4 Algebraic Limit Theorems and Error Accumulation

4.1 Statement of the Teaching Problem

Students sometimes have to work with numbers whose decimal expansions do not terminate. To make such numbers manageable, they often round these values. (Although sometimes these are truncated and not rounded values, the issues are the same and we use the term rounded throughout the chapter.) For instance, the decimal expansion of $4\sqrt{5}$ is $8.9442719099\ldots$, but students often round to write this as 8.94. In general, students feel more comfortable working with values they regard to be "numbers"—like the counting numbers or short decimal representations, whereas they feel less comfortable with expressions like $4\sqrt{5}$ or long strings of decimals. This is understandable. Decimal notation is familiar and ubiquitous, and rounded values are easy to operate with and to record.

There are many instances when teachers actually want students to round, but at what point in a computation is rounding most appropriate? A general rule is that students should round at the end of a computation, but what goes wrong when students round in the middle? Often very little.

Consider the following pedagogical situation:

A student, Adrian, sets up and solves the equation,

$$\sin(59^\circ) = \frac{x}{4\sqrt{5}}$$

by showing the following work:

$$0.85 = \frac{x}{8.94}, \text{ so } x = 0.85 \cdot 8.94 = 7.59.$$

(continued)

N. H. Wasserman et al., *Understanding Analysis and its Connections to Secondary Mathematics Teaching*, Springer Texts in Education,
https://doi.org/10.1007/978-3-030-89198-5_4

The teacher, Mr. Lee, walks around the room and observes the student's work. Mr. Lee tells Adrian:

> Remember, do not round in the middle of the problem—wait until the end.

Adrian objects to this remark:

> Well, my answer is basically the same as Veronica's. She got 7.67, and she rounded only at the end. I finished faster and I understand my way better anyway.

Mr. Lee's advice is sound—it is generally better not to use approximated values in calculations if you are able to use more precise ones. In this sense, the teacher has responded fairly by pointing out that rounding should occur at the end. But the student offers two counterarguments: the difference compared to the actual answer is relatively small, and his solution method was faster to compute and easier for him to understand. These counterarguments are legitimate; there is a need to balance demands for accuracy with other practical concerns. While Adrian's approach holds up in this case, there are mathematical constraints around the utility of his approach (TP.1). So how does a teacher respond? Are there effective ways to illustrate that the student's approach might be problematic in the general case? What are some practical ways to respond to this sort of reasonable push-back from students?

Before moving on, think about how you, as a teacher, might respond to the student in this pedagogical situation.

4.2 Connecting to Secondary Mathematics

4.2.1 Problematizing Teaching and the Pedagogical Situation

In this section, we problematize three potential responses to the student regarding the issue of rounding.

One possible response would be to agree with the student—to regard the issue of when to round as not problematic. An argument for this response could be made in relation to the teacher's mathematical aims. Perhaps the goal of the problem is solving equations, for which the student's work demonstrates good algebraic reasoning. Multiplying both sides of the equation by the same value produces an approximate solution for x. The student's solution in this case demonstrates understanding of the intended mathematics. A drawback of this approach is that "attending to precision" is part of mathematical practice (e.g., [2]). Because a more precise answer *exists* it should probably be used. The geometric context of this particular example also comes into play: when solving for missing side lengths of a right triangle, students can check their results through other relationships such as the Pythagorean Theorem. Too much imprecision could lead to inaccurate conclusions. Furthermore, even in an algebraic context such rounding might be undesirable. Imagine a student who solves the equation $7/3 = \frac{x}{6/7}$ using the decimal approximations $2.33 = \frac{x}{0.85}$, and obtains an answer of 1.9805 rather than 2. Such instances highlight the advantages of encouraging students to work with less-preferred representations of numbers like fractions. They also draw attention to the need to understand how the impact of rounding early on in a computation affects the end result.

A second response is to declare the student's solution incorrect and mandate that students avoid rounding until the end of their computations. In some ways, this simplifies the situation. It provides clear expectations for students, which can be good in teaching. However, in this case, the rule is presented arbitrarily and without an accompanying reason. To practice TP.5 means to avoid giving rules without providing an explanation. Any consequences given for not adhering to this rule may also feel artificial given the close proximity of Adrian's answer to Veronica's.

A third possible response would be to superimpose a real-world context onto the issue of rounding. A teacher might claim that in designing a spaceship even very small errors in the real world can have tremendously negative consequences. The response here focuses on making the modest discrepancy in the answer "feel" more consequential. Yet, this line of reasoning still has challenges. This argument relies on convincing students that small errors can have large effects in applied settings, but students might be skeptical. They could insist that being off by a few tenths is not problematic in most situations; or they might say that while this would be true of an engineer designing spaceships, they are not engineers designing spaceships but students in a mathematics class. Perhaps the more important point is that such a response still does not illuminate the fact that rounding can result in very large errors—it only tries to make small errors feel large. In this sense, the response only partially addresses the student's counterargument.

4.2.2 Approximation and Error Accumulation

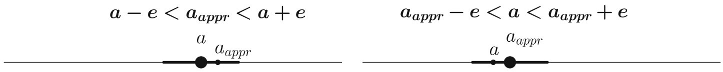
A rounded number is an approximation. This means we can think of error as we did in Chap. 3. In particular, if a_{appr} is a rounded approximation of a number a then we can consider both the *actual error*, e_{appr}, defined by

$$e_{appr} = \left|a_{appr} - a\right|$$

and the *potential error* or *error-bound*, e, which satisfies

$$\left|a_{appr} - a\right| < e.$$

Recall from Chap. 3 that the inequality $\left|a_{appr} - a\right| < e$, can be understood with two different referents. We might use a as the referent point, in which case the statement gives us the locus of points on the number line where a_{appr} is located; or we might use a_{appr} as the referent point, in which case it tells us the range of values where a is located. Both are depicted below.



Because the potential error e is a *bound* on the actual error e_{appr}, there are many possible values for e but only one for e_{appr}. It also means a value for e tends to be more readily accessible than one for e_{appr}. As an example, the potential error of a rounded decimal approximation can be computed from the number of decimal places—0.3 approximates the fraction $\frac{1}{3}$ to the tenths place, meaning we can use $e = 0.1$. (Note that this is indeed an upper bound for $e_{appr} = \left|0.3 - \frac{1}{3}\right| = \frac{1}{30}$.) Likewise, 3.14 approximates π to the hundredths place, meaning we can use $e = 0.01$, which produces the bound $3.13 < \pi < 3.15$. (Here, e_{appr} is the irrational number $\pi - 3.14$.)

In this chapter we consider not just individual approximations, as we did in Chap. 3, but what happens when we *operate* on approximated values. We consider how the error in the initial approximations accumulates, or changes, when the approximations are algebraically combined. To continue the above example, let $a = \pi$ and $b = \frac{1}{3}$. Using the notation e_a for the error-bound of a, we see that $a_{appr} = 3.14$ comes with error-bound $e_a = 0.01$. For $b = \frac{1}{3}$, the approximation $b_{appr} = 0.3$ has error-bound $e_b = 0.1$. What happens when we use these approximations to compute the sum $\pi + \frac{1}{3} \approx 3.14 + 0.3 = 3.43$? How far off could this approximated sum, 3.43, be from $\pi + \frac{1}{3}$? The potential error inequalities $|3.14 - \pi| < 0.01$ and $\left|0.3 - \frac{1}{3}\right| < 0.1$ can be arranged as

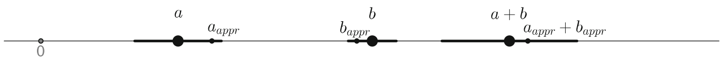
$$\pi - 0.01 < 3.14 < \pi + 0.01$$

$$\frac{1}{3} - 0.1 < 0.3 < \frac{1}{3} + 0.1$$

and summing yields

$$\left(\pi + \frac{1}{3}\right) - 0.11 < 3.43 < \left(\pi + \frac{1}{3}\right) + 0.11.$$

The approximate sum 3.43 must be within 0.11 of the actual sum. That is, the new potential error that results from adding two approximations is no worse than the sum of the initial potential errors, $0.01 + 0.1 = 0.11$. The addition of approximated values can be visualized as a linear transformation on a number line.



If e_a is the radius of the interval centered at a and e_b is the radius of the interval centered at b, then the interval centered at $a + b$ has radius $e_a + e_b$.

The same kinds of questions arise if we operate on approximations in other ways, such as subtracting, multiplying, or dividing them. To capture this idea of error accumulation more generally we give the following definition:

Definition For approximations a_{appr} of a and b_{appr} of b, each having potential errors e_a and e_b, **error accumulation** refers to the new potential error $e_{a \oplus b}$ that results from doing some operation ($\oplus$) to a_{appr} and b_{appr}.

Returning to the teaching scenario where the student solved for x by computing $x = 0.85 \cdot 8.94 = 7.59$, we can reframe the student's work as one of multiplying two approximated values. The potential error for both approximations (e_a and e_b) is 0.01. But what about the computed product, 7.59? What is *its* potential error ($e_{a \cdot b}$)? The actual answer, a little less than 7.67, suggests this new error has to be at least 0.07. Is there a way to calculate this error-bound from the potential errors for each factor? Is there a general method for calculating the error accumulation that results from other kinds of algebraic combinations?

It is to these issues that we turn next.

4.3 Connecting to Real Analysis

To connect this discussion to a real analysis course we return to the analogy introduced in Chap. 3 between the potential error inequality $\left|a_{appr} - a\right| < e_a$ and the expression $|a_n - a| < \varepsilon$ which appears in the definition for convergent sequences. For a sequence $(a_n) = (a_1, a_2, a_3, \ldots)$ that converges to a, we imagine the terms a_n getting closer to a as n gets large. For our purposes we want to think of each a_n in the sequence as an approximation of a (i.e., a_{appr_n}), so that the expression $|a_n - a| < \varepsilon$ can be interpreted to say that the approximation a_n has error-bound ε.

To study how error accumulates when we algebraically combine approximations, it turns out we can use the theorems from analysis that explain what happens when we algebraically combine convergent sequences.

4.3.1 The Algebraic Limit Theorem for sequences

The Algebraic Limit Theorem for sequences asserts what happens to convergent sequences when we add, multiply, or divide them (cf., Theorem 2.3.3 in Abbott [1]):

Theorem (Algebraic Limit Theorem) Let $\lim a_n = a$ and $\lim b_n = b$. Then,

1. $\lim (c \cdot a_n) = c \cdot a$, for all $c \in \mathbb{R}$
2. $\lim (a_n + b_n) = a + b$
3. $\lim (a_n \cdot b_n) = a \cdot b$
4. $\lim (a_n / b_n) = a/b$, provided $b \neq 0$.

The Algebraic Limit Theorem confirms that, when algebraically combining convergent sequences, things go as we might expect. For instance, if (a_n) converges to a and (b_n) converges to b, then the new multiplied sequence $(a_n b_n)$ converges to ab. The agenda of the real analysis proof is showing that the potential error of the new combined sequence can be made arbitrarily small. We do not provide the proofs (they can be found in Abbott's Theorem 2.3.3.), but we do list the four key inequalities that form the cornerstone for the proofs of each part of the Algebraic Limit Theorem. The numbering below corresponds to the numbering in the statement of the theorem:

1. $|ca_n - ca| \leq |c| \cdot |a_n - a|$
2. $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$
3. $|(a_n \cdot b_n) - (a \cdot b)| \leq |b_n| \, |a_n - a| + |a| \, |b_n - b|$
4. For N_1 sufficiently large such that, for all $n \geq N_1$, b_n is closer to b than to 0, then: $\left| \frac{1}{b_n} - \frac{1}{b} \right| \leq \frac{2}{|b|^2} \cdot |b_n - b|$.

(Note inequality (4) is really about the reciprocal $1/b_n$ rather than the quotient a_n/b_n.) These inequalities are central to proving the algebraically combined sequences converge to their respective limits, but they can also be adapted to our particular agenda of estimating the error accumulation of combined approximations. Each inequality is true for every term in the corresponding sequence, so if we suppose a_n and b_n are our approximations a_{appr} and b_{appr}, then these statements tell us something about error accumulation when a_{appr} and b_{appr} are combined in each way.

4.3.2 Implications for Error Accumulation

Take a look at each of the four inequality statements.

The left-hand side of the inequalities all have a similar form—the difference between an operated-on approximation and its theoretical value (e.g., $|ca_n - ca|$). In fact, they are all statements about potential error in our operated-on approximations. In particular, they indicate the accumulated error in the operated-on approximation is no worse than the expression on the right-hand side. The expressions on the right-hand are all in terms of the initial error of each approximation, $|a_n - a|$ and $|b_n - b|$. That is, we can interpret each inequality as a statement about how initial errors accumulate when operating on approximations.

Claim For approximations a_{appr} of a and b_{appr} of b, each having potential errors e_a and e_b:

1. the error accumulation of the **scalar product** ca_{appr} is no worse than initial error scaled by $|c|$; i.e., $e_{ca} = |c|e_a$,

2. the error accumulation of the **sum** $a_{appr} + b_{appr}$ is no worse than the sum of the initial errors; i.e., $e_{a+b} = e_a + e_b$,
3. the error accumulation of the **product** $a_{appr} \cdot b_{appr}$ is no worse than the sum of the initial error of a scaled by $|b_{appr}|$ and the initial error of b scaled by $|a|$; i.e., $e_{ab} = |b_{appr}|e_a + |a|e_b$,
4. the error accumulation of the **reciprocal** $\frac{1}{b_{appr}}$ is no worse than the initial error scaled by $\frac{2}{|b|^2}$; i.e., $e_{1/b} = \frac{2}{|b|^2}e_b$.

Re-read each inequality statement and the corresponding inequality statement from Sect. 4.3.1 and convince yourself that they mean the same thing. We will refer to these claims as "rules" for error accumulation. Notably, the sum rule arrived at in statement (2) aligns with the conclusions we found previously. Statement (4) is about the reciprocal; but by writing $\frac{a_{appr}}{b_{appr}}$ as $a_{appr} \cdot \frac{1}{b_{appr}}$, Problem 4.6 asks you to derive the corresponding quotient rule. And Problem 4.7 asks you to consider how some of these rules about error accumulation might be simplified under further assumptions.

To get a better sense of these rules let's return to our previous example.

Example Suppose we approximate π with 3.14 and $\frac{1}{3}$ with 0.3. How much potential error is there in: (i) $5 \cdot 3.14$; (ii) $3.14 \cdot 0.3$; (iii) $\frac{1}{3.14}$; (iv) $\frac{0.3}{3.14}$?

The rules provide a bound for how error *potentially* accumulates when operating on approximated values. For (i), $5 \cdot 3.14 = 15.7$ is an approximation for 5π. The scalar product rule in (1) states that the potential error is no worse than $|5| \cdot 0.01 = 0.05$—i.e., that 15.7 is within ± 0.05 of 5π. This is sensible. The rule says error potentially accumulates up to five times the original error, or $e_{5a} = 5 \cdot e_a$.

For (ii), $3.14 \cdot 0.3 = 0.942$ is an approximation for $\frac{1}{3}\pi$. From the product rule in (3), the potential error is no worse than

$$|3.14| \cdot 0.1 + |1/3| \cdot 0.01 \approx 0.3173.$$

For comparison, the actual error in this case is about 0.1052. Note the use of one approximated value, 3.14, and one theoretical value, 1/3, in the error accumulation calculation above. If a theoretical value for, say, a is not available we can overestimate it with $|a_{appr}| + e_a$.[1] In most cases, the errors of the initial approximations (e_a and e_b) are relatively small compared to the approximations (a_{appr} and b_{appr}) and so swapping theoretical values for approximated ones in the rules causes very little change to our error estimates. This means that when multiplying two approximated numbers, as a rule of thumb the error of the product accumulates by approximately $|a|$ times the error in b plus $|b|$ times the error in a.

[1] In general, we can replace statement (3) in the claim with the more conservative estimate, $e_{ab} = |b_{appr}|e_a + (|a_{appr}| + e_a)e_b$.

For (iii), the reciprocal of an approximation, $\frac{1}{3.14} \approx 0.31847$, has a potential error no worse than $\frac{2}{|\pi|^2} \cdot 0.01 \approx 0.002$. Here, the error has gotten smaller because it was scaled by $\frac{2}{|\pi|^2}$, which is a value between 0 and 1.[2]

Lastly, for (iv), $0.3/3.14 = 0.3 \cdot \frac{1}{3.14} \approx 0.0955$ approximates $\frac{1}{3\pi}$. Combining the product and the reciprocal rules, we can estimate the error in the quotient as

$$\left(\left|\frac{1}{3.14}\right| \cdot 0.1\right) + \left(\left|\frac{1}{3}\right| \cdot \frac{2}{|\pi|^2} \cdot 0.01\right) \approx 0.0325.$$

4.3.3 Visualizing the Potential Error Inequality for Products

In a proper proof of the Algebraic Limit Theorem, the primary conclusion is that although error accumulates, it does not do so uncontrollably. Even though operating on sequences might increase the error, the potential error at each stage is bounded by some knowable combination of the original errors. This means we can go out far enough in the new operated-on sequence so that the accumulated error is arbitrarily small (since we know the original errors become arbitrarily small). The theorem tells us that accumulated error converges to zero as n increases in the sequence. In the context of our discussions about approximations, we have borrowed the parts of the proof that tell us *how* the errors interact when approximations are operated upon and fashioned them into rules for error accumulation.

The derivation of the four inequalities that underlie our accumulation rules all make use of the triangle inequality. They are not especially difficult (see Abbott, Sect. 2.3), but the product statement (3) appears a bit mysterious. The product accumulation rule $e_{ab} = |b_{appr}|e_a + |a|e_b$ is based on the inequality

$$|a_{appr}b_{appr} - ab| \leq |b_{appr}||a_{appr} - a| + |a||b_{appr} - b|.$$

To visualize this inequality consider the area model in Fig. 4.1. The products $a_{appr}b_{appr}$ and ab appear as the areas of two large shaded rectangles. The absolute value of their difference $|a_{appr}b_{appr} - ab|$ is, at most, the area of what's left of these two rectangles when we remove their intersection. In the figure, this remaining area appears as the tall thin rectangle on the right with dimensions $b_{appr} \times (a_{appr} - a)$ and the long thin rectangle across the top with dimensions $a \times (b_{appr} - b)$. The fact that the sum of these two areas is an upper bound for $|a_{appr}b_{appr} - ab|$ verifies the original inequality statement.

[2] If we do not have an exact value for $|b|$, we can use the more conservative estimate, $e_{1/b} = \frac{2}{(|b_{appr}|-e_b)^2} e_b$.

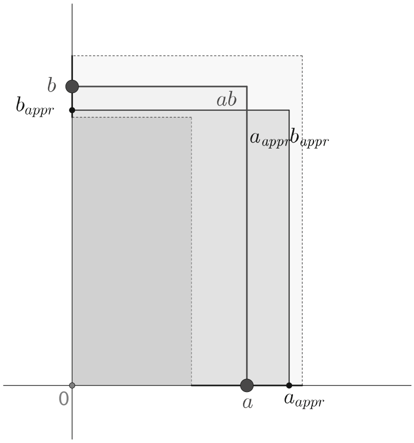


Fig. 4.1 Area model for the products ab and $a_{appr}b_{appr}$

4.4 Connecting to Secondary Teaching

In the initial teaching situation, the student's use of approximated values is not intrinsically problematic, and the teacher's advice of not rounding until the end is also sound. The student's reasonable counterarguments in the scenario emphasize that teachers should be prepared to give justifications for their advice. Simply stating a rule—"Remember, do not round in the middle of the problem, wait until the end"—without any sort of mathematical justification runs contrary to TP.5. A response to the student in this teaching situation can be informed by the insights from the real analysis proofs of the Algebraic Limit Theorem.

4.4.1 Applying Principles of Error Accumulation to Design Problems

In the original problem, the student substituted rounded values for $\sin(59°)$ and $4\sqrt{5}$ with a potential error of 0.01 for both approximations. Using these two values, the student solved the equation by multiplying them. The student's answer of 7.59 was reasonably close to the actual answer of about 7.67. The product rule from this chapter can be used to calculate the potential error accumulated in the student's calculation. Specifically, the error in the student's approximated answer is no worse than $0.85 \cdot 0.01 + 4\sqrt{5} \cdot 0.01 \approx 0.098$, which is about 10 times the initial error.

Although it is useful to determine the potential error in the student's answer, we regard it as more important to think about how the teacher might *apply* the ideas about error accumulation to respond to the student. Rather than simply telling the student the error could get big, it would be more beneficial to construct another problem for the student—one that demonstrates that rounding early in a computation can result in a relatively large error. Proceeding in this way is aligned with TP.2, using special cases to illustrate mathematical ideas.

Consider the following continuation of the teaching situation:

> Mr. Lee responds to Adrian: "I would like you to try your rounding approach on the following problem:
>
> $$\sin(59°) = \frac{x}{360\sqrt{5}}$$
>
> Tell me, how close is your answer this time?"

The changes that have been made to the problem appear to be minor, but they make an important difference. The student's approach presumed the potential error in both approximations to be the same; i.e., $e_a = e_b = e$. With this assumption, the error accumulation for a product simplifies: the original error e will accumulate by no more than a factor of $(|a| + |b_{appr}|)$. That is to say, the initial potential error of 0.01 will grow approximately by a factor of the sum of the two values being multiplied. The seemingly minor change of replacing $4\sqrt{5}$ with $360\sqrt{5}$ in the calculation is pedagogically-motivated—it is meant to increase the accumulated error. By changing the values in the equation, the potential error in solving for x in this new equation grows by a factor of $(0.85 + 360\sqrt{5}) \approx 805.83$, which is 800 times the original error! An initial rounding to the hundredths place could result in an error of more than 8. (Alternately, Mr. Lee could edit the equation to use $\tan(59°)$ since, unlike sine, the tangent function is not bounded.) In fact, Adrian's error in this new problem would be about 5.77—a difference most students would regard as non-trivial.

This example can be adapted to a wide class of problems. Students invoke decimal approximations when solving other equations with rational coefficients, such as $\frac{2}{7}x = 60\frac{6}{7}$. Solving for x in this example involves the quotient of two approximations. As is evident from the reciprocal and product accumulation rules, larger values for a or smaller values for b result in increased potential errors. This particular example has both. So the actual error ends up being 400 times the original error (presuming a student has rounded both numbers to the same number of decimal places)—a large error indeed!

In these examples, we have applied the rules for error accumulation to construct new problems, intentionally adjusting the values being approximated so as to increase the potential error accumulation in the solution. This is an example of "using a special case to illustrate a mathematical idea" (TP.2). We have designed an exercise to illustrate that using rounded values instead of actual values can lead to large errors. Special cases are important in mathematics, and they also serve a pedagogical purpose. In this case we wanted to convince a student their rounding approach could be problematic. Our example does not communicate to the student precisely when rounding leads to large errors—that's a heavier lift that requires engaging the ideas in the real analysis proofs—but it does supply a cautionary warning that rounding can be problematic.

The student's approach to rounding has some limitations, and the various constructed exercises discussed are designed specifically to reinforce the teacher's maxim to not round until the end of the problem. TP.5 suggests that teachers avoid giving rules without an accompanying mathematical explanation. In this scenario, the explanation takes the form of an exercise rather than a verbal justification. Observing the large errors that accumulate in the constructed exercises is a compelling piece of evidence in favor of the teacher's advice and may in fact be more convincing than any words the teacher could say. Providing students opportunities to *experience* ideas and not just have them *explained* is an important part of teaching.

As a final comment, we address the question of how the more advanced content of real analysis relates to the daily reality of teaching secondary school mathematics. In this chapter we have seen how the proofs for the various parts of the Algebraic Limit Theorem contain insights for understanding the nature of error accumulation. Knowledge of the ideas from the proofs empowers a teacher to engage student questions and counterarguments about the efficacy of approximations with carefully crafted examples designed to illuminate certain pitfalls. The constructed examples showcase a teacher applying ideas learned in real analysis to respond to a student but in a way that does not involve an exposition of more advanced concepts. This suggests that real analysis can be an impactful subject for teachers in ways that do not amount to teaching gifted secondary students proofs for results like the Algebraic Limit Theorem. Despite its formal reputation, analysis represents a body of ideas that can reveal new insights about day to day issues that arise in teaching.

4.4.2 The Tip of the Iceberg

Throughout this chapter we have focused on potential error rather than actual error. This is the more useful and appropriate point of focus. The fact that we are approximating suggests that there is some uncertainty in the theoretical value being approximated. This means the actual error is not typically known—or even knowable. Our rules for error accumulation are based on overestimates, or worst-case scenarios, so it is certainly possible that actual errors might decrease even as our potential error calculations increase. For example, rounding $\frac{1}{3}+\frac{2}{3}$ gives $0.33+0.67$, where each approximation has an error but the sum is perfect. In this case, it helps to give the actual error a *signed direction*. When we add, the actual errors cancel out; the potential error estimates of course do not. The triangle inequality—which is at the root of all our error accumulation rules—is an equality if and only if the errors have the same sign. This explains in part why the actual accumulated error is likely to be smaller than the potential accumulated error. The mix of positive and negative terms results in some cancellation that yields a better than expected approximation.

The calculation of the potential error meanwhile assumes the worst case where the errors stack up in one direction.

Another factor contributing to an inflated accumulated error estimate is an overestimate in the original error. Taking the rounded value of 3.14 as our estimate of π, we have been using $e = 0.01$ as our error-bound but the actual error is closer to 0.00159. Generally speaking, if our initial error-bounds are close to the actual errors, and all the actual errors have the same sign, then the error accumulation formulas tend to give values close to the actual error of the final computation.

Another direction for further investigation is how errors behave in cases beyond the scope of the Algebraic Limit Theorem. The error accumulation rules developed in this chapter apply to the scalar product, sum, product, and quotient. From these we could derive a rule for what happens to our error when we square a_{appr}^2 or cube a_{appr}^3 an approximation. New tools are required, however, to sort our how error accumulates when we take the square root $\sqrt{a_{appr}}$ or apply a function like $\sin(a_{appr})$ or $\tan(a_{appr})$. These sorts of questions are studied in depth in courses on numerical analysis, but preliminary error estimates can be derived from ideas in a real analysis course (and the theorems about continuity in particular).

This chapter is just the tip of the iceberg in terms of understanding error accumulation from approximations.

Problems

4.1 A student approximates $12/7$ as 1.714, and $7/6$ as 1.167. Use the initial potential errors (0.001 for each), and the rules about error accumulation in this chapter, to determine a bound for the potential error if the student used those approximations to compute: (i) $\frac{12}{7} + \frac{7}{6}$; (ii) $\frac{12}{7} \cdot \frac{7}{6}$; and (iii) $\frac{12}{7} \div \frac{7}{6}$. What is the actual error in each case?

4.2 In an algebra class, students are solving for the roots of the quadratic, $f(x) = x^2 - 2x - 27$. Students use the quadratic formula to find the roots to be at $x = 1 \pm 2\sqrt{7}$, and then evaluate the quadratic formula using their calculator. The calculator uses an approximation for $\sqrt{7}$ that is accurate to eight decimal places—i.e., the potential error in the calculator's approximation is 0.00000001. How much error could be introduced in the calculator's evaluation of the roots of the quadratic?

4.3 During class, a teacher recommends that students approximate π with the value 3.14 for computations. Describe a specific situation in secondary mathematics where such an approximation might lead to a large error.

4.4 (i) Design a problem of the form $Ax + B = C$, with $A, B, C \in \mathbb{R}$, to be given to the students as a multiple choice item, for which a student using the "round to the nearest hundredth" approach would almost certainly select the incorrect choice. (Your problem should include the multiple choice options.) (ii) Assuming the potential error is $e = 0.01$ for any decimal approximation, provide an analysis of the *potential error* introduced in solving the equation with rounded values. (iii) Provide an analysis of the *actual error* for the solution with rounded values in comparison to the theoretical solution, $x = \frac{C-B}{A}$. Discuss your multiple choice options in relation to this error.

4.5 A teacher gives a question in class that involves determining the perimeter and the area of a rectangle where the side lengths are $\sqrt{75}$ and $\sqrt{362}$. The teacher writes on her answer key (rounding at the end), $P = 55.37$ and $A = 164.77$. A student's calculator shows $\sqrt{75} = 8.660254038$, and $\sqrt{362} = 19.02629759$. If the student approximates these two numbers before making the perimeter and area computations—presume the student is simply "truncating" throughout the problem—use your knowledge of how error accumulates to determine the degree of accuracy that would be required for the student to get the *same* answer as the teacher. That is, should the student's original rounding for the square roots be accurate to the tenths, hundredths, thousandths, etc.? Explain whether the requisite accuracy level differs between the perimeter and area problems, and why.

4.6 Use the fact that $\frac{a_{appr}}{b_{appr}} = a_{appr} \cdot \frac{1}{b_{appr}}$ to determine a general rule for how error accumulates for a quotient of two approximations. You will need to combine how error accumulates for both reciprocals and products.

4.7 Suppose we make some additional assumptions about our approximations: (i) the theoretical values (and their approximations) are *positive* ($a, b, a_{appr}, b_{appr} > 0$); (ii) the initial potential errors are the *same* ($e_a = e_b = e$); and (iii) $a_{appr} < a$ and $b_{appr} < b$ (our approximations are *under-approximations*—such as truncating a decimal expansion). These three assumptions simplify some of the rules about error accumulation. (i) For products, we had $e_{ab} = |b_{appr}|e_a + |a|e_b$. With these additional assumptions, what is the new claim about the error accumulation of a product? (ii) In the reciprocal inequality, we have: $\left|\frac{1}{b_{appr}} - \frac{1}{b}\right| = \frac{1}{|b||b_{appr}|} \cdot \left|b_{appr} - b\right|$. With these additional assumptions, what is the new claim about reciprocals? (Note: you should no longer have a '2' in the numerator.) (iii) Building on (ii), with these additional assumptions, what is the new claim about the error accumulation of a quotient?

4.8 Suppose we allow our actual errors to be signed (positive or negative). That is, we define $a_{appr} = a + e_{appr_a}$ and $b_{appr} = b + e_{appr_b}$. Use substitution to show the product $a_{appr} \cdot b_{appr}$ has an error of $\left(a e_{appr_b} + b_{appr} e_{appr_a}\right)$ from ab, and relate this to the product rule in Sect. 4.3.2.

4.9 In Exercise 2.3.7 in Abbott's text, he asks for students to "give an example" (or argue that such a request is impossible) of, for example, "sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges." Indeed, in many sections, Abbott uses exercises similar to this. Describe what teaching principle you believe is exemplified in these exercises, and explain your reasoning.

4.10 Chapter 2 in Abbott's text is broadly about defining and understanding *sequences*. But at the beginning of the chapter (Sect. 2.1), and then later at the end of the chapter (Sect. 2.7), Abbott explicitly talks about infinite *series*. In the introduction to the chapter, Abbott uses an example: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots$. Then, he states, "The crucial question is whether or not properties of addition and equality that are well understood for finite sums remain valid when applied to infinite objects such as [the example]" (p. 40). Describe what teaching principle you believe is exemplified in his text.

References

1. Abbott, S. (2015). *Understanding analysis* (2nd ed.). New York, NY: Springer.
2. Common Core State Standards in Mathematics (CCSSM). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics