7 The Intermediate Value Theorem and Implicit Assumptions

7.1 Statement of the Teaching Problem

Effective communication, in the classroom and elsewhere, relies on shared understanding. Perhaps the simplest illustration is with vocabulary. When most people hear the word "plane" they picture a machine that flies through the air. Yet, if a student draws on that understanding while their geometry teacher discusses a "plane," very little that is said will be meaningful to the student. This so-called semantic contamination occurs when everyday definitions interfere with properly understanding mathematical definitions. While the confusion between airplanes and Cartesian planes is not hard to remedy, there are more nuanced levels of semantic contamination that can be amplified in the classroom and are more challenging to address.

When we communicate about mathematical ideas, there is a dichotomy between the rigorous language of formal definitions and the more intuitive type of discourse that focuses on big picture properties and phenomena.[1] Proving theorems requires careful attention to formal definitions, but when most of us discuss a mathematical concept like "function" or "continuity," we are likely referring to the informal collection of examples and ideas that are common from our experience. We may not be thinking about the formal definition. This is especially true for students, and the result is a type of semantic contamination that can go undiagnosed. Students frequently make assumptions about mathematical concepts that stem from a mental image that has built up over time. These assumptions tend to remain *implicit* in communication; it's hard to be explicit about assumptions of which you are not aware.

[1] Tall and Vinner [5] make a similar differentiation. They contrast a *concept definition*, by which they mean a formal definition, with a *concept image*, by which they mean all the other interesting things associated with that concept.

N. H. Wasserman et al., *Understanding Analysis and its Connections to Secondary Mathematics Teaching*, Springer Texts in Education,
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Consider the following pedagogical situation:

> Mr. Cai, a high school teacher, is illustrating the Intermediate Value Theorem by locating zeros of functions. Using the example $g(x) = x^3 - 3x^2 - 2x + 7$, Mr. Cai points out that since $g(2) = -1$ and $g(3) = 1$, the function must have at least one zero between 2 and 3.
>
> Later, as an "exit ticket," students are asked for a short summary of the key ideas. One student, Chrissy, submits the following response:
>
> > Okay, so if a function is less than 0 somewhere and greater than 0 somewhere else then we know there will be a zero somewhere between them. So in general, if $f(a) < 0$ and $f(b) > 0$, then there is at least one zero in the interval (a, b).

Chrissy's response in the exit ticket appears to be a positive reflection on the lesson—she was able to capture some of the most salient ideas about the theorem. On the other hand, assumptions left implicit can lead to problematic understandings about the mathematics; TP.1 insists these be explicitly acknowledged and revisited. One of the challenges of teaching is the ability to listen to students, interpret their statements, hear potentially implicit assumptions, probe those assumptions, and identify how to respond in order to further mathematical understanding.

Before moving on, think about how you, as a teacher, might respond to the student. What comments might you make? What questions might you ask?

7.2 Connecting to Secondary Mathematics

7.2.1 Problematizing Teaching and the Pedagogical Situation

We problematize some potential responses to the student's summary of the Intermediate Value Theorem (IVT).

A first reaction may be to commend the student. Overall, Chrissy has done a good job summarizing key parts of the IVT. Several aspects are particularly noteworthy. She has successfully generalized the essential ideas from the particular example. This is especially evident from her use of symbolism, such as using '$f(a) < 0$,' and the interval '(a, b).' The student has also noted the importance of the phrase "at least one" zero and made sure to include it in her summary. This is a critical nuance of the IVT. When $f(a) < 0$ and $f(b) > 0$, there could be multiple zeros in (a, b), but we cannot be sure, so "at least one" is the most accurate claim. The student's attention to these details suggests she understands some important ideas about finding zeros and merits a degree of validation from the teacher. But are there other aspects that should be considered? Are there any implicit assumptions or mathematical limitations in the student's statement that, according to TP.1, might need to be clarified?

In response to these questions, a second reaction could involve pointing out that a completely correct answer needs to acknowledge the role of continuity. A function must be continuous on the given interval to apply the IVT. A step function, for example, could be less than 0 at one point, greater than 0 at another, and not have any zeros in between because it "jumps" over the x-axis. In addition to affirmation, a teacher should point out that the function has to be continuous on the interval (a, b) to ensure there is a zero in that interval. Emphasizing this condition to the whole class might also be worthwhile.

Having raised the issue of continuity, the teacher's next job is to ferret out the reason for the omission. What led to this missing component in Chrissy's exit ticket response? Perhaps she is not aware of the significant role of continuity in the IVT, or maybe it was just a careless mistake. Falling somewhere between these two scenarios is the possibility of a subtle form of semantic contamination in Chrissy's use of the term "function." She might be using this term to reference something different than what the teacher, or you as a reader, imagine. Based on the examples she has seen, Chrissy may have been using "function" to mean "polynomials"; or perhaps her mental picture of function only includes continuous ones and so the term was implicitly referring to "continuous functions." What students say and write does not always align with what they understand. Interrogating students' mistakes to unearth their thinking requires asking, "What might the student be assuming in order for this statement to be logical?" The answer frequently includes the existence of implicit assumptions.

Praising Chrissy's response may let a misunderstanding linger, and correcting the response might not address the right misconception. Indeed, one of the things we will see is that, in addition to continuity, there are other assumptions being made about the IVT that are implicit in the student's statement.

7.2.2 Defining Function

Before we consider the IVT further, let's directly address any confusion about the term "function." Based on your own experiences, you probably have a particular image for the concept of function. Before we present a formal definition, pause to consider the different informal images you have for this concept. What do you think about when you consider the notion of a function? What examples? What properties? What pictures or words?

Definition A **function** f is a set of ordered pairs (x, y) such that each x is associated with a unique y. Specifically, $f = \{(x, y) | (x, y_1), (x, y_2) \in f \text{ implies } y_1 = y_2\}$. In this case, we write $f(x) = y$. The set of all x-values is the *domain*, $A = \{x | (x, y) \in f\}$, and the set of all y-values is the *range*, $f(A) = \{y | (x, y) \in f\}$. Any superset $B \supseteq f(A)$ can be the *codomain*, and we write $f : A \to B$.

The definition above is similar to other definitions of function (e.g., Abbott's [1] Definition 1.2.3). The domain A and the co-domain B do not have to be sets of real numbers. The domain could be the set of students in a class and B the set of desks. A function f could be the set of ordered pairs (x, y) where student x sits in desk y.

With a set A (the domain) and a set B (a superset of the range) we could consider different possible collections of ordered pairs (x, y) where $x \in A$ and $y \in B$. The full collection, known as the Cartesian product $A \times B$, contains all possible ordered pairs. Relative to these different possible collections, functions are a particular kind of collection—one in which each x is associated with only one y. This property is known as *univalence*. A second property associated with functions is *totality*, which means every $x \in A$ appears as the first coordinate at least once. In our previous example, totality means that every student gets a desk; univalance means that no student gets more than one. (It's entirely possible, however, that two students share the same desk.) Using this notation, another way to characterize a function is as a subset of ordered pairs from $A \times B$ that is univalent and total on A.

Real analysis and much of secondary mathematics is focused on functions with domains and ranges from the set of real numbers. The graph of such a function on the Cartesian plane $\mathbb{R}\times\mathbb{R}$ is a useful representation of its set of ordered pairs. In fact, you probably imagined particular graphs of functions as you thought about the concept earlier. The formal definition of function is quite broad. Even when we restrict our attention to real-valued functions, there is a wide array of surprising examples that fulfill the defining criteria. Figure 7.1 depicts a range of examples that meet the formal definition. Although you might be tempted to reject them based on your preconceived notion of what a function should look like, the definition demands they become part of your example space. Claims about functions must hold across all possible examples, or be appropriately amended to apply to a particular subclass.

7.3 Connecting to Real Analysis

As we discussed in Chap. 5, conditional statements ($A \implies B$) have a condition A and a consequence B; if condition A is met we necessarily have B as a consequence. Conditions are the explicit assumptions required for a proposition to hold, but precisely determining the conditions of a theorem can be a bit challenging. Mathematics is a dense language where a lot can be conveyed in a few words and symbols. Fully comprehending all the assumptions of a theorem requires careful scrutiny of the notation to appreciate what is being articulated. It also involves following through to see how the various assumptions are incorporated in the proof. To illustrate the different steps and potential pitfalls in this process, let's take a detailed look at the Intermediate Value Theorem and a corollary we call the Intermediate Zero Theorem.

Theorem (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a $c \in (a, b)$ where $f(c) = L$.

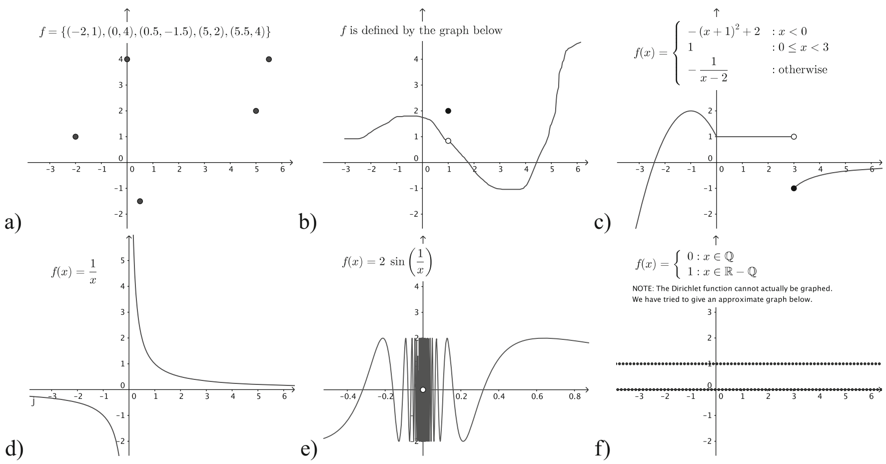


Fig. 7.1 Functions that are: (**a**) sets of discrete points; (**b**) defined by their graphs; (**c**) piecewise defined, and with "jumps"; (**d**) missing domain values, with vertical asymptotes; (**e**) missing domain values, with holes, and arbitrarily close oscillations; and (**f**) defined but cannot be graphed

Theorem (Intermediate Zero Theorem) Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists a $c \in (a,b)$ where $f(c) = 0$.

The Intermediate Zero Theorem is the specific case of the Intermediate Value Theorem when $L = 0$. This is the version relevant to the process of locating roots of functions. We use the abbreviation IVT to refer to either statement, although it is the second one we attend to more closely.

7.3.1 Differentiating Conditions in Statements

As a general rule, mathematicians try to avoid ambiguity and inefficiency. Everything required should be explicitly stated, and everything explicitly stated should be required. In terms of style, mathematicians lean toward brevity, saying what is necessary in the fewest words needed. Because mathematical concepts build on each other, phrases and concepts can include hidden implications. Unpacking the full meaning of a mathematical statement involves reviewing the relevant definitions and their implications as well as paying attention to the statement's logical structure.

Turning our attention to the IVT, the first thing to point out is the inclusion of continuity as a condition. Without it, the conclusion does not hold (see Fig. 7.2). We should also acknowledge the phrase "there exists" in the theorem's conclusion. The existence of a value c does not preclude the possibility that there could be more—existence and uniqueness are different questions. These two observations

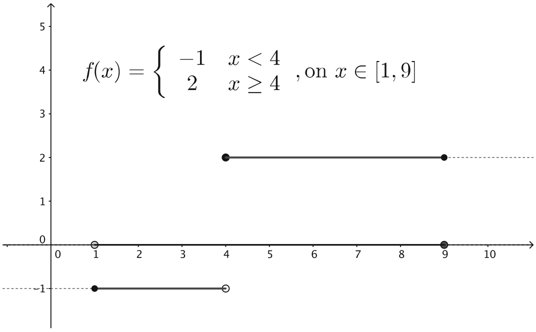


Fig. 7.2 A discontinuous function f, defined on $[1, 9]$, with $f(1) < 0 < f(9)$ but no zeros in $(1, 9)$

about the IVT were raised in our earlier discussion of the teaching scenario. What other aspects of the IVT might be interrogated?

A useful heuristic to better understand the conditions of a theorem is to ask what happens if the conditions were changed or if parts were left out.[2] Consider the assumption, $f(a) < 0 < f(b)$. As stated, it utilizes strict inequalities. What happens if we change it to $f(a) \leq 0 \leq f(b)$? You might consider some possibilities before moving on. One example to test would be when $f(a) = 0$ and $f(b) > 0$. In this case, a line between $(a, 0)$ and $(b, f(b))$ does not have any zeros in (a, b) and so the conclusion would not hold. What if both endpoints were zero? In this situation we might draw a sine curve that crosses the x-axis several times in the interval, or a parabola with roots at a and b that has no zeros in the interval. This latter example reinforces the prior observation that the IVT's conclusion no longer follows with the amended conditions. To fix this we could edit the conclusion to assert the existence of a value $c \in [a, b]$ instead of $c \in (a, b)$. This puts us on firm logical ground, but the cases when either $f(a)$ or $f(b)$ equal zero make the conclusion of the IVT rather trivial. The takeaway of this experiment is that the use of either strict or inclusive inequalities in the condition and the conclusion are no coincidence—they are linked. And the strict inequalities yield the most appropriate version of the IVT!

Let's look more carefully at one other condition: $f : [a, b] \to \mathbb{R}$. There is a component to this part of the hypothesis that is often overlooked; in particular, it says the domain is a closed interval. What happens if we remove this part of the condition, but keep everything else the same? Does the conclusion about the existence of a zero still hold?

Question *Let $f(x) : A \to \mathbb{R}$ be continuous on its domain A, and let a and b be points in the domain with $a < b$. If $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, is it true f must have at least one zero in (a, b)?*

[2] Brown and Walter [3] describe this as the "what-if-not" strategy for problem posing.

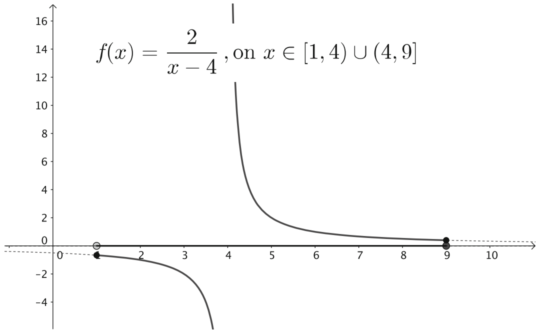


Fig. 7.3 A continuous function $f : [1, 4) \cup (4, 9] \to \mathbb{R}$, with $f(1) < 0 < f(9)$, but no zeros in $(1, 9)$

To answer the question, let's think about whether it is possible to construct a continuous function with, for example $f(a) < 0 < f(b)$, but that has no zeros in the interval (a, b). This requires us to stretch our image of a function. (The examples in Fig. 7.1 might be a guide.) We also have to sharpen our understanding of what it means to assert that a function is continuous over a given set A. In Chap. 6 we learned that, according to the standard definition of continuity, a function such as $g(x) = \frac{1}{x}$ is continuous over its natural domain $A = \{x \in \mathbb{R} : x \neq 0\}$. Likewise, the related example $f(x) = \frac{2}{x-4}$ depicted in Fig. 7.3 is continuous on the domain $A = [1, 4) \cup (4, 9]$. Note that f satisfies $f(1) < 0 < f(9)$ but has no zeros in $(1, 9)$. This example shines a spotlight on a condition in the IVT that is subtly embedded in the notation: the domain of the function must be a closed interval $[a, b]$. In our example, f is continuous at every point in A and thus continuous on A, but f is not defined at $x = 4$. Being defined at a point is a prerequisite for continuity at that point; we need to distinguish between the two conditions and interrogate them separately.

7.3.2 Use of Conditions in Proofs

Mathematical propositions are typically crafted so that all the conditions in the hypothesis are required for the conclusion to follow. Granted, there are certainly exceptions. Teachers and textbook authors sometimes include additional information to make the statements more understandable to students; sometimes additional conditions are added to simplify the proof. Generally-speaking, however, we should presume that every condition is required for the proof to go forward—that the conditions do not include unnecessary information.

Let's consider the two particular conditions of the IVT we delineated in the previous discussion: (i) f is *defined* on $[a, b]$; and (ii) f is *continuous* on $[a, b]$. Below is a standard proof of the IVT that uses the Nested Interval Property (see Abbott [1] on pp. 138–139). As you read the proof, identify where each assumption comes into play. At what point does the argument break down if f is not defined on $[a, b]$? Where does it break down if f is not continuous on $[a, b]$?

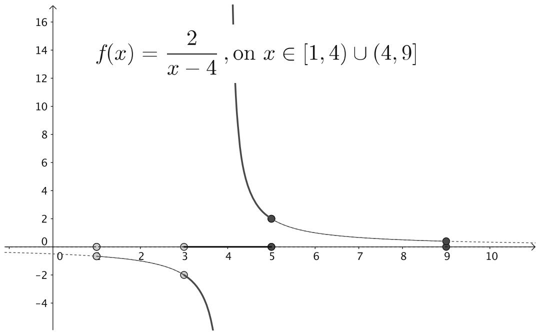


Fig. 7.4 With f not defined on $[a, b]$, the constructed sequence of nested intervals may stop

Proof We will consider the case when $f(a) < 0 < f(b)$, and begin with the interval $I_0 = [a, b]$. At its midpoint, z, test the value of the function. If $f(z) \geq 0$, set $a_1 = a$ and $b_1 = z$; if $f(z) < 0$, set $a_1 = z$ and $b_1 = b$. This process can be repeated on the new interval $I_1 = [a_1, b_1]$ to create $I_2 = [a_2, b_2]$. Continuing this procedure, we construct a sequence of nested intervals $I_n = [a_n, b_n]$ with the property that $f(a_n) < 0$ and $f(b_n) \geq 0$ for every n. Because we are bisecting each time, the length of each interval is half the length of the preceding one, which means the lengths are converging to 0.

By the Nested Interval Property (Abbott's Theorem 1.4.1), the intervals all contain at least one point c. Because c belongs to every interval and the lengths of the intervals tend to 0, the sequences of left- and right-hand endpoints both approach c, meaning $(a_n) \to c$ and $(b_n) \to c$. Therefore, $f(a_n) \to f(c)$ and $f(b_n) \to f(c)$. By construction, we know that $f(a_n) < 0$ for every n and so $f(c) = \lim f(a_n) \leq 0$ (see Theorem 2.3.4 in Abbott). Likewise, $f(b_n) \geq 0$ for every n and so $f(c) = \lim f(b_n) \geq 0$. Since $f(c) \leq 0$ *and* $f(c) \geq 0$, we know $f(c) = 0$. □

Let's focus first on the condition that f is defined at every real number in $[a, b]$. Where does the argument fail without this assumption? We want to think specifically about the values in the domain, and what it means for a function to be, or not be, defined at a particular value. The argument entails constructing a sequence of intervals $I_0, I_1, I_2, \ldots$ where at each stage we evaluate the midpoint z to determine the endpoints of the next interval. The problem occurs if z is not in the domain—if we can't compute $f(z)$ then there is no way to generate the next interval. Figure 7.4 gives an example where this occurs. The first midpoint, $z = 5$, makes $I_1 = [1, 5]$; the next midpoint, $z = 3$, makes $I_2 = [3, 5]$; but the function is not defined at the next midpoint, $z = 4$. Hitting this roadblock, the process stops. We are not able to generate the sequence of intervals I_n which are required to produce the value of c satisfying the conclusion of the theorem.

Where in this argument does the assumption of continuity on $[a, b]$ come into play? As long as f is defined at every real number in $[a, b]$, the bisection algorithm will result in an infinite sequence of intervals I_n. Nothing about this aspect of

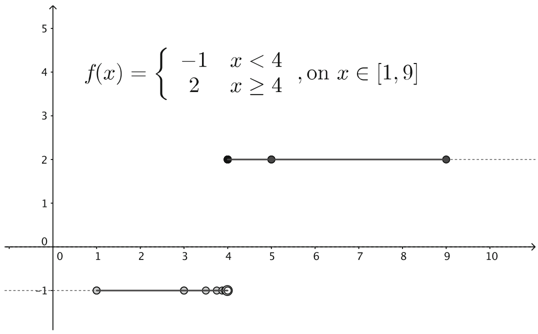


Fig. 7.5 With f defined but not continuous on $[a, b]$, the sequence of function values of the left-endpoints $f(a_n)$ does not necessarily approach $f(c)$

the proof requires continuity. It is only toward the end of the proof, when we are considering the sequences of left-endpoints and right-endpoints, that continuity is needed. With a left-endpoint sequence $(a_n) \to c$, it is the continuity of f that allows us to conclude $f(a_n) \to f(c)$. The same is true of the right-endpoint sequence (b_n). The example in Fig. 7.5 shows how this process breaks down without continuity. It illustrates that for a function with a jump discontinuity, the algorithm in the proof successfully generates a sequence of nested intervals with a unique point of intersection. In this example, the left-hand endpoint sequence (a_n) converges to $c = 4$ as does the right-hand endpoint sequence (b_n). Without continuity, however, we are no longer guaranteed that $f(a_n)$ and $f(b_n)$ both converge to $f(c)$. For this example, $f(a_n) \to -1$ while $f(b_n) \to 2$; the limits are different and, notably, not equal to 0.

7.4 Connecting to Secondary Teaching

In the initial teaching situation, the student provided a reasonable summary of the IVT, but one that made some implicit assumptions about functions. In a classroom context, we should not necessarily regard students' implicit assumptions to be wrong or unwarranted. Rather, we see them as an inherent part of the learning process. It is the teacher's responsibility to unearth these assumptions and call attention to them (TP.1). In the scenario, the student's statement is true for polynomials because these functions are *defined* and *continuous* on $\mathbb{R}$—polynomials meet the conditions of the theorem. Details like these allow teachers to illuminate the nuances of mathematical relationships and clarify why certain statements are valid. Doing this in an effective way requires expanding the kinds of examples available to students so that they understand the need for being explicit about the relevant details.

7.4.1 Implicit Assumptions in the Classroom

The Intermediate Value Theorem is frequently included as a topic of secondary school mathematics. When working with the IVT, the function needs to be both defined and continuous on the interval $[a, b]$, and we want to ensure that students understand both of these conditions. There cannot be “holes” in the domain, nor “jumps” in the function.

Through no fault of their own, secondary mathematics students are especially susceptible to making implicit assumptions about functions. At the university level, mathematics students encounter discontinuous functions frequently enough that they become an organic part of their example space. From this perspective, the hypothesis of continuity in the IVT stands out as an explicit and necessary requirement. From a secondary student’s perspective, it may feel as though there is no need to specify that the function be continuous because all the functions they work with are continuous. Adding the modifier “continuous” becomes optional if every function is assumed to have this property already. To take another example, if the term “pyramid” calls to mind only solids with square bases, why would a student think there is a need to specify “square pyramid”? As teachers, we should consider not only whether a student’s statement is valid in general, but whether it would be valid under their implicit assumptions. If so, a good response should make clear that the student’s statement is relatively correct, and then reveal the assumptions that the student left unsaid (TP.1). Doing so might involve the teacher introducing an example that does not conform to the missing assumptions, an instance of TP.2.

To make this concrete, let’s return to the original teaching scenario and consider the following response:

> The next time the class meets, Mr. Cai brings Chrissy’s exit ticket to their attention. After discussing it, Mr. Cai concludes:
>
>> Chrissy’s description is a great way to generalize how the IVT works *for polynomials*. This is true because polynomials are both *defined* and *continuous* on the real numbers. Although we will primarily use the IVT with respect to polynomials, it is important to note that the conclusions would not hold in all situations.
>
> Mr. Cai then asks the class to come up with and graph examples of functions that take on both positive and negative values but are not continuous, or not defined, everywhere. Using their examples, he asks them to check whether those examples always have a root between two values with opposite signs.

Teaching typically involves taking concepts apart in order to make the various components clear, and then organizing those components into a useful order for

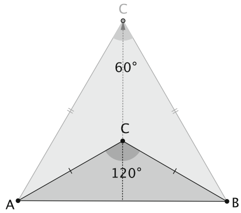


Fig. 7.6 An IVT-based argument for the existence of acute, obtuse, and right isosceles triangles

learning.[3] This task of unpacking can be more difficult than it sounds. Not only do teachers typically compress their own knowledge as they progress through more advanced topics, but by its nature mathematics is a condensed discipline. Each word and symbol in a statement like $f : [a, b] \to \mathbb{R}$ may convey some detail that needs to be fleshed out. As the notation and concepts become more familiar to us, we can lose track of the different layers and forget what it feels like to be encountering the ideas for the first time. Being attentive to this starts with paying attention to *all* of the mathematical ideas expressed in a statement and anticipating where students might reasonably make assumptions based on what they know. Teachers can then make decisions about what needs elaboration before moving forward to order, structure, and connect the individual pieces into a meaningful and coherent whole. Students' notions of a concept can be refined, and further developed, only once their sense of the concept has been expanded first.

7.4.2 Implicitly Assuming the IVT in Secondary Mathematics

To this point we have focused on the implicit assumptions students might make about the hypothesis of the IVT, but it is not unusual in secondary school mathematics classrooms for students to implicitly assume the *entirety* of the theorem. The content of the IVT is so plausible that on occasion it can be unwittingly invoked in the course of some other argument. Here is an example from a geometry class, where the students were asked whether isosceles triangles can also be acute, right, and obtuse triangles. A student in the class responds with the following argument based on the drawing in Fig. 7.6:

> I am thinking about the vertex of an isosceles triangle. If it is down low, the triangle would have an obtuse angle, and if it is dragged further up along this center line, the angle would be acute. So, somewhere in between, it must be exactly 90°. Like, if the bottom angle is 120°, and the top 60°, then 90° is right in the middle so it's probably exactly halfway between those two. So, yes, there are obtuse, acute, and right isosceles triangles.

[3] Ball and Bass [2] describe this notion of unpacking; teachers need to be able to "deconstruct [their] own mathematical knowledge into less polished and final form, where elemental components are accessible and visible" (p. 98).

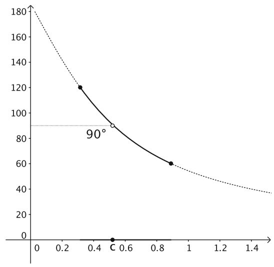


Fig. 7.7 Continuity of the function $f(h)$ which assigns an angle measure to each height

Not everything in the student's argument is correct, but the conclusion that a right isosceles triangle exists somewhere in between the two examples depicted is indeed true. The student's reasoning implicitly draws on the IVT. Let's think more about why this is the case.

First, the student has observed an angle value of 120° at one location and an angle value of 60° at another. This is akin to observing values $f(a) = 120$ at location a, and $f(b) = 60$ at location b, where f is measuring the size of the angle. The student then argues that, since 90° is between 120° and 60°, there must be a triangle where the 90° angle measure is attained. The student makes no mention of the IVT, or even of a function f, but there is an unmistakable impression that the spirit of the IVT is being invoked. Filling in the details to put this argument on solid ground requires thinking a little more about the angle-measuring function f.

To define a proper domain for f, we focus on the perpendicular bisector from the diagram in Fig. 7.6. This is the vertical line along which the student was mentally "dragging" the central vertex, and we define the *height* h to be the distance from the horizontal base to this imagined vertex. The values of h are the input values of our function $f(h)$, which we formally define to be the measure of the central angle of the isosceles triangle with vertices A, B and height h.

If the base between A and B has length 1, then height $a \approx 0.28$ yields $f(a) = 120$ and height $b \approx 0.87$ yields and $f(b) = 60$, which are the two depicted in the figure from above. (The exact values are $a = \frac{1}{2\sqrt{3}}$ and $b = \frac{\sqrt{3}}{2}$). This provides the necessary raw material to properly apply the IVT. (See Fig. 7.7.) The function f is *defined* for all values of h in the closed interval $[a, b]$. To convince ourselves that f is *continuous*, we mentally drag the vertex up and down the perpendicular bisector, just as the student did, and observe there are no jumps or holes. (For a more rigorous argument, we can deduce $f(h) = 2\arctan\left(\frac{1}{2h}\right)$ and appeal to the continuity of the inverse trigonometric functions.) Since $120 > 90 > 60$, a straightforward application of the IVT confirms the existence of a height c where $f(c) = 90$.

What the IVT does not tell us is how to compute the value of c. On this point, the student's hunch that c is "halfway between" $a \approx 0.28$ and $b \approx 0.87$ is off the mark. The student seems to have made the additional implicit assumption that $f(h)$ is linear, which we can see from Fig. 7.7 is not the case. For what it's worth, $f(c) = 90$ when $c = 0.5$, which is in the interval $[0.28, 0.87]$ but not at its midpoint.

Problems

7.1 Graph the function $f(x)$ shown below, using any preferred technological tool:

$$f(x) = \begin{cases} 2x & x < 0 \\ \frac{1}{8}x\,(x-1)^2 + x\,(x-2)\,(x-5) & 0 \le x < 5 \\ 10 & 5 \le x \end{cases}$$

First, show that $f(x)$ meets the conditions of the IVT on the interval $[-1, 5]$. Second, what range of values can you be certain that $f(x)$ takes on due to the IVT? Third, does $f(x)$ take on any additional values on the interval $[-1, 5]$? If so, what are they? How do you know? Fourth, how many zeros does the IVT guarantee $f(x)$ has on $[-1, 5]$? How many zeros does $f(x)$ actually have on this interval? Which zero(s) would the nested intervals process in the proof of the IVT find (beginning with $I_0 = [-1, 5]$)?

7.2 The following multiple choice question was on a geometry test:

> Quadrilateral $ABCD$ is a rectangle, with diagonal AC. How do the quantities $\frac{AC}{AB}$ and $\frac{AB}{AD}$ compare? a) $\frac{AC}{AB} > \frac{AB}{AD}$; b) $\frac{AC}{AB} < \frac{AB}{AD}$; c) $\frac{AC}{AB} = \frac{AB}{AD}$; d) The relationship cannot be determined from the given information. Justify your answer.

First, determine your answer to the question. Next, consider a student's written response: "The answer is (a) because $\frac{AB}{AD} < 1 < \frac{AC}{AB}$." Discuss any assumptions the student may be making about the situation. Under those assumptions is the student's statement valid? Last, describe how you would respond to the student as a teacher, making explicit under what assumptions the student's statement is correct and providing examples that do not conform to those assumptions.

7.3 The following question was on a geometry test: "A triangle has side lengths of 3 and 4 units. What do you know about the third side length?" One student drew a picture of a 3-4-5 right triangle, and wrote: "I know that $a^2 + b^2 = c^2$. And $3^2 + 4^2$ is equal to 5^2. The third side must be 5." Discuss any assumptions that the student may be making about the situation. Under those assumptions is the statement valid? Then, describe for what concept the student's "concept image" appears to be limited. Discuss how you might push the student to expand their sense of that concept so that they would recognize the limitation that arose from their implicit assumption.

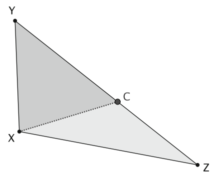
7.4 A geometry class is learning about the surface area and volume of geometric solids. The teacher provides the following formulas for the volume and surface area of a regular polygonal pyramid—a pyramid with a regular n-gon as its base:

- $V_{reg\ poly\ pyr} = \frac{1}{3}B^*h$, where B^* is the area of the n-gon's base, and h the height of the pyramid
- $SA_{reg\ poly\ pyr} = B^* + \frac{1}{2}P^*l$, where B^* is the area of the n-gon's base, P^* the perimeter of the n-gon, and l the slant height of the triangular face (the slant height is the height of a triangular face of a pyramid—the length of the line segment along a triangular face from the base to the apex of the pyramid).

The formulas given by the teacher for the volume and surface area of a pyramid have some specific assumptions. What assumptions about pyramids are explicitly included in the description of volume? In the description of surface area? Do either of the descriptions about volume or surface area have any implicit assumptions about pyramids—things unstated but assumed about pyramids for these formulas to be valid? Discuss any implicit assumptions and how they would have an impact on the pertinent formulas for volume or surface area.

7.5 A student is attempting to split a triangle XYZ into two equal-area pieces:

> Well, I'm not sure if it is *exactly* in two equal pieces as I have drawn it. But I could adjust the point (C) somewhere along that segment (YZ) and they would be.



How has the following student's statement implicitly used the Intermediate Value Theorem? Verify all necessary conditions of the IVT are met—it is okay to be somewhat informal for continuity in this case. Discuss which mathematical ideas you might make explicit to students, and how you might do so. [This exercise was adapted from a classroom example in [4].]

7.6 For hourly workers, work after 40 h typically results in an overtime rate of 1.5 times the normal hourly rate. Suppose a student observes in this situation that at 30 h, a worker would be earning an hourly rate of, say, \$20, and at 50 h they would be earning an hourly rate of \$30. The student suggests that the worker must have been earning an hourly rate of exactly \$25 at some point. First, describe how the student's reasoning has implicitly assumed the IVT. Second, the student's conclusion is false. Describe what conditions of the IVT have not been met.

7.7 In this chapter it was useful for us to think about isolating conditions. Indeed, identifying whether there were functions with a certain set of properties was productive (e.g., a function $f(x) : A \to \mathbb{R}$ that is continuous on its domain A, with $f(a) < 0 < f(b)$, but has no zero in the interval (a, b)). In Exercise 4.4.8, Abbott asks a similar kind of question: "Give an example of each of the following, or provide a short argument for why the request is impossible. (a) A continuous function defined on [0, 1] with range (0, 1)…" Questions such as these exemplify what teaching principle? Explain your reasoning.

7.8 The Intermediate Value Theorem (IVT) is a statement about continuous functions; namely, that if a function f is continuous on $[a, b]$, then on that interval the function will attain all values between $f(a)$ and $f(b)$. Abbott also defines what he calls the Intermediate Value Property (IVP), in Definition 4.5.3 [1, p. 139].

> A function f has the *intermediate value property* on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

Think about these two statements in relation to TP.3, about exposing logic. Rephrase the IVT to incorporate the IVP, and discuss any pertinent observations about the various properties of functions in terms of logic.

Turning the Tables

Reflecting on *teaching* from your *learning* in real analysis: TP.6

To reflect more on TP.6—seeking out and giving multiple explanations—we include some additional commentary about the description and proof of the Intermediate Value Theorem, as given in Abbott's text.

The teaching principle advocates using multiple explanations for the same phenomenon because some students may follow one explanation better than another. As an example of this practice, after introducing the content of the IVT, Abbott goes on to state a topological theorem about the preservation of connected sets: "Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well" (Theorem 4.5.2). He then explains the IVT is really just a special case of this theorem. In other words, there is a typical analysis approach that accounts for the phenomenon in the IVT, but there is also a topological approach that is more general. Here, we see TP.6 exemplified in terms of providing two descriptions of an observed phenomenon. Describing ideas in multiple ways—and from multiple mathematical domains—adds depth to the mathematics being studied.

We see this teaching principle again in Abbott's justification of the IVT when he provides two different proofs (pp. 138–139). The one given in this chapter is based on the Nested Interval Property, but Abbott gives another that uses the Axiom of Completeness. For this second proof, Abbott defines a set $A = \{x \in [a, b] | f(x) \leq 0\}$. Because A is bounded, the Axiom of Completeness asserts it must have a least upper bound, c. He then shows $f(c)$ must be equal to 0. The two proofs provide two different arguments for readers to follow, creating opportunities for them to make connections across the two approaches. Together, they provide a more comprehensive sense of the IVT and why it is true. In the spirit of TP.6, Abbott's text utilizes the pedagogical approach of describing a phenomenon in multiple ways and justifying it in multiple ways as well. Because of this approach, we have a richer sense not only of what the IVT means, but why it is true.

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