r)$ with $r \geq 2$, then R is Gorenstein and therefore, the classes $\mathbf{G}(3)$ and $\mathbf{G}(2i)$ for all $i \geq 1$ are empty. Christensen, Veliche, and Weyman [46, 47] gave counterexamples to this conjecture. More precisely, it is shown in the latter paper that if S is the power series algebra in three variables over a field, then for every $r \geq 3$ there is an ideal I of S with $\operatorname{type}(S/I) = 2$ such that S/I belongs to $\mathbf{G}(r)$. For counterexamples to Avramov's conjecture of arbitray type, see VandeBogert [116].

3 Friendliness and Persistence of Local Rings

In this section, let $(R, \mathfrak{m}, k)$ be a local ring.

Vanishing of Ext and Tor, and Finiteness of Homological Dimensions
Let M, N be finitely generated R-modules. Following Avramov, Iyengar, Nasseh, and Sather-Wagstaff [29], R is called *Tor-friendly* if $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$ implies that $\operatorname{pd}_R M < \infty$ or $\operatorname{pd}_R N < \infty$. We say that R is *Tor-persistent* if $\operatorname{Tor}_i^R(M, M) = 0$ for all $i \gg 0$ implies that $\operatorname{pd}_R M < \infty$. The ring R is *Ext-friendly*

if $\operatorname{Ext}[R]iMN = 0$ for all $i \gg 0$ implies that $\operatorname{pd}_R M < \infty$ or $\operatorname{id}_R N < \infty$, where id is the injective dimension. Finally, R is *Ext-persistent* if $\operatorname{Ext}[R]iMM = 0$ for all $i \gg 0$ implies $\operatorname{pd}_R M < \infty$ or $\operatorname{id}_R M < \infty$.

Friendliness and persistence have been studied in numerous works; see for instance [17, 18, 29, 70, 71, 73, 74, 76, 77, 94–97, 108, 109]. The main motivation for this section is the following result in which the proofs of parts (a), (b), (c), (e), and (f) use DG algebra techniques.

Theorem 3.1 ([29, Theorem 5.1, Lemmas 5.7 and 5.9]) *Assume there exist a local homomorphism $R \to R'$ of finite flat dimension and a deformation $R' \twoheadleftarrow Q$, i.e., a local surjection with kernel generated by a Q-regular sequence, where Q satisfies at least one of the conditions*

(a) edim Q − depth $Q \leq 3$.
(b) *Q is Gorenstein and* edim Q − depth $Q = 4$.
(c) *Q is Cohen-Macaulay, almost complete intersection,* edim Q − depth $Q = 4$, *and* $\frac{1}{2} \in Q$.
(d) *Q is complete intersection.*
(e) *Q is one link from a complete intersection.*
(f) *Q is two links from a complete intersection and is Gorenstein.*
(g) *Q is Golod.*
(h) *Q is Cohen-Macaulay and* mult $Q \leq 7$.

Then R is Tor- and Ext-persistent. Moreover, Q can be chosen to be complete, with algebraically closed residue field, and with no embedded deformation; in this case, Q is Tor-friendly.

One of the most important motivations for working on friendliness and persistence is the following conjecture that is known as the *Auslander-Reiten Conjecture* [7]. This conjecture stems from work of Nakayama [88] and Tachikawa [112] on the representation theory of Artin algebras.

Conjecture 3.2 ([7, p. 70]) Let M be a finitely generated R-module that satisfies the condition $\operatorname{Ext}[R]iMM \oplus R = 0$ for all $i > 0$. Then M is a free R-module.

3.3 It is straightforward to show that if R is Ext-persistent, then it satisfies the Auslander-Reiten Conjecture 3.2.

By Avramov et al. [29, Proposition 6.5], Tor-friendliness implies Ext-friendliness. (In the context of complexes, these two notions are equivalent; see [29, Propositions 3.2 and 6.5].) The question of whether all rings are Tor-persistent is open. However, examples of rings that are not Ext-persistent (hence, not Ext-friendly nor Tor-friendly) are straightforward to construct: for instance, $(k[x, y]/(x, y)^2) \otimes_k (k[u, v]/(u, v)^2)$.

Next, we describe some DG methods from [29, 30] used to prove Theorem 3.1.

Perfect DG Modules, Trivial Extensions, and DG Syzygies

In order to apply DG techniques in the above setting, the first tool we need is the following DG analogue of finitely generated module of finite projective dimension.

3.4 Assume that $(B, \mathfrak{m}_B)$ is a local DG algebra. A homologically finite DG B-module M is called *perfect* is it satisfies one of the following equivalent conditions (see [30] or [105]):

(i) M is quasiisomorphic to a DG B-module F such that the underlying graded $B^\natural$-module $F^\natural$ has a finite basis.
(ii) For all homologically bounded DG B-modules N, one has $\operatorname{Tor}_i^B(M, N) = 0$ for all $i \gg 0$.
(iii) $\operatorname{Tor}_i^B(M, B/\mathfrak{m}_B) = 0$ for all $i \gg 0$.

The approach described below to understanding friendliness and persistence is motivated in parts by work of Nasseh and Yoshino [97] who prove that the trivial extension $R \ltimes k$ is Tor-friendly. See 2.9 for the definition of trivial extensions. This result is generalized to the DG setting as follows.

Theorem 3.5 ([30, Theorem 4.1]) *Let A be a DG algebra that is quasiisomorphic to $B \ltimes W$, where B is a homologically bounded local DG algebra, and W is a homologically bounded DG k-module with* $\mathrm{H}(W) \neq 0$*. If M, N are homologically finite DG A-modules with* $\operatorname{Tor}_i^A(M, N) = 0$ *for all $i \gg 0$, then M or N is perfect.*

The proof of Theorem 3.5 is similar to that of [97, Theorem 3.1]. In order to translate *loc. cit.* to the DG setting, a DG version of the important notion of a syzygy was needed. This is the DG module N in the following result which we expect to be useful for other applications.

Proposition 3.6 ([30, Proposition 4.2]) *Let (A, A_+) be a local DG R-algebra. Let M be a homologically finite DG A-module. Then there exists a short exact sequence*

$$0 \to N \xrightarrow{\alpha} F \to \widetilde{M} \to 0$$

of morphisms of DG A-modules such that

(1) $M \simeq \widetilde{M}$*;*
(2) *the underlying graded $A^\natural$-module $F^\natural$ has a finite basis; and*
(3) $\operatorname{Im}(\alpha) \subseteq A_+ \cdot F$*.*

Friendliness and Persistence

An important consequence of Theorem 3.5 is the following result that is a bridge between Ext vanishing over R and its corresponding DG algebra.

Theorem 3.7 ([30, Theorem 6.3]) *Assume there exists a minimal Cohen presentation $\widehat{R} \cong P/I$ such that the minimal free resolution of $\widehat{R}$ over P has the structure of a DG algebra and the k-algebra $A =$ $\operatorname{Tor}^P(\widehat{R}, k)$ is isomorphic to the trivial extension $B \ltimes W$ of a graded k-algebra B by a graded B-module $W \neq 0$ with $B_{\geqslant 1} \cdot W = 0$. Then R is Tor-friendly.*

The proof of this result, which we outline next, relies on Avramov's machine 2.10 whence we also take our notation. To prove Theorem 3.7, one transfers Tor-vanishing over R to Tor-vanishing over the Koszul complex $K = K^R(\mathbf{x})$ by base change. Then using the quasiisomorphisms (2.10.1), one transfers Tor-vanishing over K to Tor-vanishing over A. Since the property of being perfect transfers from A to K, then to R, the DG result Theorem 3.5 gives us the desired conclusion.

Next we sketch the proof of Theorem 3.1. Using standard base-change techniques, one can assume without loss of generality that $R = R'$ and hence, R and Q have a common residue field k. Furthermore, we can assume that Q is complete, k is algebraically closed, and Q does not admit embedded deformation; see [29, Lemma 5.7]. It suffices by Avramov et al. [29, Theorems 2.2 and 6.3] and 3.3 to show that Q is Tor-friendly. Let $\widehat{Q} \cong P/J$ be a minimal Cohen presentation, and let F be a minimal free resolution of $\widehat{Q}$ over P. If Q satisfies one of the conditions (a)–(g) in Theorem 3.1, then F admits a DG-algebra structure as we mentioned in 2.8. For some of these cases, the Tor algebra $\mathrm{Tor}^P(\widehat{Q}, k)$ satisfies the assumptions of Theorem 3.7. Hence, Q is Tor-friendly in those cases by Theorem 3.7. In the remaining cases other methods are used to conclude that Q is Tor-friendly.

Geller [59] and Morra [87] are working to apply Theorem 3.7 to other rings.

4 Bass Series of Local Ring Homomorphisms of Finite Flat Dimension

In this section, let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a local ring homomorphism.

Relations Among Bass Series

Assume in this paragraph that φ is flat. Then many properties of S are controlled by the corresponding properties for R and the closed fibre[1] $S/\mathfrak{m}S$. For instance, S is Gorenstein if and only if R and $S/\mathfrak{m}S$ are both Gorenstein. More generally, the Bass series of S is related to the Bass series for R and $S/\mathfrak{m}S$ by the formula

$$I_S(t) = I_R(t) I_{S/\mathfrak{m}S}(t). \tag{4.0.1}$$

In particular, for each $i \in \mathbb{Z}$, we have $\mu_R^{i+\operatorname{depth} R} \leq \mu_S^{i+\operatorname{depth} S}$. If $S/\mathfrak{m}S$ is Gorenstein, then Grothendieck says that φ is Gorenstein [63, 7.3.1–7.3.2].

When φ is not flat, the properties in the previous paragraph can fail, e.g., for the natural surjection $R \to k$ when R is not regular, i.e., when $\operatorname{pd}_R k$ is not finite. However, Avramov, Foxby, and Lescot [19, 20, 23] recognized that the full strength of flatness is not needed:

Theorem 4.1 ([23, Theorems A, B, C]) *Assume that φ is of finite flat dimension, i.e., the R-module S has a bounded resolution by flat modules. For instance, this holds if $S = R/I$, where I is an ideal of R with finite projective dimension.*

[1] or "fiber," depending on your preference.

(a) *There is a formal Laurent series* $I_\varphi(t)$ *with non-negative integer coefficients such that*

$$I_S(t) = I_R(t) I_\varphi(t). \tag{4.1.1}$$

(b) *For each* $i \in \mathbb{Z}$*, the following inequality holds:*

$$\mu_R^{i+\operatorname{depth} R} \leq \mu_S^{i+\operatorname{depth} S}.$$

(c) *Assume further that the closed fibre* $S/\mathfrak{m}S$ *is artinian, and either* φ *is not flat or* $S/\mathfrak{m}S$ *is not a field. Then the following coefficient-wise inequality holds:*

$$I_S(t) \preccurlyeq I_R(t) \frac{-(1+t) + \sum_{i=0}^{\operatorname{fd}_R(S)} \operatorname{len}_S \left(\operatorname{Tor}_i^R(k,S)\right) t^{-i}}{1 + t - \sum_{i=0}^{\operatorname{fd}_R(S)} \operatorname{len}_S \left(\operatorname{Tor}_i^R(k,S)\right) t^{i+1}}. \tag{4.1.2}$$

Equality in (4.1.2) *holds if and only if* φ *is Golod; see 2.11.*

4.2 Here is some perspective on Theorem 4.1(c). If the closed fibre $S/\mathfrak{m}S$ is artinian, then the following coefficient-wise inequality holds:

$$P_\ell^S(t) \preccurlyeq \frac{P_k^R(t)}{1 + t - \sum_{i=0}^{\operatorname{fd}_R(S)} \operatorname{len}_S \left(\operatorname{Tor}_i^R(k,S)\right) t^{i+1}}. \tag{4.2.1}$$

The ring homomorphism φ is called a *standard Golod homomorphism* if equality holds in (4.2.1).

Assume either φ is not flat or $S/\mathfrak{m}S$ is not a field. Then φ is a Golod homomorphism if and only if it is a standard Golod homomorphism; see Avramov [12]. Hence, in the finite flat dimension setting, Theorem 4.1(c) says that equality in (4.1.2) holds if and only if equality in (4.2.1) holds if and only φ is Golod.

The proof of Theorem 4.1 uses the DG fibre introduced by Avramov [11].

The DG Fibre of φ

Assume that φ is of finite flat dimension. Let $G \xrightarrow{\simeq} k$ and $L \xrightarrow{\simeq} S$ be DG algebra resolutions over R. (Note that the free modules in L will not be finitely generated over R in general.) The *DG fibre* of φ is defined to be the local DG algebra

$$F(\varphi) := G \otimes_R S \simeq G \otimes_R L \simeq k \otimes_R L$$

where the quasiisomorphisms come from the balance property for $\operatorname{Tor}^R(k,S)$. The multiplication on $F(\varphi)$ is inherited from G, S, k, and L. The degree 0 homology module of $F(\varphi)$ is the closed fibre $S/\mathfrak{m}S$. In case that φ is flat, $F(\varphi) \simeq S/\mathfrak{m}S$.

The *Bass series* of φ, denoted $I_\varphi(t)$, is the Bass series $I_{F(\varphi)}(t)$ of the DG algebra $F(\varphi)$, which by Avramov et al. [23, Theorem A] is a formal Laurent series.

In the case where φ is flat, the formulas (4.0.1) and (4.1.1) are the same. In this case, they are a particular instance of the formula

$$I_S^{M\otimes_R S}(t) = I_R^M(t) I_{S/\mathfrak{m}S}(t)$$

where M is finitely generated over S; one verifies this formula using the isomorphism

$$\operatorname{Ext}[S]\ell M \otimes_R S \cong \operatorname{Ext}[R]kM \otimes_k \operatorname{Ext}[S/\mathfrak{m}S]\ell S/\mathfrak{m}S$$

In the general finite flat dimension case, Theorem 4.1(a) follows from a similar isomorphism. The innovative point in [23] that we want to emphasize here is the replacement of the usual closed fibre $S/\mathfrak{m}S$ by the DG fibre $F(\varphi)$.

It is worth noting that Avramov and Foxby [21] established the conclusions of Theorem 4.1 for a larger class of local ring homomorphisms using relative dualizing complexes, but this work does not use DG techniques.

Gorenstein Homomorphisms

As we mentioned above, if φ is flat with Gorenstein closed fibre, then S is Gorenstein if and only if R is Gorenstein. In case φ has finite flat dimension, one should not expect Gorensteinness of the closed fibre to guarantee the same conclusion. In part to remedy this, Avramov and Foxby [19, 20] extend Grothendieck's aforementioned notion of a Gorenstein homomorphism:

The local ring homomorphism φ is called *Gorenstein* if there is an integer a such that for all i we have $\mu_R^i = \mu_S^{i+a}$. In particular, if φ is Gorenstein, then S is Gorenstein if and only if R is Gorenstein. If φ has finite flat dimension, Gorensteinness of φ is equivalent to having the equality $\mu_R^i = \mu_S^{i+\operatorname{depth} S - \operatorname{depth} R}$ for all i by Theorem 4.1(a).

In case that φ is flat, Gorensteinness of φ is equivalent to the Gorensteinness of the closed fibre $S/\mathfrak{m}S$; see [20, (4.2) Proposition]. Hence, this notion of Gorenstein homomorphisms is a generalization of Grothendieck's Gorenstein homomorphisms.

The result *op. cit.* can be extended to the following characterization of Gorenstein homomorphisms in terms of their DG fibres.

Theorem 4.3 ([20, (4.4) Theorem]) *Assume that φ has finite flat dimension. Then φ is Gorenstein if and only if the DG fibre $F(\varphi)$ is a Gorenstein DG algebra (that is, $I_\varphi(t) = t^d$ for some integer d).*

As one might imagine, given the usefulness of the Gorenstein property for local rings, Gorenstein DG algebras have been investigated separately; see Frankild, Iyengar, and Jørgensen [56, 57].

5 Ascent Property of pd-test Modules

In this section, let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local ring homomorphism.

Pd-test Modules

A useful, classical result states that the residue field k has the ability to test for finite projective dimension: a finitely generated R-module N has finite projective dimension if and only if $\operatorname{Tor}_i^R(k, N) = 0$ for $i \gg 0$. According to the following definition, which was coined by O. Celikbas, Dao, and Takahashi [41], this says that k is a pd-test R-module.

A finitely generated R-module M is called a *pd-test module* if for every finitely generated R-module N with $\operatorname{Tor}_i^R(M, N) = 0$ for $i \gg 0$ we have $\operatorname{pd}_R N < \infty$.

It is natural to ask how the pd-test property for a finitely generated R-module M behaves under completion. This is related to the well-known fact that R is regular if and only if $\widehat{R}$ is regular. It is straightforward to show that if $\widehat{M}$ is pd-test over $\widehat{R}$, then M is pd-test over R. That is, the pd-test property descends from the completion. The question of ascent is more subtle. It was posed in [41] and answered by O. Celikbas and Sather-Wagstaff [42] using derived category techniques. The following more general ascent result is proved by Sather-Wagstaff [105].

Theorem 5.1 ([105, Theorem 4.8]) *Assume that the closed fibre $S/\mathfrak{m}S$ of φ is regular and the induced field extension $k \to \ell$ is algebraic. If a finitely generated R-module M is pd-test over R, then $S \otimes_R M$ is a pd-test module over S.*

Theorem 5.1 is proved using the following DG techniques.

Pd-test DG Modules

A homologically finite DG module M over a local DG algebra B is a *pd-test DG module* if every homologically finite DG B-module N with $\operatorname{Tor}_i^B(M, N) = 0$ for all $i \gg 0$ is perfect.

The following result is a special case of a DG version of Theorem 5.1. It plays an essential role in the proof of Theorem 5.1.

Theorem 5.2 ([105, Theorem 4.6]) *Let A be a finite-dimensional DG k-algebra with $A_0 = k$ and $\mathrm{H}_0(A) \neq 0$. Let $k \to \ell$ be an algebraic field extension, and set $B = \ell \otimes_k A$. If M is pd-test over A, then $B \otimes_A M$ is pd-test over B.*

Before applying Theorem 5.2, we sketch its proof. Assume that N is a homologically finite DG B-module such that $\operatorname{Tor}_i^B(B \otimes_A M, N) = 0$ for all $i \gg 0$. In case that $k \to \ell$ is a finite field extension, the assertion follows from a standard argument using 3.4. Now consider the general case, where $k \to \ell$ is algebraic. By truncating an appropriate resolution of N over B one can assume that N is finite-dimensional over ℓ. It then follows that the differential and scalar multiplication on N are represented by matrices consisting of finitely many elements of ℓ. Adjoining these algebraic elements to k, one obtains an intermediate field extension $k \to k' \to \ell$ such that $k \to k'$ is finite. By construction of k', with $A' = k' \otimes_k A$, there is a bounded DG A'-module L such that $N \cong B \otimes_{A'} L$. At this point, the assumption of

$\operatorname{Tor}_i^B(B \otimes_A M, N) = 0$ for all $i \gg 0$ implies that $\operatorname{Tor}_i^{A'}(A' \otimes_A M, L) = 0$ for all $i \gg 0$. Since $k \to k'$ is finite, it follows that L is perfect over A', so $N \cong B \otimes_{A'} L$ is perfect over B.

Outline of the Proof of Theorem 5.1

Assume that M is a pd-test module over R. We need to show that $S \otimes_R M$ is a pd-test module over S. Assume that $\operatorname{Tor}_i^S(S \otimes_R M, N) = 0$ for $i \gg 0$, where N is a finitely generated S-module. Standard techniques reduce to the case where R and S are complete with $S/\mathfrak{m}S = \ell$. Using the notation from 2.10 and applying [22, (1.6) Theorem] we have a minimal Cohen presentation $P' \xrightarrow{\tau'} S$ and a commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ R & \xrightarrow{\varphi} & S \end{array}$$

such that α is flat, τ' is surjective, $P'/\mathfrak{p}P' \cong \ell$, and $S \cong R \otimes_P P'$. The last isomorphism implies that $F' := F \otimes_P P' \xrightarrow{\simeq} S$ is a DG algebra resolution of S over P'. Note that $\varphi(\mathbf{x})$ minimally generates $\mathfrak{n}$. Following the process of 2.10 for the ring S, we get the next commutative diagram of morphisms of DG algebras

$$\begin{array}{ccccccccc} R & \longrightarrow & K^R & \xleftarrow{\cong} & K^P \otimes_P R & \xleftarrow{\simeq} & K^P \otimes_P F & \xrightarrow{\simeq} & k \otimes_P F = A \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & K^S & \xleftarrow{\cong} & K^{P'} \otimes_{P'} S & \xleftarrow{\simeq} & K^{P'} \otimes_{P'} F' & \xrightarrow{\simeq} & \ell \otimes_{P'} F' \end{array}$$

in which $K^R = K^R(\mathbf{x})$, $K^S = K^S(\varphi(\mathbf{x}))$, $K^P = K^P(\mathbf{y})$, and $K^{P'} = K^{P'}(\alpha(\mathbf{y}))$. Note that the DG algebra $\ell \otimes_{P'} F'$ is isomorphic to $\ell \otimes_k A$. Now, the pd-test problem between R and S can be translated through the rows of this diagram to a DG pd-test problem between A and $\ell \otimes_k A$. At this point the assertion follows from Theorem 5.2.

In case that $\ell = k(x)$ is a transcendental extension of k, the same conclusion as in the statement of Theorem 5.1 holds by a result of Tavanfar [114].

6 A Conjecture of Vasconcelos on the Conormal Module

Throughout this section, let I be an ideal of a ring R, and set $S = R/I$.

Ferrand [54] and Vasconcelos [117] show that properties of the ring S are often reflected in the properties of the *conormal module* I/I^2 over S. This section focuses on the following conjecture of Vasconcelos [119].

Conjecture 6.1 ([119, (C_1)]) If $\mathrm{pd}_R\, S$ and $\mathrm{pd}_S\, I/I^2$ are finite, then I is locally generated by a regular sequence.

This conjecture was settled in the affirmative for some special cases by Vasconcelos [120], Gulliksen and Levin [65], and Herzog [68]. The following major progress on this conjecture was made by Avramov and Herzog [25] using André-Quillen homology and DG homological methods.

Theorem 6.2 ([25, Theorem 3]) *Let k be a field of characteristic 0, and assume R is a positively graded polynomial ring over k and I is homogeneous. Then the following are equivalent:*

(i) *S is complete intersection;*
(ii) *I/I^2 is a free S-module;*
(iii) $\mathrm{pd}_S\, I/I^2 < \infty$.

In a recent paper, Briggs [34] establishes Conjecture 6.1 in its full generality.

Theorem 6.3 ([34, Theorem A]) *Conjecture 6.1 holds in general.*

6.4 Briggs' proof for Theorem 6.3 relies on methods pioneered by Avramov and Halperin [11, 24] on homotopy Lie algebras $\pi^*(\varphi)$ arising from DG constructions.

Assume without loss of generality that $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, k)$ are local. Let $\varphi\colon R \to S$ be the natural surjection. Fix a *minimal model* for φ which is a factorization $R \to A \xrightarrow{\simeq} S$, where $(A, \mathfrak{m}_A)$ is a local DG R-algebra such that:

(a) The underlying algebra $A^\natural = R[X_1, X_2, \ldots]$ is the free graded commutative R-algebra, where each X_i is a set of variables of degree i; and
(b) $\partial(\mathfrak{m}_A) \subseteq \mathfrak{m} + \mathfrak{m}_A^2$.

The DG algebra A is also denoted $R\langle X\rangle$.

A graded basis for each $\pi^i(\varphi)$ is dual to X_i, and each element $z \in \pi^2(\varphi)$ corresponds to a derivation $\theta_z\colon A \to \mathfrak{m}_A$ of degree -2 as is described in [14, 34]. Let $\overline{\theta_z}\colon A \to \mathfrak{n}$ be the composition of θ_z and the surjective quasiisomorphism $\mathfrak{m}_A \to \mathfrak{n}$. Under the assumptions of Conjecture 6.1, one can find a certain factorization of $\overline{\theta_z}$ which implies that z is radical in $\pi^2(\varphi)$; see [34, proof of Lemma 2.6 and Theorem 2.7]. Now [24, Theorem C] implies that φ is complete intersection, as desired.

7 A Conjecture of Vasconcelos on Semidualizing Modules

In this section, $(R, \mathfrak{m}, k)$ is a local ring.

Here we discuss a class of modules that are particularly well-suited for creating dualities. They were originally introduced by Foxby [55] who called them *PG modules of rank* 1. They are useful, e.g., for understanding Gorenstein dimensions, in particular, Avramov and Foxby's composition question for local ring homomorphisms of finite G-dimension [21, 106].

Semidualizing Modules

A finitely generated R-module C is called *semidualizing* if the homothety morphism $\chi_C^R\colon R \to \mathrm{Hom}[R]CC$ is an isomorphism and $\mathrm{Ext}[R]iCC = 0$ for all $i \geq 1$. A semidualizing module of finite injective dimension is called a *dualizing module*. Let $\mathfrak{S}_0(R)$ be the set of isomorphism classes of semidualizing R-modules.

This section is centered on the following conjecture posed by Vasconcelos [118].

Conjecture 7.1 ([118, p. 97]) If R is Cohen-Macaulay, then $\mathfrak{S}_0(R)$ is finite.

Note that if R is Ext-persistent, then R satisfies this conjecture. Moreover, in this case, the only semidualizing R-modules are the free module of rank 1 and a dualizing module, if one exists.

Christensen and Sather-Wagstaff [43] answered Conjecture 7.1 in the case where R contains a field. Their proof reduces to the case of a finite-dimensional algebra, then implicitly uses the following technology from geometric representation theory.

7.2 Assume that R is a finite-dimensional k-algebra, where k is algebraically closed. The R-modules of a fixed length r are parametrized by an algebraic variety Mod_r^R. One can define an action of the general linear group GL_r^k on Mod_r^R. The isomorphism class of an R-module M is the orbit $\mathrm{GL}_r^k \cdot M$, and the tangent space $\mathsf{T}_M^{\mathrm{GL}_r^k \cdot M}$ to the orbit $\mathrm{GL}_r^k \cdot M$ at M is identified with a subspace of the tangent space $\mathsf{T}_M^{\mathrm{Mod}_r^R}$. A result of Voigt [122] (see also Brion [37] or Gabriel [58]) provides an isomorphism $\mathrm{Ext}\,1MM \cong \mathsf{T}_M^{\mathrm{Mod}_r^R} / \mathsf{T}_M^{\mathrm{GL}_r^k \cdot M}$. As in work of Happel [66], it follows that if $\mathrm{Ext}\,1MM = 0$ (e.g., if M is a semidualizing module), then the orbit $\mathrm{GL}_r^k \cdot M$ is open in Mod_r^R. Since Mod_r^R is quasi-compact, it can contain only finitely many open orbits, hence, $\mathfrak{S}_0(R)$ is finite.

Using a modification of these ideas, Nasseh and Sather-Wagstaff [93] establish Conjecture 7.1 in total generality with no Cohen-Macaulay hypothesis.

Theorem 7.3 ([93, Theorem A]) *For the local ring R, the set $\mathfrak{S}_0(R)$ is finite.*

A DG Version of Voigt's Theorem and the Proof of Theorem 7.3

To prove Theorem 7.3, we work with the following DG version of semidulazing modules due to Christensen and Sather-Wagtaff [44].

Let A be a homologically degreewise noetherian DG R-algebra. A homologically finite DG A-module C is *semidualizing* if the homothety morphism $\chi_C^A\colon A \to \mathbf{R}\mathrm{Hom}_A(C, C)$ is an isomorphism in the derived category $\mathcal{D}(A)$. If $A = R$, a semidualizing DG R-module C is called a *semidualizing R-complex*. A semidualizing R-complex of finite injective dimension is called a *dualizing complex*. Let $\mathfrak{S}(A)$ denote the set of shift-isomorphism classes of semidualizing DG A-modules in $\mathcal{D}(A)$.

Theorem 7.3 is a consequence of the following result because $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$.

Theorem 7.4 ([93, 4.2 and Theorem A]) *Consider the notation of 2.10. The sets $\mathfrak{S}(A)$ and $\mathfrak{S}(R)$ are finite.*

Using Grothendieck [62, Proposition (0.10.3.1)], we can assume in Theorem 7.4 that R is complete with algebraically closed residue field. Because of Avramov's machine 2.10, it suffices to show that $\mathfrak{S}(A)$ is finite. To establish this finiteness, one uses the following DG version of 7.2 above.

The set of finite-dimensional DG A-modules M with fixed underlying graded k-vector space W is parametrized by an algebraic variety $\mathrm{Mod}^A(W)$. A product $\mathrm{GL}(W)_0$ of general linear groups acts on $\mathrm{Mod}^A(W)$ and the isomorphism class of M is the orbit $\mathrm{GL}(W)_0 \cdot M$ under this action. See [93] for more details.

The DG version of Voigt's result from 7.2 that enables us to prove Theorem 7.4 is the following.

Theorem 7.5 ([93, Theorem B]) *Let W be a finite-dimensional graded k-vector space. Given an element $M \in \mathrm{Mod}^A(W)$, there is an isomorphism*

$$\mathsf{T}_M^{\mathrm{Mod}^A(W)} / \mathsf{T}_M^{\mathrm{GL}(W)_0 \cdot M} \cong \mathrm{YExt}_A^1(M, M)$$

where $\mathrm{YExt}_A^1(M, M)$ denotes the Yoneda Ext group defined as the set of equivalence classes of short exact sequences $0 \to M \to L \to M \to 0$.

As in 7.2, it follows from Theorem 7.5 that if $\mathrm{YExt}_A^1(M, M) = 0$, then the orbit $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^A(W)$. Since $\mathrm{Mod}^A(W)$ is quasi-compact, it follows that there are only finitely many open orbits in it. Thus, it remains to show that a each semidualizing DG A-module C satisfies $\mathrm{YExt}_A^1(C, C) = 0$. This vanishing follows from work of Nasseh and Sather-Wagstaff [92].

One can actually obtain a very tight connection between the sizes of $\mathfrak{S}(A)$ and $\mathfrak{S}(R)$ using a lifting result in [91] that generalizes results of Auslander, Ding, and Solberg [6] and Yoshino [125]. See Nasseh, Ono, and Yoshino [89, 90], Nasseh and Yoshino [98], and Ono and Yoshino [99] for more general lifting results. Also, Altmann and Sather-Wagstaff [1] utilize Avramov's machine to extend results of Gerko [60] from the realm of finite-dimensional algebras to arbitrary local rings.

8 Complete Intersection Maps and the Proxy Small Property

In this section, let $\varphi \colon R \to S$ be a surjective ring homomorphism.

Here, we outline results of Briggs, Iyengar, Letz, and Pollitz [36] on questions motivated by work of Dwyer, Greenlees, and Iyengar [50] and Pollitz [103].

A triangulated subcategory $\mathcal{X}$ of the derived category $\mathcal{D}(R)$ is called *thick* if it is closed under direct summands and satisfies the following two-of-three property: for each exact triangle $L \to M \to N \to$ in $\mathcal{D}(R)$ if two of the objects are in $\mathcal{X}$, then so is the third. The thick subcategory of $\mathcal{D}(R)$ *generated by* an R-complex M is the smallest thick subcategory of $\mathcal{D}(R)$ (with respect to inclusion) that contains M. Note that an R-complex is perfect if and only if it is in the thick subcategory generated by R. If an R-complex N is in the thick subcategory generated by another R-complex M, we say that N is *finitely built from* M.

A triangulated subcategory of $\mathcal{D}(R)$ is called *localizing* if it is closed under arbitrary coproducts. Note that a localizing subcategory is thick. The localizing subcategory of $\mathcal{D}(R)$ *generated by* an R-complex M is the smallest localizing subcategory of $\mathcal{D}(R)$ that contains M. If an R-complex N is in the localizing subcategory generated by another R-complex M, we say that N is *built from* M.

A *small* complex M over a ring R is an R-complex such that $\mathrm{Hom}[\mathcal{D}(R)]M-$ commutes with arbitrary direct sums. Note that the perfect R-complexes are precisely the small R-complexes (or the small objects in $\mathcal{D}(R)$).

In [49], an R-complex M is *proxy small* if there exists a small R-complex N such that N is finitely built from M, and M is built from N. Note that every small R-complex is proxy small. Other examples of proxy small complexes include the residue field of a local ring and modules of finite complete intersection dimension over a local ring.

Let R be a local ring. The famous result of Auslander-Buchsbaum and Serre [4, 110] says that R is regular if and only if every homologically bounded R-complex is small. The paper [50] contains a partial analogue of this statement for complete intersection rings: if R is complete intersection, then every homologically bounded R-complex is proxy small. Pollitz [103] proved the converse of this by showing that if every homologically bounded R-complex is proxy small, then R is complete intersection. Pollitz's proof heavily uses DG methods relying on his version [102] of Avramov and Buchweitz's [17] support varieties over Koszul complexes. Due to space restrictions here, we do not provide further details of this construction.

In the not necessarily local setting, [50] includes a more general statement than the one mentioned above: if φ is complete intersection, then proxy smallness ascends along φ, i.e., any S-complex that is proxy small over R is proxy small over S. Briggs, Iyengar, Letz, and Pollitz [36] prove the following converse of this statement.

Theorem 8.1 ([36, Theorem B]) *Assume that φ has finite projective dimension. If proxy smallness ascends along φ, then φ is complete intersection.*

A consequence of this theorem [36, Corollary 4.1] is another proof of a fundamental result of Avramov [15, (5.7.1) Lemma] used in his solution to Quillen's conjecture discussed in Sect. 9 below. More precisely, if $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ are surjective local homomorphisms such that $\mathrm{fd}_S T < \infty$, then $\psi \circ \varphi$ is complete intersection if and only if φ and ψ are complete intersection.

The proof of Theorem 8.1 reduces to the case where $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local. Set $\widetilde{S} := R/I$, where I is an ideal generated by a maximal R-regular sequence in $\ker\varphi \setminus \mathfrak{m}\ker\varphi$. The surjection $R \to S$ is the composition of the natural surjections $R \xrightarrow{\widetilde{\varphi}} \widetilde{S} \xrightarrow{\dot{\varphi}} S$. To complete the proof it suffices to show that S is small over $\widetilde{S}$; indeed, then [38, Corollary 1.4.7] implies $\varphi = \widetilde{\varphi}$ is complete intersection, as desired.

To show that S is small over $\widetilde{S}$, let $K = K^S(\mathfrak{n})$ be the Koszul complex on a minimal generating set of $\mathfrak{n}$, and consider the restriction $\dot{\varphi}_*\colon \mathcal{D}(S) \to \mathcal{D}(\widetilde{S})$. By Dwyer et al. [50, Remark 5.6], it suffices to prove that $\dot{\varphi}_*(K)$ is a small $\widetilde{S}$-complex. This smallness follows from the next lemma which uses Hochschild cohomology for DG algebras as constructed by Avramov, Iyengar, Lipman, and Nayak [28].

Lemma 8.2 ([36, Lemma 2.5]) *Let A be a DG R-algebra, and let M and N be DG A-modules. Let α be an element of the graded Hochschild cohomology algebra* $\mathrm{HH}^*(A \mid R)$. *If N is (finitely) built from M, then the mapping cone $N//\alpha$ of an induced morphism $N \xrightarrow{\chi_N(\alpha)} \Sigma^{|\alpha|}N$ is (finitely) built from $M//\alpha$. In particular, if M is proxy small then so is $M//\alpha$.*

9 Conjectures of Quillen on André-Quillen Homology

In this section, let $\varphi\colon R \to S$ be a ring homomorphism.

Here, we describe Avramov's solution [15] to a famous conjecture of Quillen [104] and Avramov and Iyengar's significant progress [27] on a second one.

Quillen's Conjectures

The nth *André-Quillen homology* of the R-algebra S with coefficients in an S-module N is $\mathrm{D}_n(S \mid R, N) = \mathrm{H}_n\,(\mathrm{L}(S \mid R) \otimes_S N)$, where $\mathrm{L}(S \mid R)$ is the cotangent complex of φ; see André [2], Iyengar [72], and Quillen [104] for definitions and foundational properties.

The first of Quillen's conjectures that we consider deals with locally complete intersection homomorphisms. This notion was originally defined for maps that are essentially of finite type or flat. Avramov's solution of this conjecture hinges on the following generalization of this notion.

Assume in this paragraph that $\varphi\colon R \to (S, \mathfrak{n})$ is a local ring homomorphism, and let $\grave{\varphi}\colon R \to \widehat{S}$ be the composition of φ with the natural completion map $S \to \widehat{S}$. A *Cohen factorization* of $\grave{\varphi}$ is a factorization into local ring homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ such that $\dot{\varphi}$ is flat with regular closed fibre, φ' is surjective, and R' is complete. If there is a Cohen factorization $R \to R' \xrightarrow{\varphi'} \widehat{S}$ of $\grave{\varphi}$ in which $\ker \varphi'$ is generated by an R'-regular sequence, then φ is called *complete intersection at* $\mathfrak{n}$.

In general, the (not necessarily local) ring homomorphism $\varphi\colon R \to S$ is called *locally complete intersection* if it is complete intersection at all prime ideals $\mathfrak{q}$ of S, i.e., for all such $\mathfrak{q}$, the induced local ring homomorphism $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ is complete intersection at $\mathfrak{q}S_{\mathfrak{q}}$. Also, φ is *locally of finite flat dimension* if $\mathrm{fd}_R\, S_{\mathfrak{q}} < \infty$ for all prime ideals $\mathfrak{q}$ of S. In case that R has finite Krull dimension this condition is equivalent to $\mathrm{fd}_R\, S < \infty$; see Auslander and Buchsbaum [5].

Now we can state the conjectures of Quillen [104] that we are concerned with.

Conjecture 9.1 ([104, (5.6) and (5.7)]) Assume φ is essentially of finite type.

(a) If φ is locally of finite flat dimension and $\mathrm{D}_n(S \,|\, R, -) = 0$ for all $n \gg 0$, then it is locally complete intersection.
(b) If $\mathrm{D}_n(S \,|\, R, -) = 0$ for all $n \gg 0$, then $\mathrm{D}_n(S \,|\, R, -) = 0$ for all $n \geq 3$.

Avramov's Solution of Conjecture 9.1(a) via DG Techniques

Theorem 9.2 ([15, (1.3)]) *Conjecture 9.1(a) holds without the essentially of finite type assumption.*

The proof of Theorem 9.2 reduces to the case where φ is surjective and local. In this case, the proof hinges on the following spectral sequence [15, (4.2) Theorem]

$$^2E_{p,q} = \pi_{p+q}\left(\mathrm{Sym}_q^{\ell}(\Sigma\, \mathrm{L}(S \,|\, R) \otimes_S \ell)\right) \Longrightarrow \ell\langle X\rangle_{p+q}$$

where ℓ is the residue field of S, and the other notation including the DG algebra $\ell\langle X\rangle$ is from 6.4.

Very recently Briggs and Iyengar [35] improved upon Theorem 9.2 with the following. The proof of this result also uses DG technology, but we do not discuss it because of space constraints.

Theorem 9.3 ([35, Theorem A]) *If φ is locally of finite flat dimension and one has* $\mathrm{D}_n(S \,|\, R, -) = 0$ *for some $n \geq 1$, then φ is locally complete intersection.*

Conjecture 9.1(b) for Algebra Retracts

Avramov and Iyengar [27] proved Conjecture 9.1(b) in the case where S is an *algebra retract* of R, that is, where there is a ring homomorphism $\psi\colon S \to R$ such that $\varphi \circ \psi = \mathrm{id}_S$.

Theorem 9.4 ([27, Theorem I]) *Assume that S is an algebra retract of R. Then the following conditions are equivalent.*

(i) $\mathrm{D}_n(S \,|\, R, -) = 0$ *for all $n \gg 0$.*
(ii) $\mathrm{D}_n(S \,|\, R, -) = 0$ *for all $n \geq 3$.*
(iii) $\mathrm{D}_3(S \,|\, R, -) = 0$.
(iv) $\mathrm{D}_n(S \,|\, R, -) = 0$ *for some $n \geq 3$ such that $\lfloor \frac{n-1}{2} \rfloor!$ is invertible in S.*

Conjecture 9.1(b) fails in the non-noetherian case; see André [3] and Planas-Vilanova [101]. This conjecture is still open in general for noetherian rings.

In the proof of Theorem 9.4, the following notion plays an essential role. A local homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is *almost small* if the kernel of the homomorphism $\mathrm{Tor}^{\varphi}(\overline{\varphi}, \ell)\colon \mathrm{Tor}^R(k, \ell) \to \mathrm{Tor}^S(\ell, \ell)$ of graded algebras is generated by elements of degree 1.

DG techniques are crucial in the proof of Theorem 9.4. Key to this is a structure theorem [27, 4.11 Theorem] for surjective almost small homomorphisms in terms of DG algebra homomorphisms. From this one concludes [27, 5.6. Theorem] that almost small homomorphisms have finite *weak category*; a notion motivated by the works of Félix and Halperin [53]. As a result, information on the positivity and growth of deviations of almost small homomorphisms is revealed by Avramov and Iyengar [27, 5.4. Theorem]. The local version of Theorem 9.4 follows from this via a characterization of complete intersection local homomorphisms having finite weak category in terms of the vanishing of the André-Quillen homology with coefficients in the residue field; see [27, 6.4. Theorem]. A reduction to the local case then finishes the proof.

10 Finite Generation of Hochschild Homology Algebras

Throughout this section, let $\varphi\colon R \to S$ be a ring homomorphism.

We discuss work of Avramov and Iyengar [26] on finite generation of Hochschild homology algebras. In it, they prove the converse of the Hochschild-Kostant-Rosenberg Theorem using DG methods and André-Quillen homology; see [40, 72, 86] for definitions and facts that are used in this section.

The Hochschild homology algebra, denoted $\mathrm{HH}_*(S \mid R)$, is a graded commutative algebra defined using shuffle products on the Hochschild complex. This satisfies $\mathrm{HH}_0(S \mid R) = S$, and $\mathrm{HH}_1(S \mid R) = \Omega^1_{S|R}$ is the S-module of Kähler differentials. Recall that the R-algebra S is called *regular* if φ is flat and $S \otimes_R k$ is regular for each homomorphism $R \to k$ from R to a field k. Hochschild, Kostant, and Rosenberg [69] proved that if R is a perfect field and S is *smooth* over R (that is, S is a regular R-algebra and essentially of finite type), then $\mathrm{HH}_*(S \mid R)$ is a finitely generated S-algebra. Here is the aforementioned converse.

Theorem 10.1 ([26, Theorem (5.3)]) *Assume that φ is flat and essentially of finite type. If the S-algebra $\mathrm{HH}_*(S \mid R)$ is finitely generated, then S is smooth over R.*

This result settles a conjecture of Vigué-Poirrier [121] who already established it in the case where $S = R[x_1, \dots, x_n]/I$, and R is a field of characteristic 0, and I is generated by a regular sequence. It was also known for positively graded S such that $S_0 = R$ is a field of characteristic 0 by Dupont and Vigué-Poirrier [48].

The DG techniques used in the proof of Theorem 10.1 are confined to the characteristic-0 case. Here Avramov and Iyengar use a version of Avramov's machine [26, 4.2] which gives a DG algebra A where $\mathrm{H}(A)$ is the Tor algebra $\mathrm{Tor}^R(S, S)$.

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Regularity Bounds by Projection

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Dedicated to Professor David Eisenbud on the occasion of his 75th birthday.

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1 Introduction

Throughout this paper, we work over the field $k = \mathbb{C}$ of complex numbers. A variety is a separated, reduced and irreducible scheme of finite type over $\mathbb{C}$. Curves, surfaces and threefolds are always assumed to be projective and possibly singular.

The notion of Castelnuovo-Mumford regularity can be traced back to Castelnuovo's study on linear systems on a space curve [6] (see Example 2.3). Mumford formally defined the notion of regularity in [35] and applied it to simplify the construction of Quot schemes. Nowadays, Castelnuovo-Mumford regularity has become a fundamental invariant in algebraic geometry, especially in the syzygy theory. There are multiple ways to define Castelnuovo-Mumford regularity from either the geometric point of view or the algebraic one. General references include [11, Section 20.5] [12], [28, Section 1.8], and [35]. In this paper, we follow the geometric approach to define the Castelnuovo-Mumford regularity for coherent sheaves.

Definition 1.1 (Mumford [35]) Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^r$. We say $\mathscr{F}$ is m-regular if

$$H^i(\mathbb{P}^r, \mathscr{F}(m-i)) = 0, \qquad \text{for } i > 0.$$

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The minimal such number, if exits, is called the Castelnuovo-Mumford regularity, or simply regularity, of $\mathscr{F}$ and denoted by reg($\mathscr{F}$). If $X \subseteq \mathbb{P}^r$ is a closed subscheme defined by the ideal sheaf $\mathscr{I}_X$, then the regularity of X is defined to be $\mathrm{reg}(X) = \mathrm{reg}(\mathscr{I}_X)$.

It is clear from the definition that the regularity governs higher cohomology groups of a given sheaf. As an extremal case, one sees that the support of a coherent sheaf $\mathscr{F}$ has dimension zero if and only if $\mathrm{reg}\,\mathscr{F} = -\infty$. Let $X \subseteq \mathbb{P}^r$ be a projective subscheme defined by an ideal sheaf $\mathscr{I}_X$. One may attempt to define the regularity of X by the regularity $\mathrm{reg}(\mathscr{O}_X)$ of the structure sheaf $\mathscr{O}_X$ considered as an $\mathscr{O}_{\mathbb{P}^r}$-module. However, this number is weaker than $\mathrm{reg}(\mathscr{I}_X)$ because from the short exact sequence $0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_X \to 0$ we can only have $\mathrm{reg}\,\mathscr{O}_X \leq \mathrm{reg}\,\mathscr{I}_X - 1$ and in general $\mathrm{reg}\,\mathscr{I}_X$ cannot be bounded above by $\mathrm{reg}\,\mathscr{O}_X$. For instance if $X = \{x\}$ is a closed point in $\mathbb{P}^r$, then $\mathrm{reg}\,\mathscr{O}_X = -\infty$ while $\mathrm{reg}\,\mathscr{I}_X = 1$. But if X is k-normal, i.e., the restriction morphism $H^0(\mathscr{O}_{\mathbb{P}^r}(k)) \to H^0(\mathscr{O}_X(k))$ is surjective, and $\mathscr{O}_X$ is k-regular, then we can have X is $(k+1)$-regular.

The regularity of X governs the degrees of defining equations of the variety. Indeed, if X is m-regular, i.e., $\mathrm{reg}(X) \leq m$, then by the Castelnuovo's result [35, Lecture 14], we see that $\mathscr{I}_X(m)$ is globally generated, which means that X can be defined by several homogeneous polynomials of degree no more than m. An easy example is that X is 1-regular if and only if X is a linear space. Conversely, the regularity can also be bounded above by the degrees of defining equations. There is an amazing general result says that for any ideal sheaf $\mathscr{I}$ on a projective space $\mathbb{P}^r$, if $\mathscr{I}(d)$ is globally generated, then there is a doubly exponential upper bound of the form $(2d)^{2^{r-1}}$ for the regularity of $\mathscr{I}$ (see [14, 16] for characteristic zero, and see [7] for characteristic free). An example constructed by Mayr-Meyer [33] shows that such bound can actually be achieved. A slightly weaker upper bound of the form $(2d)^{r!}$ is proved by Bayer-Mumford [5] by the cohomological method. Certainly, with more information on the geometry or algebra of the scheme defined by the ideal sheaf, one would expect a better bound, even a linear bound in terms of the degree of defining equations. Typical example is when X is a complete intersection so its ideal sheaf has a Koszul resolution, from which one can obtain a linear regularity upper bound. A striking result of Betram-Ein-Lazarsfeld [3] establishes such linear bound for any nonsingular variety: if $X \subseteq \mathbb{P}^r$ is a nonsingular projective variety of codimension c defined by the equations of degree $d_1 \geq d_2 \geq \cdots \geq d_t$ $(t \geq c)$, then $\mathrm{reg}\,X \leq d_1 + d_2 + \cdots + d_c - c + 1$. The equality holds if and only if X is a complete intersection. This result has inspired research including [8, 9, 36] to seek singular varieties for which the linear bound still holds.

On the other hand, rather than using the degrees of defining equations, classical results of Castelnuovo for integral curves, completed by Gruson-Lazarsfeld-Peskine [17], suggest another form of a linear optimal regularity bound involving geometric invariants of the variety concerned. This bound was further conjectured by Eisenbud-Goto [10] for arbitrary varieties as follows.

Eisenbud-Goto Conjecture *Let $X \subseteq \mathbb{P}^r$ be a nondegenerate projective variety. Then*

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1.$$

In addition to the aforementioned work of [17] for integral curves, the conjectured bound has also been established for connected curves [15] (see also [38]), for nonsingular surfaces [27, 44], for normal surfaces with certain singularities [37, 42], and for smooth threefolds in $\mathbb{P}^r$ with $r \geq 9$ [45] and in $\mathbb{P}^5$ [26]. Slightly weaker bounds were obtained for lower dimensional smooth varieties in [24, 25], for threefolds with rational or Du Bois singularities in [42], and for scrolls over curves in [4, 43].

A breakthrough in recent research on the conjecture is the work of McCullough-Peeva [34], in which they construct some singular varieties to show the conjecture does not hold. According to the known results mentioned above, however, the bound in the conjecture still attracts considerable attention and serves as a guidance in finding optimal regularity bounds for varieties, especially for nonsingular varieties or varieties with mild singularities.

One important method in establishing regularity bounds is the generic projection, on which we focus our discussion in this paper. In his influential paper on surfaces [27], Lazarsfeld set the cohomological framework of projection method in obtaining regularity bound. This method is originated in Casteulnuovo's work and developed by many people including Pinkham [44] and Szpiro [50]. After that, Kwak extended the projection method to some low dimensional nonsingular varieties and gave a very nice summary on the method [24]. Here we will slightly generalize the setup from the classic nonsingular case to the Cohen-Macaulay case, which may be useful in the future.

The paper is organized as follows. In Sect. 2, we give the details of the construction of general projection and related it to bounding the regularity of a projective variety. In Sect. 3, we discuss the double-point formula of projection as well as the complexity of fibers of projection. In the last section, we list several regularity bounds obtained by the projection method and also discuss some other important regularity bounds.

2 Construction of Projection

In this section, we give the details how one can use projection method to obtain a regularity bound. We follow the approach used by Lazarsfeld in [27] and further refined by Kwak in his several work [24, 25]. It has become rather standard to use projection in the study of the regularity of smooth varieties, as summarized by Greenberg and Kwak. The major obstruction in this method comes from the complexity of fibers of projection. Here, we generalize the setup of this method to the case of Cohen-Macaulay varieties. We have to replace the cohomological computation due to Lazarsfeld for the smooth case by the duality argument introduced by Ein in his seminar talk to fit the Cohen-Macaulay case. Of course, one still needs to understand the complexity of fibers of projection to eventually obtain any meaningful result.

Let $X \subset \mathbb{P}^r$ be a nondegenerate closed subvariety of dimension $n \geq 1$ and degree d. So X is not contained in any hyperplane and $d \geq \operatorname{codim} X + 1$. We always assume that X is Cohen-Macaulay and therefore it has a dualizing sheaf ω_X. To define a projection from X to a linear space, we choose a linear subspace Λ in $\mathbb{P}^r$ of codimension $m \geq 1$ as the projection center such that $\Lambda \cap X = \emptyset$ (so $\dim \Lambda < \operatorname{codim} X$). Suppose that Λ is defined by the independent linear forms $l_0, l_1, \cdots, l_{m-1}$. If $\mathscr{I}_\Lambda$ is the ideal sheaf of Λ, then we see that the vector space $W = \langle l_0, \cdots, l_{m-1} \rangle$ spanned by those forms are exactly the space $H^0(\mathscr{I}_\Lambda(1))$ (in this paper, we write the chomology group $H^i(X, \mathscr{F})$ by dropping the underlying space X as $H^i(\mathscr{F})$ to save the space). So there is a surjective evaluation morphism

$$e : W \otimes \mathscr{O}_{\mathbb{P}^r}(-1) \longrightarrow \mathscr{I}_\Lambda.$$

In addition, we have a decomposition of the vector space of linear forms

$$H^0(\mathscr{O}_{\mathbb{P}^r}(1)) = W \oplus V, \text{ where } V = H^0(\mathscr{O}_\Lambda(1)).$$

Let M be the blowing-up of $\mathbb{P}^r$ along the center Λ equipped with the natural morphism $p : M \to \mathbb{P}^r$. The evaluation morphism e above yields an embedding

$$M \stackrel{i}{\hookrightarrow} \mathbb{P}(W \otimes \mathscr{O}_{\mathbb{P}^r}(-1)) \cong \mathbb{P}^r \times \mathbb{P}^{m-1},$$

where we identify $\mathbb{P}(W) = \mathbb{P}^{m-1}$. So the morphism p is the composition of the embedding i with the projection to $\mathbb{P}^r$. Let q be the composition of i with the projection to $\mathbb{P}^{m-1}$. We then form the following diagram

$$\begin{array}{ccc} M = \mathrm{Bl}_\Lambda \mathbb{P}^r & \xrightarrow{q} & \mathbb{P}^{m-1} \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}^r. & & \end{array} \tag{2.0.1}$$

Let E be the exceptional divisor of M. The calculation on tautological bundles shows that

$$\mathscr{O}_M(-E) = p^*\mathscr{O}_{\mathbb{P}^r}(-1) \otimes q^*\mathscr{O}_{\mathbb{P}^{m-1}}(1).$$

Since we have chosen $\Lambda \cap X = \emptyset$, X can be viewed in a natural way as a subvariety of the blowing-up M. The projection f of X from Λ to $\mathbb{P}^{m-1}$ is defined to be the restriction of q to X, i.e.,

$$f = q|_X : X \longrightarrow \mathbb{P}^{m-1}.$$

We may also write the projection as f_Λ if we need to emphasize the projection center Λ. The image of f is denoted by $\bar{X} = f(X)$. The crucial point in many computation is that the morphism q turns the blowing-up M to be a projectivized vector bundle over $\mathbb{P}^{m-1}$ as described in the following proposition.

Proposition 2.1 *There is a locally free sheaf*

$$\mathscr{E} = V \otimes \mathscr{O}_{\mathbb{P}^{m-1}} \oplus \mathscr{O}_{\mathbb{P}^{m-1}}(1)$$

of rank $r - m + 2$ such that $M \cong \mathbb{P}(\mathscr{E})$ with the tautological bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = p^*\mathscr{O}_{\mathbb{P}^r}(1)$. For any $k \geq 0$, one has an identification of vector bundles on $\mathbb{P}^{m-1}$,

$$q_* p^*(\mathscr{O}_{\mathbb{P}^r}(k)) = \bigoplus_{i=0}^{k} S^{k-i} V \otimes \mathscr{O}_{\mathbb{P}^{m-1}}(i).$$

Proof First of all, we claim that the morphism $q : M \to \mathbb{P}^{m-1}$ is a smooth morphism with fibers identical to $\mathbb{P}^{r-m+1}$. To see this, we use a representation $\wedge^2 W \otimes \mathscr{O}_{\mathbb{P}^r}(-2) \to W \otimes \mathscr{O}_{\mathbb{P}^r}(-1)$ of $\mathscr{I}_\Lambda$ truncated from the associated Koszul resolution of $\mathscr{I}_\Lambda$. From it, we can deduce a surjective morphism

$$\wedge^2 W \otimes p^*\mathscr{O}_{\mathbb{P}^r}(-1) \otimes q^*\mathscr{O}_{\mathbb{P}^{m-1}}(-1) \longrightarrow \mathscr{I}_M \longrightarrow 0 \tag{2.1.1}$$

where $\mathscr{I}_M$ is the ideal sheaf of M in $\mathbb{P}^r \times \mathbb{P}^{m-1}$. Let $y \in \mathbb{P}^{m-1}$ be a closed point and assume that, without loss of generality, it is defined by the forms $l_1, \cdots, l_{m-1}$. Let $\mathbb{P}^r_y$ be the fiber of the projection $\mathbb{P}^r \times \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ over the point y. Restricting the morphism in (2.1.1) onto $\mathbb{P}^r_y$, we deduce $\wedge^2 W \otimes \mathscr{O}_{\mathbb{P}^r_y}(-1) \longrightarrow \mathscr{I}_M \cdot \mathscr{O}_{\mathbb{P}^r_y} \longrightarrow 0$. Note that for any form $l_i \wedge l_j$ in $\wedge^2 W$, it maps to either 0 if $i \neq 0$ and $j \neq 0$, or l_j if $i = 0$ in the group $H^0(\mathscr{I}_M \cdot \mathscr{O}_{\mathbb{P}^r_y}(1))$. Thus in the space $\mathbb{P}^r_y$, the fiber M_y of M over y is defined by the linear forms $l_1, \cdots, l_{m-1}$, which proves the claim.

Now push down by q the short exact sequence $0 \to \mathscr{O}_M \to \mathscr{O}_M(E) \to \mathscr{O}_E(E) \to 0$ to yield a short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{m-1}} \longrightarrow \mathscr{F} \longrightarrow q_*\mathscr{O}_E(E) \longrightarrow 0.$$

As $E \cong \mathbb{P}(\mathscr{I}_\Lambda/\mathscr{I}_\Lambda^2) \cong \mathbb{P}(W \otimes \mathscr{O}_\Lambda(-1))$, it is easy to calculate that $q_*\mathscr{O}_E(E) = V \otimes \mathscr{O}_{\mathbb{P}^{m-1}}(-1)$. Since $\mathrm{Ext}^1(V \otimes \mathscr{O}_{\mathbb{P}^{m-1}}(-1), \mathscr{O}_{\mathbb{P}^{m-1}}) = 0$, the above short exact sequence splits and therefore $\mathscr{F} = \mathscr{O}_{\mathbb{P}^{m-1}} \oplus V \otimes \mathscr{O}_{\mathbb{P}^{m-1}}(-1)$. Now we can verify by a standard approach that $M \cong \mathbb{P}(\mathscr{F})$. Of course, twisting $\mathscr{F}$ to yield $\mathscr{E}$, one still has $M \cong \mathbb{P}(\mathscr{E})$ but gets the desired form of the tautological bundle as claimed in the proposition. $\square$

We look into the fibers appeared in the above construction. For a closed point $y \in \mathbb{P}^{m-1}$, write $L_y = q^{-1}(y)$ as the fiber of q over y. Similarly, we write $X_y = f^{-1}(y)$ the fiber of f over y (note that if y is not in the image $f(X)$ then X_y is empty). By the proposition above, we know that $L_y \cong \mathbb{P}(\mathscr{E} \otimes k(y))$ is the projective space $\mathbb{P}^{r-m+1}$ and $H^0(\mathscr{O}_{L_y}(k)) = S^k \mathscr{E} \otimes k(y)$, where $S^k \mathscr{E}$ means the k-th symmetric product of $\mathscr{E}$. Without loss of generality, let us assume that y is cut out by the linear forms $l_1, , \cdots , l_{m-1}$. Thinking of y as a dimension zero linear space, the skyscript sheaf $k(y)(k)$ has only one global section l_0^k. This suggests that we can formally write

$$\mathscr{O}_{\mathbb{P}^{m-1}}(k) \otimes k(y) \cong \mathbb{C} \cdot l_0^k.$$

Now the fiber L_y is also cut out by the forms $l_1, l_2, \cdots , l_{m-1}$ in the space $\mathbb{P}^r$. The base change suggests that

$$H^0(\mathscr{O}_{L_y}(k)) = S^k \mathscr{E} \otimes k(y) = S^k V \oplus S^{k-1} V \otimes \mathbb{C} \cdot l_0 \oplus \cdots \oplus \mathbb{C} \cdot l_0^k.$$

Also note that X_y is indeed the scheme-theoretical intersection of X with L_y. In addition, the projection center Λ is a hyperplane in the fiber L_y defined by the linear form l_0. As $X \cap \Lambda$ is empty, so is $X_y \cap \Lambda$ and therefore X_y must have dimension zero. Hence the projection f is a finite morphism.

Certainly, different choices of the projection center Λ would yield different projection. All such choice of the center can be parameterized by an open set of the Grassmannian variety $\mathbb{G}(r - m, r)$ (containing all linear space Λ of codimension m in $\mathbb{P}^r$ which does not touch X). In application, we would expect certain property $\mathscr{P}$ could be a general phenomenon appeared in the most choice of the projection. To be more precise, by saying that a general, or generic, projection f satisfies the property $\mathscr{P}$ we mean that there exists an open set U in the Grassmannian $\mathbb{G}(r - m, r)$ such that for all $\Lambda \in U$ the projection of X from Λ satisfies the property $\mathscr{P}$. Sometimes, the property $\mathscr{P}$ is clear from the context so we would not state it all the time. As an example, we show that a general projection is a birational map to its image if $m \geq n+2$. Indeed, let $x \in X$ be a nonsingular point of X. Let S_x be the variety swept out by secant lines passing through the point x. We see that $\dim S_x = \dim X+1 < r$. We can choose a general Λ to avoid both S_x and the tangent space T_x of X at x. Then the projection f from Λ is injective and unramified at x which imlies that it is isomorphic at x. Hence it is a birational map to its image.

The most important case in application is to project X to a hyersurface, i.e., when $m = n + 2$. We will mainly focus on this case in the rest of the paper and therefore we always assume $m = n + 2$ unless stated otherwise. For a general projection $f : X \to \mathbb{P}^{n+1}$, we have seen that f is a finite birational map. The image $\bar{X}$ is therefore a degree d hypersurface. To bound the regularity of X, the cohomological method used in [27] is to trace the natural morphism $\mathscr{O}_{\mathbb{P}^r}(k) \to \mathscr{O}_X(k)$ through the projection for certain $k \geq 0$. Indeed, pulling back by p and then pushing down by q, we obtain the morphism $q_* p^*(\mathscr{O}_{\mathbb{P}^r}(k)) \to f_* \mathscr{O}_X(k)$. As we have calculated, the sheaf $q_* p^*(\mathscr{O}_{\mathbb{P}^r}(k)) = \bigoplus_{i=0}^k S^{k-i} V \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(i)$ is actually a locally free sheaf.

Instead of taking the whole sheaf $q_* p^*(\mathscr{O}_{\mathbb{P}^r}(k))$ as in [27], Greenberg and Kwak refined Lazarsfeld's method to take certain subspaces $V_{k-i} \subseteq S^{k-i}V$ and to consider the induced morphism

$$w_k : \bigoplus_{i=0}^{k} V_{k-i} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(i) \longrightarrow f_* \mathscr{O}_X(k), \tag{2.1.2}$$

where $V_0 = S^0 V = \mathbb{C}$. Note that $f_* \mathscr{O}_X(k)$ is a Cohen-Macaulay $\mathscr{O}_{\mathbb{P}^{n+1}}$-module of codimension one, as we assume that X is Cohen-Macaulay. Thus if w_k is surjective, the kernel sheaf would be locally free by the Auslander–Buchsbaum formula. Furthermore, the duality theory (see for example [1]) gives

$$f_* \omega_X = \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{P}^{n+1}}}(f_* \mathscr{O}_X, \omega_{\mathbb{P}^{n+1}}).$$

The subjectivity of w_k plays a critical role in obtaining regularity bounds and if so we can get the following proposition.

Proposition 2.2 *Let $X \subset \mathbb{P}^r$ be a nondegenerate closed subvariety of dimension $n \geq 1$. Assume that X is Cohen-Macaulay and has the Kodaira vanishing property, i.e., $H^i(\omega_X(l)) = 0$ for $i > 0$ and $l > 0$. Assume also that the morphism w_k defined in (2.1.2) is surjective. Twist w_k by $\mathscr{O}_{\mathbb{P}^{n+1}}(-k)$ and let E be the kernel sheaf of the resulting morphism. So we obtain a short exact sequence*

$$0 \longrightarrow E \longrightarrow \bigoplus_{i=0}^{k} V_{k-i} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(i-k) \longrightarrow f_* \mathscr{O}_X \longrightarrow 0. \tag{2.2.1}$$

Then one has

(1) $\operatorname{reg} E^* \leq -2$ *and*

$$\operatorname{reg} X \leq \deg X - \dim V_1 + \sum_{i=3}^{k} (i-2) \dim V_i.$$

(2) E^* *is* (-3)*-regular if and only if the following three conditions hold*

(i) $H^1(\mathscr{O}_X) = 0$.
(ii) *the natural morphism* $V_1 \oplus H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1)) \longrightarrow H^0(\mathscr{O}_X(1))$ *is surjective.*
(iii) *the natural morphism* $H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(2)) \oplus V_1 \otimes H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1)) \oplus V_2 \longrightarrow H^0(\mathscr{O}_X(2))$ *is injective.*

In this case,

$$\operatorname{reg} X \leq \deg X - 2\dim V_1 - \dim V_2 + \sum_{i=4}^{k} (i-3) \dim V_i.$$

Proof Applying the functor $\mathscr{H}om(\ ,\omega_{\mathbb{P}^{n+1}})$ to the short exact sequence (2.2.1) and then tensoring with $\omega^*_{\mathbb{P}^{n+1}}$ yields

$$0 \longrightarrow \bigoplus_{i=0}^{k} V^*_{k-i} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(k-i) \longrightarrow E^* \longrightarrow f_*\omega_X(n+2) \longrightarrow 0. \qquad (2.2.2)$$

We show E^* is (-2)-regular first. Note that for $0 < j < n$, we have $H^j(\mathscr{O}_{\mathbb{P}^{n+1}}(k-i-2-j)) = H^j(\omega_X(n-j)) = 0$ which implies $H^j(E^*(-2-j)) = 0$. Twist the short exact sequence (2.2.2) by $(-2-n)$ and consider the associated long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(E^*(-2-n)) & \longrightarrow & H^n(\omega_X) & \xrightarrow{\tau_n} & \bigoplus_{i=0}^k V^*_{k-i}\otimes H^{n+1}(\mathscr{O}_{\mathbb{P}^{n+1}}(k-i-2-n)) & \longrightarrow & \cdots \\ & & & & \| & & \| & & \\ & & & & H^0(\mathscr{O}_X)^* & \longrightarrow & H^0(\mathscr{O}_{\mathbb{P}^{n+1}})^*. & & \end{array}$$

Thus the connection map τ_n is injective and therefore $H^n(E^*(-2-n)) = 0$. Similarly, tensor the short exact sequence (2.2.2) by $(-3-n)$ and consider the associated long exact sequence

$$\begin{array}{ccccccc} H^n(\omega_X(-1)) & \xrightarrow{\eta_n} & \bigoplus_{i=0}^k V^*_{k-i}\otimes H^{n+1}(\mathscr{O}_{\mathbb{P}^{n+1}}(k-i-3-n)) & \longrightarrow & H^{n+1}(E^*(-3-n)) & \longrightarrow & 0 \\ \| & & \| & & & & \\ H^0(\mathscr{O}_X(1))^* & \xrightarrow{\eta_n^*} & H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1))^* \oplus V_1^*\otimes H^0(\mathscr{O}_{\mathbb{P}^{n+1}})^*. & & & & \end{array}$$

Since X is nondegenerate in $\mathbb{P}^r$, the restriction morphism $H^0(\mathscr{O}_{\mathbb{P}^r}(1)) \to H^0(\mathscr{O}_X(1))$ is injective. By the choice of V_1, we see that the space $H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1)) \oplus V_1 \otimes H^0(\mathscr{O}_{\mathbb{P}^{n+1}})$ is a subspace of $H^0(\mathscr{O}_{\mathbb{P}^r}(1))$. Hence the induced restriction morphism $H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(1)) \oplus V_1 \otimes H^0(\mathscr{O}_{\mathbb{P}^{n+1}}) \to H^0(\mathscr{O}_X(1))$ is also injective. This implies that its dual morphism η_n^* is surjective. As a consequence, we obtain $H^{n+1}(E^*(-3-n)) = 0$. So we conclude that E^* is (-2)-regular.

Next, we show E is (-3)-regular. We follow the same approach as above. So we see that $H^{n-1}(E^*(-2-n)) = 0$ if and only if the condition (i) holds. $H^n(E^*(-3-n)) = 0$ if and only if the condition (ii) holds. And $H^{n+1}(E^*(-4-n)) = 0$ if and only if the condition (iii) holds.

Finally, we calculate the regularity of X. From the short exact sequence (2.2.1), we see that X is $(\operatorname{reg} E - 1)$-normal and $\mathscr{O}_X$ is $(\operatorname{reg} E - 1)$-regular which imply that $\operatorname{reg} X \le \operatorname{reg} E$. So it is enough to bound $\operatorname{reg} E$. Since $E = \wedge^{\operatorname{rank} E - 1} E^* \otimes \det E$, we have $\operatorname{reg} E \le (\operatorname{rank} E - 1)\operatorname{reg} E^* - c_1(E)$. It is easy to calculate that $\operatorname{rank} E = \sum_{i=0}^k \dim V_{k-i}$ and

$$c_1(E) = -d + c_1\Big(\bigoplus_{i=0}^{k} V_{k-i} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(i-k)\Big) = -d + \sum_{i=0}^{k}(i-k)\dim V_{k-i},$$

where $d = \deg X$. So we obtain reg $E \leq d -$reg $E^* + \sum_{i=0}^{k}(k-i+$reg $E^*) \dim V_{k-i}$. Hence if E^* is (-2)-regular, we obtain the bound in (1). If E^* is (-3)-regular, we obtain the bound in (2). □

In order to apply the above proposition, one needs to choose appropriate subspaces V_i from $S^i V$ (we always choose $V_0 \cong \mathbb{C}$) and then show the morphism w_k is surjective. All these depend on the complexity of fibers of the projection f. Indeed, for any $y \in \bar{X}$, tensoring the residue field $k(y)$ with the morphism w_k, we obtain

$$w_{k,y} : V_k \oplus V_{k-1} \otimes \mathbb{C} \cdot l_0 \oplus \cdots \oplus \mathbb{C} \cdot l_0^k \longrightarrow \mathscr{O}_{X_y}(k)$$

The domain of $w_{k,y}$ is a subspace of $H^0(\mathscr{O}_{L_y}(k))$. If we choose $V_i = S^i V$, then the domain of $w_{k,y}$ is exactly the space $H^0(\mathscr{O}_{L_y}(k))$. To get the subjectivity of w_k, one needs $w_{k,y}$ is surjective for all $y \in \bar{X}$, by base change. This in turn means that one needs to know the regularity of the fiber X_y in the linear space L_y.

Example 2.3 (Regularity of Space Curves, a Theorem of Castelnuovo) In this example, we use projection method to prove a classical result of Castelnuovo on the regularity of space curve. Details can be found for instance in [50]. It has been extended to arbitrary integral projective curves in the celebrated work [17] by Gruson-Lazarsfeld-Peskine. Here we state the theorem by using the language of Castelnuovo-Mumford regularity.

Castelnuovo's Theorem *Let $X \subset \mathbb{P}^3$ be a nondegenerate nonsingular space curve, then* reg $X \leq \deg X - 1$.

To prove it, let us project X from a general point Λ in $\mathbb{P}^3$ to $\mathbb{P}^2$ to yield a projection morphism $f : X \to \mathbb{P}^2$. As we already knew, the morphism f is always finite and birational to its image. To apply Proposition 2.2, we need to determine a positive integer k such that the morphism

$$w_k : q_* p^* \mathscr{O}_{\mathbb{P}^3}(k) \longrightarrow f_* \mathscr{O}_X(k)$$

is surjective. Here we simply choose the vectors space $V_i = S^i V$. Let y be a closed point in the image of f. By base change, the subjectivity of w_k is equivalent to the subjectivity of the morphism

$$w_{k,y} : \mathscr{O}_{L_y}(k) \longrightarrow \mathscr{O}_{X_y}(k)$$

for all y, where L_y and X_y are fibers of q and f over y respectively. We need to use the following classical geometry of space curves, which can be found in [19]. It will give us more information about the projection f.

(1) The tangent variety of X, which is a variety swept out by the tangent lines of X, has dimension 2.

(2) The trisecant lines of X, i.e., the lines intersecting X more than 3 points, do not fill up the space $\mathbb{P}^3$.

So we can arrange the center Λ such that it is not contained in the tangent variety and it does not lie in any trisecant line. As a consequence, f maps all tangent lines of X to lines in $\mathbb{P}^2$ and it does not map any three points of X to a point in $\mathbb{P}^2$. Each fiber of f is either a reduced point or consists of two reduced points. Hence for any closed point $y \in \mathbb{P}^2$, the morphism on the fibers over y

$$w_{1,y} : \mathcal{O}_{L_y}(1) \longrightarrow \mathcal{O}_{X_y}(1)$$

is surjective. So by base change, we obtain that the morphism

$$w_1 : q_* p^* \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow f_* \mathcal{O}_X(1)$$

is surjective. Finally we apply Proposition 2.2 with the choice of $V_1 = V$ and $k = 1$ to obtain that reg $X \leq \deg X - 1$.

From the proof, we can apply this method to a nondegenerate nonsingular projective curve in any space $\mathbb{P}^r$. Two stated geometric results (1) and (2) still work in this case. For arbitrary integral curve in $\mathbb{P}^r$, the situation is complicated.

Remark 2.4 It is interesting to see how the idea of projection grow up in literature. This remark comes from J. Park's personal note in which he sorts out how the projection method was developed from the original Castelnuovo's work through the work of Pinkham, Szpiro and Lazarsfeld. We keep use the notation in Example 2.3 and write $d = \deg X$.

Castelnuovo's original proof is to show that X is $(d-2)$-normal. Still use a general projection $f : X \to \mathbb{P}^2$ with the image $\bar{X}$ a degree d curve. Let D be the non-isomorphic locus of f and let $\bar{D} = f(D)$ which is a Weil divisor on $\bar{X}$ (D and $\bar{D}$ will be discussed in the next section). Then to show X is $(d-2)$-normal it is enough to show D is $(d-2)$-normal. On the other hand, it is easy to check that $\bar{D}$ is $(d-3)$-normal in $\mathbb{P}^2$. Write $\bar{D} = \sum p_i$ as the sum of distinct points. By the trisecant lemma, every $f^{-1}(p_i)$ consists of two points q_i and r_i on X. So for p_i, one can choose a degree $d-3$ hypersurface in $\mathbb{P}^2$ passing through all points in $\bar{D}$ except of p_i. Lifting this hypersurface to $\mathbb{P}^3$ and adding a hyperplane passing through q_i but not r_i to create a degree $(d-2)$ hypersurface passing all points in D except of q_i. In this way, one shows that D is $(d-2)$-normal. This idea was also used by Pinkham in [44] for nonsingular surfaces.

The above argument has a cohomological interpretation by Szpiro [50]. By the trisecant lemma again, the morphism $\mathcal{O}_{\bar{X}(-1)} \oplus \mathcal{O}_{\bar{X}} \to f_* \mathcal{O}_X$ is surjective. This further implies the induced morphism on global sections $H^0(\mathcal{O}_{\bar{X}}(d-3) \oplus \mathcal{O}_{\bar{X}}(d-2)) \to H^0(\mathcal{O}_X(d-2))$ is surjective, which means X is $(d-2)$-normal. This argument fails for higher dimensional varieties. The approach by Lazarsfeld, which we follow in this paper, is to consider the morphism w_k of (2.1.2) and its kernel bundle. The advantage is that one can control both the normality and the regularity of X at the same time.

3 Complexity of Fibers of Projections

As discussed in the previous section, in order to show the surjectivity of the morphism w_k, one needs to show the surjectivity of the morphism $w_{k,y}$ on fibers for each point $y \in \mathbb{P}^{n+1}$. It is determined by the complexity of fibers, which we shall discuss in this section. In addition, we will also discuss the double-point formula associated to a projection. The application of the formula leads to several interesting regularity bounds.

We keep using the same notation as in the previous section. Recall that $X \subseteq \mathbb{P}^r$ is a Cohen-Macaulay closed subvariety of dimension $n \geq 1$ and degree d, and $f : X \to \mathbb{P}^{n+1}$ is a general projection from the center Λ. Write $\bar{X}$ to be the image of the projection f, which is a degree d hypersurface in $\mathbb{P}^{n+1}$. For the general choice of Λ, we have seen that f is a finite birational morphism to $\bar{X}$. Certainly, the singularities of $\bar{X}$ is closely related to the singularities of X. If X is nonsingular, which we assume for the most part of this section, X is the normalization of $\bar{X}$. There is a natural inclusion of the sheaves $\mathscr{O}_{\bar{X}} \hookrightarrow f_*\mathscr{O}_X$. The conductor ideal of $\mathscr{O}_{\bar{X}}$ in $\mathscr{O}_X$ is defined by

$$\mathfrak{C} := \mathscr{H}om(f_*\mathscr{O}_X, \mathscr{O}_Y) \simeq \operatorname{ann}(f_*\mathscr{O}_X/\mathscr{O}_Y).$$

It is an ideal sheaf in both $\mathscr{O}_X$ and $\mathscr{O}_{\bar{X}}$ with the property that $\mathfrak{C} \cdot \mathscr{O}_X = \mathfrak{C} \subseteq \mathscr{O}_X$ and $f_*\mathfrak{C} = \mathfrak{C}$. The *double locus* of $\bar{X}$ is the subscheme D of $\bar{X}$ defined by the ideal $\mathfrak{C}$, and the *double locus* of f is $\Delta = \pi^{-1}(D)$ defined by the ideal $\mathfrak{C} \cdot \mathscr{O}_X$. It is clear that as sets D is the same as the singular locus of $\bar{X}$ since it has a nonsingular normalization X.

The conduct ideal and the double locus associated to a projection have been studied for a long time. The subscheme D of $\bar{X}$ is a Cohen-Macaulay subscheme of pure codimension one [47]. The hypersurface $\bar{X}$ is seminormal and the schemes D and Δ are reduced [18, Theorem 3.7], [48, Theorem 1.1, Proposition 4.1]. Among other things, the double-point formula says that the double locus Δ is actually an effective divisor in a base point free linear system (see the following theorem). The formula can be used in bounding regularity of X as well as the regularity of $\mathscr{O}_X$. Further study on the positivity of double-point divisors can be found in [39, 40].

Theorem 3.1 (Double-Point Formula) *Let X be a nonsingular projective variety $X \subseteq \mathbb{P}^r$ of dimension $n \geq 1$ and degree d. Let $f : X \to \mathbb{P}^{n+1}$ be a general projection with the image $\bar{X} = f(X)$. Let $\mathfrak{C}$ be the conductor ideal of $\mathscr{O}_{\bar{X}}$ in $\mathscr{O}_X$ defining the double locus $D \subseteq \bar{X}$ and let $\Delta = f^{-1}(D) \subseteq X$. Then one has $\omega_X \cong \mathfrak{C} \otimes f^*\omega_{\bar{X}}$. Therefore $\mathfrak{C}$ is invertible on X and Δ is an effective divisor of X satisfying*

$$\Delta \sim_{lin} (d - n - 2)H - K_X,$$

where H is the hyperplane divisor on X induced by $\mathscr{O}_{\mathbb{P}^r}(1)$. Furthermore, the linear system $|(d - n - 2)H - K_X|$ is base point free.

Proof The way to get double-point formula can be found in [21, Section V]. If one varies the projection center of f, it is not hard to see the linear system $|(d-n-2)H-K_X|$ is base point free (see also [13, Proposition 3.3]). □

Remark 3.2 (Regularity of $\mathscr{O}_X$) As a direct consequence of double point formula, one can immediately get the regularity bound for the structure sheaf. Indeed, since the linear system $|(d-n-2)H-K_X|$ is base point free, by Kodaira vanishing theorem, one can show that reg $\mathscr{O}_X \le d-1$. Along this line, using inner projection, i.e., project X from a point in X, Noma [39] shows that if $X \subseteq \mathbb{P}^r$ is nondegenerate with $n \ge 2$ and codimension ≥ 2 and is neither a scroll over a smooth projective curve, the second Veronese surface, nor a Roth variety, then the linear system $|(d-r-1)H-K_X|$ is semiample. By dealing with the exceptional cases in Noma's work, Kwak-Park [23] eventually prove the following

Theorem 3.3 (Kwak-Park) *Let $X \subseteq \mathbb{P}^r$ be a non-degenerate nonsingular projective variety of dimension n and degree d. Then* $\operatorname{reg}(\mathscr{O}_X) \le d-r+n$.

They also classify the extremal and the next extremal cases. Note that if Eisenbud-Goto conjecture is assumed to hold for nonsingular varieties, then it will imply the above regularity bound for $\mathscr{O}_X$. However, McCullough-Peeva's counterexamples to Eisenbud-Goto conjecture show that $\operatorname{reg}(\mathscr{O}_X)$ is not even bounded above by any polynomial function of d if X is not nonsingular [34].

Remark 3.4 (Regularity Bound by Mumford) Mumford shows in [5] how one can use the fact that the linear system $|(d-n-2)H-K_X|$ is free to obtain a regularity bound for X. This bound has been improved by many work including [3, 23, 41]. One can find more detailed discussion in [23]. It is interesting to review this bound to see the role played by projection.

Theorem 3.5 (Mumford) *Let $X \subseteq \mathbb{P}^r$ be a nondegenerate nonsingular projective variety of dimension $n \ge 1$ and degree d. Then*

$$\operatorname{reg} X \le (n+1)(d-2)+2.$$

To see the proof, we consider a general projection $f : X \to \mathbb{P}^{n+1}$ and use the notation introduced in Theorem 3.1 above. Let $\mathscr{I}_D$ be the ideal sheaf of D as a subscheme of $\mathbb{P}^{n+1}$. Since $\bar{X}$ is a degree d hypersurface and D is defined by the conductor ideal $\mathfrak{C}$ in $\bar{X}$ which is the same as $f_*\mathscr{O}_X(-\Delta)$, there exists a short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \to \mathscr{I}_D \to f_*\mathscr{O}_X(-\Delta) \to 0$. From it, for any integer $l \in \mathbb{Z}$, the natural morphism $H^0(\mathscr{I}_D(l)) \to H^0(\mathscr{O}_X(l-\Delta))$ is surjective. These fit into the following diagram

$$\begin{array}{ccccc} H^0(\mathscr{I}_D(l)) & \hookrightarrow & H^0(\mathscr{O}_{\mathbb{P}^{n+1}}(l)) & \hookrightarrow & H^0(\mathscr{O}_{\mathbb{P}^r}(l)) \\ \downarrow & & & & \downarrow \theta_l \\ H^0(\mathscr{O}_X(l-\Delta)) & & \xrightarrow{\cdot\Delta} & & H^0(\mathscr{O}_X(l)) \end{array}.$$

Thus multiplication with the divisor Δ maps $H^0(\mathscr{O}_X(l-\Delta))$ into $H^0(\mathscr{O}_X(l))$ and the image is contained in the image of θ_l. Write U as the subspace of $H^0(\mathscr{O}_X(-K_X+$

$(d-n-2)H)$) spanned by the divisors Δ for all possible general projections. Then the natural morphism

$$\alpha : U \otimes H^0(\mathscr{O}_X(l-\Delta)) \longrightarrow H^0(\mathscr{O}_X(l))$$

satisfies the condition that $\operatorname{im}\alpha \subseteq \operatorname{im}\theta_l$. Hence to show the surjectivity of θ_l for some l, it is enough to show the surjectivity of α. To this end, since U is base point free, we can choose general $n+1$ sections from U which generate $\mathscr{O}_X(\Delta)$ and yield an exact Koszul complex

$$0 \longrightarrow \mathscr{O}_X(l-(n+1)\Delta) \longrightarrow \cdots \longrightarrow \oplus\mathscr{O}_X(l-\Delta) \longrightarrow \mathscr{O}_X(l) \longrightarrow 0.$$

The rest of the proof is to use appropriate positivity conditions and chase through the above complex to obtain the desired vanishing in the definition of regularity.

One of the central problems about generic projection is to understand the complexity of fibers. This is still widely open. In [31] J. Mather gave a description of possible fiber algebra of projection of a nonsingular projective variety if the dimension of the variety is in a nice range. In our situation of projecting X to a hypersurface, X is in the nice range if the dimension $n \leq 14$. In this case, a $\mathbb{C}$-algebra Q that can occur as a fiber algebra in the projection (i.e., there exists a fiber X_y such that $X_y = \operatorname{Spec} Q$) if it satisfies the following conditions:

(1) Q is finite dimension as a $\mathbb{C}$-vector space.
(2) Q is the finite direct product $Q = \Pi_i Q_i$ where each Q_i is the quotient of $\mathbb{C}[[x_1, \cdots, x_n]]$.
(3) $-1 \leq \iota(Q_i) \leq 0$, where if $Q_i = \mathbb{C}[[x_1, \cdots, x_n]]/I$ for an ideal I, then $\iota(Q_i) = n-$ the number of minimal generators of I.
(4) $\delta_Q + \gamma_Q \leq n+1$ where δ_Q is the length of Q and γ_Q is a non-negative invariant defined in [30].

Mather gave a complete list of such algebra Q. Based on his list, or using dimension counting method, one even can see that fibers are curvilinear if the dimension of X is small enough.

When the dimension of X is beyond the nice range, i.e., $\dim X > 14$, results of Mather [32] imply that the number of distinct points in the fiber of a general projection of X to a hypersurface is at most $\dim X + 1$ (for details see [2, Theorem 2.1]). More sophisticated results describing fibers were established by Beheshti-Eisenbud [2, Corollary 1.2.].

Theorem 3.6 *Let $X \subseteq \mathbb{P}^r$ be a nonsingular projective variety of dimension $n \geq 1$, and let $f : X \to \mathbb{P}^{n+1}$ be a general projection. For $y \in \mathbb{P}^{n+1}$, let $X_y = f^{-1}(y)$ be the fiber over y.*

(1) *X_y contains at most $n+1$ distinct points.*
(2) *If $n \leq 14$, then the length $l(X_y) \leq n+1$.*
(3) *If $n \leq 5$, then X_y is curvilinear.*

Proof (1) is from [32] and also by Beheshti and Eisenbud [2, Theorem 2.1] and [2, Corollary 1.2.]. (2) is from [31]. (3) is by the list of fiber algebras in [31]. Or one can show it by dimension counting. See also [25]. □

It is clear now that one cannot expect the length of a fiber of a projection is bounded above by the form of "dimension +1" of the variety, since the length could be very large, as illustrated in an example by Lazarsfeld.

Example 3.7 (Lazarsfeld, Beheshti-Eisenbud) Let X be a nonsingular projective variety of dimension n with Ω^1_X nef. Consider a general projection $f : X \to \mathbb{P}^{n+1}$. It induces a morphism on tangent sheaves

$$df : T_X \longrightarrow f^*T_{\mathbb{P}^{n+1}}.$$

Note that $\Omega^1_X \otimes f^*T_{\mathbb{P}^{n+1}}$ is ample. So by Lazarsfeld [29, 7.2.1] we see that the k-th singular locus $S^k(f) = \{x \in X \mid \operatorname{rank} df_x \leq n - k\}$ is nonempty if $n \geq k(k+1)$. Take a point $x \in S^k(f)$ and let $y = f(x)$. Consider the length of the fiber algebra at the point x,

$$e_f(x) = \dim_{\mathbb{C}}(\frac{\mathscr{O}_{X,x}}{\mathfrak{m}_y \cdot \mathscr{O}_{X,x}}),$$

where $\mathfrak{m}_y$ is the maximal ideal of the local ring $\mathscr{O}_{\mathbb{P}^{n+1},y}$. It is proved in [2, Proposition 2.2] that $e_f(x) \geq \binom{k+1}{\lceil k/2 \rceil}$. Hence if we take $k = \lfloor \sqrt{n} \rfloor - 1$, then the length of the fiber over y is

$$l(f^{-1}(y)) \geq \binom{k+1}{\lceil k/2 \rceil} = \binom{\lfloor \sqrt{n} \rfloor}{\lceil \frac{\lfloor \sqrt{n} \rfloor - 1}{2} \rceil}.$$

Remark 3.8 (Invariant Defined by Beheshti-Eisenbud) In their work [2], Beheshti-Eisenbud propose a new invariant to measure the complexity of fibers of general projections. This invariant works for arbitrary projection of nonsingular projective varieties. Here we focus on the situation of projecting a variety to a hypersurface. Recall $X \subset \mathbb{P}^r$ is a nonsingular projective variety of dimension n and let $f : X \to \mathbb{P}^{n+1}$ be a general projection. For a point $y \in \mathbb{P}^{n+1}$, it is cut out by linear forms $l_0, \cdots, l_n$ and those forms cut out a linear space L of codimension $n+1$ in $\mathbb{P}^r$. Write $Z = X \cap L$ which is the fiber $f^{-1}(y)$ over the point y. Let $W = \langle l_0, \cdots, l_n \rangle$ be the vector space spanned by the forms l_i. The evaluation map gives a surjective morphism of sheaves

$$W \otimes \mathscr{O}_{\mathbb{P}^r}(-1) \longrightarrow \mathscr{I}_L \longrightarrow 0$$

where $\mathscr{I}_L$ is the ideal sheaf of L in $\mathbb{P}^r$. Tensoring with $\mathscr{O}_L$ one deduces the conormal sheaf of L in $\mathbb{P}^r$ as $N^*_{L/\mathbb{P}^r} \cong W \otimes \mathscr{O}_L(-1)$. The forms l_i define the subscheme Z

in X so one has a surjective morphism $W \otimes \mathscr{O}_X(-1) \to I_{Z/X}$. Restricting it on Z, one deduces a surjective morphism $W \otimes \mathscr{O}_Z(-1) \to N^*_{Z/X}$. Then by applying $\mathscr{H}om(\_, \mathscr{O}_Z)$, one arrives at a short exact sequence

$$0 \longrightarrow N_{Z/X} \longrightarrow W \otimes \mathscr{O}_Z(1) \xrightarrow{\alpha} Q(X, L) \longrightarrow 0$$

where $Q(X, L)$ is the quotient sheaf, which is supported along the scheme Z. The invariant defined by Beheshti-Eisenbud is the number

$$q(X, L) = l(Q(X, L)),$$

the length of the sheaf $Q(X, L)$. As the normal sheaf of L in $\mathbb{P}^r$ is $N_{L/\mathbb{P}^r} \cong W \otimes \mathscr{O}_L(1)$, combining with the morphism α, we create a short exact sequence

$$0 \longrightarrow F \longrightarrow N_{L/\mathbb{P}^r} \longrightarrow Q(X, L) \longrightarrow 0,$$

where F is the kernel sheaf. Using the deformation method, Beheshti-Eisenbud proves that $h^1(F(-1)) = 0$. Hence the induced morphism on global sections

$$H^0(N_{L/\mathbb{P}^r}(-1)) \longrightarrow H^0(Q(X, L)(-1))$$

is surjective which means that

$$q(X, L) \leq h^0(N_{L/\mathbb{P}^r}(-1)) = n + 1.$$

The good thing for the invariant $q(X, L)$ is that even n is outside the nice range of Mather, it is still bounded by $n + 1$. It was also expected in [2, Conjecture 1.4.] that one could bound the regularity of fibers by this invariant. In general, the regularity of fibers was given by the following conjecture by Beheshti-Eisenbud.

Conjecture 3.9 ([2, Conjecture 1.3]) Let $X \subset \mathbb{P}^r$ be a smooth projective variety of dimension n, and let $f : X \to \mathbb{P}^{n+1}$ be a general projection. Then for a fiber $X_y = f^{-1}(y)$, one has $\operatorname{reg} X_y \leq n + 1$.

As shown in Example 3.7, the length of a fiber in a projection could be very large. To fix this problem, one possible invariant that could be used to measure the complexity of fibers is the Loewy length. It works particularly well for low dimensional varieties with mild singularities. Hopefully it could also be related to the invariant introduced by Beheshti-Eisenbud.

Definition 3.10 For an Artinian local ring $(A, \mathfrak{m})$, we define the *Loewy length* $ll(A)$ of A to be the nonnegative number $ll(A) := \max\{i \mid \mathfrak{m}^i \neq 0\}$. If $\mathfrak{m} = 0$, i.e., A is a field, then we put $ll(A) = 0$.

For a dimension zero subscheme in $\mathbb{P}^r$, the Loewy length of the scheme is generally smaller than its length, especially when the scheme is nonreduced and has large embedding dimension. Suppose that $X \subseteq \mathbb{P}^r$ is a dimension zero subscheme. The classical result says that $\operatorname{reg} X \le l(X)$ where $l(X)$ is the length of the structure sheaf $\mathscr{O}_X$. Better bounds can be obtained if the position of X is considered. We give a similar bound in the following theorem by using Loewy length.

Theorem 3.11 ([37, Theorem 2.2]) *Let $X \subseteq \mathbb{P}^r$ be a zero dimensional subscheme supported at distinct closed points $p_1, \dots, p_t$. For each $1 \le i \le t$, set $\mu_i := ll(\mathcal{O}_{X,p_i})$ to be the Loewy length of the local ring $\mathcal{O}_{X,p_i}$. Then X is $(\mu_1 + \cdots + \mu_t + t)$-regular.*

To apply the Loewy length in regularity problem, we need to use the notation of reduction, which we recall here. Let $(R, \mathfrak{m})$ be a local Noetherian ring. An ideal $J \subseteq \mathfrak{m}$ is called a *reduction* of $\mathfrak{m}$ if $\mathfrak{m}^{k+1} = J\mathfrak{m}^k$ for some integer $k \ge 0$. Moreover, J is called a *minimal reduction* if it is a reduction minimal with respect to inclusion. If J is a reduction of $\mathfrak{m}$, then J always contains a minimal reduction [20, Theorem 8.3.5]. Furthermore, if $\dim R = n$ and $R/\mathfrak{m} = \mathbb{C}$, then there exists a nonempty Zariski open subset U in the n-th Cartesian product $(\mathfrak{m}/\mathfrak{m}^2)^n$ of the cotangent space such that if $x_1, \dots, x_n \in \mathfrak{m}$ with $(x_1 + \mathfrak{m}^2, \dots, x_n + \mathfrak{m}^2) \in U$, then $(x_1, \dots, x_n)$ is a reduction of $\mathfrak{m}$ [20, Theorem 8.6.6].

Turning to geometric setting, let $X \subseteq \mathbb{P}^r$ be a projective variety of dimension n, and $p \in X$ be a point. Let $L \subseteq \mathbb{P}^r$ be a codimension k linear subspace passing through p, and assume that L is cut out by linear forms $l_1, \cdots, l_k$ on $\mathbb{P}^r$. Locally at the point p, each form l_i gives an element $\bar{l}_i$ of $\mathfrak{m}_{X,p}$ via the quotient $\mathfrak{m}_{X,p} = \mathfrak{m}_{\mathbb{P}^r,p}/I_{X,p}$. Thus we obtain an ideal $(\bar{l}_1, \dots, \bar{l}_k) \subseteq \mathfrak{m}_{X,p}$ generated by the elements $\bar{l}_i$. We say that L is a *reduction linear subspace at* (X, p), or simply L is a *reduction at* (X, p), if the ideal $(\bar{l}_1, \dots, \bar{l}_k)$ is a reduction of $\mathfrak{m}_{X,p}$. If L is reduction at (X, p), then the intersection $X \cap L$ has dimension zero at p and $k \ge n$. If one can find a positive integer a such that $\mathfrak{m}^a_{X,p} \subseteq J$ for any minimal reduction J of $\mathfrak{m}_{X,p}$, then we have $ll(\mathcal{O}_{X \cap L,p}) \le a - 1$. Indeed, in this case, locally at the point p, the ideal $(\bar{l}_1, \dots, \bar{l}_k) \subseteq \mathfrak{m}_{X,p}$ is a reduction ideal of $\mathfrak{m}_{X,p}$, and hence, $\mathfrak{m}^a_{X,p} \subseteq (\bar{l}_1, \dots, \bar{l}_k)$ since the latter contains a minimal reduction. As $\mathcal{O}_{X \cap L,p} = \mathcal{O}_{X,p}/(\bar{l}_1, \dots, \bar{l}_k)$, we see immediately that $\mathfrak{m}^a_{X \cap L,p} = 0$ as desired. In application, the number a turns out to depend on the singularities of the local ring.

To see how we can use Loewy length to bound the regularity of fibers of a projection for singular cases, we recall rational singularities as well as Du Bois singularities. For a projective variety X, we say that X has *rational singularities* if X is normal and there exists a proper birational morphism $f : Y \to X$ from a smooth variety Y such that $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$. Let $\underline{\Omega}^\bullet_X$ be the Deligne-Du Bois complex for X, which is a generalization of the de Rham complex for a nonsingular variety (see [22, Chapter 6] for details). We say that X has *Du Bois singularities* if the natural map $\mathscr{O}_X \to \underline{\Omega}^0_X = Gr^0_{\text{filt}} \underline{\Omega}^\bullet_X$. is a quasi-isomorphism. Note that if X has rational singularities or log canonical singularities, then X has Du Bois singularities [22, Corollary 6.23 and Corollary 6.32].

Proposition 3.12 *Let $X \subset \mathbb{P}^r$ be a normal projective variety, $p \in X$ be a point, and $L \subset \mathbb{P}^r$ be a reduction linear subspace at (X, p). Then*

(1) *If* $\dim X = 2$ *and X has Du Bois singularities then* $ll(\mathcal{O}_{X \cap L, p}) \leq 2$.
(2) *If* $\dim X = 3$*, then the following hold:*

   (a) *If X has rational singularities, then* $ll(\mathcal{O}_{X \cap L, p}) \leq 2$.
   (b) *If X has Cohen-Macaulay Du Bois singularities, then* $ll(\mathcal{O}_{X \cap L, p}) \leq 3$.

Proof Let J be a minimal reduction of the maximal ideal $\mathfrak{m}_{X,p}$ of the local ring $\mathcal{O}_{X,p}$ at p. If (X, p) is a 3-dimensional rational singularity, then it follows from Briançon-Skoda type theorem (see [20, Theorem 3.2 (1)]) that $\mathfrak{m}_{X,p}^3 \subseteq J$. If (X, p) is a 3-dimensional normal Cohen-Macaulay Du Bois singularity, then [49, Lemma 3.5] implies that $\mathfrak{m}_{X,p}^4 \subseteq J$. Thus the assertions follow immediately. □

As application, we show that the conjectured bound for fibers in Conjecture 3.9 holds for surface Du Bois singularities and threefold rational singularities.

Proposition 3.13 *Let $X \subset \mathbb{P}^r$ be a closed subvariety of dimension n. Consider a general projection $f : X \to \mathbb{P}^{n+1}$. For $y \in \mathbb{P}^{n+1}$, let $X_y = f^{-1}(y)$ be the fiber over y.*

(1) *If $n = 2$ and X has normal Du Bois singularities, then* $\operatorname{reg} X_y \leq 3$.
(2) *If $n = 3$ and X has rational singularities, then* $\operatorname{reg} X_y \leq 4$.

Proof The proof uses the classical method of dimension counting and the result above. We do not give the details here. Instead, the complete proof can be found in [42]. □

4 Regularity Bounds

In this section, we discuss several regularity bounds. Some of them are optimal and can be obtained by the generic projection method. In general, obtaining a regularity bound is a difficult task. It requires better understanding of the geometry and singularities of the varieties concerned. The current method for regularity bounds is also limited and hopefully more research can be conducted in this area in the future.

The study of the regularity bounds for a curve was initiated by Castelnuovo in his work on linear system on curves (see Example 2.3). A complete result of optimal regularity bounds for integral curves was established by Gruson-Lazarsfeld-Peskine. We summarize their results in the following theorem.

Theorem 4.1 (Gruson-Lazarsfeld-Peskine [17]) *Let X be a non-degenerate reduced and irreducible projective curve in $\mathbb{P}^r$ of degree d, then*

$$\operatorname{reg} X \leq d - r + 2.$$

Furthermore, X is $(d-r+1)$-irregular with $r \geq 3$ if and only if one of the following cases

(1) *$d = r$ and X is a rational normal curve (2-regular).*
(2) *$d = r+1$ and X is either a elliptic normal curve or a rational curve (3-regular).*
(3) *$d \geq r+2$ and X is a rational curve with a $(d-r+2)$-secant line.*

Remark 4.2 The regularity bound above was further proved by Giamo [15] for a connected reduced curve which may have several irreducible components. In [38], Noma extended the above result by obtaining a regularity bound involving the arithmetic genus of the curve. The essential idea used in [17] is to resolve the defining ideal of the curve by a Eagon-Nothcott complex which is almost exact except at finitely many points. If the curve is nonsingular, then one can still use the projection method to obtain the regularity bound, as shown in Example 2.3. It would be interesting to know if the projection method can still be applied to a singular curve or a connected curve to obtain the above regularity bound.

The major breakthrough of establishing optimal regularity bounds for higher dimensional varieties was done by Lazarsfeld [27] for nonsingular surfaces (see also Pinkham [44] for surfaces in $\mathbb{P}^5$). He proves that if X is a nondegenerate nonsingular projective surface in $\mathbb{P}^r$ $(r \geq 4)$, then $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$, as expected in Eisenbud-Goto conjecture. Extending this regularity bound to singular varieties has attracted considerable attentions. Here we illustrate how one can use the projection method discussed in Sect. 2 to yield a regularity bound for surfaces with Du Bois singularities.

Theorem 4.3 (Niu-Park [42]) *Let X be a nondegenerate normal surface in $\mathbb{P}^r$ $(r \geq 4)$ with the Du Bois singularities. Then one has*

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1.$$

Proof The proof essentially follows the idea in [27] by using projection method. Hence we consider a general projection $f : X \to \mathbb{P}^3$ from a center Λ. Write $V = H^0(\mathscr{I}_\Lambda(1))$, the 4-dimensional vector space spanned by the linear equations of Λ. Since X has normal Du Bois singularities, the Kodaira vanishing theorem holds for X. For any $y \in \mathbb{P}^3$, the regularity of the fiber $X_y = f^{-1}(y)$ is no more than 3 by Proposition 3.13. By the base change, the morphism $w_2 : q_* p^*(\mathscr{O}_{\mathbb{P}^r}(2)) \to \mathscr{O}_X(2)$ is then surjective, which gives us a short exact sequence

$$0 \longrightarrow E \longrightarrow \mathscr{O}_{\mathbb{P}^3} \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(-1) \oplus S^2 V \otimes \mathscr{O}_{\mathbb{P}^3}(-2) \longrightarrow f_* \mathscr{O}_X \longrightarrow 0. \qquad (4.3.1)$$

By Proposition 2.2, we have $\operatorname{reg} E^* \leq -2$ and therefore $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$ as desired. □

Remark 4.4 (Normal Surfaces) For an arbitrary singular surface, it is still unknown if one can even have a reasonable regularity bound. This question is still widely open. However, for a normal surface, one can use Rathmann's method in his recent work [46] to establish the following regularity bound: let $X \subseteq \mathbb{P}^r$ be a non-degenerate normal surface of degree d, then one has

$$\operatorname{reg} X \leq d + \frac{r(r-3)}{2}(\pi - 1) - \frac{(r-2)(r-3)}{2}\chi,$$

where π is the sectional genus and $\chi = \chi(\mathscr{O}_X)$. As indicated in the theorem above, one may still hope that the regularity bound conjectured by Eisenbud-Goto would hold for a normal surface. It is definitely worth the effort to conduct research on this problem.

Beyond dimension two, obtaining optimal regularity bounds becomes more and more difficult, even by using projection method. Along this line, Kwak has pushed the projection method further to establish regularity bounds for nonsingular varieties of dimension ≤ 6 [24, 25]. Unfortunately, there is still no clue if those bounds are optimal (which is equivalent to give counterexamples for Eisenbud-Goto conjecture).

Theorem 4.5 (Kwak) *Let $X \subset \mathbb{P}^r$ be a nondegenerate nonsingular projective variety of dimension $3 \leq n \leq 6$. Then*

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1 + \delta_n,$$

where $\delta_3 = 1$, $\delta_4 = 4$, $\delta_5 = 9$ and $\delta_6 = 19$.

It is natural to extend Kwak's regularity bounds for singular varieties. The difficulty is that if the singular locus has positive dimension, then it is rather hard in projection to control the complexity of fibers which touch the singular locus. For a threefold with rational singularities, we can use dimension counting to achieve such control on fibers, since in this case, the singular locus has dimension at most one. However, for dimension more than 3, using dimension counting becomes unrealistic.

Theorem 4.6 (Niu-Park [42]) *Let $X \subseteq \mathbb{P}^r$ be a non-degenerate projective threefold with rational singularities. Then*

$$\operatorname{reg}(X) \leq \deg X - \operatorname{codim} X + 2.$$

In addition to establish regularity bound for arbitrary varieties, it is also interesting to look for varieties of special types for which one can obtain good regularity bounds. A scroll over a curve is one of such varieties (see also the last paragraph of [17]). It was first considered in Bertin's work [4] and further discussed in [42].

Theorem 4.7 (Niu-Park [43]) *Let C be a smooth projective curve of genus $g \geq 0$ and let E be a very ample vector bundle on C of rank n and degree d. Let $X = \mathbb{P}(E) \subseteq \mathbb{P}^r = \mathbb{P}(V)$ embedded by a base-point-free subspace $V \subseteq$*

$H^0(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(1))$. $X \subseteq \mathbb{P}^r$ is a scroll of codimension e and degree d over a smooth projective curve C of genus $g \geq 0$. Then one has

$$\operatorname{reg}(X) \leq d - e + 1 + g(e - 1).$$

Remark 4.8 If the scroll is over $\mathbb{P}^1$, then the bound above is optimal and is the one conjectured by Eisenbud-Goto. However, if the curve has positive genus, it is still not clear what bound is optimal. Maybe, the first case one should look at is a scroll over an elliptic curve.

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The Zariski-Riemann Space of Valuation Rings

Bruce Olberding

1 Introduction

In a 1961 letter to Serre, Grothendieck [10, p. 129] complained that Boubarki included valuations in Chapter VI of *Commutative Algebra*, in the middle of the book instead of the back, "among the things 'not to be read'." Valuations, wrote Grothendieck, were "a huge mess" because of "endless scales and arpeggios on compositions of valuations, baroque ordered groups, full subgroups of the above and whatever." Valuations do present a mess of various structures—fields, mappings, rings, groups and orderings. Even the very abundance of available valuations in contexts such as function fields can pose issues of hard-to-sort data. The present article isn't meant to address the opinion of valuations as an out-of-category tool for the geometer or commutative algebraist—numerous applications in algebraic geometry and commutative algebra could serve as such a defense—but instead to survey some of the attempts to find order in another of the complicated aspects of the theory, that of the topological and geometric nature of the totality of valuation rings of a field, the Zariski-Riemann space of the field.

Instead of trying to inventory the current uses for the Zariski-Riemann space, we give an idiosyncratic treatment that emphasizes some basic themes involving the view of this space as a locally ringed space, with particular emphasis on the connection between the topology and geometry of subspaces and the intersection of the valuation rings in these subspaces. The article is thus meant as an introduction to these topics, and to give a bit of intuition for the Zariski-Riemann space as a topological or locally ringed space a bit of intuition for the Zariski-Riemann space as a topological or locally ringed space.

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Let F be a field, and let k be a subring of F. A *valuation subring* of F/K is a k-subalgebra V of F such that for each $0 \neq t \in F$, if $t \notin V$ then $t^{-1} \in V$; equivalently, V has quotient field F and the ideals of V are linearly ordered with respect to set inclusion. The intersection of all valuation rings of F/k is the integral closure of k in F. (We do not assume k itself has quotient field F.) Each valuation ring arises from a valuation, a mapping from F to a totally ordered group G with a symbol ∞ adjoined. This connection with mappings permits arithmetical data to be associated to points, ideals and rings, and while these maps are fundamental in many applications, they are not explicitly needed for our purposes. For the treatment of valuation rings via valuations, we instead refer to Bourbaki [6], to—what else?—Chapter VI.

The set $\text{Zar}(F/k)$ of all valuation rings of F/k is imbued with a topology, the Zariski topology, which we will review in Sect. 3. We occasionally refer to the Zariski-Riemann space of a domain R and write $\text{Zar}(R)$. By this, we mean $\text{Zar}(Q(R)/R)$, where $Q(R)$ is the quotient field of R. In any case, with this topology, $\text{Zar}(F/k)$ is often called the *Zariski-Riemann space* of F/k, for reasons discussed by Nagata [51, p. 2]:

> The name of Riemann is added because Zariski called this space 'Riemann manifold' in the case of a projective variety, though this is not a Riemann manifold in the usual sense in differential geometry. The writer believes that the motivation for the terminology came from the case of a curve. Anyway, the notion has nearly nothing to do with Riemann, hence the name 'Zariski space' is seemingly preferable. But, unfortunately, the term 'Zariski space' has been used in a different meaning [as a Noetherian topological space for which every nonempty closed irreducible subset has a unique generic point]. Therefore we are proposing the name 'Zariski-Riemann space.'

In [82] Zariski implicitly used the Zariski-Riemann space in formulating his definition of birational correspondence, an idea he developed in order to remedy a lack of a rigorous way of relating points on two projective models of a function field K/k. As Zariski [81, p. 402] put it, "It is true that the geometers have a fairly good intuitive idea of what happens or what may happen to an algebraic variety when it undergoes a birational transformation; but the only thing they know with any certainty is what happens in a thousand and one special cases." The idea then was to use valuations to track points on successive blow-ups of projective models. (We review projective models in Sect. 2.) The starting point is Chevalley's theorem that every local domain is dominated by a valuation ring. If the local ring is itself not a valuation domain, then it will in fact be birationally dominated by infinitely many valuation rings, each representing a different direction in which to blow up the center of the valuation ring in the model. (The *center* of the valuation ring V in the model is the local ring R in the model that is *dominated* by V, meaning that $R \subseteq V$ and the maximal ideal of R is contained in the maximal ideal of V.) Next is the observation that no two points of the same projective model are the centers of the same valuation ring. Zariski referred to this as "irredunance," and in modern guise it is the valuative criterion for separateness. Then, since the model is projective, it follows that every valuation ring of F/k is centered on some point in the model. This is called "completeness" by Zariski and it expresses, via the valuative criterion of properness, the fact that a projective model X is given by a proper morphism $X \to \text{Spec}(k)$.

Piecing this together, for each projective model X of F/k, there is a surjective mapping $\text{Zar}(F/k) \to X$ that sends a valuation ring to its center in X. That the map is well defined reflects the fact that $X \to \text{Spec}(k)$ is separated; that the map is onto reflects the properness of $X \to \text{Spec}(k)$. The fact that $\text{Zar}(F/k) \to X$ is well-defined and surjective is an important feature of the valuation rings of F/k, and it is one of the things that allows for tracking points through successive blow-ups, as well as for developing a birational correspondence between models of F/k. Similarly, when considering models X of F/k that are not necessarily projective, the Zariski-Riemann space $\text{Zar}(F/k)$ reveals what's missing from X. If $X \to \text{Spec}(k)$ is not proper, the mapping $\text{Zar}(F/k) \to X$ is not well defined since there are valuation rings in $\text{Zar}(F/k)$ that do not have a center in X. These valuation rings then are useful for filling in the missing points for X. This is the idea behind Nagata compactification [51, 52], although modern treatments such as [11] perform compactification without recourse to valuation rings.

Although X is a projective model, $\text{Zar}(F/k)$ itself rarely is. However, there is a natural way to view $\text{Zar}(F/k)$ as a locally ringed space (see Sect. 3), and the map $\text{Zar}(F/k) \to X$ then induces a morphism of locally ringed spaces. As a locally ringed space, $\text{Zar}(F/k)$ is a scheme only in very special circumstances. We show in Theorem 5.2 this happens only if $\text{Zar}(F/k)$ is the normalization of a projective model. Among other things, this implies that if $\text{Zar}(F/k)$ is a scheme, then each valuation ring of F/k is a localization of the integral closure of a finitely generated k-algebra. So, since $\text{Zar}(F/k)$ is a locally ringed space that is typically not a scheme, the question arises as to what its "signature" should be.

In Zariski's case, he treated $\text{Zar}(F/k)$ as a topological space, with special focus on the quasicompactness of $\text{Zar}(F/k)$. His approach to resolving singularities in dimensions 2 and 3 included the step of replacing an infinite resolving system with a finite one, a step that compactness made possible. In Sect. 3 we discuss the topology on $\text{Zar}(F/k)$ and the significance of compactness for this space. We also pinpoint what type of topological space $\text{Zar}(F/k)$ is: A spectral space whose specialization order is a tree with a unique minimal element. The fact that $\text{Zar}(F/k)$ is spectral allows for a refinement to a Hausdorff topology, the patch topology, that is important for many applications. This is discussed in Sect. 4.

In Sect. 5, the focus shifts to the structure of $\text{Zar}(F/k)$ as a locally ringed space and conditions under which $\text{Zar}(F/k)$ is a scheme. This being rarely the case, the more useful question is which subspaces of $\text{Zar}(F/k)$ are affine schemes when given the structure of a locally ringed space in a natural way. Affineness in $\text{Zar}(F/k)$ then remains the theme for the rest of the article.

2 Projective Models

Let F be a field, and let k be a subring of F. The concept of a projective model arises naturally when considering valuation rings. Let $x_1, \ldots, x_n \in F$ and V be a valuation ring of F/k. Since the ideals of V form a chain, $(x_1, \ldots, x_n)V = x_i V$ for some i. Thus $k[x_1/x_i, \ldots, x_n/x_i] \subseteq V$. A different choice of valuation ring could produce a different choice for i, but at least one such choice is always possible, so each valuation ring V of F/k contains one of the rings $D_i := k[x_1/x_i, \ldots, x_n/x_i]$ for some i. Moreover, if $\mathfrak{M}_V$ is the maximal ideal of V and $P = \mathfrak{M}_{\mathfrak{V}} \cap D_i$, then $(D_i)_P \subseteq V$ and V dominates $(D_i)_P$. Letting i vary and collecting the localizations of the D_i at each prime ideal, we obtain a collection X of local subrings of F. This collection,

$$X = \bigcup_i \{(D_i)_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec}(D_i)\},$$

is the *projective model*[1] of F/k determined by $x_1, \ldots, x_n$. The *normalization* of the projective model X is the set of localizations at the prime ideals of the integral closures $\overline{D}_i$, $i \in \{1, 2, \ldots, n\}$, of the rings D_i.

Thus, given a projective model X of F/k, each valuation ring in $\operatorname{Zar}(F/k)$ dominates a local ring in the model, the *center* of the valuation ring in the model. This suggests viewing the local rings in X as points. Indeed, the projective model X admits a topology, the *Zariski topology*, that has as a basis of open sets the sets of the form

$$\{R \in X : x_1, \ldots, x_n \in R\}, \text{ where } x_1, \ldots, x_n \in F.$$

Since the points in X are local rings, there is an additional structure present on X. Define a sheaf $\mathcal{O}_X$ on X by $\mathcal{O}_X(U) = \bigcap_{R \in U} R$ for each open subset U of X. (If U is empty, define $\mathcal{O}_X(U) = F$.) If x is a local ring in X, then x is both a point in X and a stalk of the sheaf (the stalk at itself: $\mathcal{O}_{X,x} = x$). With this sheaf, X becomes a projective integral scheme, and there is a closed immersion of X into $\mathbb{P}_k^n = \operatorname{Proj}(k[X_1, \ldots, X_n])$; see for example [60, Remark 2.1]. In fact, the projective models of F/k are simply the projective integral schemes over $\operatorname{Spec}(k)$ whose function fields are contained in F. If the subring k has quotient field F and we choose elements $x_1, \ldots, x_n$ in k (not just F), then the projective model of F/k is the blow-up of $\operatorname{Spec}(k)$ at the ideal $(x_1, \ldots, x_n)$.

We constructed a projective model by first choosing elements $x_1, \ldots, x_n$ in F, and so each finite subset of F determines a projective model of F/k. The collection of all projective models of F/k forms a directed system under the ordering of domination: Let X, Y be a projective models of F/k. Then Y *dominates* X if for

[1] We are following [84] here by not requiring the rings D_i to have quotient field F. In other settings, the rings in the projective models are assumed to have quotient field F.

each $y \in Y$, there is $x_y \in X$ such that $\mathcal{O}_{X,x_y} \subseteq \mathcal{O}_{Y,y}$ and $\mathfrak{m}_{X,x_y} = \mathfrak{m}_{Y,y} \cap \mathcal{O}_{X,x_y}$. (Here $\mathfrak{m}_{X,x_y}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x_y}$.) The *domination map* $\delta_X^Y : Y \to X$ sends y to x_y Since X and Y are projective models, the mapping δ_X^Y is well-defined, continuous and closed [84, Lemma 5, p. 119]. (The map δ_X^Y when coupled with the obvious sheaf map is a dominant morphism of schemes.)

The projective models of F/k form an inverse system with respect to domination [84, Lemma 6, p. 119]. The important conclusion that can be deduced from all this is: *The inverse limit of this system is the Zariski-Riemann space of* F/k. But in which category?

To answer this, we need a topology on $\text{Zar}(F/k)$ and a map from $\text{Zar}(F/k)$ to each projective model of F/k. For $x_1, \ldots, x_n \in F$, let

$$\mathcal{U}(x_1, \ldots, x_n) = \{V \in \text{Zar}(F/k) : x_1, \ldots, x_n \in V\},$$

and declare the set of all the $\mathcal{U}(x_1, \ldots, x_n)$ to be a basis for a topology, the *Zariski topology* for $\text{Zar}(F/k)$. Now $\text{Zar}(F/k)$ dominates each projective model X of F/k in the sense that for each $x \in X$, there exists a valuation ring $V \in \mathfrak{X}$ such that $\mathcal{O}_{X,x} \subseteq V$ and $\mathfrak{m}_{X,x} = \mathfrak{M}_V \cap \mathcal{O}_{X,x}$, where $\mathfrak{M}_V$ is the maximal ideal of V [84, pp. 119–120]. Let $d : \text{Zar}(F/k) \to X$ be the *domination map* that sends V to x. The mapping d is surjective, continuous and closed [84, Lemma 4, p. 117]. (That d is well-defined and surjective expresses the valuative criterion for properness [29, Theorem II.4.7].)

Returning to the statement that the inverse limit of the system of projective models of F/k is the Zariski-Riemann space of F/k, we can now state this more precisely by asserting this in the category of topological spaces. We can do better and locate this statement in the category of locally ringed spaces by defining a sheaf of rings on $\text{Zar}(R/k)$ in the obvious way: If U is an open subset of $\mathfrak{X} = \text{Zar}(F/k)$, then $\mathcal{O}_{\mathfrak{X}}(U) = \bigcap_{V \in U} V$, where $\mathcal{O}_{\mathfrak{X}}(U) = F$ if U is empty. The stalks of this sheaf are the valuation rings in $\text{Zar}(F/k)$. With this additional structure, we arrive at the conclusion that the locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a "pro-scheme," a projective limit of schemes. While not stated in this terminology, the following proposition is implicit in Zariski-Samuel [84, Theorem VI.41, p. 122]. For more details, see [59, Proposition 3.3].

Theorem 2.1 (Zariski-Samuel) *As locally ringed spaces,* $\text{Zar}(F/k)$ *is the projective limit of the projective models of* F/k.

The proof of the theorem shows that each valuation ring V of F/k is the union of the local rings that are the centers of V in the projective models of F/k. This elementary observation is not hard to show directly. In some contexts it is possible to considerably restrict the types of projective models needed to obtain the valuation rings in this way. For example, if k is a two-dimensional regular local ring with quotient field F, then iterated blow-ups of closed points (quadratic transformations) suffice; see Sect. 7.

Theorem 2.1 establishes the connection between $\text{Zar}(F/k)$ and the projective models of F/k, but it does not immediately reveal the nature of $\text{Zar}(F/k)$ as either a topological space or a locally ringed space. In the next sections we focus on these two aspects of $\mathrm{Zar}(F/k)$.

3 Topology of the Zariski-Rieman Space

As the results discussed in this section bear out, the topology of the Zariski-Riemann space is now well understood in the sense that it is possible to say what kind of topological space $\mathrm{Zar}(F/k)$ is, although as of yet there does not exist a full characterization of spaces that arise as $\mathrm{Zar}(F/k)$ for some choice of F and k. This is summarized in Corollary 3.3 and the discussion that follows it. The development of this corollary is worth recounting because along the way it emphasizes different features of $\mathrm{Zar}(F/k)$. We do so in this section, with the first theme being that of compactness. A direct proof of compactness for $\mathrm{Zar}(F/k)$ can be found in Matsumura [47, Theorem 10.5, p. 74]. While this proof has the virtue of being self-contained, other arguments give more information about the nature of $\mathrm{Zar}(F/k)$, so we discuss these in some detail.

The first appearance of the compactness of the Zariski-Riemann space occurs in Zariski's 1940 article [80], where he proves versions of local uniformization for zero-dimensional valuations of function fields. As a step in the case in which the base field k is the field $\mathbb{C}$ of complex numbers, Zariski proves the compactness of the subspace X of $\mathrm{Zar}(F/k)$ consisting of valuation rings whose residue fields are $\mathbb{C}$. This is done by viewing X as a closed subspace of a product of copies of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. As a closed subset of a product of compact spaces, X is therefore compact by Tychonoff's Theorem. In this special case, X is also Hausdorff, a consequence of the fact that the valuation rings in X all have residue field $k = \mathbb{C}$. By comparison, $\mathrm{Zar}(F/k)$ is never Hausdorff if the field has a nontrivial valuation ring.

Several years later in [83], Zariski gave a different proof. This time, still working over a function field F/k but now with arbitrary base field k, he shows that the space of valuation rings whose residue field is algebraic over k is the projective limit of the closed points of the projective models[2] of F/k. Specifically, he observes that the sets of closed points in projective models of F/k form an inverse system. Since these sets of closed points are quasicompact and T_1, the inverse limit is quasicompact by a theorem of Steenrood that had been published 8 years earlier. With these observations, it remains to show that there is a homeomorphism from the space X of valuation rings in $\mathrm{Zar}(F/k)$ whose residue fields are algebraic over k to this inverse limit. However, Zariski proves something stronger, namely that the valuation rings in X are the unions of the local rings of the closed points of the projective models.

[2] In his articles in this time period, Zariski's projective models consisted only of the closed points of the projective models defined in Sect. 2. This was in keeping with the focus on classically defined projective varieties.

This is the precursor to Theorem 2.1, the fact that the locally ringed space $\mathrm{Zar}(F/k)$ of all valuation rings of F/k (not just the residually algebraic ones) is the projective limit of the projective models of F/k. But this had to wait for the right language, as well as a shift of focus to the projective model and the space $\mathrm{Zar}(F/k)$ in their entirety rather than the closed points of these spaces only, a shift that took place with the replacement of classical varieties by schemes.

By 1960 when Zariski and Samuel's text [84] was published, this shift had taken place. Now the situation is in maximum generality and applies to the Zariski-Riemann space of an arbitrary field. Specifically, F is a field and k is assumed only to be a subring of F. With these assumptions, it is shown that $\mathrm{Zar}(F/k)$ is the projective limit of the projective models with no restriction to closed points needed. The language of locally ringed spaces is not used, but the arguments imply that as locally ringed spaces, $\mathrm{Zar}(F/k)$ is a projective limit of the projective models, and so we arrive at Theorem 2.1. One small adjustment is needed to the treatment in [84]: the generic point of the projective models and the space $\mathrm{Zar}(F/k)$ is omitted in [84] from the definitions of these spaces, but these points are needed in order for the usual presheaf defined over open sets as intersections of local rings to be a sheaf. With the generic point inserted, the full strength of the claim can be asserted in the language of locally ringed spaces, as in Theorem 2.1.

Appealing again to Steenrod's theorem, quasicompactness of $\mathrm{Zar}(F/k)$ now follows from the quasicompactness of the projective models that comprise the inverse limit. Rather than derive quasicompactness this way, Zariski and Samuel give another proof. Like the arguments involving inverse limits, the proof implies something considerably stronger for which only later would there be appropriate language and context. Again, we work in full generality, where k is a subring of F. It is shown in [84, Theorem 40, p. 113] that $\mathrm{Zar}(F/k)$ is embedded in a product of copies of a three element set $\{-, 0, +\}$, where this set has the T_0 topology whose open sets are $\emptyset$, $\{0, +\}$ and $\{-, 0, +\}$. The product here is indexed by the elements x of F, and in the coordinate x, the embedding into the product sends a valuation ring V to $-$ if $x \notin V$, 0 if x is a unit in V and $+$ if $x \in \mathfrak{M}_V$. The set $\mathrm{Zar}(F/k)$ inherits a Hausdorff topology from this product that is finer than the Zariski topology. The set $\mathrm{Zar}(F/k)$ is shown to be a closed subspace of the product. The product is quasicompact since it is a product of quasicompact spaces, so as a closed subspace of a quasicompact space, $\mathrm{Zar}(F/k)$ is quasicompact in this finer topology and hence is quasicompact in the Zariski topology.[3] We will have more to say on this technique in the next section.

More can be deduced from this line of reasoning. The three element sets are finite T_0 spaces, and the projective limit of finite T_0 spaces is *spectral*, meaning (a) the space is quasicompact and T_0; (b) the quasicompact open subsets are closed under finite intersection and form an open basis; and (c) every nonempty irreducible closed subset has a generic point. Hochster [36] has shown that the topological spaces that are spectral are precisely the spaces that occur as the prime spectrum

[3] For applications of this technique to other contexts, see [61].

of a commutative ring, hence the reason for the terminology. A closed subspace of a spectral space is spectral, so the Zariski-Samuel argument proves $\mathrm{Zar}(F/k)$ is not only quasicompact but spectral in the Zariski topology. Thus we arrive at a key topological description of the Zariski-Riemann space as a spectral space.

But the fact that the Zariski-Riemann space is spectral is not stated in [84], since its publication predates the concept, nor is it obvious that this is the case from reading the proof. The first explicit proof that $\mathrm{Zar}(F/k)$ is spectral was given by Dobbs, Federer and Fontana in [12], who proved this by verifying directly for $\mathrm{Zar}(F/k)$, where k has quotient field F, each of the defining criteria for a space to be spectral. This proof contributes a different understanding of the topological features of $\mathrm{Zar}(F/k)$ because of how it directly addresses these features. For a model-theoretic proof that $\mathrm{Zar}(F/k)$ is spectral, see the appendix of [39].

In the case in which F is the quotient field of the integral domain k, Dobbs and Fontana gave in [13] a very different proof of the fact that $\mathrm{Zar}(F/k)$ is a spectral space by exhibiting an integral domain whose prime spectrum is homeomorphic to $\mathrm{Zar}(F/k)$, namely the Kronecker function ring of the domain k. We discuss this construction in some detail because of its importance in understanding $\mathrm{Zar}(F/k)$, and because it is not well known outside of multiplicative ideal theory, where it is a fundamental tool.

Following Halter-Koch [28], we state the construction in its most general form, where k is a subring of F that need not have F as its quotient field. Let T be an indeterminate for F, and define for each valuation ring $V \in \mathrm{Zar}(F/K)$, a valuation ring V^* of the field $F(T)$ by

$$V^* = V[T]_{\mathfrak{M}_{\mathfrak{V}}[T]}, \text{ where } \mathfrak{M}_V \text{ is the maximal ideal of } V.$$

Then $V = V^* \cap F$ and the maximal ideal of V^* contracts in F to the maximal ideal of V. The *Kronecker function ring* R *of* $\mathrm{Zar}(F/k)$ is the intersection of the rings V^*, where V varies over $\mathrm{Zar}(F/k)$. If the base ring k has quotient field F, then R is the classical Kronecker function ring of k with respect to integral closure of ideals (see [25, Section 26]):

$$R = \left\{ \frac{f}{g} : f, g \in D[T], g \neq 0 \text{ and } \overline{c(f)} \subseteq \overline{c(g)} \right\}.$$

Here $c(-)$ is the content of a polynomial and $\overline{I}$ denotes the integral closure of the ideal I in D. (Other operations on ideals, the e.a.b. star operations, give rise to different Kronecker functions rings [25, Section 26].)

Regardless of whether k has quotient field F, the ring R has quotient field $F(T)$ and is a *Bézout domain*, meaning that every finitely generated ideal is principal;[4]

[4] Desire for a ring extension of k with this property was what motivated Kronecker's early version of this construction, which in it original form was an alternate approach, one using divisors rather than ideals, to repairing the failure of the Fundamental Theorem of Arithmetic for classes of orders

see [16, Corollary 3.6], [28, Theorem 2.2] and [35, Corollary 2.2]. As a Bézout domain, R has the property that the valuation rings of $F(T)/R$ are exactly the localizations of the ring R at its prime ideals. (Rings with this property, *Prüfer domains*, are the subject of later sections.)

Piecing this together with facts from [13] and [35, Proposition 2.7], it is noted in [59, Proposition 4.2] (see also [16, Corollary 3.6]) that this implies the following theorem.

Theorem 3.1 *The Zariski-Riemann space of $F(T)/R$ consists of the localizations of the ring R at its prime ideals. The mapping from Spec(R) to* Zar(F/k) *that sends $P \in$ Spec(R) to $R_P \cap F$ is a closed bijective morphism of locally ringed spaces.*

And so we obtain once more that Zar(F/k) is a spectral space, this time because it is homeomorphic to the prime spectrum of a ring. Moreover, the Zariski-Riemann space of F/k is the image of an affine scheme under a morphism of locally ringed spaces. For more on the connection between topological properties of Zar(F/k) and the Kronecker function ring R, see [16] and [59].

A spectral space X admits a partial ordering induced by specialization: If $x, y \in X$ and y is in the closure of the set $\{x\}$, then x is a *generalization* of y and y is a *specialization* of x. Specialization defines a partial order given by $x \leq y$ if and only if y is a specialization of x. If $X = \text{Spec}(R)$ for a ring R, then the specialization order is simply the partial order on the set of prime ideals given by inclusion, while for Zar(F/k) the specialization order is the reverse order of set inclusion.

The posets P that arise as the set of prime ideals of a Bézout domain are characterized in [43, Theorem 3.1] as trees with unique minimal element for which (a) every chain in P has an infimum and a supremum, and (b) if $x, y \in P$ and $x < y$, then there exist $x_1, y_1 \in P$ such that $x \leq x_1 < y_1 \leq y$ and there does not exist an element of P properly between x_1 and y_1. These last two properties are always satisfied for a partially ordered set of prime ideals of a commutative ring.

Corollary 3.2 *A partially ordered set X is order isomorphic to* Zar(F/k) *with the specialization order for some field F and subring k if and only if X is a tree with unique minimal element and X satisfies* (*a*) *and* (*b*).

Proof For necessity, apply [43, Theorem 3.1] and Theorem 3.1. Conversely, suppose that X is a tree with unique minimal element and X satisfies (a) and (b). By Lewis [43, Theorem 3.1], there exists a Bézout domain R whose poset of prime ideals is isomorphic to the poset of elements of X. As partially ordered sets, the Zariski-Riemann space Zar(R) with the specialization order is order isomorphic to Spec(R). (This follows from the fact that Zar(R) is the set of localizations of the Bézout domain R at its prime ideals.) Thus Zar(R) is order isomorphic to X. □

Corollary 3.3 *Let F be a field, and let k be a subring of F. Then* Zar(F/k) *is a spectral space whose specialization order is a tree having a unique minimal element.*

in algebraic number fields. See Fontana and Loper [23] for a discussion of this and for much more on the Kronecker function ring construction.

Proof By Theorem 3.1, Zar(F/k) is homeomorphic to the prime spectrum of a Bézout domain. Now apply Corollary 3.2. □

Having a converse to the corollary—a precise topological classification of the Zariski-Riemann space—is equivalent to the difficult and longstanding problem of characterizing the spectral spaces that occur as the prime spectrum of a Bézout domain. A remarkable connection between the theory of Bézout domains and that of lattice-ordered abelian groups, the Jaffard-Kaplansky-Ohm Theorem [24, Theorem 5.3, p. 113], allows this problem to be reinterpreted as that of characterizing the prime spectrum of lattice-ordered abelian groups, which in turn translates through a duality between these groups and MV-algebras into the problem of describing the prime spectrum of an MV-algebra. It is these last two versions of the problem, which still remain open, that have received the most attention; see for example [78] and it references.

In any case, in light of Theorem 3.1, saying any more about the nature of Zar(F/k) as a spectral space is equivalent to saying more about the prime spectrum of a Bézout domain, a lattice-ordered abelian group or an MV-algebra.

Although it is beyond the scope of this article, there is an interesting extension of the topological point of view on Zar(F/k). In a series of articles, Finocchiaro, Fontana and Spirito have shown that the Zariski-Riemann space of a domain can be viewed as a subspace of the space of semi-star operations of finite type for this domain. (A semistar operation is a type of closure operation on the monoid of submodules of the quotient field of the domain.) This space is also a spectral space in which reside other spaces of star operations and local rings. This approach has a number of applications to multiplicative ideal theory. See [17, 19] for surveys of extensive work done in this direction.

4 The Patch Topology

Although rarely Hausdorff themselves, spectral spaces admit a refinement to a Hausdorff topology, called the *patch* or *constructible* topology, that plays a crucial role in the theory of spectral spaces, as well as in the topology of the Zariski-Riemann space. This topology on Zar(F/k) is the topology inherited from the product of the three element spaces discussed in Sect. 3. But a direct definition is possible too: A basis for the patch topology on a spectral space X is given by unions of sets of the form $U_1 \cup (X \smallsetminus U_2)$, where U_1, U_2 are quasicompact open sets of X. When interpreted for the Zariski-Riemann space Zar(F/k) of F/k, the patch topology has basic open sets of the form

$$\mathcal{U}(x_1, \ldots, x_k; y_1, \ldots, y_m) = \{V \in \mathrm{Zar}(F/k) : x_1, \ldots, x_k \in V, y_1, \ldots, y_m \in \mathfrak{M}_V\},$$

where $x_1, \ldots, x_k, y_1, \ldots, y_m \in F$. These open sets are also closed in the patch topology, and thus the topology has a basis of clopen sets. As such, the patch topology is zero-dimensional, Hausdorff and quasicompact. The subsets of a spectral space that are closed in the patch topology are again spectral spaces. For examples of the use of the patch topology in the context of Zariski-Riemann space, see [15, 16, 18, 35, 39, 54, 59, 61–63]

Density in the Zariski topology of $\mathrm{Zar}(F/k)$ is not very useful because it is too easy to satisfy. Each subset of $\mathrm{Zar}(F/k)$ containing F, and hence every nonempty open subset, is dense in the Zariski topology. Similarly, closure in the Zariski topology produces sets that can be too too large for applications. Having more closed and open sets, the patch topology allows for more subtle distinctions and flexibility with topological concepts. For example, finding patch dense subspaces of $\mathrm{Zar}(F/k)$ is useful because of the following observation, which is a quick consequence of the interpretation of the patch topology for $\mathrm{Zar}(F/k)$. It concerns a situation that is in the spirit of tracking points on blowups by allowing for the replacement of an arbitrary valuation ring with one from a dense subset.

Proposition 4.1 *Let k be a Noetherian subring of the field F, and let $X \subseteq Y \subseteq \mathrm{Zar}(F/k)$. Then X is patch dense in Y if and only if for all finitely generated k-subalgebras R of F and prime ideals P of R, whenever there is a valuation ring in Y that contains R and is centered on P, then there is a valuation ring in X that contains R and is centered on P.*

In particular, if X is patch dense in $\mathrm{Zar}(F/k)$, then for each projective model Y of F/k and point p of Y, whenever there is a valuation ring centered on p, then this valuation ring can be replaced with one from X that is centered on p.

Maintaining still the assumption that k is a Noetherian ring, the patch closure of a subset X in $\mathrm{Zar}(F/k)$ is the projective limit of the patch closures of the images of X in the projective models of F/k under the domination map [59, Lemma 2.8]. In fact, the patch closure of X is the set of valuation rings in $\mathrm{Zar}(F/k)$ that are centered in each projective model Y of F/k on a point y in Y for which the Zariski closure of $\{y\}$ in Y is the Zariski closure of a subset of the image of X in Y under the domination map [59, Theorem 3.4].

The following theorem for function fields, due to Kuhlmann, exhibits some important dense subspaces of $\mathrm{Zar}(F/k)$. Density in these cases is a consequence of powerful existence theorems in [39] for valuations of function fields. To state Theorem 4.2, we recall that an *Abhyankar valuation ring* of a function field F/k is a valuation ring for which $\mathrm{trdeg}_k F = r + \mathrm{trdeg}_k V/\mathfrak{M}_{\mathfrak{V}}$, where r is the rational rank of the value group of V. A special class of Abhyankar valuation rings is the set of prime divisors of F/k, those DVRs V for which $\mathrm{trdeg}_k V/\mathfrak{M}_V = \mathrm{trdeg}_k F - 1$. Throughout the rest of the paper, we refer also to the Krull dimension of V as the *rank* of V. (The rank of V is thus the rank of the value group of V as an ordered group.)

Theorem 4.2 (Kuhlmann [39, Corollaries 2, 4, 5, 6, 8, 10, 11]) *If F/k is a finitely generated field extension, then each of the following subsets of $\mathrm{Zar}(F/k)$ is dense in the patch topology: the set of Abhyankar valuation rings of F/k; the set of prime divisors of F/k; the set of valuation rings with finitely generated value group and* *residue field a finite extension of k; the set of discrete valuation rings with residue field a finite extension of k; the set of valuation rings of maximal rank with residue field a finite extension of k. If also k is perfect, then the set of discrete valuation rings with residue field k and the set of valuation rings of maximal rank and residue field k each lie patch dense in the space of all valuation rings of F/k with residue field k.*

Since the patch topology is Hausdorff, a valuation ring V in $\mathrm{Zar}(F/k)$ is a patch limit point of a subset X of $\mathrm{Zar}(F/k)$ if and only if for all $x_1, \ldots, x_n \in V$ and $y_1, \ldots, y_m \in \mathfrak{M}_V$, there are infinitely many valuation rings W in X with the x_i in W and the y_j in $\mathfrak{M}_W$. Patch limit points can also be interpreted using ultrafilter limits; see [16].

Distinguishing patch limit points of a set of valuation rings is useful when dealing with representations of an integrally closed domain. To simplify notation, define for each subset X of $\mathrm{Zar}(F/k)$:

$$A(X) = \bigcap_{V \in X} V, \quad J(X) = \bigcap_{V \in X} \mathfrak{M}_V.$$

All the valuation rings that are patch limit points of a subset X of $\mathrm{Zar}(F/k)$ also contain $A(X)$ so that $A(X) = A(\overline{X})$, where $\overline{X}$ is the patch closure of X. There are optimal choices for these patch closed subsets in the sense that there is a patch closed subset Y of $\overline{X}$ such that $A(X) = A(Y)$ and no proper patch closed subset of Y gives a representation of $A(X)$ [61, (4.2)]. (This representation need not be unique [61, Example 4.3].)

An application of this idea is taken from [63, Corollary 3.6]: Suppose the field F is countable and A is a completely integrally closed local subring of F/k that is not a valuation ring of F/K. Then there is a patch closed representation of A that in the patch topology is homeomorphic to the Cantor set.

A second example is the situation in which X is a nonempty subset of $\mathrm{Zar}(F/k)$ such that $J(X) \neq 0$. In this case, if $A(X)$ is a completely integrally closed local ring that is not a valuation ring of F/k, then there is a representation of $A(X)$ that is perfect (i.e., every point is a limit point) and patch closed in $\mathrm{Zar}(F/k)$ [63, Theorem 3.5].

Moreover, the Baire Category Theorem implies in this case that $A(X) = A(Y)$ for every co-countable subset Y of X [63, Corollary 3.7]. The underlying theme here is that there is in general a great deal of redundancy in the representation of an integrally closed domain as an intersection of valuation rings. In fact, irredunance among valuation rings in a representation is fairly special; see [55, 61] for example. Irredundance is closely connected with whether a valuation ring is isolated in the representing set with respect to the patch topology; see [61]. Being isolated in the entire Zariski-Riemann space rather than a subspace is a property so strong that it occurs only in very special circumstances:

Theorem 4.3 (Spirito [75, Theorem 3.4]) *A valuation ring $V \in \mathrm{Zar}(F/k)$ is isolated in the patch topology if and only if there are $x_1, \ldots, x_n \in F$ and a maximal* *ideal M of $k[x_1, \ldots, x_n]$ that is isolated in the patch topology such that V is the integral closure of $k[x_1, \ldots, x_n]_M$.*

So if V is isolated in the patch topology, then V appears on the normalization of a projective model of F/k. Moreover, if V has rank 1 and k has quotient field F, then V is isolated in the patch topology of $\mathrm{Zar}(F/D)$ if and only if V is a localization of k and its center on k is isolated in the patch topology of $\mathrm{Spec}(k)$ [75, Theorem 5.2].

Finally, we mention a connection between patch closure and rank that is important later for Theorem 6.7.

Theorem 4.4 ([62, Theorem 4.3]) *Let X be a nonempty subset of $\mathrm{Zar}(F/k)$ such that $J(X) \neq 0$. If $A(X)$ is a local ring that is not a valuation domain, then the patch closure of X contains a valuation ring of rank > 1.*

The preceding results are mostly concerned with subspaces of $\mathrm{Zar}(F/k)$, but we can ask similar questions about the nature of the entire space $\mathrm{Zar}(F/k)$. Spirito [75, Corollary 6.5] has shown that if D is a Noetherian local domain of dimension at least 3, then the patch topology of $\mathrm{Zar}(D)$ is perfect. Ideas behind this lead to the striking fact that for a countable local Noetherian domain D, the patch topology of $\mathrm{Zar}(D)$ is determined by very little information about D. (In the theorem, $\overline{D}$ denotes the integral closure of D.)

Theorem 4.5 (Spirito [75, Theorem 6.12]) *Let D_1 and D_2 be two countable Noetherian local domains. Then $\mathrm{Zar}(D_1)$ and $\mathrm{Zar}(D_2)$ are homeomorphic in the patch topology if and only if* (a) $\dim(D_1) = \dim(D_2) = 1$ *and* $|Max(\overline{D}_1)| = |Max(\overline{D}_2)|$*;* (b) $\dim(D_1) = \dim(D_2) = 2$*; or* (c) $\dim(D_1) \geq 3$ *and* $\dim(D_2) \geq 3$.

5 Schemes in $\mathrm{Zar}(F/k)$

The locally ringed space $\mathrm{Zar}(F/k)$ is a scheme only in special circumstances, as we will demonstrate in this section. A more interesting question, that of which subspaces of $\mathrm{Zar}(F/k)$ are affine schemes, is taken up in the next section. The purpose of this section then is discuss the extreme case in which $\mathrm{Zar}(F/k)$ is itself a scheme.

For each nonempty subset X of $\mathrm{Zar}(F/k)$, we let $\mathcal{O}_X$ be the presheaf defined on each Zariski open subset U of X by $\mathcal{O}_X(U) = \bigcap_{V \in U} V$, where $O_X(U) = F$ if U is empty. The stalks of this presheaf are the valuation rings in X, and thus the valuation rings serve as both points in the space and the stalks of the presheaf at these points. It is straightforward to see that $\mathcal{O}_X$ is a sheaf if and only if X is irreducible as a topological space.

In order for $(X, \mathcal{O}_X)$ to be a scheme over k, it is thus necessary that X be irreducible. Second, if $(X, \mathcal{O}_X)$ is a scheme, then there is an open cover $\{U_\alpha\}$ of X such that for each α, U_α is a quasicompact open subset of X for which $(U_\alpha, \mathcal{O}_{U_\alpha})$ is an affine scheme. This last condition requires that for each α, the localizations of the ring $\mathcal{O}_X(U_\alpha)$ at its prime ideals are the valuation rings in U_α, and so U_α is the Zariski-Riemann space of $\mathcal{O}_X(U_\alpha)$. This calibration between the prime ideals of $\mathcal{O}_X(U_\alpha)$ and the valuation rings in U_α is what rules out almost all Noetherian examples while ruling in an important class of non-Noetherian rings, the Prüfer rings of multiplicative ideal theory. These are precisely the integral domains for which each localization at a prime ideal is a valuation ring.

The class of Prüfer domains has been thoroughly studied; see, for example, [22, 25, 41, 42]. There exist dozens of characterizations of such rings, among them that a domain is Prüfer if and only if each localization at a maximal ideal is a valuation domain. Necessarily, a Prüfer domain is integrally closed, and it is Noetherian if and only if it a Dedekind domain. A valuation-free characterization, of which there are many, is the original definition of Prüfer's, that every non-zero finitely generated ideal is invertible. Homologically, the Prüfer domains are the domains for which torsion-free modules are flat.

This almost describes the conditions under which $(X, \mathcal{O}_X)$ is a scheme. What remains is the issue of which subspaces U of $\text{Zar}(F/k)$ can occur as the Zariski-Riemann space of a Prüfer domain with quotient field F. Necessarily, for U to be such a subspace, if a valuation ring V is in U, then all the valuation rings between V and F (which are in fact all the rings between V and F) must be in U. In other words, U is closed under generalizations. Less obviously, U must be patch closed. This is because the fact that U is the set of valuation rings containing $\mathcal{O}_X(U)$ implies that U is the intersection of the patch closed sets of the form $\mathcal{U}(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathcal{O}_X(U)$.

Thus a subspace U of $\text{Zar}(F/k)$ is the Zariski-Riemann space of a Prüfer domain with quotient field F only if U is closed in the patch topology and closed under generalizations. By way of comparison, a subset of a spectral space is closed if and only if it is patch closed and closed under *specializations*. This suggests a dual topology on a spectral space, and hence on $\text{Zar}(F/k)$ also, one whose closed sets consist of patch closed sets that are closed under generalizations. This topology, which is a familiar tool in the study of spectral spaces, appears unnamed in Hochster's paper, where he shows that a spectral space endowed with this topology is again spectral. (And so, strikingly, if a poset can occur as the poset of prime ideals of a commutative ring, so can the poset formed from reversing the ordering.) This dual topology, which is called the inverse topology in [71], is used implicitly in the next lemma.

Putting all this together, we arrive at

Lemma 5.1 *Let X be a nonempty irreducible subspace of the Zariski-Riemann space* $\text{Zar}(F/k)$ *of F/k. Then $(X, \mathcal{O}_X)$ is a scheme over k if and only if X is closed under generalizations and X has a cover $\{U_\alpha\}$ of Zariski open sets in X that are patch closed in* $\text{Zar}(F/k)$ *and have the property that $\mathcal{O}_X(U_\alpha)$ is a Prüfer domain with quotient field F.*

Proof One direction is proved above. Conversely, suppose that X is closed under generalizations and X has a cover $\{U_\alpha\}$ of Zariski open sets in X that are patch closed in $\text{Zar}(F/k)$ and have the property that $\mathcal{O}_X(U_\alpha)$ is a Prüfer domain with quotient field F. We need only verify that $(U_\alpha, \mathcal{O}_{U_\alpha})$ is an affine scheme for each

α. Since $A = \mathcal{O}_X(U_\alpha)$ is a Prüfer domain with quotient field F, all the valuation rings in U_α arise as localizations of A. The only thing left to verify then is that every localization of A at a prime ideal is in U_α. Since A is a Prüfer domain with quotient field F, then the only patch closed subsets of the Zariski-Riemann space of A that are closed under generalizations and whose valuation rings intersect to A is the Zariski-Riemman space of A itself [59, Lemma 5.4(3)]. Thus U_α is the set of all valuation rings between A and F, and since A is a Prüfer domain, every localization of A at a prime ideal is one of these, proving that $(U_\alpha, \mathcal{O}_{U_\alpha})$ is an affine scheme. □

Theorem 5.2 *The following are equivalent.*

(1) $\mathrm{Zar}(F/k)$ *is a scheme over* k.
(2) $\mathrm{Zar}(F/k)$ *is the normalization of a projective model of* F/k.
(3) *There exist* $x_1, \ldots, x_n \in F$ *such that for each* i *the integral closure of the ring* $k[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ *in* F *is a Prüfer domain.*

Proof

(1) ⇒ (2) Suppose $X = \mathrm{Zar}(F/k)$ is a scheme over k. If U is a Zariski open subset of X such that $(U, \mathcal{O}_U)$ is an affine scheme, then for each basic open subset U' of U, $(U', \mathcal{O}_{U'})$ is also an affine scheme [59, Lemma 5.4(2)]. By Lemma 5.1 and the fact that X is quasicompact, X has a finite cover $\{U_i\}$ of basic open sets U_i such that $\mathcal{O}_X(U_i)$ is a Prüfer domain with quotient field F for each i. For each i, write $U_i = \mathcal{U}(x_{i1}, \ldots, x_{in(i)})$, for some $x_{i1}, \ldots, x_{in(i)} \in F$. Since $(U_i, \mathcal{O}_{U_i})$ is an affine scheme, the integral closure $\mathcal{O}_X(U_i)$ of $k[x_{i1}, \ldots, x_{in(i)}]$ is by Lemma 5.1 a Prüfer domain with quotient field F. Let M be the union of the affine models of F/k determined by the rings $k[x_{i1}, \ldots, x_{in(i)}]$; i.e. M is the collection of localizations of these rings at their prime ideals. Since X is the set of localizations at prime ideals of the normalizations of these rings, M is, in the language of [84], a complete model. (Alternatively, as a scheme, M is proper over k by the valuative criterion.) Therefore, by the Zariski-Samuel version of Chow's Lemma [84, Lemma 7, p. 121], M is dominated by a projective model N of F/k. Let $\overline{N}$ denote the normalization of this model, and let $\overline{M}$ denote the normalization of M. If $V \in \mathrm{Zar}(F/k) = \overline{M}$, then V is dominated by a local ring in $\overline{N}$. However, the only local ring that birationally dominates a valuation ring is the valuation ring itself, so $V \in \overline{N}$ and hence $\mathrm{Zar}(F/k) \subseteq \overline{N}$. On the other hand, if R is a local ring in $\overline{N}$, then R birationally dominates a local ring in M, and since R is integrally closed, this implies that R birationally dominates a local ring in $\overline{M}$. The local rings in $\overline{M}$ are valuation rings, so the valuation ring in $\overline{M}$ dominated by R is R itself, which proves $R \in \mathrm{Zar}(F/k)$ and shows that $\overline{N} = \mathrm{Zar}(F/k)$.

(2) ⇒ (3) This is clear.

(3) ⇒ (1) Assuming (3), it suffices by Lemma 5.1 to show that each $V \in \mathrm{Zar}(F/k)$ occurs as the localization of the integral closure of one of the rings in (3). Let $V \in \mathrm{Zar}(F/k)$. Then $(x_1, \ldots, x_n)V = x_i$ for some i since V is a valuation ring. Therefore, the integral closure R of $k[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ is contained in

V. Let P be the center of V in R. Then R_P is by assumption a valuation domain, and since V dominates R_P, $V = R_P$, which verifies (1).

□

In the case of function fields in at least one variable, we obtain the unsurprising fact that the Riemann surface of a curve is the only Zariski-Riemann space that is a scheme.

Corollary 5.3 *If F/k is a finitely generated field extension, then* Zar(F/k) *is a scheme if and only if the transcendence degree of F/k is ≤ 1.*

Proof This follows from Theorem 5.2 and the fact that an integrally closed Noetherian domain is a Prüfer domain if and only if it has Krull dimension ≤ 1. □

Corollary 5.4 *If X is a k-scheme in* Zar(F/k) *that is proper over k, then $X =$* Zar(F/k) *and X is projective over k.*

Proof If X is a k-scheme in Zar(F/k), the valuative criterion shows that X is proper over k if each valuation ring in Zar(F/k) dominates a valuation ring in X. A valuation ring birationally dominates another valuation ring only if the two rings are equal. Thus X is proper over k if and only if X is the Zariski-Riemann space of F/k. The corollary now follows from Theorem 5.2 □

The next example shows it is possible for Zar(F/k) to be a projective scheme that is not affine.

Example 5.5 Suppose k is a subfield of the field F and there is a valuation ring V of F/k such that $V/\mathfrak{M}_V \cong k(T)$, where T is an indeterminate for k. (For example, if $F = k(S, T)$, with S, T indeterminates for k, and V is the localization of $k[S, \frac{S}{T}]$ at the height one prime ideal $Tk[S, \frac{S}{T}]$, then $V/\mathfrak{M}_V \cong k(\frac{S}{T})$.) Let $R = k + \mathfrak{M}_V$. The ring R has quotient field F and every integrally closed subring of F properly containing R is a Prüfer domain [14, Example 6.3]. The ring R is local but not a valuation ring, and so there is $0 \neq x \in F$ such that neither x nor x^{-1} is in V. The integral closures of the rings $R[x]$ and $R[x^{-1}]$ are Prüfer domains. Every valuation ring of F containing R contains either $R[x]$ of $R[x^{-1}]$, so by Theorem 5.2, the Zarski-Riemann space of F/R is a projective scheme that is not affine.

In [27], Green defines a scheme to be a *Prüfer scheme* if it is normal, integral, separated and the local ring at each closed point is a valuation ring. Any scheme in Zar(F/k) is separated and therefore Prüfer under Green's definition. Temkin and Tyomkin [77] introduce the notion of a Prüfer algebraic space in their study of relative Zariski-Riemann spaces. They call an integral quasicompact quasiseparated algebraic space *Prüfer* if each proper birational morphism into it is an isomorphism. An algebraic space is then *Prüfer* if it a finite disjoint union of reduced irreducible components and each component is Prüfer. By the valuative criterion for properness, any affine scheme in Zar(F/k) has the property that each proper birational morphism into it is an isomorphism, and so the affine schemes in Zar(F/k) are Prüfer in the sense of Temkin and Tyomkin also.

6 Affine Schemes in Zar(F/k)

In this section we are interested in when a subspace of Zar(F/k) has the structure of an affine scheme. While this is partly because of a desire to understand Zar(F/k) as a locally ringed space, the real motivation is birational algebra and the assembly of rings as intersections of valuation rings between an integral domain and its quotient field. In the case of many of the main examples of rings assembled in this way, the base domain k is a Noetherian domain, or even a finitely generated algebra over a field, and still the ring obtained is a Prüfer domain, often quite complicated in structure. Roughly, if Noetherian domains are what arise in F/k when k is Noetherian and we take k-subalgebras of F without too many generators (e.g., finitely many), then Prüfer domains are what arise from intersecting not too many (but in general, many more than finitely many) valuation rings of F/k. The "not too many" here is really a qualitative rather than quantitative statement, and it is in formalizing this where the topology and geometry of the Zariski-Riemann space can assist. In some cases, purely algebraic arguments can be given instead, but the point of this section is to put the topological and geometric approach front and center.

We begin with a lemma that is a slightly stronger version of a result due to Sekiguchi (see [72, Propositions 10 and 11], [59, Lemma 5.4(3) and Proposition 6.1]), and which is not surprising in light of Lemma 5.1. It shows that affine schemes in the Zariski-Riemann space are the Zariski-Riemann spaces of Prüfer domains.

Lemma 6.1 *Let X be a nonempty subspace of the Zariski-Riemann space* Zar(F/k) *of F/k. Then $(X, \mathcal{O}_X)$ is an affine scheme (and implicitly $\mathcal{O}_X$ is a sheaf) if and only if X is patch closed in* Zar(F/k) *and closed under generalizations, and $\mathcal{O}_X(X)$ is a Prüfer domain with quotient field F. In this case, X is the Zariski-Riemann space of $\mathcal{O}_X(X)$.*

Ultimately we are interested in the affineness of subspaces of Zar(F/k) because the intersection of the rings in an affine subspace X produce Prüfer domains. As discussed in Sect. 4, any subset of X that is dense in the patch topology will produce the same intersection as that of X, and so we want to be a little looser with the terminology so as to apply the adjective "affine" to patch dense subspaces of an affine scheme. To do so, we drop "scheme" and say a subset X of Zar(F/k) is *affine* if it contains F and the closure of X in Zar(F/k) with respect to the patch topology and generalizations is an affine scheme. The reason for insisting that $F \in X$ is so that X has a generic point and hence is irreducible in the Zariski topology. This in turn implies that the presheaf $\mathcal{O}_X$ is a sheaf, and hence $(X, \mathcal{O}_X)$ is a locally ringed space. Since we are interested in the intersection of rings in X, there is no harm in assuming $F \in X$.

In summary, affine subsets of Zar(F/k) are locally ringed spaces that are easily reinterpreted as affine schemes by adding some patch limit points and generalizations. More generally, any subset X of Zar(F/k) that contains F yields a locally ringed space $(X, \mathcal{O}_X)$, a fact that we will use implicitly at several points in this section when treating morphisms from X to, say, the projective line $\mathbb{P}^1_k$. In particular, "morphism" is morphism in the category of locally ringed spaces.

So now the focus shifts to affine subsets and the intersection of the rings in such sets. To simplify notation and to reflect this shift, we continue to write $A(X)$ for $\mathcal{O}_X(X)$, i.e., $A(X) = \bigcap_{V \in X} V$, as was done in Sect. 4. As expected, the geometry of an affine subset of $\mathrm{Zar}(F/k)$ trivializes. For example, if k is integrally closed in F, the Zariski-Riemann space $\mathrm{Zar}(F/k)$ is itself affine if and only if $\mathrm{Zar}(F/k) \to \mathrm{Spec}(k)$ is an isomorphism of locally ringed spaces; if and only if every projective model of F/k is affine. This last equivalence follows from the following more general theorem that can be pieced together from Lemma 6.1 above, Corollary 6.4 and Theorem 6.6 of [59] and Theorem 3.1 of [60]. By $\mathbb{P}^1_k$, we mean the projective line $\mathrm{Proj}(k[X, Y])$.

Theorem 6.2 *The following are equivalent for a subset X of $\mathrm{Zar}(F/k)$ with $F \in X$.*

(1) *X is affine.*
(2) *$A(X)$ is a Prüfer domain with quotient field F.*
(3) *Every projective model of F/k is dominated by an affine model dominated by X.*
(4) *Every k-morphism $\phi : (X, \mathcal{O}_X) \to \mathbb{P}^1_k$ factors through an affine scheme.*

Item (4) is the point of departure for the next theorem. The idea is to detect affineness via the images of morphisms into $\mathbb{P}^1_k$. If these images all land in affine schemes in $\mathbb{P}^1_k$, then X is affine by Theorem 6.2. This is the point of view taken in [60], where this approach is used to give a single explanation for a disparate set of results in the literature on when an intersection of valuation rings is a Prüfer domain.

Theorem 6.3 ([60, Corollary 3.6]) *Let X be a subset of $\mathrm{Zar}(F/k)$ with $F \in X$. If for each k-morphism $\phi : X \to \mathbb{P}^1_k$ there is a homogeneous polynomial $f \in k[T_0, T_1]$ of positive degree such that the image of ϕ is in $(\mathbb{P}^1_k)_f$, then X is affine with torsion Picard group.*

That such a subset X is affine follows from Theorem 6.2, so the additional strength here is that X has torsion Picard group, which, because of the affineness of X, amounts to $A(X)$ having torsion Picard group; i.e., for each nonzero finitely generated ideal I of $A(X)$, some power of I is principal. If there is $n > 0$ such that each homogeneous polynomial f in the statement of the theorem has degree $\leq n$, then this power is divisible by only such primes that appear as factors of n [60, Remark 3.7]. Thus if $n = 1$, every finitely generated ideal of $A(X)$ is principal and so $A(X)$ is a Bézout domain.

Theorem 6.3 can be strengthened if k a field that is algebraically closed in F. In this case, if no k-morphism $X \to \mathbb{P}^1_k$ has every closed point of $\mathbb{P}^1_k$ in its image, then X is affine with torsion Picard group [60, Theorem 4.2].

As a set, $\mathbb{P}^1_k$ is the collection of nonmaximal homogeneous prime ideals in $k[X, Y]$, so Theorem 6.3 trades the problem of determining when an intersection of valuation rings is a Prüfer ring for the problem of homogeneous prime avoidance.

With this in mind, here is how Theorem 6.3 can be applied to verify some of the main sources of examples of Prüfer intersections. We only outline the proofs; for more details see [60].

Corollary 6.4 (Nagata [50, (11.11), p. 121]) *Finite subsets of* $\mathrm{Zar}(F/k)$ *containing* F *are affine.*

Proof Any finite subset of a projective scheme is contained in an affine scheme, but more to the point here, finiteness and prime avoidance allow the choice of a linear polynomial f in Theorem 6.3, and so finite subsets X of $\mathrm{Zar}(F/k)$ are not only affine but have the additional property that $A(X)$ is a Bézout domain. □

A little weaker version of the next corollary was first proved in [65] for the case in which k is a field.

Corollary 6.5 *If* k *is a local domain and* X *is a subset of* $\mathrm{Zar}(F/k)$ *containing* F *that has cardinality less than that of the residue field of* k*, then* X *is affine and* $A(X)$ *is a Bézout domain.*

Proof Let $\phi : X \to \mathbb{P}^1_k$ be a k-morphism. Use the fact that there are more units in k than valuation rings in X to construct a homogeneous linear polynomial $f \in k[T_0, T_1]$ that is not contained in any prime ideal in the image of ϕ. Then $(\mathbb{P}^1_k)_f$ is an affine open set in $\mathbb{P}^1_k$ containing the image of ϕ. Now apply Theorem 6.3. □

The next corollary has a long pedigree and is one of the more surprising and powerful sources of Prüfer domains and hence of affine sets in $\mathrm{Zar}(F/k)$. Versions of this corollary have been proved by Dress [20], Gilmer [26, Theorem 2.2], Roquette [68, Theorem 1], Loper [45] and Rush [69, Theorem 1.4].

Corollary 6.6 *Let* X *be a subset of* $\mathrm{Zar}(F/k)$ *with* $F \in X$*. If there is a nonconstant monic polynomial* $f \in k[T]$ *that has no root in the residue field of any* $V \in X$*, then* X *is affine with torsion Picard group.*

Proof The homogenization of f behaves as the "f" in Theorem 6.3 and yields the corollary. □

In particular, if k is a non-algebraically closed subfield of the field F and $f(T) \in k[T]$ has no root in k, then the set of valuation rings of F/k whose residue fields contain no root of f is affine. An important application of this is the *absolute real holomorphy ring* of a formally real field, the intersection of the formally real valuation rings of the field. Choosing $f(X) = X^2 + 1$ in Corollary 6.6 shows that this ring is a Prüfer domain. This was first proved by Dress [20], and there is now an extensive literature on this ring and its applications. We briefly discuss this ring in the last section.

The criteria discussed so far for determining affineness use geometric criteria, either implicitly or explicitly. The next results are more topological in nature, and with mild hypotheses can be made strictly topological, as is done in Corollary 6.9.

Theorem 6.7 ([62, Main Theorem]) *A quasicompact set* X *of rank one valuation rings in* $\mathrm{Zar}(F/k)$ *whose maximal ideals do not intersect to* 0 *is affine. In this case,* *the ring $A(X)$ is a Bézout domain of Krull dimension* 1 *with nonzero Jacobson radical.*

Easy examples show that each of the hypotheses in the theorem (that X is quasicompact, that the valuation rings in X have rank 1, and that maximal ideals do not intersect to 0) is necessary; see [62, Example 5.7]. Notably, every integrally closed Noetherian domain (or more generally, Krull domain) of Krull dimension >1 is an intersection of DVRs yet is not a Prüfer domain. In this case, the intersection of the maximal ideals of the DVRs is 0.

The difficulty in applying the theorem is verifying quasicompactness. Indeed, the point of the theorem is that under the assumptions on the valuation rings in X, quasicompactness is equivalent to affineness. Theorem 6.7 follows from a more general statement given in [62, Main Lemma]: The mappings $X \mapsto A(X)$ and $A \mapsto \{A_M : M \in \text{Max}(A)\}$ define a bijection between the quasicompact sets X of rank one valuation rings in $\text{Zar}(F/k)$ with $J(X) \neq 0$ and the one-dimensional Prüfer domains A with nonzero Jacobson radical and quotient field F. An example in the next section shows one way to apply these results. In any case, restating all this in terms of affineness, we have

Corollary 6.8 *Let X be a set of rank* 1 *valuation rings in* $\text{Zar}(F/k)$ *such that* $J(X) \neq 0$. *Then $X \cup \{F\}$ is an affine scheme if and only if X is quasicompact; if and only if $X \cup \{F\}$ is closed in the patch topology.*

Proof Let $A = A(X)$. If $X \cup \{F\}$ is an affine scheme, then by Lemma 6.1, the set $\{A_M : M \in \text{Max}(A)\}$ is a quasicompact set of valuation rings in $\text{Zar}(F/k)$, and this set is X. The converse follows from Theorem 6.7. The last equivalence is a consequence of the fact that if a subset Y of $\text{Zar}(F)$ consists of rank one valuation rings and $J(Y) \neq 0$, then Y is quasicompact if and only if Y is closed in the patch topology [62, Proposition 2.4]. □

As shown in [62, Corollary 5.11], the corollary implies that if A is a domain with quotient field F and $A = A_1 \cap \cdots \cap A_n$ for one-dimensional Prüfer domains $A_1, \ldots, A_n$ with nonzero Jacobson radical, then A is a one-dimensional Prüfer domain with nonzero Jacobson radical. If also each A_i is an almost Dedekind domain (i.e., each localization of A_i at a maximal ideal is a DVR), then so is A. In general, the intersection of two one-dimensional Prüfer rings need not be Prüfer. For example, if R is a two-dimensional integrally closed Noetherian local domain, then for any prime element r of R, $R = R[1/r] \cap R_{(r)}$ and R is an intersection of a PID and a DVR. Less elementary, each integrally closed domain R that is finitely generated over a Dedekind domain or a field is an intersection of finitely many Dedekind rings between R and its quotient field [64, Theorem 3.3]. It is an open question whether *every* integrally closed domain is an intersection of two Prüfer rings in its quotient field. For more on this problem, see [64] and its references.

In the case in which k is a domain with quotient field F, Theorem 6.7 can be rephrased in purely topological terms. We have no application for this in mind, but as a conceptual matter it seems worth noting that the topology of $\text{Zar}(F/k)$ alone can distinguish affineness in some cases. Recall the specialization order from

Sect. 4, which is defined for a spectral space in terms of its topology. Interpreting the specialization order for Zar(F/k), we have that for $V_1, V_2 \in$ Zar(F/k), $V_1 \leq V_2$ if and only if $V_2 \subseteq V_1$. This is the order used in the following corollary.

Corollary 6.9 *If k is a domain with quotient field F and X is a compact Hausdorff subspace of* Zar(F/k) *whose points are minimal in* Zar(F/k) *with respect to not being the least element of* Zar(F/k)*, then X is affine and $A(X)$ is a Bézout domain.*

Proof The valuation rings in X are necessarily of rank 1 because they are minimal with respect to not being the least element F of Zar(F/k). Since k is a domain that has quotient field F and X is a set of rank one valuation rings, X is Hausdorff if and only if $J(X) \neq 0$ [62, Proposition 2.4(3)]. The corollary now follows from Theorem 6.7. □

The emphasis here has been on quasicompact subsets of rank one valuation rings. Removing the rank one restriction and strengthening the quasicompact condition to that of a subspace of Zar(F/k) being Noetherian in the Zariski topology, it is possible to deduce strong consequences about the intersection of the valuation rings in the subspace, but the resulting rings need not be Prüfer domains [56, 58, 61]. If k is a Noetherian ring, then the projective models of F/k are Noetherian spaces with respect to the Zariski topology, but this property is seldom inherited by Zar(F/k); see [74].

It is not the case that affineness is always detectable topologically. This fails dramatically. For example, suppose k is an integrally closed domain with quotient field F and k is not a Prüfer domain. Then Zar(F/k) is not affine. As discussed in Sect. 3, Zar(F/k) is homeomorphic to the Zariski-Riemman space of the Kronecker function ring of F/k, and this ring is a Prüfer domain. In this case, the same topological space occurs as the Zariski-Riemann space of a non-Prüfer domain and a Prüfer domain. In the latter case the locally ringed space is affine while in the former case the locally ringed space is not, although the base spaces are homeomorphic.

7 Example: Two-dimensional Noetherian Domains

The richness of the relationship between a subset of the Zariski-Riemann space and the intersection of its valuation rings can be illustrated by a case that at first glance seems promisingly tractable, that of the integrally closed rings between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. An integrally closed ring R between these two rings is an intersection of $\mathbb{Q}[X]$ and valuation rings of $\mathbb{Q}(X)$ containing $\mathbb{Z}[X]$. How complicated can such a ring R be?

Consider first the rings R of the form $R = V \cap \mathbb{Q}[X]$, where V is a DVR between $\mathbb{Z}[X]$ and its quotient field. Then R is either a Dedekind domain or a two-dimensional Noetherian domain, depending on whether the residue field of V is algebraic over a finite field or transcendental; see [2, Theorem 5.7] and [46, Theorem 0.1]. As pointed out in [46, p. 92], this implies that R either has no valuation overrings other than V and those of $\mathbb{Q}[X]$ (and in particular $\mathrm{Zar}(R)$ is affine), or "there is a staggering infinite collection" of valuation overrings of R other than V and the valuation overrings of $\mathbb{Q}[X]$.

Moving beyond DVRs, if V is an irrational valuation ring, meaning that V has rank one and its value group has rational rank more than 1, then Ohm [53, Section 5] has shown that R is a two-dimensional completely integrally closed domain with a unique height 2 maximal ideal, the center of V in R. This maximal ideal is the radical of a principal ideal [53, Lemma 5.6], and so R is not Noetherian. Nor is R a Prüfer domain since V is not a localization of R.

On the other hand, if V is a rational valuation ring, meaning its value group has rational rank 1, and V is not a DVR, then V is a localization of R [31, Lemma 1.3], and so V is not a Noetherian ring since V is not a DVR. This and the other possible cases, which get quite a bit more complicated, are worked out by Loper and Tartarone in [46]. For example, it is shown that in the present case where V is a rational valuation ring that R is a Prüfer v-multiplication domain [46, Theorem 5.8]. (See [25] or [46] for the definition of Prüfer v-multiplication domain.) Precisely when R is a Prüfer domain is determined by whether V is a certain limit of rings of the form $W \cap \mathbb{Q}[X]$, where W is a valuation ring constructed using key polynomials [46, Proposition 4.1]. In the case in which V has rank two, there are criteria also for when R is a Mori domain (a domain in which divisorial ideals satisfy the ascending chain condition) [46, Theorem 5.6]. For some variations on these ideas where the base ring is a two-dimensional regular local ring rather than $\mathbb{Z}[X]$, see [66].

So far we have only considered a ring of the form $R = V \cap \mathbb{Q}[X]$, but of course the driving question is a classification of all the integrally closed rings between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$, the case in which V is replaced by an intersection of possibly infinitely many valuation rings. This classification is the subject of Loper and Tartarone's work in [46], where the interplay between the types of valuation rings in the representation of such an intermediate integrally closed ring is worked out in detail. As the case of a single valuation makes clear, such a classification is nuanced and depends on the types of valuation rings that comprise the intersection.

It is worth pointing out that hidden away in the current discussion and between the rings $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is the entrance to an entire branch of non-Noetherian commutative ring theory, that of the theory of integer-valued polynomials. The classical focus of this area of research is on the ring $\mathrm{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] : f(\mathbb{Z}) \subseteq \mathbb{Z})\}$.This ring and its generalizations have been the subject of study for over a 100 years, and such rings remain an active area of investigation complete with its own set of tools. The ring $\mathrm{Int}(\mathbb{Z})$ is a completely integrally closed Prüfer domain of Krull dimension 2 that has a number of interesting properties. See [8] and [9] for much more on these rings.

We shift attention to the general setting of two-dimensional Noetherian local domains and use this case to illustrate some of the ideas considered in the previous sections, specifically affineness and compactness in $\mathrm{Zar}(F/k)$. The Zariski-Riemann space of a one-dimensional normal Noetherian domain R is a simple matter: The space consists of the localizations of R at its prime ideals. The valuation theory of a two-dimensional Noetherian domain R is much more complicated, but it is still, in a sense, reachable from the domain itself, not through localization this time but through a possibly infinite sequence of iterated blow ups of maximal ideals and normalizations. This is well known in the case in which R is a two-dimensional regular local ring because of a theorem due to Abhyankar [1] that shows that each valuation ring in $\mathrm{Zar}(R)$ is a union of a sequence of iterated local quadratic transforms, but it is perhaps not as well known in the case in which R is assumed to be a two-dimensional normal Noetherian local domain.

More formally, let $\mathfrak{m}$ denote the maximal ideal of such a ring R, and choose $x_1, \ldots, x_n$ in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m} = (x_1, \ldots, x_n)$. A *local quadratic transform* of R is a ring of the form $R' = R[x_1/x_i, \ldots, x_n/x_i]_P$, where $i \in \{1, 2, \ldots, n\}$ and P is a prime ideal of $R[x_1/x_i, \ldots, x_n/x_i]$ that contains $\mathfrak{m}$. Denote the integral closure of R' in F by $\overline{R'}$. Then $\overline{R'}$ is a Noetherian domain by the Krull-Akizuki Theorem [50, Theorem 33.2, p. 115]. If P is a prime ideal of $\overline{R'}$ containing $\mathfrak{m}$, then the Noetherian local domain $(\overline{R'})_P$ is a *normalized quadratic transform of* R. A sequence of local rings $\{R_i\}$ (finite or infinite) is a *normal sequence* over R if $R = R_0$ and R_{i+1} is a normalized quadratic transform of R_i for each i; see Zariski [79, p. 681] and Lipman [44, p. 201]. If the base ring R is a regular local ring, then the normalization step is not needed since a local quadratic transform of a regular local ring is a regular local ring.

With $(R, \mathfrak{m})$ a two-dimensional normal Noetherian local domain, Lipman [44, p. 202] has shown that the union of rings in an infinite normal sequence over R is a valuation ring. Conversely, every valuation overring of R is the union of a unique normal sequence over R [33, Proposition 2.1].[5] Our focus here is on the prime divisors V that dominate R (i.e., V is a DVR for which $\mathfrak{m} \subseteq \mathfrak{M}_V$ and $V/\mathfrak{M}_V$ has transcendence degree 1 over $R/\mathfrak{m}$). Such a prime divisor occurs as the last term in a uniquely determined finite normal sequence $\{R_i\}_{i=0}^n$ [33, Proposition 2.1]. The *level* of V is n. Using an analysis of patch limit points in $\mathrm{Zar}(R)$ and Theorem 6.7, it is shown in [33] that the prime divisors of bounded level form an affine set in $\mathrm{Zar}(R)$.

Theorem 7.1 ([33, Theorem 4.2]) *Let $d \geq 0$, and let X be a nonempty set of prime divisors that dominate R and occur at level at most d. Then X is an affine set in* $\mathrm{Zar}(R)$ *and $A(X)$ is a Bézout domain with nonzero Jacobson radical.*

The ring $A(X)$ is an almost Dedekind domain [33, Theorem 4.2(1)], meaning that the localization of $A(X)$ at each maximal ideal is a DVR. In this case, each such localization is in X. As long as X is infinite, $A(X)$ will not be a Dedekind domain. This is a consequence of the fact that a maximal ideal M of A is finitely generated if and only if A_M is a patch isolated point in X [33, Theorem 4.2(3)].

The DVRs in Theorem 7.1 dominate R and are prime divisors. Despite what Theorem 7.1 might suggest, intersections of DVRs in the Zariski-Riemann space

[5] If R is not two-dimensional, then the valuation overrings of R need not be reachable by normal sequences, even if R is a regular local ring; see [73]. The structure of a union of iterated quadratic transforms of a high-dimensional regular local ring is explored in [32, 34]. The patch topology plays an important role in the form of capturing for each such union a "boundary valuation ring" that is determinative for the structure of the union.

of a two-dimensional Noetherian domain can be complicated. A good example of this is Nagata's one-dimensional completely integrally closed local domain that is not a valuation ring [48, 49]. This ring, which had been conjectured by Krull not to exist, is assembled from rank one valuation rings in an intricate way, and it can be constructed as an overring of a polynomial ring in two variables [57, Proposition 4.4]. By contrast, a theorem due to Heinzer [30] shows that any finite character intersection of DVRs in the Zariski-Riemann space of a two-dimensional Noetherian domain is a Noetherian domain. The crucial difference here from Nagata's example is that the collection of DVRs has finite character.

8 Example: Holomorphy Rings

Prüfer domains, which as we have seen arise from affine sets in the Zariski-Riemann space, are generally easy to deal with on the local level since their localizations at maximal ideals are valuation domains. What gives the Prüfer rings their complexity are their global properties, and this complexity is both reflected in and expressive of topological features of affine sets in the Zariski-Riemann space. To illustrate this with an example, we discuss the case in which F/k is a function field in at least two variables, k is a field of characteristic 0 that is not algebraically closed and k is *existentially closed* in F, i.e., for each $m, n > 0$ and choice of polynomials $f_1, \ldots, f_n, g$ in $k[X_1, \ldots, X_m]$ such that $f_1, \ldots, f_n$ have a common zero in F^m that is not a zero of g, then $f_1, \ldots, f_n$ also have a common zero in k^m that is not a zero of g; equivalently, every finitely generated k-subalgebra A of L admits a k-homomorphism $\phi : A \to k$ [4, Theorem 1.1]. There are two examples that motivate this.

(a) If k is a real closed field and F is formally real, then k is existentially closed in F.
(b) If k is a non-algebraically closed field of characteristic 0 and F is a finitely generated field extension of k that is contained in a purely transcendental extension of F, then k is existentially closed in F.

Statement (a) can be found in [5, Proposition 4.1.1]. For (b), see [67, Proposition 1].

The *absolute k-holomorphy ring H* of F/k is the intersection of all valuation rings V of F/k such that k is existentially closed in the residue field of V. (There is at least one such valuation ring, the field F.) Since k is not algebraically closed, Corollary 6.6 implies that this intersection is a Prüfer domain with quotient field F. For related constructions of holomorphy rings, see [7, 40]. As an aside, we mention here an application of the ideas in [60, Corollary 4.4] that is related: If k is a real-closed field and F is formally real, then for any valuation rings $V_1, \ldots, V_n$ of F/k, the ring $H \cap V_1 \cap \cdots \cap V_n$ is a Prüfer domain with torsion Picard group and quotient field F.

So far we have not used the strength of the hypothesis of existential closure or the fact that we are in a function field. Where these become decisive is in the structure of the holomorphy ring. This is because existential closure guarantees an abundance of valuation rings in the affine set that defines the holomorphy ring, and the ring thus assembled as an intersection of these valuation rings has a complicated ideal theory and prime spectrum.

As in [54], we say that a valuation ring V of F/k is *good* if $P \neq P^2$ for each nonzero prime ideal of V, k is existentially closed in the residue field $V/\mathfrak{M}_V$ of V, and $V/\mathfrak{M}_V$ is a finitely generated field extension of k. Let $\mathcal{G}_{r,d}$ denote the set of good valuation rings in F/k such that V has rank r and the transcendence degree of $V/\mathfrak{M}_V$ over k is d.

In [54, Lemma 3.3], it is shown that theorems of Kuhlmann from [39] similar in spirit to Theorem 4.2 imply that if $0 \leq d < n$ and $1 \leq r \leq n-d$, where n is the transcendence degree of F over k, then the set $\mathcal{G}_{r,d}$ is dense with respect to the patch topology in the subspace of $\mathrm{Zar}(F/k)$ consisting of the valuation rings such that k is existentially closed in $V/\mathfrak{M}_V$. The consequences of this for the absolute holomorphy ring, which are worked out in [54, Theorems 3.4 and 4.7], are summarized in the following theorem. For the theorem we recall that a *Zariski-Samuel associated prime ideal* of an ideal I of a commutative ring is a prime ideal P for which $P = \sqrt{I : x}$ for some element x of the ring. (Without the finiteness condition of a Noetherian ring, the theory of associated primes is somewhat unruly. There are at least seven inequivalent definitions of an associated prime ideal for non-Noetherian rings; see [37]. All of these coincide for Noetherian rings.[6])

Theorem 8.1 *The absolute K-holomorphy ring H of F/k is a Prüfer domain having Krull dimension $n = \mathrm{trdeg}_k F$ and quotient field F, and H is the intersection of the valuation rings in $\mathcal{G}_{r,d}$ for each $0 \leq d < n$ and $1 \leq r \leq n-d$. Moreover:*

(1) *No nonzero prime ideal of H is the radical of a finitely generated ideal.*
(2) *No nonzero finitely generated ideal of H has a Zariski-Samuel associated prime ideal.*
(3) *If I is a proper nonzero finitely generated ideal of H and $0 \leq d < n$, $0 < h \leq n$ and $d + h \leq n$, then there exist infinitely many prime ideals of H of dimension d and height h that are minimal over I.*
(4) *$J^{-1} = H$ for all nonzero radical ideals J of H.*

If also the function field $F|K$ has 2 variables, then every prime ideal of H is an intersection of maximal ideals [54, Proposition 3.11]. In any case, the conclusion that we wish to draw here is that the highly non-Noetherian behavior of the prime

[6] Although familiar, this is a powerful fact about Noetherian rings. David Eisenbud [21]: "The book [of Demazure and Gabriel] seemed to be making an elementary logical mistake, leading to the statement that if I is an ideal in a commutative Noetherian ring and if every element of I annihilates some (possibly varying) element of the ring, then all of the elements of I annihilate one fixed element of the ring. Absurd! Well, actually, this is one of the main lemmas in the theory of primary decomposition. I think it was my excitement when I finally untangled the mystery that first hooked me on commutative algebra."

ideals of H reflects the abundance of the valuation rings from which the ring is assembled.

In many contexts, the finitely generated ideals in a Prüfer domain can be generated by two elements. The only known examples in which this is not so involve real holomorphy rings, where the problem of finding the number of generators of an ideal in a holomorphy ring is motivated by classical problems involving sums of squares in function fields; see for example [3, Theorem 1.21]. The first such example, given by Schülting [70] in 1979, is the fractional ideal $(1, X, Y)$ of the real holomorphy ring of $k(X, Y)$, where k is a formally real field. Despite the simple statement of the example, the proof that this fractional ideal cannot be generated by two elements is intricate and lengthy. For direct proofs of this fact, all of which involve geometric arguments, see [22, 65, 70]. Swan [76, Theorems 1 and 2] extended Schülting's example to prove that for each integer $n \geq 1$, there is Prüfer domain of Krull dimension n that has an ideal that can be generated by $n+1$ elements but not by n elements. Kucharz [38, Theorem 1] has shown that such examples are ubiquitious: If F is a formally real function field of transcendence degree n over a real closed field k, then the holomorphy ring of F/k has a finitely generated ideal that cannot be generated by n elements.

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Rational Points and Trace Forms on a Finite Algebra over a Real Closed Field

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Dedicated to Professor David Eisenbud on the occasion of his 75th birthday

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1 Introduction

The objective of this paper is to present an exposition of classical and modern results concerning the number of real or complex points in the solution space of a finite system of polynomial equations with real coefficients in arbitrary number of variables. Let $F_1, \ldots, F_m \in \mathbb{R}[X_1, \ldots, X_n]$ and assume that the residue-class $\mathbb{R}$-algebra $\mathbb{R}[X_1, \ldots, X_n]/\langle F_1, \ldots, F_m\rangle$ is finite dimensional over $\mathbb{R}$, then the set of common zeros

$$\mathrm{V}_{\mathbb{R}}(F_1, \ldots, F_m) := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid F_j(a_1, \ldots, a_n) = 0 \text{ for all } j = 1, \ldots, m\}$$

of $F_1, \ldots, F_m$ in $\mathbb{R}^n$ is finite. The converse is not true, for example, for $F_1 = X_1^2 + 1$, $\mathrm{V}_{\mathbb{R}}(F_1) = \emptyset$ is finite and $\mathbb{R}[X_1, \ldots, X_n]/\langle F_1\rangle \xrightarrow{\sim} \mathbb{C}[X_2, \ldots, X_n]$ is not finite dimensional over $\mathbb{R}$ if $n \geq 2$. However, for polynomials $F_1, \ldots, F_m \in \mathbb{C}[X_1, \ldots, X_n]$, the residue-class $\mathbb{C}$-algebra $\mathbb{C}[X_1, \ldots, X_n]/\langle F_1, \ldots, F_m\rangle$ is finite dimensional over $\mathbb{C}$ if and only if the set of common zeros $\mathrm{V}_{\mathbb{C}}(F_1, \ldots, F_m)$ of $F_1, \ldots, F_m$ in $\mathbb{C}^n$ is finite. Moreover, by the classical Hilbert's nullstellensatz

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$\mathrm{V}_{\mathbb{C}}(F_1, \ldots, F_m) \neq \emptyset$ if and only if the ideal $\langle F_1, \ldots, F_m \rangle$ generated by $F_1, \ldots, F_m$ in $\mathbb{C}[X_1, \ldots, X_n]$ is a non-unit ideal. But, this is not true over the field $\mathbb{R}$ or more generally over real closed fields. Therefore the natural questions one deals with are: when exactly $\mathrm{V}_K(F_1, \ldots, F_m) \neq \emptyset$ and how to find its cardinality, where K is an arbitrary real closed field.

Many researchers have studied these problems and devised effective algorithms. For example, already in the nineteenth century Sturm, Jacobi, Sylvester, Hermite (see [9, 10]) Hurwitz proved fundamental results for counting real points (in small number of variables $n \leq 2$) by using the signature of appropriate quadratic forms, see 3.2.

In Sect. 2, we collect standard results on symmetric bilinear and Hermitian forms over a real closed field K and its algebraic closure $\mathbb{C}_K = K[\mathrm{i}]$ with $\mathrm{i}^2 = -1$. However, for the sake of completeness, we recall them without proofs in the format they are used in later sections. With these preliminaries at the end of Sect. 2, we state the important Rigidity Theorem for quadratic forms (see [4]) which is used in Sect. 4.

In Sect. 3, we collect some elementary concepts from commutative algebra and recall the important Theorem 4.5 from [4] which relates the K-rational points of a finite dimensional algebra A over a real closed field K with the type of the trace form Tr_K^A on A and derive some consequences.

In Sect. 4, we compute the cardinality of the K-rational points of finite algebra over real closed field K. The main ingredient in this section is the Shape Lemma 4.2 which guarantees a distinguished generating set for a radical ideal $\mathfrak{A} \subseteq K[X_1, \ldots, X_n]$ if the residue-class K-algebra $K[X_1, \ldots, X_n]/\mathfrak{A}$ is finite dimensional. Using the Shape Lemma 4.2 one can reduce the problem of counting the number of K-rational points in $\mathrm{V}_K(\mathfrak{A})$ to the one variable case. In Theorem 4.5 using the results from Sect. 3, we relate type, signature and rank of a symmetric bilinear form defined by using the trace form on $A = K[X_1, \ldots, X_n]/\mathfrak{A}$ associated to elements $h \in A$ with the number of points in $\mathrm{V}_K(\mathfrak{A})$ and in $\mathrm{V}_{\overline{K}}(\mathfrak{A})$. Finally, we give a precise formulation and a proof of the following theorem of Pederson-Roy-Szpirglas which is quoted from [16]:

Theorem ([16, Theorem 2.1]) *Let K be a field, V be a finite affine algebraic variety defined by the ideal I generated by $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$, R be a real closed field such that $K \subseteq R$ and C its algebraic closure, and given one other polynomial $h \in K[X_1, \ldots, X_n]$. Then*

$$\sigma(Q_h) = \#\{\mathbf{x} \in \mathrm{V}_R(I) \mid h(\mathbf{x}) > 0\} - \#\{\mathbf{x} \in \mathrm{V}_R(I) \mid h(\mathbf{x}) < 0\},$$
$$\rho(Q_h) = \#\{\mathbf{x} \in \mathrm{V}_C(I) \mid h(\mathbf{x}) \neq 0\},$$

where σ denotes the signature and ρ the rank of the quadratic form Q_h associated to the symmetric bilinear form $B_h : A \times A \to K$, $(f, f') \mapsto \mathrm{Tr}_K^A(hff')$ on $A := K[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n \rangle$.

2 Type, Signature and Classification of Hermitian Forms

The main aim of this section is to recall the classification of symmetric and Hermitian forms on finite dimensional vector spaces over real closed field and its algebraic closure, up to congruence. Most of these results can be found in standard graduate text books, for instance see [18, Ch. V, §12], [17, Ch. IX] or [11, Ch. 11], [1, Ch. 7], or [14, Ch. XV]. However, for setting the notation, terminology and for the sake of completeness, we recall them without proofs in the format that they are used in this article.

Notation 2.1 Let K be a *real closed field*.[1] Then $\operatorname{Aut} K = \{\mathrm{id}_K\}$ and the field $\mathbb{C}_K := K[\mathrm{i}\,]$, where $\mathrm{i}^2 = -1$, of (*complex*) *numbers over* K, is the algebraic closure of K with the Galois group $\mathrm{Gal}(\mathbb{C}_K \mid K) = \{\mathrm{id}_{\mathbb{C}_K}, \kappa\}$, where $\kappa : \mathbb{C}_K \to \mathbb{C}_K$ is the (*complex*)-*conjugation* defined by $\mathrm{i} \mapsto -\mathrm{i}$.

Further, we denote by $\mathbb{K}$ either the field K with the identity map $\mathrm{id}_K : K \to K$ as the involution, or the field $\mathbb{C}_K$ with the (complex)-conjugation $\kappa : \mathbb{C}_K \to \mathbb{C}_K$ as the involution. We denote κ by the standard bar-notation, i. e. $a \mapsto \overline{a},\ \ a \in \mathbb{C}_K$.

With these notation the term "Hermitian" means "real-symmetric" if $\mathbb{K} = K$ and "complex-Hermitian" if $\mathbb{K} = \mathbb{C}_K$. Recall that a square matrix $\mathcal{C} \in \mathrm{M}_n(\mathbb{K})$ is Hermitian if $\mathcal{C} = {}^{\mathrm{t}}\overline{\mathcal{C}}$. Therefore it is real-symmetric in the case $\mathbb{K} = K$ and complex-hermitian in the case $\mathbb{K} = \mathbb{C}_K$.

Sylvester's Law of Inertia 2.2 *Let Φ be a Hermitian form on a finite dimensional $\mathbb{K}$-vector space V. Then there exists an orthogonal basis $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $n := \mathrm{Dim}_K V$ of V with respect to Φ such that the Gram's matrix $\mathcal{G}_\Phi(\boldsymbol{x}) = (\Phi(x_i, x_j))_{1 \le i,j \le n}$ of Φ with respect to the basis $\boldsymbol{x}$ is a diagonal matrix*

$$\mathcal{E}_n^{p,q} := \mathrm{Diag}\,(\underbrace{1, \ldots, 1}_{p\text{-times}}, \underbrace{-1, \ldots, -1}_{q\text{-times}}, \underbrace{0, \ldots, 0}_{(n-p-q)\text{-times}}).$$

[1] **Real closed fields** A field K is called *real closed* if it is *real*, i.e. for all $a_1, \ldots, a_n \in K$, $a_1^2 + \cdots + a_n^2 = 0$ implies $a_1 = \cdots = a_n = 0$ and if it has no nontrivial real algebraic extension $L \mid K$, $L \neq K$. For example, the field $\mathbb{R}$ of real numbers is real closed. The algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$ is real closed. The field $\mathbb{Q}$ is real, but not real closed. In 1927, Artin-Schreier proved: *A field K is real if and only if there is an order $\le$ on K such that $(K, \le)$ is an ordered field.* In particular, the characteristic of a real field is 0.

Theorem (Euler-Lagrange) *Let $(K, \le)$ be an ordered field satisfying the properties:* (i) *Every polynomial $f \in K[X]$ of odd degree has a zero in K. (ii) Every positive element in K is a square in K. Then the field $\overline{K} = K$*(i) *obtained from K by adjoining a square root* i *of* -1 *is algebraically closed. In particular, K itself is real-closed.* For a proof see [11, Ch. 11, §11.1]. (**Remark:** Since the field $\mathbb{R}$ of real numbers is ordered and satisfies the properties (i) and (ii), the Euler-Lagrange theorem proves the *Fundamental Theorem of Algebra*: *The field* $\mathbb{C} = \mathbb{R}$(i) *of complex numbers is algebraically closed* . The Euler-Lagrange Theorem has a remarkable complement:—**Theorem** (Artin–Schreier) *Let L be an algebraically closed field. If $K \subseteq L$ be a subfield of L such that $L \mid K$ is finite and $K \neq L$, then $L = K$*(i) *with* $\mathrm{i}^2 + 1 = 0$ *and K is a real-closed field.* For a proof see [11, Ch. 11, §11.7].)

Moreover, p is the maximum of the dimensions of subspaces of V on which Φ is positive definite, and q is the maximum of the dimensions of subspaces of V on which Φ is negative definite. In particular, p and q do not depend on the special choice of the orthogonal basis $x_1, \ldots, x_n$ of V, $p + q = \operatorname{rank} \Phi$ and Φ is non-degenerate if and only if $p + q = n$.

Definition 2.3 The pair (p, q) as in the Sylvester's Law of Inertia 2.2 is called the *type* of the form Φ. The natural number p is called the (*inertia* -) *index*, the natural number q is called the *Morse - index*, and the integer $p - q$ is called the *signature* of Φ. The type, signature and rank of a Hermitian form Φ are denoted by type Φ, sign Φ and rank Φ, respectively.

The *type*, *signature* and *rank* of a Hermitian matrix $\mathcal{C} = (c_{ij})_{1 \le i, j \le n} \in \mathrm{M}_n(\mathbb{K})$ are, by definition, the type, signature and rank of the Hermitian form $\Phi_{\mathcal{C}} : K^n \times K^n \to K$, $(e_i, e_j) \mapsto c_{ij}$, $1 \le i, j \le n$, defined by the matrix $\mathcal{C}$, where e_i, $1 \le i \le n$, is the standard basis of K^n, respectively.

Recall that two square matrices $\mathcal{C}, \mathcal{C}' \in \mathrm{M}_n(K)$ are said to be *congruent* if there exists an invertible matrix $\mathcal{A} \in \mathrm{GL}_n(K)$ with $\mathcal{C} = {}^{\mathrm{t}}\mathcal{A}\, \mathcal{C}'\, \overline{\mathcal{A}}$.

The matrix analog of the Sylvester's Law of Inertia 2.2 is the following:

Corollary 2.4 *Let Φ be a Hermitian form on an n-dimensional $\mathbb{K}$-vector space V with $\mathbb{K}$-basis $\boldsymbol{x} = \{x_1, \ldots, x_n\}$. Then Φ is of type (p, q) if and only if the Gram's matrix $\mathcal{G}_\Phi(\boldsymbol{x})$ is congruent to the matrix $\mathcal{E}_n^{p,q}$, i. e. there exists an invertible matrix $\mathcal{A} \in \mathrm{GL}_n(\mathbb{K})$ such that $\mathcal{G}_\Phi(\boldsymbol{x}) = {}^{\mathrm{t}}\mathcal{A}\mathcal{E}_n^{p,q}\, \overline{\mathcal{A}}$. Two Hermitian matrices $\mathcal{C}$ and $\mathcal{C}' \in \mathrm{M}_n(\mathbb{K})$ have the same type if and only if they are congruent. In particular, a Hermitian matrix $\mathcal{C} \in \mathrm{M}_n(\mathbb{K})$ has type (p, q) if and only if $\mathcal{C}$ is congruent to the matrix $\mathcal{E}_n^{p,q}$.*

If $\mathbb{K} = K$ (real closed), then one can choose[2] $\mathcal{A} \in \mathrm{GL}_n^+(K)$, i.e. $\operatorname{Det} \mathcal{A} > 0$. In the situation of Corollary 2.4, if Φ is non-degenerate, i.e. if $p + q = n$, then $\operatorname{Det} \mathcal{G}_\Phi(\boldsymbol{x}) = (-1)^q |\operatorname{Det} \mathcal{A}|^2$, i.e. $\operatorname{Sign}(\operatorname{Det} \mathcal{G}_\Phi(\boldsymbol{x})) = (-1)^q$. *Therefore, the signature of the Gram's determinant* $\operatorname{Det} \mathcal{G}_\Phi(\boldsymbol{x})$ *determines the parity of* q. From this the following useful criterion for the determination of the type follows:

Hurwitz's Criterion 2.5 (see [18, Ch. V, §12, 12.C.4]) *Let Φ be a Hermitian form on an n-dimensional $\mathbb{K}$-vector space V with a basis $\boldsymbol{x} = \{x_1, \ldots, x_n\}$. Suppose that the principal minors*

$$D_0 := 1 \quad \textit{and} \quad D_i := \begin{vmatrix} \Phi(x_1, x_1) \cdots \Phi(x_1, x_i) \\ \vdots \quad \ddots \quad \vdots \\ \Phi(x_i, x_1) \cdots \Phi(x_i, x_i) \end{vmatrix}, \qquad i = 1, \ldots, n,$$

[2] Use the following observation: Let V be an oriented vector space over a real-closed field K of dimension $n \in \mathbb{N}^+$ and Φ be a Hermitian form of type (p, q) on V. Then there exists an orientation of V represented by a basis $x_1, \ldots, x_n$ of V such that the Gram's matrix of Φ is equal to the matrix $\mathcal{E}_n^{p,q}$.

of the Gram's matrix $\mathcal{G}_\Phi(\boldsymbol{x}) = (\Phi(x_i, x_j)) \in \mathrm{M}_n(K)$ *of* Φ *with respect to the basis* $\boldsymbol{x}$ *are all non-zero. Then the type of* Φ *is* $(n-q, q)$*, where* q *is the number of sign changes*[3] *in the sequence* $1 = D_0, D_1, \ldots, D_n = \operatorname{Det} \mathcal{G}_\Phi(\boldsymbol{x})$.

Corollary 2.6 *Let* Φ *be a Hermitian form on an* n*-dimensional* $\mathbb{K}$*-vector space* V *with basis* $\boldsymbol{x} = \{x_1, \ldots, x_n\}$*. Then, with notation as in the Hurwitz's Criterion 2.5, we have:*

(1) Φ *is positive definite if and only if* $D_i > 0$ *for all* $i = 1, \ldots, n$.
(2) Φ *is negative definite if and only if* $(-1)^i D_i > 0$ *for all* $i = 1, \ldots, n$, *i.e. at every position in the sequence* $D_0, D_1, \ldots, D_n$ *there is a sign change.*

Example 2.7 Let $\{v_1, v_2\}$ be a basis of a 2-dimensional K-vector space V. For a symmetric bilinear form $\Phi = \langle -, - \rangle$ on V, let $D_1 = \langle v_1, v_1 \rangle$ and $D_2 = \operatorname{Det}\begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle \end{pmatrix} = \langle v_1, v_1\rangle \langle v_2, v_2\rangle - |\langle v_1, v_2\rangle|^2$. Then the following table shows the dependence of the sign D_1, sign D_2 and the type of Φ:

D_1	+	+	−	−	+	−	0	0	0	0
D_2	+	−	+	−	0	0	−	0	0	0
$\langle v_1, v_2\rangle$								> 0	< 0	0
Type	(2,0)	(1,1)	(0,2)	(1,1)	(1,0)	(0,1)	(1,1)	(1,0)	(0,1)	(0,0)

Note that the case $D_1 = 0$, $D_2 > 0$ is not possible.

Example 2.8 Let $z \in \mathbb{C}_K \smallsetminus K$, $\pi := (X - z)(X - \overline{z}) \in K[X]$, $A := K[X]/\langle \pi \rangle := K[x]$, where x is the image of X modulo $\langle \pi \rangle$. Further, let $H \in K[X]$, $h = h(x) \in A$ be the image of H in A and let $\Phi_h : A \times A \longrightarrow K$ be the symmetric bilinear form defined by $\Phi_h(f, g) = \operatorname{Tr}_K^A(hfg)$, $f, g \in A$. Then the Gram's matrix of Φ_h with respect to the basis $\{1, x\}$

$$\mathcal{G}_{\Phi_h}(1, x) = \begin{pmatrix} h(z) + h(\overline{z}) & h(z)\cdot z + h(\overline{z})\cdot \overline{z} \\ h(z)\cdot z + h(\overline{z})\cdot \overline{z} & h(z)\cdot z^2 + h(\overline{z})\cdot \overline{z}^2 \end{pmatrix} \in \mathrm{M}_2(K)$$

is a symmetric matrix with $D_1 = h(z) + h(\overline{z}) = 2\operatorname{Re} h(z)$ and $D_2 = \operatorname{Det} \mathcal{G}_{\Phi_h}(1, x) = h(z)\, h(\overline{z})\, (z - \overline{z})^2 = -4\,|h(z)|^2 (\operatorname{Re} z)^2 < 0$. Therefore, if $h(z) = 0$ (and hence $h(\overline{z}) = 0$ also, since $H \in K[X]$), then $\mathcal{G}_{\Phi_h}(1, x) = 0$, and if $h(z) \neq 0$, then $D_2 < 0$. Now, by the table in Example 2.7, it follows that the type of Φ_h is $(0, 0)$ if $h(z) = 0$ and $(1, 1)$ if $h(z) \neq 0$.

[3] Recall that we say that a sequence $a_0, \ldots, a_n$ of non-zero real numbers *changes the sign at the* i*-th place* if $0 \le i < n$ and $a_i\, a_{i+1} < 0$. For an arbitrary sequence of real numbers $b_0, \ldots, b_m$ by a *change of signs* means a change of signs in the sequence obtained by removing the zeros from the original sequence.

The type of a Hermitian form on a finite dimensional vector space V over $\mathbb{C}_K$ can also be determined by using the eigenvalues of the Gram's matrix, see Theorem 2.10 below. Usual proofs given in the standard text books of this fact uses the *Principal Axis Theorem for self-adjoint operators* (also known as the *Spectral Theorem*). We give here a direct proof using the following interesting Lemma 2.9:

Lemma 2.9 *Let K be a real closed field with notation as in 2.1, $\Phi : V \times V \to \mathbb{C}_K$ be a positive definite $\mathbb{C}_K$-Hermitain form on a n-dimensional $\mathbb{C}_K$-vector space V and let $f : V \to V$ be a $\mathbb{C}_K$-linear operator on V. Then there exists an orthonormal basis $\boldsymbol{x} = (x_1, \ldots, x_n)$ of V w. r. to Φ such that the matrix $\mathcal{M}_{\boldsymbol{x}}^{\boldsymbol{x}}(f)$ of f w. r. to $\boldsymbol{x}$ is an upper triangular matrix.*

Theorem 2.10 *Let K be a real closed field with notation as in 2.1 and let $\mathcal{C} \in \mathrm{M}_n(\mathbb{C}_K)$ be a Hermitian matrix. Then all the eigenvalues of $\mathcal{C}$ are in K and $\mathcal{C}$ is of type (p, q), where p is the number of positive eigenvalues and q is the number of negative eigenvalues of $\mathcal{C}$, counted with their multiplicities in the characteristic polynomial $\chi_{\mathcal{C}}$ of $\mathcal{C}$.*

Corollary 2.11 *Let K be a real closed field with notation as in 2.1 and let $\mathcal{C} \in \mathrm{M}_n(\mathbb{C}_K)$ be a Hermitian matrix. Then the characteristic polynomial $\chi_{\mathcal{C}} = c_0 + c_1 X + \cdots + c_{n-1}X^{n-1} + X^n$ belongs to $K[X]$ and $\mathcal{C}$ is of type (p, q), where p is the number of sign changes in the sequence $c_0, c_1, \ldots, c_{n-1}, c_n = 1$ and q is the number of sign changes in the sequence $c_0, -c_1, \ldots, (-1)^{n-1}c_{n-1}, (-1)^n c_n = (-1)^n$. If $c_0 = c_1 = \cdots = c_{r-1} = 0$ and $c_r \neq 0$, then $p + q = n - r$.*

Proof Note that, since all the eigenvalues of $\mathcal{C}$ are real by Theorem 2.10, indeed $\chi_{\mathcal{C}} \in K[X]$. The assertion is immediate from Theorem 2.10 and the classical Descartes' rule of signs. □

We now recall (from [4]) that "being of type (p, q)" is an open property (with respect to the *strong topology*[4]) which is an easy consequence of Hurwitz's Criterion 2.5:

[4] **Strong topology** Let K be a real closed field (see Footnote 1). Then K is equipped with the *order topology* which is determined by the base of the open intervals $]a, b[$, $a, b \in K$, $a < b$. The K-vector spaces K^n, $n \in \mathbb{N}$, are endowed with the *product topology* (with the base given by the open cuboids $]a_1, b_1[\times \cdots \times]a_n, b_n[$, $a_i < b_i$, $i = 1, \ldots, n$). With the ordered and product topology, the addition, the multiplication and the inverse are continuous functions on $K \times K$ and $K^\times = K \setminus \{0\}$, respectively. Further, polynomial functions (resp. rational functions F/G, $F, G \in K[X_1, \ldots, X_n]$, $G \neq 0$), in n variables are continuous K-valued functions on K^n (resp. on $K^n \smallsetminus \mathrm{V}_K(G)$, where $\mathrm{V}_K(G) := \{a \in K \mid G(a) = 0\}$ is a zero set of the denominator G which is closed in K^n).

The product topology on K^n transfers uniquely to every n-dimensional K-vector space by a K-linear isomorphism $f : V \to K^n$. Any other isomorphism $g : V \to K^n$ defines the same topology, since $gf^{-1} : K^n \to K^n$ and $(gf^{-1})^{-1} = fg^{-1} : K^n \to K^n$ are continuous (polynomial) maps. Therefore, polynomial and rational functions are also defined on any finite dimensional vector space V by an isomorphism $f : V \to K^n$. This topology on V may be characterized as the smallest topology for which the K-linear functions $V \to K$ are continuous and is called the *strong topology* on V, since it is stronger than the *Zariski topology* on V if $V \neq 0$.

Lemma 2.12 (cf. [4, Lemma 1.2]) *Let K be a real closed field with notation as in 2.1 and $F_{ij} \in K[T]$ be polynomials such that $F_{ij} = F_{ji}$, $1 \le i, j \le n$. Suppose that the bilinear form defined by the symmetric matrix $(F_{ij}(s))_{1 \le i,j \le n} \in \mathrm{M}_n(K)$ at $s \in K$, is non-degenerate, then there exists an $\varepsilon > 0$ such that the type of the symmetric matrices $(F_{ij}(t))_{1 \le i,j \le n}$ is the same for all $t \in]\, s - \varepsilon, s + \varepsilon\, [$. In particular, for non-degenerate symmetric bilinear forms over K, "being of type (p, q)" is an open property.*

We end this section by noting the following important Rigidity Theorem for symmetric bilinear forms (see [4]) which is proved by using Hurwitz's Criterion 2.5, Lemma 2.12 and the *Intermediate Value Theorem for polynomial functions*.[5]

Rigidity Theorem for symmetric bilinear forms 2.13 (cf. [4, 1.3]) *Let K be a real closed field with notation as in 2.1 and let $R_{ij}(t) = R_{ij}(t_1, \ldots, t_n)$, $1 \le i\,,\ j \le n$, be rational functions on a line-connected[6] subset $U \subseteq K^n$ such that the matrices $\mathcal{R}(t) = (R_{ij}(t))_{1 \le i,j \le n} \in \mathrm{M}_n(K)$, $t \in U$, are symmetric, i. e. $R_{ij} = R_{ji}$ for all $1 \le i\,,\ j \le n$ with $\mathrm{Det}\ \mathcal{R}(t) \ne 0$ for all $t \in U$. Then all the matrices $\mathcal{R}(t) \in \mathrm{M}_n(K)$, $t \in U$, have the same type (p, q), or equivalently, the same signature $p - q$.*

3 Trace Forms and Rational Points

In this section, we recall the results from [4] (based on the talk of Prof. U. Storch at IIT Bombay in November 2009) on trace forms, their invariants such as rank, type, signature and their relations with the number of rational points of a finite algebra A over a real closed field. For detailed proofs of these results, the reader is recommended to see [4, § 3].

Preliminaries 3.1 In this subsection, we recall the basic concepts from elementary commutative algebra (see [2, 13, 15]) which are used in this article.

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[5] **Intermediate Value Theorem for polynomial functions** *Let K be a real closed field and $F \in K[T]$ be a polynomial with coefficients in K such that $F(a)F(b) < 0$ for some $a, b \in K$. Then F has a zero in* $[a, b]$. In other words, the values $F(t)$, $t \in [a, b]$, have the same sign if F has no zero on $[a, b]$. In particular, every polynomial of odd degree has a zero in K. A field with this property is called a 2*-field*. Therefore, *a real closed field is a* 2*-field*. Furthermore, every monic polynomial F over a real closed field K has a positive zero in K if $F(0) < 0$ (since $F(x) > 0$ for "large" x).

[6] **Line-connected subsets** Let V be a vector space over a real closed field K. For two points $x, y \in V$, the subset $[x, y] = [y, x] := \{(1-t)x + ty \mid t \in K, 0 \le t \le 1\} \subseteq V$ is called the (closed) *line-segment* connecting x and y. For $x_0, \ldots, x_r \in V$, $r \ge 1$, the subset $[x_0, \ldots, x_r] := \cup_{i=1}^{r} [x_{i-1}, x_i]$ is called the *broken line from x_0 to x_r*. A subset $V' \subseteq V$ is called *line-connected* if for any two points $x, y \in V'$ there is a broken line from x to y which lies entirely in V'. Note that, if $K = \mathbb{R}$ and $U \subseteq V$ is open (in the strong topology, see Footnote 4), then the notion "line-connected" is equivalent to the topological notion of "connected". The only topologically connected subspaces of $K = \mathbb{Q}$ are the singletons. If V is a line, i. e. 1-dimensional, and if $x \in V$, then $V \setminus \{x\}$ is not line-connected. However, if $\mathrm{Dim}_K V \ge 2$, then $V \setminus \{x\}$ is always line-connected: If $u, w \in V \setminus \{x\}$ are arbitrary points, there is always a point $v \in V \setminus \{x\}$ such that $[u, v, w] \subseteq V \setminus \{x\}$.

Let A be an arbitrary commutative ring (with unity). The set Spec$A$ (resp. Spm A) of prime (resp. maximal) ideals in A is called the *prime* (resp. *maximal*) *spectrum* of A. The nilradical $\mathfrak{n}_A := \sqrt{0} = \cap_{\mathfrak{p}\in \operatorname{Spec} A} \mathfrak{p}$ is the intersection of all prime ideals in A. More generally, (*Formal Nullstellensatz*) $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p}\in \operatorname{Spec} A} \{\mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ for every ideal $\mathfrak{a}$ in A.

The Jacobson radical of A, $\mathfrak{m}_A := \cap_{\mathfrak{m}\in \operatorname{Spm} A} \mathfrak{m}$, is the intersection of all maximal ideals in A.

(a) The K-Spectrum and the set of K-rational points of a K-algebra. (see [15]) Let K be a field. Using the universal property of the polynomial algebra $K[X_1, \ldots, X_n]$, the affine space K^n can be identified with the set $\operatorname{Hom}_{K\text{-alg}}(K[X_1, \ldots, X_n], K)$ of K-algebra homomorphisms, where the point $a = (a_1, \ldots, a_n) \in K^n$ is identified with the substitution homomorphism $\xi_a : K[X_1, \ldots, X_n] \to K$, $X_i \mapsto a_i$. The kernel of ξ_a is $\operatorname{Ker} \xi_a = \mathfrak{m}_a = \langle X_1 - a_1, \ldots, X_n - a_n\rangle$ a maximal ideal in $K[X_1, \ldots, X_n]$ and ξ_a induces a K-algebra isomorphism $K[X_1, \ldots, X_n]/\mathfrak{m}_a \xrightarrow{\sim} K$. Moreover, every maximal ideal $\mathfrak{m}$ in $K[X_1, \ldots, X_n]$ with $K[X_1, \ldots, X_n]/\mathfrak{m} = K$ is of the type $\mathfrak{m}_a$ for a unique point $a = (a_1, \ldots, a_n) \in K^n$, where the i-th component a_i of a is determined by the congruence relation $X_i \equiv a_i \bmod \mathfrak{m}$.
The subset $K\text{-Spec}\, K[X_1, \ldots, X_n] := \{\mathfrak{m}_a \mid a \in K^n\}$ of $\operatorname{Spm} K[X_1, \ldots, X_n]$ is called the K-*spectrum* of $K[X_1, \ldots, X_n]$. We have the identifications:

$$\begin{array}{ccccc} K^n & \longleftrightarrow & \operatorname{Hom}_{K\text{-alg}}(K[X_1, \ldots, X_n], K) & \longleftrightarrow & K\text{-Spec}\, K[X_1, \ldots, X_n], \\ a & \longleftrightarrow & \xi_a & \longleftrightarrow & \mathfrak{m}_a = \operatorname{Ker} \xi_a . \end{array}$$

More generally, for any K-algebra A, the map

$$\operatorname{Hom}_{K\text{-alg}}(A, K) \longrightarrow \{\mathfrak{m} \in \operatorname{Spm} A \mid A/\mathfrak{m} = K\}, \xi \longmapsto \operatorname{Ker} \xi,$$

is bijective. Therefore, we make the following definition:
For any K-algebra A of finite type, the subset $K\text{-Spec}\, A := \{\mathfrak{m} \in \operatorname{Spm} A \mid A/\mathfrak{m} = K\}$ is called the K-*spectrum* of A and is denoted by K-Spec A.

Further, if $A \xrightarrow{\sim} K[X_1, \ldots, X_n]/\mathfrak{A}$ is a representation of the finite K-algebra A, then the K-algebraic set $\operatorname{V}_K(\mathfrak{A}) := \{a \in K^n \mid F(a) = 0 \text{ for all } F \in \mathfrak{A}\}$ defined by the ideal $\mathfrak{A}$ is called the set of K-*rational points* of A.

Under the above bijective maps, we have the identification $\operatorname{V}_K(\mathfrak{A}) = \operatorname{Hom}_{K\text{-alg}}(A, K) = K\text{-Spec}\, A$. For example, since $\mathbb{C}$ is an algebraically closed field, $\operatorname{Spm} \mathbb{C}[X] = \mathbb{C}\text{-Spec}\, \mathbb{C}[X]$, but $\mathbb{R}\text{-Spec}\, \mathbb{R}[X] \subsetneqq \operatorname{Spm} \mathbb{R}[X]$. In fact, the maximal ideal $\mathfrak{m} := \langle X^2 + 1\rangle \in \operatorname{Spm} \mathbb{R}[X]$ does not belong to $\mathbb{R}\text{-Spec}\, \mathbb{R}[X]$. More generally, a field K is algebraically closed if and only if $\operatorname{Spm} K[X] = K\text{-Spec}\, K[X]$, see [2, 13] or [8, Theorem 2.10, HNS 3].

(b) Local components of a finite algebra. Let A be a finite algebra over a field K, i.e. A finite dimensional as a K-vector space of dimension $\operatorname{Dim}_K A$. Then $\operatorname{Spm} A = \operatorname{Spec} A$ (since any finite K-algebra which is an integral domain is a field). Moreover, from the Chinese Remainder Theorem, it follows that $\operatorname{Spm} A$ *is a finite set*. In particular, $\#\operatorname{Spm} A \leq \operatorname{Dim}_K A$ and equality holds if and only if A is isomorphic to the product K-algebra $K^{\operatorname{Dim}_K A}$).

Further, let $\operatorname{Spm} A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. Then the unit group $A^\times$ of A is $A \setminus \bigcup_{i=1}^r \mathfrak{m}_i$ and the canonical homomorphism $A \longrightarrow \prod_{i=1}^r A_{\mathfrak{m}_i}$ is injective (where $A_\mathfrak{p}$ denotes the localization of A at a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$). In our special case, it is also surjective and hence an isomorphism, cf. [17, Corollary 55.16]. Therefore, *A is the direct product of the local finite K-algebras $A_i := A_{\mathfrak{m}_i}$, $i = 1, \ldots, r$*, which are called the *local components* of A. Furthermore, we have: $\operatorname{Dim}_K A = \sum_{i=1}^r \operatorname{Dim}_K A_i = \sum_{i=1}^r \ell(A_i) \cdot [K_i : K]$, where, for $i = 1, \ldots, r$, $K_i = A/\mathfrak{m}_i$ is the residue class field of A at $\mathfrak{m}_i$ and $\ell(A_i)$ the (finite) length of A_i, i.e. the length ℓ of a composition series $0 = \mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_\ell = A_i$ with $\mathfrak{a}_{j+1}/\mathfrak{a}_j \cong A/\mathfrak{m}_i$, $j = 0, \ldots, \ell - 1$.

For example, if K is a 2-*field*,[7] then $[K_i : K]$ is even if K_i is a non-trivial field extension of K and, in particular, *K-Spec $A \neq \emptyset$ if $\operatorname{Dim}_K A$ is odd*.

Further, $\mathfrak{m}_A = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \mathfrak{n}_A$ and $\mathfrak{m}_A = \mathfrak{n}_A = 0$, i.e. *$A$ is reduced, if and only if $A = K_1 \times \cdots \times K_r$ is the product of its residue class fields*. Moreover, if all the field extensions K_i of K are separable, then A is called a (finite) *separable* K-algebra.

The trace form 3.2 Let A be a finite algebra over a field K. The *trace form* on A over K is the symmetric K-bilinear form $\operatorname{Tr} := \operatorname{Tr}_K^A : A \times A \to K$, $(f, f') \mapsto \operatorname{Tr}_K^A(ff')$ on A. It is a classical tool used to study the K-algebra A.

The decomposition of $A = A_1 \times \cdots \times A_r$ into its local components (cf. 3.1 (b)) yields the orthogonal decomposition

$$\operatorname{Tr}_K^A = \operatorname{Tr}_K^{A_1} \oplus \cdots \oplus \operatorname{Tr}_K^{A_r}$$

of the trace form, i.e. for every $f = (f_1, \ldots, f_r)$, $f' = (f'_1, \ldots, f'_r) \in A = A_1 \times \cdots \times A_r$, we have $\operatorname{Tr}_K^A(ff') = \operatorname{Tr}_K^{A_1}(f_1 f'_1) + \cdots + \operatorname{Tr}_K^{A_r}(f_r f'_r)$.

The degeneration space $A^\perp = A^{\perp_{\operatorname{Tr}}} = \{f \in A \mid \operatorname{Tr}(Af) = 0\}$ is an ideal in A.

In Theorem 4.5, we will use the *trace form* $\Phi_h : A \times A \to K$, $(f, f') \mapsto \operatorname{Tr}_K^A(hff')$, *on A associated to an element $h \in A$*. Note that if K is a real closed field and if $L \mid K$ is a finite field extension, then (since Tr_K^L is non-degenerate) for every $h \in L$, $h \neq 0$, the symmetric bilinear forms Φ_h on L are non-degenerate and $\Phi_1 = \operatorname{Tr}_K^L$ and $\Phi_{-1} = -\operatorname{Tr}_K^L$.

For finite reduced $\mathbb{R}$-algebras $A = \mathbb{R}[X_1, \ldots, X_n]/\mathfrak{A}$, $n \leq 2$, the trace forms associated to elements of A has been studied by Hermite, see [9, 10].

Lemma 3.3 (cf. [4, Lemma 3.1]) *Let A be a finite algebra over an arbitrary field K and let $A^\perp$ be the degeneration space of the trace form Tr_K^A. Then the nilradical $(\mathfrak{m}_A =)\, \mathfrak{n}_A \subseteq A^\perp$. Moreover, equality holds if and only if all the residue class*

[7] A field K is called a 2-*field* if every polynomial of odd degree over K has a zero in K.

fields of A are separable over K, i. e. if and only if the reduction $A_{\text{red}} = A/\mathfrak{n}_A$ is a separable K-algebra. —In particular, the trace form is non-degenerate if and only if A is a separable K-algebra.

Corollary 3.4 *Let A be a finite separable algebra over an arbitrary field K. Then*

$$\operatorname{rank}\ \operatorname{Tr}_K^A = \operatorname{Dim}_K(A/\mathfrak{m}_A) = \sum_{i=1}^{r} [K_i : K].$$

Moreover, if K is an ordered field, then:

$$\operatorname{type}\ \operatorname{Tr}_K^A = \sum_{i=1}^{r} \operatorname{type} \operatorname{Tr}_K^{K_i} \quad \textit{and} \quad \operatorname{sign} \operatorname{Tr}_K^A = \sum_{i=1}^{r} \operatorname{sign}\ \operatorname{Tr}_K^{K_i}.$$

Now, we state the following important and classical criterion for the existence of K-rational points for real closed fields which is proved in [4].

Theorem 3.5 ([4, Theorem 3.2]) *Let A be a finite algebra over a real closed field K. Then:*

$$\operatorname{sign}\ \operatorname{Tr}_K^A = \#\, K\text{-}\operatorname{Spec} A.$$

In particular, K is a residue class field of A if and only if sign $\operatorname{Tr}_K^A \neq 0$.

Example 3.6 Let K be a real closed field and $\mathbb{C}_K = K[\mathrm{i}]$, $\mathrm{i}^2 = -1$ (the algebraic closure of K). The Gram's matrix of the trace form $\operatorname{Tr}_K^{\mathbb{C}_K}$ of $\mathbb{C}_K$ over K with respect to the basis $\{1, \mathrm{i}\}$ is the matrix

$$\begin{pmatrix} \operatorname{Tr}(1) & \operatorname{Tr}(\mathrm{i}) \\ \operatorname{Tr}(\mathrm{i}) & \operatorname{Tr}(-1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Therefore type $\operatorname{Tr}_K^{\mathbb{C}_K} = (1, 1)$ and sign $\operatorname{Tr}_K^{\mathbb{C}_K} = 0$.

Corollary 3.7 *Let A be a finite algebra over a real closed field K. Then the trace form Tr_K^A is positive definite if and only if A is separable over K and A splits over K, i. e. there exists an isomorphism of K-algebras $A \xrightarrow{\sim} K^{\operatorname{Dim}_K A}$.*

Corollary 3.8 *Let K be a real closed field and $f \in K[X]$ be a monic polynomial. Then all zeros of f (in $\overline{K}$) belong to K and are simple if and only if the trace form Tr_K^A of the K-algebra $A := K[X]/\langle f \rangle$ is positive definite.*

4 Counting Rational Points of Finite Affine Algebraic Sets

In this section we will apply results from Sect. 3 on trace forms to count the rational points of finite affine algebraic sets over real closed fields. Our method is a modern version of old results of Hermite and Sylvester who had used signatures of quadratic forms to count real zeros of polynomials in one variable, see [9, 10, 19]. We use elementary commutative algebra to treat the multivariate versions of these problems.

Notation and Consequences 4.1 Throughout this section, we use the following notation and assumptions and their consequences:

Let K be a real closed field with notation as in 2.1 and $\mathfrak{A}$ be a non-unit radical ideal in the polynomial ring $K[X_1, \ldots, X_n]$ over K, $\mathrm{V}_K(\mathfrak{A}) := \{a \in K^n \mid F(a) = 0 \text{ for all } F \in \mathfrak{A}\}$ be the affine algebraic set in K^n defined by $\mathfrak{A}$ and $A := K[X_1, \ldots, X_n]/\mathfrak{A}$. Further, let $\langle\mathfrak{A}\rangle = \mathfrak{A}\mathbb{K}[X_1, \ldots, X_n]$ be the extended ideal in the polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$ over $\mathbb{K}$, $\mathrm{V}_{\mathbb{K}}(\langle\mathfrak{A}\rangle) := \{a \in \mathbb{K}^n \mid F(a) = 0 \text{ for all } F \in \mathfrak{A}\}$ be the affine algebraic set in $\mathbb{K}^n$ defined by $\langle\mathfrak{A}\rangle$ and $A_{\mathbb{K}} := \mathbb{K} \otimes A = \mathbb{K}[X_1, \ldots, X_n]/\langle\mathfrak{A}\rangle$. Then A is a reduced and hence $A_{\mathbb{K}}$ is also a reduced[8] (since K is perfect).

Polynomials in $K[X_1, \ldots, X_n]$ are denoted by capital letters $F, G, H, \ldots$ and their images in the K-algebra A are denoted by small letters $f, g, h, \ldots$.

Every element $f \in A$ defines a (regular or polynomial) function on $\mathrm{V}_K(\mathfrak{A})$, namely $f : \mathrm{V}_K(\mathfrak{A}) \longrightarrow K, a \longmapsto f(a)$. Further, if $f, g \in A$, then, clearly:

$f = g$ on $\mathrm{V}_K(\mathfrak{A}) \iff f = g$ in $A \iff F \equiv G \pmod{\mathfrak{A}}$, i. e. $F - G \in \mathfrak{A}$.

We assume that A is finite dimensional K-vector space and put $\mathrm{Dim}_K\, A := m \in \mathbb{N}^+$. Then $A_{\mathbb{K}}$ is finite dimensional over $\mathbb{K}$ with $\mathrm{Dim}_{\mathbb{K}} = m$ and $\mathrm{V}_K(\mathfrak{A}) \subseteq \mathrm{V}_{\mathbb{K}}(\mathfrak{A})$ are finite sets with $\#\big(\mathrm{V}_{\mathbb{K}}(\mathfrak{A})\big) = m$ (since $\mathbb{K}$ is algebraically closed). Further, since $\mathfrak{A} \subseteq K[X_1, \ldots, X_n]$, it follows that if $\mathbf{a} \in \mathrm{V}_{\mathbb{K}}(\mathfrak{A})$, then its conjugate $\bar{\mathbf{a}} \in \mathrm{V}_{\mathbb{K}}(\mathfrak{A})$, too. Therefore, after renumbering we assume that:

4.1.a $\mathrm{V}_K(\mathfrak{A}) = \{\mathbf{a}_1, \ldots, \mathbf{a}_r\} \subseteq \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) = \{\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{a}_{r+1}, \bar{\mathbf{a}}_{r+1}, \ldots, \mathbf{a}_{r+s}, \bar{\mathbf{a}}_{r+s}\}$, where $r := \#\mathrm{V}_K(\mathfrak{A})$, $r + s = \#\,\mathrm{Spm}\, A$ and $m := r + 2s = \mathrm{Dim}_K\, A = \mathrm{Dim}_{\mathbb{K}}\, A_{\mathbb{K}} = \#\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$.

Furthermore, since K is a real closed field, $\mathrm{Char}\, K = 0$. In particular, K is infinite and hence by a linear change of coordinates (over K) (for instance, $Y_i = X_i$ for all $i = 1, \ldots, n-1$ and $Y_n = X_n + \sum_{i=1}^{n-1} X_i\, t^i$ for suitable $t \in K$ avoiding

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8 *Let $L \mid K$ be a separable field extension and A be a noetherian reduced K-algebra. Then $L \otimes_K A$ is also reduced.*

PROOF Since A is noetherian and reduced, $\mathrm{Ass}_A A$ is finite and the natural ring homomorphism $A \to \prod_{\mathfrak{p} \in \mathrm{Ass}_A A} A/\mathfrak{p}$ is injective. Therefore, without loss of generality, we may assume that $L \mid K$ is finite and A is an integral domain. Further, since $L \otimes_K A \subseteq L \otimes_K \mathrm{Q}(A)$, where $\mathrm{Q}(A)$ is the quotient field of A, it is enough to prove that $L \otimes_K A \subseteq L \otimes_K \mathrm{Q}(A)$ is reduced. Since $L \mid K$ is separable, $L = K[x] == K[X]/\langle \mu_{x,K} \rangle$, where $\mu_{x,K} \in K[X]$ is the minimal monic polynomial of the primitive element $x \in L$ which splits into distinct linear factors over the algebraic closure $\overline{K}$ of K. Therefore $\mu_{x,K}$ also splits into distinct linear factors over the algebraic closure of $\mathrm{Q}(A)$, too and $L \otimes_K \mathrm{Q}(A) = K[X]/\langle \mu_{x,K} \rangle \otimes_K \mathrm{Q}(A) = \mathrm{Q}(A)[X]/\langle \mu_{x,K} \rangle$ is reduced. □

finitely many $t \in K$), we may assume that $\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$ is in *general X_n-position*, or the ideal $\mathfrak{A}$ is in *general X_n-position* (The intention is to separate all zeros in an algebraic closure of K by their last coordinate), i. e.:

4.1.b The n-th coordinates $a_{i\,n}$ of the points $\mathbf{a}_i = (a_{i\,1}, \ldots, a_{i\,n}) \in \mathbb{K}^n$, $i = 1, \ldots, m$ are distinct.

Note that $\mathrm{V}_K(\mathfrak{A}) = \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \cap K^n$ is the set of K-rational points of

$$\mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \xrightarrow{\sim} \mathbb{K}\text{-Spec}\,A_{\mathbb{K}} = \mathrm{Spm}\,A_{\mathbb{K}} = \mathrm{Spec}\,A_{\mathbb{K}}$$

(the first equality follows from Hilbert's Nullstellensatz, see [13] or [8, Theorem 2.10, HNS 3]) and $\mathrm{V}_K(\mathfrak{A}) \xrightarrow{\sim} K\text{-Spec}\,A \subseteq \mathrm{Spm}\,A = \mathrm{Spec}\,A$, see 3.1 (a). Further, since A and $A_{\mathbb{K}}$ are reduced, the local components (see 3.1 (b)) of A corresponding to the K-rational points $\mathbf{a}_i \in \mathrm{V}_K(\mathfrak{A})$, $i = 1, \ldots, r$, are isomorphic to K and corresponding to the points in $\{\mathbf{a}_{r+j}\,,\, \overline{\mathbf{a}}_{r+j} \mid j = 1, \ldots, s\} = \mathrm{Spm}\,A \smallsetminus K\text{-Spec}\,A$ are isomorphic to $\mathbb{K}$, but local components of $A_{\mathbb{K}}$ corresponding to all the points in $\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$ are all isomorphic to $\mathbb{K}$. Therefore the explicit structures of the K-algebra A and the $\mathbb{K}$-algebra $A_{\mathbb{K}}$ are determined by the algebra isomorphisms which are defined by the substitutions:

4.1.c $A \xrightarrow{\sim} K^r \times \mathbb{K}^s\,,\; h \mapsto (h\,(\mathrm{mod}\,\mathfrak{m}))_{\mathfrak{m}\in\mathrm{Spm}\,A}$, where r, s are as in 4.1.a and

$A_{\mathbb{K}} \xrightarrow{\sim} \mathbb{K}^m,\; f \mapsto (f(\mathbf{a}))_{\mathbf{a}\in\mathrm{V}_{\mathbb{K}}(\mathfrak{A})}$, where $m := r + 2s$.

Note that $m = \mathrm{Dim}_K\,A = \mathrm{Dim}_{\mathbb{K}}\,A_{\mathbb{K}} = \#\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$. Furthermore, the following eigenvector theorem (see [5, Ch. 2, §4, Theorem 4.5]) follows directly from 4.1.c:

4.1.d For every $h \in A$, the eigenvalues of the K-linear map $\lambda_h : A \to A,\; f \mapsto hf$ are the values $h(\mathbf{a}_1), \ldots, h(\mathbf{a}_r),\; h(\mathbf{a}_{r+1}), h(\overline{\mathbf{a}}_{r+1}) \ldots, h(\mathbf{a}_{r+s}), h(\overline{\mathbf{a}}_{r+s})$ of the function $h : \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \to \mathbb{K}$.

For more efficient determination of the type and signature of the trace form Tr_K^A, we need a nice basis of A over K. The following crucial key observation, so-called Shape Lemma (see [5, 6, 12]), guarantees a distinguished generating set for a radical ideal $\mathfrak{A}$ in $K[X_1, \ldots, X_n]$ whose residue-classs K-algebra $K[X_1, \ldots, X_n]/\mathfrak{A}$ is finite. We give a proof of the Shape Lemma by using the natural action of the Galois group $\mathrm{Gal}(\overline{K}|K)$ on $\mathrm{V}_{\overline{K}}(\mathfrak{A})$.

Shape Lemma 4.2 *Let K be an infinite perfect field and let $\mathfrak{A} \subseteq K[X_1, \ldots, X_n]$ be a non-unit radical ideal. Suppose that the K-algebra $A := K[X_1, \ldots, X_n]/\mathfrak{A}$ is a finite dimensional vector space over K with* $\mathrm{Dim}_K\,A = m \in \mathbb{N}^+$. *Then there exist polynomials $g_1, \ldots, g_{n-1},\; g_n \in K[X]$ (where X is an indeterminate over K) such that $g_n \neq 0$ square free of degree m and that $\mathfrak{A}$ is generated by $X_1 - g_1(X_n)\,, \ldots,\; X_{n-1} - g_{n-1}(X_n),\; g_n(X_n)$. In particular, $\mathbf{x} = \{1, x_n, \ldots, x_n^{m-1}\}$ is a K-basis of A, where x_n is the image of X_n in A.*

Proof Let $\overline{K}$ be an algebraic closure of K. Then, since K is perfect and A is reduced, $A_{\overline{K}} := \overline{K}[X_1, \ldots, X_n]/\langle\mathfrak{A}\rangle$ is reduced, too (see Footnote No. 8). Further, it is an m-dimensional $\overline{K}$-vector space and $\mathrm{V}_{\overline{K}}(\mathfrak{A}) := \{a \in \overline{K}^{\,n} \mid F(a) = 0 \text{ for all } F \in$

$\mathfrak{A}\}$ is a finite algebraic set in $\overline{K}^N$ with $\#\,\mathrm{V}_{\overline{K}}(\mathfrak{A}) = \mathrm{Dim}_{\overline{K}}\,A_{\overline{K}} = m$. Further, since K is infinite, by a linear change of coordinates (over K), we may assume that (see the argument as in 4.1.b) the n-th projection map $q_n : \mathrm{V}_{\overline{K}}(\mathfrak{A}) \to \overline{K}$, $(a_1, \ldots, a_n) \mapsto a_n$ is injective. Furthermore, since K is perfect, the field extension $\overline{K}|K$ is a Galois extension. Let $\mathrm{Gal}(\overline{K}|K)$ be its Galois group. The Galois group $\mathrm{Gal}\,(\overline{K}|K)$ operates on $\mathrm{V}_{\overline{K}}(\mathfrak{A})$ and its images $q_i(\mathrm{V}_{\overline{K}}(\mathfrak{A}))$ under the i-th projections $q_i : \mathrm{V}_{\overline{K}}(\mathfrak{A}) \to \overline{K}$, $(a_1, \ldots, a_n) \mapsto a_i$, $i = 1, \ldots, n$, with the natural operations:

$$\mathrm{Gal}\,(\overline{K}|K) \times \mathrm{V}_{\overline{K}}(\mathfrak{A}) \longrightarrow \mathrm{V}_{\overline{K}}(\mathfrak{A}),\ (\sigma, (a_1, \ldots, a_n)) \longmapsto (\sigma(a_1), \ldots, \sigma(a_n)) \ \text{ and}$$

$$\mathrm{Gal}\,(\overline{K}|K) \times q_i(\mathrm{V}_{\overline{K}}(\mathfrak{A})) \longrightarrow q_i(\mathrm{V}_{\overline{K}}(\mathfrak{A})),\ \ (\sigma, a_i) \longmapsto \sigma(a_i)\,,\ i = 1, \ldots, n$$

Obviously,

$$(*) \qquad\qquad W := q_n(\mathrm{V}_{\overline{K}}(\mathfrak{A})) = W_1 \uplus \cdots \uplus W_\ell$$

is the union of orbits of this operation and each orbit $W_k = \mathrm{V}_{\overline{K}}(\pi_k)$ is the zero set of the irreducible polynomial $\pi_k \in K[X]$, $k = 1, \ldots, \ell$, see [11] or [17, Ch. XI, §93, 93.2]. Therefore, since K is perfect, the polynomial $g_n := \pi_1 \cdots \pi_\ell \in K[X]$ is square free and $W = \mathrm{V}_{\overline{K}}(g_n)$, $\deg\, g_n = \#\mathrm{W} = \#\mathrm{V}_{\overline{K}}(\mathfrak{A}) = m$, since q_n is injective (see 4.1.b).

4.2.a For all $a_n \in q_n(\mathrm{V}_{\overline{K}}(\mathfrak{A}))$, there exist polynomials $g_i \in K[X]$ with $\deg g_i < \deg g_n = m$, $i = 1, \ldots, n-1$, such that $(g_1(a_n), \ldots, g_{n-1}(a_n), a_n)$ is the unique point lying over a_n.

To prove 4.2.a, let $a_n \in q_n(\mathrm{V}_{\overline{K}}(\mathfrak{A}))$ and $(a_1, \ldots, a_{n-1},\ a_n)$ be the unique point (since q_n is injective, see 4.1.b) lying over a_n. Renumbering in (*) above, we may assume that $W_1 = \{\sigma_j(a_n) \mid j = 1, \ldots, d,\ \sigma_1 = \mathrm{id}_{\overline{K}}\}$ is the orbit of a_n. Then $\#\,W_1 = d$ and for every $i = 1, \ldots, n-1$, the orbit of a_i is contained in $\{\sigma_j(a_i) \mid j = 1, \ldots, d\}$ (note that the elements $\sigma_j(a_i)$, $j = 1, \ldots, d$, may not be distinct).

Now, since $\sigma_j(a_n)$, $j = 1, \ldots, d$, are distinct elements in $\overline{K}$, by *Lagrange's Interpolation Formula*,[9] for each $i = 1, \ldots, n-1$, there exists a polynomial $g_i \in \overline{K}[X]$, $\deg\, g_i < d < \deg\, g_n$, such that $g_i(\sigma_j(a_n)) = \sigma_j(a_i)$ for all $j = 1, \ldots, d$. Moreover, $g_1, \ldots, g_{n-1} \in K[X]$, since $\sigma(g_i) = g_i$ for every $\sigma \in \mathrm{Gal}(\overline{K} \mid K)$.

Finally we claim the equality $\mathfrak{A}' := \langle X_1 - g_1(X_n), \ldots, X_{n-1} - g_{n-1}(X_n), g_n(X_n)\rangle = \mathfrak{A}$. To prove this note that the substitution homomorphism

$$K[X_1, \ldots, X_{n-1}, X_n] \to K[X_n],\, X_i \mapsto g_i(X_n),\, i = 1, \ldots, n-1 \text{ and } X_n \mapsto X_n,$$

[9] **Lagrange's Interpolation Formula:** *Let K be a field and let $x_1, \ldots, x_d \in K$ be distinct elements. Then for arbitrary elements $y_1, \ldots, y_d \in K$, there exists a polynomial $g \in K[X]$ of degree* $\deg\, g < d$ *such that $g(x_i) = y_i$ for every $i = 1, \ldots, d$.* For a proof consider the polynomial $g := \sum_{i=1}^{d} \frac{y_i}{z_i} \prod_{j \neq i} (X - x_j)$, where $z_i := \prod_{j \neq i} (x_i - x_j)$.

induces a K-algebra isomorphism $K[X_1,\ldots,X_n]/\mathfrak{A}' \xrightarrow{\sim} K[X_n]/\langle g_n\rangle$ and $K[X_1,\ldots,X_n]/\mathfrak{A}'$ is reduced, since g_n is separable over K. Therefore $\mathfrak{A}'$ is a radical ideal. Further, from 4.2.a it follows that $\mathrm{V}_{\overline{K}}(\mathfrak{A}') = \mathrm{V}_{\overline{K}}(\mathfrak{A})$. Now, use Hilbert's Nullstellensatz (see [2, 13] or [15, Theorem 2.10 , HNS 2]) to conclude the equality $\mathfrak{A}' = \mathfrak{A}$. □

Remark 4.3 The Shape Lemma 4.2 appeared first time in [6] which may be regarded as a natural generalization of the Primitive Element Theorem. Further, it gives a very useful presentation of the radical ideal $\mathfrak{A}$ which allows to find the solution space $\mathrm{V}_{\overline{K}}(\mathfrak{A})$ immediately, namely:

$$\mathrm{V}_{\overline{K}}(\mathfrak{A}) = \{(g_1(a),\ldots,g_{n-1}(a),a) \in \overline{K}^n \mid g_n(a) = 0\}.$$

In other words, the last coordinates are zeros of g_n and for a fixed last coordinate a_n, all the other coordinates are determined by evaluation of polynomials $g_{n-1},\ldots,g_1$ at a_n: $g_n(a_n) = 0,\ a_{n-1} = g_{n-1}(a_n),\ \ldots,\ a_1 = g_1(a_n)$. This simple shape of the solution space $\mathrm{V}_{\overline{K}}(\mathfrak{A})$ is quite convenient to work with. The primary decomposition of $\mathfrak{A}$ is given by the prime factorization of the polynomial g_n. Under the conditions on the polynomials $g_1,\ldots,g_{n-1},g_n \in K[X]$ as in the proof of the Shape Lemma 4.2, one can easily verify that $X_1 - g_1(X_n),\ldots,X_{n-1} - g_{n-1}(X_n),\ g_n(X_n)$ form a reduced (= minimal) Gröbner basis of the radical ideal $\mathfrak{A}$ relative to the lexicographic order $X_1 > X_2 > \cdots > X_n$. For a different proof of the Shape Lemma 4.2 see [12, Theorem 3.7.25] and a detailed recipe for solving systems of polynomial equations efficiently using the Shape Lemma 4.2 is also given in [12, Theorem 3.7.26]. The Shape Lemma 4.2 also appeared in [5, Ex. 16, § 4, Ch. 2].

Consequence and identifications 4.4 Let K be a real closed field, $\mathbb{K} := \mathbb{C}_K = K[\mathrm{i}]$, $\mathrm{i}^2 = -1$, the algebraic closure of K (see 2.1) and let $\mathfrak{A} \subseteq K[X_1,\ldots,X_n]$ a radical ideal. Suppose that $A := K[X_1,\ldots,X_n]/\mathfrak{A}$ is a finite dimensional K-vector space.

Let $g_1,\ldots,g_{n-1},g := g_n \in K[X]$ be the polynomials as in the statement of the Shape Lemma 4.2 and let $\varphi : A \xrightarrow{\sim} K[X]/\langle g\rangle$ be the K-algebra isomorphism induced by the substitution homomorphism $K[X_1,\ldots,X_{n-1},X_n] \to K[X_n]$, $X_i \mapsto g_i(X_n)$, $i = 1,\ldots,n-1$ and $X_n \mapsto X_n$ (see the proof of the Shape Lemma 4.2). Then, since g is square-free and K is a real closed field (see Footnote 1), $g = (X - a_1)\cdots(X - a_r)\pi_1\cdots\pi_s$, $a_i \in K$, $i = 1,\ldots r$ and $\pi_j = (X - z_j)(X - \overline{z}_j) \in K[X]$, $z_j \in \mathbb{K} \smallsetminus K$, $j = 1,\ldots,s$, where r, s and $m = r + 2s$ as in 4.1.a, since φ is a K-algebra isomorphism.

We use the above K-algebra isomorphism φ to identify A with $K[X]/\langle g\rangle$ and $\mathfrak{A}$ with $\langle g\rangle$. Let x be the image of X in A. Then $\boldsymbol{x} := \{1,x,\ldots,x^{m-1}\}$ is a K-basis of A and with the above identification, we have the equalities $\mathrm{V}_K(\mathfrak{A}) = \mathrm{V}_K(g) = \{a_1,\ldots,a_r\} \subseteq \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) = \mathrm{V}_{\mathbb{K}}(g) = \{a_1,\ldots,a_r,z_1,\overline{z}_1,\ldots,z_s,\overline{z}_s\}$, $r + 2s = m$.

Further, for $H \in K[X_1,\ldots,X_n]$, we put $h(X) := H(g_1(X),\ldots,g_{n-1}(X),X) \in K[X]$. Then using the above identifications, we have $h(x) \in A$, and the values $H(\mathbf{a}_i) \in K$, $i = 1,\ldots,r$, and $H(\mathbf{a}_{r+j})$, $H(\overline{\mathbf{a}}_{r+j}) \in \mathbb{K}$, $j = 1,\ldots,s$ are identified with the values $h(a_i) \in K$, $i = 1, \dots, r$, and $h(z_j)$, $h(\overline{z}_j) \in \mathbb{K}$, $j = 1, \dots, s$, respectively.

Theorem 4.5 *With the notation and consequences as in 4.1 and 4.4, let* $H \in K[X_1, \dots, X_n]$, *h be the image of H in A and let* $\Phi_h : A \times A \to K$, $(f, f') \mapsto \mathrm{Tr}_K^A(hff')$, *be the trace form on A associated to the element* $h \in A$ (see 3.2). *Then*:

(a) *The Gram's matrix* $\mathcal{G}_{\Phi_h}(\boldsymbol{x})$ *of* Φ_h *with respect to the K-basis* $\boldsymbol{x}$, *is a symmetric matrix in* $\mathrm{M}_m(K)$. *Moreover,* $\mathcal{G}_{\Phi_h}(\boldsymbol{x}) = \mathcal{V}\,\mathcal{D}_h\,{}^{\mathrm{t}}\mathcal{V}$, *where* $\mathcal{V} \in \mathrm{GL}_m(\mathbb{K})$ *is the Vandermonde's matrix*[10] *of the elements* $a_1, \dots, a_r, z_1, \dots, z_s, \overline{z}_1, \dots, \overline{z}_s \in \mathbb{K}$ *and* $\mathcal{D}_h \in \mathrm{M}_m(\mathbb{K})$ *is the diagonal matrix with diagonal entries* $h(a_1), \dots, h(a_r), h(z_1), \dots, h(z_s), h(\overline{z}_1), \dots, h(\overline{z}_s)$.

(b) *Let* $p_H := \#\{\mathbf{a} \in \mathrm{V}_K(\mathfrak{A}) \mid H(\mathbf{a}) > 0\}$, $q_H := \#\{\mathbf{a} \in \mathrm{V}_K(\mathfrak{A}) \mid H(\mathbf{a}) < 0\}$ *and let* $s_H := \#\{j \mid j = 1, \dots, s, h(z_j) \neq 0\} = \frac{1}{2}\,\#\left[\mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \smallsetminus \left(\mathrm{V}_K(\mathfrak{A}) \cup \mathrm{V}_{\mathbb{K}}(H)\right)\right]$. *Then:*

$$\text{type } \Phi_h = (p_H + s_H\,,\ q_H + s_H) \ \textit{and}$$

$$\text{rank } \Phi_h = p_H + q_H + 2s_H = \#\{\mathbf{a} \in \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \mid H(\mathbf{a}) \neq 0\} = \#\left(\mathrm{V}_{\mathbb{K}}(\mathfrak{A}) \smallsetminus \mathrm{V}_{\mathbb{K}}(H)\right).$$

In particular, $\text{sign } \Phi_h = p_H - q_H$.

Proof Recall from 4.1 that:

$$\mathrm{V}_K(\mathfrak{A}) = \{\mathbf{a}_1, \dots, \mathbf{a}_r\} \subseteq \mathrm{V}_{\mathbb{K}}(\mathfrak{A}) = \{\mathbf{a}_1, \dots, \mathbf{a}_r\,,\ \mathbf{a}_{r+1}\,,\ \overline{\mathbf{a}}_{r+1}\,,\ \dots\,,\ \mathbf{a}_{r+s}\,,\ \overline{\mathbf{a}}_{r+s}\},$$

where $r := \#\,\mathrm{V}_K(\mathfrak{A})$, $r+s = \#\ \mathrm{Spm}\ A$ and $m = r+2s = \mathrm{Dim}_K\, A = \mathrm{Dim}_{\mathbb{K}}\, A_{\mathbb{K}} = \#\,\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$ and that $\mathrm{V}_{\mathbb{K}}(\mathfrak{A})$ is in general X_n-position, see 4.1.a and 4.1.b.

(a) From the indentifications in 4.4, it follows that for $0 \le k\,,\ \ell \le m-1$, the (k, ℓ)-entry in the Gram's matrix $\mathcal{G}_{\Phi_h}(1, x, \dots, x^{m-1}) = \left(\Phi_h(x^k,\ x^\ell)\right)_{0 \le k\,,\,\ell \le m-1}$ is:

4.5.1

$$\begin{aligned}\mathrm{Tr}_K^A(h(x)\,x^{k+\ell-2}) &= \sum_{z \in \mathrm{V}_{\mathbb{K}}(g)} h(z)\,z^{k+\ell-2} \\ &= \sum_{i=1}^{r} h(a_i)\,a_i^{k+\ell-2} + \sum_{j=1}^{s}\Big(h(z_j)\,z_j^{k+\ell-2} + h(\overline{z}_j)\,\overline{z}_j^{k+\ell-2}\Big).\end{aligned}$$

[10] **Vandermonde's matrix** For elements $a_1, \dots, a_m$ is a field K, the matrix $\mathcal{V}(a_1, \dots, a_m) := (a_i^j)_{\substack{1 \le i \le m \\ 0 \le j \le m-1}} \in \mathrm{M}_m(K)$ is called the *Vanderminde's matrix of the elements* $a_1, \dots, a_m$. The elements $a_1, \dots, a_m$ are pairwise distinct if and only if $\mathcal{V}(a_1, \dots, a_m) \in \mathrm{GL}_m(K)$.

Now, by the Fundamental Theorem on Symmetric Polynomials (see [17, Theorem 54.13], the right hand side of 4.5.1 is a polynomial in the coefficients of $h(X)$ and $g(X)$ (with coefficients in $\mathbb{Z}$) and hence belongs to K. Therefore $\mathcal{G}_{\Phi_h}(1, x, \ldots, x^{m-1})$ is a symmetric matrix in $\mathrm{M}_m(K)$. Furthermore, using the equation 4.5.1, the equality $\mathcal{G}_{\Phi_h}(1, x, \ldots, x^{m-1}) = \mathcal{V}\,\mathcal{D}_h\,{}^{\mathrm{t}}\mathcal{V}$, where $\mathcal{V}$ and $\mathcal{D}_h$ are as in the statement of **(a)**, can be easily verified.

(b) The assertion about the rank follows from the equality $\operatorname{rank}\Phi_h = \operatorname{rank}\mathcal{G}_{\Phi}(\boldsymbol{x}) = \operatorname{rank}\mathcal{D}_h$, since $\mathcal{V} \in \mathrm{GL}_m(\mathbb{K})$. Further, the local decomposition $A \xrightarrow{\sim} K^r \times \mathbb{K}^s$ (see 4.1.b) yields the orthogonal decomposition (see 3.2)

$$\Phi_h = (\Phi_h)_1^K \oplus \cdots \oplus (\Phi_h)_r^K \oplus (\Phi_h)_1^{\mathbb{K}} \oplus \cdots \oplus (\Phi_h)_s^{\mathbb{K}},$$

where $(\Phi_h)_i^K = \Phi_h|K$, is the restriction of Φ_h to the real component at $a_i \in K$ with Gram's matrix $\mathcal{G}_{(\Phi_h)_i^K}(1) = (h(a_i)) \in \mathrm{M}_1(K)$, $i = 1, \ldots, r$ and $(\Phi_h)_j^{\mathbb{K}} = \Phi_h|\mathbb{K}$, is the restriction of Φ_h to the non-real component $K[X]/\langle\pi_j\rangle \xrightarrow{\sim} \mathbb{K}$ at $\mathfrak{m}_j = \langle\pi_j\rangle \in \operatorname{Spm} A \smallsetminus K\text{-}\operatorname{Spec} A$, $j = 1, \ldots, s$. Furthermore, clearly,

$$\operatorname{type}(\Phi_h)_i^K = \operatorname{sign}(\Phi_h)_i^K = \operatorname{sign} h(a_i) = \operatorname{sign} H(\mathbf{a}_i) \quad \text{for all } i = 1, \ldots, r$$

and for each $j = 1, \ldots, s$, by Example 2.8 (since $\pi_j = (X - z_j)(X - \overline{z}_j)$, $z_j \in \mathbb{K} \smallsetminus K$), we have $(\Phi_h)_j^{\mathbb{K}} = 0$ if $h(z_j) = 0$ and $\operatorname{type}(\Phi_h)_j^{\mathbb{K}} = (1, 1)$ for all $j = 1, \ldots, s$. Therefore, by Corollary 3.4, we have:

$$\operatorname{type}\ \Phi_h = \sum_{i=1}^{r} \operatorname{type}(\Phi_h)_i^K + \sum_{j=r+1}^{r+s} \operatorname{type}(\Phi_h)_j^{\mathbb{K}} = (p_H + s\,,\ q_H + s)$$

and hence $\operatorname{sign}\ \Phi_h = p_H - q_H$. □

Remark 4.6 The Gram's matrix $\mathcal{G}_{\Phi_h}(1, x, \ldots, x^{m-1}) = \big(\mathrm{Tr}_K^A(h(x)\,x^{k+\ell-2})\big)_{0 \le k\,,\,\ell \le m-1}$ in the Theorem 4.5 (a) is a so-called *Hankel matrix* $\mathcal{H}(a_0, \ldots, a_{2m-2}) := (a_{k+\ell-2})_{0 \le k\,,\,\ell \le m-1} \in \mathrm{M}_m(K)$ of the sequence $a_0, \ldots, a_{2m-2} \in K$. There are efficient methods to determine rank, type and signature of a Hankel matrix by using a theorem of Frobenius, for a proof see [7, Ch. X § 10, Theorem 24].

Corollary 4.7 (Hermite) *Let K be a real closed field, $g \in K[X]$, $\deg g = m \ge 1$, and $A := K[X]/\langle g\rangle$. Then the* type $\mathrm{Tr}_K^A = (r + s\,,\ s)$, *where* $\mathrm{Tr}_K^A : A \times A \to K$, $(f, f') \mapsto \mathrm{Tr}_K(ff')$ *is the trace form on A, $r = \#\mathrm{V}_K(g)$ is the number of zeros of g in K and s is the half of the number of zeros of g in the algebraic closure $\mathbb{K}$ of K which are not in K. In particular,* $\operatorname{sign}\mathrm{Tr}_K^A = r = \#\,\mathrm{V}_K(g)$.

Proof Using the notation as in the Theorem 4.5, note that $\mathrm{Tr}_K^A = \Phi_1$ is the trace form associated to the constant polynomial $1 \in K[X]$. Therefore, by 4.5 (b), $p_1 =$

$r = \#\mathrm{V}_K(g)$, $q_1 = 0$ and type $\mathrm{Tr}_K^A = (p_1+s, q_1+s)$. Further, sign $\mathrm{Tr}_K^A = p_1 - q_1 = r = \#\,\mathrm{V}_K(g)$. Of course, the assertion also follows directly from Theorem 3.5. □

With the notation as in 4.1, our main goal is to relate the cardinality $\#\,\mathrm{V}_K(\mathfrak{A})$ with the signatures of the trace forms on the finite K-algebra A associated to its elements, see 3.2.

Notation 4.8 With the notation as in 4.1 and 4.4. Further, let $H \in K[X_1, \ldots\ldots, X_n]$ and $\mathrm{V}_K(H) := \{\mathbf{a} \in K^n \mid H(\mathbf{a}) = 0\}$ be the hypersurface in the affine n-space K^n defined by H. Then the complement of $\mathrm{V}_K(H)$ in K^n is the union of line-connected subsets (in the strong topology on K^n, see Footnote 4) on which H takes either all positive values or all negative values. With this we have the decomposition $K^n = \mathrm{V}_K(H) \uplus H^+ \uplus H^-$, where $H^+ := \{\mathbf{a} \in K^n \mid H(\mathbf{a}) > 0\}$ and $H^- := \{\mathbf{a} \in K^n \mid H(\mathbf{a}) < 0\}$.

Further, since $\mathrm{V}_K(\mathfrak{A}) = \left(\mathrm{V}_K(\mathfrak{A}) \cap H^+\right) \biguplus \left(\mathrm{V}_K(\mathfrak{A}) \cap H^-\right) \biguplus (\mathrm{V}_K(\langle \mathfrak{A}, H\rangle))$, we have:

4.8.a $\#\,\mathrm{V}_K(\mathfrak{A}) = \#\,\left(\mathrm{V}_K(\mathfrak{A}) \cap H^+\right) + \#\,\left(\mathrm{V}_K(\mathfrak{A}) \cap H^-\right) + \#\,(\mathrm{V}_K(\langle \mathfrak{A}, H\rangle))$, and hence to compute $\#\mathrm{V}_K(\mathfrak{A})$, we can use arbitrary polynomial $H \in K[X_1, \ldots, X_n]$ and compute the cardinalities $\#\,\mathrm{V}_K(\mathfrak{A}) \cap H^+$, $\#\,\mathrm{V}_K(\mathfrak{A}) \cap H^-$ and $\#\,\mathrm{V}_K(\langle \mathfrak{A}, H\rangle)$.

More precisely, we have:

Theorem 4.9 *With the notation as in 4.1 and 4.8. For $H \in K[X_1, \ldots\ldots, X_n]$, let $p_H = \#\,\mathrm{V}_K(\mathfrak{A}) \cap H^+$, $q_H = \#\,\mathrm{V}_K(\mathfrak{A}) \cap H^-$ be as in Theorem 4.5) and let h denote the image of H in $A = K[X_1, \ldots\ldots, X_n]/\mathfrak{A}$ and*

$$\Phi_h : A \times A \to K,\ (f, g) \mapsto \mathrm{Tr}_K^A(hfg) (resp. \Phi_{h^2} : A \times A \to K,\ (f, g) \mapsto \mathrm{Tr}_K^A(h^2 fg))$$

be the trace form on A associated to the element h (resp. h^2). Then:

(a) (Pederson-Roy-Szpirglas [16, Theorem 2.1], see also [5, Ch. 2, § 5, Theorem 5.2])

$$\text{sign } \Phi_h = p_H - q_H.$$

(b)

$$\text{sign } \Phi_{h^2} = p_H + q_H.$$

(c) *Let $\mathfrak{B} := \langle \mathfrak{A}, H\rangle$ be the ideal (in $K[X_1, \ldots, X_n]$) generated by $\mathfrak{A}$ and H. Then the K-algebra $B := K[X_1, \ldots, X_n]/\mathfrak{B}$ is finite over K and* sign $\mathrm{Tr}_K^B = \#\,\mathrm{V}_K(\mathfrak{B})$.

(d) *The three signatures* sign Φ_h, sign Φ_{h^2} *and* sign Tr_K^B *uniquely determine the natural numbers p_H, q_H and $\#\,\mathrm{V}_K(\mathfrak{B}) = \mathrm{V}_K(\mathfrak{A}) \cap \mathrm{V}_K(H)$. In particular, they determine the cardinality $\#\,\mathrm{V}_K(\mathfrak{A}) = p_H + q_H + \#\,\mathrm{V}_K(\mathfrak{B})$.*

Proof

(a): Proved in Theorem 4.5 (b).

(b): Since $H^2(\mathbf{a}) = H(\mathbf{a})\, H(\mathbf{a}) > 0$ for every $\mathbf{a} \in H^+ \cup H^-$ and $\mathrm{V}_K(H^2) = \mathrm{V}_K(H)$, from Theorem 4.5 (b) it follows that $\operatorname{sign}\,\Phi_{h^2} = p_H + q_H$.

(c): Since the K-algebra B is a homomorphic image of the K-algebra A, B is also finite over K. The equality $\operatorname{sign}\,\mathrm{Tr}_K^B = \#\mathrm{V}_K(\mathfrak{B})$ is immediate from Theorem 4.5 (a) ($H = 1$) or Theorem 3.5.

(d): Immediate from the formula 4.8.a for $\#\mathrm{V}_K(\mathfrak{A})$ and the parts (a), (b) and (c) above.

□

Remark 4.10 The role of a polynomial $H \in K[X_1, \ldots, X_n]$ in the Theorem 4.9 is to count the exact number of points in $\mathrm{V}_K(\mathfrak{A})$ which lie in the *signed components* H^+ and H^- determined by H. The system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_H \\ p_H \\ q_H \end{pmatrix} = \begin{pmatrix} \operatorname{sign}\,\mathrm{Tr}_K^A \\ \operatorname{sign}\,\Phi_h \\ \operatorname{sign}\,\Phi_{h^2} \end{pmatrix},$$

where $r_H := \#\,(\mathrm{V}_K(\mathfrak{A}, H)) = \#(\mathrm{V}_K(\mathfrak{A}) - p_H - q_H$, describes the results of the Theorem 4.9: the first row combines the equality 4.8.a with 3.5, the second row is the part (a) and the third row is the part (b). This is a first idea used in the efficient parallel algorithm developed by Ben-Or, Kozen, and Reif (see [3]), where they considered a family of polynomials $H_1, \ldots, H_k \in K[X_1, \ldots, X_n]$ and developed an efficient algorithm to count the number of points in $\mathrm{V}_K(\mathfrak{A})$ which lie in each of the sign-component determined by some conjunction of conditions: $H_1 \varepsilon_1 \wedge \cdots \wedge H_k\, \varepsilon_k$, where $\varepsilon_j \in \{< 0\,, = 0\,, > 0\}$, $j = 1, \ldots, k$. For example, if $H_1 = X_1$, $H_2 = X_2 \in \mathbb{R}[X_1, X_2]$, then the sign-components are X_1-axis, X_2-axis and four quadrants.

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Hermite Reciprocity and Schwarzenberger Bundles

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Dedicated to David Eisenbud on the occasion of his 75th birthday.

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1 Introduction

The goal of this article is to uncover a close relationship between Hermite reciprocity and cohomological properties of Schwarzenberger bundles, and to highlight their importance by connecting to a series of recent results in the literature. Specifically, we discuss the applications of Hermite reciprocity to proving Green's conjecture for rational cuspidal curves [1], and for canonical ribbons [23]. We also explain how to recover the description of the class group of a Hankel determinantal ring and its property of having rational singularities [5], by working on the natural desingularizations, via Schwarzenberger bundles, of the secant varieties to rational normal curves.

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Classically, Hermite reciprocity is the statement that the composition of two symmetric powers is commutative when applied to a 2-dimensional vector space, that is, there exists a $\mathrm{GL}_2(\mathbf{C})$-equivariant isomorphism [16, Exercise 11.34]

$$\mathrm{Sym}^m(\mathrm{Sym}^n \mathbf{C}^2) \cong \mathrm{Sym}^n(\mathrm{Sym}^m \mathbf{C}^2).$$

The existence of such an isomorphism can be proven combinatorially by computing the characters of both sides. However, the isomorphism is not unique (it can be chosen independently on each isotypic component), and it may not exist if we replace $\mathbf{C}$ with a field $\mathbf{k}$ of arbitrary characteristic, when the representations involved are no longer completely reducible. To get a correct statement for arbitrary fields, one has to replace Sym^m with the divided power D^m. We explain in Sect. 2 how to construct a commutative diagram of explicit $\mathrm{SL}_2(\mathbf{k})$-equivariant isomorphisms (see also [16, Exercise 11.35])

$$\begin{array}{ccc} & \bigwedge^m(\mathrm{Sym}^{m+n-1}\mathbf{k}^2) & \\ \simeq \swarrow & & \searrow \simeq \\ \mathrm{D}^m(\mathrm{Sym}^n \mathbf{k}^2) & \xrightarrow{\quad\simeq\quad} & \mathrm{Sym}^n(\mathrm{D}^m \mathbf{k}^2) \end{array}$$

and we loosely refer to any one of them as **Hermite (reciprocity) isomorphisms**. Notice that we have relaxed the requirement of GL_2-equivariance to SL_2-equivariance: this is only done to avoid twisting by appropriate powers of the **determinant representation** $\det(\mathbf{k}^2) = \bigwedge^2 \mathbf{k}^2$ of $\mathrm{GL}_2(\mathbf{k})$. For instance, to make the diagonal isomorphisms in the above diagram respect the GL_2-action, one would need to tensor the bottom representations with $(\det(\mathbf{k}^2))^{\otimes\binom{m}{2}}$.

We give two constructions of Hermite isomorphisms: the first one uses only elementary multilinear algebra, while the second one uses Schwarzenberger bundles and only some elementary facts of algebraic geometry. A third construction that passes through (a truncation of) the ring of symmetric polynomials is explained in [1, Section 3]. The fact that all of the constructions agree is a consequence of a strong compatibility between the Hermite isomorphisms as we vary the parameters. For instance, if we vary n then we get

$$\bigoplus_{n\geq 0} \bigwedge^m(\mathrm{Sym}^{m+n-1}\mathbf{k}^2) \simeq \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathrm{D}^m \mathbf{k}^2) \tag{1.1}$$

where the right side is manifestly a polynomial ring, the symmetric algebra $\mathrm{Sym}(\mathrm{D}^m \mathbf{k}^2)$. A natural action of $\mathrm{D}^m \mathbf{k}^2$ on the left side makes it into a free $\mathrm{Sym}(\mathrm{D}^m \mathbf{k}^2)$-module of rank one, with generating set $\bigwedge^m(\mathrm{Sym}^{m-1}\mathbf{k}^2) \simeq \mathbf{k}$.

The geometric approach to Hermite reciprocity, suggested by Rob Lazarsfeld, is based on a construction considered in [30, Section 2]. Specializing it to the case of $\mathbf{P}^1$, we get an incidence correspondence

$$\begin{array}{ccc} Z & \xrightarrow{\pi_1} & \mathrm{Hilb}^m(\mathbf{P}^1) \simeq \mathbf{P}^m \\ {\scriptstyle \pi_2}\downarrow & & \\ \mathbf{P}^1 & & \end{array} \tag{1.2}$$

where Z consists of pairs (Ξ, p), where Ξ is a subscheme of $\mathbf{P}^1$ of length m and $p \in \Xi$. Letting

$$\mathcal{E} = \pi_{1*}(\pi_2^*(\mathcal{O}_{\mathbf{P}^1}(m+n-1))),$$

one has that $\mathcal{E}$ is a vector bundle of rank m on $\mathbf{P}^m$, with $\det(\mathcal{E}) = \bigwedge^m \mathcal{E} = \mathcal{O}_{\mathbf{P}^m}(n)$. It is noted in [30, (2.11)] that there exists an isomorphism

$$\bigwedge^m \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m+n-1)) \simeq \mathrm{H}^0(\mathbf{P}^m, \det(\mathcal{E})),$$

and upon identifying the left side with $\bigwedge^m(\mathrm{Sym}^{m+n-1}\mathbf{k}^2)$ and the right side with $\mathrm{Sym}^n(\mathrm{D}^m\mathbf{k}^2)$, one discovers an instance of Hermite reciprocity. We treat this in more detail in Sect. 2.2, where we note that $\mathcal{E} = \mathcal{E}^m_{n-1}$ is a **Schwarzenberger bundle**, with presentation [25, Proposition 2]

$$0 \longrightarrow \mathrm{Sym}^{n-1}\mathbf{k}^2(-1) \longrightarrow \mathrm{Sym}^{m+n-1}\mathbf{k}^2 \otimes \mathcal{O}_{\mathbf{P}^m} \longrightarrow \mathcal{E} \longrightarrow 0.$$

In the diagram (1.2), in can be shown that $Z \simeq \mathbf{P}^1 \times \mathbf{P}^{m-1}$, and that the morphism π_1 is defined by a linear series of type (1, 1). As such, $\mathcal{E}$ arises as a special case of the construction of [14, Theorem 6.1], and is therefore a **supernatural vector bundle**. In Sect. 3 we give a similar realization, which appears to be new, for all of the exterior powers $\bigwedge^i \mathcal{E}$. We remark here that the natural generalizations of the Schwarzenberger bundles, obtained by replacing $\mathbf{P}^1$ in (1.2) with a higher genus curve, have been also notably used in the Ein–Lazarsfeld proof of the gonality conjecture [6] and in the recent work of Ein–Niu–Park on secant varieties of nonsingular curves [8].

In the final chapters we discuss two applications of Hermite reciprocity and Schwarzenberger bundles. The first is to the study of secant varieties of rational normal curves, and is developed in Sect. 4. These secant varieties have desingularizations given by the total space of Schwarzenberger bundles (see [25, Section 2], [22, Section 6], [8, Section 3]). We review some basic known properties of the secant varieties using what we have developed, such as showing that they are normal Cohen–Macaulay varieties which have rational singularities, and showing that the minimal free resolution of their homogeneous coordinate rings are given by the Eagon–Northcott complex of a Hankel matrix. We also discuss the classification of the rank one maximal Cohen–Macaulay modules on these varieties and show that there is always a distinguished one which is an Ulrich module and is self-dual. Most of the properties of secant varieties that we present are classical, but the results on rational singularities and on the rank one MCMs have been obtained only recently [5], based on a more algebraic approach. In Sect. 5, we recast Hermite reciprocity in terms of the self-duality of the rank one Ulrich module, and also show that it characterizes the unique global section of an appropriate twist of a symmetric power of $\mathcal{E}$.

The second application of the ideas behind Hermite reciprocity is to give a purely algebraic proof of Green's conjecture for generic curves of genus g, and is discussed in Sect. 6. There are two different approaches [1, 23], each of which involves studying a mapping cone and showing that certain comparison maps are surjective. Surprisingly, in both cases, by considering all g at once, the Tor groups involved in these comparison maps can be given the structure of finitely generated modules over polynomial rings, in the spirit of (1.1). This hidden structure is most easily observed using Hermite reciprocity, and we explain the calculations leading up to this observation, referring the reader to the original articles for the rest of the technical aspects.

Notation Throughout we fix a field $\mathbf{k}$. Since all of our results are independent of characteristic and compatible with change of rings, one can work over a general commutative ring with the proper rephrasing. We will make use of standard multilinear functors: D^m denotes the m-th divided power functor, which is the subspace of symmetric tensors in the m-th tensor power, Sym^m denotes the m-th symmetric power functor, and $\bigwedge^m$ denotes the m-th exterior power functor. For any finite dimensional vector space V, we have a natural identification $\mathrm{D}^m(V) = \mathrm{Sym}(V^\vee)^\vee$ (see [1, Section 3.1] for a quick discussion of the relevant constructions).

2 Hermite Reciprocity

Consider a 2-dimensional vector space U (which we identify with its dual space $U^\vee$ after picking a volume form, i.e., a nonzero element of $\bigwedge^2 U$). We let $\mathrm{SL} = \mathrm{SL}(U) \cong \mathrm{SL}_2(\mathbf{k})$ denote the special linear group, i.e., the group of invertible operators on U of determinant 1. The careful reader can upgrade all of the statements in this paper to account for the general linear group $\mathrm{GL}(U)$ by inserting appropriate powers of determinant characters, but we avoided this to simplify the notation. In this section, we give two different constructions of the isomorphisms in the following result and then we show that they agree with each other, and with the isomorphisms constructed in [1, Section 3].

Theorem 2.1 (Hermite Reciprocity) *We have natural* $\mathrm{SL}(U)$*-equivariant isomorphisms:*

$$\bigwedge^m (\mathrm{Sym}^n U) \cong \mathrm{Sym}^{n-m+1}(\mathrm{D}^m U), \tag{2.1a}$$

$$\mathrm{D}^m(\mathrm{Sym}^{n-m} U) \cong \mathrm{Sym}^{n-m}(\mathrm{D}^m U). \tag{2.1b}$$

2.1 An Algebraic Construction

Given vector spaces A and B, we have a canonical map

$$\mathrm{D}^m A \otimes \bigwedge^m B \to \bigwedge^m (A \otimes B) \tag{2.2}$$

given by realizing $\bigwedge^m B$ as the skew-symmetric tensors in $B^{\otimes m}$. More precisely, if $\sum_\alpha a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_m}$ is invariant and $\sum_\beta b_{\beta_1} \otimes \cdots \otimes b_{\beta_m}$ is skew-invariant, then $\sum_{\alpha,\beta} (a_{\alpha_1} \otimes b_{\beta_1}) \otimes \cdots \otimes (a_{\alpha_m} \otimes b_{\beta_m})$ is also skew-invariant. Similarly, we have a canonical map

$$\mathrm{D}^m A \otimes \mathrm{D}^m B \to \mathrm{D}^m(A \otimes B).$$

In particular, we can define multiplication maps

$$\mathrm{D}^m U \otimes \bigwedge^m (\mathrm{Sym}^d U) \to \bigwedge^m (\mathrm{Sym}^{d+1} U) \tag{2.3a}$$

$$\mathrm{D}^m U \otimes \mathrm{D}^m(\mathrm{Sym}^d U) \to \mathrm{D}^m(\mathrm{Sym}^{d+1} U) \tag{2.3b}$$

by using the canonical maps above followed by either the functor $\bigwedge^m$ or D^m applied to the multiplication map $U \otimes \mathrm{Sym}^d U \to \mathrm{Sym}^{d+1} U$. If we do it twice, the resulting map is invariant under swapping the copies of $\mathrm{D}^m U$, so $\bigoplus_{d \geq 0} \bigwedge^m(\mathrm{Sym}^d U)$ and $\bigoplus_{d \geq 0} \mathrm{D}^m(\mathrm{Sym}^d U)$ acquire the structure of modules over $\mathrm{Sym}(\mathrm{D}^m U)$.

Proposition 2.4 *If* U *has basis* $\{1, x\}$, *then* $\bigoplus_{d \geq 0} \bigwedge^m(\mathrm{Sym}^d U)$ *is a free* $\mathrm{Sym}(\mathrm{D}^m U)$*-module of rank one. If* $m \geq 1$ *then the generator is* $x^{m-1} \wedge x^{m-2} \wedge \cdots \wedge 1$ *in degree* $m-1$, *and if* $m = 0$ *then the generator is in degree* 0. *Similarly,* $\bigoplus_{d \geq 0} \mathrm{D}^m(\mathrm{Sym}^d U)$ *is a free* $\mathrm{Sym}(\mathrm{D}^m U)$*-module of rank one generated in degree* 0.

Proof The $m = 0$ case is obvious, so we assume that $m > 0$. We claim that the multiplication $\mathrm{D}^m U \otimes \bigwedge^m(\mathrm{Sym}^d U) \to \bigwedge^m(\mathrm{Sym}^{d+1} U)$ is surjective if $d \geq m-1$. First, $\mathrm{D}^m U$ has a basis

$$\{x^{(k)} \mid 0 \leq k \leq m\}$$

where $x^{(k)}$ is the sum over all $\binom{m}{k}$ ways of tensoring k copies of x and $m-k$ copies of 1. The multiplication map is then described by

$$x^{(k)} \otimes (x^{d_1} \wedge \cdots \wedge x^{d_m}) \mapsto \sum_{\substack{S \subseteq \{1,\dots,m\} \\ |S|=k}} x^{d'_1} \wedge \cdots \wedge x^{d'_m}$$

where $d'_j = d_j$ if $j \notin S$ and $d'_j = d_j + 1$ if $j \in S$. Now consider an element of the form $x^{d_1} \wedge \cdots \wedge x^{d_m}$ with $d_1 > \cdots > d_m$. Let $j \geq 0$ be maximal such that $d_j = d + 2 - j$ (we set $j = 0$ if $d_i \neq d + 2 - i$ for all i). We show how $x^{d_1} \wedge \cdots \wedge x^{d_m}$ is in the image of the multiplication map by induction on j. If $j = 0$, then $d + 1 \notin \{d_1, \dots, d_m\}$, so we can multiply $x^{d_1} \wedge \cdots \wedge x^{d_m}$ by $x^{(0)}$. Otherwise, if $j > 0$, multiply $x^{d_1-1} \wedge x^{d_2-1} \cdots x^{d_j-1} \wedge x^{d_{j+1}} \cdots x^{d_m}$ by $x^{(j)}$. This is a sum of the term we want together with terms covered by our induction hypothesis, so the claim is proven.

It follows from the prove above that $\bigoplus_{d \geq 0} \bigwedge^m (\mathrm{Sym}^d\, U)$ is a cyclic $\mathrm{Sym}(\mathrm{D}^m\, U)$-module, generated by $x^{m-1} \wedge x^{m-2} \wedge \cdots \wedge 1$ in degree $m-1$. Since

$$\dim \bigwedge^m (\mathrm{D}^{m-1+d}\, U) = \binom{m+d}{d} = \dim \mathrm{Sym}^d(\mathrm{D}^m(U)) \text{ for all } d \geq 0,$$

it follows that $\bigoplus_{d \geq 0} \bigwedge^m (\mathrm{Sym}^d\, U)$ is a free module, as desired.

The symmetric case is similar: we just need to show that the multiplication map $\mathrm{D}^m\, U \otimes \mathrm{D}^m(\mathrm{Sym}^d\, U) \to \mathrm{D}^m(\mathrm{Sym}^{d+1}\, U)$ is surjective. We have a basis of $\mathrm{D}^m(\mathrm{Sym}^d\, U)$ consisting of the sum of the unique permutations of $x^{d_1} \otimes \cdots \otimes x^{d_m}$ where $d \geq d_1 \geq \cdots \geq d_m \geq 0$, which we denote by $x^{d_1} \cdots x^{d_m}$. The product of $x^{(k)}$ with $x^{d_1} \cdots x^{d_m}$ is almost as before, namely, it is a sum (with coefficients) of $x^{d'_1} \cdots x^{d'_m}$ where $d + 1 \geq d'_1 \geq \cdots d'_m \geq 0$ and $(d'_1, \dots, d'_m)$ is obtained by adding 1 to k of the d_i and sorting. The exact coefficients will not be relevant, rather we will show that $x^{d_1} \cdots x^{d_m}$ is in the image of the multiplication map by induction on how many exponents are equal to $d + 1$. If there are none, then this is the product of $x^{(0)}$ and $x^{d_1} \cdots x^{d_m} \in \mathrm{D}^m(\mathrm{Sym}^d\, U)$. Otherwise, if $d_k = d + 1 > d_{k+1}$, then consider the product of $x^{(k)}$ with $x^d \cdots x^d x^{d_{k+1}} \cdots x^{d_m} \in \mathrm{D}^m(\mathrm{Sym}^d\, U)$. It will contain $x^{d+1} \cdots x^{d+1} x^{d_{k+1}} \cdots x^{d_m}$ with coefficient 1, and all other terms will have less than k exponents equal to $d + 1$. These latter terms are in the image of the multiplication map by induction, so we are done. □

Proof of Theorem 2.1 By Proposition 2.4, the multiplication map

$$\mathrm{Sym}^{n-m+1}(\mathrm{D}^m\, U) \otimes \bigwedge^m (\mathrm{Sym}^{m-1}\, U) \to \bigwedge^m (\mathrm{Sym}^n\, U)$$

is an isomorphism, and $\bigwedge^m(\mathrm{Sym}^{m-1} U) \simeq \mathbf{k}$ is a trivial $\mathrm{SL}(U)$-representation. Similarly, the multiplication map

$$\mathrm{Sym}^{n-m}(\mathrm{D}^m U) \otimes \mathrm{D}^m(\mathrm{Sym}^0 U) \to \mathrm{D}^m(\mathrm{Sym}^{n-m} U)$$

is an isomorphism and $\mathrm{D}^m(\mathrm{Sym}^0 U) \simeq \mathbf{k}$. □

2.2 Via Schwarzenberger Bundles

We let $\mathbf{P}^m = \mathrm{Proj}(\mathrm{Sym}(\mathrm{D}^m U))$. For $d \geq 1$, consider the composition $\mathrm{Sym}^d U \otimes \mathcal{O}_{\mathbf{P}^m}(-1) \to \mathrm{Sym}^d U \otimes \mathrm{Sym}^m U \to \mathrm{Sym}^{d+m} U$. This has locally constant rank, so we have a locally free sheaf $\mathcal{E}_d^m$, the **Schwarzenberger bundle** on $\mathbf{P}^m$ defined by the short exact sequence

$$0 \longrightarrow \mathrm{Sym}^d U(-1) \longrightarrow \mathrm{Sym}^{d+m} U \otimes \mathcal{O}_{\mathbf{P}^m} \longrightarrow \mathcal{E}_d^m \longrightarrow 0. \tag{2.5a}$$

It follows from (2.5a) that

$$\mathrm{rank}(\mathcal{E}_d^m) = m \quad \text{and} \quad \det(\mathcal{E}_d^m) = \mathcal{O}_{\mathbf{P}^m}(d+1). \tag{2.5b}$$

Proof of Theorem 2.1 The m-th exterior power of (2.5a) with $d = n - m$ gives a resolution $\mathcal{F}_\bullet^n$ of $\det(\mathcal{E}_{n-m}^m) \cong \mathcal{O}_{\mathbf{P}^m}(n-m+1)$ by locally free sheaves, where

$$\mathcal{F}_i^n = \bigwedge^{m-i}(\mathrm{Sym}^n U) \otimes \mathrm{D}^i(\mathrm{Sym}^{n-m} U) \otimes \mathcal{O}_{\mathbf{P}^m}(-i), \text{ for } i = 0, \ldots, m.$$

Since the sheaves $\mathcal{F}_i^n$ have no cohomology for $i = 1, \ldots, m$, it follows that

$$\bigwedge^m(\mathrm{Sym}^n U) = \mathrm{H}^0(\mathbf{P}^m, \mathcal{F}_0^n) = \mathrm{H}^0(\mathbf{P}^m, \det(\mathcal{E}_{n-m}^m)) = \mathrm{Sym}^{n-m+1}(\mathrm{D}^m U),$$

proving (2.1a). To prove (2.1b), we note that $\det(\mathcal{E}_{n-m}^m) \otimes \mathcal{O}_{\mathbf{P}^m}(-1) \cong \mathcal{O}_{\mathbf{P}^m}(n-m)$. Since the sheaves $\mathcal{F}_i^n(-1)$ have no cohomology for $i = 0, \ldots, m-1$, it follows that

$$\mathrm{D}^m(\mathrm{Sym}^{n-m} U) = \mathrm{H}^m(\mathbf{P}^m, \mathcal{F}_m^n(-1)) = \mathrm{H}^0(\mathbf{P}^m, \det(\mathcal{E}_{n-m}^m)(-1)) = \mathrm{Sym}^{n-m}(\mathrm{D}^m U).$$

□

Remark 2.6 We have a natural identification between $\mathbf{P}^m$ and $\mathrm{Hilb}^m(\mathbf{P}^1)$ (the Hilbert scheme of m points on $\mathbf{P}^1$), where a point $[f] \in \mathbf{P}^m$ with $0 \neq f \in \mathrm{Sym}^m U$ corresponds to the zero locus of $f \in \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(m))$. The incidence correspondence

$$Z = \{([f], [p]) \in \mathbf{P}^m \times \mathbf{P}^1 \mid f(p) = 0\},$$

is defined by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^1}(-1, -m) \longrightarrow \mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^1} \longrightarrow \mathcal{O}_Z \longrightarrow 0, \tag{2.6a}$$

where the defining equation of Z is given by the unique $\mathrm{SL}(U)$-invariant subspace in

$$\mathrm{H}^0(\mathbf{P}^m \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^1}(1, m)) = \mathrm{D}^m\, U \otimes \mathrm{Sym}^m\, U.$$

In coordinates, if $z_0, \dots, z_m$ are the coordinate functions in $\mathbf{P}^m$, and x, y are those in $\mathbf{P}^1$, then Z is defined by the equation

$$z_0 x^m + z_1 x^{m-1} y + \cdots + z_i x^{m-i} y^i + \cdots + z_m y^m = 0.$$

We write $\pi_1 \colon Z \longrightarrow \mathbf{P}^m$ and $\pi_2 \colon Z \longrightarrow \mathbf{P}^1$ for the natural projections. Given $n \geq m$, it follows from (2.6a) that we have

$$\mathcal{E}^m_{n-m} = \pi_{1*}(\pi_2^* \mathcal{O}_{\mathbf{P}^1}(n)) = \pi_{1*}(\mathcal{O}_Z(0, n)).$$

In Sect. 3 we generalize the isomorphism above to all exterior powers of Schwarzenberger bundles, showing that they can be realized as direct images of line bundles on a product of projective spaces.

2.3 The Isomorphisms Agree

We now prove that the Hermite reciprocity isomorphisms given in Sects. 2.1 and 2.2 agree with each other and also with the one given by Aprodu et al. [1, Lemma 3.3]. We focus on (2.1a). The proof for (2.1b) is similar, so we omit it. To do this, we check that there is a commutative diagram

$$\begin{array}{ccc} \bigwedge^m(\mathrm{Sym}^n\, U) \otimes \mathrm{D}^m\, U & \longrightarrow & \bigwedge^m(\mathrm{Sym}^{n+1}\, U) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^{n-m+1}(\mathrm{D}^m\, U) \otimes \mathrm{D}^m\, U & \longrightarrow & \mathrm{Sym}^{n-m+2}(\mathrm{D}^m\, U) \end{array} \tag{2.7}$$

where the vertical maps are isomorphisms induced by the isomorphism in Sect. 2.2, the bottom map is the natural multiplication, and the top map is (2.3a). Since the Hermite reciprocity isomorphisms are characterized by the commutativity of (2.7), as seen in the proof of [1, Lemma 3.3], we conclude by verifying the following.

Proposition 2.8 *The diagram (2.7) is commutative.*

Proof The following square commutes (the vertical maps are the usual multiplication maps)

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Sym}^{n-m} U \otimes U(-1) & \longrightarrow & \mathrm{Sym}^{n} U \otimes U \ , \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Sym}^{n-m+1} U(-1) & \longrightarrow & \mathrm{Sym}^{n+1} U \end{array}$$

which induces a map on cokernels

$$\mathcal{E}^{m}_{n-m} \otimes U \longrightarrow \mathcal{E}^{m}_{n+1-m}. \tag{2.8a}$$

This in turn gives rise to natural maps

$$\bigwedge^{m} \mathcal{E}^{m}_{n-m} \otimes \mathrm{D}^{m} U \longrightarrow \bigwedge^{m} (\mathcal{E}^{m}_{n-m} \otimes U) \longrightarrow \bigwedge^{m} \mathcal{E}^{m}_{n+1-m}, \tag{2.8b}$$

which lifts to a map of resolutions $\mathcal{F}^{n}_{\bullet} \otimes \mathrm{D}^{m} U \longrightarrow \mathcal{F}^{n+1}_{\bullet}$, as follows. For each $i = 0, \dots, m$, the map $\mathcal{F}^{n}_{i} \otimes \mathrm{D}^{m} U \longrightarrow \mathcal{F}^{n+1}_{i}$ is given as the composition

$$\begin{aligned} &\bigwedge^{m-i} (\mathrm{Sym}^{n} U) \otimes \mathrm{D}^{i} (\mathrm{Sym}^{n-m} U) \otimes \mathrm{D}^{m} U \\ &\quad \to \left(\bigwedge^{m-i} (\mathrm{Sym}^{n} U) \otimes \mathrm{D}^{m-i} U \right) \otimes \left(\mathrm{D}^{i} (\mathrm{Sym}^{n-m} U) \otimes \mathrm{D}^{i} U \right) \end{aligned}$$

$$\to \bigwedge^{m-i} (\mathrm{Sym}^{n} U \otimes U) \otimes \mathrm{D}^{i} (\mathrm{Sym}^{n-m} U \otimes U) \to \bigwedge^{m-i} (\mathrm{Sym}^{n+1} U) \otimes \mathrm{D}^{i} (\mathrm{Sym}^{n-m+1} U),$$

where all maps are induced by multiplication and comultiplication. The map on global sections

$$\mathrm{H}^{0}(\mathbf{P}^{m}, \mathcal{F}^{n}_{0} \otimes \mathrm{D}^{m} U) \longrightarrow \mathrm{H}^{0}(\mathbf{P}^{m}, \mathcal{F}^{n+1}_{0})$$

is the top map in (2.7), while the bottom map in (2.7) is the map induced from (2.8b) by taking global sections, from which the commutativity follows. □

2.4 Compatibility of Hermite Isomorphisms

We end our discussion of Hermite reciprocity by constructing one last isomorphism and discussing its compatibility with the ones from Theorem 2.1. We define α as the composition

$$\alpha\colon \mathrm{D}^m(\mathrm{Sym}^{n-m} U) \xrightarrow{\simeq} \mathrm{D}^m(\mathrm{Sym}^{n-m} U) \otimes \bigwedge^m(\mathrm{Sym}^{m-1} U) \longrightarrow \bigwedge^m(\mathrm{Sym}^{n-1} U)$$

where the first isomorphism follows from the identification $\mathbf{k} \cong \bigwedge^m(\mathrm{Sym}^{m-1} U)$ given by $1 \mapsto x^{m-1} \wedge \cdots \wedge x \wedge 1$, while the second map is induced by (2.2) and the multiplication $\mathrm{Sym}^{n-m} U \otimes \mathrm{Sym}^{m-1} U \longrightarrow \mathrm{Sym}^{n-1} U$.

Theorem 2.9 *The map α is an isomorphism, and we have a commutative diagram*

$$\begin{array}{ccc} & \mathrm{D}^m(\mathrm{Sym}^{n-m} U) & \\ {}^{\alpha}\swarrow & & \searrow^{\gamma} \\ \bigwedge^m(\mathrm{Sym}^{n-1} U) & \xrightarrow{\quad\beta\quad} & \mathrm{Sym}^{n-m}(\mathrm{D}^m U) \end{array}$$

where β comes from (2.1a) and γ from (2.1b).

Proof Fix m and consider the direct sum of the terms in the triangle over all $n \geq m$ to get

$$\begin{array}{ccc} & \bigoplus_{n\geq m} \mathrm{D}^m(\mathrm{Sym}^{n-m} U) & \\ {}^{\alpha'}\swarrow & & \searrow^{\gamma'} \\ \bigoplus_{n\geq m} \bigwedge^m(\mathrm{Sym}^{n-1} U) & \xrightarrow{\quad\beta'\quad} & \mathrm{Sym}(\mathrm{D}^m U) \end{array}$$

where all 3 terms are free $\mathrm{Sym}(\mathrm{D}^m U)$-modules of rank one, and the maps α', β', γ' are linear with respect to the $\mathrm{Sym}(\mathrm{D}^m U)$-action. Let $*$ denote the $\mathrm{Sym}(\mathrm{D}^m U)$-action in all cases. For $f \in \mathrm{Sym}(\mathrm{D}^m U)$, we have $\gamma'^{-1}(f) = f * 1$ where $1 \in \mathbf{k} \cong \mathrm{D}^m(\mathrm{Sym}^0 U)$, and $\beta'^{-1}(f) = x^{m-1} \wedge \cdots \wedge x \wedge 1 \in \bigwedge^m(\mathrm{Sym}^{m-1} U)$. By construction, $\alpha'(1) = x^{m-1} \wedge \cdots \wedge x \wedge 1$, so the triangle commutes. □

3 Exterior Powers of Schwarzenberger Bundles

The goal of this section is to prove that exterior powers of $\mathcal{E}_d^m$ arise as special cases of the construction of supernatural vector bundles from [14, Section 6].

Theorem 3.1 *For $0 \leq i \leq m$ consider the multiplication map*

$$\mu\colon \mathbf{P}^i \times \mathbf{P}^{m-i} \longrightarrow \mathbf{P}^m.$$

We have an isomorphism

$$\bigwedge^{i} \mathcal{E}_d^m \simeq \mu_* \mathcal{O}(d+m-i+1, 0),$$

and in particular the bundle $\bigwedge^i \mathcal{E}_d^m$ *has supernatural cohomology, with root sequence*

$$-1, -2, \ldots, -(m-i), -(m-i+d+2), \ldots, -(m+d+1).$$

If $i = 0$ or $i = m$ then there is nothing to prove, so we may assume that $m \geq 2$ and $0 < i < m$. Inside the product $\mathbf{P} = \mathbf{P}^i \times \mathbf{P}^m$, consider the locus (see also [13, Section 2.1.8])

$$Z = \{(f, g) \mid f \text{ divides } g\}.$$

with the reduced scheme structure. We have an isomorphism

$$\phi \colon \mathbf{P}^i \times \mathbf{P}^{m-i} \simeq Z \subset \mathbf{P}, \quad (f, h) \mapsto (f, fh),$$

and under this isomorphism we have

$$\phi^*(\mathcal{O}_Z(a, b)) = \mathcal{O}_{\mathbf{P}^i \times \mathbf{P}^{m-i}}(a+b, b).$$

The divisibility $f|g$ is equivalent to the existence of a form $h \in \operatorname{Sym}^{m-i} U$ and a scalar $c \in \mathbf{k}$, not both 0, such that $hf + cg = 0$. It follows that Z can (set-theoretically) be realized as the degeneracy locus (i.e., where the map fails to have full rank) of a map of vector bundles

$$\operatorname{Sym}^{m-i} U(-1, 0) \oplus \mathcal{O}_{\mathbf{P}}(0, -1) \longrightarrow \operatorname{Sym}^m U \otimes \mathcal{O}_{\mathbf{P}},$$
$$(h \otimes f, c \otimes g) \mapsto hf + cg.$$

We write $\alpha \colon \mathbf{P} \longrightarrow \mathbf{P}^i$ for the first projection, and observe that if we restrict the domain of the morphism above to the first summand then we get from (2.5a) an injective map

$$\operatorname{Sym}^{m-i} U(-1, 0) \hookrightarrow \operatorname{Sym}^m U \otimes \mathcal{O}_{\mathbf{P}},$$

with cokernel given by $\alpha^*(\mathcal{E}_{m-i}^i)$.

Lemma 3.2 *Z is the zero scheme of the induced map*

$$\mathcal{O}_{\mathbf{P}}(0, -1) \longrightarrow \alpha^*(\mathcal{E}_{m-i}^i).$$

Proof Since ϕ is a closed immersion, and $\phi^*(\mathcal{O}_{\mathbf{P}}(1,1)) = \mathcal{O}_{\mathbf{P}^i \times \mathbf{P}^{m-i}}(2,1)$, we can compute the degree of Z with respect to $\mathcal{O}_{\mathbf{P}}(1,1)$ as the m-fold self-intersection of $\mathcal{O}_{\mathbf{P}^i \times \mathbf{P}^{m-i}}(2,1)$. This is the coefficient of $s^i t^{m-i}$ in $(2s+t)^m$, which is $2^i\binom{m}{i}$.

Since $\operatorname{codim} Z = i$, the cohomology class of the zero locus of this section is the top Chern class of $\alpha^*(\mathcal{E}^i_{m-i})(0,1)$. Writing the Chow ring of $\mathbf{P}$ as $\mathbb{Z}[s,t]/(s^{i+1}, t^{m+1})$ (using [13, Theorem 2.10]), the top Chern class of $\alpha^*(\mathcal{E}^i_{m-i})(0,1)$ is by Eisenbud and Harris [13, Proposition 5.17]

$$\sum_{j=0}^{i} c_j(\mathcal{E}^i_{m-i}, s) t^{i-j}$$

where $c_j(\mathcal{E}^i_{m-i}, s)$ denotes the jth Chern class of $\alpha^*(\mathcal{E}^i_{m-i})$. Using (2.5a) and [13, Theorem 5.3(c)], the Chern polynomial of $\alpha^*(\mathcal{E}^i_{m-i})$ is (see also [13, Section 9.3.3])

$$(1-s)^{m-i+1} = \sum_{j=0}^{i} \binom{m-i+j}{j} s^j.$$

Since Z has dimension m, the degree of the zero locus with respect to $\mathcal{O}_{\mathbf{P}}(1,1)$ is the coefficient of $s^i t^m$ in $(1-s)^{m-i+1}(s+t)^m$, which is

$$\sum_{j=0}^{i} \binom{m-i+j}{j}\binom{m}{i-j} = \sum_{j=0}^{i} \binom{m}{i}\binom{i}{j} = 2^i\binom{m}{i}.$$

This agrees with the degree of Z, and hence we conclude that Z is scheme-theoretically the zero locus of the claimed map of vector bundles. □

Proof of Theorem 3.1 We will prove the result by induction on m. From Lemma 3.2, we obtain an exact Koszul resolution (using (2.5b))

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(0,-i) \longrightarrow \alpha^*(\mathcal{E}^i_{m-i})(0,-i+1) \longrightarrow \cdots \longrightarrow$$
$$\alpha^*(\det(\mathcal{E}^i_{m-i})) \longrightarrow \mathcal{O}_Z(m-i+1,0) \longrightarrow 0.$$

If we let $\beta \colon \mathbf{P} \longrightarrow \mathbf{P}^m$ denote the second projection, then $\mu = \beta \circ \phi$, and in particular

$$\mathcal{F} := \mu_*(\mathcal{O}_{\mathbf{P}^i \times \mathbf{P}^{m-i}}(d+m-i+1,0)) = \beta_*(\mathcal{O}_Z(d+m-i+1,0))$$

is resolved by the push-forward along β of the earlier Koszul complex twisted by $\mathcal{O}_{\mathbf{P}}(d,0)$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(d,-i) \longrightarrow \alpha^*(\mathcal{E}^i_{m-i})(d,-i+1) \longrightarrow \cdots \longrightarrow \alpha^*(\det(\mathcal{E}^i_{m-i}))(d,0). \tag{3.3}$$

By induction, we know that $(\bigwedge^j \mathcal{E}^i_{m-i})(d)$ has no higher cohomology, and

$$\begin{aligned}\mathrm{H}^0(\mathbf{P}^i, \bigwedge^j(\mathcal{E}^i_{m-i})(d)) &= \mathrm{Sym}^{m-j+1+d}(\mathrm{D}^j\, U) \otimes \mathrm{Sym}^d(\mathrm{D}^{i-j}\, U) \\ &\cong \bigwedge^j(\mathrm{Sym}^{m+d}\, U) \otimes \mathrm{D}^{i-j}(\mathrm{Sym}^d\, U),\end{aligned}$$

where the last isomorphism follows from Hermite reciprocity. Since β is a finite map, we get that β_* is exact, and hence the sheaf $\mathcal{F}$ is resolved by a complex

$$\cdots \longrightarrow \bigwedge^j(\mathrm{Sym}^{m+d}\, U)\otimes \mathrm{D}^{i-j}(\mathrm{Sym}^d\, U)(-i+j) \longrightarrow \cdots \longrightarrow \bigwedge^i(\mathrm{Sym}^{m+d}\, U)\otimes \mathcal{O}_{\mathbf{P}^m}.$$

We claim that the rightmost differential in the above complex can be identified with the rightmost differential in the i-th exterior power of the 2-term resolution

$$\mathrm{Sym}^d\, U(-1) \longrightarrow \mathrm{Sym}^{m+d}\, U \otimes \mathcal{O}_{\mathbf{P}^m}$$

of $\mathcal{E}^m_d$. Once shown, this implies that $\mathcal{F} \simeq \bigwedge^i \mathcal{E}^m_d$.

To prove the claim, in the exact sequence (3.3), replace each term $\alpha^*(\bigwedge^j \mathcal{E}^i_{m-i})$ by its resolution $\alpha^*(\bigwedge^j(\mathrm{Sym}^{m-i}\, U(-1) \to \mathrm{Sym}^m\, U))$. Then we get a double complex mapping to the complex in question, and we take sections of the rightmost two terms to get:

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ \bigwedge^{i-1}(\mathrm{Sym}^{m+d}\, U) \otimes \mathrm{Sym}^d\, U(-1) & \longrightarrow & \bigwedge^{i}(\mathrm{Sym}^{m+d}\, U) \otimes \mathcal{O}_{\mathbf{P}^m} \\ \uparrow & & \uparrow \\ \bigwedge^{i-1}(\mathrm{Sym}^{m}\, U) \otimes \mathrm{Sym}^d(\mathrm{D}^i\, U)(-1) & \longrightarrow & \bigwedge^{i}(\mathrm{Sym}^{m}\, U) \otimes \mathrm{Sym}^d(\mathrm{D}^i\, U) \otimes \mathcal{O}_{\mathbf{P}^m} \\ \uparrow & & \uparrow \end{array}$$

The vertical maps from the second row to the top row are surjective and the second row comes from the ith exterior power of the 2-term complex $\mathcal{O}(-1) \to \mathrm{Sym}^m\, U$ tensored with $\mathrm{Sym}^d(\mathrm{D}^i\, U)$. This implies that the differentials in the first row are determined by the second row and the vertical maps, so it suffices to show that the claimed differential for the first row gives a commutative square.

For $j = i, i-1$, the vertical map

$$\bigwedge^j(\mathrm{Sym}^{m+d}\, U)\otimes \mathrm{Sym}^d(\mathrm{D}^i\, U)(-i+j) \to \bigwedge^j(\mathrm{Sym}^{m+d}\, U)\otimes \mathrm{D}^{i-j}(\mathrm{Sym}^d\, U)(-i+j)$$

factors as

$$\begin{aligned}
&\bigwedge^{j}(\mathrm{Sym}^{m+d} U) \otimes \mathrm{Sym}^{d}(\mathrm{D}^{i} U)(-i+j) \\
&\to \bigwedge^{j}(\mathrm{Sym}^{m+d} U) \otimes \mathrm{Sym}^{d}(\mathrm{D}^{j} U) \otimes \mathrm{Sym}^{d}(\mathrm{D}^{i-j} U)(-i+j) \\
&\to \bigwedge^{j}(\mathrm{Sym}^{m+d} U) \otimes \mathrm{D}^{i-j}(\mathrm{Sym}^{d} U)(-i+j)
\end{aligned}$$

where in the second map we use the action of $\mathrm{Sym}(\mathrm{D}^j U)$ on $\bigoplus_{n\geq 0} \bigwedge^j(\mathrm{Sym}^n U)$ from the previous section on the first two factors (which we denote by $*$). The second factor is the identity map in both cases.

It suffices to consider the case $d = 1$ due to the associativity of $*$. The square becomes

$$\begin{array}{ccc}
\bigwedge^{i-1}(\mathrm{Sym}^{m+1} U) \otimes U(-1) & \longrightarrow & \bigwedge^{i}(\mathrm{Sym}^{m+1} U) \otimes \mathcal{O}_{\mathbf{P}^m} \\
\uparrow & & \uparrow \\
\bigwedge^{i-1}(\mathrm{Sym}^{m} U) \otimes \mathrm{D}^{i} U(-1) & \longrightarrow & \bigwedge^{i}(\mathrm{Sym}^{m} U) \otimes \mathrm{D}^{i} U \otimes \mathcal{O}_{\mathbf{P}^m}
\end{array}.$$

Pick $\omega \otimes x^{(j)} \otimes f \in \bigwedge^{i-1}(\mathrm{Sym}^m U) \otimes \mathrm{D}^i U(-1)$. The bottom path is

$$\omega \otimes x^{(j)} \otimes f \mapsto \omega \wedge f \otimes x^{(j)} \mapsto x^{(j)} * (\omega \wedge f),$$

while the top path is

$$\begin{aligned}
\omega \otimes x^{(j)} \otimes f &\mapsto x^{(j-1)} * \omega \otimes x \otimes f + x^{(j)} * \omega \otimes 1 \otimes f \\
&\mapsto (x^{(j-1)} * \omega) \wedge xf + (x^{(j)} * \omega) \wedge f
\end{aligned}$$

where by convention, $x^{(-1)} = 0$. The two final quantities agree, which proves the claim. □

4 Secant Varieties of Rational Normal Curves

In this section we give an SL-equivariant construction of the rank one maximal Cohen–Macaulay modules over a Hankel determinantal ring B, and recover the description of the divisor class group of B from [5, Section 3], as well as the property of B having rational singularities. We formulate our results and arguments geometrically, using the usual identification of $\mathrm{Spec}(B)$ with the affine cone $\widehat{\Sigma}$ over a secant variety of a rational normal curve. In the process we recover well-known properties of $\widehat{\Sigma}$, such as normality and the Cohen–Macaulay property, along with the explicit description of its equations and syzygy modules. The key ingredients that we employ are the desingularization of $\widehat{\Sigma}$ via Schwarzenberger bundles, as explained in [22, Section 6], and the Kempf–Weyman technique for constructing syzygies, as explained in [33, Chapter 5]. We will assume that $\mathbf{k}$ is algebraically closed in order to make valid set-theoretic arguments involving the $\mathbf{k}$-points of our varieties, but the careful reader may wish to rephrase the justifications in order to remove this hypothesis.

Before going into more details, we establish some notation used throughout the section. If $\mathcal{F}$ is a coherent locally free sheaf on a variety X, we consider the sheaf of (graded) algebras

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}) = \mathcal{O}_X \oplus \mathcal{F} \oplus \mathrm{Sym}^2(\mathcal{F}) \oplus \cdots$$

and write $\mathbb{P}_X(\mathcal{F})$ for $\underline{\mathrm{Proj}}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}))$. Similarly, we write $\mathbb{A}_X(\mathcal{F})$ for $\underline{\mathrm{Spec}}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}))$. When X is understood from the context, we simply write $\mathrm{Sym}(\mathcal{F})$, $\mathbb{P}(\mathcal{F})$, and $\mathbb{A}(\mathcal{F})$.

We let $\mathbf{P}^n = \mathbb{P}(\mathrm{Sym}^n U)$, and note that its $\mathbf{k}$-points $[f] \in \mathbf{P}^n$ are represented by non-zero elements $f \in \mathrm{D}^n U$ up to scaling. Every $u \in U$ gives rise to a symmetric tensor

$$u^{(n)} = u \otimes u \otimes \cdots \otimes u \in \mathrm{D}^n U,$$

the **n-th divided power** of u. We get an SL-equivariant map

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^n, \quad [u] \longrightarrow [u^{(n)}],$$

called the degree n **Veronese embedding** of $\mathbf{P}^1$. We denote its image by Γ, which is a **rational normal curve** of degree n, and write Σ_k for the k-secant variety of Γ. Recall that this is the Zariski closure of the union of all linear spaces $\mathrm{Span}(x_1, \ldots, x_k)$, ranging over all choices of points $x_1, \ldots, x_k \in \Gamma$. In particular, we have $\Sigma_1 = \Gamma$. We consider the affine space $\mathbf{A}^{n+1} = \mathbb{A}(\mathrm{Sym}^n U)$, and write $\widehat{\Sigma}_k \subset \mathbf{A}^{n+1}$ for the affine cone over Σ_k. We let B denote the coordinate ring of $\widehat{\Sigma}_k$ (or the homogeneous coordinate ring of Σ_k), which is called a Hankel determinantal ring in [5]. The following theorem summarizes some of the basic properties of $\widehat{\Sigma}_k$:

Theorem 4.1 *The variety $\widehat{\Sigma}_k$ is normal, Cohen–Macaulay, and has rational singularities. Its divisor class group is isomorphic to $\mathbb{Z}/(n-2k+2)\mathbb{Z}$.*

The conclusions of Theorem 4.1 are not new. Our goal to provide a unified proof of this theorem based on properties of Schwarzenberger bundles and the Kempf–Weyman geometric technique. For a more in-depth study of other aspects of the theory, the reader can consult [4, 5, 9, 18, 32] or [13, Section 10.4].

We consider the Schwarzenberger bundle $\mathcal{E} = \mathcal{E}^k_{n-k}$ on $\mathbf{P}^k = \mathbb{P}(\mathrm{D}^k U)$, along with the diagram of spaces and maps

$$\begin{array}{ccccc} & & Y = \mathbb{A}_{\mathbf{P}^k}(\mathcal{E}) & & \\ & \swarrow^{\pi} & \downarrow{\scriptstyle \iota} & \searrow^{\psi} & \\ \mathbf{A}^{n+1} & \xleftarrow[q]{} & \mathbf{A}^{n+1} \times \mathbf{P}^k & \xrightarrow[p]{} & \mathbf{P}^k \end{array}$$

where ι is the closed immersion induced by the surjection $\mathrm{Sym}^n U \otimes \mathcal{O}_{\mathbf{P}^k} \twoheadrightarrow \mathcal{E}$ in (2.5a), the maps p, q are the projections to the two factors, and $\pi = q \circ \iota$. The map $\psi = p \circ \iota$ is the structure map of the geometric vector bundle Y over $\mathbf{P}^k$, whose $\mathbf{k}$-points correspond to pairs $(f, [g])$, where f belongs to the dual of the fiber of $\mathcal{E}$ at $[g] \in \mathbf{P}^k$. To make this more explicit, we use the perfect pairing

$$\langle -, - \rangle \colon \ \mathrm{D}^d U \times \mathrm{Sym}^d U \longrightarrow \mathbf{k},$$

which exists for each $d \geq 0$ and is SL-equivariant. It will be important to note that if we think of $P \in \mathrm{Sym}^d U$ as a homogeneous polynomial of degree d on $U^\vee = U$, then for each $u \in U$, the evaluation of P at u is computed by

$$P(u) = \langle u^{(d)}, P \rangle. \tag{4.2a}$$

We can construct more generally a **contraction map**

$$\langle -, - \rangle \colon \ \mathrm{D}^d U \times \mathrm{Sym}^r U \longrightarrow \mathrm{D}^{d-r} U \text{ for } d \geq r \geq 0,$$

induced by the comultiplication $\mathrm{D}^d U \to \mathrm{D}^{d-r} U \otimes \mathrm{D}^r U$ and the pairing $\mathrm{D}^r U \times \mathrm{Sym}^r U \longrightarrow \mathbf{k}$. Suppose now that $[g] \in \mathbf{P}^k$, where $0 \neq g \in \mathrm{Sym}^k U$. If we restrict (2.5a) to the fiber at $[g]$ and dualize, we can identify the fiber of $\mathcal{E}^\vee$ at $[g]$ with the kernel of the contraction

$$\langle -, g \rangle \colon \ \mathrm{D}^n U \longrightarrow \mathrm{D}^{n-k} U. \tag{4.2b}$$

This yields the explicit description of the $\mathbf{k}$-points in Y as

$$Y = \{(f, [g]) : 0 \neq g \in \mathrm{Sym}^k U, \ f \in \ker\langle -, g \rangle \colon \ \mathrm{D}^n U \longrightarrow \mathrm{D}^{n-k} U\}. \tag{4.3}$$

The connection between Schwarzenberger bundles and secant varieties is given as follows.

Lemma 4.4 *The image of π is $\widehat{\Sigma}_k \subseteq \mathbf{A}^{n+1}$.*

Proof Consider a general point $(f, [g]) \in Y$, where $g \in \mathrm{Sym}^k U$ is a homogeneous polynomial with distinct roots $u_1, \dots, u_k \in U$. It follows from (4.2a) that $u_1^{(n)}, \dots, u_k^{(n)}$ belong to the kernel of the map (4.2b), and since they are linearly independent, they must generate the fiber of $\mathcal{E}^\vee$ at $[g]$. We conclude that $f \in$

$\mathrm{Span}\{u_1^{(n)},\ldots,u_k^{(n)}\}$, hence $f\in\widehat{\Sigma}_k$. Conversely, since every general point $f\in\widehat{\Sigma}_k$ belongs to $\mathrm{Span}\{u_1^{(n)},\ldots,u_k^{(n)}\}$ for some distinct $u_1,\ldots,u_k\in U$, it follows by considering $g\in\mathrm{Sym}^k U$ with roots $u_1,\ldots,u_k$, that $f\in\mathrm{Im}(\pi)$. This shows that $\mathrm{Im}(\pi)$ is dense in $\widehat{\Sigma}_k$, but since π is a projective morphism, it follows that $\mathrm{Im}(\pi)=\widehat{\Sigma}_k$. □

We will see shortly that π is in fact birational, and therefore it provides a resolution of singularities of $\widehat{\Sigma}_k$. We make the usual identification between quasi-coherent sheaves on affine space and their global sections, and let

$$S=\mathcal{O}_{\mathbf{A}^{n+1}},\quad B=\mathcal{O}_{\widehat{\Sigma}_k},\quad \tilde{B}=\pi_*\mathcal{O}_Y.$$

Proposition 4.5 *We have that $\tilde{B}=B$ has an* SL*-equivariant minimal graded free resolution $F_\bullet$ over S, whose terms are given by*

$$F_0=S,\quad F_i=\mathrm{D}^{i-1}(\mathrm{Sym}^k U)\otimes\bigwedge^{i+k}(\mathrm{Sym}^{n-k}U)\otimes S(-i-k)\ \textit{for } i=1,\ldots,n-2k+1.$$

Proof Since the natural map $S\longrightarrow\tilde{B}$ factors through B, in order to prove that $\tilde{B}=B$, it suffices to check that S surjects onto $\tilde{B}$. We do so by applying [33, Theorem 5.1.2] with $V=\mathbf{P}^k$, $\mathcal{V}=\mathcal{O}_{\mathbf{P}^k}$, $X=\mathbf{A}^{n+1}$, $\eta=\mathcal{E}$, and $\xi=\mathrm{Sym}^{n-k}U(-1)$. We get a complex $F_\bullet$ of free S-modules, with

$$\begin{aligned}F_i&=\bigoplus_{j\geq 0}\mathrm{H}^j\left(\mathbf{P}^k,\bigwedge^{i+j}(\mathrm{Sym}^{n-k}U(-1))\right)\otimes S(-i-j)\\&=\bigoplus_{j\geq 0}\mathrm{H}^j\left(\mathbf{P}^k,\mathcal{O}_{\mathbf{P}^k}(-i-j)\right)\otimes\bigwedge^{i+j}(\mathrm{Sym}^{n-k}U)\otimes S(-i-j),\end{aligned}$$

whose homology groups vanish in positive degrees, and satisfy

$$\mathrm{H}_{-i}(F_\bullet)=\mathrm{H}^i(Y,\mathcal{O}_Y)=\bigoplus_{d\geq 0}\mathrm{H}^i(\mathbf{P}^k,\mathrm{Sym}^d\mathcal{E}),\ \text{for } i\geq 0.$$

To identify the terms in the complex $F_\bullet$, we note that a line bundle $\mathcal{O}_{\mathbf{P}^k}(d)$ has no intermediate cohomology and

$$\mathrm{H}^0(\mathbf{P}^k,\mathcal{O}_{\mathbf{P}^k}(d))=\begin{cases}\mathrm{Sym}^d(\mathrm{D}^k U) & \text{if } d\geq 0,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathrm{H}^k(\mathbf{P}^k,\mathcal{O}_{\mathbf{P}^k}(d))=\begin{cases}\mathrm{D}^{-d-k-1}(\mathrm{Sym}^k U) & \text{if } d\leq -k-1,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $F_0 = S$, and the only other non-zero terms are

$$F_i = \mathrm{D}^{i-1}(\mathrm{Sym}^k U) \otimes \bigwedge^{i+k}(\mathrm{Sym}^{n-k} U) \otimes S(-i-k) \text{ for } i = 1, \dots, n-2k+1.$$

We conclude that $\mathrm{H}_i(F_\bullet) = 0$ for $i < 0$, and that $F_\bullet$ gives a minimal free resolution of $\tilde{B}$. Since $F_0 = S$, the natural map $S \longrightarrow \tilde{B}$ is surjective, hence $\tilde{B} = B$, as desired. □

To make the results of Proposition 4.5 even more explicit, consider for a moment the general situation of a **k**-linear map

$$\varphi\colon V_1 \longrightarrow V_0 \otimes W,$$

where W, V_0, V_1 are finite dimensional **k**-vector spaces of dimensions $k+1, n+1$, and $m+1$ respectively. After choosing bases for W, V_0, V_1, we can represent φ either as an $(m+1) \times (n+1)$ matrix A of linear forms in $\mathrm{Sym}(W) \simeq \mathbf{k}[y_0, \dots, y_k]$, or as an $(m+1) \times (k+1)$ matrix A' of linear forms in $\mathrm{Sym}(V_0) = \mathbf{k}[z_0, \dots, z_n]$. Such matrices occur frequently, for instance in the study of Rees algebras, when one is the presentation matrix of an ideal, while the other is the **Jacobian dual** [26, 29]. Of interest to us is the encoding of φ as a morphism of sheaves on $\mathbb{P}W \simeq \mathbf{P}^k$

$$V_1 \otimes \mathcal{O}_{\mathbb{P}W}(-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}W}. \tag{4.6}$$

Taking the $(k+1)$-st symmetric power yields a Koszul complex $\mathcal{K}_\bullet$

$$0 \longrightarrow \bigwedge^{k+1} V_1 \otimes \mathcal{O}_{\mathbb{P}W}(-k-1) \longrightarrow \cdots \longrightarrow V_1 \otimes \mathrm{Sym}^k V_0 \otimes \mathcal{O}_{\mathbb{P}W}(-1) \longrightarrow \mathrm{Sym}^{k+1} V_0 \otimes \mathcal{O}_{\mathbb{P}W} \longrightarrow 0.$$

The intermediate sheaves in the above complex have no cohomology, and the hypercohomology spectral sequence involves precisely one interesting map

$$\begin{array}{ccc} \mathrm{H}^k\left(\mathbb{P}W, \bigwedge^{k+1} V_1 \otimes \mathcal{O}_{\mathbb{P}W}(-k-1)\right) & \longrightarrow & \mathrm{H}^0(\mathbb{P}W, \mathrm{Sym}^{k+1} V_0 \otimes \mathcal{O}_{\mathbb{P}W}) \\ \| & & \| \\ \bigwedge^{k+1} V_1 \simeq \bigwedge^{k+1} V_1 \otimes \bigwedge^{k+1} W^\vee & & \mathrm{Sym}^{k+1} V_0 \end{array} \tag{4.7}$$

If we think of the basis of $\bigwedge^{k+1} V_1$ as being indexed by collections of $(k+1)$ rows of the matrix A', then this map associates to every such collection the corresponding (maximal) $(k+1) \times (k+1)$ minor of A'.

We specialize this discussion to the case when $V_1 = \mathrm{Sym}^{n-k} U$ (where $m = n-k+1$), $V_0 = \mathrm{Sym}^n U$, $W = \mathrm{D}^k U$, and the map φ is induced by the dual of the contraction map discussed earlier. If we choose the standard monomial bases on the three vector spaces, and denote them $x_\bullet$ on V_1, $z_\bullet$ on V_0, $y_\bullet$ on W, then we have

$$\varphi(x_i) = \sum_{j=0}^{k} z_{i+j} \otimes y_j \text{ for } i = 0, \dots, n-k. \tag{4.8}$$

The matrix A' then takes the form

$$A' = \begin{bmatrix} z_0 & z_1 & \cdots & z_k \\ z_1 & z_2 & \cdots & z_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-k} & z_{n-k+1} & \cdots & z_n \end{bmatrix}$$

which is called a **Hankel** (or **catalecticant**) matrix. The map (4.6) is the presentation of $\mathcal{E}$ from (2.5a), and therefore $\mathcal{K}_\bullet$ is a resolution of $\operatorname{Sym}^{k+1} \mathcal{E}$. The map (4.7) is the degree $(k+1)$ component of the differential $d_1 : F_1 \longrightarrow F_0$ (whose cokernel is $B_{k+1} = \mathrm{H}^0(\mathbf{P}^k, \operatorname{Sym}^{k+1} \mathcal{E})$). Since F_1 is generated by its degree $(k+1)$ component, it follows that the image of d_1 (which is the defining ideal of $\widehat{\Sigma}_k$) is the ideal I generated by the $(k+1) \times (k+1)$ minors of the Hankel matrix A'. The complex $F_\bullet$ is the **Eagon–Northcott complex** associated with A'.

Corollary 4.9 *The variety Σ_k is arithmetically Cohen–Macaulay. Its dimension and degree are computed by:*

$$\dim(\Sigma_k) = 2k - 1, \quad \deg(\Sigma_k) = \binom{n-k+1}{k} \textit{ for } 1 \le k \le \frac{n+1}{2}.$$

Proof We will prove shortly that the Hilbert series of B can be expressed in lowest terms as

$$\mathrm{HS}_B(t) = \frac{\sum_{i=0}^{k} \binom{n-2k+i}{i} \cdot t^i}{(1-t)^{2k}}. \tag{4.9a}$$

This implies that $\dim(\Sigma_k) + 1 = \dim(B) = 2k$, and setting $t = 1$ in the numerator we obtain

$$\deg(\Sigma_k) = \sum_{i=0}^{k} \binom{n-2k+i}{i} = \binom{n-k+1}{k}$$

(the sum is the number of monomials of degree $\le k$ in $(n+1-2k)$ variables, which by homogenization, is the number of monomials of degree exactly k in $(n+2-2k)$ variables, which is the right side). Since B has codimension $(n+1-2k)$, equal to the projective dimension as computed by the resolution in Proposition 4.5, it follows that B is a Cohen–Macaulay module, hence Σ_k is arithmetically Cohen–Macaulay.

To prove (4.9a), we use the minimal free resolution of B from Proposition 4.5 to obtain

$$\mathrm{HS}_B(t) = \frac{1+\sum_{i=1}^{n-2k+1}(-1)^i\binom{i-1+k}{k}\binom{n-k+1}{i+k}\cdot t^{i+k}}{(1-t)^{n+1}}.$$

If we write $F(t)$ for the numerator, then (4.9a) is equivalent to $F(t) = G(t)$, where

$$G(t) = (1-t)^{n+1-2k}\cdot\sum_{i=0}^{k}\binom{n-2k+i}{i}\cdot t^i.$$

Since $F(t)$ and $G(t)$ have constant term 1, to show they coincide it suffices to check that $F'(t) = G'(t)$. We have

$$\begin{aligned}
F'(t) &= \sum_{i=1}^{n-2k+1}(-1)^i\binom{i-1+k}{k}\binom{n-k+1}{i+k}(i+k)\cdot t^{i+k-1}\\
&= \sum_{i=1}^{n-2k+1}(-1)^i\binom{n-k}{k}\binom{n-2k}{i-1}(n-k+1)\cdot t^{i+k-1}\\
&\overset{j=i-1}{=} -(n-k+1)\binom{n-k}{k}\cdot t^k\cdot\sum_{j=0}^{n-2k}(-1)^j\binom{n-2k}{j}\cdot t^j\\
&= -(n-k+1)\binom{n-k}{k}\cdot t^k\cdot(1-t)^{n-2k}.
\end{aligned}$$

Using the product rule for $G'(t)$ and dividing by $(1-t)^{n-2k}$, we obtain

$$\frac{G'(t)}{(1-t)^{n-2k}} = -(n-2k+1)\cdot\left(\sum_{i=0}^{k}\binom{n-2k+i}{i}\cdot t^i\right) + (1-t)\cdot\left(\sum_{i=0}^{k}\binom{n-2k+i}{i}i\cdot t^{i-1}\right).$$

In the above sum, the coefficient of t^i vanishes for $0 \le i \le k-1$ due to the identity

$$-(n-2k+1)\binom{n-2k+i}{i} + \binom{n-2k+i+1}{i+1}(i+1) - \binom{n-2k+i}{i}i = 0,$$

while the coefficient of t^k is

$$-(n-2k+1)\binom{n-k}{k} - \binom{n-k}{k}k = -(n-k+1)\binom{n-k}{k}.$$

This shows that $F'(t) = G'(t)$, concluding the proof. □

Corollary 4.10 *The morphism π is birational onto $\widehat{\Sigma}_k$. The variety $\widehat{\Sigma}_k$ is normal with rational singularities.*

Proof Since $\dim(Y) = \dim(\widehat{\Sigma}_k)$, it follows that $\pi : Y \longrightarrow \widehat{\Sigma}_k$ is generically finite. Combining this with the equality $\pi_* \mathcal{O}_Y = \mathcal{O}_{\widehat{\Sigma}_k}$ established in Proposition 4.5, we see that π is birational [27, Tag 03H2], hence it provides a resolution of singularities of $\widehat{\Sigma}_k$. The fact that $\widehat{\Sigma}_k$ is normal with rational singularities is now a consequence of [33, Theorem 5.1.3(c)], and of the description of the sheaf cohomology groups of $\mathcal{O}_Y$ explained in the proof of Proposition 4.5. $\square$

Returning to the explicit presentation (4.8) of the bundle $\mathcal{E}$, and using the affine coordinates $z_0, \dots, z_n$ on $\mathbf{A}^{n+1}$, and the projective coordinates $y_0, \dots, y_k$ on $\mathbf{P}^k$ as before, it follows that Y is defined as a subvariety in $\mathbf{A}^{n+1} \times \mathbf{P}^k$ by the condition

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_k \\ z_1 & z_2 & \cdots & z_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-k} & z_{n-k+1} & \cdots & z_n \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = \vec{0}.$$

Representing each element of $\mathbf{A}^{n+1}$ as a Hankel matrix M, and each element of $\mathbf{P}^k$ as a non-zero column vector $\vec{v}$ up to scaling, we get

$$Y = \{(M, [\vec{v}]) : M \cdot \vec{v} = \vec{0}\}, \text{ and } \pi : Y \longrightarrow \mathbf{A}^{n+1} \text{ is given by } \pi(M, [\vec{v}]) = M.$$

Since the generic element $M \in \widehat{\Sigma}_k$ has rank k, we have that $\ker(M)$ is one-dimensional, hence the non-zero vector $\vec{v}$ in $\ker(M)$ is uniquely defined up to scaling. This gives a more concrete interpretation of the fact that π is birational onto $\widehat{\Sigma}_k$. It also shows that the locus where π fails to be an isomorphism is identified with the set of Hankel matrices of rank $\leq (k-1)$, that is, with $\widehat{\Sigma}_{k-1}$. We are now ready to prove the final conclusion of Theorem 4.1.

Proposition 4.11 *The class group of $\widehat{\Sigma}_k$ is isomorphic to $\mathbb{Z}/(n-2k+2)\mathbb{Z}$.*

Proof We let $U = \widehat{\Sigma}_k \setminus \widehat{\Sigma}_{k-1}$, define

$$Z = \pi^{-1}(\widehat{\Sigma}_{k-1}) \subset Y,$$

and note that as remarked earlier that π establishes an isomorphism between $Y \setminus Z$ and U. We will show that Z is a divisor on Y, defined by a section of $\psi^*(\mathcal{O}_{\mathbf{P}^k}(-n + 2k - 2))$ (see also [22, Proposition 6.14]). Since $\widehat{\Sigma}_{k-1}$ has codimension two inside $\widehat{\Sigma}_k$, it follows that

$$\mathrm{Cl}(\widehat{\Sigma}_k) = \mathrm{Cl}(U) = \mathrm{Cl}(Y \setminus Z).$$

Moreover, we have an exact sequence

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(Y \setminus Z) \longrightarrow 0,$$

where the first map sends 1 to $[Z]$ [27, Tag 02RX]. Since Y is a vector bundle over $\mathbf{P}^k$, it follows that $\mathrm{Cl}(Y) = \mathbb{Z}$, generated by the pullback of a hyperplane class along ψ [27, Tag 02TY] and [27, Tag 0BXJ]. Since Z is cut out by a section of $\psi^*(\mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$, $[Z]$ generates the subgroup $(n-2k+2)\mathbb{Z} \subset \mathbb{Z} = \mathrm{Cl}(Y)$, from which the desired conclusion follows.

To prove that Z has the desired properties, we consider the natural map on $\mathbf{P}^k$

$$\mathrm{Sym}^{k-1} U \otimes \mathcal{E}^k_{n-2k+1} \longrightarrow \mathcal{E}, \tag{4.11a}$$

defined analogously to (2.8a) by the multiplication $\mathrm{Sym}^{n-k+1} U \otimes \mathrm{Sym}^{k-1} U \longrightarrow \mathrm{Sym}^n U$. Pulling back along ψ and applying the natural map $\psi^*(\mathcal{E}) \longrightarrow \mathcal{O}_Y$, we obtain a morphism

$$\mathrm{Sym}^{k-1} U \otimes \mathcal{O}_Y \longrightarrow \psi^*(\mathcal{E}) \otimes \psi^*\left(\mathcal{E}^k_{n-2k+1}\right)^\vee \longrightarrow \psi^*\left(\mathcal{E}^k_{n-2k+1}\right)^\vee$$

of vector bundles of rank k on Y. Since $\det(\mathcal{E}^k_{n-2k+1}) = \mathcal{O}_{\mathbf{P}^k}(n-2k+2)$, we have that $\det(\Delta)$ defines a section of $\psi^*(\mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$, so to conclude we need to check that Z is the degeneracy locus of Δ. The restriction of Δ to the fiber at a point $(f, [g]) \in Y$ as in (4.3) is given by a map

$$\mathrm{Sym}^{k-1} U \longrightarrow \ker\left(\mathrm{D}^{n-k+1} U \xrightarrow{\langle -, g\rangle} \mathrm{D}^{n-2k+2} U\right), \quad h \mapsto \langle f, h\rangle.$$

This map drops rank precisely when there exists a non-zero $h \in \mathrm{Sym}^{k-1} U$ with $\langle f, h\rangle = 0$, that is, when $(f, [h]) \in \mathbb{A}_{\mathbf{P}^{k-1}}(\mathcal{E}^{k-1}_{n-k+1})$. By Lemma 4.4, this happens if and only if $f \in \widehat{\Sigma}_{k-1}$, or equivalently $(f, [g]) \in Z$.

Thus, $\det(\Delta)$ defines Z set-theoretically. Hence $n-2k+2$ gives an upper bound on the size of the class group of $\widehat{\Sigma}_k$. But we will construct $n-2k+2$ different representatives in Lemma 4.13, so in fact, $\det(\Delta)$ must define Z scheme-theoretically as well. □

It is noted in [5, Remark 3.7] that each class in $\mathrm{Cl}(\widehat{\Sigma}_k)$ is represented by a rank one MCM (maximal Cohen–Macaulay) module, and an explicit construction is given in terms of ideals of minors of Hankel matrices. For an SL-equivariant realization of these MCM modules we argue using the Kempf–Weyman geometric technique. We define

$$M_r = \pi_*(\psi^*(\mathcal{O}_{\mathbf{P}^k}(r))) = \mathrm{H}^0(Y, \psi^*(\mathcal{O}_{\mathbf{P}^k}(r))) \text{ for } r = 0, \dots, n-2k+1, \tag{4.12}$$

and prove the following (note that $M_0 = B$).

Lemma 4.13 *Up to isomorphism, the rank one maximal Cohen–Macaulay B-modules are $M_0, \ldots, M_{n-2k+1}$. The minimal number of generators of M_r is*

$$\mu(M_r) = \binom{r+k}{k},$$

and M_{n-2k+1} is an Ulrich module.

We recall that an MCM module M is called **Ulrich** (short for **maximally generated maximal Cohen–Macaulay**) if $\mu(M)$ equals the multiplicity of M, see [3, 19] for basic information and references.

Proof Since M_r is the direct image of a line bundle on the desingularization Y of $\widehat{\Sigma}_k$, it follows that M_r is a rank one B-module. To see that M_r is Cohen–Macaulay, we apply [33, Corollary 5.1.5]. With the notation in loc. cit., if $\mathcal{V} = \mathcal{O}_{\mathbf{P}^k}(r)$ then we have that $\mathcal{V}^\vee = \mathcal{O}_{\mathbf{P}^k}(2n-k-r)$. We then have to check that

$$\mathrm{R}^i\pi_*(\psi^*\mathcal{V}^\vee) = 0 \text{ for } i > 0.$$

Equivalently, we have to show that for each $d \geq 0$,

$$\mathrm{H}^i(\mathbf{P}^k, \mathrm{Sym}^d\,\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^k}(2n-k-r)) = 0 \text{ for } i > 0.$$

By our assumption on r, we have $2n-k-r+1 \geq 0$. The 2-step resolution (2.5a) of $\mathcal{E}$ induces for each $d \geq 0$ a resolution of $\mathrm{Sym}^d(\mathcal{E})$, where the i-th step is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_{\mathbf{P}^k}(-i)$. This implies that $\mathrm{Sym}^d(\mathcal{E})$ is a 0-regular sheaf. Therefore it is also $(2n-k-r+1)$-regular, hence the desired vanishing statement holds.

To calculate $\mu(M_r)$, we consider the minimal free resolution $F^r_\bullet$ of M_r. As in the proof of Proposition 4.5, we have using [33, Theorem 5.1.2] that

$$\begin{aligned} F^r_0 &= \bigoplus_{j\geq 0} \mathrm{H}^j\left(\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(-j+r)\right) \otimes \bigwedge^j(\mathrm{Sym}^{n-k} U) \otimes S(-j) \\ &= \mathrm{H}^0\left(\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(r)\right) \otimes S = \mathrm{Sym}^r(\mathrm{D}^k\,U) \otimes S. \end{aligned}$$

The space of minimal generators of M_r is then $\mathrm{Sym}^r(\mathrm{D}^k\,U)$, which has dimension $\binom{r+k}{k}$. Since the numbers $\mu(M_r)$ are distinct, it follows that the modules M_r are pairwise non-isomorphic. Since B is normal, the class group of B is isomorphic to the group of isomorphism classes rank 1 reflexive B-modules [27, Tag 0EBM]. Since the class group of B has size $(n-2k+2)$, it follows that every rank 1 MCM B-module is isomorphic to M_r for some r.

Since M_r is a rank one MCM module supported on $\widehat{\Sigma}_k$, it follows from Corollary 4.9 that its multiplicity is $\binom{n-k+1}{k}$. This quantity agrees with $\mu(M_r)$ precisely when $r = n-2k+1$, proving that M_{n-2k+1} is Ulrich. ◻

5 Self-Duality for the Rank One Ulrich Module

In this section we analyze the Ulrich module M_{n-2k+1} in Lemma 4.13, and prove that it is self-dual. To explain this, we note that the definition (4.12) can be extended to arbitrary $r \in \mathbb{Z}$, but the resulting modules will typically fail to be Cohen–Macaulay. With the notation in Lemma 4.13, if $\mathcal{V} = \mathcal{O}_{\mathbf{P}^k}(n-2k+1)$ then $\mathcal{V}^\vee = \mathcal{O}_{\mathbf{P}^k}(-1)$, so in fact M_{-1} is MCM and is the dual of M_{n-2k+1}. We will show that the section $\det(\Delta)$ constructed in the proof of Proposition 4.11 gives rise to an isomorphism $M_{-1} \simeq M_{n-2k+1}$ which is given at the level of minimal generators by Hermite reciprocity.

Theorem 5.1 *The graded modules M_{-1} and M_{n-2k+1} are generated in a single degree, with spaces of minimal generators*

$$\bigwedge^k (\mathrm{Sym}^{n-k} U) \text{ and } \mathrm{Sym}^{n-2k+1}(\mathrm{D}^k U)$$

respectively. There is an isomorphism $M_{-1} \simeq M_{n-2k+1}$ which restricts to (2.1a) on the space of minimal generators.

Proof We begin by describing the minimal free resolutions $F_\bullet^{n-2k+1}$ and $F_\bullet^{-1}$ of M_{n-2k+1} and M_{-1} respectively. We obtain using [33, Theorem 5.1.2] that (up to a shift in grading)

$$\begin{aligned} F_i^{n-2k+1} &= \mathrm{H}^0\left(\mathbf{P}^k, \bigwedge^i (\mathrm{Sym}^{n-k} U(-1)) \otimes \mathcal{O}_{\mathbf{P}^k}(n-2k+1)\right) \otimes S(-i) \\ &= \bigwedge^i (\mathrm{Sym}^{n-k} U) \otimes \mathrm{Sym}^{n-2k+1-i}(\mathrm{D}^k U) \otimes S(-i), \text{ and} \end{aligned}$$

$$\begin{aligned} F_i^{-1} &= \mathrm{H}^k\left(\mathbf{P}^k, \bigwedge^{i+k} (\mathrm{Sym}^{n-k} U(-1)) \otimes \mathcal{O}_{\mathbf{P}^k}(-1)\right) \otimes S(-i) \\ &= \bigwedge^{i+k} (\mathrm{Sym}^{n-k} U) \otimes \mathrm{D}^i(\mathrm{Sym}^k U) \otimes S(-i), \text{ for } i = 0, \dots, n-k+1. \end{aligned}$$

By taking $i = 0$ we obtain the desired description for the spaces of minimal generators of M_{n-2k+1} and M_{-1}. Since $\mu(M_{n-2k+1}) = \mu(M_{-1})$ and M_{n-2k+1}, M_{-1} are rank one MCMs, it follows from Lemma 4.13 that they must be isomorphic. Before describing the isomorphism, we make explicit the graded structure of the two modules, which is induced by the usual identification of $\mathcal{O}_Y$ with the sheaf of graded algebras $\mathrm{Sym}(\mathcal{E})$ on $\mathbf{P}^k$. We have that

$$(M_{n-2k+1})_d = \mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^d\,\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^k}(n-2k+1)), \text{ and}$$
$$(M_{-1})_d = \mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^{d+k}\,\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^k}(-1)).$$

To construct an isomorphism, we note that the section $\det(\Delta)$ of $\psi^*(\mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$ in Proposition 4.11 defines an injective morphism of invertible sheaves on Y

$$\delta\colon \psi^*(\mathcal{O}_{\mathbf{P}^k}(n-2k+1)) \longrightarrow \psi^*(\mathcal{O}_{\mathbf{P}^k}(-1)).$$

Making the identification of the source and target with sheaves of graded $\mathrm{Sym}(\mathcal{E})$-modules, we see that in degree d the map is given by

$$\delta_d\colon \mathrm{Sym}^d\,\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^k}(n-2k+1) \longrightarrow \mathrm{Sym}^{d+k}\,\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^k}(-1).$$

Since δ_d is injective, the same is true for $\mathrm{H}^0(\mathbf{P}^k, \delta_d)\colon (M_{n-2k+1})_d \longrightarrow (M_{-1})_d$. Since the source and target are finite dimensional vector spaces, we see that δ induces an isomorphism between M_{n-2k+1} and M_{-1}. To conclude, we need to check that δ_0 induces (2.1a).

Note that $\delta_0 \otimes \mathcal{O}_{\mathbf{P}^k}(1)$ is constructed from (4.11a), as the composition

$$\mathcal{O}(n-2k+2) \simeq \bigwedge^k(\mathcal{E}^k_{n-2k+1}) \simeq \bigwedge^k(\mathrm{Sym}^{k-1}\,U) \otimes \bigwedge^k(\mathcal{E}^k_{n-2k+1}) \longrightarrow \mathrm{Sym}^k(\mathcal{E}).$$

The presentation (2.5a) induces a resolution $\mathcal{F}_\bullet$ of $\bigwedge^k(\mathcal{E}^k_{n-2k+1})$, with

$$\mathcal{F}_i = \mathrm{D}^i(\mathrm{Sym}^{n-2k+1}\,U) \otimes \bigwedge^{k-i}(\mathrm{Sym}^{n-k+1}\,U) \otimes \mathcal{O}_{\mathbf{P}^k}(-i),$$

and a resolution $\mathcal{G}_\bullet$ of $\mathrm{Sym}^k(\mathcal{E})$, with

$$\mathcal{G}_i = \bigwedge^{i}(\mathrm{Sym}^{n-k}\,U) \otimes \mathrm{Sym}^{k-i}(\mathrm{Sym}^n\,U) \otimes \mathcal{O}_{\mathbf{P}^k}(-i).$$

We lift $\delta_0 \otimes \mathcal{O}_{\mathbf{P}^k}(1)$ to a map of complexes $\mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$, using the comultiplication

$$\mathbf{k} \simeq \bigwedge^k(\mathrm{Sym}^{k-1}\,U) \longrightarrow \bigwedge^i(\mathrm{Sym}^{k-1}\,U) \otimes \bigwedge^{k-i}(\mathrm{Sym}^{k-1}\,U),$$

and the natural maps

$$\mathrm{D}^i(\mathrm{Sym}^{n-2k+1} U) \otimes \bigwedge^i(\mathrm{Sym}^{k-1} U) \longrightarrow \mathrm{D}^i(\mathrm{Sym}^{n-2k+1} U \otimes \mathrm{Sym}^{k-1} U)$$
$$\longrightarrow \mathrm{D}^i(\mathrm{Sym}^{n-k} U)$$

and

$$\bigwedge^{k-i}(\mathrm{Sym}^{n-k+1} U) \otimes \bigwedge^{k-i}(\mathrm{Sym}^{k-1} U) \longrightarrow \mathrm{Sym}^{k-i}(\mathrm{Sym}^{n-k+1} U \otimes \mathrm{Sym}^{k-1} U)$$
$$\longrightarrow \mathrm{Sym}^{k-i}(\mathrm{Sym}^n U).$$

We get a morphism of exact complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_k(-1) & \longrightarrow \cdots \longrightarrow & \mathcal{F}_0(-1) & \longrightarrow & \mathcal{O}_{\mathbf{P}^k}(n-2k+1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \delta_0 & & \\ 0 & \longrightarrow & \mathcal{G}_k(-1) & \longrightarrow \cdots \longrightarrow & \mathcal{G}_0(-1) & \longrightarrow & \mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-1) & \longrightarrow & 0 \end{array}$$

and note that the sheaves $\mathcal{F}_i(-1)$ and $\mathcal{G}_i(-1)$ have no cohomology for $0 \le i \le k-1$, while $\mathcal{F}_k(-1)$, $\mathcal{G}_k(-1)$ only have top cohomology. Taking hypercohomology we obtain a commutative square

$$\begin{array}{ccc} \mathrm{H}^k(\mathbf{P}^k, \mathcal{F}_k(-1)) = \mathrm{D}^k(\mathrm{Sym}^{n-2k+1} U) & \xrightarrow{\ \gamma\ } & (M_{n-2k+1})_0 = \mathrm{Sym}^{n-2k+1}(\mathrm{D}^k U) \\ \alpha \downarrow & & \downarrow \mathrm{H}^0(\mathbf{P}^k, \delta_0) \\ \mathrm{H}^k(\mathbf{P}^k, \mathcal{G}_k(-1)) = \bigwedge^k(\mathrm{Sym}^{n-k} U) & = & (M_{-1})_0 \end{array}$$

The vertical map α is by construction the isomorphism in Sect. 2.4, while γ is the Hermite isomorphism (2.1b), as constructed in Sect. 2.2. We wrote the bottom map as an equality, because this is how we identify $(M_{-1})_0$ with $\bigwedge^k(\mathrm{Sym}^{n-k} U)$. It follows from Theorem 2.9 that

$$\mathrm{H}^0(\mathbf{P}^k, \delta_0) = \alpha \circ \gamma^{-1} = \beta^{-1},$$

is the inverse of the Hermite isomorphism (2.1a), concluding our proof. □

In [28, Proposition 1.2], it is shown that $\mathrm{Sym}^2(\mathcal{E}^2_{n-2}) \otimes \mathcal{O}_{\mathbf{P}^2}(-n+2)$ has non-zero global sections. We show that in fact it has only one non-zero section up to scaling, and that this property extends to higher rank Schwarzenberger bundles as follows.

Proposition 5.2 *For $n \ge 2k-1$ we have that $\mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$ is one-dimensional.*

Proof The proof of Theorem 5.1 shows that $\mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-n + 2k - 2))$ contains at least one non-zero section, and that any such section yields an isomorphism $M_{n-2k+1} \simeq M_{-1}$. Suppose now that s_1, s_2 are non-zero sections, write $M = M_{n-2k+1}$, and let $s = s_1^{-1} \circ s_2$ denote the induced automorphism of M. Since M is a rank one reflexive module, we have an isomorphism $\mathrm{Hom}_B(M, M) \simeq B$. Moreover, identifying M with a fractional ideal, we can represent s as the multiplication by an element in the fraction field $\mathrm{Frac}(B)$, which we also denote by s. Since $sM \subseteq M$ and M is a finitely generated B-module, it follows from the Cayley–Hamilton theorem that s is integral over B. By Theorem 4.1, B is normal, hence $s \in B$. Since M is graded, and multiplication by s is an isomorphism on M, it follows that s must have degree 0, that is, $s \in \mathbf{k}$. It follows that s_1, s_2 are proportional, concluding our proof. □

The following remark explains how to think of the unique, up to scaling, non-zero section of $\mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-n+2k-2)$ as Hermite reciprocity.

Remark 5.3 Using the resolution $\mathcal{G}_\bullet$ of $\mathrm{Sym}^k(\mathcal{E})$ in the proof of Theorem 5.1, we can identify $\mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$ as the kernel of the natural map

$$\mathrm{H}^k(\mathbf{P}^k, \mathcal{F}_k(-n+2k-2)) \longrightarrow \mathrm{H}^k(\mathbf{P}^k, \mathcal{F}_{k-1}(-n+2k-2)).$$

We have

$$\begin{aligned}\mathrm{H}^k(\mathbf{P}^k, \mathcal{F}_k(-n+2k-2)) &= \bigwedge^k(\mathrm{Sym}^{n-k} U) \otimes \mathrm{D}^{n-2k+1}(\mathrm{Sym}^k U) \\ &= \mathrm{Hom}_{\mathbf{k}}\left(\mathrm{Sym}^{n-2k+1}(\mathrm{D}^k U), \bigwedge^k(\mathrm{Sym}^{n-k} U)\right).\end{aligned}$$

Our results can be summarized as saying that $\mathrm{H}^0(\mathbf{P}^k, \mathrm{Sym}^k(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^k}(-n+2k-2))$ is naturally identified with the subspace of $\mathrm{Hom}_{\mathbf{k}}\left(\mathrm{Sym}^{n-2k+1}(\mathrm{D}^k U), \bigwedge^k(\mathrm{Sym}^{n-k} U)\right)$ spanned by the inverse of the Hermite isomorphism (2.1a).

The isomorphism between M_{n-2k+1} and M_{-1} lifts to an isomorphism between their free resolutions, leading up to a series of SL-isomorphisms that extend Theorem 2.1.

Corollary 5.4 *For each $i = 0, \ldots, n-k+1$ we have* SL*-equivariant isomorphisms*

$$\bigwedge^i(\mathrm{Sym}^{n-k} U) \otimes \mathrm{Sym}^{n-2k+1-i}(\mathrm{D}^k U) \simeq \bigwedge^{i+k}(\mathrm{Sym}^{n-k} U) \otimes \mathrm{D}^i(\mathrm{Sym}^k U).$$

Proof The two sides of the isomorphism above correspond to the minimal generators of the free modules F_i^{n-2k+1} and F_i^{-1} in the resolutions of M_{n-2k+1} and M_{-1} constructed in the proof of Theorem 5.1. □

6 Syzygies of Canonical Curves

In this section we summarize some recent applications of Hermite reciprocity to Green's conjecture for generic canonical curves (see [17] for the original reference, and [7, 10] and [11, Chapter 9] for some expository accounts). The first proof is due to Voisin [30, 31] (using different methods), and a streamlined version of Voisin's arguments was recently explained in [21]. The approaches we discuss here are more elementary and also deal with the problem in positive characteristics. Throughout the section we work over an algebraically closed field $\mathbf{k}$ and assume that $\mathrm{char}(\mathbf{k}) \neq 2$.

Let C be a smooth curve of genus $g \geq 3$ over $\mathbf{k}$, and let ω_C be its canonical bundle. The canonical ring $S_C = \bigoplus_{n\geq 0} \mathrm{H}^0(C, \omega_C^{\otimes n})$ is finitely generated over the g-dimensional polynomial ring $S = \mathrm{Sym}(\mathrm{H}^0(C, \omega_C))$ and hence we can define the graded Betti numbers

$$\beta_{i,j}(C, \omega_C) = \dim_{\mathbf{k}} \mathrm{Tor}_i^S(\mathbf{k}, S_C)_j.$$

Green's conjecture asserts that, when $\mathrm{char}(\mathbf{k}) = 0$, we have $\beta_{i,i+2}(C, \omega_C) = 0$ for $i < \mathrm{Cliff}(C)$, where $\mathrm{Cliff}(C)$ is the Clifford index of C. We do not need the definition of the Clifford index here, but instead note that for most curves, we have $\mathrm{Cliff}(C) = d - 2$ where d is the gonality of C (recall that the **gonality** of an algebraic curve C is the minimum degree of a non-constant map from C to $\mathbf{P}^1$; here "most" means that it holds for a Zariski open subset of the locus of curves of each fixed gonality in the moduli space of curves).

To show that Green's conjecture holds generically, that is, for a non-empty Zariski open subset of curves in the moduli space of curves, one can appeal to degeneration techniques and check that it is satisfied by a single (smoothable) curve. Several examples of such curves have been proposed over the years, including rational cuspidal curves suggested independently by Buchweitz–Schreyer and O'Grady, and ribbon curves put forward by Bayer and Eisenbud [2]. We discuss these examples separately in Sects. 6.1 and 6.2 below. Examples of Schreyer show that Green's conjecture does not hold in positive characteristic [24], but a careful analysis of cuspidal and ribbon curves allows one to keep track effectively of the characteristics where Green's conjecture holds generically.

6.1 Rational Cuspidal Curves

This section follows [1]. A rational curve with g simple cusps has genus g and can be smoothed out, i.e., there exist flat families whose special fiber is a rational cuspidal curve C and whose generic fiber is smooth. The upper bound for the Clifford index of a genus g curve is $\lfloor (g-1)/2 \rfloor$. This means that it suffices to show that $\beta_{i,i+2}(C, \omega_C) = 0$ for $i < \lfloor (g-1)/2 \rfloor$. We can realize C as a hyperplane section of the tangential variety of the rational normal curve in its g-uple embedding, so the computation can equivalently be done for the tangential variety. The advantage of the latter is that it has SL_2-symmetry.

To be precise, the **tangential variety** T_g is the union of all tangent lines to the rational normal curve Γ in $\mathbf{P}^g = \mathbb{P}(\mathrm{Sym}^g U)$. We have an inclusion of sheaves over $\mathbf{P}^1 = \mathbb{P}(U)$ given by $(\mathrm{Sym}^{g-2} U)(-2) \to \mathrm{Sym}^g U \otimes \mathcal{O}_{\mathbf{P}^1}$ whose cokernel $\mathcal{J}$ is the **bundle of principal parts** of $\mathcal{O}_{\mathbf{P}^1}(g)$ [13, Section 7.2]. This induces a map $\mathbf{P}(\mathcal{J}) \to \mathbf{P}^g$ which is birational onto T_g.

It turns out that there is a short exact sequence

$$0 \to \mathbf{k}[T_g] \to \widetilde{\mathbf{k}[T_g]} \to \omega_\Gamma(-1) \to 0,$$

where $\mathbf{k}[T_g]$ is the homogeneous coordinate ring of T_g, $\widetilde{\mathbf{k}[T_g]}$ is its normalization, and ω_Γ is the canonical module of the homogeneous coordinate ring of Γ. Hence we can use [33, Theorem 5.1.2] with $V = \mathbf{P}^1$, $X = \mathbb{A}(\mathrm{Sym}^g U)$, $\xi = (\mathrm{Sym}^{g-2} U)(-2)$, and $\mathcal{V} = \mathcal{O}_{\mathbf{P}^1}$ to compute Tor groups for $\widetilde{\mathbf{k}[T_g]}$ over $S = \mathrm{Sym}(\mathrm{Sym}^g U)$:

$$\mathrm{Tor}_i^S(\mathbf{k}, \widetilde{\mathbf{k}[T_g]})_{i+1} = \mathrm{D}^{2i} U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{g-2} U) \qquad \text{for } i = 0, \dots, g-2.$$

All other Tor groups vanish except $\mathrm{Tor}_0^S(\mathbf{k}, \widetilde{\mathbf{k}[T_g]})_0 = \mathbf{k}$. The canonical module ω_Γ can be realized as $\bigoplus_{n \geq 0} \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(ng + g - 2))$, so again using [33, Theorem 5.1.2] with $V = \mathbf{P}^1$, $X = \mathbb{A}(\mathrm{Sym}^g U)$, $\xi = (\mathrm{Sym}^{g-1} U)(-1)$, and $\mathcal{V} = \mathcal{O}_{\mathbf{P}^1}(g-2)$, its Tor groups can be computed:

$$\mathrm{Tor}_i^S(\mathbf{k}, \omega_\Gamma(-1))_{i+1} = \mathrm{Tor}_i^S(\mathbf{k}, \omega_\Gamma)_i = \bigwedge^i(\mathrm{Sym}^{g-1} U) \otimes \mathrm{Sym}^{g-2-i}(U) \qquad \text{for } i = 0, \dots, g-2.$$

Using the long exact sequence on Tor, our goal then is to show that the map

$$\mathrm{Tor}_i^S(\mathbf{k}, \widetilde{\mathbf{k}[T_g]})_{i+1} \to \mathrm{Tor}_i^S(\mathbf{k}, \omega_\Gamma(-1))_{i+1} \tag{6.1}$$

is surjective for $i \leq \lfloor (g-1)/2 \rfloor$. Using our computations above, the terms are

$$\mathrm{D}^{2i} U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{g-2} U) \to \bigwedge^i(\mathrm{Sym}^{g-1} U) \otimes \mathrm{Sym}^{g-2-i}(U).$$

Describing these maps directly is quite subtle! What is done in [1] is to realize

$$\bigwedge^i(\mathrm{Sym}^{g-1} U) \otimes \mathrm{Sym}^{g-2-i}(U) = \ker\left(\mathrm{D}^{i+1} U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{g-1} U) \xrightarrow{(2.3a)} \bigwedge^{i+1}(\mathrm{Sym}^g U)\right),$$

and rewrite (6.1) as the (middle) homology of a 3-term complex

$$\mathrm{D}^{2i}\, U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{g-2}\, U) \to \mathrm{D}^{i+1}\, U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{g-1}\, U) \xrightarrow{(2.3a)} \bigwedge^{i+1}(\mathrm{Sym}^{g}\, U),$$

where the first map comes from the inclusion $\mathrm{D}^{2i}\, U \to \mathrm{D}^{i+1}\, U \otimes \mathrm{D}^{i+1}\, U$, followed by (2.3a). Taking the direct sum over all g as in Sect. 2.1, and using the Hermite isomorphism, this gets identified with a 3-term complex of free modules over $\tilde{S} = \mathrm{Sym}(\mathrm{D}^{i+1}\, U)$:

$$\mathrm{D}^{2i}\, U \otimes \tilde{S}(-2) \longrightarrow \mathrm{D}^{i+1}\, U \otimes \tilde{S}(-1) \longrightarrow \tilde{S},$$

which a subcomplex of the Koszul complex for the maximal ideal of $\tilde{S}$. The (middle) homology is a **Weyman module** $W^{(i+1)}$, which is a special case of a **Koszul module** considered in [1]. Using general vanishing results for Koszul modules, for which we refer the reader to [1, Section 2], one gets that (6.1) is surjective when $i \leq \lfloor (g-1)/2 \rfloor$ (and $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq \frac{g+2}{2}$), as desired.

6.2 *Ribbon Curves*

This section follows [23]. A ribbon structure on a variety X is a non-reduced scheme whose structure sheaf is a square-zero extension of $\mathcal{O}_X$ by some line bundle on X [2, Section 1]. We will be interested in ribbon structures on $\mathbf{P}^1$, which are called **rational ribbon curves**. They offer more flexibility than cuspidal curves, in that they allow us to keep track not only of the genus, but also of the gonality.

Given an integer $a \leq (g-1)/2$, we consider the projective space

$$\mathbf{P}^g = \mathbb{P}(\mathrm{Sym}^a\, U \oplus \mathrm{Sym}^{g-1-a}\, U).$$

We have the Veronese embeddings $v_1 \colon \mathbf{P}^1 \to \mathbb{P}(\mathrm{Sym}^a\, U)$ and $v_2 \colon \mathbf{P}^1 \to \mathbb{P}(\mathrm{Sym}^{g-1-a}\, U)$ into the corresponding subspaces, and for each $x \in \mathbf{P}^1$ we get a line joining $v_1(x)$ and $v_2(x)$. The **rational normal scroll** $\mathcal{S}(a, g-1-a)$ is the union of these lines as we vary over $x \in \mathbf{P}^1$ [12]. Let B be the coordinate ring of $\mathcal{S}(a, g-1-a)$. There is a ribbon structure $\mathcal{X}(a, g-1-a)$ on $\mathcal{S}(a, g-1-a)$, called a **K3 carpet**, whose coordinate ring A fits into a short exact sequence

$$0 \to \omega_B \to A \to B \to 0$$

where ω_B is the canonical module of B. Most importantly, a hyperplane section of the K3 carpet $\mathcal{X}(a, g-1-a)$ gives a genus g canonically embedded ribbon that can be smoothed out to a curve of Clifford index a. As in Sect. 6.1, we can reduce to the study of the syzygies of the K3 carpet, which like T_g, has SL_2-symmetry.

The multiplication map

$$U \otimes (\mathrm{Sym}^{a-1} U \oplus \mathrm{Sym}^{g-2-a} U) \to \mathrm{Sym}^{a} U \oplus \mathrm{Sym}^{g-1-a} U$$

gives a $2 \times (g-1)$ matrix whose entries are linear forms in $\mathbf{P}^g$, and the Eagon–Northcott complex for this matrix gives a minimal free resolution for B. Hence, if we let

$$S = \mathrm{Sym}(\mathrm{Sym}^{a} U \oplus \mathrm{Sym}^{g-1-a} U)$$

denote the coordinate ring of $\mathbf{P}^g$, then we have for $i \geq 1$

$$\mathrm{Tor}_i^S(\mathbf{k}, B)_{i+1} = \mathrm{D}^{i-1} U \otimes \bigwedge^{i+1}(\mathrm{Sym}^{a-1} U \oplus \mathrm{Sym}^{g-2-a} U),$$

$$\mathrm{Tor}_i^S(\mathbf{k}, \omega_B)_{i+2} = \mathrm{Sym}^{g-3-i} U \otimes \bigwedge^{i}(\mathrm{Sym}^{a-1} U \oplus \mathrm{Sym}^{g-2-a} U).$$

As in the previous section, using the long exact sequence on Tor, our goal becomes to show that the connecting homomorphism

$$\mathrm{Tor}_{i+1}^S(\mathbf{k}, B)_{i+2} \to \mathrm{Tor}_i^S(\mathbf{k}, \omega_B)_{i+2} \tag{6.2}$$

is surjective for $i < a$. However, it is better to view all the Tor groups as being bigraded, using the usual decomposition of exterior powers

$$\bigwedge^{n}(V \oplus W) = \bigoplus_{n'+n''=n} \bigwedge^{n'} V \otimes \bigwedge^{n''} W,$$

with $V = \mathrm{Sym}^{a-1} U$ and $W = \mathrm{Sym}^{g-2-a} U$. The map then looks like

$$\bigoplus_{\substack{u+v=i \\ u,v\geq -1}} \mathrm{D}^{i} U \otimes \bigwedge^{u+1}(\mathrm{Sym}^{a-1} U) \otimes \bigwedge^{v+1}(\mathrm{Sym}^{g-2-a} U) \to \tag{6.3}$$

$$\bigoplus_{u+v=i} \mathrm{Sym}^{g-3-i} U \otimes \bigwedge^{u}(\mathrm{Sym}^{a-1} U) \otimes \bigwedge^{v}(\mathrm{Sym}^{g-2-a} U), \tag{6.4}$$

and we can concentrate on a specific (u, v)-bigraded component, while taking the direct sum over all a, g. Via Hermite reciprocity as in Sect. 2.1, the source (6.3) becomes a free module over $\tilde{S} = \mathrm{Sym}(\mathrm{D}^{u+1} U \oplus \mathrm{D}^{v+1} U)$, more precisely $\mathrm{D}^{u+v} U \otimes \tilde{S}(-1, -1)$. The target (6.4) also becomes a finitely generated $\tilde{S}$-module, though this is not at all obvious! In fact, it can be identified with the (middle) homology of

$$\mathrm{D}^{u+v+2}\,U \otimes \tilde{S}(-1,-1) \longrightarrow \mathrm{D}^{u+1}\,U \otimes \tilde{S}(-1,0) \oplus \mathrm{D}^{v+1}\,U \otimes \tilde{S}(0,-1) \longrightarrow \tilde{S},$$

where the maps are now completely transparent ($\mathrm{D}^{u+v+2}\,U$ embeds into $\mathrm{D}^{u+1}\,U \otimes \mathrm{D}^{v+1}\,U$ and $\mathrm{D}^{v+1}\,U \otimes \mathrm{D}^{u+1}\,U$ via comultiplication). Letting $Q_{u,v} = \mathrm{D}^{u+v+2}\,U \oplus \mathrm{D}^{u+v}\,U$ (or rather an appropriate SL_2-equivariant extension of the summands) leads to a subcomplex of the (bi-graded) Koszul complex of $\tilde{S}$:

$$Q_{u,v} \otimes \tilde{S}(-1,-1) \longrightarrow \mathrm{D}^{u+1}\,U \otimes \tilde{S}(-1,0) \oplus \mathrm{D}^{v+1}\,U \otimes \tilde{S}(0,-1) \longrightarrow \tilde{S}.$$

Its middle homology is the **bi-graded Weyman module** $W^{(u+1,v+1)}$, a special instance of a **bi-graded Koszul module**, for which appropriate vanishing theorems are established in [23, Section 3]. Based on this, it can be shown that the desired surjectivity of (6.2) holds when $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq a$, but we refer to [23] for details.

We finish by noting that this improves the approach via cuspidal curves in two ways. First, the bound on the characteristic for generic curves is slightly better than the one given by rational cuspidal curves since there we have $a = \lfloor (g-1)/2 \rfloor$, and in particular it confirms a conjecture from [15]. Second, this allows us to prove that generic Green's conjecture holds for each gonality, and not just the maximum value.

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Generation in Module Categories and Derived Categories of Commutative Rings

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1 Generation Problem

In this article, we consider the following problem.

Problem 1.1 Let R be a commutative noetherian ring. Let M, N be objects of the module category mod R (resp. the derived category $\mathrm{D}^{\mathrm{b}}(R)$). Then:

(1) Clarify whether M can be built out of N by taking short exact sequences (resp. exact triangles) etc.
(2) If M can be built out of N, then compute the number of required short exact sequences (resp. exact triangles).

Problem 1.1 naturally arises for the purpose to understand the structure of the module category mod R and the derived category $\mathrm{D}^{\mathrm{b}}(R)$. The author has been studying Problem 1.1 for more than 10 years. Item (1) of Problem 1.1 will be done by *classifying* the subcategories closed under short exact sequences (resp. exact triangles) etc. The number appearing in item (2) of Problem 1.1 corresponds to *dimensions* of subcategories.

The organization of this article is as follows. In Sect. 2, we recall the basic definitions and fundamental properties, which are used later. In Sects. 3 and 4, we discuss classification and dimensions of subcategories, respectively.

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2 Preliminaries

The following notation is used throughout this article.

Notation 2.1

(1) Let R be a commutative noetherian ring with identity.
(2) We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. We denote by $\mathrm{D^b}(R)$ the bounded derived category of $\operatorname{mod} R$, that is, the derived category of bounded complexes of finitely generated R-modules.
(3) By module, we mean finitely generated module. By subcategory, we mean full subcategory closed under isomorphism.
(4) Recall that an R-module M is called *maximal Cohen–Macaulay* if

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \dim R_\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. Here, the depth of the zero module over a local ring is ∞ by definition, so an R-module M is maximal Cohen–Macaulay if and only if $\operatorname{depth} M_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp} M$. Denote by $\mathrm{MCM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay modules.
(5) Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Denote by $\operatorname{Spec}_0 R$ the *punctured spectrum* of R, namely,

$$\operatorname{Spec}_0 R = \operatorname{Spec} R \setminus \{\mathfrak{m}\}.$$

Denote by $\operatorname{Sing} R$ the *singular locus* of R, which is by definition the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not regular. Denote by $\mu(-)$ the number of elements in the minimal system of generators, that is to say,

$$\mu(M) = \dim_k(M \otimes_R k)$$

for each R-module M. Denote by $\operatorname{edim} R$ the *embedding dimension* of R, i.e.,

$$\operatorname{edim} R = \mu(\mathfrak{m}) = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Denote by $\operatorname{codim} R$ the *(embedding) codimension* of R, that is,

$$\operatorname{codim} R = \operatorname{edim} R - \operatorname{depth} R.$$

By $\mathrm{e}(-)$ we denote the (Hilbert–Samuel) multiplicity, namely,

$$\mathrm{e}(I) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(R/I^{n+1})$$

for an $\mathfrak{m}$-primary ideal I of R, and set $\mathrm{e}(R) = \mathrm{e}(\mathfrak{m})$. By $\ell\ell(-)$ we denote the *Loewy length*, namely,

$$\ell\ell(M) = \inf\{n \geq 0 \mid \mathfrak{m}^n M = 0\}$$

for an R-module M. Note that $\ell\ell(M) < \infty$ if and only if M has finite length.

(6) For an additive category $\mathcal{C}$, the bounded (resp. right bounded) homotopy category is denoted by $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$ (resp. $\mathrm{K}^{-}(\mathcal{C})$), i.e., the homotopy category of bounded (resp. right bounded) complexes of objects in $\mathcal{C}$.

(7) For an abelian category $\mathcal{A}$, we denote by $\operatorname{proj} \mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of projective objects, and we set $\operatorname{proj} R = \operatorname{proj}(\operatorname{mod} R)$.

(8) The *(first) syzygy* of an object $M \in \mathcal{A}$ is by definition the kernel of an epimorphism from a projective object of $\mathcal{A}$ to M, and denoted by ΩM. For an integer $n \geq 1$ we inductively define the nth syzygy of M by $\Omega^n M = \Omega(\Omega^{n-1} M)$, and set $\Omega^0 M = M$. For each $M \in \mathcal{A}$ and each $n \geq 0$ the object $\Omega^n M$ is uniquely determined up to direct summands which are projective objects.

(9) For an additive category $\mathcal{C}$ and a subcategory $\mathcal{X}$ of $\mathcal{C}$, the *additive closure* of $\mathcal{X}$ is defined as the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands, and denoted by $\operatorname{add} \mathcal{X}$. Note that for an object $M \in \mathcal{A}$ one has

$$M \in \operatorname{add} \mathcal{X} \iff \begin{cases} \text{there exist a finite number of objects} \\ X_1, \ldots, X_n \in \mathcal{X} \text{ such that } M \text{ is} \\ \text{(isomorphic to) a direct summand of} \\ \text{the direct sum } X_1 \oplus \cdots \oplus X_n. \end{cases}$$

When $\mathcal{X}$ consists of a single object X, we write $\operatorname{add} X$. Hence, we have

$$\operatorname{add} R = \operatorname{proj} R.$$

Next we recall the definition of a resolving subcategory.

Definition 2.2 Let $\mathcal{A}$ be an abelian category with enough projective objects. A subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *resolving* if it satisfies the following conditions.

(a) $\mathcal{X}$ contains $\operatorname{proj} \mathcal{A}$.

(b) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{A}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.

(c) $\mathcal{X}$ is closed under extensions. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

of objects of $\mathcal{A}$, if $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$.

(d) $\mathcal{X}$ is closed under kernels of epimorphisms. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

of objects of $\mathcal{A}$, if $M, N \in \mathcal{X}$, then $L \in \mathcal{X}$.

Remark 2.3

(1) Condition (d) in Definition 2.2 can be replaced with the following condition.

(d)' $\mathcal{X}$ is closed under syzygies. That is, for any $X \in \mathcal{X}$ one has $\Omega X \in \mathcal{X}$.

(2) When $\mathcal{A} = \operatorname{mod} R$, condition (a) in Definition 2.2 can be replaced with the following condition.

(a)' R belongs to $\mathcal{X}$.

(3) The subcategory $\operatorname{proj} \mathcal{A}$ is the smallest resolving subcategory of $\mathcal{A}$, while the biggest one is $\mathcal{A}$ itself.

Here are some examples of a resolving subcategory of the abelian category $\operatorname{mod} R$ with enough projective objects.

Example 2.4

(1) If R is a Cohen–Macaulay ring, then $\mathrm{MCM}(R)$ is a resolving subcategory of $\operatorname{mod} R$. (The converse also holds true.)
(2) Set $(-)^* = \operatorname{Hom}_R(-, R)$. Recall that an R-module M is called *totally reflexive* if the canonical map $M \to M^{**}$ is an isomorphism (i.e., M is reflexive) and

$$\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$$

for all positive integers i. The subcategory $\mathcal{G}(R)$ of $\operatorname{mod} R$ consisting of totally reflexive modules is resolving.
(3) Denote by $\operatorname{mod}_0 R$ the subcategory of $\operatorname{mod} R$ consisting of modules which are locally free on the punctured spectrum of R. Then $\operatorname{mod}_0 R$ is a resolving subcatgeory of $\operatorname{mod} R$.

Next we recall the definitions of thick subcategories of an abelian category and a triangulated category.

Definition 2.5

(1) Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. A subcategory $\mathcal{X}$ of $\mathcal{C}$ is called *thick* if it satisfies the following conditions.

(a) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{A}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.
(b) $\mathcal{X}$ is closed under short exact sequences in $\mathcal{C}$. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{A}$ with $L, M, N \in \mathcal{C}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

(2) Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{T}$ of $\mathcal{X}$ is called *thick* if it satisfies the following conditions.

  (a) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{T}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.
  (b) $\mathcal{X}$ is closed under exact triangles. That is, for an exact triangle

$$L \to M \to N \to \Sigma L$$

in $\mathcal{T}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

Remark 2.6 Every thick subcategory of the abelian category mod R that contains R is a resolving subcategory of mod R.

Here are several examples of a thick subcategory.

Example 2.7

(1) The homotopy category $\mathrm{K^b}(\mathrm{proj}\, R)$ of projective modules is a thick subcategory of the triangulated category $\mathrm{D^b}(R)$.
(2) The category $\mathcal{G}(R)$ of totally reflexive modules is a thick subcategory of the category $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay modules.
(3) Set

$$\mathrm{MCM}_0(R) = \mathrm{MCM}(R) \cap \mathrm{mod}_0(R).$$

Then $\mathrm{MCM}_0(R)$ is a thick subcategory of $\mathrm{MCM}(R)$.
(4) Denote by fl R (resp. fpd R) the subcategory of mod R consisting of modules of finite length (resp. modules of finite projective dimension). Both fl R and fpd R are thick subcategories of mod R.

Finally, we recall the definition of a singularity category.

Definition 2.8 The Verdier quotient

$$\mathrm{D_{sg}}(R) = \frac{\mathrm{D^b}(R)}{\mathrm{K^b}(\mathrm{proj}\, R)}$$

of the derived category $\mathrm{D^b}(R)$ by the homotopy category $\mathrm{K^b}(\mathrm{proj}\, R)$ is called the *singularity category* or *stable derived category* of R. Note by definition that $\mathrm{D_{sg}}(R)$ is a triangulated category as well.

The singularity category has been introduced by Buchweitz [19]. There are many studies on singularity categories by Orlov [44–47] in connection with the Homological Mirror Symmetry Conjecture.

3 Classification of Subcategories

The study of classification of subcategories has started by Gabriel [28] in the 1960s, who classified the Serre subcategories of the module category of a commutative noetherian ring. In the 1990s, Auslander and Reiten [7] classified the contravariantly finite resolving subcategories of the module category of an artin algebra of finite global dimension. In the 2000s, Hovey [31] classified the wide subcategories of the module category of the quotient of a regular coherent ring by a finitely generated ideal.

For triangulated categories, a lot of classification theorems have been obtained for thick subcategories. Devinatz, Hopkins and Smith [26] and Hopkins and Smith [30] classified the thick subcategories of compact objects in the stable homotopy category, and then Hopkins and Neeman [29, 41] classified the thick subcategories of the derived category of perfect complexes over a commutative noetherian ring. Thomason [56] extended this to quasi-compact quasi-separated schemes. Benson, Carlson and Rickard [15] classified the thick tensor ideals of the stable category of finite dimensional representations of a finite group. Benson, Iyengar and Krause [16] extended this to the derived category, while Friedlander and Pevtsova [27] and Benson, Iyengar, Krause and Pevtsova [17] extended it to finite group schemes.

Furthermore, Balmer [10] defined the Balmer spectrum of a tensor triangulated category, and classified the thick tensor ideals of a tensor triangulated category by using the topological structure of the Balmer spectrum. This result is the fundation of *tensor triangular geometry*, which was invented by Balmer himself and introduced in his ICM lecture [12]. This theory spreaded to commutative algebra, algebraic geometry, modular representation theory, stable homotopy theory, motif theory, noncommutative topology, symplectic geometry and so on, and various results have been obtained; see [9–14] and references therein.

Thus, classification theory of subcategories is a research theme shared by a lot of areas of mathematics, and has been studied actively and widely through the interactions between those areas.

Here, we consider an example to explain how powerful classification of subcategories is.

Example 3.1 Let $R = k[x, y]$ be a polynomial ring in two variables x, y over a field k. For an R-module M we write[1]

$$\langle M\rangle = \left\{ N \in \operatorname{mod} R \,\middle|\, \begin{array}{c} N \text{ can be built out of } M \text{ by taking} \\ \text{direct summands, extensions and syzygies} \end{array} \right\}.$$

[1] The notation $\langle - \rangle$ here is only to simply explain this example, which is different from the one appearing in Definition 4.1.

(1) There exists an exact sequence

$$0 \to (x, y)/(x^2, y) \to R/(x^2, y) \to R/(x, y) \to 0$$

of R-modules. Note that $(x, y)/(x^2, y)$ is isomorphic to $R/(x, y)$, and (x^2, y) is the first syzygy of $R/(x^2, y)$. Hence

$$R/(x^2, y) \in \langle R/(x, y)\rangle$$

follows.

(2) Suppose that $R/(xy)$ belongs to $\langle R/(x)\rangle$. Then localization at the prime ideal (y) of R shows that $(R/(xy))_{(y)}$ beolongs to $\langle (R/(x))_{(y)}\rangle$. Here, $(R/(xy))_{(y)}$ is isomorphic to the residue field $R_{(y)}/yR_{(y)}$, while we have $(R/(x))_{(y)} = 0$. It is deduced that $R_{(y)}/yR_{(y)}$ is a projective $R_{(y)}$-module, which is a contradiction. Thus,

$$R/(xy) \notin \langle R/(x)\rangle$$

follows.

(3) There exists an exact sequence

$$0 \to R/(xy) \xrightarrow{f} R/(x) \oplus R/(xy^2) \xrightarrow{g} R/(xy) \to 0 \tag{3.1.1}$$

of R-modules, where f and g are defined by

$$f(\overline{a}) = \begin{pmatrix} \overline{a} \\ \overline{ay} \end{pmatrix}, \quad g(\begin{pmatrix} \overline{b} \\ \overline{c} \end{pmatrix}) = \overline{c - by}$$

Thus

$$R/(x) \in \langle R/(xy)\rangle$$

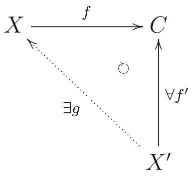
follows.

In general, it is quite difficult to find such an exact sequence as (3.1.1), and also there is no way to see at the beginning whether such an exact sequence exists or not. This problem will be settled if we can classify all the subcategories of mod R closed under direct summands, extensions and syzygies, that is to say, all the resolving subcategories of mod R. We will actually do this later; see Example 3.22.

In what follows, we consider classifications of subcategories of the module category mod R, the derived category $\mathrm{D}^{\mathrm{b}}(R)$ and the singularity category $\mathrm{D}_{\mathrm{sg}}(R)$ of a commutative noetherian ring R. We begin with recalling the definition of a contravariantly finite subcategory.

Definition 3.2 Let $\mathcal{C}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.

(1) Let $f : X \to C$ be a morphism (in $\mathcal{C}$) from an object $X \in \mathcal{X}$ to an object $C \in \mathcal{C}$. We say that f is a *right $\mathcal{X}$-approximation* of C if for every object $X' \in \mathcal{X}$ and every morphism $f' : X' \to C$ there exists a morphism $g : X' \to X$ such that $f' = fg$.



(2) We say that $\mathcal{X}$ is *contravariantly finite* if every object of $\mathcal{C}$ admits a right $\mathcal{X}$-approximation.

Remark 3.3

(1) The name "contravariantly finite" comes from the fact that for each object $C \in \mathcal{C}$ the contravaiant functor $\mathrm{Hom}_{\mathcal{C}}(-, M)$ from $\mathcal{C}$ to the category of abelian groups is a finitely generated object of the functor category of $\mathcal{C}$.
(2) Dual notions also exist. Namely, a *left $\mathcal{X}$-approximation* and a *covariantly finite* subcategory are defined dually (but we do not use them in this article).

We state a couple of examples of a contravariantly finite subcategory.

Example 3.4

(1) Let X be an R-module. Then the additive closure $\mathrm{add}\, X$ is a contravariantly finite subcategory of $\mathrm{mod}\, R$.

Indeed, take any object $M \in \mathcal{C}$. Then $\mathrm{Hom}_R(X, M)$ is a finitely generated R-module. Choose a system of generators $f_1, \ldots, f_n$ of $\mathrm{Hom}_R(X, M)$. Consider the homomorphism

$$f = (f_1, \ldots, f_n) : X^{\oplus n} \to M.$$

The module $X^{\oplus n}$ belongs to $\mathrm{add}\, X$. Let $g : Y \to M$ be any homomorphism of R-modules such that $Y \in \mathrm{add}\, X$. Then Y is a direct summands of $X^{\oplus m}$ for some $m \geq 0$. Let

$$\pi = (\pi_1, \ldots, \pi_m) : X^{\oplus m} \twoheadrightarrow Y$$

be a splitting of the inclusion map $\theta : Y \hookrightarrow X^{\oplus m}$. Then each $g\pi_i$ belongs to $\mathrm{Hom}_R(X, M)$, and

$$g\pi_i = \sum_{j=1}^{n} a_{ji} f_j$$

for some $a_{ji} \in R$. We have $g\pi = f \cdot A$, where $A = (a_{ij})$ is an $n \times m$ matrix. We get $g = g\pi\theta = fA\theta$, and thus g factors through f. This shows that f is a right (add X)-approximation of M.

(2) Let R be a Cohen–Macaulay local ring with a canonical module. Then $\mathrm{MCM}(R)$ is a contravariantly finite subcategory of $\operatorname{mod} R$. This is a direct consequence of the so-called *Cohen–Macaulay approximation theorem* due to Auslander and Buchweitz [6].

To be more precise, let M be an R-module. Then the Cohen–Macaulay approximation theorem asserts that there exists an exact sequence

$$0 \to Y \to X \xrightarrow{f} M \to 0$$

of R-modules such that X is maximal Cohen–Macaulay and Y has finite injective dimension. We claim that the map f is a right $\mathrm{MCM}(R)$-approximation of M. In fact, let X' be any maximal Cohen–Macaulay R-module. Applying the functor $\mathrm{Hom}_R(X', -)$ to the above short exact sequence induces an exact sequence

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X', f)} \mathrm{Hom}_R(X', M) \to \mathrm{Ext}^1_R(X', Y).$$

Since X' is maximal Cohen–Macaulay and Y has finite injective dimension, we have $\mathrm{Ext}^1_R(X', Y) = 0$. This implies that the map $\mathrm{Hom}_R(X', f)$ is surjective. Thus the claim follows.

The contravariantly finite resolving subcategories of the module category of a Gorenstein ring can be determined completely, as follows. In view of Remark 2.3 and Examples 3.4 and 2.4, we observe that those three subcategories which appear in the theorem are contravariantly finite resolving subcategories.

Theorem 3.5 ([53, Theorem 1.2]) *Let R be a henselian local ring. If R is Gorenstein, then the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are the following three subcategories of* $\operatorname{mod} R$.

$$\begin{cases} \operatorname{proj} R, \\ \mathrm{MCM}(R), \\ \operatorname{mod} R. \end{cases}$$

This theorem is a consequence of the following more complicated result. Here, pd_R and id_R stand for the projective dimension and the injective dimension, respectively. A typical example of an R-module G as below is a nonfree totally reflexive R-module, or more generally, an R-module of infinite projective dimension but of finite Gorenstein dimension in the sense of Auslander and Bridger [5].

Proposition 3.6 ([53, Theorem 1.3]) *Let R be a henselian local ring with residue field k. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *such that the R-module k has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ with* $\operatorname{pd}_R G = \infty$ *and* $\operatorname{Ext}_R^i(G, R) = 0$ *for $i \gg 0$. Let M be an R-module such that for each $X \in \mathcal{X}$ satisfies* $\operatorname{Ext}_R^{\gg 0}(X, M) = 0$ *for $i \gg 0$. Then* $\operatorname{id}_R M < \infty$.

This proposition together with the theorem called "Bass' conjecture" yields the following corollary, which deduces Theorem 3.5.

Corollary 3.7 ([53, Theorem 1.4]) *Let R be a henselian local ring. Let $\mathcal{X} \neq$* $\operatorname{mod} R$ *be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$*. Assume that there exists an R-module $G \in \mathcal{X}$ with* $\operatorname{pd}_R G = \infty$ *and* $\operatorname{Ext}_R^i(G, R) = 0$ *for $i \gg 0$. Then R has to be Cohen–Macaulay, and one obtains an equality* $\mathcal{X} = \operatorname{MCM}(R)$.

This corollary yields as a by product another proof of the following result due to Christensen, Piepmeyer, Striuli and the author [20].

Corollary 3.8 ([53, Corollary 1.5]) *Let R be a complete local ring over an algebraically closed field of characteristic zero. Then the following are equivalent.*

(1) *The local ring R is a simple hypersurface singularity.*
(2) *There exist at least one but only finitely many isomorphism classes of nonfree indecomposable totally reflexive R-modules.*

Sketch of Proof of Corollary 3.8 Suppose that there exist only finitely many isomorphism classes of indecomposable totally reflexive R-modules. Then there exists a totally reflexive R-module G such that $\mathcal{G}(R) = \operatorname{add} G$, and Example 3.4(1) implies that the resolving subcategory $\mathcal{G}(R)$ of $\operatorname{mod} R$ is contravariantly finite. Applying Corollary 3.7, we observe that R is Gorenstein and $\mathcal{G}(R) = \operatorname{MCM}(R)$. Hence R has finite representation type. It is known that a Gorenstein complete local ring of finite representation type over an algebraically closed field of characteristic zero is nothing but a simple hypersurface singularity. ■

To state our next result, we recall the definitions of several notions.

Definition 3.9

(1) Let I be an ideal of R. We say that I is *quasi-decomposable* if I contains an R-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ such that the R-module $I/(\boldsymbol{x})$ is decomposable.
(2) Let X be a subset of $\operatorname{Spec} R$. We say that X is *specialization-closed* if for every $\mathfrak{p} \in X$ and every $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ one has $\mathfrak{q} \in X$. It is well-known and easy to see that X is specialization-closed if and only if it is a (possibly infinite) union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.
(3) Let $\mathbb{P}$ be a property of local rings. Let X be a subset of $\operatorname{Spec} R$. We say that X satisfies $\mathbb{P}$ if for all $\mathfrak{p} \in X$ the local ring $R_\mathfrak{p}$ satisfies the property $\mathbb{P}$.
(4) Let $(R, \mathfrak{m})$ be a local ring, and let I be an ideal of R. We say that I is a *Burch ideal* if $\mathfrak{m}I \neq \mathfrak{m}(I : \mathfrak{m})$. We call R a *Burch ring* if there exist a maximal $\widehat{R}$-

regular sequence $\boldsymbol{x} = x_1, \dots, x_t$, a regular local ring S and a Burch ideal J of S such that $\widehat{R}/(\boldsymbol{x}) \cong S/J$. Here, $\widehat{R}$ stands for the $\mathfrak{m}$-adic completion of R.

(5) Let R be a Cohen–Macaulay local ring. Then, as is well-known (and easy to see), the inequality

$$\mathrm{e}(R) \geq \operatorname{codim} R + 1$$

holds. We say that R has *minimal multiplicity* if the equality holds. When the residue field of R is infinite, R has minimal multiplicity if and only if there exists a parameter ideal Q of R such that $\mathfrak{m}^2 = Q\mathfrak{m}$.

The following classification theorem on resolving subcategories and thick subcategories holds.

Theorem 3.10 ([21, 40, 52, 54]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring. Suppose that it satisifies one of the following three conditions.*

(a) *The local ring R is a hypersurface.*
(b) *The maximal ideal $\mathfrak{m}$ of R is quasi-decomposable, and $\operatorname{Spec}_0 R$ is either a hypersurface or has minimal multiplicity.*
(c) *The local ring R is a Burch ring, and $\operatorname{Spec}_0 R$ is either a hypersurface or has minimal multiplicity.*

Then there are one-to-one correspondences:

$$\begin{aligned}
&\left\{\begin{matrix}\textit{Thick subcategories of } \mathrm{MCM}(R)\\ \textit{containing } R\end{matrix}\right\}\\
\overset{1-1}{\longleftrightarrow}\ &\left\{\begin{matrix}\textit{Thick subcategories of } \operatorname{mod} R\\ \textit{containing } R\end{matrix}\right\}\\
\overset{1-1}{\longleftrightarrow}\ &\left\{\begin{matrix}\textit{Thick subcategories of } \mathrm{D}^{\mathsf{b}}(R)\\ \textit{containing } R\end{matrix}\right\}\\
\overset{1-1}{\longleftrightarrow}\ &\left\{\begin{matrix}\textit{Resolving subcategories of } \operatorname{mod} R\\ \textit{contained in } \mathrm{MCM}(R)\end{matrix}\right\}\\
\overset{1-1}{\longleftrightarrow}\ &\left\{\textit{Thick subcategories of } \mathrm{D}_{\mathrm{sg}}(R)\right\}\\
\overset{1-1}{\longleftrightarrow}\ &\left\{\begin{matrix}\textit{Specialization-closed subsets of } \operatorname{Spec} R\\ \textit{contained in } \operatorname{Sing} R\end{matrix}\right\}.
\end{aligned}$$

A local ring with quasi-decomposable maximal ideal is nothing but a local ring that deforms to a fiber product over the residue field. The class of local rings satisfying conditions (b) and (c) in Theorem 3.10 contains the class of Cohen–Macaulay local rings with minimal multiplicity, so that it contains the class of non-Gorenstein rational singularities of dimension two.

Theorem 3.10 can be thought of as a higher-dimensional version of the theorem of Benson, Carlson and Rickard which is mentioned before. The bijections giving the one-to-one correspondences can be described explicitly.

Key roles are played in the proof of the above theorem by the following two results.

Lemma 3.11 ([21, Proposition 7.6], [40, Lemma 4.4], [52, Proposition 5.9]) *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension d. Suppose that it satisifies one of the following three conditions.*

(a) *The local ring R is a hypersurface.*
(b) *The maximal ideal $\mathfrak{m}$ is quasi-decomposable, and* $\operatorname{Spec}_0 R$ *is either a hypersurface or has minimal multiplicity.*
(c) *The local ring R is a Burch ring, and* $\operatorname{Spec}_0 R$ *is either a hypersurface or has minimal multiplicity.*

Let M be a nonfree maximal Cohen–Macaulay R-module. Then the d-th syzygy $\Omega^d k$ of the R-module k belongs to the resolving closure of M.

Lemma 3.12 ([52, Theorem 2.4]) *Let R be a Cohen–Macaulay local ring of dimension d. Let M be an R-module of depth t. Assume that M is locally free on the punctured spectrum of R. Then M belongs to the extension closure of the R-module $\bigoplus_{i=t}^{d} \Omega^i k$.*

Here, the *resolving closure* of an R-module M means the smallest resolving subcategory of $\operatorname{mod} R$ containing M. The *extension closure* of M means the smallest subcategory of $\operatorname{mod} R$ which contains M and is closed under direct summands and extensions.

Applying the above lemmas, we can also improve a theorem of Keller, Murfet and Van den Bergh [37] on maximal Cohen–Macaulay modules over a completion, and recover a theorem of Huneke and Wiegand [34] and a theorem of Nasseh and Sather-Wagstaff [39] on rigidity of vanishing of Tor. Recall that a local ring R is said to have an *isolated singularity* if $R_\mathfrak{p}$ is a regular local ring for all nonmaximal prime ideals $\mathfrak{p}$ of R.

Corollary 3.13 (Keller, Murfet and Van den Bergh, [52, Corollary 3.8]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring whose $\mathfrak{m}$-adic completion $\widehat{R}$ has an isolated singularity (e.g., let R be an excellent Cohen–Macaulay local ring with an isolated singularity). Then the natural functor*

$$\mathrm{D_{sg}}(R) \to \mathrm{D_{sg}}(\widehat{R})$$

is an equivalence up to direct summands.

Corollary 3.14 (Huneke and Wiegand, [52, Corollary 7.3]) *Let R be a hypersurface local ring. Let M and N be R-modules. Suppose that*

$$\operatorname{Tor}_n^R(M, N) = \operatorname{Tor}_{n+1}^R(M, N) = 0$$

for some $n \geq 0$. Then either M or N has finite projective dimension.

Corollary 3.15 (Nasseh and Sather-Wagstaff, [40, Corollary 6.2]) *Let $R = S \times_k T$ be a fiber product, where S and T are local rings with common residue field k and $S \neq k \neq T$. Let M and N be R-modules.*

(1) *Assume that either S or T has depth zero and*

$$\mathrm{Tor}_n^R(M, N) = 0$$

for some $n \geq 5$. Then either M or N is free.

(2) *Assume that*

$$\mathrm{Tor}_n^R(M, N) = \mathrm{Tor}_{n+1}^R(M, N) = 0$$

for some $n \geq 5$. Then either $\mathrm{pd}_R\, M \leq 1$ or $\mathrm{pd}_R\, N \leq 1$.

The Ext version of the above corollary is also obtained; see [40, Corollary 6.3]. Furthermore, we can get similar vanishing results on Tor and Ext for local rings with quasi-decomposable maximal ideal and for Burch rings; see [40, Corollaries 6.5 and 6.6] and [21, Corollary 7.13 and Remark 7.14].

Stevenson [50, 51] classified the thick subcategories of the singularity category and the derived category of a complete intersection (more precisely, a quotient of a regular ring by a regular sequence), using Theorem 3.10(a) and a theorem of Orlov [45]. In the following, we explain Stevenson's classification theorem of the thick subcategories of the singularity category.

Let R be the residue ring of a regular local ring $(S, \mathfrak{n})$ by an S-regular sequence $\boldsymbol{x} = x_1, \ldots, x_c$. We may assume that the x_i are all in $\mathfrak{n}^2$, so that $c = \mathrm{codim}\, R$. Then the *generic hypersurface* of R is defined as the graded ring

$$G = \frac{S[y_1, \ldots, y_c]}{(x_1 y_1 + \cdots + x_c y_c)},$$

where $y_1, \ldots, y_c$ are indeterminates over S with degree 1 and the elements of S have degree 0. The classification theorem of Stevenson is stated as follows.

Theorem 3.16 (Stevenson) *Let R be the quotient of a regular local ring S by an S-regular sequence $\boldsymbol{x} = x_1, \ldots, x_c$. Let G be the generic hypersurface of R. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathrm{D_{sg}}(R)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of the singular locus of } \mathrm{Proj}\, G\end{matrix}\right\}.$$

To state our next theorem, we need to introduce a certain $\mathbb{N}$-valued function on the set of prime ideals.

Definition 3.17 A function $f : \operatorname{Spec} R \to \mathbb{N}$ is called *grade-consistent* if it satisfies the following two conditions.

(1) For all prime ideals $\mathfrak{p}, \mathfrak{q}$ of R with $\mathfrak{p} \subseteq \mathfrak{q}$, one has $f(\mathfrak{p}) \leq f(\mathfrak{q})$.
(2) For all prime ideals $\mathfrak{p}$ of R one has $f(\mathfrak{p}) \leq \operatorname{grade} \mathfrak{p}$.

Using grade-consistent functions and specialization-closed subsets of the singular locus of Proj G where G is the generic hypersurface, we can completely classify the resolving subcategories of the category of finitely generated modules over a local complete intersection.

Theorem 3.18 ([23, Theorem 1.5]) *Let R be a quotient of a regular local ring by a regular sequence. Let G be the generic hypersurface of R. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix} \textit{Resolving subcategories} \\ \textit{of } \operatorname{mod} R \end{matrix}\right\}$$
$$\xleftrightarrow{1-1} \left\{\begin{matrix} \textit{Grade-consistent functions} \\ \textit{on } \operatorname{Spec} R \end{matrix}\right\} \times \left\{\begin{matrix} \textit{Specialization-closed subsets} \\ \textit{of the singular locus of } \operatorname{Proj} G \end{matrix}\right\}.$$

The bijections giving the one-to-one correpondence in the above theorem can be described explicitly.

Let us explain a bit how to obtain Theorem 3.18. It is a consequence of the combination of the following Propositions 3.19 and 3.20 with Stevenson's Theorem 3.16. One can view Proposition 3.20 as a category version of the Cohen–Macaulay approximation theorem due to Auslander and Buchweitz [6].

Proposition 3.19 ([23, Theorem 1.2]) *There is a one-to-one correspondence*

$$\left\{\begin{matrix} \textit{Resolving subcategories} \\ \textit{of } \operatorname{mod} R \\ \textit{contained in } \operatorname{fpd} R \end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix} \textit{Grade-consistent functions} \\ \textit{on } \operatorname{Spec} R \end{matrix}\right\}.$$

Proposition 3.20 ([23, Theorem 7.4]) *Let R be a locally complete intersection ring. There exists a one-to-one correspondence*

$$\left\{\begin{matrix} \textit{Resolving subcategories} \\ \textit{of } \operatorname{mod} R \end{matrix}\right\}$$
$$\xleftrightarrow{1-1} \left\{\begin{matrix} \textit{Resolving subcategories} \\ \textit{of } \operatorname{mod} R \\ \textit{contained in } \operatorname{fpd} R \end{matrix}\right\} \times \left\{\begin{matrix} \textit{Resolving subcategories} \\ \textit{of } \operatorname{mod} R \\ \textit{contained in } \operatorname{MCM}(R) \end{matrix}\right\}.$$

Applying Proposition 3.19, we also obtain the following corollary.

Corollary 3.21 ([23, Theorem 1.7]) *The following are equivalent for two modules M and N over a regular ring R.*

(1) *One can build N out of M by taking direct summands, extensions and syzygies.*
(2) *One has* $\mathrm{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$ *for each prime ideal* $\mathfrak{p}$ *of* R.

Corollary 3.21 recovers and categorifies the main theorem of the ICM lecture of Auslander [3] in 1962. Also, it gives an answer to Problem 1.1.

Now Example 3.1 can be explained as follows by using the above corollary.

Example 3.22 Let $R = k[x, y]$ be the polynomial ring in two variables x, y over a field k.

(1) Put $M = R/(x, y)$ and $N = (x^2, y)$. Consider the maximal ideal $\mathfrak{m} = (x, y)$ of R. Then it holds that

$$\mathrm{pd}_{R_\mathfrak{m}} N_\mathfrak{m} = 1 \le 2 = \mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m},$$
$$\mathrm{pd}_{R_\mathfrak{p}} N_\mathfrak{p} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \text{ with } \mathfrak{p} \neq \mathfrak{m}.$$

It is observed that

$$\mathrm{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$$

for all prime ideals $\mathfrak{p}$ of R. By virtue of Corollary 3.21, we see that N can be built out of M by taking direct summands, extensions and syzygies.

(2) Put $M = R/(x)$ and $N = R/(xy)$. Consider the prime ideal $\mathfrak{p} = (y)$ of R. We see that $\mathrm{pd}_{R_\mathfrak{p}}(R/(xy))_\mathfrak{p} = 1$, while $\mathrm{pd}_{R_\mathfrak{p}}(R/(x))_\mathfrak{p} = -\infty$ as $\mathfrak{p}$ does not belong to the support of $R/(x)$. It follows that

$$\mathrm{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \not\le \sup\{\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}.$$

Applying Corollary 3.21, we observe that N cannot be built out of M by taking direct summands, extensions and syzygies.

(3) Put $M = R/(xy)$ and $N = R/(x)$. If $\mathfrak{p}$ is a prime ideal of R with $\mathrm{pd}_{R_\mathfrak{p}}(R/(x))_\mathfrak{p} = 1$, then we must have $\mathfrak{p} = (x)$, and $\mathrm{pd}_{R_\mathfrak{p}}(R/(xy))_\mathfrak{p} = 1$. It is easy to observe from this that

$$\mathrm{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$$

for all prime ideals $\mathfrak{p}$ of R. Thanks to Corollary 3.21, we see that N can be built out of M by taking direct summands, extensions and syzygies.

As the final topic of this section, we consider classification of subcategories of the category $\mathrm{D}^-(R)$, the right bounded derived category of mod R, that is, the derived category of right bounded complexes of finitely generated R-modules. This is a tensor triangulated category with tensor product $- \otimes_R^{\mathbf{L}} -$. The category $\mathrm{D}^-(R)$ is equivalent as a tensor triangulated category to the homotopy category $\mathrm{K}^-(\mathrm{proj}\, R)$.

We define a *compact ideal* of $\mathrm{D}^{-}(R)$ as a thick tensor ideal (i.e., a thick subcategory closed under $X \otimes_R^{\mathbf{L}} -$ for each $X \in \mathrm{D}^{-}(R)$) generated by bounded complexes. We can completely classify the compact ideals of $\mathrm{D}^{-}(R)$.

Theorem 3.23 ([38, Theorem A]) *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Compact ideals}\\ \textit{of } \mathrm{D}^{-}(R)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of } \operatorname{Spec} R\end{matrix}\right\}.$$

Denote by $\mathrm{D}^{perf}(R)$ the derived category of *perfect complexes* over R, that is, bounded complexes of finitely generated projective R-modules, or in other words, complexes of finite projective dimension. The category $\mathrm{D}^{perf}(R)$ is also a tensor triangulated category with tensor product $- \otimes_R^{\mathbf{L}} -$. The category $\mathrm{D}^{perf}(R)$ is equivalent as a tensor triangulated category to $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R)$. Restricting the above theorem, we recover the celebrated Hopkins–Neeman theorem [41, Theorem 1.5] stated below.

Corollary 3.24 (Hopkins–Neeman) *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathrm{D}^{perf}(R)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of } \operatorname{Spec} R\end{matrix}\right\}.$$

The proof of Theorem 3.23 also extends the Hopkins–Neeman smash nilpotence theorem on $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R) \cong \mathrm{D}^{perf}(R)$ to $\mathrm{K}^{-}(\operatorname{proj} R) \cong \mathrm{D}^{-}(R)$. For the details, we refer the reader to [38, Theorem 2.7].

4 Dimensions of Subcategories

The notion of the dimension of a triangulated category has been introduced by Rouquier [49]. Bondal and Van den Bergh [18] proved that the bounded derived category of coherent sheaves on a smooth proper commutative/noncommutative algebraic variety has finite dimension, and by using it proved that a contravariant cohomological functor of finite type to the category of vector spaces is representable. Rouquier [48] applied the notion of the dimension of a triangulated category to representation dimension. Representation dimension has been introduced by Auslander [4] to measure how far a given artin algebra is from finite representation type, and many representation theorists including Oppermann [43] have investigated it so far. Rouquier computed the dimension of the singularity category of an exterior algebra of a vector space to give the first example of an artinian ring of representation dimension more than three.

On the other hand, Rouquier [49] proved that the bounded derived category of coherent sheaves on a separated scheme of finite type over a perfect field has finite dimension. Recently, Neeman [42] proved that the bounded derived category of coherent sheaves on a separated scheme that is essentially of finite type over a separated excellent scheme of dimension at most two has finite dimension. This clarifies that even in the mixed characteristic case the derived category has finite dimension in many cases.

In what follows, we consider the dimensions of the derived category $\mathrm{D^b}(R)$ and the singularity category $\mathrm{D_{sg}}(R)$, and analogues for abelian categories. We begin with stating the definitions of the dimension and radius of a subcategory of a triangulated category or an abelian category.

Definition 4.1 Let $\mathcal{T}$ be a triangulated category.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\langle\mathcal{X}\rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts. That is,

$$\langle\mathcal{X}\rangle = \mathrm{add}\{\Sigma^i X \mid i \in \mathbb{Z},\ X \in \mathcal{X}\}.$$

When $\mathcal{X}$ consists of a single object X, we simply write $\langle X\rangle$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects M admitting an exact triangle

$$X \to M \to Y \to \Sigma X$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $\mathcal{X} \diamond \mathcal{Y} = \langle\langle\mathcal{X}\rangle * \langle\mathcal{Y}\rangle\rangle$.

(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$, and set

$$\langle\mathcal{C}\rangle_r = \begin{cases} 0 & (r = 0), \\ \langle\mathcal{C}\rangle & (r = 1), \\ \langle\mathcal{C}\rangle_{r-1} \diamond \mathcal{C} = \langle\langle\mathcal{C}\rangle_{r-1} * \langle\mathcal{C}\rangle\rangle & (r \geq 2). \end{cases}$$

When $\mathcal{C}$ consists of a single object C, we simply write $\langle C\rangle_r$.

(4) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We define the *dimension* and *radius* of $\mathcal{X}$ as follows.

$$\dim \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} = \langle G\rangle_{n+1} \text{ for some } G \in \mathcal{T}\}$$

$$\mathrm{radius}\, \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} \subseteq \langle G\rangle_{n+1} \text{ for some } G \in \mathcal{T}\}$$

Definition 4.2 Let $\mathcal{A}$ be an abelian category with enough projective objects.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote by $[\mathcal{X}]$ the smallest subcategory of $\mathcal{A}$ containing $\mathrm{proj}\,\mathcal{A}$ and $\mathcal{X}$ and closed under finite direct sums, direct summands and syzygies. That is,

$$[\mathcal{X}] = \mathrm{add}(\mathrm{proj}\,\mathcal{A} \cup \{\Omega^i X \mid i \geq 0,\ X \in \mathcal{X}\}).$$

When $\mathcal{X}$ consists of a single object X, we simply write $[X]$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\mathcal{A}$ consisting of objects $M \in \mathcal{A}$ admitting a short exact sequence

$$0 \to X \to M \to Y \to 0$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We put $\mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]]$.

(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$, and put

$$[\mathcal{C}]_r = \begin{cases} 0 & (r = 0), \\ [\mathcal{C}] & (r = 1), \\ [\mathcal{C}]_{r-1} \bullet \mathcal{C} = [[\mathcal{C}]_{r-1} \circ [\mathcal{C}]] & (r \geq 2). \end{cases}$$

When $\mathcal{C}$ consists of a single object C, we simply write $[C]_r$.

(4) Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We define the *dimension* and *radius* of $\mathcal{X}$ as follows.

$$\dim \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} = [G]_{n+1} \text{ for some } G \in \mathcal{A}\}$$

$$\operatorname{radius} \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} \subseteq [G]_{n+1} \text{ for some } G \in \mathcal{A}\}$$

The following theorem describes the relationship between the dimension of a subcategory and an isolated singularity.

Theorem 4.3 ([24, Theorem 1.1]) *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring. Consider the following four conditions.*

(a) *The subcategory* $\mathrm{MCM}_0(R)$ *of the abelian category* mod R *has finite dimension.*

(b) *The ideal*

$$\bigcap_{i>0} \bigcap_{M,N \in \mathrm{MCM}_0(R)} \operatorname{Ann}_R \operatorname{Ext}_R^i(M, N)$$

of the local ring R *is* $\mathfrak{m}$*-primary.*

(c) *The ideal*

$$\bigcap_{i>0} \bigcap_{M,N \in \mathrm{MCM}_0(R)} \operatorname{Ann}_R \operatorname{Tor}_i^R(M, N)$$

of the local ring R *is* $\mathfrak{m}$*-primary.*

(d) *The local ring* R *is an isolated singularity.*

Then the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *hold. If* R *is equicharacteristic and excellent, then the four conditions* (a), (b), (c), (d) *are equivalent.*

When R is Gorenstein, a similar assertion holds for the stable category $\underline{\mathrm{MCM}}_0(R)$ of $\mathrm{MCM}_0(R)$, which is a triangulated category.

Let $\mathcal{A}$ be an abelian category with enough projective objects. By definition, there is an inequality

$$\operatorname{radius} \mathcal{X} \leq \dim \mathcal{X}$$

for all subcategories $\mathcal{X}$ of $\mathcal{A}$. Applying the above theorem, we see that the equality does not necessarily hold.

Example 4.4 Let $R = k[\![x, y]\!]/(x^2)$ be a homomorphic image of a formal power series ring over a field k. Then for the prime ideal $\mathfrak{p} = (x)$ the local ring $R_{\mathfrak{p}}$ is not regular, so R does not have an isolated singularity. According to Theorem 4.3, the subcategory $\mathrm{MCM}_0(R)$ of $\operatorname{mod} R$ has infinite dimension. On the other hand, it is observed from [2, Theorem 1.1] that $\mathrm{MCM}(R)$ has dimension (at most) one. Hence $\mathrm{MCM}_0(R)$ has radius (at most) one, and in particular, the strict inequality

$$\operatorname{radius} \mathrm{MCM}_0(R) < \dim \mathrm{MCM}_0(R)$$

holds.

Applying Theorem 4.3 to the case where $\mathrm{MCM}_0(R)$ has dimension zero, we immediately obtain the following corollary.

Corollary 4.5 ([24, Corollary 1.2]) *Let R be a Cohen–Macaulay local ring. Suppose that the number*

$$\# \left\{ M \in \mathrm{MCM}(R) \;\middle|\; \begin{array}{c} M \text{ is indecomposable, and} \\ M \text{ is locally free on the pctured spectrum of } R \end{array} \right\} \Big/ \cong$$

is finite. Then R is an isolated singularity.

In fact, under the assumption of the above corollary, we can choose a finite number of modules $M_1, \ldots, M_n \in \mathrm{MCM}_0(R)$ whose isomorphism classes form those of the indecomposable maximal Cohen–Macaulay R-modules that are locally free on the puctured spectrum of R. Then setting

$$M = M_1 \oplus \cdots \oplus M_n,$$

we observe that $\mathrm{MCM}_0(R) = [M] = [M]_1$. Hence we obtain $\dim \mathrm{MCM}_0(R) = 0 < \infty$. Applying Theorem 4.3, we deduce that the Cohen–Macaulay local ring R has an isolated singularity.

Corollary 4.5 improves the following celebrated theorem [33].

Corollary 4.6 (Auslander–Huneke–Leuschke–Wiegand) *Let R be a Cohen–Macaulay local ring. Suppose that R has finite representation type. Then R is of an isolated singularity.*

Recall that a Cohen–Macaulay ring R is said to have *finite representation type* provided that there exist only a finite number of isomorphism classes of indecomposable maximal Cohen–Macaulay modules over R.

Concerning the radius of a resolving subcategory, we have the following conjecture.

Conjecture 4.7 Let R be a Cohen–Macaulay local ring. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Suppose that $\mathcal{X}$ has finite radius. Then $\mathcal{X}$ is contained in the subcategory $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay modules.

This conjecture holds true in the case where R is a complete intersection.

Theorem 4.8 ([22, Theorem I]) *Let R be a local complete intersection. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$*. If $\mathcal{X}$ has finite radius, then all the modules belonging to $\mathcal{X}$ are maximal Cohen–Macaulay.*

The proof of this theorem is long and contains a lot of ideas. Here we would like to explain roughly how the theorem is proved. Recall that the *(Auslander) transpose* of an R-module M, which is denoted by $\operatorname{Tr} M$, is defined as follows. Take an exact sequence

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

with $P_0, P_1 \in \operatorname{proj} R$. Then $\operatorname{Tr} M$ is by definition the cokernel of the R-dual f^* of the map f. Hence there is an exact sequence

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} M \to 0.$$

***Proof** (Sketch of Proof of Theorem 4.8)* Let $(R, \mathfrak{m})$ be a complete intersection local ring of dimension d. We may assume $d > 0$. Suppose that $\mathcal{X}$ contains an R-module M which is not maximal Cohen–Macaulay. It follows from [2] that M has reducible complexity, and using this, we observe that the resolving closure of M contains an R-module N such that

$$0 < \operatorname{pd}_R N < \infty.$$

Hence N belongs to $\mathcal{X}$. Using a technique given in [55], we may assume that N is locally free on the punctured spectrum of R. Further replacing it with a syzygy, we may assume $\operatorname{pd}_R N = 1$. Note that $\operatorname{Ext}_R^1(N, R)$ is a nonzero R-module with finite length. We find a nonzero element σ in the socle of $\operatorname{Ext}_R^1(N, R)$. We get a short exact sequence

$$\sigma : 0 \to R \to L \to N \to 0.$$

Since $\mathcal{X}$ is resolving, it contains L. An exact sequence

$$0 \to k \to \operatorname{Ext}_R^1(N, R) \to \operatorname{Ext}_R^1(L, R) \to 0$$

is induced, which shows

$$\ell_R(\operatorname{Ext}_R^1(L, R)) = \ell_R(\operatorname{Ext}_R^1(N, R)) - 1.$$

It is observed that one may assume $\operatorname{Ext}_R^1(N, R) \cong k$. There are isomorphisms $\operatorname{Tr} N \cong \operatorname{Ext}_R^1(N, R) \cong k$, and hence $\operatorname{Tr} k = N \in \mathcal{X}$. Therefore $\operatorname{Tr} K$ belongs to $\mathcal{X}$ for all R-modules K of finite length. In particular,

$$\operatorname{Tr}(R/\mathfrak{m}^i) \in \mathcal{X}$$

for all $i > 0$.

Suppose that $\mathcal{X}$ has finite radius. Then there exist an R-module G and an integer $n > 0$ such that $\mathcal{X} \subseteq [G]_n$. The module $\operatorname{Tr}(R/\mathfrak{m}^i)$ belongs to $[G]_n$ for all $i > 0$. We may assume that R is complete. We see that

$$\begin{aligned} \mathfrak{m}^i &= \operatorname{Ann}_R R/\mathfrak{m}^i \\ &= \operatorname{Ann}_R \operatorname{Ext}_R^1(\operatorname{Tr} R/\mathfrak{m}^i, R) \\ &\supseteq \bigcap_{t>0} \operatorname{Ann}_R \operatorname{Ext}_R^t(\operatorname{Tr}(R/\mathfrak{m}^i), R) \\ &\supseteq (\operatorname{Ann}_R \operatorname{Ext}_R^j(G, R))^n \text{ for all } 1 \le j \le d. \end{aligned}$$

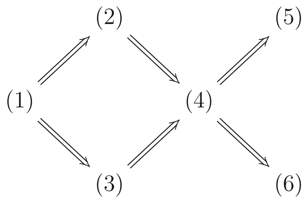
Applying Krull's intersection theorem, we observe that $\operatorname{Ann}_R \operatorname{Ext}_R^j(G, R)$ is nilpotent, and contained in every minimal prime ideal $\mathfrak{p}$ of R. It follows that $\operatorname{Ext}_{R_\mathfrak{p}}^j(G_\mathfrak{p}, R_\mathfrak{p}) \neq 0$ for all $1 \le j \le d$. This contradicts the fact that $R_\mathfrak{p}$ is an artinian Gorenstein ring. ■

The above proof actually shows that Theorem 4.8 holds for every local ring R and an R-module M of finite complete intersection dimension. As a corollary of this statement, we get the following result.

Corollary 4.9 *Let R be a Gorenstein local ring. Consider the following six conditions.*

(1) *The ring R is a hypersurface.*
(2) *The ring R is a complete intersection.*
(3) *Every resolving subcategory in* $\operatorname{MCM}(R)$ *is closed under R-duals.*
(4) *Every resolving subcategory in* $\operatorname{MCM}(R)$ *is closed under cosyzygies.*
(5) *The ring R is AB.*
(6) *The ring R satisfies Conjecture 4.7.*

Then the implications



hold true.

Here, a local ring R is called *AB* if there exists a constant C, depending only on R, such that if $\mathrm{Ext}^i(M, N) = 0$ for all $i \gg 0$, then $\mathrm{Ext}^i(M, N) = 0$ for all $i > C$. This notion is introduced by Huneke and Jorgensen [32]. The *(first) cosyzygy* $\Omega^{-1}M$ of a maximal Cohen–Macaulay module M over a Gorenstein local ring R is defined by a short exact sequence

$$0 \to M \to F \to \Omega^{-1}M \to 0$$

of maximal Cohen–Macaulay R-modules with F free. For each $n \geq 0$, the nth cosyzygy $\Omega^{-n}M$ is defined similarly to the nth syzygy. For each maximal Cohen–Macaulay R-module M and each integer $n \geq 0$, the nth cosyzygy $\Omega^{-n}M$ is uniquely determined up to free summands.

Applying the theorem of Rouquier [49] stated before to an affine scheme implies that $\mathsf{D}^{\mathsf{b}}(R)$ has finite dimension if R is essentially of finite type over a perfect field. The author [1] proved that the same statement holds true for a complete local ring R over a perfect field. The following theorem improves this.

Theorem 4.10 ([36, Theorem 1.4]) *Let R be either*

(i) *an equicharacteristic excellent local ring, or*
(ii) *a ring that is essentially of finite type over a field.*

Then $\mathsf{D}^{\mathsf{b}}(R)$ has finite dimension.

Theorem 4.10 is, as far as the author knows, the strongest result on finite dimension of the derived category of a local ring containing a field.

To prove the above theorem, first we need to make a simplified version of Definition 4.2.

Definition 4.11 Let $\mathcal{A}$ be an abelian category.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote by $|\mathcal{X}|$ the smallest subcategory of $\mathcal{A}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands. That is,

$$|\mathcal{X}| = \mathrm{add}\,\mathcal{X}.$$

When $\mathcal{X}$ consists of a single object X, we simply write $|X|$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we put $\mathcal{X} * \mathcal{Y} = ||\mathcal{X}| \circ |\mathcal{Y}||$.
(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$, and put

$$|\mathcal{C}|_r = \begin{cases} 0 & (r = 0), \\ |\mathcal{C}| & (r = 1), \\ |\mathcal{C}|_{r-1} * \mathcal{C} = ||\mathcal{C}|_{r-1} \circ |\mathcal{C}|| & (r \geq 2). \end{cases}$$

When $\mathcal{C}$ consists of a single object C, we simply write $|C|_r$.

Next, we need to introduce the notion of a cohomology annihilator.

Definition 4.12 For an integer $n \geq 0$ we set

$$\mathrm{ca}^n(R) = \left\{ a \in R \mid a \operatorname{Ext}^n_R(M, N) = 0 \text{ for all } M, N \in \operatorname{mod} R \right\}$$

and call this the nth *cohomology annihilator* of R.

Also, we need the following two technical lemmas. For an integer $n \geq 0$ we denote by $\Omega^n(\operatorname{mod} R)$ the subcategory of $\operatorname{mod} R$ consisting of nth syzygies of R-modules.

Lemma 4.13 ([36, Theorem 4.3]) *Let R have Krull dimension d. Suppose that there exist an R-module G and integers $s, n \geq 0$ such that $\Omega^s(\operatorname{mod} R) \subseteq |G|_n$. Then there is an equality*

$$\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{s+d+1}(R)).$$

In particular, $\operatorname{Sing} R$ *is closed.*

Lemma 4.14 ([36, Theorems 5.1 and 5.2]) *Let R have Krull dimension d.*

(1) *Suppose that there exists aninteger $s > 0$ such that $\mathrm{ca}^s(R/\mathfrak{p}) \neq 0$ for all prime ideals $\mathfrak{p}$ of R. Then there exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^{s+d-1}(\operatorname{mod} R) \subseteq |G|_n.$$

(2) *Suppose that for all prime ideals $\mathfrak{p}$ of R there exists an integer $s \leq \dim R/\mathfrak{p} + 1$ such that $\mathrm{ca}^s(R/\mathfrak{p}) \neq 0$. Then there exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^d(\operatorname{mod} R) \subseteq |G|_n.$$

Using the above two lemmas, we can show the following proposition.

Proposition 4.15 ([36, Theorem 5.3]) *Let R be a d-dimensional excellent equicharacteristic local ring.*

(1) *There is an equality*

$$\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{2d+1}(R)).$$

(2) *There exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^{3d+1}(\operatorname{mod} R) \subseteq |G|_n.$$

Proof of Proposition 4.15

(1) First we consider the case where R is complete. Fix a prime ideal $\mathfrak{p}$ of R. By virtue of a result of Gabber [35, IV, Théorème 2.1.1], the integral domain $R/\mathfrak{p}$ admits a separable Noether normalization. Then it follows from [57] that $\mathrm{ca}^{\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$, which is shown by using the sum of the Noether differents of $R/\mathfrak{p}$. Lemma 4.14(2) yields $\Omega^d(\operatorname{mod} R) \subseteq |G|_n$ for some R-module G and some integer $n \geq 0$. It follows from Lemma 4.13 that $\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{2d+1}(R))$.

Next we consider the case where R is excellent. By the definition of excellence, there exists an ideal I of R such that $\operatorname{Sing} R = \mathrm{V}(I)$. Then

$$I\widehat{R} \subseteq P \iff I \subseteq P \cap R \iff P \cap R \in \operatorname{Sing} R \iff P \in \operatorname{Sing} R$$

as formal fibers are regular. Hence

$$\mathrm{V}(I\widehat{R}) = \operatorname{Sing} \widehat{R} = \mathrm{V}(\mathrm{ca}^{2d+1}(\widehat{R}))$$

by the complete case. Since $\widehat{R}$ is faithfully flat over R, we obtain $\mathrm{V}(I) = \mathrm{V}(\mathrm{ca}^{2d+1}(R))$.

(2) Fix a prime ideal $\mathfrak{p}$ of R. By (1), we have

$$0 \notin \operatorname{Sing} R/\mathfrak{p} = \mathrm{V}(\mathrm{ca}^{2\dim R/\mathfrak{p}+1}(R/\mathfrak{p})).$$

Hence $\mathrm{ca}^{2\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$. Then it is easy to see that $\mathrm{ca}^{2d+1}(R/\mathfrak{p}) \neq 0$. Lemma 4.14(1) yields $\Omega^{3d}(\operatorname{mod} R) \subseteq |G|_n$ for some R-module G and an integer n. ■

Proof of Theorem 4.10 Using Proposition 4.15(2), we easily see that the derived category $\mathrm{D}^{\mathsf{b}}(R)$ has finite dimension. ■

So far, we have stated results on finiteness of the dimension and radius. The following theorem concretely gives an upper bound by using well-known invariants. For a complete local ring

$$R = \frac{k[\![x_1, \ldots, x_n]\!]}{(f_1, \ldots, f_t)}$$

over a field k, the *Jabobian ideal* of R is by definition the ideal of R generated by the c-minors of the Jacobian matrix of $f_1, \dots, f_t$, where $c = \operatorname{codim} R$.

Theorem 4.16 ([25, Theorem 1.1]) *Let R be a complete equicharacteristic Cohen–Macaulay local ring with an isolated singularity. Let J be the Jacobian ideal of R. Then there is an inequality*

$$\dim \mathrm{D_{sg}}(R) < (\mu(J) - \dim R + 1) \cdot \ell\ell(R/J).$$

If the residue field of R is infinite, the inequality

$$\dim \mathrm{D_{sg}}(R) < \mathrm{e}(J).$$

holds as well.

In the above theorem, one can replace J with any $\mathfrak{m}$-primary ideal of R contained in the sum of the Noether differents of R.

The first inequality of Theorem 4.16 immediately recovers the following result due to Ballard, Favero and Katzarkov [8, Proposition 4.11].

Corollary 4.17 (Ballard, Favero and Katzarkov, [25, Corollary 1.4]) *Let k be a field, and let $R = k[[x_1, \dots, x_n]]/(f)$ be a hypersurface complete local ring. Suppose that R has an isolated singularity. Let*

$$J = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) R$$

be the Jacobian ideal of R. Then the inequality

$$\dim \mathrm{D_{sg}}(R) < 2\, \ell\ell(R/J)$$

holds true.

We end this article by giving an outline of the proof of Theorem 4.16.

Sketch of Proof of Theorem 4.16 We can show the following statements.

(a) Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $I = (x_1, \dots, x_n)$ be an $\mathfrak{m}$-primary ideal of R. Let M be an R-module. Set $t = \operatorname{depth} R$ and $l = \ell\ell(R/I)$. Then the Koszul complex $\mathrm{K}(\boldsymbol{x}, M)$ belongs to $\langle k \rangle_{(n-t+1)l}$ in $\mathrm{D^b}(R)$.
(b) Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R. Let M be an R-module. Suppose that for all $1 \le i \le n$ the multiplication map $M \xrightarrow{x_i} M$ is zero in $\mathrm{D_{sg}}(R)$. Then M is a direct summand of the Koszul complex $\mathrm{K}(\boldsymbol{x}, M)$ in $\mathrm{D_{sg}}(R)$.

(c) Let R be a complete equicharacteristic Cohen–Macaulay local ring. Let x be an element in J. Let M be a maximal Cohen–Macaulay R-module. Then the multiplication map $M \xrightarrow{x} M$ is zero in $\mathrm{D_{sg}}(R)$.

(d) Suppose that the residue field of R is infinite. Then one can choose a minimal reduction Q of J as a parameter ideal of R. It holds that

$$(\nu(Q) - d + 1) \cdot \ell\ell(R/Q) = \ell\ell(R/Q) \leq \ell(R/Q) = \mathrm{e}(J).$$

The first inequality in the theorem follows from (a), (b) and (c), while the second one is obtained by (d). ■

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Existence and Constructions of Totally Reflexive Modules

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Dedicated to David Eisenbud on the occasion of his 75th birthday

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1 Introduction

A homological invariant named Gorenstein dimension (G-dimension for short) for finitely generated modules over a commutative Noetherian ring was introduced in [2] as a generalization of projective dimension, and it was further developed in [3]. G-dimension can be used to give a homological characterization of Gorenstein rings, similar to the way projective dimension characterizes regular rings by the famous theorem of Serre [32] and Auslander-Buchsbaum [4], which states that a ring is regular if an only if every finitely generated module has finite projective dimension. We refer the reader to [14] for an extensive survey of G-dimension.

Totally reflexive modules, also called modules of G-dimension zero, are the building blocks for the theory of G-dimension, the same way that projective modules are building blocks for the theory of projective dimension. The category of totally reflexive modules over a ring reveals subtle information about the ring. For example, it is shown in [15] that the property of having finitely many indecomposable totally reflexive modules up to isomorphism characterizes simple hypersurface singularities among all complete local rings.

The following conventions will be in effect throughout this paper.

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Notations and Definitions 1.1

1. $(R, \mathfrak{m}, \mathsf{k})$ *denotes a commutative Noetherian ring, which is either local with maximal ideal* $\mathfrak{m}$*, or graded with a unique maximal homogeneous ideal* $\mathfrak{m}$*, and* $\mathsf{k} = \mathsf{R}/\mathfrak{m}$*. All R-modules we consider will be finitely generated.*
2. *The socle of R,* $\mathrm{Soc}(R)$ *is the ideal* $(0 :_R \mathfrak{m}) = \{r \in R \mid r\,\mathfrak{m} = 0\}$
3. *If R is Artinian and M is a finitely generated R-module,* $\lambda(M)$ *denotes the length of M.*
4. *If* $R = \mathsf{k} \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_s$ *is a standard graded Artinian algebra over a field* k*, we say that R has Hilbert function* $(1, n_1, n_2, \ldots, n_s)$*, where* $n_i = \dim_{\mathsf{k}}(R_i)$ *for* $1 \le i \le s$*.*
5. *The embedding dimension of R is* $\dim_{\mathsf{k}}(\mathfrak{m}/\mathfrak{m}^2)$
6. *The parameter* $\nu(R)$ *is defined by*

$$\nu(R) = \inf\{i \mid \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1}) < \binom{n-1+i}{i}\},$$

   where n is the embedding dimension of R. If S is defined as $S = P/J$*, where* $P = k[x_1, \ldots, x_n]$*, and J is a homogeneous ideal, then* $\nu(S)$ *is the lowest degree of a minimal generator of J.*
7. *The canonical module of R is denoted* $\omega(R)$*.*

We recall the following definitions:

Definition 1.2

1. A complex of R-modules (in homological notation) is a sequence

$$\mathcal{M}: \qquad \cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots \tag{1}$$

   where M_i are R-modules, and d_i are R-module homomorphisms, such that $d_i d_{i+1} = 0$ for all i. We say that the complex $\mathcal{M}$ is minimal if $\mathrm{im}(d_{i+1}) \subseteq \mathfrak{m}M_i$ for all i.
2. The ith homology of $\mathcal{M}$ is

$$H_i(\mathcal{M}) := \frac{\ker(d_i)}{\mathrm{im}(d_{i+1})}$$

3. The complex $\mathcal{M}$ is called *acyclic* if $H_i(\mathcal{M}) = 0$ for all i
4. $(\ )^*$ denotes the functor $\mathrm{Hom}_R(\ , R)$. The dual complex $\mathcal{M}^*$ is the (cohomological) complex $\mathrm{Hom}_R(\mathcal{M}, R)$, with maps $d_i^* : M_i^* \to M_{i+1}^*$.
5. The complex $\mathcal{M}$ is called *totally acyclic* if both $\mathcal{M}$ and $\mathcal{M}^*$ are acyclic.
6. A finitely generated R-module M is called *totally reflexive* if there exists a totally acyclic complex $\mathcal{M}$ such that every M_i is a finitely generated free module and $M = \ker(d_0)$.

The definition in part (6.) above is usually stated with the requirement that M_i is finitely generated projected instead of free; the two notions are equivalent when $(R, \mathfrak{m})$ is a local ring.

Note that if a module M is itself free, we can take $\mathcal{M}$ be to the complex

$$\cdots \to 0 \to M \xrightarrow{d} M \to 0 \to 0 \to \cdots$$

with d being the identity function. Thus, every finitely generated free module is totally reflexive.

Observation 1.3 *If $M = \ker(d_0)$, then for each integer i we can consider the complex $\mathcal{M}(i)$ shifted by i, such that M is the kernel of the i*th *differential in $\mathcal{M}(i)$.*

*Thus, every totally reflexive module can also be obtained as an i*th *syzygy, for every i. In other words, M is an infinite syzygy. It follows that if R is Cohen–Macaulay, every totally reflexive module is maximal Cohen–Macaulay.*

The following gives an equivalent characterization of the total reflexivity property:

Proposition 1.4 *Let M be a finitely generated R-module. The following are equivalent:*

1. *M is totally reflexive*
2. *M is reflexive (i.e. $M \cong M^{**}$) , and*

$$\operatorname{Ext}^i_R(M, R) = 0 = \operatorname{Ext}^i_R(M^*, R)$$

for all $i > 0$.

It is natural to ask whether the total reflexivity property can be verified by checking the vanishing of finitely many of the Ext modules in Proposition 1.4. Yoshino [37] studied certain situations where the vanishing of $\operatorname{Ext}^i_R(M, R)$ for all $i > 0$ implies total reflexivity. However, an example is provided in [24] to show that this does not hold in general. More specifically, there exists a local Artinian ring R and a reflexive R-module M such that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$, but $\operatorname{Ext}^i_R(M^*, R) \neq 0$ for all $i > 0$.

We recall the definition of G-dimension.

Definition 1.5 Let $M \neq 0$ be a finitely generated R-module. The G-dimension of M, $\text{G-dim}_R(M)$, is the smallest integer $n \geq 0$ such that there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

such that G_i is totally reflexive for all $0 \leq i \leq n$.

If no such n exists, then we say that $\text{G-dim}_R(M) = \infty$.

Since finitely generated free modules are totally reflexive, we have $\text{G-dim}_R(M) \leq \text{pd}_R(M)$.

The following result established in [3] gives the characterization of Gorenstein rings in terms of G-dimension:

Theorem 1.6 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. The following conditions are equivalent:*

1. *Every finitely generated R-module has finite G-dimension*
2. *k has finite G-dimension*
3. *R is Gorenstein*

Moreover, if R is Gorenstein, then $G\text{-dim}_R(M) \leq \dim(R)$ for every finitely generated R-module M. In particular, if R is Artinian Gorenstein, then every finitely generated R-module is totally reflexive.

We also have an analog of the Auslander-Buchsbaum formula for G-dimension:

Theorem 1.7 ([3], 4.13) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module. If $G\text{-dim}_R(M) < \infty$, then*

$$G - \dim_R(M) = \text{depth}(R) - \text{depth}(M).$$

In particular, $G\text{-dim}_R(M) = \text{pd}_R(M)$ whenever $\text{pd}_R(M) < \infty$.

Enochs and Jenda [19] studied the notion of Gorenstein projective modules, which extends the theory of G-dimension beyond the setting of finitely generated modules over Noetherian rings. They also introduced a theory of Gorenstein injective dimension.

A different homological invariant called *complete intersection dimension* (or c.i. dimension), which fits in between G-dimension and projective dimension, was introduced in [7]. More precisely,

$$\text{G} - \dim_R(M) \leq \text{CI} - \dim_R(M) \leq \text{pd}_R(M),$$

and equalitites hold to the left of any finite value in the above inequalitites. In a different direction, projective dimension and Gorenstein dimension have been extended to complexes (see [5, 13]). A notion of ring homomorphisms of finite Gorenstein dimension was introduced in [6]. We will not pursue these directions in the present survey.

In view of the commonalites between the theory of projective dimension and the theory of G-dimension, it seems to relevant to ask for a measure of the difference between the category of projective modules and the category of totally reflexive modules over a given ring R. In the case when R is Gorenstein, but not regular, every finitely generated module has finite G-dimension, while the finite projective dimension property is quite special. In fact, from a representation theoretic point of view, the category of finitely generated projective modules over a local ring is trivial, because every projective module is free. The category of totally reflexive modules can be a lot more complicated in general; however, there are rings for which the two categories coincide.

Definition 1.8 A Noetherian local ring $(R, \mathfrak{m})$ is called G-regular if every totally reflexive R-module is free.

The following striking result from [15] can be thought of as a counterpart of the work of Buchweitz, Greuel, and Schreyer [12], Herzog [18], and Yoshino [31], characterizing Gorenstein rings of finite Cohen–Macaulay representation type as simple singularities.

Theorem 1.9 ([15], Theorem B) *If the set of isomorphism classes of indecomposable totally reflexive R-modules is finite, then R is either Gorenstein or G-regular.*

In other words, every non-Gorenstein ring which is not G-regular has infinitely many non-isomorphic indecomposable totally reflexive module. Direct sums of totally reflexive modules are totally reflexive, which explains why the focus is on the indecomposable ones.

The following two questions, prompted by the result of Theorem 1.9, constitute the main focus of this survey:

Question 1.10 Among all non-Gorenstein local rings, which ones are G-regular?

If one non-free totally reflexive module is given, Theorem 1.9 states that there are infinitely many such indecomposable modules. However, the proof of Theorem 1.9 is non-constructive. We ask:

Question 1.11 Given one indecomposable non-free totally reflexive module M over a non-Gorenstein ring R, how can one construct infinitely many non-isomorphic such modules?

If M is indecomposable totally reflexive, then so is every syzygy of M. Thus, one can obtain the desired infinite family from one totally reflexive module that has a non-periodic resolution. Unfortunately, many of the totally reflexive modules that we are able to find in practice have periodic resolutions with period two, which means that all their syzygies are isomorphic to the first two.

We will present a number of constructions that give rise tototally reflexive modules over certain classes of rings. We usually focus on building minimal totally acyclic complexes; totally reflexive modules can then be obtained as syzygies in such complexes. The minimal property of the complex implies that the complex is not split, and therefore the syzygy modules are not free.

The case of Artinian rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ has been studied extensively. In this case, it turns out that once there is one non-free totally reflexive module, the property of a module being totally reflexive holds generically. This will be explained in more detail in Sect. 3. On the other hand, once the assumption $\mathfrak{m}^3 = 0$ is removed, the totally reflexive property is no longer generical.

We will also present large classes of rings that are G-regular.

2 Totally Reflexive Modules Over Rings That Have an Embedded Deformation

Definition 2.1 A local ring $(Q, \mathfrak{n})$ is an *embedded deformation* of R if there exists a Q-regular sequence $\mathbf{x} = x_1, \ldots, x_c$ contained in $\mathfrak{n}^2$ such that $R \cong Q/(\mathbf{x})$.

It is shown in [7] that every ring that has an embedded deformation has non-free totally reflexive modules.

Theorem 2.2 ([7], Theorem 3.2) *Let $\mathfrak{a} = (a_1, \ldots, a_m)$ and $\mathfrak{b}$ be ideals in a commutative ring Q, and $\mathbf{x} = x_1, \ldots, x_c$ a Q-regular sequence contained in $\mathfrak{ab}$. Let $R = Q/(\mathbf{x})$.*

There exists an exact complex of free R-modules $(\mathbf{T}, \delta)$ such that $\delta(\mathbf{T}) \subseteq (\mathfrak{a} + \mathfrak{b})\mathbf{T}$, there is a chain isomorphism $\mathrm{Hom}_R(\mathbf{T}, R) \cong \Sigma\mathbf{T}$, and

$$\mathrm{rank}_R(T_n) = \begin{cases} 2^{cm}\dbinom{n+c-1}{c-1} & \text{for } n \geq 0 \\ 2^{cm}\dbinom{-n+c-2}{c-1} & \text{for } n < 0 \end{cases}$$

In the statement above, $\Sigma\mathbf{T}$ stands for the shifted complex, $(\Sigma\mathbf{T})_n = T_{n-1}$. Applying the statement to the case when $(Q, \mathfrak{n})$ is a local ring, and $\mathfrak{a} = \mathfrak{b} = \mathfrak{n}$, the resulting complex $\mathbf{T}$ has $\delta(\mathbf{T}) \subset \mathfrak{n}T$, hence it is a minimal complex and its syzygy modules are not free. Note that $\mathrm{Hom}_R(\mathbf{T}, R) \cong \Sigma\mathbf{T}$ implies that the dual complex $\mathbf{T}^*$ is also exact, and therefore $\mathbf{T}$ is totally reflexive.

The syzygies of the complex $\mathbf{T}$ above are in fact shown to have finite CI-dimension in [7], which is a stronger property than total reflexivity.

3 Yoshino's Conditions for Rings with $\mathfrak{m}^3 = 0$

The simplest Artinian local rings $(R, \mathfrak{m})$ are the ones that satisfy $\mathfrak{m}^2 = 0$. Every syzygy in a minimal complex over these rings is a vector space. Therefore, if R has non-free totally reflexive modules, then the residue class field $\mathsf{k} = \mathsf{R}/\mathfrak{m}$ must be totally reflexive, which means that R is Gorenstein (from Theorem 1.6). We conclude that all local rings with $\mathfrak{m}^2 = 0$ are either Gorenstein or G-regular.

The next simplest case is local rings with $\mathfrak{m}^3 = 0$, which has been extensively studied by numerous authors. Theorem 3.1 in [36] gives necessary conditions for the existence of non-free totally reflexive modules for this case. It also gives information about the totally reflexive R-modules, in case they exist.

Theorem 3.1 *Let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0$. Assume that R contains a field k isomorphic to $R/\mathfrak{m}$, and assume that there is a non-free totally reflexive R-module M. Then:*

(1) *R has a natural structure of homogeneous graded ring $R = R_0 \oplus R_1 \oplus R_2$ with $R_0 = \mathsf{k}$, $\dim_{\mathsf{k}}(R_1) = r + 1$, and $\dim_{\mathsf{k}}(R_2) = r$, where r is the type of R. In other words, R has Hilbert function $(1, r + 1, r)$.*
(2) *$(0 :_R \mathfrak{m}) = \mathfrak{m}^2$. In other words, there is no linear element in the socle of R.*
(3) *R is a Koszul algebra.*
(4) *M has a natural structure of graded R-module, and, if M is indecomposable, then the minimal free resolution of M has the form*

$$\cdots \to R(-n-1)^b \to R(-n)^b \to \cdots \to R(-1)^b \to R^b \to M \to 0$$

In other words, the resolution of M is linear with constant Betti numbers.

The requirement that R is a Koszul algebra means that the minimal free R-resolution of $\mathsf{k} = \mathsf{R}/\mathfrak{m}$ is linear (every map is represented by a matrix with linear entries). If R is represented as a quotient P/I of a polynomial ring P, the fact that R is Koszul implies that I must be homogeneous generated by quadratics. The converse is not true in general; however, if I has a Gröbner basis consisting of quadratics (in particular if I is generated by quadratic monomials), then R is Koszul. More information on the subject of Koszul algebras can be found in [20].

We emphasize the fact that the conditions in Theorem 3.1 are necessary for the existence of non-free totally reflexive module, but they are far from sufficient.

The fiber product construction provides examples of rings that satisfy Yoshino's conditions from Theorem 3.1, but do not have non-free totally reflexive modules. We describe this construction below.

Definition 3.2 Let $(S, \mathfrak{m}_S, \mathsf{k})$ and $(T, \mathfrak{m}_T, \mathsf{k})$ be local Noetherian rings with the same residue field k. Let $S \xrightarrow{\pi_S} \mathsf{k} \xleftarrow{\pi_T} T$ denote the canonical projections of S and T onto the residue field.

The *fiber product* of S and T over k is the ring

$$R := S \times_{\mathsf{k}} T = \{(s, t) \in S \times T \mid \pi_S(s) = \pi_T(t)\}$$

More explicitly, if $S = \mathsf{k}[x_1, \ldots, x_n]/I$ and $T = \mathsf{k}[y_1, \ldots, y_m]/J$, then

$$S \times_{\mathsf{k}} T \cong \frac{P}{IP + JP + (x_1, \ldots, x_n)(y_1, \ldots, y_m)}$$

where $P = \mathsf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

It is shown in [31] that fiber product rings are always G-regular, unless they are Gorenstein. However, we can construct fiber product rings with $\mathfrak{m}^3 = 0$ that satisfy the conditions in Theorem 3.1, and are not Gorenstein.

It is shown in [29], Fact 2.9 that if S and T are Artinian rings and $R = S \times_{\mathsf{k}} T$, then $\lambda(R) = \lambda(S) + \lambda(T) - 1$. It is clear that if $\mathfrak{m}_S^3 = \mathfrak{m}_T^3 = 0$, then $\mathfrak{m}_R^3 = 0$. If S has Hilbert function $(1, n, k)$ and T has Hilbert function $(1, m, l)$, then $R = S \times_{\mathsf{k}} T$

will have Hilbert function $(1, n+m, k+l-1)$. Condition (1) in Theorem 3.1 is satisfied provided that $n+m=k+l$.

For a concrete example, consider $R = S \times_{\mathsf{k}} T$, with

$$S = \frac{\mathsf{k}[x_1, x_2, y_1, y_2]}{(x_1, x_2)^2 + (y_1, y_2)^2}, \quad T = \frac{\mathsf{k}[z_1, z_2, w_1, w_2]}{(z_1, z_2)^2 + (w_1, w_2)^2 + (z_1 w_1)}$$

The Hilbert function of S is $(1, 4, 4)$, and the Hilbert function of T is $(1, 4, 3)$, which implies that the Hilbert function of R is $(1, 8, 7)$. Moreover, R is a Koszul algebra because its defining ideal is generated by quadratic monomials. In order to verify condition (2) in (3.1), note that every linear element in R can be written as the image of $f+g$ with $f \in \mathsf{k}[x_1, x_2, y_1, y_2]$, $g \in \mathsf{k}[z_1, z_2, w_1, w_2]$, and it is a socle element in R if and only if the images of f in S and g in T are socle elements. It can be easily checked that S and T do not have linear elements in their socles. Therefore, all the conditions (1)–(3) from Theorem 3.1 hold.

4 Exact Zero Divisors

The easiest examples of non-free totally reflexive modules are obtained when the ring has exact zero divisors, which were introduced in [21]:

Definition 4.1 A pair (x, y) of elements of R is called a pair of **exact zero divisors** if $\operatorname{Ann}_R(x) = (y)$ and $\operatorname{Ann}_R(y) = (x)$.

Example 4.2

1. $(\overline{x}, \overline{y})$ is a pair of exact zero divisors in $R = \mathsf{k}[x, y]/(xy)$
2. $(\overline{x}^{n-1}, \overline{x})$ is a pair of exact zero divisors in $R = \mathsf{k}[x]/(x^n)$.

Observation 4.3 *If (x, y) is a pair of exact zero divisors, then the complex*

$$\cdots \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} \cdots$$

is totally acyclic.

Since $R/(x)$ and $R/(y)$ are syzygies in this complex, it follows that they are totally reflexive modules.

Observation 4.4 *If R is an Artinian local ring and $x, y \in R$ are such that $\operatorname{Ann}_R(x) = (y)$, then (x, y) are a pair of exact zero divisors.*

Proof The assumption that $\operatorname{Ann}_R(x) = (y)$ implies that (x) is isomorphic to $R/(y)$ as an R-module. Therefore

$$\lambda((x)) + \lambda((y)) = \lambda(R).$$

We can re-write this as

$$\lambda((y)) = \lambda(R/(x)).$$

Since $xy = 0$, we have an inclusion $(y) \subseteq \operatorname{Ann}_R(x)$, and the equality follows by noting that the two modules have the same length. □

Therefore, an element x in an Artinian ring is an exact zero divisor provided that $\operatorname{Ann}_R(x)$ is a principal ideal. The exact zero divisor property is easy to verify in practice for a given $x \in R$, whereas the total reflexivity of a given module in general requires infinitely many verifications.

There are many interesting properties of exact zero divisors in the case of Artinian rings with $\mathfrak{m}^3 = 0$. Recalling Theorem 3.1, a ring with $\mathfrak{m}^3 = 0$ that has exact zero divisors must be a Koszul algebra. Therefore we can write $R = \mathsf{k}[x_1, \dots, x_e]/\mathfrak{q}$, where $\mathfrak{q}$ is an ideal generated by quadratic forms. Moreover, R must have Hilbert function $(1, e, e-1)$. For a fixed value of e, the algebras defined by quadratic equations that have Hilbert function $(1, e, e-1)$ can be parametrized by points in a Grassmanian variety. Let W denote the $e(e+1)/2$-dimensional vector space spanned by all quadratic monomials $x_i x_j$ with $1 \le i \le j \le e$, and let V denote the subspace of W spanned by the generators of $\mathfrak{q}$. Condition (1) in Theorem 3.1 implies $\dim_{\mathsf{k}}(W/V) = \dim_k(R_2) = e-1$, and therefore $\dim_{\mathsf{k}}(V) = (e^2 - e + 2)/2$. Letting $n = e(e+1)/2, m = (e^2 - e + 2)/2$, the ring R corresponds to the point in the Grassmaninan variety $\operatorname{Grass}_{\mathsf{k}}(n, m)$ given by the subspace V of W.

Theorem 8.4 and Remark 8.8 in [16] show that exact zero divisors are ubiquitous in such rings:

Theorem 4.5 *Let* k *be an infinite field, and* $e \ge 2$*. Let* $n = e(e+1)/2, m = (e^2 - e + 2)/2$*.*

(a) *There is a non-empty open set of* $\operatorname{Grass}_k(n, m)$ *such that for every point in that set corresponding to an* m*-dimensional subspace* V *of* W*, the algebra* $R = \mathsf{k}[x_1, \dots, x_e]/\mathfrak{q}$*, where* $\mathfrak{q}$ *is the ideal generated by* V*, has exact zero divisors.*
(b) *Let* $R = \mathsf{k}[x_1, \dots, x_e]/\mathfrak{q}$ *satisfy the conditions in Theorem 3.1. If* R *has a pair of exact zero divisors, then a generic linear form in* R *is an exact zero divisor.*

In part (b) of the statement above, we are thinking of linear forms as parameterized by points in k^n corresponding to the coefficients of the linear form.

A related property was studied in [17]. The terminology *Conca generator* for this property was introduced in [8].

Definition 4.6 Assume $R = \mathsf{k} \oplus \mathsf{R}_1 \oplus \mathsf{R}_2$ is graded ring. An element $x \in R_1$ is called a *Conca generator* if $x^2 = 0$ and $R_2 = xR_1$.

Conca showed that if R has a Conca generator, then it is G-quadratic, i.e. its defining ideal has a Gröbner basis of quadratics (recall that this implies Koszul). He also showed that a generic quadratic algebra R with $\dim_{\mathsf{k}}(R_2) < \dim_{\mathsf{k}}(R_1)$ has a Conca generator.

Note that in the case $\dim_k(R_2) = \dim_k(R_1) - 1$, if x is a Conca generator, then (x, x) is a pair of exact zero divisors. This is because the kernel of the linear map $R_1 \to R_2$ given by multiplication by x has dimension 1, and therefore $\operatorname{Ann}_R(x) = (x)$.

We also note that the existence of exact zero divisors in rings with $\mathfrak{m}^3 = 0$ is related to the weak Lefshetz property.

Definition 4.7 For $R = \oplus_{n \in \mathbf{N}} R_n$ a graded k-algebra, we say that R has the **weak Lefshetz property** if there is a linear form $l \in R_1$ (equivalently, for every generic linear form $l \in R_1$) such that the multiplication by l viewed as a linear map $R_n \to R_{n+1}$ has maximal rank (i.e. it is either injective or surjective) for all n.

In the case of graded rings $R = k \oplus R_1 \oplus R_2$, the Weak Lefschetz property simply means that there exists $l \in R_1$ such that $R_2 = lR_1$. If we have $\dim_k(R_2) = \dim_k(R_1) - 1$, this is equivalent to $\operatorname{Ann}_R(l)$ being a principal ideal, i.e. l is an exact zero divisor. We refer the reader to [28] for a survey on the topic of the weak Lefschetz property.

The following result extends Condition (1) from Theorem 3.1 to graded rings that do not have $\mathfrak{m}^3 = 0$.

Theorem 4.8 ([26], Theorem 2.9) *Let S be a standard graded Artinian algebra. Suppose that (x, y) is a pair of homogeneous exact zero divisors, and let $D =$* $\deg(x) + \deg(y)$.

Then the Hilbert series of $S, H_S(t) = \sum_{i=0}^{\infty} \dim_k(S_i)t^i \in Z[t]$ is divisible by $(t^D - 1)/t - 1$.

Equivalently, for each $0 \leq a < b \leq d - 1$, we have

$$\sum_{i \equiv a \,(\mathrm{mod} D)} \dim_k(S_i) = \sum_{i \equiv b \,(\mathrm{mod} D)} \dim_k(S_i)$$

The following Corollary of Theorem 4.8 gives a counterpart to Theorem 4.5, showing that ubiquity of exact zero divisors no longer holds when the assumption $\mathfrak{m}^3 = 0$ is removed. In fact, generic algebras with $\mathfrak{m}^3 \neq 0$ do not have homogeneous exact zero divisors. Before giving the statement, we recall the relevant definitions:

Definition 4.9 A standard graded Artinian algebra

$$R = k \oplus R_1 \oplus \cdots \oplus R_d$$

is called *level* if $\operatorname{Soc}(R) = R_d$.

For R as above, d is called the socle degree, and $r := \dim_k(R_d)$ is called the socle dimension.

Definition 4.10 A standard graded level Artinian algebra with given codimension e, socle degree d and socle dimension r is called *compressed* if it has maximal Hilbert function among all standard graded level Artinian algebras with given codimension, socle degree and socle dimension.

The level algebras with given values of e, r, d can be parametrized using Macaulay inverse systems. A generic level algebra with fixed e, r, d is compressed. We refer the reader to [23] for details about compressed algebras.

Theorem 4.11 *Let R be a compressed level algebra with codimnesion e, socle dimension r and socle degree d. Then R does not have homogeneous exact zero divisors unless R is Gorenstein (i.e. $r = 1$) with $e \leq 2$ or $d = 3$, or R has Hilbert function $(1, e, e-1)$.*

5 Constructing Totally Reflexive Modules from Exact Zero Divisors

Exact zero divisors have been used to construct more complicated totally reflexive modules in [16].

For $w, x, y, z \in R$, and for all $n \geq 1$, consider the $n \times n$ matrix

$$\Theta_n(w, x, y, z) = \begin{pmatrix} w & y & 0 & 0 & 0 & \cdots \\ 0 & x & z & 0 & 0 & \cdots \\ 0 & 0 & w & y & 0 & \cdots \\ 0 & 0 & 0 & x & z & \cdots \\ 0 & 0 & 0 & 0 & w & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Theorem 5.1 ([16], Theorem 3.1) *Let $(R, \mathfrak{m})$ be a local ring and assume that w, x are elements in $\mathfrak{m} \setminus \mathfrak{m}^2$, that form an exact pair of zero divisors. Assume further that y and z are elements in $\mathfrak{m} \setminus \mathfrak{m}^2$ with $yz = 0$ and that one of the following conditions holds:*

(a) The elements w, x, and y are linearly independent modulo $\mathfrak{m}^2$.
(b) One has $w \in (x) + \mathfrak{m}^2$ and $y, z \in (x) + \mathfrak{m}^2$.

For every $n \geq 1$, the R-module $M_n(w, x, y, z)$ is indecomposable, totally reflexive, and non-free. Moreover, $M_n(w, x, y, z)$ has constant Betti numbers equal to n, and its minimal free resolution is periodic of period at most 2.

This result is usually applied in the case when $\mathfrak{m}^3 = 0$ (if $\mathfrak{m}^3 \neq 0$, one of the two elements x, w in a pair of exact zero divisors will typically be in $\mathfrak{m}^2$, causing condition (a) above to fail).

The theoretical importance of the construction given in Theorem 5.1 lies in the fact that it reveales the structure of the category of totally reflexive modules to be quite complex for rings with $\mathfrak{m}^3 = 0$ that have exact zero divisors. The following results, modeled on the Brauer-Thrall conjectures for modules of finite length over a finitely dimensional algebra, are proved in [16] using the construction described above.

Theorem 5.2 *Assume* $(R, \mathfrak{m})$ *is a local ring with* $\mathfrak{m}^3 = 0$*. If there exist an exact zero divisor in R, then there exists a family* $\{M_n\}_n$ *of indecomposable totally reflexive modules such that* $\lambda(M_n) = n\lambda(R)$*.*

Moreover, if the residue field $\mathsf{k} = \mathsf{R}/\mathfrak{m}$ *is algebraically closed, then for each* $n \geq 1$ *there exists a family* $\{M_{n,a}\}_{a\in\mathsf{k}}$ *of indecomposable totally reflexive modules such that* $\lambda(M_{n,a}) = n\lambda(R)$ *for each* $a \in \mathsf{k}$*.*

6 Lifting Totally Reflexive Modules

A common theme in commutative algebra is the transfer of properties between a ring R and a quotient $R/(x)$, when $x \in R$ is a regular element.

The following is an easy consequence of the definition of totally reflexive modules:

Observation 6.1 *Let* $(R, \mathfrak{m})$ *be a local ring, and* $x \in \mathfrak{m}$ *a regular element.*

(a) *If M is a totally reflexive R-module, then* $M/(x)M$ *is a totally reflexive* $R/(x)$*-module.*
(b) *If R has non-free totally reflexive modules, then so does* $R/(x)$*.*

It is natural to ask whether the converse of part (b) above holds. If $x \in \mathfrak{m}^2$, then Theorem 1.4 in [7] provides a construction of non-free totally reflexive modules over $R/(x)$, regardless of whether the original ring R has non-free totally reflexive modules or not. This shows that the converse of Observation 6.1(b) does not hold in general.

It turns out that the converse holds under the additional assumption $x \notin \mathfrak{m}^2$. More precisely, we have:

Theorem 6.2 *Let* $(R, \mathfrak{m})$ *be a local ring, and* $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ *a regular element. Then R has non-free totally reflexive modules if and only if* $R/(x)$ *does.*

This was proved in [34] using non-constructive methods. In [10], an explicit construction is given for lifting a totally reflexive $R/(x)$-module to a totally reflexive R-module in the case when R is a standard graded algebra over a field.

Construction 6.3 *Let* $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ *be a standard graded algebra, and* $x \in S_1$ *an S-regular element. Let* $R = S/(x)$*.*

Given a complex

$$\cdots \longrightarrow R^{b_{i+1}} \xrightarrow{\delta_{i+1}} R^{b_i} \xrightarrow{\delta_i} R^{b_{i-1}} \xrightarrow{\delta_{i-1}} \cdots \qquad (1)$$

of free R-modules, we construct a complex

$$\cdots \longrightarrow S^{2b_{i+1}} \xrightarrow{\epsilon_{i+1}} S^{2b_i} \xrightarrow{\epsilon_i} S^{2b_{i-1}} \xrightarrow{\epsilon_{i-1}} \cdots \qquad (2)$$

as follows:

For each i, let $\tilde{\delta}_i : S^{b_i} \to S^{b_{i-1}}$ be any lifting of δ_i to S. We think of maps between free modules as matrices. Since $\delta_i \delta_{i+1} = 0$, all the entries of $\tilde{\delta}_i \tilde{\delta}_{i+1}$ are mutliples of x. For each i, there exists a matrix M_{i+1} with entries in S such that

$$\tilde{\delta}_i \tilde{\delta}_{i+1} = x M_{i+1}$$

We define

$$\epsilon_i := \begin{bmatrix} \tilde{\delta}_i & x I_{b_{i-1}} \\ M_i & \tilde{\delta}_{i-1} \end{bmatrix}, \quad \text{if } i \text{ is even}$$

and

$$\epsilon_i := \begin{bmatrix} \tilde{\delta}_i & -x I_{b_{i-1}} \\ -M_i & \tilde{\delta}_{i-1} \end{bmatrix}, \quad \text{if } i \text{ is odd,}$$

where each entry above represents a block, and $I_{b_{i-1}}$ is the identity matrix.

It is shown in [10] that if (1) is totally acyclic, then so is (2).

Since the existence of totally reflexive R-modules can be established by looking at the analogous problem for the ring $R/(x)$, the rest of this survey will focus on the case on Artinian local rings.

7 When Does a Generic Matrix Give Rise to a Totally Reflexive Module?

We have seen in Theorem 4.5 that for graded algebra $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ that has exact zero divisors, a generic choice of an $x \in R_1$ is an exact zero divisor. One can ask whether a similar phenomenon continues to hold for totally reflexive modules which are not cyclic.

Under the assumption that $\mathfrak{m}^3 = 0$, Yoshino's result guarantees that every totally reflexive module has a linear resolution with constant Betti numbers. Such a module can be described as the cokernel of a square matrix A with entries in R_1. For a fixed $b \geq 1$, the $b \times b$ matrices with entries in R_1 can be parametrized by points in the affine space $\mathsf{k}^{b^2 n}$, where n is the embedding dimension of R, by associating a vector of coefficients in k^n to each linear form in R_1, and then arranging the vectors coming from the b^2 entries of the matrix into a vector in $\mathsf{k}^{b^2 n}$. The following was proved in [10]:

Theorem 7.1 *Let $(R, \mathfrak{m})$ be a standard graded non-Gorenstein algebra with $\mathfrak{m}^3 = 0$ of embedding dimension n. Assume that there exists a non-free totally reflexive R-module, which can be described as the cokernel of a $b \times b$ matrix of linear forms for* *some* $b \geq 1$. *Then there is a countable intersection of Zariski open sets in* $\mathsf{k}^{b^2 n}$ *such that for every matrix* A *corresponding to a vector in this intersection, the module* $\operatorname{coker}(A)$ *is totally reflexive.*

This phenomenom is specific to rings that have $\mathfrak{m}^3 = 0$. Example 3.3 in [11] shows that for a fixed $b \geq 1$, the cokernel of a generic matrix with linear entries in the ring $R = k[x, y, z_1, \ldots, z_n]/(x^2, y^2, (z_1, \ldots, z_n)^2)$ is not a totally reflexive R-module.

8 Other Constructions of Totally Reflexive Modules

We saw in Sect. 5 that exact zero divisors are used as building blocks for constructing totally reflexive modules. However, there are rings that have do not have exact zero divisors, but still have totally reflexive modules.

While there is no systematic way to construct totally reflexive modules over an arbitrary ring in the absence of zero divisors, we present a construction that allows us to put together totally reflexive modules over two rings, and obtain a totally reflexive module over the new ring, which is a connected sum of the two original rings. This construction is studied in [35] for rings with $\mathfrak{m}^3 = 0$ that satisfy the conditions in (3.1).

Construction 8.1 *Let* $P_1 = \mathsf{k}[x_1, \ldots, x_m]$, $P_2 = \mathsf{k}[y_1, \ldots, y_n]$ *be two polynomial rings over a field k,* $I_1 \subseteq P_1$ *and* $I_2 \subseteq P_2$ *be homogeneous ideas generated by quadratics.*

Let $f \in P_1$ *be a quadratic such that the image of* f *in* $R_0 = P_1/I_1$ *is in the socle of* R_0*, and let* $g \in P_2$ *be quadratic such that the image of* g *in* $S_0 = P_2/I_2$ *is in the socle of* S_0.

Let

$$R = \frac{P}{I_1 P + I_2 P + (x_i y_j \mid 1 \leq i \leq m, 1 \leq j \leq n) + (f - g)}$$

Note that R is a *connected sum* of R_0 and S_0 in the sense of [1]. Although connected sums have been primarily studied in the case of Gorenstein rings, our interest here is in the non-Gorenstein case. The ring R can also be described as a quotient of the fiber product:

$$R = \frac{R_0 \times_{\mathsf{k}} S_0}{(f - g)},$$

where the fiber product is the ring

$$R_0 \times_{\mathsf{k}} S_0 = \frac{P}{I_1 P + I_2 P + (x_i y_j \mid 1 \leq i \leq n, 1 \leq j \leq m)}$$

Although the construction above describes R as a connected sum of R_0 and S_0, the totally reflexive modules over R are actually obtained from totally reflexive modules over the rings $R_1 := P_1/I_1 + (f)$ and $S_1 := P_2/I_2 + (g)$ rather than R_0 and S_0.

Adopt the notation from (8.1). Let $\mathfrak{m}_0$ denote the ideal generated by the images of $x_1, \ldots, x_m$ in R_0, and $\mathfrak{n}_0$ denote the ideal generated by the images of $y_1, \ldots, y_n$ in S_0.

Assume $\mathfrak{m}_0^3 = \mathfrak{n}_0^3 = 0$. Recall that a totally reflexive module is a syzygy in a totally acyclic complex. Under the assumption $\mathfrak{m}^3 = 0$, all maps in a totally acyclic complex can be represented by square matrices with linear entries (from Theorem 3.1).

A map $d : R^b \to R^c$ with linear entries induces maps $A : R_1^b \to R_2^c$ and $B : S_1^b \to S_2^b$ as follows. Thinking of maps as matrices, each entry in d_{ij} in the matrix corresponding to d can be written as $a_{ij} + b_{ij}$, where a_{ij} is the image of a linear form in $k[x_1, \ldots, x_m]$, and b_{ij} is the image of a linear form in $k[y_1, \ldots, y_n]$. We take the matrix A that has the images of the corresponding a_{ij}'s in R_1 as entries, and B the matrix that has the images of the corresponding b_{ij}'s in S_1 as entries. Abusing notation, we will write $d = A + B$, where A, B are the maps described above.

The following result from [35] gives the relationship between totally acyclic complexes over R and totally acyclic complexes over the rings R_1 and S_1:

Theorem 8.2 ([35], Theorem 5.3) *Let*

$$\cdots \xrightarrow{d_{i+1}} R^b \xrightarrow{d_i} R^b \xrightarrow{d_{i-1}} \cdots \tag{1}$$

be a complex of R-modules.

Write $d_i = A_i + B_i$ where $A_i : R_1^b \longrightarrow R_1^b$ and $B_i : S_1^b \longrightarrow S_1^b$ are as explained above.

Then

$$\cdots \xrightarrow{A_{i+1}} R_1^b \xrightarrow{A_i} R_1^b \xrightarrow{A_{i-1}} \cdots \tag{2}$$

$$\cdots \xrightarrow{B_{i+1}} S_1^b \xrightarrow{B_i} S_1^b \xrightarrow{B_{i-1}} \cdots \tag{3}$$

are also complexes.

Assume moreover that

$$(f)R_0^b \subseteq \operatorname{im}(\tilde{A}_i) \text{ and } (g)S_0^b \subseteq \operatorname{im}(\tilde{B}_i) \text{ for all } i \tag{4}$$

where $\tilde{A}_i$ is a lifting of A_i to R_0 and $\tilde{B}_i$ is a lifting of B_i to S_0.

Then:

(a) *The complex (1) is acyclic if and only if both complexes (2) and (3) are acyclic.*

(b) *The complex (1) is totally acyclic if and only if both complexes (2) and (3) are totally acyclic.*

We think of this result as saying that R has totally reflexive modules if and only if R_1 and S_1 have totally reflexive modules that can be "glued" together, in the sense that (4) holds for the maps in their resolutions. Condition (4) is implied whenever (1) is totally acyclic, provided that R is not Gorenstein, so it is a necessary and sufficient condition for two totally acyclic complexes (2) and (3) to give rise to a totally acyclic complex (1).

For a fixed $b \geq 1$, it is possible for R_1 and S_1 to have totally reflexive modules with Betti numbers equal to b, but not satisfy (4) for any totally acyclic complexes consisting of free modules of rank b. If this is the case, then R will not have totally reflexive modules with Betti numbers equal to b.

However, under some additional assumptions, we one can use a totally acyclic complex that does not satisfy (4) as a building block to construct totally acyclic complexes consisting of free modules of larger rank that do satisfy (4). In particular, we can exhibit examples of rings that do not have exact zero divisors, but have non-free totally reflexive modules using this construction.

Example 8.3 Let

$$R_0 = \frac{k[x_1, x_2, y_1, y_2, y_3]}{(x_1, x_2)^2 + (y_1, y_2, y_3)^2 + (x_1 y_2)}, \; S_0 = \frac{k[x_3, x_4, x_5, y_4, y_5]}{(x_3, x_4, x_5)^2 + (y_4, y_5)^2 + (x_3 y_4)}$$

Let $f = x_1 y_1 \in R_0$, $g = x_4 y_4 \in S_0$, and

$$R = \frac{k[x_1, \ldots, x_5, y_1, \ldots, y_5]}{J},$$

where

$$J = (x_1, \ldots, x_5)^2 + (y_1, \ldots, y_5)^2 + (x_1, x_2, y_1, y_2, y_3) \cdot (x_3, x_4, x_5, y_4, y_5)$$
$$+(x_1 y_2, x_3 y_4, x_1 y_1 - x_4 y_4)$$

The rings R_1, S_1 have exact zero divisors, but condition (4) fails for every pair of exact zero divisors in R_1 and in S_1. Indeed, a pair of exact zero divisors in R_1 consists of elements of the form $l_x + l_y, l_x - l_y$ where l_x is a linear combination of x_1, x_2 and l_y is a linear combination of y_1, y_2, y_3. When lifted to R_0, we have $(\tilde{l}_x + \tilde{l}_y)(\tilde{l}_x - \tilde{l}_y) = \tilde{l}_x^2 - \tilde{l}_y^2 = 0$. Therefore, the ring R does not have exact zero divisors.

However, Example 4.5 in [35] shows that there are totally acyclic complexes consisting of free modules of rank 2 over R_1 and S_1 that do satisfy (4), and therefore the ring R has totally reflexive modules with Betti numbers equal to two.

9 G-Regular Rings

Recall that a ring R is called G-regular if every totally reflexive R-module is free. As explained in Sect. 6, we focus our attention on the case of Artinian rings.

The goal of this section is to describe classes of Artinian rings that are G-regular. Gorenstein rings are never G-regular, because every maximal Cohen–Macaulay module over a Gorenstein ring is totally reflexive. For this reason, all the results in this section will assume that R is non-Gorenstein.

Numerous classes of G-regular rings have been established by various authors. A stronger property called *strong G-regularity* was identified in [11]. It turns out that this property is present in all the classes of G-regular rings that have been established in literature. We do not know an example of a G-regular ring that is not strongly G-regular.

Definition 9.1 Let $(R, \mathfrak{m})$ be a local ring and T be a finitely generated R-module.

(a) T is called a *test module* for R if the only finitely generated R-modules N with $\operatorname{Tor}_i(T, N) = 0$ for all $i \geq 1$ are the free R-modules.
(b) R is called a *strongly G-regular ring* if it is Artinian and the canonical module ω_R is a test module.

Observation 9.2 *If R is strongly G-regular, then it is G-regular.*

Proof Let M be a totally reflexive R-module. Then $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$. By Matlis duality, it follows that $\operatorname{Tor}^R_i(M, \omega_R) = 0$ for all $i > 0$, and therefore M is a free module. □

The prototypical test module is the reside class field $R/\mathfrak{m}$. Some properties of test modules have been studied in [12]. The following examples of test modules have been established and used to prove strong G-regularity in [25]:

Proposition 9.3

(a) *If $(R, \mathfrak{m})$ is a local ring with $\nu(R) \geq 3$, then any finitely generated module T with $\mathfrak{m}^2 = 0$ is a test module.*
(b) *If $R = P/J$ is an Artinian ring, where $P = k[x_1, \ldots, x_n]$ and $\nu(R) \gg 0$, then $T = R/\mathfrak{a}R$ is a test module for any ideal $\mathfrak{a} \subset P$ generated by a maximal regular sequence in P (recall that the parameter $\nu(R)$ has been defined in (1.1)).*

Most of the results that establish strong G-regularity rely on finding a direct summand of a syzygy of the canonical module, and using the following easy observation:

Observation 9.4

(a) *If M has a syzygy which is a test module, then M is a test module.*
(b) *If M has a direct summand which is a test module, then M is a test module.*

Theorem 9.6 below collects results from literature about various classes of rings that are strongly G-regular. Although the strongly G-regular terminology has not been used in the original statements, an examination of the proofs allows us to conclude that it is present in all the classes of rings listed below. For each statement, we give a reference to the original statement about G-regularity, and indicate the reason why strong G-regularity also holds. Before stating the Theorem, we need the following:

Definition 9.5 If $R = S/J$ is a quotient of an Artinian Gorenstein ring S, then $c_S(R) := \lambda(S) - \lambda(R)$. The Gorenstein colength of R is the minimum of $c_S(R)$ when S is a Gorenstein Artinian ring mapping onto R.

Theorem 9.6 *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian non-Gorenstein local ring. Assume one of the following:*

(a) $\mathfrak{m}^2 = 0$
(b) $\mathfrak{m}^3 = 0$ *and* $\dim_{\mathsf{k}}(\mathfrak{m}/\mathfrak{m}^2) \neq \dim_{\mathsf{k}}(\mathfrak{m}^2) + 1$
(c) *R is a fiber product $S \times_{\mathsf{k}} T$ where S and T are Artinian k-algebras.*
(d) *R is a Golod ring.*
(e) *$R = S/J$ where $(S, \mathfrak{n})$ is Gorenstein local ring $(S, \mathfrak{n})$ with embedding dimension at least two, J is a proper ideal of S, $c_S(R) \leq 4$ and $\nu(S) \gg 0$*
(f) *$R = S/J$ where $(S, \mathfrak{n})$ is a standard graded Artinian Gorenstein algebra over a field, J is a proper ideal of S, $c_S(R) = 5$ and $\nu(S) \gg 0$*
(g) *$R = P/I$ where $(P, \mathfrak{m})$ is a regular local ring and $I \subseteq P$ is a proper ideal such that $\mathfrak{m}I \neq \mathfrak{m}(I :_P \mathfrak{m})$.*

Then R is strongly G-regular.

Proof

(a) The statement that R is G-regular is Corollary 2.5 in [36]. Let S be the first syzygy of ω_R. We have $S \subseteq \mathfrak{m}F$ for some free R-module F, which implies $\mathfrak{m}S = 0$. Therefore S is a vector space, and $R/\mathfrak{m}$ is a direct summand of S. Strong G-regularity follows from Observation (9.4).
(b) The statement that R is G-regular is part of Theorem 3.1 in (3.1). Strong G-regularity follows from Proposition 2.8 in [22].
(c) Strong G-regularity follows from Theorem 1.1 in [30]. Additionally, it follows from Theorem 3.6 in [31] that $\mathfrak{m}$ is a direct summand of a syzygy of ω_R.
(d) G-regularity is proved in Example 3.5.2 in [9]. The argument in [9] actually shows strong G-regularity. Assume that M is a finitely generated non-free R-module such that $\mathrm{Tor}_i^R(M, \omega_R) = 0$ for all $i \geq 1$. By Matlis duality, this is equivalent to $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geq 1$. Let

$$\mathcal{P}: \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be the minimal free resolution of M. Lescot proved in [27] Theorem 6.5 that $\mathrm{rank}_R(P_{n+1}) > \mathrm{rank}_R(P_n)$ for all $n > 0$. In particular, $\mathrm{rank}_R(P_2) > \mathrm{rank}_R(P_1)$.

The dual complex

$$\mathcal{P}^* : 0 \to M^* \to P_0^* \to P_1^* \to \cdots \to P_n^* \to P_{n+1}^* \cdots$$

is also exact.

Let $n \gg 0$, and choose N to be the cokernel of the map $P_n^* \to P_{n+1}^*$. The beginning of the minimal free resolution of N over R is given by

$$\cdots P_0^* \to P_1^* \to P_2^* \to \cdots \to P_n^* \to P_{n+1}^* \to N \to 0$$

and Lescot's theorem implies that $\mathrm{rank}_R(P_1^*) > \mathrm{rank}_R(P_2^*)$. This is a contradiction.

(e) and (f) are proved in [25], Theorem 7.5.

(g) G-regularity is proved in [18], Theorem 7.7. A brief analysis of the proof convinces us that strong G-regularity also hold. Indeed, the assumption that R is not Gorenstein implies that ω_R is not free. Lemma 7.4 and Proposition 4.2 in [18] show that there is a short exact sequence

$$0 \to \mathrm{S}yz^1(\omega_R) \to K \to \omega_R^n \to 0$$

for some $n \geq 1$, where $\mathrm{S}yz^1(\omega_R)$ denotes the first syzygy of ω_R and K is a module with the property that the second syzygy of K has a direct summand isomorphic to $R/\mathfrak{m}$, and therefore K is a test module. If N is an R-module with $\mathrm{Tor}_i^R(N, \omega_R) = 0$, the short exact sequence above implies that $\mathrm{Tor}_i^R(N, K) = 0$ for all $i \geq 1$, and therefore N is free. □

Note 9.7 *The condition $\nu(S) \gg 0$ in parts (e) and (f) of Theorem 9.6 amounts to saying that J is contained in a large enough power of the ideal generated by the variables.*

This requirement $\nu(S) \gg 0$ can be made more explicit for certain values $c_S(R)$. For example, if $c_S(R) = 1$, it is shown in [33] that there is a direct summand of a syzygy of ω_R which is isomorphic to the residue class field $R/\mathfrak{m}$. Thus, the requirement $\nu(S) \gg 0$ is not needed in this case.

A further motivation for studying the strong G-regular property is given in Theorem 2.9 in [11]:

Theorem 9.8 *Let $(R, \mathfrak{m}_R)$, $(S, \mathfrak{m}_S)$ be local Artinian algebras over a field k, and let $T = R \otimes_{\mathsf{k}} S$.*

Assume that R is Gorenstein and S is strongly G-regular.

If $\mathcal{F}$ is a totally acyclic complex over T, then

$$\mathcal{F}_R := \mathcal{F} \otimes_T (R \otimes_{\mathsf{k}} S/\mathfrak{m}_S)$$

is a totally acyclic complex over R.

In other words, under the assumptions in Theorem 9.8, every totally acyclic complex over T must specialize to a totally acyclic complex over R.

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Local Cohomology—An Invitation

Uli Walther and Wenliang Zhang

David Eisenbud
has had a transformative impact on the interface of commutative algebra with homological algebra and algebraic geometry. With admiration and gratitude for his research accomplishments, his prodigious mentoring activities, and his outstanding service as emissary of mathematics to the world at large, at MSRI and elsewhere, we dedicate this article to him on occasion of his 75th birthday.

This article is a mixture of an introduction to local cohomology, and a survey of the recent advances in the area, *with a view towards*[1] relations to other parts of mathematics. It thus proceeds at times rather carefully, with definitions and examples, and sometimes is more cursory, aiming to give the reader an impression about certain parts of the mathematical landscape. As such, it is more than a reference list but less than a monograph. One possible use we envision is as a guide for a novice, such as a beginning graduate student, to get an idea what the general thrust of local cohomology is, and where one can read more about certain topics.

While the article is rather much longer than originally anticipated, several active areas that interact with local cohomology have been left out. For instance, we refer

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[1] Pun intended

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the reader to [95, 247, 248] for connections with dualizing complexes which are not discussed in this article. What we have put into the article is driven by personal preferences and lack of expertise; we apologize to those offended by our choices.

Over time, several excellent survey articles on local cohomology and related themes have been written, and we strongly recommend the reader study the following ones. One should name [181] on the state of the art 20 years since, the article [211] specifically geared at Lyubeznik numbers, and the survey [120].

In the more expository direction, we and many others have been fortunate to be able to study Hochster's unpublished notes (available on his website) and Huneke's point of view in [136]. These notes come with our highest recommendations and have strongly influenced us and this article. For a treatment de-emphasizing Noetherianness we point at [270].

We close this thread of thoughts with mentioning the books concerned with local cohomology as main subject: the original account of Grothendieck as recorded by Hartshorne [95], the classic [50] by Brodmann and Sharp, and the outgrowth [139] of a summer school on local cohomology.

Some words on the prerequisites for reading this article are in order. Inasmuch as pure commutative algebra is concerned, we imagine the reader be familiar with the contents of the book by Atiyah and Macdonald [4] or an appropriate subset of the book by Eisenbud [69]. For homological algebra one should know about injective and projective resolutions, Ext and Tor and the principles of derived functors, and perhaps a bit about spectral sequences at the level of Rotman [230]. Hartshorne's opus [99] covers all that is needed on varieties, schemes and sheaves in chapters 1–3.

1 Introduction

Notation 1.1 Throughout, A will denote a commutative Noetherian ring. On occasion, A will be assumed to be local; then its maximal ideal is denoted by $\mathfrak{m}$ and the residue field by $\Bbbk$.

We reserve the symbol R for the case that A is regular, while M will generally denote a module over A. ◇

Definition 1.2 For an ideal $I \subseteq A$ the (left-exact) *section functor with support in I* (also called the *I-torsion functor*) $\Gamma_I(-)$ and the *local cohomology functors* $H^\bullet_I(-)$ *with support in I* are

$$\Gamma_I : M \rightsquigarrow \{m \in M | \exists \ell \in \mathbb{N}, I^\ell m = 0\}$$

and its right derived functors $H^\bullet_I(-)$. Since $\Gamma_I(-)$ is left exact, $\Gamma_I(-)$ agrees with $H^0_I(-)$. ◇

Local cohomology was invented by Grothendieck, at least in part, for the purpose of proving Lefschetz and Barth type theorems (comparisons between a smooth ambient variety and a possibly singular subvariety). The idea rests on the fact, already exploited by Serre in [256], that the geometry of projective varieties is encoded in the algebra of its coordinate ring. Grothendieck makes it clear in his Harvard seminar that, for this purpose, studying general properties of the concept of local cohomological dimension is of great importance [95, p. 79].

Definition 1.3 The *local cohomological dimension* $\operatorname{lcd}_A(I)$ of the A-ideal I is

$$\operatorname{lcd}_A(I) = \max\{k \in \mathbb{N} \mid H^k_I(A) \neq 0\}.$$

One can show, using long exact sequences and direct limits, that $H_I^{>\operatorname{lcd}_A(I)}(M)$ vanishes for every A-module M. ◇

It is an essential feature of the theory of local cohomology and its applications that there are several different ways of calculating $H^k_I(M)$ for any A-module M, all compatible with natural functors. We review briefly three other approaches; for a more complete account we refer to [139].

1.1 Koszul Cohomology

Let $x \in A$ be a single element and consider the multiplication map $A \xrightarrow{x} A$ by x, also referred to as the *cohomological Koszul complex* $K^\bullet(A; x)$, so the displayed map is a morphism from position 0 to position 1 in the complex. We write $H^i(A; x)$ for the cohomology modules of this complex.

Replacing x with its own powers, one arrives at a tower of commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{x} & A \\ \downarrow 1 & & \downarrow x \\ A & \xrightarrow{x^2} & A \\ \downarrow 1 & & \downarrow x \\ A & \xrightarrow{x^3} & A \\ \downarrow 1 & & \downarrow x \\ \vdots & & \vdots \end{array} \tag{1.1.0.1}$$

which induces maps on the cohomology level, $x\colon H^i(A; x^\ell) \longrightarrow H^i(A; x^{\ell+1})$ and hence a direct system of cohomology modules over the index set $\mathbb{N}$. It is an instructive exercise (using the fact that $\mathbb{N}$ is an index set that satisfies: for all $n, n' \in \mathbb{N}$ there is $N \in \mathbb{N}$ exceeding both n, n') to check that the direct limit $\varinjlim_\ell H^k(A; x^\ell)$ agrees with the local cohomology module $H^k_{(x)}(A)$.

If M is an A-module and the ideal I is generated by $x_1, \ldots, x_m$ then to each such generating set there is a *cohomological Koszul complex*

$$K^\bullet(M; x_1, \ldots, x_m) := M \otimes_A \bigotimes_{i=i}^{m} K^\bullet(A; x_i)$$

whose cohomology modules are denoted $H^\bullet(M; x_1, \ldots, x_m)$. Again, one can verify that replacing each x_i by powers of themselves leads to a tower of complexes whose direct limit has a cohomology that functorially equals the local cohomology $H^\bullet_I(M)$. In particular, it is independent of the chosen generating set for I.

1.2 The Čech Complex

Inspection shows that the direct limit of the tower $A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots$ is functorially equal to the localization $A[x^{-1}]$ which we also write as A_x. Thus, the limit complex to the tower (1.1.0.1) is the localization complex $A \longrightarrow A_x$. In greater generality, the module that appears in the limit complex $\check{C}^\bullet(M; x_1, \ldots, x_m)$ of the tower $K^\bullet(M; x_1, \ldots, x_m) \longrightarrow K^\bullet(M; x_1^2, \ldots, x_m^2) \longrightarrow K^\bullet(M; x_1^3, \ldots, x_m^3) \longrightarrow \cdots$ in cohomological degree k is the direct sum of all localizations of M at k of the m elements $x_1, \ldots, x_m$. Hence,

$$\check{C}^\bullet(M; x_1, \ldots, x_m) = \varinjlim_\ell K^\bullet(M; x_1^\ell, \ldots, x_m^\ell)$$

and a corresponding statement links the cohomology modules on both sides.

The point of view of the Čech complex provides a useful link to projective geometry. Indeed, suppose $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ is the homogeneous ideal defining the projective variety X in $\mathbb{P}^n_{\mathbb{K}}$. Then the *cohomological dimension* $\mathrm{cd}(U)$ of $U = \mathbb{P}^n_{\mathbb{K}} \setminus X$, the largest integer k for which $H^k(U, -)$ is not the zero functor on the category of quasi-coherent sheaves on U, equals $\mathrm{lcd}_R(I) - 1$. This follows from the exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow \bigoplus_{k \in \mathbb{Z}} \Gamma(\mathbb{P}^n_{\mathbb{K}} \setminus X, \tilde{M}(k)) \longrightarrow H^1_I(M) \longrightarrow 0 \tag{1.2.0.1}$$

and the isomorphisms $\bigoplus_{k \in \mathbb{Z}} H^i(\mathbb{P}^n_{\mathbb{K}} \setminus X, \tilde{M}(k)) = H^{i+1}_I(M)$ for any R-module M with associated quasi-coherent sheaf $\tilde{M}$.

1.3 Limits of Ext-modules

Again, let $I = (x_1, \ldots, x_m)$ be an ideal of A. The natural projections $A/I^{\ell+1} \longrightarrow A/I^{\ell}$ lead to a natural tower of morphisms $\operatorname{Ext}^k_A(A/I, M) \longrightarrow \operatorname{Ext}^k_A(A/I^2, M) \longrightarrow \operatorname{Ext}^k_A(A/I^3, M) \longrightarrow \cdots$. An exercise involving δ-functors (also known as connected sequences of functors) shows that the direct limit of this system functorially agrees with $H^k_I(M)$.

We have thus the functorial isomorphisms

$$\begin{aligned} H^k_I(M) &\simeq \varinjlim_{\ell} H^k(M; x_1^{\ell}, \ldots, x_m^{\ell}) \simeq H^k \check{C}^{\bullet}(M; x_1, \ldots, x_m) \\ &\simeq \varinjlim_{\ell} \operatorname{Ext}^k_A(A/I^{\ell}, M) \end{aligned}$$

for all choices of generating sets $x_1, \ldots, x_m$ for I.

Remark 1.4

(1) The derived functor version of local cohomology shows that $H^{\bullet}_I(-)$ and $H^{\bullet}_J(-)$ are the same functor whenever I and J have the same radical.
(2) It follows easily from the Čech complex interpretation that local cohomology satisfies a local-to-global principle: for any multiplicatively closed subset S of A one has $S^{-1} \cdot H^i_I(M) = H^i_{I(S^{-1}A)}(S^{-1}M)$, and so in particular $H^i_I(M) = 0$ if and only if $H^i_{IA_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$.
(3) If I is (up to radical) a complete intersection in the localized ring $A_{\mathfrak{p}}$, then $H^k_I(A) \otimes_A A_{\mathfrak{p}}$ is zero unless $k = \operatorname{ht}(IA_{\mathfrak{p}})$. If R is a regular local ring and I reduced then I is a complete intersection in every smooth point. It follows that for equidimensional I the support of $H^k_I(R)$ with $k > \operatorname{ht}(I)$ only contains primes $\mathfrak{p}$ contained in the singular locus of I.
(4) It is in general a difficult question to predict how the natural maps $\operatorname{Ext}^k_A(A/I^{\ell}, M) \longrightarrow H^k_I(M)$ and $H^k(M; x_1^{\ell}, \ldots, x_m^{\ell}) \longrightarrow H^k_I(M)$ behave; some information can be found in [28, 72, 202, 281].
(5) If $\phi\colon A' \longrightarrow A$ is a ring morphism, and if M is an A-module and I' an ideal of A', then there is a functorial isomorphism between $H^k_{I'}(\phi_* M)$ and $\phi_*(H^k_{I'A}(M))$, where ϕ_* denotes restriction of scalars from A to A'. The easiest way to see this is by comparison of the two Čech complexes involved.

◇

Remark 1.5 Let I be an ideal of a Noetherian commutative ring A. A sequence of ideals $\{I_k\}$ is called *cofinal* with the sequence of powers $\{I^k\}$ if, for all $k \in \mathbb{N}$, there are $\ell, \ell' \in \mathbb{N}$ such that both $I_{\ell} \subseteq I^k$ and $I^{\ell'} \subseteq I_k$.

Sequences $\{I_k\}$ cofinal with $\{I^k\}$ are of interest in the study of local cohomology since

$$\varinjlim_{k} \operatorname{Ext}^i_R(R/I_k, M) \cong \varinjlim_{k} \operatorname{Ext}^i_R(R/I^k, M) = H^i_I(M).$$

This provides one with the flexibility of using sequences of ideals other than $\{I^n\}$. In characteristic $p > 0$, the sequence of ideals defined next plays an extraordinary part in the story.

Let A be a Noetherian commutative ring of prime characteristic p and I be an ideal of A. The *e-th Frobenius power* of I, denoted by $I^{[p^e]}$, is defined to be the ideal generated by the p^e-th powers of all elements of I. Since the Frobenius endomorphism $A \xrightarrow{a \mapsto a^p} A$ is a ring homomorphism, $I^{[p^e]} = (f_1^{p^e}, \dots, f_t^{p^e})$ for every set of generators $\{f_1, \dots, f_t\}$ of I.

It is straightforward to check that $\{I^{[p^e]}\}$ is cofinal with $\{I^k\}$ since A is Noetherian and thus I is finitely generated. ◇

1.4 Local Duality

Matlis duality over a complete local ring $(A, \mathfrak{m}, \Bbbk)$ provides a one-to-one correspondence between the Artinian and the Noetherian modules over A; in both directions it is given by the functor

$$D(M) := \operatorname{Hom}_A(M, E_A(\Bbbk))$$

of homomorphisms into the injective hull of the residue field.[2] Of course, one can in principle apply $D(-)$ to any module, but the property $D(D(M)) = M$ is likely to fail when M does not enjoy any finiteness condition.

A natural question is what the result of applying $D(-)$ to $H^i_{\mathfrak{m}}(A)$ should be or, more generally, how to describe $D(H^i_{\mathfrak{m}}(M))$ for Noetherian A-modules M. It turns out that when A "lends itself to duality", then this question has a pleasing answer:

Theorem 1.6 *Suppose* $(A, \mathfrak{m}, \Bbbk)$ *is a local Gorenstein ring. Then*

$$D(H^i_{\mathfrak{m}}(M)) \cong \operatorname{Ext}_A^{\dim(A)-i}(M, A)$$

for every finitely generated A*-module* M.

The original version is due to Grothendieck [95], and then expanded in Hartshorne's opus [94]. As it turns out, there are extensions of local duality to Cohen–Macaulay rings with a dualizing module, and yet more generally to rings with a dualizing complex. Duality on formal or non-Noetherian schemes and other generalizations are discussed in [16].

In particular, Chapter 4 of [94] contains a discussion on Cousin complexes and their connection to local cohomology, that we do not have the space to give justice to. Further accounts in this direction can be found in [164, 247, 258, 259].

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[2] Strictly speaking, one should write $D_A(-)$, but in all cases the underlying ring will be understood from the context.

2 Finiteness and Vanishing

2.1 Finiteness Properties

In general, local cohomology modules are not finitely generated. For instance, the Grothendieck nonvanishing theorem says:

Theorem 2.1 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and* M *be a finitely generated* A*-module. Then* $H_{\mathfrak{m}}^{\dim(M)}(M) \neq 0$*. Moreover, if* $\dim(M) > 0$ *then* $H_{\mathfrak{m}}^{\dim(M)}(M)$ *is not finitely generated.*

Finiteness is more unusual yet than this theorem indicates. For example, over a ring R of polynomials over $\mathbb{C}$, a local cohomology module $H_I^k(R)$ is a finite R-module precisely if $I = 0$ and $k = 0$, or if $H_I^k(R) = 0$. This lack of finite generation prompted people to look at other types of finiteness properties, and in this section we survey various fruitful avenues of research that pertain to finiteness.

In [91, exposé 13, 1.2] Grothendieck conjectured that, if I is an ideal in a Noetherian local ring A, then $\mathrm{Hom}_A(A/I, H_I^j(A))$ is finitely generated. Hartshorne refined this finiteness of $\mathrm{Hom}_A(A/I, H_I^j(A))$ and introduced the notion of *cofinite modules* in [102].

Definition 2.2 Let A be a Noetherian commutative ring and $I \subseteq A$ an ideal. An A-module M is called *I-cofinite* if $\mathrm{Supp}_A(M) \subseteq V(I)$ and $\mathrm{Ext}_A^i(A/I, M)$ is finitely generated for all i. ◇

In [102] Hartshorne constructed the following example which answered Grothendieck's conjecture on finiteness of $\mathrm{Hom}_A(A/I, H_I^j(M))$ in the negative.

Example 2.3 Let $\mathbb{k}$ be a field and put $A = \frac{\mathbb{k}[x,y,u,v]}{(xu-yv)}$. Set $\mathfrak{a} = (x, y)$ and $\mathfrak{m} = (x, y, u, v)$. Then $\mathrm{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{a}}^2(A))$ is not finitely generated and hence neither is $\mathrm{Hom}_A(A/\mathfrak{a}, H_{\mathfrak{a}}^2(A))$.

We note in passing, that while the socle dimension of $H_{\mathfrak{a}}^2(A)$ is infinite, it is nonetheless a finitely generated module over the ring of $\mathbb{k}$-linear differential operators on A, [130]. ◇

The ring A in Hartshorne's example is not regular; one may ask whether local cohomology modules $H_I^i(R)$ of a Noetherian regular ring R are I-cofinite. Huneke and Koh showed in [111] that this is not the case even for a polynomial ring over a field.

Example 2.4 Let $\mathbb{k}$ be a field of characteristic 0 and let $R = \mathbb{k}[x_{1,1}, \ldots, x_{2,3}]$ be the polynomial ring over $\mathbb{k}$ in 6 variables. Set I to be the ideal generated by the 2×2 minors of the matrix (x_{ij}).

The geometric origins and connections of this example, including a discussion of the interaction of the relevant local cohomology groups with de Rham cohomology and D-modules, can be found in Examples 2.14, 4.8 and Remark 4.9 below. In particular, Example 4.8 discusses that $H^3_I(R)$ is isomorphic to the injective hull of $\Bbbk$ over R, which means that $\operatorname{Hom}_R(R/I, H^3_I(R))$ is the injective hull of $\Bbbk$ over R/I and thus surely not finitely generated. ◇

Huneke and Koh further proved in [111] that:

Theorem 2.5 *Let R be a regular local ring and I be an ideal in R. Set b to be the biggest height of any minimal prime of I and set $c = \operatorname{lcd}_R(I)$, compare Definition 1.3.*

(1) If R contains a field of characteristic $p > 0$ and if $j > b$ is an integer such that $\operatorname{Hom}_R(R/I, H^j_I(R))$ is finitely generated, then $H^j_I(R) = 0$.
(2) If R contains $\mathbb{Q}$ then $\operatorname{Hom}_R(R/I, H^c_I(R))$ is not finitely generated.

In Example 2.4, it turns out that the socle $\operatorname{Hom}_R(R/\mathfrak{m}, H^3_I(R))$ of $H^3_I(R)$ is finitely generated. It is natural to ask whether the socle of local cohomology of a Noetherian regular ring is always finitely generated; as a matter of fact this was precisely [133, Conjecture 4.3].

In [133], Huneke proposed a number of problems on local cohomology which guided the study of local cohomology modules for decades.

Problem 2.6 (Huneke's List)

1. When is $H^j_I(M) = 0$?
2. When is $H^j_I(M)$ finitely generated?
3. When is $H^j_I(M)$ Artinian?
4. If M is finitely generated, is the number of associated primes of $H^j_I(M)$ always finite?

◇

Huneke remarked that all of these problems are connected with another question

5. *What annihilates the local cohomology module $H^j_I(M)$?*

More concretely, Huneke conjectured:

Conjecture 2.7 (Conjectures 4.4 and 5.2 in [133]) Let R be a regular local ring and I be an ideal. Then

(1) the Bass numbers $\operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), H^j_I(R_\mathfrak{p}))$ are finite for all i, j, and prime ideals $\mathfrak{p}$, and
(2) the number of associated primes of $H^j_I(R)$ is finite for all j.

◇

Later in [176] Lyubeznik conjectured further that the finiteness of associated primes holds for local cohomology of all Noetherian regular rings. Substantial progress has been made on these conjectures. If the regular ring has prime characteristic $p > 0$, then these conjectures were completely settled by Huneke and

Sharp in [125]; in equi-characteristic 0, Lyubeznik proved these conjectures for two large classes of regular rings in [176]; for complete unramified regular local rings of mixed characteristic, these conjectures were first settled by Lyubeznik in [179] (different proofs can also be found in [209] and [27]). The finiteness of associated primes of local cohomology was also proved in [27] for smooth $\mathbb{Z}$-algebras. We summarize these results as follows.

Theorem 2.8 *Assume that R is*

(1) a Noetherian regular ring of characteristic $p > 0$, or
(2) a complete regular local ring containing a field of characteristic 0, or
(3) regular of finite type over a field of characteristic 0, or
(4) an unramified regular local ring of mixed characteristic, or
(5) a smooth $\mathbb{Z}$-algebra.

Then the Bass numbers and the number of associated primes of $H^j_I(R)$ are finite for every ideal I of R and every integer j.

Remark 2.9 When R is a smooth $\mathbb{Z}$-algebra, then finiteness of Bass numbers was not addressed in [27]. However, one can conclude readily from the unramified case in [179] as follows. The Zariski-local structure theorem for smooth morphism says that $\mathbb{Z} \longrightarrow R$ factors as a composition of a polynomial extension and a finite etale morphism, which implies that locally R is an unramified regular local ring of mixed characteristic. Since the finiteness of Bass numbers is a local problem, the desired conclusion follows from the results in [179]. ◇

Conjecture 2.7 is still open when R is a ramified regular local ring of mixed characteristic. Theorem 2.8(1) was proved in [125] using properties of the Frobenius endomorphism; this approach was later conceptualized by Lyubeznik to his theory of F-modules in [177]. The proof of Theorem 2.8(2)–(5) uses D-modules (i.e. modules over the ring of differential operators). Both F-modules and D-modules will be discussed in the sequel.

For a non-regular Noetherian ring A, if $\dim(A) \leqslant 3$ ([190]), or if A is a 4-dimensional excellent normal local domain ([112]), then the number of associated primes of $H^j_I(M)$ is finite for every finitely generated A-module M, for every ideal I and for all integers j. Once the restriction on $\dim(A)$ is removed, then the number of associated primes of local cohomology modules can be infinite; such examples have been discovered in [148, 264, 269]. Note that all these examples are hypersurfaces; the hypersurface in [269] has rational singularities.

As local cohomology modules may have infinitely many associated primes in general, one may ask a weaker question ([112, p. 3195]):

Question 2.10 *Let A be a Noetherian ring, I be an ideal of A and M be a finitely generated A-module. Does $H^j_I(M)$ have only finitely many minimal associated primes? Or equivalently, is the support of $H^j_I(M)$ Zariski-closed?* ◇

It is stated in [112] that "this question is of central importance in the study of cohomological dimension and understanding the local-global properties of local cohomology".

When $\dim(A) \leqslant 4$, then Question 2.10 has a positive answer due to [112]. If $\mu(I)$ denotes the number of generators of I and A has prime characteristic p, it was proved and attributed to Lyubeznik in [149] that $H_I^{\mu(I)}(A)$ has a Zariski-closed support. When $A = R/(f)$ where R is a Noetherian ring of prime characteristic p with isolated singular closed points, it was proved independently in [163] and in [116] that $H_I^j(A)$ has a Zariski-closed support for every ideal I and integer j.

A classical result in commutative algebra (*c.f.* [33, Theorem 3.1.17]) says that if A is a Noetherian local ring and M is a finitely generated A-module M that has finite injective dimension, then

$$\dim(M) \leqslant \operatorname{injdim}_A(M) = \operatorname{depth}(A)$$

where $\operatorname{injdim}_A(M)$ denotes the injective dimension of M over A. Interestingly, for local cohomology modules over regular rings, the inequality seems to be reversed. More precisely, the following was proved in [125] and [176]

Theorem 2.11 *Assume that R is*

(1) a Noetherian regular ring of characteristic $p > 0$, or
(2) a complete regular local ring of characteristic 0, or
(3) regular of finite type over a field of characteristic 0.

Then

$$\operatorname{injdim}_R(H_I^j(R)) \leqslant \dim(\operatorname{Supp}_R(H_I^j(R)))$$

for every ideal I and integer j.

In [223], Puthenpurakal showed that if $R = \Bbbk[x_1, \dots, x_n]$ where $\Bbbk$ is a field of characteristic 0 then $\operatorname{injdim}_R(H_I^j(R)) = \dim(\operatorname{Supp}_R(H_I^j(R)))$ for every ideal I. Later this was strengthened in [322, Theorem 1.2] as follows: assume that either R is a regular ring of finite type over an infinite field of prime characteristic p and M is an F-finite F-module, or $R = \Bbbk[x_1, \dots, x_n]$ where $\Bbbk$ is a field of characteristic 0 and M is a holonomic[3] D-module. Then

$$\operatorname{injdim}_R(M) = \dim(\operatorname{Supp}_R(M)).$$

Subsequently [291] proved that, if M is either a holonomic D-module over a formal power series ring R with coefficients in a field of characteristic 0, or an F-

[3] The notions of holonomic D-modules and F-finite F-modules will be explained in the sequel; local cohomology modules with argument R and R as discussed here are primary examples of those.

finite F-module over a Noetherian regular ring R of prime characteristic p, then

$$\dim(\operatorname{Supp}_R(M)) - 1 \leqslant \operatorname{injdim}_R(M) \leqslant \dim(\operatorname{Supp}_R(M)).$$

When the regular ring R does not contain a field, the bounds on injective dimension of local cohomology modules of R are different. In [325], Zhou proved that, *if* $(R, \mathfrak{m})$ is an unramified regular local ring of mixed characteristic and I is an ideal of R, then $\operatorname{injdim}_R(H^j_I(R)) \leqslant \dim(\operatorname{Supp}_R(H^j_I(R)))+1$ and $\operatorname{injdim}_R(H^i_{\mathfrak{m}} H^j_i(R)) \leqslant 1$. Moreover, it may be the case that $\operatorname{injdim}_R(H^i_{\mathfrak{m}} H^j_i(R)) = 1$, as shown in [65, 117].

2.2 Vanishing

Problem 1 in Huneke's list of problems in [133] asks: when is $H^j_I(M) = 0$? Vanishing results on local cohomology modules have a long and rich history. Note that $H^j_I(M) = 0$ for all $j > t$ and all A-modules M if and only if $H^j_I(A) = 0$ for $j > t$. Recall the notion of local cohomological dimension from Definition 1.3. For a Noetherian local ring A, we set

$$\operatorname{mdim}(A) = \min\{\dim(A/Q) \mid Q \text{ is a minimal prime of } A\}$$

and we write $\operatorname{embdim}(A)$ for the embedding dimension (the number of generators of the maximal ideal) of a local ring A. For an ideal I of A, we set

$$c_A(I) = \operatorname{embdim}(A) - \operatorname{mdim}(A/I).$$

Note that if A is regular then $c_A(I)$ is called the big height, i.e. the biggest height of any minimal prime ideal of I.

We now summarize the most versatile vanishing theorems on local cohomology.

- (Grothendieck Vanishing) Let A be a Noetherian ring and M be a finitely generated R-module. Then $H^j_I(M) = 0$ for all integers $j > \dim(M)$ and ideals I. In particular, this implies that $\operatorname{lcd}_A(I) \leqslant \dim(A)$ for all ideals I.
- (Hartshorne–Lichtenbaum Vanishing) Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Then $\operatorname{lcd}_A(I) \leqslant \dim(A)-1$ if and only if $\dim(\hat{A}/(I\hat{A}+P)) > 0$ for every minimal prime P of $\hat{A}$ such that $\dim(\hat{A}/P) = \dim(A)$, where $\hat{A}$ denotes the completion of A. In particular, this implies that if A is a complete local domain and $\sqrt{I} \neq \mathfrak{m}$ then $\operatorname{lcd}_A(I) \leqslant \dim(A) - 1$. *cf.* [96].
- (Faltings Vanishing) Let R be a complete equi-characteristic regular local ring with a separably closed residue field. Then

$$\operatorname{lcd}_R(I) \leqslant \dim(R) - \left\lfloor \frac{\dim(R)-1}{c_R(I)} \right\rfloor,$$

cf. [75]. [4] This is bound is sharp according to [172].

- (Second Vanishing Theorem) Let R be a complete regular local ring that contains a separably closed coefficient field and I be an ideal. Then $\operatorname{lcd}_R(I) \leqslant \dim(R)-2$ if and only if $\dim(R/I) \geqslant 2$ *and* the punctured spectrum of R/I is connected. A version of this vanishing theorem for projective varieties was first obtained by Hartshorne in [96, Theorem 7.5] who coined the name 'Second Vanishing Theorem'. The local version stated here was left as a problem by Hartshorne in [96, p. 445]. Subsequently, this theorem was proved in prime characteristic in [220], in equi-characteristic 0 in [213] (a unified proof for equi-characteristic regular local rings can be found in [113]), and for unramified regular local rings in mixed characteristic in [324].
- (Peskine–Szpiro Vanishing) Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of prime characteristic p and I be an ideal. Then $\operatorname{lcd}_R(I) \leqslant \dim(R) - \operatorname{depth}(R/I)$, *cf.* [220].
- (Vanishing via action of Frobenius) Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic p and I be an ideal. Set $d = \dim(R)$. Then $H^j_I(R) = 0$ if and only if the Frobenius endomorphism on $H^{d-j}_{\mathfrak{m}}(R/I)$ is nilpotent. *cf.* [183].

There have been various extensions of the vanishing theorems mentioned above. Most notably, [113] initiated an investigation on finding bounds of local cohomological dimension under topological and/or geometric assumptions. For instance, [113, Theorem 3.8] asserts that if A is a complete local ring containing a field and I is a formally geometrically irreducible ideal such that $0 < c_A(I) < \dim(A)$ then

$$\operatorname{lcd}_A(I) \leqslant \dim(A) - 1 - \left\lfloor \frac{\dim(A)-2}{c_A(I)} \right\rfloor. \tag{2.2.0.1}$$

Furthermore, if A/I is normal then

$$\operatorname{lcd}_A(I) \leqslant \dim(A) - \left\lfloor \frac{\dim(A)+1}{c_A(I)+1} \right\rfloor - \left\lfloor \frac{\dim(A)}{c_A(I)+1} \right\rfloor.$$

The bound on cohomological dimension in (2.2.0.1) was later extended to reducible ideals in [184] as follows.

Theorem 2.12 *Let $(A, \mathfrak{m}, \Bbbk)$ be a d-dimensional local ring containing $\Bbbk$. Assume $d > 1$. Let c be a positive integer, let $t = \lfloor (d-2)/c \rfloor$ and $v = d-1-\lfloor (d-2)/c \rfloor$. Let I be an ideal of A with $c(I\hat{A}) \leqslant c$. Let B be the completion of the strict Henselization of the completion of A. Let $I_1, \dots, I_n$ be the minimal primes of IB and let $P_1, \dots, P_m$ be the primes of B such that* $\dim(B/P_i) = d$. *Let Δ_i be*

[4] This is the floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z}, k \leqslant x\}$.

the simplicial complex on n vertices $\{1, 2, \dots, n\}$ *such that a simplex* $\{j_0, \dots, j_s\}$ *belongs to* Δ_i *if and only if* $I_{j_0} + \cdots + I_{j_s} + P_i$ *is not* $\mathfrak{m}B$*-primary. Let* $\tilde{H}_{t-1}(\Delta_i; k)$ *be the* $(t-1)$*st singular homology group of* Δ_i *with coefficients in* k*. Then* $\operatorname{lcd}_A(I) \leqslant v$ *if and only if* $\tilde{H}_{t-1}(\Delta_i; k) = 0$ *for every* i*.*

We ought to point out that the simplicial complex introduced in Theorem 2.12 has spurred a line of research on connectedness dimension, *cf.* [66, 155, 210, 299].

Also, [66] shows that the same bound as in (2.2.0.1) holds when A is a complete regular local ring containing a field such that A/I has positive dimension and satisfies Serre's condition (S_2). This result is in the spirit of a question raised by Huneke in [133].

Question 2.13 (Huneke) *Let* R *be a complete regular local ring with separably closed residue field and* I *be an ideal of* R*. Assume that* R/I *satisfies Serre's conditions* (S_i) *and* (R_j)*. What is the maximal possible cohomological dimension for such an ideal?* ◇

In the same spirit as Huneke's Question 2.13, one may ask about the possibility of an implication

$$[\operatorname{depth}_R(R/I) \geqslant t] \quad \overset{?}{\Longrightarrow} \quad [\operatorname{lcd}_R(I) \leqslant \dim(R) - t]. \tag{2.2.0.2}$$

In prime characteristic p, such implication holds due to Peskine–Szpiro Vanishing. On the other hand, Peskine–Szpiro Vanishing can fail in characteristic 0.

Example 2.14 Let R be the polynomial ring in the variables $x_{1,1}, \dots, x_{2,3}$ over the field $\mathbb{K}$, localized at $\boldsymbol{x} = (x_{1,1}, \dots, x_{2,3})$. Let I be the ideal of maximal minors of the matrix $(x_{i,j})$. Then I is the radical ideal associated to the 4-dimensional locus V of the 2×3 matrices of rank one, which agrees with the image of the map $\mathbb{K}^2 \times \mathbb{K}^3 \longrightarrow \mathbb{K}^{2\times 3}$ that sends $((s, t), (x, y, z))$ to (xs, ys, zs, xt, yt, zt). In particular, I is the prime ideal associated to the image of the Segre embedding of $\mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^2_{\mathbb{K}} \hookrightarrow \mathbb{P}^5_{\mathbb{K}}$.

Thus I is 3-generated of height $2 = 6 - 4$, and in fact R/I is Cohen–Macaulay of depth 4. Since the origin is the only singular point of V, the local cohomology groups $H^k_I(R)$ are supported at the origin for $k \neq 2$ and zero for $k \notin \{2, 3\}$. Cohen–Macaulayness of R/I forces the vanishing of $H^k_I(R)$ for $k \neq 2$ in prime characteristic, but if the characteristic of $\mathbb{K}$ is zero then $H^3_I(R)$ is actually nonzero. For a computational discussion involving D-modules see [92, 212, 215, 306, 310]. We will return to this situation in Example 4.8. ◇

In Example 2.14, $\operatorname{depth}(R/I) = 4$ but $H^3_I(R) \neq 0$. This shows that the implication (2.2.0.2) can fail in characteristic 0 when $t \geqslant 4$. When $t \leqslant 2$, the implication (2.2.0.2) holds due the Second Vanishing Theorem and the Hartshorne–Lichtenbaum Theorem. The case $t = 3$ is not completely settled, but there have been positive results. In [298], continuing his work on the number of defining equations in [297], Varbaro proved that if a homogeneous ideal I in a polynomial ring

$R = \mathbb{k}[x_1, \dots, x_n]$ over a field $\mathbb{k}$ satisfies $\operatorname{depth}(R/I) \geqslant 3$ then $\operatorname{lcd}_R(I) \leqslant n - 3$. He also conjectured:

Conjecture 2.15 (Varbaro) *Let R be a regular local ring containing a field and I be an ideal of R. If* $\operatorname{depth}(R/I) \geqslant 3$*, then* $\operatorname{lcd}_R(I) \leqslant \dim(R) - 3$. ◇

The fact that Implication (2.2.0.2) can fail for complex projective threefolds (Example 2.14) raises the question what exact features are responsible for failure when $t = 3$. Clearly, more knowledge about the singularity is required than just $\operatorname{depth}_R(R/I)$.

Dao and Takagi prove Conjecture 2.15 in [66] when R is essentially of finite type over a field. More specifically, they show the following facts about the inequality $\operatorname{lcd}_R(I) \leqslant \dim(R) - 3$. Suppose R is a regular local ring essentially of finite type over its algebraically closed residue field of characteristic zero. Take an ideal I such that R/I has depth 2 or more and $H^2_{\mathfrak{m}}(R/I)$ is a $\mathbb{K}$-vector space (i.e., it is killed by $\mathfrak{m}$). Then $\operatorname{lcd}_R(I) \leqslant \dim R - 3$ if and only if the torsion group of $\operatorname{Pic}(\operatorname{Spec}(R/I))$ is finitely generated on the punctured completed spectrum. In case that the depth of R/I is at least 4, one even has $\operatorname{lcd}_R(I) \leqslant \dim(R) - 4$ if and only if the Picard group is torsion on the punctured completed spectrum of R/I. In Example 2.14, the depth of R/I is four, but the Picard group on the punctured spectrum is not torsion but $\mathbb{Z}$. Conjecture 2.15 remains open in general.

Both the Hartshorne–Lichtenbaum Vanishing Theorem and the Second Vanishing Theorem may viewed as topological criteria for vanishing and have applications to topology of algebraic varieties (*cf.* [175] and [136]). It would be desirable to have an analogue of the Second Vanishing Theorem for non-regular rings. In [181, p. 144] Lyubeznik asked the following questions.

Question 2.16 *Let $(A, \mathfrak{m})$ be a complete local domain with a separably closed residue field.*

(1) Find necessary and sufficient conditions on I such that $\operatorname{lcd}_A(I) \leqslant \dim(A) - 2$.
(2) Let I be a prime ideal. Is it true that $\operatorname{lcd}_A(I) \leqslant \dim(A) - 2$ *if and only if $(P + I)$ is not primary to the maximal ideal for any prime ideal P of height 1?*

◇

Question 2.16(1) remains open. It turns out that Question 2.16(2) has a negative answer due to [138, Proposition 7.7]:

Example 2.17 Let $A = \frac{\mathbb{C}[[x,y,z,u,v]]}{(x^3+y^3+z^3, z^2-ux-vy)}$ and $I = (x, y, z)$. Then

(1) $\dim(A) = 3$ and $\operatorname{ht}(I) = 1$;
(2) $I + P$ is *not* primary to the maximal ideal for every height-1 prime ideal P;
(3) $H^2_I(A) \neq 0$.

◇

Given the connections between local cohomology and sheaf cohomology (*cf.* (1.2.0.1)), vanishing of sheaf cohomology can be interpreted in terms of local cohomology. The classical Kodaira Vanishing Theorem asserts that: *If X is smooth projective variety over a field $\mathbb{K}$ of characteristic 0, then $H^i(X, \mathcal{O}(j)) = 0$ for $i < \dim(X)$ and all $j < 0$.* This result has an equivalent formulation in terms of local cohomology: *If R is a standard*[5] *graded domain over a field $\mathbb{K}$ of characteristic 0 such that $\mathrm{Proj}(R)$ is smooth, then $H^j_{\mathfrak{m}}(R)_{<0} = 0$ for all $j < \dim(R)$, where $\mathfrak{m}$ is the homogeneous maximal ideal of R.* For ideal-theoretic interpretations and connections with tight closure and Frobenius, we refer the interested reader to [126, 267].

The Kodaira Vanishing Theorem fails for singular varieties in characteristic 0 and also fails for smooth varieties in characteristic p. It is proved in [28] that, if one focuses on the range $i < \mathrm{codim}(\mathrm{Sing}(X))$, then the Kodaira Vanishing Theorem can be extended to thickenings of local complete intersections. More precisely:

Theorem 2.18 *Let X be a closed local complete intersection subvariety of $\mathbb{P}^n_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic 0 and let I be its defining ideal. Let X_t denote the scheme defined by I^t. Then*

$$H^i(X_t, \mathcal{O}_{X_t}(j)) = 0$$

for all $i < \mathrm{codim}(\mathrm{Sing}(X))$, all $t \geqslant 1$, and all $j < 0$.

Or, equivalently, let $S = \mathbb{K}[x_0, \ldots, x_n]$ and I be as above. Then

$$H^\ell_{\mathfrak{m}}(S/I^t)_{<0} = 0$$

for $\ell < \mathrm{codim}(\mathrm{Sing}(X)) + 1$ and all $t \geqslant 1$.

A natural question is whether the restriction on $\mathrm{codim}(\mathrm{Sing}(X))$ can be relaxed or even removed. The following example from [29] shows that this is not the case.

Example 2.19 Let $R = \mathbb{K}[x, y, u, v, w]$ where $\mathbb{K}$ is a field of characteristic 0. Fix an integer $c \geqslant 2$ and set $I := (uy - vx, vy - wx) + (u, v, w)^c$. Then one can check that

(1) $X = \mathrm{Proj}(R/I)$ is local complete intersection in $\mathbb{P}^4_{\mathbb{K}}$;
(2) $H^2_{\mathfrak{m}}(R/I^t)_{-ct+1} \neq 0$;
(3) $H^2_{\mathfrak{m}}(R/I^t)_{\leqslant -ct} = 0$.

◇

Example 2.19 indicates that, if one removes the restriction on the homological degree by $\mathrm{codim}(\mathrm{Sing}(X))$, the best vanishing result one can hope for is an asymptotic vanishing bounded by a linear function of t. Such an asymptotic vanishing turns out to be true, as shown in [29].

[5] A standard graded algebra over a field $\mathbb{K}$ is a graded quotient of a polynomial ring over $\mathbb{K}$ with the standard grading.

Theorem 2.20 *Let X be a closed local complete intersection subscheme of $\mathbb{P}^n$ over a field of arbitrary characteristic. Then there exists an integer $c \geqslant 0$ such that for each $t \geqslant 1$ and $i < \dim(X)$, one has*

$$H^i(X_t, \mathcal{O}_{X_t}(j)) = 0, \ \forall\, j < -ct.$$

When $\mathrm{Proj}(R/I)$ is a local complete intersection (here $R = \mathbb{K}[x_0, \ldots, x_n]$ and I is a homogeneous ideal of R), the local cohomology modules $H^j_{\mathfrak{m}}(R/I^t)$ have finite length for $j < \dim(R/I)$ and consequently $H^j_{\mathfrak{m}}(R/I^t)_{\ell \ll 0} = 0$. This is one of the underlying reasons for the vanishing in Theorems 2.18 and 2.20. Once the local complete intersection assumption is dropped, $H^j_{\mathfrak{m}}(R/I^t)$ may not have finite length and hence the vanishing may fail. However, since $H^j_{\mathfrak{m}}(R/I^t)$ are Artinian (even when $j = \dim(R/I)$), the socles $\mathrm{Hom}_R(R/\mathfrak{m}, H^j_{\mathfrak{m}}(R/I^t))$ are finite dimensional and vanish in all sufficiently negative degrees. Therefore, one can ask:

Question 2.21 *Let $R = \mathbb{K}[x_0, \ldots, x_n]$ and I be a homogeneous ideal of R. For each $j \geqslant 0$, does there exist an integer c such that*

$$\mathrm{Hom}_R(R/\mathfrak{m}, H^j_{\mathfrak{m}}(R/I^t))_\ell = 0$$

for all $t \geqslant 1$ and all $\ell < -ct$? ◇

For related questions and applications, we refer the interested reader to [323].

2.3 Annihilation of Local Cohomology

We now turn to the question: what annihilates the local cohomology module $H^j_I(M)$?

If R is a Noetherian regular ring of prime characteristic p, then Huneke and Koh proved in [111] that $\mathrm{ann}_R(H^j_I(R)) \neq 0$ if and only if $H^j_I(R) = 0$. The same conclusion for Noetherian regular rings of characteristic 0 was established implicitly in [176]. The aforementioned result due to Huneke–Koh was later generalized to strongly F-regular domains in [30]. Inspired by the results due to Huneke–Koh and Lyubeznik, Lynch [171] conjectured that $\dim(A/\mathrm{ann}_A(H^\delta_I(A))) = \dim(A/H^0_I(A))$ for every Noetherian local ring A, where $\delta = \mathrm{lcd}_A(I)$. This conjecture turns out to be false in general, *cf.* [17] and [286]. Note that the rings in the counterexamples in [17] and [286] are not equidimensional. In [120, Question 6] Hochster asks the following.

Question 2.22 *If A is a Noetherian local domain and I is an ideal of cohomological dimension c, is $H^c_I(A)$ a faithful A-module?* ◇

In [110], Hochster and Jeffries answer this question in the affirmative in the following cases:

- $\mathrm{ch}(A) = p > 0$ and c equals the *arithmetic rank* of I, see Sect. 4.1 below;
- A is a pure subring of a regular ring containing a field.

In [65], Datta, Switala and Zhang answer Question 2.22 in the negative by the following (equidimensional) example.

Example 2.23 Let $R = \mathbb{Z}_2[x_0, \dots, x_5]$ and let I be the ideal of R generated by the 10 monomials

$$\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_4, x_0x_3x_5, x_0x_4x_5, x_1x_2x_5, x_1x_3x_4, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5\}.$$

Then $\mathrm{cd}(I) = 4$, but $\mathrm{ann}_R(H^4_I(R))$ is the ideal generated by $2 \in R$. ◇

When $(A, \mathfrak{m})$ is a local ring, the annihilation of $H^j_{\mathfrak{m}}(A)$ is particularly interesting for $j < \dim(A)$, and has a wide range of applications. We recall that an element $x \in A^\circ$ is called a *uniform local cohomology annihilator* of A if $xH^j_{\mathfrak{m}}(A) = 0$ for $j < \dim(A)$, where $A^\circ = A \setminus \bigcup_{\mathfrak{p} \in \min(A)} \mathfrak{p}$. Since $H^j_{\mathfrak{m}}(A)$ may not be finitely generated, it is not clear whether such a uniform annihilator should exist. Surprisingly, in [327] Zhou proved that if A is an excellent local ring then A admits a uniform local cohomology annihilator if and only if A is equidimensional. If x is a uniform local cohomology annihilator then A_x is Cohen–Macaulay (*cf.* [106, 326]); in fact, there is a deep connection between the existence of uniform local cohomology annihilators and the Cohen–Macaulay locus. To explain this connection, we need to recall some definitions from [134]. For a Noetherian ring A, a finite complex of finitely generated free A-modules

$$G_\bullet : 0 \longrightarrow G_n \xrightarrow{f_n} G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{f_1} G_0$$

is said

- to satisfy the *standard condition on rank* if $\mathrm{rank}(f_i) + \mathrm{rank}(f_{i-1}) = \mathrm{rank}(G_{i-1})$ for $1 \leqslant i \leqslant n$ and $\mathrm{rank}(f_n) = \mathrm{rank}(G_n)$, where the rank of a map is the determinantal rank;
- to satisfy the *standard condition on height* if $\mathrm{ht}(I(f_i)) \geqslant i$ for all i, where $I(f_j)$ is the ideal generated by the rank-size minors of f_j which is viewed as a matrix.

For a Noetherian ring A, we denote by $\mathrm{CM}(A)$ the set of elements $x \in A$ such that for all finite complexes $G_\bullet$ of finitely generated free A-modules satisfying the standard conditions on rank and height, $xH_i(G_\bullet) = 0$ for $i \geqslant 1$. Huneke conjectured in [134] that if A is an equidimensional excellent Noetherian ring then $\mathrm{CM}(A)$ is not contained in any minimal prime of A. Zhou proved this conjecture in [327] by showing the following theorem.

Theorem 2.24 *Let A be an excellent local ring A. Then A admits a uniform local cohomology annihilator if and only if* $\mathrm{CM}(A)$ *is not contained in any minimal prime of A.*

One may consider the uniform annihilation of local cohomology in a different direction.

Question 2.25 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of characteristic p and I be an ideal of A. Does there exist a constant B such that*

$$\mathfrak{m}^{Bp^e} H^0_{\mathfrak{m}}(A/I^{[p^e]}) = 0$$

for all $e \geqslant 1$? ◇

The special case of Question 2.25 where I is primary to a prime ideal of height $\dim(A)-1$ was explicitly asked by Hochster and Huneke in [106]; a positive answer to this special case would have significant consequences in tight closure theory, especially to the notion of F-regularity. Question 2.25 is wide open to the best of our knowledge. The graded analog, when A is a standard graded ring over a field of characteristic p and I is homogeneous, has also attracted attention. When $\dim(A/I) = 1$, the graded version was settled independently in [135] and [304]. Let A be a standard graded ring over a field and let M be a finitely generated graded A-module. We set

$$a_j(M) := \max\{\ell \mid H^j_{\mathfrak{m}}(M)_\ell \neq 0\}$$

for each integer j. Since $H^j_{\mathfrak{m}}(M)$ is Artinian, $a_j(M) < \infty$ for each j. Hence for a homogeneous ideal J, if $a_0(A/J) \leqslant t$, then $\mathfrak{m}^{t+1} H^0_{\mathfrak{m}}(A/J) = 0$. Consequently, if there is an integer B such that $a_0(A/I^{[p^e]}) \leqslant Bp^e$ for all e, then $\mathfrak{m}^{Bp^e+1} H^0_{\mathfrak{m}}(A/I^{[p^e]}) = 0$ for all e. In general, it is an open question whether there exists an integer B (independent of e) such that $a_0(R/I^{[p^e]}) \leqslant Bp^e$ for all e. On the other hand, $a_0(M)$ may be considered as a partial Castelnuovo–Mumford regularity since the regularity of M is defined as

$$\mathrm{reg}(M) := \max\{a_j(M) + j \mid 0 \leqslant j \leqslant \dim(M)\},$$

so that $a_0(M) \leqslant \mathrm{reg}(M)$. Therefore, one may ask for a stronger conclusion on the linear growth of $\mathrm{reg}(A/I^{[p^e]})$ with respect to p^e. Indeed, the following was asked in [147, p. 212].

Question 2.26 *Let A be a standard graded ring over a field of characteristic p and I be a homogeneous ideal. Does there exist a constant C such that*

$$\mathrm{reg}(A/I^{[p^e]}) \leqslant Cp^e$$

for all e? ◇

Some progress has been made: for cases of small singular locus see [46, 55] and [321]; for rings of finite Frobenius representation type, see [160].

At the crux of the homological conjectures stands the existence of big Cohen–Macaulay algebras: the assertion that each Noetherian complete local domain $(A, \mathfrak{m})$ admits an algebra (not necessarily Noetherian) in which every system of parameters of A becomes a regular sequence. A beautiful result of Hochster–Huneke in [107] says that if A is an excellent Noetherian local domain of characteristic p then its absolute integer closure[6] A^+ is a big Cohen–Macaulay A-algebra. In [114], Huneke and Lyubeznik gave a much simpler proof using annihilation of local cohomology.

Theorem 2.27 *Let A be a commutative Noetherian domain that contains a field of characteristic p, let $\mathbb{K}$ be its field of fractions and $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. Let I be an ideal of A and let α be an element in $H^i_I(R)$ such that the elements*[7] *$\alpha, \alpha^p, \dots, \alpha^{p^t}, \dots$ belong to a finitely generated submodule of $H^i_I(A)$. Then there is a module-finite extension A' of A inside $\overline{\mathbb{K}}$ such that the natural map $H^i_I(A) \longrightarrow H^i_I(A')$ induced by $A \longrightarrow A'$ sends α to 0.*

Since the module-finite extension A' is constructed using the equations satisfied by α, Theorem 2.27 is referred in the literature as an "equational lemma". Using Theorem 2.27, Huneke and Lyubeznik proved

Theorem 2.28 *Let $(A, \mathfrak{m})$ be a commutative Noetherian domain that contains a field of characteristic p, let $\mathbb{K}$ be its field of fractions and $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. Assume, furthermore, that A is a homomorphic image of a Gorenstein local ring. For every module-finite extension A' of A inside $\overline{\mathbb{K}}$, there exists module-finite extension $A' \subseteq A''$ inside $\overline{\mathbb{K}}$ such that the natural maps*

$$H^i_{\mathfrak{m}}(A') \longrightarrow H^i_{\mathfrak{m}}(A'')$$

are the zero map for each $i < \dim(A)$.

In particular,

(1) $H^i_{\mathfrak{m}}(A^+) = 0$ for $i < \dim(A)$;
(2) every system of parameter of A is a regular sequence on A^+.

The Huneke–Lyubeznik equational lemma (or equivalently, the technique of annihilating local cohomology with finite extensions) in characteristic p has found many applications, for instance [34] and [52]. In equi-characteristic 0, such annihilation of local cohomology is not possible once the dimension is at least 3: every module-finite extension of a normal domain must split in equi-characteristic

[6] The absolute integral closure of an integral domain A is defined to be the integral closure of A in the algebraic closure of the field of fractions of A.

[7] Here α^p denotes $f(\alpha)$ where f is the natural action of Frobenius on $H^i_I(A)$ induced by the Frobenius endomorphism on A.

0. The situation in mixed characteristic has long been a mystery. However, in a very surprising turn of events, Bhatt proved in [35, Theorem 5.1] the following:

Theorem 2.29 *Let* $(A, \mathfrak{m})$ *be an excellent Noetherian local domain with mixed characteristic* $(0, p)$ *and let* A^+ *be an absolute integral closure of* A*. Then*

(1) $H^i_{\mathfrak{m}}(A^+/pA^+) = 0$ *for* $i < \dim(A/pA)$ *and* $H^i_{\mathfrak{m}}(A^+) = 0$ *for* $i < \dim(A)$.
(2) Every system of parameters of A *is a Koszul regular sequence*[8] *on* A^+.
(3) If A *admits a dualizing complex, then there exists a module-finite extension* $A \longrightarrow B$ *with* $H^i_{\mathfrak{m}}(A/pA) \longrightarrow H^i_{\mathfrak{m}}(B/pB)$ *being the 0 map for all* $i < \dim(A/pA)$.

For other connections between annihilators of local cohomology modules and homological conjectures, we refer the reader to [229, 246].

3 *D*- and *F*-Structure

In this section we discuss some special structures that local cohomology have. In positive characteristic the Frobenius endomorphism is the main tool, while in any case they have a structure over the ring of differential operators.

3.1 $\mathcal{D}$*-Modules*

Following Grothendieck's approach in [90], we reproduce the definition of differential operators as follows. Let A be a commutative ring. The *differential operators*

$$\mathcal{D}(A) = \bigcup_{j \in \mathbb{N}} \mathcal{D}_j(A)$$

on A (which is to say, the differential operators from A to A) are classified by their *order* j (a natural number), and defined inductively as follows. The differential operators $\mathcal{D}_0(A)$ of *order zero* are precisely the multiplication maps $\tilde{a}\colon A \longrightarrow A$ where $a \in A$; for each positive integer j, the differential operators $\mathcal{D}_j(A)$ of *order less than or equal to* j are those additive maps $P\colon A \longrightarrow A$ for which the commutator

$$[\tilde{a}, P] = \tilde{a} \circ P - P \circ \tilde{a}$$

[8] A sequence of elements $z_1 \dots, z_t$ in a commutative ring C is a Koszul regular sequence if $H_i(K_\bullet(C; z_1, \dots, z_t)) = 0$ for $i > 0$ where $K_\bullet(C; z_1, \dots, z_t)$ is the Koszul complex of C on $z_1, \dots, z_t$.

is a differential operator on A of order less than or equal to $j-1$. If P' and P'' are differential operators of orders at most j' and j'' respectively, then $P' \circ P''$ is again a differential operator and its order is at most $j'+j''$. Thus, the differential operators on R form an $\mathbb{N}$-filtered subring $\mathcal{D}(R)$ of $\operatorname{End}_{\mathbb{Z}}(R)$, and the order filtration is (by definition) increasing and exhaustive.

When A is an algebra over the central subring $\Bbbk$, we define $\mathcal{D}(A, \Bbbk)$ to be the subring of $\mathcal{D}(A)$ consisting of those elements of $\mathcal{D}(A)$ that are $\Bbbk$-linear. Thus, $\mathcal{D}(A, \mathbb{Z}) = \mathcal{D}(A)$ and $\mathcal{D}(A, \Bbbk) = \mathcal{D}(A) \cap \operatorname{End}_{\Bbbk}(A)$. It turns out that if A is an algebra over a perfect field $\mathbb{F}$ of prime characteristic, then $\mathcal{D}(A, \mathbb{F}) = \mathcal{D}(A)$, see, for example, [177, Example 5.1 (c)].

By a $\mathcal{D}(A, \Bbbk)$-module, we mean a *left* $\mathcal{D}(A, \Bbbk)$-module, unless we expressly indicate a right module. The standard example of a $\mathcal{D}(A, \Bbbk)$-module is A itself. Using the quotient rule, localizations A' of A also carry a natural $\mathcal{D}(A, \Bbbk)$-structure and the formal quotient rule induces a natural map $\mathcal{D}(A, \Bbbk) \longrightarrow \mathcal{D}(A', \Bbbk)$. Suppose $\mathfrak{a}$ is an ideal of A. The Čech complex on a generating set for $\mathfrak{a}$ is a complex of $\mathcal{D}(A, \Bbbk)$-modules; it then follows that each local cohomology module $H^k_{\mathfrak{a}}(A)$ is a $\mathcal{D}(A, \Bbbk)$-module.

More generally, if M is a $\mathcal{D}(A, \Bbbk)$-module, then each local cohomology module $H^k_{\mathfrak{a}}(M)$ is also a $\mathcal{D}(A, \Bbbk)$-module. This was used by Kashiwara as early as 1970 as inductive tool in algebraic analysis via reduction of dimension [144] and was introduced to commutative algebra in [176, Examples 2.1 (iv)].)

If R is a polynomial or formal power series ring in the variables $x_1, \dots, x_n$ over a commutative ring $\Bbbk$, then $\frac{1}{t_i!}\frac{\partial^{t_i}}{\partial x_i^{t_i}}$ can be viewed as a differential operator on R even if the integer $t_i!$ is not invertible. In these cases, $\mathcal{D}(R, \Bbbk)$ is the free R-module with basis elements

$$\frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_n!}\frac{\partial^{t_n}}{\partial x_n^{t_n}} \qquad \text{for } (t_1, \dots, t_n) \in \mathbb{N}^n,$$

see [90, Théorème 16.11.2]. When R is a polynomial ring or formal power series ring over a field $\Bbbk$ of characteristic 0, then the ring of differential operators

$$\mathcal{D}(R, \Bbbk) = R\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle,$$

is known as the *Weyl algebra*, a simple ring in the sense that it has no non-trivial two-sided ideals.

If $\Bbbk$ is a field and if A is a singular $\Bbbk$-algebra then the structure of $\mathcal{D}(R, A)$ can be very complicated, even in characteristic zero. For example, the ring of differential operators on the cone over an elliptic curve is not Noetherian and also not generated by homogeneous operators of bounded finite degree, [32, 131]. In most cases, differential operators on singular spaces are completely mysterious, except for toric varieties, Stanley-Reisner rings and hyperplane arrangements, see [121, 142, 200, 203, 293, 294].

3.1.1 Characteristic 0

As references for background reading in this section we recommend [37, 132, 144, 145, 159].

Let $\Bbbk$ denote a field of characteristic 0 and fix $n \in \mathbb{N}$. Let R denote either $\Bbbk[x_1, \ldots, x_n]$ or $\Bbbk[[x_1, \ldots, x_n]]$, and let $\mathcal{D}$ denote $\mathcal{D}(R, \Bbbk)$, unless specified otherwise. The partial differential operator $\frac{\partial}{\partial x_i}$ is denoted by ∂_i for each variable x_i.

Note that here the order of $r\partial_1^{e_1} \cdots \partial_n^{e_n}$ ($r \in R$) equals simply $\sum_i e_i$. The order (i.e., the filtration level) of an element $\sum_{c_{\alpha,\beta} \neq 0} c_{\alpha,\beta} \boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \in \mathcal{D}$ is the maximum of the orders $|\beta|$ of its terms $\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta$. Then we have $\mathcal{D}_j = \{r\partial_1^{e_1} \cdots \partial_n^{e_n} \mid r \in R, \sum_i e_i \leqslant j\}$, an increasing and exhaustive filtration of $\mathcal{D}$, called the *order filtration* of $\mathcal{D}$.

Using the order filtration $\{\mathcal{D}_j\}$, one can form the associated graded ring,

$$\operatorname{gr}(\mathcal{D}) := \mathcal{D}_0 \oplus \frac{\mathcal{D}_1}{\mathcal{D}_0} \oplus \cdots .$$

Since the only nonzero commutators of pairs of generators in $\mathcal{D}$ are the $[\partial_i, x_i] = 1 \in \mathcal{D}_0$, it follows that $\operatorname{gr}(\mathcal{D})$ is isomorphic to a (commutative) ring of polynomials $R[\xi_1 \ldots, \xi_n]$ where ξ_i is the image of ∂_i in $\mathcal{D}_1/\mathcal{D}_0$. Note that $\operatorname{gr}(\mathcal{D})$ is naturally the coordinate ring on the cotangent space of $\Bbbk^n$, if R is a ring of polynomials. We use this to construct varieties from $\mathcal{D}$-modules as follows.

Definition 3.1 Let M be a $\mathcal{D}$-module. A *filtration* of M *with respect to the order filtration* $\{\mathcal{D}_j\}$ is a sequence of R-submodules $\{F_i M\}$ such that

(1) $F_0 M \subseteq F_1 M \subseteq \cdots \subseteq F_i M \subseteq F_{i+1} M \subseteq \cdots$;
(2) $\bigcup_i F_i M = M$;
(3) $\mathcal{D}_j \cdot F_i M \subseteq F_{i+j} M$.

Such filtration is called a *good filtration* if the associated graded module $\operatorname{gr}^F(M) := F_0 M \oplus \frac{F_1 M}{F_0 M} \oplus \cdots$ is finitely generated over $\operatorname{gr}(\mathcal{D})$. $\diamond$

Every finitely generated $\mathcal{D}$-module admits a good filtration $\{F_i M\}$; for instance, if M can be generated by $m_1, \ldots, m_d$, then setting $F_i M := \sum_j \mathcal{D}_i m_j$ produces a good filtration of M. Set J to be the radical of $\operatorname{ann}_{\operatorname{gr}(\mathcal{D})}(\operatorname{gr}^F(M))$. This ideal J is independent of the good filtration $\{F_i M\}$ (*cf.* [37, 1.3.4], [57, 11.1]), and is called the *characteristic ideal* of M. The characteristic ideal of M induces the notion of dimension of M (as a $\mathcal{D}$-module) and characteristic variety of M.

Definition 3.2 Let M be a $\mathcal{D}$-module with good filtration and let J be its characteristic ideal. The dimension of M is defined as

$$d(M) := \dim(\operatorname{gr}(\mathcal{D})/J).$$

The *characteristic variety* $\mathrm{Ch}(M)$ of M is defined as the subvariety of $\mathrm{Spec}(\mathrm{gr}(\mathcal{D}))$ defined by J. The set of the irreducible components of $\mathrm{Ch}(M)$, paired with their multiplicities in $\mathrm{gr}(M)$ is called the *characteristic cycle* of M. $\diamond$

It turns out that dimensions cannot be small:

Theorem 3.3 (Bernstein Inequality) *Let M be a nonzero finitely generated $\mathcal{D}$-module. Then*

$$n \leqslant d(M) \leqslant 2n.$$

The nonzero modules of minimal dimension form a category with many good features.

Definition 3.4 A finitely generated $\mathcal{D}$-module M is called *holonomic* if $d(M) = n$ or $M = 0$. $\diamond$

Example 3.5

(1) Set $F_i R = R$ for all $i \in \mathbb{N}$. Then one can check that $\{F_i R\}$ is a good filtration on R and $\mathrm{gr}^F(R) \cong R$. Hence

$$J = \sqrt{\mathrm{ann}_{\mathrm{gr}(\mathcal{D})}(\mathrm{gr}^F(R))} = (\xi_1, \ldots, \xi_n).$$

This shows that $d(R) = n$. Therefore, R is a holonomic $\mathcal{D}$-module.

(2) Denote $H^n_{\mathfrak{m}}(R)$ by E and set $\eta = \left[\frac{1}{x_1 \cdots x_n}\right]$, the class of the given fraction inside E. Set $F_i E = \mathcal{D}_i \cdot \eta$. Then one can check that $\{F_i E\}$ is a good filtration of E and $\mathrm{gr}^F(E) \cong \Bbbk[\xi_1, \ldots, \xi_n]$ where ξ_i denotes the image of ∂_i in $\mathcal{D}_1/\mathcal{D}_0$. Hence

$$J = \sqrt{\mathrm{ann}_{\mathrm{gr}(\mathcal{D})}(\mathrm{gr}^F(E))} = (x_1, \ldots, x_n).$$

This shows that $d(E) = n$. Therefore, $E = H^n_{\mathfrak{m}}(R)$ is a holonomic $\mathcal{D}$-module. $\diamond$

We collect next some of the basic properties of holonomic $\mathcal{D}$-modules.

Theorem 3.6

(1) Holonomic $\mathcal{D}$-modules form an Abelian subcategory of the category of $\mathcal{D}$-modules that is closed under the formation of submodules, quotient modules and extensions ([37, 1.5.2]*).*

(2) If M is holonomic, then so is the localization M_f for every $f \in R$ ([37, 3.4.1]*). Consequently, each local cohomology module $H^j_I(M)$ of M is holonomic.*

(3) Each holonomic $\mathcal{D}$-module admits a finite filtration in the category of $\mathcal{D}$-modules in which each composition factor is a simple $\mathcal{D}$-module ([37, 2.7.13]*).*

(4) A simple holonomic $\mathcal{D}$-module has only one associated prime ([37, 3.3.16]*).*

Certain finiteness properties of $H_I^j(R)$ are enjoyed by arbitrary holonomic $\mathcal{D}$-modules. In the following list, the first is a special case of Kashiwara equivalence; the latter were established in [176, Theorem 2.4].

Theorem 3.7 *Let $R = \Bbbk[[x_1, \dots, x_n]]$ and let $\mathfrak{m}$ denote the maximal ideal. Let M be a finitely generated $\mathcal{D}$-module.*

(1) If $\dim(\mathrm{Supp}_R(M)) = 0$*, then M is a direct sum of copies of $\mathcal{D}/\mathcal{D}\mathfrak{m}$.*
(2) $\mathrm{injdim}_R(M) \leqslant \dim(\mathrm{Supp}_R(M))$.
(3) If M is finitely generated (as a $\mathcal{D}$-module), then M has finitely many associated primes (as an R-module).
(4) If M is holonomic, then the Bass numbers of M are finite.

Similar statements hold when $R = \Bbbk[x_1, \dots, x_n]$.

Remark 3.8 Let $S = \Bbbk[y_1, \dots, y_{2n}]$ be the polynomial ring over $\Bbbk$ in $2n$ variables. When $R = \Bbbk[x_1, \dots, x_n]$, we have seen that $\mathrm{gr}(\mathcal{D}) \cong S$. The Poisson bracket on S is defined as follows:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_{n+i}} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial y_{n+i}} \frac{\partial f}{\partial y_i}\right).$$

An ideal $\mathfrak{a}$ of S is said to be *closed under the Poisson bracket* if $\{f, g\} \in \mathfrak{a}$ whenever $f, g \in \mathfrak{a}$.

The Poisson bracket is closely related to *symplectic structures* on $\mathbb{C}^{2n}$ and involutive subvarieties of $\mathbb{C}^{2n}$. A symplectic structure ω on $\mathbb{C}^{2n}$ is a non-degenerate skew-symmetric form; the standard one is given by

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. Fix a symplectic structure ω on $\mathbb{C}^{2n}$. Given any subspace W of $\mathbb{C}^{2n}$, its skew-orthogonal complement is defined as

$$W^{\perp} := \{\vec{v} \in \mathbb{C}^{2n} \mid \omega(\vec{w}, \vec{v}) = 0 \; \forall \; \vec{w} \in W\}.$$

A subspace W is called *involutive* if $W^{\perp} \subseteq W$. A subvariety X of $\mathbb{C}^{2n}$ is called *involutive* if the tangent space $T_x X \subseteq \mathbb{C}^{2n}$ is a involutive subspace for every smooth point $x \in X$. One can show that an affine variety $X \subseteq \mathbb{C}^{2n}$ is involutive with respect to the standard symplectic structure on $\mathbb{C}^{2n}$ if and only if its (radical) defining ideal $I(X)$ is closed under the Poisson bracket. ◇

The following was conjectured in [88] by Guillemin–Quillen–Sternberg and proved in [265] for sheaves of differential operators with holomorphic coefficients on a complex analytic manifold by Kashiwara–Kawai–Sato. The first algebraic proof was discovered by Gabber in [79].

Theorem 3.9 *Let $R = \Bbbk[x_1, \ldots, x_n]$ and M be a holonomic $\mathcal{D}$-module. Then the characteristic ideal J of M is closed under the Poisson bracket on $gr(\mathcal{D})$.*

Again, let R be either $\Bbbk[x_1, \ldots, x_n]$ or $\Bbbk[[x_1, \ldots, x_n]]$. Then each $\mathcal{D}$-module M admits a (global) *de Rham complex*. This is a complex of length n, denoted $\Omega^\bullet_R \otimes M$ (or simply $\Omega^\bullet_R$ in the case $M = R$), whose objects are R-modules but whose differentials are merely $\Bbbk$-linear. It is defined as follows [37, §1.6]: for $0 \leqslant i \leqslant n$, $\Omega^i_R \otimes M$ is a direct sum of $\binom{n}{i}$ copies of M, indexed by i-tuples $1 \leqslant j_1 < \cdots < j_i \leqslant n$. The summand corresponding to such an i-tuple will be written $M\, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_i}$. The $\Bbbk$-linear differentials $\mathrm{d}^i : \Omega^i_R \otimes M \to \Omega^{i+1}_R \otimes M$ are defined by

$$\mathrm{d}^i(m\, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_i}) = \sum_{s=1}^{n} \partial_s(m)\, \mathrm{d}x_s \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_i},$$

with the usual exterior algebra conventions for rearranging the wedge terms, and extended by linearity to the direct sum. We remark that in the polynomial case we are simply using the usual Kähler differentials to build this complex, whereas in the formal power series case, we are using the $\mathfrak{m}$-adically continuous differentials (since in this case the usual module $\Omega^1_{R/\Bbbk}$ of Kähler differentials is not finitely generated over R). An alternative way is to view $\Omega^\bullet_R \otimes M$ as a representative of $\omega_R \otimes^L_{\mathcal{D}} M$, where ω_R is the *right* $\mathcal{D}$-module $\mathcal{D}/(\partial_1, \ldots, \partial_n)\mathcal{D}$ which is as R-module simply R.

The cohomology objects $H^i(M \otimes \Omega^\bullet_R)$ are $\Bbbk$-spaces and called the *de Rham cohomology spaces* of the left $\mathcal{D}$-module M, and are denoted $H^i_{\mathrm{dR}}(M)$. The simplest de Rham cohomology spaces (the 0th and nth) of M take the form

$$H^0_{\mathrm{dR}}(M) = \{m \in M \mid \partial_1(m) = \cdots = \partial_n(m) = 0\} \subseteq M$$
$$H^n_{\mathrm{dR}}(M) = M/(\partial_1 \cdot (M) + \cdots + \partial_n \cdot (M)).$$

The de Rham cohomology spaces are not finite dimensional in general, even for finitely generated M. The following theorem is (for the Weyl algebra) a special case of fact that the $\mathcal{D}$-module theoretic direct image functor preserves holonomicity, [132, Section 3.2]. It can be found in [37, 1.6.1]) for the polynomial case and in [302, Prop. 2.2] for the formal power series case.

Theorem 3.10 *Let M be a holonomic $\mathcal{D}$-module. The de Rham cohomology spaces $H^i_{\mathrm{dR}}(M)$ are finite-dimensional over $\Bbbk$ for all i.*

Let E denote $H^n_{\mathfrak{m}}(R)$. If $R = \Bbbk[[x_1, \ldots, x_n]]$, then we use $D(-)$ to denote $\mathrm{Hom}_R(-, E)$ (this is the Matlis dual; it should not be confused with the holonomic duality functor $\mathbb{D}$ which is quite different). If $R = \Bbbk[x_1, \ldots, x_n]$, we consider the following "natural" grading on R and on $\mathcal{D}$:

$$\deg(x_i) = 1,\ \deg(\partial_i) = -1,\ i = 1, \ldots, n.$$

Note that this is really a grading on $\mathcal{D}$ since the relations $[\partial_i, x_i] = 1$ are homogeneous of degree zero. Then E inherits a grading from setting $\deg([\frac{1}{x_1 \dots x_n}]) = -n$. In this graded setting, we use ${}^*\mathrm{Hom}_R$ to denote the graded Hom and use $D^*(-)$ to denote ${}^*\mathrm{Hom}_R(-, E)$ (the graded Matlis dual).

It turns out that $D(-)$ is a functor on the category of $\mathcal{D}$-modules, that is compatible with de Rham cohomology. The following theorem is a combination of [288, Theorem 5.1] and [290, Theorem A].

Theorem 3.11

(1) Let $R = \Bbbk[[x_1, \dots, x_n]]$ and M be a holonomic $\mathcal{D}$-module. Then

$$H^i_{\mathrm{dR}}(M)^\vee \cong H^{n-i}_{\mathrm{dR}}(D(M)),\ i = 1, \dots, n,$$

where $(-)^\vee$ denotes the $\Bbbk$-dual of a $\Bbbk$-vector space.

(2) Let $R = \Bbbk[x_1, \dots, x_n]$ and M be a graded $\mathcal{D}$-module. Assume that $\dim_{\Bbbk}(H^i_{\mathrm{dR}}(M)) < \infty$. Then

$$(H^i_{\mathrm{dR}}(M))^\vee \cong H^{n-i}_{\mathrm{dR}}(D^*(M)).$$

As shown in [290, Example 3.14], $D(M)$ may not be holonomic even if M is. The duality statements in Theorem 3.11 show that the (graded) Matlis duals of holonomic $\mathcal{D}$-modules still have finite dimensional de Rham cohomology.

Remark 3.12 The idea of applying Matlis duality to local cohomology modules already appears in the work of Ogus and Hartshorne. For example, Proposition 2.2 in [213] states that in a local Gorenstein ring A with dualizing functor $D(-)$, the dual $D(H^i_I(A))$ of the local cohomology module $H^i_I(A)$ is equal to the local cohomology module $H^{\dim(A)-i}_{\mathfrak{P}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ where $\mathfrak{X}$ is the completion of Spec(A) along I, and $\mathfrak{P}$ its closed point.

In much greater generality, *Greenlees–May duality* [87] states that (the derived functor of sections with support in I) $R\Gamma_I(-)$ and (the derived functor of completion along I) $L\Lambda^I(-)$ are adjoint functors. See also [16, 164]. ◇

We briefly discuss algorithmic aspects. The Weyl algebra is both left and right Noetherian and has a Poincaré–Birkhoff–Witt basis of a polynomial ring in $2n$ variables; this makes it possible to extend the usual Gröbner basis techniques to $\mathcal{D}$-modules, see for example [81].

When R is a polynomial ring over the rational numbers, algorithms have been formulated that compute:

(1) the local cohomology modules $H^i_I(R)$ in [306], but see also [39, 212, 215];
(2) the characteristic cycles and Bass numbers of $H^j_I(R)$ when I is a monomial ideal in [5, 6];
(3) an algorithm to compute the support of local cohomology modules in [13].

In a nutshell, the algorithms are based on the fact that the modules that appear in a Čech complex $\check{C}^\bullet(R; f_1, \dots, f_m)$ are holonomic and sums of modules generated by fractions of the form $(f_{i_1} \cdots f_{i_t})^e$ for sufficiently small $e \in \mathbb{Z}$. In general, $e = -n$ is sufficient by [242], but in the spirit of computability, it is desirable to know the largest e that may be used. This number turns out to be the smallest integer root of the *Bernstein–Sato polynomial* $b_f(s)$ of the polynomial f in question. Indeed, as was shown by Bernstein in [31], for every polynomial $f \in R$ there is a linear differential operator P depending polynomially on the additional variable s such that

$$P(x_1, \dots, x_n, \partial_1, \dots, \partial_n, s) \bullet f^{s+1} = b_{P,f}(s) \cdot f^s,$$

where $0 \neq b_{P,f}(s) \in \Bbbk[s]$ with $\Bbbk$ a field of definition for f. Since $\Bbbk[s]$ is a PID, Bernstein's theorem implies there is a monic generator for the ideal of all $b_{P,f}(s)$ that arise this way; this then is called the *Bernstein–Sato polynomial* $b_f(s)$. It was shown to factor over the rational numbers in [146, 188] and is a fascinating invariant of f as it relates to monodromy of the Milnor fiber, multiplier ideals, (Igusa, topological, motivic) zeta functions, the log-canonical threshold and various other geometric notions with differential background. See [158, 312] for more details and [12] for a generalization of Bernstein-Sato polynomials to direct summands of polynomial rings.

The polynomial $b_f(s)$ can be computed as the intersection of a left ideal (derived from $f_1, \dots, f_k$) inside a Weyl algebra with one more variable t, with a "diagonal subring" $\mathbb{Q}[t\partial_t]$. The idea of how to compute this intersection, and then to give a presentation for the corresponding localization R_f, is due to Oaku. In [306] it was realized how to read off the D-structure of the resulting local cohomology $H^1_f(R)$ and the process was scaled up to non-principal ideals. The algorithm in [215] is different in nature and exploits the fact that local cohomology can be seen as certain Tor-modules along the geometric diagonal in $2n$-space. It is, however, still based on the computation of certain b-functions that generalize the notion of a Bernstein–Sato polynomial. To understand conceptually how exactly the singularity structure of I influences the structure of the $\mathcal{D}$-module $H^k_I(R)$ remains a question of great interest.

3.1.2 $\mathcal{D}$-Modules and Group Actions

We start with discussing the ideal determining the space of matrices of bounded rank, and then outline more recent developments that consider more general actions by Lie groups.

Let for now $\mathbb{K}$ be a field, choose natural numbers $m \leqslant n$ and set $R = \mathbb{K}[x_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n]$. Let $I_{m,n,t}$ be the ideal generated by the t-minors of the matrix (x_{ij}). Then R/I is Cohen–Macaulay and I has height $(m-t+1)(n-t+1)$, compare [48, 49].

Thus, in characteristic $p > 0$ one has vanishing $H^k_{I_{m,n,t}}(R)$ for any $k \neq (m-t+1)(n-t+1)$, because of the Frobenius (via the Peskine-Szpiro vanishing result in Sect. 2.2). In characteristic zero, by [49], $\mathrm{lcd}_R(I) = mn - t^2 + 1$. Therefore, $\mathrm{lcd}_R(I) - \mathrm{depth}(I, R) = (m+n-2t)(t-1) > 0$, unless $m = n = t$ or $t = 1$.

Bruns and Schwänzl also proved in all characteristics that a determinantal variety is cut out set-theoretically by $mn - t^2 + 1$ equations, and no fewer. In fact, these equations can be chosen to be homogeneous; their methods rest on results involving étale cohomology. In particular, $I_{m,n,t}$ is a set-theoretic complete intersection if and only if $n = m = t$. The same questions for the case of symmetric and skew-symmetric matrices were answered completely in [19] by Barile. In many but not all cases the number of defining equations agree with the local cohomological dimension.

Consider now the integral version of $I_{m,n,p}$ inside $R_{\mathbb{Z}} = \mathbb{Z}[x_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n]$. By [168], $H^k_{I_{m,n,t}}(R_{\mathbb{Z}})$ is a vector space over $\mathbb{Q}$ when k exceeds the height of $I_{m,n,t}$. Similar results are shown for the case of generic matrices that are symmetric or anti-symmetric. As a corollary, $H^{mn-t^2+1}_{\mathfrak{a}}(A)$ vanishes for *every* commutative ring A of dimension less than mn where $\mathfrak{a}$ is the ideal of t-minors of *any* $m \times n$ matrix over A. The initial version of this result ($m = 2 = n - 1 = t$) appeared in [112].

If $\mathbb{K}$ is algebraically closed, Barile and Macchia study in [42] the number of elements needed to generate the ideal of t-minors of a matrix X up to radical, if the entries of X outside some fixed $t \times t$-submatrix are algebraically dependent over $\mathbb{K}$. They prove that this number drops at least by one with respect to the generic case; under suitable assumptions, it drops at least by k if X has k zero entries.

Notation 3.13 We now specialize the base field to $\mathbb{C}$ and let G be a connected linear algebraic group acting on a smooth connected complex algebraic variety X. ◇

Suppose $R = \mathbb{C}[x_1, \dots, x_n]$ and G is an algebraic Lie group acting algebraically on $X = \mathbb{C}^n$. There is a natural map

$$\psi : \mathfrak{g} \longrightarrow Der(\mathbb{C}^n)$$

from the Lie algebra to the global vector fields on $\mathbb{C}^n$, i.e., the derivations inside the Weyl algebra $\mathcal{D} = \mathcal{D}(R, \mathbb{C})$.

The induced action $\star$ of G on R can be extended to an action on $\mathcal{D}$ that we also denote by $\star$. If M is a $\mathcal{D}$-module with a G-action, it is *equivariant* if the actions of G on $\mathcal{D}$ and M are compatible:

$$(g \star P) \bullet (g \star m) = g \star (P \bullet m)$$

for all $g \in G$, $P \in \mathcal{D}$, $m \in M$.

Differentiating the G-action on M one obtains an action of $\mathfrak{g}$ on M. One can now ask whether the Lie algebra element γ acts on M via differentiation of the G-action the same way that $\psi(\gamma)$ acts on M as element of $\mathcal{D}$. This is not necessarily the case.

Example 3.14 Let $G = \mathbb{C}^*$ act on $\mathbb{C}$ by standard multiplication. The Lie algebra $\mathrm{Lie}(G)$ has an equivariant generator γ that via ψ becomes $x\partial_x \in \mathcal{D}$.

Let $M = \mathcal{D}/(x\partial_x - \lambda)$, with G-action inherited from the standard G-action on $\mathcal{D}$: $g \star x = g^{-1}x$, $g \star \partial_x = g\partial_x$. Since $x\partial_x - \lambda$ is g-invariant, this is indeed a G-action on M. Since $1 \in \mathcal{D}$ is G-invariant, the effect of γ on $\bar{1} \in M$ should be zero. On the other hand, $\psi(\gamma) \cdot \bar{1} = \bar{\lambda}$. Thus, the two actions agree if and only if $\lambda = 0$.

Now note that there are other ways to act with G on M. Indeed, a $\mathbb{C}^*$-action is the same as the choice of a $\mathbb{Z}$-grading on M. Our choice above was $\deg(\bar{1}) = 0$; we now consider the choice $\deg(\bar{1}) = k \in \mathbb{Z}$. This corresponds to $g \star \bar{1} = g^k\bar{1}$, so that γ must act on $\bar{1}$ as multiplication by k. We conclude that the two actions of γ agree if and only if λ is an integer and the degree of $\bar{1}$ is λ. ◇

Definition 3.15 The $\mathcal{D}$-module M is *strongly equivariant* if the differential action of G on M agrees with the effect of ψ on M. In other words, $\gamma \star m = \psi(\gamma)m$ for all $\lambda \in \mathfrak{g}, m \in M$. ◇

Remark 3.16 Strong G-equivariance of a group acting on a variety X can be also phrased as follows, see [132, Dfn. 11.5.2]: let π and μ be the projection and multiplication maps

$$\pi\colon G \times X \longrightarrow X,$$
$$\mu\colon G \times X \longrightarrow X,$$

respectively. Then M is strongly equivariant if there is a $\mathcal{D}_{G\times X}$-isomorphism

$$\tau\colon \pi^*M \longrightarrow \mu^*M$$

that satisfies the usual compatibility conditions on $G \times G \times X$, see [300, Prop. 2.6]. If such τ exists, it is unique. ◇

Strongly G-equivariant $\mathcal{D}_X$-modules are rather special D-modules. A G-equivariant morphism of smooth varieties with G-action automatically preserves G-equivariance under direct and inverse images (since G is connected, see [301, before Prop. 3.1.2]). If G has finitely many orbits on X, strong equivariance implies that the underlying $\mathcal{D}$-module is *regular holonomic*; this is a growth condition of the solution sheaf of the module and a critical component of the Riemann–Hilbert correspondence. In this case, the simple and strongly equivariant $\mathcal{D}_X$-modules are labeled by pairs consisting of a G-orbit G/H and a finite-dimensional irreducible representation of the component group of H (in other words, a simple G-equivariant local system on the orbit), [132, Prop. 11.6.1]. For example, if $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$, these simple modules are the modules $H^{|S|}_{I_S}(R)$ where $R = \mathbb{C}[x_1, \ldots, x_n]$, $S \in 2^{[n]}$ and $I_S = (\{x_s \mid s \in S\})$.

If I is an ideal of R and Y the corresponding variety, then I is G-stable if and only if Y is. In this case, the localization of a strongly equivariant module M at an equivariant $g \in R$ is also strongly equivariant. It follows that all local cohomology modules $H^i_I(M)$ are as well. In particular, this holds when M is R or a local cohomology module of R obtained in this way.

In [238], the authors initiated the study of the GL-equivariant decomposition of the local cohomology modules of determinantal ideals characteristic zero. The main result of the paper is a complete and explicit description of the character of this representation. An important consequence is a complete and explicit description of exactly which local cohomology modules $H^j_{I_{m,n,t}}(R)$ vanish and which do not, in the case $t = n$. This was then refined and extended to Pfaffians in [238]. The restriction $t = n$ was removed in [236]. Generalizations to symmetric and skew-symmetric matrices were published in [207, 237].

In [227], Raicu obtains results on the structure of the G-invariant simple D-modules and their characters for rank-preserving actions on matrices, extending work of Nang [206, 207]. Remarkably, for the case of symmetric matrices, this provides a correction to a conjecture of Levasseur. Raicu's methods produce composition factors for certain local cohomology modules. In [165] then this was taken the furthest, to give character formulæ for iterated local cohomology modules.

A more general approach was used in [165, 169] in order to study decompositions and categories of equivariant modules in the category of D-modules, specifically with regards to quivers. These arise when when G acts on X with finitely many orbits and more particularly when X is a spherical vector space and G is reductive and connected. This leads to the study of the "representation type" of the underlying quiver (shown to be finite or tame) and the quivers are described explicitly for all irreducible G-spherical vector spaces of connected reductive groups using the classification of Kac. An early paper on this regarding the determinantal case was [205]. More recently, cases of exceptional representations and their quivers have been studied: [166, 218].

Remark 3.17 Invariant theory has also recently been aimed at singularity invariants such as multiplier and test ideals [137], and F-pure thresholds [201]. ◇

3.1.3 Coefficient Fields of Arbitrary Characteristic

Let here $\mathbb{k}$ be a field and set $R = \mathbb{k}[x_1, \dots, x_n]$ or $R = \mathbb{k}[[x_1, \dots, x_n]]$. We have seen that $\mathcal{D} = \mathcal{D}(R, \mathbb{k})$ is the free R-module with basis

$$\frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_n!}\frac{\partial^{t_n}}{\partial x_n^{t_n}} \qquad \text{for } (t_1, \dots, t_n) \in \mathbb{N}^n$$

When $\mathrm{ch}(\mathbb{k}) = p > 0$, the ring $\mathcal{D}$ is no longer left or right Noetherian. However, some desirable properties of $\mathcal{D}$-modules in characteristic 0 extend to the finite characteristic case.

Theorem 3.18 *Let $R = \Bbbk[x_1, \ldots, x_n]$ or $R = \Bbbk[[x_1, \ldots, x_n]]$, where $\Bbbk$ is a field. Then*

(1) $\operatorname{injdim}_R(M) \leqslant \dim(\operatorname{Supp}_R(M))$ *for every $\mathcal{D}$-module M ([180]).*
(2) R_f has finite length in the category of $\mathcal{D}$-modules for each $f \in R$ ([178, 185]).
Consequently, local cohomology modules $H^j_I(R)$ have finite length in the category of $\mathcal{D}$-modules.

In general, it is a difficult problem to calculate the length $\ell_{\mathcal{D}}(H^j_I(R))$, or even just $\ell_{\mathcal{D}}(R_f)$. Some results are in [292] and [36]. The following upper bounds were obtained in [157].

Theorem 3.19 *Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \ldots, x_n]$.*

(1) For each $f \in R$,

$$\ell_{\mathcal{D}}(R_f) \leqslant (\deg(f) + 1)^n.$$

(2) Assume an ideal I can be generated by $f_1, \ldots, f_t$. Then

$$\ell_{\mathcal{D}}(H^j_I(R)) \leqslant \sum_{1 \leqslant i_1 \cdots \leqslant i_j \leqslant t} (\deg(f_{i_1}) + \cdots + \deg(f_{i_j}) + 1)^n - 1.$$

Example 3.20 Let $R = \Bbbk[x_1, x_2, x_3]$ and $f = x_1^3 + x_2^3 + x_3^3$. Then

$$\ell_{\mathcal{D}}(H^1_{(f)}(R)) = \begin{cases} 1 \text{ if } \operatorname{ch}(\Bbbk) \equiv 2 \pmod 3; \\ 1 \text{ if } \operatorname{ch}(\Bbbk) = 3; \\ 2 \text{ if } \operatorname{ch}(\Bbbk) \equiv 1 \pmod 3; \\ 2 \text{ if } \operatorname{ch}(\Bbbk) = 0. \end{cases}$$

◇

If $\operatorname{ch}(\Bbbk) = 0$ and $R = \Bbbk[x_1, \ldots, x_n]$ or $R = \Bbbk[[x_1, \ldots, x_n]]$, then R_f can always be generated by $1/f^n$ as a $\mathcal{D}$-module, but may not be generated by $1/f$. For instance, let f, R be as in Example 3.20, then $1/f$ generates a proper $\mathcal{D}$-submodule of R_f in characteristic 0. On the other hand, in characteristic p, the situation is quite different as shown in [9], and generalized to rings of F-finite representation type in [295].

Theorem 3.21 *Let $\Bbbk$ be a field of characteristic $p > 0$ and let $R = \Bbbk[x_1, \ldots, x_n]$ or $R = \Bbbk[[x_1, \ldots, x_n]]$. Then R_f can be generated by $1/f$ as a $\mathcal{D}$-module for every $f \in R$.*

We have seen that, when $\Bbbk$ is a field, $R = \Bbbk[x_1, \ldots, x_n]$, and M is a $\mathcal{D}$-module, then $\operatorname{injdim}_R(M) \leqslant \dim(\operatorname{Supp}_R(M))$. Thus, if $\dim(\operatorname{Supp}_R(M)) = 0$, then M must be an injective R-module. Let I be a homogeneous ideal of R and assume that $\operatorname{Supp}_R(H^j_I(R)) = \{\mathfrak{m}\}$ where $\mathfrak{m} = (x_1 \ldots, x_n)$. Then $H^j_I(R) \cong \oplus H^n_{\mathfrak{m}}(R)^{\mu_j}$, a direct sum of finitely many copies of $H^n_{\mathfrak{m}}(R)$. Since both $H^j_I(R)$ and $H^n_{\mathfrak{m}}(R)$ are graded, a natural question is whether this isomorphism is degree-preserving. To answer this question, the notion of Eulerian graded $\mathcal{D}$-modules was introduced in [204].

Recall that $R = \Bbbk[x_1, \dots, x_n]$ and $\mathcal{D} = \mathcal{D}(R, \Bbbk)$ are naturally graded via:

$$\deg(x_i) = 1, \qquad \deg(\partial_i) = -1.$$

Definition 3.22 Denote the operator $\frac{1}{t_i!}\frac{\partial^{t_i}}{\partial x_i^{t_i}}$ by $\partial_i^{[t_i]}$.

The *t-th Euler operator* E_t is defined as

$$E_t := \sum_{\substack{t_1+t_2+\cdots+t_n=t\\ t_1\geqslant 0,\dots,t_n\geqslant 0}} x_1^{t_1}\cdots x_n^{t_n}\partial_1^{[t_1]}\cdots\partial_n^{[t_n]}.$$

In particular E_1 is the usual Euler operator $\sum_{i=1}^n x_i\partial_i$.

A graded $\mathcal{D}$-module M is called *Eulerian*, if each homogeneous element $z \in M$ satisfies

$$E_t \cdot z = \binom{\deg(z)}{t} \cdot z$$

for every $t \geqslant 1$. ◇

We collect some basic properties of Eulerian graded $\mathcal{D}$-modules as follows.

Theorem 3.23 *Let M be an Eulerian graded $\mathcal{D}$-module. Then*

(1) Graded $\mathcal{D}$-submodules of M and graded $\mathcal{D}$-quotients of M are Eulerian.
(2) If S is a homogeneous multiplicative system in R, then $S^{-1}M$ is Eulerian. In particular, M_g is Eulerian for every homogeneous $g \in R$.
(3) The local cohomology modules $H^j_I(M)$ are Eulerian for every homogeneous ideal I.
(4) The degree-shift $M(\ell)$ is Eulerian if and only if $\ell = 0$.

It follows from Theorem 3.23 that, if $\operatorname{Supp}_R(H^j_I(R)) = \{\mathfrak{m}\}$ for a homogeneous ideal I, then $H^j_I(R) \cong \oplus H^n_{\mathfrak{m}}(R)^{\mu_j}$ is a degree-preserving isomorphism. Consequently,

$$H^j_I(R)_{\geqslant -n+1} = 0. \tag{3.1.3.1}$$

This turns out to be a source of vanishing results for sheaf cohomology. For example, (3.1.3.1) is one of the ingredients in [28] to prove Theorem 2.18 which is an extension of Kodaira vanishing to a non-reduced setting.

Extensions of Eulerian $\mathcal{D}$-modules may not be Eulerian as shown in [204, Remark 3.6]. In [224] the notion of generalized Eulerian $\mathcal{D}$-module in characteristic 0 was introduced as follows. Fix integers $w_1, \dots, w_n$ and set

$$\deg(x_i) = w_i \ \ \deg(\partial_i) = -w_i$$

A graded $\mathcal{D}$-module M is called *generalized Eulerian* if, for every homogeneous element $m \in M$, there is an integer a (which may depend on m) such that

$$(E_1 - \deg(m))^a \cdot m = 0.$$

It was shown that the category of generalized Eulerian $\mathcal{D}$-modules is closed under extension. This notion of generalized Eulerian $\mathcal{D}$-modules turns out to be useful in calculating de Rham cohomology of local cohomology modules in characteristic 0 (*cf.* [221, 224, 239]).

In characteristic p, the fact that $H^j_I(R) \cong \oplus H^n_{\mathfrak{m}}(R)^{\mu_j}$ is a degree-preserving isomorphism when $\mathrm{Supp}_R(H^j_I(R)) = \{\mathfrak{m}\}$ was also established in [320] using F-modules, a technique that we discuss next.

3.2 F-Modules

Let A be a Noetherian commutative ring of characteristic p. Then A is equipped with the Frobenius endomorphism

$$F : A \xrightarrow{a \mapsto a^p} A.$$

The Frobenius endomorphism plays a very important role in the study of rings of characteristic p. For instance, in [161], regularity of A is characterized by the flatness of the Frobenius endomorphism.

Definition 3.24 (Peskine–Szpiro Functor) Let A be a Noetherian commutative ring of characteristic p. For each A-module M, denote by F_*M the A-bimodule whose underlying Abelian group is the same as M, whose left A-module structure is the usual one: $a \cdot z = az$ for each $z \in F_*M$, and whose right A-module structure is given via the Frobenius F: $z \cdot a := a^p z$ for each $z \in F_*M$.

The Peskine–Szpiro functor $F_A(-)$ from the category of left A-modules to itself is defined via

$$F_A(M) := F_*A \otimes_A M$$

for each A-module M, where the tensor product uses the right A-structure on F_*A.

Geometrically, consider the morphism of spectra induced by the Frobenius $F: A \longrightarrow A$. Then the right A-module structure of $F_*(M)$ is obtained via restriction of scalars along F, and hence agrees with the pushforward of M. On the other hand, $F_A(M)$ is the pullback of a module under the Frobenius. ◇

If A is regular, then it follows from [161] that F_*A is a flat A-module and hence $F_A(-)$ is an exact functor.

Remark 3.25 Let R be a Noetherian regular ring of characteristic p and I be an ideal of R.

(1) We have

$$F_R(R^m) \cong R^m,$$
$$F_R(R/I) \cong R/I^{[p]}.$$

Here $I^{[p]}$ is the Frobenius power from Remark 1.5

(2) Moreover,

$$F_R(\operatorname{Ext}^j_R(R/I, R)) \cong \operatorname{Ext}^j_R(F_R(R/I), F_R(R)) \cong \operatorname{Ext}^j_R(R/I^{[p]}, R).$$

The natural surjection $R/I^{[p]} \longrightarrow R/I$ induces

$$\beta : \operatorname{Ext}^j_R(R/I, R) \longrightarrow \operatorname{Ext}^j_R(R/I^{[p]}, R)$$

and by iteration produces a directed system

$$\operatorname{Ext}^j_R(R/I, R) \xrightarrow{\beta} \operatorname{Ext}^j_R(R/I^{[p]}, R) \xrightarrow{F_R(\beta)} \operatorname{Ext}^j_R(R/I^{[p^2]}, R) \cdots$$

which agrees with

$$\operatorname{Ext}^j_R(R/I, R) \xrightarrow{\beta} F_R(\operatorname{Ext}^j_R(R/I, R)) \xrightarrow{F_R(\beta)} F_R^2(\operatorname{Ext}^j_R(R/I, R)) \cdots$$

Since $\{I^{[p^e]}\}_{e \geqslant 0}$ and $\{I^t\}_{t \geqslant 0}$ are cofinal (that is, the two families of ideals define the same topology on the ring), the direct limit of this direct system is $H^j_I(R)$.

(3) The previous items suggest that $H^j_I(R)$ may be built from the finitely generated R-module $\operatorname{Ext}^j_R(R/I, R)$ using Frobenius, and hence it is natural to expect some properties of $H^j_I(R)$ to be reflected in $\operatorname{Ext}^j_R(R/I, R)$. Indeed, it was proved in [125] that

$$\operatorname{Ass}_R(H^j_I(R)) \subseteq \operatorname{Ass}_R(\operatorname{Ext}^j_R(R/I, R)), \ \ \mu^i_{\mathfrak{p}}(H^j_I(R)) \leqslant \mu^i_{\mathfrak{p}}(\operatorname{Ext}^j_R(R/I, R))$$

for every prime ideal $\mathfrak{p}$, where $\mu^i_{\mathfrak{p}}(M)$ denotes the i-th Bass number of an R-module M with respect to $\mathfrak{p}$. (This was generalized to rings of F-finite representation type in [295]).

Based on the idea of building $H^j_I(R)$ using $\operatorname{Ext}^j_R(R/I, R)$, [163] describes a practical algorithm to calculate the support of $H^j_I(R)$; this algorithm has been implemented in *Macaulay2* [92]. ◇

3.2.1 F-Modules

In order to conceptualize the approach in [125], Lyubeznik introduced the theory of F-modules in [177]. Throughout 3.2.1, R is a regular (not necessarily local) Noetherian ring of characteristic $p > 0$, and I is an ideal of R.

Definition 3.26 An *F-module* over R (or *F_R-module*) is a pair (M, θ_M) where M is an R-module and $\theta_M : M \xrightarrow{\sim} F_R(M)$ is an R-module isomorphism, called the *structure morphism*. (When the underlying ring is understood, we sometimes refer simply to M as an "F-module".) The category of F_R-modules will be denoted by $\mathcal{F}_R$ (or $\mathcal{F}$ when R is clear from the context).

If R is graded, a *graded F-module* is an F-module M such that M is graded and the structure isomorphism $M \longrightarrow F_R(M)$ is degree-preserving. ◇

Example 3.27 One can check that $F_*R \otimes_R R \xrightarrow{r' \otimes r \mapsto r' r^p} R$ is an R-linear isomorphism. Hence R is an F-module; consequently so are all free R-modules.

Given any $g \in R$, one can check that $F_*R \otimes_R R_g \xrightarrow{r' \otimes \frac{r}{g^t} \mapsto \frac{r' r^p}{g^{tp}}} R_g$ is an R-linear isomorphism. Hence R_g is an F-module.

When $R = \Bbbk[x_1, \dots, x_n]$ with standard grading, then for each graded R-module M we define a grading on $F_R(M) = F_*R \otimes_R M$ via

$$\deg(r' \otimes m) = \deg(r') + p \deg(m)$$

for all homogeneous $r' \in R$ and $m \in M$.

In this setting, $F_*R \otimes_R R \xrightarrow{r' \otimes r \mapsto r' r^p} R$ is a degree-preserving R-linear isomorphism and so R is a graded F-module. Likewise, if $g \in R$ is homogeneous, then $F_*R \otimes_R R_g \xrightarrow{r' \otimes \frac{r}{g^t} \mapsto \frac{r' r^p}{g^{tp}}} R_g$ is a degree-preserving R-linear isomorphism and hence R_g is a graded F-module. ◇

Definition 3.28 Let (M, θ_M) be an F-module. We say that M is *F-finite* if there exists a finitely generated R-module M' and an R-linear map $\beta : M' \to F_R(M')$ such that

$$\varinjlim(M' \xrightarrow{\beta} F_R(M') \xrightarrow{F^*\beta} F^2_R(M') \longrightarrow \cdots) \cong M, \qquad (3.2.1.1)$$

and the structure morphism θ_M is induced by taking the direct limit over ℓ of $F_R^\ell(\beta) : F_R^\ell(M') \to F_R^{\ell+1}(M')$. In this case we call M' a *generator* of M and β a *generating morphism*. A generator M' of an F-finite F-module M is called a *root* if the generating morphism $\beta : M' \to F_R(M')$ is injective.

A graded F-finite F-module is defined to be an F-finite F-module for which the modules and morphisms in (3.2.1.1) can be chosen to be homogeneous. ◇

Example 3.29 From Remark 3.25, one can see that every local cohomology module $H_I^j(R)$ is an F-finite F-module since it is the direct limit of

$$\mathrm{Ext}_R^j(R/I, R) \xrightarrow{\beta} F_R(\mathrm{Ext}_R^j(R/I, R)) \xrightarrow{F_R(\beta)} F_R^2(\mathrm{Ext}_R^j(R/I, R)) \cdots$$

and $\mathrm{Ext}_R^j(R/I, R)$ is finitely generated.

When $R = \Bbbk[x_1, \ldots, x_n]$ and I is a homogeneous ideal of R, the local cohomology modules $H_I^j(R)$ are graded F-finite F-modules. ◇

There is a fruitful analogy between (F-finite) F-modules and (holonomic) $\mathcal{D}$-modules. We collect some basic properties of F-modules, which are parallel to those of $\mathcal{D}$-modules, as follows.

Theorem 3.30 *Let R be a Noetherian regular ring of characteristic $p > 0$.*

(1) If M is an F-module, then $\mathrm{injdim}_R(M) \leqslant \dim(\mathrm{Supp}_R(M))$, [177, 1.4].
(2) F-finite F-modules form a full Abelian subcategory of the category of R-modules that is closed under the formation of submodules, quotient modules, and extensions, [177, 2.8].
(3) If M is an F-finite F-module, then so is the localization M_g for each $g \in R$, [177, 2.9].
(4) A simple F-module has a unique associated prime, [177, 2.12].
(5) F-finite F-modules have finite length in the category of F-modules, [177, 3.2].

Remark 3.31 The theory of F-modules plays a crucial role in the extension of the Riemann–Hilbert correspondence to characteristic p by Emerton and Kisin [70], which is beyond the scope of this survey. ◇

3.2.2 $A\{f\}$-Modules: Action of Frobenius

Let A be a Noetherian commutative ring of characteristic p. We will use $A\{f\}$ to denote the associative A-algebra with one generator f and relations $fa = a^p f$ for all $a \in A$.

Remark 3.32 Let M be an A-module M. The following are equivalent.

(1) M is an $A\{f\}$-module.
(2) M admits an additive map $f : M \longrightarrow M$ such that $f(am) = a^p f(m)$ for every $a \in A$ and $m \in M$; this f is called a *Frobenius action* on M.

(3) M admits an A-linear map $M \longrightarrow F_*M$ where $F\colon A \longrightarrow A$ is the Frobenius endomorphism on A.
(4) M admits an A-linear map $F_*A \otimes_A M \longrightarrow M$ where $F : A \longrightarrow A$ is the Frobenius endomorphism on A.

In (2), we still use f to denote the Frobenius action since multiplication on the left by f on M is indeed a Frobenius action for each $A\{f\}$-module M.

Of course, the standard example of a Frobenius action is A with the p-th power map. Note that the image $f(M)$ is in general just a group, but acquires the structure of a $\Bbbk$-space when $\Bbbk$ is perfect.

The Frobenius on A induces a natural Frobenius action on each $H^i_{\mathfrak{a}}(A)$ for every ideal $\mathfrak{a}$; hence $H^i_{\mathfrak{a}}(A)$ is an $A\{f\}$-module. In this paper, we always consider $H^i_{\mathfrak{a}}(A)$ as an $A\{f\}$-module with the Frobenius action f induced by the Frobenius endomorphism on A. For this reason, some authors denote by F (instead of f) the Frobenius action on $H^i_{\mathfrak{a}}(A)$ induced by the Frobenius endomorphism on A. ◇

Definition 3.33 Given an $A\{f\}$-module M with Frobenius action $f : M \longrightarrow M$, the intersection

$$M_{\mathrm{st}} := \bigcap_{t\geqslant 1} f^t(M)$$

is called the *f-stable part* of M.

An element $z \in M$ is called *f-nilpotent* if $f^t(z) = 0$ for some integer t.

An $A\{f\}$-module M is called *f-torsion* if every element in M is in the kernel of some iterate of f, and it is called *f-nilpotent* if there is an integer t such that $f^t(M) = 0$. ◇

Remark 3.34 When $M = H^i_{\mathfrak{a}}(A)$ is a local cohomology module of A, the notions of f-torsion and f-nilpotent are also denoted by F-torsion and F-nilpotent, respectively, since the Frobenius action f is induced by the Frobenius endomorphism on A.

Assume $(A, \mathfrak{m}, \Bbbk)$ is a local ring and $x_1 \dots, x_d$ is a full system of parameters. Then the Frobenius action f on $H^d_{\mathfrak{m}}(A)$ can be described as follows. Let $\eta = [\frac{a}{x_1^{n_1}\cdots x_d^{n_d}}]$ be an element in $H^d_{\mathfrak{m}}(A)$, then

$$f(\eta) = [\frac{a^p}{x_1^{n_1 p}\cdots x_d^{n_d p}}].$$

◇

An $A\{f\}$-module that is also an Artinian A-module is called a *cofinite* $A\{f\}$-module. Cofinite $A\{f\}$-modules enjoy an amazing property.

Theorem 3.35 *Let A be a local ring of characteristic $p > 0$. Assume that M is an f-torsion cofinite $A\{f\}$-module. Then M must be f-nilpotent.*

Theorem 3.35 was first proved by Hartshorne and Speiser in [124]. There, Hartshorne and Speiser created a version of some of Ogus' results from [213] in characteristic $p > 0$. Their motivating question was to determine when the cohomology of every coherent sheaf on the complement of a projective variety be a finite dimensional vector space. Hartshorne and Speiser use the Frobenius endomorphism on $\mathcal{O}_{\hat{X}}$ to supply the information given by the connection used by Ogus in characteristic zero, and $\mathbb{Z}/p$-étale cohomology turns up in place of de Rham cohomology. Theorem 3.35 was later generalized by Lyubeznik in [177] (using the $\mathcal{H}_{R,A}$-functor discussed in the sequel). It has found applications in [26, 54, 154] in the study of singularities and invariants defined by Frobenius.

Theorem 4.6 in [183] reads as follows: if $\Bbbk$ is an algebraically closed field of positive characteristic, and if $(A, \mathfrak{m}, \Bbbk)$ is a complete local ring with connected punctured spectrum and $\Bbbk \subseteq A$, then $H^1_{\mathfrak{m}}(A)$ is f-torsion. Lyubeznik derives this via a comparison with local cohomology in a complete regular local ring that surjects onto A. In [283], this result is sharpened to a numerical statement over an algebraically closed coefficient field: the number of connected components of the punctured spectrum of A is one more than the dimension of the f-stable part of $H^1_{\mathfrak{m}}(A)$.

A general study of Frobenius operators started with [167] and later was carried out by various authors: aside from Sharp's article [260] we should point at [261] by the same author, [26] which develops the notion of *Cartier modules* (which are approximately modules with a Frobenius action), and [80]. The article [160] contains positive results on finiteness dual to [255] as well as examples of failure.

Definition 3.36 Let $(A, \mathfrak{m}, \Bbbk)$ be a local ring of characteristic $p > 0$. Given a cofinite $A\{f\}$-module W, a prime ideal $\mathfrak{p}$ is called a *special prime* of W if it is the annihilator of an $A\{f\}$-submodule of W. ◇

It is proved in [260, Corollary 3.7] and [68, Theorem 3.6] that if the Frobenius action $f : M \longrightarrow M$ on the $A\{f\}$-module M is injective then M admits only finitely many special primes. This will be useful when we discuss the F-module length of local cohomology modules in the sequel.

Definition 3.37 ([68]) Let $(A, \mathfrak{m})$ be a Noetherian local ring of characteristic p. Let $f : H^j_{\mathfrak{m}}(A) \longrightarrow H^j_{\mathfrak{m}}(A)$ denote the Frobenius action induced by the Frobenius on A.

A submodule N of $H^j_{\mathfrak{m}}(A)$ is called *F-stable* if $f(N) \subseteq N$.

The ring A is called FH-*finite* if $H^j_{\mathfrak{m}}(A)$ admits only finitely many F-stable submodules for each $0 \leqslant j \leqslant \dim(A)$.

Also, A is called *F-injective* if the natural Frobenius action $f : H^j_{\mathfrak{m}}(A) \longrightarrow H^j_{\mathfrak{m}}(A)$ is injective for each integer $j \leqslant \dim(A)$. ◇

The Frobenius action on local cohomology modules connects with a very important type of singularities, that of F-rationality, which we recall next.

Definition 3.38 Let A be a Noetherian ring of characteristic p, let A° denote the complement of the union of minimal primes in A and let $\mathfrak{a}$ be an ideal of A. An element $a \in A$ is in the *tight closure* of $\mathfrak{a}$ if there is a $c \in A^\circ$ such that $ca^{p^e} \in \mathfrak{a}^{[p^e]}$ for all $e \gg 0$. Let $\mathfrak{a}^*$ denote the set of elements $a \in A$ that are in the tight closure of $\mathfrak{a}$; it is an ideal of A. An ideal $\mathfrak{a}$ is called *tightly closed* if $\mathfrak{a} = \mathfrak{a}^*$.

A local ring A is called *F-rational* if $\mathfrak{a} = \mathfrak{a}^*$ for every parameter ideal $\mathfrak{a}$. ◇

In her work to relate F-rationality (an algebraic notion) to rational singularity (a geometric notion), Smith [266] proves the following characterization of F-rationality using a Frobenius action.

Theorem 3.39 *Let $(A, \mathfrak{m})$ be a d-dimensional excellent local domain of characteristic p. Then A is F-rational if and only if A is normal, Cohen–Macaulay, and $H^d_{\mathfrak{m}}(A)$ contains no non-trivial F-stable submodules.*

In independent work of Smith, Mehta–Srinivas, and Hara, F-rationality was shown to be the algebraic counterpart to the notion of rational singularities [101, 198, 266]. The purpose of these studies was to establish a parallelism between the concept of a rational singularity in characteristic zero, and invariants based on the Frobenius for its models in finite (large) characteristic. The development of such connections has a fascinating and distinguished history, and we recommend the recent and excellent survey article [296] by two experts in the field.

A related construction goes back to [73]. For an element $x \in A$ and a parameter ideal I of A let $I(x)$ be the ideal of elements $c \in A$ that multiply x^{p^e} into $I^{[p^e]}$ for all large e (*cf.* Definition 3.38). Enescu shows in [73] that if A is F-injective and Cohen–Macaulay, then the set of maximal elements in $\{I(x) : x \notin I\}$ does not depend on I, is finite and consists only of prime ideals. These are called *F-stable primes*, and the collection of them is denoted by $\mathrm{FS}(R)$. Enescu shows further that for an F-injective Cohen–Macaulay complete local ring A, the F-stable primes can be expressed in terms of F-unstability, introduced by Fedder and Watanabe. Enescu and Sharp continued the study of properties of F-stable primes in [74, 260].

Along with FH-finiteness goes another property of rings that will come back to us later:

Definition 3.40 A is called *F-pure* if $(A \xrightarrow{a \mapsto a^p} A) \otimes_A M$ is injective for all A-modules M. ◇

Remark 3.41 For background to this remark we refer to the excellent article [296].

A standard question on "deformation" in commutative algebra is to ask "If a quotient $A/(x)$ of A by a regular element has a nice property, is A forced to share it?".

It turns out that F-purity does not deform in this sense, [76, 262]. The reader familiar with the concepts of F-regularity and F-rationality may know that F-rationality deforms [108] while F-regularity does not [263] although it does so for $\mathbb{Q}$-Gorenstein rings [3, 108]. Very recently, Polstra and Simpson proved in [222] that F-purity deforms in $\mathbb{Q}$-Gorenstein rings.

It is still an open question whether F-injectivity deforms, but some progress has been made. Fedder showed in [76] that F-injectivity deforms when the ring is Cohen–Macaulay. In [115], it was proved that if R/xR is F-injective and

$H^j_{\mathfrak{m}}(A/(x^\ell)) \longrightarrow H^j_{\mathfrak{m}}(A/(x))$ is surjective for all $\ell > 1$ and j then A is F-injective. Ma and de Stefani established deformation when the local cohomology modules $H^\bullet_{\mathfrak{m}}(A)$ have secondary decompositions that are preserved by the Frobenius [63]. ⋄

In [68] it is proved that face rings of finite simplicial complexes are FH-finite. They showed further that an F-pure and quasi-Gorenstein local ring is FH-finite, and raised the question whether all F-pure and Cohen–Macaulay local rings are FH-finite. Ma answered this question in the affirmative by proving the following result in [186].

Theorem 3.42 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of characteristic p. If A is F-pure, then A and all power series rings over A are FH-finite.*

In the paper he also proved that if A is F-pure (even just on the punctured spectrum) then $H^\bullet_{\mathfrak{m}}(A)$ is a finite length $A\{F\}$-module, and he also established that the finite length property is stable under localization. With Quy, he introduced more recently in [197] the notions *F-full* (when the Frobenius action is surjective) and *F-anti-nilpotent* (when the action is injective on every $A\{F\}$-subquotient of local cohomology). They established that F-anti-nilpotence implies F-fullness and equals FH-finiteness of [68]. Inspired by ideas from [115], they prove the interesting fact that both F-anti-nilpotence and F-fullness do deform.

The action of the Frobenius also ties in naturally with the action of the Frobenius on the cohomology of projective varieties via the identification (1.2.0.1). For example, the Segre product of a smooth elliptic curve E with $\mathbb{P}^1_{\mathbb{K}}$ has F-injective coordinate ring (recall Definition 3.40) if and only if the curve is ordinary (the group $H^1(E; \mathcal{O}_E)$ is the degree zero part of $H^2_{\mathfrak{m}}(A)$ and the Frobenius action is the induced one; here A is the coordinate ring of E). Compare Example 4.2.

Hartshorne and Speiser in [124], and Fedder and Watanabe in [78] studied F-actions on local cohomology with regards to vanishing of cohomology on projective varieties, and with regards to singularity types of local rings respectively.

According to [273], a local ring $(A, \mathfrak{m})$ is *F-nilpotent* if the Frobenius action is nilpotent on $H^{<\dim(A)}_{\mathfrak{m}}(A)$ and $0^*_{H^{\dim(A)}_{\mathfrak{m}}(A)}$ (the tight closure of the zero submodule of $H^{\dim(A)}_{\mathfrak{m}}(A)$), and Srinivas and Takagi show that A is F-injective and F-nilpotent if and only it is F-rational. In [219], Polstra and Quy characterize F-nilpotence as (under mild hypotheses) being equivalent to the equality of tight and Frobenius closure for all parameter ideals. This work extends the finite length case discussed in [187] and is somewhat surprising since the complementary notion of F-injectivity is not equivalent to the Frobenius-closedness of all parameter ideals, [226], but only implied by it.

Ma also shows in [187], in his setting of finite length lower local cohomology, that F-injectivity implies the ring being Buchsbaum (a generalization of Cohen–Macaulay, [279]), and that the analogous statement in characteristic zero is true in the sense that, if A is a normal standard graded $\mathbb{K}$-algebra with $\mathbb{K} \supseteq \mathbb{Q}$ that is Du Bois and has finite length lower local cohomology, then A is Buchsbaum. (A singularity X embedded inside a smooth scheme over the complex numbers is du Bois, following Schwede's paper [249], if and only if an embedded resolution

$\pi : Y \longrightarrow X$ of $X = \mathrm{Spec}(A)$ with reduced total transform E leads to an isomorphism $\mathcal{O}_X = R\pi_*(\mathcal{O}_E)$. Initially, Du Bois singularities arose from Hodge-theoretic filtrations of the de Rham complex in [58]; they include normal crossings and quotient singularities). Du Bois singularities are closely related to (and conjecturally equivalent to) singularities of dense F-injective type. Recall that, a finite $\mathbb{Z}$-algebra $A_{\mathbb{Z}}$ is of *dense F-injective type* if its reductions A_p modulo p are F-injective for infinitely many primes $p \in \mathbb{Z}$. Schwede proved in [252] that if a finite $\mathbb{Z}$-algebra $A_{\mathbb{Z}}$ is of dense F-injective type then the complex model $A_{\mathbb{C}} = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ is Du Bois. The other implication remains an open problem and was proved to be equivalent to the Weak Ordinarity Conjecture (see [53] for details).

We close this section with a brief discussion on the very interesting topic of the interaction of the Frobenius with Hodge theory, crossing characteristics. Suppose A is a finitely generated graded $\mathbb{C}$-algebra, and set $X := \mathrm{Proj}(A)$. It is known that certain aspects of the Hodge theory of X are encoded in the combinatorics of the resolution of singularities of X, [1, 59, 60]. In this context, Srinivas and Takagi proposed and studied in [273] the following local conjecture.

Conjecture 3.43 *If $\mathfrak{x}$ is a normal isolated singularity on the n-dimensional $\mathbb{C}$-scheme X then the local ring at $\mathfrak{x}$ is of F-nilpotent type if and only if for all $i < \dim(X)$, the zeroth graded piece $\mathrm{Gr}^0_F(H^i_{\{\mathfrak{x}\}}(X^{\mathrm{an}}, \mathbb{C}))$ of the Hodge filtration is zero.* $\diamond$

How much is still unknown in this fascinating area between characteristics can be seen from the fact that the following conjectural statement is still open: let V be an $(n-1)$-dimensional projective simple normal crossings variety in characteristic zero; then the Frobenius action on $H^i(V_p, \mathcal{O}_{V_p})$ is not nilpotent for an infinite set of reductions V_p modulo p of V. Srinivas and Takagi [273] prove the case $n-1=2$ of this and derive from it the case $n=3$ for the conjecture above.

3.2.3 The Lyubeznik Functor $\mathcal{H}_{R,A}$

Assume that A is a homomorphic image of a Noetherian regular ring R. The approach of building $H^j_I(R)$ using a finitely generated R-module results in a very useful functor $\mathcal{H}_{R,A}$ from the category of cofinite $A\{f\}$-modules to the category of F_R-finite F_R-modules.

Remark 3.44 Let $R = \Bbbk[[x_1, \ldots, x_n]]$ and $E = H^n_{(x_1,\ldots,x_n)}(R)$. Denote as before the Matlis dual functor $\mathrm{Hom}_R(-, E)$ by $D(-)$. Then there is a functorial R-module isomorphism

$$\tau : D(F_R(M)) \cong F_R(D(M))$$

for all Artinian R-modules M.

Let A be a homomorphic image of R. Let M be an $A\{f\}$-module. One can check that

$$\alpha : F_R(M) \xrightarrow{r\otimes m \mapsto rf(m)} M \tag{3.2.3.1}$$

is an R-module homomorphism. Now, assume that M is a cofinite $A\{f\}$-module. Taking the Matlis dual of α, we have an R-module homomorphism

$$\beta = \tau \circ D(\alpha) : D(M) \longrightarrow F_R(D(M)),$$

and hence we have a direct system of Noetherian R-modules:

$$D(M) \xrightarrow{\beta} F_R(D(M)) \xrightarrow{F_R(\beta)} F_R^2(D(M)) \longrightarrow \cdots$$

Analogously, let $R = \Bbbk[x_1, \ldots, x_n]$ and denote the graded Matlis dual functor ${}^*\mathrm{Hom}_R(-, E)$ by $D^*(-)$. There is a functorial graded R-module isomorphism

$$\tau : D^*(F_R(M)) \cong F_R(D^*(M))$$

for all Artinian graded R-modules M.

Let M be a graded $A\{f\}$-module. One can check that then (3.2.3.1) is a graded R-module homomorphism. Now, assume that M is a cofinite graded $A\{f\}$-module. Taking the graded Matlis dual of α, we have a graded R-module homomorphism

$$\beta = \tau \circ D^*(\alpha) : D^*(M) \longrightarrow F_R(D^*(M)),$$

and hence we have a direct system of graded Noetherian R-modules:

$$D^*(M) \xrightarrow{\beta} F_R(D^*(M)) \xrightarrow{F_R(\beta)} F_R^2(D^*(M)) \longrightarrow \cdots$$

◇

Definition 3.45 Let R be a complete regular local ring R of characteristic p and let A be a homomorphic image of R. For each cofinite $A\{f\}$-module M, we define

$$\mathcal{H}_{R,A}(M) := \varinjlim(D(M) \xrightarrow{\beta} F_R(D(M)) \xrightarrow{F_R(\beta)} F_R^2(D(M)) \longrightarrow \cdots)$$

The graded version $\mathcal{H}^*_{R,A}$ is defined analogously on homogeneous input. ◇

Example 3.46 Let $R = \Bbbk[[x_1, \ldots, x_n]]$ (or $R = \Bbbk[x_1, \ldots, x_n]$ respectively) and let I be an ideal of R (homogeneous, if $R = \Bbbk[x_1, \ldots, x_n]$). Set $A = R/I$. Hence

$H^j_{\mathfrak{m}}(A)$ is an $A\{f\}$-module according to Remark 3.32. Since $H^j_{\mathfrak{m}}(A)$ is Artinian, it is a cofinite $A\{f\}$-module. Then one can check that

$$\mathcal{H}_{R,A}(H^j_{\mathfrak{m}}(A)) \cong H^{n-j}_I(R)$$

(which reads

$$\mathcal{H}^*_{R,A}(H^j_{\mathfrak{m}}(A)) \cong H^{n-j}_I(R)$$

when $R = \Bbbk[x_1, \ldots, x_n]$). ◇

Remark 3.47 The functor $\mathcal{H}_{R,A}$ (resp. $\mathcal{H}^*_{R,A}$) from the category of cofinite (graded) $A\{f\}$-modules to the category of (graded) F-finite F-modules is contravariant, additive, and exact.

Given a cofinite (graded) $A\{f\}$-module M, $\mathcal{H}_{R,A}(M) = 0$ (or $\mathcal{H}^*_{R,A}(M) = 0$ respectively) if and only if the additive map $\varphi : M \longrightarrow M$ in Remark 3.32 is nilpotent.

Now Lyubeznik's vanishing theorem in characteristic p follows from Example 3.46: $H^{n-j}_I(R) = 0$ if and only if the natural Frobenius (induced by the Frobenius on R) on $H^j_{\mathfrak{m}}(A)$ is nilpotent. ◇

The nilpotence of the action of Frobenius on $H^j_{\mathfrak{m}}(A)$ prompts the following definition (*cf.* [183, Definition 4.1]).

Definition 3.48 Let $(A, \mathfrak{m})$ be a local ring of characteristic p. The F-depth of A is the smallest i such that $H^i_{\mathfrak{m}}(A)$ is not f-nilpotent, where f is the natural action of Frobenius on $H^i_{\mathfrak{m}}(A)$ induced by the Frobenius endomorphism on A. ◇

Remark 3.49 One can show that (*cf.* [183, §4])

(1) $\operatorname{depth}(A) \leqslant F\text{-depth}(A) \leqslant \dim(A)$,
(2) $F\text{-depth}(A) = F\text{-depth}(\hat{A})$,
(3) $F\text{-depth}(A) = F\text{-depth}(A_{\mathrm{red}})$ where $A_{\mathrm{red}} = A/\sqrt{(0)}$.

In terms of F-depth, the vanishing theorem via Frobenius in characteristic p can be restated as follows: *let* $(R, \mathfrak{m})$ *be a regular local ring of characteristic p and I be an ideal. Then*

$$\operatorname{lcd}_R(I) = \dim(R) - F\text{-depth}(R/I).$$

(Compare also the corresponding statement in characteristic zero, Theorem 4.12). ◇

In general, F-depth(A) can be different from depth(A) as shown in the following example (*cf.* [183, §5]).

Example 3.50 Let $\Bbbk$ be a perfect field of characteristic p and let $\mathcal{C} \subseteq \mathbb{P}^2_{\Bbbk}$ denote the Fermat curve defined by $x^3 + y^3 + z^3$. Let $R = \Bbbk[x_0, \ldots, x_5]$ and $I \subseteq R$ be the defining ideal of $\mathcal{C} \times \mathbb{P}^1_{\Bbbk} \subseteq \mathbb{P}^5_{\Bbbk}$. Set $A = (R/I)_{\mathfrak{m}}$ where $\mathfrak{m} = (x_0, \ldots, x_5)$.

If $3 \mid (p-2)$, then

$$F\text{-depth}(A) = 3 > 2 = \text{depth}(A).$$

See also Example 4.2. ◇

Since F-finite F-modules have finite length in the category of F-modules, it is natural to ask whether one can compute the length, especially for local cohomology modules. It turns out that F-module length of local cohomology modules is closely related to singularities defined by the Frobenius, and Lyubeznik's functor $\mathcal{H}_{R,A}$ is a useful tool for studying this length. To illustrate this, let R be a regular local ring of characteristic p. That $\mathcal{H}_{R,A}$ sets up a link between the length of $H_I^{\text{ht}(I)}(R)$ and the singularities of $A = R/I$ was first discovered in [40]; this was later extended and strengthened in [157] as follows, see also [36]

Theorem 3.51 *Let $R = \Bbbk[[x_1, \dots, x_n]]$ (or $R = \Bbbk[x_1, \dots, x_n]$), and set $\mathfrak{m} = (x_1, \dots, x_n)$. Let $A = R/I$ be a reduced and equidimensional (and graded, if $R = \Bbbk[x_1, \dots, x_n]$) ring of dimension $d \geqslant 1$. Let c denote the number of minimal primes of A.*

(1) If A has an isolated non-F-rational point at $\mathfrak{m}$ and $\Bbbk$ is separably closed, then

$$\ell_{\mathcal{F}_R}(H_I^{n-d}(R)) = \dim_{\Bbbk}(H_{\mathfrak{m}}^d(A)_{\text{st}}) + c.$$

(2) If the non-F-rational locus of A has dimension $\leqslant 1$ and $\Bbbk$ is separably closed, then

$$\ell_{\mathcal{F}_R}(H_I^{n-d}(R)) \leqslant \sum_{\dim(A/\mathfrak{p})=1} \dim_{\kappa(\mathfrak{p})}(H_{\mathfrak{p}\widehat{A_{\mathfrak{p}}}}^{d-1}(A_{\mathfrak{p}})_{\text{st}}) + \dim_{\Bbbk}(H_{\mathfrak{m}}^d(A)_{\text{st}}) + c,$$

(3) If A is F-pure, then $\ell_{\mathcal{F}_R}(H_I^{n-d}(R))$ is at least the number of special primes of $H_{\mathfrak{m}}^d(A)$. Moreover, if A is F-pure and quasi-Gorenstein, then $\ell_{\mathcal{F}_R}(H_I^{n-d}(R))$ is precisely the number of special primes of $H_{\mathfrak{m}}^d(A)$.

It remains an open problem whether one can extend Theorem 3.51 to the case of a higher dimensional non-F-rational locus.

Recently, in [10], Àlvarez Montaner, Boix and Zarzuela computed $\ell_{\mathcal{F}}(H_I^j(R))$ and $\ell_{\mathcal{D}}(H_I^j(R))$ when R is a polynomial ring over a field and I is generated by square-free monomials and pure binomials (i.e. I is a toric face ideal).

3.3 Interaction Between D-Modules and F-Modules

In characteristic p, the theories of D-modules and F-modules are entwined; it has been fruitful to consider local cohomology modules from both perspectives.

Remark 3.52 Let $\Bbbk$ be a field of characteristic p and let $R = \Bbbk[x_1, \dots, x_n]$ or $R = \Bbbk[[x_1, \dots, x_n]]$. It is clear from the definition that, if (M, θ) is an F-module, the map

$$\alpha_e : M \xrightarrow{\theta} F_R(M) \xrightarrow{F_R(\theta)} F_R^2(M) \xrightarrow{F_R^2(\theta)} \cdots \longrightarrow F_R^e(M)$$

induced by θ is also an isomorphism.

This induces a $\mathcal{D} = \mathcal{D}(R, \Bbbk)$-module structure on M. To specify the induced $\mathcal{D}$-module structure, it suffices to specify how $\partial_1^{[i_1]} \cdots \partial_n^{[i_n]}$ acts on M. Choose e such that $p^e \geqslant (i_1 + \cdots + i_n) + 1$. Given $z \in M$, we consider $\alpha_e(z)$ and we will write it as $\sum r_j \otimes z_j$ with $r_j \in F_*^e R$ and $z_j \in M$. Then define

$$\partial_1^{[i_1]} \cdots \partial_n^{[i_n]} \cdot z := \alpha_e^{-1}(\sum \partial_1^{[i_1]} \cdots \partial_n^{[i_n]} r_j \otimes z_j);$$

that this is legal is due to a simplification in the product rule in characteristic p: $(x^p g)' = x^p (g)'$.

Therefore, every F-module is also a $\mathcal{D}$-module. ◇

When $R = \Bbbk[x_1, \dots, x_n]$ with its standard grading, the $\mathcal{D}$-module structure on each graded F-module as in Remark 3.52 is also graded. Moreover, [204] proves the following:

Theorem 3.53 *Let $R = \Bbbk[x_1, \dots, x_n]$. Every graded F-module is an Eulerian graded $\mathcal{D}$-module.*

Since every F-module is a $\mathcal{D}$-module, given an F-finite F-module M, one may compare $\ell_{\mathcal{F}}(M)$ and $\ell_{\mathcal{D}}(M)$. A quick observation is that, since each filtration of M in $\mathcal{F}$ is also a filtration in $\mathcal{D}$, one always has

$$\ell_{\mathcal{F}}(M) \leqslant \ell_{\mathcal{D}}(M).$$

It turns out that this inequality can be strict.

Theorem 3.54 (Proposition 7.5 in [157]) *Let p be a prime number such that $7 \mid (p-4)$. Let $R = \overline{\mathbb{F}}_p[x, y, z, t]$ and $f = tx^7 + ty^7 + z^7$. Then*

$$\ell_{\mathcal{F}}(H^1_{(f)}(R)) = 3 < 7 = \ell_{\mathcal{D}}(H^1_{(f)}(R)).$$

On the other hand, the equality holds when hypotheses are added:

Theorem 3.55 *Let R, I, A be as in Theorem 3.51. If A has an isolated non-F-rational point at $\mathfrak{m}$ and $\Bbbk$ is separably closed, then*

$$\ell_{\mathcal{F}}(H_I^{n-d}(R)) = \ell_{\mathcal{D}}(H_I^{n-d}(R)).$$

F-modules and $\mathcal{D}$-modules are deeply connected via a generating property. The following is a special case of [9, Corollary 4.4].

Theorem 3.56 *Let $\Bbbk$ be a field of characteristic p such that $[\Bbbk : \Bbbk^p] < \infty$ and let $R = \Bbbk[x_1, \ldots, x_n]$ or $R = \Bbbk[[x_1, \ldots, x_n]]$. Let M be an F-finite F-module. If $z_1, \ldots, z_t \in M$ generate a root of M, then $z_1, \ldots, z_t \in M$ generate M as a $\mathcal{D}$-module.*

Theorem 3.56 plays a crucial role in proving that $1/g$ generates R_g as a $\mathcal{D}$-module in [9], and also in proving the finiteness of associated primes of local cohomology of smooth $\mathbb{Z}$-algebras in [27].

4 Local Cohomology and Topology

In this section we discuss the interaction of local cohomology with various themes of topological flavor. The interactions can typically be seen as a failure of flatness in some family witnessed by specific elements of certain local cohomology.

We start with a classical discussion of the number of defining equations for a variety, then elaborate on the more recent developments that originate from this basic question. We survey interactions with topology in characteristic zero, and with the Frobenius map in positive characteristic. We discuss a collection of applications of local cohomology to various areas: hypergeometric functions, the theory of Milnor fibers, the Bockstein morphism from topology. We close with a discussion on a set of numerical invariants based on local cohomology modules introduced by Lyubeznik.

4.1 Arithmetic Rank

The main object of interest here is described in our first definition.

Definition 4.1 The *arithmetic rank* $\operatorname{ara}_A(I)$ of the A-ideal I is the minimum number of generators for an ideal with the same radical as I:

$$\operatorname{ara}_A(I) = \min\{\ell \in \mathbb{N} \mid \exists x_1, \ldots, x_\ell \in A,\ \sqrt{I} = \sqrt{(x_1, \ldots, x_\ell)}\}.$$

Here, $\sqrt{-}$ denotes the radical of the given ideal. ◇

The arithmetic rank of an ideal has been of interest to algebraists for as long as they have looked at ideals. In a polynomial ring over an algebraically closed field it answers the question by how many hypersurfaces the affine variety defined by I is cut out. The problem of finding this number has a long history that is detailed excellently in [173, 181]. Some ground-breaking contributions before the turn of the millennium included [49, 67, 96, 97, 124, 174, 213, 220, 268], and [162] contains a gentle introduction to the problem.

4.1.1 Some Examples and Conjectures

Local cohomology is sensitive to arithmetic rank and relative dimension. Indeed, it follows from the Čech complex point of view that

$$\max\{k \in \mathbb{N} \mid H^k_I(A) \neq 0\} = \mathrm{lcd}_A(I) \leqslant \mathrm{ara}_A(I),$$

while a standard theorem in local cohomology asserts that

$$\min\{k \in \mathbb{N} \mid H^k_I(A) \neq 0\} = \mathrm{depth}_A(I, A),$$

where $\mathrm{depth}_A(I, M)$ is the length of the longest M-regular sequence in I. If A is a Cohen–Macaulay ring, $\mathrm{depth}_A(I, A)$ is the height of the ideal.

There are examples where the arithmetic rank exceeds the local cohomological dimension, but it is often not easy to verify this since the determination of $\mathrm{lcd}_A(I)$ and $\mathrm{ara}_A(I)$ is tricky.

Example 4.2

(1) Let E be an elliptic curve over any field of characteristic $p > 0$, and consider the Segre embedding $E \times \mathbb{P}^1_{\mathbb{K}} \hookrightarrow \mathbb{P}^5_{\mathbb{K}}$. The curve E is *supersingular* if the Frobenius acts as zero on the one-dimensional space $H^1(E, \mathcal{O}_E)$. It is known that if E is defined over the integers then there are infinitely many p for which the reduction E_p is supersingular [71], and infinitely many primes for which it is ordinary. For example, for $E = \mathrm{Var}(x^3 + y^3 + z^3)$, supersingularity is equivalent to $p - 2$ being a multiple of 3. By [124, Ex. 3], the local cohomological dimension of the ideal defining $E \times \mathbb{P}^1_{\mathbb{K}}$ in $\mathbb{P}^5_{\mathbb{K}}$ equals three if and only if E is supersingular (and it is 4 otherwise). However, by [280], the arithmetic rank is always four, independently of supersingularity (and even in characteristic zero).

(2) Let $I \subseteq R = \mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be the ideal describing the image of the second Veronese map from $\mathbb{P}^2$ to $\mathbb{P}^5$ over the complex numbers. Then $\mathrm{lcd}_R(I) = 3 = \mathrm{ht}(I)$. On the other hand, as will be discussed in Example 4.10, the arithmetic rank of I is 4, not 3. The underlying method, de Rham cohomology, is the topic of Sect. 4.2. Replacing de Rham arguments with étale cohomology, similar results hold in prime characteristic, Example 4.14. This is an example where the étale cohomological dimension of the projective complement U surpasses the sum of the dimension and the cohomological dimension, $\mathrm{ecd}(U) = 6 > 2 + 3 = \mathrm{cd}(U) + \dim(U)$, compare also the discussion around Conjecture 4.13. ◇

Finding the arithmetic rank in concrete cases can be extremely difficult; some of the long-standing open problems in this area include general questions about "large" ambient spaces, but also about concrete curves:

- Hartshorne's conjecture ([97]: If $Y = \operatorname{Proj}(R/I)$ is a smooth s-dimensional subvariety of $\mathbb{P}^n_{\mathbb{C}}$, and $s > \frac{2n}{3}$, is then Y a global complete intersection (i.e., is Y the zero set of $\operatorname{codim}(Y)$ many projective hypersurfaces that, at each point of Y, are smooth and meet transversally)?
- Is the *Macaulay curve* in $\mathbb{P}^3_{\mathbb{K}}$, parameterized as $\{(s^4, s^3t, st^3, t^4)\}_{s/t\in\mathbb{P}^1_{\mathbb{K}}}$, a set-theoretic complete intersection (i.e., does the defining ideal have arithmetic rank 2, realized by homogeneous generators)?
  This question is specific to characteristic zero, as in prime characteristic p, Hartshorne proved in [100] that the Macaulay curve is a set-theoretic complete intersection for each p.

The (degree 5) Plücker embedding of the (6-dimensional) Grassmann variety $\operatorname{Gr}_{\mathbb{C}}(2, 5)$ of affine $\mathbb{C}$-planes in $\mathbb{C}^5$ into $\mathbb{P}^9_{\mathbb{K}}$ is not contained in a hyperplane, so Bezout's Theorem indicates that we are not looking at a complete intersection. Thus, the factor $2/3$ in the Hartshorne Conjecture is, in a weak sense, optimal. Asymptotically, the coefficient must be at least $1/2$, but Hartshorne writes in [97]: "I do not know any infinite sequences of examples of noncomplete intersections which would justify the fraction of the conjecture as $n \to \infty$". On the other hand, even less is known when $\dim(Y)$ is small. For example, scores of articles have been devoted to the study of monomial curves in $\mathbb{P}^3_{\mathbb{C}}$; in larger ambient spaces [303] contains a criterion for estimating arithmetic rank in terms of ideal transforms, the functors $\varinjlim_\ell \operatorname{Ext}^\bullet_A(I^\ell, -)$.

Some of the major vanishing theorems in local cohomology came out of an unsuccessful attempt to use local cohomology in order to show that certain curves in $\mathbb{P}^3_{\mathbb{K}}$ cannot be defined set-theoretically by two equations. For example, let $I \subseteq R = \mathbb{K}[x_1, \dots, x_4]$ define an irreducible projective curve. In order for the arithmetic rank of I to be 2, $H^3_I(R)$ and $H^4_I(R)$ should both be zero. That $H^4_I(R)$ vanishes follows from the Hartshorne–Lichtenbaum theorem. That $H^3_I(R)$ is also always zero is the *Second Vanishing Theorem* discussed in Sect. 2.2, in its incarnations due to Ogus, Peskine–Szpiro, Hartshorne, and Huneke–Lyubeznik. In particular, the desired obstruction to $\operatorname{ara}_A(I) = 2$ cannot not materialize, but the attempt led to the discovery of the Second Vanishing theorem.

On the positive side, Moh proved in [195] that in *positive* characteristic every monomial curve in $\mathbb{P}^3_{\mathbb{K}}$ is defined set-theoretically by two binomials; compare also [51, 56, 77, 100, 231]. The construction of the two binomials uses heavily the Frobenius and, as one might expect, the equations that work in one characteristic do not work in another [41]. In characteristic zero, Kneser proved that a curve in $\mathbb{P}^3_{\mathbb{K}}$ is cut out by three equations if it has a $\mathbb{K}$-rational point, and monomial space curves are cut out by three binomials by [41], but nothing better is known at this point.

There is recent progress on arithmetic rank and local cohomological dimension in toric and monomial situations.

In [297], Varbaro shows that if X is a general smooth hypersurface of projective n-space of degree less than $2n$ then the arithmetic rank of the natural embedding of the Segre product of X with a projective line is at most $2n$. This generalizes an observation that appeared in [280] where X is an elliptic curve. Moreover, Varbaro continues, if X is a smooth conic in the projective plane then its Segre product with projective m-space has arithmetic rank exactly $3m$, as long as the characteristic is not 2.

Toric varieties, by which we mean here the spectra of semigroup rings $\mathbb{K}[S]$ where $S \subseteq \mathbb{Z}^d$ is a finitely generated semigroup, provide a standard testing ground for theories and conjectures. Note that, for example, the Macaulay curve falls into this category.

Barile and her coauthors have studied the question whether a toric variety is a complete or almost complete intersection in [21, 22, 43, 44]. Building on this, [23] shows that certain toric ideals of codimension two are not complete intersections, and that their arithmetic rank is equal to 3. The combinatorial condition with arithmetic flavor of being p-glued has been shown to be pertinent here. A semigroup can be p-glued for exactly one prime p, [38]. That such examples might be possible is explained in part by the fact that the depth of the semigroup ring may depend on the chosen characteristic: Hochster's theorem from [118] indicates for example how Cohen–Macaulayness can toggle with p.

Monomial ideals and their local cohomology have been studied by Àlvarez-Montaner and his collaborators, see [196] for notes to a lecture series. At the heart of this work stands the Galligo–Granger–Maisonobe correspondence between perverse sheaves and hypercubes detailed in [82]. Morally, this is similar to the quiver encoding from Sect. 3.1.2 and will receive a second look in Sect. 4.4; compare specifically [7] on the category of regular holonomic $\mathcal{D}$-modules with support on a normal crossing divisor and variation zero, and [11].

In [278] a technique is given how to find generators (up to radical) for ideals that are intersections of ideals with given generators. Application to monomial ideals relates to systematic search for the arithmetic rank of certain intersections, compare [20]. Goresky and MacPherson noted in [86, 141] a formula on the singular cohomology of the complement of a complex subspace arrangement. The article [316] generalizes the formula to subspace arrangements over any separably closed field using étale cohomology and sheaf theory. These results are then applied to determine the arithmetic rank of monomial ideals. In [315], Yan studies a question of Lyubeznik on the arithmetic rank of certain resultant systems and again uses étale cohomology to get some lower bounds. More recently, Kimura and her collaborators have produced a wealth of new information on arithmetic rank of monomial ideals, *cf.* [156] and its bibliography tree.

4.1.2 Endomorphisms of Local Cohomology

As always, $(A, \mathfrak{m}, \Bbbk)$ is a Noetherian local ring and $\mathfrak{a}$ an ideal of A. In this subsection we discuss some challenges that have arisen in the last two decades, connecting the question of finding the arithmetic rank to problems about $\mathcal{D}$-modules, with the focus on the question of determining whether a given ideal be a complete intersection.

We recall that the local cohomologcal dimension $\mathrm{lcd}_A(\mathfrak{a})$ is a lower bound for the arithmetic rank $\mathrm{ara}_A(\mathfrak{a})$ and that the two invariants may not be equal, Example 4.2. Nonetheless, as work primarily of Hellus and Stückrad shows, local cohomology modules contain information that can lead to the determination of arithmetic rank. However, decoding it successfully is at this point a serious challenge.

The story starts with a result of Hellus from [103]. Denote $E = E_A(\mathbb{k})$ the injective hull of the residue field. Suppose $f_1, \dots, f_c$ are elements of $\mathfrak{a}$, and write for simplicity $\mathfrak{b}_i$ for the A-ideal generated by $f_1, \dots, f_i$. Assuming that $\mathrm{lcd}_A(\mathfrak{a}) = c$, Hellus showed that these elements generate $\mathfrak{a}$ up to radical if and only if f_i operates surjectively on $H^{c+1-i}_{\mathfrak{a}}(A/\mathfrak{b}_{i-1})$ for $1 \leqslant i \leqslant c$. This has the following corollary pertaining to set-theoretic complete intersections: if $f_1, \dots, f_c$ is an A-regular sequence (in our situation this means that $H^i_{\mathfrak{a}}(A) = 0$ unless $i = c$), then the sequence generates $\mathfrak{a}$ up to radical if and only if they form a regular sequence on $D(H^c_{\mathfrak{a}}(A))$ where, as before,

$$D(M) := \mathrm{Hom}_A(M, E)$$

is the Matlis dual. This is discussed from a new angle in [122]

This motivates (when only one $H^i_{\mathfrak{a}}(A)$ is nonzero) the study of the multiplication operators $f_i \colon D(H^c_{\mathfrak{a}}(A)) \longrightarrow D(H^c_{\mathfrak{a}}(A))$, and in particular the associated primes of $D(H^c_{\mathfrak{a}}(A))$. In fact, Hellus offers the following conjecture: if $(A, \mathfrak{m}, \mathbb{k})$ is local Noetherian,

$$\text{Is } \mathrm{Ass}_A(D(H^i_{\mathfrak{b}_i}(A))) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid H^i_{\mathfrak{b}_i}(A/\mathfrak{p}) \neq 0\}\ ? \qquad (4.1.2.1)$$

(One always has the inclusion $\subseteq$ above, and in the equi-characteristic case, the set $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid f_1, \dots, f_i \text{ is part of an s.o.p. for } A/\mathfrak{p}\}$ is contained in $\mathrm{Ass}_A(D(H^i_{\mathfrak{b}_i}(A)))$—but this may not be an equality. In mixed characteristic, a similar statement can be made). Hellus proceeds to show that this conjecture is equivalent to $\mathrm{Ass}_A(D(H^i_{\mathfrak{b}_i}(A)))$ being stable under *generalization*, and also gives the following reformulation:

Problem 4.3 For all Noetherian local *domains* $(A, \mathfrak{m}, \mathbb{k})$ and for all $f_1, \dots, f_c \in A$, show that the nonvanishing of $H^i_{(f_1,\dots,f_i)}(A)$ implies that the zero ideal is associated to $D(H^i_{(f_1,\dots,f_i)}(A))$. ◇

Remark 4.4 A significant part of Problem 4.3 was resolved positively in [170]. Namely, if R is a regular Noetherian local ring of prime characteristic, then $\mathrm{Ass}_R(D(H^i_I(R)))$ contains $\{0\}$, as long as $H^i_I(R)$ is nonzero. In fact, it is shown for all F-finite F-modules M that $\{0\}$ has to be associated to at least one of M, $D(M)$. The proof is an explicit construction of an element that is not torsion.

Motivated by their work in prime characteristic, they conjectured in [170, Conjecture 1] that, if $(R, \mathfrak{m})$ is a regular local ring and I is an ideal such that $H^i_I(R) \neq 0$, then $(0) \in \mathrm{Ass}_R(D(H^i_I(R)))$. ◇

Remark 4.5 Let $R = \mathbb{Z}_2[[x_0, \dots, x_5]]$ and let I be the monomial ideal as in Example 2.23. It follows from [65, Remark 5.3] that the arithmetic rank of I is 4; equivalently there are $f_1, \dots, f_4 \in R$ such that $H^4_I(R) = H^4_{(f_1,\dots,f_4)}(R)$. By [65, Proposition 5.5], $H^4_I(R) \cong E_{\bar{R}}(\bar{R}/\mathfrak{m}))$, where $\bar{R} = R/(2)$ and $\mathfrak{m} = (2, x_0, \dots, x_5)$. Hence

$$D(H^4_{(f_1,\dots,f_4)}(R)) = D(H^4_I(R)) \cong \bar{R}.$$

Consequently the zero ideal is not associated to $D(H^4_{(f_1,\dots,f_4)}(R))$. This answers Hellus' question in Problem 4.3 in the negative for unramified regular local rings of mixed characteristic, and provides a counterexample to the conjecture of Lyubeznik and Yildirim in mixed characteristic. ◇

In [105], an example is given where arithmetic rank and local cohomological dimension differ. What is special here is that $\mathrm{lcd}_A(\mathfrak{a}) = 1$; Hellus gives a criterion for the arithmetic rank to be one, based on the *prime avoidance property* of $\mathrm{Ass}_A(D(H^1_{\mathfrak{a}}(A)))$. In the same year and journal [104], he shows for Cohen–Macaulay rings the curious identity $H^c_{\mathfrak{a}}(D(H^c_{\mathfrak{a}}(R))) = D(R)$, provided that $c = \mathrm{lcd}_A(\mathfrak{a})$ is also the grade of $\mathfrak{a}$. This was subsequently generalized in [152].

In [129], Hellus and Stückrad continue their study of associated primes of, and regular sequences on, $D(H^c_{\mathfrak{a}}(A))$. They show that $H^m_{(f_1,\dots,f_m)}(A)$ always surjects onto $H^{m+n}_{(f_1,\dots,f_m,g_1,\dots,g_n)}(A)$ for $m > 0$ and derive from this some insights about the inclusion (4.1.2.1), and about Problem 4.3 when A is a complete domain and $\mathfrak{a}$ a 1-dimensional prime. In [127] the same authors show that in a complete local ring, when $\mathfrak{a}$ has the local cohomological behavior of a complete interection (i.e., $H^i_{\mathfrak{a}}(A) = 0$ unless $i = c$), then the natural map $A \longrightarrow \mathrm{End}_A(H^c_{\mathfrak{a}}(A))$ is an isomorphism. (In general, this map is not surjective and has a kernel). In particular, no element of A annihilates $H^c_{\mathfrak{a}}(A)$. By results mentioned above, this means that if $\mathfrak{a}$ behaves local cohomologically like a complete intersection and if $f_1, \dots, f_c$ is an A-regular sequence in $\mathfrak{a}$, then $D(H^c_{\mathfrak{a}}(D(H^c_{\mathfrak{a}}(A))))$ is an ideal of A which, if computable, predicts whether $\mathfrak{a}$ is a complete intersection. For more on $\mathrm{End}_A(H^c_{\mathfrak{a}}(A))$, see [153, 251, 253, 254].

In [128] it is investigated which ideals behave like a complete intersection from the point of local cohomology, by establishing relations to iterated local cohomology functors which then lead to Lyubeznik numbers (see Sect. 4.4). For example, if $\mathfrak{a} = (f_1, \dots, f_c)$ is an ideal of dimension d in a local Gorenstein ring, and if $\mathfrak{a}$ is a complete intersection outside the maximal ideal, then $[H^i_{\mathfrak{a}}(A) = 0$ unless $i = c]$ precisely when $[H^d_{\mathfrak{m}}(H^c_{\mathfrak{a}}(A)) = E_A(\mathbb{k})$ and $H^i_{\mathfrak{m}}(H^c_{\mathfrak{a}}(A)) = 0$ for $i \neq d]$. In particular, the complete intersection property of $\mathfrak{a}$ is then completely detectable from $H^c_{\mathfrak{a}}(A)$ alone. A new version of some of these ideas is given in a recent work of Hartshorne and Polini, who introduce and investigate coregular sequences and codepth in [122].

4.2 Relation with de Rham and étale Cohomology

4.2.1 The Čech–de Rham Complex

Suppose $I \subseteq R_{\mathbb{K}} := \mathbb{K}[x_1, \dots, x_n]$ is generated by $f_1, \dots, f_m$ and assume that $\mathbb{K}$ is a field containing $\mathbb{Q}$. The finitely many coefficients of $f_1, \dots, f_m$ all lie in some finite extension field $\Bbbk$ of $\mathbb{Q}$, and because of flatness one has $H^i_I(\mathbb{K}[x_1, \dots, x_n]) = H^i_{I_{\Bbbk}}(\Bbbk[x_1, \dots, x_n]) \otimes_{\Bbbk} \mathbb{K}$, where $I_{\Bbbk} = (f_1, \dots, f_m)R_{\Bbbk} = I \cap R_{\Bbbk}$ with $R_{\Bbbk} = \Bbbk[x_1, \dots, x_n]$.

The finite extension $\Bbbk$ can be embedded into $\mathbb{C}$ and then, by flatness again, $H^i_I(\mathbb{C}[x_1, \dots, x_n]) = H^i_{I_{\Bbbk}}(\Bbbk[x_1, \dots, x_n]) \otimes_{\Bbbk} \mathbb{C}$. It follows that most aspects of the behavior of local cohomology in characteristic zero can be studied over the complex numbers.

Convention 4.6 In this subsection, $\Bbbk = \mathbb{C}$ and I is an ideal of $R = \mathbb{C}[x_1, \dots, x_n]$. The advantage of working over $\mathbb{C}$ is that one has access to topological notions and tools. ◇

The arithmetic rank of the ideal I is the smallest number of principal open affine sets U_f that cover the complement $U_I = \mathbb{C}^n \setminus \mathrm{Var}(I)$. Any U_f arises also as the closed affine set defined by $f \cdot x_0 = 1$ inside $\mathbb{C}^n \times \mathbb{C}^1_{x_0}$.

Complex affine space as well as all its Zariski closed subsets are *Stein spaces*. This is a complex analytic condition that includes separatedness by holomorphic functions, and a convexity condition about compact sets under holomorphic functions. It implies, among other things, that a Stein space of complex dimension n has the homotopy type of an n-dimensional CW-complex. In particular, a Stein space S of complex dimension n cannot have singular cohomology $H^i_{\mathrm{Sing}}(S; -)$ beyond degree n. That complex affine varieties have this latter property is the *Andreotti–Frankel Theorem*. (For example, a Riemann surface is Stein exactly when it is not compact). In the "spirit of GAGA", [256, 257], Stein spaces are the notion that corresponds to affine varieties.

Now consider the complement $U_I = U_{f_1} \cup \cdots \cup U_{f_m}$ of the variety $\mathrm{Var}(I)$. It follows from the Mayer–Vietoris principle that $H^i_{\mathrm{Sing}}(U; -) = 0$ for all $i > n + m - 1$ and all coefficients. Being Stein is not a local property:

Example 4.7 Let $I = (x, y) \subseteq \mathbb{C}[x, y]$. Then U_I is homotopy equivalent to the 3-sphere and in particular cannot be Stein. ◇

The Čech complex on a set of generators for I is always a complex in the category of $\mathcal{D}$-modules. Let $\varphi \colon X \longrightarrow Y$ be a morphism of smooth algebraic varieties. We refer to [132] for background and details on the following continuation of the discussion on functors on D-modules in Sect. 3.

There are (both regular and exceptional) direct and inverse image functors between the categories of bounded complexes of $\mathcal{D}$-modules on X and Y. These functors preserve the categories of complexes with holonomic cohomology. In particular, one can apply them to the structure sheaf, or to local cohomology modules and Čech complexes.

If $\iota\colon U \hookrightarrow X$ is an open embedding and M a $\mathcal{D}_U$-module, then the direct image of M under ι as $\mathcal{D}$-module agrees with the direct image as $\mathcal{O}$-module. For example, in both categories there is an exact triangle

$$\mathrm{R}\Gamma_{X\setminus U}(-) \longrightarrow \mathrm{id} \longrightarrow \iota_*((-)|_U) \xrightarrow{+1} .$$

Let $X = \mathbb{C}^n$ and choose $\varphi\colon X \longrightarrow Y$ be the projection to a point Y. Write

$$\omega_X = \mathcal{D}_X/(\partial_1, \dots \partial_n) \cdot \mathcal{D}_X;$$

this gives the canonical sheaf of the manifold X a right $\mathcal{D}_X$-structure in a functorial way. Then under $\iota\colon U \hookrightarrow X$, $\mathcal{O}_U$ turns into a complex of sheaves that is represented on global sections by the Čech complex on generators of the ideal $I = (f_1, \dots, f_m)$ describing $X \setminus U$. The $\mathcal{D}$-module direct image under φ corresponds to the functor $\omega_X \otimes^L_{\mathcal{D}_X} (-)$ whose output is a complex of vector spaces. Applying this functor to the Čech complex for I invites the inspection of a *Čech–de Rham spectral sequence* starting with $\mathrm{Tor}^{\mathcal{D}_X}_\bullet(\omega_X, H^\bullet_I(\mathcal{O}_X))$. With $R = \Gamma(X, \mathcal{O}_X)$, $\omega_R = \Gamma(X, \omega_X)$, and $D = \Gamma(X, \mathcal{D}_X)$, the Grothendieck Comparison Theorem [89] asserts that on global sections, the abutment of the spectral sequence is the reduced de Rham cohomology of U,

$$E_2^{i,j} = \mathrm{Tor}^D_{n-j}(\omega_R, H^i_I(R)) \Rightarrow \tilde{H}^{i+j-1}_{\mathrm{dR}}(U; \mathbb{C}). \tag{4.2.1.1}$$

We note in passing that there are algorithmic methods that can compute the pages of this spectral sequence as vector spaces over $\mathbb{C}$, see [214, 215, 307, 308]. In the sequence (4.2.1.1), the Tor-groups involved vanish for the index exceeding $\dim X$, and so the spectral sequence operates clearly inside the rectangle $0 \leqslant i \leqslant \mathrm{lcd}_R(I)$, $0 \leqslant j \leqslant n$. However, it is actually limited to a much smaller, triangular region, compare [239].

This now opens the door to direct comparisons between local cohomology groups of high index and singular cohomology groups of high index; the de Rham type arguments in the following example are written down in [123, 168], but are folklore and were known to the authors of [213] and [98]. For example, Theorem 2.8 in [213] shows that in a regular local ring R over $\mathbb{Q}$ with closed point $\mathfrak{p}$, the vanishing of local cohomology $H^j_I(R)$ for all $j > r$ implies the vanishing of the local de Rham cohomology groups $H^i_{\mathfrak{p}}(\mathrm{Spec}(R/I))$ for all $i < \dim(R) - r$ (and is in fact equivalent to it if one already knows that the support of $H^j_I(R)$ is inside $\mathfrak{p}$ for $j > r$).

Example 4.8 We continue Example 2.14 with $\mathbb{K} = \mathbb{C}$. The open set $U = \mathbb{C}^6 \setminus \mathrm{Var}(I)$ consists of the set of 2×3 complex matrices of rank two. The closed set $V = \mathrm{Var}(I)$ is smooth outside the origin, as one sees from the $GL(2, \mathbb{C})$-action.

Since $\dim(R/I) = 4$, the height of I is 2 and so $H_I^{2+1}(R)$ must be, if nonzero, supported at the origin only, by Remark 1.4.

Since $H_I^3(R)$ is also a holonomic D-module, D being the ring of $\mathbb{C}$-linear differential operators on R, Kashiwara equivalence ([132, §1.1.6]) asserts that $H_I^3(R)$ is a finite direct sum of λ copies of E, the R-injective hull of the residue field at the origin. The number λ can be evaluated as follows.

Since I is 3-generated, $H_I^{>3}(R) = 0$ and the Čech–de Rham spectral sequence shows that $H^i(U;\mathbb{C})$ vanishes for $i > 6+3-1 = 8$. Moreover, an easy exercise shows that $\operatorname{Tor}^D_{n-j}(\omega_R, E) = 0$ unless $j = 0$, and in that case returns one copy of $\mathbb{C}$ so that the only possibly nonzero E_2-entry in the spectral sequence (4.2.1.1) in column 3 is the entry $E_2^{3,6} = \mathbb{C}^\lambda$. The workings of the spectral sequence make it clear that all differentials into and out of position $(3, 6)$ on all pages numbered 2 and up vanish. So, $\mathbb{C}^\lambda = E_2^{3,6} = E_\infty^{3,6} = H^8(U;\mathbb{C})$. We now compute this group explicitly via the following argument taken from Mel Hochster's unpublished notes on local cohomology.

Let A be a point of U, representing a rank two 2×3 matrix. Consider the deformation that scales the top row to length 1, followed by the deformation (based on gradual row reduction) that makes the bottom row perpendicular to the top row and then scales it to length 1 as well. Then the top row varies in the 5-sphere, and for each fixed top row the bottom row varies in a 3-sphere. Let M be this retract of U and note that, projecting to the top row, it is the total space of an S^3-bundle over S^5. Both base and fiber are orientable, and the base is simply connected. Thus, M is an orientable compact manifold of dimension 8 which forces $1 = \dim_{\mathbb{C}} H^8(M;\mathbb{C}) = \dim_{\mathbb{C}} H^8(U;\mathbb{C}) = \lambda$. ◇

Remark 4.9 Already Ogus proved in [213] results that relate the local cohomology module $H_I^3(R)$ of Example 4.8 to topological information. We discuss this in and after Theorem 4.12 below. In brief, the non-vanishing of $H_I^3(R)$ is "to be blamed" on the failure of the restriction map $H^2_{\mathrm{dR}}(\mathbb{P}^5_{\mathbb{C}}) \longrightarrow H^2_{\mathrm{dR}}(Y)$ to be surjective. Here, Y is the image of the Segre map and $\dim_{\mathbb{C}}(H^2_{\mathrm{dR}}(Y)) = \dim_{\mathbb{C}}(H^2_{\mathrm{dR}}(\mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^2_{\mathbb{K}})) = 2$ by the Künneth theorem. ◇

Example 4.10 (Compare [213, Exa. 4.6]) Let $\iota\colon \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \mathbb{P}^5_{\mathbb{C}}$ be the second Veronese morphism, denote the target by X, the image by Z and write $U := X \setminus Z$. There is a long exact sequence of singular (local) cohomology

$$H^p_Z(X;-) \longrightarrow H^p(X;-) \longrightarrow H^p(U;-) \xrightarrow{+1}$$

and a natural identification $H^p_Z(X;-) \cong (H_c^{2\dim X - p}(Z;-))^\vee$ with compactly supported cohomology, for any coefficient field, compare [140, §6.6]. Via Poincaré duality, this allows to identify the map $H^p_Z(X;-) \longrightarrow H^p(X;-)$ as the dual to $H_{2\dim X-p}(Z;-) \longrightarrow H_{2\dim X-p}(X;-)$. Now take $\mathbb{Z}/2\mathbb{Z}$ as coefficients. Then, since ι^* sends the generator of $H^2(X;\mathbb{Z})$ to twice the generator of $H^2(Z;\mathbb{Z})$, the long exact sequence shows that $H^8(U;\mathbb{Z}/2\mathbb{Z})$ is nonzero. Thus, U cannot be covered by three affine sets and $\text{ara}_A(I) \geqslant 4$. (In fact, $\text{ara}_A(I) = 4$ as one finds easily from experiments). ◇

4.2.2 Algebraic de Rham Cohomology

In [98], Hartshorne defines and develops for (possibly singular) schemes over a field of characteristic zero a purely algebraic (co)homology theory that he connects to singular cohomology via comparison theorems. In a nutshell, the de Rham cohomology $H^q_{\text{dR}}(Y)$ of Y embedded into a smooth scheme X is the q-th hypercohomology on X of the de Rham complex on X, completed along Y. Similarly, the de Rham homology $H_q(Y)$ of Y is the $(2\dim(X) - q)$-th local hypercohomology group with support in Y of the de Rham complex on X. (We add here a pointer to Remark 3.12). Hartshorne develops many tools of singular (co)homology: Mayer–Vietoris sequences, Thom–Gysin sequences, Poincaré duality, and a local (relative) version. With it, he shows foundational finiteness as well as Lefschetz type theorems.

One of the most remarkable applications of his theory as it relates to local cohomology is worked out in the thesis of Ogus, and based on the following definition.

Definition 4.11 ([213, Dfn. 2.12]) Let Y be a scheme over a field of characteristic zero. The *de Rham depth* dR-depth(Y) of Y is the greatest integer d such that for every point $\mathfrak{y} \in Y$ (closed or not) one has

$$H^i_{\mathfrak{y}}(Y) = 0 \qquad \text{for } i < d - \dim(\overline{\{y\}}).$$

◇

This number never exceeds the dimension of Y as one sees by looking at a closed point $\mathfrak{y}$. Ogus uses it in the following fundamental result; we point here at Remark 3.49 for the corresponding result in positive characteristic and note the formal similarities both of de Rham and F-depth, and the corresponding results on local cohomological dimension.

Theorem 4.12 ([213, Thm. 2.13]) *If Y is a closed subset of a smooth Noetherian scheme X of dimension n over a field $\Bbbk$ of characteristic zero, then for each $d \in \mathbb{N}$ one has*

$$[\text{lcd}(X, Y) \leqslant n - d] \Leftrightarrow [\text{dR-depth}(Y) \geqslant d].$$

In particular, if $Y = \text{Spec}(R/I)$ for some regular $\Bbbk$-algebra R then $n - \text{lcd}_R(I) =$ dR-depth(Y) is intrinsic to Y and does not depend on X.

Now let Y be a projective variety over the field $\Bbbk$ of characteristic zero, embedded into $\mathbb{P}^n_{\Bbbk}$. Let R be the coordinate ring of $\mathbb{P}^n_{\Bbbk}$ and I the defining ideal of Y; of course, these are not determined by Y. Then Ogus obtains in [213, Thm. 4.1] the equivalences

$$[\mathrm{lcd}(\mathbb{P}^n_{\Bbbk}, Y) \leqslant r] \Leftrightarrow [\mathrm{Supp}_R(H^i_I(R)) \subseteq \mathfrak{m} \text{ for } i > r]$$
$$\Leftrightarrow [\text{dR-depth}(Y) \geqslant n - r].$$

In particular, for any such embedding, the smallest integer r such that $H^{>r}_I(R)$ is Artinian is intrinsic to Y.

One might wonder whether a similar result holds for $\mathrm{lcd}(R, I)$ itself. With the same notations as in the previous theorem, Ogus proves in [213, Thm. 4.4]:

$$[\mathrm{cd}(\mathbb{P}^n_{\Bbbk} \smallsetminus Y) < r]$$

(that is, $\mathrm{lcd}(R, I) \leqslant r$) is equivalent to

$$\Big[[\text{dR-depth}(Y) \geqslant n - r] \text{ and } [H^i_{\mathrm{dR}}(\mathbb{P}^n_{\Bbbk}) \twoheadrightarrow H^i_{\mathrm{dR}}(Y) \text{ for } i < n - r]\Big].$$

Note that these restriction maps are always injective, and surjectivity is preserved under Veronese maps.

4.2.3 Lefschetz and Barth Theorems

Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a projective variety and $H \subseteq \mathbb{P}^n_{\mathbb{C}}$ a hyperplane. Setting $Y = X \cap H$, the *Lefschetz hyperplane theorem* states that under suitable hypotheses the natural restriction map

$$\rho^i_{X,Y} \colon H^i(X; \mathbb{C}) \longrightarrow H^i(Y; \mathbb{C}) \tag{4.2.3.1}$$

is an isomorphism for $i < \dim(Y)$ and injective for $i = \dim(Y)$. In the original formulation by Lefschetz, X is supposed to be smooth and H should be generic (which then entails Y being smooth). Inspection showed that the relevant condition is that the affine scheme $X \setminus Y$ be smooth, since then the relative groups $H^i(X, Y; \mathbb{C})$ are zero in the required range.

It is clear that one can iterate this procedure and derive similar connections between the cohomology of X and the cohomology of complete intersections on X that are well-positioned with respect to the singularities of X. (Recall that any hypersurface section can be cast as a hyperplane section via a suitable Veronese embedding of X).

A rather more difficult problem is to establish connections when Y is not a complete intersection. At the heart of the problem is the issue that in general $X \setminus Y$ will not be affine and thus might allow more complicated cohomology.

In [18], Barth developed theorems that connect, for $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ smooth (and of small codimension), the surjectivity of $\rho^i_{\mathbb{P}^n_{\mathbb{C}},Y}$ to the surjectivity of corresponding restrictions $\rho^i_{\mathbb{P}^n_{\mathbb{C}},Y}(\mathcal{F})$ of coherent sheaves $\mathcal{F}$ and hence to the cohomological dimension of $\mathbb{P}^n_{\mathbb{C}} \setminus Y$ and the arithmetic rank of the defining ideal of Y. More precisely, he proved that surjectivity of $\rho^i_{\mathbb{P}^n_{\mathbb{C}},Y}(\mathcal{F})$ occurs for $i \leqslant 2\dim(Y) - n$ and proved for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$ that surjectivity of $\rho^i_{\mathbb{P}^n_{\mathbb{C}},Y}(\mathcal{F})$ is equivalent to surjectivity of $\rho^i_{\mathbb{P}^n_{\mathbb{C}},Y}$ in the sense of Eq. (4.2.3.1) above. As a corollary, he obtained a more general form of the Lefschetz Hyperplane Theorem: if $\mathbb{P}^n_{\mathbb{C}} \supseteq X, Y$ are smooth with $\dim(X) = a$, $\dim(Y) = b$ then

$$\rho^i_{Y,X\cap Y}\colon H^i(Y;\mathbb{C}) \longrightarrow H^i(X\cap Y;\mathbb{C})$$

is an isomorphism for $i \leqslant \min(2b-n, a+b-n-1)$. It is worth looking at the special case when X is the ambient projective space. For $i = 0$ the theorem then generalizes the fundamental fact that a smooth subvariety of pure dimension a is connected whenever $2a \geqslant n$. But it also gives obstructions for embedding varieties into projective spaces of given dimension, since it forces the singular cohomology groups $H^i(Y;\mathbb{C})$ to agree with those of $\mathbb{P}^n_{\mathbb{C}}$ in the range $i \leqslant 2\dim(Y) - n$. For example, an Abelian variety Y of dimension b cannot be embedded in $Y = \mathbb{P}^{2b-1}_{\mathbb{C}}$ since with such embedding the map $H^1(\mathbb{P}^{2b-1};\mathbb{C}) \longrightarrow H^1(Y;\mathbb{C})$ should be surjective.

Barth uses the special unitary group action on $\mathbb{P}^n_{\mathbb{C}}$ to "spread" the classes on Y to classes on $\mathbb{P}^n_{\mathbb{C}}$ near Y. In order to glue them, he then needs a suitable cohomological triviality of the complement of Y. In [213], Ogus gives an algebraic version of Barth's transplanting technique, and succeeds (in his Sect. 4) in proving various statements that connect the isomorphy of the restriction maps of de Rham cohomology of two schemes $X \subseteq Y \subseteq \mathbb{P}^n_{\Bbbk}$ to the de Rham depths of X, Y and $X \smallsetminus Y$.

In [268], Speiser studies in varying characteristics the cohomological dimension of the complement C_Y of the diagonal in $Y \times Y$. As a stepping stone he studies $C_{\mathbb{P}^n_{\mathbb{K}}}$ for arbitrary fields. In any characteristic, the diagonal scheme is the set-theoretic intersection of $2n-1$ very ample divisors. However, a big difference appears for cohomological dimension: $\operatorname{cd}(C_{\mathbb{P}^n_{\mathbb{K}}}) = 2n-2$ when $\mathbb{Q} \subseteq \mathbb{K}$, but $\operatorname{cd}(C_{\mathbb{P}^n_{\mathbb{K}}}) = n-1$ in positive characteristic. The discrepancy is due to the Peskine–Szpiro Vanishing since the diagonal comes with a Cohen–Macaulay coordinate ring.

In characteristic zero, Speiser's results imply that the diagonal of projective space is cut out set-theoretically by $2n-1$ and no fewer hypersurfaces. More generally, for Cohen–Macaulay Y, he shows in [268, Thm. 3.3.1] a similar vanishing result about C_Y in positive characteristic over algebraically closed fields: the cohomological dimension of $Y \times Y \setminus \Delta$ is bounded by $2n-2$ whenever $Y \subseteq \mathbb{P}^n$ is a Cohen–Macaulay scheme of dimension $s \geqslant (n+1)/2$.

4.2.4 Results via étale Cohomology

Suppose U is an open subset of affine space $X = \mathbb{C}^n$ whose closed complement $V = X \setminus U$ is defined by the ideal I in the appropriate polynomial ring R. We have seen in (4.2.1.1) that the local cohomological dimension $\operatorname{lcd}_R(I)$ is related to the de Rham cohomology via the vanishing

$$[H^i_{\mathrm{dR}}(U;\mathbb{C})=0] \text{ whenever } [i \geqslant \operatorname{lcd}(I)+n-1=\operatorname{cd}(U)+n]. \qquad (4.2.4.1)$$

We mention here a variant of this in arbitrary characteristic, involving étale cohomology. This is a cohomology theory that interweaves topological data with arithmetic information. We refer to [191, 193] for guidance on étale cohomology.

One significant difference to the de Rham case is that the basic version of étale cohomology involves coefficients that are torsion (i.e., sheaves with stalk $\mathbb{Z}/\ell\mathbb{Z}$) of order not divisible by $p = \operatorname{ch}(\Bbbk)$.

In many aspects, over a separably closed field $\Bbbk$, étale cohomology behaves quite similar to de Rham or singular cohomology over the complex numbers. For example, on non-singular projective varieties there is a version of Poincaré duality, there is a Künneth theorem, and if a variety is defined over $\mathbb{Z}$ then its model over $\mathbb{C}$ has singular cohomology group ranks equal to the corresponding étale cohomology ranks of the reductions modulo p for most primes p.

The étale cohomology groups on a scheme X vanish beyond $2\dim X$, and even beyond $\dim(X)$ if X is affine, similar to the Andreotti–Frankel Theorem. So, it makes sense to talk of étale cohomological dimension $\operatorname{ecd}(-)$, the largest index of a non-vanishing étale cohomology group. The Mayer–Vietoris principle implies that if V is a variety inside affine n-space $X \neq V$ over the algebraically closed field $\Bbbk$, cut out by the ideal I, then with $U = X \setminus V$ one has

$$\operatorname{ecd}(U) \leqslant n + \operatorname{ara}_A(I) - 1. \qquad (4.2.4.2)$$

Note that $\operatorname{ara}_A(I) \geqslant \operatorname{lcd}_R(I) = \operatorname{cd}(U)+1$.

In [181], Lyubeznik formulates the following conjecture.

Conjecture 4.13 *Over a separably closed field* $\Bbbk$,

$$\operatorname{ecd}(U) \geqslant \dim(U) + \operatorname{cd}(U).$$

◇

In this conjecture, U need not be the complement of an affine variety or even smooth. Comparison with (4.2.4.1) shows that (for complements of varieties in affine or projective spaces) the conjecture can be interpreted to say that étale cohomology always provides a better lower bound for arithmetic rank than local cohomological dimension does. At present, this conjecture seems wide open. Varbaro shows in [297] that it holds over $\mathbb{C}$ in the case that U is the complement in projective space $\mathbb{P}^n_{\mathbb{C}} \setminus V$ of a smooth variety V with $\operatorname{cd}(\mathbb{P}^n \setminus V) > \operatorname{codim}_{\mathbb{P}^n}(V) - 1$.

Example 4.14 We continue Example 4.10. For $\mathbb{K} = \mathbb{C}$ and all other field coefficients of characteristic not equal to 2, one has $H^8(U;\mathbb{K}) = 0$. Thus, we cannot conclude that $\operatorname{lcd}_R(I) \geqslant 4$ in the way we concluded in Example 4.8. In fact, as

Ogus [213, Exa. 4.6] proved, $\mathrm{cd}(U) = 2$ (and so $\mathrm{lcd}_R(I) = 3$) and, in particular, $\mathrm{ecd}(U) > \dim(U) + \mathrm{cd}(U)$.

In finite characteristic different from 2, if one replaces "singular" by "étale", the same formal arguments as in Example 4.10 show that the arithmetic rank of the defining ideal of the Plücker embedding is 4 while (since the coordinate ring is Cohen–Macaulay) $\mathrm{lcd}_R(I) = 3$.

In characteristic 2, the arithmetic rank drops to 3 and the ideal is generated up to radical by $\{t_{xx}t_{yy} - t_{xy}t_{xy}, t_{xx}t_{zz} - t_{xz}t_{xz}, t_{yy}t_{zz} - t_{yz}t_{yz}\}$ since, for example, $t_{xx}t_{zz}(t_{xx}t_{yy} - t_{xy}t_{xy}) + t_{xy}t_{xy}(t_{xx}t_{zz} - t_{xz}t_{xz}) + t_{xx}t_{xx}(t_{yy}t_{zz} - t_{yz}t_{yz}) = (t_{xx}t_{yz} - t_{xy}t_{xz})^2$ in characteristic 2. ◇

Remark 4.15 In [297, Rmk. 2.13], Varbaro points out that Example 4.14 shows that the étale cohomological dimension of the complement of an embedding of $\mathbb{P}^2_{\Bbbk}$ into $\mathbb{P}^5_{\Bbbk}$ depends on the embedding: for a subspace embedding it is at most $3+4$ since the subspace is covered by three affine spaces of dimension 5, but for the Veronese it is 8 (compare also [19] for arithmetic rank consequences that highlight variable behavior in varying characteristic). This contrasts with his Theorem 2.4, which states that the quasi-coherent cohomological dimension is independent of the embedding (intrinsic to the given smooth projective subvariety).

Ogus proved in [213, Ex. 4.6] for any Veronese map of a projective space in characteristic zero that the local cohomological dimension agrees with the height of the defining ideal. In positive characteristic, the same follows from Peskine–Szpiro [220, Prop. III.4.1]. In [216], Pandey shows that this is even true over the integers, and by extension then over every commutative Noetherian ring. ◇

Now, recall Speiser's result from Sect. 4.2.3, on the arithmetic rank $2n - 1$ of the diagonal of $\mathbb{P}^n_{\mathbb{K}} \times \mathbb{P}^n_{\mathbb{K}}$. In [297] Varbaro shows that it remains true in every characteristic as long as $\mathbb{K}$ is separably closed; note, however, that the cohomological dimension of the complement is much smaller in finite characteristic, always equal to $n - 1$. The main ingredient comes from Künneth theorems on étale cohomology.

There are Lefschetz and Barth type results for étale cohomology. For example, in [175, Prop 9.1], Lyubeznik proves the following: assume $\mathbb{K}$ to be separably closed, of any characteristic, and pick two varieties $Y \subseteq X$ with $X \setminus Y$ smooth. If $\mathrm{ecd}(U) < 2\dim(X) - r$ then $H^i_{\mathrm{et}}(X, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H^i_{\mathrm{et}}(Y, \mathbb{Z}/\ell\mathbb{Z})$ is an isomorphism for $i < r$ and injective for $i = r$.

In the [297], Varbaro also investigates the interaction of étale cohomological dimension with intersections: let $\mathbb{K}$ be an algebraic closed field of arbitrary characteristic and let X and Y be two smooth projective varieties of dimension at least 1. Set $Z = X \times Y \subseteq \mathbb{P}^N_{\mathbb{K}}$ (any embedding) and $U = P^N \setminus Z$. Then $\mathrm{ecd}(U) \geqslant 2N - 3$. In particular, if $\dim Z \geqslant 3$ then Z cannot be a set-theoretic complete intersection by (4.2.4.2).

4.3 *Other Applications of Local Cohomology to Geometry*

4.3.1 Bockstein Morphisms

In this subsection we discuss a construction that originates (to our knowledge) in topology but can, in principle, be used as a tool to study any linear functor in prime characteristic.

For this we need the following concept. A collection of functors $\{\mathrm{F}^\bullet\}$ is a *covariant δ-functor* (in the sense of Grothendieck) if for each short exact sequence of A-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ one obtains a functorial long exact sequence

$$\ldots \longrightarrow \mathrm{F}^i(M') \longrightarrow \mathrm{F}^i(M) \longrightarrow \mathrm{F}^i(M'') \longrightarrow \mathrm{F}^{i+1}(M') \longrightarrow \cdots$$

Now suppose that for some A-module M, multiplication by $f \in A$ induces an injection $0 \longrightarrow M \xrightarrow{f} M \xrightarrow{\pi} M/fM \longrightarrow 0$. If $\mathrm{F}^\bullet$ is a covariant δ-functor that is A-linear (i.e., each F^i is additive, and $\mathrm{F}^i(M \xrightarrow{a \cdot h} N) = \mathrm{F}^i(M) \xrightarrow{a \cdot \mathrm{F}^i(h)} \mathrm{F}^i(N)$ for all $a \in A$ and all $h \in \mathrm{Hom}_A(M, N)$) then there is an induced long exact sequence

$$\cdots \longrightarrow \mathrm{F}^i(M/fM) \xrightarrow{\delta_f^{\mathrm{F},i}} \mathrm{F}^{i+1}(M) \xrightarrow{f\cdot} \mathrm{F}^{i+1}(M) \xrightarrow{\pi_f^{\mathrm{F},i+1}} \mathrm{F}^{i+1}(M/fM) \xrightarrow{\delta_f^{\mathrm{F},i+1}} \cdots$$

Now one can define a sequence of *Bockstein* morphisms

$$\beta_f^{\mathrm{F},i} : \mathrm{F}^i(M/fM) \longrightarrow \mathrm{F}^{i+1}(M/fM)$$

as the composition

$$\beta_f^{\mathrm{F},i} = \pi_f^{\mathrm{F},i+1} \circ \delta_f^{\mathrm{F},i}.$$

Remark 4.16

(1) Clearly, f, i, F and the A/fA-module M/fM are ingredients of a Bockstein morphism. However, while the notation does not indicate this, is also depends on A and the avatar $M \xrightarrow{f} M$ for M/fM (or at least an infinitesimal avatar $0 \longrightarrow M/fM \longrightarrow M/f^2M \longrightarrow M/fM \longrightarrow 0$). Bocksteins are not intrinsic but arise from a specialization.
(2) It is possible to modify the constructions to include contravariant functors, or A-modules N on which f acts surjectively.

◇

The original version of a Bockstein morphism appeared in topology, where $A = \mathbb{Z}$, f is a prime number, M is an Abelian group without p-torsion, and $\mathrm{F}^\bullet$ is singular homology (or cohomology) with coefficients M on a fixed space X.

Generally, in this context there is a Bockstein spectral sequence that arises from the short exact sequence of singular chains on X with coefficients in M, M and M/pM respectively. It starts with $E^1_{i,j} = H_{i+j}(X; M/pM)$, the differential on the E^1-page is the Bockstein morphism, and it converges to the tensor product of $\mathbb{Z}/p\mathbb{Z}$ with the free part of $H_{i+j}(X; M)$.

In [285], Bockstein maps were introduced and studied in local cohomology. So, A is a Noetherian $\mathbb{Z}$-algebra, $I = (\boldsymbol{g} = g_1, \ldots, g_m) \subseteq A$ is an ideal, and M is a p-torsion free A-module. In this setup there are several δ-functors $\mathrm{F}^\bullet$ that arise naturally: the local cohomology functor $\mathrm{F}^i = H^i_I(-)$ with support in I, the extension functors $\mathrm{F}^i = \mathrm{Ext}^i_A(A/I^\ell, -)$, the Koszul cohomology functors $\mathrm{F}^i = H^i(-; \boldsymbol{g})$. It is shown in [285] that in the same way that these three δ-functors allow natural transformations, the three families of Bockstein morphisms are compatible. Several examples are given, based (for example) on the arithmetic of elliptic curves and on subspace arrangements.

One result of [285] states that when A is a polynomial ring over $\mathbb{Z}$ containing the ideal I, then the Bockstein on $H^\bullet_I(R/pR)$ is zero except for a finite set of primes p. On a more topological note, the same article investigates the interplay between Bocksteins on local cohomology and those on singular homology in the context of Stanley–Reisner rings. More precisely, let $R = \mathbb{Z}[\boldsymbol{x}]$ be the $\mathbb{Z}^n$-graded polynomial ring on the vertices of the simplicial complex Δ on n vertices, and let $\mathfrak{m} = (\boldsymbol{x})$ be the graded maximal ideal. Hochster linked the multi-graded components of the local cohomology $H^\bullet_\mathfrak{m}(M \otimes_\mathbb{Z} R/I)$ with the singular cohomology with coefficients in M of a certain simplicial subcomplex of Δ determined by the chosen multi-degree, [119]. Then [285] shows that the topological Bocksteins on these links are compatible with the local cohomology Bocksteins via Hochster's identification, and that it behaves well with respect to local duality.

It follows easily from the definitions that the composition of Bocksteins $\beta^{\mathrm{F},i+1}_f \circ \beta^{\mathrm{F},i}_f$ is zero; this is the origin of the Bockstein spectral sequence mentioned above. Its ingredients are the *Bockstein cohomology modules* $\ker(\beta^{\mathrm{F},i+1}_f)/\operatorname{im}(\beta^{\mathrm{F},i}_f)$. In [225], this notion is used to study the extended Rees ring $A[It, t^{-1}]$ of an $\mathfrak{m}$-primary ideal in the local ring $(A, \mathfrak{m})$ as M, using t for f and F is the local cohomology with support in $\mathfrak{m}$. The accomplishment consists in vanishing theorems for local cohomology of the associated graded ring $\mathrm{gr}_I(A)$, extending earlier such results of Narita, and Huckaba–Huneke [109, 208].

4.3.2 Variation of Hodge Structures and GKZ-Systems

Here we give a brief motivation of A-hypergeometric systems and explain how local cohomology of toric varieties enters the picture. We recommend [233, 271, 277] for more detailed information and literature sources.

Notation 4.17 Let $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = A \subseteq \mathbb{Z}^{d\times n}$ satisfy the following properties:

(1) the cone $C_A := \mathbb{R}_{\geqslant 0} A$ spanned by the columns of A inside $\mathbb{R}^d$ is d-dimensional and its lineality (the dimension of the largest real vector space that it contains) is zero;
(2) there exists a $\mathbb{Z}$-linear functional

$$h\colon \mathbb{Z}^d \longrightarrow \mathbb{Z}$$

such that $h(\mathbf{a}_j) = 1$ for $1 \leqslant j \leqslant n$;
(3) the semigroup $\mathbb{N}A := \sum_{j=1}^n \mathbb{N}\mathbf{a}_j$ agrees with the intersection $\mathbb{Z}^d \cap C_A$.

◇

The graded (via h) semigroup ring

$$S_A := \mathbb{C}[\mathbb{N}A]$$

gives rise to a projective toric variety $Y_A \subseteq \mathbb{P}^{n-1}_{\mathbb{C}}$ of dimension $d-1$ and its cone $X_A = \operatorname{Spec}(S_A) \subseteq \mathbb{C}^n$. They can be viewed as (partial) compactifications of the $(d-1)$-torus

$$\mathbb{T} := \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^d, \mathbb{C}^*)}_{=:\tilde{\mathbb{T}}} / \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathbf{a}_0, \mathbb{C}^*),$$

and $\tilde{\mathbb{T}}$ respectively, where $\mathbf{a}_0$ is a suitable element of $\mathbb{Z}^d \cap C_A$ that induces h in the sense that $h(\mathbf{a}_j)$ is the dot product $\langle \mathbf{a}_0, \mathbf{a}_j\rangle$.

A global section $F_{A,\mathfrak{x}} \in \Gamma(Y_A, \mathcal{O}_{Y_A}(1))$ is an element $\sum \mathfrak{x}_j t^{\mathbf{a}_j}$ of the Laurent polynomial ring $\mathbb{C}[t_1^{\pm}, \dots, t_d^{\pm}]$ that is equivariant under the action of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathbf{a}_0, \mathbb{C}^*)$. Its vanishing defines a hypersurface $Z_{\mathfrak{x}}$ inside $\mathbb{T}$ with complement $U_{\mathfrak{x}} = \mathbb{T} \setminus Z_{\mathfrak{x}}$. Batyrev initiated the study of the Hodge theory of these objects in his search for mirror symmetry on toric varieties and their hypersurfaces [24]. As is explained in Stienstra's article [277], for understanding the weight filtration on the cohomology of $Z_{\mathfrak{x}}$ it is useful to study Hodge aspects of the cohomology $H^\bullet(\tilde{\mathbb{T}}, \tilde{Z}_{\mathfrak{x}}; \mathbb{C})$ relative to the affine cone

$$\tilde{Z}_{\mathfrak{x}} := \tilde{\mathbb{T}} \cap \operatorname{Var}(F_{A,\mathfrak{x}} - 1).$$

A powerful tool in this endeavor is the idea of letting the section vary and studying these cohomology groups as a family, viewing the coefficients of the Laurent polynomial as parameters. For this, read the parameter $\mathfrak{x}$ of $F_{A,\mathfrak{x}}$ as a point in $\mathbb{C}^n$. For any face τ of the cone C_A let $F^\tau_{A,\mathfrak{x}}$ be the subsum of $F_{A,\mathfrak{x}}$ of terms with support on τ. Then $\mathfrak{x}$ is *non-degenerate* if the singular locus of $F^\tau_{A,\mathfrak{x}}$ does not meet $\tilde{\mathbb{T}}$ for any τ, including the case $\tau = A$.

For non-degenerate $\mathfrak{x}$, $H^i(\tilde{\mathbb{T}}, \tilde{Z}_{\mathfrak{x}}; \mathbb{C})$ is nonzero only when $i = d$ and there is a natural identification of $H^d(\tilde{\mathbb{T}}, \tilde{Z}_{\mathfrak{x}}; \mathbb{C})$ with the stalk of the solutions of a certain natural D-module that we describe next.

For what is to follow, we assume that A satisfies condition 4.17(1) but not necessarily 4.17(2) and 4.17(3), unless indicated expressly.

Let D_A be the Weyl algebra $\mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ with subring $O_A = \mathbb{C}[x_1, \dots, x_n]$, and let L_A be the $\mathbb{Z}$-kernel of A. Define two types of operators

$$E_i := \sum_{j=1}^{n} a_{i,j} x_j \partial_j \qquad \text{(Euler operators);}$$

$$\Box_{\mathbf{u}} := \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \qquad \text{(box operators).}$$

Here, $1 \leqslant i \leqslant d$ and $\mathbf{u} \in L_A$ with $(\mathbf{u}_+)_i = \max\{\mathbf{u}_i, 0\}$ and $(\mathbf{u}_-)_i = \max\{-\mathbf{u}_i, 0\}$. Then choose a parameter vector $\beta \in \mathbb{C}^d$ and define the *hypergeometric ideal*

$$H_A(\beta) = D_A \cdot \{E_i - \beta_i\}_{i=1}^d + D_A \cdot \{\Box_{\mathbf{u}}\}_{\mathbf{u} \in L_A}$$

and the *hypergeometric module*

$$M_A(\beta) := D_A / H_A(\beta)$$

to A and β. These modules were defined by Gelfand, Graev, Kapranov and Zelevinsky in a string of articles including [83, 93] during their investigations of Aomoto type integrals. The modules are always holonomic [2], and they are regular holonomic if and only if A satisfies Condition 4.17.(2), [282]. We refer to [271] for extensive background on hypergeometric functions, their associated differential equations, and how they relate to hypergeometric modules $M_A(\beta)$ via a dehomogenization technique investigated in [45]. The article [233] is a gentle introduction to hypergeometric D-modules, combined with a survey on recent applications to Hodge theory.

Let $R_A = \mathbb{C}[\partial_1, \dots, \partial_n]$; while this is a subring of operators of D_A, one can also view it as a polynomial ring in its own right. The ideal

$$I_A := R_A \cdot \{\Box_{\mathbf{u}}\}_{\mathbf{u} \in L_A}$$

that forms part of the defining equations for $H_A(\beta)$ is called the *toric ideal*; its variety in $\hat{\mathbb{C}}^n = \operatorname{Spec} R_A$ is the toric variety X_A. We use here the "hat" to distinguish the copy of complex n-space that arises as $\operatorname{Spec} R_A$ from that which arises as $\operatorname{Spec} O_A$. The two are domain and target of the Fourier–Laplace transform FL(−) which, on elements of D_A, amounts to $x_j \mapsto \partial_j$, $\partial_j \mapsto -x_j$.

Local cohomology arises in two ways in the study of $M_A(\beta)$: in connection with the dimension of the space of solutions, and in the limitations of a functorial description of $M_A(\beta)$ via a D-module theoretic pushforward.

For any holonomic D_A-module M there is a Zariski open set of $\mathbb{C}^n$ on which M is a connection; we call the rank of this connection the *rank* of M. For $M_A(\beta)$, this open set is determined by the non-vanishing of the *A-discriminant*, a generalization of the discriminant of a polynomial. In particular, it does not depend on β; we denote it U_A. If A satisfies Condition 4.17.(3) then the connection on U_A has rank equal to the simplicial volume in $\mathbb{R}^d$ of the convex hull of the origin and the columns of A, [2, 83, 93]. Indeed, the hypothesis implies that the semigroup ring S_A is Cohen–Macaulay by Hochster's theorem [118], and this allows a certain spectral sequence to degenerate, which determines the rank. In fact, one can even produce the solutions often in explicit forms, by writing down suitable hypergeometric series and proving convergence [83, 271].

In the absence of Condition 4.17.(3), the situation can be more interesting since then there may be choices of β with the effect of changing the rank [194]. That the possibility of changing rank exists at all was discovered in [272]. A certain Koszul-like complex based on the Euler operators E_i that appeared in [194] can be used to substitute for the (now not degenerating) spectral sequence.

A natural question is which parameters β will show a change in rank. Because of basic principles, the rank at special β can only go up [194]. Since S_A is A-graded via $\deg_A(\partial_j) = \mathbf{a}_j \in \mathbb{Z}A$, so are its local cohomology modules $H^i_{\boldsymbol{\partial}}(S_A)$ supported at the homogeneous maximal ideal. Set

$$\mathcal{E}_A := \overline{\bigcup_{i=0}^{d-1} \deg_A(H^i_{\boldsymbol{\partial}}(S_A))}^{Zariski},$$

the Zariski closure of the union of all A-degrees of nonzero elements in a local cohomology module with $i < d$. Note that the union of these degrees can be seen as witnesses to the failure of S_A being Cohen–Macaulay: the union is empty if and only if S_A has full depth. In generalization of the implication of equal rank for all β in the Cohen–Macaulay case, it is shown in [194] that

$$[\mathrm{rk}(M_A(\beta) > \mathrm{vol}(A)] \Leftrightarrow [\beta \in \mathcal{E}_A].$$

Consider now the monomial map

$$\begin{aligned} \varphi = \varphi_A \colon \tilde{\mathbb{T}} &\longrightarrow \hat{\mathbb{C}}^n, \\ \mathfrak{t} &\mapsto (\mathfrak{t}^{\mathbf{a}_1}, \dots, \mathfrak{t}^{\mathbf{a}_n}) \end{aligned}$$

induced by A. The map is an isomorphism onto the image by Condition 4.17.(1), and its closure is the toric variety X_A. On $\tilde{\mathbb{T}}$ one has for each β the (regular) connection $\mathcal{L}_\beta = D_{\mathbb{T}}/D_{\mathbb{T}} \cdot \{t_i \partial_{t_i} + \beta_i\}_1^d$. In [84], Gelfand, Kapranov and Zelevinsky proved that if β is sufficiently generic then the Fourier–Laplace transform $\mathrm{FL}(M_A(\beta))$ agrees with the $\mathcal{D}$-module direct image $\varphi_+(\mathcal{L}_\beta)$, where the set of "good" β forms the complement of a countably infinite and locally finite hyperplane arrangement called the *resonant* parameters, and given by all L_A-shifts of the bounding hyperplanes of the cone C_A. In [284] this result was refined and completed to an equivalence

$$[M_A(\beta) = \varphi_+(\mathcal{L}_\beta)] \Leftrightarrow [\beta \text{ is not strongly resonant}].$$

Here, following [284], $\beta \in \mathbb{C}^d$ is *strongly resonant* if and only if there is a finitely generated R_A-submodule of $\bigoplus_{j=1}^n H^1_{\partial_j}(S_A)$ containing β in the Zariski closure of its A-degrees. (Since the local cohomology modules here are not coherent, being strongly resonant is more special than being in the Zariski closure of the A-degrees of the direct sum). Some further improvements have been made in [275, 276].

Remark 4.18 As it turns out, when Conditions 4.17 are in force in full strength, then certain $M_A(\beta)$, including the case $\beta = \mathbf{0}$, are not just a regular D_A-module but in fact carry a mixed Hodge module structure in the sense of Saito, [241]. The Hodge and weight filtrations of hypergeometric systems have been studied in [228, 232, 235], showing connections to intersection homology of toric varieties. See [233] for a survey. ◇

4.3.3 Milnor Fibers and Torsion in the Jacobian Ring

Let f be a non-unit in $R = \mathbb{C}[x_1, \ldots, x_n]$ and put $X := \mathbb{C}^n = \operatorname{Spec}(R)$. By the ideal J_f we mean the ideal generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$; this ideal varies with the choice of coordinate system in which we calculate. In contrast, the Jacobian ideal $\operatorname{Jac}(f) = J_f + (f)$ is independent.

If $\mathfrak{x} \in \operatorname{Var}(f)$, let $B(\mathfrak{x}, \varepsilon)$ denote the ε-ball around $\mathfrak{x} \in \operatorname{Var}(f) \subseteq \mathbb{C}^n$. Milnor [192] proved that the diffeomorphism type of the open real manifold

$$M_{f,\mathfrak{x},t,\varepsilon} = B(\mathfrak{x}, \varepsilon) \cap \operatorname{Var}(f - t)$$

is independent of ε, t as long as $0 < |t| \ll \varepsilon \ll 1$. Abusing language, for $0 < t \ll \varepsilon \ll 1$ denote by $M_{f,\mathfrak{x}}$ the fiber of the bundle

$$B(\mathfrak{x}, \varepsilon) \cap \{\mathfrak{y} \in \mathbb{C}^n \mid 0 < |f(\mathfrak{y})| < t\} \longrightarrow f(\mathfrak{y}).$$

If f has an isolated singularity at $\mathfrak{x}$ then the Milnor fiber $M_{f,\mathfrak{x}}$ is a bouquet of $(n-1)$-spheres, and $H^{n-1}(M_{f,\mathfrak{x}}; \mathbb{C})$ can be identified non-canonically with the Jacobian ring $R/\operatorname{Jac}(f)$ as vector space; in particular, the Jacobian ring "knows" the number of spheres in the bouquet.

We call f *quasi-homogeneous* under the weight $(w_1, \ldots, w_n) \in \mathbb{Q}^n$ if $\sum_{i=1}^n w_i \frac{\partial}{\partial x_i}(f) = f$. In this case, the Jacobian ring acquires a $\mathbb{Q}[s]$-module structure where s acts via the Euler homogeneity, compare [188]. This is actually true for general isolated singularities, not just in the presence of homogeneity, and the eigenvalues of the action of s turn out to be the non-trivial roots of the local Bernstein–Sato polynomial of f at $\mathfrak{x}$. The s-action comes then from the

Gauß–Manin connection. Compare also [143, 189, 242, 243, 312]. Compare [274] for details on the Hodge structure on the cohomology of the Milnor fiber.

For non-isolated singularities, most of this must break down, since $R/\operatorname{Jac}(f)$ is not Artinian in that case. Suppose from now on that f is homogeneous, and that $\mathfrak{x}$ is the origin. Note that now $\operatorname{Jac}(f) = J_f$; we abbreviate $M_{f,\mathfrak{x}}$ to $M_f = \operatorname{Var}(f-1)$. The *Jacobian module*

$$H^0_{\mathfrak{m}}(R/J_f) = \{g + J_f \mid \exists k \in \mathbb{N}, \forall i, x_i^k g \in J_f\}$$

has been studied in [217, 305] for various symmetry properties and connections with geometry. Note that this finite length module agrees with the Jacobian ring in the case of an isolated singularity, it can hence be considered a generalization of it in more general settings.

If

$$\eta = \sum_i x_i \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$$

denotes the canonical $(n-1)$-form on X, then (via residues) every class in $H^{n-1}(M_f;\mathbb{C})$ is of the form $g\eta$ for suitable $g \in R$, and if $g \in R$ is the smallest degree homogeneous polynomial such that $g\eta$ represents a chosen class in $H^{n-1}(M_f;\mathbb{C})$ then $-\deg(g\eta)/\deg(f)$ is a root of the Bernstein–Sato polynomial of f, [311]. Suppose the singular locus of f is (at most) 1-dimensional. Then by [244, 245, 313], with $1 \leqslant k \leqslant d$ and $\lambda = \exp(2\pi\sqrt{-1}k/d)$, the following holds:

$$\dim_{\mathbb{C}}[H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))]_{d-n+k} \leqslant \dim_{\mathbb{C}} \operatorname{gr}^{\mathrm{Hodge}}_{n-2}(H^{n-1}(M_f;\mathbb{C})_\lambda),$$

where the right hand side indicates the λ-eigenspace of the associated graded object to the Hodge filtration on $H^{n-1}(M_f;\mathbb{C})$. Dimca and Sticlaru have used this inequality to study nearly free divisors and pole order filtrations, [61, 62]. It would be interesting to find more general inequalities of this type. The above estimate is based on local cohomology of logarithmic forms introduced in [240]; such modules have been calculated in [64] for generic hyperplane arrangements. See [312] for more connections to monodromy and zeta-functions.

4.4 *Lyubeznik Numbers*

Let $(R, \mathfrak{m}, \Bbbk)$ be a commutative regular local Noetherian ring of dimension n that contains its residue field. For any ideal I of R, Lyubeznik proved in [176] that the $\Bbbk$-dimension

$$\lambda_{i,j}(R,I) := \dim_{\Bbbk}(\operatorname{Ext}^i_R(\Bbbk, H^{n-j}_I(R)))$$

is for each $i, j \in \mathbb{N}$ only a function of R/I and so does not depend on the presentation of R/I as a quotient of a regular local ring.

In his seminal paper, Lyubeznik also showed that $\lambda_{i,j}(R, I)$ agrees with the socle dimension in $H^i_{\mathfrak{m}}(H^{n-j}_I(R))$, and hence with the i-th Bass number of $H^{n-j}_I(R)$ with respect to $\mathfrak{m}$. In fact, $H^i_{\mathfrak{m}}(H^{n-j}_I(R))$ is the direct sum of $\lambda_{i,j}(R/I)$ many copies of $E_R(\Bbbk)$, the injective hull of $\Bbbk$ when viewed as R-module.

It follows from the local cohomology interpretation that $\lambda_{i,j}(R, I) = \lambda_{i,j}(\hat{R}, I\hat{R})$ is invariant under completion. By the Cohen structure theorems, every complete local Noetherian ring containing its residue field is the quotient of a complete regular Noetherian local ring containing its residue field. One can thus define for every local Noetherian ring A the (i, j)-*Lyubeznik number*

$$\lambda_{i,j}(A) := \lambda_{i,j}(R, I)$$

via any surjection $R \twoheadrightarrow R/I = \hat{A}$ from a complete regular ring R onto the completion of A.

Notation 4.19 Throughout this subsection, $(R, \mathfrak{m}, \Bbbk)$ is a regular local ring containing its residue field, $\hat{R}$ its completion along $\mathfrak{m}$, and I an ideal of R such that $A = R/I$. Set $d := \dim(A)$. Field extensions $R \rightsquigarrow \mathbb{K} \otimes_{\Bbbk} R$ have no impact on the Lyubeznik numbers, so that one can always assume $\Bbbk$ to be algebraically or separably closed if necessary. Moreover, since $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$, one may assume that A is reduced. ◇

By Grothendieck's vanishing theorem, $\lambda_{i,j}(A)$ is zero if $j < 0$, and by the depth sensitivity of local cohomology, $\lambda_{i,j}(A) = 0$ if $j > \dim(A)$, [139]. By construction, the dimension of the support of $H^{n-j}_I(R)$ is contained in the variety of I, so that $\lambda_{i,j}(A) = 0$ for all $i > d$.

We can thus write $\Lambda(A)$ for the *Lyubeznik table*

$$\Lambda(A) := \begin{pmatrix} \lambda_{0,0}(A) & \dots & \lambda_{0,d}(A) \\ \vdots & & \vdots \\ \lambda_{d,0}(A) & \dots & \lambda_{d,d}(A) \end{pmatrix}$$

It has been shown in [125] in the case $\operatorname{char}(R) > 0$, and then in [176] when $\mathbb{Q} \subseteq \Bbbk$ that the injective dimension of $H^k_I(R)$ is always bounded above by the dimension of its support. However, it is standard that the support of $H^{n-j}_I(R)$ is contained in a variety of dimension at most j. This implies that the nonzero entries of $\Lambda(A)$ are on or above the main diagonal of $\Lambda(A)$.

There is a Grothendieck spectral sequence

$$H^i_{\mathfrak{m}}(H^j_I(R)) \Longrightarrow H^{i+j}_{\mathfrak{m}}(R). \tag{4.4.0.1}$$

It follows directly from this spectral sequence that

- the alternating sum $\sum_{i,j}(-1)^{i+j}\lambda_{i,j}(A)$ equals 1;
- $\lambda_{0,d}(A) = \lambda_{1,d}(A) = 0$ for all A unless $\dim(A) \leqslant 1$;
- if R/I is a complete intersection, then $\lambda_{i,j}(A)$ vanishes unless $i = j = d$. (We say that *the Lyubeznik table is trivial*).
- Moreover, following [15] let $\rho_j(A) := -\delta_{0,j} + \sum_{i=0}^{d-j} \lambda_{i,i+j}(A)$ be the reduced sum along the j-th super-diagonal in $\Lambda(A)$, where δ denotes the Kronecker-δ. Then $\rho_d(A)$ is always zero, and) non-vanishing of $\rho_j(A)$ implies that of either $\rho_{j-1}(A)$ or $\rho_{j+1}(A)$, compare [210].

In characteristic $p > 0$, the (iterated) Frobenius functor sends a free resolution of the ideal I to a free resolution of the Frobenius power $I^{[p^e]}$. As the Frobenius powers of I are cofinal with the usual powers, $H^k_I(R) = 0$ whenever k exceeds the projective dimension of R/I. In particular, if I is perfect (i.e., R/I is Cohen–Macaulay), the Lyubeznik table of R/I is trivial in positive characteristic. In characteristic zero, this is not so; for example, the Lyubeznik table for the (perfect) ideal of the 2×2 minors of a 2×3 matrix of indeterminates over $\Bbbk \supseteq \mathbb{Q}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

as one sees from the fact that I is 3-generated and $H^3_I(R) = E_R(\Bbbk)$, compare Example 2.14, or the computations in [306] (Single dots indicate a zero entry).

Definition 4.20 The *highest Lyubeznik number* of A is $\lambda_{d,d}(A)$. ◇

It follows directly from the spectral sequence that for $d \leqslant 1$, only $\lambda_{d,d}(A)$ is nonzero (and thus equal to 1).

Lyubeznik proved in [176] that $\lambda_{d,d}(A)$ is always positive. For 2-dimensional complete local rings, with separably closed residue field, it was shown in [150, 309] that the Lyubeznik table is independent of the 1-dimensional components of I. Indeed, one has:

$$\Lambda(A) = \begin{pmatrix} 0 & t-1 & 0 \\ \cdot & 0 & 0 \\ \cdot & \cdot & t \end{pmatrix}$$

where t is the number of components of the punctured spectrum of A. In any dimension d, the number $\lambda_{d,d}(A)$ is 1 if A is analytically normal [176] or has Serre's condition S_2 [151]. On the other hand, $\lambda_{d,d}(A)$ can be 1 without A being Cohen–Macaulay or even S_2, [151].

More generally, consider the *Hochster–Huneke graph* of A: the vertices of $\mathrm{HH}(A)$ are the d-dimensional primes of A and an edge links two such primes if the height of their sum is 1. Then Zhang, generalizing the case $d \leqslant 2$ from [309] and the case $\mathrm{char}(\Bbbk) > 0$ from [182], proved (in a characteristic-independent way) in [318] that $\lambda_{d,d}(A)$ agrees with the number of connected components of $\mathrm{HH}(A)$. The main result in [318] has been extended to mixed characteristic in [324]. See also [199, 250] for more on the relationship between connectedness and the structure of local cohomology.

4.4.1 Combinatorial Cases and Topology

If I is a monomial ideal, then Alvarez, Vahidi and Yanagawa [8, 14, 15, 317] have obtained the following results:

- Lyubeznik numbers of monomial ideals relate to linear strands of the minimal free resolution of their Alexander duals;
- If A is sequentially Cohen–Macaulay (i.e., every $\mathrm{Ext}^i_R(A, R)$ is zero or Cohen–Macaulay of dimension i) then both in characteristic $p > 0$ and also if I is monomial then the Lyubeznik table is trivial.
- there are Thom–Sebastiani type results for Lyubeznik tables of monomial ideals in disjoint sets of variables.
- Lyubeznik numbers of Stanley–Reisner rings are topological invariants attached to the underlying simplicial complex.

In a different direction, consider the case when $I_{r,s,t}$ is the ideal generated by the $(t+1) \times (t+1)$ minors of an $r \times s$ matrix of indeterminates over the field $\Bbbk$. In positive characteristic, the Cohen–Macaulayness of R/I implies triviality of the Lyubeznik table. In characteristic zero, however, these numbers carry interesting combinatorial information related to representations of the general linear group. Lörincz and Raicu proved in [165] the following. Write the Lyubeznik numbers into a bivariate generating function

$$L_{r,s,t}(q, w) := \sum_{i,j \geqslant 0} \lambda_{i,j}(A_{r,s,t}) \cdot q^i \cdot w^j$$

with $A_{r,s,t} = \mathbb{C}[\{x_{i,j} | 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}]/I_{r,s,t}$, with $r > s > t$. Then

$$L_{r,s,t}(q, w) = \sum_{i=0}^{t} q^{i^2+i(r-s)} \cdot \binom{s}{i}_{q^2} \cdot w^{t^2+2t+i(r+s-2t-2)} \cdot \binom{s-1-i}{t-i}_{w^2}.$$

Here, the subscripts to the binomial coefficient indicate the Gaussian q-binomial expression $\binom{a}{b}_c = \frac{(1-c^a)(1-c^{a-1})\cdot\ldots\cdot(1-c^{a-b+1})}{(1-c^b)(1-c^{b-1})\cdot\ldots\cdot(1-c)}$.

There is a similar formula for the case $r = s > t$.

We now turn to topological interpretations on Lyubeznik tables. The earliest such results were formulated by García López and Sabbah. Suppose A has an isolated singularity at $\mathfrak{m}$. Then $H_I^{n-j}(R)$ is $\mathfrak{m}$-torsion for $n - j \neq d$. Hence, by the spectral sequence (4.4.0.1), $\Lambda(A)$ is concentrated in the top row and the rightmost column, and there are equalities $\lambda_{0,j} + \delta_{j+1,d} = \lambda_{j+1,d}$, using again Kronecker notation. It is shown in [85] that, if the coefficient field is $\mathbb{C}$, then $\lambda_{0,j}$ equals the $\mathbb{C}$-dimension of the topological local cohomology group of the analytic space Spec(V) with support in the vertex $\mathfrak{m}$.

This result was then generalized by Blickle and Bondu as follows. Suppose (over $\mathbb{C}$) that the constant sheaf on the spectrum of A is self-dual in the sense of Verdier outside the vertex $\mathfrak{m}$. This is the case when $\mathfrak{m}$ is an isolated singularity, but it also occurs in more general cases. For example, on a hypersurface $f = 0$ this condition is equivalent to the Bernstein–Sato polynomial of f having no other integral root but -1, and -1 occurring with multiplicity one, [292]. Blickle and Bondu prove in [25] that in this situation the same interpretation of $\Lambda(A)$ can be made as in the article by García López and Sabbah. In parallel, they also show that if the field has finite characteristic, a corresponding interpretation can be made in terms of local étale cohomology with supports at the vertex.

Lyubeznik numbers also contain information on connectedness of algebraic varieties. For example, as mentioned before, for $\dim(A) = 2$ over a separably closed field, the Lyubeznik table is entirely characterized by the number of connected components of the 2-dimensional part of the punctured spectrum.

Suppose A is equidimensional, with separably closed coefficient field $\Bbbk$. Denote by $\kappa(A)$ the *connectedness dimension of* A, the smallest dimension t of a subvariety Y in Spec(A) whose removal leads to a disconnection. Núñez-Betancourt, Spiroff and Witt discuss in [210] the relationship between the number $\kappa(A)$ and the vanishing of certain Lyubeznik numbers. Their results generalize a consequence of the Second Vanishing Theorem that can be phrased as: $H_I^{n-1}(R) = 0$ if and only if $\kappa(A) \neq 0$. To be precise, they show for an equidimensional ring A:

- $[\kappa(A) \geqslant 1] \Longleftrightarrow [\lambda_{0,1}(A) = 0]$;
- $[\kappa(A) \geqslant 2] \Longleftrightarrow [\lambda_{0,1}(A) = \lambda_{1,2}(A) = 0]$;
- for $i < \dim(A)$, $[\kappa(A) \geqslant i] \Longleftarrow [\lambda_{0,1}(A) = \cdots = \lambda_{i-1,i}(A) = 0]$.

Earlier, Dao and Takagi, inspired by remarks of Varbaro, showed that over any field, Serre's condition S_3 implies that $\lambda_{d-1,d} = 0$, [66], while in increasing generality it was shown in [182, 309, 318] that $[\kappa(A) \geqslant \dim(A) - 1] \Leftrightarrow [\lambda_{d,d}(A) = 1]$. In [239] are some other results on the effect of Serre's conditions (S_t) on $\Lambda(A)$.

4.4.2 Projective Lyubeznik Numbers

Suppose $X = X_{\Bbbk}$ is a projective variety of dimension $d - 1$, with embedding $\iota\colon X \hookrightarrow \mathbb{P} := \mathbb{P}_{\Bbbk}^{n-1}$ via sections of the line bundle $\mathcal{L} = \iota^*(\mathcal{O}_{\mathbb{P}}(1))$. With this embedding comes a global coordinate ring $\Gamma_*(\mathbb{P})$ of $\mathbb{P}$ and a homogeneous ideal defining the cone $C(X)$ over X in the corresponding affine space. Let R be the localization of $\Gamma_*(\mathbb{P})$ at the vertex, and let I be the ideal defining the germ of $C(X)$ in R. A natural question is to ask:

Problem 4.21 To what extent are the Lyubeznik numbers of R/I dependent on the embedding ι? ◇

Certainly, if two such cones (R, I) and (R', I') arise from one another by an automorphism of $\mathbb{P}$, then the attached Lyubeznik tables are equal. It is less clear from the definitions whether two embeddings that produce the same sheaf $\mathcal{L}$ on X, or at least the same element in the Picard group, should give the same Lyubeznik tables. And even more difficult is the question whether ι, ι' should give rise to equal Lyubeznik tables when $\mathcal{L}_\iota \neq \mathcal{L}_{\iota'}$ in the Picard group.

We say that $\Lambda(X)$ *(or just* $\lambda_{i,j}(X)$*) is projective* if each cone derived from a projective embedding of X produces the same Λ-table (or at least the same $\lambda_{i,j}$). Positive known results include the following:

- If $\dim(X) \leqslant 1$ then $\Lambda(X)$ is projective by [309], since then each cone ring is at most 2-dimensional, and connectedness of the punctured d-dimensional spectrum of R/I is equivalent to connectedness of the $(d-1)$-dimensional part of X.
- If X is smooth and $\mathbb{k} = \mathbb{C}$, then each cone has an isolated singularity, so that the Lyubeznik numbers can be expressed in terms of topological local cohomology as in [85]. Switala proves in [287] that these data are actually intrinsic to X, appearing as cokernels of the cup product with the Chern class of the embedding on singular cohomology of X. By independence of Lyubeznik numbers under field extensions, this also works when just $\mathbb{Q} \subseteq \mathbb{k}$.
- Since $\lambda_{0,1}(A) = 0$ is equivalent to $H^{n-1}_I(R) = 0$, which in turn is equivalent to connectedness of the punctured spectrum of A, $\lambda_{0,1}(X)$ is projective.
- Similarly, the simultaneous vanishing of $\lambda_{0,1}, \lambda_{1,2}, \ldots, \lambda_{i-1,i}$ is projective since it measures by [210] the connectedness dimension of the cone, which corresponds to connectedness dimension of X itself.

Consider the module $\mathcal{E}_{i,j}(\iota) := \operatorname{Ext}^{n-i}_R(\operatorname{Ext}^{n-j}_R(R/I, \Omega_R), \Omega_R)$ where Ω_R is the canonical module of R. In [319], Zhang proves that in finite characteristic, the degree zero part of $\mathcal{E}_{i,j}(\iota)$ supports a natural action of Frobenius, whose stable part is independent of ι and has $\mathbb{k}$-dimension $\lambda_{i,j}(R/I)$. In particular, Λ is projective in positive characteristic.

In characteristic zero, after base change to $\mathbb{C}$, the modules $H^i_{\mathfrak{m}}(H^{n-j}_I(R))$ have a natural structure as mixed Hodge modules. This has been exploited in [234] to prove that in this setting, on the level of constructible sheaves via the Riemann–Hilbert correspondence,

$$\lambda_{i,j}(R/I) = \dim_{\mathbb{C}} H^i \tau^! {}^p\mathcal{H}^{-j}(\mathbb{D}\mathbb{Q}_C).$$

Here, $\mathbb{Q}_C$ is the constant sheaf on the cone $C = C(X)$ under any embedding of X, $\mathbb{D}$ is Verdier duality (corresponding to holonomic duality), ${}^p\mathcal{H}$ is taking perverse cohomology (corresponding to usual cohomology for D-modules via the Riemann–Hilbert correspondence) and $\tau^!$ is the exceptional inverse image for constructible sheaves under the embedding τ of the vertex into the cone. One can then recast this as the dimension of the cohomology of a certain related sheaf on the punctured cone, and this cohomology is the middle term in an exact sequence whose other terms are kernels and cokerels of the Chern class of $\mathcal{L}_\iota$ on certain sheaves on X. These sheaves are relatives of, but not always equal to, intersection cohomology of X. This difference is then exploited to construct examples of (reducible) varieties whose Lyubeznik numbers are not projective. In [314], the construction was modified to yield irreducible ones with non-projective Λ-table.

The construction of [234] starts with a variety whose Picard number is greater than one, and from it constructs a suitable X. In [239] it is shown that if the rational Picard group of X is $\mathbb{Q}$ then almost all Lyubeznik numbers of X are projective. In particular, this applies to determinantal varieties so that the Lőrincz–Raicu computation in [165] determines the vast majority of the entries of the Lyubeznik tables for such varieties under all embeddings.

Remark 4.22 A similar set (to Lyubeznik numbers) of invariants is introduced in [47] (but see also [289]). It is shown that if I is an ideal in a polynomial ring over the complex numbers then the Čech-to-de Rham spectral sequence whose abutment is the reduced singular cohomology of the complement of the variety of I has terms on page two that do not depend on the embedding of the variety of I into an affine space, at least when suitably re-indexed. Using algebraic de Rham cohomology, this is actually shown over all fields of characteristic zero. These *Čech–de Rham numbers* are further investigated in [239] from the viewpoint of projectivity since, if I is homogeneous, one can ask to what extent these numbers are defined by the associated projective variety (rather than the affine cone). Reichelt et al. [239] studies their behavior under Veronese maps, and the degeneration of the spectral sequence. $\diamond$

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Which Properties of Stanley–Reisner Rings and Simplicial Complexes are Topological?

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1 Introduction

A (finite abstract) simplicial complex Δ is a subset of the power set 2^Ω for some finite non-empty groundset Ω such that $A \subseteq B \in \Delta$ implies $A \in \Delta$. All simplicial complexes will be non-empty. The simplicial complex $\{\emptyset\}$ is allowed. We call an $F \in \Delta$ a face of Δ and an inclusionwise maximal face a facet of Δ. We also write $\bar{F}$ for the simplicial complex 2^F and $\partial \bar{F}$ for the simplicial complex $\bar{F} \setminus \{F\}$.

Let $\mathbb{K}$ be a field and $S_\Omega = \mathbb{K}[x_\omega : \omega \in \Omega]$ be a polynomial ring over $\mathbb{K}$. For a subset $A \subseteq \Omega$ we write $\mathbf{x}_A$ for $\prod_{\omega\in A} \mathbf{x}_\omega$. The Stanley-Reisner ring or face ring $\mathbb{K}[\Delta]$ of Δ is the quotient S_Ω / I_Δ of S_Ω by the Stanley-Reisner ideal $I_\Delta = (\mathbf{x}_A : A \not\in \Delta, A \subseteq \Omega)$. The set of monomials $\mathbf{x}_N$ for (inclusionwise) minimal non-faces N of Δ is a minimal monomial generating set of Δ.

Relabeling the vertices of Δ preserves the isomorphism type of $\mathbb{K}[\Delta]$. Hence ring theoretic properties and invariants of $\mathbb{K}[\Delta]$ are determined by the combinatorics of Δ and by $\mathbb{K}$. In this survey we will focus on properties and invariants of $\mathbb{K}[\Delta]$ and Δ determined by the topology of the geometric realization of Δ (and the field $\mathbb{K}$).

Basic algebraic topology (see e.g. [9]) teaches us that every simplicial complex comes with a topological space which is called its geometric realization. Recall, that for the definition one chooses points $p_\omega \in \mathbb{R}^d$ for some d, such that for $F \in \Delta$ the p_ω, $\omega \in F$, are affinely independent and $\mathrm{conv}(F) \cap \mathrm{conv}(F') = \mathrm{conv}(F \cap F')$ for $F, F' \in \Delta$. Here for $F \in \Delta$ we denote by $\mathrm{conv}(F)$ the geometric $(\#F - 1)$-simplex which is the set of all convex combinations $\sum_{\omega\in F} \lambda_\omega p_\omega$ for $\lambda_\omega \geq 0$, $\omega \in F$, and $\sum_{\omega\in F} \lambda_\omega = 1$. Then $|\Delta| = \bigcup_{F\in\Delta} \mathrm{conv}(F)$ considered as a subspace of $\mathbb{R}^d$ is a geometric realization of Δ. From algebraic topology we know that all geometric

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realizations are homeomorphic. Given a geometric realization $|\Delta|$ of Δ we write $|\bar{F}|$ for the subspace $\mathrm{conv}(F)$ of $|\Delta|$.

Clearly, the combinatorics of two simplicial complexes with homeomorphic geometric realization can be quite different. Nevertheless, there are surprising results demonstrating that not few properties of Δ or ring theoretic invariants and properties of $\mathbb{K}[\Delta]$ depend only on $\mathbb{K}$ and the homeomorphism type of $|\Delta|$. These are usually called topological invariants or topological properties of Δ or $\mathbb{K}[\Delta]$.

In this article we survey properties and invariants that are topological and give counterexamples for some others. We do not claim completeness but we do our best to at least mention as many related results as possible. We also try to give an overview of the methods from topological combinatorics used in the proofs. For that reason we for example provide two proofs of Munkres' result on the topological invariance of depth Theorem 3.4. As far we know historically this result is the first result on topological invariance. We assume the reader to be familiar with basic algebraic topology (see e.g., [9]) and some methods from topological combinatorics (see e.g., [2] or [20]). When proofs use heavy machinery from commutative algebra we will confine ourselves to a brief outline of the proof and references. For definitions and facts from commutative algebra used but not defined in the paper we refer the reader to [7].

2 Dimension

In this section we study the Krull dimension of $\mathbb{K}[\Delta]$. For that we need to consider $\mathbb{K}[\Delta]$ as a standard graded $\mathbb{K}$-algebra. As a $\mathbb{K}$-vectorspace we have $\mathbb{K}[\Delta] = \bigoplus_{r=0}^{\infty} A_r$ where A_r is the $\mathbb{K}$-vectorspace of cosets $m + I_\Delta$ of monomials m in S_Ω of degree r. Now by $A_0 = \mathbb{K}$, $A_r A_s \subseteq A_{r+s}$ and the fact that $\mathbb{K}[\Delta]$ is generated by A_1 as a $\mathbb{K}$-algebra it follows, that $\mathbb{K}[\Delta]$ is a standard graded algebra.

Before we can demonstrate that the Krull dimension is a topological invariant we need to introduce some combinatorial invariants of simplicial complexes and relate them to the dimensions of the vectorspaces A_r, $r \geq 0$.

Recall that the dimension of a face F of Δ is given by $\dim(\Delta) = \#F - 1$. We write $\dim(\Delta) = \max_{F\in\Delta} \dim(F)$ for the dimension of Δ and set $f_i = \#\{F \in \Delta : \dim(F) = i\}$ for all $i \geq -1$. The f-vector of Δ is the vector $\mathfrak{f}^\Delta = (f_{-1}, \ldots, f_{\dim(\Delta)})$ whose entries are the non-zero f_i.

We now show how the f-vector of a simplicial complex determines the Hilbert-series of $\mathbb{K}[\Delta]$. Recall that the Hilbert-series of $\mathbb{K}[\Delta]$ is $\mathrm{Hilb}(\mathbb{K}[\Delta]) = \sum_{r=0}^{\infty} \dim_{\mathbb{K}}(A_r) t^r$, where $\dim_{\mathbb{K}}(A_r)$ denotes the $\mathbb{K}$-Vectorspace dimension of A_r. It is well known (see [7, Exercise 10.11]) that the Hilbert-series of any standard graded $\mathbb{K}$-algebra is a rational function of the form $\frac{h(t)}{(1-t)^d}$ where $d = \dim(\mathbb{K}[\Delta])$ is the Krull-dimension of $\mathbb{K}[\Delta]$ and $h(t)$ a polynomial with $h(1) \neq 0$.

Theorem 2.1 *Let* Δ *be a simplicial complex with* f*-vector* $\mathfrak{f} = (f_{-1}, \dots, f_{\dim(\Delta)})$ *then*

$$\mathrm{Hilb}(\mathbb{K}[\Delta]) = \frac{\sum_{i=0}^{\dim(\Delta)+1} t^i \, (1-t)^{\dim(\Delta)+1-i} \, f_{i-1}}{(1-t)^{\dim(\Delta)+1}}.$$

In particular, $\dim(\mathbb{K}[\Delta]) = \dim(\Delta) + 1$.

Proof Since I_Δ is an ideal generated by monomials, it follows that a polynomial from $\mathbb{K}[\Delta]$ lies in I_Δ if and only if each monomial appearing with non-zero coefficient in the polynomial lies in I_Δ. Thus the cosets $m + I_\Delta$ of the degree r monomials $m \notin I_\Delta$ form a basis of A_r. Now $m + I_\Delta = I_\Delta$ if and only if m is divisible by $\mathbf{x}_N$ for a minimal non-face N. Thus $m + I_\Delta \neq I_\Delta$ if and only if the support $\mathrm{supp}(m) = \{\omega : x_\omega \text{ divides } m\}$ of m lies in Δ. If $i \geq 0$ then for each i-dimensional face $F \in \Delta$ there are $\binom{r-1}{i}$ monomials of degree $r - (i+1)$ in the variables x_ω, $\omega \in F$. If $i = -1$ the unique (-1)-dimensional face $\emptyset$ of Δ corresponds to monomials with empty support and hence contributes only the unique basis element of A_0. It follows that for $r \geq 0$

$$\dim_{\mathbb{K}}(A_r) = \sum_{i=0}^{r-1} \binom{r-1}{i} f_i = \sum_{i=0}^{\infty} \binom{r-1}{i} f_i$$

for arbitrary choices of f_i when $i > \dim(\Delta)$. It follows that

$$\begin{aligned}
\mathrm{Hilb}(\mathbb{K}[\Delta]) &= f_{-1} + \sum_{r=1}^{\infty} \Big(\sum_{i=0}^{\infty} \binom{r-1}{i} f_i \Big) t^i \\
&= f_{-1} + \sum_{i=0}^{\infty} \Big(\sum_{r=1}^{\infty} \binom{r-1}{i} t^r \Big) f_i \\
&= f_{-1} + \sum_{i=0}^{\infty} \frac{t^{i+1}}{(1-t)^{i+1}} f_i \\
&= \frac{\sum_{i=0}^{\dim(\Delta)+1} t^i \, (1-t)^{\dim(\Delta)+1-i} \, f_{i-1}}{(1-t)^{\dim(\Delta)+1}}.
\end{aligned}$$

In the representation of the Hilbert-series as a rational function the numerator polynomial evaluates to $f_{\dim(\Delta)} \neq 0$ at $t = 1$. Thus the Krull dimension of $\mathbb{K}[\Delta]$ is given by the power of $(1-t)$ in the denominator and hence is $\dim(\Delta) + 1$. □

In particular, we see that proving the topological invariance of the Krull dimension of $\mathbb{K}[\Delta]$ and the dimension of Δ is equivalent. Before we deduce the topological invariance of both dimensions, we prove the following lemma. It will serve as the key argument in the proof of the invariance, which could also be deduced by much simpler means. But the lemma will prove to be useful later in more complicated situations. We will use the following notation. We write $\mathrm{link}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}$ for the link of F in Δ and $\mathrm{star}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta\}$ for the (closed) star of F in Δ. For two simplicial complexes Δ and Δ' on disjoint ground sets we denote by $\Delta * \Delta' = \{F \cup F' : F \in \Delta, F' \in \Delta'\}$ the join of Δ and Δ'. Using the textbook definition (see [9, p.9]) of the join operation, we have that the join of the topological spaces $|\Delta| * |\Delta'|$ and $|\Delta * \Delta'|$ are homeomorphic if $\Delta, \Delta' \neq \{\emptyset\}$. In case we (for example) have $\Delta = \{\emptyset\}$ the textbook definition implies $|\Delta| * |\Delta'| = \emptyset$ and $|\Delta * \Delta'| = |\Delta'|$. In order to avoid case distinctions we set $|\Delta| * |\Delta'| = |\Delta'|$ in this case and proceed analogously in case $\Delta' = \{\emptyset\}$. Note that $\mathrm{star}_\Delta(F) = \bar{F} * \mathrm{link}_\Delta(F)$ and hence $|\mathrm{star}_\Delta(F)| = |\bar{F}| * |\mathrm{link}_\Delta(F)|$. For a face F of Δ we write $\Delta \setminus F$ for the simplicial complex $\{G \in \Delta : F \not\subseteq G\}$ and for a point x in $|\Delta|$ we write $|\Delta| - x$ for $|\Delta| \setminus \{x\}$. For a simplicial complex Δ we write $\widetilde{H}_i(\Delta, \mathbb{K})$ for the ith reduced simplicial homology groups of Δ with coefficients in $\mathbb{K}$ and for a space X we write $\widetilde{H}_i(X, \mathbb{K})$ for the ith reduced singular homology group of X with coefficients in $\mathbb{K}$. Of course it is well known that $\widetilde{H}_i(\Delta, \mathbb{K}) = \widetilde{H}_i(|\Delta|, \mathbb{K})$. For two simplicial complexes $\Gamma \subseteq \Delta$ we write $H_i(\Delta, \Gamma, \mathbb{K})$ for the simplicial homology of the pair (Δ, Γ) with coefficients in $\mathbb{K}$ and $H_i(X, A, \mathbb{K})$ for the singular homology with coefficients in $\mathbb{K}$ of a pair (X, A) of topological spaces.

Lemma 2.2 *Let Δ be a simplicial complex, F a face of Δ and x a point in the relative interior of $|\bar{F}|$. Then $|\Delta \setminus F|$ is a deformation retract of $|\Delta| - x$ and*

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}). \tag{1}$$

In particular, we have that

$$\dim(\Delta) = \max\Big\{ j \ : \ \text{exists } x \in |\Delta| \text{ such that } H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0 \Big\}. \tag{2}$$

Proof Assume our geometric realization is given by points $p_\omega \in \mathbb{R}^d$, $\omega \in \Omega$. Since x is from the relative interior of $|\bar{F}|$ it follows that $x = \sum_{\omega \in F} \lambda_\omega p_\omega$ with all $\lambda_\omega > 0$ and $\sum_{\omega \in F} p_\omega = 1$. Let $y = \sum_{\omega \in \Omega} \mu_\omega p_\omega \in |\Delta|$ given as a convex combination with $\{\omega : \mu_\omega > 0\} \in \Delta$. From the fact that each barycentric coordinate defines a continuous map on $|\Delta|$ it follows that $f : y \mapsto \min_{\omega \in F} \frac{\mu_\omega}{\lambda_\omega}$ is a continuous map on $|\Delta|$. Clearly $0 \leq f(y) \leq 1$, $f(y) = 1$ if and only if $y = x$ and $f(y) = 0$ if and only if $y \in |\Delta \setminus F|$. Define the map $g : |\Delta| - x \to |\Delta \setminus F|$ as follows. For $y \in |\Delta| - x$ set $g(y) = \frac{1}{1-f(y)}(y - f(y)x)$. One easily checks that $g(y) \in |\Delta \setminus F|$ and $g(y) = y$ for $y \in |\Delta \setminus F|$. Continuity follows from the continuity of f. Now the standard interpolation between f and the identity of $|\Delta|$ shows the claim (see [12, Lemma 2.2] for detailed calculations).

By excising $|\Delta| - |\mathrm{star}_\Delta(F)|$ we get

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = H_j(|\mathrm{star}_\Delta(F)|, |\mathrm{star}_\Delta(F)| - x, \mathbb{K}).$$

Since $\mathrm{star}_\Delta(F)$ is contractible, it is acyclic. Thus by the long exact sequence in reduced homology we get that $H_j(|\mathrm{star}_\Delta(F)|, |\mathrm{star}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{star}_\Delta(F)| - x, \mathbb{K})$. Since $\mathrm{star}_\Delta(F) \setminus F = \partial \bar{F} * \mathrm{link}_\Delta(F)$ we know from the first part that $|\partial \bar{F}| * |\mathrm{link}_\Delta(F)|$ is a deformation retract of $|\mathrm{star}_\Delta(F)| - x$. Thus

$$\widetilde{H}_{j-1}(|\mathrm{star}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\partial \bar{F})| * |\mathrm{link}_\Delta(F)|, \mathbb{K}).$$

Now $\partial \bar{F}$ is the boundary of an $\dim(F)$-simplex and hence a triangulation of an $(\dim(F) - 1)$-sphere. From

$$\begin{aligned}\widetilde{H}_{j-1}(|\partial \bar{F})| * |\mathrm{link}_\Delta(F)|, \mathbb{K}) &= \widetilde{H}_{j-1-(\dim(F)-1+1)}(|\mathrm{link}_\Delta(F)|, \mathbb{K}) \\ &= \widetilde{H}_{j-\dim(F)-1}(|\mathrm{link}_\Delta(F)|, \mathbb{K}) \\ &= \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K})\end{aligned}$$

we now deduce (1).

For (2) consider the following argumentation. Let F be a face of Δ. Pick a point x in the relative interior of $|\bar{F}|$. If F is a facet of dimension $\dim(F) = \dim(\Delta)$. It follows that $\mathrm{link}_\Delta(F) = \{\emptyset\}$. From (2) we deduce $H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$ if and only if $j = \dim(F) = \dim(\Delta)$. For an arbitrary face F of Δ the we deduce from $\dim(\mathrm{link}_\Delta(F)) = \dim(\Delta) - \dim(F) - 1$ that $\widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for $j > \dim(\Delta)$. This implies (2) □

We can now deduce the topological invariance of dimension and Krull dimension.

Theorem 2.3 *Let Δ and Δ' are simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then the Krull dimensions of $\mathbb{K}[\Delta]$ (resp. the dimensions of Δ) and of $\mathbb{K}[\Delta']$ (resp. Δ') coincide.*

Proof By Theorem 2.1 it suffices to argue that for two simplicial complexes Δ and Δ' with homeomorphic geometric realizations we have $\dim(\Delta) = \dim(\Delta')$.

From the facts that $|\Delta|$ and $|\Delta'|$ are homeomorphic and that homeomorphic spaces have isomorphic homology it follows that:

$$\begin{aligned}\dim(\Delta) &\overset{(2)}{=} \max\left\{j \ : \ \text{exists } x \in |\Delta| \text{ such that } H_j(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0\right\} \\ &\overset{|\Delta|\cong|\Delta'|}{=} \max\left\{j \ : \ \text{exists } x \in |\Delta'| \text{ such that } H_j(|\Delta'|, |\Delta'| - x, \mathbb{K}) \neq 0\right\} \\ &= \dim(\Delta')\end{aligned}$$

□

The last property which we study in this section is the purity condition. A simplicial complex Δ is called pure if all facets have the same dimension.

Theorem 2.4 *Let Δ and Δ' simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is pure if and only if Δ' is pure.*

Proof From Lemma 2.2 we know that for a point x from the relative interior of $|\bar{F}|$ for a face F of Δ we have that

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}). \tag{3}$$

On the right hand side there can only be a non-zero contribution if $j \geq \dim(F)$. Moreover, there is a non-trivial contribution for $j = \dim(F)$ if and only if F is a facet. Assume Δ is pure and F is a face of Δ. Then there is a facet G of dimension $\dim(\Delta)$ such that $F \subseteq G$. Thus for any x in the relative interior of $|\bar{F}|$ and every open neighborhood $x \in U \subseteq |\Delta|$ there is a $y \in U$ such that y is in the relative interior of $|\bar{G}|$. In particular, for every x in the relative interior of $|\bar{F}|$ and every open neighborhood U of x in $|\Delta|$ there is a $y \in U$ such that $H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = \mathbb{K}$. Assume Δ' is not pure then there is a face G of dimension $< \dim(\Delta)$. But then for every x form the relative interior of $|\bar{G}|$ there is a small neighbourhood which only contains points y from $|\bar{G}|$. For them $H_{\dim(\Delta)}(|\Delta|, |\Delta| - y, \mathbb{K}) = \widetilde{H}_{\dim(\Delta)-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ as $\mathrm{link}_\Delta(G) = \{\emptyset\}$. □

3 Minimal Free Resolution and Depth

In this section we review results from [13] which show the topological invariance of the depth of $\mathbb{K}[\Delta]$ using a formula by Hochster for the Betti-number of its free resolution. Recall that the depth $\mathrm{depth}(\mathbb{K}[\Delta])$ is the maximal number d of elements $f_1, \ldots, f_d \in \mathbb{K}[\Delta]$ of positive degree such that f_i is a non-zerodivisor on $\mathbb{K}[\Delta]/(f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, d$ and $\mathbb{K}[\Delta]/(f_1, \ldots, f_d) \neq 0$ (see [7, p.424ff]). We follow Munkres' approach and study this invariant through its relation to minimal free resolutions. In the next paragraphs we review some basic material on minimal free resolutions. In particular, we will easily see that the minimal free resolution as a whole is far from being a topological invariant of $\mathbb{K}[\Delta]$.

A free resolution of $\mathbb{K}[\Delta]$ over S_Ω is an exact sequence:

$$\cdots \xrightarrow{\partial_{i+1}} S_\Omega^{b_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} S_\Omega^{b_1} \xrightarrow{\partial_1} S_\Omega^{b_0} \xrightarrow{\partial_0} \mathbb{K}[\Delta] \to 0$$

where all maps are S_Ω-module homomorphisms. It is well known that there is a free resolution which minimizes all the b_i simultaneously and that this resulution has $b_i = 0$ for $i > |\Omega|$. Such a resolution is unique up to isomorphism and is called the minimal free resolution of $\mathbb{K}[\Delta]$ over S_Ω and the corresponding b_i are called the Betti-numbers of $\mathbb{K}[\Delta]$ as an S_Ω-module. We will write $\beta_i(\mathbb{K}[\Delta])$ or β_i for these b_i.

For our purposes we need a more refined structure of the free resolution. For that we use the multigraded structure of S_Ω which is inherited by $\mathbb{K}[\Delta]$. For a monomial $\prod_{\omega\in\Omega} x_\omega^{\alpha_\omega}$ we call $(\alpha_\omega)_{\omega\in\Omega}$ its multidegree. For $\alpha = (\alpha_\omega)_{\omega\in\Omega} \in \mathbb{N}^\Omega$ we write $\mathbf{x}^\alpha$ for $\prod_{\omega\in\Omega} x_\omega^{\alpha_\omega}$. Then as vectorspaces

$$S_\Omega = \bigoplus_{\alpha\in\mathbb{N}^\Omega} x_\omega^\alpha \mathbb{K}$$

and

$$\mathbb{K}[\Delta] = \bigoplus_{\alpha\in\mathbb{N}^\Omega} A_\alpha$$

where $A_\alpha = 0$ if $\alpha \neq (0)_{\omega\in\Omega}$ and $\mathbf{x}^\alpha \in I_\Delta$ and $\mathbf{x}^\alpha + I_\Delta$ otherwise. We can speak of the scalar multiples of x^α in S_Ω as the α-graded part of S_Ω and of A_α as the α-graded part of $\mathbb{K}[\Delta]$. For $\alpha \in \mathbb{N}^\Omega$ we write $S_\Omega(-\alpha)$ to denote the multigrading on S_Ω where the multiples $\mathbf{x}^{\alpha'}$ form the $\alpha' + \alpha$ graded part. Clearly, $S_\Omega(-\alpha)$ is an $\mathbb{N}^\Omega$-graded S_Ω-module. A multigraded free resolution of $\mathbb{K}[\Delta]$ over S_Ω is an exact sequence:

$$\cdots \xrightarrow{\partial_{i+1}} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\Omega(-\alpha)^{b_{i,\alpha}} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\Omega(-\alpha)^{b_{1,\alpha}}$$
$$\xrightarrow{\partial_1} \bigoplus_{\alpha\in\mathbb{N}^\Omega} S_\omega(-\alpha)^{b_{0,\alpha}} \xrightarrow{\partial_0} \mathbb{K}[\Delta] \to 0$$

where all maps are multigraded S_Ω-module homomorphisms. Again it is well known that there is a free resolution which minimizes all the $b_{i,\alpha}$ simultaneously and which satisfies $b_{i,\alpha} = 0$ for $i > |\Omega|$. This resolution is unique up to multigraded isomorphism and is called the multigraded minimal free resolution of $\mathbb{K}[\Delta]$ over S_Ω and the corresponding $b_{i,\alpha}$ are called the multigraded Betti-numbers of $\mathbb{K}[\Delta]$ as an S_Ω-module. We will write $\beta_{i,\alpha}(\mathbb{K}[\Delta])$ or $\beta_{i,\alpha}$ for these $b_{i,\alpha}$.

It is also well known that $\beta_{i,\alpha} = 0$ unless $\alpha \in \{0,1\}^\Omega$. This for example follows from the fact that the Taylor-resolution (see [7, Exercise 17.11]) is a free resolution with $b_{i,\alpha} = 0$ for any α with an entry ≥ 2. Note that the Taylor-resolution is non-minimal in most cases. We can identify $\alpha \in \{0,1\}^\Omega$ with the set W of all ω with $\alpha_\omega = 1$. We then write $\beta_{i,W}$ for $\beta_{i,\alpha}$ (resp. $\beta_{i,W}(\mathbb{K}[\Delta])$ for $\beta_{i,\alpha}(\mathbb{K}[\Delta])$).

The connection between the structure of the minimal free resolution of $\mathbb{K}[\Delta]$ and the geometry of Δ is provided through the following formula by Hochster. For its formulation we denote for $W \subseteq \Omega$ by $\Delta_W = \{F \in \Delta \ : \ F \subseteq W\}$ the restriction of Δ to W.

Theorem 3.1 (Hochster Formula [10]) *Let Δ be a simplicial complex over ground set Ω and let $W \subseteq \Omega$. Then for $i \geq 0$ the multigraded Betti-number $\beta_{i,W}(\mathbb{K}[\Delta])$ is given as*

$$\beta_{i,W}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \left(\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \right).$$

The following is an immediate corollary.

Corollary 3.2 *Let Δ and Δ' be simplicial complexes over Ω and Ω' respectively. If $|\Delta|$ and $|\Delta'|$ are homotopy equivalent then $\beta_{i+\#\Omega,\Omega}(\mathbb{K}[\Delta]) = \beta_{i+\#\Omega',\Omega'}(\mathbb{K}[\Delta'])$ for all $i \geq 0$.*

Proof By Theorem 3.1 we have

$$\begin{aligned}
\beta_{i+\#\Omega,\Omega}(\mathbb{K}[\Delta]) &= \dim_{\mathbb{K}} \left(\widetilde{H}_{\#\Omega-i-\#\Omega-1}(\Delta_\Omega, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(\Delta_\Omega, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(|\Delta|, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(|\Delta'|, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} \left(\widetilde{H}_{i-1}(\Delta'_{\Omega'}, \mathbb{K}) \right) \\
&= \dim_{\mathbb{K}} (\widetilde{H}_{\#\Omega'-i-\#\Omega'-1}(\Delta'_{\Omega'}, \mathbb{K}) \Big) \\
&= \beta_{i+\#\Omega',\Omega'}(\mathbb{K}[\Delta'])
\end{aligned}$$

□

On the other hand the set of topologies that arise among the restrictions Δ_W for subsets W of the ground set can be very different for simplicial complexes with homeomorphic geometric realization.

For example consider for a simplicial complex Δ over ground set Ω and its barycentric subdivision $\mathrm{sd}(\Delta)$; that is the simplicial complex on group set $\Delta \setminus \{\emptyset\}$ with simplices $\{F_0, \ldots, F_i\}$ being sets of non-empty faces of Δ which if suitable numbered satisfy $F_0 \subset F_1 \subset \cdots \subset F_i$. It is well known that $|\Delta|$ and $|\mathrm{sd}(\Delta)|$ are homeomorphic. Indeed the geometric realizations can be chosen such that $|\Delta| = |\mathrm{sd}(\Delta)|$ by the following construction. Assume the geometric realization $|\Delta| \subseteq \mathbb{R}^d$ has simplices that are convex hulls of points $p_\omega \in \mathbb{R}^d$, $\omega \in \Omega$. For $F \in \Delta \setminus \{\emptyset\}$ set $p_F = \frac{1}{\#F} \sum_{\omega \in F} p_\omega$. Then one can show that for a face $\{F_0, \ldots, F_i\}$ of $\mathrm{sd}(\Delta)$ the p_{F_i}, $i = 0, \ldots, i$ are affinely independent and define a geometric realization $|\mathrm{sd}(\Delta)|$ of $\mathrm{sd}(\Delta)$. When speaking of a simplicial complex and its barycentric subdivision we will assume that the geometric realizations are chosen in that way. In particular, $|\Delta| = |\mathrm{sd}(\Delta)|$.

Let $\Delta = \partial 2^{\{1,\ldots,n\}}$ be the boundary of the $(n-1)$-simplex. For any $W \subseteq \{1,\ldots,n\}$, $W \neq \emptyset, \{1,\ldots,n\}$, we have that Δ_W is a simplex and hence contractible and acyclic. For $\mathrm{sd}(\Delta)$ any restriction to $W = \{F, F'\}$ for $F, F \in \Delta$ such that $F \not\subseteq F'$ and $F' \not\subseteq F$ is a 0-sphere and hence has homology of rank 1 in dimension 0. Similarly, for any face $F \in \Delta \setminus \{\emptyset\}$ and $W = \partial \bar{F}$ we have that Δ_W is a triangulation of a $(\dim(F) - 1)$-sphere and hence has homology of rank 1 concentrated in dimension $\dim(F) - 1$.

Finally, we recall the relation of the depth of $\mathbb{K}[\Delta]$ to its minimal free resolution. The following is the Auslander-Buchsbaum formula (see [7, Theorem 19.9]) in our context. Recall that the projective dimension of $\mathbb{K}[\Delta]$ is the maximal i for which $\beta_i(\mathbb{K}[\Delta]) \neq 0$.

Theorem 3.3 (Auslander–Buchsbaum Formula) *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \#\Omega - \mathrm{pd}(\mathbb{K}[\Delta]).$$

Theorem 3.3 allowed Munkres to use Theorem 3.1 in order to deduce the topological invariance of the depth from the invariance of the difference of the cardinality of the ground set and the projective dimension. For that let us introduce a homological version of depth. The following homological version of depth which is obviously a topological invariant of a simplicial complex Δ over ground set Ω

$$\mathrm{hdepth}(\Delta) = \min_i \left\{ \begin{array}{c} \widetilde{H}_i(|\Delta|, \mathbb{K}) \neq 0 \text{ or} \\ H_i(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0 \text{ for some } x \in |\Delta| \end{array} \right\} + 1.$$

Theorem 3.4 *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{pd}(\mathbb{K}[\Delta]) = \#\Omega - \mathrm{hdepth}(\Delta).$$

In particular, if Δ' is a simplicial complex over ground set Ω' such that $|\Delta|$ and $|\Delta'|$ are homeomorphic then $\#\Omega - \mathrm{pd}(\mathbb{K}[\Delta]) = \#\Omega' - \mathrm{pd}(\mathbb{K}[\Delta'])$.

Clearly, the second part of the theorem is an immediate consequence of the first. We will prove the first part in the next section.

Finally, by Theorem 3.3 the following theorem is equivalent to Theorem 3.4

Theorem 3.5 *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \mathrm{hdepth}(\Delta).$$

In particular, if Δ' is a simplicial complex such that $|\Delta|$ and $|\Delta'|$ are homeomorphic then $\mathrm{depth}(\mathbb{K}[\Delta]) = \mathrm{depth}(\mathbb{K}[\Delta'])$.

We will present independent proofs of the two equivalent theorems Theorems 3.4 and 3.5. The first in Sect. 4 proves Theorem 3.4 and follows the lines of Munkres' proof. For this proof one has to develop tools from topological combinatorics which are of independent interest. In Sect. 5 we prove Theorem 3.5 in a rather straightforward manner but use deep facts about local cohomology.

4 Munkres' Proof of Theorems 3.4 and 3.5

First we define a covering of the barycentric subdivision of a simplicial complex which carries a lot of structural information but which is not covered by most texts on methods in topological combinatorics. For a simplicial complex Δ and a face $F \in \Delta$ we denote by $\mathrm{dblock}_\Delta(F)$ the subcomplex of $\mathrm{sd}(\Delta)$ which consists of all subsets of faces of the form $\{F = F_0 \subset\subseteq \cdots \subset F_i\}$. The simplicial complex $\mathrm{dblock}_\Delta(F)$ is called the dual block to F. By definition, $\mathrm{dblock}_\Delta(F)$ is a subcomplex of $\mathrm{star}_{\mathrm{sd}(\Delta)}(\{F\})$. As we have observed before as a star of a simplicial complex $\mathrm{star}_{\mathrm{sd}(\Delta)}(\{F\}) = \overline{\{F\}} * \mathrm{link}_{\mathrm{sd}(\Delta)}(\{F\})$. The dual block has a similar decomposition as $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$, where $\mathrm{lblock}_\Delta(F)$ consists of all $\{F_1 \subset \cdots \subset F_i\} \in \mathrm{sd}(\Delta)$ for which F is a proper subset of F_1. In particular, as a cone $|\mathrm{dblock}_\Delta(F)|$ is contractible and hence acyclic. We can also decompose $\mathrm{link}_{\mathrm{sd}(\Delta)}(\{F\}) = \mathrm{sd}(\partial \bar{F}) * \mathrm{lblock}_\Delta(F)$. Thus

$$\mathrm{star}_\Delta(F) = \overline{\{F\}} * \mathrm{sd}(\partial \bar{F}) * \mathrm{lblock}_\Delta(F). \tag{1}$$

Thus the pairs $(\mathrm{star}_{\mathrm{sd}(\Delta)}(F), \mathrm{link}_{\mathrm{sd}(\Delta)}(F))$ and $(\mathrm{dblock}_\Delta(F), \mathrm{lblock}_\Delta(F))$ exhibit analogous structural properties. The following lemma, which is an analog of Lemma 2.2, shows that these structural similarities lead to analogous homological behavior.

Lemma 4.1 *Let Δ be a simplicial complex and $F \in \Delta \setminus \{\emptyset\}$ a face of Δ. For an point x in the relative interior of $|\bar{F}|$ we have*

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-\dim(F)-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}).$$

Proof By excising $|\Delta| \setminus |\mathrm{dblock}_\Delta(F)|$ we obtain

$$H_j(|\Delta|, |\Delta| - p, \mathbb{K}) = H_j(|\mathrm{dblock}_\Delta(F)|, |\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}).$$

Since $\mathrm{dblock}_\Delta(F)$ is contractible the long exact sequence in homology shows

$$H_j(|\mathrm{dblock}_\Delta(F)|, |\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{dblock}_\Delta(F)| - x, \mathbb{K}).$$

Using (1) we obtain

$$|\mathrm{dblock}_\Delta(F)| = |\overline{\{F\}}| * |\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|.$$

Since x is taken from the relative interior of $|\bar{F}|$ and $|\bar{F}| = |\mathrm{sd}(\bar{F})| = |\overline{\{F\}}| * |\mathrm{sd}(\partial \bar{F})|$ we can see analogous to the proof of Lemma 2.2 that $|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|$ is a deformation retract of $|\mathrm{dblock}_\Delta(F)| - x = |\{\bar{F}\}| * |\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)| - x$. Thus

$$H_j(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_{j-1}(|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|, \mathbb{K}).$$

From the fact that $|\mathrm{sd}(\partial \bar{F})|$ is a $(\dim(F) - 1)$-sphere we infer

$$\begin{aligned}\widetilde{H}_{j-1}(|\mathrm{sd}(\partial \bar{F})| * |\mathrm{lblock}_\Delta(F)|, \mathbb{K}) &= \widetilde{H}_{j-\dim(F)-1}(|\mathrm{lblock}_\Delta(F)|, \mathbb{K}) \\ &= \widetilde{H}_{j-\dim(F)-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}).\end{aligned}$$

□

Next we study collections of dual blocks. Let Δ be a simplicial complex. For a face $F \in \Delta$ set $m_F = \max_{F \subseteq G \in \Delta} \dim(G)$. It follows from $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$ and (1) that $\dim(\mathrm{dblock}_\Delta(F)) = m_F - \dim(F) \leq \dim(\Delta) - \dim(F)$. We call $m_F - \dim(F)$ also the codimension of F in Δ and set $\mathrm{fcodim}(F) = \mathrm{fdim}(\mathrm{dblock}_\Delta(F)) = \dim(\Delta) - \dim(F)$ which we call the formal codimension of F and the formal dimension of $\mathrm{dblock}_\Delta(F)$.

We collect in $\mathfrak{Db}_\Delta$ all $\mathrm{dblock}_\Delta(F)$ for $F \in \Delta \setminus \{\emptyset\}$. We say that a collection $\mathfrak{C} \subseteq \mathrm{dblock}_\Delta(F)$ is a block-subcomplex if $\mathrm{dblock}_\Delta(F) \in \mathfrak{C}$ and $F \subseteq G \in \Delta$ implies that $\mathrm{dblock}_\Delta(G) \in \mathfrak{C}$. For a block-subcomplex $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ we write $\mathfrak{C}^{\langle k \rangle}$ for the collection of all $\mathrm{dblock}_\Delta(F) \in \mathfrak{C}$ for $F \in \Delta \setminus \{\emptyset\}$ such that $\mathrm{fdim}(\mathrm{dblock}_\Delta(F)) \leq k$ or equivalently $\mathrm{fcodim}(F) \leq k$. Note that $\mathfrak{C}^{\langle k \rangle}$ is also a block-subcomplex. If $\mathfrak{C}$ is a block-subcomplex then we call the set $\mathrm{Face}_{\mathfrak{C}} = \{F \in \Delta \ : \ \mathrm{dblock}_\Delta(F) \in \mathfrak{C}\}$ the face set of $\mathfrak{C}$. Clearly, $\mathfrak{C} = \{\mathrm{dblock}_\Delta(F) \ : \ F \in \mathrm{Face}_{\mathfrak{C}}\}$.

For a block-subcomplex $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ we write $|\mathfrak{C}|$ for

$$|\bigcup_{\mathrm{dblock}_\Delta(F) \in \mathfrak{C}} \mathrm{dblock}_\Delta(F)| \subseteq |\mathrm{sd}(\Delta)| = |\Delta|.$$

Lemma 4.2 *Let Δ be a simplicial complex and $\mathfrak{C} \subseteq \mathfrak{Db}_\Delta$ a block-subcomplex. Then for a number $k \geq 0$ we have*

$$\begin{aligned}&H_i\left(\mathfrak{Db}_\Delta^{\langle k \rangle} \cup \mathfrak{C}, \mathfrak{Db}_\Delta^{\langle k-1 \rangle} \cup \mathfrak{C}, \mathbb{K}\right) \\ &\quad = \bigoplus_{\substack{F \in \Delta \setminus \mathrm{Face}_{\mathfrak{C}} \\ \mathrm{fcodim}(F) = k}} H_i\left(\mathrm{dblock}_\Delta(F), \partial \mathrm{sd}(\bar{F}) * \mathrm{lblock}_\Delta(F), \mathbb{K}\right)\end{aligned}$$

Proof For F, $F' \in \Delta$ we have that $\mathrm{dblock}_\Delta(F) \cap \mathrm{dblock}_\Delta(F') =$ if $F \cup F' \notin \Delta$ and $\mathrm{dblock}_\Delta(F \cup F')$ otherwise. Since $\mathrm{dblock}_\Delta(F) = \overline{\{F\}} * \mathrm{lblock}_\Delta(F)$ it then follows that

$$\left|\mathfrak{Db}_\Delta^{\langle k\rangle} \cup \mathfrak{C}\right| / \left|\mathfrak{Db}_\Delta^{\langle k-1\rangle} \cup \mathfrak{C}\right|$$

is a wedge of the suspensions of $|\mathrm{lblock}_\Delta(F)|$ for F of formal codimension k and such that $F \notin \mathrm{Face}_{\mathfrak{C}}$. Hence

$$H_i\left(\mathfrak{Db}_\Delta^{\langle k\rangle} \cup \mathfrak{C}, \mathfrak{Db}_\Delta^{\langle k-1\rangle} \cup \mathfrak{C}, \mathbb{K}\right) = \bigoplus_{\substack{F \in \Delta \setminus \mathrm{Face}_{\mathfrak{C}} \\ \mathrm{fcodim}(F)=k}} \widetilde{H}_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}).$$

Since $\mathrm{dblock}_\Delta(F)$ is contractible and hence acyclic it follows that

$$\widetilde{H}_{i-1}(\mathrm{lblock}_\Delta(F), \mathbb{K}) = H_i(\mathrm{dblock}_\Delta(F), \mathrm{lblock}_\Delta(F), \mathbb{K}).$$

This completes the proof. □

Consider a subcomplex $\Gamma \subseteq \Delta$ of a simplicial complex Δ such that $\Gamma \neq \{\emptyset\}$. Note that in this situation $\Gamma \setminus \{\emptyset\}$ is a subset of the ground set of $\mathrm{sd}(\Delta)$. Moreover, $\mathrm{sd}(\Gamma)$ is a subcomplex of $\mathrm{sd}(\Delta)$. Now if Γ is a proper subcomplex then $\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}$ is the subcomplex of $\mathrm{sd}(\Delta)$ with simplices $\{F_0 \subset \cdots \subset F_i\}$ such that $F_0, \ldots, F_i \in \Delta \setminus \Gamma$. We write $\mathfrak{Db}_{\Delta\setminus\Gamma}$ for the set of simplicial complexes $\mathrm{dblock}_\Delta(F)_{\Delta\setminus\Gamma}$ for $F \in \Delta \setminus \Gamma$. Clearly, $\mathfrak{Db}_{\Delta\setminus\Gamma}$ is a block-subcomplex of $\mathfrak{Db}_\Delta$.

Lemma 4.3 *Let Δ be a simplicial complex and $\Gamma \subset \Delta$ a proper subcomplex $\neq \{\emptyset\}$. Assume that for some $0 \leq M \leq \dim(\Delta)$ we have that $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ for all $x \in |\Gamma|$ and $0 \leq i < M$. Then*

(i) $H_j(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K}) = 0$ *for* $0 \leq j < M - \dim(\Gamma)$.
(ii) $H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K})$ *is isomorphic to the cokernel of*

$$\begin{array}{c} H_{M-\dim(\Gamma)+1}\left(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)+1\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|\right) \\ \Big\downarrow \partial^* \\ H_{M-\dim(\Gamma)}\left(|\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, |\mathfrak{Db}_\Delta^{\langle \dim(\Delta)-\dim(\Gamma)-1\rangle} \cup \mathfrak{Db}_{\Delta-\Gamma}|\right) \end{array} . \tag{2}$$

Proof Claim 1: For $i \leq j$ we have

$$H_i(|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+1\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle} \cup \mathfrak{Db}_{\Delta\setminus\Gamma}|, \mathbb{K}) = 0 \tag{3}$$

◁ ***Proof of Claim*** By Lemma 4.2 we know that the homology group on the left hand side of (3) decomposes as a direct sum of groups $H_i(\text{dblock}(F), \text{lblock}(F), \mathbb{K}) = \widetilde{H}_{i-1}(\text{lblock}_\Delta(F), \mathbb{K})$ for faces $F \in \Delta$ of formal codimension

$$\text{fcodim}(F) = \dim(\Delta) - \dim(F) = j + \dim(\Delta) - M + 1$$

that are not in Γ. By Lemma 4.1 we have that $H_{i-1}(\text{lblock}_\Delta(F), \mathbb{K}) = H_{i+\dim(F)}(|\Delta|, |\Delta| - x, \mathbb{K})$ for any x in the interior of $|\bar{F}|$. By assumption this group vanishes for $i + \dim(F) < M$. Now

$$i + \dim(F) = i + \dim(\Delta) - (j + \dim(\Delta) - M + 1) = M + (i - j) - 1.$$

Since $M + (i - j) - 1 < M$ for $i \leq j$ the assertion follows. ▷

Claim 2: For $i \leq j$ and $\ell \geq 1$ we have

$$H_i\left(|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle}|, \mathbb{K}\right) = 0. \tag{4}$$

In particular,

$$H_i\left(|\Delta|, |\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M\rangle}|, \mathbb{K}\right) = 0. \tag{5}$$

◁ ***Proof of Claim*** Since $|\mathfrak{Db}_\Delta^{\langle j+\dim(\Delta)-M+\ell\rangle}| = |\Delta|$ for $\ell \geq M - j$ we get (5) as a direct consequence of (4).

We prove (4) by induction on ℓ. For $\ell = 1$ the assertion coincides with Claim 1.

Let $\ell \geq 2$. Set $K = j + \dim(\Delta) - M$ and consider the long exact sequence

$$\begin{array}{ccc} \cdots \longrightarrow H_i\left(|\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}, \mathbb{K}\right)|, \mathbb{K}) & \longrightarrow & H_i\left(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}\right) \\ & & \downarrow \\ \cdots \longleftarrow H_i(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}) & \longleftarrow & H_i\left(|\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}\right) \end{array}$$

of the triple

$$(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K+\ell-1\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|).$$

By induction we can deduce the vanishing all homology groups except for

$$H_i(|\mathfrak{Db}_\Delta^{\langle K+\ell\rangle}|, |\mathfrak{Db}_\Delta^{\langle K\rangle}|, \mathbb{K}).$$

The fact that the sequence is exact then implies also the vanishing of this group. ▷

Claim 3: For $i < \dim(\Delta) - \dim(\Gamma)$ we have

$$\mathfrak{Db}_{\Delta}^{\langle i \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma} = \mathfrak{Db}_{\Delta \setminus \Gamma}.$$

◁ ***Proof of Claim*** For a face F of Γ the formal dimension of $\mathrm{dblock}_{\Delta}(F)$ is at least $\dim(\Delta) - \dim(\Gamma)$. This shows that $|\mathfrak{Db}_{\Delta}^{\langle i \rangle}| \subseteq |\mathfrak{Db}_{\Delta \setminus \Gamma}|$ for $i < \dim(\Delta) - \dim(\Gamma)$ and implies the assertion. ▷

Now we are in position to prove part (i) and (ii) of the lemma.

◁ ***Proof of (i)*** For $i < \dim(\Delta) - \dim(\Gamma)$ we have

$$\begin{aligned} H_i(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta \setminus \Gamma}|, \mathbb{K}) &= H_i(|\mathfrak{Db}_{\Delta}|, |\mathfrak{Db}_{\Delta \setminus \Gamma}|, \mathbb{K}) \\ &\overset{\text{Claim 3}}{=} H_i(|\mathfrak{Db}_{\Delta}|, |\mathfrak{Db}_{\Delta}^{\langle j \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}|, \mathbb{K}) \\ &\overset{\text{Claim 2}}{=} 0. \end{aligned}$$

▷

◁ ***Proof of (ii)*** Let ℓ be such that $|\mathfrak{Db}_{\Delta}^{\langle j + \dim(\Delta) - M + \ell \rangle}| = |\mathfrak{Db}_{\Delta}|$. Setting $i = j = M - \dim(\Gamma)$ in Claim 2 we obtain:

$$H_{M - \dim(\Gamma)}\Big(\Big(|\mathfrak{Db}_{\Delta}|, |\mathfrak{Db}_{\Delta}^{\langle \dim(\Delta) - \dim(\Gamma) \rangle}|, \mathbb{K}\Big) = 0. \tag{6}$$

Setting $i = j = M - \dim(\Gamma) + 1$ in Claim 2 we obtain

$$H_{M - \dim(\Gamma) + 1)}\Big(\Big(|\mathfrak{Db}_{\Delta}|, |\mathfrak{Db}_{\Delta}^{\langle \dim(\Delta) - \dim(\Gamma) + 1 \rangle}|, \mathbb{K}\Big) = 0. \tag{7}$$

Using long exact sequences of triples in rows and columns and (7) to obtain the 0 on the top of the first column and (6) to obtain the 0 at the end of the second row we derive the following commutative diagram with exact rows and columns. In the diagram we write D for $\dim(\Delta)$ and G for $\dim(\Gamma)$.

$$\begin{array}{ccccccc} 0 & & & & & & \\ \uparrow & & & & & & \\ H_{M-G+1}\left(\begin{smallmatrix} |\Delta|, \\ |\mathfrak{Db}_{\Delta}^{\langle D-G \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}| \end{smallmatrix}, \mathbb{K}\right) & \longrightarrow & H_{M-G+1}\left(\begin{smallmatrix} |\mathfrak{Db}_{\Delta}^{\langle D-G \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}|, \\ |\mathfrak{Db}_{\Delta \setminus \Gamma}| \end{smallmatrix}, \mathbb{K}\right) & \longrightarrow & H_{M-G}\left(\begin{smallmatrix} |\mathfrak{Db}_{\Delta}|, \\ |\mathfrak{Db}_{\Delta \setminus \Gamma}| \end{smallmatrix}, \mathbb{K}\right) & \longrightarrow & 0 \\ \uparrow & & \| & & & & \\ H_{M-G+1}\left(\begin{smallmatrix} |\mathfrak{Db}_{\Delta}^{\langle D-G+1 \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}|, \\ |\mathfrak{Db}_{\Delta}^{\langle D-G \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}| \end{smallmatrix}, \mathbb{K}\right) & \longrightarrow & H_{M-G}\left(\begin{smallmatrix} |\mathfrak{Db}_{\Delta}^{\langle D-G \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}|, \\ |\mathfrak{Db}_{\Delta}^{\langle D-G-1 \rangle} \cup \mathfrak{Db}_{\Delta \setminus \Gamma}| \end{smallmatrix}, \mathbb{K}\right) & & & & \end{array}$$

Note that the equality in the second column is a consequence of Claim 3. Since $|\mathrm{sd}(\Delta)| = |\mathfrak{Db}_{\Delta}|$ and $|\mathrm{sd}(\Delta)_{\Gamma - \Delta}| = |\mathfrak{Db}_{\Delta - \Gamma}|$ it suffices to show that by the exactness of the second row it follows that $H_{M-G}\Big(|\mathfrak{Db}_{\Delta}|, |\mathfrak{Db}_{\Delta - \Gamma}|, \mathbb{K}\Big)$ is isomorphic to the image of (2). By the exactness of the diagram above it follows that $H_{M-G}\Big(|\mathfrak{D}\mathfrak{b}_{\Delta}|, |\mathfrak{D}\mathfrak{b}_{\Delta-\Gamma}|, \mathbb{K}\Big)$ is isomorphic to the cokernel of the left map in the second row of the above diagram. From the fact that the diagram is commutative and the exactness of the first column the assertion then follows. ▷ □

Lemma 4.4 *Let Δ be a simplicial complex over ground set Ω such that $\Delta \neq 2^{\Omega}$. Assume further that M is a number such that for all $x \in |\Delta|$ and all $i < M$ we have $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$. Then the following are equivalent:*

(i) *There is an $x \in |\Delta|$ for which $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(ii) *There is a subcomplex $\Gamma \subseteq \Delta$, $\Gamma \neq \{\emptyset\}$ such that for every x in the relative interior of $|\Gamma|$ we have $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(iii) *There is a face $F \neq \emptyset$ of Δ such that for every x in the relative interior of $|\bar{F}|$ we have $H_M(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$.*
(iv) *There is face $F \neq \emptyset$ of Δ such that such that for $\Gamma = \bar{F}$ we have*

$$H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta-\Gamma}|, \mathbb{K}) \neq 0.$$

Proof The implications (iii) ⇔ (ii) ⇒ (i) and (v) ⇒ (iv) are valid for trivial reasons.

First we show (i) ⇒ (iii). By Lemma 4.1 we know that the homology groups $H_M(|\Delta|, |\Delta| - x, \mathbb{K})$ are isomorphic whenever x is chosen from the relative interior of $|\bar{F}|$ for a fixed face F of Δ. This implies the assertion.

Before we show (iv) ⇔ (iii) we analyze

$$H_{M-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta\setminus\Gamma}|, \mathbb{K})$$

more closely in case $\Gamma = \bar{F}$ for a non-empty face F of Δ. By Lemma 4.3(ii) the homology group is isomorphic to the cokernel of the map from (2). By Lemma 4.2 we know that

(A)

$$H_{M-\dim(\Gamma)+1}(|\mathfrak{D}\mathfrak{b}_{\Delta}^{\langle\dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{D}\mathfrak{b}_{\Delta-\Gamma}|,$$
$$|\mathfrak{D}\mathfrak{b}_{\Delta}^{\langle\dim(\Delta)-\dim(\Gamma)-1\rangle} \cup \mathfrak{D}\mathfrak{b}_{\Delta-\Gamma}|, \mathbb{K})$$

is isomorphic to a direct sum of homology groups $H_{M-\dim(\Gamma)}(\mathrm{lblock}_{\Delta}(G), \mathbb{K})$ for $G \in \Gamma$ of formal codimension $\mathrm{fcodim}(G) = \dim(\Delta) - \dim(\Gamma)$ or equivalently of dimension $\dim(\Gamma)$. By $\Gamma = \bar{F}$ only $F \in \Gamma$ satisfies this condition and it follows that the homology group is isomorphic to $H_{M-\dim(\Gamma)}(\mathrm{lblock}_{\Delta}(F), \mathbb{K})$.

(B)

$$H_{M-\dim(\Gamma)}(|\mathfrak{D}\mathfrak{b}_{\Delta}^{\langle\dim(\Delta)-\dim(\Gamma)+1\rangle} \cup \mathfrak{D}\mathfrak{b}_{\Delta\setminus\Gamma}|,$$
$$|\mathfrak{D}\mathfrak{b}_{\Delta}^{\langle\dim(\Delta)-\dim(\Gamma)\rangle} \cup \mathfrak{D}\mathfrak{b}_{\Delta\setminus\Gamma}|, \mathbb{K})$$

is isomorphic to a direct sum of homology groups $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(G), \mathbb{K})$ for $G \in \Gamma$ of formal codimension $\mathrm{fcodim}(G) = \dim(\Delta) - \dim(\Gamma) + 1$ or equivalently of dimension $\dim(\Gamma) - 1$. By $\Gamma = \bar{F}$ it follows that the homology group is isomorphic to the direct sum $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(F \setminus \{\omega\}), \mathbb{K})$ for $\omega \in F$.

Now we can prove (iv) $\Rightarrow$ (iii). By assumption the cokernel of (2) is non-trivial. Thus it follows from (A) that $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(F), \mathbb{K})$ is non-trivial. By Lemma 4.1 the latter is isomorphic to $H_M(|\Delta|, |\Delta| - x, \mathbb{K})$ for any x in the interior of $|\bar{F}|$. This implies (iii)

To prove (iii) $\Rightarrow$ (iv) By Lemma 4.1 each group $H_{M-\dim(\Gamma)-1}(\mathrm{lblock}_\Delta(F \setminus \{\omega\}), \mathbb{K})$ for $\omega \in F$ is isomorphic to $H_{M-1}(|\Delta|, |\Delta| - x, \mathbb{K})$ for x in the relative interior of $|\overline{F \setminus \{\omega\}}|$. By assumption the latter group is trivial. Thus by (A) and (B) the cokernel of (2) is isomorphic to $H_{M-\dim(\Gamma)}(\mathrm{lblock}_\Delta(F), \mathbb{K})$. By Lemma 4.1 the latter is isomorphic to $H_M(|\Delta|, |\Delta| - x, \mathbb{K})$ for x in the relative interior of $|\bar{F}|$. Thus it is non-trivial by the hypothesis of (iii). Now the assertion follows. □

We now show that hdepth is a homological version of depth.

Lemma 4.5 *Let Δ be a simplicial complex over ground set Ω and $T \subseteq \Omega$. Then*

(i) $\widetilde{H}_{j-\#T}(|\Delta| \setminus |\Delta_T|, \mathbb{K}) = 0$ *for* $j < \mathrm{hdepth}(\Delta) - 1$.
(ii) $\widetilde{H}_{j-\#T}(|\Delta| \setminus |\Delta_T|, \mathbb{K}) = 0$ *for* $j \leq \mathrm{hdepth}(\Delta) - 1$ *if* $\Delta_T \neq \bar{T}$.

Proof If $T = \emptyset$ then $\Delta_T = \{\emptyset\} = \bar{\emptyset}$ and $\widetilde{H}_{j-\#T}(|\Delta| \setminus |\Delta_T|, \mathbb{K}) = \widetilde{H}_j(|\Delta|, \mathbb{K})$ which vanishes for $j < \mathrm{hdepth}(\Delta) - 1$ by definition.

If $T = \Omega$ then $\Delta_T = \Delta$ and then $\widetilde{H}_{j-\#T}(|\Delta| \setminus |\Delta_T|, \mathbb{K}) = \widetilde{H}_{j-\#\Omega}(\emptyset, \mathbb{K}) = 0$ for all $j - \#\Omega \neq -1$. Now $\mathrm{hdepth}(\Delta) \leq \#\Omega$ and therefore for $j < \mathrm{hdepth}(\Delta) - 1$ we have $j - \#\Omega < -1$. If $\Delta \neq 2^\Omega = \bar{\Omega}$ then $\mathrm{hdepth}(\Delta) < \#\Omega$ and for $j \leq \mathrm{hdepth}(\Delta) - 1$ we have $j - \#\Omega < -1$.

Let $T \neq \emptyset, \Omega$. Consider the long exact sequence

$$\cdots \to H_{i+1}(|\Delta|, |\Delta| \setminus |\Delta_T|, \mathbb{K}) \to \widetilde{H}_i(|\Delta| \setminus |\Delta_T|, \mathbb{K}) \to \widetilde{H}_i(|\Delta|, \mathbb{K}) \to \cdots .$$

The group on the right hand side vanishes for $i < \mathrm{hdepth}(\Delta) - 1$ by definition. By Lemma 4.3(i) and the definition of $\mathrm{hdepth}(\Delta)$ the group on the left hand side vanishes for $i + 1 < \mathrm{hdepth}(\Delta) - 1 - \dim(\Delta_T)$. Therefore, $\widetilde{H}_i(|\Delta| - |\Delta_T|, \mathbb{K}) = 0$ for $i + 1 < \mathrm{hdepth}(\Delta) - 1 - \dim(\Delta_T)$. Since $\dim(\Delta_T) \leq \#T - 1$ with equality if and only if $\Delta_T = \bar{T}$ the assertions (i) and (ii) follow. □

We are now in position to prove the following proposition which will immediately implies Theorem 3.4.

Proposition 4.6 *Let Δ be a simplicial complex on ground set Ω.*
Then

(i) $\mathrm{hdepth}(\Delta) = \#\Omega - \max_i\{\beta_i(\mathbb{K}[\Delta]) \neq 0\} = \#\Omega - \mathrm{pd}(\mathbb{K}[\Delta])$.
(ii) *Let $\emptyset \neq W \in \Delta$ and assume that $H_{\mathrm{hdepth}(\Delta)-|T|}(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$ for some x in the relative interior of $|\bar{W}|$ then $H_i(|\Delta_W|, \mathbb{K}) = 0$ for $i \neq 0$.*

Proof (i)

Case 1: $\Delta = \bar{F}$ for some $F \subseteq \Omega$ is a full simplex.

Then $\mathbb{K}[\Delta] = \mathbb{K}[x_\omega : \omega \in \Omega \setminus F]$ and by simple homological algebra $\beta_i(\mathbb{K}[\Delta]) = 0$ for $i > \#\Omega - \#F$ and $\beta_{\#\Omega-\#F}(\mathbb{K}[\Delta]) = 1$ for $i = \#\Omega - \#F$. Thus $\mathrm{pd}(\mathbb{K}[\Delta] = \#\Omega - \#F$. Thus we need to show that $\mathrm{hdepth}(\Delta) = \#F$.

If $F = \emptyset$ then $\widetilde{H}_i(|\Delta|, \mathbb{K}) = 0$ for $i > -1$ and $\mathbb{K}$ for $i = -1$. Obviously, there are no x in the relative interior of $|\Delta|$. Thus $\mathrm{hdepth}(\Delta) = (-1) + 1 = 0 = \#F$.

Now assume that $F \neq \emptyset$. Since $\Delta = \bar{F}$ is a full simplex we have $\widetilde{H}_i(|\Delta|, \mathbb{K}) = H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ for all x in the boundary of the simplex and all $i \geq -1$. If x is in the relative interior of $|\Delta|$ then $H_i(|\Delta|, |\Delta| - x, \mathbb{K})$ is 0 for $i < \dim(F)$ and $\mathbb{K}$ for $i = \dim(F)$. Thus $\mathrm{hdepth}(\Delta) = \dim(F) + 1 = \#F$.

Since in both cases $\mathrm{hdepth}(\Delta) = \#F$ the assertion (i) follows.

Case 2: Δ is not a full simplex.

By Hochster's formula Theorem 3.1 we know that

$$\beta_{i,W} = \dim_{\mathbb{K}}\left(\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K})\right) \text{ and therefore}$$

$$\begin{aligned}\mathrm{pd}(\mathbb{K}[\Delta]) &= \max_i\{\beta_i(\mathbb{K}[\Delta]) \neq 0\} \\ &= \max_i\{\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0 \text{ for some } W \subseteq \Omega\}.\end{aligned}$$

Recall that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) = \widetilde{H}_{\#W-i-1}(|\Delta| - |\Delta_{\Omega\setminus W}|, \mathbb{K})$. We apply Lemma 4.5 to $T = \Omega \setminus W$ and deduce that $\widetilde{H}_{j-\#\Omega+\#W}(\Delta_W, \mathbb{K}) = 0$ for $j < \mathrm{hdepth}(\Delta) - 1$. It follows that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) = 0$ for $i > \#\Omega - \mathrm{hdepth}(\Delta)$. Hence we infer $\mathrm{pd}(\mathbb{K}[\Delta]) \leq \#\Omega - \mathrm{hdepth}(\Delta)$.

It remains to show that there is a $W \subseteq \Omega$ such that $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0$ for $i = \#\Omega - \mathrm{hdepth}(\Delta)$.

If $\widetilde{H}_{\mathrm{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) \neq 0$ then for $W = \Omega$ one has $\widetilde{H}_{\mathrm{hdepth}(\Delta)-1}(\Delta_W, \mathbb{K}) \neq 0$. Thus for $i = \#W - \mathrm{hdepth}(\Delta)$ one has $\widetilde{H}_{\#W-i-1}(\Delta_W, \mathbb{K}) \neq 0$ and the assertion follows.

If $\widetilde{H}_{\mathrm{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) = 0$ then there is some $x \in \Delta$ such that $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) \neq 0$ for $i = \mathrm{hdepth}(\Delta) - 1$ and $H_i(|\Delta|, |\Delta| - y, \mathbb{K}) = 0$ for $i < \mathrm{hdepth}(\Delta) - 1$ and any $y \in |\Delta|$. Thus we can apply Lemma 4.4 for $M = \mathrm{hdepth}(\Delta) - 1$. It follows that There is face $T \neq \emptyset$ of Δ such that such that for $\Gamma = \bar{T}$ we have

$$\begin{aligned}&H_{\mathrm{hdepth}(\Delta)-1-\dim(\Gamma)}(|\mathrm{sd}(\Delta)|, |\mathrm{sd}(\Delta)_{\Delta-\Gamma}|, \mathbb{K}) \\ &\quad = H_{\mathrm{hdepth}(\Delta)-\#T}(|\Delta|, |\Delta| \setminus |\Delta_T|, \mathbb{K}) \neq 0.\end{aligned}$$

For $W = \Omega \setminus T$ we obtain that $H_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, |\Delta_W|, \mathbb{K}) \neq 0$. Since $-\#\Omega + \#W \leq 0$ we know from $\widetilde{H}_{\mathrm{hdepth}(\Delta)-1}(\Delta, \mathbb{K}) = 0$ that $H_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) =$

$0 = H_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, \mathbb{K}) = 0$. The long exact sequence

$$\begin{array}{ccccc} \cdots \longrightarrow & \widetilde{H}_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|,\mathbb{K}) & \longrightarrow & H_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|,|\Delta_W|,\mathbb{K}) \\ & & & \downarrow \\ \cdots \longleftarrow & \widetilde{H}_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|,\mathbb{K}) & \longleftarrow & \widetilde{H}_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W-1}(|\Delta_W|,\mathbb{K}) \end{array}$$

now shows that

$$0 \neq H_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W}(|\Delta|, |\Delta_W|, \mathbb{K}) \cong \widetilde{H}_{\mathrm{hdepth}(\Delta)-\#\Omega+\#W-1}(|\Delta_W|, \mathbb{K}).$$

Thus for $i = \#\Omega - \mathrm{hdepth}(\Delta)$ we obtain $\widetilde{H}_{\#W-i-1}(|\Delta_W|, \mathbb{K}) \neq 0$. □

5 Local Cohomology Proof of Theorems 3.4 and 3.5

In this section we use local cohomology (see [4] for definitions and basic properties) to prove Theorem 3.5 and hence Theorem 3.4. This verification is much shorter than the one from Sect. 4 but builds on substantially more deep theory from commutative algebra. Topologically, the simplification comes from the fact that here we can work with links, which are easier to control than the induced subcomplexes used in the previous section.

Let $\mathbb{K}[\Delta] = \bigoplus_{r=0}^{\infty} A_r$ be the vectorspace decomposition of $\mathbb{K}[\Delta]$ as a standard graded algebra as in Sect. 2. We write $\mathfrak{m} = \bigoplus_{r=1}^{\infty} A_r$ for the unique graded maximal ideal of $\mathbb{K}[\Delta]$ and $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ for the ith local cohomology module of $\mathbb{K}[\Delta]$. The local cohomology $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ is itself a graded module and the following formula by Hochster expresses its Hilbert series in homological terms (see e.g. [18, Theorem 4.1]).

Theorem 5.1 (Hochster Formula for Local Cohomology) *Let Δ be a simplicial complex over ground set Ω. Then*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])) = \sum_{F \in \Delta} \dim_{\mathbb{K}} \left(\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_{\Delta}(F), \mathbb{K}) \right) \frac{1}{(t-1)^{\#F}}.$$

Local cohomology is a powerful tool which encodes many invariants of a module. Here the following fact will be important. We formulate this very general fact for Stanley-Reisner rings only (see [4, Chapter 6] for more details).

Theorem 5.2 *Let Δ be a simplicial complex over ground set Ω. Then*

$$\dim(\mathbb{K}[\Delta]) = \max_i H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]) \neq 0$$

and

$$\mathrm{depth}(\mathbb{K}(\Delta)) = \min_i H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]) \neq 0.$$

Using Theorem 5.1 we immediately obtain the following corollary.

Corollary 5.3 *Let* Δ *be a simplicial complex over ground set* Ω. *Then*

$$\dim(\mathbb{K}[\Delta]) = \max_i\{\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0 \textit{ for some } F \in \Delta\}$$

and

$$\mathrm{depth}(\mathbb{K}[\Delta]) = \min_i\{\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0 \textit{ for some } F \in \Delta\}.$$

Now we already in position to prove Theorem 3.5.

Proof or Theorem 3.5 If $F = \emptyset$ then $\mathrm{link}_\Delta(F) = \Delta$ and

$$\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F), \mathbb{K}) = \widetilde{H}_{i-1}(\Delta, \mathbb{K}) = \widetilde{H}_{i-1}(|\Delta|, \mathbb{K}). \tag{1}$$

If $F \neq \emptyset$ and x is a point from the relative interior of $|\bar{F}|$ then by Lemma 2.2 we have:

$$\widetilde{H}_{i-\dim(F)-2}(\mathrm{link}_\Delta(F), \mathbb{K}) = H_{i-1}(|\Delta|, |\Delta| - x, \mathbb{K}). \tag{2}$$

The minimal i for which at least one of homology groups on the right hand side of (1) or (2) is non-zero is exactly $\mathrm{hdepth}(\Delta) - 1$. Thus Theorem 3.5 follows. □

6 Cohen–Macaulay, Gorenstein, Buchsbaum

A ring R is called Cohen–Macaulay if $\dim(R) = \mathrm{depth}(R)$, i.e. its depth equals it Krull dimension. As an immediate consequence of Theorem 2.3 and Theorem 3.5 we obtain the following result by Munkres (see [13, Corollary 3.4]).

Theorem 6.1 (Munkres) *Let* Δ *and* Δ' *be simplicial complexes such that* $|\Delta|$ *and* $|\Delta'|$ *are homeomorphic. Then* $\mathbb{K}[\Delta]$ *is Cohen–Macaulay if and only if* $\mathbb{K}[\Delta']$ *is Cohen–Macaulay.*

As a further consequence of Theorem 3.5 and (1) and (2) we obtain the following criterion for Cohen–Macaulayness by Reisner [14].

Theorem 6.2 (Reisner's Criterion) *Let* Δ *be a simplicial complex over ground set* Ω. *Then* $\mathbb{K}[\Delta]$ *is Cohen–Macaulay if and only if for all* $F \in \Delta$

$$\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0 \textit{ for all } i < \dim(\mathrm{link}_\Delta(F)).$$

In particular, if $\mathbb{K}[\Delta]$ *is Cohen–Macaulay then so is* $\mathbb{K}[\mathrm{link}_\Delta(F)]$ *for all* $F \in \Delta$.

For a Cohen–Macaulay $\mathbb{K}[\Delta]$ the Betti-number $\beta_{\mathrm{pd}(\mathbb{K}[\Delta])}(\mathbb{K}[\Delta])$ is called the Cohen–Macaulay type of $\mathbb{K}[\Delta]$. The Cohen–Macaulay $\mathbb{K}[\Delta]$ of type 1 are called Gorenstein.

Example 6.3 Let $\Omega = \{1, 2, 3, 4\}$ and Δ the simplicial complex over Ω with facets $\{1, 2, 3\}$ and $\{2, 3, 4\}$. Then $\mathbb{K}[\Delta] = S/I_\Delta = S_\Omega/(x_1x_4)$ and

$$0 \to S_\Omega \xrightarrow{(x_1x_4)} S_\Omega \xrightarrow{m\mapsto m+I_\Delta} \mathbb{K}[\Delta] \to 0$$

is the minimal free resolution. In particular, $\mathrm{pd}(\mathbb{K}[\Delta]) = 1$ and $\beta_1(\mathbb{K}[\Delta]) = 1$. Thus by Theorem 3.3 we have $\mathrm{depth}(\mathbb{K}[\Delta]) = \#\Omega - 1 = 3$. Since $\dim(\Delta) = 2$ it follows that $\dim(\mathbb{K}[\Delta]) = 3$. Thus $\mathbb{K}[\Delta]$ is Cohen–Macaulay and of type 1 and therefore $\mathbb{K}[\Delta]$ is Gorenstein.

Now consider Δ' over ground set $\Omega' = \{1, 2, 3, 4, 5\}$ with facets $\{1, 2, 3\}$, $\{2, 3, 4\}$ and $\{1, 2, 5\}$. Then $\mathbb{K}[\Delta] = S_{\Omega'}/I_{\Delta'} = S_{\Omega'}/(x_1x_4, x_3x_5, x_4x_5)$. It can be checked the minimal free resolution is given by

$$0 \to S_{\Omega'}^2 \xrightarrow{\begin{pmatrix} x_3x_4 & -x_1x_4 & 0 \\ 0 & x_4 & -x_3 \end{pmatrix}} S_{\Omega'}^3 \xrightarrow{\begin{pmatrix} x_1x_4 \\ x_3x_5 \\ x_4x_5 \end{pmatrix}} S_{\Omega'} \xrightarrow{m\mapsto m+I_{\Delta'}} \mathbb{K}[\Delta'] \to 0.$$

In particular, $\mathrm{pd}(\mathbb{K}[\Delta']) = 2$ and $\beta_2(\mathbb{K}[\Delta]) = 2$. Thus by Theorem 3.3 we have $\mathrm{depth}(\mathbb{K}[\Delta']) = \#\Omega' - 2 = 3$. Since $\dim(\Delta') = 2$ it follows that $\dim(\mathbb{K}[\Delta']) = 3$. Thus $\mathbb{K}[\Delta]$ is again Cohen–Macaulay but of type 2 and hence not Gorenstein.

Both $|\Delta|$ and $|\Delta'|$ are homeomorphic to a 2-ball. It follows that the Gorenstein property is not topological.

The following will allow us to deduce the topological invariance of a property which is slightly stronger than Gorenstein. A simplicial complex Δ is called Gorenstein* (over $\mathbb{K}$) if $\mathbb{K}[\Delta]$ is Gorenstein and $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \neq 0$. To study the topological invariance of the Gorenstein* property, we need a few more definitions. For a simplicial complex Δ we define its core core(Δ) as the induced subcomplex $\Delta_{\mathrm{core}(\Omega)}$ where core(Ω) is the set of all $\omega \in \Omega$ such that $\mathrm{star}_\Delta(\omega) \neq \Delta$. It follows that $\Delta = 2^{\Omega\setminus\mathrm{core}(\Omega)} * \mathrm{core}(\Delta)$ and $\dim(\Delta) = \dim(\Delta_{\mathrm{core}(\Omega)}) + \#\Omega - \#\mathrm{core}(\Omega)$.

Theorem 6.4 *Let Δ be a simplicial complex over ground set Ω. Then the following are equivalent.*

(i) $\mathbb{K}[\Delta]$ *is Gorenstein.*
(ii) *For all $F \in \mathrm{core}(\Delta)$ we have*

$$\widetilde{H}_i(\mathrm{link}_{\mathrm{core}(\Delta)}(F), \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \\ 0 & \text{if } i < \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \end{cases}.$$

(iii) *For all* $x \in |\mathrm{core}(\Delta)|$ *we have*

$$\widetilde{H}_i(|\mathrm{core}(\Delta)|, \mathbb{K}) = H_i(|\mathrm{core}(\Delta)|, |\mathrm{core}(\Delta)| - x, \mathbb{K})$$
$$= \begin{cases} \mathbb{K} & \text{if } i = \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \\ 0 & \text{if } i < \dim(\mathrm{link}_{\mathrm{core}(\Delta)}(F)) \end{cases}.$$

Proof The equivalence of (ii) and (iii) again follows from Lemma 2.2.

The equivalence of (i) and (ii) is much harder and was originally proved in [16]. A detailed proof of this fact can be found in [5, Section 5.5.]. □

It follows that if Δ is a simplicial complex for which $\mathbb{K}[\Delta]$ is Gorenstein then $\mathbb{K}[\mathrm{core}(\Delta)]$ is Gorenstein as well. Condition (ii) from Theorem 6.4 then implies for $F = \emptyset$ that $\widetilde{H}_{\dim(\mathrm{core}(\Delta))}(\mathrm{core}(\Delta), \mathbb{K}) \neq 0$ and hence $\mathrm{core}(\Delta)$ is Gorenstein*. Thus any simplicial complex Δ for which $\mathbb{K}[\Delta]$ is Gorenstein has a decomposition $\Delta = 2^{\Omega\setminus\mathrm{core}(\Omega)} * \mathrm{core}(\Delta)$ and $\mathrm{core}(\Delta)$ is Gorenstein*.

Corollary 6.5 *Let* Δ *be a simplicial complex over ground set* Ω *and* Δ' *a simplicial complex over ground set* Ω' *such that*

- $\mathrm{core}(\Delta) = \Delta$ *and* $\mathrm{core}(\Delta') = \Delta'$,
- $|\Delta|$ *is homeomorphic to* $|\Delta'|$.

Then $\mathbb{K}[\Delta]$ *is Gorenstein** *if and only if* $\mathbb{K}[\Delta']$ *is Gorenstein**.

Proof The result follows from Theorem 6.4(iii) and the fact that $\mathrm{core}(\Delta) = \Delta$ and $\mathrm{core}(\Delta') = \Delta'$. □

Next we consider the Buchsbaum property of $\mathbb{K}[\Delta]$. We refer the reader to [19] for the general theory. For its definition we need the concept of a weak $\mathbb{K}[\Delta]$-sequence. A sequence $f_1, \ldots, f_r$ of elements from the maximal graded ideal of $\mathbb{K}[\Delta]$ is called a weak $\mathbb{K}[\Delta]$ sequence if $\mathfrak{m}((f_1, \ldots, f_{i-1} : f_i) \subseteq (f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, r$. Now $\mathbb{K}[\Delta]$ is called Buchsbaum if every system of parameters is a weak $\mathbb{K}[\Delta]$-sequence. The following is an analog of Reisner's criterion for Buchsbaum rings proved by Schenzel in [15].

Theorem 6.6 *Let* Δ *be a simplicial complex over ground set* Ω*. Then the following are equivalent.*

(i) $\mathbb{K}[\Delta]$ *is Buchsbaum.*
(ii) *For all* $F \in \Delta$, $F \neq \emptyset$ *we have* $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ *for* $i < \dim(\mathrm{link}_\Delta(F))$.
(iii) *For all* $x \in |\Delta|$ *we have* $H_i(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ *for* $i < \dim(\Delta)$.

The equivalence of (ii) and (iii) is again in immediate consequence of Lemma 2.2. The equivalence of (i) and (ii) is Theorem 3.2 in [15]. Its proof first shows a characterization of Buchsbaum $\mathbb{K}[\Delta]$ as those $\mathbb{K}[\Delta]$ for which the localization at all prime ideals different from the graded maximal ideal is Cohen–Macaulay. Using this characterization the equivalence can be reduced to Reisner's criterion Theorem 6.2.

Since condition (iii) from Theorem 6.6 is obviously a topological property, we obtain the following immediate corollary.

Corollary 6.7 *Let Δ and Δ' be simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then $\mathbb{K}[\Delta]$ is Buchsbaum if and only if $\mathbb{K}[\Delta']$ is Buchsbaum.*

7 *n*-Purity, *n*-Cohen–Macaulay and *n*-Buchsbaum

A simplicial complex Δ over ground set Ω is called n-pure if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\Delta_{\Omega \setminus W}$ is pure and $\dim(\Delta) = \dim(\Delta_{\Omega\setminus W})$. In particular, 1-pure is the usual pure property.

For $n \geq 3$ the n-pure property is not topological.

Example 7.1 Let Δ be the simplicial complex over ground set $\Omega = \{1, \ldots, n+2\}$ for some $n \geq 1$ with facets $\{i, j\}$ for $1 \leq i < j \leq n$. Then Δ is $(n+1)$-pure. The deletion of an vertex set of size $< n+1$ leaves a connected 1-dimensional simplicial complex. Consider $\Delta' = \mathrm{sd}(\Delta)$ on ground set $\Omega' = 2^{\Omega} \setminus \{\emptyset\}$. Clearly, $|\Delta|$ and $|\Delta'|$ are homeomorphic. We set $W = \{\{1\}, \{2\}\}$ and get that $\Delta'_{\Omega'\setminus W}$ is a simplicial complex with two connected components. One component is a connected 1-dimensional simplicial complex and the other the 0-dimensional complex $\overline{\{\{1, 2\}\}}$. In particular, $\Delta'_{\Omega'\setminus W}$ is not pure. Thus Δ' is not $(n+1)$-pure for $n+1 \geq 3 > 2 = \#W$.

Theorem 7.2 *Let Δ be a pure simplicial complex over ground set Ω. Then the following are equivalent:*

(i) *Δ is 2-pure.*
(ii) *If F is a face of Δ such that $H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = 0$ for all x in the relative interior of $|\bar{F}|$ then $\dim(F) \leq \dim(\Delta) - 2$.*
(iii) *For any simplicial complex Δ' such that $|\Delta'|$ and $|\Delta|$ are homeomorphic and for all faces F of Δ' and all x from the relative interior of $|\bar{F}|$ we have $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x, \mathbb{K}) = 0$.*

Proof
(i) $\Rightarrow$ (ii)

Let F be a face of Δ of dimension $\dim(\Delta) - 1$. Since Δ is pure there must be a facet G of dimension $\dim(\Delta)$ containing F. Let ω be the unique vertex in $G \setminus F$. Since $\Delta_{\Omega\setminus\{\omega\}}$ is of the same dimension as Δ it follows that there must be at least a second facet containing F. In particular, writing 0 as $\dim(\Delta) - \dim(F) - 1$ we get

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_0(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0$$

for every x in the relative interior of $|\bar{F}|$.

Let F be a face of Δ of dimensions $\dim(\Delta)$. It follows that

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_{-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = \mathbb{K} \neq 0$$

for any x in the relative interior of $|\bar{F}|$.

These two facts imply (ii).

(ii) $\Rightarrow$ (i)

By assumption, for a face F of dimension $\dim(\Delta) - 1$ we have that

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) = \widetilde{H}_0(\mathrm{link}_\Delta(F), \mathbb{K}) \neq 0.$$

As a consequence there are at least two facets containing F. This implies that for any $\omega \notin F$ there is a facet of dimension $\dim(\Delta)$ containing F in $\Delta_{\Omega\setminus\{\omega\}}$. In particular, $\Delta_{\Omega\setminus\{\omega\}}$ is pure.

(iii) $\Rightarrow$ (ii)

This is obvious.

(ii) $\Rightarrow$ (iii)

Since $|\Delta'|$ is homeomorphic to $|\Delta|$ it follows from Theorem 2.4 that Δ' is pure of the same dimension as Δ. Assume there is a face F of Δ' such that $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x, \mathbb{K}) = 0$ for some x from the relative interior of $|\bar{F}|$ and $\dim(F) \geq \dim(\Delta') - 1$. If $\dim(F) = \dim(\Delta')$ then

$$H_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K}) \overset{Lemma\ 2.2}{=} \widetilde{H}_{-1}(\mathrm{link}_\Delta(F), \mathbb{K}) = \mathbb{K} \neq 0.$$

Thus we have $\dim(F) = \dim(\Delta') - 1$. Note that our assumptions imply that for any x' from the relative interior of $|F|$ we have $H_{\dim(\Delta')}(|\Delta'|, |\Delta'| - x', \mathbb{K}) = 0$. But then (ii) shows that the homeomorphic image of the relative interior of $|\bar{F}|$, which is an open $\dim(F)$-ball, must be covered by the relative interiors of $|\bar{G}|$ for faces G of Δ of dimension $\leq \dim(\Delta) - 2 < \dim(F)$. The latter is impossible in the geometric realization of a simplicial complex. Thus (iii) follows. □

The next corollary immediately follows from the fact that condition (iii) in Theorem 7.2 only depends on the homeomorphism type of the geometric realization.

Corollary 7.3 *Let Δ and Δ' be two pure simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-pure if and only if Δ' is 2-pure.*

A simplicial complex Δ over ground set Ω is called n-Cohen–Macaulay (over $\mathbb{K}$) if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\mathbb{K}[\Delta_{\Omega\setminus W}]$ is Cohen–Macaulay and $\dim(\Delta) = \dim(\Delta_{\Omega\setminus W})$. In particular, 1-Cohen–Macaulay is the usual Cohen–Macaulay property of $\mathbb{K}[\Delta]$.

If Δ and Δ' are the simplicial complexes from Example 7.1 then the arguments in the example show that for $n \geq 2$ we have that $\mathbb{K}[\Delta]$ is $(n+1)$-Cohen–Macaulay but Δ' is not. Thus for $n \geq 3$ the property of being n-Cohen–Macaulay is not topological.

In the thesis of J. Walker [21, Theorem 9.8] it is proved that 2-Cohen–Macaulayness is indeed a topological property. Following the ideas from [12] we will provide a proof of this result below. As a preparation we need to study properties of links.

Lemma 7.4 *If F is a face of Δ such that for any face $F \subseteq G \in \Delta$ we have $\widetilde{H}_i(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_\Delta(G))$ then for any face $G' \in \mathrm{link}_\Delta(F)$ we have that $\widetilde{H}_i(\mathrm{link}_{\mathrm{link}_\Delta(F)}(G'), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_{\mathrm{link}_\Delta(F)}(G'))$.*

In particular, it follows that

(i) *if $\mathbb{K}[\Delta]$ is Cohen–Macaulay then so is $\mathbb{K}[\mathrm{link}_\Delta(F)]$ for every $F \in \Delta$.*
(ii) *if $\mathbb{K}[\Delta]$ is Buchsbaum, then $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen–Macaulay for every $\emptyset \neq F \in \Delta$.*
(iii) *if $\mathbb{K}[\Delta]$ is 2-Cohen–Macaulay then so is $\mathbb{K}[\mathrm{link}_\Delta(F)]$.*

Proof If $G' \in \mathrm{link}_\Delta(F)$ then $G = F \cup G' \in \Delta$. Then

$$\begin{aligned}\mathrm{link}_\Delta(G) &= \{H \subseteq \Omega \ : \ H \cap G = \emptyset \text{ and } H \cup G \in \Delta\} \\ &= \{H \subseteq \Omega \ : \ H \cap G' = \emptyset \text{ and } H \cup G' \in \mathrm{link}_\Delta(F)\} \\ &= \mathrm{link}_{\mathrm{link}_\Delta(F)}(G').\end{aligned}$$

This implies the first assertion of the lemma. The claims (i) about the Cohen–Macaulay and (ii) about the Buchsbaum property follow from Theorem 6.2 and Theorem 6.6. For (iii) we argue as follows. By (i) we already know that $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen–Macaulay for all $F \in \Delta$. Let $\omega \in \Omega$ and set $W = \Omega \setminus \{\omega\}$. If $F \cap W \neq \emptyset$ then there is nothing to show. If $F \cap W = \emptyset$ then

$$\begin{aligned}(\mathrm{link}_\Delta(F))_W &= \{G \subseteq W \ : \ G \cap F = \emptyset \text{ and } G \cup F \in \mathrm{link}_\Delta(F)\} \\ &= \{G \subseteq W \ : \ G \cap F = \emptyset \text{ and } G \cup F \in \mathrm{link}_{\Delta_W}(F)\} \quad = \mathrm{link}_{\Delta_W}(F).\end{aligned}$$

Now the facts that Δ_W is Cohen–Macaulay and $\dim(\Delta_W) = \dim(\Delta)$ imply the claim. □

As a last prerequisite for a topological characterization of 2-Cohen–Macaulayness we need the following simple fact about chain complexes.

Lemma 7.5 *Let Δ be a simplicial complex and $H \subseteq K$ faces of Δ. Then there is a commutative diagram*

$$\begin{array}{ccc}
\widetilde{H}_i(\mathrm{link}_\Delta(H),\mathbb{K}) & \longrightarrow & H_i(\mathrm{link}_\Delta(H),\mathrm{link}_\Delta(H)\setminus(K\setminus H),\mathbb{K}) \\
\| & & \| \\
 & & \widetilde{H}_{i+\dim(H)-\dim(K)}(\mathrm{link}_\Delta(K),\mathbb{K}) \\
\| & & \| \\
H_{i+\dim(H)+1}(\Delta,\Delta\setminus H,\mathbb{K}) & \longrightarrow & H_{i+\dim(H)+1}(\Delta,\Delta\setminus K,\mathbb{K})
\end{array}$$

where the maps in the rows are given by the long exact sequences of the

pair $(\mathrm{link}_\Delta(H), \mathrm{link}_\Delta(H) \setminus (K \setminus H))$ *and the triple* $(\Delta, \Delta \setminus K, \Delta \setminus H)$

and the maps in the columns are isomorphisms.

Proof Consider for a simplicial complex Δ' a face E of Δ'. For $i \geq -1$ let $C_{i+\dim(E)+1}(\Delta', \Delta' \setminus E, \mathbb{K})$ be the simplicial chain group in dimension $i+\dim(E)+1$ and the reduced simplicial chain group $\widetilde{C}_i(\mathrm{link}_\Delta(E), \mathbb{K})$ in dimension i. The first chain group has as a basis the faces E' of Δ' such that $E \subseteq E'$ and $\dim(E') = i + \dim(E) + 1$, the second a has as a basis faces $E'' \in \mathrm{link}_\Delta(E)$ with $\dim(E'') = i$. Now mapping E'' to $E'' \cup E$ establishes a bijection of the two bases which after choosing appropriate orientations extends to an isomorphism of chain complexes.

This fact explains all isomorphism in the columns of the asserted diagram. It is then easily checked that these isomorphisms commute with the exact sequences of the pair and the triple. The assertion then follows (see [12, Theorem 2.1] for more details). □

Now we are in position to state and prove a result which will immediately imply the result by Walker [21, Theorem 9.8]. For the formulation and the proof of the next theorem we again mostly follow [12].

Theorem 7.6 *Let Δ be a simplicial complex on ground set Ω such that $\mathbb{K}[\Delta]$ is Cohen–Macaulay. Then the following are equivalent*

(i) *Δ is 2-Cohen–Macaulay.*
(ii) *For all $\emptyset \neq F \in \Delta$ the map*

$$\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) \tag{1}$$

from the long exact sequence of the pair $(\Delta, \Delta \setminus F)$ is surjective.
(iii) *For all $\emptyset \neq F \in \Delta$ we have $\widetilde{H}_{\dim(\Delta)-1}(\Delta \setminus F, \mathbb{K}) = 0$.*
(iv) *For all $x \in |\Delta|$ we have $\widetilde{H}_{\dim(\Delta)-1}(|\Delta| - x, \mathbb{K}) = 0$.*

Proof

(i) $\Rightarrow$ (ii)

We prove the assertion by induction on $\dim(F)$ for arbitrary Δ for which $\mathbb{K}[\Delta]$ is 2-Cohen–Macaulay. If $\dim(F) = 0$ then $\Delta \setminus F = \Delta_{\Omega \setminus F}$ which is Cohen–Macaulay of dimension $\dim(\Delta)$ by assumption. It follows that $\widetilde{H}_{\dim(\Delta)-1}(\Delta \setminus F, \mathbb{K}) = 0$. Hence by the exactness of the long exact sequence of the pair $(\Delta, \Delta \setminus F)$ the map in (1) must be surjective.

Now let F be a face of dimension $\dim(F) > 0$ and let $\omega \in F$ be some fixed element. We set $G = F \setminus \{\omega\}$. From Lemma 7.4 we know that $\mathrm{link}_\Delta(G)$ is 2-Cohen–Macaulay of dimension $\dim(\Delta) - \dim(G) - 1$. Hence by induction we know that the map

$$\widetilde{H}_{i-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathbb{K}) \to H_{i-\dim(G)-1}(\mathrm{link}_\Delta(G), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K})$$

is surjective. Thus by Lemma 7.5 for $H = G$ and $K = F$ we obtain that the map

$$H_i(\Delta, \Delta \setminus G, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$$

is surjective. Again by induction we know that the map

$$\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta, \setminus G, \mathbb{K})$$

is surjective.

By the naturality of the maps it follows that the composition map $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$ is surjective.

(ii) $\Rightarrow$ (i)

Let $\omega \in \Omega$ and $F \in \Delta$.

If $F \cup \{\omega\} \notin \Delta$ then $\mathrm{link}_{\Delta_{\Omega \setminus \{\omega\}}}(F) = \mathrm{link}_\Delta(F)$. Since Δ is Cohen–Macaulay it follows from Theorem 6.2 that $\widetilde{H}_i(\mathrm{link}_{\Delta_{\Omega \setminus \{\omega\}}}(F), \mathbb{K}) = 0$ for $i < \dim(\mathrm{link}_{\Delta_{\Omega \setminus \{\omega\}}}) = \dim(\mathrm{link}_\Delta(F))$.

We are left with the case when $G = F \cup \{\omega\} \in \Delta$. For that consider the commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_{\dim(\Delta)}(\Delta) & \longrightarrow & H_{\dim(\Delta)}(\Delta, \Delta \setminus G, \mathbb{K}) \\
\downarrow & \nearrow & \\
H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) & &
\end{array}$$

with maps induced by the long exact sequence of the pairs $(\Delta, \Delta \setminus G)$, $(\Delta, \Delta \setminus F)$ and the triple $(\Delta, \Delta \setminus F, \Delta \setminus G)$. The map in the first row is surjective by (ii). Thus the diagonal map is surjective too. By Lemma 7.5 for $H = F$ and $K = G$ we deduce that the map

$$\widetilde{H}_{\dim(\Delta)-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathbb{K}) \to H_{\dim(\Delta)-\dim(F)-1}(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K})$$

is surjective as well. Since $\mathbb{K}[\mathrm{link}_\Delta(F)]$ is Cohen–Macaulay it follows by Theorem 6.2 that $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for $i < \dim(\Delta) - \dim(F) - 1$. By Lemma 7.5 $H_i(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K}) = \widetilde{H}_{i-1}(\mathrm{link}_\Delta(G), \mathbb{K})$. Since $\mathbb{K}[\mathrm{link}_\Delta(G)]$ is also Cohen–Macaulay again by Theorem 6.2 we obtain $\widetilde{H}_{i-1}(\mathrm{link}_\Delta(G), \mathbb{K}) = 0$ for $i - 1 < \dim(\Delta) - \dim(G) - 1 = \dim(\Delta) - \dim(F) - 2$.

Hence in the long exact sequence of the pair $(\mathrm{link}_\Delta(F), \mathrm{link}_\Delta(F) \setminus \{\omega\})$. We have that

$$\widetilde{H}_i(\mathrm{link}_{\Delta\setminus\{\omega\}}(F), \mathbb{K}) = \widetilde{H}_i(\mathrm{link}_\Delta(F) \setminus \{\omega\}, \mathbb{K}) = 0$$

for $i < \dim(\Delta) - \dim(F) - 1$.

Now it follows from Theorem 6.2 that $\mathbb{K}[\Delta \setminus \{\omega\}]$ is Cohen–Macaulay and hence Δ is 2-Cohen–Macaulay.

(ii) ⇔ (iii)

Consider the exact sequence

$$\begin{aligned} \cdots \to \widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K}) \\ \to \widetilde{H}_{\dim(\Delta)}(\Delta \setminus F, \mathbb{K}) \to \widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) \to \end{aligned}$$

Since $\mathbb{K}[\Delta]$ is Cohen–Macaulay we know by Theorem 6.2 that $\widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) = 0$. It follows that $\widetilde{H}_{\dim(\Delta)}(\Delta \setminus F, \mathbb{K}) = 0$ if and only if the map $\widetilde{H}_{\dim(\Delta)}(\Delta, \mathbb{K}) \to H_{\dim(\Delta)}(\Delta, \Delta \setminus F, \mathbb{K})$ is surjective.

(iii) ⇔ (iv)

We know by Lemma 2.2 that $|\Delta_{\Omega\setminus F}|$ is a deformation retract of $|\Delta| - x$ for x in the relative interior of $|\bar{F}|$. In particular, the homology groups of the two spaces coincide. □

The next corollary is an immediate consequence of the fact that condition (iv) of Theorem 7.6 depends only on the homeomorphism type of $|\Delta|$.

Corollary 7.7 (Walker) *Let Δ and Δ' be two simplicial complexes for which $\mathbb{K}[\Delta]$ and $\mathbb{K}[\Delta']$ are Cohen–Macaulay and such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-Cohen–Macaulay if and only if Δ' is 2-Cohen–Macaulay.*

A simplicial complex Δ over ground set Ω is called n-Buchsbaum (over $\mathbb{K}$) if for any subset $W \subseteq \Omega$ of cardinality $\#W < n$ we have that $\mathbb{K}[\Delta_{\Omega\setminus W}]$ is Buchsbaum and $\dim(\Delta) = \dim(\Delta_{\Omega\setminus W})$. In particular, 1-Buchsbaum is the usual Buchsbaum property for $\mathbb{K}[\Delta]$.

Analogous to the case of the Cohen–Macaulay property n-Buchsbaum is not a topological property for $n \geq 3$.

For the $n = 2$ there is the following result [12, Theorem 4.3].

Theorem 7.8 *Let Δ be a simplicial complex such that $\mathbb{K}[\Delta]$ is Buchsbaum. Then the following are equivalent.*

(i) *Δ is 2-Buchsbaum.*
(ii) *For any $x \in |\Delta|$ and any neighbourhood U of x in Δ there exists an open set V such that*
  (a) *$x \in V \subseteq U$.*
  (b) *The inclusion $|\Delta| \setminus V \hookrightarrow |\Delta| - x$ induces an isomorphisms*

$$\widetilde{H}_i(|\Delta| - V, \mathbb{K}) \to \widetilde{H}_i(|\Delta| - x, \mathbb{K})$$

  for all $i \geq 0$.
  (c) *For any $y \in V$ we have $\widetilde{H}_{\dim(\Delta)-1}(|\Delta| - y, \mathbb{K}) = 0$.*

The proof of Theorem 7.8 in [12] is based on arguments similar to those used in the proof of Theorem 7.6. But the deduction becomes more technical and more involved. We refer the reader to the paper [12] for details. Condition (ii) of the preceding result obvious only depends on the homeomorphism type of $|\Delta|$. Therefore, Theorem 7.8 immediately implies the following corollary (see [12, Corollary 4.4]).

Corollary 7.9 (Miyazaki) *Let Δ and Δ' be two simplicial complexes for which $\mathbb{K}[\Delta]$ and $\mathbb{K}[\Delta']$ are Buchsbaum and such that $|\Delta|$ and $|\Delta'|$ are homeomorphic. Then Δ is 2-Buchsbaum if and only if Δ' is 2-Buchsbaum.*

Building on condition (iv) of Theorem 7.6 one can define the class of Buchsbaum* simplicial complexes. A simplicial complex Δ such that $\mathbb{K}[\Delta]$ is Buchsbaum is called Buchsbaum* (over $\mathbb{K}$) if $\widetilde{H}_{\dim(\Delta)-1}(|\Delta|, \mathbb{K}) = \widetilde{H}_{\dim(\Delta)-1}(|\Delta|, \mathbb{K})$ for all $x \in \Delta$. By definition the Buchsbaum* property depends only on the homeomorphism type of $|\Delta|$. In the following results (see [1, Proposition 2.5, 2.8]) the relation of this property to the properties Gorenstein*, 2-Cohen–Macaulay and 2-Buchsbaum is clarified.

Lemma 7.10 *Let Δ be a simplicial complex.*

(i) *If $\mathbb{K}[\Delta]$ is Cohen–Macaulay then*

$$\Delta \textit{ is } 2\textit{-Cohen–Macaulay} \Leftrightarrow \Delta \textit{ Buchsbaum}^*.$$

(ii) *If $\mathbb{K}[\Delta]$ is Gorenstein then*

$$\Delta \textit{ Gorenstein}^* \Leftrightarrow \Delta \textit{ Buchsbaum}^*.$$

(iii) *If Δ is Buchsbaum* then Δ is 2-Buchsbaum.*

The statement in (i) is immediate from the fact that by Theorem 6.2 we have that $\widetilde{H}_{\dim(\Delta)-1}(\Delta, \mathbb{K}) = 0$ for a Cohen–Macaulay Δ. Statements (ii) and (iii) follow by arguments similar to those used in the proof of Theorem 7.6

8 Other Properties

In this section we go over other properties of $\mathbb{K}[\Delta]$ studied in the literature for which the question of whether the property is topological or not was considered. We do not think that the list is exhaustive but we have included all results known to us.

An interesting strengthening of the Cohen–Macaulay property was studied in [11]. Here a simplicial complex Δ for which $\mathbb{K}[\Delta]$ is Cohen–Macaulay is called uniformly Cohen–Macaulay (over $\mathbb{K}$) if $\mathbb{K}[\Delta \setminus F]$ is Cohen–Macaulay and $\dim(\Delta \setminus F) = \dim(\Delta)$ for every facet F of Δ. The authors show the following topological characterization in [11, Theorem 1.1].

Theorem 8.1 *Let Δ be a Cohen–Macaulay simplicial complex. Then the following are equivalent.*

(i) *Δ is uniformly Cohen–Macaulay.*
(ii) *For every $x \in |\Delta|$ the map $\widetilde{H}_{\dim(\Delta)}(|\Delta|, \mathbb{K}) \to \widetilde{H}_{\dim(\Delta)}(|\Delta|, |\Delta| - x, \mathbb{K})$ from the long exact sequence of the pair $(|\Delta|, |\Delta| - x)$ is an inclusion.*

Clearly, condition (ii) from the theorem depends only on homeomorphism type of $|\Delta|$ and hence the property is topological.

In commutative algebra the Cohen–Macaulay property of a ring is equivalent to the ring having Serre's property (S_d) for the Krull dimension d of the ring. We refer the reader to [5, p. 62] for the definition of property (S_r) in general. It can be shown, again using Hochster's formula Theorem 5.1 on the local cohomology of $\mathbb{K}[\Delta]$, that $\mathbb{K}[\Delta]$ has property (S_r) if and only if $\widetilde{H}_i(\mathrm{link}_\Delta(F), \mathbb{K}) = 0$ for all $i < \min\{r - 1, \dim(\Delta) - \dim(F) - 1\}$. Clearly for $r = d$ we recover Reisner's criterion Theorem 6.2 for the Cohen–Macaulay property of $\mathbb{K}[\Delta]$. In [22, Theorem 4.4] Yanagawa showed that the property (S_r) is topological for any r, which is a vast generalization of Munkres' result Theorem 6.1 on Cohen–Macaulayness.

Theorem 8.2 (Yanagawa) *Let Δ and Δ' be two simplicial complexes such that $|\Delta|$ and $|\Delta'|$ are homeomorphic and $r \geq 0$ a number. Then $\mathbb{K}[\Delta]$ has property (S_r) if and only of $\mathbb{K}[\Delta']$ has property (S_r).*

The original proof from [22, Theorem 4.4] uses quite heavy machinery from commutative algebra. Recently a short proof was given in [8, Corollary 3].

Equally natural as these weakenings and strengthenings of the Cohen–Macaulay condition are generalizations of the Cohen–Macaulay condition towards pairs of simplicial complexes. A pair (Δ, Γ) of simplicial complexes consists of two simplicial complexes over the same ground set Ω such that Γ is a subcomplex of Δ. For a relative simplicial complex (Δ, Γ) its Stanley-Reisner ideal is the ideal

$I_{\Delta,\Gamma}$ in $\mathbb{K}[\Delta]$ generated by the monomials $\mathbf{x}_F$ for $F \in \Delta \setminus \Gamma$. The relative simplicial complex (Δ, Γ) is called Cohen–Macaulay (over $\mathbb{K}$), if the S_Ω module $I_{\Delta,\Gamma}$ is. As for rings the equality of depth and dimension defines Cohen–Macaulayness for modules.

In [18] Stanley deduces the topological invariance of the Cohen–Macaulay property from results in [17, Corollary 5.4]. Topological invariance here means that the property only depends on the homeomorphism type of the pair $(|\Delta|, |\Gamma|)$. The proof heavily relies on a relative version of Reisner's criterion Theorem 6.2.

Theorem 8.3 (Stanley) *Let* (Δ, Γ) *and* (Δ', Γ') *be two pairs of simplicial complexes such that* $(|\Delta|, |\Gamma|)$ *and* $(|\Delta|', |\Gamma|')$ *are homeomorphic pairs of spaces. Then* $I_{\Delta,\Gamma}$ *is Cohen–Macaulay if and only if* $I_{\Delta',\Gamma'}$ *is Cohen–Macaulay.*

In the 90s motivated by a series of interesting non-pure simplicial complexes arising in combinatorics, Stanley [18, p. 87] defined the notion of a sequentially Cohen–Macaulay module. We do not want to work with the general definition here. Using [18, Proposition 2.11] we rather define sequential Cohen–Macaulayness for Stanley-Reisner rings $\mathbb{K}[\Delta]$ only. Let Δ be a simplicial complex. For a number $0 \leq i \leq \dim(\Delta)$ let Δ_i be the simplicial complex of all $F \in \Delta$ such that there is a facet $G \in \Delta$ satisfying $\dim(G) = i$ and $F \subseteq G$. Then one calls $\mathbb{K}[\Delta]$ sequentially Cohen–Macaulay (over $\mathbb{K}$) if for all $0 \leq i \leq \dim(\Delta)$ the relative simplicial complex

$$(\Delta_i, \Delta_i \cap (\Delta_{i+1} \cup \cdots \cup \Delta_{\dim(\Delta)})$$

is Cohen–Macaulay over $\mathbb{K}$ (see [3, 6] for equivalent formulations).

Stanley's result on the sequential Cohen–Macaulay property follows from Theorem 8.3.

In [21, Theorem 4.1.6] Wachs provides an obviously topological property which is equivalent to sequential Cohen–Macaulayness.

Theorem 8.4 *Let* Δ *be a simplicial complex. Then* $\mathbb{K}[\Delta]$ *is sequentially Cohen–Macaulay if and only if for all* $0 \leq j < i \leq \dim(\Delta)$ *and* $x \in |\Delta_i|$ *we have*

$$\widetilde{H}_j(|\Delta_i|, \mathbb{K}) = H_j(|\Delta_i|, |\Delta_i| - x, \mathbb{K}) = 0.$$

So either using Theorem 8.3 or using Theorem 8.4 we get the Stanley's result as a corollary.

Corollary 8.5 (Stanley) *Let* Δ *and* Δ' *be simplicial complexes such that* $|\Delta|$ *and* $|\Delta'|$ *are homeomorphic. Then* $\mathbb{K}[\Delta]$ *is sequentially Cohen–Macaulay if and only if* $\mathbb{K}[\Delta']$ *is.*

In a similar fashion sequential versions have been attached to other properties of Δ or $\mathbb{K}[\Delta]$. In [8, Corollary 7] the topological invariance of the sequential (S_r) properties is proved. In [3, Proposition 2.4] sequential connectivity and sequential acyclicity are shown to be topological properties.

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