The Riemann Integral and Area-Preserving Transformations

**12**

12.1 Statement of the Teaching Problem

Mathematics requires rigorous justification for its propositions. A challenge in teaching mathematics is presenting students with proofs that also provide conceptual insights (TP.5). Doing so requires that the instructor have the dexterity to think about a problem in different ways and determine when one approach might be more effective than another.

One area where this flexibility is especially valuable is generating the formulas for area of figures like triangles, parallelograms, circles, and ellipses. The notion of area starts as an enumeration of *square units* and a consequence of this bedrock idea is that a rectangle has area $A_{rect} = lw$. A standard strategy for determining the area of shapes such as triangles and parallelograms is to cut these new shapes into pieces that can be reassembled into a rectangle—or into some other planar shape whose area has already been determined.

Consider the following pedagogical situation:

> A geometry teacher, Mr. Williams, shows the class how to cut the triangle off the left side of a parallelogram and move it to the right to see that the area of a parallelogram is given by $A_{par} = bh$.

(continued)
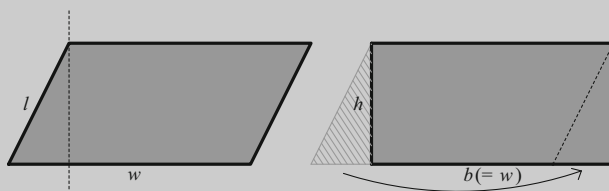
Mr. Williams then decides to have the class look at the "oblique" parallelogram below, created by sliding the top segment of the original parallelogram quite far to the right while maintaining the base w and the height h.

Mr. Williams starts to use the "cut-reassemble" argument he used before to find the new area but gets stuck because moving the sliced off triangle no longer makes a rectangle. He then then wonders if there is another way to explain to his students that the area of this parallelogram is still $A_{par} = bh$.
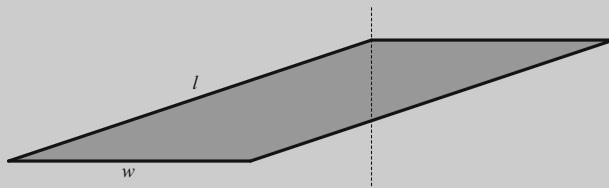

In the first example, the "cut-reassemble" algorithm results in a rectangle with dimensions $w \times h_w$ where h_w is the height of the parallelogram when we take w as the base. This yields the area formula $A_{par} = wh_w = bh$. The problem comes in the second example where the line cutting the original parallelogram no longer meets the base. The same area formula certainly holds, but we are momentarily without a proof of this fact. (Some might be surprised to read that the areas of the three figures above are all the same!) This is a situation where the teacher would benefit from having multiple explanations for the concept at hand (TP.6). Figuring out how the argument might be modified or what other justifications might exist requires us to think flexibly about area-preserving transformations.

Before moving on, think about how you might adapt the "cut-reassemble" argument to this "oblique" parallelogram, or if there is some other approach you could use to justify the area formula.

12.2 Connecting to Secondary Mathematics

12.2.1 Problematizing Teaching and the Pedagogical Situation

One solution for dealing with the "oblique" parallelogram is to change the direction of the line cutting it. The base of the figure need not be orthogonal to the page (as shown below).
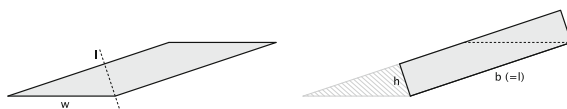
It turns out it will always be the case that one or the other cut in a (non-rectangular) parallelogram will intersect the base.[1] Although this is a way to salvage the "cut-reassemble" approach, students may have trouble recognizing that a different cut could work and that the base need not be orthogonal to the page. Another drawback is that the resulting rectangle has different dimensions than the one we got from before. The non-vertical cut produces a rectangle with dimensions $l \times h_l$ rather than $w \times h_w$, obscuring the interesting fact that both parallelograms have the same area.

A second option is to use several "cut-reassemble" steps.
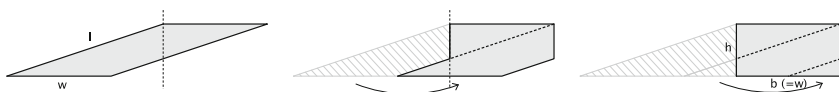
Although this produces a valid argument, the multiple steps add to the complexity and mask any conceptual insight about *why* the areas of the two parallelograms come out equal. The proof works but it feels a bit like forcing a square peg into a round hole. The "cut-reassemble" approach depicted in the teaching situation gives the impression that only *one* cut is necessary. By giving students justifications that are applicable only to specific diagrams, we may be encouraging them to see proof methods as not applicable to the entire classes of objects the diagrams are intended to represent.

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[1] For w and l of the (non-rectangular) parallelogram, and the *acute* angle θ between them ($0 < \theta < \pi/2$), the vertical cut (with w as the base) will meet the base so long as $l \leq w \sec \theta$. If this is not the case then $l > w \sec \theta \implies l > w \cos \theta$, since $\sec \theta > \cos \theta$ on $0 < \theta < \pi/2$, which means $w \leq l \sec \theta$, and so the non-vertical cut (with l as the base) *must* meet the base.
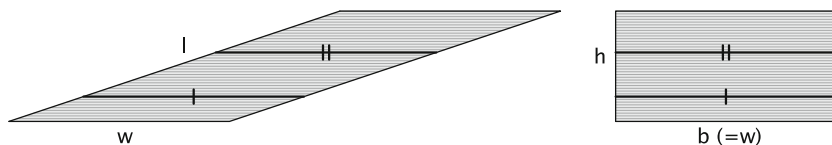
Fig. 12.1 Cavalieri's (2D) principle justifying the area of the "oblique" parallelogram in relation to a $w \times h_w$ rectangle

12.2.2 Area-Preserving Transformations and Cavalieri's Principle

The "cut-reassemble" approach to finding area formulas is based on the idea of *area-preserving transformations*. Because we move the pieces of the original region in a rigid way that does not alter their shape, putting the pieces back together (without gaps or overlaps) creates a different region with the same area. It is this area-preserving nature that allows us to determine the area of an unknown region in terms of the area of a known one. (This point should recall TP.4—modeling more complex objects with simpler ones.) But there are other area-preserving transformations that provide other means to justifying areas.

The Common Core State Standards in Mathematics—which currently guide the content taught to U.S. secondary school students—contains the following standard: *"Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures"* [2]. In this instance, Cavalieri's principle is being invoked to solve a three dimensional problem about volume, but it is equally valid in two dimensions where it can be used in conjunction with area:

Cavalieri's (2D) Principle Suppose two regions in a plane lie between two parallel lines in that plane. If every line parallel to these two boundary lines intersects both regions in line segments of equal length, then the two regions have equal areas.

Cavalieri's principle asks us to conceptualize a region in the plane as being "composed" of a stack of parallel line segments. If we have another region "composed" of line segments with the same lengths, this new region should have the same area as the original. As a first example, Cavalieri's principle offers a quick and efficient way to conclude that the areas of the parallelogram and rectangle (in Fig. 12.1) are equal, based on the fact that the line segments made by the parallel lines are always congruent. Notice how this might relate to the dilemma in the teaching situation.

Although the statement of Cavalieri's principle has a static feel—comparing lengths of line segments—a more dynamic interpretation can be given in terms of a specific kind of transformation that we make precise:

Definition Consider a region in the plane composed of parallel line segments. A **segment-skewing transformation** is one that translates each line segment along

the line to which it belongs in such a way that the composite transformation yields a new region in the plane.[2]

   With this new vocabulary, we can restate Cavalieri's principle by saying that *a segment-skewing transformation is area-preserving*. Analogous to the "cut-reassemble" technique where we move the pieces of the original in an area-preserving way, we can imagine transforming the "oblique" parallelogram in Fig. 12.1 into the rectangle by applying a segment-skewing transformation that slides each horizontal segment the proper distance to the right. Cavalieri's principle tells us the area of the transformed region is the same as the original.

12.3   Connecting to Real Analysis

The primary connection between Cavalieri's principle and real analysis is though the theory of integration. Specifically, we look at how the Riemann integral can be used as justification for this principle, as well as how Cavalieri's principle provides some geometric insight into various integration rules.

### 12.3.1 Justification for Cavalieri's Principle via Integration

Why is Cavalieri's principle true? Considering a planar region to be composed of line segments introduces legitimate conceptual difficulties. Unlike the pieces in the "cut-reassemble" approach, these individual line segments have no area on their own yet come together to create a region that does possess an area. At issue is how to define what we mean by area for planar regions that aren't rectilinear, and this is where the Riemann integral can offer us some assistance.

   Let f be a continuous function taking positive values on the interval $[a, b]$. Intuitively, the integral $\int_a^b f(x)dx$ represents the area of the region S bounded by $x = a$, $x = b$, $f(x)$ and the x-axis (see below). To compute this rigorously we can set $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$ and conclude:

$$\text{Area}(S) = \int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x.^3$$

[2] We note that a segment-skewing transformation is related to, but more flexible than, a *shear* transformation.

[3] This calculation of Area(S) is not the definition of the Riemann integral but a consequence of it when the function is continuous. The treatment of the integral in Abbott employs upper and lower sums. The quantity in this expression is a Riemann sum evaluated for a specific tagged-partition. Knowing that the integral exists, it follows that this particular sequence of approximations must converge to the proper value. See Abbott, Theorem 8.1.2.
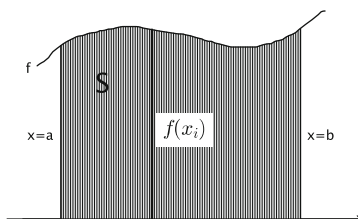
As with all of the other formulas for area in this chapter, the fundamental ingredient for defining the integral is the area of a rectangle; in this case $f(x_i) \cdot \Delta x$. The Riemann integral defines 'Area(S)' to be the limit of the sums of increasingly more, and increasingly thinner, rectangles. To connect this to Cavalieri's principle, imagine these tall thin rectangles as looking more and more like the parallel line segments from Cavalieri's principle, but this time oriented vertically.

Now let's apply a segment-skewing transformation to S (in the vertical direction) by adding a continuous function $t(x)$ to $f(x)$. That is, we'll let $g(x) = f(x) + t(x)$ and define the new region T between $x = a$ and $x = b$ to be the one bounded above by $g(x)$ and below by $t(x)$. Because $f(x) = g(x) - t(x)$, the corresponding vertical segments that make up S and T have the same length. Cavalieri's principle then asserts that Area(S) = Area(T), and in this setting we have the Riemann integral to help us look under the hood to see why this is the case. Specifically,

$$\text{Area}(S) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} [g(x_i) - t(x_i)] \cdot \Delta x = \text{Area}(T).$$
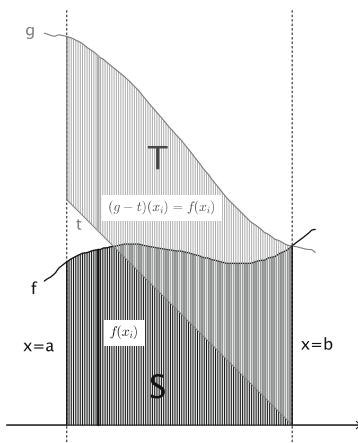
With a little polish, this example can be upgraded to a full proof of Cavalieri's principle in two dimensions. All we need to do is let S be more general by allowing

it to be bounded from below by an arbitrary continuous function (rather than the x-axis). Restricting our attention to continuous functions seems reasonable since all these curves are meant to be the boundaries of planar figures like parallelograms and ellipses. The assumption of continuity ensures the functions are Rieamnn integrable (see Theorem 7.2.9 in Abbott [1]) and the rest is smooth sailing.

12.3.2 Cavalieri's Principle and Integral Properties

Having used the theory of the Riemann integral to ground Cavalieri's principle on a solid mathematical footing, we can flip the script and see how Cavalieri's principle provides geometric insight for some important properties of the integral.

Consider the property

$$\int_a^b (f - g) = \int_a^b f - \int_a^b g.$$

This equation expresses a relationship between *three* functions, $f, g, (f - g)$, and the areas under their graphs (below). (We presume $f(x) > g(x)$ for all $x \in [a, b]$.)
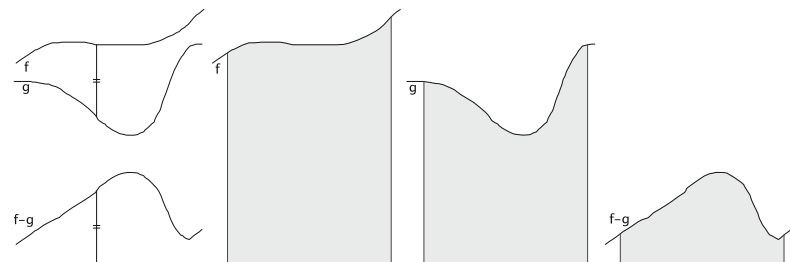
While this rule conveys a fact about the algebra of integrals, the geometric justification amounts to a segment-skewing transformation. Specifically, translate each vertical segment in the original planar region between f and g down to the x-axis (see Fig. 12.2). The boundary curves of this new region are the function $f - g$ and the x-axis. The area-preserving nature of the transformation means the area of the first region, $\int_a^b f - \int_a^b g$, must be equal to that of the second, $\int_a^b (f - g)$.

Cavalieri's principle helps us understand why this integral property is sensible, from a geometric and not just algebraic standpoint. Other integral properties such as

$$\int_a^b kf = k \int_a^b f \quad \text{or} \quad \int_a^b f + \int_a^b g = \int_a^b (f + g)$$
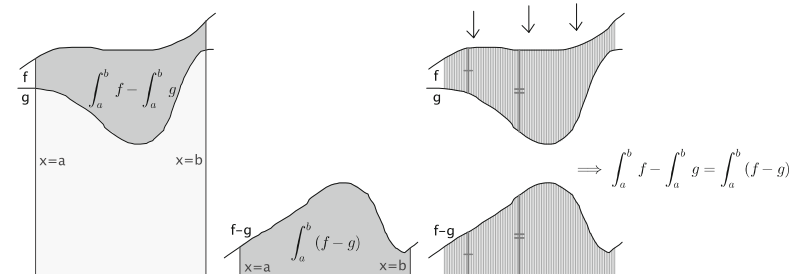
Fig. 12.2 Segment-skewing the area between functions f and g to form the area under $f - g$

have similar geometric justifications in relation to segment-skewing transformations. They each rely on recognizing a region in the plane as having an area equal to some other region(s). (Problems 12.6 and 12.7 ask you to look at these.)

12.4   Connecting to Secondary Teaching

Although the "cut-reassemble" algorithm can be used to justify the area formulas for many shapes treated in secondary school mathematics, the segment-skewing approach has its particular benefits. One rationale is simply knowing a variety of approaches to a given problem (TP.6). Having multiple explanations affords teachers the ability to relate to students who might require a different approach to a problem. Some students will respond better to one method than another! On occasions where a justification becomes difficult in a particular situation—like the "oblique" parallelogram example from the teaching scenario—it is always important to have an alternative.

Incorporating the segment-skewing approach to area also provides points of connection to other content. Cavalieri's principle is typically used for finding the volume of three-dimensional solids. Becoming familiar with Cavalieri's principle in the plane provides a foundation for appreciating its use in use in three-dimensional space. Even better, segment-skewing foreshadows fundamental ideas in calculus; namely, how the Riemann integral approaches area. This way of thinking about area is not just different, it is productive. Exposing students to Cavalieri's principle lays a groundwork on which they can eventually build an understanding of calculus.

12.4.1 Areas of Polygons

Let's return to the "oblique" parallelogram in the initial teaching situation and suppose the teacher proceeds as follows:

> Mr. Williams acknowledges that with the "oblique" parallelogram, the "cut-reassemble" approach would take multiple steps. So instead he asks students to think about its area in the following way:
>
> First, I want you to imagine a bunch of pencils lined up that fill in the "oblique" parallelogram's area (like in Fig. 12.1).
>
> If we push those pencils sideways to create a new shape (like in Fig. 12.1), then we haven't changed the area. Doing so produces a rectangle that is $w \times h_w$, and so the area is still $A_{par} = A_{rect} = wh_w = bh$.

Mr. Williams's use of a segment-skewing transformation provides a relatively simple explanation for why the area of the "oblique" parallelogram is still $A_{par} = bh$. Moreover, it provides another means by which to transform the parallelogram into the same $w \times h_w$ rectangle as had been accomplished with the prior "cut-reassemble" approach. This resolution aligns with TP.5 because it provides a justification for the area formula in this unusual case, and it aligns with TP.6 because it provides an alternative explanation. Introducing students to segment-skewing transformations gives them an alternate method for conceptualizing and justifying the areas of planar figures.

To sharpen Mr. Williams's pencil analogy, consider the parallelogram bounded between $x = 0$, $x = h$, $f(x) = mx + w$, and $g(x) = mx$. Subtracting $g(x)$ transforms the parallelogram into a rectangle with height $w = f(x) - g(x)$, and Cavalieri (or, equivalently, properties of the integral) does the rest.
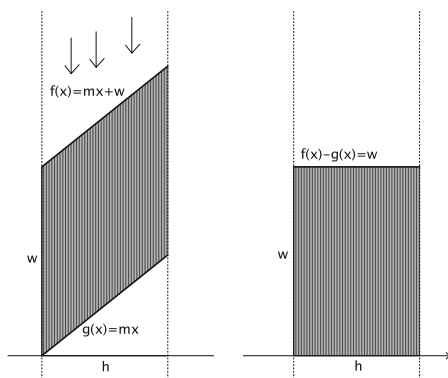
Cavalieri's principle can also be used to study other polygons. A segment-skewing transformation can transform any triangle into a right triangle with the same base and height, thereby confirming the area formula $A_{tri} = \frac{1}{2}bh$. This exercise also

provides students with another way to understand what is meant by the "height" or "altitude" of a triangle.

To determine the area of a trapezoid we can use a combination of segment-skewing and the "cut-reassemble" algorithm. As a first step we apply a segment-skewing transformation in the direction parallel to the base so that we end up with a right trapezoid. Then we split the right trapezoid into a rectangle and a right triangle to compute

$$A_{trap} = A_{rec} + A_{tri} = b_1 h + \frac{1}{2}(b_2 - b_1)h = \frac{1}{2}b_1 h + \frac{1}{2}b_2 h.$$

The punchline is that Cavalieri's principle is a valuable tool in the toolkit, for teachers and students. It provides additional opportunities for reasoning about geometric shapes and offers another way of thinking about area and area-preservation that is important for more advanced mathematical studies.

12.4.2 Areas of Ellipses

As one final class of examples, consider the case of an ellipse. The area of an ellipse is given by $A_{ellip} = \pi ab$ where a and b are the two axes. This formula is less familiar to students, and most textbooks give little insight as to why it is sensible. Cavalieri's principle provides the sort of conceptual insight that is frequently missing.[4]

We start with a circle of radius 1, which we know has an area of π square units. Suppose our ellipse has a major axis of 4 in the x direction and a minor axis of 2 in the y direction. Starting with the unit circle, apply a horizontal dilation with a factor of 4 to form the major axis. Stretching horizontally creates three new sets of crescent-shaped areas, each of which by Cavalieri's principle has an area equal to the original unit circle (Fig. 12.3a). (To visualize this, imagine the original unit circle cut in half vertically and composed of horizontal line segments; push those outward to create the two adjacent crescent-shaped regions.) Now that we have stretched along the major axis, we can stretch the resultant region in the vertical direction by a factor of 2 to form the minor axis. Vertically stretching this region, which has an area of 4π, creates a set of crescent-shaped areas that has an area equal to the original region (Fig. 12.3b). Hence, $A_{ellip} = (4\pi) \cdot 2 = 8\pi$. This argument can be generalized to an arbitrary a and b and provides the needed insight into why the area of an ellipse is $A_{ellip} = \pi ab$.

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[4] We restrict our discussion to natural numbers a and b, although the argument can be extended to the real numbers with a slight modification to Cavalieri's principle.
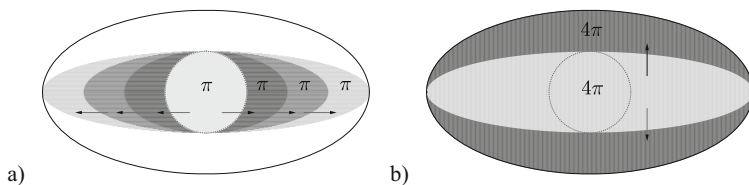
Fig. 12.3 The area formula for an ellipse via segment-skewing (**a**) horizontally, and (**b**) vertically

12.4.3 Transformations That Are Not Continuous

One condition of Cavalieri's principle was that the composite transformation yield a new planar region. But will this always be the case? What happens, for example, when the segment-skewing transformation applied to a region is not continuous? Does it make sense to discuss Cavalieri's principle with respect to sets in the plane created in this way? Although we get a well-defined collection of points when we translate by a discontinuous transformation, it's not so obvious whether this new set has a well-defined area. Let's probe this condition further (TP.1).

Consider the planar region S depicted previously that lies under the function $f(x)$. Now translate S by adding the function

$$t(x) = \begin{cases} 0 & \text{if } x \in [a, c) \cup (c, b] \\ 1 & \text{if } x = c \end{cases}.$$

Because $t(x)$ is zero everywhere except at $x = c$, our new set of points looks like S with the exception of a single segment raised up by one unit.
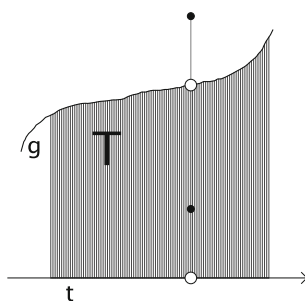
The geometric object does not constitute a planar region, which means Cavalieri's principle does not technically apply. Yet, it is reasonable to think that if we *were* to assign an area to this new object, it should be the same as the area of the original region. It turns out, Riemann integration does just this. Unlike

the derivative, which is not defined at a point of discontinuity (see Chap. 9), the Riemann integral is defined for functions with a discontinuity.

Recall that S is the region between $f(x)$ and the x-axis. Letting T be the new set obtained by adding $t(x)$ to the points of S, the Riemann integral assigns T the area $\int_a^b g(x)dx - \int_a^b t(x)dx$ where $g(x) = f(x) + t(x)$. By properties of the integral,

$$\text{Area}(T) = \int_a^b g(x)dx - \int_a^b t(x)dx = \int_a^b (f+t)(x)dx - \int_a^b t(x)dx =$$

$$= \int_a^b f(x)dx + \int_a^b t(x)dx - \int_a^b t(x)dx = \int_a^b f(x)dx = \text{Area}(S).$$

That is to say, the Riemann integral assigns the new geometric object T the same area as S, thereby *expanding* the class of segment-skewing transformations that can be considered area-preserving. This works because $t(x)$ is a Riemann integrable function on $[a, b]$ even though it has a point of discontinuity. When a non-Riemann-integrable transformation is applied to S—such as Dirichlet's function restricted to $[a, b]$—we cannot use the Riemann integral to find the resulting area. This leads to a curious state of affairs: although we might instinctively feel this new unruly set of points in the plane should still have area S (since it is composed of translated vertical line segments), Cavalieri's principle doesn't apply and the Riemann integral can no longer help. Not all segment-skewing transformations produce geometric objects for which the Riemann integral assigns the same area. Shortcomings of this nature in the Riemann integral eventually led to new definitions of integration, each motivated in part by the desire to create a mathematically sound definition of area applicable to larger classes of sets. (For further discussion of the strengths and weaknesses of the Riemann integral as well as an introduction to some alternative integrals, see Abbott's Section 7.6, 7.7, and 8.1.)

Problems

12.1 In this chapter, we explored an "oblique" parallelogram. Consider an "oblique" triangle—a triangle whose vertex is not over the base (i.e., the altitude falls outside the triangle). First, try to produce a "cut-reassemble" argument for why the area of an "oblique" triangle is still $A_{tri} = \frac{1}{2}bh$. Second, give a segment-skewing argument for its area. In both cases, make sure to connect each argument all the way back to the area of a rectangle.

12.2 One way that a teacher could justify the area formula for a kite, $A_{kite} = \frac{1}{2}d_1d_2$ (with d_1 and d_2 being the lengths of the two diagonals) is to use a "cut-reassemble" transformation to turn the kite into a rectangle with base d_1 and height $\frac{1}{2}d_2$. As an alternative, a segment-skewing transformation could be used to turn the kite into a triangle, with base d_1 and height d_2. Provide an explanation for the area formulas for a kite that you might give as a geometry teacher using this approach; make sure

to include a description for how this transformation gives a justification of the area formula for a kite.

12.3 Stretch a semicircle vertically by a factor of two. How does the crescent-shaped area that was added by this stretching compare to the original semicircle's area? Give a detailed explanation for your reasoning.

12.4 On an international test, a problem similar to this one appeared:

Farmer Joe and Farmer Bob share a fence between their properties (see figure). Currently, their properties are exactly equal in terms of area. However, they would like to create a new fence—one that is straight—but that retains their properties having equal areas. How might you determine a straight fence between their properties that would not result in the loss of any area for either farmer?
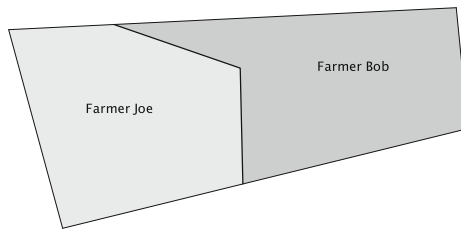
Use a "segment-shifting" transformation to determine how to construct the straight fence in a way that preserves the area of the two properties.

12.5 Transform a circle of radius 1, first, by segment-skewing in the vertical direction (until you have a flat base) and then, second, by segment-skewing the result of your first transformation in the horizontal direction. What is the resultant shape? Use this transformation to provide a justification that could be used by a geometry teacher for the fact that a circle with double the radius (i.e., 2:1) has an area that is four times the original's area (i.e., 4:1). (More generally, with similar planar figures, if the ratio of side lengths is $a : b$, the ratio of areas is $a^2 : b^2$.)

12.6 Provide a geometric justification that you might give as a calculus teacher, connected to segment-skewing transformations and Cavalieri's principle, for the following integral rule: $\int_a^b kf = k \int_a^b f$ (consider only the case for $k \in \mathbb{N}$). [Presume $f : [a, b] \to \mathbb{R}^+$]

12.7 Provide a geometric justification that you might give as a calculus teacher, connected to segment-skewing transformations and Cavalieri's principle, for the integral rule: $\int_a^b f + \int_a^b g = \int_a^b (f + g)$. [Presume $f, g : [a, b] \to \mathbb{R}^+$]

12.8 Consider the triangular region S between the lines $x = 0$, $x = h$, $f(x) = (m - \frac{b}{h})x + b$, and $g(x) = mx$ (for $m, b, h \in \mathbb{R}^+$). (i) Provide an algebraic approach (relying on integral properties from calculus) to find the area of S; (ii) Provide a geometric justification for the area of S based on a segment-skewing transformation.

12.9 Draw an arbitrary quadrilateral $ABCD$. Explain at least two ways that you could transform the quadrilateral $ABCD$ into a triangle, while preserving the area. That is, how to create a triangle that has an area equal to the drawn quadrilateral.

12.10 Consider a regular hexagon and a regular octagon, with side length s. Provide a segment-skewing transformation that would turn them both into trapezoidal-like figures (one is a trapezoid, the other a trapezoid on top of a rectangle). Find the appropriate dimensions (you'll need to use some trigonometry), and then determine the areas of the regular hexagon and regular octagon using the fact that $A_{trap} = \frac{1}{2}h(b_1 + b_2)$.

12.11 In his Sect. 7.3, Abbott describes some of the nuances around integrating functions with discontinuities. In particular, he gives an inductive argument for the Riemann-integrability of functions with a *finite* number of discontinuities. He then gives Dirichlet's function and Thomae's function as two cases with an *infinite* number of discontinuities. He writes:

> ...so we conclude that Dirichlet's function is *not* integrable...[but] we should realize that Dirichlet's function is discontinuous at *every* point in [0, 1]. It would be useful to investigate a function where the discontinuities are infinite in number but do not necessarily make up all of [0, 1]. Thomae's function... is one such example...[and] we will see that Thomae's function *is* Riemann-integrable...

First, describe how Abbott's use of these two examples is related to TP.2. What mathematical purpose did each serve? Second, in the last section of this chapter we looked at one segment-skewing transformation t that had a *singular* discontinuity. Describe what the conclusions from Abbott's examples mean for the kinds of segment-skewing transformations t that would still be area-preserving using Riemann integration.

References

1. Abbott, S. (2015). *Understanding analysis* (2nd ed.). New York, NY: Springer.
2. Common Core State Standards in Mathematics (CCSSM). (2010). Retreived from: http://www.corestandards.org/the-standards/mathematics.