Chapter 1
Introduction

The classical theory of nonlinear partial differential equations assumes that the involved nonlinear terms are of power-law type, or in other words they satisfy growth and coercivity conditions of polynomial type. This leads to the well-known framework of Sobolev spaces. Notwithstanding their powerful properties, they sometimes turn out not to be sufficient to describe some physical phenomena. The studies undertaken in this book concern the existence of solutions to abstract elliptic and parabolic equations, as well as systems of equations which come from applications in the continuum mechanics of non-Newtonian fluids and porous structures.

Our goal is to provide a methodology which allows us to consider such problems with rather general growth conditions of the highest order term. Namely, when the leading part of the differential operator is governed by possibly inhomogeneous (dependent on the spatial variable), fully anisotropic (of different growth in various directions of a gradient of the unknown) convex function without polynomial growth restrictions. Such a formulation requires a general framework for the function space setting. For this reason we consider our PDE problems in Orlicz and Musielak–Orlicz spaces.

The advantage of using such an approach is twofold. Firstly, it provides a unified framework for numerous settings that are developed in the literature: classical Lebesgue spaces, variable exponent spaces, Orlicz spaces, weighted Lebesgue spaces, double-phase spaces, among others. A setting that allows us to treat all these approaches is to the benefit of our understanding of the subtleties of various theories. Secondly, the motivation behind this setting appears in the applied content of the book. These kinds of spaces, which at first glance may seem too sophisticated, indeed allow us to include various properties of materials, like anisotropic character, space inhomogeneity and rheology, which are more general than of power-law type. Non-Newtonian fluids are described in Chapter 7. We mention below some particular examples of materials where such phenomena occur. For instance, there are colloids in which the formation of chains or column-like structures in the fluid can be observed as a response to the application of an electric or magnetic field. The second example corresponds to the homogenization of elliptic boundary value problems described in Chapter 6.

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5_1

The recent development of advanced body protection is concerned with so-called *liquid body armor* – a solution which provides a flexible and light weight armor which stiffens under impact. This can be achieved by soaking existing armor materials with special fluids. We mention two types of fluids used for liquid body armor: magnetorheological fluids and shear thickening fluids. Their common feature is that they are both colloids and consequently react strongly in response to a stimulus. Thus using them, for example, to impregnate kevlar armor means that far fewer layers of kevlar are necessary, which improves the flexibility and significantly reduces the weight of the protection. Kevlar material soaked with the described fluids has the ability to transfer from flexible to completely rigid. The rheological properties of the fluid, such as its viscosity or shape, change rapidly within ca. 0.02 seconds, which makes it highly effective. One can easily observe the anisotropic character of the fluid when the magnetic field is applied. This structure hinders the movement of the fluid in the direction perpendicular to the magnetic field. Shear thickening fluid is a liquid with suspended tiny particles which slightly repel each other. The particles are able to float easily throughout the liquid, but once a high shear stress is applied the repulsive forces among the particles are overwhelmed and the particles aggregate, forming so-called hydroclusters. This example corresponds both to the anisotropic character and exponential growth of an operator used for modeling the phenomena. Besides the abovementioned application such fluids are widely used elsewhere: advanced automotive solutions (viscosity clutch, suspension shock absorbers), seismic protection, and for various medical purposes (the resistance of materials to needle or knife puncture).

Another example that we want to recall refers to the study of homogenization for elliptic systems, and captures the process whereby a porous structure is created by the influence of an electric field. Here the steady-state pore growth occurs in a situation when the geometrical features of a growing porous film do not depend on time. Such a process is expected when the applied electric field is constant in time. An example is the spatially irregular formation of porous structures in oxides of metals appearing in the process of anodization. Note that the process of anodization is widely applied, as an oxide film significantly improves resistance to corrosion and provides better adhesion for various substances than bare metal itself.

To demonstrate the generality of the framework let us recall the definition of an N-function, which in particular will later determine the behavior of differential operators and the functional space setting.

Suppose $Z \subset \mathbb{R}^N$ is a bounded set. A function $M : Z \times \mathbb{R}^d \to \mathbb{R}_+$ is called an *N-function* if it is a Carathéodory function (i.e. measurable with respect to $z \in Z$ and continuous with respect to the last variable), $M(z,0) = 0$, $\xi \mapsto M(z,\xi)$ is convex and even for a.e. $z \in Z$, and there exist two convex functions $m_1, m_2 : [0,\infty) \to [0,\infty)$, positive on $(0,\infty)$, such that $m_1(0) = 0 = m_2(0)$ and both are superlinear at zero and at infinity, that for a.a. $z \in Z$ allow us to estimate the N-function M as follows $m_1(|\xi|) \le M(z,\xi) \le m_2(|\xi|)$.

This definition comprises various features of an N-function that directly correspond to the characteristics of the above described processes and that we want to particularly emphasize:

- Full anisotropy. Namely, an N-function may be dependent on the whole vector ξ in $\mathbb{R}^d$. In particular, it may possess growth which is not a function of the length of ξ, nor the sum of one-dimensional functions of each of its coordinates ξ_i.
- Inhomogeneity, i.e. an N-function may depend on the spatial variable $z \in Z$.
- Rapid or slow growth. Namely, the growth of an N-function does not have to be restricted by any polynomial function, e.g. M can be of type $L \log L$ or have exponential growth at infinity.

An N-function defines a modular ϱ_M of a measurable function $\xi : Z \to \mathbb{R}^d$, namely

$$\varrho_M(\xi) := \int_Z M(z,\xi(z))\,\mathrm{d}z.$$

The set of all measurable functions $\xi : Z \to \mathbb{R}^d$ such that $\varrho_M(\xi)$ is finite is called a *generalized Musielak–Orlicz class*, which we denote by $\mathcal{L}_M(Z;\mathbb{R}^d)$. Note that such a set may fail to be invariant under multiplication by scalars. The smallest linear space containing the Musielak–Orlicz class is called a *generalized Musielak–Orlicz space* and we denote it by $L_M(Z;\mathbb{R}^d)$. The generalized Musielak–Orlicz space equipped with the *Luxemburg norm*

$$||\xi||_{L_M} := \inf\left\{\lambda > 0 : \int_Z M\left(z, \frac{\xi(z)}{\lambda}\right)\,\mathrm{d}z \le 1\right\}$$

is a Banach space.

As we have already mentioned, the Musielak–Orlicz space setting captures some important function spaces, widely studied recently. To emphasize the wide spectrum of the framework we list examples of function spaces, together with the appropriately identified N-function, that fall into this regime:

- classical Lebesgue spaces $L^p(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^p$, where $p \in (1,\infty)$,
- classical (homogeneous) Orlicz spaces $L_M(Z;\mathbb{R}^d)$, isotropic when $M(z,\xi) = M(|\xi|)$ as well as anisotropic when $M(z,\xi) = M(\xi)$ (M is a homogeneous N-function); e.g. $L_M = L \log L$ when $M(\xi) = |\xi| \log(\mathrm{e} + |\xi|)$, or $L_M = L_{\exp}$ when $M(\xi) = \exp(|\xi|) - 1 + |\xi|$,
- weighted Lebesgue spaces $L^p_\omega(Z;\mathbb{R}^d)$ with $M(z,\xi) = \omega(z)|\xi|^p$, where $p \in (1,\infty)$ and $\omega : Z \to (0,\infty)$ is measurable,
- generalized Lebesgue spaces with variable exponent $L^{p(\cdot)}(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^{p(z)}$, where $p : Z \to [p_-, p_+]$, $1 < p_- \le p_+ < \infty$, is measurable,
- double phase spaces $L_M(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^p + a(z)|\xi|^q$, where $a : Z \to [0,\infty)$ is measurable and $1 < p < q < \infty$,
- many others, e.g. $L_M(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^{p(z)} \log(1 + |\xi|)$, where $p : Z \to [1, p_+]$, $p_+ < \infty$, is measurable, or weighted Orlicz spaces $M(z,\xi) = \omega(z) M_0(\xi)$, where $\omega : Z \to (0,\infty)$ is measurable and M_0 is a homogeneous N-function.

Studies on PDEs involving an operator exhibiting Orlicz growth go back to Talenti [307], Donaldson [122], and Gossez [173, 174, 175] with later results due to Benkirane, Elmahi and Meskine [33, 130, 131], Mustonen and Tienari [263], Lieberman [235], and Cianchi [90]. The mathematical theory of classical Orlicz spaces, important from the point of view of functional analysis and applications in the theory of partial differential equations, is presented by Adams and Fournier in [5], see also [220, 281]. The framework of generalized Lebesgue spaces with variable exponent for problems of functional analysis and the theory of PDEs is studied in [100] by Cruz-Uribe and Fiorenza and [115] by Diening et al. A broad overview of results in this framework is available in [194]. The first monograph on Musielak–Orlicz spaces where the N-functions depend on the spatial variable but are isotropic was written by Nakano [265], whereas a comprehensive reference for the foundations of the theory was provided by Musielak [262]. We note that Musielak–Orlicz spaces provide a natural framework for the so-called (p,q)-growth problems that received special attention starting from the pioneering works of Marcellini [246, 247] and for the non-uniformly elliptic problems studied since [226] by Ladyzhenskaya and Ural'tseva and [210] by Ivanov. On the other hand, the cornerstones for partial differential equations in fully anisotropic Orlicz spaces were laid by Klimov [219] and Cianchi [91, 93]. Our aim is not only to capture all the mentioned types of growth and provide a unified theory, as described in the survey [71], but also to prepare a toolkit for analysis within the setting which simultaneously combines the inhomogeneous, Orlicz and fully anisotropic properties.

A substantial part of our investigations concerns the scenario where the growth of the highest order term cannot be compared with a polynomial function. In other words, the N-function used to describe the growth and which defines the space setting does not satisfy the so-called Δ_2-condition. Recall that we say that an N-function $M : Z\times\mathbb{R}^d \to [0,\infty)$ satisfies the Δ_2-*condition* if there exists a constant $c > 0$ and a nonnegative integrable function $h : Z \to \mathbb{R}$ such that

$$M(z,2\xi) \le cM(z,\xi)+h(z) \qquad \text{for a.e. } z \in Z.$$

This property implies that the corresponding Musielak–Orlicz space is separable. For further considerations it is meaningful to ask whether the Δ_2-condition is satisfied not only by M, but also by its conjugate

$$M^*(z,\eta) := \sup_{\xi\in\mathbb{R}^d} \{\xi\cdot\eta - M(z,\xi)\}.$$

The Musielak–Orlicz space is reflexive provided both M and M^* satisfy the Δ_2-condition. In particular, this implies that M is trapped between two power-type functions.

Let us now briefly describe the types of problems arising in the mathematical theory of PDEs which will be influential to us and for which we will attempt to develop functional analytic methods in the setting of general Musielak–Orlicz spaces. For an abstract elliptic system one can consider, for an unknown $u : \Omega \to \mathbb{R}^d$, the following equation

$$-\operatorname{div} \mathbf{A}(x,\nabla u) = \mathbf{f} \tag{1.1}$$

with zero Dirichlet boundary condition on the bounded domain $\Omega \subset \mathbb{R}^N$ and where $\mathbf{f}$ is a given function having appropriate regularity. The function $\mathbf{A} : \Omega\times\mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ is assumed to be a Carathéodory function which satisfies the following growth and coercivity condition

$$\mathbf{A}(x,\xi)\cdot\xi \geq d_1\Big\{M(x,d_2\xi)+M^*\big(x,d_3\mathbf{A}(x,\xi)\big)\Big\} \text{ for a.a. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^{d\times N}, \tag{1.2}$$

where $d_1, d_2, d_3 > 0$ and $\mathbf{A}$ is monotone, i.e.

$$(\mathbf{A}(x,\xi_1)-\mathbf{A}(x,\xi_2))\cdot(\xi_1-\xi_2)\geq 0 \quad \text{for a.a. } x\in\Omega \text{ and all } \xi_1,\xi_2\in\mathbb{R}^{d\times N}. \tag{1.3}$$

Let us note that the above conditions may be formulated in more general way

$$\begin{aligned} M(x,c_1\xi) &\leq \mathbf{A}(x,\xi)\cdot\xi, \\ c_2M^*(x,c_3\mathbf{A}(x,\xi)) &\leq M(x,c_4\xi) \end{aligned} \tag{1.4}$$

for some $c_1, c_2, c_3, c_4 > 0$. This relation is discussed in detail in Section 3.8.2. Taking $M(x,\xi)=\frac{1}{p}|\xi|^p$ with $p\in(1,\infty)$, we have $M^*(x,\xi)=\frac{1}{p'}|\xi|^{p'}$ with p' being Hölder conjugate, and the classical form of the growth and coercivity condition for the Leray–Lions operator in L^p spaces is reflected [237, 232].

Problem (1.1) is studied in various directions. Firstly we concentrate on weak solutions. The analysis is conducted under different assumptions on the N-function. We emphasize the influence of its properties on the methods used in existence proofs. Here there are three pathways that we follow: assuming the Δ_2-condition on the N-function M; assuming the Δ_2-condition on the conjugate N-function M^*; and finally, a continuity-type assumption on the N-function M with respect to the space variable. For simplicity, the last result is presented for a scalar equation. All these results are contained in Chapter 4. Then, in Chapter 5, we turn our attention to less regular data, i.e. merely integrable. Immediately we fall into the regime of renormalized solutions.

For a parabolic problem we consider the corresponding equation, namely

$$\partial_t u - \operatorname{div}\mathbf{a}(t,x,\nabla u) = f. \tag{1.5}$$

We focus on the case with only integrable data, which again requires us to study a special notion of solution. Chapter 5 includes a study of well posedness – existence and uniqueness – in the class of renormalized solutions. We make use of the discussion of weak solutions to parabolic problem presented in Chapter 4 in the consecutive part on renormalized solutions.

The next area of great interest is the homogenization process for families of strongly nonlinear elliptic systems with homogeneous Dirichlet boundary conditions under very general assumptions on the N-functions. Here the differential operator takes the form $\mathbf{a}(\frac{x}{\varepsilon},\nabla u)$ and we investigate the passage to the limit when $\varepsilon\to 0$. The growth and the coercivity of the elliptic operator is assumed to be described by a condition of type (1.2), related to (1.4). In particular, the homogenization process changes the underlying function spaces and the nonlinear elliptic operator at each step, since the governing N-function depends on the spatial variable x.

Further, we consider a large class of problems which arise from the mechanics of non-Newtonian fluids with non-standard rheology. We want to include the phenomena of viscosity changing under various stimuli like shear rate, or a magnetic or electric field. This forces us to use inhomogeneous anisotropic Musielak–Orlicz spaces. Our investigations are directed towards the existence of weak solutions. The system of equations describing incompressible non-Newtonian fluid flow may take the following form

$$\begin{aligned} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S}(x, \mathbf{D}\mathbf{u}) + \nabla \pi = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } (0,T) \times \Omega, \end{aligned} \tag{1.6}$$

where $\mathbf{u}$ denotes the velocity field of a fluid; π is a pressure; Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary; $T < \infty$; $\mathbf{f}$ is a given body force; and $\mathbf{D}\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetric part of the gradient of the velocity field. The first equation is the momentum equation and the second one is the incompressibility condition. We assume a no-slip boundary condition (zero Dirichlet boundary condition).

In order to close the system we have to state a constitutive relation, rheology, which describes the relation between $\mathbf{S}$ and $\mathbf{D}\mathbf{u}$. In our considerations we do not want to assume that $\mathbf{S}$ has only a polynomial structure, which would not suffice to describe the nonstandard behavior of the fluid. Motivated by the significant shear thickening phenomenon we want to investigate the processes where the growth is faster than polynomial and possibly different in various directions of the shear stress. Hence an N-function defining a functional space does not satisfy the Δ_2-condition and is possibly anisotropic. The viscosity of the fluid is not assumed to be constant and it can depend on density and the full symmetric part of the velocity gradient. Therefore we formulate the growth conditions of the stress tensor in an analogous way as in (1.2) or (1.4).

In particular, we investigate, with various degrees of generality of the N-function, the flow of inhomogeneous heat-conducting fluids, which depends also on density and temperature. This means that the above system needs to be supplemented with two equations: balance of mass (the continuity equation) and the heat equation. Moreover, the stress tensor then also depends on density and temperature. The other problem we study is the system describing fluid-structure interaction where the motion of rigid bodies immersed in the fluid is taken into account. Moreover, if the model allows us to skip the convective term, we are able also to consider shear thinning fluids, in which case M^* may not satisfy the Δ_2-condition.

Since our considerations on PDE problems concentrate on growth and coercivity of (1.2) type, we employ Musielak–Orlicz spaces defined by means of an N-function M. Let us emphasize that we do not want to assume that M satisfies the Δ_2-condition or that it is sandwiched between two polynomials. Consequently, we lose a wide range of useful properties of function spaces. The lack of numerous basic properties results in many subtle but deep difficulties which require significantly more sophisticated methods than in the classical case.

An important aspect of a Sobolev-type space related to an N-function M which sets it apart from a classical Sobolev space is the issue of density of regular functions. The classical theorem of Meyers and Serrin [253] tells us that $C^\infty \cap W^{m,p}$ is dense in $W^{m,p}$ in the strong topology for $1 \leq p < \infty$. An extension of this fact to the classical, i.e. homogenous and isotropic, Orlicz (or rather Orlicz–Sobolev) spaces was investigated by Gossez [174]. He proved that the related density result holds, however not with respect to the strong, but with respect to the so-called modular topology. An analogous fact in Musielak–Orlicz spaces holds only provided the asymptotic behavior of the modular function is sufficiently balanced, see Section 3.7.

It is worth pointing out that even some partial information on the behavior of an N-function enables us to simplify the tools needed for the proofs of existence of solutions. Knowing that M^* satisfies the Δ_2-condition tells us that the weak sequential stability of a considered PDE problem (i.e. passing from an approximate problem to the solutions of the original problem) can be proved by means of weak-$*$ convergence. However, once we want to relax this assumption, an essential tool that comes into play is an approximation by smooth functions with respect to the modular topology. To show the density of smooth functions in the modular topology it is necessary to specify an appropriate balance of asymptotical behavior of M with respect to small changes of z and big values of $|\xi|$, relating to the log-Hölder continuity of variable exponent or a closeness condition on powers in double phase spaces.

In the case of parabolic problems additional difficulties appear. One of them is the lack of an integration by parts formula, cf. [165] and [123]. Such a tool is essential for testing the equation with a solution and using monotonicity methods. Let us recall the well-known Newton's formula in the Bochner space setting. For $0 \le t_0 < t_1 \le T$ and $q \in (1,\infty)$, $q' = q/(q-1)$ we set $v \in L^q(0,T;X)$, $\partial_t v \in L^{q'}(0,T;X^*)$, where X is a reflexive, separable Banach space and X^* is its dual. Then there exists a Hilbert space H such that $X \subset H = H^* \subset X^*$ and the following formula holds

$$\int_{t_0}^{t_1} \langle \partial_t v, v \rangle_{X^*,X} \,\mathrm{d}t = \frac{1}{2}\|v(t_1)\|_X^2 - \frac{1}{2}\|v(t_0)\|_X^2.$$

To extend this formula to any generalization of classical Orlicz spaces we would essentially need that C^∞-functions are dense in $L_M((0,T)\times\Omega)$ and that

$$L_M((0,T)\times\Omega) = L_M(0,T;L_M(\Omega)).$$

Even for classical Orlicz spaces (homogeneous and isotropic) these hold only in particular cases, e.g. the former only holds if M, M^* satisfy the Δ_2-condition. In order to provide the factorization property we recall the result of [123], which is stated for classical Orlicz spaces with homogeneous and isotropic $M = M(|\xi|)$ and therefore we rather cannot expect a better result for more general N-functions.

Let I be a time interval, $\Omega \subset \mathbb{R}^d$, $M : [0,\infty) \to [0,\infty)$ an N-function, $L_M(I\times\Omega)$ the Orlicz space on $I\times\Omega$, and $L_M(I;L_M(\Omega))$ the vector-valued Orlicz space on I. Then

$$L_M(I\times\Omega) = L_M(I;L_M(\Omega))$$

if and only if there exist constants $k_0, k_1 > 0$ such that

$$k_0 M^{-1}(s) M^{-1}(r) \le M^{-1}(sr) \le k_1 M^{-1}(s) M^{-1}(r) \tag{1.7}$$

for every $s \ge 1/|I|$ and $r \ge 1/|\Omega|$.

One can show that (1.7) means that M must be equivalent to some power p, $1 < p < \infty$. Hence, if (1.7) should hold, very strong assumptions must be satisfied by M. Surely they would force $L_M((0,T)\times\Omega)$ to be separable and reflexive.

Besides the lack of integration by parts formula there are many other obstacles resulting from the general (Orlicz) type of growth of the modular function. Among others we mention the Korn inequality, which is a basic tool in continuum mechanics, providing bounds on the full velocity gradient in terms of its symmetric part. However, in homogenous Orlicz spaces $L_M(\Omega)$ it holds only if M and M^* satisfy the Δ_2-condition. In order to overcome this problem for more general growths we need to construct different types of estimates.

Furthermore, classical results of harmonic analysis are not available in their full strength. For instance, a tool which has already become standard in fluid mechanics, however missing in our setting, is the method of Lipschitz truncations [159], which is widely used to deal with low regularity of gradients of solutions in the convective term. The only available results where the Lipschitz truncations method is applied in the Musielak–Orlicz setting are in the isotropic and homogeneous case where M and M^* satisfy the Δ_2-condition [61] and in variable exponent spaces [116].

A lot of facts which hold in the isotropic case are no longer true in the anisotropic setting, but this is subtle and hard to capture in a brief summary. One of the most preeminent examples is that in a fully anisotropic setting, the meaning of the Sobolev embedding is essentially different than in the isotropic setting. In fact, the anisotropic energy of a gradient of a function is expected to improve integrability of the real-valued function itself. In the case of the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$, $\vec{p} = (p_1,\dots,p_d)$, besides the obvious embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_m^*}(\Omega)$ with $p_m = \min\{p_1,\dots,p_d\}$, when $p_m < N$ and p_m^* is a Sobolev conjugate of p_m, that is $p_m^* = Np_m/(N-p_m)$, one can use symmetrization techniques to get

$$W^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_0^*}(\Omega)$$

with p_0 being the harmonic mean of $p_i s$, $p_0 < N$, and p_0^* is a Sobolev conjugate of p_0. This result turns out to be the optimal embedding into an isotropic Orlicz target space. Such an embedding is known for fully anisotropic Orlicz spaces [91], but – due to inhomogeneity – it fails in general Musielak–Orlicz spaces. Let us stress here that we refrain from using these kinds of techniques, taking care, as much as possible, to use straightforward formulations of the involved results.

The goal of this monograph is to systematize the methods available for anisotropic Musielak–Orlicz spaces which are useful in the theory of partial differential equations. To this end we present in detail the analytical tools, stressing the importance and challenges resulting from inhomogeneity, anisotropy, and from relaxing the growth conditions.