5 Divergence Criteria and Logic in Communication

5.1 Statement of the Teaching Problem

Euclid's *Elements* [2] is the most influential mathematics textbook in history. With a short set of postulates and some common notions and definitions, Euclid deduced thirteen volumes of geometrical propositions. His approach was axiomatic; it applied *logical* principles, which are now central to mathematics. The challenge is that teaching mathematics does not, and cannot, operate purely within this logical structure.

Teaching is an act of bridging between what a learner knows and does not yet know. To achieve this, teachers must communicate ideas in everyday language that students understand rather than in a strictly formal mathematical language. Finding a balance between these two modes of communication can be challenging. Consider the following two statements: "A square is a regular quadrilateral" and "A square is a rectangle." Grammatically, these follow an identical structure ("a [blank] is a [blank]"). Before reading on, think about how you interpreted them. What do you think the first one means? What about the second?

Logically, the two statements are different. In the first, "is" represents a *bidirectional* ($\iff$) relation, which characterizes a definition or an 'if-and-only-if' statement. In the second, "is" represents a *directional* ($\implies$) relation, which is used to articulate a property or an 'if-then' statement. A description of squares is that they are all rectangles, but the relationship is *not* true in reverse. For a different example, consider the statement: "You can have dessert if you finish your dinner." This is intended to communicate that the only way to have dessert is to finish your dinner. But this is an 'if-and-only-if', or bidirectional, interpretation (Dessert $\iff$ Dinner). Technically, the grammatical phrasing uses an 'if-then' structure (Dinner $\implies$ Dessert). In this literal interpretation we cannot be sure what happens if you do not finish your dinner. You may or may not get dessert—all we know is what happens if you *do* finish your dinner. Indeed, not attending to the

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direction of directional statements is commonplace in normal conversation; some might actually interpret the statement to mean the converse (Dinner $\Longleftarrow$ Dessert). This is because logical relationships can get lost in the translation to everyday language—a translation which is a necessary part of teaching.

Consider the following pedagogical situation:

> A geometry teacher, Ms. Rojas, has been teaching students about special quadrilaterals. One of those special quadrilaterals, a trapezoid, was defined to be "a quadrilateral with exactly one pair of parallel sides."
>
> Ms. Rojas is now discussing isosceles trapezoids. Throughout her explanations and in her responses to student questions, Ms. Rojas makes the following three statements:
>
> 1. "A trapezoid is isosceles if the non-parallel opposing sides are congruent"
> 2. "An isosceles trapezoid is a quadrilateral with congruent diagonals"
> 3. "A trapezoid is isosceles if the diagonals are congruent"

In these three statements, Ms. Rojas appears to give a definition for an isosceles trapezoid as well as some properties. The language she uses is not overly formal, and the fact that she describes the main ideas in multiple ways is a positive aspect (TP.6). But one of the challenges of teaching is being aware of the intended mathematical relationships, the way those relationships are expressed, and the possible ways they might be interpreted by students in the class. This is especially important when discussing relationships that are not logically equivalent.

Before moving on, think more about each statement. How could you rewrite each one using a more formal logical structure, and would the meaning that arises from that structure align with what you believe the teacher is trying to communicate?

5.2 Connecting to Secondary Mathematics

5.2.1 Problematizing Teaching and the Pedagogical Situation

We problematize the three isosceles trapezoid statements by considering the different ways they may be interpreted, including some which convey potentially contradictory meanings.

The primary logical relationships we discuss in this chapter are conditional and biconditional statements:

Definition A **conditional** statement is *directional*, of the form 'if A then B', or $A \Longrightarrow B$, where A is referred to as the *condition* and B the *consequence*.

Definition A **biconditional** statement is *bidirectional*, of the form 'A if-and-only-if B', or $A \iff B$, where A and B are interpreted as *equivalent*; that is, $A \implies B$ and $A \impliedby B$.

As discussed, a statement has possible logical meanings across each of three categories: the *intended* relationship, the *expressed* relationship, and the *interpreted* relationship. To frame the scope of the challenge, we could categorize different logical possibilities in the table below. We introduce the table not for the purpose of discussing every possibility, but for situating the few examples we do discuss.

Speaker		Listener		
Intended	Expressed	Interpreted		
		$\Longrightarrow$	$\Longleftrightarrow$	$\Longleftarrow$
$\Longrightarrow$	$\Longrightarrow$			
$\Longrightarrow$	$\Longleftrightarrow$			
$\Longrightarrow$	$\Longleftarrow$			
$\Longleftrightarrow$	$\Longrightarrow$			
$\Longleftrightarrow$	$\Longleftrightarrow$			
$\Longleftrightarrow$	$\Longleftarrow$			

For example, the first row of the table represents when the speaker intends a directional statement and expresses a directional statement. The three blank cells to the right allow for the possibility that the listener interprets the statement accurately (first column), as a biconditional (second column), or in the reverse (third column).

In the first statement from the teaching situation, the teacher intended to give a definition. Definitions are necessarily biconditional statements—if we call something A then it has property B, and if we see something with property B we call it A. In this case, the definition is:

$$\text{Isos. Trap.} \iff \text{Non-parallel Opp. sides Cong.}$$

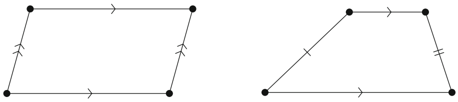
The teacher's statement, however, is expressed in the form of a directional relationship:

$$\text{Non-parallel Opp. sides Cong.} \implies \text{Isos. Trap.}$$

The intended relationship is $\iff$, but it has been expressed as $\implies$, which corresponds to row four in the table above.[1] A listener might interpret the relationship as it was expressed ($\implies$) which has a different logical meaning than was intended. The condition expressed is 'if Non-parallel Opp. sides Cong.' This is a *criterion* for isosceles trapezoids: any trapezoid that meets this condition is isosceles. However,

[1] This is very common in giving definitions in mathematics; indeed, some definitions given in this book also have been expressed in this way.

with the directional interpretation, we cannot conclude anything about trapezoids whose non-parallel opposing sides are *not* congruent—who knows, some of these might also be isosceles trapezoids. Directional implications tell us nothing about when the condition is not met. With this interpretation, we could not determine whether either of the following quadrilaterals is an isosceles trapezoid.



For the quadrilateral on the left, non-parallel opposing sides do not exist. For the quadrilateral on the right, non-parallel opposing sides do exist but they are not congruent. In our scenario, the teacher's definition of trapezoid rules out the parallelogram on the left, but statement (1), as expressed, leaves ambiguous whether the trapezoid on the right is isosceles. We refer to this as the **'iff' confusion**: a biconditional relationship is expressed as a conditional one, and so it is difficult to interpret whether the intended relationship goes both ways.

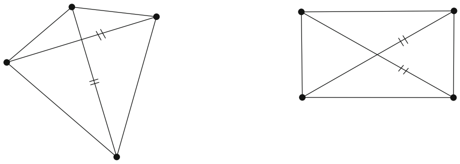
Now consider statement (2), which seems to describe a property of isosceles trapezoids: their diagonals are congruent. But the statement uses the word "is". Interpreted as a conditional statement, we get:

$$\text{Isos. Trap.} \implies \text{Quad. with Cong. Diag.}$$

This interpretation matches the actual relationship. The problem is that the word "is" can also be interpreted to indicate a biconditional relationship:

$$\text{Isos. Trap.} \iff \text{Quad. with Cong. Diag.}$$

Indeed, the grammatical structure of the statement—an isosceles trapezoid is not just a quadrilateral but one with congruent diagonals—matches how definitions are often given (e.g., a square is not just a quadrilateral, but a particular kind, a regular one). Interpreted this way, the statement is false. Two counterexamples, quadrilaterals that have congruent diagonals but are not isosceles trapezoids, are shown below.[2]



[2] The rectangle is an unusual case, explored more in the Chap. 6. For now it suffices to say that, according to the definition given in class, the rectangle would not be a trapezoid at all.

The logical meaning of statement (2) hinges on the interpretation of "is," which might express an intended relationship of $\Longrightarrow$, or $\Longleftrightarrow$. We refer to this as the **'is' confusion**: a logical connector expressed by "is" makes interpretation difficult because the actual relationship could be conditional or biconditional.

Statement (3) can be paired with statement (1) because, grammatically, they have the same structure (which raises the same 'iff' concerns). It can also be paired with statement (2) because the teacher is communicating about congruent diagonals in both statements. Upon closer inspection, however, statement (3) is distinct because the property of congruent diagonals is part of the condition, which is the *reverse* of the second statement. To be honest, it is hard to say precisely what the intended condition of statement (3) actually is: it could be 'Quad. with Cong. Diag.' or possibly 'Trap. with Cong. Diag.'? Let's consider both:

$$\text{Isos. Trap.} \Longleftarrow \text{Quad. with Cong. Diag.}$$

$$\text{Isos. Trap.} \Longleftarrow \text{Trap. with Cong. Diag.}$$

With the first interpretation, the statement is false by the previous counterexamples. This interpretation would be the *converse* of statement (2) and, as we can see, expresses something logically distinct. The second interpretation is true. It indicates that congruent diagonals in a trapezoid is sufficient to conclude the trapezoid is isosceles. In conjunction with the second statement, it could serve as another way to define an isosceles trapezoid—a biconditional relationship, 'Isos. Trap. $\Longleftrightarrow$ Trap. with Cong. Diag.' If the teacher meant for statement (3) to have this latter interpretation, then its expressed directional structure does not capture the intended bidirectional meaning. If the teacher intended statement (3) to be a recasting of statement (2), then the teacher actually switched the condition; the intended relationship ($\Longrightarrow$) was expressed in the reverse ($\Longleftarrow$). This confusion could also happen during interpretation: something expressed as $\Longrightarrow$ being interpreted as $\Longleftarrow$. We refer to this as the **'converse' confusion**: a conditional (directional) relationship is expressed or interpreted in the reverse, switching the condition and the consequence.

As teachers who necessarily employ informal language to communicate formal mathematical ideas, it is important to be aware of how easily the intended logical relationships can get lost in the way they are expressed and interpreted.

5.2.2 Common Variants of Conditional Statements

There are several common variants of a conditional statement $A \Longrightarrow B$, some of which incorporate negation ($\neg$). Each are distinct, although pairs of them have the same truth value, meaning they are logically equivalent.

1. *Statement*: $A \Longrightarrow B$
2. *Converse*: $B \Longrightarrow A$

3. *Inverse*: $\neg A \implies \neg B$
4. *Contrapositive*: $\neg B \implies \neg A$

The converse was mentioned previously—reversing the condition and the consequence. This can confuse logical communication because, as we have seen, a statement and its converse are not logically equivalent. On the other hand, even though they look different, a statement and its contrapositive *are* logically equivalent. If A implies B, then not having property B implies not having property A.

Claim A conditional statement and its contrapositive are logically equivalent.[3]

5.3 Connecting to Real Analysis

The isosceles trapezoid statements highlight some of the challenges of mathematical communication, and these same kinds of challenges certainly arise in real analysis. In this section, we consider several statements about convergence of sequences. To connect this to the previous discussion, the strategy is to transport the same scrutiny about the expressed and intended mathematical relationships to a new domain. Whether the topic is geometry or advanced calculus, paying attention to logic can improve our understanding of the relationships—especially in the way directional statements communicate *descriptions of* or *criteria for* mathematical concepts.

5.3.1 Convergence Theorems

As an initial exercise, read through the following real analysis statements. Some are stated in an atypical way. As you do so, try to clarify the logical structure of the intended mathematical relationship ($\implies$ or $\iff$) and think about what other interpretations might be possible.

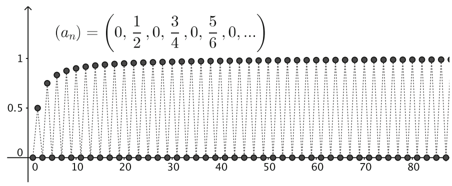
Real Analysis Statements

1. A sequence is convergent if all of its subsequences converge to the same limit
2. A convergent sequence is a sequence that's bounded
3. A sequence is convergent if it is monotone and bounded

[3] The converse and inverse have this same relationship, so they, too, are logically equivalent.

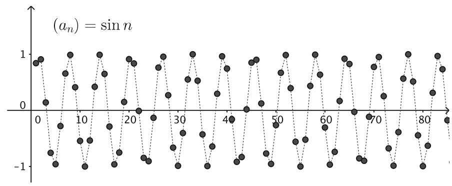
To showcase possible points of confusion, the three statements have been grammatically constructed to mirror the isosceles trapezoid statements. The content has changed, so the truth values might change as well, but notice that we are faced with the same ambiguity about the intended directional nature of each claim.

Statement (1) is true as a biconditional relationship. In this sense, the *'iff' confusion* applies since statement (1) is currently expressed as a directional statement. As written, the statement has the condition 'if All Subseq. Converg. to Same Lim.' and the consequence 'Seq. is Converg.' This is a true statement, albeit a rather trivial one since one of the subsequences would be the sequence itself. This may be the intended meaning but perhaps not. Interpreted this way, statement (1) says nothing about a sequence where the condition is not met. For instance, the sequence $(a_n) = \left(0, \frac{1}{2}, 0, \frac{3}{4}, 0, \frac{5}{6}, 0, \ldots\right)$, depicted below, does not meet the condition since not all its subsequences converge to the same limit.



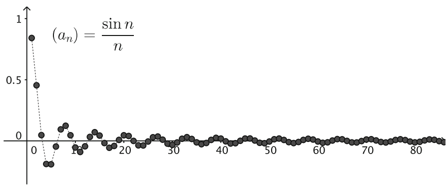
Does (a_n) converge? It does not, but statement (1) interpreted as a directional if-then proposition cannot be invoked to reach this conclusion. In fact, the relationship in statement (1) is biconditional, and this may be how it was intended. The converse proposition, 'if a sequence converges then all of its subsequences converge to the same limit' is a significant result in analysis that we discuss momentarily.

Statement (2) demonstrates the *'is' confusion*. The meaning changes depending on whether we interpret "is" as conditional ($\Longrightarrow$) or biconditional ($\Longleftrightarrow$). Interpreting statement (2) as a bidirectional statement means convergence and boundedness are equivalent—that 'Converg. $\Longleftrightarrow$ Bound.' This is not true. The sequence $(a_n) = \sin n$ is bounded between -1 and 1 but does not converge (see below). So it is not true that 'Converg. $\Longleftarrow$ Bound.'



However, if we interpret “is” to be directional we get: ‘Converg. $\Longrightarrow$ Bound.’ This is true. Convergence is the condition and boundedness is a *description of* convergent sequences—they all have this property.

Statement (3) grammatically mirrors statement (1), but like the first statement, the intended relationship is directional, not bidirectional. It also has similarities to statement (2) in that they both describe relationships between convergent sequences and bounded ones. Being a directional statement, the *‘converse’ confusion* might surface; one might mistakenly interpret the statement in the reverse: ‘Converg. $\Longrightarrow$ Mon. and Bound.’ This implication is not true. Although we know convergent sequences must be bounded, they do not have to be monotone to be convergent. The sequence (a_n) where $a_n = \frac{\sin n}{n}$ converges to 0 (and hence is bounded), but it is not monotonic.



The intended statement is ‘Converg. $\Longleftarrow$ Mon. and Bound.’ In contrast to statement (2), convergence is the consequence. Being monotonic and bounded is a sufficient *criterion for* convergent sequences.

These three statements can be recast as theorems that should be familiar from analysis (cf., Abbott’s Theorems 2.3.2, 2.4.2, and 2.5.2). In this more formal setting, note how the statements are carefully crafted to clarify the logical relationships:

Theorems

1. **(Subsequence Convergence Theorem)** If a sequence converges, then all subsequences converge to the same limit.
2. **(Boundedness Theorem)** If a sequence converges, then it is bounded.
3. **(Monotone Convergence Theorem)** If a sequence is monotone and bounded, then it converges.

We do not provide proofs of these theorems (see Abbott’s [1] text). Instead, we want to study their logical structure by exploring the inverse, converse, and contrapositives. The result is a robust investigation of the complementary notions of convergence and divergence, as well as a heightened clarity for the inner workings of propositional logic in general.

Before reading on, formulate the converse, inverse, and contrapositive of each theorem. In each case ask yourself, “Is this true?” and “What does this tell me about divergent sequences?”

5.3.2 Logical Implications About Divergence

Because a contrapositive statement is logically equivalent to the original, it often provides additional insight. Here, it supplies ways of understanding *divergent* sequences. The contrapositive of each theorem is given below. Because the original theorems are true, these statements are true as well.

Contrapositive Statements

1. **(Divergence Subsequences Criterion)** If not all subsequences converge to the same limit, then the sequence diverges.
2. **(Divergence Boundedness Criterion)** If a sequence is unbounded, then it diverges.
3. **(Divergence Description)** If a sequence diverges, then it is not monotone or not bounded.

Focusing on the property of divergence (Div.), let's take a moment to differentiate between the condition and the consequence. Directional statements of the form 'Div. $\implies B$' (with divergence as the condition) provide a *description of* divergent sequences; whereas '$A \implies$ Div.' (with divergence as the consequence) provide a *criterion for* divergent sequences.

The first two contrapositive statements each give a criterion for divergent sequences. The first says that if not all subsequences converge to the same limit—i.e., if there exists two subsequences that converge to different limits or one sequence that does not converge at all—then we can conclude the original sequence is divergent. The second gives a different criterion which says that if the sequence is unbounded then it diverges. Consider the previously discussed sequence

$$(a_n) = \left(0, \tfrac{1}{2}, 0, \tfrac{3}{4}, 0, \tfrac{5}{6}, 0, \ldots\right).$$

How might we justify that it diverges? This sequence is bounded, so the non-boundedness criterion from the second statement does not apply. Recognizing (a_n) is not monotonic suggests possibly using statement (3), but this statement is not a criterion for divergence since divergence appears in the condition of the statement. The first contrapositive statement is the one we want! Observing that $(0, 0, 0, 0, 0 \ldots)$ is a subsequence whose limit is 0, and $\left(\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots\right)$ is a subsequence whose limit is 1, statement (1) justifies the conclusion that (a_n) diverges.

Because the third contrapositive statement has divergence in the condition, it articulates a description of divergent sequences (*not* a criterion for them). If we know a sequence diverges, the third statement guarantees the sequence will be either non-monotone or unbounded—or both. Consider the divergent sequence

$$(b_n) = \left(0, 1, 2, \tfrac{1}{4}, 0, 1, 2, \tfrac{1}{8}, 0, 1, 2, \tfrac{1}{12}, \ldots\right).$$

Because it diverges, (b_n) must be non-monotonic or unbounded. A little inspection reveals that (b_n) is not monotonic. (Other divergent sequences like $(c_n) = (\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots)$ are monotonic, but unbounded.) But we cannot guarantee, for example, that there are no convergent subsequences in (b_n)—$(0, 0, 0, \ldots)$ is one; nor can we guarantee there would not even be subsequences that converged to the same limit—$\left(\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \ldots\right)$ also converges to 0. Which is to say the property in statement (1) does not describe something true of all divergent sequences, even though it gives a condition for divergence. If we flip the question around and ask for a proof that (b_n) really diverges, then we would have to turn our attention back to statements (1) and (2). Because it has two subsequences converging to different limits—e.g., $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$—(b_n) diverges by the criterion in statement (1).

5.4 Connecting to Secondary Teaching

In the initial teaching situation, the teacher made several statements, each expressing some intended mathematical relationship. By considering students' potential interpretations, we highlighted some of the challenges of classroom communication. When the intended, expressed, and interpreted logical meaning don't all match then confusions ('iff', 'is', 'converse') arise—the proper mathematical relationship can literally be lost in translation.

The teacher is responsible for being attentive to the way everyday communication intersects with mathematical meaning. We have argued that teachers need to use everyday language in class and that they should not always state concepts in formal terms. To be consistent with TP.3, this means paying special attention to logical relationships as they might be interpreted, and not just as they were intended. Informal does not mean imprecise, and the principles of logic can still be invoked in a useful way. Distinguishing between the condition and the consequence of a directional relationship determines whether it communicates a description or a criterion, and switching to the contrapositive also has the potential to add new insight.

5.4.1 Counterexamples

Let's look at these ideas by continuing the teaching scenario from the beginning of the chapter.

Ms. Rojas (who previously defined a trapezoid to be a quadrilateral with exactly one pair of parallel sides) gives statement (1) as a definition: "A trapezoid is isosceles if the non-parallel opposing sides are congruent." After proving that the diagonals of an isosceles trapezoid are congruent, Ms. Rojas makes statement (2):

> This is the same as saying that an isosceles trapezoid is a quadrilateral with congruent diagonals.

Giving this some thought, a student named Adya suggests the statement is incorrect. Adya goes to the board and draws a rectangle which she proposes is a counterexample.

Ms. Rojas first asks Adya for some **content** clarification:

> You drew a rectangle. How are you relating a rectangle and an isosceles trapezoid?

After Adya responds, Ms. Rojas follows up with a **logic** question:

> You drew a quadrilateral with congruent diagonals. Tell me more about why this is a counterexample?

Sometimes confusion is about mathematical content, but the theme of this chapter is that it can also result from a misunderstanding of logical relationships. As a teacher, addressing both *content* and *logic* matters; pursuing one without the other might not get to the heart of the confusion. Both are evident in Ms. Rojas's response.

Ms. Rojas's first content question is motivated by a desire to understand whether Adya thinks a rectangle is a special case of an isosceles trapezoid. The follow-up logic question explores whether she is interpreting the statement to be one of expressing equivalence ($\Longleftrightarrow$) of the two parts, or of expressing a property ($\Longrightarrow$) of isosceles trapezoids. Adya's responses can help identify the actual source of confusion.

Let's look at what this means in terms of two student profiles.

Student Profile 1 A student who thinks a rectangle is an isosceles trapezoid.

Suppose in response to the first content question, the student explains the rectangle by saying it is an example of an isosceles trapezoid. In this case, a teacher can point out that because a trapezoid was defined to have exactly one pair of parallel sides, a rectangle is in fact not a trapezoid at all (and hence not an isosceles trapezoid). This would also be an opportunity to make a point about counterexamples more generally. Since a rectangle has congruent diagonals, it would not qualify as a counterexample. A proper counterexample would require finding a quadrilateral whose diagonals are not congruent that still managed to be an isosceles trapezoid.

Student Profile 2 A student who thinks the statement is bidirectional (because of the “is”), and that a rectangle is not an isosceles trapezoid.

Suppose in response to the second (logic) question, the student explains she is trying to show that a quadrilateral with congruent diagonals and an isosceles trapezoid are not the same thing. In this case, the teacher can start by affirming that, if this were a bidirectional statement, a rectangle would be an excellent counterexample. A rectangle is indeed a quadrilateral with congruent diagonals that is not an isosceles trapezoid. The teacher can then go on to address the ‘is’ confusion. Acknowledging the potential for misunderstanding, the teacher can clarify that her statement was intended to communicate a conditional relationship—specifically, that isosceles trapezoids have the property of being quadrilaterals with congruent diagonals. The statement was *not* trying to suggest the converse—that all quadrilaterals with congruent diagonals are isosceles trapezoids. It provides a *description of*, not a *criterion for*, isosceles trapezoids.

Even though the proposed counterexample was slightly off the mark in this case, we can still observe the way its use falls under the umbrella of TP.2. Counterexamples are an especially valuable type of special case because they can be used to test and scrutinize mathematical ideas—or, more pointedly, to demonstrate the falseness of a claim with a single example.

5.4.2 Converses

Let’s consider a different continuation of the teaching scenario where, this time, the teacher’s statement brings up the logical issue of the converse:

Ms. Rojas gives statement (1) as the definition for isosceles trapezoid and, after proving that the diagonals of an isosceles trapezoid are congruent, she makes statement (3):

> This is the same as saying that a trapezoid is isosceles if the diagonals are congruent.

In response, Adya asks the following question:

> Are you saying that any trapezoid with congruent diagonals is isosceles?

We look at how this might play out in terms of two teacher profiles.

Teacher Profile 1 The teacher intended to restate that isosceles trapezoids have congruent diagonals.

As teachers, it is important to interrogate our own statements as well as those of our students. Although students are known to reverse the condition and the consequence of a statement, in this case, it is the teacher who has accidentally done the switching. Recognizing the error, the teacher should affirm to the student that, no, this was not what she intended to communicate. In terms of logic, it provides an opportunity to point out the difference between a statement and its converse, and to give examples of each. In this particular case, the class could then investigate whether congruent diagonals imply a trapezoid is isosceles, and then return to statement (3) and rephrase it in more precise language.

Teacher Profile 2 The teacher intended to give a new definition for an isosceles trapezoid.

In this instance, the teacher needs to clarify her use of the phrase, "This is the same as saying. . . " Logically, the teacher's directional statement is not equivalent to the observation that isosceles trapezoids have congruent diagonals—in fact, it is the converse, and it needs to be independently verified. Once she proves the converse to be true, the teacher can then assert a biconditional relationship in which $A \implies B$ and $B \implies A$. This means congruent diagonals in a trapezoid are a defining feature—a *criterion for*, and a *description of*, every isosceles trapezoid.

5.4.3 Grammatical Variation

As a final point, we consider this discussion in relation to TP.6. Some logical ambiguity is an inevitable and necessary part of teaching. No matter how hard we try, we cannot avoid some measure of confusion that the mixture of formal and informal communication brings to the classroom.

After reading this chapter, you might conclude that the best solution is to avoid semantic ambiguity at all costs—to always give rigorous, logically formulated statements. Under this scenario, teachers should state all conditional claims in the form "if A then B," and all biconditional claims in the form "A if and only if B." A statement such as "A square is a regular quadrilateral" is banned in favor of the more logically clear "A shape is a square if and only if the shape is a regular quadrilateral." The problem, from our point of view, is that teachers must act as a bridge for students' learning; always remaining on the formal mathematical side of the bridge does not work. We would argue for a different strategy. Rather than providing only logically precise statements, teachers should phrase and re-phrase mathematical ideas in multiple ways. We should intentionally *vary* the grammatical structure in order to flesh out the intended mathematical meanings. It is fine to say "A square is a regular quadrilateral" on one occasion if we complement it on other occasions with statements like "All quadrilaterals that are regular are squares", "Being a square implies being a regular quadrilateral, and vice versa," and "The set of squares is the same as the set of regular quadrilaterals." Re-phrasing the relationship in different forms communicates nuances that students might miss from a single formulation. This kind of grammatical variation reinforces TP.6—that a teacher should have multiple ways to explain the same idea—and it has additional relevance in this context. By considering how logic underpins mathematical interpretation, we are inviting students to enter into characteristically mathematical ways of thinking and learning.

Problems

5.1 Consider the statements, "A square is a regular quadrilateral" and "A square is a rectangle." (i) For each, write the correct mathematical relationship as a conditional ($\Longrightarrow$) if-then statement or a biconditional ($\Longleftrightarrow$) if-and-only-if statement. Then write the converse, inverse, and contrapositive statements of any conditional statements. (ii) Next, suppose a student interprets the "is" in a way that is opposite the intended meaning in each statement—the 'is' confusion. Describe how you might respond to a student who is confused about each statement? Make sure to address both content and logic issues in your response.

5.2 Consider the three isosceles trapezoid statements from the teaching situation, re-written to capture the logical structure of a directional relationship:

1. If a trapezoid has opposing sides that are non-parallel and congruent, then it is an isosceles trapezoid
2. If a quadrilateral is an isosceles trapezoid, then it is a quadrilateral with congruent diagonals
3. If a quadrilateral is a trapezoid and has congruent diagonals, then it is an isosceles trapezoid.

Write the contrapositives for each statement. Discuss which contrapositive statements provide a 'description of' non-isosceles-trapezoids, as well as what that description would be; and which provide a 'criterion for' being non-isosceles-trapezoids, as well as what that criterion would be.

5.3 In class, an Algebra II teacher states, "If two functions are inverses of each other, then their graphs are reflections over the line $y = x$." On one of the practice problems in class, a student looks at a graph, and says: "Well, their graphs are reflections over the line $y = x$ so they are inverse functions." First, create a counterexample to the student's claim. Then, explain how you, as the teacher, would respond to the student and why. Make sure to address both content and logic issues in your response.

5.4 Read the short description of the classroom situation below:

Teacher: The slope of the graphed line is 2, which means the coefficient in the equation is 2.
Student: What's a coefficient?
Teacher: A coefficient is the number in front of x in the equation. The coefficient changes the slope of a function.

First, write a description of two different ways that someone might *interpret* the teacher's response to the student. Second, rephrase the above dialogue, as it has been expressed, into a definition for coefficient and an if-and-only-if statement about slopes. Third, decide whether the definition for coefficient as expressed is appropriate, and whether the if-and-only-if statement is true—show some examples you used to help in your decision. If the definition is not appropriate, or the if-and-only-if statement is not true, modify them so they are valid.

5.5 A high school teacher is helping students learn to solve quadratic equations. The example $(x+2)(x+3) = 0$ is on the board. The teacher states, "Well, we know that if either $(x + 2)$ or $(x + 3)$ is zero, then the product will be zero. So, to solve equations like this and find values for x which make the product 0, we write $x+2 = 0$ or $x + 3 = 0$, which gives us $x = -2$ or $x = -3$ as our solutions." Translate the teacher's first sentence and second sentence into formal logical statements (the statements are not false). (Note, the second sentences can be framed in similar terms as the first.) Explain the logical error the teacher has made—that is, why the first sentence does not provide a logical justification for the second sentence about how to solve quadratic equations.

5.6 This problem builds on the previous Problem 5.5. Suppose a high school teacher is helping students learn to solve quadratic inequalities, such as $(x+2)(x+3) > 0$. The following statement is true: "If $(x + 2) > 0$ and $(x + 3) > 0$, then we know $(x+2)(x+3) > 0$." What is the logical error about this statement that would be analogous to the one the teacher made in the previous exercise? Explain why the problem is magnified in this context. Rephrase this statement to be a logically correct and complete statement for solving quadratic inequalities. Describe how this might inform your teaching of solving inequalities to secondary students.

5.7 The ideas of logic underpin the algebraic work of solving equations. For this problem, we offer some explanation before asking you to complete the task.

Explanation When we write an equation with variable expressions, or are given one, we can consider such equations, and the algebraic solving process, as logical statements about solution sets. Here, we consider single-variable equations, because they are very common in secondary mathematics. We can interpret the equation $2x + 1 = 15$ as the statement, "x is a solution to the equation $2x + 1 = 15$". Algebraic solving processes are then logical statements about solution sets: (if possible) we want to write another equation that has the same solution set as the original. Consider the following example:

Equation	Logical statement
(1) $2x + 1 = 15$	"x is a solution to the equation $2x + 1 = 15$"
(2) $2x = 14$	"x is a solution to (1) $\Longleftrightarrow$ x is a solution to (2)"
(3) $x = 7$	"x is a solution to (2) $\Longleftrightarrow$ x is a solution to (3)"

What we find here is that the algebraic steps are connected by if-and-only-if ($\Longleftrightarrow$) statements about the solution sets. This is because adding or subtracting to both sides of an equation, and dividing by a non-zero value to both sides of an equation, are algebraic steps that preserve the solution set—they are some of the axioms of equality. And although we often see this sequence of algebraic steps from top to bottom, i.e., in the order that we write them, what is especially important is the sequence of logical statements from *bottom to top*. This is because we want the end product, statement (3), to tell us something about the initial problem, statement (1), and vice versa. What ends up being important is the logical chain(s) we can form about solutions sets. In this case: (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1), which means "if x is a solution to $x = 7$, then x is a solution to $2x + 1 = 15$"; *and* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), which means "if x is a solution to $2x + 1 = 15$, then x is a solution to $x = 7$." This essentially means that solutions to $2x + 1 = 15$ are *identical* to solutions to $x = 7$, which clearly has exactly one solution. However, not all steps in algebraic solution processes are connected by if-and-only-if statements.

Problem Prompt Consider the following algebraic solution:

Equation	Logical statement
(1) $(x + 3)^2 = 4$	"x is a solution to the equation $(x + 3)^2 = 4$"
(2) $x + 3 = 2$	"x is a solution to (1) $\Longleftarrow$ x is a solution to (2)"
(3) $x = -1$	"x is a solution to (2) $\Longleftrightarrow$ x is a solution to (3)"

Look carefully at the logical statements in the right-hand column. First, explain why step (2) is connected by a directional implication ($\Longleftarrow$) and not if-and-only-if ($\Longleftrightarrow$). Then write the logical chains and conclusions we can make between $(x + 3)^2 = 4$ and $x = -1$ in *both directions*. Do we have all the solutions? Can we have any extraneous solutions? How do you know? What happened?

5.8 This problem builds on ideas from the previous Problem 5.7. For the following algebraic solution, write the corresponding logical statements for each of the algebraic steps, and write an interpretation for what these mean in terms of the solution set. [Note, you should end up discussing the idea of *extraneous solutions*.] Describe how you would respond to a student who asks why, and at what point, the extraneous solution came into the solving process.

Equation	Logical statement
(1) $\sqrt{x+3} = x - 9$	
(2) $x + 3 = (x - 9)^2$	
(3) $x + 3 = x^2 - 18x + 81$	
(4) $0 = x^2 - 19x + 78$	
(5) $0 = (x - 6)(x - 13)$	
(6) $x - 6 = 0$ or $x - 13 = 0$	
(7) $x = 6$ or $x = 13$	

5.9 Two teachers were trying to explain solving systems of linear equations and they said the following:

> Teacher A: All solutions of a system of linear equations can be found via either elimination or substitution.
> Teacher B: If a system of linear equations has one solution, you can find it via elimination or substitution.

(i) Write each statement as an if-then solution. (ii) Describe what the mathematical differences are between these two statements. (iii) Based on the mathematical differences between the statements, state which would be better to tell students and why (you might refer to TP.1 in your response).

5.10 In the language of real analysis, a *sequence* and a *series* are different mathematical objects. A series is an infinite sum of real numbers—that is, it replaces all the commas separating terms in a sequence by addition signs. Yet, in Definition 2.4.3, Abbott first defines an infinite series, and then states: "We define the corresponding *sequence of partial sums* (s_m) by $s_m = b_1 + b_2 + b_3 + \ldots + b_m$, and say that the series converges to B if the sequence (s_m) converges to B" [1, p. 57]. That is, he describes a way to turn a series into a sequence (a previously-studied object). Explain what teaching principle you would say this describes.

References

1. Abbott, S. (2015). *Understanding analysis* (2nd ed.). New York, NY: Springer.
2. Euclid., Heath, T. L., & Densmore, D. (2002). *Euclid's Elements: All thirteen books complete in one volume, the Thomas L. Heath translation*. Santa Fe, NM: Green Lion Press.