



# Evacuating from $\ell_p$ Unit Disks in the Wireless Model

## (Extended Abstract)

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**Abstract.** The search-type problem of evacuating 2 robots in the wireless model from the (Euclidean) unit disk was first introduced and studied by Czyzowicz et al. [DISC'2014]. Since then, the problem has seen a long list of follow-up results pertaining to variations as well as to upper and lower bound improvements. All established results in the area study this 2-dimensional search-type problem in the Euclidean metric space where the search space, i.e. the unit disk, enjoys significant (metric) symmetries.

We initiate and study the problem of evacuating 2 robots in the wireless model from  $\ell_p$  unit disks,  $p \in [1, \infty)$ , where in particular robots' moves are measured in the underlying metric space. To the best of our knowledge, this is the first study of a search-type problem with mobile agents in more general metric spaces. The problem is particularly challenging since even the circumference of the  $\ell_p$  unit disks have been the subject of technical studies. In our main result, and after identifying and utilizing the very few symmetries of  $\ell_p$  unit disks, we design *optimal evacuation algorithms* that vary with  $p$ . Our main technical contributions are two-fold. First, in our upper bound results, we provide (nearly) closed formulae for the worst case cost of our algorithms. Second, and most importantly, our lower bounds' arguments reduce to a novel observation in convex geometry which analyzes trade-offs between arc and chord lengths of  $\ell_p$  unit disks as the endpoints of the arcs (chords) change position around the perimeter of the disk, which we believe is interesting in its own right. Part of our argument pertaining to the latter property relies on a computer assisted numerical verification that can be done for non-extreme values of  $p$ .

**Keywords:** Search · Evacuation · Wireless model ·  $\ell_p$  metric space · Convex and computational geometry

## 1 Introduction

In the realm of mobile agent computing, search-type problems are concerned with the design of searchers' (robots') trajectories in some known search space to locate a

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hidden object. Single searcher problems have been introduced and studied as early as the 60's by the mathematics community [11, 12], and later in the late 80's and early 90's by the theoretical computer science community [8]. The previously studied variations focused mainly on the type of search domain, e.g. line or plane or a graph, and the type of computation, e.g. deterministic or randomized. Since search was also conducted primarily by single searchers, termination was defined as the first time the searcher hit the hidden object. In the last decade with the advent of robotics, search-type problems have been rejuvenated within the theoretical computer science community, which is now concerned with novel variations including the number of searchers (mobile agents), the communication model, e.g. face-to-face or wireless, and robots' specifications, e.g. speeds or faults, including crash-faults or byzantine faults. As a result of the multi-searcher setup, termination criteria are now subject to variations too, and these include the number or the type of searchers that need to reach the hidden item (for a more extended discussion with proper citations, see Sect. 1.1).

One of the most studied search domains, along with the line, is that of a circle, or a disk. In a typical search-type problem in the disk, the hidden item is located on the perimeter of the unit circle, and searchers start in its center. Depending on the variation considered, and combining all specs mentioned above, a number of ingenious search trajectories have been considered, often with counter-intuitive properties. Alongside the hunt for upper bounds (as the objective is always to minimize some form of cost, e.g. time or traversed space or energy) comes also the study of lower bounds, which are traditionally much more challenging to prove (and which rarely match the best known positive results).

Search on the unbounded plane as well as in other 2-dimensional domains, e.g. triangles or squares, has been considered too, giving rise to a long list of treatments, often with fewer tight (optimal) results. While the list of variations for searching on the plane keeps growing, there is one attribute that is common to all previous results where robots' trajectories lie in  $\mathbb{R}^2$ , which is the underlying Euclidean metric space. In other words, distances and trajectory lengths are all measured with respect to the Euclidean  $\ell_2$  norm. Not only the underlying geometric space is well understood, but it also enjoys symmetries, and admits standard and elementary analytic tools from trigonometry, calculus, and analytic geometry.

We deviate from previous results, and to the best of our knowledge, we initiate the study of a search-type problem with mobile agents in  $\mathbb{R}^2$  where the underlying metric space is induced by any  $\ell_p$  norm,  $p \geq 1$ . The problem is particularly challenging since even "highly symmetric" shapes, such as the unit circle, enjoy fewer symmetries in non-Euclidean spaces. Even more, robot trajectories are measured with respect to the underlying metric, giving rise to technical mathematical expressions for measuring the performance of an algorithm. In particular, we consider the problem of reaching (evacuating from) a hidden object (the exit) placed on the perimeter of the  $\ell_p$  unit circle. Our unit-speed searchers start from the center of the circle, placed at the origin of the Cartesian plane  $\mathbb{R}^2$ , and are controlled by a centralized algorithm that allows them to communicate their findings instantaneously. Termination is determined by the moment that the last searcher reaches the exit, and the

performance analysis is evaluated against a deterministic worst case adversary. For this problem we provide *optimal evacuation algorithms*. Apart from the novelty of the problem, our contributions pertain to (a) a technical analysis of search (optimal) algorithms that have to vary with  $p$ , giving rise to our upper bounds, and to (b) an involved geometric argument that also uses, to the best of our knowledge, a novel observation on convex geometry that relates a given  $\ell_p$  unit circle's arcs to its chords, giving rise to our matching lower bounds.

## 1.1 Related Work

Our contributions make progress in Search-Theory, a term that was coined after several decades of celebrated results in the area, and which have been summarized in books [3,5,6,53]. The main focus in that area pertains to the study of (optimal) searchers' strategies who compete against (possibly hidden) hider(s) in some search domain. An even wider family of similar problems relates to exploration [4], terrain mapping, [48], and hide-and-seek and pursuit-evasion [49].

The traditional problem of searching with one robot on the line [8] has been generalized with respect to the number of searchers, the type of searchers, the search domain, and the objective, among others. When there are multiple searchers and the objective is that all of them reach the hidden object, the problem is called an *evacuation problem*, with the first treatments dating back to over a decade ago [10,35]. The evacuation problem that we study is a generalization of a problem introduced by Czyzowicz et al. [21] and that was solved optimally. In that problem, a hidden item is placed on the (Euclidean) unit disk, and is to be reached by two searchers that communicate their findings instantaneously (wireless model). Variations of the problem with multiple searchers, as well as of another communication model (face-to-face) was considered too, giving rise to a series of follow-up papers [15,25,32]. Searching the boundary of the disc is also relevant to so-called Ruckle-type games, and closely related to our problem is a variation mentioned in [9] as an open problem, in which the underlying metric space is any  $\ell_p$ -induced space,  $p \geq 1$ , as in our work.

The search domain of the unit circle that we consider is maybe one of the most well studied, together with the line [18]. Other topologies that have been considered include multi-rays [16], triangles [20,27], and graphs [7,14]. Search for a hidden object on an unbounded plane was studied in [47], later in [34,46], and more recently in [1,33].

Search and evacuation problems with faulty robots have been studied in [22,39,50] and with probabilistically faulty robots in [13]. Variations pertaining to the searcher's speeds appeared in [36,38] (immobile agents), in [45] (speed bounds) and in [26] (terrain dependent speeds). Search for multiple exits was considered in [28,51], while variation of searching with advice appeared in [41]. Some variations of the objective include the so-called priority evacuation problem [23,30] and its generalization of weighted searchers [40]. Randomized search strategies have been considered in [11,12] and later in [42] for the line, and more recently in [19] for the disk. Finally, turning costs have been studied in [31] and an objective of minimizing a notion pertaining to energy (instead of time) was studied in [29,44], just to name

a few of the developments related to our problem. The reader may also see recent survey [24] that elaborates more on selected topics.

## 1.2 High Level of New Contributions and Motivation

The algorithmic problem of searching in arbitrary metric spaces has a long history [17], but the focus has been mainly touching on database management. In our work, we extend results of a search-type problem in mobile agent computing first appeared in [21]. More specifically, we provide optimal algorithms for the search-type problem of evacuating two robots in the wireless model from the  $\ell_p$  unit disk, for  $p \geq 1$  (previously considered only for the Euclidean space  $p = 2$ ). The novelty of our results is multi-fold. First, to the best of our knowledge, this is the first result in mobile agent computing in which a search problem is studied and optimally solved in  $\ell_p$  metric spaces. Second, both our upper and lower bound arguments rely on technical arguments. Third, part of our lower bound argument relies on an interesting property of unit circles in convex geometry, which we believe is interesting in its own right.

The algorithm we prove to be optimal for our evacuation problem is very simple, but it is one among infinitely many natural options one has to consider for the underlying problem (one for each deployment point of the searchers). Which of them is optimal is far from obvious, and the proof of optimality is, as we indicate, quite technical.

Part of the technical difficulty of our arguments arises from the implicit integral expression of arc lengths of  $\ell_p$  circles. Still, by invoking the Fundamental Theorem of Calculus we determine the worst case placement of the hidden object for our algorithms. Another significant challenge of our search problem pertains to the limited symmetries of the unit circle in the underlying metric space. As a result, it is not surprising that the behaviour of the provably optimal algorithm does depend on  $p$ , with  $p = 2$  serving as a threshold value for deciding which among two types of special algorithms is optimal. Indeed, consider an arbitrary contiguous arc of some fixed length of the  $\ell_p$  unit circle with endpoints  $A, B$ . In the Euclidean space, i.e. when  $p = 2$ , the length of the corresponding chord is invariant of the locations of  $A, B$ . In contrast, for the unit circle hosted in any other  $\ell_p$  space, the slope of the chord  $AB$  does determine its length. The relation to search and evacuation is that the arc corresponds to a subset of the search domain which is already searched, and points  $A, B$  are the locations of the searchers when the exit is reported. Since searchers operate in the wireless model in our problem (hence one searcher will move directly to the other searcher when the hidden object is found), their trajectories are calculated so that their  $\ell_p$  distance is the minimum possible for the same elapsed search time.

Coming back to the  $\ell_p$  unit disks, we show an interesting property which may be of independent interest (and which we did not find in the current literature). More specifically, and in part using computer assisted numerical calculations for a wide range of values of  $p$ , we show that for any arc of fixed length, the placement of its endpoint  $A, B$  that minimizes the  $\ell_p$  length of chord  $AB$  is when  $AB$  is parallel to the  $y = 0$  or  $x = 0$  lines, for  $p \leq 2$ , and when  $AB$  is parallel to the  $y = x$  or  $y = -x$  lines for  $p \geq 2$ . The previous fact is coupled by a technical extension of a result first sketched

in [21], according to which at a high level, as long as searchers have left any part of the unit circle of cumulative length  $\alpha$  unexplored (not necessarily contiguous), then there are at least two unexplored points of arc distance *at least*  $\alpha$ .

All omitted proofs from this extended abstract can be found in the full version of the paper [37].

## 2 Problem Definition, Notation and Nomenclature

For a vector  $x = (x_1, x_2) \in \mathbb{R}^2$ , we denote by  $\|x\|_p$  the vector's  $\ell_p$  norm, i.e.  $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ . The  $\ell_p$  unit circle is defined as  $\mathcal{C}_p := \{x \in \mathbb{R}^2 : \|x\|_p = 1\}$ , see also Fig. 2a for an illustration. We equip  $\mathbb{R}^2$  with the metric  $d_p$  induced by the  $\ell_p$  norm, i.e. for  $x, y \in \mathbb{R}^2$  we write  $d_p(x, y) = \|x - y\|_p$ . Similarly, if  $r : [0, 1] \rightarrow \mathbb{R}^2$  is an injective and continuously differentiable function, its  $\ell_p$  length is defined as  $\mu_p(r) := \int_0^1 \|r'(t)\|_p dt$ . As a result, a unit speed robot can traverse  $r([0, 1])$  in metric space  $(\mathbb{R}^2, d_p)$  in time  $\mu_p(r)$ .

We proceed with a formal definition of our search-type problem. In problem  $WE_p$  (*Wireless Evacuation in  $\ell_p$  space,  $p \geq 1$* ), two unit-speed robots start at the center of a unit circle  $\mathcal{C}_p$  placed at the origin of the metric space  $(\mathbb{R}^2, d_p)$ . Robots can move anywhere in the metric space, and they operate according to a centralized algorithm. An *exit* is a point  $P$  on the perimeter of  $\mathcal{C}_p$ . An *evacuation algorithm*  $A$  consists of robots trajectories, either of which may depend on the placement of  $P$  only after at least one of the robots passes through  $P$  (*wireless model*).<sup>1</sup> For each exit  $P$ , we define the evacuation cost of the algorithm as the first instance that the last robot reaches  $P$ . The *cost of algorithm*  $A$  is defined as the supremum, over all placements  $P$  of the exit, of the evacuation time of  $A$  with exit placement  $P$ . Finally, the *optimal evacuation cost of  $WE_p$*  is defined as the infimum, over all evacuation algorithms  $A$ , of the cost of  $A$ .

Next we show that  $\mathcal{C}_p$  has 4 axes of symmetry (and of course  $\mathcal{C}_2$  has infinitely many, i.e. any line  $ax + by = 0, a, b \in \mathbb{R}$ ).

**Lemma 1.** *Lines  $y = 0, x = 0, y = x, y = -x$  are all axes of symmetry of  $\mathcal{C}_p$ . Moreover, the center of  $\mathcal{C}_p$  is its point of symmetry.*

*Proof.* Reflection of point  $P = (a, b)$  across lines  $y = 0, x = 0, y = x, y = -x$  give points  $P_1 = (a, -b), P_2 = (-a, b), P_3 = (b, a), P_4 = (-b, -a)$ , respectively. It is easy to see that setting  $\|P\|_p = 1$  implies that  $\|P_i\|_p = 1, i = 1, 2, 3, 4$ .

<sup>1</sup> An underlying assumption is also that robots can distinguish points  $(x, y)$  by their coordinates, and they can move between them at will. As a byproduct, robots have a sense of orientation. This specification was not mentioned explicitly before for the Euclidean space, since all arguments were invariant under rotations (which is not the case any more). However, even in the  $\ell_2$  case this specification was silently assumed by fixing the cost of the optimal offline algorithm to 1 (a searcher that knows the location of the exit goes directly there), hence all previous results were performing competitive analysis by just doing worst case analysis.

We use the generalized trigonometric functions  $\sin_p(\cdot), \cos_p(\cdot)$ , as in [52], which are defined as  $\sin_p(\phi) := \sin(\phi)/N_p(\phi)$ ,  $\cos_p(\phi) := \cos(\phi)/N_p(\phi)$ , where  $N_p(\phi) := (|\sin(\phi)|^p + |\cos(\phi)|^p)^{1/p}$ . By introducing  $\rho_p(\phi) := (\cos_p(\phi), \sin_p(\phi))$ , which is injective and continuously differentiable function in each of the 4 quadrants, we have the following convenient parametric description of the  $\ell_p$  unit circle;  $\mathcal{C}_p = \{\rho_p(\phi) : \phi \in [0, 2\pi)\}$ . In particular, set  $Q_1 = [0, \pi/2), Q_2 = [\pi/2, \pi), Q_3 = [\pi, 3\pi/2), Q_4 = [3\pi/2, 2\pi)$ , and define for each  $U \subseteq \mathcal{C}_p$  its *length* (measure) as

$$\mu_p(U) = \sum_{i=1}^4 \int_{t \in Q_i: \rho_p(t) \in U} \left\| \rho'_p(t) \right\|_p dt.$$

It is easy to see that  $\mu_p(\cdot)$  is indeed a measure, hence it satisfies the principle of inclusion-exclusion over  $\mathcal{C}_p$ . Also, by Lemma 1 it is immediate that for every  $U \subseteq \mathcal{C}_p$ , and for  $\bar{U} = \{\rho_p(t + \pi) : \rho(t) \in U\}$ , we have that  $\mu_p(U) = \mu_p(\bar{U})$  (both observations will be used later in Lemma 7). As a corollary of the same lemma, we also formalize the following observation.

**Lemma 2.** *For any  $\phi \in \{k \cdot \pi/4 : k = 0, 1, 2, 3, 4\}$  and  $\theta \in [0, \pi]$ , let  $U_+ = \{\rho_p(\phi + t) : t \in [0, \theta]\}$  and  $U_- = \{\rho_p(\phi - t) : t \in [0, \theta]\}$ . Then, we have that  $\mu_p(U_+) = \mu_p(U_-)$ .*

The perimeter of the  $\ell_p$  unit circle can be computed as

$$\mu_p(\mathcal{C}_p) = \sum_{i=1}^4 \int_{Q_i} \left\| \rho'_p(t) \right\|_p dt = 4 \int_0^{\pi/2} \left\| \rho'_p(t) \right\|_p dt := 2\pi_p.$$

By Lemma 2, we also have  $\int_0^{\pi/2} \left\| \rho'_p(t) \right\|_p dt = 2 \int_0^{\pi/4} \left\| \rho'_p(t) \right\|_p dt = \pi_p/2$ . Clearly  $\mu_2(\mathcal{C}_2)/2 = \pi_2 = \pi = 3.14159\dots$ , while the rest of the values of  $\pi_p$ , for  $p \geq 1$ , do not have known number representation, in general. However, it is easy to see that  $\pi_1 = \pi_\infty = 4$ . More generally we have that  $\pi_p = \pi_q$  whenever  $p, q \geq 1$  satisfy  $1/p + 1/q = 1$  [43]. As expected,  $\pi_2 = \pi$  is also the minimum value of  $\pi_p$ , over  $p \geq 1$  [2], see also Fig. 2b for the behavior of  $\pi_p$ .

For every  $\phi, \theta \in [0, 2\pi)$ , let  $A = \rho(\phi), B = \rho(\phi + \theta)$  be two points on the  $\ell_p$  unit circle. The *chord*  $\overline{AB}$  is defined as the line segment with endpoints  $A, B$ . From the previous discussion we have  $\mu_p(\overline{AB}) = d_p(A, B)$ . The *arc*  $\widehat{AB}$  is defined as the curve  $\{\rho(\phi + t) : t \in [0, \theta]\}$ , hence arcs identified by their endpoints are read counter-clockwise. The length of the same arc is computed as  $\mu_p(\widehat{AB})$ .

Finally, the *arc distance* of two points  $A, B \in \mathcal{C}_p$  is defined as  $\widehat{d}_p(A, B) := \min\{\mu_p(\widehat{AB}), \mu_p(\widehat{BA})\}$ , which can be shown to be a metric. By definition, it follows that  $\widehat{d}_p(A, B) \in [0, \pi_p]$ .

Next we present an alternative parameterization of the  $\ell_p$  unit circle that will be convenient for some of our proofs. We define

$$r_p(s) := \left( -s, (1 - |s|^p)^{1/p} \right), \quad (1)$$

and we observe that  $r_p(s) \in \mathcal{C}_p$ , for every  $s \in [-1, 1]$ . It is easy to see that as  $s$  ranges from  $-1$  to  $1$ , we traverse the upper 2 quadrants of the unit circle with the same direction as  $\rho_p(t)$ , when  $t$  ranges from  $0$  to  $\pi$ . Moreover, for every  $t \in [0, \pi]$ , there exists

unique  $s = s(t)$ , with  $s \in [-1, 1]$  such that  $\rho_p(t) = r_p(s)$ , and  $s(t)$  strictly increasing in  $t$  with  $s(0) = -1, s(\pi/4) = -2^{-1/p}, s(\pi/2) = 0, s(3\pi/4) = 2^{-1/p}$  and  $s(\pi) = 1$ .

### 3 Algorithms for Evacuating 2 Robots in $\ell_p$ Spaces

First we present a family of algorithms  $\text{Wireless-Search}_p(\phi)$  for evacuating 2 robots from the  $\ell_p$  unit circle  $\mathcal{C}_p$ . The family is parameterized by  $\phi \in \mathbb{R}$ , see also Fig. 4a for two examples, Algorithm  $\text{Wireless-Search}_{1.5}(0)$  and  $\text{Wireless-Search}_3(\pi/4)$ .

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#### Algorithm 1. $\text{Wireless-Search}_p(\phi)$

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- 1: Both robots move to point  $\rho_p(\phi)$ .
  - 2: Robots follow trajectories  $\rho_p(\phi \pm t)$ ,  $t \geq 0$ , till the exit is found and communicated.
  - 3: Finder stays put, and non-finder moves to finder's location along the shortest chord (line segment).
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Our goal is to prove the following.

#### Theorem 1

*For all  $p \in [1, 2]$ , Algorithm  $\text{Wireless-Search}_p(0)$  is optimal.*

*For all  $p \in [2, \infty)$ , Algorithm  $\text{Wireless-Search}_p(\pi/4)$  is optimal.*

Figure 4b depicts the performance of our algorithms as  $p \geq 1$  varies. Our analysis is formal, however we do rely on computer-assisted numerical calculations to verify certain analytical properties in convex geometry (see proof of Lemma 5 on page 14, and proof of Lemma 9 on page 9) that effectively contribute a part of our lower bound argument for bounded values of  $p$ , as well as  $p = \infty$ . For large values of  $p$ , e.g.  $p \geq 1000$ , where numerical verification is of limited help, we provide provable upper and lower bounds that differ by less than 0.042%, multiplicatively (or less than 0.0021, additively).

Recall that as  $\phi$  ranges in  $[0, 2\pi)$ , then  $\rho_p(\phi)$  ranges over the perimeter of  $\mathcal{C}_p$ . In particular, for any execution of Algorithm 1, the exit will be reported at some point  $\rho_p(\phi \pm t)$ , where  $t \in [0, \pi]$ . Since in the last step of the algorithm, the non-finder has to traverse the line segment defined by the locations of robots when the exit is found, we may assume without loss of generality that the exit is always found at some point  $\rho_p(\phi \pm t)$ , where  $t \in [0, \pi]$ , say by robot #1. Note that even though Algorithm 1 is well defined for all  $[0, 2\pi)$  (in fact all reals), due to Lemma 1 it is enough to restrict to  $\phi \in [0, \pi/4]$ .

In the remaining of this section, we denote by  $\mathcal{E}_{p,\phi}(\tau)$  the evacuation time of Algorithm  $\text{Wireless-Search}_p(\phi)$ , given that the exit is reported after robots have spent time  $\tau$  searching in parallel. We also denote by  $\delta_{p,\phi}(\tau)$  the distance of the two robots at the same moment, assuming that no exit has been reported previously. Hence,

$$\mathcal{E}_{p,\phi}(\tau) = 1 + \tau + \delta_{p,\phi}(\tau). \quad (2)$$

Since  $\mu_p(\mathcal{C}_p) = 2\pi_p$  and the two robots search in parallel, an exit will be reported for some  $\tau \in [0, \pi_p]$ . Hence, the worst case evacuation time  $E_{p,\phi}$  of Algorithm Wireless-Search $_p(\phi)$  is given by<sup>2</sup>

$$E_{p,\phi} := \max_{\tau \in [0, \pi_p]} \mathcal{E}_{p,\phi}(\tau).$$

### 3.1 Worst Case Analysis of Algorithm Wireless-Search $_p(\phi)$

It is important to stress that parameter  $t$  in the description of robots' trajectories in Algorithm Wireless-Search $_p(\phi)$  does not represent the total elapsed search time. Even more, and for an arbitrary value of  $\phi$ , it is not true that robots occupy points  $\rho_p(\phi \pm t)$  simultaneously. To see why, recall that from the moment robots deploy to point  $\rho_p(\phi)$ , they need time  $\alpha_{1,2}(\phi, t) := \mu_p(\{\rho_p(\phi \pm s) : s \in [0, t]\})$  in order to reach points  $\rho_p(\phi \pm t)$ . Moreover,  $\alpha_1(\phi, t) \neq \alpha_2(\phi, t)$ , unless  $\phi = k \cdot \pi/4$  for some  $k = 0, 1, 2, 3$ , as per Lemma 2. We summarize our observation with a lemma.

**Lemma 3.** *Let  $\phi \in \{0, \pi/4\}$ , and consider an execution of Algorithm 1. When one robot is located at point  $\rho_p(\phi + t)$ , for some  $t \in [-\pi, \pi]$ , then the other robot is located  $\rho_p(\phi - t)$ , and in particular  $\alpha_1(\phi, t) = \alpha_2(\phi, t)$ .*

Now we provide worst case analysis of two Algorithms for two special cases of metric spaces. The proof is a warm-up for the more advanced argument we employ later to analyze arbitrary metric spaces.

**Lemma 4.**  $E_{1,0} = E_{\infty, \pi/4} = 5$ .

*Proof.* First we study Algorithm Wireless-Search $_1(0)$  for evacuating 2 robots from the  $\ell_1$  unit disk. By (2), if the exit is reported after time  $\tau$  of parallel search, then  $\mathcal{E}_{1,0}(\tau) = 1 + \tau + \delta_{1,0}(\tau)$ . Note that  $\pi_1 = 4$ , so the exit is reported no later than parallel search time 4. First we argue that  $\mathcal{E}_{1,0}(\tau)$  is increasing for  $\tau \in [0, 2]$ . Indeed, in that time window robot #1 is moving from point  $(1, 0)$  to point  $(0, 1)$  along trajectory  $(1 - \tau/2, \tau/2)$  (note that this parameterization induces speed 1 movement). By Lemma 3, robot #2 at the same time is at point  $(1 - \tau/2, -\tau/2)$ . It follows that  $\delta_{1,0}(\tau) = \tau$ , so indeed  $\mathcal{E}_{1,0}(\tau)$  is increasing for  $\tau \in [0, 2]$ . Finally we show that  $\mathcal{E}_{1,0}(\tau) = 5$ , for all  $\tau \in [2, 4]$ . Indeed, note that for the latter time window, robot #1 moves from point  $(0, 1)$  to point  $(-1, 0)$  along trajectory  $(-(\tau - 2)/2, 1 - (\tau - 2)/2)$ . By Lemma 3, robot #2 at the same time is at point  $(-(\tau - 2)/2, -1 + (\tau - 2)/2)$ . It follows that  $\delta_{1,0}(\tau) = |-(\tau - 2)/2 + (\tau - 2)/2| + |1 - (\tau - 2)/2 + 1 - (\tau - 2)/2| = 4 - \tau$ , and hence  $\mathcal{E}_{1,0}(\tau) = 1 + \tau + \delta_{1,0}(\tau) = 5$ , as wanted.

Next we study Algorithm Wireless-Search $_{\infty}(\pi/4)$  for evacuating 2 robots from the  $\ell_{\infty}$  unit disk. By (2), if the exit is reported after time  $\tau$  of parallel search, then  $\mathcal{E}_{\infty, \pi/4}(\tau) = 1 + \tau + \delta_{\infty, \pi/4}(\tau)$ . As before,  $\pi_{\infty} = 4$ , so the exit is reported no later than parallel search time 4. We show again that  $\mathcal{E}_{\infty, \pi/4}(\tau)$  is increasing for  $\tau \in [0, 2]$ . Indeed, in that time window robot #1 is moving from point  $(1, 1)$  to point  $(-1, 1)$  along trajectory  $(1 - \tau, 1)$  (note that this induces speed 1 movement). By Lemma 3, robot #2 at the

<sup>2</sup> For arbitrary algorithms one should define the cost as the supremum over all exit placements. Since in Algorithm Wireless-Search $_p(\phi)$  the searched space remains contiguous and its boundaries keep expanding with time, the maximum always exists.



same time is at point  $(1, 1 - \tau)$ . It follows that  $\delta_{\infty, \pi/4}(\tau) = \max\{|1 - \tau - 1|, |1 - 1 + \tau|\} = \tau$ , so indeed  $\mathcal{E}_{\infty, \pi/4}(\tau)$  is increasing for  $\tau \in [0, 2]$ . Finally we show that  $\mathcal{E}_{\infty, \pi/4}(\tau) = 5$ , for all  $\tau \in [2, 4]$ . Indeed, note that for the latter time window, robot #1 moves from point  $(-1, 1)$  to point  $(-1, -1)$  along trajectory  $(-1, 1 - (\tau - 2))$ . By Lemma 3, robot #2 at the same time is at point  $(1 - (\tau - 2), -1)$ . It follows that  $\delta_{\infty, \pi/4}(\tau) = \max\{|-1 - 1 + (\tau - 2)|, |1 - (\tau - 2) + 1|\} = 4 - \tau$ , and hence  $\mathcal{E}_{\infty, \pi/4}(\tau) = 1 + \tau + \delta_{\infty, \pi/4}(\tau) = 5$ , as wanted.

It is interesting to see that the algorithms of Lemma 4 outperform algorithms with different choices of  $\phi$ . For example, it is easy to see that  $E_{1, \pi/4} \geq 6$ . Indeed, note that Algorithm Wireless-Search<sub>1</sub>( $\pi/4$ ) deploys robots at point  $(1/2, 1/2)$ . Robot reaches point  $(0, 1)$  after 1 unit of time, and it reaches point  $(-1, 0)$  after an additional 2 units of time. The other robot is then at point  $(0, -1)$ , at an  $\ell_1$  distance of 2. So, the placement of the exit at point  $(-1, 0)$  induces cost  $1 + 1 + 2 + 2 = 6$ . A similar argument shows that  $E_{\infty, 0} \geq 6$  too.

We conclude this section with a summary of our positive results, introducing at the same time some useful notation.

**Theorem 2.** *Let  $w_p$  be the unique root to equation  $w^p + 1 = 2(1 - w)^p$ , and define*

$$s_p := \begin{cases} \left( (2^p - 1)^{\frac{1}{p-1}} + 1 \right)^{-1/p}, & p \in (1, 2] \\ \left( w_p^{p/(p-1)} + 1 \right)^{-1/p}, & p \in (2, \infty). \end{cases}$$

*For every  $p \in (1, 2]$ , the placement of the exit inducing worst case cost for Algorithm Wireless-Search<sub>p</sub>(0) results in the total explored portion of  $\mathcal{C}_p$  with measure*

$$e_p^- := \pi_p + 2 \int_0^{s_p} \left( z^{p^2-p} (1 - z^p)^{1-p} + 1 \right)^{1/p} dz.$$

*Also, when the exit is reported, robots are at distance  $\gamma_p^- := 2(1 - s_p^p)^{1/p}$ .*

*For every  $p \in [2, \infty)$ , the placement of the exit inducing worst case cost for Algorithm Wireless-Search<sub>p</sub>( $\pi/4$ ) results in the total explored portion of  $\mathcal{C}_p$  with measure*

$$e_p^+ := \pi_p + 2 \int_{2^{-1/p}}^{s_p} \left( z^{p^2-p} (1 - z^p)^{1-p} + 1 \right)^{1/p} dz.$$

*Also, when the exit is reported, robots are at distance  $\gamma_p^+ := 2^{1/p} \left( (1 - s_p^p)^{1/p} + s_p \right)$ .*

*We also set  $e_p$  (and  $\gamma_p$ ) to be equal to  $e_p^-$  (and  $\gamma_p^-$ ) if  $p \leq 2$ , and equal to  $e_p^+$  (and  $\gamma_p^+$ ) if  $p > 2$ , and in particular  $e_p \in (\pi_p, 2\pi_p]$ .*

Quantities  $e_p, \gamma_p$ , and some of their properties are depicted in Figs. 3a, 3b, and discussed in Sect. 4. One can also verify that  $\lim_{p \rightarrow 2^-} e_p^- = \lim_{p \rightarrow 2^+} e_p^+ = 4\pi/3$ , and that  $\lim_{p \rightarrow 2^-} \gamma_p^- = \lim_{p \rightarrow 2^+} \gamma_p^+ = \sqrt{3}$ . In order to justify that indeed  $e_p \in (\pi_p, 2\pi_p]$ , recall that by robots' positions during the first  $\pi_p/2$  search time (after robots reach perimeter in time 1) is an increasing function. Since the rate of change of time is constant (it remains strictly increasing) for the duration of the algorithm, it follows that the evacuation cost of our algorithms remains increasing till some additional search time. Since robots search in parallel and in different parts of  $\mathcal{C}_p$ , and since

$e_p$  is the measure of the combined explored portion of the unit circle, it follows that for  $e_p > 2\pi_p/2 = \pi_p$ . At the same time, the unit circle has circumference  $2\pi_p$ , hence  $e_p \leq 2\pi_p$ .

In other words,  $\gamma_p^-$  is the length of chord with endpoints on  $\mathcal{C}_p$ ,  $p \in (1, 2]$ , defining an arc of length  $e_p^-$ . Similarly,  $\gamma_p^+$  is the length of a chord with endpoints on  $\mathcal{C}_p$ ,  $p \in (2, \infty)$ , defining an arc of length  $e_p^+$ . Unlike the Euclidean unit disks, in  $\ell_p$  unit disks, arc and chord lengths are not invariant under rotation. In other words, arbitrary chords of length  $\gamma_p^-, \gamma_p^+$  do not necessarily correspond to arcs of length  $e_p^-, e_p^+$ , respectively, and vice versa. The claim extends also to the  $\ell_1, \ell_\infty$  spaces. For a simple example, consider points  $A = \rho_1(\pi/4) = (1/2, 1/2), B = \rho_1(3\pi/4) = (-1/2, 1/2), C = \rho_1(0) = (1, 0), D = \rho_1(\pi/2) = (0, 1)$ . It is easy to see that  $d_p(A, B) = 1$  and  $d_p(C, D) = 2$ , while  $\widehat{d}_p(A, B) = \widehat{d}_p(C, D) = \pi_1/2 = 2$ , in other words two arcs of the same length identify chords of different length. We are motivated to introduce the following definition.

**Definition 1.** For every  $p \in [1, \infty)$ , and for every  $u \in [0, 2\pi_p)$ , we define

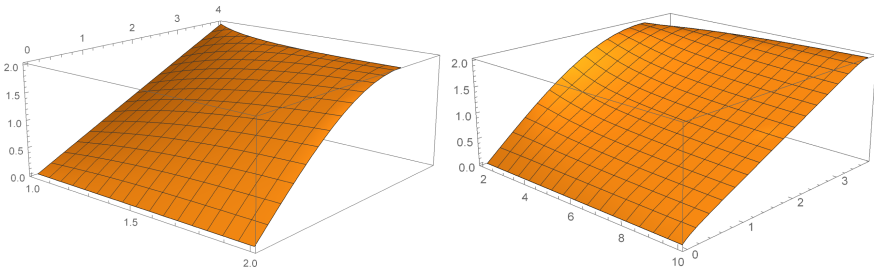
$$\mathcal{L}_p(u) := \min_{A, B \in \mathcal{C}_p} \left\{ \|A - B\|_p : \mu_p(\widehat{AB}) = u \right\}.$$

In other words  $\mathcal{L}_p(u)$  is the length of the shortest line segment (chord) with endpoints in  $\mathcal{C}_p$  at arc distance  $u$  (and corresponding to an arc of measure  $u$ ), and hence  $\mathcal{L}_p(u) = \mathcal{L}_p(2\pi_p - u)$  for every  $u \in (0, 2\pi_p)$ . As a special example, note that  $\mathcal{L}_2(u) = 2 \sin(u/2)$ , as well as  $\mathcal{L}_p(\pi_p) = 2$ , for all  $p \in [1, \infty)$ .

**Lemma 5.** For every  $p \in (1, \infty)$ , function  $\mathcal{L}_p(u)$  is increasing in  $u \in [0, \pi_p]$ .

The intuition behind Lemma 5 is summarized in the following proof sketch. Assuming, for the sake of contradiction, that the lemma is false, there must exist an interval of arc lengths, and some  $p \geq 1$  for which  $\mathcal{L}_p(u)$  is strictly decreasing. By first-order continuity of  $\|A - B\|_p$ , and in the same interval of arc-lengths, chord  $\|A - B\|_p$  must be decreasing in  $\mu_p(\widehat{AB})$  even when points  $A, B$  are conditioned to define a line with a fixed slope (instead of admitting a slope that minimizes the chord length). However, the last statement gives a contradiction. Indeed, consider points  $A, B, A', B'$  such that  $\widehat{d}_p(A, B) = u, \widehat{d}_p(A', B') = u'$ , with  $u < u' \leq \pi_p$ . It should be intuitive that  $d_p(A, B) \leq d_p(A', B) \leq d_p(A', B')$ .

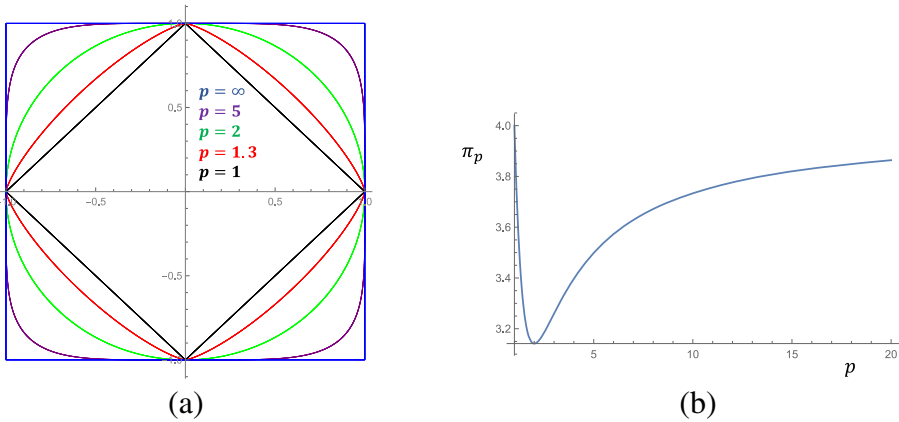
For fixed values of  $p$ , Lemma 5 can also be verified with confidence of at least 6 significant digits in MATHEMATICA. Due to precision limitations, the values of  $p$  cannot be too small, neither too big, even though a modified working precision can handle more values of  $p$ . With standard working precision, any  $p$  in the range between 1.001 and 45 can be handled within a few seconds. As we argue later, for large values of  $p$ , Lemma 5 bears less significance, since in that case we have an alternative way to prove the (near) optimality of algorithms  $\text{Wireless-Search}_p(\phi)$ , as per Theorem 1. Next we provide a visual analysis of function  $\mathcal{L}_p$  that effectively justifies Lemma 5, see Fig. 1.



**Fig. 1.** Figures depict  $\mathcal{L}_p(u)$  for various values of  $p$  and for  $u \in [0, \pi_p]$ . Left-hand side figure shows increasing function  $\mathcal{L}_p(u)$  for  $p \in (1, 2]$ . Right-hand side figure shows increasing function  $\mathcal{L}_p(u)$  for  $p \in [2, 10]$ . Recall that  $\mathcal{L}_2(u) = 2 \sin(u/2)$ ,  $\mathcal{L}_p(\pi_p) = 2$ , for all  $p \in [1, \infty)$ , as well as that  $\pi_1 = \pi_\infty = 4$  and  $\pi_p < 4$  for  $p \in (1, \infty)$ .

### 4 Visualization of Key Concepts and Results

In this section we provide visualizations of some key concepts, along with visualizations of our results. The Figures are referenced in various places in our manuscript but we provide self-contained descriptions.

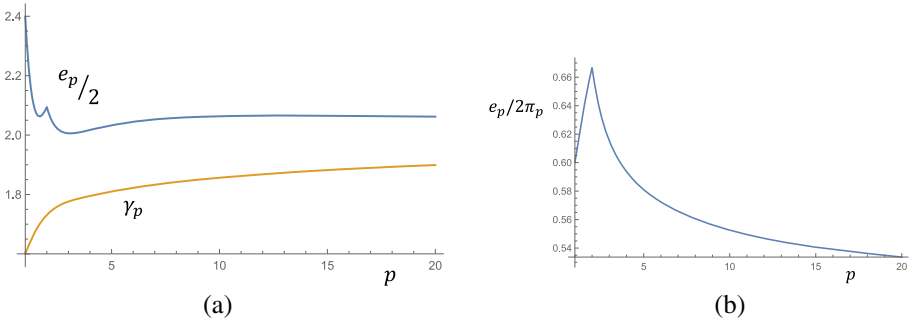


**Fig. 2.** (a) Unit circles  $\mathcal{C}_p$ , for  $p = 1, 1.3, 2, 5, \infty$ , induced by the  $\ell_p$  norm. Circles are nested, starting from  $p = 1$  for the inner diamond-shaped circle, moving monotonically to  $p = \infty$  for the outer square-shaped circle. (b) The behavior of  $\pi_p$  as  $p$  ranges from 1 up to  $\infty$ , where  $\pi_1 = \pi_\infty = 4$  and  $\pi_2 = \pi$  is the smallest value of  $\pi_p$ .

Figures 3a and 3b depict quantities pertaining to algorithm  $\text{Wireless-Search}_p(\phi)$  (where  $\phi = 0$ , if  $p \in [1, 2)$  and  $\phi = \pi/4$ , if  $p \in (2, \infty)$ ) for the placement of the hidden exit inducing the worst case cost. Moreover Fig. 3a depicts quantities  $e_p/2, \gamma_p$ , as per Theorem 2. In particular, for each  $p$ , quantity  $e_p/2$  is the time a searcher

has spent searching the perimeter of  $\mathcal{C}_p$  till the hidden exit is found (in the worst case). Therefore,  $e_p$  represents the portion of the perimeter that has been explored till the exit is found. Quantity  $\gamma_p$  is the distance of the two robots at the moment the hidden exit is found so that the total cost of the algorithm is  $1 + e_p/2 + \gamma_p$ . By [21] we know that  $e_2 = 4\pi/3$  and  $\gamma_2 = \sqrt{3}$ . Our numerical calculations also indicate that  $\lim_{p \rightarrow 1} e_p = 12/5$ ,  $\lim_{p \rightarrow 1} \gamma_p = 8/5$ , and  $\lim_{p \rightarrow \infty} e_p = \lim_{p \rightarrow \infty} \gamma_p = 2$ .

Figure 3b depicts quantities  $e_p/2\pi_p$ , which equals the explored portion of the unit circle  $\mathcal{C}_p$ , relative to its circumference, of Algorithm Wireless-Search $_p(\phi)$  (where  $\phi = 0$ , if  $p \in [1, 2)$  and  $\phi = \pi/4$ , if  $p \in (2, \infty)$ ) when the worst case cost inducing exit is found. By [21] we know that  $e_2/2\pi_2 = (4\pi/3)/2\pi = 2/3$ . Interestingly, quantity  $e_p/2\pi_p$  is maximized when  $p = 2$ , that is in the Euclidean plane searchers explore the majority of the circle before the exit is found, for the placement of the exit inducing worst case cost. Also, numerically we obtain that  $\lim_{p \rightarrow 1} e_p/2\pi_p = 3/5$ , and  $\lim_{p \rightarrow \infty} e_p/2\pi_p = 1/2$ . The reader should contrast the limit valuations with Lemma 4 according which in both cases  $p = 1, \infty$  the cost of our search algorithms is constant and equal to 5 for all placements of the exit that are found from the moment searchers have explored half the unit circle and till the entire circle is explored.

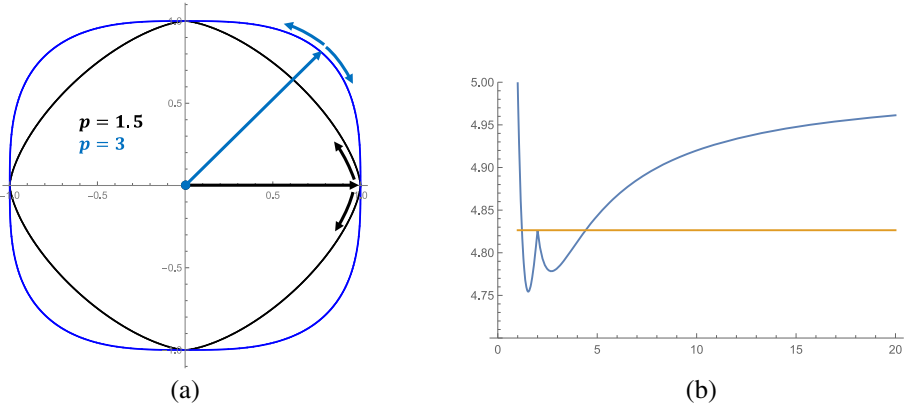


**Fig. 3.** (a) Perimeter search time  $e_p/2$  and distance  $\gamma_p$  between searchers when worst case cost inducing exit is found as a function of  $p$ , see also Theorem 2. (b) Explored portion  $e_p/2\pi_p$  as a function of  $p$ .

Figure 4b shows the worst case performance analysis of Algorithm Wireless-Search $_p(\phi)$  (where  $\phi = 0$ , if  $p \in [1, 2)$ ), which is also optimal for problem WE $_p$ . As per Lemma 4, the evacuation cost is 5 for  $p = 1$  and  $p = \infty$ . The smallest (worst case) evacuation cost when  $p \in [1, 2]$  is 4.7544 and is attained at  $p \approx 1.5328$ . The smallest (worst case) evacuation cost when  $p \in [2, \infty]$  is 4.7784 and is attained at  $p \approx 2.6930$ . As per [21], the cost is  $1 + \sqrt{3} + 2\pi/3 \approx 4.82644$  for the Euclidean case  $p = 2$ .

## 5 Lower Bounds and the Proof of Theorem 1

First we prove a weak lower bound that holds for all  $\ell_p$  spaces,  $p \geq 1$  (see also Fig. 2b for a visualization of  $\pi_p$ ).



**Fig. 4.** (a) Figure depicts robots' trajectories for algorithms  $\text{Wireless-Search}_{1.5}(0)$  and  $\text{Wireless-Search}_3(\pi/4)$ . The inner unit circle corresponds to  $p = 1.5$  and the outer to  $p = 3$ . (b) Blue curve depicts the worst case evacuation cost of Algorithm  $\text{Wireless-Search}_p(\phi)$ , where  $\phi = 0$ , if  $p \in [1, 2)$ , as a function of  $p$ . Yellow line (constant function) is the optimal evacuation cost in the Euclidean metric space.

**Lemma 6.** For every  $p \in [1, \infty)$ , the optimal evacuation cost of  $\text{WE}_p$  is at least  $1 + \pi_p$ .

*Proof.* The circumference of  $\mathcal{C}_p$  has length  $2\pi_p$ . Two unit speed robots can reach the perimeter of  $\mathcal{C}_p$  in time at least 1. Since they are searching in parallel, in additional time  $\pi_p - \epsilon$ , they can only search at most  $2\pi_p - 2\epsilon$  measure of the circumference. Hence, there exists an unexplored point  $P$ . Placing the exit at  $P$  shows that the evacuation time is at least  $1 + \pi_p - 2\epsilon$ , for every  $\epsilon > 0$ .

In particular, recall that  $\pi_1 = \pi_\infty = 4$ , and hence no evacuation algorithm for  $\text{WE}_1$  and  $\text{WE}_\infty$  has cost less than 5. As a corollary, we obtain that Algorithms  $\text{Wireless-Search}_1(0)$  and  $\text{Wireless-Search}_\infty(\pi/4)$  are optimal, hence proving the special cases  $p = 1, \infty$  of Theorem 1. The remaining cases require a highly technical treatment.

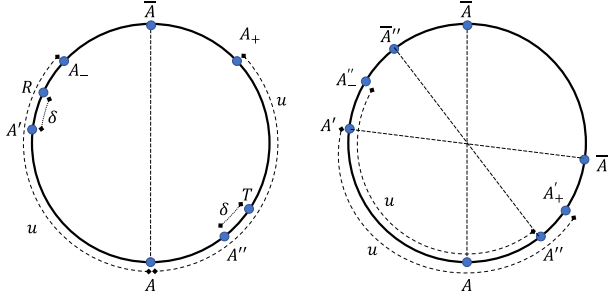
The following is a generalization of a result first proved in [21] for the Euclidean metric space (see Lemma 5 in the Appendix of the corresponding conference version). The more general proof is very similar.

**Lemma 7.** For every  $V \subseteq \mathcal{C}_p$ , with  $\mu_p(V) \in (0, \pi_p]$ , and for every small  $\epsilon > 0$ , there exist  $A, B \in V$  with  $\widehat{d}_p(A, B) \geq \mu_p(V) - \epsilon$ .

*Proof.* For the sake of contradiction, consider some  $V \subseteq \mathcal{C}_p$ , with  $\mu_p(V) \in (0, \pi_p]$ , and some small  $\epsilon > 0$ , such no two points both in  $V$  have arc distance at least  $\mu_p(V) - \epsilon$ . Below we denote the latter quantity by  $u$ , and note that  $u \in (0, \pi_p)$ , as well as that  $\mu_p(V) = u + \epsilon > u$ . We also denote by  $V^c$  the set  $\mathcal{C}_p \setminus V$ . The argument below is complemented by Fig. 5.

Since  $V$  is non-empty, we consider some arbitrary  $A \in V$ . We define the set of antipodal points of  $V$

$$N := \{B \in \mathcal{C}_p : \exists C \in V, \widehat{d}_p(B, C) = \pi_p\}$$



**Fig. 5.** An abstract  $\ell_p$  unit circle, for some  $p \geq 1$ , depicted as a Euclidean unit circle for simplicity. On the left we depict points  $A, \bar{A}, A_-, A_+, A', A'', T, R$ . On the right we keep only the points relevant to our final argument, and add also points  $A'', A'_+, \bar{A}', \bar{A}''$ .

Note that  $N \cap V = \emptyset$  as otherwise we have a contradiction, i.e. two points in  $V$  with arc distance  $\pi_p > u = \mu_p(V) - \epsilon$ . In particular, we conclude that  $N \subseteq V^c$ , and hence by Lemma 1 we have  $\mu_p(N) = \mu_p(V) = u + \epsilon$ .

Let  $\bar{A}$  be the point antipodal to  $A$ , i.e.  $\widehat{d}_p(A, \bar{A}) = \pi_p$ . Next, consider points  $A_-, A_+ \in \mathcal{C}_p$  at anti-clockwise and clockwise arc distance  $u$  from  $A$ , that is  $\widehat{d}_p(A, A_-) = \widehat{d}_p(A_+, A) = u$ . All points in  $\widehat{A_+ A_-}$  are by definition at arc distance at least  $u$  from  $A$ . In particular,  $\bar{A} \in \widehat{A_+ A_-}$  and  $A_- \in \widehat{\bar{A} A}, A_+ \in \widehat{\bar{A} A}$ . We conclude that  $V \cap \widehat{A_+ A_-} = \emptyset$ , as otherwise we have  $A \in V$  together with some point in  $V \cap \widehat{A_+ A_-}$  make two points with arc distance at least  $u$ . Note that this implies also that  $\widehat{A_+ A_-} \subseteq V^c$ .

Consider now the minimal, inclusion-wise, arc  $\widehat{TR} \subseteq V^c$ , containing  $\widehat{A_+ A_-}$ . Such arc exists because  $A_-, A_+ \in \widehat{A_+ A_-} \subseteq V^c$ . In particular, since  $A \in V$ , we have that  $R \in \widehat{A_- A}$  and  $T \in \widehat{A A_+}$ .

For some arbitrarily small  $\delta > 0$ , with  $\delta < \min\{u, \epsilon/2\}$ , let  $A', A'' \in V$  such that  $\widehat{d}(RA') = d(A''T) = \delta$ . Such points  $A', A''$  exist, as otherwise  $\widehat{TR}$  would not be minimal. Clearly, we have  $A' \in \widehat{RA}$  and  $A'' \in \widehat{AT}$ .

Since  $A' \in \widehat{RA} \subseteq \widehat{AA}$ , its antipodal point  $\bar{A}'$  lies in  $\widehat{AA}$ . Similarly, since  $A'' \in \widehat{AT} \subseteq \widehat{AA}$ , its antipodal point  $\bar{A}''$  lies in  $\widehat{AA}$ . Finally, we consider point  $A''_+$  at clockwise arc distance  $u$  from  $A''$ , and point  $A'_+$  at anti-clockwise distance  $u$  from  $A'$ , that is  $\widehat{d}(A''_+, A'') = \widehat{d}(A'_+, A') = u$ . We observe that  $A''_+ \in \widehat{\bar{A}'' A}$  and  $A'_+ \in \widehat{\bar{A}' A}$ .

Recall that  $A'' \in V$ , hence  $\widehat{\bar{A}'' A} \subseteq V^c$ , as otherwise any point in  $\widehat{\bar{A}'' A} \cap V$  together with  $A''$ , at arc distance at least  $u$ , would give a contradiction. Similarly, since  $A' \in V$ , hence  $\widehat{\bar{A}' A} \subseteq V^c$ , as otherwise any point in  $\widehat{\bar{A}' A} \cap V$  together with  $A'$ , at arc distance at least  $u$ , would give a contradiction.

Lastly, abbreviate  $\widehat{\bar{A}'' A}, \widehat{\bar{A}' A}$  by  $X, Y$ , respectively. Note that  $\mu_p(X) = \mu_p(\widehat{\bar{A}'' A} \setminus \widehat{A''_+ A''}) = \mu_p(\widehat{\bar{A}'' A}) - \mu_p(\widehat{A''_+ A''}) = \pi_p - u$ . Similarly,  $\mu_p(Y) = \mu_p(\widehat{\bar{A}' A} \setminus \widehat{A'_+ A'}) = \mu_p(\widehat{\bar{A}' A}) - \mu_p(\widehat{A'_+ A'}) = \pi_p - u$ . Recall that  $A''_+ \in \widehat{\bar{A}'' A}$  and  $A'_+ \in \widehat{\bar{A}' A}$ , and hence sets

$X, Y$  intersect either at point  $A$  or have empty intersection. As a result  $\mu_p(X \cap Y) = 0$ , as well as  $\mu_p(N \cap X \cap Y) = 0$ .

Recall that  $\mu_p(\widehat{RA'}) = \mu_p(\widehat{A''T}) = \delta$ , and so by Lemma 1 we also have  $\mu_p(X \cap N) = \mu_p(Y \cap N) = \delta$ . But then, using inclusion-exclusion for measure  $\mu_p$ , we have

$$\begin{aligned} \mu_p(N \cup X \cup Y) &= \mu_p(N) + \mu_p(X) + \mu_p(Y) - \mu_p(N \cap X) - \mu_p(N \cap Y) - \mu_p(X \cap Y) + \mu_p(N \cap X \cap Y) \\ &= u + \epsilon + \pi - u + \pi - u - \delta - \delta - 0 + 0 \\ &= 2\pi_p - u + \epsilon - 2\delta \\ &> 2\pi_p - u \\ &> 2\pi_p - \mu_p(V) \\ &= \mu_p(V^c). \end{aligned}$$

Hence  $\mu_p(N \cup X \cup Y) > \mu_p(V^c)$ . On the other hand, recall that  $N, X, Y \subseteq V^c$ , hence  $N \cup X \cup Y \subseteq V^c$ , hence  $\mu_p(N \cup X \cup Y) \leq \mu_p(V^c)$ , which is a contradiction.

We are now ready to prove a general lower bound for  $\text{WE}_p$  which we further quantify later.

**Lemma 8.** *For every  $p \in (1, \infty)$ , the optimal evacuation cost of  $\text{WE}_p$  is at least  $1 + e_p/2 + \mathcal{L}_p(e_p)$ .*

*Proof.* Consider an arbitrary evacuation algorithm  $\mathcal{A}$ . We show that the cost of  $\mathcal{A}$  is at least  $1 + e_p + \mathcal{L}_p(e_p)$ . By Theorem 2, we have that  $e_p \in (\pi_p, 2\pi_p]$ . Let  $\epsilon > 0$  be small enough, were in particular  $\epsilon < e_p - \pi_p$ . We let evacuation algorithm  $\mathcal{A}$  run till robots have explored exactly  $e_p - \epsilon$  part of  $\mathcal{C}_p$ .

The two unit speed robots need time 1 to reach the perimeter of  $\mathcal{C}_p$ . Since moreover they (can) search in parallel (possibly different parts of the unit circle), they need an additional time at least  $(e_p - \epsilon)/2$  in order to explore measure  $e_p - \epsilon$ . The unexplored portion  $V$  of  $\mathcal{C}_p$  has therefore measure  $u := 2\pi_p - (e_p - \epsilon)$ , where  $u \in (0, \pi_p)$ .

By Lemma 7, there are two points  $A, B \in V$  that are at an arc distance  $v \geq u - \epsilon = 2\pi_p - e_p$ . By definition, both points  $A, B$  are unexplored. We let algorithm  $\mathcal{A}$  run even more and till the moment any one of the points  $A, B$  is visited by some robot, and we place the exit at the other point (even if points are visited simultaneously), hence algorithm  $\mathcal{A}$  needs an additional time  $d_p(A, B)$  to terminate, for a total cost at least  $1 + e_p/2 - \epsilon/2 + d_p(A, B)$ . But then, note that  $d_p(A, B) \geq \mathcal{L}_p(v) \geq \mathcal{L}_p(2\pi_p - e_p)$ , where the first inequality is due Definition 1 and the second inequality due to Lemma 5, and the claim follows by recalling that  $\mathcal{L}_p(2\pi_p - e_p) = \mathcal{L}_p(e_p)$ .

Recall that, for every  $p \in (1, \infty)$ , the evacuation algorithms we have provided for  $\text{WE}_p$  have cost  $1 + e_p/2 + \gamma_p$ . At the same time, Lemma 8 implies that no evacuation algorithm has cost less than  $1 + e_p/2 + \mathcal{L}_p(e_p)$ . So, the optimality of our algorithms, that is, the proof of Theorem 1, is implied directly by the following lemma, which is verified numerically.

**Lemma 9.** *For every  $p \in (1, \infty)$ , we have  $\mathcal{L}_p(e_p) = \gamma_p$ .*

## 6 Discussion

We provided tight upper and lower bounds for the evacuation problem of two searchers in the wireless model from the unit circle in  $\ell_p$  metric spaces,  $p \geq 1$ . This is just a starting point of revisiting well studied search and evacuation problems in general metric spaces that do not enjoy the symmetry of the Euclidean space. In light of the technicalities involved in the current manuscript, we anticipate that the pursuit of the aforementioned open problems will also give rise to new insights in convex geometry and computational geometry.

## References

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