6 Continuity and Definitions

6.1 Statement of the Teaching Problem

Definitions play a fundamental role in mathematics. Because mathematical objects do not exist in a physical sense—they are abstract—definitions are necessary in order for us to have a proper sense of the objects we are studying. In most aspects of life, definitions are *extracted* from a collection of pre-existing examples so that the definition flows from an attempt to describe the objects being defined. Perfect precision is not typically a requirement. Debating whether a hot dog is a sandwich or a stool is a chair illustrates that what constitutes a "sandwich" is pretty flexible and there might not be a definition for "chair" at all. In the words of U.S. Supreme Court Justice Potter Stewart, "I'll know it when I see it" is usually good enough for daily life. Not so in mathematics. While definitions in mathematics are often extracted from a collection of examples, once they have been established they become *stipulative*—the definition precisely bounds and specifies a concept so that any object which meets the defining criteria is considered an example.[1]

Much work goes into the process of crafting definitions. It can be difficult to generate a definition that unambiguously captures a specific set of objects—one that extracts the most salient characteristics and matches our intentions and intuitions. For example, the history of mathematics is full of different kinds of functions (e.g., polynomials, trigonometric functions, logarithms), but it was only in the last century that mathematicians attempted to formulate a proper definition of "function," and there remains a range of options about how the definition should be phrased. (We give a definition for this text in the next chapter.) In some cases, a function is defined to be "a mapping relating a set of input values to a set of output values where each input is related to exactly one output." Another common definition is "a collection of ordered pairs (x, y) where x comes from a set X, y comes from a set Y, and no

[1] See Edwards and Ward [2] for further discussion about definitions in mathematics classrooms.

N. H. Wasserman et al., *Understanding Analysis and its Connections to Secondary Mathematics Teaching*, Springer Texts in Education,
https://doi.org/10.1007/978-3-030-89198-5_6

two ordered pairs have the same first coordinate." Is one better? Are they the same? Is either an improvement over the intuitive idea that a function is just a formula that relates x to y?

Whereas arguing about whether a hot dog is a sandwich is a harmless way to pass the time, proving theorems about functions requires that there be no confusion about what qualifies as a function. Once a definition is agreed upon, it becomes the foundation for mathematical study. Intuition can still be a guide, but any implications or properties that follow must flow logically from the definition.[2] Unambiguous definitions are paramount to the deductive process of mathematics, but they are not set in stone or handed down from on high. Crafting rigorous definitions is a human endeavor and, as such, there is not always agreement on what they should be.

Consider the following pedagogical situation:

Trapezoids have different definitions. Texas uses an *exclusive* definition:

1. A trapezoid is defined as a quadrilateral with *exactly one pair* of parallel sides

New York uses an *inclusive* definition:

2. A trapezoid is defined as a quadrilateral with *at least one pair* of parallel sides

Ms. Abara, a geometry teacher in Texas, has already taught her students about different kinds of quadrilaterals when a new student named Lena arrives from New York. After a few days, Ms. Abara senses that Lena disagrees with the other students about whether or not certain quadrilaterals are trapezoids. Ms. Abara is trying to figure out how best to resolve this issue with Lena, as well as how to talk about isosceles trapezoids with the class. How might she respond?

Having two definitions for a trapezoid is not necessarily problematic. Many concepts are defined in multiple ways, and definitions that appear to be different can sometimes turn out to be logically *equivalent*, meaning they specify the same set of objects. For example, the two definitions for "function" given above are essentially equivalent. The first uses less formal language than the second, but an object deemed a function according to one would also be a function according to the other. Two definitions being equivalent is not necessarily problematic in a classroom. A

[2] The relationship between definition and theorem is not quite this uni-directional. The key point is not necessarily about a chronological ordering, but a semantic one—theorems draw on definitions.

textbook typically chooses one as the definition and explains how the other is a consequence of the first, based on how the author wants the content to be structured. However, two definitions are *competing* when they specify different sets of objects for the same concept. This situation poses more of a challenge in teaching. How definitions are stated can potentially clarify or obscure the mathematical objects being studied. Moreover, the definition is the starting point, and so it determines the properties and implications that follow and subsequent definitions all have to be crafted with respect to the original choice.

Before moving on, think about which definition of trapezoid from the pedagogical situation is most familiar. Are these two definitions equivalent or competing? How would you define an isosceles trapezoid based on each definition? Which definition is better, in your opinion, and how does your choice reflect the characteristics you value in a mathematical definition?

6.2 Connecting to Secondary Mathematics

6.2.1 Problematizing Teaching and the Pedagogical Situation

The two definitions for trapezoid are competing definitions because they specify different sets of objects. We elaborate on why and problematize two ways a teacher might respond to the issues in the pedagogical scenario.

One way to approach competing definitions is simply to pick one, assert it as the correct definition, and reject the other. Although this would resolve the tension, it seems odd to say that one state's definition is "incorrect." It would make more sense to say that different state education boards make different choices, and in this class we will use a particular definition. Moreover, declaring that one definition is wrong misrepresents how definitions operate within mathematics. Such a response does not convey the "human construct" nature of definitions. Some definitions are more normative, or standard, and there is a temptation to declare less normative definitions to be incorrect, but at some level they are not wrong, just different. They designate a distinct set of objects for study. Simply opting for one definition over the other misses an opportunity to highlight the stipulative nature of definitions in mathematics. Once a definition is given, the collection of objects characterized by that definition becomes the domain of study. Our personal opinion as to what objects should qualify no longer matters. We have to adapt to consider all possible objects that fulfill the definition, even if the resulting collection is different from what we might have preferred or expected. In the teaching scenario, why the definitions are competing has to do with how trapezoids relate to parallelograms; either *no* parallelograms are trapezoids or *all* parallelograms are trapezoids.

The teacher's response in the pedagogical situation also has implications for how to define the concept of an *isosceles* trapezoid. This is a bit surprising. A first impression is that what makes a trapezoid isosceles should be evident by applying the criteria for isosceles to the objects designated by either definition of trapezoid. The problem is that definitions for subsequent concepts build on earlier definitions,

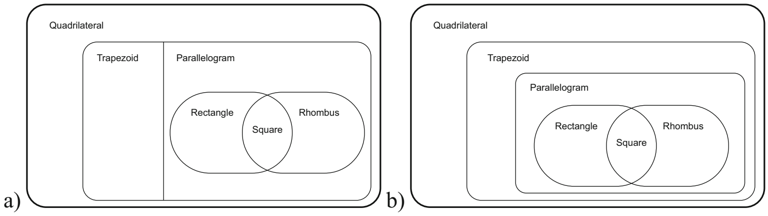


Fig. 6.1 Comparing the (**a**) exclusive and (**b**) inclusive definitions of trapezoid

and this causes some complications. Consider a standard definition for isosceles trapezoid, given in Chap. 5, which asserts that "a trapezoid is isosceles if the non-parallel opposing sides are congruent." This definition assumes the exclusive definition of trapezoid. The criterion for isosceles does not make sense if we have two sets of parallel sides, which is a possibility in the inclusive definition. What do we do if there are no non-parallel sides to consider?

Before reading on, think about how you might resolve the issue of defining an isosceles trapezoid so that it makes sense for both definitions of trapezoid.

6.2.2 Trapezoids

The essential difference between the two definitions is the way trapezoids relate to parallelograms. This is illustrated in the Venn diagram in Fig. 6.1.

The exclusive definition captures the intuitive idea that a trapezoid can be created by slicing off the top of a triangle. By adopting this definition trapezoids are required to have non-parallel sides and so parallelograms are not trapezoids. Categorizing in this way, a quadrilateral with at least one set of parallel sides is either a trapezoid or a parallelogram—it falls into one category or the other, but not both. They are *disjoint*.

If we adopt the inclusive definition, though, trapezoids encompass parallelograms. Parallelograms are a nested *subset* of trapezoids. In this categorization, every parallelogram is simultaneously a trapezoid. Although this categorization might feel unusual at first, it is a familiar way to structure definitions. "All squares are rectangles but not all rectangles are squares" expresses the same type of nested relationship. If we consider number sets, the natural numbers are a subset of the integers, which are a subset of the rationals, and so on. With this nested structure, if a property is true for the objects in a set, then it necessarily applies to a nested subset. As an example, a trapezoid's area is found by the formula $A_{trap} = \frac{1}{2}(b_1 + b_2)h$, where b_1 and b_2 are the lengths of two of its parallel sides and h is the perpendicular height between those sides. With the exclusive definition, it cannot be assumed that this formula will also be true for the area of a parallelogram. But using the inclusive definition, a parallelogram's area must be given by the same formula. With the extra condition $b_1 = b_2 = b$, the formula gives $A_{par} = \frac{1}{2}(2b)h = bh$, illustrating

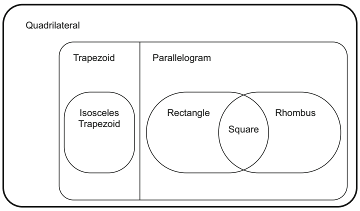


Fig. 6.2 Defining isosceles trapezoid with the exclusive definition of trapezoid

how nested categories can highlight connections between related objects and their properties.

6.2.3 Isosceles Trapezoids

Defining an isosceles trapezoid as one in which "non-parallel opposing sides are congruent" works well with the exclusive definition for trapezoid. It makes isosceles trapezoids a proper subset of trapezoids (Fig. 6.2). Because parallelograms are not trapezoids, there are no questions about whether a parallelogram is an isosceles trapezoid.

This definition does not work well with the inclusive definition of trapezoid; it does not resolve the question of whether a parallelogram is an isosceles trapezoid. We consider two possibilities for defining an isosceles trapezoid in the inclusive case.

As a starting point, we could try to adapt our current criterion for an isosceles trapezoid to work with the exclusive definition. What happens if we drop the "non-parallel" stipulation and rephrase the criterion as: "*A trapezoid is isosceles if it has a pair of opposing sides that are congruent*." For any quadrilateral that qualifies as a trapezoid under the exclusive definition (i.e., a trapezoid that is not a parallelogram), we get the same conclusion we did before. Meaning any trapezoid that was previously classified as isosceles retains that designation with this adapted criteria.[3] So far so good. Now let's see what happens when we move to the inclusive definition and consider a trapezoid that is also a parallelogram. Notably, our amended definition of isosceles is meaningful in this context since we can assign a truth value to the existence of a pair of congruent opposing sides. Because every parallelogram has (two) pairs of congruent opposing sides, all parallelograms are isosceles trapezoids. In terms of our Venn diagram, this means we would add an additional nested set to specify isosceles trapezoids with the inclusive definition (see Fig. 6.3a).

But does this classification of isosceles trapezoids agree with what our intuition tells us to expect? This is hard to say, and there are likely varying opinions on the matter. Extending the notions of congruence and symmetry familiar from isosceles triangles, though, we can make a list of several properties that we might naturally associate with isosceles trapezoids:

[3] If it is the parallel sides that were the ones congruent, it would be a parallelogram and hence not a trapezoid.

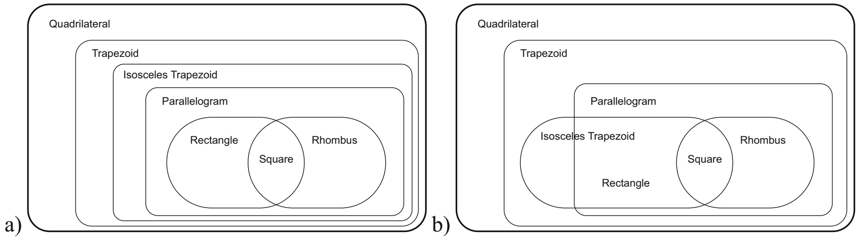


Fig. 6.3 Two approaches to defining isosceles trapezoids, with the inclusive definition of trapezoid

- opposite sides congruent
- base angles congruent (i.e., a pair of consecutive angles that are congruent)
- a line of symmetry
- diagonals congruent

To explore the possible differences in these characterizations, let's fashion another possible definition of isosceles based on the second item in the list: "*A trapezoid is isosceles if it has a pair of consecutive angles (or base angles) that are congruent*." Again, for any quadrilateral that qualifies as a trapezoid under the exclusive definition (i.e., a trapezoid that is not a parallelogram), we get the same conclusion we did with the original formulation. That is, we could prove a theorem that says, "A trapezoid has a pair of non-parallel sides that are congruent if and only if it has a pair of congruent consecutive angles." But now using the inclusive definition, where parallelograms are trapezoids, we need to consider which types of parallelograms have a pair of congruent consecutive angles. Rectangles and squares have pairs of congruent consecutive angles, but other parallelograms do not. For consecutive angles of a parallelogram to be congruent, the sides need to perpendicular. This approach to defining isosceles trapezoids, which is the more normative approach with an inclusive definition of trapezoid, configures subsets of quadrilaterals quite differently than before (Fig. 6.3b).

What intuition about isosceles trapezoids does this second definition capture? Which definition feels like the "right" one in this inclusive context?

6.3 Connecting to Real Analysis

Continuous functions are a central point of study in an analysis course. Throughout high school and university mathematics, the concept of continuity gets described in ways that range from informal to excessively precise. Here, we consider four definitions of continuity, some of which are likely familiar to you. The initial question is whether the definitions are equivalent or competing. Do these definitions classify the same set of functions as continuous? Be sure to consider atypical functions as you sort through the four proposals.

(Possible) Definition For each proposed definition we consider a real-valued function $f : A \to \mathbb{R}$, where A is a subset of $\mathbb{R}$.

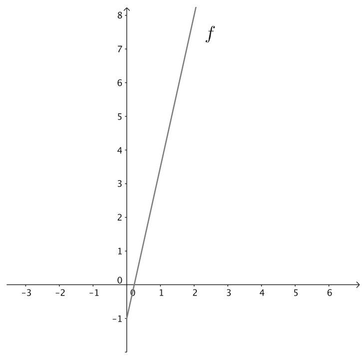
1. The function f is **continuous on** A if its graph can be drawn without lifting up one's pencil.
2. The function f is **continuous on** A if for every $c \in A$, $\lim_{x\to c} f(x) = f(c)$.
3. The function f is **continuous on** A if for every $c \in A$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$ (and $x \in A$) then $|f(x) - f(c)| < \varepsilon$.
4. The function f is **continuous on** A if $\lim_{n\to\infty} f(x_n) = f(c)$ for every sequence (x_n) (with $x_n \in A$) that converges to some $c \in A$.

Most of us come to the table with a set of expectations for what continuity entails. Polynomials are continuous; so are sine and cosine curves. Continuous functions should not have holes or jumps. Which of these definitions capture our sense of what continuity should be? Which functions are included and which are ruled out? Are there stipulations in some of these definitions that might surprise us or push against our intuition? Remembering the pitfalls we experienced defining isosceles trapezoids, are there situations where the definitions don't make sense? Is the given criteria precise enough that it can be evaluated at all?

6.3.1 Considering Various Definitions of Continuity

To get a better sense of each of these four proposed definitions for continuity, let's try them out on some example functions.

Example Consider the function f depicted in the graph



Tasked with deciding whether f is continuous from just this graphical information, Definition (1) feels like an appropriate and straightforward way to proceed. The graph can be drawn without lifting up one's pencil, so f appears to be continuous. But how compelling is this argument?

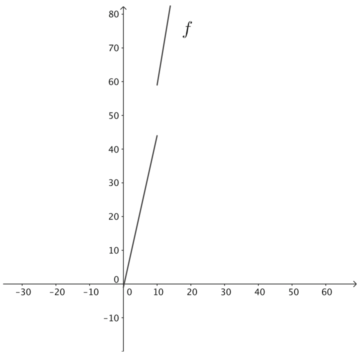
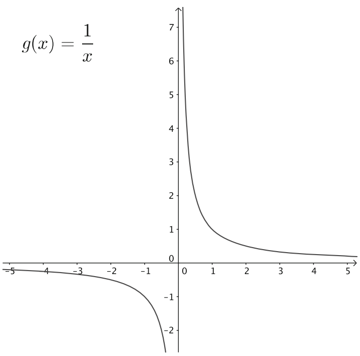


Fig. 6.4 Zoomed out graph of f, discontinuous by Definition (1)

This example illustrates a weakness in the first definition. Relying on a graph for our definition of continuity is inherently limiting. There are functions that cannot be graphed in a meaningful way. One example would be the discontinuous Dirichlet function, $g(x) = \begin{cases} 1 : x \in \mathbb{Q} \\ 0 : x \notin \mathbb{Q} \end{cases}$. How can we apply Definition (1) when g cannot be graphed? An example of an ungraphable function that is continuous is the Weierstrass Function. Its progressively finer layers of oscillations outstrip the resolution of any graphing device (the function has a self-replicating, fractal, nature). Generally speaking, graphs are visual summaries—useful for our intuition but, by their nature, incomplete. Unless the domain of a function is a finite set of points, a graph can only provide a partial description. Returning to the function in the graph above, suppose $f(x) = 1.5x \cdot \lfloor 0.1x + 3 \rfloor - 1$ (for $x \in \mathbb{R}^+$). Indeed, this is what generated the graph of f. However, now look at a plot of this function, zoomed out, in Fig. 6.4. By expanding the viewing window, we see that f has a "jump." This leads to the conclusion that f is discontinuous because we have to pick up our pencil to draw it. The moral of this story is that Definition (1) has some severe limitations—criteria based on a function's graph is difficult to evaluate consistently.

Example Consider the function $g(x) = \frac{1}{x}$ depicted in the graph



By Definition (1), $g(x)$ would be discontinuous because drawing the graph requires us to lift our pencil. The problem occurs at $x = 0$, but $x = 0$ is not even in the domain of g! In this example, Definition (1) holds g accountable for its behavior at a point where g is not even defined. Notably, the continuity of a function is typically defined with respect to a specified domain A. In the previous example, f had domain $\mathbb{R}^+$. Although we judged f to be discontinuous, this was due to the jump in the graph and not because it happened not to be defined for $x \leq 0$.

This points to an interesting quality, and perhaps a counterintuitive implication, of Definitions (2), (3), and (4). Each of these latter three definitions explicitly requires us to investigate the behavior of the function in question at individual points $c \in A$. Each definition is imbued with a particular way to define continuity at a point, and defines a function to be "continuous on A" if it is continuous at each point of A. The definitions do not consider what happens at points outside the intended domain. For $g(x) = 1/x$, the natural choice for the domain is $A = \{x \in \mathbb{R} : x \neq 0\}$, and by Definitions (2), (3), and (4), it turns out that g is indeed continuous on A. If we choose an arbitrary $c \neq 0$, the specified criteria in (2), (3), or (4) is met at c and therefore the function is continuous. (This requires some thought and you are encouraged to pause and think about why this is true in each case.)

It may feel a bit strange to assert that $g(x) = 1/x$ is continuous when it just looks so discontinuous. This is an example of what it means to adopt a formal definition and then live by all its stipulations. The feeling that g is not continuous, which arises from a natural sympathy for the sentiments in Definition (1), must be set aside in favor of the desire to structure our theory of continuity in a rigorous way. That said, it is still the case that some calculus books refer to g as having an "infinite discontinuity" at $x = 0$. One way to make that statement align with our formal definitions is to add $c = 0$ to the domain of g. For instance, we could set $g(0) = 0$ so that A is now all of $\mathbb{R}$. Setting $c = 0$ in Definition (2), we can observe $g(0) \neq \lim_{x \to 0} g(x)$ because the limit does not exist. This implies g is no longer continuous. Switching to the criteria in Definition (4) yields the same conclusion. Figure 6.5 depicts two sequences in the domain of g that both approach $c = 0$: (x_n) from the left and (z_n) from the right. (Those sequences are depicted on the x-axis.) However, the associated sequences $g(x_n)$ and $g(z_n)$ (depicted on the y-axis) diverge to infinity

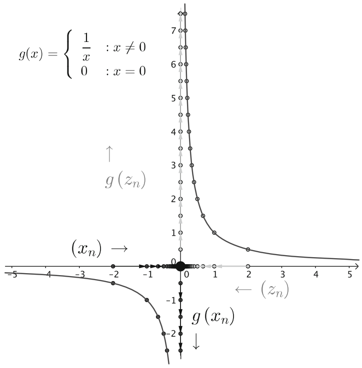


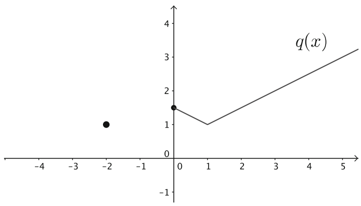
Fig. 6.5 The new function g is not discontinuous at $x = 0$ by Definition (4)

rather than approach $g(0) = 0$. By Definition (4), g is no longer continuous when its domain is expanded to include 0. (Definition (3) results in the same conclusion as well.)

Example As a final example, consider the function

$$q(x) = \begin{cases} \frac{1}{2}|x - 1| + 1 & : x \geq 0 \\ 1 & : x = -2 \end{cases}$$

which has the following graph:



For $q(x)$, we will consider continuity at two particular points of the domain. The first is $c = 0$, which is an *endpoint* of part of the domain. To use Definition (2) we need to consider the functional limit as x approaches 0. We know $q(0) = 1.5$, so the question is whether $\lim_{x \to 0} q(x) = 1.5$. The answer hinges on the definition of functional limit, and in particular whether it exists at an endpoint like $c = 0$. Using the definition in Abbott [1] (4.2.1), we can confirm that everything checks out—the functional limit exists and q is continuous at 0. In fact, q is continuous on the set $A = \{x \in \mathbb{R} : x \geq 0\}$.

But what happens at the point $c = -2$? This is an *isolated point* of the domain of $q(x)$, and Abbott's definition stipulates that functional limits can only be considered at *limit points*, which $c = -2$ is not. This is significant. If we adopt Definition

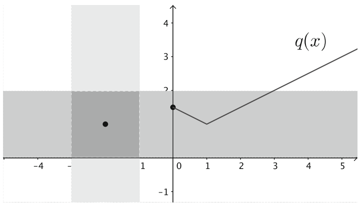


Fig. 6.6 The function q is continuous at the isolated point $c = -2$ using Definition (3)

(2), then continuity requires that $\lim_{x\to -2} q(x) = q(-2)$ which is not true. The functional limit is not defined and so using Definition (2) we get that q is *not* continuous on this larger domain that includes the isolated point. This conclusion is different from the one that emerges from adopting either Definition (3) or (4). The logical structure of these latter two statements implies that functions *are* continuous at isolated points of their domain. Definition (2) is thus a competing definition, specifying a different set of functions to be continuous than from Definitions (3) and (4), which turn out to be equivalent to each other.

To understand why these definitions are competing, look carefully at the wording of Definition (3) as it relates to the isolated point $c = -2$, and note especially the parenthetical reminder $(x \in A)$. Given an arbitrary $\varepsilon > 0$, the definition requires us to find a δ neighborhood centered at -2 such that all the points in this neighborhood *that are also in the domain* have y-values within ε of $q(-2) = 1$. Choosing $\delta = 1$ results in the neighborhood $(-3, -1)$, and the only domain point contained in this interval is $c = -2$. With no other x-values to worry about, we conclude that q is continuous. (See Fig. 6.6.) Take a moment to confirm that Definition (4) is structured in a similar way so that a function is determined to be continuous at any isolated points of its domain.

6.3.2 Choosing a Definition

Exploring the examples in this section has revealed some of the strengths and weakness, as well as the logical distinctions, that exist among our four proposed definitions for continuity. From this more informed point of view, how might we settle on the best choice to be our official definition?

Definition (1), although intuitively helpful, must be ruled out on the grounds that it is simply too informal. A proper definition should precisely delineate a set of mathematical objects, and it is not at all clear how to apply the statement in Definition (1) to numerous functions we would like to categorize. As the first two examples show, it can also lead to categorizations that are potentially contradictory.

Definition (2) is appropriately formal, but it categorizes any function with an isolated point in its domain as discontinuous. This competes with Definition (3) (and (4) as well) which is crafted so that isolated points turn out to be points of continuity. How should we decide between these two options? On the one hand, we might feel that isolated points don't innately feel "continuous." The graph of a function defined on the positive integers would amount to a sequence of disconnected dots which is certainly a far cry from a graph that can be drawn without picking up a pencil. This is a reasonable argument for adopting Definition (2), but there are other considerations. The most significant is the way the chosen definition sets the stage for the conclusions that follow. The study of continuous functions is connected to a network of other ideas that are articulated in the theorems of analysis. Details aside, many elegantly stated results such as "continuous functions restricted to compact sets are uniformly continuous," would become laden with awkward disclaimers if we adopted Definition (2). Although it initially pushes against our intuition, classifying functions to be continuous at isolated points turns out to be the more organic way to build the larger theory.

This brings us to Definitions (3) and (4), which we've discussed are equivalent (cf., Abbott's proof of Theorem 4.3.2). Precisely the same set of functions meet their respective criteria for continuity, and both criteria are used widely throughout a course in analysis. This suggests we have a genuine choice to make; *either* could serve as the definition for continuity and then we could prove the other as a theorem to be used as needed. This choice between equivalent definitions means we could use whichever one we found to be most appropriate for the course, the students, etc. While this is true—and there are analysis textbooks that take both approaches—the fact that two statements are logically equivalent does not mean they are equivalent in every respect. There are other considerations, too. When building a mathematical theory, there is an implied hierarchy between a definition and a theorem—definitions are the more primitive foundation on which theorems are built. The distinction is sometimes more art than science and making this distinction usually requires looking forward to see what lies ahead. In the case of continuity, for example, the $\varepsilon - \delta$ criterion in Definition (3) is most amenable to defining the concept of "uniform continuity" referenced in the above result.

Settling on the right definition can be a deliberative and subtle process, but a good sign that you are heading in the right direction is when there is an organic—we might even say poetic—connection between the definition and the theorems that follow. As the mathematician G.H. Hardy famously said, "Beauty is the first test!"

6.4 Connecting to Secondary Teaching

The takeaway from our discussion thus far is that, as teachers, we need to be purposeful about the definitions and explanations we use with students. We have a choice, and that choice has implications for the theorems and definitions that follow. Returning to the competing trapezoid definitions from the beginning of the chapter, the one we choose determines whether parallelograms are distinct from trapezoids or whether they form a nested subset. This arrangement then leads to consequences of its own, shaping the trajectory for the class.

Of our six teaching principles, TP.2 is the most pertinent to this discussion. Whether it's trapezoids, continuity, or some other concept, creating a range of special cases is the best way to probe a proposed definition. This means constructing examples that meet the criteria in the definition as well as those that don't. It also means creating examples that fall near the boundary, barely qualifying as one of the defined objects or falling just shy. These kinds of "near" or "minimal" examples are especially valuable for marking out the scope of a proposed definition and getting a head start on determining the kinds of theorems that it engenders.

6.4.1 Defining Isosceles Trapezoids

Two definitions for the same concept are logically different (i.e., competing) if they specify distinct sets of objects. While many objects will meet the criteria of both definitions, objects that meet one criteria but not the other are the important cases that separate the two definitions. For the two competing ways to define a trapezoid—the exclusive and inclusive definitions—so-called "common" trapezoids with exactly one pair of parallel sides fit both definitions while the set of parallelograms satisfies only the inclusive definition. Illustrating the impact of TP.2, these examples shape our understanding of the tension between the competing definitions and should be at the forefront when we consider the best way to extend the theory of trapezoids with additional definitions and theorems.

Consider the following continuation of the previous teaching scenario:

Ms. Abara gathers the class together and reviews the definition of trapezoid:

> In our class, we have defined a trapezoid as a quadrilateral with *exactly one pair* of parallel sides.

She continues by drawing attention to parallelograms and rectangles to illustrate the key difference:

> What this means is that trapezoids and parallelograms are separate. Parallelograms, including rectangles, are not trapezoids according to the definition we are using.

She then introduces a new definition for isosceles trapezoids:

> Isosceles trapezoids are essentially about symmetry. With our definition of trapezoid, we could define them in terms of the non-parallel opposing sides being congruent. But for class purposes, we will define them in terms of another symmetry: a trapezoid is isosceles if the base angles are congruent.

Recognizing that her new student from New York was introduced to a competing definition of trapezoid, Ms. Abara clarifies the definition used in her Texas-based class and then gives an example to shed light on the difference. To define isosceles trapezoids, the teacher focuses on "symmetry" as the essential feature and states that various definitions based on symmetry might be possible. Ultimately, Ms. Abara defines an isosceles trapezoid in a manner that makes sense with either definition of trapezoid. Although a system of definitions can build on one another, using a definition that works equally well across multiple definitions is a worthy consideration. Ms. Abara's definition using congruence of base angles, rather than congruence of two non-parallel opposing sides, is meaningful and effective for either the inclusive or the exclusive definition of trapezoid. A related version of this type of consideration that happens in a geometric context is whether a definition in Euclidean geometry still makes sense in a non-Euclidean setting. Problem 6.8 asks you to think about this issue.

The process of defining terms and generating examples is an important component of students' mathematical education. Students should also *experience* this aspect of mathematics in an active way. As teachers, it is also important to engage students in this process, recognizing them as independent thinkers capable of refining their own definitions and appreciating the objects their definitions describe.

6.4.2 The Relationship Between Definitions and Theorems

If we choose the definition of isosceles trapezoid to be a trapezoid with congruent base angles, the next logical step is to prove a theorem which states that the opposite sides of an isosceles trapezoid are congruent as well. The definitions established by the teacher lay out the logical trajectory for the class, which can have implications for how students view the larger theory.

To appreciate how the definition-theorem relationship can impact student understanding, consider two possible ways we might choose to define a rectangle. A first definition could be: "A rectangle is an equiangular quadrilateral." It is important to distinguish between what this definition explicitly *assumes* about rectangles and what it logically *implies*. In this case, a rectangle is assumed to be a quadrilateral with four congruent angles. This definition embeds rectangles as a subset of quadrilaterals, but there is no mention yet of right angles or parallel sides. The fact that all rectangles turn out to be parallelograms is a theorem that has to be proved from the definition.[4]

A second definition could be: "A rectangle is a parallelogram with one right angle." Defining a rectangle in this way means we are nesting rectangles as a subset of parallelograms, which is itself a subset of quadrilaterals. This is conceptually

[4] 'Equiang. Quad. $\implies$ Par.': Through the construction of a diagonal, the four angles sum to the interior angles of two triangles. In Euclidean geometry, this sum is two straight angles; a fourth of two straight angles is a right angle, and so each angle is a right angle. This makes the same side interior angles supplementary, which means both pairs of opposite sides are parallel. Hence, a rectangle is a parallelogram—and with at least one right angle.

different from from where we started before. In this case, being equiangular is not part of the definition for rectangle but becomes a theorem we can prove.[5]

These two definitions are not competing—we get the same set of rectangles with either one. So is one better than the other? Which criterion feels more fundamental to the nature of rectangles? Or perhaps we should reject them both in favor of asserting "a rectangle is a quadrilateral with four right angles." This latter statement is not as primitive as either of the other two proposed definitions—it assumes more than is necessary to specify the same set of mathematical objects—but there is an argument that this is what a rectangle really is. Without resolving this debate here, we note that the proofs showing these three definitions are logically equivalent utilize ideas specific to Euclidean geometry. In a non-Euclidean context the definitions can lose their equivalence and start to compete, raising the stakes considerably for deciding which one ought to be the definition of a rectangle to begin with.

For many secondary topics there are a variety of definitions that can be chosen. As teachers, it is important to think through these choices and the implications for how the ideas would then progress. There are significant ramifications for students that result from the different ways teachers sequence the definitions with the theorems and properties that follow.

Problems

6.1 In the previous Chap. 5, we looked at several isosceles trapezoid statements. There, we were assuming the *exclusive* definition of trapezoid. This chapter introduced the *inclusive* definition. Look at several statements or theorems about trapezoids from geometry. Determine the truth value of each, depending on which of the two definitions of trapezoid is used. If possible, give two example theorems that would be true under one definition but not true under the other.

6.2 Zero is an even number. However, students often suggest that zero is neither even nor odd. Which of the following would still be true if all other integers (positive and negative) except zero retain their even or odd status? Justify your response for each statement.

1. even + even = even
2. odd + odd = even
3. even + odd = odd
4. even × even = even
5. odd × odd = odd
6. even × odd = even

[5] 'Par. One Rt. Angle $\Longrightarrow$ Equiang. Quad.': Because lines are parallel in a rectangle, and in Euclidean geometry the same side interior angles are supplementary, this means another angle is a right angle. By repetition, we can conclude the rectangle has four right angles, and so equiangular.

6.3 In geometry, the distance between a line and a point not on the line is defined as the distance along a *perpendicular* line. In statistics, the distance between a line (of best fit) and a point (not on the line) is defined as the distance along a *vertical* line. Are these two definitions equivalent or competing definitions? If they are equivalent, provide a justification. If they are competing, provide an example where they would be different and, if possible, one where they would be the same. Then, discuss why geometry and statistics might define the "distance" between two such points in the way they do.

6.4 A class is asked to prove the following definitions of rectangle are equivalent:

1. A quadrilateral is a rectangle if it is a parallelogram with four right angles.
2. A quadrilateral is a rectangle if it is a parallelogram with one right angle.
3. A quadrilateral is a rectangle if it is a quadrilateral with four right angles.

One student submits: "Assume we have a quadrilateral $ABCD$ that is a parallelogram with four right angles. If we accept Definition (1), then we call it a rectangle. But, obviously, if it has four right angles then it has one right angle, so it also fulfills Definition (2). In addition, all parallelograms are quadrilaterals, so it also fulfills Definition (3). Also, we know that adjacent angles of a parallelogram are supplementary, meaning if there is one right angle in a parallelogram, then we actually know that all four are right." Respond to the following: (i) as the teacher, how would you respond to the student's written work?; (ii) discuss what, if any, errors are present, and what, based on what the student has submitted, would still be needed to complete the question.

6.5 Consider teaching a course in geometry. Give two different ways to structure a sequence of definitions for special quadrilaterals: trapezoids, parallelograms, rectangles, rhombuses, and kites. Give both a precise definition for each, as well as the sequential order you would discuss them with students. Provide a justification for each of the two possible approaches. Then, consider having to teach about area: Which sequence of definitions do you think would be better, or worse, for teaching students about area formulas for quadrilaterals? Explain your reasoning.

6.6 A common definition for the absolute value of a number is piece-wise: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Think about whether this definition makes sense for numbers, x, that are Natural numbers? Integers? Rational numbers? Real numbers? Complex numbers? If the definition does not make sense for a number set, explain why not. Now think about an alternate definition: $|x|$ is the distance (measured in the typical way) from the "origin" (0 on a number line, $(0, 0)$ in the plane, etc.). For which number sets would this definition make sense? Discuss which definition you might use with a class of students? Why?

6.7 Think about how you would define the "perimeter" of a (2D) shape (an idea we pick up on in Chap. 10). Compare and contrast the following two possible definitions: (i) The perimeter of a shape is the sum of all the side lengths (on the edge that encloses it); (ii) The perimeter of a shape is the distance around the edge that encloses it. Draw several different kinds of shapes studied in secondary mathematics. Which definition would you use to "define" perimeter? Why? Even though only one is being used as the definition, would the other description of perimeter be discussed with students in any way? If so, how and when might it be discussed?

6.8 In Euclidean geometry, we can define a rectangle in a variety of equivalent ways. Consider the three possibilities below. Which of these definitions makes the most sense in a non-Euclidean geometry context? (In non-Euclidean geometry, the interior angle sum of a triangle does not have to be 180°. Also, there might be multiple lines through a point all parallel to a given line, or there could be no parallel lines through this point that are parallel to the given line.) Explain your reasoning.

- A rectangle is a parallelogram with one right angle
- A rectangle is a quadrilateral with three right angles
- A rectangle is an equiangular quadrilateral

6.9 Consider the following two definitions for a real-valued function f defined on domain A.

- An *increasing function* $f : A \to \mathbb{R}$ is a function such that for $x_1, x_2 \in A$ with $x_1 < x_2$, $f(x_1) \leq f(x_2)$.
- A *decreasing function* $f : A \to \mathbb{R}$ is a function such that for $x_1, x_2 \in A$ with $x_1 < x_2$, $f(x_1) \geq f(x_2)$.

If you can, sketch a function that is increasing but not decreasing. One that is decreasing but not increasing. One that is both increasing and decreasing. One that is neither increasing nor decreasing. Explain why your functions meet the required specifications.

6.10 Consider Abbott's Exercise 4.2.10, about the use of left- and right-hand limits in introductory calculus:

> Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."
>
> (a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:
>
> $$\lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L$$
>
> (b) Prove that $\lim_{x \to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

What purpose does part (b) of that exercise serve in terms of a calculus teacher's ability to only discuss functional limits as being in terms of right-hand and left-hand limits? What teaching principle would you consider this example as illustrating?

Turning the Tables

Reflecting on *teaching* from your *learning* in real analysis: TP.4

As another way of connecting to issues of teaching and learning, the "Turning the Tables" sections scattered throughout the text provide additional commentary on some of the ways our teaching principles are exemplified in learning real analysis. Here, we consider TP.4: modeling more complex objects with simpler ones.

Real analysis has many excellent examples of this principle. In Chap. 3, we constructed sequences of rational numbers that converged to a real number. In that example, real numbers with infinite and irregular decimal expansions are being modeled, or approximated, by simpler rational numbers whose decimal expansions terminate. In Chap. 9 we will discuss how the derivative conceptualizes tangent lines as a sequence of secant lines, and in Chap. 12 we'll see how the Riemann integral models the complex region under a curve with a collection of simpler rectangles. These examples collectively illustrate how complex mathematical theories such as a calculus are constructed out of simpler building blocks and make a compelling case for how TP.4 can inform our approach to teaching.

The nested definitions discussed in the present chapter reinforce the central role of TP.4 in the way mathematics is structured. Here, we consider a particular definition from real analysis. The sequential criterion for continuity—given earlier as Definition (4)—states that a function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for every sequence (x_n) in A converging to c, it follows that $f(x_n)$ converges to $f(c)$. In this characterization, continuity is being defined in terms of *sequences* and *limits of sequences*—both concepts that were previously defined. In this manner, the concept of continuous functions is building on prior ideas. In the spirit of TP.4, we are conceptualizing continuous functions—something relatively complex—by using the simpler device of convergent sequences.

Constructing new concepts and definitions from previously-defined ones is fundamental to how mathematics is organized. Although the primary and more practical significance of TP.4 is best realized in specific examples such as approximating real numbers with rational sequences, the hierarchical nesting of concepts that characterizes mathematics showcases the broad relevance of TP.4. Explicitly making connections to previous concepts as we introduce new ones is a staple of good teaching and another point of connection to TP.4 in the classroom.

References

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