2 Equivalent Real Numbers and Infinite Decimals

2.1 Statement of the Teaching Problem

Mathematical ideas advance through progressively more powerful number systems:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

In this chain, $\mathbb{N}$ represents the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. With each step, students are faced with a heightened degree of abstraction. Natural numbers are concrete—they are easy to instantiate in the real world—whereas negative numbers, rational numbers, irrational numbers, and imaginary numbers are increasingly harder to conceptualize. One particular way students experience this escalation in abstraction is through the challenge of *representation*. As students move up to new number sets, not only do they need to learn different representation systems, such as representing real numbers as fractions or decimals, but one number might have multiple representations within a single system.

Consider the set of rational numbers, $\mathbb{Q}$. By definition, rational numbers are those that can be expressed as a quotient of integers; $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$. In contrast to the natural numbers, each rational number can be represented in multiple *different* ways. In fact, every one has an infinite class of equivalent representations. For instance, $\frac{1}{4}$ can be represented as $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, and so on. Along with the loss of uniqueness that is inherent to fractional notation comes the added challenge that a rational number like $\frac{1}{4}$ can also be expressed as the decimal 0.25.

N. H. Wasserman et al., *Understanding Analysis and its Connections to Secondary Mathematics Teaching*, Springer Texts in Education,
https://doi.org/10.1007/978-3-030-89198-5_2

Consider the following pedagogical situation:

> Ms. Schmidt asks her students to find the following sum:
>
> $$\frac{4}{5} + \frac{1}{4}$$
>
> One student, Mila, suggests that the answer is 1.05, having quickly converted to decimal representations to find the sum. "This is how I normally do it," she explains, "since fractions are harder to work with. There's only one way to write a decimal and you can add and subtract them in the normal way."

Mila's approach works well in this problem. Although she arrives at the sum in a way that the teacher did not expect, her process used meaningful mathematics and resulted in the correct answer. Indeed, the fact that there are often different ways to get to the same answer in mathematics is something to be celebrated! Yet in doing so Mila may have avoided the mathematical ideas intended by Ms. Schmidt, and some of the ideas Mila expresses have limitations that need to be addressed (TP.1). Figuring out how to respond to students when they approach problems in an unanticipated way is one of the challenges of teaching.

Before moving on, think about how you, as a teacher, might respond to the pedagogical situation just presented. What would you do next?

2.2 Connecting to Secondary Mathematics

2.2.1 Problematizing Teaching and the Pedagogical Situation

To unpack this pedagogical situation, we first problematize two responses to the student—ones that might be similar to some of your initial reactions.

One possible response is positive reinforcement. Mila has managed to solve a problem in a way that is potentially easier than other approaches and perhaps not expected by the teacher. Such a solution is commendable. Changing the numerical representation indicates she is being resourceful and not simply applying a memorized algorithm about how to add fractions with uncommon denominators. Furthermore, the solution may be a shortcut but it is not trivial. Switching from fractional to decimal form demonstrates a degree of mathematical competence. Nonetheless, there are some disadvantages to just offering congratulations. Specifically, the student has avoided developing proficiency with fractions, including the fundamental and necessary skill of writing fractions so they have common denominators. There are situations where converting fractions to decimals prior to performing an operation would be undesirable, or even impossible.

Another possible response is to ask the student to solve a different problem—one that reveals the limitations of the student's approach. Consider, for example, asking the student to find the sum, $\frac{1}{3} + \frac{1}{7}$. This introduces *infinite* decimal representations into the problem, significantly complicating matters. The unfamiliar nature of infinite decimals might be enough to encourage the student to return to fractional notation and attempt a solution using a common denominator. On the other hand, the student might be sufficiently happy with an approximate decimal answer like $0.333+0.143 \approx 0.476$. In the student's defense, approximations are often sufficient in applications (and on multiple choice exams!), and 0.476 is arguably more quantitatively accessible than the exact fractional answer of $\frac{10}{21}$. Still, there is a qualitative distinction between an approximate solution and a precise one that demands some attention. Not only might this distinction motivate the student to reconsider how they *operate* with fractions, it might prompt them to reconsider how they *conceptualize* rational numbers more generally.

2.2.2 Equivalence Classes and Decimal Representations

Moving from the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ to the integers $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ to the rational numbers $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ is motivated primarily by the desire to undertake more robust arithmetical operations. Addition and multiplication can be carried out in $\mathbb{N}$ without leaving the set, but we need $\mathbb{Z}$ if we want to properly define subtraction and $\mathbb{Q}$ if want to do division. The standard notation for numbers in these sets conveys the (accurate) impression that each successive set is constructed from the preceding ones; but something qualitatively different occurs in the description of $\mathbb{Q}$. Specifically, two distinct expressions like $\frac{1}{4}$ and $\frac{2}{8}$ can represent the same rational number. More formally, we say that two expressions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $ad = bc$. The collection of all such fractional expressions that share this property can then be bundled together into an *equivalence class*, and all members of this equivalence class are associated with a single rational number. The equivalence class for $\frac{1}{4}$ is the set $\left\{\frac{1}{4}, \frac{-1}{-4}, \frac{2}{8}, \frac{-2}{-8}, \frac{3}{12}, \ldots\right\}$. While this set contains an infinite number of different fractional expressions, they all represent the same rational number.

The non-uniqueness of fractional representation is a familiar feature of the notation, and also a potential source of consternation for some students. In the pedagogical situation described above, the student's solution was to sidestep this issue altogether by changing from fractions to decimals. Although $\frac{1}{4}$, $\frac{2}{8}$, and $\frac{3}{12}$ are all different looking expressions, we write all of these with the same decimal, 0.25. At first glance, decimal representations do not appear to have the same equivalence issue that fractional ones do. Maybe—as the student claimed—there is only one decimal representation for any number. Stop and think for a minute: Do you think this is true?

One technical way to assert decimal expansions are non-unique is to point out that 0.25 can also be written as 0.250 or 0.2500. This trick of adding zeros can be applied to any terminating decimal, and a similar strategy can be used for rational numbers with repeating decimal representations; for example, representing $\frac{1}{6}$ as $0.1\overline{6}$ or $0.1\overline{66}$ or $0.1\overline{666}$. But this should feel unsatisfying, at least in mathematics.[1] While superficially different, $0.1\overline{6}$ and $0.1\overline{66}$ both describe the exact same infinite string of digits. They are certainly not as different as $\frac{1}{4}$, $\frac{2}{8}$, and $\frac{3}{12}$. To avoid this uninteresting technicality, let's agree to the following:

Statement A decimal representation implicitly refers to an *infinite* string of digits.

Because some numbers require an infinite decimal expansion, it's best to level the playing field and agree that every decimal expression contains an infinite string of digits. Thus 0.25, 0.250, and $0.25\overline{0}$ are all shorthand for the same infinite decimal 0.25000 . . .. Likewise $0.1\overline{66}$ and $0.1\overline{666}$ are each ways to denote the infinite decimal 0.1666 . . ..

With this convention, it may seem that decimal representations are indeed unique. In the next section we will explore this conjecture by looking more closely at examples like $0.25\overline{0}$ that have an infinite string of 0s in the tail. But here is the crux of the matter. Whereas using fractions to represent rational numbers emphasizes their algebraic properties, using decimals emphasizes their geometric properties. A decimal expansion is a description of a location, like an "address" on the real number line; when that address has an infinite number of directions to follow it is no longer precisely clear what it should mean. It makes intuitive sense, as long as you do not think about it too hard. Real analysis is the result of what happens when you do think about it too hard. Negotiating with infinite processes is the business of real analysis, and making rigorous sense of infinite decimal representations requires some better tools for determining whether two real numbers might in fact be equal, even if they are expressed using different decimal representations.

Could two different infinite addresses actually give directions to the same point on the number line?

2.3 Connecting to Real Analysis

The "real" in "real analysis" refers to the set of real numbers, $\mathbb{R}$, but what exactly are these numbers? The extension of the natural numbers $\mathbb{N}$ to the integers $\mathbb{Z}$ is relatively tangible in the sense that constructing a model of the bigger set—$\mathbb{Z}$ in this case—using the smaller set $\mathbb{N}$ as raw material is relatively intuitive. The same is true of the extension of $\mathbb{Z}$ up to $\mathbb{Q}$. To construct the rational numbers we take all possible quotients of integers.

The extension of $\mathbb{Q}$ to $\mathbb{R}$ is more conceptually challenging. When the ancient Greeks realized that certain geometric lengths like $\sqrt{2}$ and $\sqrt[3]{5}$ could not be

[1] In other fields, such as science, 0.25 and 0.250 are used to indicate a difference in measurement accuracy.

expressed as a ratio of integers—i.e., were irrational—their response was to prioritize geometry over arithmetic. Some 2000 years later, the 19th century efforts to firm up the logical foundations of calculus finally reached the point where a proper construction of $\mathbb{R}$ from $\mathbb{Q}$ was required, and several rigorous models were proposed. The details of these constructions will take us too far afield, but it's not too inaccurate to say that $\mathbb{R}$ is the result of filling in the gaps of $\mathbb{Q}$. Wherever $\mathbb{Q}$ has a hole, a new "irrational" number is defined and inserted into the number line to plug up the gap. Thus $\mathbb{R}$ is the disjoint union of the familiar rational numbers $\mathbb{Q}$ with these newly minted irrational numbers $\mathbb{I} = \mathbb{R} - \mathbb{Q}$.

This brings us to the issue of how to represent an arbitrary real number. Every rational number has a complete (and finite) description in the form $\frac{a}{b}$ where a and b are integers. A select few irrational numbers, which have acquired some degree of importance, have their own special notation, such as π, $\sqrt{2}$ and ϕ. But the common language used to describe both rational and irrational numbers is decimal notation. Every real number, rational or irrational, can be represented as a decimal.

Through the standard division algorithm for dividing b into a, a given rational number $\frac{a}{b}$ can be converted into a decimal. For rational numbers, this process results in a decimal expansion of a very specific form: it either terminates or begins cycling through a fixed periodic pattern. And those decimals that terminate are simply a particular kind of fixed periodic pattern—one with an infinite string of 0s at the end. Thus, rational numbers are precisely the real numbers with decimal expansions that are eventually periodic. Irrational numbers, then, are those with non-repeating decimal expansions. When we write an expression like $\pi = 3.141592\ldots$ or $\sqrt{2} = 1.4142135\ldots$, we have to acknowledge the insufficiency of these descriptions. The decimal representations keep going, and although there are algorithms for finding each successive digit, there is no tidy way to express the entirety of these expansions. Such is the nature of infinity and such is the reason for real analysis.

A preliminary task of real analysis is to verify our intuition that every real number can be represented as an infinite decimal and, conversely, that every infinite decimal describes a well-defined real number. This is a significant exercise. Each digit in a decimal expansion specifies the location of a number with progressively greater accuracy. The property of $\mathbb{R}$ that guarantees there really is at least one real number at that location is *completeness* in the form of the Nested Interval Property (cf., Theorem 1.4.1 from Abbott [1]). The property of $\mathbb{R}$ that guarantees there is at most one number at that location is the *Archimedean Property* (cf., Theorem 1.4.2 from Abbott). (Both the Nested Interval Property and the Archimedean Property can be derived from the standard Axiom of Completeness; cf., Abbott, p. 15.) Leaving the important details to a real analysis course, what emerges is confirmation that infinite decimal expansions can indeed be used to represent each and every real number.

While this is comforting, the thorny issue that remains is whether decimal representations are unique. Just as $\frac{1}{4}$ and $\frac{2}{8}$ are equivalent descriptions of the same rational number, can two distinct decimal expressions be equivalent in the same way?

2.3.1 An ε-Approach for Defining Equivalence of Real Numbers

The confirmation that real numbers correspond to decimal expansions, and vice-versa, brings with it the added benefit of employing geometric intuition to understand properties of $\mathbb{R}$. Given $a, b \in \mathbb{R}$, the expression $|a - b|$ provides a notion of the distance between a and b on the real number line. One trait for any reasonable notion of distance—defined on any collection of objects—is that *two objects should be equal if and only if the distance between them is equal to zero*. Now this statement may appear to be so obvious that it seems useless, but we can employ it to obtain a surprisingly helpful criterion for when two real numbers are the same (cf., Theorem 1.2.6 in Abbott):

Theorem Two real numbers a and b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$.

Why is this criterion—that for every real number $\varepsilon > 0$, $|a - b| < \varepsilon$—logically equivalent to asserting that the distance between a and b is equal to 0? Before reading on, consider how you might prove this.

The proof amounts to considering two possibilities for the distance $|a-b|$. By the definition of the absolute value function, the distance between a and b will either be greater than 0 or equal to 0; that is, (i) $|a - b| > 0$, or (ii) $|a - b| = 0$. The strategy of the proof is to rule out (i) as a possibility so that the only option left is (ii).

Proof The first implication to prove is: $a = b \implies \forall \varepsilon > 0, |a - b| < \varepsilon$. This direction is sensible. If two real numbers a and b are equal (which is the condition), then the distance between them is 0 according to our definition—they occupy the same position on the number line. Because $|a - b| = 0$, it must be less than any positive ε, and we are finished.

The second implication is more interesting: $\forall \varepsilon > 0, |a - b| < \varepsilon \implies a = b$. Supposing the two real numbers are such that $|a - b| < \varepsilon$ for any positive ε (the condition), we have that every positive distance is too big to be the distance between them. So, the distance between a and b is smaller than 0.1, smaller than 0.01, smaller than 0.001, etc. Having ruled out the possibility that the distance between them is greater than 0, we conclude that the distance between them is precisely equal to 0, and a and b must be the same number. □

This theorem gives us a new way of thinking about when two real numbers might be equal—one that is particularly useful for interpreting infinite decimal representations.[2]

[2] Again, these decimal expansions—and the geometric intuition that comes with them—rely on a version of completeness that includes the Archimedean Property.

2.3.2 Implications for Real Numbers

Two real numbers are equal precisely when the distance between them is zero, and the distance between a and b in $\mathbb{R}$ is equal to $|a - b|$. But how is the operation of subtraction carried out on two infinite decimals?

As a simple example, the standard algorithm for subtraction can be used to determine the precise distance between 0.25 and 0.249. Because $|0.25 - 0.249| = 0.001$, which is greater than 0, we can confirm what we've long believed—that the real numbers represented by 0.25 and 0.249 (or 0.25000 . . . and 0.249000 . . . using infinite decimal representations) are indeed different numbers. But how far apart exactly are, for example, $\pi \approx 3.141\ldots$ and $\phi \approx 1.618\ldots$? The first step in the standard algorithm is to subtract the two digits furthest to the right. But this does not make sense with infinite decimals. What digit is furthest to the right in π? Or ϕ? If we try starting from the left instead we run into trouble because we can never be sure about the need to borrow. In our $\pi - \phi$ example, if we subtract starting on the left, we have to borrow in the second step! Now, π and ϕ are of course different real numbers, which we could show using inequalities[3], but the point is that infinite decimals are not always amenable to being subtracted and this can muddy the water around deciding whether two real numbers are the same.

To make this concrete, consider the two infinite decimals $a = 0.250000\ldots$ and $b = 0.249999\ldots$. What is the distance between these two numbers? The subtraction algorithm loses meaning in this case, so how else might we determine $|a-b|$? Better yet, what can we determine *about* this distance? As a starting point, the distance cannot be 0.001—it must be smaller than this because $0.24\overline{9}$ is closer to 0.25 than 0.249 (which is 0.001 away). By a similar argument, $|a - b|$ is smaller than 0.0001 and also smaller than 0.00001. In fact, even though there seems to be no way to find the exact distance between these two real numbers by algorithmically computing $a - b$, we can rule out the possibility that $|a - b|$ is a positive number. Whatever positive number $\varepsilon > 0$ is proposed, we can argue $|a - b| < \varepsilon$. (Stop reading and sketch out a justification for this last statement.)

Based on the theorem, the only option left to conclude is that, perhaps counter-intuitively, $0.25\overline{0}$ and $0.24\overline{9}$ occupy the same position on the number line. The challenge with an infinite decimal representation is that if we think about $0.24\overline{9}$ as an infinite progression of numbers—as a process of "getting closer and closer to" a number—we have begun in the wrong spot. Instead, we should start by thinking of the infinite decimal $0.24\overline{9}$ as representing *one* number and occupying *one* position on the number line. The goal then is to figure out where. But this is the easy part. Once we recognize the infinite decimal $0.24\overline{9}$ as a properly defined real number, there is only one choice for its location—$0.24\overline{9}$ must be another name for the real number better known as 0.25.

The moral of this example is that decimal representations are not unique. In addition to illustrating normative approaches in an analysis course (i.e., formal

[3] As an example, we could show $\phi < 1.7 < 3.1 < \pi$, meaning $\pi - \phi > 3.1 - 1.7$, or $|\pi - \phi| > 1.4$.

Table 2.1 Infinite decimal representations for several rational and irrational numbers

Real number		
Rational number		**Irrational number**
Terminating decimal	Non-terminating rational	Non-terminating irrational
$\frac{1}{4} = 0.25\overline{0}$ or $0.24\overline{9}$	$\frac{1}{3} = 0.\overline{3}$	$\pi = 3.1415\ldots$
$\frac{7}{8} = 0.875\overline{0}$ or $0.874\overline{9}$	$\frac{5}{11} = 0.\overline{45}$	$1 + \sqrt{2} = 2.4142\ldots$
$\frac{2}{1} = 2.\overline{0}$ or $1.\overline{9}$	$\frac{4119}{9990} = 0.4\overline{123}$	$\phi^2 = 2.6180\ldots$

definitions and arguments involving ε), this conclusion is an important part of understanding the set of real numbers. While it is customary and largely appropriate to conflate the real numbers with the collection of all infinite decimal representations, there is a set of examples where different decimal expansions describe the same real number. That said, this curious phenomenon is more the exception than the rule. The previous example involves a terminating decimal, which turns out to be a necessary and sufficient ingredient for non-uniqueness.

Claim For any real number that can be expressed as a terminating decimal (which therefore makes it a rational number), there are precisely two infinite decimal representations—one of them ending in a string of 9s and the other in a string of 0s. Every other real number (which could be a rational with a non-terminating decimal or an irrational number) has a unique infinite decimal representation.

Table 2.1 summarizes these conclusions. We note the fact that the infinite decimals that are equal (in the terminating decimals column) are those ending in 0s or 9s, which is particular to using *base-10* numbers; Problem 2.10 at the end of the chapter asks you to think further about this.

2.4 Connecting to Secondary Teaching

In the initial teaching situation, the student's strategy of converting fractions to decimals to find the sum was clever, although it likely avoided the mathematics intended by the teacher. This in itself may not be problematic, but the student's approach has some limitations as well. Fractions with decimal expansions that do not terminate create arithmetic challenges, and decimal representations do not completely avoid the issue of equivalent representations that fractions present. Leaving these issues unaddressed would run contrary to teaching principle TP.1. Ideas from real analysis provide insight into some of the mathematical challenges that arise with decimal notation—particularly, because decimal notation necessarily requires us to consider the infinite.

2.4.1 Exploring Infinite Decimals with Students

One of the keys in responding to the student in the teaching situation is to think more generally about whether the student's strategy of switching from fractions to decimals would work in every case. The student's comment about there being only "one way to write a decimal" also needs to be tested. Doing so—as we just did—necessitates thinking about situations in which the approach might not work, or the comment not hold up. This work on the part of the teacher involves identifying assumptions and limitations. As TP.1 suggests, it is insufficient to simply identify such assumptions or limitations—the teacher must go further to explicitly acknowledge and revisit them with students.

One way to push back on the student's assumptions is to construct new problems that reveal their limitations. Finding examples that illustrate particular mathematical ideas is aligned with TP.2. The examples should problematize the student's approach for operating with rational numbers, as well as address some of the difficulties conceptualizing decimals themselves. You might choose examples to do each separately or examples that do both at the same time. Imagine the following scenario:

> In response, Ms. Schmidt asks Mila and the class to first try using decimals to add the following rational numbers, and then try adding them as fractions:
>
> $$\frac{1}{3} + \frac{2}{3}$$
>
> $$\frac{1}{30} + \frac{1}{15}$$
>
> $$\frac{5}{44} + \frac{3}{22}$$

Before reading on, make a list of what issues involving representations and arithmetic operations arise in each example.

In the first problem, students are likely familiar with these fractions as decimal representations: $\frac{1}{3} = 0.3333\ldots$ and $\frac{2}{3} = 0.6666\ldots$. Such an example provides an opportunity to think about the challenges of adding infinite decimals. Similar to the subtraction algorithm, the addition algorithm also goes from right to left, which is problematic for infinite decimals. In this particular example, however, it is relatively straightforward to visualize adding each column to obtain an infinite string of 9s. In this example, the student's contention that it is always easier to add decimal representations is turned on its head— here, the computation is messier with decimals! Another challenge to the student and the class arises from adding the fractions. Clearly $\frac{1}{3}+\frac{2}{3}=1$, which forces students to consider their previous answer of $0.\overline{9}$ and the potentially disorienting conclusion that it must therefore be equal to 1. Now, it is also true that calculators will round $\frac{2}{3}$ to something like 0.66666667, and so students might argue that the decimals do not *really* add up to $0.\overline{9}$, they add up to 1 (as desired).

The decimal expansion of the fractions in the second example, $\frac{1}{30}$ and $\frac{1}{15}$, are likely less familiar. A calculator might help students write them as $0.03333\ldots$ and $0.06666\ldots$, which puts us back in essentially the same position as the previous example, with the sum being $0.09999\ldots$. Working with the fractional representations, we find a common denominator and compute $\frac{1}{30}+\frac{2}{30}=\frac{3}{30}=\frac{1}{10}$. Students would recognize this answer as the decimal 0.1, prompting the conclusion that 0.1 is equivalent to $0.0\overline{9}$. Providing multiple instances of this phenomenon reinforces the reality of non-unique decimal representations, but, as before, students may object by rounding the 'last' 6 into a 7 to avoid the dissonant idea that $0.09999\ldots. = 0.1$.

The last example also uses fractions with unfamiliar decimal representations, and it generates a more complicated addition task. A calculator yields $\frac{5}{44}$ as $0.11363636\ldots$, and $\frac{3}{22}$ as $0.13636363\ldots$. The addition feels more challenging due to the alternating 3s and 6s, and the first few place-value sums being different than the rest. (Determining the difference between these two infinite decimals is even more challenging—you are asked to do this in Problem 2.3.) Still, the result of addition is relatively easy to imagine: $0.249999\ldots$. If instead we work with the fraction representations, find a common denominator, add, and then reduce, the result is $\frac{1}{4}$. Similar to the first two problems, this sum draws attention to the challenges of doing arithmetic with infinite decimals and the idea that two infinite decimals can be equivalent. In this example, the rounding issue is less likely to occur; rounding up to 0.113637 feels unlikely because the next decimal is a 3, which would round down to 0.113636. As this example illustrates, we can create other examples of two rational numbers whose sum results in an infinite tail of 9s by making sure the aligned columns in the repeating parts sum to 9—e.g., $0.11\overline{756}+0.13\overline{243}$ (see Problem 2.2).

Asking students to compute examples like these prods them to consider the challenges of operating with rational numbers in their decimal form and explore issues about equivalent representations. As in these examples, there are times when decimals (not fractions) can be harder to work with! Indeed, why decimals like 0.25 and $0.24\overline{9}$ are equivalent can be more challenging to explain than fractions that are equivalent like $\frac{1}{4}=\frac{2}{8}=\frac{3}{12}$.

2.4.2 The Progression of Number Sets

The nested chain of number systems

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

is a robust example of creating more complex objects out of simpler ones. Because each set is defined in terms of the previous one, we come to understand each more complicated number system in terms of the more primitive ones that come before it. This idea is connected to TP.4. In the present context we are exploring TP.4 as it applies in mathematics rather than teaching, but we typically mirror this kind of progression in secondary school mathematics as well. Each extension results in a number system with additional capabilities, but as we have seen there are some trade-offs as we move up the hierarchy. As new qualities are gained, others are lost. Some of these trade-offs are related to notational complexity. A recurring theme of this chapter is that in the step from $\mathbb{Z}$ to $\mathbb{Q}$ we lose uniqueness of representations, and from $\mathbb{Q}$ to $\mathbb{R}$ we encounter the delicate subtleties implicit in the infinite nature of decimal notation.

Beyond notation, the number systems themselves exhibit an interesting give-and-take of algebraic, order, and set-theoretic properties. In terms of algebra, $\mathbb{N}$ is closed under addition and multiplication, $\mathbb{Z}$ allows for subtraction, and $\mathbb{Q}$ makes it possible to properly define both subtraction and division. Moving to $\mathbb{R}$ allows for some new operations like square roots of positive numbers, and the complex numbers $\mathbb{C}$ can accommodate square roots of any number. As their algebraic dexterity increases, the sets become more crowded. The sets $\mathbb{N}$ and $\mathbb{Z}$ are discrete—each element has a unique successor, or "next largest" element in the ordering. This property is lost in the step up to $\mathbb{Q}$, which is still ordered but not in this discrete way. Given any two rational numbers a and b, the rational number $\frac{a+b}{2}$ sits in between them, as do infinitely more. This shows that the elements of $\mathbb{Q}$ are densely nestled together with no intervals of empty space. In every interval on the number line it is always possible to find rational numbers that are arbitrarily close together, but $\mathbb{Q}$ is still permeated by holes. The property of completeness that defines the step up to $\mathbb{R}$ fills in these holes, but it comes with a host of other implications. One of the most profound is that $\mathbb{R}$ is no longer a so-called "countable" set. While $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are infinite, there is a rigorous way to articulate that the infinity characterizing the size of $\mathbb{R}$ is of a distinct and higher magnitude. Although we have not met the complex numbers in this chapter, a price for moving from $\mathbb{R}$ up to $\mathbb{C}$ is that a meaningful ordering is no longer possible.

Taken together, as each successive number system is constructed, we see that it inherits some of the properties of its predecessor but also sacrifices others in pursuit of some other form of added dexterity. As the numbers change, the operations on them change, and the properties of each set need to be re-examined. When students transition from one number system to another, it is a good idea to address the possibility that properties they understood from a previous set may no longer apply: How do we need to think about arithmetic differently when we change to a new number set? Do the same conceptions of number, or operation, hold for these new objects? Do the same procedures work? How are these procedures dependent on the representation system being used? As students wrestle with different number systems and different notations for them, we need to remind them what is gained, and what is lost, in each case.

Problems

2.1 Write all possible infinite decimal representations for the following real numbers: (i) 0.8; (ii) 0.142; (iii) $\frac{15}{8}$; (iv) $\frac{9}{11}$; (v) $\pi/4$

2.2 Show that the sum of $0.11\overline{756}$ and $0.13\overline{243}$ yields an infinite tail of 9s. Determine fractions for these decimals, and show that the sum of the fractions is one-fourth. Find another pair of decimals whose sum would result in 0.249999 . . .. Determine fractions for these decimals, and show that the sum of the fractions is one-fourth.

2.3 Use the infinite decimal representations of $\frac{3}{22} = 0.1\overline{363}$ and $\frac{5}{44} = 0.11\overline{36}$ to determine the difference. Compare your infinite decimal answer with what you would find the difference to be as a fraction.

2.4 Integers are 'signed' numbers. They afford the ability to differentiate numbers by adding a sign, $-$ or $+$. One result is they give (at least) two different ways to express a positive number. The representation '$+4$' would be one way to express the number 4. (i) What would be another way to express 4 using 'signed' numbers? [Hint: how might you express the additive inverse of '-4'?] (ii) Describe how this second representation of a positive number poses problems for students, and how you as a teacher might help address the problem.

2.5 Complex numbers are part of secondary mathematics. They are often written in the form $z = a + bi$ (with $a, b \in \mathbb{R}$) and plotted as the point (a, b) on the plane ($\mathbb{R}^2$). But complex numbers, and points on the plane, can also be referenced using a central angle, θ (rotation around (0, 0) from the positive x-axis), and a radius, r (signed distance from (0, 0))—like polar coordinates. In this way, complex numbers are written as $z = r(\cos\theta + i\sin\theta)$. If we allow our angle to be $\theta \in [0, 2\pi)$, and our radius r to be a real number (positive or negative), then there is an equivalence class on $\mathbb{C}$. What is the one other way you could express the complex number, $z = 3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$?

2.6 A student is trying to understand the idea that 0.999 . . . is equal to 1, and 0.24999 . . . is equal to 0.25, and so forth. In trying to generalize, the students asks: "So, then is 0.24777 . . . equal to 0.248?" (i) How would you respond to the student? (ii) Use the theorem in the real analysis section to justify whether these two numbers are equal or not.

2.7 A teacher asks a class to convert $\frac{8}{9}$ into a decimal. One student uses his calculator and says that the calculator has given 0.8888888889. The teacher responds, "Well, that's close, but there's actually never a 9, it's a bunch of 8s. The calculator is just rounding at the end." The student replies, "Well if it was 0.8888888888, then if I were to add $\frac{1}{9}$, which the calculator says is 0.1111111111, it would give me 0.9999999999. But it should be 1, that's why the calculator put the 9 at the end so that it would add to 1.0000000000." (i) Describe the mathematical ideas about real numbers that are being discussed. (ii) Provide a description for how you would respond to the student.

2.8 A pre-calculus teacher is trying to explain the limit of the following function: $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)$. In particular, that the the limit is *equal to*, and not just *close to*, 1. Describe how the idea of $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)=1$ is similar to the idea that $0.\overline{9}=1$. What is different about the situations mathematically?

2.9 Consider the following two proofs that $0.999\ldots = 1$. Discuss: (i) what assumptions are being made about infinite decimals in each proof; and (ii) in what contexts, if any, you might find either of these proofs useful for your own teaching.

Proof 1	Proof 2
Let $x = 0.999\ldots$	$0.\underbrace{999\ldots9}_{n} = 1-\left(\frac{1}{10}\right)^n$
$10x = 9.999\ldots$	For a decimal with an infinite amount
$10x - x = 9x = 9.999\ldots - 0.999\ldots = 9$	of terms, we can use a limit:
Since $9x = 9$, then $x = 1$. Therefore	$0.999\ldots = \lim_{n\to\infty}\left[1-\left(\frac{1}{10}\right)^n\right]$
$x = 0.999\ldots$ and $x = 1$, so $0.999\ldots = 1$.	$= 1-\lim_{n\to\infty}\left(\frac{1}{10}\right)^n = 1-0 = 1.$

2.10 In this chapter, we saw that $0.24\overline{9} = 0.25\overline{0}$, or, more generally, that an infinite string of '9's was equivalent to 'bumping up' to the next decimal digit with an infinite string of 0s. This is in fact specific to *base-10* numbers. Now, consider base-6 numbers, which express numbers only using the numerals 0, 1, 2, 3, 4, and 5. Fractional decimals are similarly interpreted in that base. For example, $0.13000\ldots$, in base-6, represents 1 sixth $(1\cdot\frac{1}{6})$ plus 3 sixths-squared $(3\cdot\frac{1}{6^2})$. (i) What other base-6 infinite decimal would be equivalent to $0.13000\ldots$? Think about the criteria given in the theorem in the real analysis section for when two real numbers are equal. (ii) Describe the two infinite decimal expressions that would be equivalent for a general base-b number system.

2.11 Abbott's Sect. 1.1 begins the first chapter with a discussion of there being no *rational number* whose square is 2; later, in Sect. 1.4, this idea is revisited with the real numbers—that there is a *real number* whose square is 2. As an example of TP.2, how would you describe the mathematical idea that this particular example serves to illustrate? Now, discuss this same idea in relation to TP.1. [If you would like to do some further reading, the July 2020 edition of *Mathematics Teacher Educator* is about teaching and "mathematical statements that expire."]

2.12 In Sect. 1.2, Abbott gives an intuitive idea about sets: "Intuitively speaking, a *set* is any collection of objects" (p. 5). Afterwards, on p. 7, Abbott writes:

> Admittedly, there is something imprecise about the definition of set presented at the beginning of this discussion. The defining sentence begins with the phrase "Intuitively speaking," which might seem an odd way to embark on a course of study that purportedly intends to supply a rigorous foundation for the theory of functions of a real variable. In some sense, however, this is unavoidable. Each repair of one level of the foundation reveals something below it in need of attention. The theory of sets has been subjected to intense scrutiny over the past century precisely because so much of modern mathematics rests on this foundation. But such a study is really only advisable once it is understood why our naive impression about the behavior of sets is insufficient. For the direction in which we are heading, this will not happen, although an indication of some potential pitfalls is given in Sect. 1.7.

Describe the teaching principle that you believe Abbott is illustrating in this paragraph. Then, describe any way Abbott's use of this teaching principle here helps you think more generally about how this principle might be implemented in teaching.

Turning the Tables

Reflecting on *teaching* from your *learning* in real analysis: TP.2

Although the primary agenda of this supplemental textbook is to connect the content of a course in real analysis to secondary mathematics teaching, it would be a missed opportunity to ignore the teaching of analysis as a case study. An especially rich source of insight into good teaching comes from our personal experiences as students. Because users of this book are most likely currently engaged as students, it makes sense to pause from time to time and explore how a student's perspective in a challenging course like real analysis might illuminate the six teaching principles at the core of this book. We do so primarily by thinking about teaching as it is evident in an analysis text.

TP.2 is about the use of specific cases. One particular type of specific case are boundary cases, which are designed to test—or showcase—the limits of a definition, theorem, procedure, or proof. We touched on TP.2 briefly in response to the pedagogical situation in this chapter. To develop this principle further, and to think about it in the context of your own real analysis learning, consider an example Abbott introduces in Sect. 1.2: the *Dirichlet function*,

$$g(x) = \begin{cases} 1 : x \in \mathbb{Q} \\ 0 : x \notin \mathbb{Q} \end{cases}$$

He introduces this example immediately after defining function. The intent is to introduce a boundary case that illustrates the "unruliness" of functions (one that was also historically important in mathematics for the same reason). What is particularly effective about this example is that the Dirichlet function is regularly referenced throughout the remainder of the textbook. The Dirichlet function is useful to have in one's "example space" because it challenges our expectations of what a function is; in Abbott's words, "examples such as this one will provide us with an invaluable testing ground for the many conjectures we encounter" (p. 8). By introducing this example, Abbott is indicating the Dirichlet function to be so qualitatively different from other functions that it should refine how we think about functions and how we expect them to behave. This kind of example is the epitome of TP.2 in the way it shapes future thinking and learning. Be on the lookout for other instances of this teaching principle in your own learning of real analysis.

References

1. Abbott, S. (2015). *Understanding analysis* (2nd ed.). New York, NY: Springer.