# 9 Differentiability and the Secant Slope Function

## 9.1 Statement of the Teaching Problem

The familiar concept of the *slope* of a line can be understood from multiple points of view. Computationally it is given by the formula $\frac{y_2-y_1}{x_2-x_1}$, where $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the line. Geometrically it represents how much the $y$-value increases or decreases for each 1-unit increase in the $x$-value. Physically the slope represents the rate of change of $y$ with respect to $x$. For instance, if $y$ is the height of an object (in meters) at time $x$ (measured in seconds), then the slope is the velocity of the object measured in meters per second.

The core idea of differential calculus is to generalize the fundamental notion of slope to functions that are not straight lines. With linear functions, the slope is a *global* feature—it is constant and can be computed using any two points on the line. For non-linear functions we have to shift our understanding of slope from a global feature to a *local* one. We ask, "What is the slope, or rate of change, at a particular point on the function?" This is the impetus behind the concept of the derivative. Like the notion of slope, the derivative can be viewed from multiple points of view. It has a formal definition that incorporates the computational formula of slope, which we review momentarily. Geometrically, the derivative can be understood as the "slope" as a point on a curve. Likewise, the physical interpretation of the derivative is in regard to a rate of change, but there is the significant complication in that this rate of change can be different at different points on the function. This is what it means to say the slope is a local feature. First we must specify a point on the function, and then we set ourselves the task of finding the "slope" at this particular point.

This brings us face to face with the central dilemma of defining the derivative. Intuitively, the derivative is the "slope" at a single point on the graph, but the familiar algebraic formula for computing slope requires two points.

Consider the following pedagogical situation:



N. H. Wasserman et al., *Understanding Analysis and its Connections to Secondary Mathematics Teaching*, Springer Texts in Education,
https://doi.org/10.1007/978-3-030-89198-5_9

A calculus teacher, Mr. Petrov, is having students practice calculating the derivative and uses a problem from the textbook:

$$\text{Calculate } f'(1) \text{ for } f(x) = \begin{cases} 2x + 1, & x \geq 1 \\ 2x - 1, & x < 1 \end{cases}$$

After some discussion, the class ends up split into two groups claiming two different answers. The first side argues $f'(1) = 2$ because derivatives are about slope, and the slope is 2 for both parts of this piecewise definition. The second side argues $f'(1)$ is not defined, because they remember something about differentiable functions needing to be continuous, and the function is discontinuous at $x = 1$.

Although both groups give justifications for their conclusions, both cannot be correct. These kinds of disagreements are common in teaching. Identifying which group has the right answer is a necessary part of the teacher's responsibility; but the more important challenge is finding a justification that explains why the conclusion is valid (TP.5). In situations where students cannot agree, both sides likely have something meaningful—yet incomplete—to contribute to the conversation. In this scenario, slope and continuity are both connected to the derivative, and each concept turns out to be essential for finding a satisfying resolution to the debate.

Before reading on, which group do you think has come to the right conclusion? Do you feel their reasoning is sufficient? If you were the teacher, how would you respond to the students in the class?

## 9.2 Connecting to Secondary Mathematics

### 9.2.1 Problematizing Teaching and the Pedagogical Situation

Let's start with a closer look at how each side arrived at its conclusion.

The group with the answer $f'(1) = 2$ is evidently thinking about the derivative as slope. Vertically-translated lines like $y = 2x$, $y = 2x + 1$, and $y = 2x + 2$ all have the same slope. Interpreting the derivative as the slope of the curve, this group reasoned that even though the function has a discontinuity—it "jumps" up—the slope does not change with this vertical shift and hence the derivative should not change. This conclusion emerges naturally from looking at each piece of the function individually. If we define the first part of $f$ to be $f_1(x) = 2x + 1$ for $x \geq 1$, then it would appear that $f_1'(x) = 2$ for $x \geq 1$ (and at least certainly for $x > 1$). Likewise, if we define the second part to be $f_2(x) = 2x - 1$ for $x < 1$, then its derivative is clearly $f_2'(x) = 2$ for $x < 1$. Since approaching $x = 1$ from the right side and the left side both point to the conclusion $f'(1) = 2$, there is a compelling case to be made that $f'(1)$ should indeed equal 2.

The second group concludes the derivative is not defined and appears to be reasoning based on a theorem about derivatives. Their argument relies on the recollection that differentiability requires continuity (Diff. $\Longrightarrow$ Cont.) or, in logical terms, "If a function is differentiable at a point then it is continuous at that point." The group is actually invoking the contrapositive of this result: "If a function is not continuous at a point, it is not differentiable either." When working with directional statements like these, it can be easy to confuse the contrapositive with its converse, and there is also the possibility that the group has misremembered the theorem altogether. Perhaps it is that continuous functions are differentiable (Cont. $\Longrightarrow$ Diff.)? When relationships are memorized, the chain of reasoning can only go so far. Regardless of whether they have the right answer, the students have not provided any further reasoning to support their conclusion, which suggests their thinking is based solely on a recollection of some rule and not a robust understanding of how continuity impacts the existence of the derivative.

Pause to consider whether either of these two answers seem more correct than the other now. Was there anything that changed your mind?

### 9.2.2 Recognizing Computations as Singular Objects

Many mathematical ideas start out as processes, and then those processes turn into objects of study themselves. Some suggest learning mathematics is akin to development through such conceptual transitions.[1] A computation is initially understood as a procedure on constituent *parts*, but when we start reasoning about the resultant computation as a *whole* without actually executing the procedure, we are recognizing the computation as a singular object.[2]

Consider a basic computation like $2 + 5$. Children first learn about addition in terms of physical actions. They might put a group of 2 marbles together with a group of 5 marbles and then count all the marbles—so the answer is 7. The symbol '+' signifies that two groups of objects should be joined together, defining addition as the process of joining *two individual numbers*, $(2) + (5)$. But as our understanding of addition progresses, there comes a moment when the computation becomes a *singular object* itself—the "sum." We do not need to go through the process to arrive at 7; instead we regard the entire calculation as a singular expression, $(2+5)$. The calculation is understood as an object, interchangeable with 7 because they both represent the sum.

Another example of this phenomenon, especially pertinent to discussing the derivative, is the computation of slope. Initially, the formula $\frac{y_2-y_1}{x_2-x_1}$ is interpreted

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[1] APOS theory [2] posits that mathematical learning transitions through phases; understanding mathematical concepts as actions (A), then processes (P), then objects (O), then schemas (S).

[2] Sfard [3] describes this process as "reification."

by students as a series of three computations—calculate $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$, and then divide. But eventually it is important to see the entire expression $\left(\frac{y_2-y_1}{x_2-x_1}\right)$ holistically as representing the slope of the line through two given points.

## 9.3 Connecting to Real Analysis

To connect this discussion to the content of real analysis content, let's look carefully at the formal definition of the derivative and appreciate how it incorporates the notion of slope. For a given function $g(x)$ and a given value $c$ in the domain of $g$, the derivative is formally defined as

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

The first critical step in understanding this definition is recognizing the quotient $\frac{g(x)-g(c)}{x-c}$ as the slope calculation between two points—a given point $(c, g(c))$ and another point $(x, g(x))$ somewhere else on the function. The second step is appreciating the role of the limiting operation $\lim_{x \to c}$. This limit really asks, "What happens to the slope value as the point $(x, g(x))$ becomes 'very close' to the fixed point $(c, g(c))$?" This is how calculus solves the riddle of computing the slope at a single point on a curve. The idea is to compute the old-fashioned slope through two points on the curve and then think about what happens as those two points get closer together. This is where the tools of real analysis such as limits, and whether they exist, can be brought to bear.

### 9.3.1 The Secant Slope Function

Throughout the subsequent sections, we will use the example

$$g(x) = \tfrac{1}{10}x^3 - \tfrac{3}{10}x^2 - \tfrac{9}{5}x + 5.$$

Like the elementary school student learning about addition for the first time, it is natural at first to view the definition of $g'(c)$ as a computational process involving constituent parts. First, we must pick a fixed value, $c$. Let's start with $c = 2$ as an example, which means we can compute $g(c) = g(2) = 1$. So both $c$ and $g(c)$ are real numbers. To compute $\frac{g(x)-g(c)}{x-c}$, we might give ourselves a selection of $x$ values. Taking $x = 1$, for instance, yields $g(1) = 3$. This gives us two points on the graph of $g$—the fixed point $(2, 1)$, at which we are trying to find the derivative, and a second point $(1, 3)$. The slope through these two points is $\frac{3-1}{1-2} = \frac{2}{-1} = -2$. Table 9.1 displays the results of repeating this calculation for a selection of other $x$-

**Table 9.1** Computing the slopes of individual secant lines with $c = 2$

| $x$ | $(x, g(x))$ | $(2, g(2))$ | $g(x) - g(2)$ | $x - 2$ | $\frac{g(x)-g(2)}{x-2}$ |
|---|---|---|---|---|---|
| 1 | (1,3) | (2,1) | 2 | $-1$ | $-2$ |
| 1.5 | (1.5, 1.96...) | (2,1) | 0.96... | $-0.5$ | $-1.92...$ |
| 2 | (2,1) | (2,1) | 0 | 0 | und. |
| 2.5 | (2.5, 0.18...) | (2,1) | $-0.81...$ | 0.5 | $-1.62...$ |
| 3 | (3, $-0.4$) | (2,1) | $-1.4$ | 1 | $-1.4$ |
| 4 | | | | | |
| 5 | | | | | |

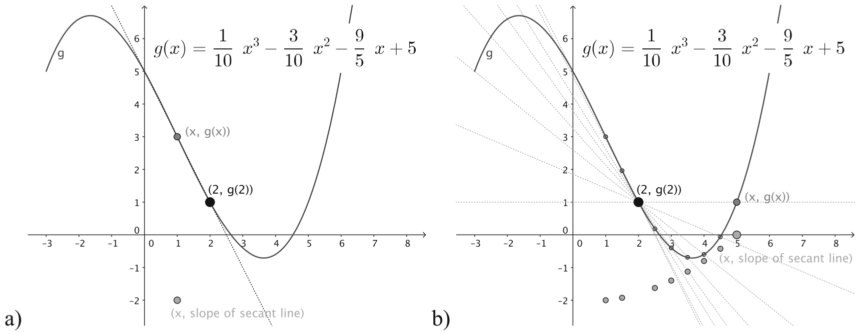


**Fig. 9.1** Secant line(s) and slope(s), with $c = 2$, for a) $x = 1$, and b) $x = 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5$, and 5

values. Note that for $x = 2$ the calculation breaks down. (Do you see why?) Before moving on, complete the table for $x = 4$ and $x = 5$ (while keeping $c = 2$).

A line through two points on a curve is called a *secant line*. The secant line through (1, 3) and (2, 1) on $g$ is depicted in Fig. 9.1a. This corresponds to our using $x = 1$ and results in a secant line with slope $-2$. To capture this result, Fig. 9.1a also contains the point $(1, -2)$. Figure 9.1b plots all of the secant lines associated with the $x$-values in the table, as well as the points corresponding to the secant line slopes for each value of $x$. Turning our attention to the definition of $g'(2)$, we see that we are most interested in computations when the $x$-values are close to $c = 2$. At $x = 1.5$ we found a slope of approximately $-1.92$; at $x = 2.5$ the slope is roughly $-1.62$. These values give us a sense of the slope as we approach $c = 2$.

This process of computing individual slopes is a useful way to begin to understand the definition of the derivative. However, focusing too intently on individual computations does not capture the big picture. A proper understanding requires looking holistically at this process across all $x$-values, secant lines, and slopes. Instead of interpreting $\frac{g(x)-g(c)}{x-c}$ as a series of arithmetic computations, it is more helpful to view the expression as a singular object. Notably, since $g(x)$ and $x - c$ are functions, the expression is a quotient of two functions, and so the "object" it represents is a (new) *function*. This means thinking about $\left(\frac{g(x)-g(c)}{x-c}\right)$ as a function

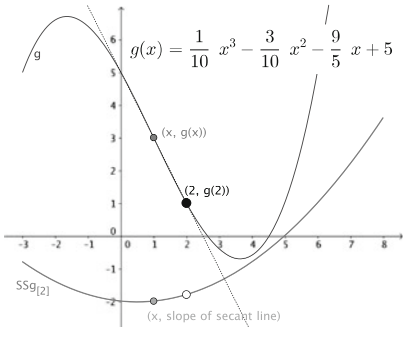


**Fig. 9.2** The secant slope function of $g$ with $c = 2$

of $x$.[3] The case where $c = 2$ which we are currently considering yields the function $\left(\frac{g(x)-1}{x-2}\right)$. What is this new function? The $x$ inputs are the locations of the second point $(x, g(x))$ (the first point for us is always $(2, 1)$), and the outputs are the slope values of the resulting secant lines. We refer to as this new function as the *secant slope function*. Its' construction is specific to a given point $c$ and function $g(x)$; the notation we use to denote it, $SSg_{[c]}(x)$, specify both of these givens.

**Definition** For a given function $g(x)$, and a given point $c$ (in the domain of $g$), the **Secant Slope Function** $\boldsymbol{SSg_{[c]}(x)}$ is $\frac{g(x)-g(c)}{x-c}$, which takes each input $x$ (in the domain of $g$), and outputs the slope of the secant line between $(x, g(x))$ and $(c, g(c))$.

Figure 9.2 depicts the graph of the secant slope function for our example with $c = 2$, which is $SSg_{[2]}(x) = \frac{g(x)-1}{x-2}$. Comparing Figures 9.2 and 9.1b highlights the fact that the secant slope function is the composite result of *all* of the individual calculations from different $x$-values. Indeed, since $x$ can vary, we might even imagine the second point on the function "moving," and at each new $x$-value representing the slope of the corresponding secant line.

We are now at a better vantage point from which to think about the '$\lim_{x\to c}$' portion of the definition of $g'(c)$. From the more primitive computational point of view, the best we can do is gather empirical evidence about the limit from a handful of slope computations when $x$ is close to $c$. Transitioning to the idea that $\frac{g(x)-g(c)}{x-c}$ is not a multi-step computational process but a singular function provides new insight. In this light, the derivative at $c$ is understood as the functional limit as $x$ approaches $c$ of this new secant slope function; that is, $g'(c) = \lim_{x\to c} SSg_{[c]}(x)$. Even without engaging the formal $\varepsilon - \delta$ definition of functional limits, it is more straightforward to make a determination on whether the limit of the secant slope function exists as $x$ approaches $c$. Every $SSg_{[c]}(x)$ has a hole when $x = c$ (why?), but now we see that

[3] It is helpful to remind yourself $c$ is a constant in this expression (even though we will also eventually want to think of many different $c$-values).

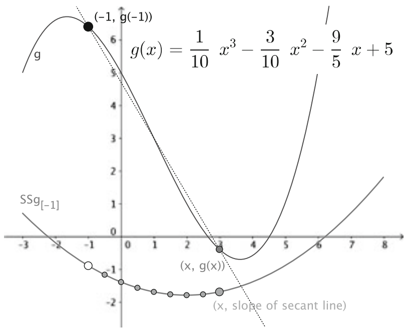


**Fig. 9.3** Graph of $SSg_{[-1]}(x)$, where $g'(-1)$ is found by evaluating '$\lim_{x\to -1}$' of this function

$g'(c)$ is equivalent to whatever value is the functional limit $\lim_{x\to c} SSg_{[c]}(x)$, if it exists, which is clearer to conceptualize. In the case of our particular example, this limit on our secant slope function comes out to be $-1.8$.

### 9.3.2 The Derivative as a Function

Having reached a more robust understanding of the definition of the derivative at a given point $c$ in the domain of $g$, we make one last conceptual leap and consider $c$ to be a variable so that $g'(c)$ becomes a function in its own right. For $c = 2$ we found $g'(2) = -1.8$. To find $g'(c)$ for some other value of $c$ we return to the secant slope function $SSg_{[c]}(x)$ and note that this function will be different for different values of $c$. Figure 9.3 demonstrates this difference; it shows the secant slope function for the same function $g$ but with a new value $c = -1$. To compute $g'(-1)$ we take the limit of $SSg_{[-1]}(x)$ as $x$ approaches $-1$ to find $g'(-1) = -0.9$.

In general, each value of $c$ determines a specific secant slope function via the formula

$$SSg_{[c]}(x) = \frac{g(x) - g(c)}{x - c},$$

from which we compute $g'(c)$ by taking the limit as $x$ approaches $c$. The derivative function $g'$ is then a record or compilation of *all* these functional limits obtained from *all* of these different secant slope functions. Table 9.2 shows the computations for generating the derivative function at a few selected values of $c$, and Fig. 9.4 depicts $g'$. The figure also displays a particular point $(c, g'(c))$ on the derivative function, which is the result of evaluating $\lim_{x\to c} SSg_{[c]}(x)$. To illustrate this relationship, the secant slope function $SSg_{[c]}$ for this singular value of $c = 2$ is also included.

Before moving on, try to replicate the processes in this section until you feel comfortable with the idea of a secant slope function. Using $g(x) = x^2$ as an example, set $c = 1$ and sketch the secant lines you get using $x = 3, 2, 2.5$ and $2.1$. Find an algebraic expression for $SSg_{[1]}(x)$ and simplify as much as possible.

**Table 9.2** Computations for $g'(x)$ from secant slope functions, for $x = 2, 1, 0, -1$

| $c$ | $SSg_{[c]}(x)$ | $\lim_{x\to c}$ | $g'(c)$ | $(x, g'(x))$ |
|---|---|---|---|---|
| 2 | $SSg_{[2]}(x) = \frac{g(x)-1}{x-2}$ | $\lim_{x\to 2} SSg_{[2]}(x) = -1.8$ | $-1.8$ | $(2, -1.8)$ |
| 1 | $SSg_{[1]}(x) = \frac{g(x)-3}{x-1}$ | $\lim_{x\to 1} SSg_{[1]}(x) = -2.1$ | $-2.1$ | $(1, -2.1)$ |
| 0 | $SSg_{[0]}(x) = \frac{g(x)-5}{x-0}$ | $\lim_{x\to 0} SSg_{[0]}(x) = -1.8$ | $-1.8$ | $(0, -1.8)$ |
| $-1$ | $SSg_{[-1]}(x) = \frac{g(x)-6.4}{x+1}$ | $\lim_{x\to -1} SSg_{[-1]}(x) = -0.9$ | $-0.9$ | $(-1, -0.9)$ |

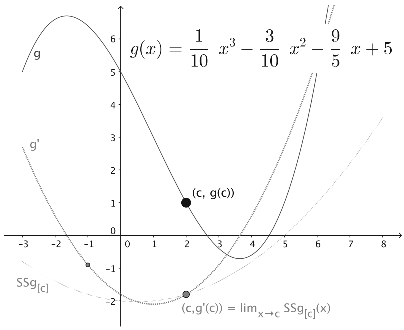


**Fig. 9.4** Graph of $g'(x)$; the point labeled $(c, g'(c))$ is the $\lim_{x\to c} SSg_{[c]}(x)$ (for $c = 2$)

What is $\lim_{x\to 1} SSg_{[1]}(x)$? Try this again for $c = 0$. Find $SSg_{[c]}(x)$ for an arbitrary $c$ and use it to find $g'(c)$. How about $SSg_{[c]}(x)$ for $g(x) = x^3$? You should be very comfortable with secant slope functions before reading on!

### 9.3.3 Derivatives and Continuity

After defining the derivative at a point, it is common in a real analysis course to state and prove the following result (Theorem 5.2.3 in Abbott's [1] text).

**Theorem** If $g$ is differentiable at a point $c$, then $g$ is continuous at $c$ as well.

***Proof*** By assumption, $g'(c)$ exists, meaning the $\lim_{x\to c} \frac{g(x)-g(c)}{x-c}$ exists. Based on the Algebraic Limit Theorem for functional limits, we have:

$$\begin{aligned}\lim_{x\to c}(g(x) - g(c)) &= \lim_{x\to c}\left(\frac{g(x)-g(c)}{x-c}\right)(x-c)\\ &= \lim_{x\to c}\left(\frac{g(x)-g(c)}{x-c}\right)\cdot \lim_{x\to c}(x-c)\\ &= g'(c)\cdot 0\\ &= 0.\end{aligned}$$

It follows that $\lim_{x \to c}(g(x) - g(c)) = 0$, and thus $\lim_{x \to c} g(x) = g(c)$. This shows $g$ is continuous at $c$. □

The proof just given is a standard one for this theorem (it's the one in Abbott, for instance), but it engages in a bit of algebraic trickery. Beginning with the expression $g(x) - g(c)$, we first multiply by 1 written in the form $\frac{x-c}{x-c}$. (We don't have to worry about the case where $x - c = 0$ because the limit as $x$ approaches $c$ is independent of what happens when $x = c$.) The rest of the proof is a straightforward application of the Algebraic Limit Theorem for functional limits (Corollary 4.2.4 in Abbott). Although it is concise and properly substantiates the claim, this proof does not offer much intuition for *why* the theorem is true.

In search of a more enlightening proof, let's consider the logically equivalent contrapositive statement:

**Corollary** If $g$ is not continuous at a point $c$, then $g$ is not differentiable at $c$ either.

This statement goes to the heart of the teaching challenge described at the beginning of the chapter. Finding an intuitively appealing argument for this corollary would not only help resolve the debate from the classroom scenario, but it would provide the basis for a new proof of the original theorem. Secant slope functions provide just such a conceptual explanation for why functions are not differentiable at points where they are not continuous.

In his textbook, Abbott [1, p. 142] explains that discontinuities fall into three categories: (i) *removable* discontinuities; (ii) *jump* discontinuities; and (iii) *essential* discontinuities. Let's consider the case of a removable discontinuity. (In Problems 9.6 and 9.7 you will look at the other two categories.) Consider the following function which has a removable discontinuity at $c = 2$:

$$f(x) = \begin{cases} x + 1, & x \neq 2 \\ 1, & x = 2 \end{cases}.$$

In Fig. 9.5, a particular secant line, as well as the secant slope function $SSf_{[2]}(x)$, is depicted. As you look at the figure, make sure you understand the values being plotted by the secant slope function! Remember that the removable discontinuity—the point $(c, f(c)) = (2, 1)$—is the fixed point in constructing secant lines, and $f'(2)$ is defined to be the limit as $x \to 2$ of the constructed secant slope function. In the figure we can see the limit of that secant slope function *does not exist* as we approach 2. Because the point of discontinuity at (2, 1) lies on every every secant line, the lines have slopes that tend toward infinity when the second point $x$ gets closer to 2. From the right the slopes tend toward positive infinity; from the left they tend toward negative infinity. These are reflected in the graph of $SSf_{[2]}(x)$. Either of these conditions is enough to conclude that $\lim_{x \to 2} SSf_{[2]}(x)$ does not exist and so $f'(2)$ does not either.

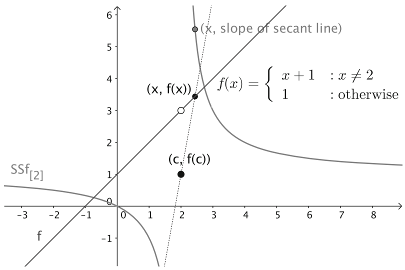


**Fig. 9.5** $SSf_{[2]}(x)$ for a function $f$ with a removable discontinuity at $c = 2$

The existence of $f'(c)$ depends on $SSf_{[c]}(x)$ being well-behaved as $x$ approaches $c$. What this example reveals is that a discontinuity at $c$ creates the possibility for $SSf_{[c]}(x)$—i.e., the slopes $\frac{f(x)-f(c)}{x-c}$—to get unboundedly large as $x$ gets closer to $c$. With values of $\frac{f(x)-f(c)}{x-c}$ heading off to infinity, there is no way for $\lim_{x\to c} SSf_{[c]}(x)$ to exist, which by definition means $f'(c)$ is not defined at that value.

## 9.4 Connecting to Secondary Teaching

In the initial teaching situation, students split into two dissenting factions about how to determine the derivative at a point of discontinuity. Results from the previous section show that the correct answer is the derivative does not exist, but we still need to consider how to help the class as a whole come to agreement and understand why this answer is correct. Simply stating that differentiable functions must be continuous and hence the derivative is undefined runs contrary to TP.5—we would have given a rule without giving any kind of mathematical justification. Providing the standard proof for the theorem that differentiability implies continuity might convince a few students that the answer is correct, but still not help them conceptually understand what's really happening.

### 9.4.1 Navigating Disagreement in the Classroom

Sometimes all it takes to resolve a disagreement is to have students look again at the problem in order to clarify what is being asked or to identify a small error. Revisiting the definition for a central concept is also a reasonable heuristic when there is disagreement; doing so affords an opportunity to resolve potential misconceptions. But definitions can be sufficiently complex that coordinating the parts along with the whole is difficult. This challenge was illustrated in this chapter by having to view the quotient $\frac{g(x)-g(c)}{x-c}$ not only as a calculation for slope, but as a holistic function as well.

In the teaching situation from this chapter, both groups of students drew on important ideas about the derivative and neither could convince the other group of its conclusion. Here is a potential continuation of the scenario that highlights

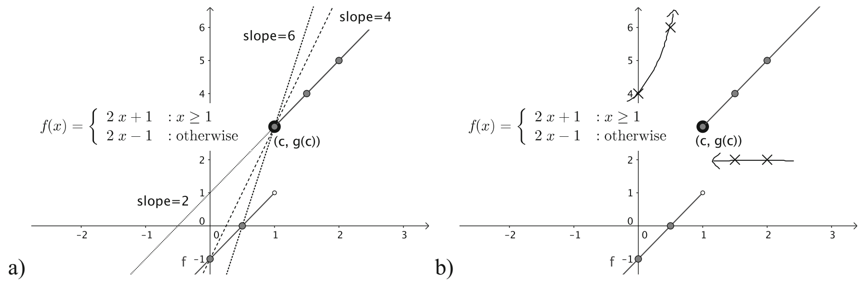


**Fig. 9.6** For the function $f$, a) secant lines and slopes with $c = 1$, and b) a sketch of $SSf_{[1]}$

some useful practices for navigating disagreements and providing meaningful justifications.

Mr. Petrov plots the function $f(x) = \begin{cases} 2x+1, & x \geq 1 \\ 2x-1, & x < 1 \end{cases}$ on the board, and rewrites the formal definition of $f'(c)$. He circles the point $(1, 3)$ on $f$, and asks four students to come up to the board:

> For each $x$-value, one of you please plot the point on the function: at $x = 2$, $x = 1.5$, $x = 0.5$, and $x = 0$. Then, draw the secant line connecting the circled point at $(1, 3)$ to your other point on $f$.

After the students mark the secant lines (depicted in Fig. 9.6a), Mr. Petrov asks the class to give an approximate value for the slope of each secant line. They give correct slopes for each: 2, 2, 6, and 4 respectively.

He then marks an ‘X’ at the corresponding points: $(2, 2)$, $(1.5, 2)$, $(0.5, 6)$, and $(0, 4)$ (see Fig. 9.6b). Sketching lines through the Xs, he says:

> Slope is a key idea for determining the derivative. From the right-hand side, the slopes of the secant lines are always 2, but from the left-hand side the secant lines get steeper and steeper because one of the two points is *always* $(1, 3)$. According to the definition, to determine $f'(1)$ we need to look at *this* (sketched) function and evaluate the limit at $x = 1$. Notice the limit does not exist. This is why the derivative is not defined at the point of discontinuity.

With the goal of having both groups of students come to a shared understanding, Mr. Petrov affirms parts of both arguments. In disagreements, each side often has valid but incomplete points. This means student thinking can be leveraged by endorsing some aspects of their arguments while also still pushing for additional refinement. Mr. Petrov confirms the importance of slope for the first camp, asking students to plot lines and identify their slopes. Notably, however, they are slopes of secant lines rather than the sought after slope of the curve at $c = 2$. He also reinforces the observation that the secant line slopes are 2 on the right-hand side of the function before pointing out the problem that arises on the left. With regard to the second camp, the teacher does not simply state the theorem the group recalled. Rather than a rule, Mr. Petrov crafts a hands-on explanation for why $f'(2)$ does not exist at that point of discontinuity that entails a thoughtful engagement with the limit definition of the derivative. Without using the term "secant slope function," Mr. Petrov employs this very concept to push the students to think more deeply about how continuity is related to the existence of the derivative.

Incorporating the notion of a secant slope function with respect to a debate about the derivative does more than just substantiate a particular claim—it affords a conceptual explanation for *why* the claim is true. In Mr. Petrov's response, the sketched secant slope function provides the intuition for why the derivative cannot exist at a point of discontinuity. Anchoring one point of the secant line to a point of discontinuity leads to the recognition that the secant slope function is unbounded in any neighborhood of the discontinuity; concrete "jumps" in $y$-values, over arbitrarily small $x$-values, mean the slope values will tend to infinity. More so than with algebraic arguments, the visual impact of steeper and steeper secant lines, together with an unbounded secant slope function, provides a powerful and, often, preferred source of insight for learners, helping meet the expectations of TP.5.

## Problems

**9.1** Draw the graph of the indicated secant slope function, explain why the secant slope function is not defined at the given $c$ value, and explain how to use the graphs of the secant slopes functions to justify the derivative value at $c$ even though the secant slope function is not defined there.

a) Draw $SSf_{[1]}(x)$ for $f(x) = 3x$ (i.e., with $c = 1$).
b) Draw $SSg_{[1]}(x)$ for $g(x) = x^2$ (i.e., with $c = 1$).
c) Draw $SSh_{[0]}(x)$ for $h(x) = \sin(x)$ (i.e., with $c = 0$).

**9.2** In the example function used in this chapter, $g(x) = \frac{1}{10}x^3 - \frac{3}{10}x^2 - \frac{9}{5}x + 5$, we stated that the secant slope function when $c = 2$ was given by $\frac{g(x)-g(2)}{x-2}$. Algebraically, make the substitutions and express $SSg_{[2]}(x)$ as a rational function. Confirm for each input $x$ that this function outputs the slope of the secant line connecting $(x, g(x))$ with $(2, 1)$. You might also try to factor the numerator to confirm there is indeed a removable discontinuity at $x = 2$.

**9.3** On a linear function $f(x) = mx + b$, demonstrate that for a given point $(c, f(c))$ the secant slope function is the horizontal line $y = m$ (but with a hole at $x = c$). You might approach this both graphically, sketching several secant lines, and algebraically. Use this fact to justify that the functional limit of each $SSf_{[c]}(x)$ as $x \to c$ is always the constant $m$.

**9.4** A common refrain in calculus is the idea that all "smooth curves" are differentiable. Consider the function $f(x) = x^{1/3}$, which is a "smooth curve" and the point (0, 0) on that curve. First, sketch the secant slope function for $c = 0$, which is $SSf_{[0]}(x)$. Second, explain why the limit of this secant slope function does not exist as $x$ approaches 0, which means $f'(0)$ is undefined. Third, TP.1 suggests that we acknowledge and revisit limitations in such statements. Describe how you would talk about how and when "smooth curves" are differentiable with a class of calculus students. Provide an explanation and any corresponding visuals that you would use to help a group of students understand when and why a point on a "smooth" curve might not be differentiable.

**9.5** Consider another function with a *removable discontinuity*. Suppose

$$f(x) = \begin{cases} 3 \text{ if} x \neq 1 \\ 4 \text{ if} x = 1 \end{cases}.$$

Use a secant slope function to present a graphical argument, in connection with the definition of derivative, for why $f(x)$ is not differentiable at $x = 1$—one you might be able to use while teaching a calculus class.

**9.6** Consider a function with a *jump discontinuity*. Suppose

$$g(x) = \begin{cases} 1 \text{ if} x \leq 2 \\ 3 \text{ if} x > 2 \end{cases}.$$

Use a secant slope function to present a graphical argument, in connection with the definition of derivative, for why $g(x)$ is not differentiable at $x = 2$—one you might be able to use while teaching a calculus class.

**9.7** Consider a function with an *essential discontinuity*. Suppose

$$h(x) = \begin{cases} \frac{1}{(x-2)^2} & \text{if} x \neq 2 \\ 1 & \text{if} x = 2 \end{cases}.$$

Use a secant slope function to present a graphical argument, in connection with the definition of derivative, for why $h(x)$ is not differentiable at $x = 2$—one you might be able to use while teaching a calculus class.

**9.8** We often talk about differentiability in terms of a curve being smooth; meaning that functions are not differentiable at "sharp" points. Consider a function $f$ that has a sharp point at $x = c$ (such as $f(x) = |x|$). Sketch out a general argument (i.e., don't limit yourself to the absolute value function!) for why a function with a sharp point at $x = c$ would not have a derivative at that point. Draw on the secant slope function at that point in your explanation. Discuss how you would talk about differentiability at sharp points with a class of calculus students. If you find it helpful, you can consider a sharp point as being where two lines intersect (although there are also other types of "sharp" points).

**9.9** Consider the following piece-wise defined function: $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. After sketching out this function, which group from the original pedagogical situation—the "slope" group, or the "discontinuity" group—might argue that $f'(0)$ is undefined, and which might argue that $f'(0) = 0$? Explain your reasoning. Now, determine $f'(0)$ from the definition. You might sketch out the associated secant slope function to help determine whether the relevant limit exists and, if so, its value.

**9.10** In introducing the derivative in Section 5.1, Abbott states:

> A particularly useful class of examples for this discussion are functions of the form:
>
> $$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

What teaching principle do you think this illustrates? To what "discussion" is Abbott referring—meaning, for what mathematical point about derivatives is this class of examples useful?

## Turning the Tables

### Reflecting on *teaching* from your *learning* in real analysis: TP.1

As another opportunity to reflect on teaching, we explore an aspect of learning real analysis that exemplifies another one of our teaching principles.

TP.1 is about acknowledging and revisiting assumptions and mathematical limitations in mathematics teaching and learning. We use one exercise in Abbott's textbook to ground the discussion. Exercise 5.2.10 (p. 154) states:

> A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

In this exercise, Abbott gives what we will refer to as a "Calculus 1" mantra: *a differentiable function is increasing if its derivative is positive*. He then suggests this statement is not completely accurate; that there are limitations in the statement that should be acknowledged. The exercise asks students to probe the statement by considering the function, $g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if} x \neq 0 \\ 0 & \text{if} x = 0 \end{cases}$. Essentially, the function $g(x)$ provides a situation for which a point on a differentiable function, $(0, 0)$, has a positive derivative at 0, but where the function is not increasing on any interval that includes 0—meaning, the Calculus 1 mantra needs some work. Abbott then notes in the exercise, if $f'(x) > 0$ for all $x$ in an interval $(a, b)$ (and not just for *one* point), then it is true that the function will be increasing on that interval. That is, as stated, the Calculus 1 mantra is not quite correct (even though it is helpful), but it would be correct with a slight modification (a modification that's probably beyond what most students in Calculus 1 are ready for).

We elaborate a few points related to TP.1. First, special examples, like the function $g$ in Abbott's exercise, can be used to illustrate mathematical constraints or limitations. In Chap. 2 we discussed boundary examples, ones that probe the "edges" of a concept. The current chapter included a number of possible boundary examples: the jump discontinuity in the pedagogical situation; the function $f(x) = x^{\frac{1}{3}}$ at $x = 0$ in one of the homework problems (9.4); and now Abbot's function $g(x)$ in the section above. In each case, the example illustrated the "edges" of behavior, and made a statement's limitations evident. Such examples are one way that TP.1 might be connected to TP.2. Second, limited statements still often contain kernels of truth. Here, the Calculus 1 mantra simply needed to be adjusted to ensure we are talking about the derivative being positive on all points in an interval—not just a single one. The point being that an extra clarification, or additional assumption, can often turn statements with limitations into more accurate ones. Third, enacting TP.1 in the classroom does not mean all statements be perfectly accurate but rather, at some point, that the

(continued)

limitations be discussed. Teachers need to communicate to students in ways they can understand, which often requires simplification—the opposite of precision. To be precise, we include all qualifications and assumptions; to be simple, we give broad descriptions and analogies. TP.1 suggests teaching need not be only one or the other. Derivatives should be talked about as slopes; they should be connected to a function being increasing; but the story cannot end there. Precisely "when" limitations are acknowledged is less of the point than "that" they are acknowledged.

## References

1. Abbott, S. (2015). *Understanding analysis* (2nd ed.). New York, NY: Springer.
2. Dubinsky, E. (2014). Actions, Processes, Objects, Schemas (APOS) in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Dordrecht, The Netherdlands: Springer.
3. Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44–55.