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Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces

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To our families

Preface

Anisotropic and inhomogeneous spaces, which are at the core of the present study, may appear exotic at first. However, the reader should abandon this impression once they realize how many phenomena in their nearest surroundings can be described by partial differential equations in Musielak–Orlicz spaces. Even when driving a car one relies on viscous clutch or dynamical shock absorbers in the suspension system. Inhabitants of seismic regions, perhaps without realizing it, entrust their safety to magnetorheological dampers which are filled with a fluid that absorbs the shock by becoming more viscous when vibrations are detected. Finally the phenomenon of blood flow in the human body is another example of a process falling into the mathematical framework presented in the current monograph.

The idea to incorporate inhomogeneity by describing it in the language of variable exponent spaces or weighted spaces is now a well-established approach. It has been further extended to double-phase spaces; however, more is needed to describe the most technically advanced material, which we intend to cover in an Orlicz setting, allowing also for inhomogeneity. The resulting structure is then referred to as the Musielak–Orlicz formulation.

The theory of Musielak–Orlicz spaces provides a unified framework for variable exponent, Orlicz, weighted, and double-phase spaces. Despite the intense research in each of these directions, exhaustive studies of partial differential equation methods in Musielak–Orlicz spaces are still in short supply.

The majority of research in this field so far has concentrated on isotropic spaces where the modular function has a growth comparable with a polynomial or is trapped between two power-type functions and, hence, where one can use powerful tools inherited from the classical setting of Lebesgue and Sobolev spaces. However, in the case of slowly or very rapidly growing modular functions, we encounter analytical difficulties that substantially restrict good properties of the space, such as separability or reflexivity.

There is a growing community interested in various aspects of Musielak–Orlicz spaces. We aim to provide them with a manual for everyday use, but at the same time, we hope to make the subject accessible to all specialists in PDEs. We stress that there exist multiple useful methods in the literature, which until now have been widely dispersed over numerous papers, and hence have not been easily accessible.

Our goal is to give a systematic and careful presentation of the analytical tools of partial differential equations posed in the Musielak–Orlicz setting, stressing the importance and challenges resulting from the generality of the growth requirements. We provide full and detailed proofs, fix the gaps in some existing proofs, provide proofs of previously announced results, and arrange the material in a way which will enable those unfamiliar with this branch of mathematics to get a heuristic insight into the subject.

We start with brief introduction to the subject followed by two extensive chapters on the foundations of the theory useful in the analysis of PDEs. We provide a comprehensive study of the problem of density of smooth functions in Musielak–Orlicz spaces. As a basic application we present existence results for general elliptic and parabolic problems, which for bounded data will result in weak solutions and in the case of merely integrable data in a renormalized solutions regime. We also attempt to view various problems from different perspectives, and draw the reader's attention to how the interplay between different properties of function spaces (or rather structural functions, called N-functions) influence the proof techniques. This will be presented in the case of weak solutions to elliptic problems. Lastly we turn more to problems that are inspired by applications in materials science and concentrate on the theory of homogenization of elliptic systems and well-posedness of problems arising in fluid dynamics.

Warsaw,
July 2021

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Contents

Part I
Overture

We start by preparing the framework for PDEs in the Musielak–Orlicz setting.

Musielak–Orlicz spaces generalize many different spaces, each featuring non-standard growth, and shares the difficulties faced by each of them. Even more, the spaces we want to study simultaneously combine the inhomogeneous, Orlicz and fully anisotropic properties. Thus, the theory of differential equations within this setting presents various obstructions from the point of view of functional analysis.

This part is devoted to the careful presentation of the basics of this theory. We collect and systematize a lot of known results which previously have been widely distributed over the literature, and we fix the gaps in some available proofs. Furthermore, there are some results provided here that have only been announced but not proved before in this generality.

Providing a broad view of the subject, we do not restrict ourselves to the tools necessary for the applications in Part II. In particular, for instance, we compare two analytical situations: the growth restrictions imposed on the function defining the norm and the balance conditions imposed on the asymptotic regularity of this function. Our aim is to provide a clear parallel between these approaches, stressing the importance and challenges resulting from relaxing the growth requirements that will be useful in our analysis of PDEs.

Chapter 1
Introduction

The classical theory of nonlinear partial differential equations assumes that the involved nonlinear terms are of power-law type, or in other words they satisfy growth and coercivity conditions of polynomial type. This leads to the well-known framework of Sobolev spaces. Notwithstanding their powerful properties, they sometimes turn out not to be sufficient to describe some physical phenomena. The studies undertaken in this book concern the existence of solutions to abstract elliptic and parabolic equations, as well as systems of equations which come from applications in the continuum mechanics of non-Newtonian fluids and porous structures.

Our goal is to provide a methodology which allows us to consider such problems with rather general growth conditions of the highest order term. Namely, when the leading part of the differential operator is governed by possibly inhomogeneous (dependent on the spatial variable), fully anisotropic (of different growth in various directions of a gradient of the unknown) convex function without polynomial growth restrictions. Such a formulation requires a general framework for the function space setting. For this reason we consider our PDE problems in Orlicz and Musielak–Orlicz spaces.

The advantage of using such an approach is twofold. Firstly, it provides a unified framework for numerous settings that are developed in the literature: classical Lebesgue spaces, variable exponent spaces, Orlicz spaces, weighted Lebesgue spaces, double-phase spaces, among others. A setting that allows us to treat all these approaches is to the benefit of our understanding of the subtleties of various theories. Secondly, the motivation behind this setting appears in the applied content of the book. These kinds of spaces, which at first glance may seem too sophisticated, indeed allow us to include various properties of materials, like anisotropic character, space inhomogeneity and rheology, which are more general than of power-law type. Non-Newtonian fluids are described in Chapter 7. We mention below some particular examples of materials where such phenomena occur. For instance, there are colloids in which the formation of chains or column-like structures in the fluid can be observed as a response to the application of an electric or magnetic field. The second example corresponds to the homogenization of elliptic boundary value problems described in Chapter 6.

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The recent development of advanced body protection is concerned with so-called *liquid body armor* – a solution which provides a flexible and light weight armor which stiffens under impact. This can be achieved by soaking existing armor materials with special fluids. We mention two types of fluids used for liquid body armor: magnetorheological fluids and shear thickening fluids. Their common feature is that they are both colloids and consequently react strongly in response to a stimulus. Thus using them, for example, to impregnate kevlar armor means that far fewer layers of kevlar are necessary, which improves the flexibility and significantly reduces the weight of the protection. Kevlar material soaked with the described fluids has the ability to transfer from flexible to completely rigid. The rheological properties of the fluid, such as its viscosity or shape, change rapidly within ca. 0.02 seconds, which makes it highly effective. One can easily observe the anisotropic character of the fluid when the magnetic field is applied. This structure hinders the movement of the fluid in the direction perpendicular to the magnetic field. Shear thickening fluid is a liquid with suspended tiny particles which slightly repel each other. The particles are able to float easily throughout the liquid, but once a high shear stress is applied the repulsive forces among the particles are overwhelmed and the particles aggregate, forming so-called hydroclusters. This example corresponds both to the anisotropic character and exponential growth of an operator used for modeling the phenomena. Besides the abovementioned application such fluids are widely used elsewhere: advanced automotive solutions (viscosity clutch, suspension shock absorbers), seismic protection, and for various medical purposes (the resistance of materials to needle or knife puncture).

Another example that we want to recall refers to the study of homogenization for elliptic systems, and captures the process whereby a porous structure is created by the influence of an electric field. Here the steady-state pore growth occurs in a situation when the geometrical features of a growing porous film do not depend on time. Such a process is expected when the applied electric field is constant in time. An example is the spatially irregular formation of porous structures in oxides of metals appearing in the process of anodization. Note that the process of anodization is widely applied, as an oxide film significantly improves resistance to corrosion and provides better adhesion for various substances than bare metal itself.

To demonstrate the generality of the framework let us recall the definition of an N-function, which in particular will later determine the behavior of differential operators and the functional space setting.

Suppose $Z \subset \mathbb{R}^N$ is a bounded set. A function $M : Z \times \mathbb{R}^d \to \mathbb{R}_+$ is called an *N-function* if it is a Carathéodory function (i.e. measurable with respect to $z \in Z$ and continuous with respect to the last variable), $M(z,0) = 0$, $\xi \mapsto M(z,\xi)$ is convex and even for a.e. $z \in Z$, and there exist two convex functions $m_1, m_2 : [0,\infty) \to [0,\infty)$, positive on $(0,\infty)$, such that $m_1(0) = 0 = m_2(0)$ and both are superlinear at zero and at infinity, that for a.a. $z \in Z$ allow us to estimate the N-function M as follows $m_1(|\xi|) \le M(z,\xi) \le m_2(|\xi|)$.

This definition comprises various features of an N-function that directly correspond to the characteristics of the above described processes and that we want to particularly emphasize:

- Full anisotropy. Namely, an N-function may be dependent on the whole vector ξ in $\mathbb{R}^d$. In particular, it may possess growth which is not a function of the length of ξ, nor the sum of one-dimensional functions of each of its coordinates ξ_i.
- Inhomogeneity, i.e. an N-function may depend on the spatial variable $z \in Z$.
- Rapid or slow growth. Namely, the growth of an N-function does not have to be restricted by any polynomial function, e.g. M can be of type $L\log L$ or have exponential growth at infinity.

An N-function defines a modular ϱ_M of a measurable function $\xi : Z \longrightarrow \mathbb{R}^d$, namely

$$\varrho_M(\xi) := \int_Z M(z, \xi(z))\,\mathrm{d}z.$$

The set of all measurable functions $\xi : Z \longrightarrow \mathbb{R}^d$ such that $\varrho_M(\xi)$ is finite is called a *generalized Musielak–Orlicz class*, which we denote by $\mathcal{L}_M(Z;\mathbb{R}^d)$. Note that such a set may fail to be invariant under multiplication by scalars. The smallest linear space containing the Musielak–Orlicz class is called a *generalized Musielak–Orlicz space* and we denote it by $L_M(Z;\mathbb{R}^d)$. The generalized Musielak–Orlicz space equipped with the *Luxemburg norm*

$$||\xi||_{L_M} := \inf\left\{\lambda > 0 : \int_Z M\left(z, \frac{\xi(z)}{\lambda}\right)\,\mathrm{d}z \le 1\right\}$$

is a Banach space.

As we have already mentioned, the Musielak–Orlicz space setting captures some important function spaces, widely studied recently. To emphasize the wide spectrum of the framework we list examples of function spaces, together with the appropriately identified N-function, that fall into this regime:

- classical Lebesgue spaces $L^p(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^p$, where $p \in (1,\infty)$,
- classical (homogeneous) Orlicz spaces $L_M(Z;\mathbb{R}^d)$, isotropic when $M(z,\xi) = M(|\xi|)$ as well as anisotropic when $M(z,\xi) = M(\xi)$ (M is a homogeneous N-function); e.g. $L_M = L\log L$ when $M(\xi) = |\xi|\log(\mathrm{e} + |\xi|)$, or $L_M = L_{\exp}$ when $M(\xi) = \exp(|\xi|) - 1 + |\xi|$,
- weighted Lebesgue spaces $L^p_\omega(Z;\mathbb{R}^d)$ with $M(z,\xi) = \omega(z)|\xi|^p$, where $p \in (1,\infty)$ and $\omega : Z \longrightarrow (0,\infty)$ is measurable,
- generalized Lebesgue spaces with variable exponent $L^{p(\cdot)}(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^{p(z)}$, where $p : Z \longrightarrow [p_-, p_+]$, $1 < p_- \le p_+ < \infty$, is measurable,
- double phase spaces $L_M(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^p + a(z)|\xi|^q$, where $a : Z \longrightarrow [0,\infty)$ is measurable and $1 < p < q < \infty$,
- many others, e.g. $L_M(Z;\mathbb{R}^d)$ with $M(z,\xi) = |\xi|^{p(z)}\log(1 + |\xi|)$, where $p : Z \longrightarrow [1, p_+]$, $p_+ < \infty$, is measurable, or weighted Orlicz spaces $M(z,\xi) = \omega(z)M_0(\xi)$, where $\omega : Z \longrightarrow (0,\infty)$ is measurable and M_0 is a homogeneous N-function.

Studies on PDEs involving an operator exhibiting Orlicz growth go back to Talenti [307], Donaldson [122], and Gossez [173, 174, 175] with later results due to Benkirane, Elmahi and Meskine [33, 130, 131], Mustonen and Tienari [263], Lieberman [235], and Cianchi [90]. The mathematical theory of classical Orlicz spaces, important from the point of view of functional analysis and applications in the theory of partial differential equations, is presented by Adams and Fournier in [5], see also [220, 281]. The framework of generalized Lebesgue spaces with variable exponent for problems of functional analysis and the theory of PDEs is studied in [100] by Cruz-Uribe and Fiorenza and [115] by Diening et al. A broad overview of results in this framework is available in [194]. The first monograph on Musielak–Orlicz spaces where the N-functions depend on the spatial variable but are isotropic was written by Nakano [265], whereas a comprehensive reference for the foundations of the theory was provided by Musielak [262]. We note that Musielak–Orlicz spaces provide a natural framework for the so-called (p,q)-growth problems that received special attention starting from the pioneering works of Marcellini [246, 247] and for the non-uniformly elliptic problems studied since [226] by Ladyzhenskaya and Ural'tseva and [210] by Ivanov. On the other hand, the cornerstones for partial differential equations in fully anisotropic Orlicz spaces were laid by Klimov [219] and Cianchi [91, 93]. Our aim is not only to capture all the mentioned types of growth and provide a unified theory, as described in the survey [71], but also to prepare a toolkit for analysis within the setting which simultaneously combines the inhomogeneous, Orlicz and fully anisotropic properties.

A substantial part of our investigations concerns the scenario where the growth of the highest order term cannot be compared with a polynomial function. In other words, the N-function used to describe the growth and which defines the space setting does not satisfy the so-called Δ_2-condition. Recall that we say that an N-function $M : Z \times \mathbb{R}^d \to [0,\infty)$ satisfies the Δ_2-*condition* if there exists a constant $c > 0$ and a nonnegative integrable function $h : Z \to \mathbb{R}$ such that

$$M(z,2\xi) \le cM(z,\xi) + h(z) \qquad \text{for a.e. } z \in Z.$$

This property implies that the corresponding Musielak–Orlicz space is separable. For further considerations it is meaningful to ask whether the Δ_2-condition is satisfied not only by M, but also by its conjugate

$$M^*(z,\eta) := \sup_{\xi\in\mathbb{R}^d} \{\xi\cdot\eta - M(z,\xi)\}.$$

The Musielak–Orlicz space is reflexive provided both M and M^* satisfy the Δ_2-condition. In particular, this implies that M is trapped between two power-type functions.

Let us now briefly describe the types of problems arising in the mathematical theory of PDEs which will be influential to us and for which we will attempt to develop functional analytic methods in the setting of general Musielak–Orlicz spaces. For an abstract elliptic system one can consider, for an unknown $u : \Omega \to \mathbb{R}^d$, the following equation

$$-\operatorname{div} \mathbf{A}(x,\nabla u) = \mathbf{f} \tag{1.1}$$

with zero Dirichlet boundary condition on the bounded domain $\Omega \subset \mathbb{R}^N$ and where $\mathbf{f}$ is a given function having appropriate regularity. The function $\mathbf{A} : \Omega\times\mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ is assumed to be a Carathéodory function which satisfies the following growth and coercivity condition

$$\mathbf{A}(x,\xi)\cdot\xi \geq d_1\Big\{M(x,d_2\xi)+M^*\big(x,d_3\mathbf{A}(x,\xi)\big)\Big\} \quad \text{for a.a. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^{d\times N}, \tag{1.2}$$

where $d_1,d_2,d_3>0$ and $\mathbf{A}$ is monotone, i.e.

$$(\mathbf{A}(x,\xi_1)-\mathbf{A}(x,\xi_2))\cdot(\xi_1-\xi_2)\geq 0 \quad \text{for a.a. } x\in\Omega \text{ and all } \xi_1,\xi_2\in\mathbb{R}^{d\times N}. \tag{1.3}$$

Let us note that the above conditions may be formulated in more general way

$$\begin{aligned} M(x,c_1\xi) &\leq \mathbf{A}(x,\xi)\cdot\xi,\\ c_2M^*(x,c_3\mathbf{A}(x,\xi)) &\leq M(x,c_4\xi) \end{aligned} \tag{1.4}$$

for some $c_1,c_2,c_3,c_4>0$. This relation is discussed in detail in Section 3.8.2. Taking $M(x,\xi)=\frac{1}{p}|\xi|^p$ with $p\in(1,\infty)$, we have $M^*(x,\xi)=\frac{1}{p'}|\xi|^{p'}$ with p' being Hölder conjugate, and the classical form of the growth and coercivity condition for the Leray–Lions operator in L^p spaces is reflected [237, 232].

Problem (1.1) is studied in various directions. Firstly we concentrate on weak solutions. The analysis is conducted under different assumptions on the N-function. We emphasize the influence of its properties on the methods used in existence proofs. Here there are three pathways that we follow: assuming the Δ_2-condition on the N-function M; assuming the Δ_2-condition on the conjugate N-function M^*; and finally, a continuity-type assumption on the N-function M with respect to the space variable. For simplicity, the last result is presented for a scalar equation. All these results are contained in Chapter 4. Then, in Chapter 5, we turn our attention to less regular data, i.e. merely integrable. Immediately we fall into the regime of renormalized solutions.

For a parabolic problem we consider the corresponding equation, namely

$$\partial_t u - \operatorname{div}\mathbf{a}(t,x,\nabla u) = f. \tag{1.5}$$

We focus on the case with only integrable data, which again requires us to study a special notion of solution. Chapter 5 includes a study of well posedness – existence and uniqueness – in the class of renormalized solutions. We make use of the discussion of weak solutions to parabolic problem presented in Chapter 4 in the consecutive part on renormalized solutions.

The next area of great interest is the homogenization process for families of strongly nonlinear elliptic systems with homogeneous Dirichlet boundary conditions under very general assumptions on the N-functions. Here the differential operator takes the form $\mathbf{a}(\frac{x}{\varepsilon},\nabla u)$ and we investigate the passage to the limit when $\varepsilon\to 0$. The growth and the coercivity of the elliptic operator is assumed to be described by a condition of type (1.2), related to (1.4). In particular, the homogenization process changes the underlying function spaces and the nonlinear elliptic operator at each step, since the governing N-function depends on the spatial variable x.

Further, we consider a large class of problems which arise from the mechanics of non-Newtonian fluids with non-standard rheology. We want to include the phenomena of viscosity changing under various stimuli like shear rate, or a magnetic or electric field. This forces us to use inhomogeneous anisotropic Musielak–Orlicz spaces. Our investigations are directed towards the existence of weak solutions. The system of equations describing incompressible non-Newtonian fluid flow may take the following form

$$\begin{aligned} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S}(x,\mathbf{Du}) + \nabla\pi = \mathbf{f},& \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } (0,T)\times\Omega,& \end{aligned} \tag{1.6}$$

where $\mathbf{u}$ denotes the velocity field of a fluid; π is a pressure; Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary; $T < \infty$; $\mathbf{f}$ is a given body force; and $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the symmetric part of the gradient of the velocity field. The first equation is the momentum equation and the second one is the incompressibility condition. We assume a no-slip boundary condition (zero Dirichlet boundary condition).

In order to close the system we have to state a constitutive relation, rheology, which describes the relation between $\mathbf{S}$ and $\mathbf{Du}$. In our considerations we do not want to assume that $\mathbf{S}$ has only a polynomial structure, which would not suffice to describe the nonstandard behavior of the fluid. Motivated by the significant shear thickening phenomenon we want to investigate the processes where the growth is faster than polynomial and possibly different in various directions of the shear stress. Hence an N-function defining a functional space does not satisfy the Δ_2-condition and is possibly anisotropic. The viscosity of the fluid is not assumed to be constant and it can depend on density and the full symmetric part of the velocity gradient. Therefore we formulate the growth conditions of the stress tensor in an analogous way as in (1.2) or (1.4).

In particular, we investigate, with various degrees of generality of the N-function, the flow of inhomogeneous heat-conducting fluids, which depends also on density and temperature. This means that the above system needs to be supplemented with two equations: balance of mass (the continuity equation) and the heat equation. Moreover, the stress tensor then also depends on density and temperature. The other problem we study is the system describing fluid-structure interaction where the motion of rigid bodies immersed in the fluid is taken into account. Moreover, if the model allows us to skip the convective term, we are able also to consider shear thinning fluids, in which case M^* may not satisfy the Δ_2-condition.

Since our considerations on PDE problems concentrate on growth and coercivity of (1.2) type, we employ Musielak–Orlicz spaces defined by means of an N-function M. Let us emphasize that we do not want to assume that M satisfies the Δ_2-condition or that it is sandwiched between two polynomials. Consequently, we lose a wide range of useful properties of function spaces. The lack of numerous basic properties results in many subtle but deep difficulties which require significantly more sophisticated methods than in the classical case.

An important aspect of a Sobolev-type space related to an N-function M which sets it apart from a classical Sobolev space is the issue of density of regular functions. The classical theorem of Meyers and Serrin [253] tells us that $C^\infty \cap W^{m,p}$ is dense in $W^{m,p}$ in the strong topology for $1 \le p < \infty$. An extension of this fact to the classical, i.e. homogenous and isotropic, Orlicz (or rather Orlicz–Sobolev) spaces was investigated by Gossez [174]. He proved that the related density result holds, however not with respect to the strong, but with respect to the so-called modular topology. An analogous fact in Musielak–Orlicz spaces holds only provided the asymptotic behavior of the modular function is sufficiently balanced, see Section 3.7.

It is worth pointing out that even some partial information on the behavior of an N-function enables us to simplify the tools needed for the proofs of existence of solutions. Knowing that M^* satisfies the Δ_2-condition tells us that the weak sequential stability of a considered PDE problem (i.e. passing from an approximate problem to the solutions of the original problem) can be proved by means of weak-$*$ convergence. However, once we want to relax this assumption, an essential tool that comes into play is an approximation by smooth functions with respect to the modular topology. To show the density of smooth functions in the modular topology it is necessary to specify an appropriate balance of asymptotical behavior of M with respect to small changes of z and big values of $|\xi|$, relating to the log-Hölder continuity of variable exponent or a closeness condition on powers in double phase spaces.

In the case of parabolic problems additional difficulties appear. One of them is the lack of an integration by parts formula, cf. [165] and [123]. Such a tool is essential for testing the equation with a solution and using monotonicity methods. Let us recall the well-known Newton's formula in the Bochner space setting. For $0 \le t_0 < t_1 \le T$ and $q \in (1,\infty)$, $q' = q/(q-1)$ we set $v \in L^q(0,T;X)$, $\partial_t v \in L^{q'}(0,T;X^*)$, where X is a reflexive, separable Banach space and X^* is its dual. Then there exists a Hilbert space H such that $X \subset H = H^* \subset X^*$ and the following formula holds

$$\int_{t_0}^{t_1} \langle \partial_t v, v\rangle_{X^*,X}\,\mathrm{d}t = \frac{1}{2}\|v(t_1)\|_X^2 - \frac{1}{2}\|v(t_0)\|_X^2.$$

To extend this formula to any generalization of classical Orlicz spaces we would essentially need that C^∞-functions are dense in $L_M((0,T)\times\Omega)$ and that

$$L_M((0,T)\times\Omega) = L_M(0,T;L_M(\Omega)).$$

Even for classical Orlicz spaces (homogeneous and isotropic) these hold only in particular cases, e.g. the former only holds if M, M^* satisfy the Δ_2-condition. In order to provide the factorization property we recall the result of [123], which is stated for classical Orlicz spaces with homogeneous and isotropic $M = M(|\xi|)$ and therefore we rather cannot expect a better result for more general N-functions.

Let I be a time interval, $\Omega \subset \mathbb{R}^d$, $M : [0,\infty) \to [0,\infty)$ an N-function, $L_M(I\times\Omega)$ the Orlicz space on $I\times\Omega$, and $L_M(I;L_M(\Omega))$ the vector-valued Orlicz space on I. Then

$$L_M(I\times\Omega) = L_M(I;L_M(\Omega))$$

if and only if there exist constants $k_0, k_1 > 0$ such that

$$k_0 M^{-1}(s)M^{-1}(r) \le M^{-1}(sr) \le k_1 M^{-1}(s)M^{-1}(r) \tag{1.7}$$

for every $s \ge 1/|I|$ and $r \ge 1/|\Omega|$.

One can show that (1.7) means that M must be equivalent to some power p, $1 < p < \infty$. Hence, if (1.7) should hold, very strong assumptions must be satisfied by M. Surely they would force $L_M((0,T)\times\Omega)$ to be separable and reflexive.

Besides the lack of integration by parts formula there are many other obstacles resulting from the general (Orlicz) type of growth of the modular function. Among others we mention the Korn inequality, which is a basic tool in continuum mechanics, providing bounds on the full velocity gradient in terms of its symmetric part. However, in homogenous Orlicz spaces $L_M(\Omega)$ it holds only if M and M^* satisfy the Δ_2-condition. In order to overcome this problem for more general growths we need to construct different types of estimates.

Furthermore, classical results of harmonic analysis are not available in their full strength. For instance, a tool which has already become standard in fluid mechanics, however missing in our setting, is the method of Lipschitz truncations [159], which is widely used to deal with low regularity of gradients of solutions in the convective term. The only available results where the Lipschitz truncations method is applied in the Musielak–Orlicz setting are in the isotropic and homogeneous case where M and M^* satisfy the Δ_2-condition [61] and in variable exponent spaces [116].

A lot of facts which hold in the isotropic case are no longer true in the anisotropic setting, but this is subtle and hard to capture in a brief summary. One of the most preeminent examples is that in a fully anisotropic setting, the meaning of the Sobolev embedding is essentially different than in the isotropic setting. In fact, the anisotropic energy of a gradient of a function is expected to improve integrability of the real-valued function itself. In the case of the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$, $\vec{p} = (p_1,\ldots,p_d)$, besides the obvious embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_m^*}(\Omega)$ with $p_m = \min\{p_1,\ldots,p_d\}$, when $p_m < N$ and p_m^* is a Sobolev conjugate of p_m, that is $p_m^* = Np_m/(N-p_m)$, one can use symmetrization techniques to get

$$W^{1,\vec{p}}(\Omega) \hookrightarrow L^{p_0^*}(\Omega)$$

with p_0 being the harmonic mean of p_is, $p_0 < N$, and p_0^* is a Sobolev conjugate of p_0. This result turns out to be the optimal embedding into an isotropic Orlicz target space. Such an embedding is known for fully anisotropic Orlicz spaces [91], but – due to inhomogeneity – it fails in general Musielak–Orlicz spaces. Let us stress here that we refrain from using these kinds of techniques, taking care, as much as possible, to use straightforward formulations of the involved results.

The goal of this monograph is to systematize the methods available for anisotropic Musielak–Orlicz spaces which are useful in the theory of partial differential equations. To this end we present in detail the analytical tools, stressing the importance and challenges resulting from inhomogeneity, anisotropy, and from relaxing the growth conditions.

Chapter 2
N-Functions

Several PDE problems with solutions in Musielak–Orlicz spaces are described in later chapters of this monograph. As our particular concern lies in the anisotropic and inhomogeneous character of problems, the functional setting needs careful introduction. We shall also collect properties of spaces which in many cases differ essentially from standard Lebesgue and Sobolev spaces. The notion of an N-function provides a foundation to define the function spaces. Its features influence the properties of Musielak–Orlicz spaces and in turn lead to various proof techniques.

As the concept of an N-function plays such an important role, we devote an entire chapter to it. Sections 2.1 and 2.2 rapidly recount the most essential facts. These two sections provide a minimum of knowledge for readers wishing to reach the part directly treating PDEs as soon as possible.

For the readers interested in studying the more subtle properties of N-functions and similar classes of convex functions we provide Section 2.3. We collect there numerous studies on the fine differences between isotropic and anisotropic types of functions. The comparison with the setting of classical Orlicz spaces is important in view of the vast literature on the subject. Some of the results presented in that section do not have a direct application in later chapters of this monograph, but appear to have a significant value for researchers working on regularity aspects of PDEs or harmonic analysis.

2.1 Elementary Facts

N-functions are a special class of convex functions, and thus in the first step we discuss various properties of convex functions that will be useful to us later.

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5_2

2.1.1 Properties of convex functions

Definition 2.1.1 (Convex function). A function $f : \mathbb{R}^d \to \mathbb{R}$ is called a *convex function* if for every $x, y \in \mathbb{R}^d$ and every $t \in [0,1]$ we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Moreover, f is strictly convex if for every $x, y \in \mathbb{R}^d$, such that $x \neq y$, and every $t \in (0,1)$ the above inequality is strict.

The following inequality, although simple, will be one of our most exploited tools.

Lemma 2.1.2 (Discrete Jensen's inequality) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function and $\alpha_i \ge 0$, where $i = 1, \ldots, n$ with $n \in \mathbb{N}$, be such that $\sum_{i=1}^n \alpha_i = 1$. Then for any $x_i \in \mathbb{R}^d$,*

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \le \sum_{i=1}^n \alpha_i f(x_i). \tag{2.1}$$

Proof. The proof is by induction with respect to n. For $n = 1$ the statement is obvious. To proceed the induction step we first observe

$$\sum_{i=1}^{n+1} \alpha_i x_i = \alpha_{n+1} x_{n+1} + \sum_{i=1}^n \alpha_i x_i = \alpha_{n+1} x_{n+1} + \left(\sum_{i=1}^n \alpha_i\right) \sum_{i=1}^n \frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)} x_i.$$

Obviously

$$\frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)} \ge 0 \quad \text{and} \quad \sum_{i=1}^n \frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)} = 1.$$

Convexity of f implies that

$$f\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) \le \alpha_{n+1} f(x_{n+1}) + \left(\sum_{i=1}^n \alpha_i\right) f\left(\sum_{i=1}^n \frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)} x_i\right).$$

Using the induction hypothesis on the right-hand side of the above inequality yields

$$f\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) \le \alpha_{n+1} f(x_{n+1}) + \left(\sum_{i=1}^n \alpha_i\right) \sum_{i=1}^n \frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)} f(x_i) = \sum_{i=1}^{n+1} \alpha_i f(x_i)$$

and the proof is complete. □

Remark 2.1.3. Let $U \subset \operatorname{conv}\{x_1, \ldots, x_n\}$, i.e., $\forall x \in U, x = \sum_{i=1}^n \alpha_i x_i$ for $\alpha_i \ge 0$ and $\sum_{i=1}^n \alpha_i = 1$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a convex function, then

$$\sup_{x \in U} f(x) \le \max_{i \in \{1, \ldots, n\}} f(x_i).$$

Lemma 2.1.4 *A convex function $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous, i.e. it is Lipschitz continuous on every compact subset of $\mathbb{R}^d$.*

Proof. To show that f is Lipschitz on every compact subset of $\mathbb{R}^d$ it suffices to show that it holds on a closed ball $B(r)$ centered at the origin with an arbitrary radius $r > 0$. Set $x, y \in B(r)$, and

$$z = x + \left(\tfrac{1}{\alpha} - 1\right)(x - y) \tag{2.2}$$

with

$$\alpha = \tfrac{|x-y|}{|x-y|+r} < 1. \tag{2.3}$$

Observe that

$$|z| \le |x| + |\tfrac{1}{\alpha} - 1| \cdot |x - y| = |x| + \tfrac{r}{|x-y|} \cdot |x - y| \le 2r$$

and thus $z \in B(2r)$. Moreover from (2.2) it immediately follows that

$$z = \tfrac{1}{\alpha}x - (\tfrac{1}{\alpha} - 1)y$$

and consequently

$$x = \alpha z + (1-\alpha)y \qquad \text{with} \qquad \alpha \in (0,1),$$

which follows from (2.3). As f is convex we have

$$f(x) = f(\alpha z + (1-\alpha)y) \le \alpha f(z) + (1-\alpha) f(y) = f(y) + \alpha(f(z) - f(y)). \tag{2.4}$$

We set

$$K := \sup_{x \in B(2r)} f(x) - \inf_{x \in B(2r)} f(x). \tag{2.5}$$

Suppose first that K is bounded (we will momentarily check that this condition does indeed hold). Thus for every $z, y \in B(2r)$ we have $f(z) - f(y) \le K$. Using this observation and (2.3) in (2.4) we get the following

$$f(x) \le f(y) + \alpha K = f(y) + K\tfrac{|x-y|}{|x-y|+r} \le f(y) + \frac{K}{r}|x - y|.$$

Thus $f(x) - f(y) \le \frac{K}{r}|x - y|$. Since the role of x and y is symmetric, we infer that

$$|f(x) - f(y)| \le \frac{K}{r}|x - y|. \tag{2.6}$$

To complete the proof we only need to show that K defined by (2.5) is bounded. Observe firstly that for every ball there exist $d+1$ points $x_i, i = 1, \ldots, d+1$ such that the ball is contained in $\operatorname{conv}\{x_1, \ldots, x_{d+1}\}$. Then by Jensen's inequality (Lemma 2.1.2) we obtain that for $q \in \operatorname{conv}\{x_1, \ldots, x_{d+1}\}$ we have

$$f(q) = f\left(\sum_{i=1}^{d+1} \omega_i x_i\right) \le \sum_{i=1}^{d+1} \omega_i f(x_i) \le (d+1) \max_{i=1,\ldots,d+1} f(x_i) < C,$$

where $\omega_i \geq 0$, $i = 1, \dots, d+1$, $\sum_{i=1}^{d+1} \omega_i = 1$. In particular, for every $q \in B(2r)$ we have $f(q) \leq C$ and thus $\sup_{x \in B(2r)} f(x) \leq C$. This bound together with a simple argument using the convexity of f

$$2f(0) \leq f(x) + f(-x) \implies f(0) - f(x) \leq f(-x) - f(0)$$

allows us to conclude that

$$\begin{aligned} -\inf_{x \in B(2r)} \{f(x) - f(0)\} &= \sup_{x \in B(2r)} \{f(0) - f(x)\} \leq \sup_{x \in B(2r)} \{f(-x) - f(0)\} \\ &= \sup_{x \in B(2r)} \{f(x)\} - f(0). \end{aligned}$$

Thus

$$\inf_{x \in B(2r)} \{f(x)\} - f(0) \geq -\sup_{x \in B(2r)} \{f(x)\} + f(0),$$

which implies that

$$\inf_{x \in B(2r)} f(x) \geq -\sup_{x \in B(2r)} \{f(x)\} + 2f(0) > -\infty,$$

and the argument is complete. □

For a geometrical interpretation of some properties of convex functions it is useful to recall the notion of an epigraph.

Definition 2.1.5 (Epigraph). The *epigraph* of $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\operatorname{epi} f := \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R} : \ f(x) \leq \alpha\}.$$

Using the above definition, one formulates a useful characterization of convexity.

Proposition 2.1.6 *A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if its epigraph* epi f *is a convex set.*

Proof. Assume that f is a convex function and consider two points $(x, \alpha), (y, \beta) \in \operatorname{epi} f$. We thus want to show that for all $t \in (0, 1)$

$$t(x, \alpha) + (1-t)(y, \beta) \in \operatorname{epi} f. \tag{2.7}$$

Since the points belong to the graph, and since f is convex, the following estimates hold

$$t\alpha + (1-t)\beta \geq t f(x) + (1-t) f(y) \geq f(tx + (1-t)y), \tag{2.8}$$

which directly gives the conclusion (2.7).

Let us now assume that epi f is a convex set. As the points $(x, f(x))$ and $(y, f(y))$ for any $x, y \in \mathbb{R}^d$ obviously belong to epi f, we also have for all $t \in (0, 1)$

$$t(x, f(x)) + (1-t)(y, f(y)) \in \operatorname{epi} f, \tag{2.9}$$

or equivalently

$$(tx + (1-t)y, t f(x) + (1-t) f(y)) \in \operatorname{epi} f. \tag{2.10}$$

This means, by definition, that $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$ and thus f is convex. □

Definition 2.1.7 (Lower semi-continuity). A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *lower semi-continuous at* $x \in \mathbb{R}^d$ if for every $x_n \to x$ it holds that $\liminf_{n\to\infty} f(x_n) \ge f(x)$. We say that a function is *lower semi-continuous* if it is lower semi-continuous at every point of its domain.

Lemma 2.1.8 *A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semi-continuous if and only if its epigraph is closed.*

Proof. Suppose f is lower semi-continuous and $(x_n, y_n) \in \operatorname{epi} f$ is such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ for $n \to \infty$. Then for every n we have $y_n \ge f(x_n)$ and

$$\bar{y} = \liminf_{n\to\infty} y_n \ge \liminf_{n\to\infty} f(x_n) \ge f(\bar{x}).$$

Therefore, $(\bar{x}, \bar{y}) \in \operatorname{epi} f$ and $\operatorname{epi} f$ is closed.

To prove the converse, assume that $\operatorname{epi} f$ is closed. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to \bar{x}$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f(x_{n_k}) = \liminf_{n\to\infty} f(x_n).$$

Then $(x_{n_k}, f(x_{n_k})) \to (\bar{x}, \lim_{k\to\infty} f(x_{n_k}))$ for $k \to \infty$ and since $\operatorname{epi} f$ is closed, we conclude that

$$(\bar{x}, \lim_{k\to\infty} f(x_{n_k})) \in \operatorname{epi} f$$

and, by definition,

$$f(\bar{x}) \le \lim_{k\to\infty} f(x_{n_k}) = \liminf_{n\to\infty} f(x_n),$$

which means that f is lower semi-continuous. □

Let us stress an easy, but fundamental fact.

Lemma 2.1.9 *Suppose $\{f_n\}_n$ is a family of convex functions, $f_n : \mathbb{R}^d \to [0,\infty)$. Moreover, assume that* $\sup_n f_n(x) < \infty$ *for each $x \in \mathbb{R}^d$. Then*

(i) $x \mapsto \sup_n f_n(x)$ *is a convex function,*
(ii) $x \mapsto \inf_n f_n(x)$ *may fail to be convex.*

Proof. The case (i) follows directly from the definition. Indeed, for every $x, y \in \mathbb{R}^d$ and every $t \in [0,1]$

$$\begin{aligned}\sup_n f_n(tx+(1-t)y) &\le \sup_n (tf_n(x)+(1-t)f_n(y)) \\ &\le t\sup_n f_n(x)+(1-t)\sup_n f_n(y).\end{aligned}$$

For (ii) it suffices to consider two different linear functions. □

Definition 2.1.10. By an *affine minorant* of $f : \mathbb{R}^d \to \mathbb{R}$ we mean any affine function $g : \mathbb{R}^d \to \mathbb{R}$ such that $g(x) \le f(x)$ for every $x \in \mathbb{R}^d$. We define $\operatorname{am}(f)$ – the set of all affine minorants of f and $E_f(x) := \sup_{g\in\operatorname{am}(f)} g(x)$.

Lemma 2.1.11 *If $f : \mathbb{R}^d \to [0,\infty)$ is convex, then $E_f = f$. Moreover, $E_f(x) = \max_{g\in \mathrm{am}(f)} g(x)$.*

Proof. Consider the set $C = \{(x,\alpha) : f(x) < \alpha\}$ and $\bar{x} \in \mathbb{R}^d$ such that $(\bar{x}, f(\bar{x})) \notin C$. Continuity of the function f, provided by Lemma 2.1.4, allows us to conclude that C is an open set. Let $v = (-\xi, -t)$ be a functional on $\mathbb{R}^{d+1}$ given by Theorem 8.30 applied to the set C and the point $(\bar{x}, f(\bar{x})) \notin C$. We consider the hyperplane

$$A = \{(x,y) \in \mathbb{R}^{d+1} : \xi x + ty = \xi\bar{x} + tf(\bar{x})\}. \tag{2.11}$$

In view of this fact, when we take $\gamma \in \mathbb{R}$ such that $(\bar{x},\gamma) \in C$, we have

$$-\xi\bar{x} - t\gamma < -\xi\bar{x} - tf(\bar{x}), \tag{2.12}$$

thus $t\gamma > tf(\bar{x})$, which implies that necessarily $t > 0$. As

$$\xi x + t\gamma > \xi\bar{x} + tf(\bar{x}) \tag{2.13}$$

holds for all $\gamma > f(x)$ and the graph of $f(x)$ may be approximated by a sequence of elements from C, it follows that for every $x \in \mathbb{R}^d$ we have

$$\xi x + tf(x) \geq \xi\bar{x} + tf(\bar{x}),$$

which, since $t > 0$, may also be written as

$$f(x) \geq -\tfrac{\xi}{t}x + \tfrac{\xi\bar{x}+tf(\bar{x})}{t} =: g(x).$$

It is immediate to verify that $f(\bar{x}) = g(\bar{x})$ and g is affine. Since $\bar{x} \in \mathbb{R}^d$ was chosen arbitrarily, the proof is complete. □

We are in position to prove Jensen's inequality involving a probability measure.

Theorem 2.1.12 (Jensen's inequality, general) *Suppose μ is a probability measure on $\mathbb{R}^N$, while $f : \mathbb{R}^d \to \mathbb{R}$ is convex. If $\xi : \mathbb{R}^N \to \mathbb{R}^d$ is μ-integrable, then*

$$f\left(\int_{\mathbb{R}^N} \xi(y)\, \mathrm{d}\mu(y)\right) \leq \int_{\mathbb{R}^N} f(\xi(y))\, \mathrm{d}\mu(y).$$

Proof. Let $\xi_0 = \int_{\mathbb{R}^N} \xi(y)\, \mathrm{d}\mu(y) \in \mathbb{R}^d$. The convexity of f implies that there exists an affine minorant $g : \mathbb{R}^d \to \mathbb{R}$ such that $f(\xi_0) = g(\xi_0)$ and $g(\xi) = \eta\cdot\xi + b$ for some $\eta \in \mathbb{R}^d, b \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^d$, see Lemma 2.1.11. Therefore $f(\xi) \geq \eta\cdot\xi + b$ for every $\xi \in \mathbb{R}^d$ and $f(\xi_0) = \eta\cdot\xi_0 + b$. Consequently,

$$\begin{aligned} f\left(\int_{\mathbb{R}^N} \xi(y)\, \mathrm{d}\mu(y)\right) &= f(\xi_0) = \eta\cdot\xi_0 + b = \eta\cdot\int_{\mathbb{R}^N} \xi(y)\, \mathrm{d}\mu(y) + b\int_{\mathbb{R}^N} \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^N} (\eta\cdot\xi(y) + b)\, \mathrm{d}\mu(y) \leq \int_{\mathbb{R}^N} f(\xi(y))\, \mathrm{d}\mu(y). \end{aligned}$$

□

Definition 2.1.13. We define the *subdifferential* $\boldsymbol{\partial} f : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$ of a convex function $f : \mathbb{R}^d \longrightarrow [0,\infty)$ at a point x_0 as

$$\boldsymbol{\partial} f(x_0) := \{y \in \mathbb{R}^d : \quad f(x) - f(x_0) \geq y \cdot (x - x_0) \quad \text{for all } x \in \mathbb{R}^d\}.$$

Remark 2.1.14. Directly from the definition it follows that $0 \in \boldsymbol{\partial} f(x_0)$ if and only if f attains a minimum in x_0.

Lemma 2.1.15 *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function. Then $\boldsymbol{\partial} f(x_0)$ is a nonempty, convex and closed set for every $x_0 \in \mathbb{R}^d$.*

Proof. First we will prove that $\boldsymbol{\partial} f(x)$ is nonempty. With this aim, observe that the convexity of f implies that there exists an affine minorant $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}^d$ it holds that $f(x) \geq g(x)$ and $f(x_0) = g(x_0)$. Each such affine function can be written as $g(x) = y(x - x_0) + f(x_0)$ for some $y \in \mathbb{R}^d$ and thus

$$f(x) - f(x_0) \geq y(x - x_0),$$

hence we have found an element of the set $\boldsymbol{\partial} f(x_0)$.

To infer convexity, observe that for $y_1, y_2 \in \boldsymbol{\partial} f(x_0)$ and every $t \in [0,1]$ we have

$$t(f(x) - f(x_0)) \geq t y_1 (x - x_0)$$

and

$$(t-1)(f(x) - f(x_0)) \geq (t-1) y_2 (x - x_0).$$

Adding these two inequalities yields

$$f(x) - f(x_0) \geq (t y_1 + (1-t) y_2)(x - x_0),$$

which means that $t y_1 + (1-t) y_2 \in \boldsymbol{\partial} f(x_0)$, and thus $\boldsymbol{\partial} f(x_0)$ is convex.

To show that $\boldsymbol{\partial} f(x_0)$ is closed we only need to consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \longrightarrow y$ for $n \longrightarrow \infty$ and

$$f(x) - f(x_0) \geq y_n (x - x_0),$$

where passing to the limit we get that $y \in \boldsymbol{\partial} f(x_0)$. □

Lemma 2.1.16 *Assume $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is convex and for $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ it holds that $x_n \longrightarrow x$ for $n \longrightarrow \infty$. Suppose $g_n \in \boldsymbol{\partial} f(x_n)$ is such that $g_n \longrightarrow g$ for $n \longrightarrow \infty$. Then $g \in \boldsymbol{\partial} f(x)$.*

Proof. Fix $y \in \mathbb{R}^d$ and notice that by lower semicontinuity of f we have

$$f(y) \geq \liminf_{n\to\infty} f(x_n) + \liminf_{n\to\infty} g_n \cdot (y - x_n) \geq f(x) + g \cdot (y - x). \qquad \square$$

Lemma 2.1.17 *If $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then there exists a uniquely defined $\boldsymbol{\partial} f^0(x) \in \boldsymbol{\partial} f(x)$ such that for every $g \in \boldsymbol{\partial} f(x)$ it holds that $|\boldsymbol{\partial} f^0(x)| \leq |g|$.*

Proof. Since by Lemma 2.1.15 $\boldsymbol{\partial} f(x)$ is a nonempty, convex, and closed set, and $|\cdot|^2$ is strictly convex, we know that there exists a unique solution to the problem $\inf_{g\in\boldsymbol{\partial} f(x)} |g|^2$. □

Lemma 2.1.18 *If $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then $\boldsymbol{\partial} f(x)$ is monotone.*

Proof. Let $x, y \in \mathbb{R}^d$ and $g_x \in \boldsymbol{\partial} f(x)$, $g_y \in \boldsymbol{\partial} f(y)$. By definition of subdifferential we have

$$f(y) \geq f(x) + g_x \cdot (y - x) \quad \text{and} \quad f(x) \geq f(y) + g_y \cdot (x - y).$$

When we add these inequalities, we get that

$$(g_y - g_x) \cdot (y - x) \geq 0,$$

which is the desired monotonicity formula. □

Definition 2.1.19. The *Moreau–Yosida approximation* of a convex function $f : \mathbb{R}^d \to \mathbb{R}$ with an index $\lambda > 0$ is defined as

$$f_\lambda(x) := \inf_{y \in \mathbb{R}^d} \{ \tfrac{1}{2\lambda} |x - y|^2 + f(y) \}. \tag{2.14}$$

For any $\lambda > 0$ and any $x \in \mathbb{R}^d$ by $J_\lambda(x)$ we denote the point where the function $y \mapsto \frac{1}{2\lambda}|x - y|^2 + f(y)$ attains its minimum. Then J_λ is the resolvent of the maximal monotone operator $\boldsymbol{\partial} f$, i.e. $J_\lambda x = (I + \lambda \boldsymbol{\partial} f)^{-1} x$.

Lemma 2.1.20 *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then for any $\lambda > 0$ the Moreau–Yosida approximation f_λ of f satisfies the following properties:*

(i) *Let us define $A_\lambda(x) := \frac{1}{\lambda}(x - J_\lambda(x))$. Then A_λ is Lipschitz continuous with a Lipschitz constant $\frac{1}{\lambda}$ and $\nabla f_\lambda(x) = A_\lambda(x)$ for all $x \in \mathbb{R}^d$.*
(ii) *The function f_λ is convex.*
(iii) *If $\lambda \searrow 0$ then $f_\lambda \nearrow f$.*
(iv) *For every $x \in \mathbb{R}^d$ it holds that $\nabla f_\lambda(x) \to \boldsymbol{\partial} f^0(x)$ as $\lambda \to 0$, where $\boldsymbol{\partial} f^0(x)$ is an element of minimal norm of the closed convex set $\boldsymbol{\partial} f(x)$.*

Proof. Note that the infimum of $y \mapsto \frac{1}{2\lambda}|x - y|^2 + f(y)$ is attained at a point $\bar{y}$ where

$$\tfrac{1}{\lambda}(\bar{y} - x) + \boldsymbol{\partial} f(\bar{y}) \ni 0.$$

Since by the definition $\bar{y} = J_\lambda(x)$, we deduce that

$$-\tfrac{1}{\lambda}(J_\lambda(x) - x) \in \boldsymbol{\partial} f(J_\lambda(x)). \tag{2.15}$$

(i) In order to prove that A_λ is Lipschitz with Lipschitz constant $\frac{1}{\lambda}$, let us take arbitrary $x, y \in \mathbb{R}^d$. Then

$$x - J_\lambda(x) \in \lambda \boldsymbol{\partial} f(J_\lambda(x)) \quad \text{and} \quad y - J_\lambda(y) \in \lambda \boldsymbol{\partial} f(J_\lambda(y)).$$

Our aim now is to show that $x \mapsto J_\lambda(x)$ is Lipschitz. By Lemma 2.1.18 the subdifferential $\boldsymbol{\partial} f$ is monotone, thus we have that

$$0 \leq \Big(\big(x - J_\lambda(x)\big) - \big(y - J_\lambda(y)\big) \Big) \cdot (J_\lambda(x) - J_\lambda(y)) =: I^1.$$

On the other hand, we have

$$0 \le \Big(\big(x - J_\lambda(x)\big) - \big(y - J_\lambda(y)\big)\Big) \cdot \Big(\big(x - J_\lambda(x)\big) - \big(y - J_\lambda(y)\big)\Big) =: I^2.$$

By adding the last two inequalities we get that

$$0 \le I^2 \le I^1 + I^2 = \Big(\big(x - J_\lambda(x)\big) - \big(y - J_\lambda(y)\big)\Big) \cdot (x - y).$$

Hence

$$|(x - J_\lambda(x)) - (y - J_\lambda(y))| \le |x - y| \tag{2.16}$$

and consequently

$$|J_\lambda(x) - J_\lambda(y)| \le 2|x - y|.$$

Furthermore, by (2.16)

$$|A_\lambda(x) - A_\lambda(y)| = |\tfrac{1}{\lambda}(x - J_\lambda(x)) - \tfrac{1}{\lambda}(y - J_\lambda(y))| \le \tfrac{1}{\lambda}|x - y|, \tag{2.17}$$

which means that A_λ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$.

To prove that $A_\lambda(x) = \nabla f_\lambda(x)$ it suffices to show that

$$\lim_{y \to x} \frac{f_\lambda(y) - f_\lambda(x) - (y - x) \cdot A_\lambda(x)}{|x - y|} = 0. \tag{2.18}$$

Since f is convex and we have (2.15), it holds that

$$f(J_\lambda(y)) - f(J_\lambda(x)) \ge \tfrac{1}{\lambda}(x - J_\lambda(x)) \cdot (J_\lambda(y) - J_\lambda(x)). \tag{2.19}$$

Therefore for any $x, y \in \mathbb{R}^d$ by definition of J_λ we have

$$\begin{aligned} f_\lambda(y) - f_\lambda(x) &= f(J_\lambda(y)) + \tfrac{1}{2\lambda}|J_\lambda(y) - y|^2 - f(J_\lambda(x)) - \tfrac{1}{2\lambda}|J_\lambda(x) - x|^2 \\ &\ge \tfrac{1}{2\lambda}\Big(2(x - J_\lambda(x)) \cdot (J_\lambda(y) - J_\lambda(x)) + |J_\lambda(y) - y|^2 - |J_\lambda(x) - x|^2\Big). \end{aligned} \tag{2.20}$$

Since

$$\begin{aligned} 2(x - J_\lambda(x)) \cdot (J_\lambda(y) - J_\lambda(x)) &= 2(y - x) \cdot (x - J_\lambda(x)) \\ &\quad + 2\Big((J_\lambda(y) - y) - (J_\lambda(x) - x)\Big) \cdot \Big((J_\lambda(y) - y) - (J_\lambda(x) - x)\Big) \\ &\quad + 2\Big((J_\lambda(y) - y) - (J_\lambda(x) - x)\Big) \cdot (y - J_\lambda(y)) \\ &= 2(y - x) \cdot (x - J_\lambda(x)) + \Big|(J_\lambda(y) - y) - (J_\lambda(x) - x)\Big|^2 \\ &\quad - |J_\lambda(y) - y|^2 + |J_\lambda(x) - x|^2, \end{aligned}$$

we can continue estimating from (2.20) to get

$$\begin{aligned} f_\lambda(y) - f_\lambda(x) &\ge \tfrac{1}{2\lambda}\Big(2(y - x) \cdot (x - J_\lambda(x)) + \Big|(J_\lambda(y) - y) - (J_\lambda(x) - x)\Big|^2\Big) \\ &\ge \tfrac{1}{\lambda}(y - x) \cdot (x - J_\lambda(x)) = (y - x) \cdot A_\lambda(x). \end{aligned} \tag{2.21}$$

By changing the role of the variables x and y we have that also $f_\lambda(x) - f_\lambda(y) \geq \frac{1}{\lambda}(x-y)\cdot(y - J_\lambda(y))$ and, consequently, we have

$$f_\lambda(y) - f_\lambda(x) \leq \tfrac{1}{\lambda}(y-x)\cdot(y - J_\lambda(y)). \tag{2.22}$$

On the other hand, by the parallelogram law we note that for any $a, b \in \mathbb{R}^N$ we have

$$a\cdot(a-b) = \tfrac{1}{2}\big(|a|^2 + |a-b|^2 - |b|^2\big). \tag{2.23}$$

Combining (2.21), (2.22), (2.23) for $a = x-y$ and $b = J_\lambda(y) - J_\lambda(x)$, and (2.16) we get

$$\begin{aligned}0 &\leq f_\lambda(y) - f_\lambda(x) - \tfrac{1}{\lambda}(y-x)(x - J_\lambda(x)) \leq \tfrac{1}{\lambda}(y-x)(y - J_\lambda(y) - x + J_\lambda(x))\\ &= \tfrac{1}{2\lambda}\Big(|y-x|^2 + \big|(y - J_\lambda(y)) - (x - J_\lambda(x))\big|^2 - |J_\lambda(x) - J_\lambda(y)|^2\Big) \leq \tfrac{1}{\lambda}|y-x|^2,\end{aligned}$$

from which we infer (2.18).

(ii) As we know that $\nabla f_\lambda(x) = A_\lambda(x)$, formula (2.21) implies convexity of f_λ.

(iii) Let $y \in \boldsymbol{\partial} f(x)$. Then by the definition of $J_\lambda(x)$, we see that $\frac{1}{\lambda}(x - J_\lambda x) \in \boldsymbol{\partial} f(J_\lambda(x))$. Thus, by monotonicity of $\boldsymbol{\partial} f$ we infer that for any $y \in \boldsymbol{\partial} f(x)$ we have

$$\big(y - \tfrac{1}{\lambda}(x - J_\lambda(x))\big)\cdot(x - J_\lambda(x)) > 0.$$

Therefore,

$$\tfrac{1}{\lambda}\big((x - J_\lambda(x))\big)\cdot(x - J_\lambda x) \leq y\cdot(x - J_\lambda(x))$$

and

$$\tfrac{1}{\lambda}|x - J_\lambda(x)|^2 \leq |y|\,|x - J_\lambda(x)|.$$

Then $|A_\lambda(x)| = |\frac{1}{\lambda}(x - J_\lambda x)| \leq |y|$ for any $y \in \boldsymbol{\partial} f(x)$. In turn, by Lemma 2.1.17, there exists an element $\boldsymbol{\partial} f^0(x)$ of minimal norm of $\boldsymbol{\partial} f(x)$, such that

$$|A_\lambda(x)| \leq |\boldsymbol{\partial} f^0(x)| \quad \text{for any } x \in \mathbb{R}^d \tag{2.24}$$

and thus

$$|x - J_\lambda(x)| \leq \lambda|\boldsymbol{\partial} f^0(x)|. \tag{2.25}$$

Consequently for any $x \in \mathbb{R}^d$ we have

$$J_\lambda(x) \longrightarrow x \text{ as } \lambda \longrightarrow 0. \tag{2.26}$$

We notice that for $\lambda_1 < \lambda_2$ we have

$$\begin{aligned}f_{\lambda_1}(x) &= f(J_{\lambda_1}(x)) + \tfrac{1}{2\lambda_1}|x - J_{\lambda_1}(x)|^2\\ &\geq f(J_{\lambda_1}(x)) + \tfrac{1}{2\lambda_2}|x - J_{\lambda_1}(x)|^2\\ &\geq \inf_{y\in\mathbb{R}^d}\Big\{f(y) + \tfrac{1}{2\lambda_2}|x-y|^2\Big\} = f_{\lambda_2}(x).\end{aligned}$$

Moreover, by definition of f_λ we see that $f_\lambda(x) \le f(x) + \frac{1}{2\lambda}|x-x|^2 = f(x)$. Therefore, $\{f_\lambda(x)\}_\lambda$ is convergent for $\lambda \searrow 0$. By lower semicontinuity of f and (2.26) we have

$$\begin{aligned}\liminf_{\lambda\searrow 0} f_\lambda(x) &= \liminf_{\lambda\searrow 0}\left\{f(J_\lambda(x)) + \tfrac{1}{2\lambda}|x - J_\lambda(x)|^2\right\}\\ &\ge \liminf_{\lambda\searrow 0} f(J_\lambda(x)) \ge f(x).\end{aligned}$$

(iv) We take $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \searrow 0$ and by (2.24) we infer that $\{A_{\lambda_n}\}_{n\in\mathbb{N}}$ is a bounded sequence. We choose a subsequence $\{A_{\lambda_{n_k}}\}_{k\in\mathbb{N}}$ convergent to some $A^\infty \in \mathbb{R}^d$. Moreover, by (2.26) we get that $J_{\lambda_{n_k}}(x) \to x$ and $A_{\lambda_{n_k}}(x) \in \boldsymbol{\partial} f(J_{\lambda_{n_k}}(x))$. Therefore, Lemma 2.1.16 enables us to deduce that $A^\infty \in \boldsymbol{\partial} f(x)$. Since (2.24) implies that $|A^\infty| \le |\boldsymbol{\partial} f^0(x)|$, by Lemma 2.1.17 we conclude that $A^\infty = \boldsymbol{\partial} f^0(x)$. □

2.1.2 Carathéodory functions

As we intend later to work on inhomogeneous problems, which means that the considered convex function additionally depends on a variable $z \in Z \subset \mathbb{R}^N$, we introduce the notion of Carathéodory functions.

Definition 2.1.21 (Carathéodory function). Let $Z \subset \mathbb{R}^N$. A function

$$M : Z \times \mathbb{R}^d \to [0,\infty)$$

is called a *Carathéodory function* if $z \mapsto M(z,\xi)$ is measurable for every ξ and $\xi \mapsto M(z,\xi)$ is continuous for a.a. $z \in Z$.

Lemma 2.1.22 *If $M : Z \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory function and $\xi : Z \to \mathbb{R}^d$ is measurable, then the composition $z \mapsto M(z,\xi(z))$ is measurable.*

Proof. Firstly we will prove the assertion for simple functions. Assume thus that ξ can be written as follows

$$\xi = \sum_{k=1}^m v_k \mathbb{1}_{E_k},$$

for some $m \in \mathbb{N}$ and measurable disjoint sets $E_k \subset Z$, $k = 1,\ldots,m$, such that $\bigcup_{k=1}^m E_k = \Omega$, and $v_k \in \mathbb{R}^d$. We notice that for every $t \in \mathbb{R}$

$$\big\{z \in Z : M(z,\xi(z)) > t\big\} = \bigcup_{k=1}^m \big\{z \in E_k : M(z,v_k) > t\big\}.$$

The measurability of $M(z,v_k)$ for every fixed v_k implies that the right-hand side is a measurable set, and hence, so is the left-hand side. Therefore $z \mapsto M(z,\xi(z))$ is also measurable.

In order to deal with the case of measurable ξ, recall that every measurable function ξ can be approximated by simple functions ξ_k in the sense that

$$M(z,\xi_k(z)) \xrightarrow[k\to\infty]{} M(z,\xi(z)) \quad \text{for a.a. } z \in Z.$$

Finally, the right-hand side is measurable as an almost everywhere limit of measurable functions. □

Most often we focus on Carathéodory functions which additionally satisfy

$$\begin{aligned} &M(z,\xi) = 0 \iff \xi = 0 \\ \text{and} \quad &\xi \mapsto M(z,\xi) \text{ is even and convex for a.a. } z \in Z. \end{aligned} \tag{2.27}$$

Such functions share the following properties.

Lemma 2.1.23 *Let $Z \subset \mathbb{R}^N$. For a Carathéodory function $M : Z \times \mathbb{R}^d \to [0,\infty)$ satisfying* (2.27) *the following conditions hold:*

(i) *For a fixed $\xi \in \mathbb{R}^d$ and $\varepsilon \in [0,1]$*

$$M(z,\varepsilon\xi) \le \varepsilon M(z,\xi). \tag{2.28}$$

(ii) *For a fixed $\xi \in \mathbb{R}^d$ and $\alpha > 1$*

$$\alpha M(z,\xi) \le M(z,\alpha\xi). \tag{2.29}$$

(iii) *If a continuous function $m : [0,\infty) \to [0,\infty)$ satisfies* (2.27) *with $M(z,\xi) = m(|\xi|)$, then m is strictly monotone.*

Proof. As $M(z,0) = 0$, the first statement for $\varepsilon \in [0,1]$ follows immediately from convexity

$$M(z,\varepsilon\xi) \le \varepsilon M(z,\xi) + (1-\varepsilon)M(z,0) = \varepsilon M(z,\xi).$$

In the same manner we show (ii) in the case $\alpha > 1$

$$M(z,\xi) \le \left(1 - \tfrac{1}{\alpha}\right) M(z,0) + \tfrac{1}{\alpha} M(z,\alpha\xi). \tag{2.30}$$

In view of (ii) it is easy to verify that (iii) holds. Indeed, let $s_1, s_2 \in [0,\infty)$ and $s_1 < s_2$. Thus there exists an $\alpha > 1$ such that $s_2 = \alpha s_1$ and

$$m(s_1) < \alpha m(s_1) \le m(\alpha s_1) = m(s_2).$$ □

As a corollary of Theorem 2.1.12 we obtain Jensen's inequality for inhomogeneous functions. By considering a measure absolutely continuous with respect to the Lebesgue measure and having density ϱ, we get the following version.

Corollary 2.1.24 *Suppose $Z, U \subset \mathbb{R}^N$ are open bounded sets and $M : Z \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory function convex with respect to the second variable. Let $\varrho \in L^1(U)$, $\varrho \ge 0$, be such that $\int_U \varrho(x)\,\mathrm{d}x = 1$ and let $\xi : \mathbb{R}^d \to \mathbb{R}^d$ be integrable with weight ϱ. Then for a.a. $z \in Z$*

$$M\left(z, \int_U \xi(y)\,\varrho(y)\,\mathrm{d}y\right) \le \int_U M(z,\xi(y))\,\varrho(y)\,\mathrm{d}y.$$

In the sequel a typical choice of function ϱ is the standard regularizing kernel, i.e. $\varrho \in C^\infty(\mathbb{R}^N)$, $\operatorname{supp}\varrho \subset\subset B(0,1)$ and $\int_{\mathbb{R}^N} \varrho(x)\,\mathrm{d}x = 1$, $\varrho(x) = \varrho(-x)$.

From Lemma 2.1.2 we conclude a discrete Jensen's inequality for Carathéodory functions. The following version has a slightly different formulation, which will often be used later in many estimates. Comparing it with Lemma 2.1.2 observe that $\alpha_i = \frac{\lambda_i}{\lambda}$, where α_i comes from Lemma 2.1.2 and λ and λ_i are the quantities that appear in the proceeding corollary.

Corollary 2.1.25 *Suppose $Z \subset \mathbb{R}^N$ is an open bounded set and $M : Z\times\mathbb{R}^d \to [0,\infty)$ is a Carathéodory function convex with respect to the second variable. Let a vector $\xi \in \mathbb{R}^d$ be decomposed as*

$$\xi = \sum_{i=1}^n \lambda_i \xi^i \qquad \textit{and} \qquad \lambda = \sum_{i=1}^n \lambda_i$$

with some $\xi^i \in \mathbb{R}^d$ and $\lambda_i > 0$ for every $i = 1,\dots,n$, and $n \in \mathbb{N}$. Then for a.a. $z \in Z$

$$M\left(z, \frac{\xi}{\lambda}\right) \leq \sum_{i=1}^n \frac{\lambda_i}{\lambda} M\left(z,\xi^i\right).$$

For Carthéodory functions $M : Z\times\mathbb{R}^d \to [0,\infty)$ which are convex with respect to the second variable we shall employ subdifferentials with respect to this second variable, freezing the dependence on the first variable. To highlight this, a notation analogous to partial derivatives with lower index ξ is used. We thus define a subdifferential $\boldsymbol{\partial}_\xi M : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ of a Carathéodory function in the same way as for convex functions, here for a.a. $z \in Z$ as a subdifferential of a function $\xi \mapsto M(z,\xi)$, i.e.

$$\boldsymbol{\partial}_\xi M(z,\xi_0) := \{\eta \in \mathbb{R}^d : \quad M(z,\xi) - M(z,\xi_0) \geq \eta\cdot(\xi-\xi_0) \quad \text{for all } \ \xi \in \mathbb{R}^d\}. \tag{2.31}$$

In the next lemma we use the notion of Moreau–Yosida approximation introduced by Definition 2.1.19. Here, in the context of Carathéodory functions, the Moreau–Yosida approximation is only with respect to the second variable, i.e.

$$M_\lambda(z,\xi) := \inf_{\eta\in\mathbb{R}^d} \{\tfrac{1}{2\lambda}|\xi-\eta|^2 + M(z,\eta)\}. \tag{2.32}$$

The properties prescribed in Lemma 2.1.20 also hold for a.a. $z \in Z$ for a function $\xi \mapsto M(z,\xi)$ and thus we will not repeat them here. The only fact that we want to pay attention to, and that is used later, is the issue of measurability.

Lemma 2.1.26 *Suppose $Z \subset \mathbb{R}^N$ is an open bounded set and $M : Z\times\mathbb{R}^d \to [0,\infty)$ is a Carathéodory function convex with respect to the second variable. For any $\lambda > 0$ by M_λ we mean the Moreau–Yosida approximation of M. Then $z \mapsto M_\lambda(z,\xi)$ and $z \mapsto \nabla_\xi M_\lambda(z,\xi)$ are measurable functions for all $\xi \in \mathbb{R}^d$.*

Proof. In the first step we will show that $z \mapsto M_\lambda(z,\xi)$ is measurable. Observe that

$$M_\lambda(z,\xi) = \inf_{\eta\in\mathbb{R}^d} \{\tfrac{1}{2\lambda}|\xi-\eta|^2 + M(z,\eta)\} = \inf_{\eta\in\mathbb{Q}^d} \{\tfrac{1}{2\lambda}|\xi-\eta|^2 + M(z,\eta)\}.$$

The result is now clear, since an infimum of a countable family of measurable functions is also measurable.

Observe that $z \mapsto \nabla_\xi M_\lambda(z,\xi)$ is measurable if and only if $z \mapsto \frac{\partial}{\partial \xi_i} M_\lambda(z,\xi)$ is measurable for all $i \in \{1,\dots,d\}$. For M_λ we have

$$\frac{\partial}{\partial \xi_i} M_\lambda(z,\xi) = \lim_{h\to 0} \frac{M_\lambda(z,\xi+he_i) - M_\lambda(z,\xi)}{h}.$$

Moreover, $z \mapsto \frac{M_\lambda(z,\xi+he_i)-M_\lambda(z,\xi)}{h}$ is measurable. As the pointwise limit of measurable functions is measurable, $z \mapsto \frac{\partial}{\partial \xi_i} M_\lambda(z,\xi)$ is measurable and the proof is complete. □

Lemma 2.1.27 *Let $Z \subset \mathbb{R}^N$ be an open bounded set, $\xi : Z \to \mathbb{R}^d$ be measurable and $M : Z \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory convex function. Then $z \mapsto \boldsymbol{\partial}_\xi M^0(z,\xi(z))$ is measurable, where $\boldsymbol{\partial}_\xi M^0(z,\xi(z))$ is an element of minimal norm of $\boldsymbol{\partial}_\xi M(z,\xi(z))$.*

Proof. Recall again that the notation M_λ is used for the Moreau–Yosida approximation, see (2.32). By Lemma 2.1.26 we know that $z \mapsto \nabla_\xi M_\lambda(z,\xi)$ is a measurable function for all $\xi \in \mathbb{R}^d$, and Lemma 2.1.20 yields that $\xi \mapsto \nabla_\xi M_\lambda(z,\xi)$ is continuous. Thus $z \mapsto \nabla_\xi M_\lambda(z,\xi(z))$ is measurable as a composition of a measurable function and a Carathéodory function, see Lemma 2.1.22. Since, again by Lemma 2.1.20, $\nabla_\xi M_\lambda(z,\xi(z)) \to \boldsymbol{\partial}_\xi M^0(z,\xi(z))$ a.e. as $\lambda \to 0$, it follows that $z \mapsto \boldsymbol{\partial}_\xi M^0(z,\xi(z))$ is also measurable. □

2.1.3 The conjugate function

The fundamental role in the analysis of the Musielak–Orlicz setting is played by the conjugate function, often also called the complementary function, the Young conjugate function, or the Legendre transform.

Definition 2.1.28 (Conjugate function). The *conjugate function* $M^* : Z \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ to a Carathéodory function $M : Z \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$M^*(z,\eta) := \sup_{\xi \in \mathbb{R}^d} (\xi \cdot \eta - M(z,\xi)), \qquad \text{for } \eta \in \mathbb{R}^d \text{ and a.a. } z \in Z.$$

Remark 2.1.29. If $M(z,0) = 0$ for a.a. $z \in Z$, then $M^* : Z \times \mathbb{R}^d \to [0,\infty]$. Indeed, $M^*(z,\eta) = \sup_{\xi\in\mathbb{R}^d} (\xi \cdot \eta - M(z,\xi)) \geq \{\eta \cdot 0 - M(z,0)\} = 0$.

In most of the considerations in this section, as well as in the overall setting, we assume that a Carathéodory function is superlinear at infinity, see Definition 8.17. This assumption is particularly useful when talking about conjugate functions due to the following fact.

Lemma 2.1.30 *Let $M : Z \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function. If $\xi \mapsto M(z,\xi)$ is for a.a. $z \in Z$ superlinear at infinity and $M(z,0) = 0$ for a.a. $z \in Z$, then the function $\xi \mapsto (\xi \cdot \eta - M(z,\xi))$ attains its maximum and, consequently, M^* is finite-valued.*

Proof. Observe that

$$\xi \cdot \eta - M(z,\xi) \le |\xi| \cdot |\eta| - M(z,\xi) = |\xi| \left(|\eta| - \tfrac{M(z,\xi)}{|\xi|} \right).$$

The right-hand side tends to $-\infty$ as $|\xi|$ tends to ∞. Thus these two properties: continuity of $\xi \mapsto (\xi \cdot \eta - M(z,\xi))$ and $\lim_{|\xi|\to\infty} (\xi \cdot \eta - M(z,\xi)) = -\infty$ together with Remark 2.1.29 imply that the function $\eta \mapsto \sup_{\xi \in \mathbb{R}^d} \{\xi \cdot \eta - M(z,\xi)\}$ attains its maximum and thus M^* is finite-valued. □

Remark 2.1.31. If $\xi \mapsto M(z,\xi)$ is convex and superlinear at infinity, then $\xi \mapsto (-\xi \cdot \eta + M(z,\xi))$ is also convex and $\xi \mapsto (\xi \cdot \eta - M(z,\xi))$ is concave. Note that in analogy to Remark 2.1.14 it holds that $\xi_0 \in \arg\max_{\xi \in \mathbb{R}^d} (\xi \cdot \eta - M(z,\xi))$ whenever $0 \in \boldsymbol{\partial}_\xi (-\xi_0 \cdot \eta + M(z,\xi_0))$, which is equivalent to $\eta \in \boldsymbol{\partial}_\xi M(z,\xi_0)$. Recall that $\boldsymbol{\partial}_\xi$ denotes the subdifferential with respect to the variable ξ defined in (2.31).

Lemma 2.1.32 (Fenchel–Young inequality) *If M is a Carathéodory function and M^* its conjugate, the following inequality holds*

$$\xi \cdot \eta \le M(z,\xi) + M^*(z,\eta) \qquad \text{for all } \xi,\eta \in \mathbb{R}^d \text{ and a.a. } z \in Z. \tag{2.33}$$

Proof. Directly from the definition of the conjugate function (Definition 2.1.28) we get

$$\begin{aligned} \xi \cdot \eta &= M(z,\xi) + \xi \cdot \eta - M(z,\xi) \le M(z,\xi) + \sup_{\zeta \in \mathbb{R}^d} (\zeta \cdot \eta - M(z,\zeta)) \\ &= M(z,\xi) + M^*(z,\eta). \end{aligned}$$

□

Remark 2.1.33. Suppose $\xi \mapsto M(z,\xi)$ is convex and superlinear at infinity. Then, by the arguments of Remark 2.1.31, the equality in (2.33) holds for any η belonging to the subdifferential $\boldsymbol{\partial}_\xi M(z,\xi_0)$. If additionally $\xi \mapsto M(z,\xi)$ is differentiable, then the subdifferential $\boldsymbol{\partial}_\xi M$ is single-valued and equal to $\{\nabla_\xi M\}$ – the set consisting of a gradient with respect to the variable ξ. Consequently, the equality in (2.33) holds for $\eta = \nabla_\xi M(z,\xi_0)$, that is

$$\xi_0 \cdot \nabla_\xi M(z,\xi_0) = M(z,\xi_0) + M^*(z, \nabla_\xi M_z(\xi_0)) \qquad \text{for all } \xi_0 \in \mathbb{R}^d.$$

Lemma 2.1.34 *Let $M : Z \times \mathbb{R}^d \to [0,\infty)$ be a Carathéodory function such that $\xi \mapsto M(z,\xi)$ is superlinear at infinity for a.a. $z \in Z$. Then the conjugate function to M is convex with respect to the second variable, i.e. $M^*(z,\cdot)$ is convex for a.a. $z \in Z$.*

Proof. Let $s \in [0,1]$. Observe that

$$\begin{aligned} M^*(z, s\eta_1 + (1-s)\eta_2) &= \sup_{\zeta \in \mathbb{R}^d} \{(s\eta_1 + (1-s)\eta_2) \cdot \zeta - (s + 1 - s) M(z,\zeta)\} \\ &= \sup_{\zeta \in \mathbb{R}^d} \{s\eta_1 \cdot \zeta - sM(z,\zeta) + (1-s)\eta_2 \cdot \zeta - (1-s)M(z,\zeta)\} \\ &\le sM^*(z,\eta_1) + (1-s)M^*(z,\eta_2). \end{aligned}$$

□

Lemma 2.1.35 *Let $M : Z\times\mathbb{R}^d \to [0,\infty)$ be a Carathéodory function such that $\xi\mapsto M(z,\xi)$ is superlinear at infinity for a.a. $z\in Z$. Then the conjugate function to M is also a Carathéodory function.*

Proof. The definition of M^* directly implies $M^*(z,0)=0$ for a.a. $z\in Z$. First observe that $z\mapsto \big(\eta\cdot\xi - M(z,\xi)\big)$ is measurable. Furthermore, by the density argument, $M^*(z,\eta)=\sup_{\xi\in\mathbb{Q}^d}\big(\eta\cdot\xi - M(z,\xi)\big)$. Hence, as a supremum of a countable family of measurable functions, $z\mapsto M^*(z,\eta)$ is also measurable. Since M^* is finite-valued due to Remark 2.1.30 and $M^*(z,\cdot)$ is convex for a.a. $z\in Z$ due to Lemma 2.1.34, the function $\eta\mapsto M^*(z,\eta)$ is locally Lipschitz for a.a. $z\in Z$ (see Lemma 2.1.4), and hence continuous. □

Lemma 2.1.36 *Let $M : Z\times\mathbb{R}^d \to [0,\infty)$ be a Carathéodory function such that $\xi\mapsto M(z,\xi)$ is superlinear at infinity for a.a. $z\in Z$ and which is even with respect to the second variable. Then the conjugate function to M is also even with respect to the second variable for a.a. $z\in Z$, i.e. $M^*(z,\eta)=M^*(z,-\eta)$ for a.a. $z\in Z$.*

Proof. We have

$$
\begin{aligned}
M^*(z,\eta) &= \sup_{\zeta\in\mathbb{R}^d}\{-\eta\cdot\zeta - M(z,-\zeta)\}\\
&= \sup_{\zeta\in\mathbb{R}^d}\{(-\eta)\cdot\zeta - M(z,\zeta)\} = M^*(z,-\eta).
\end{aligned}
$$
□

Lemma 2.1.37 *Let $M_1,\ M_2 : Z\times\mathbb{R}^d \to [0,\infty)$ be Carathéodory functions for which $\xi\mapsto M_1(z,\xi)$ and $\xi\mapsto M_2(z,\xi)$ are superlinear at infinity for a.a. $z\in Z$. If for a.a. $z\in Z$ and all $\xi\in\mathbb{R}^d$ we have*

$$
M_1(z,\xi)\le M_2(z,\xi), \tag{2.34}
$$

then

$$
M_2^*(z,\eta)\le M_1^*(z,\eta) \tag{2.35}
$$

for every $\eta\in\mathbb{R}^d$ and a.a. $z\in Z$.

Proof. If $M_1(z,\xi)\le M_2(z,\xi)$, then for every $\eta\in\mathbb{R}^d$

$$
\xi\cdot\eta - M_2(z,\xi)\le \xi\cdot\eta - M_1(z,\xi).
$$

We take the supremum on the both sides to get the assertion. □

Remark 2.1.38. Suppose $\xi\mapsto M(z,\xi)$ is convex and superlinear at infinity. Then for every $\xi_0\in\mathbb{R}^d$ and $\eta\in\boldsymbol{\partial}_\xi M(z,\xi_0)$ we have

$$
M^*(z,\eta)\le \xi_0\cdot\eta
$$

and

$$
M^*(z,\eta)\le 2M(z,2\xi_0).
$$

Indeed, when we fix $\eta\in\boldsymbol{\partial}_\xi M(z,\xi_0)$, by the Remarks 2.1.31, 2.1.33 and Lemma 2.1.32 we get

$$M^*(z,\eta) = \eta\cdot\xi_0 - M(z,\xi_0) \le \eta\cdot\xi_0 \le M(z,2\xi_0) + M^*(z,\tfrac{1}{2}\eta)$$
$$\le M(z,2\xi_0) + \tfrac{1}{2}M^*(z,\eta),$$

where the last inequality is justified by Jensen's inequality. Now it suffices to rearrange terms to get the claim.

2.1.4 The second conjugate function

Let us now consider the second conjugate of a Carathéodory function M

$$M^{**}(z,\xi) = (M^*(z,\xi))^*, \tag{2.36}$$

so the conjugate of the conjugate of M.

Lemma 2.1.39 *For any Carathéodory function $M : Z\times\mathbb{R}^d \to \mathbb{R}$ the second conjugate function M^{**} is convex with respect to ξ and we have*

$$M^{**}(z,\xi) \le M(z,\xi) \quad \text{for all } \xi\in\mathbb{R}^d \text{ and a.a. } z\in Z.$$

Proof. The first conjugate M^* is already convex with respect to the second variable as a supremum of affine functions. Therefore, the second conjugate is convex as well. Moreover, we have for a.a. $z\in Z$ that

$$\begin{aligned}
M^{**}(z,\xi) &= \sup_\eta \{\xi\cdot\eta - M^*(z,\eta)\}\\
&= \sup_{\eta\in\mathbb{R}^d,\ a\in\mathbb{R}} \{\xi\cdot\eta - a,\ \text{such that}\ a \ge M^*(z,\eta)\}\\
&= \sup_{\eta\in\mathbb{R}^d,\ a\in\mathbb{R}} \left\{\xi\cdot\eta - a,\ \text{such that}\ a \ge \sup_\zeta\,(\eta\cdot\zeta - M(z,\zeta))\right\}\\
&= \sup_{\eta\in\mathbb{R}^d,\ a\in\mathbb{R}} \left\{\xi\cdot\eta - a,\ \text{such that}\ a \ge \eta\cdot\zeta - M(z,\zeta) \quad \forall\,\zeta\in\mathbb{R}^d\right\}\\
&\le \sup_{\eta\in\mathbb{R}^d,\ a\in\mathbb{R}} \{\xi\cdot\eta - a,\ \text{such that}\ \xi\cdot\eta - a \le M(z,\xi)\}\\
&\le M(z,\xi).
\end{aligned}$$

Hence, M^{**} is a convex minorant of M. □

We give below a trivial corollary of the convexity of the second conjugate and the lack of convexity of the infimum of convex functions, which however seems to be a surprisingly frequent mistake in the literature.

Corollary 2.1.40 *When M is convex with respect to the second variable, then due to Lemma 2.1.9, we infer that*

(i) $\operatorname{ess\,inf}_{z\in Z} M(z,\xi)$ *in general is not convex and Jensen's inequality does not apply;*
(ii) $\xi \mapsto \left(\operatorname{ess\,inf}_{z\in Z} M(z,\xi)\right)^{**}$ *is convex and Jensen's inequality can be applied.*

Theorem 2.1.41 (Fenchel–Moreau) *If a Carathéodory function* $M : Z \times \mathbb{R}^d \to [0,\infty)$ *is convex with respect to the second variable, then*

$$M^{**}(z,\xi) = M(z,\xi).$$

Note that we assume here that M takes only nonnegative values purely for the simplicity of the presentation and because this is the most general form of the result we shall need later on.

Proof. Having Lemma 2.1.39 it suffices to prove that $M^{**}(z,\xi) \geq M(z,\xi)$. Take an arbitrary affine minorant of $\xi \mapsto M_z(\xi) = M(z,\xi)$, namely an affine function $f_z(\xi) = \eta \cdot \xi + b$, such that $f_z(\xi) \leq M_z(\xi)$ for all $\xi \in \mathbb{R}^d$. By definition $\eta \cdot \xi - M_z(\xi) \leq -b$. Then $M_z^*(\eta) = \sup_\xi(\eta \cdot \xi - M_z(\xi)) \leq -b$ and for all $\xi \in \mathbb{R}^d$

$$\eta \cdot \xi - M_z^*(\eta) \geq \eta \cdot \xi + b = f_z(\xi).$$

Hence, $M_z^{**}(\xi) = (M_z^*)^*(\xi) = \sup_\eta(\eta \cdot \xi - M_z^*(\eta)) \geq f_z(\xi)$. By Lemma 2.1.11 a convex and lower semicontinuous function is equal to a supremum over its affine minorants, so we conclude that $M^{**}(z,\xi) \geq M(z,\xi)$ and the proof is complete. □

Corollary 2.1.42 *For any Carathéodory function* $M : Z \times \mathbb{R}^d \to \mathbb{R}$ *the second conjugate function* M^{**} *is its greatest convex minorant.*

Proof. By Lemma 2.1.39 we know that M^{**} is a convex minorant of M. We prove that it is the greatest one by contradiction. We suppose that there exists a convex function $\bar{M} \not\equiv M^{**}$, for which

$$M^{**}(z,\xi) \leq \bar{M}(z,\xi) \leq M(z,\xi) \quad \text{for a.e. } z \in Z.$$

Due to Lemma 2.1.37, for every fixed z we have

$$M^*(z,\xi) \leq \left(\bar{M}(z,\xi)\right)^* \leq (M^{**}(z,\xi))^* = M^*(z,\xi),$$

where we used Theorem 2.1.41. Consequently, $M^* \equiv \bar{M}^*$. Again by Theorem 2.1.41 we infer

$$\bar{M}^{**}(z,\xi) = \bar{M}(z,\xi) = M^{**}(z,\xi).$$

This, however, contradicts with the choice of $\bar{M}$ and, consequently, M^{**} has to be the greatest convex function smaller than or equal to M. □

2.2 Definition of an N-Function

Having introduced convex functions, our main exposition now focuses on inhomogeneous and anisotropic functions, which are at the foundation of the definition of function spaces. To introduce them we first define a Young function.

Definition 2.2.1 (Young function). A function $m : [0,\infty) \to [0,\infty)$ is called a *Young function* if it satisfies the following conditions:

1. $m(s) = 0 \iff s = 0$.
2. m is convex.
3. m is superlinear at zero and at infinity, i.e.

$$\lim_{s\to 0^+} \frac{m(s)}{s} = 0 \qquad \text{and} \qquad \lim_{s\to\infty} \frac{m(s)}{s} = \infty.$$

Even though most texts call such a mapping an N-function, we reserve this name for a z-dependent and anisotropic function, which is the most important object for the presented theory. The definition presented above agrees with many textbooks, see e.g. [221, 244], we are however aware that the name Young function is sometimes used in the literature for a more general notion than the one here, see the definition and the bibliographical note in [281, Section 1.3], where, following Young's original works, it is understood as a convex function $m : [0,\infty) \to [0,\infty)$ satisfying $m(0) = 0$ and $\lim_{s\to\infty} m(s) = \infty$.

We are now ready to define an N-function.

Definition 2.2.2 (N-function). Suppose $Z \subset \mathbb{R}^N$ is a bounded connected set. A function $M : Z\times\mathbb{R}^d \to [0,\infty)$ is called an *N-function* if it satisfies the following conditions:

1. M is a Carathéodory function (i.e. measurable with respect to z and continuous with respect to the second variable);
2. $M(z,0) = 0$ and $\xi \mapsto M(z,\xi)$ is a convex function for a.a. $z \in Z$;
3. $M(z,\xi) = M(z,-\xi)$ for a.a. $z \in Z$ and all $\xi \in \mathbb{R}^d$;
4. there exist two Young functions $m_1, m_2 : [0,\infty) \to [0,\infty)$ such that for a.a. $z \in Z$

$$m_1(|\xi|) \le M(z,\xi) \le m_2(|\xi|). \tag{2.37}$$

We say that a Carathéodory function $M : Z\times\mathbb{R}^d \to [0,\infty)$ is inhomogeneous and anisotropic, where

- *inhomogeneity* means dependence on the spatial variable $z \in Z$,
- *anisotropy* means dependence on ξ, not necessarily via $|\xi|$.

Remark 2.2.3 (Notation). Note that if an N-function is homogeneous (independent of z) and isotropic, then the above definition reduces to the definition of a Young function (Definition 2.2.1). In order to stress the difference, we shall denote Young functions by lower case letters (e.g. $m, m_i, \overline{m}$) and upper case letters for general N-functions. Nonetheless, we sometimes allow some ambiguity and call both m and M an N-function, even though they are defined on different domains.

Lemma 2.2.4 *If M is an N-function, then the conjugate function M^* is also an N-function.*

Proof. Since M is an N-function, $M(z,0) = 0$ for a.a. $z \in Z$ and M is a Carathéodory function, then the definition of the conjugate M^* (Definition 2.1.28) directly implies $M^*(z,0) = 0$ for a.a. $z \in Z$. The fact that M^* is a Carathéodory function is motivated in Lemma 2.1.35. Symmetry is provided in Lemma 2.1.36, convexity of M^* is justified by Lemma 2.1.34. Finally, Lemma 2.1.37 ensures that the conjugate is trapped between m_1^* and m_2^*. □

An important characteristic of an N-function is its rate of growth. If this growth is moderate, a significant part of the analytical background presented in the next chapter is a rather straightforward extension of structures well-known for L^p spaces. This growth is prescribed by the so-called Δ_2-condition, which indeed comprises the core for various useful properties of function spaces and operators. For the PDE problems considered in this monograph an overall impediment will be that the Δ_2-condition is not assumed.

Definition 2.2.5 (Δ_2-condition). We say that an N-function $M : Z\times\mathbb{R}^d \to [0,\infty)$ satisfies the Δ_2-*condition* (denoted $M \in \Delta_2$) if there exists a constant $c_{\Delta_2} > 0$ and a nonnegative integrable function $h : Z \to \mathbb{R}$ such that

$$M(z,2\xi) \le c_{\Delta_2} M(z,\xi) + h(z) \qquad \text{for a.a. } z \in Z \text{ and all } \xi \in \mathbb{R}^d. \tag{2.38}$$

Remark 2.2.6. One also finds for an N-function M the so-called Δ_2-*condition far from the origin* (denoted $M \in \Delta_2^\infty$), which means that there exists a $c_0 \ge 0$ such that (2.38) holds for all $\xi \in \mathbb{R}^d$ with $|\xi| \ge c_0$. However if Z is a bounded set, this condition is equivalent to the Δ_2-condition. Indeed, for $|\xi| < c_0$ we can estimate $M(z,2\xi) \le m_2(2c_0)$ and thus the function h can be modified $\tilde{h}(z) := h(z)+m_2(2c_0)$. Notice that $\tilde{h}$ is also an integrable function.
Sometimes the Δ_2-condition is understood to mean (2.38) with $h \equiv 0$ to distinguish it from Δ_2^∞.

2.3 Refined Properties of N-Functions

This section provides a deeper insight into properties of N-functions and collects numerous examples which illustrate them. Particular attention is paid here to delicate differences between isotropic and anisotropic functions.

2.3.1 Examples of N-functions

Let us present some examples of N-functions with links to subsections of Section 3.8.1 briefly describing their applications to PDEs and the calculus of variations. The main model function captured by Definition 2.2.2 is

$$M_0(z,\xi) = |\xi|^p, \quad 1 < p < \infty.$$

Then the simplest choice of m_1 and m_2 is $m_1(|\xi|) = M_0(z,\xi) = m_2(|\xi|)$ for all $\xi \in \mathbb{R}^d$. It will be explained in further chapters that the Musielak–Orlicz space generated by M_0 and its Sobolev-type version are the classical Lebesgue spaces $L^p(Z)$ and Sobolev space $W^{1,p}(Z)$, respectively. See Section 3.8.1 for more details on the fact that an example of an operator whose growth is governed by M_0 is the classical p-Laplacian. One can consider the so-called Zygmund functions

$$M_1(z,\xi) = |\xi|^p \log^\alpha(1+|\xi|),$$

where

$$1 < p < \infty \text{ and } \alpha \in \mathbb{R} \qquad \text{or} \qquad p = 1 \text{ and } \alpha > 0,$$

when again taking $m_1(|\xi|) = M_1(z,\xi) = m_2(|\xi|)$ is allowed. For more information on the Orlicz and Orlicz–Sobolev spaces generated by such functions and related differential operators, see Section 3.8.1.5.

2.3.1.1 Inhomogeneity

Since by inhomogeneity we mean z-dependence of M, the basic inhomogeneous example is

$$M_2(z,\xi) = |\xi|^{p(z)}, \quad 1 < p_- \le p(\cdot) \le p_+ < \infty,$$

where $p : Z \longrightarrow [1,\infty)$ is a measurable function. As a supremum of convex functions is always convex, it is allowed to take $m_2(|\xi|) = \sup_{z\in Z} M_2(z,\xi)$ (unless it blows up for a finite argument). It is possible to take

$$m_1(|\xi|) = \begin{cases} |\xi|^{p_+} & \text{if } |\xi| < t_1, \\ m_a(|\xi|) & \text{if } t_1 \le |\xi| \le t_2, \\ |\xi|^{p_-} & \text{if } |\xi| > t_2, \end{cases} \quad \text{and} \quad m_2(|\xi|) = \begin{cases} |\xi|^{p_-} & \text{if } |\xi| \le 1, \\ |\xi|^{p_+} & \text{if } |\xi| > 1, \end{cases} \tag{2.39}$$

where t_1, t_2 and an affine function m_a are chosen to ensure that m_1 is convex and $m_1(|\xi|) \le \min\{|\xi|^{p_-}, |\xi|^{p_+}\}$. In fact, one can take

$$t_1 = \left(\frac{1}{p_+}\right)^{\frac{1}{p_+-1}} \left[\frac{(p_+-1){p_-}^{p_-/(p_--1)}}{(p_--1){p_+}^{p_+/(p_+-1)}}\right]^{\frac{p_--1}{p_+-p_-}}$$

and

$$t_2 = \left(\frac{1}{p_-}\right)^{\frac{1}{p_--1}} \left[\frac{(p_+-1){p_-}^{p_-/(p_--1)}}{(p_--1){p_+}^{p_+/(p_+-1)}}\right]^{\frac{p_+-1}{p_+-p_-}}.$$

An affine function m_a crossing points $(t_1, t_1^{p_+})$ and $(t_2, t_2^{p_-})$ is given by a formula

$$m_a(t) = t\frac{t_2^{p_+} - t_1^{p_-}}{t_2 - t_1} + \frac{t_1^{p_-} - t_2^{p_+}}{t_2 - t_1}\frac{t_2 + t_1}{2} + \frac{t_1^{p_-} + t_2^{p_+}}{2}.$$

Then $m_a(t) \le \min\{|\xi|^{p_-}, |\xi|^{p_+}\}$ and m_1 defined in (2.39) is convex.

Some studies concern the related function of variable exponent type

$$M_3(z,\xi) = \frac{1}{p(z)}\left((1+|\xi|^2)^{\frac{p(z)}{2}} - 1\right), \quad 1 < p_- \le p(\cdot) \le p_+ < \infty,$$

which is also an N-function according to Definition 2.2.2 and leads to analysis in the same functional space as M_2. For more information on the variable exponent case, see Section 3.8.1.3.

Mixing the above ideas of logarithmically perturbed growth and varying the first variable leads us to investigate the following N-function

$$M_4(z,\xi) = |\xi|^{p(z)} \log^{\alpha(z)}(1+|\xi|), \quad 1 < p_- \le p(\cdot) \le p_+ < \infty, \ 0 \le \alpha(\cdot) \in L^\infty(Z),$$

where p, α are measurable and scalar functions. The functions m_1 and m_2 can be found in a similar way as in (2.39). We give more information on the space generated by M_4 in Section 3.8.1.6.

Another important function falling into the realm of Definition 2.2.2 is

$$M_5(z,\xi) = |\xi|^p + a(z)|\xi|^q, \quad 1 < p < q < \infty, \ 0 \le a(\cdot) \in L^\infty(Z).$$

In this case one can take $m_1(|\xi|) = |\xi|^p$ and $m_2(|\xi|) = |\xi|^p + \|a\|_{L^\infty}|\xi|^q$. Properties of M_5 and related N-functions are described in Section 3.8.1.4 together with applications.

As Definition 2.2.2 does not restrict our attention to functions growing more slowly than a polynomial, we can consider

$$M_6(z,\xi) = |\xi|\left(\mathrm{e}^{|\xi|^{p(z)}} - 1\right), \quad \text{where } 1 \le p(\cdot) \in L^\infty(Z).$$

For more information on this setting, see Section 3.8.1.6.

2.3.1.2 Anisotropy

The examples provided in the previous section illustrate what we understand by inhomogeneity of an N-function. Here we explain what anisotropy means. Let us recall that we say $M(x,\xi)$ is anisotropic if it is a function of ξ but not necessarily of $|\xi|$. In the isotropic setting (namely when $M(z,\xi) = M(z,|\xi|)$) we have the following integral representation

$$M(z,s) = \int_0^s M^\bullet(z,r)\,\mathrm{d}r \tag{2.40}$$

with a nondecreasing function $M^\bullet : Z \times [0,\infty) \to [0,\infty)$ called the *density* of M. The basic example of an inhomogeneous and anisotropic function satisfying (2.27) is

$$M(z,\xi) = \sum_{i=1}^d |\xi^i|^{p_i(z)}, \qquad 1 < p_- \le p_i(\cdot) \le p_+ < \infty \ \text{ for } \ i \in \{1,\ldots,d\}.$$

However, an anisotropic function is *not* necessarily described by its behavior in each direction separately. A function M which admits a decomposition

$$M(z,\xi) = \sum_{i=1}^d M_i(z,\xi^i), \quad \xi = (\xi^1,\ldots,\xi^d) \in \mathbb{R}^d, \quad M_i : Z \times \mathbb{R} \to [0,\infty), \tag{2.41}$$

is called *orthotropic*. If M fails to admit such a decomposition, we call it *fully anisotropic*. The classical two-dimensional example of a fully anisotropic function provided by Trudinger in [316] is

$$M(z,\xi) = |\xi^1 - \xi^2|^\alpha + |\xi^1|^\beta \log^\delta(c + |\xi^1|), \qquad \alpha,\beta \geq 1,$$

where $\delta \in \mathbb{R}$ if $\beta > 1$, or $\delta > 0$ if $\beta = 1$, with $c > 1$ large enough to ensure convexity.

It should be strongly emphasized here that the family of fully anisotropic functions is far more robust. The strong property of monotonicity of a form

$$\begin{aligned} \text{if} \quad \xi = (\xi^1,\dots,\xi^d),\ \eta = (\eta^1,\dots,\eta^d),\ \text{and}\ |\xi^i| \leq |\eta^i|, \\ \text{then} \quad M(z,\xi) \leq M(z,\eta) \end{aligned} \tag{2.42}$$

fails in general. In fact, it suffices to take $M : Z \times \mathbb{R}^2 \to [0,\infty)$ given by

$$M(z,\xi) = |\xi^1|^2 + |\xi^2|^2 + |\xi^1 - \xi^2|^2 \exp(|\xi^1 - \xi^2|).$$

Indeed, for $(2,0),(3,3) \in \mathbb{R}^2$ we have

$$M(z,(2,0)) = 4(1+\exp(2)) > 20 > 18 = M(z,(3,3)).$$

In [83] there is an example of a function between $|\xi|^p$ and $|\xi|^p \log^\alpha(1+|\xi|)$ ($p > 1$, $\alpha > 0$), for which after any linear and invertible change of variables the orthotropic decomposition is impossible even up to equivalence.

Remark 2.3.1. The decomposition (2.41) and the strong property of monotonicity (2.42) are useful tools, which are not available in general, and which significantly simplify proofs, e.g. of the density of simple functions in the space (cf. Theorem 3.4.11 and Theorem 3.4.16). As a matter of fact, the proofs already simplify when the function M admits an even more general decomposition than (2.41). Suppose $L_j : \mathbb{R}^d \to \mathbb{R}^d$ are linear functions for $j = 1,\dots,D$, $D \geq d$, such that $\mathrm{lin}\{\mathrm{Im}\, L_j\}_{j=1}^D = \mathbb{R}^d$. Then an example of such a decomposition holds provided $M_j : Z \times [0,\infty) \to [0,\infty)$ for every j and

$$M(z,\xi) = \sum_{j=1}^{D} M_j(z,|L_j\xi|). \tag{2.43}$$

As observed above, this type of function does not necessarily satisfy (2.42).

Note that (2.43) captures the situation when M admits a decomposition in directions other than cardinal. Namely, consider an arbitrary basis of $\mathbb{R}^d$, denoted $(\bar{e}_1,\dots,\bar{e}_d)$, write $\xi = (\bar{\xi}_1,\dots,\bar{\xi}_d)$ in the coordinates of this basis, and let M admit the decomposition $M(z,\xi) = \sum_{i=1}^d M_i(z,|\bar{\xi}^i|)$.

2.3.1.3 N-functions satisfying growth conditions

In the available literature, a significant part of research in the Musielak–Orlicz setting so far has been conducted for *doubling* M, i.e. when the Δ_2-condition (see

Definition 2.2.5) is imposed both on M and on M^*. This is sometimes denoted by

$$\Delta_2(\{M, M^*\}) < \infty.$$

This condition implies that both M and M^* are sandwiched between inhomogeneous power-type functions even in the anisotropic case. We want to stress that it is possible that $M \in \Delta_2$, but $M^* \notin \Delta_2$. Indeed,

$$M(z,\xi) = (1+|\xi|)\log(1+|\xi|) - |\xi| \in \Delta_2,$$

but

$$M^*(z,\eta) = \exp(|\eta|) - |\eta| - 1 \notin \Delta_2.$$

Let us present some examples of inhomogeneous and possibly anisotropic modular functions.

Example 2.3.2 (Doubling N-functions).

- $M(z,|\xi|) = |\xi|^{p(z)}$, where $1 < p_- \le p(\cdot) \le p_+ < \infty$; covering the variable exponent case with possibly non-regular exponent;
- $M(z,|\xi|) = |\xi|^{p(z)} \log^{\alpha(z)}(\mathrm{e}+|\xi|)$, where $1 < p_- \le p(\cdot) \le p_+ < \infty$ and $\alpha \ge 0$, or $1 \le p_- \le p(\cdot) \le p_+ < \infty$ and $\alpha(\cdot) \ge \alpha_- > 0$;
- $M(z,\xi) = \sum_i a_i(z)|\xi^i|^{p_i(z)}$, where $1 < (p_i)_- \le p_i(\cdot) \le (p_i)_+ < \infty$, the weight functions $a_i(\cdot) \ge (a_i)_- > 0$ are bounded in Z; this case covers the anisotropic weighted variable exponent case with possibly non-regular exponent;
- $M_1(z,|\xi|) = |\xi|^p + a(z)|\xi|^q$ or $M_2(z,|\xi|) = |\xi|^p + a(z)|\xi|^p \log(e+|\xi|)$, where $1 < p < q < \infty$ and a weight function $a : Z \to [0,\infty)$ is bounded and possibly touching zero; covering the case of the double-phase space;
- $M_1(z,|\xi|) = |\xi|^{p(z)} + a(z)|\xi|^{q(z)}$ or $M_2(z,|\xi|) = |\xi|^{p(z)} + a(z)|\xi|^{p(z)} \log(e+|\xi|)$, where $1 < p_- \le p(\cdot) < q(\cdot) \le q_+ < \infty$ and a weight function $a : Z \to [0,\infty)$ is bounded and possibly touching zero; covering the case of variable exponent double-phase space;
- $M(z,\xi) = M_0(\xi) + \sum_{i=1}^k a_i(z)M_i(\xi)$, $k \in \mathbb{N}$, or $M(z,\xi) = M_0(\xi) + \sum_{i=1}^N a_i(z)M_i(\xi^i)$, where the Orlicz modular functions $M_i, M_i^* \in \Delta_2$, while the weight functions $a_i : Z \to [0,\infty)$ are bounded and possibly touching zero; covering the anisotropic weighted Orlicz case under the most common nonstandard growth conditions.

Example 2.3.3 (Non-doubling N-functions).

- $M(z,\xi) = a(z)\,(\exp(|\xi|) - 1 + |\xi|)$ with a bounded weight $a : Z \to (c,\infty)$, $c > 0$;
- $M(z,\xi) = a(z)|\xi|\log(\mathrm{e}+|\xi|) + b(z)|\xi|^p$ with $1 < p < \infty$ and nonnegative weights $a, b \in L^\infty(Z)$, where b vanishes on a subset of positive measure, but there is no subset of Z of positive measure where both a, b disappear;
- $M(z,\xi) = M_1(\xi) + a(z)M_2(\xi)$ with bounded and possibly touching zero weight $a : Z \to [0,\infty)$ relating to the double phase space, but with $M_i \notin \Delta_2$ for $i = 1$ or $i = 2$. Recall that $M \notin \Delta_2$ can be trapped between two power-type functions;
- $M(z,\xi) = a(z)\big(\exp(|\xi^1|) - 1\big) + |\xi^2| \cdot |\xi|^{p(z)}$, $\xi = (\xi^1, \dots, \xi^d)$ with a bounded and possibly touching zero weight $a : Z \to [0,\infty)$ and variable exponent $1 < p_- \le p(\cdot) \le p_+ < \infty$. This is an example of an anisotropic modular function;

- $M(z,\xi) = a(z)|\xi^1|^{p_1(z)}(1+|\log(1+|\xi|)|)+\exp(|\xi^2|^{p_2(z)})-1$, when $(\xi^1,\xi^2)\in\mathbb{R}^2$ and $p_i : Z \to [1,\infty]$. This is also an example of an anisotropic modular function;
- See the example in Remark 2.3.4 of a non-doubling N-function between $|\xi|^p$ and $|\xi|^q$ for any $1 < p < q < \infty$.

The following example shows that comparison with two power-type functions is not enough for the Δ_2-condition. The following construction comes from [78], another one can be found in [49].

Example 2.3.4. For arbitrary $1 < p < q < \infty$, there exists a continuous, increasing, and convex function $m : [0,\infty) \to [0,\infty)$ which is trapped between power type functions $t \mapsto t^p$ and $t \mapsto t^q$ and does not satisfy the Δ_2-condition, nor (2.54).

We shall construct $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ so that the desired function is given by the following formula

$$m(t) = \begin{cases} \text{affine } t \in (a_i,b_i),\\ t^p \qquad \text{otherwise.}\end{cases}$$

To describe $\{a_i\}_{i\in\mathbb{N}}$ let us introduce yet another sequence $\{k_i\}_{i\in\mathbb{N}}$ and fix $a_i = 2^{k_i}$ for every $i \in \mathbb{N}$. Let $k_1 \in \mathbb{N}$ be large enough to satisfy both

$$k_1 > 2^p \qquad \text{and} \qquad \left(\frac{k_1-1}{q}\right)^{\frac{1}{k_1}} \le 2^{q-p}. \tag{2.44}$$

Define

$$m(t) = 2^{pk_1} + 2^{(p-1)k_1}(k_1-1)(t-2^{k_1}) \qquad \text{for} \qquad t \in (a_1,b_1),$$

where $b_1 > a_1$ is an intersection point of the chord

$$f_1(t) = 2^{pk_1} + 2^{(p-1)k_1}(k_1-1)(t-2^{k_1})$$

and $t \mapsto t^p$. Note that $(2.44)_1$ ensures that

$$2^{pk_1} + 2^{(p-1)k_1}(k_1-1)(2^{k_1+1}-2^{k_1}) = k_1 2^{pk_1} > (2^{k_1+1})^p,$$

so in particular $2^{k_1+1} < b_1$ and $m(2^{k_1+1}) = k_1 2^{pk_1}$. On the other hand, $(2.44)_2$ implies that the slope of the line given by f_1 equals $2^{(p-1)k_1}(k_1-1)$ and is smaller than the derivative of $t \mapsto t^q$ in a_1. Combining it with $t^p|_{2^{k_1}} < t^q|_{2^{k_1}}$ we get that $B(t) < t^q$ on (a_1,b_1).

Let k_2 be the smallest natural number such that $a_2 = 2^{k_2} \ge b_1$ and set $m(t) = t^p$ on (b_1,a_2). We repeat the construction of the chord. Note that since $k_2 > k_1$, the condition (2.44) with k_1 substituted with k_2 is satisfied. Thus, the chord is between $t \mapsto t^p$ and $t \mapsto t^q$. Further iterating the construction we obviously obtain a continuous, increasing, and convex function, whose graph lies between the same power-type functions. Moreover, we also get the sequences $\{a_i\}_i$, $\{b_i\}_i$, and $\{k_i\}_i$ such that $k_i \to \infty$, $2a_i < b_i \le a_{i+1}$ and

$$m(a_i) = a_i^p \qquad \text{and} \qquad m(2a_i) = k_i a_i^p = k_i m(a_i),$$

which contradicts the Δ_2-condition. Moreover, taking $\{y_i\}_{i\in\mathbb{N}}$ with $y_i \in (a_i, b_i)$ one can check that $i_B \le 1$, which violates (2.54).

2.3.2 Conjugation and degeneracy

The real aim of this section is to systematize various conditions formulated in the literature dedicated to inhomogeneous problems. We want to equip the reader with a useful set of tools which will enable them to compare various formulations and better understand the relations among different results.

We concentrate on the relations between nondegeneracy and limit conditions imposed on anisotropic and inhomogeneous functions M and M^*, when they are convex with respect to the second variable. We assemble the following conditions:

(i) nondegeneracy at the origin given by

$$\exists r_0 > 0 \;\; \forall r < r_0 \;\; \exists c(r) > 0 \;\; \forall \xi : \; |\xi| = r \quad \operatorname{ess\,inf}_{z\in Z} M(z,\xi) > c(r); \tag{2.45}$$

(ii) nondegeneracy at infinity reading

$$\exists R_0 > 0 \;\; \forall R > R_0 \;\; \exists C(R) > 0 \;\; \forall \xi : \; |\xi| = R \quad \operatorname{ess\,sup}_{z\in Z} M(z,\xi) < C(R); \tag{2.46}$$

(iii) the limit at the origin

$$\lim_{|\xi|\to 0} \operatorname{ess\,sup}_{z\in Z} \frac{M(z,\xi)}{|\xi|} = 0; \tag{2.47}$$

(iv) the limit at infinity

$$\lim_{|\xi|\to \infty} \operatorname{ess\,inf}_{z\in Z} \frac{M(z,\xi)}{|\xi|} = \infty. \tag{2.48}$$

Remark 2.3.5. Note that $\operatorname{ess\,sup}_{z\in Z} M(z,\xi)$ is a convex function, therefore due to Remark 2.1.3, condition (ii) is equivalent to

$$\forall\, \xi \in \mathbb{R}^d \quad \operatorname{ess\,sup}_{z\in Z} M(z,\xi) < \infty.$$

The interplay between conditions (i)–(iv) imposed on M and M^* is described by the following series of lemmas.

Lemma 2.3.6 *Suppose $M : Z \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory function satisfying* (2.46)*, such that $M(z,0) = 0$ and $\xi \mapsto M(z,\xi)$ is a convex function for a.a. $z \in Z$. Then M^* satisfies* (2.48).

Proof. We prove the claim by contradiction, that is we suppose that (2.48) fails for M^* and get that M cannot be finite-valued in the sense of (2.46).

If (2.48) is false, then there exists a $c_0 > 0$, a sequence of vectors from the unit sphere $\{\zeta_k\}_{k\in\mathbb{N}} \subset S^{d-1}(0,1)$, a sequence of positive numbers $\{a_k\}_{k\in\mathbb{N}}$ satisfying

$a_k \nearrow \infty$ as $k \to \infty$ and a sequence of sets $\{A_k\}_{k\in\mathbb{N}}$ of positive measure, such that

$$M^*(z, a_k\zeta_k) \le c_0|a_k\zeta_k| = c_0 a_k \quad \text{for all } z \in A_k. \tag{2.49}$$

By the compactness of the unit sphere, we can assume without relabeling that a subsequence of $\{\zeta_k\}_{k\in\mathbb{N}}$ converges to $\zeta \in S^{d-1}(0,1)$. Then of course

$$\lim_{k\to\infty} \zeta \cdot \zeta_k = 1.$$

We fix an arbitrary $R > c_0$. Since M is convex with respect to the second variable, due to the Fenchel–Moreau theorem (Theorem 2.1.41), for a.a. $z \in A_k$ we can write

$$M(z, R\zeta) = M^{**}(z, R\zeta) = \sup_{\eta\in\mathbb{R}^d} (R\zeta\cdot\eta - M^*(z,\eta)) \ge R\zeta\cdot(a_k\zeta_k) - M^*(z, a_k\zeta_k).$$

This can be estimated further from below due to (2.49) and the fact that $R > c_0$. We obtain for a.a. $z \in A_k$

$$C(R) > M(z, R\zeta) \ge a_k(R\zeta\cdot\zeta_k - c_0) \ge c\,a_k \quad \text{with some } c = c(R, c_0) > 0,$$

but $a_k \nearrow \infty$, which yields the desired contradiction. □

Lemma 2.3.7 *If $M : Z\times\mathbb{R}^d \to [0,\infty)$ is a Carathéodory function, $M(z,0) = 0$ and $\xi \mapsto M(z,\xi)$ is convex for a.a. $z \in Z$ and* (2.45) *holds for M, then M^* satisfies* (2.47).

Proof. Let us note that for fixed $\xi \in \mathbb{R}^d$ and $t > 0$

$$t \mapsto \frac{M(z, t\xi)}{|t\xi|} \qquad \text{is nondecreasing.}$$

Therefore, as an infimum of nondecreasing functions

$$d(t) = \inf_{\xi:\ |\xi|=1} \operatorname{ess\,inf}_{z\in Z} \frac{M(z,t\xi)}{|t\xi|} \qquad \text{is nondecreasing}$$

and by (2.45) also

$$d(t) \ge \frac{c(t)}{t} > 0.$$

Then for a.a. $z \in Z$ and all $\xi \in \mathbb{R}^d$ we have $M(z,\xi) \ge d(|\xi|)$. We shall consider $M^*(z,\eta)/|\eta|$ for small η. For $\eta, \xi \in \mathbb{R}^d$ such that $|\eta| \le d(R)$ and $|\xi| \ge R$ we have

$$\eta\cdot\xi - M(z,\xi) \le d(R)|\xi| - d(|\xi|)|\xi| = |\xi|(d(R) - d(|\xi|)) \le 0.$$

Therefore, for η with $|\eta| \le d(R)$ it holds that

$$\frac{M^*(z,\eta)}{|\eta|} = \sup_{\xi\in\mathbb{R}^d}\left(\xi\cdot\frac{\eta}{|\eta|} - \frac{M(z,\xi)}{|\eta|}\right) = \sup_{\xi:\ |\xi|\le R} \xi\cdot\frac{\eta}{|\eta|} \le R.$$

If $\bar{d} := \lim_{s\to 0^+} d(s) = 0$, then $d(R) \to 0$ if and only if $R \to 0$ and the proof is complete. If $\bar{d} > 0$, then for η with $|\eta| \le \bar{d}$ we have $M^*(z,\xi) = 0$, so (2.47) for M^* holds. □

Lemma 2.3.8 *If* $M : Z\times\mathbb{R}^d \to [0,\infty)$ *is a Carathéodory function,* $M(z,0) = 0$ *and* $\xi \mapsto M(z,\xi)$ *is convex for a.a.* $z \in Z$ *and* (2.48) *holds for* M*, then* M^* *satisfies* (2.46).

Proof. Using the definition and the Cauchy–Schwarz inequality we get that

$$M^*(z,\eta) = \sup_{\xi\in\mathbb{R}^d} \big(\xi\cdot\eta - M(z,\xi)\big) \le \sup_{\xi\in\mathbb{R}^d} \big(|\xi|\,|\eta| - M(z,\xi)\big) = \sup_{\xi\in\mathbb{R}^d} \left(|\xi| \left[|\eta| - \frac{M(z,\xi)}{|\xi|}\right]\right).$$

From (2.48) we get that for every $\eta \in \mathbb{R}^d$ there exists an $R > 0$ such that

$$\operatorname{ess\,inf}_{z\in Z} \frac{M(z,\xi)}{|\xi|} \ge 2|\eta| \quad \text{for } |\xi| > R$$

and, consequently, for those ξ the expression in the last square brackets above is negative. Therefore, the supremum has to be achieved within the range of ξ such that $|\xi| \le R$. Continuing the above estimations we get

$$M^*(z,\eta) \le \sup_{\xi:\ |\xi|\le R} \left(|\xi| \left[|\eta| - \frac{M(z,\xi)}{|\xi|}\right]\right) \le \sup_{\xi:\ |\xi|\le R} |\xi|\,|\eta| \le R|\eta| < \infty. \qquad \square$$

Lemma 2.3.9 *If* $M : Z\times\mathbb{R}^d \to [0,\infty)$ *is a Carathéodory function,* $M(z,0) = 0$ *and* $\xi \mapsto M(z,\xi)$ *is convex for a.a.* $z \in Z$ *and* (2.47) *holds for* M*, then* M^* *satisfies* (2.45).

Proof. We choose a sequence of positive numbers $\{b_k\}_{k\in\mathbb{N}}$ such that $b_k \searrow 0$ as $k \to \infty$. By definition of the conjugate and fixing arbitrary R we take any ξ with $|\xi| = R$ and we can find $k_0(R)$ large enough such that for all $k > k_0(R)$ we have

$$M^*(z,\xi) \ge b_k|\xi| \left(|\xi| - \frac{M(z,b_k\xi)}{|b_k\xi|}\right) \ge b_k R \left(|\xi| - \operatorname{ess\,sup}_{z\in Z} \frac{M(z,b_k\xi)}{|b_k\xi|}\right) \ge \frac{b_k}{2} R^2.$$

□

Any N-function M satisfies nondegeneracy conditions (2.45), (2.46), (2.47), and (2.48), due to properties of the minorant m_1 and majorant m_2 (see Definition 2.2.2). To state the converse let us note that every even convex Carthéodory function $M : Z\times\mathbb{R}^d \to [0,\infty)$ is trapped between two homogeneous and isotropic convex functions

$$m_1(s) = \Big(\inf_{\eta:\,|\eta|=s} \operatorname{ess\,inf}_{z\in Z} M(z,\eta)\Big)^{**} \quad \text{and} \quad m_2(s) = \sup_{\eta:\,|\eta|=s} \operatorname{ess\,sup}_{z\in Z} M(z,\eta).$$

Moreover, we have the following fact.

Proposition 2.3.10 *Suppose* $M : Z \times \mathbb{R}^d \to [0,\infty)$ *is an even Carathéodory function,* $M(z,0) = 0$ *and* $\xi \mapsto M(z,\xi)$ *is convex for a.a.* $z \in Z$*, then there exist nondecreasing convex functions* $m_1, m_2 : [0,\infty) \to [0,\infty]$*, such that* $m_1(0) = 0 = m_2(0)$ *and for a.a.* $z \in Z$ *and every* $\xi \in \mathbb{R}^d$

$$m_1(|\xi|) \le M(z,\xi) \le m_2(|\xi|).$$

If we assume additionally that M *satisfies* (2.45), (2.46), (2.47) *and* (2.48) *then each* m_i *is an increasing and finite-valued function satisfying*

$$\lim_{s\to 0} \frac{m_i(s)}{s} = 0 \qquad \textit{and} \qquad \lim_{s\to\infty} \frac{m_i(s)}{s} = \infty, \qquad i \in \{1,2\},$$

which means that M *is an* N*-function.*

Proof. Let us define $\underline{m}_1(s) = \inf_{\eta:\,|\eta|=s} \operatorname{ess\,inf}_{z\in Z} M(z,\eta)$. Then $m_1(s) = (\underline{m}_1)^{**}(s)$. Moreover, Lemma 2.1.37 implies that there exist nondecreasing convex functions $m_{1,*}, m_{2,*} : [0,\infty) \to [0,\infty]$ such that $m_{1,*}(0) = 0 = m_{2,*}(0)$ and for a.a. $z \in Z$ and every $\xi \in \mathbb{R}^d$

$$m_{1,*}(|\xi|) \le M^*(z,\xi) \le m_{2,*}(|\xi|),$$

because it suffices to take $m_{1,*} = m_2^*$ and $m_{2,*} = m_1^*$.

Since M satisfies (2.45), we have $m_2(s) > m_1(s) \ge 0$ for every $s > 0$. Notice that since M satisfies (2.46), the functions m_1 and m_2 are finite-valued. Hence, by invoking Lemma 2.3.6 we can conclude that $\lim_{s\to 0} \frac{m_2^*(s)}{s} = \lim_{s\to 0} \frac{m_{1,*}(s)}{s} = 0$ and $\lim_{s\to 0} \frac{m_1^*(s)}{s} = \lim_{s\to 0} \frac{m_{2,*}(s)}{s} = 0$. Directly by (2.45) we have $m_2(s) > 0$ whenever $s > 0$. To prove the same property for m_1 we also use (2.45). In fact, by the argument of the proof of Lemma 2.3.7 we deduce that if $\underline{m}(s) = \inf_{\eta:\,|\eta|=s} \operatorname{ess\,inf}_{z\in Z} M(z,\eta)$, then the function $s \mapsto \underline{m}(s)/s$ is nondecreasing. The degeneracy can occur only close to the origin. For fixed $r_0 > t > s/2 > 0$ we have that

$$\frac{\underline{m}(t)}{t} \ge \frac{\underline{m}\left(\frac{s}{2}\right)}{\frac{s}{2}} \ge c\left(\tfrac{s}{2}\right) > 0.$$

Then

$$\underline{m}(t) \ge t\frac{\underline{m}\left(\frac{s}{2}\right)}{\frac{s}{2}} > \left(t - \tfrac{s}{2}\right)_+ \frac{\underline{m}\left(\frac{s}{2}\right)}{\frac{s}{2}} > 0,$$

so by taking $t = s$ we get $\underline{m}(s) > \underline{m}\left(\frac{s}{2}\right) > 0$. Since $t \mapsto \left(t - \frac{s}{2}\right)_+ \frac{\underline{m}\left(\frac{s}{2}\right)}{\frac{s}{2}}$ is an affine (and thus convex) minorant of $\underline{m}$, whereas $(\underline{m})^{**}$ is its greatest convex minorant (Corollary 2.1.42), we infer that $m_1(s) = (\underline{m})^{**}(s) > \underline{m}\left(\frac{s}{2}\right) > 0$.

As a consequence of (2.47) and Lemma 2.3.9 we get that M^* satisfies (2.45). By the same reasoning as above we conclude that $m_{1,*}, m_{2,*}$ are increasing and each of them vanishes at zero only. Further, Lemma 2.3.7 gives that $\lim_{s\to 0} \frac{m_2(s)}{s} = 0 = \lim_{s\to 0} \frac{m_1(s)}{s}$.

Having (2.48) imposed on M, Lemma 2.3.8 implies that M^* does not degenerate at infinity in the sense of (2.46). Then $m_{1,*} = m_2^*$ and $m_{2,*} = m_1^*$ are finite-valued and Lemma 2.3.6 yields that $\lim_{s\to\infty} \frac{m_1(s)}{s} = \infty$ and $\lim_{s\to\infty} \frac{m_2(s)}{s} = \infty$. □

2.3.3 Remarks on isotropic functions

An important isotropic relation between M and M^* is given below. This result shows a way of comparing our growth and coercivity assumption with the assumptions which appear in the literature, see the comments in Section 3.8.2. Moreover, it is extensively used in the regularity theory, see e.g. [22, 95, 72, 77, 114, 191, 235].

Lemma 2.3.11 (Isotropic case) *Let $M : Z\times[0,\infty) \to [0,\infty)$ be an N-function (Definition 2.2.2) and let M^* be the conjugate function to M (Definition 2.1.28). Then for a.a. $z \in Z$ and every $r > 0$ we have*

$$M^*\left(z, \frac{M(z,r)}{r}\right) \le M(z,r) \le M^*\left(z, 2\frac{M(z,r)}{r}\right).$$

Proof. Note that for almost every $z \in Z$ and every $r > 0$ Definition 2.1.28 of the conjugate function implies

$$\begin{aligned} M^*\left(z, \frac{M(z,r)}{r}\right) &= \sup_{s>0}\left\{\left(\frac{M(z,r)}{r} - \frac{M(z,s)}{s}\right)s\right\} \\ &= \sup_{s\in(0,r]}\left\{\left(\frac{M(z,r)}{r} - \frac{M(z,s)}{s}\right)s\right\} \\ &\le \sup_{s\in(0,r]}\left\{\frac{M(z,r)}{r}s\right\} = M(z,r). \end{aligned} \tag{2.50}$$

On the other hand

$$M^*\left(z, 2\frac{M(z,r)}{r}\right) = \sup_{s>0}\left\{2\frac{M(z,r)}{r}s - M(z,s)\right\},$$

where we can estimate the supremum from below by its value at $s = r$, getting

$$M^*\left(z, 2\frac{M(z,r)}{r}\right) \ge 2M(z,r) - M(z,r) = M(z,r).$$ □

The above lemma has the following significant direct consequence in the isotropic Orlicz setting. Note how it justifies calling the conjugate function ‘complementary’.

Corollary 2.3.12 (Isotropic Orlicz case) *For every N-function $m : [0,\infty) \to [0,\infty)$ we have*

$$m^*\left(\frac{m(t)}{t}\right) \le m(t) \le m^*\left(2\frac{m(t)}{t}\right),$$

equivalently

$$t \le (m^*)^{-1}(t) m^{-1}(t) \le 2t.$$

Sometimes it would be useful to treat $\xi \mapsto \operatorname{ess\,inf}_{z\in Z} M(z,\xi)$ as a convex function. In the isotropic case this function is close to convex, which is illustrated by the following lemma. This fact can be used to skip Proposition 3.7.5 in an alternative proof of Theorem 3.7.8 as an isotropic version of Theorem 3.7.7, see Remark 3.7.12 for an explanation. Notice, however, that this fact is essentially false if M is anisotropic.

Lemma 2.3.13 (Isotropic case) *Suppose $M : Z \times [0,\infty) \to [0,\infty)$ is an N-function, μ is a probability measure on Z, and f is μ-integrable over Z, then*

$$\operatorname{ess\,inf}_{z\in Z} M\left(z, \frac{1}{2}\int_Z f \,\mathrm{d}\mu\right) \le \int_Z \operatorname{ess\,inf}_{z\in Z} M(z,f)\,\mathrm{d}\mu. \tag{2.51}$$

Proof. We define $\overline{M}(s) = \operatorname{ess\,inf}_{z\in Z} M(z,s)$ and notice that as an infimum of nondecreasing functions $t \mapsto M(\cdot,t)/t$ is nondecreasing, so

$$\frac{\overline{M}(t)}{t} \ge \frac{\overline{M}(s)}{s} \qquad \forall\, t > s.$$

Then

$$\overline{M}(t) \ge \tfrac{t}{s}\overline{M}(s) \qquad \forall\, t > s$$

which is equivalent to

$$\overline{M}(t) - \overline{M}(s) \ge \left(\tfrac{t}{s} - 1\right)\overline{M}(s) \qquad \forall\, t > s.$$

Hence

$$\overline{M}(t) \ge \left(\tfrac{t}{s} - 1\right)\overline{M}(s),$$

which holds true also for $t \le s$ as the term in the bracket becomes nonpositive and $\overline{M}$ takes only nonnegative values. Note that $t \mapsto \left(\tfrac{t}{s} - 1\right)\overline{M}(s)$ is an affine, and hence convex, minorant of $\overline{M}(t)$. On the other hand, $(\overline{M})^{**}(t)$ is the greatest convex minorant of $\overline{M}(t)$ (Corollary 2.1.42), thus

$$(\overline{M})^{**}(t) \ge \left(\tfrac{t}{s} - 1\right)\overline{M}(s) \quad \forall\, t > s.$$

When we choose $t = 2s > s$, we obtain

$$(\overline{M})^{**}(2s) \ge \left(\tfrac{2s}{s} - 1\right)\overline{M}(s) = \overline{M}(s).$$

Since $(\overline{M})^{**}$ is already convex, we may apply Jensen's inequality (Theorem 2.1.12) in the following way

$$\overline{M}\left(\frac{1}{2}\int_Z f\,\mathrm{d}\mu\right) \le (\overline{M})^{**}\left(\int_Z f\,\mathrm{d}\mu\right) \le \int_Z (\overline{M})^{**}(f)\,\mathrm{d}\mu \le \int_Z \overline{M}(f)\,\mathrm{d}\mu,$$

which ends the proof. □

Remark 2.3.14. There is no anisotropic analogue of Lemma 2.3.13. This results from the fact that $\inf_{z\in Z} M(z,\xi)$ can be arbitrarily far from its second conjugate $(\inf_{z\in Z} M(z,\xi))^{**}$. This is visible already in the orthotropic case, see (2.41). To verify this, it suffices to consider a function M defined for $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ that at two points z_1, z_2 have different asymptotics in the cardinal directions, i.e. for $\xi_1 \to \infty$ and $\xi_2 \to \infty$.

According to Lemma 2.3.11, we have the following consequence.

Corollary 2.3.15 (Isotropic case) *Suppose $M : Z\times[0,\infty) \to [0,\infty)$ is an N-function such that $M, M^* \in \Delta_2$, both with $h = 0$, and $\varepsilon > 0$ is arbitrary, then there exists a $c = c(\varepsilon, M)$ such that*

$$M(z,t)\tfrac{s}{t} \le \varepsilon M(z,t) + cM(z,s).$$

2.3.4 Consequences of the Δ_2-condition

It is known that a doubling, homogeneous and isotropic N-function is trapped between two power functions with powers called *Simonenko's indexes*, [294]. We prove an inhomogeneous and anisotropic version of this fact provided $\xi \mapsto M(z,\xi)$ is $C^1(\mathbb{R}^d)$ for a.a. $z \in Z$ and indicate the powers. We consider the following generalization of Simonenko's indexes i_M and s_M, defined as follows

$$i_M(z) = \liminf_{|\xi|\to\infty} \frac{\xi\cdot\nabla_\xi M(z,\xi)}{M(z,\xi)} \qquad \text{and} \qquad s_M(z) = \limsup_{|\xi|\to\infty} \frac{\xi\cdot\nabla_\xi M(z,\xi)}{M(z,\xi)}, \tag{2.52}$$

where ∇_ξ denotes the gradient with respect to the second variable. See [294, 153] and [281, Chapter II] for more details on the indexes in the homogeneous and isotropic case, and [21] for the same in the homogeneous but anisotropic case. Such indices in the homogeneous case find application in the regularity theory in the construction of auxiliary functions, cf. e.g. [22, 72, 85].

Lemma 2.3.16 *Suppose M is an N-function, Z is bounded in $\mathbb{R}^N$, and i_M, s_M are given by* (2.52). *Then*

(i) *$M \in \Delta_2$ if and only if $s_M(\cdot) \le s_+$ for some $s_+ < \infty$;*
(ii) *$M^* \in \Delta_2$ if and only if $i_M(\cdot) \ge i_-$ for some $i_- > 1$.*

Proof. The proof follows the ideas of [21].

We fix $\zeta \in \mathbb{R}^d \setminus \{0\}$ and for $t \ge 0$ we define

$$A_z(t) = M(z,t\zeta).$$

Notice that $A_z \in C^1[0,\infty)$ and $A_z'(t) = \zeta\cdot\nabla_\xi M(z,t\zeta)$. In fact, due to the nondegeneracy conditions imposed on M, A_z' is a nonnegative and strictly increasing function.

(i) If $s_M(z) < s_+ < \infty$, then for every $\varepsilon > 0$ it holds that

$$\frac{\xi \cdot \nabla_\xi M(z,\xi)}{M(z,\xi)} < s_M(z) + \varepsilon.$$

For ζ and $t \geq 1$ we have

$$\frac{A_z'(t)}{A_z(t)} = \frac{t\zeta \cdot \nabla_\xi M(z,t\zeta)}{tM(z,t\zeta)} \leq \frac{s_M(z)+\varepsilon}{t}$$

and consequently $A_z(t) \leq t^{s_M(z)+\varepsilon} A_z(1)$. Picking $t = 2$ we get

$$M(z,2\zeta) \leq 2^{s_M(z)+\varepsilon} M(z,\zeta)$$

and, since s_M is separated from infinity, we get that M satisfies the Δ_2-condition. Suppose now that $M \in \Delta_2$, that is, that there exist $c > 0$ and $0 \leq h \in L^1(Z)$ such that $M(z,2\xi) \leq cM(z,\xi) + h(z)$. Let us restrict ourselves to the full-measure subset of Z where h is finite-valued. Since A_z' is a nonnegative and nondecreasing function, for all $\zeta \in \mathbb{R}^d$ we have

$$M(z,2\zeta) = A_z(2) = \int_0^2 A_z'(t)\,\mathrm{d}t \geq \int_1^2 A_z'(t)\,\mathrm{d}t > A_z'(1) = \zeta \cdot \nabla_\xi M(z,t\zeta).$$

Due to the Δ_2-condition, we have

$$cM(z,\zeta) + h(z) \geq \zeta \cdot \nabla_\xi M(z,t\zeta).$$

After dividing by $M(z,\zeta)$ and taking limsup over $|\zeta| \to \infty$ on both sides, we obtain

$$c + \limsup_{|\zeta|\to\infty} \frac{h(z)}{M(z,\zeta)} \geq s_M(z).$$

Note that the additional term disappears as $h(z) < \infty$.

(ii) Assume $i_M(z) \geq i_- > 1$. Then for every $\varepsilon \in (0, i_M(z))$ there exists an $R_\varepsilon > 0$ such that for all ξ it holds that

$$\frac{\xi \cdot \nabla_\xi M(z,\xi)}{M(z,\xi)} > i_M(z) - \varepsilon > 1.$$

Therefore, for $t \geq 1$

$$\frac{A_z'(t)}{A_z(t)} \geq \frac{i_M(z)-\varepsilon}{t} \qquad \text{and} \qquad A_z(t) \geq t^{i_M(z)-\varepsilon} A_z(1)$$

and we can estimate

$$\begin{aligned}M^*(z,2\zeta) &= \sup_{\eta\in\mathbb{R}^d}\{2\zeta\cdot\eta - M(z,\eta)\}\\ &\le \sup_{\eta\in\mathbb{R}^d}\{2t\zeta\cdot\tfrac{\eta}{t} - t^{i_M(z)-\varepsilon}M(z,\tfrac{\eta}{t})\}\\ &= \sup_{\eta\in\mathbb{R}^d}\{2t\zeta\cdot\eta - t^{i_M(z)-\varepsilon}M(z,\eta)\}\\ &= t^{i_M(z)-\varepsilon}\sup_{\eta\in\mathbb{R}^d}\{2t^{1-i_M(z)+\varepsilon}\zeta\cdot\eta - M(z,\eta)\}.\end{aligned}$$

Take $t = 2^{\frac{1}{i_M(z)-1-\varepsilon}}$ and observe that

$$M^*(z,2\zeta) \le 2^{\frac{i_M(z)-1}{i_M(z)-1-\varepsilon}} M^*(z,\zeta).$$

Since $(i_M - 1)$ is separated from zero, we get that M^* satisfies the Δ_2-condition.

Now we consider the case $M^* \in \Delta_2$ with a constant $2/k$. Instead of $h(z)$ we may take

$$\bar{h}(z) = \sup_{\zeta:\ |\zeta|\le R} M^*(z,\zeta) + h(z)$$

and treat M^* as Δ_2 everywhere. Then using the Fenchel–Moreau theorem (Theorem 2.1.41) we have for all sufficiently large ξ

$$\begin{aligned}2kM(z,\xi) &\le 2k\sup_{\eta\in\mathbb{R}^d}\{\xi\cdot\eta - \tfrac{1}{2k}M^*(z,2\eta) + \bar{h}\}\\ &= \sup_{\eta\in\mathbb{R}^d}\{2k\xi\cdot\eta - M^*(z,2\eta)\} + \bar{h} = M(z,k\xi) + \bar{h}(z)\}.\end{aligned}$$

Then $M\left(z,\frac{2\xi}{k}\right) \le \frac{1}{2k}M\left(z,2\xi\right) + \frac{h(z)}{2k}$ and due to convexity we arrive at

$$\begin{aligned}M(z,\xi) &= M\left(z,\left[\tfrac{1}{2k}+\tfrac{k-1}{2k}\right]2\xi\right) \le \tfrac{1}{2}M\left(z,\tfrac{2\xi}{k}\right) + \tfrac{1}{2}M\left(z,\tfrac{k-1}{k}2\xi\right)\\ &\le \tfrac{1}{4k}M\left(z,2\xi\right) + \tfrac{\bar{h}(z)}{4k} + \tfrac{k-1}{2k}M\left(z,2\xi\right) = \widetilde{C}M\left(z,2\xi\right) + \tfrac{\bar{h}(z)}{4k},\end{aligned}$$

where $\widetilde{C} = \frac{2k-1}{4k} > 1$.
Notice that

$$\begin{aligned}&\int_0^1 A_z'(t)\,\mathrm{d}t = M(z,\zeta) \le \tfrac{1}{\widetilde{C}}M(z,2\zeta) + \widetilde{h}(z) = \tfrac{1}{\widetilde{C}}\int_0^2 A_z'(t)\,\mathrm{d}t + \widetilde{h}(z),\\ &\int_1^2 A_z'(t)\,\mathrm{d}t \le A_z'(2) = \zeta\cdot\nabla_\xi M(z,2\zeta).\end{aligned}$$

Therefore,

$$\int_0^2 A_z'(t)\,\mathrm{d}t = \int_0^1 A_z'(t)\,\mathrm{d}t + \int_1^2 A_z'(t)\,\mathrm{d}t$$
$$\leq \tfrac{1}{\widetilde{C}} \int_0^2 A_z'(t)\,\mathrm{d}t + \widetilde{h}(z) + \zeta \cdot \nabla_\xi M(z, 2\zeta)$$

and finally,

$$1 < 2\left(1 - \frac{1}{\widetilde{C}}\right) \leq \frac{2\widetilde{h}(z)}{M(z,2\zeta)} + \frac{2\zeta \cdot \nabla_\xi M(z,2\zeta)}{M(z,2\zeta)}.$$

Taking liminf over $|2\zeta| \to \infty$, the first term on the rightmost side vanishes and we get that indeed $1 < i_- \leq i_M(\cdot)$. □

Passing to the isotropic case we have the following direct consequences of the above fact and Lemmas 2.1.37 and 2.3.11.

Lemma 2.3.17 (Isotropic case) *If $M : Z \times [0,\infty) \to [0,\infty)$ is an N-function such that $M, M^* \in \Delta_2$, and the Hölder conjugate exponents to i_M and s_M are denoted by $i_M' = \frac{i_M}{i_M - 1}$ and $s_M' = \frac{s_M}{s_M - 1}$, respectively, then for a.e. $z \in Z$*

$$\begin{aligned} &s \mapsto \tfrac{M(z,s)}{s^{i_M}} \text{ is nondecreasing,} \qquad s \mapsto \tfrac{M(z,s)}{s^{s_M}} \text{ is nonincreasing,} \\ &s \mapsto \tfrac{M^*(z,s)}{s^{s_M'}} \text{ is nondecreasing,} \qquad s \mapsto \tfrac{M^*(z,s)}{s^{i_M'}} \text{ is nonincreasing.} \end{aligned} \tag{2.53}$$

We infer the following anisotropic consequence of the doubling conditions.

Corollary 2.3.18 *Any N-function $M : Z \times \mathbb{R}^d \to [0,\infty)$ such that $M, M^* \in \Delta_2$ is between two isotropic N-functions of power type.*

Lemma 2.3.19 (Isotropic case) *If $M : Z \times [0,\infty) \to [0,\infty)$ is an N-function such that $M, M^* \in \Delta_2$, then up to c_1, c_2 depending only on i_M, s_M we have*

$$c_1 M^*(z, M'(z,|\xi|)) \leq M(z,|\xi|) \leq c_2 M^*(z, M'(z,|\xi|)),$$

where $'$ stands for the right derivative acting on the second variable.

Note that Corollary 2.3.18 states that the growth of a modular function satisfying the Δ_2-condition, whose conjugate also satisfies the Δ_2-condition, is between inhomogeneous power-type functions. Let us concentrate for a moment on the homogeneous and isotropic case with $M(z,\xi) = m(|\xi|)$. We point out that the condition

$$1 < i_m = \inf_{t>0} \frac{tm'(t)}{m(t)} \leq \sup_{t>0} \frac{tm'(t)}{m(t)} = s_m < \infty \tag{2.54}$$

is not equivalent to comparison with power-type functions. The assumption (2.54) is equivalent to $\Delta_2(\{m, m^*\}) < \infty$; it requires regularity of the growth and restricts its rate at the same time.

Chapter 3
Musielak–Orlicz Spaces

Now that the key properties of N-functions have been established, we are equipped with a basic toolkit for identifying related Musielak–Orlicz and Musielak–Orlicz–Sobolev spaces. Here we present a study of their properties.

3.1 Definitions and Fundamental Properties

In the sequel $Z \subset \mathbb{R}^N$ is a bounded set.

Definition 3.1.1 (Modular). By a *modular* we mean a functional ρ_M defined on the set of measurable functions $\xi : Z \to \mathbb{R}^d$ given by the following formula

$$\rho_M(\xi) := \int_Z M(z, \xi(z))\, \mathrm{d}z,$$

where M is an N-function.

According to this definition we shall always be interested in the theory for measurable functions. Therefore, throughout this section we assume that

$$\xi, \eta, \zeta : Z \to \mathbb{R}^d \qquad \text{are measurable.}$$

Lemma 3.1.2 *Suppose M is an N-function. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a decreasing sequence converging to λ_0 and $\xi : Z \to \mathbb{R}^d$ be measurable. Moreover, suppose there exists a $c > 0$ such that for all $n \in \mathbb{N}$*

$$\int_Z M\left(z, \frac{\xi(z)}{\lambda_n}\right) \mathrm{d}z \le c.$$

Then

$$\int_Z M\left(z, \frac{\xi(z)}{\lambda_0}\right) \mathrm{d}z \le c.$$

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5_3

Proof. Consider a nonnegative sequence $a_n(z) := M\left(z, \frac{\xi(z)}{\lambda_n}\right)$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a decreasing sequence converging to λ_0. To show that $a_n(z) \le a_{n+1}(z)$ for all $n \in \mathbb{N}$ and a.a. $z \in Z$ observe that using the simple properties shown in Lemma 2.1.23 we conclude that

$$a_n(z) = M\left(z, \frac{\xi(z)}{\lambda_n}\right) \le \frac{\lambda_n}{\lambda_{n+1}} M\left(z, \frac{\xi(z)}{\lambda_n}\right) \le M\left(z, \frac{\lambda_n}{\lambda_{n+1}} \cdot \frac{\xi(z)}{\lambda_n}\right) = M\left(z, \frac{\xi(z)}{\lambda_{n+1}}\right) = a_{n+1}(z).$$

Since M is a Carathéodory function, which implies the almost everywhere convergence of a_n, we conclude from the Monotone Convergence theorem that

$$\int_Z M\left(z, \frac{\xi(z)}{\lambda_0}\right) \mathrm{d}z = \lim_{n\to\infty} \int_Z M\left(z, \frac{\xi(z)}{\lambda_n}\right) \mathrm{d}z \tag{3.1}$$

and thus the assertion holds. □

Definition 3.1.3 (Classes of functions). Let $M : Z \times \mathbb{R}^d \to [0, \infty)$ be an N-function and $Z \subset \mathbb{R}^N$ be bounded. We shall deal with the following classes of functions.

(i) $\mathcal{L}_M(Z;\mathbb{R}^d)$ — the *generalized Musielak–Orlicz class* is the set of all measurable functions $\xi : Z \to \mathbb{R}^d$ such that $\rho_M(\xi) < \infty$,
(ii) $L_M(Z;\mathbb{R}^d)$ — the *generalized Musielak–Orlicz space* is the smallest linear space containing $\mathcal{L}_M(Z;\mathbb{R}^d)$,
(iii) $E_M(Z;\mathbb{R}^d)$ — the largest linear space contained in $\mathcal{L}_M(Z;\mathbb{R}^d)$.

Remark 3.1.4. The convexity of M implies that $\mathcal{L}_M(Z;\mathbb{R}^d)$ is a convex set.

Remark 3.1.5. If $d = 1$, then we omit the target space and write $L_M(Z) := L_M(Z;\mathbb{R})$. When defining a norm we often just write L_M in the index, as in (3.4), however when necessary, for the sake of clarity, we may include information on the domain, possibly omitting the target space.

Remark 3.1.6. Directly from the definition it follows that

$$E_M(Z;\mathbb{R}^d) \subset \mathcal{L}_M(Z;\mathbb{R}^d) \subset L_M(Z;\mathbb{R}^d).$$

Remark 3.1.7. Changing M on a set of measure zero does not change the considered space. Indeed, this follows from the integral form of a modular (Definition 3.1.1) and its fundamental meaning in defining Musielak–Orlicz spaces.

The spaces $L_M(\Omega;\mathbb{R}^d)$ and $E_M(\Omega;\mathbb{R}^d)$ given by Definition 3.1.3 can be characterized in an equivalent way, which is presented in the following lemma.

Lemma 3.1.8 *Let $M : Z \times \mathbb{R}^d \to [0, \infty)$ be an N-function.*

(i) *The space $L_M(Z;\mathbb{R}^d)$ is equal to the set of all measurable functions $\xi : Z \to \mathbb{R}^d$ such that*

$$\int_Z M\left(z, \frac{\xi(z)}{\lambda}\right) \mathrm{d}z \to 0 \ \text{as} \ \lambda \to \infty. \tag{3.2}$$

(ii) *The space* $E_M(Z;\mathbb{R}^d)$ *is equal to the set of all measurable functions* $\xi : Z \longrightarrow \mathbb{R}^d$ *such that*

$$\int_Z M\left(z, \frac{\xi(z)}{\lambda}\right)\, \mathrm{d}z < \infty \ \textit{for all}\ \ \lambda > 0. \tag{3.3}$$

Proof. (i) Let $\xi \in L_M(Z,\mathbb{R}^d)$. Observe that the linear space is the smallest linear space containing some set, so that every element of the space can be represented as a linear combination of elements of this set. In this case it means that for all $\xi \in L_M(Z,\mathbb{R}^d)$ there exist $n = n(\xi) \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\zeta_1, \dots \zeta_n \in \mathcal{L}_M(Z;\mathbb{R}^d)$ such that

$$\xi = \sum_{i=1}^{n} \lambda_i \zeta_i.$$

From the convexity of $\mathcal{L}_M(Z;\mathbb{R}^d)$ it follows that

$$\frac{\xi}{\sum_{i=1}^n \lambda_i} = \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \zeta_i \in \mathcal{L}_M(Z;\mathbb{R}^d),$$

which means that

$$\int_Z M\left(z, \frac{\xi}{\sum_{i=1}^n \lambda_i}\right)\, \mathrm{d}z < \infty.$$

However

$$\lim_{\lambda\to\infty} \int_Z M\left(z, \frac{\xi(z)}{\lambda}\right)\, \mathrm{d}z = \lim_{\substack{t\to\infty\\ t\geq 1}} \int_Z M\left(z, \frac{\xi(z)}{t\sum_{i=1}^n \lambda_i}\right)\, \mathrm{d}z \leq \lim_{\substack{t\to\infty\\ t\geq 1}} \frac{1}{t} \int_Z M\left(z, \frac{\xi(z)}{\sum_{i=1}^n \lambda_i}\right) = 0$$

and thus ξ satisfies (3.2).

To show the opposite direction, let ξ be a measurable function satisfying (3.2). Then there exists a $\lambda_0 > 0$ such that $\int_Z M\left(z, \frac{\xi(z)}{\lambda_0}\right)\, \mathrm{d}z < \infty$. Note that then $\frac{\xi(z)}{\lambda_0} \in \mathcal{L}_M(Z;\mathbb{R}^d)$ and, hence, $\xi \in L_M(Z;\mathbb{R}^d)$.

(ii) If $\xi \in E_M(Z,\mathbb{R}^d)$, then for all $\lambda > 0$ we have $\frac{\xi}{\lambda} \in \mathcal{L}_M(Z;\mathbb{R}^d)$, which means that ξ satisfies (3.3).

To prove the opposite direction, consider the set

$$X := \left\{\xi : \int_Z M\left(z, \frac{\xi(z)}{\lambda}\right)\, \mathrm{d}z < \infty \text{ for all } \lambda > 0\right\}.$$

Obviously $E_M \subset X \subset \mathcal{L}_M(Z;\mathbb{R}^d)$. Once we show that X is a linear space, then – since E_M is the largest linear space contained in $\mathcal{L}_M(Z;\mathbb{R}^d)$ – these two must coincide. Indeed, if $\xi_1, \xi_2 \in X$ and $\gamma \in \mathbb{R}$, then

$$\begin{aligned}\int_Z M\left(z, \frac{\xi_1+\gamma\xi_2}{\lambda}\right)\, \mathrm{d}z &= \int_Z M\left(z, \frac{2(\xi_1+\gamma\xi_2)}{2\lambda}\right)\, \mathrm{d}z \\ &\leq \frac{1}{2}\int_Z M\left(z, \frac{2\xi_1}{\lambda}\right)\, \mathrm{d}z + \frac{1}{2}\int_Z M\left(z, \frac{2\gamma\xi_2}{\lambda}\right)\, \mathrm{d}z < \infty. \qquad \square\end{aligned}$$

Lemma 3.1.9 (Luxemburg norm) *Suppose M is an N-function, then the mapping $\|\cdot\|_{L_M} : L_M(Z;\mathbb{R}^d) \to [0,\infty)$ given by*

$$\|\xi\|_{L_M} := \inf\left\{\lambda > 0 : \int_Z M\left(z, \tfrac{\xi(z)}{\lambda}\right) \,\mathrm{d}z \le 1\right\} \tag{3.4}$$

defines a norm. We call it the Luxemburg norm*, see [241].*

Remark 3.1.10. Observe that the Luxemburg norm is the so-called Minkowski functional generated by a convex, absorbing and balanced set

$$A = \left\{\xi : Z \to \mathbb{R}^d \ \text{ measurable } : \int_Z M(z,\xi(z)) \,\mathrm{d}z \le 1\right\}.$$

Proof (of Lemma 3.1.9). The first observation is that for all $\xi \in L_M(Z;\mathbb{R}^d)$ we have $\|\xi\|_{L_M} < \infty$, which follows directly from (3.2).

Next we divide the proof into three steps showing that the usual axioms of a norm are satisfied.

1°. $\|\xi\|_{L_M} = 0 \iff \xi = 0$ a.e. in Z.
Since M is an N-function, $M(z,\xi) = 0 \iff \xi = 0$. If $\xi = 0$, then for all $\lambda > 0$ we have $\int_Z M\left(z, \frac{\xi}{\lambda}\right) \,\mathrm{d}z = 0$. An infimum over such λ's is obviously equal to zero, and thus $\|\xi\|_{L_M} = 0$.

If $\|\xi\|_{L_M} = 0$, then $\int_Z M\left(z, \frac{\xi}{\lambda}\right) \,\mathrm{d}z \le 1$ for all $\lambda > 0$. But since $\lambda \in (0,1]$, we have by (2.29)

$$\int_Z M\left(z, \tfrac{\xi}{\lambda}\right) \,\mathrm{d}z \ge \tfrac{1}{\lambda} \int_Z M(z,\xi) \,\mathrm{d}z$$

and thus the right-hand side needs to vanish (otherwise it becomes infinite when λ tends to zero), which holds true for $M(z,\xi(z)) = 0$ for a.a. $z \in Z$, and this implies that $\xi \equiv 0$.

2°. $\|\alpha\xi\|_{L_M} = |\alpha|\|\xi\|_{L_M}$, $\alpha \in \mathbb{R}$.
Using that $M(z,\xi) = M(z,-\xi)$ for a.a $z \in Z$ and all $\xi \in \mathbb{R}^d$ we notice that

$$\begin{aligned}
\|\alpha\xi\|_{L_M} &= \inf\left\{\lambda > 0 : \int_Z M\left(z, \tfrac{\alpha\xi(z)}{\lambda}\right) \,\mathrm{d}z \le 1\right\} \\
&= \inf\left\{|\alpha|\tilde{\lambda} > 0 : \int_Z M\left(z, \tfrac{\alpha\xi(z)}{|\alpha|\tilde{\lambda}}\right) \,\mathrm{d}z \le 1\right\} \\
&= |\alpha| \inf\left\{\tilde{\lambda} > 0 : \int_Z M\left(z, \tfrac{\xi(z)}{\tilde{\lambda}}\right) \,\mathrm{d}z \le 1\right\} = |\alpha|\, \|\xi\|_{L_M}.
\end{aligned}$$

3°. Triangle inequality $\|\xi_1 + \xi_2\|_{L_M} \le \|\xi_1\|_{L_M} + \|\xi_2\|_{L_M}$.
By the definition of the Luxemburg norm and by Lemma 3.1.2

$$\int_Z M\left(z, \tfrac{\xi(z)}{\|\xi\|_{L_M}}\right) \,\mathrm{d}z \le 1 \tag{3.5}$$

holds. Consider ξ_1, ξ_2 such that $\|\xi_1\|_{L_M}, \|\xi_2\|_{L_M} < \infty$. Then according to Jensen's inequality we obtain

$$\begin{aligned}
&\int_Z M\left(z, \tfrac{\xi_1(z)+\xi_2(z)}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}}\right) \mathrm{d}z \\
&= \int_Z M\left(z, \tfrac{\|\xi_1\|_{L_M}}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}} \tfrac{\xi_1(z)}{\|\xi_1\|_{L_M}} + \tfrac{\|\xi_2\|_{L_M}}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}} \tfrac{\xi_2(z)}{\|\xi_2\|_{L_M}}\right) \mathrm{d}z \\
&\le \tfrac{\|\xi_1\|_{L_M}}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}} \int_Z M\left(z, \tfrac{\xi_1(z)}{\|\xi_1\|_{L_M}}\right) \mathrm{d}z + \tfrac{\|\xi_2\|_{L_M}}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}} \int_Z M\left(z, \tfrac{\xi_2(z)}{\|\xi_2\|_{L_M}}\right) \mathrm{d}z.
\end{aligned}$$

Therefore, using (3.5), we conclude that

$$\int_Z M\left(z, \tfrac{\xi_1(z)+\xi_2(z)}{\|\xi_1\|_{L_M}+\|\xi_2\|_{L_M}}\right) \mathrm{d}z \le 1$$

and directly from the definition of the Luxemburg norm

$$\|\xi_1+\xi_2\|_{L_M} \le \|\xi_1\|_{L_M} + \|\xi_2\|_{L_M}. \qquad \square$$

Remark 3.1.11. If for some positive constant $c > 0$ and for a measurable function $\xi : Z \to \mathbb{R}^d$ it holds that $\|\xi\|_{L_M} \le c$, then $\xi \in L_M(Z;\mathbb{R}^d)$. Indeed, we then have $\frac{\|\xi\|_{L_M}}{c} \le 1$ and by (3.5) and (2.28)

$$\int_Z M\left(z, \tfrac{\xi(z)}{c}\right) \mathrm{d}z = \int_Z M\left(z, \tfrac{\xi(z)\|\xi\|_{L_M}}{c\|\xi\|_{L_M}}\right) \mathrm{d}z \le \frac{\|\xi\|_{L_M}}{c} \int_Z M\left(z, \tfrac{\xi(z)}{\|\xi\|_{L_M}}\right) \mathrm{d}z \le 1.$$

Thus since $\frac{\xi}{c} \in \mathcal{L}_M(Z;\mathbb{R}^d)$, it follows from (3.2) that $\xi \in L_M(Z;\mathbb{R}^d)$.

Lemma 3.1.12 (Orlicz norm) *Suppose M is an N-function, then the mapping $|||\cdot|||_{L_M} : L_M(Z;\mathbb{R}^d) \to [0,\infty)$ given by*

$$|||\xi|||_{L_M} := \sup\left\{\int_Z \eta\cdot\xi \,\mathrm{d}z : \int_Z M^*(z,\eta)\,\mathrm{d}z \le 1\right\} \tag{3.6}$$

defines a norm. We call it the Orlicz norm.

Proof. As in the case of the Luxemburg norm we check the three norm axioms. We will not concentrate now on showing that the Orlicz norm is bounded for all elements of $L_M(Z;\mathbb{R}^d)$, since it will immediately become clear in the next lemma, see (3.8) and the proof that follows afterwards.

1°. Obviously, if $\xi = 0$ a.e in Z, then $\int_Z \eta\cdot\xi\,\mathrm{d}z = 0$ and thus $|||\xi|||_{L_M} = 0$.
Assume now that $|||\xi|||_{L_M} = 0$. Note that

$$\int_Z M^*\left(z, \tfrac{\xi}{|\xi|}\right) \mathrm{d}z \le \int_Z m_1^*(1)\,\mathrm{d}z \le |Z| m_1^*(1).$$

If $|Z|m_1^*(1) \le 1$, then using the definition of the Orlicz norm, we can estimate

$$|||\xi|||_{L_M} \ge \int_Z \tfrac{\xi}{|\xi|}\cdot\xi\,\mathrm{d}z = \int_Z |\xi|\,\mathrm{d}z$$

and thus $\xi = 0$ a.e. If $|Z|m_1^*(1) > 1$ we estimate as follows

$$\int_Z M^*\left(z, \tfrac{\xi}{|Z|m_1^*(1)|\xi|}\right)\,\mathrm{d}z \le \tfrac{1}{|Z|m_1^*(1)}\int_Z M^*\left(z, \tfrac{\xi}{|\xi|}\right)\,\mathrm{d}z \le 1$$

and then

$$|||\xi|||_{L_M} \ge \int_Z \tfrac{\xi}{|Z|m_1^*(1)|\xi|}\cdot\xi\,\mathrm{d}z = |Z|m_1^*(1)\int_Z |\xi|\,\mathrm{d}z,$$

which again allows us to conclude that $\xi = 0$ a.e.

2°. Observe first that since $M^*(z,\eta) = M^*(z,-\eta)$, we have

$$\sup\left\{\int_Z \xi\cdot\eta\,\mathrm{d}z : \int_Z M(z,\eta)\,\mathrm{d}z \le 1\right\} = \sup\left\{\int_Z |\xi\cdot\eta|\,\mathrm{d}z : \int_Z M(z,\eta)\,\mathrm{d}z \le 1\right\}.$$

This observation tells us that, for a scalar $\alpha \in \mathbb{R}$,

$$\begin{aligned}|||\alpha\xi|||_{L_M} &= \sup\left\{|\alpha|\int_Z |\eta\cdot\xi|\,\mathrm{d}z : \int_Z M^*(z,\eta)\,\mathrm{d}z \le 1\right\}\\ &= |\alpha|\sup\left\{\int_Z \eta\cdot\xi\,\mathrm{d}z : \int_Z M^*(z,\eta)\,\mathrm{d}z \le 1\right\} = |\alpha|\,|||\xi|||_{L_M}.\end{aligned}$$

3°. It may happen that there is no element realizing the supremum, but at worst for every $\varepsilon > 0$ there exists an η_ε such that $\int_Z M^*(z,\eta_\varepsilon)\,\mathrm{d}z \le 1$ and

$$|||\xi+\zeta|||_{L_M} \le \int_Z (\xi+\zeta)\cdot\eta_\varepsilon\,\mathrm{d}z + \varepsilon. \tag{3.7}$$

By definition of the norm we estimate further

$$\int_Z (\xi+\zeta)\cdot\eta_\varepsilon\,\mathrm{d}z = \int_Z \xi\cdot\eta_\varepsilon\,\mathrm{d}z + \int_Z \zeta\cdot\eta_\varepsilon\,\mathrm{d}z \le |||\xi|||_{L_M} + |||\zeta|||_{L_M}$$

and conclude that

$$|||\xi+\zeta|||_{L_M} \le |||\xi|||_{L_M} + |||\zeta|||_{L_M} + \varepsilon.$$

As the above holds for any $\varepsilon > 0$, we have

$$|||\xi+\zeta|||_{L_M} \le |||\xi|||_{L_M} + |||\zeta|||_{L_M}.$$

□

Lemma 3.1.13 (Equivalence of Luxemburg and Orlicz norms) *Suppose $Z \subset \mathbb{R}^N$ is an open bounded set and $M : Z\times\mathbb{R}^d \to [0,\infty)$ is an N-function. Then for all $\xi \in L_M(Z;\mathbb{R}^d)$ it holds that*

$$\|\xi\|_{L_M} \le |||\xi|||_{L_M} \le 2\|\xi\|_{L_M}.$$

Proof. First we observe that $|||\xi|||_{L_M} \le 2\|\xi\|_{L_M}$. Indeed, using the Fenchel–Young inequality we have

$$
\begin{aligned}
|||\xi|||_{L_M} &= \sup\left\{ \|\xi\|_{L_M} \int_Z \eta \cdot \frac{\xi}{\|\xi\|_{L_M}} \,\mathrm{d}z : \int_Z M^*(z,\eta)\,\mathrm{d}z \le 1 \right\} \\
&\le \sup\left\{ \|\xi\|_{L_M} \left[\int_Z M^*(z,\eta)\,\mathrm{d}z + \int_Z M\left(z, \frac{\xi}{\|\xi\|_{L_M}}\right) \mathrm{d}z \right] : \int_Z M^*(z,\eta)\,\mathrm{d}z \le 1 \right\} \\
&\le 2\|\xi\|_{L_M}.
\end{aligned}
$$

Next we will show that

$$\|\xi\|_{L_M} \le |||\xi|||_{L_M}. \tag{3.8}$$

Assume first that $|||\xi|||_{L_M} = 1$. To obtain estimate (3.8) it is crucial to prove that

$$\int_Z M(z,\xi(z))\,\mathrm{d}z \le 1 \tag{3.9}$$

and then $\|\xi\|_{L_M} \le 1$. Then (3.8) will follow directly from the definition of the Luxemburg norm. We justify this as follows: $\|\xi\|_{L_M} = \inf\{\lambda > 0, \int_Z M(z, \frac{\xi}{\lambda})\,\mathrm{d}z \le 1\}$ implies that $\|\xi\|_{L_M} \le \lambda$ for all such λ and also for $\lambda = 1$, if condition (3.9) holds.

We concentrate on showing (3.9). By Lemma 2.1.27 the function

$$z \mapsto \eta(z) = \boldsymbol{\partial}_\xi M^0(z,\xi(z)) \tag{3.10}$$

is measurable. Recall that $\boldsymbol{\partial}_\xi M^0(z,\xi(z))$ is the element of minimal norm of $\boldsymbol{\partial}_\xi M(z,\xi(z))$.

To prove (3.9) assume first that the condition

$$\int_Z M^*(z,\eta(z))\,\mathrm{d}z \le 1 \tag{3.11}$$

is satisfied for η defined by (3.10). Recall that for $\eta(z) = \boldsymbol{\partial}_\xi M^0(z,\xi(z))$

$$\xi(z)\cdot\eta(z) = M(z,\xi(z)) + M^*(z,\eta(z)). \tag{3.12}$$

From the condition $\int_Z M^*(z,\eta(z))\,\mathrm{d}z \le 1$ and the definition of the Orlicz norm we conclude directly that

$$\int_Z \xi(z)\cdot\eta(z)\,\mathrm{d}z \le |||\xi|||_{L_M}. \tag{3.13}$$

Collecting (3.11)–(3.13) we estimate

$$
\begin{aligned}
\int_Z M(z,\xi(z))\,\mathrm{d}z &\le \int_Z M(z,\xi(z))\,\mathrm{d}z + \int_Z M^*(z,\eta(z))\,\mathrm{d}z \\
&= \int_Z \xi(z)\cdot\eta(z)\,\mathrm{d}z \le |||\xi|||_{L_M} = 1,
\end{aligned}
$$

which completes the proof of (3.9). In the remaining part we concentrate on showing that (3.11) holds for η given by (3.10).

Let us introduce a truncation of ξ in the usual way

$$\xi_n(z) = \begin{cases} \xi(z) & \text{if } |\xi(z)| \le n, \\ 0 & \text{if } |\xi(z)| > n. \end{cases} \tag{3.14}$$

Obviously

$$|||\xi_n|||_{L_M} \le |||\xi|||_{L_M}.$$

Consider now $\boldsymbol{\partial}_\xi M^0(z,\xi)$ and introduce the notation $\eta_n(z) := \boldsymbol{\partial}_\xi M^0(z,\xi_n)$, i.e.

$$\eta_n(z) = \begin{cases} \boldsymbol{\partial}_\xi M^0(z,\xi_n) & \text{if } |\xi(z)| \le n, \\ 0 & \text{if } |\xi(z)| > n. \end{cases} \tag{3.15}$$

There are a couple of simple observations about the sequence $\{\eta_n\}_{n\in\mathbb{N}}$, which we list below. Firstly, cf. Remark 2.1.38,

$$\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z \le 2\int_Z M(z,2\xi_n)\,\mathrm{d}z \le 2|Z|m_2(2n) < \infty \quad \forall n \in \mathbb{N}. \tag{3.16}$$

Secondly, as $\{M^*(z,\eta_n(z))\}_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative functions, by the Monotone Convergence Theorem

$$\lim_{n\to\infty}\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z = \int_Z M^*(z,\eta(z))\,\mathrm{d}z.$$

Thus once we show that

$$\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z \le 1 \quad \forall n \in \mathbb{N}, \tag{3.17}$$

then immediately

$$\int_Z M^*(z,\eta(z))\,\mathrm{d}z \le 1, \tag{3.18}$$

which is the remaining property. With this aim, assume the opposite to (3.17), namely

$$1 < \int_Z M^*(z,\eta_n(z))\,\mathrm{d}z. \tag{3.19}$$

By (2.28) and (3.16)

$$\int_Z M^*\left(z, \frac{\eta_n(z)}{\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z}\right)\mathrm{d}z \le \frac{1}{\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z}\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z = 1.$$

Thus

$$\int_Z \left|\xi_n(z)\frac{\eta_n(z)}{\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z}\right| \le |||\xi_n|||_{L_M} \le 1$$

and finally by (3.16)

$$\int_Z |\xi_n(z)\cdot\eta_n(z)|\,\mathrm{d}z \le \int_Z M^*(z,\eta_n(z))\,\mathrm{d}z < \infty. \tag{3.20}$$

Recall that here the Fenchel–Young inequality is satisfied as an equality

$$\xi_n(z)\cdot\eta_n(z) = M(z,\xi_n(z)) + M^*(z,\eta_n(z)) \ge 0$$

for a.a. $z \in Z$ and by (3.20) we have

$$\int_Z M(z,\xi_n(z))\,\mathrm{d}z + \int_Z M^*(z,\eta_n(z))\,\mathrm{d}z = \int_Z |\xi_n(z)\cdot\eta_n(z)|\,\mathrm{d}z \le \int_Z M^*(z,\eta_n(z))\,\mathrm{d}z < \infty.$$

Since $\int_Z M(z,\xi_n(z))\,\mathrm{d}z \ge 0$ the above estimate implies that $\int_Z M(z,\xi_n(z))\,\mathrm{d}z = 0$, and consequently $\xi_n(z) = 0$ for a.a. $z \in Z$. However then also $\eta_n(z) = 0$ a.e. in Z and consequently $\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z = 0$, which contradicts (3.19) and so we conclude that

$$\int_Z M^*(z,\eta_n(z))\,\mathrm{d}z \le 1$$

and (3.11) holds. To complete the proof we only need to include the case when the norm $|||\xi|||_{L_M} > 0$ is not necessarily equal to 1. Then, however, $\left|\left|\left|\frac{\xi}{|||\xi|||_{L_M}}\right|\right|\right|_{L_M} = 1$ and using the first part of the proof

$$\left\|\frac{\xi}{|||\xi|||_{L_M}}\right\|_{L_M} \le \left|\left|\left|\frac{\xi}{|||\xi|||_{L_M}}\right|\right|\right|_{L_M} = 1,$$

which implies that

$$\|\xi\|_{L_M} \le |||\xi|||_{L_M}.$$

□

Lemma 3.1.14 **(**L_M **vs.** $\mathcal{L}_M$**)** *Let M be an N-function.*

(i) *If $\xi \in L_M(Z;\mathbb{R}^d)$ and $\|\xi\|_{L_M} \le 1$, then*

$$\int_Z M(z,\xi(z))\,\mathrm{d}z \le \|\xi\|_{L_M}.$$

(ii) *If $\xi \in L_M(Z;\mathbb{R}^d)$ and $\|\xi\|_{L_M} > 1$, then*

$$\int_Z M(z,\xi(z))\,\mathrm{d}z \ge \|\xi\|_{L_M}.$$

Proof. (i) Recall that due to (3.5)

$$\int_Z M\left(z, \frac{\xi(z)}{\|\xi\|_{L_M}}\right)\,\mathrm{d}z \le 1$$

and that if $\|\xi\|_{L_M} \le 1$, then by virtue of (2.29)

$$\int_Z M\left(z, \frac{\xi(z)}{\|\xi\|_{L_M}}\right)\,\mathrm{d}z \ge \frac{1}{\|\xi\|_{L_M}} \int_Z M(z,\xi(z))\,\mathrm{d}z.$$

(ii) If $\|\xi\|_{L_M} > 1$, then for $\varepsilon > 0$ sufficiently small also $\|\xi\|_{L_M} - \varepsilon > 1$ and by the definition of the Luxemburg norm

$$\int_Z M\left(z, \frac{\xi(z)}{\|\xi\|_{L_M}-\varepsilon}\right)\,\mathrm{d}z > 1.$$

By (2.28)

$$\frac{1}{\|\xi\|_{L_M}-\varepsilon}\int_Z M\left(z,\xi(z)\right)\,\mathrm{d}z \geq \int_Z M\left(z,\frac{\xi(z)}{\|\xi\|_{L_M}-\varepsilon}\right)\,\mathrm{d}z,$$

and since ε was arbitrary, the claim follows. □

Lemma 3.1.15 (Generalized Hölder inequality) *Let M be an N-function. Suppose $\xi \in L_M(Z;\mathbb{R}^d)$ and $\eta \in L_{M^*}(Z;\mathbb{R}^d)$. Then*

$$\left|\int_Z \xi\cdot\eta\,\mathrm{d}z\right| \leq 2\|\xi\|_{L_M}\|\eta\|_{L_{M^*}}. \tag{3.21}$$

Proof. We apply the Young inequality to $\bar{\xi} := \xi/\|\xi\|_{L_M}$ and $\bar{\eta} := \eta/\|\eta\|_{L_{M^*}}$, obtaining

$$\begin{aligned}\left|\int_Z \bar{\xi}\cdot\bar{\eta}\,\mathrm{d}z\right| &\leq \int_Z M(z,\bar{\xi}(z))\,\mathrm{d}z + \int_Z M^*(z,\bar{\eta}(z))\,\mathrm{d}z\\ &= \int_Z M(z,\xi/\|\xi\|_{L_M})\,\mathrm{d}z + \int_Z M^*(z,\eta/\|\eta\|_{L_{M^*}})\,\mathrm{d}z,\end{aligned}$$

which by the definition of the norm (3.4) is less than or equal to 2. Multiplying both sides by $\|\xi\|_{L_M}\cdot\|\eta\|_{L_{M^*}}$ we obtain (3.21). □

As a direct consequence of Lemmas 3.1.13 and 3.1.15 we infer another version of the generalized Hölder inequality.

Corollary 3.1.16 *Suppose $\xi \in L_M(Z;\mathbb{R}^d)$ and $\eta \in L_{M^*}(Z;\mathbb{R}^d)$. Then for an absolute constant $C > 0$ one has*

$$\left|\int_Z \xi\cdot\eta\,\mathrm{d}z\right| \leq C|||\xi|||_{L_M}\|\eta\|_{L_{M^*}},$$

as well as

$$\left|\int_Z \xi\cdot\eta\,\mathrm{d}z\right| \leq C\|\xi\|_{L_M}|||\eta|||_{L_{M^*}}.$$

Theorem 3.1.17 (L_M is a Banach space) *Suppose Z is bounded in $\mathbb{R}^N$ and M is an N-function, then $L_M(Z;\mathbb{R}^d)$ equipped with the Orlicz norm $|||\cdot|||_{L_M}$ is a Banach space, i.e. each Cauchy sequence contained in $L_M(Z;\mathbb{R}^d)$ converges in the norm $|||\cdot|||_{L_M}$ and the limit is an element of $L_M(Z;\mathbb{R}^d)$.*

Proof. Since $|||\xi|||_{L_M} < \infty$ for all $\xi \in L_M(Z;\mathbb{R}^d)$ and $|||\cdot|||_{L_M}$ is a norm, thus $(L_M, |||\cdot|||_{L_M})$ is a normed space, cf. Lemma 3.1.12

We shall concentrate on proving the completeness of $L_M(Z;\mathbb{R}^d)$. Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $L_M(Z;\mathbb{R}^d)$ equipped with the Orlicz norm, i.e. such that for every $\varepsilon > 0$ there exists an N_ε such that for all $l,m > N_\varepsilon$ we have

$$\sup\left\{\int_Z \eta\cdot(\xi_l-\xi_m)\,\mathrm{d}z:\ \int_Z M^*(z,\eta)\,\mathrm{d}z\le 1\right\}<\varepsilon. \tag{3.22}$$

It suffices to show that it is convergent in the norm topology to some element of $L_M(Z;\mathbb{R}^d)$.

Consider $\eta\in L^\infty(Z;\mathbb{R}^d)$ with $\|\eta\|_{L^\infty}\le 1$ and observe that

$$\int_Z M^*(z,\eta)\,\mathrm{d}z\le\int_Z m_1^*(|\eta(z)|)\,\mathrm{d}z\le|Z|\cdot m_1^*(\|\eta\|_{L^\infty}).$$

Then for $|Z|\cdot m_1^*(1)\le 1$ it holds that

$$\int_Z M^*(z,\eta(z))\,\mathrm{d}z\le 1.$$

If $|Z|\cdot m_1^*(1)>1$, then

$$\int_Z M^*\left(z,\frac{\eta(z)}{|Z|\cdot m_1^*(1)}\right)\,\mathrm{d}z\le\frac{1}{|Z|\cdot m_1^*(1)}\int_Z M^*(z,\eta(z))\,\mathrm{d}z\le 1.$$

Thus choosing

$$\eta^{l,m}(z)=\begin{cases}\frac{1}{\lambda}\frac{\xi_l(z)-\xi_m(z)}{|\xi_l(z)-\xi_m(z)|} & \text{if } \xi_m\ne\xi_l,\\ 0 & \text{otherwise}\end{cases}$$

with $\lambda=\max\{1,|Z|\cdot m_1^*(1)\}$ and taking into account above estimates we conclude that

$$\int_Z M^*(z,\eta^{l,m}(z))\,\mathrm{d}z\le 1.$$

Since $\|\eta\|_{L^\infty}\le\frac{1}{\lambda}$, condition (3.22) implies that for all $l,m>N_\varepsilon$ we have

$$\int_Z|\xi_l(z)-\xi_m(z)|\,\mathrm{d}z\le\varepsilon\lambda.$$

Hence $\{\xi_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1(Z;\mathbb{R}^d)$. We denote its L^1-limit by ξ. Then by Fatou's Lemma

$$\begin{aligned}\int_Z|(\xi(z)-\xi_m(z))\cdot\eta(z)|\,\mathrm{d}z&=\int_Z\lim_{l\to\infty}|(\xi_l(z)-\xi_m(z))\cdot\eta(z)|\,\mathrm{d}z\\&\le\liminf_{l\to\infty}\int_Z|(\xi_l(z)-\xi_m(z))\cdot\eta(z)|\,\mathrm{d}z\\&\le\|\eta\|_{L^\infty}\liminf_{l\to\infty}\int_Z|\xi_l(z)-\xi_m(z)|\,\mathrm{d}z\le\frac{1}{\lambda}\varepsilon\lambda=\varepsilon.\end{aligned}$$

Therefore for every $\varepsilon>0$ and $k\in\mathbb{N}$ sufficiently large

$$|||\xi-\xi^k|||_{L_M}<\varepsilon$$

thus

$$|||\xi|||_{L_M}\le|||\xi^k|||_{L_M}+\varepsilon<\infty.$$

By equivalence of the Luxemburg and Orlicz norms, cf. Lemma 3.1.13,

$$\|\xi\|_{L_M} < \infty$$

and using Remark 3.1.11 we conclude that $\xi \in L_M(Z;\mathbb{R}^d)$. This ends the proof of completeness. □

Corollary 3.1.18 *Since the Orlicz norm and Luxemburg norm are equivalent, see Lemma 3.1.13, $L_M(Z;\mathbb{R}^d)$ with the Luxemburg norm is also a Banach space.*

Lemma 3.1.19 *Suppose M is an N-function and $\{\xi_n\}_{n=1}^{\infty} \subset L_M(Z;\mathbb{R}^d)$, $\xi \in L_M(Z;\mathbb{R}^d)$. Then*

$$\|\xi_n - \xi\|_{L_M} \xrightarrow[n\to\infty]{} 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \int_Z M\left(z, \frac{\xi_n(z) - \xi(z)}{\lambda}\right) \mathrm{d}z = 0 \quad \textit{for all } \lambda > 0.$$

Proof. Without loss of generality we may assume that $\xi = 0$. Otherwise we consider

$$\tilde{\xi}_n := \xi_n - \xi.$$

Let $\{\xi_n\}_{n=1}^{\infty} \subset L_M(Z;\mathbb{R}^d)$ be such that $\|\xi_n\|_{L_M} \to 0$ as $n \to \infty$. Then for any $\lambda > 0$ obviously also $\left\|\frac{\xi_n}{\lambda}\right\|_{L_M} \to 0$ and thus there exists an n_λ such that for all $n \geq n_\lambda$ $\left\|\frac{\xi_n}{\lambda}\right\|_{L_M} \leq 1$. Lemma 3.1.14 (i) then ensures that

$$\int_Z M\left(z, \frac{\xi_n(z)}{\lambda}\right) \mathrm{d}z \leq \left\|\frac{\xi_n}{\lambda}\right\|_{L_M}$$

for any $\lambda > 0$ and since the right-hand side converges to 0 as $n \to \infty$, the claim follows.

To prove the reverse implication, assume that

$$\lim_{n\to\infty} \int_Z M\left(z, \frac{\xi_n(z)}{\lambda}\right) \mathrm{d}z = 0 \quad \text{for all } \lambda > 0.$$

Then there exists an n_λ such that for all $n \geq n_\lambda$

$$\int_Z M\left(z, \frac{\xi_n(z)}{\lambda}\right) \mathrm{d}z \leq 1,$$

which implies, by the definition of the Luxemburg norm, that $\|\xi_n\|_{L_M} \leq \lambda$, i.e.

$$\forall \lambda > 0 \quad \exists n_\lambda \quad \forall n \geq n_\lambda \quad \|\xi_n\|_{L_M} \leq \lambda,$$

which completes the proof. □

Lemma 3.1.20 *Suppose M is an N-function. Then the space $E_M(Z;\mathbb{R}^d)$ is a closed subspace of $L_M(Z;\mathbb{R}^d)$, and consequently $E_M(Z;\mathbb{R}^d)$ is complete.*

Proof. Consider a sequence $\{\xi_n\}_{n=1}^{\infty} \subset E_M(Z;\mathbb{R}^d)$ such that $\|\xi_n - \xi\|_{L_M} \to 0$ as $n \to \infty$. Observe that for all $\lambda > 0$

$$\tfrac{\xi}{2\lambda} = \tfrac{\xi-\xi_n}{2\lambda} + \tfrac{\xi_n}{2\lambda}.$$

The convexity of M yields that

$$\int_Z M\left(z, \tfrac{\xi}{2\lambda}\right) \mathrm{d}z \le \frac{1}{2}\int_Z M\left(z, \tfrac{\xi-\xi_n}{\lambda}\right) \mathrm{d}z + \le \frac{1}{2}\int_Z M\left(z, \tfrac{\xi_n}{\lambda}\right) \mathrm{d}z.$$

Lemma 3.1.19 ensures that the first term vanishes since $\|\xi_n - \xi\|_{L_M} \to 0$, whereas the second term is bounded. Thus for all $\lambda > 0$,

$$\int_Z M\left(z, \tfrac{\xi}{2\lambda}\right) \mathrm{d}z < \infty,$$

and consequently, by Lemma 3.1.8 (ii), we conclude that $\xi \in E_M(Z;\mathbb{R}^d)$. □

Lemma 3.1.21 *Suppose M is an N-function. Then the space $E_M(Z;\mathbb{R}^d)$ is the closure of $L^\infty(Z;\mathbb{R}^d)$ in the Luxemburg norm.*

Proof. Since $E_M(Z;\mathbb{R}^d)$ is a closed space, we only need to show the density of the space $L^\infty(Z;\mathbb{R}^d)$ in $E_M(Z;\mathbb{R}^d)$. Let $\xi_n \in L^\infty(Z;\mathbb{R}^d)$ for all $n \in \mathbb{N}$ be defined as

$$\xi_n(z) = \begin{cases} \xi(z) & \text{if } |\xi(z)| \le n, \\ 0 & \text{if } |\xi(z)| > n. \end{cases} \tag{3.23}$$

Then for all $\lambda > 0$

$$0 \le M\left(z, \tfrac{\xi_n(z)-\xi(z)}{\lambda}\right) \le M(z, \tfrac{\xi}{\lambda})$$

and

$$\int_Z M(z, \tfrac{\xi}{\lambda}) \mathrm{d}z < \infty.$$

Moreover, $(\xi_n - \xi) \to 0$ almost everywhere as $n \to \infty$. Thus

$$M\left(z, \tfrac{\xi_n(z)-\xi(z)}{\lambda}\right) \le m_2\left(\tfrac{|\xi_n(z)-\xi(z)|}{\lambda}\right) \to 0 \quad \text{a.e. in } Z.$$

By the Lebesgue Dominated Convergence theorem

$$\lim_{n\to\infty} \int_Z M\left(z, \tfrac{\xi_n(z)-\xi(z)}{\lambda}\right) \mathrm{d}z = 0$$

holds for all $\lambda > 0$ and consequently, by Lemma 3.1.19,

$$\|\xi_n - \xi\|_{L_M} \to 0.$$ □

Remark 3.1.22. If Z is unbounded then $E_M(Z;\mathbb{R}^d)$ is defined as the closure in the L_M-norm of the set of essentially bounded and compactly supported functions [5]. Note that in the sequel, we will study only the case of bounded Z.

3.2 Embeddings $L_{M_1} \subset L_{M_2}$ and $L_{M_1} \subset E_{M_2}$

When a modular function dominates another one in a certain sense, we have some easy results on embeddings. Recall Z is assumed to be bounded here. We say that $L_{M_1}(Z;\mathbb{R}^d)$ is *continuously embedded* in $L_{M_2}(Z;\mathbb{R}^d)$ and write

$$L_{M_1}(Z;\mathbb{R}^d) \subset L_{M_2}(Z;\mathbb{R}^d)$$

when for every $\xi \in L_{M_1}(Z;\mathbb{R}^d)$ it holds that $\xi \in L_{M_2}(Z;\mathbb{R}^d)$ and there exists a constant C independent of ξ such that

$$\|\xi\|_{L_{M_2}} \le C\|\xi\|_{L_{M_1}}. \tag{3.24}$$

Lemma 3.2.1 (Embedding $L_{M_1} \subset L_{M_2}$) *If Z is a bounded open domain and $M_1, M_2 : Z\times\mathbb{R}^d \to [0,\infty)$ are N-functions, then the following conditions are equivalent.*

(i) *There exist $a,b > 0$ for which $M_2(z,a\xi) \le bM_1(z,\xi)+h(z)$ with some nonnegative $h \in L^1(Z)$, for a.a. $z \in Z$ and all $\xi \in \mathbb{R}^d$.*
(ii) *The embedding $L_{M_1}(Z;\mathbb{R}^d) \subset L_{M_2}(Z;\mathbb{R}^d)$ is continuous.*

Proof. We will later only be interested in the implication from (i) to (ii), which we therefore prove with great care, whereas for the converse we only provide a sketch of the proof. For more details, see [262].
(i) $\Longrightarrow$ (ii) The proof of this part is divided into two steps.
Step 1. For an N-function M we introduce the notation $M^a(z,\xi) := M(z,a\xi)$. We will show that

$$L_M(Z;\mathbb{R}^d) = L_{M^a}(Z;\mathbb{R}^d) \quad \text{for all } a > 0, \tag{3.25}$$

and

$$\|a\xi\|_{L_M} = \|\xi\|_{L_{M^a}} \quad \text{for all } a > 0. \tag{3.26}$$

If $\xi \in L_M(Z;\mathbb{R}^d)$, then equivalently ξ can be written as a linear combination of elements of $\mathcal{L}_M(Z;\mathbb{R}^d)$, i.e. $\xi = \sum_{i=1}^{n_\xi}\lambda_i\xi$ for $\lambda_i \in \mathbb{R}$ and $\xi \in \mathcal{L}_M(Z;\mathbb{R}^d)$, which is equivalent to $\xi = \sum_{i=1}^{n_\xi}\lambda_i a\frac{\xi}{a}$ with $\lambda_i \in \mathbb{R}$ and $\frac{\xi}{a} \in \mathcal{L}_{M^a}(Z;\mathbb{R}^d)$. The latter is equivalent to the fact that $\xi \in L_{M^a}(Z;\mathbb{R}^d)$ and thus (3.25) is proved. To prove (3.26) observe that

$$\begin{aligned}\|\xi\|_{L_{M^a}} &= \inf\left\{\lambda > 0 : \int_Z M\left(z,\frac{a\xi}{\lambda}\right)\,\mathrm{d}z \le 1\right\} = \inf\left\{a\tilde{\lambda} > 0 : \int_Z M\left(z,\frac{a\xi}{a\tilde{\lambda}}\right)\,\mathrm{d}z \le 1\right\}\\ &= \inf\left\{\tilde{\lambda} > 0 : \int_Z M\left(z,\frac{\xi}{\tilde{\lambda}}\right)\,\mathrm{d}z \le 1\right\} = a\|\xi\|_{L_M}.\end{aligned}$$

Step 2. By the first step of the proof, it is sufficient to show that if there exists a $b > 0$ for which $M_2(z,\xi) \le bM_1(z,\xi)+h(z)$ with some nonnegative $h \in L^1(Z)$, for a.a. $z \in Z$ and all $\xi \in \mathbb{R}^d$, then (ii) holds. To show that inequality (3.24) holds assume first that ξ is such that $\|\xi\|_{L_{M_1}} = 1$. Notice that

$$\int_Z M_2(z,\xi(z))\, \mathrm{d}z \leq b \int_Z M_1(z,\xi(z))\, \mathrm{d}z + \int_Z h(z)\, \mathrm{d}z$$
$$\leq b\|\xi\|_{L_{M_1}} + \|h\|_{L^1(Z)} \leq c_1\|\xi\|_{L_{M_1}}$$

with a constant $c_1 = 1 + b + \|h\|_{L^1(Z)}$. Since $c_1\|\xi\|_{L_{M_1}} \geq 1$, with the help of Lemma 2.1.23 (i) and the above estimate,

$$\int_Z M_2\left(z, \frac{\xi(z)}{c_1\|\xi\|_{L_{M_1}}}\right) \mathrm{d}z \leq \frac{1}{c_1\|\xi\|_{L_{M_1}}} \int_Z M_2(z,\xi(z))\, \mathrm{d}z$$
$$\leq \frac{1}{c_1\|\xi\|_{L_{M_1}}} \cdot c_1\|\xi\|_{L_{M_1}} = 1.$$

Directly from the definition of the Luxemburg norm $\|\xi\|_{L_{M_2}}$ it follows that

$$\|\xi\|_{L_{M_2}} \leq c_1\|\xi\|_{L_{M_1}}.$$

In the general case, when $\|\xi\|_{L_{M_1}} \neq 1$, by considering $\tilde{\xi} := \frac{\xi}{\|\xi\|_{L_M}}$, we obtain

$$\|\tfrac{\xi}{\|\xi\|_{L_M}}\|_{L_{M_2}} \leq c_1 \left\|\tfrac{\xi}{\|\xi\|_{L_M}}\right\|_{L_{M_1}}$$

and thus (3.24) also follows.

(ii) $\Longrightarrow$ (i) We observe that it suffices to prove that there exists an n_0 such that for every $n \geq n_0$ we have

$$h_n(z) := \sup_{\xi\in\mathbb{R}^d} \Big(M_2(z,2^{-n}\xi) - 2^n M_1(z,\xi)\Big) \in L^1(Z). \tag{3.27}$$

Indeed, since $\{h_n\}_{n=n_0}^{\infty}$ is a nonincreasing sequence and given arbitrary n_0, we can choose $a = 2^{-n_0}$, $b = 2^{n_0}$, $h(z) = h_{n_0}(z) \in L^1(Z)$. Notice that

$$h(z) \geq \sup_{\xi\in\mathbb{R}^d} \Big(M_2(z,a\xi) - bM_1(z,\xi)\Big)$$

and (i) follows. We prove (3.27) by contradiction. In fact, we assume that if (3.27) fails, then there exists a function $\zeta \in L_{M_1}$ such that $\zeta \notin L_{M_2}$, which contradicts the assumption that $L_{M_1}(Z) \subset L_{M_2}(Z)$.

Fix a family of simple vector-valued functions

$$\xi_{\vartheta,i}(z) = \vartheta \mathbb{1}_{Z_i}(z),$$

where $\vartheta \in \mathbb{R}^d$ and $\{Z_i\}_{i\in\mathrm{I}}$ is a partition of Z, i.e. $Z = \bigcup_{i\in\mathrm{I}} Z_i$, into pairwise disjoint sets. Then we show that

$$h_n(z) = \sup_{\vartheta\in\mathbb{Q}^d} \Big(M_2(z,2^{-n}\xi_{\vartheta,i}(z)) - 2^n M_1(z,\xi_{\vartheta,i}(z))\Big) \quad \text{for} \quad z \in Z_i.$$

Let us fix z and identify i such that $z \in Z_i$. We consider $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}^d$ for which

$$M_2(z,2^{-n}\eta_k)-2^nM_1(z,\eta_k)\geq h_n(z)-2^k.$$

Then there exists $\{\vartheta_k\}_{k\in\mathbb{N}}\subset\mathbb{Q}^d$ such that

$$\begin{cases}M_2(z,2^{-n}\vartheta_k)\geq M_2(z,2^{-n}\eta_k)-2^{-k},\\ M_1(z,\vartheta_k)\leq M_1(z,\eta_k)+2^{-n-k}.\end{cases}$$

Therefore, for any i and k we get

$$\begin{aligned}M_2(z,2^{-n}\xi_{\vartheta,i}(z))-2^nM_1(z,\xi_{\vartheta,i}(z))&\geq M_2(z,2^{-n}\eta_k(z))-2^{-k}\\ &\quad-2^nM_1(z,\eta_k(z))-2^{-k}\\ &\geq h_n(z)-3\cdot 2^{-k},\end{aligned}$$

which due to the arbitrariness of k implies that

$$h_n(z)=\sup_{l\in\mathbb{N}}\Big(M_2(z,2^{-n}\xi_l(z))-2^nM_1(z,\xi_l(z))\Big)\quad\text{for}\quad z\in Z_n,$$

where $\{\xi_l\}_{l\in\mathbb{N}}$ is an arbitrarily relabelled sequence $\{\xi_{\vartheta,i}\}_{i\in\mathbb{N}}$ with $\xi_1=(0,\ldots,0)$ and in turn h_n is measurable. Let us define another measurable function

$$b_{m,n}(z)=\max_{1\leq l\leq m}\Big(M_2(z,2^{-n}\xi_l(z))-2^nM_1(z,\xi_l(z))\Big)$$

and notice that since $\xi_1=(0,\ldots,0)$, $b_{m,n}$ is nonnegative almost everywhere in Z. Moreover, it is nondecreasing in m. Since we are in the case when (3.27) fails, the function h_n is not integrable. Thus for every n there exists an m_n such that we have $\int_Z b_{m,n}\,\mathrm{d}z\geq 2^n$. Setting $b_n=b_{m_n,n}$ we get

$$\int_Z b_n\,\mathrm{d}z\geq 2^n\qquad\text{for every}\ \ n\in\mathbb{N}.$$

Let us define

$$\begin{aligned}B_{n,l}&=\{z\in Z: M_2(z,2^{-n}\xi_l)-2^nM_1(z,\xi_l(z))=b_n(z)\},\\ B_n&=Z\setminus\bigcup_{l=1}^{m_n}B_{n,l},\\ \widetilde{\xi}_l&=\xi_l\mathbb{1}_{B_{n,l}\setminus\bigcup_{j=1}^{l-1}B_{n,j}}(z).\end{aligned}$$

We have

$$b_n(z)=M_2(z,2^{-n}\widetilde{\xi}_n(z))-2^nM_1(z,\widetilde{\xi}_n(z))\geq 0$$

and

$$\int_Z M_2(z,2^{-n}\widetilde{\xi}_n(z))\,\mathrm{d}z=\int_Z 2^nM_1(z,\widetilde{\xi}_n(z))\,\mathrm{d}z+\int_Z b_n(z)\,\mathrm{d}z\geq\int_Z b_n(z)\,\mathrm{d}z>2^n.$$

Then there exists a partition $\{A_j\}_{j\in\mathrm{J}}$ of Z such that for $\{n_j\}_{j\in\mathrm{J}}$ we have

$$\int_{A_j} M_2(z, 2^{-n}\widetilde{\xi}_{n_j}(z))\, \mathrm{d}z = 1.$$

We define

$$\zeta(z) := \sum_{j=1}^{\infty} \widetilde{\xi}_{n_j}(z) \mathbb{1}_{A_j}(z)$$

and notice that

$$\begin{aligned}\int_Z M_1(z, \zeta(z))\, \mathrm{d}z &= \sum_{j=1}^{\infty} 2^{-n_j} \left(\int_{A_j} M_2(z, 2^{-n}\widetilde{\xi}_{n_j}(z))\, \mathrm{d}z - \int_{A_j} b(z)\, \mathrm{d}z \right) \\ &\leq \sum_{j=1}^{\infty} 2^{-n_j} \int_{A_j} M_2(z, 2^{-n}\widetilde{\xi}_{n_j}(z))\, \mathrm{d}z \leq 1.\end{aligned}$$

On the other hand, if $\lambda \geq \|\zeta\|_{L_{M_2}}$ then there exists an n such that

$$\begin{aligned}\int_Z M_2\left(z, \frac{\zeta(z)}{\lambda}\right) \mathrm{d}z &\geq \int_Z M_2\left(z, \frac{\zeta(z)}{2^n}\right) \mathrm{d}z \\ &= \sum_{j=1}^{\infty} \int_{A_j} M_2(z, 2^{-n}\widetilde{\xi}_{n_j}(z))\, \mathrm{d}z = \infty.\end{aligned}$$

This gives the expected contradiction, because we assumed that $L_{M_1}(Z;\mathbb{R}^d) \subset L_{M_2}(Z;\mathbb{R}^d)$, but $\zeta \in L_{M_1}(Z;\mathbb{R}^d)$ and $\zeta \notin L_{M_2}(Z;\mathbb{R}^d)$. □

If we need to use tools available in the homogeneous and isotropic setting we have the following corollary of the above proof.

Corollary 3.2.2 *If Z is a bounded open domain, M is an N-function and $m_1 \leq M \leq m_2$ are as in Definition 2.2.2, then*

$$L_{m_2}(Z;\mathbb{R}^d) \subset L_M(Z;\mathbb{R}^d) \subset L_{m_1}(Z;\mathbb{R}^d).$$

To get the embedding of the space L_{M_1} into another Musielak–Orlicz space where the bounded functions are dense in norm i.e. into E_{M_2}, the function M_2 is assumed to grow significantly faster than M_1.

Definition 3.2.3. Let $M_1, M_2 : Z \times \mathbb{R}^d \to [0,\infty)$ be N-functions. We say that M_2 grows *significantly faster* than M_1 if for every $c > 0$ it holds that

$$\lim_{t\to\infty} \left(\inf_{\eta:\,|\eta|=1} \operatorname{ess\,inf}_{z\in Z} \frac{M_2(z, ct\eta)}{M_1(z, t\eta)} \right) = \infty.$$

Proposition 3.2.4 (Embedding $L_{M_1} \subset E_{M_2}$) *If Z is a bounded open domain and $M_1, M_2 : Z \times \mathbb{R}^d \to \mathbb{R}^d$ are N-functions such that M_2 grows significantly faster than M_1, then*

$$L_{M_2}(Z;\mathbb{R}^d) \subset E_{M_1}(Z;\mathbb{R}^d). \tag{3.28}$$

Proof. Since M_2 grows significantly faster than M_1, for every $c > 0$ there exists a t_c such that for every $t > t_c$, every vector $\zeta \in \partial B(0,1) \subset \mathbb{R}^d$, and almost every $z \in Z$

we have

$$M_1(z,t\zeta) \le M_2(z,ct\zeta).$$

We fix an arbitrary $\xi \in L_{M_2}(Z;\mathbb{R}^d)$ and an arbitrary $\beta > 0$ such that

$$\int_Z M_2(z,\beta\xi)\,\mathrm{d}z < \infty.$$

Our aim is to show that

$$\int_Z M_1(z,\lambda\xi)\,\mathrm{d}z < \infty \qquad \text{for every } \lambda > 0,$$

which in view of Lemma 3.1.8, (ii) is enough to have $\xi \in E_{M_1}(Z;\mathbb{R}^d)$.

Let us fix an arbitrary $\lambda > 0$ and set $c = \beta/\lambda$. For this c, we have chosen t_c. Define

$$A = \{z : \lambda|\xi(z)| > t_c\}.$$

On this set we have

$$M_1(z,\lambda\xi(z)) = M_1\left(z,\lambda|\xi(z)|\frac{\xi(z)}{|\xi(z)|}\right) \le M_2\left(z,c\lambda|\xi(z)|\frac{\xi(z)}{|\xi(z)|}\right) = M_2(z,c\lambda\xi(z))$$

and thus

$$\int_A M_1(z,\lambda\xi(z))\,\mathrm{d}z \le \int_A M_2(z,c\lambda\xi(z))\,\mathrm{d}z = \int_A M_2(z,\beta\xi(z))\,\mathrm{d}z.$$

Since λ is arbitrary we have $\xi\mathbb{1}_A \in E_{M_1}(Z;\mathbb{R}^d)$. On the other hand, since $L^\infty(Z;\mathbb{R}^d) \subset E_{M_1}(Z;\mathbb{R}^d)$ the definition of A ensures that also $\xi\mathbb{1}_{Z\setminus A} \in E_{M_1}(Z;\mathbb{R}^d)$. Recalling that $E_{M_1}(Z;\mathbb{R}^d)$ is a linear space, we conclude the proof. □

3.3 Function Spaces in View of the Δ_2-Condition

Let us emphasize that if $M \in \Delta_2$ (Definition 2.2.5) then we are equipped with much stronger tools. In particular, certain functional inequalities have a simpler form or at least a simpler proof, see Chapter 9. However, for various properties of Musielak–Orlicz spaces it is enough to consider growth conditions. Recall that Z is always assumed to be bounded. Indeed, if $M \in \Delta_2$, then

$$E_M(Z;\mathbb{R}^d) = \mathcal{L}_M(Z;\mathbb{R}^d) = L_M(Z;\mathbb{R}^d) \tag{3.29}$$

and so $L_M(Z;\mathbb{R}^d)$ is separable.

Lemma 3.3.1 *Let Z be bounded and M be an N-function. Then $\mathcal{L}_M(Z;\mathbb{R}^d)$ is a linear space if and only if $M \in \Delta_2$.*

Proof. Let M satisfy the Δ_2-condition. We will check below that $\mathcal{L}_M(Z;\mathbb{R}^d)$ is invariant under pointwise addition and scalar multiplication. Since $M(z,\xi)$ is a convex function of ξ and $\xi, \eta \in \mathcal{L}_M(Z;\mathbb{R}^d)$,

$$\begin{aligned}\int_Z M(z,\xi(z)+\eta(z))\,\mathrm{d}z &= \int_Z M\left(z, 2\frac{\xi+\eta}{2}\right)\mathrm{d}z\\ &\le \frac{1}{2}\int_Z M(z,2\xi)\,\mathrm{d}z + \frac{1}{2}\int_Z M(z,2\eta)\,\mathrm{d}z\\ &\le 2\sup_{\zeta:|\zeta|=c_0}\int_Z M(z,\zeta)\,\mathrm{d}z + 2\int_Z h(z)\,\mathrm{d}z\\ &\quad + \frac{k}{2}\left(\int_{\{z\in Z:|\xi(z)|>c_0\}} M(z,\xi)\,\mathrm{d}z + \int_{\{z\in Z:|\eta(z)|>c_0\}} M(z,\eta)\,\mathrm{d}z\right) < \infty.\end{aligned}$$

Let $n \in \mathbb{N}$ be such that $|\lambda| \le 2^n$, then

$$\begin{aligned}\int_Z M(z,\lambda\xi(z))\,\mathrm{d}z &\le \int_Z M(z,(\mathrm{sgn}\lambda)2^n\xi)\,\mathrm{d}z\\ &\le \sup_{\zeta:|\zeta|=c_0}\int_Z M(z,\zeta)\,\mathrm{d}z + k^n\int_{\{z\in Z:|\xi(z)|>c_0\}} M(z,\xi)\,\mathrm{d}z + n\int_Z h(z)\,\mathrm{d}z < \infty.\end{aligned}$$

Next let us assume that M does not satisfy the Δ_2-condition. With no loss of generality we can assume $c_0 \ge 1$. Then there exists a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ of measurable functions $\xi_j : Z \to \mathbb{R}^d$ such that

$$M(z,2\xi(z)) \ge 2^j M(z,\xi_j(z)) \quad \text{for a.a. } z \in Z \text{ and } |\xi_j| \ge c_0 > 0 \text{ and } j \in \mathbb{N}.$$

Then let us construct a partition $\{Z_j\}_{j\in N}$ of the set Z such that

$$\int_{Z_j} M(z,\xi_j(z))\,\mathrm{d}z = 2^{-j}\sup_{\zeta:|\zeta|=c_0}\int_Z M(z,\zeta)\,\mathrm{d}z.$$

Defining

$$\xi(z) = \sum_{j=1}^{\infty}\xi_j(z)\mathbb{1}_{Z_j}$$

we deduce that

$$\begin{aligned}\int_Z M(z,2\xi(z))\,\mathrm{d}z &= \int_Z M\left(z, 2\sum_{j=1}^{\infty}\xi_j(z)\mathbb{1}_{Z_j}\right)\mathrm{d}z = \sum_{j=1}^{\infty}\int_{Z_j} M(z,2\xi_j(z))\,\mathrm{d}z\\ &\ge \sum_{j=1}^{\infty}2^j\int_{Z_j} M(z,\xi_j(z))\,\mathrm{d}z = \sum_{j=1}^{\infty}\sup_{\zeta:|\zeta|=c_0}\int_Z M(z,\zeta)\,\mathrm{d}z.\end{aligned}$$

We note that the right-hand side is infinite, so $2\xi \notin \mathcal{L}_M(Z;\mathbb{R}^d)$. □

Theorem 3.3.2 *Let Z be bounded and M be an N-function. Then*

$$M \in \Delta_2 \qquad \Longleftrightarrow \qquad E_M(Z;\mathbb{R}^d) = L_M(Z;\mathbb{R}^d).$$

Proof. '$\Longleftarrow$' If $\xi \in L_M(Z;\mathbb{R}^d)$, then also $2\xi \in L_M(Z;\mathbb{R}^d)$. Therefore, denoting $M(z,2\xi)$ by $\widetilde{M}(z,\xi)$, we get

$$L_M(Z;\mathbb{R}^d) \subset L_{\widetilde{M}}(Z;\mathbb{R}^d).$$

Thus, due to the definition of the Luxemburg norm and Proposition 3.2.1, we notice that M dominates $\widetilde{M}$ in the sense that

$$\widetilde{M}(z,\xi) \le c\,M(z,\xi) + h(z) \qquad \text{for all } \xi$$

with some $c, c_0 > 0$ and nonnegative locally integrable h.

Summing up, we get $M \in \Delta_2$. Indeed,

$$M(z,2\xi) = \widetilde{M}(z,\xi) \le c\,M(z,\xi) + h(z).$$

'$\Longrightarrow$' We fix $\xi \in L_M(Z;\mathbb{R}^d)$. Then there exists a $\lambda > 0$ such that

$$\int_\Omega M(z,\xi/\lambda)\,\mathrm{d}z < \infty.$$

Using Lemma 3.1.8 we shall show that under the Δ_2-condition the above quantity is finite for every $\mu \in (0,\infty)$ in the place of λ.

From convexity and local integrability we get that close to the origin $M(z,\cdot)$ can be estimated from above by a linear function. Then

$$\int_Z M(z,\xi)\,\mathrm{d}z \le c_1 + \int_{\{|\xi|>c_0\}} M(z,\xi)\,\mathrm{d}z.$$

For every $\mu > 0$ there exists an $m \in \mathbb{N}$ such that $\mu \ge \lambda/2^m$. The convexity of M and the Δ_2-condition ensure that

$$\begin{aligned}
&\int_Z M(z,\xi/\mu)\,\mathrm{d}z \\
&\le c_1 + \frac{\lambda}{2^m\mu}\int_{\{|\xi|>c_0\}} M\left(z,\frac{2^m\xi}{\lambda}\right)\mathrm{d}z \\
&\le c_1 + \frac{\lambda}{2^m\mu}\left[c_{\Delta_2}^m \int_{\{|\xi|>c_0\}} M\left(z,\frac{\xi}{\lambda}\right)\mathrm{d}z + \left(c_{\Delta_2}^{m-1} + \cdots + 1\right)\|h\|_{L^1(Z)}\right] < \infty.
\end{aligned}$$

Therefore, Lemma 3.1.8 gives the claim. □

Remark 3.3.3. If both $M, M^* \in \Delta_2$, then L_M is reflexive. Indeed, reflexivity results from

$$L_M = L_{M^{**}} = (E_{M^*})^* \overset{M^*\in\Delta_2}{=\!=\!=} (L_{M^*})^* = (E_M)^{**} \overset{M\in\Delta_2}{=\!=\!=} (L_M)^{**}.$$

The first equality holds due to Theorem 2.1.41, the second and the fourth due to Theorem 3.5.3, while the third and the fifth follow from Theorem 3.3.2.

3.4 Topologies

There are several types of different topologies that can be used with various aims. Since L_M need not be dual to L_{M^*}, nor vice versa, we shall distinguish the topology $\sigma(L_M, L_{M^*})$ from the weak-$*$ topology in L_M, namely $\sigma(L_M, E_{M^*})$. The fact that it satisfies the classical definition of weak-$*$ topology follows from Theorem 3.5.3.

We say that $\{\xi_n\}_{n\in\mathbb{N}} \subset L_M$ is $\sigma(L_M, L_{M^*})$-convergent to $\xi \in L_M$ if for any $\eta \in L_{M^*}$

$$\int_Z \xi_n \cdot \eta \,\mathrm{d}z \xrightarrow[n\to\infty]{} \int_Z \xi \cdot \eta \,\mathrm{d}z. \tag{3.30}$$

We say that $\{\xi_n\}_{n\in\mathbb{N}} \subset L_M$ is weakly-$*$ convergent (i.e. $\sigma(L_M, E_{M^*})$-convergent) to $\xi \in L_M$ if for any $\eta \in E_{M^*}$

$$\int_Z \xi_n \cdot \eta \,\mathrm{d}z \xrightarrow[n\to\infty]{} \int_Z \xi \cdot \eta \,\mathrm{d}z. \tag{3.31}$$

We say that $\{\xi_n\}_{n\in\mathbb{N}}$ is norm-convergent to ξ (strongly) in L_M if

$$\|\xi_n - \xi\|_{L_M} \xrightarrow[n\to\infty]{} 0. \tag{3.32}$$

Obviously strong convergence (3.32) implies both convergences (3.30) and (3.31).

Remark 3.4.1. Before discussing the topologies in Musielak–Orlicz spaces we stress that it is important for the reader to understand that $\sigma(L_M, L_{M^*})$ is not a weak topology, which results from the fact that $L_{M^*} \subsetneq (L_M)^*$ as long as the Δ_2-condition is not satisfied. Indeed, notice that then $E_M \subset L_M$ and E_M is a proper closed subspace, so a bounded linear functional $\ell \in (E_M)^* = L_{M^*}$ can be extended by the Hahn–Banach theorem (Theorem 8.29) to a functional $\tilde{\ell} \in (L_M)^*$ and this extension is not unique.

Besides the strong (norm) and weak-type topologies in the theory of Orlicz and Musielak–Orlicz spaces we can consider more relevant topology, namely – the modular topology.

3.4.1 The modular topology and uniform integrability

For the definitions of convergence in measure and uniform integrability, see Definitions 8.16 and 8.18, respectively, in Chapter 8.

Let us present an anisotropic version of the classical de la Vallée Poussin theorem.

Lemma 3.4.2 *Suppose M is an N-function and let $\{\xi_n\}_{n=1}^\infty$ be a sequence of measurable functions $\xi_n : Z \to \mathbb{R}^d$ satisfying*

$$\sup_{n\in\mathbb{N}} \int_Z M(z,\xi_n(z))\,\mathrm{d}z < \infty.$$

Then the sequence $\{\xi_n\}_{n=1}^\infty$ is uniformly integrable in $L^1(Z;\mathbb{R}^d)$.

Proof. By the definition of an N-function M there exists a function $m_1 : [0,\infty) \to [0,\infty)$ satisfying

$$\sup_{n\in\mathbb{N}} \int_Z m_1(|\xi_n(z)|)\,\mathrm{d}z \le \sup_{n\in\mathbb{N}} \int_Z M(z,\xi_n(z))\,\mathrm{d}z < c. \tag{3.33}$$

Recall that by Lemma 2.1.23 part (iii) the continuous function m_1 is increasing. Moreover, due to the definition $m_1(s)/s \to \infty$ as $s \to \infty$, and thus (3.33) implies that condition (ii) of Lemma 8.19 is satisfied, which is equivalent to the uniform integrability of $\{\xi_n\}_{n=1}^\infty$. □

Definition 3.4.3 (Modular convergence). Suppose M is an N-function. We say that a sequence $\{\xi_n\}_{n=1}^\infty$ *converges modularly* to ξ in $L_M(Z;\mathbb{R}^d)$, written

$$\xi_n \xrightarrow[n\to\infty]{M} \xi,$$

if there exists a $\lambda > 0$ such that

$$\int_Z M\left(z, \frac{\xi_n - \xi}{\lambda}\right) \mathrm{d}z \to 0, \quad \text{when } n \to \infty.$$

Norm convergence always implies modular convergence. In view of Lemma 3.1.19 and Theorem 3.3.2, modular convergence implies norm convergence if and only if $M \in \Delta_2$.

Theorem 3.4.4 (Generalized Vitali's convergence theorem) *Suppose M is an N-function and let $\{\xi_n\}_{n=1}^\infty$ be a sequence of measurable functions such that $\xi_n : Z \to \mathbb{R}^d$. The following conditions are equivalent:*

(i) *the sequence $\{\xi_n\}_{n=1}^\infty$ converges modularly to ξ in $L_M(Z;\mathbb{R}^d)$;*

(ii) *the sequence $\{\xi_n\}_{n=1}^\infty$ converges in measure and there exists a $\lambda > 0$ such that*

$$\left\{ M\left(z, \frac{\xi_n}{\lambda}\right) \right\}_{n=1}^\infty \textit{ is uniformly integrable in } L^1(Z).$$

Proof.
i) $\Longrightarrow$ ii) Let $\lambda_1 > 0$ be the constant from the definition of modular convergence (Definition 3.4.3) and m_1 be a minorant of M from the definition of an N-function (Definition 2.2.2). We observe that using Lemma 2.1.23 part (iii) and the Chebyshev inequality (Theorem 8.28) we have

$$
\begin{aligned}
|\{z \in Z:\ |\xi_n(z)-\xi(z)| > \varepsilon\}| &= \left|\left\{z \in Z:\ m_1\left(\frac{|\xi_n(z)-\xi(z)|}{\lambda_1}\right) > m_1\left(\frac{\varepsilon}{\lambda_1}\right)\right\}\right| \\
&\le \frac{1}{m_1(\varepsilon/\lambda_1)} \int_{\left\{z\in Z:\ m_1\left(\frac{|\xi_n-\xi|}{\lambda_1}\right) > m_1\left(\frac{\varepsilon}{\lambda_1}\right)\right\}} m_1\left(\frac{|\xi_n-\xi|}{\lambda_1}\right)\, \mathrm{d}z \\
&\le \frac{1}{m_1(\varepsilon/\lambda_1)} \int_{\{z\in Z:\ |\xi_n-\xi|>\varepsilon\}} M\left(z, \frac{\xi_n-\xi}{\lambda_1}\right)\, \mathrm{d}z \\
&\le \frac{1}{m_1(\varepsilon/\lambda_1)} \int_Z M\left(z, \frac{\xi_n-\xi}{\lambda_1}\right)\, \mathrm{d}z,
\end{aligned}
$$

where the integral from the last line tends to zero as $n \to \infty$ and thus $\xi_n \xrightarrow[n\to\infty]{} \xi$ in measure.

For an arbitrary measurable set $Z' \subset Z$ Jensen's inequality implies

$$
\begin{aligned}
\int_{Z'} M\left(z, \frac{\xi_n}{\lambda}\right) \mathrm{d}z &\le \frac{1}{2}\left[\int_{Z'} M\left(z, \frac{2(\xi_n-\xi)}{\lambda}\right) \mathrm{d}z + \int_{Z'} M\left(z, \frac{2\xi}{\lambda}\right) \mathrm{d}z\right] \\
&\le \frac{1}{2}\left[\int_{Z} M\left(z, \frac{2(\xi_n-\xi)}{\lambda}\right) \mathrm{d}z + \int_{Z'} M\left(z, \frac{2\xi}{\lambda}\right) \mathrm{d}z\right]
\end{aligned}
$$

and choosing $\lambda = 2\lambda_1$ we obtain

$$
\int_{Z'} M\left(z, \frac{\xi_n}{\lambda}\right) \mathrm{d}z \le \frac{1}{2}\left[\int_{Z} M\left(z, \frac{\xi_n-\xi}{\lambda_1}\right) \mathrm{d}z + \int_{Z'} M\left(z, \frac{\xi}{\lambda_1}\right) \mathrm{d}z\right]. \tag{3.34}
$$

Let $\varepsilon > 0$ be arbitrary. For all $n > n_0$ with n_0 chosen large enough, in view of the modular convergence, we conclude that

$$
\int_Z M\left(z, \frac{\xi_n-\xi}{\lambda_1}\right) \mathrm{d}z < \varepsilon.
$$

The term $M\left(z, \frac{\xi}{\lambda_1}\right)$ is independent of n and thus uniformly integrable. Hence there exists a $\delta > 0$ such that for all Z' such that $|Z'| < \delta$, we have

$$
\int_{Z'} M\left(z, \frac{\xi}{\lambda_1}\right) \mathrm{d}z < \varepsilon.
$$

Therefore (3.34) yields

$$
\sup_{n>n_0} \int_{Z'} M\left(z, \frac{\xi_n}{\lambda_1}\right) \mathrm{d}z \le \varepsilon.
$$

As a supremum over a finite family of functions is always uniformly integrable, we immediately conclude that

$$
\sup_{n\in\mathbb{N}} \int_{Z'} M\left(z, \frac{\xi_n}{\lambda_1}\right) \mathrm{d}z \le \varepsilon,
$$

which completes this part of the proof.

(ii) $\Longrightarrow$ (i) We want to show that there exists a $\lambda_1 > 0$ such that for all $\varepsilon > 0$ there exists an n_0 such that for all $n > n_0$

$$\int_Z M\left(z, \frac{\xi_n - \xi}{\lambda_1}\right) \mathrm{d}z < \varepsilon. \tag{3.35}$$

Note that if $\xi_n \xrightarrow[n\to\infty]{} \xi$ in measure, then obviously $|\xi_n - \xi| \xrightarrow[n\to\infty]{} 0$ in measure, and for any $\lambda_2 > 0$ also $m_2\left(\frac{|\xi_n - \xi|}{\lambda_2}\right) \xrightarrow[n\to\infty]{} 0$ in measure, where $m_2 : [0,\infty) \to [0,\infty)$ is a function given by (2.37). Then, also from (2.37), one concludes that

$$M\left(z, \frac{\xi_n - \xi}{\lambda_2}\right) \xrightarrow[n\to\infty]{} 0 \text{ in measure for all } \lambda_2 > 0. \tag{3.36}$$

We use the notation $A_n := \{z \in Z : M\left(z, \frac{\xi_n - \xi}{\lambda_2}\right) \le \tilde{\varepsilon}\}$ Thus simply by (3.36)

$$\forall \tilde{\varepsilon} > 0 \quad \forall \delta > 0 \quad \exists\, n_0 \in \mathbb{N} \quad \forall n > n_0 \quad |Z \setminus A_n| < \delta \tag{3.37}$$

for any $\lambda_2 > 0$. For the moment we know that there exists a $\lambda > 0$ such that the sequence $\left\{M\left(z, \frac{\xi_n}{\lambda}\right)\right\}_{n=1}^{\infty}$ is uniformly integrable, however we need to establish the uniform integrability of $\left\{M\left(z, \frac{\xi_n - \xi}{\lambda_1}\right)\right\}_{n=1}^{\infty}$ for some λ_1. By convexity we have

$$M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) \le \frac{1}{2}\left[M\left(z, \frac{\xi_n}{\lambda}\right) + M\left(z, \frac{\xi}{\lambda}\right)\right].$$

The first term on the right-hand side is uniformly integrable by the assumption and the second term is independent of n and thus obviously also uniformly integrable, which means that the left-hand side is uniformly integrable, i.e.

$$\forall\, \hat{\varepsilon} > 0 \quad \exists\, \hat{\delta} > 0 \quad \text{such that} \int_{Z'} M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) dz < \hat{\varepsilon} \quad \forall Z' \subset Z,\ |Z'| \le \hat{\delta}. \tag{3.38}$$

We will use this information to estimate

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \int_Z M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) \mathrm{d}z \\ &\le \sup_{n>n_0} \int_{A_n} M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) \mathrm{d}z + \sup_{n>n_0} \int_{Z\setminus A_n} M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) \mathrm{d}z. \end{aligned} \tag{3.39}$$

Firstly in (3.37) we choose $\tilde{\varepsilon} = \frac{\varepsilon}{2|Z|}$ and $\lambda_2 = 2\lambda$ to provide a simple estimate

$$\sup_{n>n_0} \int_{A_n} M\left(z, \frac{\xi_n - \xi}{2\lambda}\right) \mathrm{d}z \le \int_Z \frac{\varepsilon}{2|Z|} \mathrm{d}z = \frac{\varepsilon}{2}. \tag{3.40}$$

To estimate the second term in (3.39) we choose $\tilde{\varepsilon} = \frac{\varepsilon}{2}$ and $\delta = \hat{\delta}$ in (3.37) to obtain (3.35) with $\lambda_1 = 2\lambda$ and complete the proof. □

Remark 3.4.5. Note that the above proof still holds for $M(x,\xi) = |\xi|$ even though it is not an N-function anymore. For $M(x,\xi) = |\xi|^p$, $1 \le p < \infty$, Theorem 3.4.4 in fact retrieves the classical Vitali's convergence theorem (Theorem 8.23).

Lemma 3.4.6 *Let M be an N-function, $\{\xi_n\}_{n=1}^\infty \subset L_M(Z;\mathbb{R}^d)$, and $\{\eta_n\}_{n=1}^\infty \subset L_{M^*}(Z;\mathbb{R}^d)$. Suppose $\xi_n \xrightarrow[n\to\infty]{M} \xi$ in $L_M(Z;\mathbb{R}^d)$ and $\eta_n \xrightarrow[n\to\infty]{M} \eta$ in $L_{M^*}(Z;\mathbb{R}^d)$. Then*

$$\int_Z \xi_n \cdot \eta_n \,\mathrm{d}z \xrightarrow[n\to\infty]{} \int_Z \xi \cdot \eta \,\mathrm{d}z.$$

Proof. Theorem 3.4.4 ensures that modular convergence of the sequences $\{\xi_n\}_{n=1}^\infty$ and $\{\eta_n\}_{n=1}^\infty$ implies that they converge in measure. Obviously the sequence of products $\{\xi_n \cdot \eta_n\}_{n=1}^\infty$ also converges in measure. In the next step we concentrate on showing uniform integrability of $\{\xi_n \cdot \eta_n\}_{n=1}^\infty$, which is however equivalent to the uniform integrability of the sequence

$$\left\{\frac{\xi_n}{\lambda_1} \cdot \frac{\eta_n}{\lambda_2}\right\}_{n=1}^\infty$$

with arbitrary $\lambda_1, \lambda_2 > 0$. Again using Theorem 3.4.4, from the modular convergence we also infer the uniform integrability of the sequences

$$\left\{M\left(z, \frac{\xi_n}{\lambda_1}\right)\right\}_{n=1}^\infty \quad \text{and} \quad \left\{M^*\left(z, \frac{\eta_n}{\lambda_2}\right)\right\}_{n=1}^\infty,$$

for some $\lambda_1, \lambda_2 > 0$. Keeping the same constants we estimate with the help of the Fenchel–Young inequality

$$\left|\frac{\xi_n}{\lambda_1} \cdot \frac{\eta_n}{\lambda_2}\right| \le M\left(z, \frac{\xi_n}{\lambda_1}\right) + M^*\left(z, \frac{\eta_n}{\lambda_2}\right).$$

As the right-hand is uniformly integrable, so is the left-hand side. Finally, we complete the proof using the classical Vitali convergence theorem (Theorem 8.23). □

Corollary 3.4.7 *Let M be an N-function and $\{\xi_n\}_{n=1}^\infty, \xi \in L_M(Z;\mathbb{R}^d)$. If $\xi_n \xrightarrow[n\to\infty]{M} \xi$ modularly in $L_M(Z;\mathbb{R}^d)$ then, up to a subsequence, $\xi_n \xrightarrow[n\to\infty]{} \xi$ in $\sigma(L_M, L_{M^*})$.*

Lemma 3.4.8 *Suppose ϱ is a regularizing kernel (i.e. a nonnegative measurable function such that $\int_{\mathbb{R}} \varrho(s)\,\mathrm{d}s = 1$) and define $\varrho^j(s) = j\varrho(js)$ for $j \in \mathbb{N}$. Let $(t,x) \in \Omega_T = (0,T)\times\Omega$ with $\Omega \subset \mathbb{R}^N$ and let M be an N-function independent of the variable t, namely $M(t,x,\xi) = M(x,\xi) : \Omega\times\mathbb{R}^d \to [0,\infty)$. Denoting the convolution in the variable t by $*$, for $j \to \infty$ we have*

(i) *for any $\psi \in L^1(\Omega_T;\mathbb{R}^d)$ the sequence $(\varrho^j * \psi)(t,x) \to \psi(t,x)$ in measure;*
(ii) *for any $\psi \in \mathcal{L}_M((\Omega_T;\mathbb{R}^d)$ the sequence $\{M(x,\varrho^j * \psi)\}_{j\in\mathbb{N}}$ is uniformly integrable.*

Proof. Observe that for a.e. $x \in \Omega$ the function $\psi(\cdot,x)$ is in $L^1(0,T)$ and $\rho^j * \psi(\cdot,x) \to \psi(\cdot,x)$ in $L^1(0,T)$. Therefore $\rho^j * \psi \to \psi$ in measure on the set Ω_T as $j \to \infty$, which proves (i).

In order to show (ii) we make use of the characterization of uniform integrability given by (8.1) in Chapter 8, that is, we show that for every $\varepsilon > 0$ there exists an $R > 0$ for which

$$\sup_{j\in\mathbb{N}} \int_{(0,T)\times\Omega} \big(M(x,\rho^j * \psi) - R\big)_+ \,\mathrm{d}x\,\mathrm{d}t \le \varepsilon.$$

We extend ψ by 0 for $t \notin (0,T)$. Since $\xi \mapsto (M(\cdot,\xi) - R)_+$ is a convex function, by Jensen's inequality we have

$$\big(M(x,\rho^j * \psi) - R\big)_+ \le \int_{\mathbb{R}} (M(x,\psi(t-s,x)) - R)_+ \rho^j(s)\,\mathrm{d}s$$

and, consequently,

$$\begin{aligned}&\int_\Omega \int_0^T \big(M(x,\rho^j * \psi) - R\big)_+ \,\mathrm{d}t\,\mathrm{d}x\\ &\qquad \le \int_\Omega \int_{\mathbb{R}} \left(\int_{\mathbb{R}} M(x,\psi(t-s,x))\rho^j(s) - R\right)_+ \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Therefore, by Fubini's theorem and the Young convolution inequality (Lemma 8.26) the following holds for all $R > 0$

$$\begin{aligned}&\int_{\Omega_T} \big(M(x,\rho^j * \psi) - R\big)_+ \,\mathrm{d}x\,\mathrm{d}t\\ &\qquad \le \int_\Omega \int_0^T \left(\int_{\mathbb{R}} M(x,\psi(t-s,x))\rho^j(s)\,\mathrm{d}s - R\right)_+ \mathrm{d}t\,\mathrm{d}x\\ &\qquad \le \int_\Omega \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (M(x,\psi(t-s,x)) - R)_+ \rho^j(s)\,\mathrm{d}s\right) \mathrm{d}x\,\mathrm{d}t\\ &\qquad = \int_\Omega \big\|(M(x,\psi(\cdot,x)) - R)_+ * \rho^j\big\|_{L^1(R)} \,\mathrm{d}x\\ &\qquad \le \int_\Omega \|\rho\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} (M(x,\psi(t,x)) - R)_+ \,\mathrm{d}t\,\mathrm{d}x\\ &\qquad = \int_{(0,T)\times\Omega} (M(x,\psi(t,x)) - R)_+ \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\psi \in \mathcal{L}_M((0,T)\times\Omega)$, the term on the left-hand side of the above is bounded. Hence taking the supremum over $j \in \mathbb{N}$, the term on the right-hand side is arbitrarily small when $R > 0$ is chosen sufficiently large. This finishes the proof. □

3.4.2 Modular density of simple functions and separability of E_{M^*}

Let us present the basic results concerning modular density.

Definition 3.4.9 (Simple function). A function $f : \mathbb{R}^d \to \mathbb{R}$ is called a *simple function* if the image of f is finite.

Before discussing the issues of density with respect to the modular topology, it may be useful to pause and make sure that a diagonal argument, widely used in metric spaces, that is used hereafter in the context of the modular topology, is indeed valid in this setting.

Lemma 3.4.10 (Diagonal argument for modular convergence) *Suppose a family $\{\xi_{k,l}\}_{k,l\in\mathbb{N}} \subset L_M(Z;\mathbb{R}^d)$, a sequence $\{\xi_l\}_{l\in\mathbb{N}} \subset L_M(Z;\mathbb{R}^d)$, and $\xi \in L_M(Z;\mathbb{R}^d)$ satisfy the following conditions*

(i)
$$\int_Z M\left(z, \tfrac{\xi_{k,l}-\xi_l}{\alpha}\right) \,\mathrm{d}z \xrightarrow[k\to\infty]{} 0$$

with α independent of l,

(ii)
$$\int_Z M\left(z, \tfrac{\xi_l-\xi}{\lambda}\right) \,\mathrm{d}z \xrightarrow[l\to\infty]{} 0.$$

Then there exists a sequence $\{\xi_{k(l),l}\}_{l\in\mathbb{N}}$ such that

$$\int_Z M\left(z, \tfrac{\xi_{k,l}-\xi}{2\max\{\lambda,\alpha\}}\right) \,\mathrm{d}z \xrightarrow[l\to\infty]{} 0.$$

Proof. Observe that

$$\begin{aligned}
\int_Z M\left(z, \tfrac{\xi_{l,k}-\xi}{2\max\{\lambda,\alpha\}}\right) \mathrm{d}z &= \int_Z M\left(z, \tfrac{\xi_{l,k}-\xi_l}{2\max\{\lambda,\alpha\}} + \tfrac{\xi_l-\xi}{2\max\{\lambda,\alpha\}}\right) \mathrm{d}z \\
&\le \tfrac{1}{2}\int_Z M\left(z, \tfrac{\xi_{l,k}-\xi_l}{\max\{\lambda,\alpha\}}\right) \mathrm{d}z + \tfrac{1}{2}\int_Z M\left(z, \tfrac{\xi_l-\xi}{\max\{\lambda,\alpha\}}\right) \mathrm{d}z \\
&\le \tfrac{1}{2}\int_Z M\left(z, \tfrac{\xi_{l,k}-\xi_l}{\alpha}\right) \mathrm{d}z + \tfrac{1}{2}\int_Z M\left(z, \tfrac{\xi_l-\xi}{\lambda}\right) \mathrm{d}z.
\end{aligned} \tag{3.41}$$

Choosing l such that $\int_Z M\left(z, \tfrac{\xi_l-\xi}{\lambda}\right) \mathrm{d}z < \varepsilon$ and then $k := k(l)$ such that

$$\int_Z M\left(z, \tfrac{\xi_{l,k(l)}-\xi_l}{\alpha}\right) \mathrm{d}z < \varepsilon$$

we conclude that $\int_Z M\left(z, \tfrac{\xi_{l,k(l)}-\xi}{2\max\{\lambda,\alpha\}}\right) \mathrm{d}z < \varepsilon$. □

Theorem 3.4.11 (Modular density of simple functions) *Suppose M is an N-function. Then the set of measurable simple functions with range in $\mathbb{Q}^d$ is dense in $L_M(Z;\mathbb{R}^d)$ with respect to the modular topology.*

Proof. Let us fix an arbitrary $\xi \in L_M(Z;\mathbb{R}^d)$. For any $l \in \mathbb{N}$ we define

$$Z_l = \{z \in Z : \ |\xi(z)| \le l\}.$$

By the Chebyshev inequality (Theorem 8.28) we obtain

$$|Z \setminus Z_l| \le \|\xi\|_{L^1(Z;\mathbb{R}^d)}/l.$$

Defining $\xi_l = \xi \mathbb{1}_{Z_l}$, we notice that for almost every $z \in Z$ it holds that $|\xi_l(z)| \le |\xi(z)|$ and $M(z,\xi_l(z)) \le M(z,\xi(z))$. Then for every $\lambda > 1$ and $\alpha \ge \|\xi\|_{L_M}/2$ we have

$$\begin{aligned}\int_Z M\left(z, \frac{\xi_l - \xi}{2\lambda\alpha}\right) \mathrm{d}z &= \int_{Z\setminus Z_l} M\left(z, \frac{\xi}{2\lambda\alpha}\right) \mathrm{d}z \\ &\le \frac{1}{\lambda} \int_{Z\setminus Z_l} M\left(z, \frac{\xi}{2\alpha}\right) \mathrm{d}z \xrightarrow[l\to\infty]{} 0.\end{aligned} \tag{3.42}$$

The function ξ_l is measurable and therefore defined almost everywhere. We choose its representant $\widetilde{\xi}_l$, which is defined everywhere in Z.

Fix arbitrary $l, k \in \mathbb{N}$ and let $Q = [-l,l]^d$. We split Q into $\{Q_i^k\}_{i=1}^{N(k)}$ – a family of $N(k)$ cubes Q_i^k of diameter $\frac{1}{k}$. We construct them using the dyadic decomposition of Q, distributing the boundary parts so that the obtained cubes Q_i^k are pairwise disjoint, not necessarily open or closed, but obviously Borel sets and such that $Q = \bigcup_{i=1}^{N(k)} Q_i^k$. We define

$$E_i^k = \widetilde{\xi}_l^{-1}(Q_i^k).$$

Then we have $Z = \bigcup_{i=1}^{N(k)} E_i^k$. Since Q_i^k is Borel and $\widetilde{\xi}_l$ is measurable, the set E_i^k is measurable as well. Note that the family $\{E_i^k\}_{i=1}^{N(k)}$ is a division of Z into pairwise disjoint measurable sets.

For any i, k, l we choose an arbitrary

$$\zeta_i^{l,k} \in Q_i^k$$

with rational coordinates. Note that this is possible because $\operatorname{int} Q_i^k \ne \emptyset$. We set

$$\xi_{l,k} = \sum_{i=1}^{N(k)} \zeta_i^{l,k} \mathbb{1}_{E_i^k}.$$

For every $z \in E_i^k$ we have $\xi_{l,k}(z) = \zeta_i^{l,k}$ and then

$$|\zeta_i^{l,k} - \widetilde{\xi}_l(z)| \le \operatorname{diam} \overline{Q_i^k} \le \tfrac{1}{k}.$$

In turn, we have $\xi_{l,k}(z) \longrightarrow \xi_l(z)$ as $k \longrightarrow \infty$ for almost every $z \in Z$. On the other hand, for every $z \in E_i^k$ we have

$$M\left(z, \frac{\xi_{l,k}(z)}{\alpha}\right) = M\left(z, \frac{\zeta_i^{l,k}}{\alpha}\right) \le m_2\left(\frac{|\zeta_i^{l,k}|}{\alpha}\right) \le m_2\left(\frac{l}{\alpha}\right),$$

where m_2 is given in the definition of an N-function. Then due to Jensen's inequality we have

$$M\left(z, \frac{\xi_{l,k}-\xi_l}{2\alpha}\right) \le \frac{1}{2} M\left(z, \frac{\xi_{l,k}}{\alpha}\right) + \frac{1}{2} M\left(z, \frac{\xi_l}{\alpha}\right)$$
$$\le \frac{1}{2} m_2\left(\frac{l}{\alpha}\right) + \frac{1}{2} M\left(z, \frac{\xi}{\alpha}\right),$$

where on the right-hand side we have a sum of functions integrable over Z. Then since M is continuous with respect to the second variable the Lebesgue dominated convergence theorem implies

$$\int_Z M\left(z, \frac{\xi_{l,k}-\xi_l}{2\alpha}\right) \mathrm{d}z \xrightarrow[k\to\infty]{} 0. \tag{3.43}$$

Since $\{\xi_l\}_{l\in\mathbb{N}}$ is a bounded sequence, it also satisfies condition (ii) of Lemma 3.4.10 with $\lambda = 2\alpha$ and thus the diagonal argument completes the proof. □

Corollary 3.4.12 *The set of measurable simple functions in Z is dense in*

(i) $E_M(Z;\mathbb{R}^d)$ *with respect to the norm (strong) topology,*
(ii) $L_M(Z;\mathbb{R}^d)$ *with respect to the modular topology.*

Proof. Note that (ii) is a rephrasing of Theorem 3.4.11. To obtain (i) from it we make use of the characterization of E_M from Lemma 3.1.8 and the condition equivalent to the norm convergence from Lemma 3.1.19. □

Lemma 3.4.13 *There exists a countable set of simple functions S such that every measurable simple function* $\xi : Z \to \mathbb{Q}^d$ *can be approximated in the modular topology by elements of S.*

Proof. Since we can choose a countable family of open sets which generate the σ-algebra of Borel sets, a simple function can be approximated in measure by functions from a countable set. □

Theorem 3.4.14 (Strong separability of E_M) *Let Z be a bounded subset of* $\mathbb{R}^d$ *and let M be an N-function. The space* $E_M(Z;\mathbb{R}^d)$ *is separable with respect to the strong topology.*

Proof. By Corollary 3.4.12 the space $E_M(Z;\mathbb{R}^d)$ contains a dense set, which has a countable dense subset due to Lemma 3.4.13. □

Corollary 3.4.12 together with Lemma 3.4.13 also give us the following.

Corollary 3.4.15 (Modular separability of L_M) *Let Z be a bounded subset of* $\mathbb{R}^d$ *and let M be an N-function. The space* $L_M(Z;\mathbb{R}^d)$ *is separable with respect to the modular topology.*

Let us note that in some cases the proof of Theorem 3.4.11 can be much simpler. We prove the following result under the condition of anisotropy, which can be expressed by decomposition, describing each of the directions separately, including the isotropic case. Note that in case the decomposition from Remark 2.3.1 holds the proof is essentially the same.

Theorem 3.4.16 (Modular density of simple functions – orthotropic case) *Suppose $M : Z\times\mathbb{R}^d \to [0,\infty)$ is an N-function that admits the representation* (2.41)*, i.e. for a.a. $z \in Z$ and all $\xi = (\xi^1,\dots,\xi^d) \in \mathbb{R}^d$,*

$$M(z,\xi) = \sum_{i=1}^{d} M_i(z,\xi^i),$$

where $M_i : Z\times[0,\infty) \to [0,\infty)$, $i = 1,\dots,d$, are isotropic N-functions. Then the set of measurable simple functions integrable on Z is dense in $L_M(Z;\mathbb{R}^d)$ with respect to the modular topology.

Proof. We denote the set of simple functions integrable on Z by L^S. By Lemma 3.1.21 we infer that $L^S \subset E_M(Z;\mathbb{R}^d)$. We shall proceed with the directions separately. Fix an arbitrary $i \in \{1,\dots,d\}$. Let a nonzero $\xi^i \in L_{M_i}(Z;\mathbb{R})$ and $\lambda > 0$ be such that $M_i\left(z,\xi^i(z)/\lambda\right) \in L^1(Z)$. Suppose for a moment that for every z we have $\xi^i(z) \in [0,\infty)$. Take a sequence

$$\{\xi^i_n(z)\}_{n\in\mathbb{N}} \subset L^S \text{ such that } 0 \le \xi^i_n(z) \nearrow \xi^i(z) \text{ when } n \to \infty$$

for almost every $z \in Z$ and each coordinate $i = 1,\dots,d$. Then

$$M_i\left(z,\frac{\xi^i_n(z)-\xi^i(z)}{2\lambda}\right) \xrightarrow[n\to\infty]{} 0 \quad \text{for a.a. } z \in Z$$

and due to Jensen's inequality

$$\begin{aligned} M_i\left(z,\frac{\xi^i_n(z)-\xi^i(z)}{2\lambda}\right) &\le \frac{1}{2}\left[M_i\left(z,\frac{\xi^i_n(z)}{\lambda}\right)+M_i\left(z,\frac{\xi^i(z)}{\lambda}\right)\right] \\ &\le M_i\left(z,\frac{\xi^i(z)}{\lambda}\right), \end{aligned}$$

which is integrable. Hence, the Lebesgue dominated convergence theorem gives the desired convergence. To dispense with the assumption $\xi^i(z) \in [0,\infty)$ we decompose each of the coordinates into positive and negative parts which belong to $L_{M_i}(Z;\mathbb{R})$. □

Remark 3.4.17. Let us note that the above proof cannot be directly used in order to solve the issue in the fully anisotropic case, since the fact that $\xi^i_n \le \xi^i$ for every $i = 1,\dots,d$ *does not imply* that $M(z,\xi_n) \le M(z,\xi)$.

3.5 Duality $(E_M)^* = L_{M^*}$

This section is devoted to the issue of duality. For a study of the isotropic Orlicz case we refer to [5, Section 8] and for related results in anisotropic Musielak–Orlicz spaces we refer to [326, Section 2].

Lemma 3.5.1 *Whenever $\eta \in L_{M^*}(Z;\mathbb{R}^d)$, the linear functional F_η given by*

$$F_\eta(\xi) = \int_Z \xi(z) \cdot \eta(z)\,\mathrm{d}z \tag{3.44}$$

is well defined for all $\xi \in E_M(Z;\mathbb{R}^d)$ and belongs to $(E_M(Z;\mathbb{R}^d))^*$. Moreover, its norm in this space, which is defined by

$$\|F_\eta\|_{(E_M)^*} = \sup\left\{|F_\eta(\xi)| : \ \xi \in E_M(Z;\mathbb{R}^d),\ \|\xi\|_{L_M} \le 1\right\}, \tag{3.45}$$

satisfies

$$\|F_\eta\|_{(E_M)^*} = \|\eta\|_{L_{M^*}}. \tag{3.46}$$

Proof. Recall that by Lemma 3.1.14 (i) for all $\xi \in L_M(Z;\mathbb{R}^d)$

$$\|\xi\|_{L_M} \le 1 \implies \int_Z M(z,\xi)\,\mathrm{d}z \le 1. \tag{3.47}$$

The definition of the Luxemburg norm implies that the converse implication also holds, thus we infer that

$$\int_Z M(z,\xi)\,\mathrm{d}z \le 1 \iff \|\xi\|_{L_M} \le 1. \tag{3.48}$$

This observation allows us to rewrite the definition of an Orlicz norm in $L_{M^*}(Z;\mathbb{R}^d)$ as follows

$$|||\eta|||_{L_{M^*}} = \sup\left\{\int_Z \xi(z) \cdot \eta(z)\,\mathrm{d}z : \xi \in L_M(Z;\mathbb{R}^d), \|\xi\|_{L_M} \le 1\right\}, \tag{3.49}$$

which implies that

$$|||\eta|||_{L_{M^*}} \ge \|F_\eta\|_{(E_M)^*}.$$

To show the opposite inequality we define for $\xi \in L_M(Z;\mathbb{R}^d)$ a sequence

$$\xi_n(z) = \begin{cases} \xi(z) & \text{if } |\xi_n(z)| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $\|\xi_n\|_{L_M} \le \|\xi\|_{L_M}$ for each $n \in \mathbb{N}$. Lemma 3.1.21 implies that $\{\xi_n\}_{n\in\mathbb{N}} \subset E_M(Z;\mathbb{R}^d)$. If $n \to \infty$, then $\xi_n \xrightarrow{M} \xi$ in $L_M(Z;\mathbb{R}^d)$. Consequently, by Lemma 3.4.6, the convergence

$$\int_Z \eta(z) \cdot \xi_n(z)\,\mathrm{d}z \to \int_Z \eta(z) \cdot \xi(z)\,\mathrm{d}z$$

holds and thus

$$|||\eta|||_{L_{M^*}} = \sup\left\{\int_Z \xi(z) \cdot \eta(z)\,\mathrm{d}z : \ \xi \in E_M(Z;\mathbb{R}^d),\ \|\xi\|_{L_M} \le 1\right\}, \tag{3.50}$$

whereas the right-hand side is equal to $\|F_\eta\|_{(E_M)^*}$, which completes the proof. □

Corollary 3.5.2 *From Lemma 3.1.13 and Lemma 3.5.1 we infer that*

$$\|\eta\|_{L_{M^*}} \le \|F_\eta\|_{(E_M)^*} \le 2\|\eta\|_{L_{M^*}}. \tag{3.51}$$

The fundamental structural theorem on the predual space to the Musielak–Orlicz space reads as follows. Its proof is based on the ideas of [5, 326].

Theorem 3.5.3 (Duality $(E_M)^* = L_{M^*}$**)** *If $Z \subset \mathbb{R}^d$ is a bounded set and M is an N-function, then the generalized Musielak–Orlicz space $L_{M^*}(Z;\mathbb{R}^d)$ is the dual space to $E_M(Z;\mathbb{R}^d)$.*

Proof. We already noticed in (3.44)–(3.51) that any $\eta \in L_{M^*}(Z;\mathbb{R}^d)$ defines a bounded linear functional F_η on $E_M(Z;\mathbb{R}^d)$. We start with the observation that inequality (3.51) shows that F cannot be represented by a function from a broader space than $L_{M^*}(Z;\mathbb{R}^d)$. It suffices to show that every bounded linear functional on $E_M(Z;\mathbb{R}^d)$ has the form F_η from (3.44) for a certain $\eta \in L_{M^*}(Z;\mathbb{R}^d)$.

Let us fix $F \in \big(E_M(Z;\mathbb{R}^d)\big)^*$ and define a vector-valued measure $\mu = (\mu_1, \dots, \mu_d)$ on the measurable subsets Y of Z by setting

$$\mu_i(Y) = e_i \cdot \mu(Y) = F(e_i \mathbb{1}_Y).$$

We start by showing that μ_i for $i = 1, \dots, d$ is indeed a signed Borel measure, $\mu_i : \mathcal{B}(Z) \to \mathbb{R}$, where by $\mathcal{B}(Z)$ we mean the smallest σ-algebra that contains the open sets of Z. We need to check that the following conditions hold:
1° $\mu_i(\emptyset) = 0$,
2° if $\{Y_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(Z)$ such that $Y_i \cap Y_j = \emptyset$ for $i \ne j$, then

$$\mu_i\left(\bigcup_{j=1}^{\infty} Y_j\right) = \sum_{j=1}^{\infty} \mu_i(Y_j)$$

holds for all $i = 1, \dots, d$.

The first condition immediately follows from the definition of μ_i. To prove the second one consider first the finite sums. Indeed,

$$\mu_i\left(\bigcup_{j=1}^{n} Y_j\right) = F(e_i \mathbb{1}_{\bigcup_{j=1}^n Y_j}) = F\left(\sum_{j=1}^{n} e_i \mathbb{1}_{Y_j}\right) = \sum_{j=1}^{n} \mu_i(Y_j).$$

Consider next the positive and negative part of μ_i, where $|\mu_i| = (\mu_i)_+ - (\mu_i)_-$. We distinguish the sets where μ_i is positive and negative

$$Y_j^+ = \begin{cases} Y_j & \text{if } \mu_i(Y_j) \ge 0, \\ \emptyset & \text{if } \mu_i(Y_j) < 0, \end{cases} \qquad Y_j^- = \begin{cases} Y_j & \text{if } \mu_i(Y_j) < 0, \\ \emptyset & \text{if } \mu_i(Y_j) \ge 0. \end{cases}$$

Then for all $n \in \mathbb{N}$ it holds that

$$\sum_{j=1}^{n} (\mu_i(Y_j))_+ = \mu_i\left(\bigcup_{j=1}^{n} Y_j^+\right) \le \|F\|_{(E_M)^*} \|e_i \mathbb{1}_{\bigcup_{j=1}^n Y_j^+}\|_{E_M} \le \|F\|_{(E_M)^*} \|e_i \mathbb{1}_Z\|_{E_M}.$$

In the same manner

$$-\sum_{j=1}^{n}(\mu_i(Y_j))_- = -\mu_i\left(\bigcup_{j=1}^{n} Y_j^-\right) \le \|F\|_{(E_M)^*}\|e_i \mathbb{1}_{\bigcup_{j=1}^{n} Y_j^-}\|_{E_M} \le \|F\|_{(E_M)^*}\|e_i \mathbb{1}_Z\|_{E_M}.$$

Finally

$$\sum_{j=1}^{n} |\mu_i(Y_j)| \le 2\|F\|_{(E_M)^*}\|e_i \mathbb{1}_Z\|_{E_M} < \infty$$

and thus the series converges and

$$\mu_i\left(\bigcup_{j=1}^{\infty} Y_j\right) = \lim_{n\to\infty}\mu_i\left(\bigcup_{j=1}^{n} Y_j\right) = \lim_{n\to\infty}\sum_{j=1}^{n}\mu_i(Y_j) = \sum_{j=1}^{\infty}\mu_i(Y_j).$$

In order to apply the Radon–Nikodym theorem in the next step we will show that the measure is absolutely continuous with respect to Lebesgue measure. Let Y be an arbitrary Borel subset of Z. In the estimates we will use $m_2 : [0,\infty) \to [0,\infty)$ – a majorant of M from the definition of an N-function. We notice that

$$\begin{aligned}
1 &= \int_Y \frac{1}{|Y|}\,\mathrm{d}z = \int_Y m_2 \circ m_2^{-1}\left(\frac{1}{|Y|}\right)\mathrm{d}z \\
&\ge \int_Z M\left(z, m_2^{-1}(1/|Y|)e_i\right)\mathrm{d}z \ge \int_Y M\left(z, m_2^{-1}(1/|Y|)e_i\right)\mathrm{d}z \\
&= \int_Z M\left(z, m_2^{-1}(1/|Y|)e_i \mathbb{1}_Y\right)\mathrm{d}z
\end{aligned}$$

and therefore, due to the definition of the Luxemburg norm we infer that

$$\frac{1}{m_2^{-1}(1/|Y|)} \ge \|e_i \mathbb{1}_Y\|_{L_M}.$$

Hence

$$|e_i \cdot \mu(Y)| = |F(e_i \mathbb{1}_Y)| \le \|F\|_{(E_M)^*}\|e_i \mathbb{1}_Y\|_{L_M} \le \frac{\|F\|_{(E_M)^*}}{m_2^{-1}(1/|Y|)}$$

and

$$|\mu_i(Y)| \le |e_i \cdot \mu(Y)| \le \frac{\|F\|_{(E_M)^*}}{m_2^{-1}(1/|Y|)}.$$

Since the right-hand side tends to zero when $|Y| \to 0$, the measure μ_i is absolutely continuous with respect to Lebesgue measure. Hence the Radon–Nikodym theorem (Theorem 8.14) implies that μ_i has the form

$$\mu_i(Y) = \int_Y \eta(z)\,\mathrm{d}z$$

for some $\eta \in L^1(Z;\mathbb{R}^d)$. Then obviously

$$F(\xi) = \int_Z \xi(z)\eta(z)\,\mathrm{d}z$$

for every measurable simple function ξ.

In the remaining part of the proof we show that $\eta \in L_{M^*}$. By Corollary 3.4.12 for any $\xi \in E_M(Z;\mathbb{R}^d)$ we can find a sequence of measurable, simple functions $\{\xi_n\}_{n\in\mathbb{N}}$ converging to ξ in the norm topology of $E_M(Z;\mathbb{R}^d)$. Therefore there exists a subsequence such that ξ_n converges a.e. in Z. Let us define a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ as follows

$$\eta_k(z) = \begin{cases} k\frac{\eta}{|\eta|} & \text{for } |\eta| \geq k, \\ \eta & \text{for } |\eta| < k. \end{cases} \tag{3.52}$$

Due to the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}
\int_Z \xi(z)\cdot\eta_k(z)\,\mathrm{d}z &= \lim_{n\to\infty}\int_Z \xi_n(z)\cdot\eta_k(z)\,\mathrm{d}z \\
&\leq \lim_{n\to\infty}\int_Z \xi_n(z)\mathbb{1}_{\{\xi_n(z)\cdot\eta(z)\geq 0\}}(z)\cdot\eta(z)\,\mathrm{d}z \\
&\leq \lim_{n\to\infty}\|\xi_n\mathbb{1}_{\{\xi_n\cdot\eta\geq 0\}}\|_{E_M}\|F\|_{(E_M)^*} \\
&\leq \lim_{n\to\infty}\|\xi_n\|_{E_M}\|F\|_{(E_M)^*} \\
&\leq \|\xi\|_{E_M}\|F\|_{(E_M)^*}.
\end{aligned}$$

With the help of (3.50) we get that $|||\eta_k|||_{L_{M^*}} \leq \|F\|_{(E_M)^*}$. Lemma 3.1.13 then implies that $\|\eta_k\|_{L_{M^*}} \leq \|F\|_{(E_M)^*}$. From the definition of the Luxemburg norm, for all $k\in\mathbb{N}$

$$\int_Z M^*\left(\frac{\eta_k(z)}{\|F\|_{(E_M)^*}}\right)\mathrm{d}z \leq 1.$$

By Fatou's lemma

$$\int_Z M^*\left(z,\frac{\eta(z)}{\|F\|_{(E_M)^*}}\right)\mathrm{d}z \leq \liminf_{k\to\infty}\int_Z M^*\left(z,\frac{\eta_k(z)}{\|F\|_{(E_M)^*}}\right)\mathrm{d}z \leq 1.$$

We know that the functional F_η, with $\eta \in L_{M^*}(Z;\mathbb{R}^d)$, given by (3.44) is bounded on $E_M(Z;\mathbb{R}^d)$. Since F_η and F achieve the same values on the set of measurable simple functions and, due to Theorem 3.4.11, this set is dense in $E_M(Z;\mathbb{R}^d)$, we infer that $F_\eta = F$ on $E_M(Z;\mathbb{R}^d)$. □

Remark 3.5.4. In view of Hölder's inequality (Lemma 3.1.15), the spaces $L_M(Z;\mathbb{R}^d)$ and $L_{M^*(Z;\mathbb{R}^d)}$ are sometimes called associate spaces [34]. This means that $\langle\xi,\eta\rangle = \int_Z \xi\cdot\eta\,\mathrm{d}z$ is well defined for $\xi \in L_M(Z;\mathbb{R}^d)$ and $\eta \in L_{M^*}(Z;\mathbb{R}^d)$. Observe that $(E_{M^*}(Z;\mathbb{R}^d))^* = L_M(Z;\mathbb{R}^d)) \subset (L_{M^*}(Z;\mathbb{R}^d))^*$ since $E_{M^*}(Z;\mathbb{R}^d)$ is a closed subspace of $L_{M^*}(Z;\mathbb{R}^d)$. Later, in Theorem 3.5.3, we prove that $L_M(Z;\mathbb{R}^d)$ is in general not a dual space to $L_{M^*}(Z;\mathbb{R}^d)$ and vice versa.

Corollary 3.5.5 *If i_M and s_M given by* (2.52) *satisfy*

$$1 < i_- \leq i_M \leq s_M \leq s_+ < \infty,$$

then the space equipped with the modular function M is reflexive.

Proof. Inequality $1 < i_- \leq i_M \leq s_M \leq s_+ < \infty$ implies that $M, M^* \in \Delta_2$. Then Remark 3.3.3 gives the claim. □

3.6 Function Spaces in PDEs

In applications of Musielak–Orlicz spaces to the theory of partial differential equations we frequently face the situation that a gradient of a function is an element of a Musielak–Orlicz class or space. For this purpose we introduce the so-called Musielak–Orlicz–Sobolev spaces. Below we divide the description into different types of domains that correspond to PDE problems studied in further chapters.

Ω – open and bounded set. Similarly as in the case of Musielak–Orlicz spaces, we distinguish among different objects in a manner which is analogous to the way we defined the spaces $E_M(\Omega;\mathbb{R}^d)$ and $L_M(\Omega;\mathbb{R}^d)$. Thus, we introduce the following notation

$$W_0^1 E_M(\Omega;\mathbb{R}^d) := \overline{C_c^\infty(\Omega;\mathbb{R}^d)}^{\|\cdot\|_{W_0^1 L_M(\Omega)}},$$

with $\Omega \subset \mathbb{R}^N$, where we endow the spaces with the norm $\|\mathbf{v}\|_{W_0^1 L_M(\Omega)} := \|\nabla \mathbf{v}\|_{L_M(\Omega)}$. The fact that it is indeed a norm is a direct consequence of the Poincaré inequality in $W^{1,1}(\Omega)$.

Next, we introduce the weak-$*$ closures of compactly supported smooth functions, i.e.,

$$\begin{aligned} W_0^1 L_M(\Omega;\mathbb{R}^d) := \{&\mathbf{u} \in W_0^{1,1}(\Omega;\mathbb{R}^d) :\ \nabla \mathbf{u} \in L_M(\Omega;\mathbb{R}^{d\times N}) \\ &\text{and } \exists \{\mathbf{u}^n\}_{n=1}^\infty \subset C_c^\infty(\Omega;\mathbb{R}^d) :\ \nabla \mathbf{u}^n \overset{*}{\rightharpoonup} \nabla \mathbf{u} \text{ in } L_M(\Omega;\mathbb{R}^{d\times N})\}. \end{aligned} \tag{3.53}$$

The above spaces are referred to as the Musielak–Orlicz–Sobolev spaces. Notice that when a gradient is considered in the anisotropic space, the function itself can be assumed to belong to various different isotropic spaces. In the anisotropic Orlicz–Sobolev case we can use symmetrization techniques to get an optimal Sobolev embedding [93], but in anisotropic and inhomogeneous Musielak–Orlicz–Sobolev spaces there is no such result.

Thus, as we will see in further chapters, the spaces prescribed by (3.53) may be too small in principle and therefore we introduce a different class of Musielak–Orlicz–Sobolev spaces by

$$V_0^M(\Omega) := \left\{ \mathbf{v} \in W_0^{1,1}(\Omega;\mathbb{R}^d) :\ \nabla \mathbf{v} \in L_M(\Omega;\mathbb{R}^{d\times N}) \right\}.$$

These spaces will be again equipped with the norms $\|\mathbf{v}\|_{V_0^M(\Omega)} = \|\nabla \mathbf{v}\|_{L_M(\Omega)}$, which makes them Banach spaces. Whether the considered functions are scalar- or vector-valued ($d = 1$ or $d > 1$) will be clear from the context and this ambiguity does not affect the clarity of presentation.

For some purposes we will need to employ classical Orlicz–Sobolev spaces generated by an isotropic homogeneous N-function $m : [0,\infty) \to [0,\infty)$ defined as

$$W^1L_m(\Omega) = \{u \in W^{1,1}(\Omega) : \ u, |\nabla u| \in L_m(\Omega)\}, \tag{3.54}$$

where the Orlicz space L_m is defined as L_M in Definition 3.1.3 with $m = M$, which is equipped with Luxemburg norm from (3.4). On the other hand, for a function $u \in W^1L_m(\Omega)$ we define the norm

$$\|u\|_{W^1L_m(\Omega)} = \|u\|_{L_m(\Omega)} + \|\nabla u\|_{L_m(\Omega)}.$$

On substituting $L_m(\Omega)$ with $E_m(\Omega)$ or $\mathcal{L}_m(\Omega)$ in (3.54), we can define spaces $W^1E_m(\Omega)$ and $W^1\mathcal{L}_m(\Omega)$, respectively. In the case of doubling m, i.e. when $m, m^* \in \Delta_2$, all of them coincide, so we use the notation $W^{1,m}(\Omega) := W^1E_m(\Omega) = W^1\mathcal{L}_m(\Omega) = W^1L_m(\Omega)$. The space $W_0^1L_m(\Omega)$ is defined as the weak-$*$ closure of $C_c^\infty(\Omega)$ in $W^1L_m(\Omega)$.

Space-time cylinder $\Omega_T := (0,T) \times \Omega$. For parabolic problems, we employ the following spaces

$$\begin{aligned} V_T^M(\Omega) &:= \{u \in L^1(0,T; W_0^{1,1}(\Omega)) : \ \nabla u \in L_M(\Omega_T; \mathbb{R}^N)\}, \\ V_T^{M,\infty}(\Omega) &:= \{u \in L^\infty(0,T; L^2(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega)) : \ \nabla u \in L_M(\Omega_T; \mathbb{R}^N)\} \\ &= V_T^M(\Omega) \cap L^\infty(0,T; L^2(\Omega)), \end{aligned}$$

which are Banach spaces according to the same arguments.

Periodic case. Let $Y = (0,1)^d$. For the purpose of the last chapter we recall the definition of the Sobolev space of periodic functions

$$W_{per}^{1,1}(Y; \mathbb{R}^d) := \overline{\{\mathbf{v} \in C_{per}^\infty(Y; \mathbb{R}^d) : \int_Y \mathbf{v} = 0\}}^{\|\cdot\|_{1,1}}.$$

Due to the Poincaré inequality, we always choose an equivalent norm on $W_{per}^{1,1}$ as $\|\mathbf{v}\|_{1,1} := \|\nabla \mathbf{v}\|_1$. Further, we define the corresponding spaces in the periodic setting

$$W_{per}^1 E_M(Y) := \overline{\left\{\mathbf{v} \in C_{per}^\infty(Y; \mathbb{R}^d) : \int_Y \mathbf{v} = 0\right\}}^{\|\cdot\|_{W_{per}^1 L_M(Y)}},$$

where we endow the spaces with the norm $\|\mathbf{v}\|_{W_{per}^1 L_M(Y)} := \|\nabla \mathbf{v}\|_{L_M(Y)}$. Again we introduce the weak-$*$ closures of smooth periodic functions

$$W^1_{per}L_M(Y;\mathbb{R}^d) := \{\mathbf{u} \in W^{1,1}_{per}(Y;\mathbb{R}^d) : \nabla\mathbf{u} \in L_M(Y;\mathbb{R}^{d\times N}) \\ \text{and } \exists\{\mathbf{u}^n\}_{n=1}^\infty \subset C^\infty_{per}(Y;\mathbb{R}^d) \text{ such that } \int_Y \mathbf{u}^n = 0 \\ \text{and } \nabla\mathbf{u}^n \overset{*}{\rightharpoonup} \nabla\mathbf{u} \text{ in } L_M(Y;\mathbb{R}^{d\times N})\}.$$

and their larger analogues

$$V^M_{per}(Y) := \left\{\mathbf{v} \in W^{1,1}_{per}(Y;\mathbb{R}^d) : \nabla\mathbf{v} \in L_M(Y;\mathbb{R}^{d\times N})\right\}$$

equipped with the norm $\|\mathbf{v}\|_{V^M_{per}(Y)} = \|\nabla\mathbf{v}\|_{L_M(Y)}$, which makes them Banach spaces.

Divergence-free functions. We define the spaces of mappings having zero divergence, both in a bounded set and in a periodic setting, as

$$E^{\mathrm{div}}_M(\Omega;\mathbb{R}^{d\times N}) := \overline{\{C^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^{d\times N})\}}^{\|\cdot\|_{L_M(\Omega)}},$$

$$E^{\mathrm{div}}_{M,per}(Y;\mathbb{R}^{d\times N}) := \overline{\{C^\infty_{per,\mathrm{div}}(Y;\mathbb{R}^{d\times N})\}}^{\|\cdot\|_{L_M(Y)}},$$

and

$$L^{\mathrm{div}}_M(\Omega;\mathbb{R}^{d\times N}) := \{\mathbf{T} \in L_M(\Omega;\mathbb{R}^{d\times N}) : \exists\{\mathbf{T}^n\}_{n=1}^\infty \subset E^{\mathrm{div}}_M(\Omega;\mathbb{R}^{d\times N}) \\ \text{such that } \mathbf{T}^n \overset{*}{\rightharpoonup} \mathbf{T} \text{ in } L_M(\Omega;\mathbb{R}^{d\times N})\},$$
$$L^{per,\mathrm{div}}_M(Y;\mathbb{R}^{d\times N}) := \{\mathbf{T} \in L^{per}_M(Y;\mathbb{R}^{d\times N}) : \exists\{\mathbf{T}^n\}_{n=1}^\infty \subset E^{per,\mathrm{div}}_M(Y;\mathbb{R}^{d\times N}) \\ \text{such that } \mathbf{T}^n \overset{*}{\rightharpoonup} \mathbf{T} \text{ in } L_M(Y;\mathbb{R}^{d\times N})\},$$

which are again Banach spaces.

Truncations. In many cases, the solutions to considered PDE problems do not belong to the spaces defined above, but their truncations do. The symmetric truncation T_k at level k is defined as follows

$$T_k(f)(s) := \begin{cases} f(s) & |f(s)| \le k, \\ k\frac{f(s)}{|f(s)|} & |f(s)| \ge k. \end{cases} \tag{3.55}$$

We may naturally expect a solution to an elliptic isotropic problem to belong to

$$\mathcal{T}V^M_0(\Omega) = \{u \text{ is measurable in } \Omega : \\ T_k(u) \in W^{1,1}_0(\Omega),\ \nabla T_k(u) \in L_M(\Omega;\mathbb{R}^N) \text{ for every } k > 0\}.$$

Since for every $u \in W^{1,1}(\Omega)$, there exist a unique measurable function $Z_u : \Omega \to \mathbb{R}^N$ such that

$$\nabla(T_t(u)) = \mathbb{1}_{\{|u|<t\}}Z_u \quad \text{a.e. in } \Omega, \text{ for every } t > 0, \tag{3.56}$$

see [31, Lemma 2.1]. Thus, in the sequel we call Z_u the generalized gradient of u and, abusing the notation, for u in the space of truncations, we write simply ∇u instead of Z_u.

This will be particularly important when we compare renormalized or entropy solutions to SOLA. Indeed, the notion of SOLA takes into account only $u \in W^{1,1}_{loc}(\Omega)$, which in the case of equations involving the p-Laplace operator requires us to restrict to $p > 2 - 1/N$. See Section 5.1.1 for more information on this topic.

3.7 Density and Approximation

One of the important features of inhomogeneous settings that play a significant role in the analysis of PDEs posed in Musielak–Orlicz spaces are problems involving the density of smooth functions. This is closely related to the so-called Lavrentiev phenomenon, cf. [229, 336, 337], which originally described the situation when the infimum of the variational problem over the regular functions (e.g. smooth or Lipschitz) is strictly greater than the infimum taken over the set of all functions satisfying the same boundary conditions. Naturally, the Lavrentiev phenomenon was generalized to the situation where functions from a certain space cannot be approximated by regular ones. The key issue is therefore to choose an appropriate topology that will be useful. Recall that Section 3.4 explains various possible choices of topologies, whereas the modular convergence is defined in Definition 3.4.3. In view of the gap between E_M and L_M (Definition 3.1.3) and the fact that simple functions are dense in L_M only in the modular topology (Theorem 3.4.11), this notion of topology, rather than the norm topology, is expected to be relevant in the further analysis.

In general, smooth functions are not dense in the norm topology, even in the reflexive Musielak–Orlicz spaces. It is known that the variable exponent spaces (with $M_{\mathrm{v}}(x,\xi) = |\xi|^{p(x)}$) can exhibit the Lavrentiev phenomenon if $p(\cdot)$ is not regular enough (see e.g. [337, Example 3.2], where p is a step function). The canonical assumption ensuring density of smooth functions in the norm topology in the variable exponent spaces is the log-Hölder continuity of the exponent $p(\cdot)$. The double-phase spaces (with $M_{\mathrm{dp}}(x,\xi) = |\xi|^p + a(x)|\xi|^q$ or mild transition $M_{\text{dp-mild}}(x,\xi) = |\xi|^p(1 + a(x)\log(\mathrm{e}+|\xi|))$ can also support the Lavrentiev phenomenon. See [137, 136], where the authors provide the result that a closeness condition for the exponents sufficient for density is governed by regularity of the weight. There are also examples of exponents and functions that cannot be approximated [337, 16, 137, 155]. In the case of M_{dp} when $a \in C^{0,\alpha}$ an easy proof from [137] shows that smooth functions are dense provided $q/p \le 1 + \alpha/N$. Due to [24] in the case of $M_{\text{dp-mild}}$ it suffices to deal with log-Hölder continuous a. The mentioned cases are fully covered by our conditions. However the present studies show that for $p < N$ the result from [137] was not optimal. The optimal range is $q < p + \alpha$, see [62]. Another formalism that also captures them all is described in [191], but unlike in our analysis the growth of the modular function there is always assumed to be comparable with a doubling one and isotropic.

Summing up, in the Musielak–Orlicz setting, even in the already mentioned examples of reflexive spaces, equipped with the not sufficiently regular modular function, there exist functions that cannot be approximated in the strong norm topology by smooth functions. In such a case the strong closure of the smooth functions coincides with the modular closure, but in general this is not true. In the nonreflexive spaces (when the modular function is an N-function of arbitrary growth) the relevant topology to be considered for weak gradients is not the norm topology, but the modular topology. See Section 3.4 for its basic properties. In his seminal paper [175] Gossez proves that the classical theorem due to Meyers and Serrin on the strong density of smooth functions in Sobolev spaces [253] can be proved in Orlicz–Sobolev spaces too, but the density has to be considered with respect to the modular topology. This result has been extended to the isotropic Musielak–Orlicz setting in [7], under restrictions on the modular function, with sharp results in the special cases of variable exponent and double-phase spaces. Its fully anisotropic counterparts are provided in [179] and [52] under various balance conditions.

For an open and bounded domain $\Omega \subset \mathbb{R}^N$ we consider approximation of scalar functions $u : \Omega \to \mathbb{R}$ with $\nabla u \in L_M(\Omega;\mathbb{R}^N)$ where we deal with an N-function (Definitions 2.2.2)

$$M : \Omega \times \mathbb{R}^N \to [0,\infty).$$

A key assumption describes the interplay between the asymptotic behavior with respect to each of the variables separately that ensures the modular density of smooth functions, namely it balances the behavior of $M = M(x,\nabla u)$ for large $|\nabla u|$ and small changes of the first variable x. Note that because of the nature of the condition in the pure Orlicz case, i.e. when

$$M(x,\xi) = M(\xi),$$

where the fully anisotropic case is included, the balance conditions do not carry any information and can be skipped. Therefore, the results on approximation we present hold in general in anisotropic Orlicz spaces without any growth restrictions of doubling type.

We study the approximation properties of the Sobolev-type spaces

$$V_0^M(\Omega) := \{u \in W_0^{1,1}(\Omega) : \ \nabla u \in L_M(\Omega;\mathbb{R}^N)\}.$$

For this we need to study the local behavior of $M : \Omega \times \mathbb{R}^N \to [0,\infty)$, so we consider

$$M_{x,\varepsilon}(\xi) := \operatorname{ess\,inf}_{y \in B(x,\varepsilon) \cap \Omega} M(y,\xi) \tag{3.57}$$

for $\varepsilon > 0$ and $x \in \Omega$ and recall that $(M_{x,\varepsilon})^{**}$ stands for the second conjugate, see (2.36). Recall that the second conjugate of a function is its greatest convex minorant (Corollary 2.1.42).

3.7.1 Condition I (general growth)

We will present the approximation results and proofs first in the anisotropic setting and then their significantly simplified form in the isotropic setting. The phenomenon of anisotropy is discussed in Section 2.3.1.2.

Let us start with the formulations of the conditions and examples.

Anisotropic case
For an N-function $M : \Omega\times\mathbb{R}^N \to [0,\infty)$ we study the following condition.

(Me) Assume that there exists a function $\Theta : [0,\infty)^2 \to [0,\infty)$ such that $\Theta(\cdot,s)$ and $\Theta(x,\cdot)$ are nondecreasing functions and for all $x,y\in\overline{\Omega}$ and ξ such that $|\xi|>1$, and a constant $c>0$,

$$M(y,\xi) \le \Theta(|x-y|,|\xi|)(M_{x,\varepsilon})^{**}(\xi) \text{ with } \limsup_{\varepsilon\to0^+}\Theta(\varepsilon,c\varepsilon^{-N}) < \infty,$$

where $(M_{x,\varepsilon})^{**}$ is the second conjugate to $M_{x,\varepsilon}$, which by Corollary 2.1.42 coincides with its greatest convex minorant.

Isotropic case
For an N-function $M : \Omega\times[0,\infty) \to [0,\infty)$ we study the following condition.

(Me^i) Assume that there exists a function $\Theta^i : [0,\infty)^2 \to [0,\infty)$ such that $\Theta^i(\cdot,s)$ and $\Theta^i(x,\cdot)$ are nondecreasing functions and for all $x,y\in\overline{\Omega}$, $s>1$, and a constant $c>0$,

$$M(y,s) \le \Theta^i(|x-y|,s)M(x,s) \text{ with } \limsup_{\varepsilon\to0^+}\Theta^i(\varepsilon,c\varepsilon^{-N}) < \infty.$$

We point out that this balance condition does not entail continuity of an N-function.

Remark 3.7.1. Observe in particular that

$$\Theta(\tau,s) \ge 1 \text{ for all } (\tau,s)\in[0,1/2]\times[0,\infty).$$

Note that in general the function M satisfying *(Me^i) is not continuous* with respect to its first variable. Actually, only if

$$\lim_{\varepsilon\to0^+}\Theta(\varepsilon,s) = 1$$

for all $s\ge0$, then the mapping $x\mapsto M(x,s)$ is a continuous function on $\overline{\Omega}$.

Example 3.7.2. We have the following examples of pairs M and Θ satisfying *(Me)* or *(Me^i)*, which are therefore admissible in our results on the density of smooth functions.

1. **Orlicz.** If $M(x,\xi)=M(\xi)$ is independent of x, then it obviously satisfies *(Me)* by choosing

$$\Theta(\tau,s)\equiv1.$$

The fully anisotropic case is included.

2. **Variable exponent.** Suppose that $M_{\mathrm{v}}(x,s)=|s|^{p(x)}$, $1<p_-\le p(\cdot)\le p_+<\infty$, satisfies *(Me^i)* with
$$\Theta(\tau,s)=\max\{s^{\omega(\tau)},s^{-\omega(\tau)}\},\tag{3.58}$$
where $\omega(\tau)=c/(\log(1/\tau))$ is the modulus of continuity of p. This is ensured when p is log-Hölder continuous, i.e. when there exists a $c>0$ such that
$$|p(x)-p(y)|\le -\frac{c}{\log(|x-y|)}\qquad\text{for}\qquad |x-y|<\frac{1}{2}.$$
For comments on the sharpness, see [100] or [115].
3. **Borderline double-phase.** When $M_{\mathrm{dp-mild}}(x,s)=|s|^p+a(x)|s|^p\log(e+|s|)$ (cf. [24]), condition *(Me^i)* is satisfied with
$$\Theta(\tau,s)=1+\omega_a(\tau)\log(e+s),\tag{3.59}$$
where $\omega_a(\tau)$ is the modulus of continuity of a. For this it is enough to deal with log-Hölder continuous a.
4. **Orlicz double-phase.** Suppose $M(x,\xi)=M_1(\xi)+a(x)M_2(\xi)$, where M_1,M_2 are (possibly anisotropic) homogeneous N-functions (without prescribed growth) such that
$$M_1(\xi)\le cM_2(\xi)\quad\text{for}\quad \xi\in\mathbb{R}^N\text{ such that }|\xi|>1\text{ and some }c>0,$$
the function $a:\Omega\longrightarrow[0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a. Then one can consider
$$\Theta(\tau,s)=1+\omega_a(\delta)\frac{\overline{M}_2(s)}{\underline{M}_1(s)},\tag{3.60}$$
where $\underline{M}_1(s):=\inf_{\xi:\,|\xi|=s}M_1(\xi)$ and $\overline{M}_2(s):=\sup_{\xi:\,|\xi|=s}M_2(\xi)$. The function M satisfies *(Me)* if
$$\limsup_{\delta\to0}\omega_a(\delta)\frac{\overline{M}_2(\delta^{-N})}{\underline{M}_1(\delta^{-N})}<\infty.$$
5. **Musielak–Orlicz.** The function $M(x,\xi)=\sum_{i=1}^K k_i(x)M_i(|\xi|)+M_0(x,|\xi|)$ satisfies *(Me^i)* if for all $i=1,\cdots,K$ there exist functions $k_i:\Omega\longrightarrow[0,\infty)$ and Θ^i satisfying
$$k_i(x)\le\Theta^i(|x-y|)k_i(y)\qquad\text{with}\qquad \lim_{\varepsilon\to0^+}\Theta^i(\varepsilon)<\infty,$$
whereas $M_0(x,\xi)$ satisfies *(Me^i)* with Θ_0. Then, we can take
$$\Theta(\tau,s)=\sum_{i=0}^K\Theta_i(\tau,s).$$
Similar examples of orthotropic M satisfying *(Me)* are provided by $M(x,\xi)=\sum_{i=1}^N k_i(x)M_i(\xi^i)+M_0(x,|\xi|)$.

Proof.

1. This case is a direct consequence of *(Me)*, which in the homogeneous case does not carry any information (there is nothing to balance).
2. We have

$$\frac{M_{\mathrm{v}}(x,s)}{M_{\mathrm{v}}(y,s)} = s^{p(x)-p(y)},$$

thus *(Me)* is satisfied with

$$\Theta(\tau,s) = s^{\sigma(\tau)} \text{ if } s \geq 1 \quad \text{and} \quad \Theta(\tau,s) = s^{-\sigma(\tau)} \text{ if } s < 1,$$

where $\sigma : [0,\infty) \to [0,\infty)$ with $\limsup_{\varepsilon\to 0} \sigma(\varepsilon) = 0$.

3. If $M_{\mathrm{dp-mild}}(x,s) = s^p(1+a(x)\log(\mathrm{e}+s))$, we have

$$\begin{aligned}\frac{M_{\text{dp-mild}}(x,s)}{M_{\text{dp-mild}}(y,s)} &= \frac{1+a(x)\log(\mathrm{e}+s)}{1+a(y)\log(\mathrm{e}+s)}\\ &= \frac{1+a(y)\log(\mathrm{e}+s)+(a(x)-a(y))\log(\mathrm{e}+s)}{1+a(y)\log(\mathrm{e}+s)}\\ &= 1+\frac{a(x)-a(y)}{1+a(y)}\log(\mathrm{e}+s) \leq 1+\omega_a(|x-y|)\log(\mathrm{e}+s).\end{aligned}$$

4. We compute

$$\begin{aligned}\frac{M(x,s)}{M(y,s)} &= \frac{\sum_{i=1}^k k_i(x)M_i(s)+M_0(x,s)}{\sum_{j=0}^k k_j(y)M_j(s)+M_0(y,s)}\\ &\leq \frac{\sum_{i=1}^k \Theta_i(|x-y|)k_i(y)M_i(s)}{\sum_{j=0}^k k_j(y)M_j(s)} + \frac{M_0(x,s)}{M_0(y,s)}\\ &\leq \sum_{i=1}^k \Theta_i(|x-y|)\frac{k_i(y)M_i(s)}{\sum_{j=1}^k k_j(y)M_j(s)} + \Theta_0(|x-y|,|s|)\\ &\leq \sum_{j=1}^k \Theta_j(|x-y|)\frac{\sum_{i=1}^k k_i(y)M_i(s)}{\sum_{j=1}^k k_j(y)M_j(s)} + \Theta_0(|x-y|,|s|)\\ &= \sum_{j=1}^k \Theta_j(|x-y|)+\Theta_0(|x-y|,|s|)\\ &= \Theta(|x-y|,|s|).\end{aligned}$$

□

3.7.2 Condition II (at least power-type growth)

When the modular function has at least power-type growth, i.e. if

$$M(x,s) \geq c|s|^p \qquad \text{with some } p>1 \text{ and } c>0, \tag{3.61}$$

we can relax *(Me)*, resp. *(Me^i)*, to cover the known range of the double-phase spaces, where the Lavrentiev phenomenon is absent (according to [137, Theorem 3]). Note that the difference with the case of arbitrary growth (i.e. *(Me)* or *(Me^i)*) lays in the rate of balance of Θ, resp. Θ^i. Let us recall that $M_{x,\varepsilon}$ is defined in (3.57).

Anisotropic case
For an N-function $M : \Omega\times\mathbb{R}^N \to [0,\infty)$ we study the following condition.

(Me)$_p$ Assume that M satisfies (3.61) and there exists a function $\Theta : [0,\infty)^2 \to [0,\infty)$ such that $\Theta(\cdot,s)$ and $\Theta(x,\cdot)$ are nondecreasing functions and for all $x,y\in\overline{\Omega}$, ξ such that $|\xi|>1$, and for a constant $c>0$,

$$M(y,\xi) \le \Theta(|x-y|,|\xi|)(M_{x,\varepsilon})^{**}(\xi) \;\; \text{with} \;\; \limsup_{\varepsilon\to 0^+}\Theta(\varepsilon, c\varepsilon^{-\frac{N}{p}}) < \infty.$$

Isotropic case
For an N-function $M : \Omega\times[0,\infty) \to [0,\infty)$ we study the following condition.

(Me^i)$_p$ Assume that M satisfies (3.61) and there exists a function $\Theta^i : [0,\infty)^2 \to [0,\infty)$ such that $\Theta^i(\cdot,s)$ and $\Theta^i(x,\cdot)$ are nondecreasing functions and for all $x,y\in\overline{\Omega}$, $s>1$, and for a constant $c>0$,

$$M(y,s) \le \Theta^i(|x-y|,s)M(x,s) \quad \text{with} \quad \limsup_{\varepsilon\to 0^+}\Theta^i(\varepsilon, c\varepsilon^{-\frac{N}{p}}) < \infty.$$

Example 3.7.3. We have the following isotropic examples of pairs M and Θ satisfying *(Me)*$_p$, which are therefore admissible in our results on the density of smooth functions.

1. **Double phase.** Consider $1<p\le q$ and a nonnegative $a\in C^{0,\alpha}_{loc}(\Omega)$ with $\alpha\in(0,1]$, then $M_{\mathrm{dp}}(s)=s^p+a(x)s^q$ satisfies *(Me^i)*$_p$ with
$$\Theta^i(\tau,s) = C_a\tau^\alpha|s|^{q-p}+1 \tag{3.62}$$
with a proper limit whenever
$$\tfrac{q}{p} \le 1+\tfrac{\alpha}{N}, \tag{3.63}$$
this being the sharp range for regularity of minimizers due to [98].
2. **Variable exponent double phase.** Consider $1<p_-\le p(\cdot)\le q(\cdot)\le q_+<\infty$ and a nonnegative $a\in C^{0,\alpha}_{loc}(\Omega)$ with $\alpha\in(0,1]$, then $M_{\mathrm{v-dp}}(x,s)=s^{p(x)}+a(x)s^{q(x)}$ satisfies *(Me^i)*$_p$ with
$$\begin{aligned}\Theta^i(\tau,s) = {} & \max\{s^{\omega_p(\tau)}, s^{-\omega_p(\tau)}\}\\ & + \max\{s^{\omega_q(\tau)}, s^{-\omega_q(\tau)}\}\left(C_a\tau^\alpha|s|^{\sup_{x\in\Omega}\left(q(x)-p(x)\right)}+1\right)\end{aligned} \tag{3.64}$$
whenever
$$p,q \;\text{ are log-Hölder continuous} \quad \text{and} \quad \sup_{x\in\Omega}\left(q(x)-p(x)\right) \le \tfrac{\alpha p_-}{N}.$$

In the constant exponent case (i.e. when p, q are constant functions) this condition is equivalent to (3.63).

3. **Orlicz double phase.** Suppose $M(x,\xi) = M_1(\xi) + a(x)M_2(\xi)$, where M_1, M_2 are (possibly anisotropic) homogeneous N-functions (without prescribed growth) such that

$$|\xi|^p \le c_1 M_1(\xi) \le c_2 M_2(\xi) \text{ for } \xi \in \mathbb{R}^N \text{ such that } |\xi| > 1 \text{ and } c_1, c_2 > 0,$$

the function $a : \Omega \to [0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a. Then we can take Θ as in (3.60) and M satisfies *(Me)$_p$* if

$$\limsup_{\delta\to 0} \omega_a(\delta) \frac{\overline{M}_2(\delta^{-N/p})}{\underline{M}_1(\delta^{-N/p})} < \infty,$$

where $\underline{M}_1(s) := \inf_{\xi:\,|\xi|=s} M_1(\xi)$ and $\overline{M}_2(s) := \sup_{\xi:\,|\xi|=s} M_2(\xi)$. This condition is essentially less restrictive than the related one from Example 3.7.2.

3.7.3 Between isotropic and anisotropic conditions

In this section we show how isotropic conditions imply anisotropic conditions. Later we shall restrict ourselves to analysis in the anisotropic setting, since the isotropic case follows from these results.

Theorem 3.7.4 *Isotropic conditions are sufficient to get their anisotropic versions. Namely, for $M : \Omega \times \mathbb{R}^N \to [0,\infty)$ we have*

(i) *if M satisfies (Me^i), then M satisfies (Me);*
(ii) *if M satisfies (Me^i)$_p$, then M satisfies (Me)$_p$.*

The above theorem is a direct consequence of the following geometrical observation.

Proposition 3.7.5 *Let Ω be an open subset of $\mathbb{R}^N$, $M_{x,\varepsilon}$ be defined by* (3.57)*, and an N-function M satisfy (Me^i) or (Me^i)$_p$. Let $\varepsilon > 0$ be an arbitrary (small) number. Then, for all x, $y \in \Omega$ such that $y \in B(x, \varepsilon/2)$ we have*

$$\frac{M(y,s)}{(M_{x,\varepsilon})^{**}(s)} \le 4(\Theta^i(\varepsilon, s))^2. \tag{3.65}$$

Proof. From *(Me^i)* (resp. *(Me^i)$_p$*), for all $x, y \in \Omega$ such that $|x-y| \le \frac{1}{2}$ one has

$$M(x,s) \le \Theta^i(|x-y|, s) M(y,s), \tag{3.66}$$

with $\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-N}) < \infty$ (resp. $\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-N/p}) < \infty$).

Moreover, M is locally Lipschitz with respect to s, so by virtue of *(Me^i)* (resp. *(Me^i)$_p$*), we have

$$\sup_{y \in B(x,\varepsilon/2)} M(y,R) \le \Theta^i(\varepsilon,R)M(x,R).$$

Therefore, we obtain

$$\sup_{y \in B(x,\varepsilon/2),\, s<R} |\boldsymbol{\partial}_s M(y,s)| \le \Theta^i(1/2,R)\frac{M(x,R)}{R}.$$

Thus, both M and $M_{x,\varepsilon}$ are continuous in s. When we fix an arbitrary $y \in \overline{B(x,\varepsilon/2)}$, we may estimate

$$A := \frac{M(y,s)}{(M_{x,\varepsilon})^{**}(s)} \le 4(\Theta^i(\varepsilon,s))^2. \tag{3.67}$$

Let us start by writing

$$A = \frac{M(y,s)}{M_{x,\varepsilon}(s)} \cdot \frac{M_{x,\varepsilon}(s)}{(M_{x,\varepsilon})^{**}(s)} = A_1 \cdot A_2$$

and noting that for any fixed $s \ne 0$ there is a sequence $\{y_s^n\}_{n\in\mathbb{N}} \subset \overline{B(x,\varepsilon/2)}$ such that for every $n > n(s)$ we have

$$M_{x,\varepsilon}(s) \ge M(y_s^n,s) - \frac{1}{n}.$$

If necessary taking larger n, we can further estimate

$$M_{x,\varepsilon}(s) \ge \frac{1}{2}M(y_s^n,s). \tag{3.68}$$

Therefore, for a.e. $y \in \overline{B\big(x,\varepsilon/2\big)}$ we have

$$A_1 = \frac{M(y,s)}{M_{x,\varepsilon}(s)} \le 2\frac{M(y,s)}{M(y_s^n,s)} \le 2\Theta^i(|y-y_s^n|,s) \le 2\Theta^i(\varepsilon,s), \tag{3.69}$$

due to (3.66), (3.68) and the monotonicity of Θ^i. As for A_2, let us remark that if $M_{x,\varepsilon}$ is convex in s, then by the Fenchel–Moreau theorem (Theorem 2.1.41) we have $M_{x,\varepsilon} = (M_{x,\varepsilon})^{**}$ and then $A_2 = 1$. Otherwise there exist $s_1 < s_2$ such that for every $s \in (s_1,s_2)$ we have $M_{x,\varepsilon}(s) > (M_{x,\varepsilon})^{**}(s)$ and $M_{x,\varepsilon}(s_i) = (M_{x,\varepsilon})^{**}(s_i)$, $i = 1,2$. Then for every $t \in [0,1]$ we have

$$(M_{x,\varepsilon})^{**}(ts_1 + (1-t)s_2) = tM_{x,\varepsilon}(s_1) + (1-t)M_{x,\varepsilon}(s_2).$$

Let us consider $\{y_{s_1}^n\}_{n\in\mathbb{N}}$, $\{y_{s_2}^n\}_{n\in\mathbb{N}}$ defined similarly to $\{y_s^n\}_{n\in\mathbb{N}}$ and estimate

$$(M_{x,\varepsilon})^{**}(ts_1 + (1-t)s_2) \ge tM(y_{s_1}^n,s_1) + (1-t)M(y_{s_2}^n,s_2) - \frac{1}{n}.$$

We can assume without loss of generality that

$$M(y_{s_1}^n,s_1) < M(y_{s_2}^n,s_1)$$

because otherwise we arrive at $M \le (M)^{**}$, that is $A_2 = 1$. Hence,

$$\begin{aligned} A_2 &= \frac{M_{x,\varepsilon}(ts_1+(1-t)s_2)}{(M_{x,\varepsilon})^{**}(ts_1+(1-t)s_2)} \\ &\le \frac{M(y^n_{s_2}, ts_1+(1-t)s_2)}{tM(y^n_{s_1},s_1)+(1-t)M_{x,\varepsilon}(y^n_{s_2},s_2)-\frac{1}{n}} \\ &\le \frac{tM(y^n_{s_2},s_1)+(1-t)M(y^n_{s_2},s_2)}{tM(y^n_{s_1},s_1)+(1-t)M(y^n_{s_2},s_2)-\frac{1}{n}} \quad =: \; h(t). \end{aligned}$$

For $t \in (0,1)$ we see that

$$\begin{aligned} h'(t) &= \frac{(M(y^n_{s_2},s_1)-M(y^n_{s_1},s_1))M(y^n_{s_2},s_2)}{(t(M(y^n_{s_1},s_1)-M(y^n_{s_2},s_2))+M(y^n_{s_2},s_2))^2} \\ &\quad + \frac{(M(y^n_{s_2},s_2)-M(y^n_{s_2},s_1))}{n(t(M(y^n_{s_1},s_1)-M(y^n_{s_2},s_2))+M(y^n_{s_2},s_2))^2} \; > \; 0. \end{aligned}$$

Hence the maximum of h is attained at $t=1$, which implies

$$A_2 \le \frac{M(y^n_{s_2},s_1)}{M(y^n_{s_1},s_1)-\frac{1}{n}}.$$

We can restrict ourselves to n sufficiently large so that

$$A_2 \le 2\frac{M(y^n_{s_2},s_1)}{M(y^n_{s_1},s_1)} \le 2\Theta^i(|y^n_{s_2}-y^n_{s_1}|,s_1) \le 2\Theta^i(\varepsilon,s_1) \le 2\Theta^i(\varepsilon,s). \tag{3.70}$$

Note that here we applied (3.66). Combining (3.69) with (3.70) gives (3.67). □

Remark 3.7.6. There is no analogue of Proposition 3.7.5 for anisotropic N-functions. This is because $\inf_{z\in Z} M(z,\xi)$ can be arbitrarily far from its second conjugate $(\inf_{z\in Z} M(z,\xi))^{**}$, cf. Remark 2.3.14. For more information on anisotropy, see Section 2.3.1.2.

3.7.4 Density results

We are in position to prove the main result on elliptic smooth approximation of functions from

$$\begin{aligned} \mathcal{T}V_0^M(\Omega) = \{&u \text{ is measurable in } \Omega : \\ &T_k(u) \in W_0^{1,1}(\Omega),\;\; \nabla T_k(u) \in L_M(\Omega) \text{ for every } k>0\}. \end{aligned}$$

Let us recall that the symmetric truncation at level k is defined by (3.55), *(Me)* in Section 3.7.1, and *(Me)$_p$* in Section 3.7.2.

Theorem 3.7.7 (Approximation) *Suppose Ω is a bounded Lipschitz domain and an N-function M satisfies (Me) or (Me)$_p$. Then for every $u \in \mathcal{T}V_0^M(\Omega)$, there exist a sequence of functions $u_\delta \in C_c^\infty(\Omega)$ such that for $\delta \to 0$ we have*

$$u_\delta \to u \ \ in \ \ L^1(\Omega) \quad and \quad \nabla u_\delta \xrightarrow{M} \nabla u \ \ in \ \ L_M(\Omega;\mathbb{R}^N).$$

Moreover, there exists a $c = c(\Omega)$ such that $\|u_\delta\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}$.

By the virtue of Theorem 3.7.4, the above result has an isotropic version with more intuitive assumptions.

Theorem 3.7.8 (Approximation – Isotropic case) *Suppose Ω is a bounded Lipschitz domain and an N-function M satisfies (Me^i) or (Me^i)$_p$. Then for every $u \in \mathcal{T}V_0^M(\Omega)$, there exist a sequence of functions $\{u_\delta\}_{\delta>0} \subset C_c^\infty(\Omega)$ such that for $\delta \to 0$ we have*

$$u_\delta \to u \ \ in \ \ L^1(\Omega) \quad and \quad \nabla u_\delta \xrightarrow{M} \nabla u \ \ in \ \ L_M(\Omega;\mathbb{R}^N).$$

Moreover, there exists a $c = c(\Omega)$ such that $\|u_\delta\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}$.

We do not need to restrict our attention to $\mathcal{T}V_0^M$ in the above theorems. Due to the following fact, Theorems 3.7.7 and 3.7.8 hold true for $u \in V_0^M(\Omega)$ as well.

Lemma 3.7.9 *If M is an N-function, $u \in W_0^{1,1}(\Omega)$ and $\nabla u \in L_M(\Omega;\mathbb{R}^N)$, then for $k \to \infty$ we have $T_k u \to u$ in $W^{1,1}(\Omega)$ and $\nabla T_k u \xrightarrow{M} \nabla u$ in $L_M(\Omega;\mathbb{R}^N)$.*

Proof. Obviously for $k \to \infty$ we have $\nabla T_k u \to \nabla u$ in measure. Moreover, there holds a pointwise estimate $M(\cdot,\nabla T_k u) = M(\cdot,\nabla u)\mathbb{1}_{|u|\leq k} \leq M(\cdot,\nabla u)$ a.e. in Ω and $M(\cdot,\nabla u) \in L^1(\Omega)$. Therefore, $\{M(\cdot,\nabla T_k u)\}_{k>0}$ is a uniformly integrable sequence and the Vitali convergence theorem (Theorem 3.4.4) gives the claim. □

We need to prepare a framework for proving the approximation results. We construct an approximate sequence based on the convolution, then we provide a uniform estimate on a star-shaped domain and we conclude the proof of Theorem 3.7.7. Let

$$\kappa_\delta := 1 - \tfrac{2\delta}{R}. \tag{3.71}$$

For a measurable function $\xi : \mathbb{R}^N \to \mathbb{R}^N$ with $\operatorname{supp}\xi \subset \Omega$, we define

$$\xi_\delta(x) := \int_\Omega \rho_\delta(x-y)\xi\big(\tfrac{y}{\kappa_\delta}\big)\,\mathrm{d}y, \tag{3.72}$$

where $\rho_\delta(x) = \rho(x/\delta)/\delta^N$ is a standard regularizing kernel on $\mathbb{R}^N$ (i.e. $\rho \in C^\infty(\mathbb{R}^N)$, $\operatorname{supp}\rho \subset\subset B(0,1)$ and $\int_\Omega \rho(x)\,\mathrm{d}x = 1$, $\rho(x) = \rho(-x)$), such that $0 \leq \rho \leq 1$. Notice that $\xi_\delta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and that this transformation preserves the L^∞ norm. Recall the isotropic and homogeneous N-functions m_1 and m_2 sandwiching M that come from definition of an N-function.

Proposition 3.7.10 *Suppose M is an N-function satisfying condition (Me) or $(Me)_p$, and Ω is a bounded star-shaped domain with respect to a ball B_R with radius $R > 0$. For a measurable function $\xi : \mathbb{R}^N \to \mathbb{R}^N$ with* $\operatorname{supp}\xi \subset \Omega$ *let ξ_δ be given by* (3.72). *Then there exist constants $C, \delta_1 > 0$ independent of δ such that for all $\delta < \delta_1$*

$$\int_\Omega M(x,\xi_\delta(x))\,\mathrm{d}x \le \int_{\{m_1(|\xi(\cdot)|)\le 1\}} m_2(|\xi(x)|)\,\mathrm{d}x + C\int_\Omega M(x,\xi(x))\,\mathrm{d}x \quad (3.73)$$

for all $\xi \in \mathcal{L}_M(\Omega;\mathbb{R}^N)$ and m_1, m_2 being a minorant and majorant, respectively, of an N-function, see Definition 2.2.2.

Proof. We present the proof only in the case when Ω is a star-shaped domain with respect to a ball centered at the origin. For the general case one should change variables moving the center of B_R to the origin, then proceed with the proof as below, and then reverse the change of variables.

Fix $\xi \in L_M(\Omega;\mathbb{R}^N)$ and note that without loss of generality it can be assumed that

$$\|\xi\|_{L^1(\Omega;\mathbb{R}^N)} \le \tfrac{1}{2^N}. \quad (3.74)$$

On the other hand, if $(Me)_p$ is in power, we may assume that $\|\xi\|_{L^p(\Omega;\mathbb{R}^N)} \le \widetilde{c}$ with absolute constant $\widetilde{c} > 0$ (we will choose it soon). Notice that

$$\begin{aligned}\int_\Omega M(x,\xi_\delta(x))\,\mathrm{d}x &\le \int_{\{M(\cdot,\xi_\delta(\cdot))\le 1\}} M(x,\xi_\delta(x))\,\mathrm{d}x \\ &\quad + \int_{\{M(\cdot,\xi_\delta(\cdot))\ge 1\}} M(x,\xi_\delta(x))\,\mathrm{d}x \\ &\le \int_{\{m_1(|\xi_\delta(\cdot)|)\le 1\}} m_2(|\xi_\delta(x)|)\,\mathrm{d}x \\ &\quad + \int_{\{M(\cdot,\xi_\delta(\cdot))\ge 1\}} M(x,\xi_\delta(x))\,\mathrm{d}x \\ &=: \mathsf{I}_\delta + \mathsf{J}_\delta.\end{aligned}$$

To deal with I_δ we notice that $\{m_1(|S_\delta\xi(\cdot)|) \le 1\} = \{m_2(|S_\delta\xi(\cdot)|) \le c\}$ for $c = m_2 \circ m_1^{-1}(1)$ and we have the following pointwise estimate

$$m_2(|\xi_\delta(\cdot)|)\mathbb{1}_{\{m_1(|\xi_\delta(\cdot)|)\le 1\}}(\cdot) \le c.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\limsup_{\delta\searrow 0} \mathsf{I}_\delta = \limsup_{\delta\searrow 0} \int_{\{m_1(|\xi_\delta(\cdot)|)\le 1\}} m_2(|\xi_\delta(x)|)\,\mathrm{d}x = \int_{\{m_1(|\xi(\cdot)|)\le 1\}} m_2(|\xi(x)|)\,\mathrm{d}x.$$

Thus, we concentrate now on J_δ. For every $\delta \in (0, R/2)$ it holds that

$$\overline{\kappa_\delta\Omega + \delta B(0,1)} \subset \Omega. \quad (3.75)$$

Indeed, since Ω is star-shaped with respect to $B(0,R)$, if we take arbitrary $x \in \Omega$ and $y \in B(0,1)$, then $\kappa_\delta x + (1-\kappa_\delta)Ry = \kappa_\delta x + 2\delta y \in \Omega$. Therefore, for $\delta \in (0, R/4)$ we have $\xi_\delta \in C_c^\infty(\Omega)$.

We consider a family of N-dimensional cubes covering the set Ω. Namely, a family $\{Q_j^\delta\}_{j=1}^{N_\delta}$ consisting of closed cubes with edges of length 2δ, such that

$$\operatorname{int} Q_j^\delta \cap \operatorname{int} Q_i^\delta = \emptyset \quad \text{for} \quad i \neq j \ \text{ and } \ \Omega \subset \bigcup_{j=1}^{N_\delta} Q_j^\delta.$$

Moreover, for each cube Q_j^δ we define the cube $\widetilde{Q}_j^\delta$ centered at the same point q_j and with parallel corresponding edges of length 4δ.

According to condition *(Me)* or *(Me)$_p$*, the relation between $M(x,\xi)$ and

$$M_j^\delta(\xi) := \operatorname{ess\,inf}_{x \in \widetilde{Q}_j^\delta \cap \Omega} M(x,\xi) \tag{3.76}$$

is as follows

$$\frac{M(x,\xi)}{(M_j^\delta)^{**}(\xi)} \le \Theta(\delta, |\xi|) \quad \text{for a.e. } x \in Q_j^\delta \ \text{ and all } \xi \in \mathbb{R}^N \text{ such that } \ |\xi| > 1, \tag{3.77}$$

where by $(M_j^\delta)^{**}(\xi) = ((M_j^\delta(\xi))^*)^*$ we denote the second conjugate, which according to Corollary 2.1.42 coincides with the greatest convex minorant of M_j^δ.

We have

$$\begin{aligned} \mathsf{J}_\delta &= \sum_{j=1}^{N_\delta} \int_{Q_j^\delta \cap \{M(\cdot,\xi_\delta(\cdot)) \ge 1\}} M(x, \xi_\delta(x))\, \mathrm{d}x \\ &= \sum_{j=1}^{N_\delta} \int_{Q_j^\delta \cap \{M(\cdot,\xi_\delta(\cdot)) \ge 1\}} \frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))} (M_j^\delta)^{**}(\xi_\delta(x))\, \mathrm{d}x. \end{aligned} \tag{3.78}$$

Let us fix an arbitrary cube and take $x \in Q_j^\delta$. Our aim now is to show the following uniform bound

$$\frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))} \le C \tag{3.79}$$

for sufficiently small $\delta > 0$, $x \in Q_j^\delta \cap \Omega$, with C independent of δ, x, j and ξ. For sufficiently small δ, due to (3.77), we obtain

$$\frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))} \le \Theta(\delta, |\xi_\delta(x)|). \tag{3.80}$$

To estimate the right-hand side of (3.80) we recall the definition of ξ_δ given in (3.72). For all $x \in \Omega$ and each $\delta > 0$ we have $\rho_\delta(x-y) \le 1/\delta^N$. Having (3.74), we observe that

$$|\xi_\delta(x)| \le \frac{1}{\delta^N}\int_\Omega |\xi(y/\kappa_\delta)|\,\mathrm{d}y \le \frac{\kappa_\delta^N}{\delta^N} \le \delta^{-N}. \tag{3.81}$$

Note that in the case of *(Me)*$_p$ we just estimate $|\xi_\delta(x)| \le \delta^{-N/p}$ using the Hölder inequality. Indeed,

$$|\xi_\delta(x)| \le \left(\int_\Omega |\xi(y/\kappa_\delta)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_\Omega \rho_\delta^{\frac{p}{p-1}}\,\mathrm{d}y\right)^{\frac{p-1}{p}} \le \frac{1}{\widetilde{c}\delta^{N/p}}\|\xi\|_{L^p(\Omega)} \le \delta^{-\frac{N}{p}},$$

where we chose $\widetilde{c}$ for the second inequality to hold. The last estimate is true, as we used $\widetilde{c}$ as the normalization constant.

We combine this with (3.80), (3.81), and by recalling *(Me)* (resp. *(Me)*$_p$) to get

$$\frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))} \le \Theta(\delta,\delta^{-N}) < C,$$

$$\left(\text{resp. } \frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))} \le \Theta(\delta,\delta^{-N/p}) < C\right)$$

for all $\delta < \delta_1$ with some $\delta_1 > 0$. Thus, we obtain a uniform bound of (3.79).

Now, starting from (3.78), noting (3.79) and the fact that on $\{M(\cdot,\xi_\delta(\cdot)) \ge 1\}$ we have $(M_j^\delta)^{**}(\xi) > 0$, we observe that

$$\begin{aligned}\int_\Omega M(x,\xi_\delta(x))\,\mathrm{d}x &= \sum_{j=1}^{N_\delta}\int_{Q_j^\delta\cap\{M(\cdot,\xi_\delta(\cdot))\ge 1\}} \frac{M(x,\xi_\delta(x))}{(M_j^\delta)^{**}(\xi_\delta(x))}(M_j^\delta)^{**}(\xi_\delta(x))\,\mathrm{d}x\\ &\le C\sum_{j=1}^{N_\delta}\int_{Q_j^\delta\cap\{M(\cdot,\xi_\delta(\cdot))\ge 1\}} (M_j^\delta)^{**}(\xi_\delta(x))\,\mathrm{d}x =: \mathsf{J}_\delta^1.\end{aligned}$$

We will carefully estimate the right-hand side above changing an indicator of a cube

$$\begin{aligned}\mathsf{J}_\delta^1 &\le C\sum_{j=1}^{N_\delta}\int_{Q_j^\delta\cap\Omega}(M_j^\delta)^{**}\left(\int_{B(0,\delta)}\rho_\delta(y)\xi\left(\frac{x-y}{\kappa_\delta}\right)\mathrm{d}y\right)\mathbb{1}_{Q_j^\delta\cap\Omega}(x)\,\mathrm{d}x\\ &\le C\sum_{j=1}^{N_\delta}\int_{\mathbb{R}^N}(M_j^\delta)^{**}\left(\int_{B(0,\delta)}\rho_\delta(y)\xi\left(\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{Q_j^\delta\cap\Omega}(x)\,\mathrm{d}y\right)\mathrm{d}x\\ &\le C\sum_{j=1}^{N_\delta}\int_{\mathbb{R}^N}(M_j^\delta)^{**}\left(\int_{\mathbb{R}^N}\rho_\delta(y)\xi\left(\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{\widetilde{Q}_j^\delta\cap\Omega}(x-y)\,\mathrm{d}y\right)\mathrm{d}x =: \mathsf{J}_\delta^2.\end{aligned}$$

The function $(M_j^\delta)^{**}$ is convex, so by applying Jensen's inequality the right-hand side above can be estimated by the following quantity

$$\mathsf{J}_\delta^2 \le C\sum_{j=1}^{N_\delta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\delta(y)(M_j^\delta)^{**}\left(\xi\left(\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{\widetilde{Q}_j^\delta\cap\Omega}(x-y)\right)\mathrm{d}y\,\mathrm{d}x$$

$$\leq C\|\rho_\delta\|_{L^1(B(0,\delta);\mathbb{R}^N)} \sum_{j=1}^{N_\delta} \int_{\mathbb{R}^N} (M_j^\delta)^{**} \left(\xi\left(\frac{z}{\kappa_\delta}\right) \mathbb{1}_{\widetilde{Q}_j^\delta \cap \Omega}(z) \right) \mathrm{d}z \tag{3.82}$$

$$\leq C \sum_{j=1}^{N_\delta} \int_{\widetilde{Q}_j^\delta \cap \Omega} (M_j^\delta)^{**} \left(\xi\left(\frac{z}{\kappa_\delta}\right)\right) \mathrm{d}z$$

$$\leq C \sum_{j=1}^{N_\delta} \int_{\widetilde{Q}_j^\delta \cap \Omega} M\left(\frac{z}{\kappa_\delta}, \xi\left(\frac{z}{\kappa_\delta}\right)\right) \mathrm{d}z =: \mathsf{J}_\delta^3,$$

where we applied Young's convolution inequality (Lemma 8.26), uniform boundedness of $\|\rho_\delta\|_{L^1(B(0,\delta);\mathbb{R}^N)}$, the fact that $(M_j^\delta)^{**}(\xi) = 0$ if and only if $\xi = 0$ and that $(M_j^\delta)^{**}$ is (the greatest convex) minorant of M_j^δ (Corollary 2.1.42). To estimate it further we substitute $x := z/\kappa_\delta$ and observe that

$$\kappa_\delta \widetilde{Q}_j^\delta \subset Q_j^{c_\Omega \delta} \tag{3.83}$$

for $c_\Omega = 4(1 + \mathrm{diam}\Omega/R)$. Indeed, since q_j is the center of $\widetilde{Q}_j^\delta$, we have

$$\kappa_\delta \widetilde{Q}_j^\delta = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : \; |x_i - (q_j)_i/\kappa_\delta| \leq 2\tfrac{\delta}{\kappa_\delta} \text{ for every } i = 1, \ldots, N\}.$$

We note that for every $i = 1, \ldots, N$ we have

$$|x_i - (q_j)_i| \leq |x_i - (q_j)_i/\kappa_\delta| + |(q_j)_i(1 - 1/\kappa_\delta)| \leq 2\delta\left(\tfrac{1}{\kappa_\delta} + \tfrac{1-\kappa_\delta}{\kappa_\delta}\mathrm{diam}\Omega\right) \leq c_\Omega \delta,$$

where we used (3.71) and the fact that $\kappa_\delta \geq 1/2$, which is true for $\delta < R/4$. Therefore (3.83) is justified and we infer that

$$\mathsf{J}_\delta^3 \leq C \sum_{j=1}^{N_\delta} \int_{Q_j^{c_\Omega \delta}} M(x, \xi(x))\, \mathrm{d}x \leq C(N) \int_\Omega M(x, \xi(x))\, \mathrm{d}x.$$

The last inequality comes from the computation of a sum, taking into account the measure of repeating parts of cubes. We get (3.73) by summing the above estimates. □

Now we are in a position to prove an approximation result.

Proof (of Theorem 3.7.7). Since Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, by Lemma 8.2 the set $\overline{\Omega}$ can be covered by a finite family of sets $\{G_i\}_{i \in I}$ such that each

$$\Omega_i := \Omega \cap G_i$$

is a star-shaped domain with respect to the balls $\{B^i\}_{i \in I}$, respectively. Then

$$\Omega = \bigcup_{i \in I} \Omega_i.$$

Let us introduce a partition of unity θ_i, i.e.

$$0 \le \theta_i \le 1, \quad \theta_i \in C_c^\infty(G_i), \quad \sum_{i\in I} \theta_i(x) = 1 \;\text{ for }\; x \in \Omega,$$

which exists due to Lemma 8.3. Fix an arbitrary $\varphi \in \mathcal{T}V_0^M(\Omega)$. We are going to show that there exists a constant $\lambda > 0$ such that

$$\lim_{\delta\to 0^+} \int_\Omega M\left(x, \frac{\nabla(\varphi_\delta) - \nabla\varphi}{\lambda}\right) \mathrm{d}x = 0,$$

where $\varphi \mapsto \varphi_\delta$ is defined in (3.72). We note that $\varphi \in \mathcal{T}V_0^M(\Omega)$ and for each $i \in I$ we have

$$\nabla(\theta_i\varphi) \;=\; \varphi\nabla\theta_i + \theta_i\nabla\varphi \;\in\; L_M(\Omega;\mathbb{R}^N).$$

Furthermore, $\sum_{i\in I} \nabla(\theta_i\varphi) = \nabla\varphi$. Since

$$\int_\Omega M\left(x, \frac{\nabla(\varphi_\delta) - \nabla\varphi}{\lambda}\right) \mathrm{d}x \le \sum_{i\in I} \frac{\lambda^i}{\lambda} \int_{\Omega_i} M\left(x, \frac{\nabla(\theta_i\varphi)_\delta - \nabla(\theta_i\varphi)}{\lambda^i}\right) \mathrm{d}x$$

for some $\lambda_i > 0$ such that $\lambda = \sum_i \lambda^i$, and there is finite number of Ω_is, it suffices to prove convergence to zero of each integral from the right-hand side.

Let us consider a family of measurable sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} E_n = \Omega_i$ and a simple vector-valued function

$$E^n(x) := \sum_{j=0}^{n} \mathbb{1}_{E_j}(x)\eta_j(x),$$

where $\{\eta_j\}_{j=0}^n$ is a family of vectors such that $\{E^n\}_{n\in\mathbb{N}}$ converges modularly to $\nabla(\theta_i\varphi)$ with λ_3 (cf. Definition 3.4.3) whose existence is ensured by Theorem 3.4.11. Note that

$$\nabla(\theta_i\varphi)_\delta - \nabla(\theta_i\varphi) = (\nabla(\theta_i\varphi)_\delta - (E^n)_\delta) + ((E^n)_\delta - E^n) + (E^n - \nabla(\theta_i\varphi)).$$

By Jensen's inequality we get that

$$\begin{aligned}
&\int_{\Omega_i} M\left(x, \frac{\nabla(\theta_i\varphi)_\delta - \nabla(\theta_i\varphi)}{\lambda^i}\right) \mathrm{d}x \\
&\quad \le \frac{\lambda_1^i}{\lambda} \int_{\Omega_i} M\left(x, \frac{\nabla(\theta_i\varphi)_\delta - (E^n)_\delta}{\lambda_1^i}\right) \mathrm{d}x + \frac{\lambda_2^i}{\lambda^i} \int_{\Omega_i} M\left(x, \frac{(E^n)_\delta - E^n}{\lambda_2^i}\right) \mathrm{d}x \\
&\qquad + \frac{\lambda_3^i}{\lambda^i} \int_{\Omega_i} M\left(x, \frac{E^n - \nabla(\theta_i\varphi)}{\lambda_3^i}\right) \mathrm{d}x \\
&\quad =: L_1^{n,\delta} + L_2^{n,\delta} + L_3^{n,\delta},
\end{aligned} \tag{3.84}$$

where $\lambda^i = \sum_{j=1}^3 \lambda_j^i$, $\lambda_j^i > 0$. We have λ_3^i fixed already. Let us take $\lambda_1^i = \lambda_3^i$.

Note that

$$L_1^{n,\delta} = \frac{\lambda_1^i}{\lambda^i} \int_{\Omega_i} M\left(x, \left(\frac{E^n - \nabla(\theta_i \varphi)}{\lambda_1^i}\right)_\delta\right) \mathrm{d}x.$$

Due to Proposition 3.7.10 the mapping $\theta_i \varphi \mapsto (\theta_i \varphi)_\delta$ is uniformly bounded from $L_M(\Omega_i; \mathbb{R}^N)$ to $L_M(\Omega_i; \mathbb{R}^N)$ and we can estimate

$$\begin{aligned} 0 \leq L_1^{n,\delta} \leq & \int_{\left\{m_1\left(\frac{|E^n - \nabla(\theta_i \varphi)|}{\lambda_1^i}\right) \leq 1\right\}} m_2\left(\frac{|E^n - \nabla(\theta_i \varphi)|}{\lambda_1^i}\right) \mathrm{d}x \\ & + C \int_{\Omega_i} M\left(x, \frac{E^n - \nabla \varphi}{\lambda_3^i}\right) \mathrm{d}x =: K^n, \end{aligned}$$

where $\lim_{n\to\infty} K^n = 0$. Consequently, $\lim_{n\to\infty} \lim_{\delta\to 0^+} L_1^{n,\delta} = 0$ as well.

Let us concentrate on the convergence of mollified step functions to a step function E^n, that is on $L_2^{n,\delta}$. Jensen's inequality and then Fubini's theorem lead to

$$\begin{aligned} \frac{\lambda^i}{\lambda_2^i} L_2^{n,\delta} &= \int_{\Omega_i} M\left(x, \frac{E^n(x) - (E^n)_\delta(x)}{\lambda_2^i}\right) \mathrm{d}x \\ &= \int_{\Omega_i} M\left(x, \frac{1}{\lambda_2^i} \int_{B(0,\delta)} \varrho_\delta(y) \cdot \right. \\ &\quad \left. \cdot \sum_{j=0}^n \left[\mathbb{1}_{E_j}(x)\eta_j(x) - \mathbb{1}_{E_j}\left(\frac{x-y}{\kappa_\delta}\right) \eta_j\left(\frac{x-y}{\kappa_\delta}\right)\right] \mathrm{d}y \right) \mathrm{d}x \\ &\leq \int_{B(0,\delta)} \varrho_\delta(y) \left(\int_{\Omega_i} M\left(x, \frac{1}{\lambda_2^i} \cdot \right.\right. \\ &\quad \left.\left. \cdot \sum_{j=0}^n \left[\mathbb{1}_{E_j}(x)\eta_j(x) - \mathbb{1}_{E_j}\left(\frac{x-y}{\kappa_\delta}\right) \eta_j\left(\frac{x-y}{\kappa_\delta}\right)\right]\right) \mathrm{d}x \right) \mathrm{d}y. \end{aligned} \tag{3.85}$$

Since the shift operator in L^1 is continuous, we have pointwise convergence

$$\sum_{j=0}^n \left[\mathbb{1}_{E_j}(x)\eta_j(x) - \mathbb{1}_{E_j}\left(\frac{x-y}{\kappa_\delta}\right) \eta_j\left(\frac{x-y}{\kappa_\delta}\right)\right] \xrightarrow{\delta\to 0} 0.$$

Moreover, when we fix arbitrary $\lambda_2^i > 0$ we have

$$\begin{aligned} & M\left(x, \frac{1}{\lambda_2^i} \sum_{j=0}^n \left[\mathbb{1}_{E_j}(x)\eta_j(x) - \mathbb{1}_{E_j}\left(\frac{x-y}{\kappa_\delta}\right) \eta_j\left(\frac{x-y}{\kappa_\delta}\right)\right]\right) \\ & \qquad \leq \sup_{\zeta\in\mathbb{R}^N:\ |\zeta|=1} M\left(x, \frac{2}{\lambda_2^i} \sum_{j=0}^n \|\eta_j\|_{L^\infty(E_j)} \zeta\right) < \infty. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem the right-hand side of (3.85) converges to zero.

Thus all terms in (3.84) are arbitrarily small and, hence, we get modular convergence of the approximate sequence. The modular convergence of gradients implies their strong L^1-convergence and the Poincaré inequality gives the claim. □

Proof (of Theorem 3.7.8). Due to Theorem 3.7.4 we get that growth conditions from Theorem 3.7.8 imply growth conditions required by Theorem 3.7.7. □

Let us comment on possible modifications of the proofs of Theorems 3.7.7 and 3.7.8.

Remark 3.7.11 (Extending the range of admissible modular functions I). Using ideas of [63], see also [62], one can prove approximation result of Theorems 3.7.7 and 3.7.8 under a less restrictive condition than *(Me)* from Section 3.7.1 (resp. *(Me)*$_p$ from Section 3.7.2), namely

(Me)* Assume that there exists a function $\Theta : [0,\infty)^2 \to [0,\infty)$ such that $\Theta(\cdot, s)$ and $\Theta(x,\cdot)$ are nondecreasing functions and for all $x, y \in \overline{\Omega}$ and ξ such that $|\xi| > 1$, and a constant $c > 0$,

$$M(y,\xi) \leq \Theta(|x-y|,|\xi|)(M_{x,\varepsilon})^{**}(\xi) \quad \text{with} \quad \limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-1}) < \infty,$$

where $(M_{x,\varepsilon})^{**}$ is the second conjugate to $M_{x,\varepsilon}$.

With this purpose one should modify Proposition 3.7.10 using the idea of [63, Lemma 2.5] to mollify not any function $\xi \in \mathcal{L}_M(\Omega;\mathbb{R}^N)$, but specifically a gradient of a truncation of a fixed function $\nabla T_k(u) \in \mathcal{L}_M(\Omega;\mathbb{R}^N)$ with some added bounded function φ. Namely, one should consider

$$\xi_\varepsilon := (\nabla T_k(u) + \varphi)_\varepsilon,$$

where the subscript ε always means convolution with a regularizing kernel ρ_ε (see (3.72) with $\delta = \varepsilon$). The key point of the reasoning is to notice that, because of the properties of the convolution, instead of (3.81), for sufficiently small ε one can estimate

$$|\xi_\varepsilon| \leq c|(T_k(u))_\varepsilon| \cdot |\nabla \rho_\varepsilon| + |\varphi_\varepsilon| \leq \frac{ck + \|\varphi\|_{L^\infty}}{\varepsilon}.$$

Consequently, (3.79) can be achieved under condition *(Me)**. Condition *(Me)** is less restrictive than *(Me)* and *(Me)*$_p$ in the case when $p < N$ and $N > 1$. Indeed, the essential point is finiteness of the appropriate limit, which in the case of *(Me)* is

$$\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-N}) < \infty,$$

for *(Me)*$_p$

$$\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-\frac{N}{p}}) < \infty,$$

while for $(Me)^*$

$$\limsup_{\varepsilon\to 0^+} \Theta(\varepsilon, c\varepsilon^{-1}) < \infty.$$

To observe how these different conditions *(Me)* or *(Me)$_p$* and condition *(Me)** behave in particular examples, consider the case of a double-phase function. Condition *(Me)** for $M(x,s) = s^p + a(x)s^q$, $a \in C^{0,\alpha}$ implies that assuming $q \le p+\alpha$ is sufficient for the density of smooth functions.

Remark 3.7.12 (Alternative proof in the isotropic case). Let us note that the proof of Theorem 3.7.8 in the isotropic case can be modified by the use of Lemma 2.3.13 instead of Theorem 3.7.4, which has a more complicated proof. In fact, in the isotropic case Lemma 2.3.13 ensures that

$$M_j^\delta(\xi) = \operatorname{ess\,inf}_{x\in\widetilde{Q}_j^\delta\cap\Omega} M(x,\xi),$$

despite not being convex, supports Jensen's inequality with the intrinsic constant 1/2. Therefore, in (3.78) one can directly divide and multiply by $M_j^\delta(\xi)$ instead of $(M_j^\delta)^{**}(\xi)$ and proceed with all the above steps, only taking into account minor modifications due to the appearance of the intrinsic constant. Indeed, in such a situation for proving a counterpart of (3.79) we notice that

$$\frac{M(x,\xi)}{M_j^\delta(\xi)} \le \sup_{x,y\in\widetilde{Q}_j^\delta\cap\Omega} \left(\frac{M(x,\xi)}{M(y,\xi)}\right) \le \Theta(\operatorname{diam}\widetilde{Q}_j^\delta, |\xi|).$$

Remark 3.7.13 (Extending the range of admissible modular functions II). In the fully anisotropic setting $(M_{x,\varepsilon})^{**}(\xi)$ can be a priori arbitrarily far from $M_{x,\varepsilon}(\xi)$ no matter how small ε is, see Remarks 2.3.14. To have better control on the anisotropy one can use ideas of [52] and assume condition

(B) there exists a constant $C_M > 1$ such that for every ball $B \subset \Omega$ with $|B| \le 1$, $x \in B$, and for all $\xi \in \mathbb{R}^N$ such that $|\xi| > 1$ and $M(x, C_M\xi) \in [1, \frac{1}{|B|}]$,

$$\sup_{y\in B} M(y,\xi) \le M(x, C_M\xi).$$

Before applying this assumption one can estimate M by its supremum over a small ball, which is convex, already in the first line of (3.78) and the rest of the proof becomes significantly simplified. Note that this condition is a general growth and anisotropic version of a commonly used assumption (A2) from [191]. As shown in [52], (B) embraces a far broader class of admissible spaces than *(Me)* or *(Me)$_p$* in terms of admissible growth, local properties, and anisotropy.

3.8 Operators and Related Musielak–Orlicz Spaces

Let us give an overview of the functional settings and comment on the expected growth conditions to be imposed on the operators.

3.8.1 Special instances

In order to explain the use of the unconventional functional framework in PDEs, we shall refer to nonlinear gradient-driven diffusion equations of the form

$$\begin{cases} \partial_t u - \operatorname{div} \mathbf{a}(t,x,\nabla u) = 0, \\ u(x,0) = u_0 \end{cases} \tag{3.86}$$

with $\mathbf{a} : [0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ exhibiting growth described by means of more classical or more innovative cases of fully anisotropic and inhomogeneous N-functions. It would be useful to keep in mind the example of a certain substance spreading from a river towards its banks, where we aim to model its diffusion throughout media like sand or clay having various seepage properties.

PDEs with the leading part of the operator having a power-type growth like the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ have received deep attention for decades already, and they also arise in the modelling of classical, fast or slow diffusion ($p = 2$, $1 < p < 2$, or $2 < p < \infty$, respectively). The analysis in the Sobolev space setting is very well understood. The polynomial growth case has been developed in a wide range of directions, including the variable exponent, anisotropic, convex, weighted, and double-phase approaches , which make it possible to describe increasingly more complicated processes and materials. The Musielak–Orlicz spaces unify all of the mentioned types of spaces. We refer to the recent survey [71] for a brief presentation of the subsettings together with the difficulties each of them carries in the analysis of PDEs, as well to the very recent survey [256] concentrating on the calculus of variations within this setting. Here, we present a very concise overview of the spaces included in the Musielak–Orlicz framework in connection with the PDEs described in this monograph.

3.8.1.1 Sobolev and weighted Sobolev spaces

It is already classical to involve the Laplace or p-Laplace operator

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

in the modeling of various processes of diffusion-type (which also have interpretations in the life or social sciences). The classical Sobolev spaces provide a natural setting to study solutions to elliptic and parabolic partial differential equations involving these operators. For this we refer to any lecture notes or book on partial differential equations, cf. [223, 227]. In the example of our river we expect $p > 2$ as the sand plays the role of a porous medium and the diffusion is made slower. In order to study processes inhomogeneous in space, e.g. when our medium is not the same in the whole space, one idea is to consider the weighted ω-p-Laplacian

$$\Delta_p^\omega u = \operatorname{div}\big(\omega(x)|\nabla u|^{p-2}\nabla u\big)$$

involving various types of singularities of the weight function ω. In turn, the appropriate setting is provided by the weighted Sobolev spaces equipped with $M(x,\nabla u)=\omega(x)|\nabla u|^p$, see [143, 201, 317].

To ensure basic reasonable properties of the weighted Lebesgue space L^p_ω we shall specify the appropriate classes of weight functions ω. According to Kufner and Opic [222], the weight should satisfy the B_p-condition, i.e. be a positive a.e., Borel measurable, real function such that $\omega'=\omega^{-1/(p-1)}(x)\in L^1_{loc}(\Omega)$. If $\omega\in B_p(\Omega)$, then the weighted space $L^p_\omega(\Omega)$ is continuously embedded in $L^1_{loc}(\Omega)$, and consequently functions from the related weighted space of a Sobolev type have well-defined distributional derivatives. Note that the condition B_p is weaker than the A_p-condition, cf. [258]. One can consider weighted Sobolev spaces with different weights, e.g. for $\omega_0,\omega_1\in B_p(\Omega)$

$$W^{1,p}_{(\omega_0,\omega_1)}(\Omega):=\left\{f\in W^{1,1}_{loc}(\Omega):\ \ f\in L^p_{\omega_0}(\Omega),\ \nabla f\in L^p_{\omega_1}(\Omega;\mathbb{R}^N)\right\},$$

but one-weighted spaces (when $\omega_1=\omega_2$) are studied more often.

Turesson's book [317] consists of a comprehensive study on the case of A_p-weights. It provides weighted analogues of multiple results from the theory of non-weighted Sobolev spaces applied to PDEs and from non-weighted potential theory, which are not addressed here, but should not be ignored. PDEs in the weighted setting are considered e.g. in [50, 51, 66, 86, 141, 142, 151, 255].

3.8.1.2 Anisotropic Sobolev spaces

The phenomenon of anisotropy is described in Section 2.3.1.2. Briefly one should think about it as the situation when the energy density is not the same in distinguished directions. We refer, for example, to the process of diffusion which is expected to be more intense, or lower, in some directions due to some forces.

To describe anisotropy one can use different exponents in various directions by involving the anisotropic $\vec{p}$-Laplacian

$$\Delta_{\vec{p}}u=\operatorname{div}\left(\sum_{i=1}^N|u_{x_i}|^{p_i-2}u_{x_i}\right)\quad\text{with}\quad \vec{p}=(p_1,\dots,p_N)$$

and thus, the relevant space for solutions is equipped with $M(x,\nabla u)=\sum_i|u_{x_i}|^{p_i}$ and it is given by

$$W^{1,\vec{p}}(\Omega):=\left\{f\in W^{1,1}_{loc}(\Omega):\ \ f\in L^{p_0}(\Omega),\ f_{x_i}\in L^{p_i}(\Omega),\ \text{for}\ \ i=1,\dots,N\right\},$$

where p_0 is a harmonic mean of $p_1,\dots,p_N$. See [314, 91] for the embedding result.

One can consider anisotropic weighted Sobolev spaces equipped with weights associated to distinct coordinates directions, i.e. for $\omega_0,\dots,\omega_N\in B_p(\Omega)$, we define

$$W^{1,\vec{p}}_{(\omega_0,\dots,\omega_N)}(\Omega):=\left\{f\in W^{1,1}_{loc}(\Omega):\ f\in L^{p_0}_{\omega_0}(\Omega),\ f_{x_i}\in L^{p_i}_{\omega_i}(\Omega)),\ \text{for}\ \ i=1,\dots,N\right\}.$$

However, it is more common to consider one-weighted spaces ($\omega_0 = \cdots = \omega_N$).

In the anisotropic setting a fundamental role is played by the powerful tool of symmetrization, an idea which started with the seminal papers [250, 306, 323] and was developed further in the Orlicz setting. For some regularity, existence, and nonexistence results we refer to e.g. [9, 53, 54, 83, 104, 156, 157, 236, 301, 318], while for other estimates on solutions we refer to [10, 319]. Very weak solutions to anisotropic PDEs with irregular data are studied starting from [45].

3.8.1.3 Variable exponent Sobolev spaces

To describe the setting in which the energy density is inhomogeneous in the space variable, one can consider the operators

$$\Delta_{p(x)} u = \mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) \quad \text{or} \quad \widetilde{\Delta}_{p(x)} u = \mathrm{div}\big(p(x)|\nabla u|^{p(x)-2}\nabla u\big).$$

Therefore, the relevant setting is described with the use of $M(x,\nabla u) = |\nabla u|^{p(x)}$. In turn, the solutions to problems involving such operators are in the variable exponent Sobolev space given by

$$W^{1,p(\cdot)}(\Omega) = \{f \in W^{1,1}_{loc}(\Omega) : \ \ f, |\nabla f| \in L^{p(\cdot)}(\Omega)\}.$$

The settings of the variable exponent Lebesgue and Sobolev spaces have been deeply examined. They are well described from the theoretical point of view in the books by Cruz-Uribe and Fiorenza [100] and by Diening, Harjulehto, Hästö, and Růžička [115]. Typical applications of variable exponent equations include models of electrorheological fluids [3, 279, 287], image restoration processing [70], elasticity equations [338], and the thermistor model [339].

Since the setting has been exhaustively explored, it will not be the focus of our considerations. Let us mention only a few articles on the basic properties of solution or minimizers to variational problems such as existence [119, 145, 240, 254, 276], regularity results [1, 2, 76, 105, 305], uniqueness of solutions [147], nonexistence [4, 128], as well as a qualitative analysis of eigenvalue problems [277]. The existence to problems with data below duality are studied in isotropic spaces in [30, 324] and in anisotropic spaces in [28, 29]. Seminal work on homogenization in this setting [336] have lately found multi-valued counterparts [270]. Finally, let us refer to the survey [194] which summarizes developments in the theory of PDEs within this setting, comprehensively covering the issues of existence and regularity.

Provided $1 < p_- \leq p(\cdot) \leq p_+ < \infty$, the variable exponent Lebesgue spaces are reflexive, which implies that the modular and the norm topologies coincide. Inhomogeneity of the variable exponent spaces implies though that the density of smooth functions depends on the regularity of the modular function. Namely, when the exponent is not regular enough, there exist functions that cannot be approximated by smooth functions. Thus, we meet the so-called Lavrentiev phenomenon, which plays a prominent role in the calculus of variations, see [337, 340] and also the beginning of Section 3.7. Typically to ensure density of smooth functions the assumption imposed on the exponent is log-Hölder continuity. Therefore, PDEs considered in this setting are usually formulated with at least log-Hölder exponents, which excludes dramatic changes of the energy density. In the river example we can allow for diffusion through media of completely different saturation like sand, soil or clay, as long as the transition between them is smooth enough.

3.8.1.4 Double-phase spaces

The investigation of problems with the growth trapped between two power-type functions was initiated by Marcellini [246, 247]. A particular case of such an approach involves operators of the form

$$\operatorname{div}\Big(\big(|\nabla u|^{p-2}+a(x)|\nabla u|^{q-2}\big)\nabla u\Big) \quad \text{or} \quad \operatorname{div}\Big(\big(1+a(x)\log(\mathrm{e}+|\nabla u|)\big)|\nabla u|^{p-2}\nabla u\Big)$$

with $1 < p,q < \infty$ and a weight function $a : \Omega \to [0,\infty)$ which can vanish. Such operators can be used in the description of diffusion-type processes in a space, where certain subdomains are distinguished from others. For instance, one might use this structure to describe a composite material having on $\{x \in \Omega : a(x) = 0\}$ an energy density with p-growth, but on $\{x \in \Omega : a(x) > 0\}$ a growth of order q. The problem should be posed in a space equipped with the modular function

$$M(x,s) = |s|^p + a(x)|s|^q \quad \text{or} \quad M(x,s) = |s|^p(1+a(x)\log(\mathrm{e}+|s|^q), \quad \text{respectively.}$$

This case is related to the variable exponent spaces with an exponent which is a step function rather than the weighted Sobolev spaces. The key feature of this setting is that the regularity of the weight function a dictates the coercivity of the energy density, saying how far apart the exponents $q > p$ have to be ensure modular approximation. This case is more closely related to variable exponent than to weighted spaces. Again, in the river example we imagine sand with energy density p and soil with energy density q with a modulating weight a whose regularity governs the transition between the phases.

The double-phase spaces originally appeared in the context of homogenization and the Lavrentiev phenomenon (see Zhikov's pioneering work [337] and the more recent [340]). Recently the regularity theory of minimizers to variational functionals has received interest, starting from [137, 98, 97, 24, 25]. See also [23, 75, 106, 107]. In this context the optimal approximation in the modular topology is strictly connected to the regularity results [256]. Lately, attention has focussed on problems exhibiting a variable exponent modification of double-phase energy [68, 257, 12, 278] or an Orlicz modification [65, 17].

Let us note that the double phase spaces with bounded $a \ge 0$ and $1 < p,q < \infty$ are reflexive no matter if the interplay of the parameters is uncontrolled or how irregular the weight is. Thus, for all possible choices of parameters the modular and the norm topologies coincide.

3.8.1.5 Isotropic and anisotropic Orlicz spaces

The power-type growth conditions of the classical Lebesgue or Sobolev case can be generalized in another direction by considering an operator of the form

$$\operatorname{div}\Big(\frac{m(|\nabla u|)}{|\nabla u|^2}\nabla u\Big)$$

with a convex function m satisfying a doubling condition. This idea goes back to Talenti [306, 307] and with general growth to Donaldson [122, 123] and Gossez [173, 174, 175]. Let us refer to later results of Benkirane, Elmahi and Meskine [33, 131] and Lieberman [235]. For a comprehensive existence theory for data in the dual space we refer to [263] by Mustonen and Tienari concerning elliptic existence and to [130] by Elmahi and Meskine for the corresponding parabolic results.

For the basics of the isotropic Orlicz spaces and a geometric introduction to this setting we refer to the short book by Krasnosel'skii and Rutickii [220]. The classical, very comprehensive book of Rao and Ren [281] systematises the framework, while the book [5] highlights clearly the crucial points of the theory relevant to differential equations. The applied motivation for the Orlicz setting include the modeling of non-Newtonian fluids [55] and of elastodynamics [268]. A good example here is wet sand on a low river bank or sea shore, which is shear-thickening. Under these conditions, a runner will leave dry footprints on the wet surface that dissolve slowly, and one can consider the diffusion process there.

For recent results on existence, potential theory, and regularity we refer e.g. to [22, 26, 56, 77, 85, 72, 74, 78, 90, 95, 114, 199], while for nonexistence to [213]. For the embedding results the classical reference is [315] by Trudinger, while the optimal embeddings are provided by Cianchi in [89] for the isotropic and in [91] for the anisotropic case. See [90] for a broad and deep overview of embedding results.

We recall again that Section 2.3.1.2 describes anisotropy. For the foundations of research on the anisotropic Orlicz results we refer to the fundamental works on symmetrization theory [91, 93] and existence and uniqueness of PDEs in this setting to [9, 83, 182].

3.8.1.6 The general Musielak–Orlicz setting

All the above mentioned challenges are faced while examining problems involving operators of the form

$$\operatorname{div}\Big(\frac{M(x,\nabla u)}{|\nabla u|^2}\nabla u\Big),$$

when M is an inhomogeneous and fully anisotropic N-function from Definition 2.2.2.

The investigation of the general isotropic approach started with the pioneering monograph of Nakano [265] and articles by Skaff [296, 297], Hudzik and Kamińska [206, 207, 208, 214, 215]. The monograph of Musielak [262] describes the prominent role played by the functional analysis of Musielak–Orlicz spaces. See the newest monographs on the topic [191, 251]. The cornerstones of the theory of PDEs in this setting come from the Russian school [226, 210], where they are called non-uniformly elliptic problems. The applications to modeling start from Ball's classic paper [19] on elasticity. For more recent results we refer to [218] on thermo-visco-elasticity and [180, 181, 183, 184, 326, 328] on the theory of non-Newtonian fluids. Nowadays the most intensively investigated fields also include potential theory [190, 87], harmonic analysis [35, 113, 198, 197], regularity theory [26, 82, 112, 192, 195, 196, 199], the variational approach to PDEs [252], and homogenization [59, 60]. We want to stress the available embeddings of [99, 144, 243]. Excluding Lavrentiev's phenomenon is elaborated on in [7, 52]. Weak solutions to parabolic problems in spaces changing with time are studied in [304, 80, 63]. Existence for measure data problems in reflexive spaces is studied in [73]. Renormalized solutions to L^1-data problems in nonreflexive anisotropic Musielak–Orlicz spaces are considered in the elliptic setting in [109, 179, 186, 187, 233] and in the parabolic setting in [79, 81, 188]. For more, see the surveys [71, 256, 248].

3.8.2 The meaning of the growth and coercivity conditions

We want to study operators which have a more relaxed growth than those presented in Sections 3.8.1.1–3.8.1.6, while keeping the functional setting for the solution. Let us describe what type of nonstandard growth and coercivity conditions can be found in the literature and what they imply.

Let us concentrate on a vector field $\mathbf{a} : Z \times \mathbb{R}^d \to \mathbb{R}^d$ which is a Carathéodory function and is monotone in the sense that for all $\xi, \eta \in \mathbb{R}^d$ and a.a. $x \in \Omega$ we have

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq 0. \tag{3.87}$$

We assume further that

(i) there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$M(x, c_1\xi) \leq \mathbf{a}(x,\xi) \cdot \xi, \tag{3.88}$$

$$c_2 M^*(x, c_3\mathbf{a}(x,\xi)) \leq M(x, c_4\xi) \tag{3.89}$$

OR

(ii) there exist $d_1 \in (0,1)$ and $d_2, d_3 > 0$ such that

$$d_1\Big(M(x, d_2\xi) + M^*\big(x, d_3\mathbf{a}(x,\xi)\big)\Big) \leq \mathbf{a}(x,\xi) \cdot \xi. \tag{3.90}$$

Proposition 3.8.1 *If M is an N-function, then (ii) implies (i).*

Proof. Suppose that (3.90) holds. By Jensen's inequality

$$M(x, d_1 d_2 \xi) \leq d_1 M(x, d_2\xi),$$

so (3.88) is satisfied with $c_1 = d_1 d_2$. Let us take any $c_4 > 1/(d_1 d_3) > 1/d_3$ and notice that

$$d_1\Big(M(x,d_2\xi)+M^*\big(x,d_3\mathbf{a}(x,\xi)\big)\Big) \le \big(\tfrac{1}{c_4}\mathbf{a}(x,\xi)\big)\cdot\big(c_4\xi\big)$$
$$\le M(x,c_4\xi)+M^*\Big(x,\tfrac{d_3}{c_4d_3}\mathbf{a}(x,\xi)\Big)$$
$$\le M(x,c_4\xi)+\tfrac{1}{c_4d_3}M^*\left(x,d_3\mathbf{a}(x,\xi)\right),$$

where that last inequality holds due to Jensen's inequality. By rearranging terms in the last display we infer that

$$\Big(d_1-\tfrac{1}{c_4d_3}\Big)M^*\left(x,d_3\mathbf{a}(x,\xi)\right)\le M(x,c_4\xi).$$

By fixing $c_2=d_1-\frac{1}{c_4d_3}$ and $c_3=d_3$ we get (3.89). □

The aim of imposing assumptions on the growth and coercivity of the operator is to place the solution in the controlled functional regime so that its gradient lives in the Musielak–Orlicz space L_M. Then the operator evaluated in the gradient is expected to live in the associate space L_{M^*}. Recall however that L_M and L_{M^*} are dual to each other only provided $M,M^*\in\Delta_2$, cf. Remark 3.3.3.

Lemma 3.8.2 *Suppose M is an N-function and $\mathbf{a}:Z\times\mathbb{R}^d\to\mathbb{R}^d$ is a Carathéodory function satisfying* (3.88)–(3.89)*, monotone in the sense of* (3.87)*, and $\mathbf{a}(\cdot,\zeta)\cdot\zeta\in L^1(\Omega)$. Then there exists a $C>0$ dependent only on the parameters from* (3.88)*,* (3.89) *and $\|\mathbf{a}(\cdot,\zeta)\cdot\zeta\|_{L^1(\Omega)}$ such that*

$$\|\mathbf{a}(\cdot,\zeta)\|_{L_{M^*}(\Omega)}<C.$$

Proof. The duality $(E_M)^*=L_{M^*}$ is proved in Theorem 3.5.3, thus we can equip L_M with the norm

$$\|\eta\|_{(E_M)^*}=\frac{1}{\lambda}\sup\left\{\int_\Omega\eta\cdot\xi\;\mathrm{d}x:\quad\|\xi\|_{L_{M^*}}\le\lambda\right\}\tag{3.91}$$

for some $\lambda>0$ comparable to the Luxemburg norm given by (3.4), cf. Lemma 3.1.12. Our aim is to find a bound on $\|\mathbf{a}(x,\zeta)\|_{(E_M)^*}$.

First we observe that due to the monotonicity of the operator for any $\xi\in E_M$ we have that

$$\mathbf{a}(x,\zeta)\cdot\xi\le\mathbf{a}(x,\zeta)\cdot\zeta-\mathbf{a}(x,\xi)\cdot(\zeta-\xi).$$

On the other hand, by the coercivity condition (3.88) and the assumption

$$\int_\Omega M(x,c_1\zeta)\;\mathrm{d}x\le\int_\Omega\mathbf{a}(x,\zeta)\cdot\zeta\;\mathrm{d}x=\|\mathbf{a}(x,\zeta)\cdot\zeta\|_{L^1(\Omega)}.$$

We estimate

$$-\int_\Omega\mathbf{a}(x,\xi)\cdot(\zeta-\xi)\;\mathrm{d}x=-\int_\Omega\frac{2}{c_3c_1}\left(c_3\mathbf{a}(x,\xi)\right)\cdot\left(\frac{\zeta-\xi}{2/c_1}\right)\;\mathrm{d}x$$
$$\le\frac{2}{c_3c_1}\int_\Omega M^*\left(x,c_3\mathbf{a}(x,\xi)\right)+M\left(x,\frac{\zeta-\xi}{2/c_1}\right)\;\mathrm{d}x$$

$$\leq \frac{2}{c_3c_1}\int_\Omega \frac{1}{c_2}M(x,c_4\xi)+M(x,c_1\zeta)+M(x,c_1\xi)\ \mathrm{d}x$$
$$\leq \frac{2}{c_3c_1}\left[\left(\frac{1}{c_2}+1\right)\int_\Omega M(x,\max\{c_1,c_4\}\xi)+\|\mathbf{a}(x,\zeta)\cdot\zeta\|_{L^1(\Omega)}\right].$$

Here we used the Fenchel–Young inequality (Lemma (2.1.32)), (3.89), and convexity. Note that by Lemma 3.1.14 if $\eta\in L_M(Z;\mathbb{R}^d)$ with $\|\eta\|_{L_M}\leq 1$, then it satisfies $\varrho_M(\eta)\leq\|\eta\|_{L_M}$. Thus, provided $\max\{c_1,c_4\}\|\xi\|_{L_{M^*}}\leq 1$, we can actually estimate

$$\int_\Omega \mathbf{a}(x,\zeta)\cdot\xi\ \mathrm{d}x \leq \int_\Omega \mathbf{a}(x,\zeta)\cdot\zeta-\mathbf{a}(x,\xi)\cdot(\zeta-\xi)\ \mathrm{d}x$$
$$\leq \|\mathbf{a}(x,\zeta)\cdot\zeta\|_{L^1(\Omega)}+\frac{2}{c_3c_1}\left[\left(\frac{1}{c_2}+1\right)\|\xi\|_{L_{M^*}}\max\{c_1,c_4\}+\|\mathbf{a}(x,\zeta)\cdot\zeta\|_{L^1(\Omega)}\right].$$

Let us consider $\eta=\mathbf{a}(x,\zeta)$ and $\lambda=\max\{c_1,c_4\}$ in (3.91) to get

$$\|\mathbf{a}(x,\zeta)\|_{(E_M)^*}\leq\frac{1}{\max\{c_1,c_4\}}\sup\left\{\int_\Omega \mathbf{a}(x,\zeta)\cdot\xi\ \mathrm{d}x:\quad \|\max\{c_1,c_4\}\xi\|_{L_{M^*}}\leq 1\right\}$$
$$\leq\frac{2}{c_1c_3\max\{c_1,c_4\}}\left((c_3c_1+1)\|\mathbf{a}(x,\zeta)\cdot\zeta\|_{L^1(\Omega)}+\frac{1}{c_2}+1\right),$$

which completes the proof. □

Remark 3.8.3. In the current monograph we have decided to restrict to the case when **a** is a function, however a lot of facts could be presented for multi-valued mappings. We list a few examples of such results:

- existence of renormalized solutions to elliptic problems, see [109],
- existence of weak solutions to parabolic problems, see [304, 303],
- existence of weak solutions to the non-Newtonian fluid model, see [61].

In the classical L^p-setting both conditions (i) and (ii) are equivalent to the classical growth and coercivity conditions of Leray and Lions [232] ensuring pseudomonotonicity of the involved operator. Note that in the case when $M=c_1|\xi|^p$ the coercivity condition (3.88) as well as (3.90) directly imply

$$c_1^p|\xi|^p\leq\mathbf{a}(x,\xi)\cdot\xi. \tag{3.92}$$

Moreover, (3.89) yields $|\mathbf{a}(x,\xi)|^{p/(p-1)}\leq c|\xi|^p$, leading further to the condition

$$|\mathbf{a}(x,\xi)|\leq c_2^p|\xi|^{p-1}. \tag{3.93}$$

The reverse implication follows trivially. On the other hand, to get (3.93) from (3.90) it suffices to use Young's inequality in the following way

$$d_1\left(|d_2\xi|^p+|d_3\mathbf{a}(x,\xi)|^{p/(p-1)}\right)\leq\mathbf{a}(x,\xi)\cdot\xi$$
$$\leq d_1\left(|(2/d_2)\xi|^p+|(d_3/2)\mathbf{a}(x,\xi)|^{p/(p-1)}\right).$$

After absorbing one term and then dropping the other nonnegative one on the left-hand side, we get an inequality of the form (3.89). We have already seen that this is sufficient for (3.93). Of course, the converse is also true, that is, if the classical conditions (3.92) and (3.93) hold true, then we have (i) as well as (ii). For comments on conditions (3.92) and (3.93), see also [264].

Conditions of the form (i) are considered in the classical Orlicz setting without growth restrictions, when $M(x,\xi) = m(|\xi|)$ is homogeneous and isotropic by e.g. [175, 173, 263, 132, 9, 78]. In the classical Orlicz case when $m, m^* \in \Delta_2$ the mentioned growth and coercivity conditions can also be expressed in the following way

$$m(|\xi|) \le \mathbf{a}(x,\xi)\cdot\xi \qquad \text{and} \qquad |\mathbf{a}(x,\xi)| \le cm'(|\xi|), \tag{3.94}$$

where m' is the left-derivative of m, cf. [95]. Sometimes when $m, m^* \in \Delta_2$ in these conditions one uses $m(s)/s$ instead of $m'(s)$, but note that $m'(s) \simeq m(s)/s$, see Lemma 2.3.16. Inhomogeneity does not present any obstacles in this type of formulation. For instance, the assumptions

$$c_2 M(x,|\xi|) \le \mathbf{a}(x,\xi)\cdot\xi \qquad \text{and} \qquad \mathbf{a}(x,\xi) \le c_1 M(x,|\xi|)/|\xi|$$

with $M, M^* \in \Delta_2$ are employed in [82, 73]. Following [226, 210] and the recent [256, 248] we call problems under conditions related to the above non-uniformly elliptic. That is, if $f : \Omega\times\mathbb{R}^N \to \mathbb{R}$ is smooth and one investigates the operator $-\mathrm{div}\mathbf{a}(x,Du) = D_\xi f(x,\xi)$, then the ellipticity condition reads $D^2_{\xi\xi} f(x,\xi)\zeta\cdot\zeta > 0$ for $\xi,\zeta \in \mathbb{R}^N$. Provided the growth of f is governed by the same doubling function m from below and from above, we are back in the regime of (3.94). If m depends on x, one is deprived of uniform control over ellipticity. For detailed comments, see the recent survey [256].

Notice that when $m, m^* \in \Delta_2$, Lemma 2.3.19 yields

$$m^*\big(m'(s)\big) \le cm(s). \tag{3.95}$$

This is equivalent to

$$m'(s) \le (m^*)^{-1}\big(cm(s)\big).$$

Note that (3.95) does not hold in the case when both $m, m^* \notin \Delta_2$, e.g. for $m(s) = s\log(1+s)$ or $m(s) = s\exp(s)$. In fact, outside the doubling case it is commonly assumed that

$$m(|\xi|) \le \mathbf{a}(x,\xi)\cdot\xi \qquad \text{and} \qquad |\mathbf{a}(x,\xi)| \le c_1(m^*)^{-1}\big(c_2 m(|\xi|)\big),$$

as is done in [130, 175, 173, 263], rather than (3.94).

In order to involve *anisotropic* structure it is more relevant to consider

$$m(\xi) \le \mathbf{a}(x,\xi)\cdot\xi \qquad \text{and} \qquad m^*\big(c_1\,\mathbf{a}(x,\xi)\big) \le c_2 m(\xi),$$

related to (i) (see [9]), or to hide them both in one assumption

$$c\Big(m(\xi) + m^*\big(\mathbf{a}(x,\xi)\big)\Big) \le \mathbf{a}(x,\xi)\cdot\xi$$

related to (ii) (see [186, 179]). We show that the condition (ii) implies (i), but the reverse is not true in general.

Part II
PDEs

This part is devoted to the application of Musielak–Orlicz spaces to partial differential equations. Although the inhomogeneity and anisotropy of the underlying space deprives us of many classical tools, we provide a deep study of nonlinear PDEs under very general conditions. We concentrate on weak solutions to elliptic and parabolic problems with bounded data, renormalized solutions to elliptic and parabolic problems with L^1 data, homogenization of elliptic problems, as well as the theory of non-Newtonian fluids.

Chapter 4 is devoted to the existence of weak solutions. There are two alternative fundamental tools used in existence proofs, based either on weak-$*$ convergence or on modular convergence. Each of them directly depends on the properties of an N-function. We will see that there are definite advantages to using assumptions on its growth (Δ_2-condition), however this information can be replaced by continuity-type assumptions on an N-function.

Chapter 5 concerns elliptic and parabolic partial differential equations of a simple structure as in Chapter 4, but with merely integrable data. Consequently, weak solutions are not well-defined and we are forced to understand them in a very weak sense by employing the notion of renormalized solutions. The proof of existence in the parabolic case is particularly delicate because the modular function is allowed to be inhomogeneous both in the time and in the space variable.

In Chapter 6 we study the theory of homogenization for families of strongly nonlinear elliptic problems. The growth and the coercivity of an elliptic operator is again assumed to be prescribed by an inhomogeneous anisotropic N-function. The overall impediment is the dependence of an N-function on a spatial variable, as consequently in each step of the homogenization process the underlying function spaces change. For that reason the presented approach is far from just being a simple extension of an analogous problem in the standard L^p-spaces. We characterize the notion of weak-$*$ and strong two-scale convergence in the setting of Musielak–Orlicz spaces, which is here the method for proving the convergence of the homogenization process.

Chapter 7 concerns a large class of problems arising from the mechanics of incompressible non-Newtonian fluids with nonstandard rheology. We concentrate there on the phenomenon of viscosity changing under various stimuli like shear rate, and magnetic or electric fields. We study the case when the relation between the viscous stress tensor and shear stress may be anisotropic, inhomogeneous and not necessarily of polynomial type.

Chapter 4
Weak Solutions

4.1 Elliptic Equations

This section gathers different results on the existence of weak solutions to elliptic problems. In the first subsection we formulate assumptions on the operator, however in various considered cases the assumptions on an N-function will differ, which prevents the possibility of a universal approach to all the problems. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N > 1$. Given a function $\mathbf{f} : \Omega \to \mathbb{R}^d$, we consider the following system

$$\begin{cases} -\operatorname{div} \mathbf{A}(x, \nabla \mathbf{u}) = \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{u}(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathbf{u} : \Omega \to \mathbb{R}^d$ and the operator $\mathbf{A} : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ is controlled by an anisotropic and inhomogeneous modular function M. The considerations in Sections 4.1.4 and 4.2 are, only for simplicity, restricted to the case of scalar equations, i.e., $d = 1$ and thus $u : \Omega \to \mathbb{R}$. In this way we avoid proving the existence of approximate solutions in Section 4.1.4, and we may use the result for scalar equations [263]. However the method is not restricted to the scalar case and the proof could be easily rewritten for a system of equations. In those sections we will switch to the lower case notation $\mathbf{a}$ for an operator instead of $\mathbf{A}$ to highlight it and to be consistent with the notational convention used in this book. Function spaces in which we consider our solutions are defined and discussed in Section 3.6.

4.1.1 Assumptions on the operator

Let us recall that an N-function M is defined in Definition 2.2.2, while its conjugate M^* in Definition 2.1.28. We assume that $\mathbf{A}$ satisfies the following conditions.

(A1e) $\mathbf{A} : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ is a Carathéodory function;
(A2e) **Growth and coercivity.** There exist an N-function $M : \Omega \times \mathbb{R}^{d\times N} \to [0, \infty)$ and constants $c_1, c_2, c_3, c_4 > 0$, such that for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$ we

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have

$$M(x, c_1\xi) \leq \mathbf{A}(x,\xi)\cdot\xi, \tag{4.2}$$
$$c_2 M^*(x, c_3\mathbf{A}(x,\xi)) \leq M(x, c_4\xi). \tag{4.3}$$

(A3e) **Monotonicity.** For all $\xi,\eta \in \mathbb{R}^{d\times N}$ and a.a. $x \in \Omega$ we have

$$(\mathbf{A}(x,\xi) - \mathbf{A}(x,\eta))\cdot(\xi - \eta) \geq 0.$$

Growth and coercivity conditions were discussed in more detail in Section 3.8.2. The proof of the first presented existence result is carried out under condition *(A2e)*, which is more general than condition (3.90), which often appears in the literature. For the sake of clarity of presentation, in further chapters, as well as in further parts of this chapter, conditions (4.2)–(4.3) are simplified to just one constant $c_{\mathbf{a}}$.

*(A2e)** There exist an N-function $M : \Omega\times\mathbb{R}^N \to [0,\infty)$ and a constant $c_{\mathbf{a}} > 0$ such that for a.a. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ we have

$$c_{\mathbf{a}} M^*(x, \mathbf{a}(x,\xi)) \leq M(x,\xi) \qquad \text{and} \qquad M(x,\xi) \leq \mathbf{a}(x,\xi)\xi. \tag{4.4}$$

In (4.4) we intentionally used notation appropriate later for the scalar equation, not a system, because in such a framework these conditions shall be used, see Theorem 4.1.5.

Monotonicity may not be sufficient to show uniqueness of solutions and in some cases even to show existence of solutions. Thus, assumption *(A3e)* in this case is substituted with a more rigorous condition

(A3e)* **Strict monotonicity.** For all $\xi,\eta \in \mathbb{R}^{d\times N}$ and a.a. $x \in \Omega$ we have

$$(\mathbf{A}(x,\xi) - \mathbf{A}(x,\eta))\cdot(\xi - \eta) > 0.$$

The above set of assumptions is not complete. In particular, until now all we know about M is that it is an anisotropic inhomogeneous N-function. However such a generic condition is not sufficient and further properties either on

(i) the growth in the second variable,

or

(ii) continuity in the first variable

shall be prescribed. These two options build the structure of the current chapter. After a short section on the monotonicity trick, which can be proved independently of assumptions on an N-function, the next two sections are devoted to these cases. Section 4.1.3 concerns the situation when information is given on the growth of an N-function or on its conjugate. This allows us to justify using weak-$*$ convergence techniques in the proof. Section 4.1.4 solves the problem of existence in the case when some kind of continuity of an N-function in the first variable is assumed. We will see that this is an assumption in the spirit of a log-Hölder continuity condition. The core of that proof is built by approximation and the modular density techniques investigated in Section 3.7. Throughout these two sections we present two different approximation techniques for problem (4.1) that can be applied in existence proofs: the Galerkin method in Section 4.1.3 and through adding a regularizing term in Section 4.1.4. Note that neither of these two cases demands great care in the choice of approximation method, indeed any of the presented methods could be used.

4.1.2 The monotonicity trick in the elliptic case

When considering PDEs with the operators described above, the obstruction, which we will encounter, is that weak-$*$ convergence is badly behaved with respect to nonlinearities, and even if such convergence is improved to modular convergence, strong convergence still may not be achieved. However, the property of monotonicity allows us to identify limits of nonlinear terms using the following technique, often called the monotonicity trick, e.g. in [178, 179, 186, 326].

Theorem 4.1.1 (Monotonicity trick in the elliptic case) *Suppose* $\mathbf{A} : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ *satisfies conditions (A1e)–(A2e) with an N-function* $M : \Omega \times \mathbb{R}^{d\times N} \to [0,\infty)$. *Assume further that there exist*

$$\boldsymbol{\mathcal{A}} \in L_{M^*}(\Omega;\mathbb{R}^{d\times N}) \quad \textit{and} \quad \boldsymbol{\xi} \in L_M(\Omega;\mathbb{R}^{d\times N})$$

such that

$$\int_\Omega \big(\boldsymbol{\mathcal{A}} - \mathbf{A}(x,\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}x \ge 0 \quad \textit{for all } \boldsymbol{\eta} \in \mathbb{R}^{d\times N}. \tag{4.5}$$

Then

$$\mathbf{A}(x,\boldsymbol{\xi}) = \boldsymbol{\mathcal{A}} \qquad \textit{a.e. in } \Omega.$$

Proof. Let us define

$$\Omega_K = \{x\in\Omega : \ |\boldsymbol{\xi}(x)| \le K\} \quad \text{for any} \quad K \in \mathbb{N}.$$

Fix arbitrary $0 < j < i$ and notice that $\Omega_j \subset \Omega_i$. We apply (4.5) with

$$\boldsymbol{\eta} = \boldsymbol{\xi}\mathbb{1}_{\Omega_i} + h\mathbf{z}\mathbb{1}_{\Omega_j},$$

where $h \in (0,1)$ and $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^{d\times N})$, giving

$$\int_\Omega (\boldsymbol{\mathcal{A}} - \mathbf{A}(x,\boldsymbol{\xi}\mathbb{1}_{\Omega_i} + h\mathbf{z}\mathbb{1}_{\Omega_j}))\cdot(\boldsymbol{\xi} - \boldsymbol{\xi}\mathbb{1}_{\Omega_i} - h\mathbf{z}\mathbb{1}_{\Omega_j})\,\mathrm{d}x \ge 0.$$

Notice that this is equivalent to

$$\int_{\Omega\setminus\Omega_i} \big(\boldsymbol{\mathcal{A}} - \mathbf{A}(x,0)\big)\cdot\boldsymbol{\xi}\,\mathrm{d}x + h\int_{\Omega_j} (\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z}) - \boldsymbol{\mathcal{A}})\cdot\mathbf{z}\,\mathrm{d}x \ge 0. \tag{4.6}$$

The first expression above tends to zero when $i\to\infty$. Indeed, *(A2e)* implies $\mathbf{A}(x,0) = 0$, moreover $\boldsymbol{\mathcal{A}} \in L_{M^*}(\Omega;\mathbb{R}^{d\times N})$ and $\boldsymbol{\xi} \in L_M(\Omega;\mathbb{R}^{d\times N})$, and the Hölder inequality (3.21) gives $\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\xi} \in L^1(\Omega)$. Then we take into account shrinking the domain of integration to get the desired convergence to 0. In particular, we can drop these expressions in (4.6) and divide the remaining expression by $h > 0$, to obtain

$$\int_{\Omega_j} (\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z}) - \boldsymbol{\mathcal{A}}) \cdot \mathbf{z}\, \mathrm{d}x \geq 0.$$

Note that

$$\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z}) \xrightarrow[h\to 0]{} \mathbf{A}(x,\boldsymbol{\xi}) \quad \text{a.e. in} \quad \Omega_j.$$

Due to *(A2e)*$_2$ we have

$$c_2 \sup_{h\in(0,1)} \int_{\Omega_j} M^*\left(x, c_3\mathbf{A}\left(x,\boldsymbol{\xi}+h\mathbf{z}\right)\right)\, \mathrm{d}x \leq \sup_{h\in(0,1)} \int_{\Omega_j} M\left(x, c_4(\boldsymbol{\xi}+h\mathbf{z})\right)\, \mathrm{d}x.$$

The right-hand side is bounded, because $\{\boldsymbol{\xi}+h\mathbf{z}\}_{h\in(0,1)}$ is uniformly bounded in

$$L^\infty(\Omega_j;\mathbb{R}^{d\times N}) \subset L_M(\Omega;\mathbb{R}^{d\times N}).$$

Indeed, on Ω_j by definition $|\boldsymbol{\xi}| \leq j$. Hence, Theorem 3.4.2 gives the uniform integrability of the family $\{\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z})\}_{h\in(0,1)}$. Since $|\Omega_j| < \infty$, we can apply the Vitali convergence theorem (Theorem 8.23) to get

$$\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z}) \xrightarrow[h\to 0]{} \mathbf{A}(x,\boldsymbol{\xi}) \quad \text{in} \quad L^1(\Omega_j;\mathbb{R}^{d\times N}).$$

Thus

$$\int_{\Omega_j} (\mathbf{A}(x,\boldsymbol{\xi}+h\mathbf{z}) - \boldsymbol{\mathcal{A}}) \cdot \mathbf{z}\, \mathrm{d}x \xrightarrow[h\to 0]{} \int_{\Omega_j} (\mathbf{A}(x,\boldsymbol{\xi}) - \boldsymbol{\mathcal{A}}) \cdot \mathbf{z}\, \mathrm{d}x.$$

Consequently,

$$\int_{\Omega_j} (\mathbf{A}(x,\boldsymbol{\xi}) - \boldsymbol{\mathcal{A}}) \cdot \mathbf{z}\, \mathrm{d}x \geq 0,$$

for any $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^{d\times N})$. The choice of

$$\mathbf{z} = \begin{cases} -\frac{\mathbf{A}(x,\boldsymbol{\xi})-\boldsymbol{\mathcal{A}}}{|\mathbf{A}(x,\boldsymbol{\xi})-\boldsymbol{\mathcal{A}}|} & \text{if} \quad \mathbf{A}(x,\boldsymbol{\xi}) - \boldsymbol{\mathcal{A}} \neq 0, \\ 0 & \text{if} \quad \mathbf{A}(x,\boldsymbol{\xi}) - \boldsymbol{\mathcal{A}} = 0, \end{cases}$$

leads to

$$\int_{\Omega_j} |\mathbf{A}(x,\boldsymbol{\xi}) - \boldsymbol{\mathcal{A}}|\, \mathrm{d}x \leq 0.$$

Hence

$$\mathbf{A}(x,\boldsymbol{\xi}) = \boldsymbol{\mathcal{A}} \quad \text{a.e.} \quad \text{in} \quad \Omega_j.$$

Since j is arbitrary, we have the equality a.e. in Ω and (4.68) is satisfied. □

4.1.3 Elliptic problems in cases $M \in \Delta_2$ or $M^* \in \Delta_2$

In this section we concentrate on the case when no information on the regularity of an N-function with respect to x is given. This possible irregularity needs to be compensated somehow as it closes the possibility of using arguments based on modular convergence, and indeed this tool is replaced by weak-$*$ compactness. This argumentation holds once we have information on the growth of an N-function M or its conjugate M^*, particularly that one of them satisfies the Δ_2-condition. As these two cases are not analogous, they are considered separately.

We start with the case of the assumption of the Δ_2-condition on the conjugate M^*. The existence result is formulated in the following theorem.

Theorem 4.1.2 *Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Assume that an operator $\mathbf{A}$ satisfies (A1e)–(A3e) and $M : \Omega \times \mathbb{R}^{d\times N} \to [0,\infty)$ is an N-function such that M^* satisfies the Δ_2-condition. Assume that $\mathbf{f} = \operatorname{div}\mathbf{F}$ and $\mathbf{F} \in E_{M^*}(\Omega;\mathbb{R}^{d\times N})$. Then there exists a weak solution to problem* (4.1)*, which is a function*

$$\mathbf{u} \in W^1_0 L_M(\Omega;\mathbb{R}^d)$$

such that

$$\int_\Omega \mathbf{A}(x,\nabla\mathbf{u}(x)) \cdot \nabla\boldsymbol{\varphi}(x)\,\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\nabla\boldsymbol{\varphi}(x)\,\mathrm{d}x \tag{4.7}$$

is satisfied for all $\boldsymbol{\varphi} \in W^1_0 L_M(\Omega;\mathbb{R}^N)$. If in addition (A3e) is satisfied, then the weak solution is unique.*

Proof. The proof of existence of solutions uses the Galerkin method. Firstly, we construct solutions of finite-dimensional approximations to (4.1) and then pass to the limit. Consider a set of smooth linearly independent functions $\{\mathbf{w}^i\}_{i=1}^\infty$, which may, for example, be the set of eigenfunctions of the $-\Delta$ operator with Dirichlet boundary condition and project the original problem to the space spanned by $\{\mathbf{w}^i\}_{i=1}^k$ for some fixed $k \in \mathbb{N}$.

Let us then define $\mathbf{u}^k = \sum_{i=1}^k \alpha_i^k \mathbf{w}^i$ for $k \in \mathbb{N}$, where $\alpha_i^k \in \mathbb{R}$ solve the system

$$\int_\Omega \mathbf{A}\left(x,\nabla\mathbf{u}^k\right)\cdot\nabla\mathbf{w}^j\,\mathrm{d}x = \int_\Omega \mathbf{F}\cdot\nabla\mathbf{w}^j\,\mathrm{d}x \tag{4.8}$$

for all $j = 1,\dots,k$.

Existence of solutions to the approximate problem. The existence of $\boldsymbol{\alpha} = (\alpha_1^k,\dots,\alpha_k^k) \in \mathbb{R}^k$ satisfying (4.8) follows from Lemma 8.53. To show that the assumptions of the lemma are satisfied for $j = 1,\dots,k$, we define a mapping $\mathbf{s} : \mathbb{R}^k \to \mathbb{R}^k$ as

$$s_j(\boldsymbol{\alpha}) = \int_\Omega \mathbf{A}\left(x,\sum_{i=1}^k \alpha_i^k\nabla\mathbf{w}^i\right)\cdot\nabla\mathbf{w}^j - \mathbf{F}\cdot\nabla\mathbf{w}^j\,\mathrm{d}x. \tag{4.9}$$

Let us define $\omega(\boldsymbol{\alpha}) := \sum_{i=1}^k \alpha_i^k\mathbf{w}^i$. To show that $\mathbf{s}$ is continuous we choose a sequence $\{\boldsymbol{\alpha}^n\}_{n=1}^\infty$ such that $\boldsymbol{\alpha}^n \to \boldsymbol{\alpha}$ in $\mathbb{R}^k$. Then

$$|s_j(\boldsymbol{\alpha}^n) - s_j(\boldsymbol{\alpha})| = \left| \int_\Omega (\mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha}^n)) - \mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha}))) \cdot \nabla \mathbf{w}^j \, \mathrm{d}x \right|$$

for all $j = 1, \dots, k$. We define

$$h_j^n := (\mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha}^n)) - \mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha}))) \cdot \nabla \mathbf{w}^j.$$

Obviously, we have for almost all $x \in \Omega$ that $h_j^n \to 0$ as $n \to \infty$. Condition *(A2e)* implies

$$c_2 M^*(x, c_3 \mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha}^n)) \le M(x, c_4 \nabla\omega(\boldsymbol{\alpha}^n)) \tag{4.10}$$

and

$$\begin{aligned} M(x, c_4 \nabla\omega(\boldsymbol{\alpha}^n)) &\le \sum_{i=1}^k \frac{\alpha_i^{k,n}}{|\boldsymbol{\alpha}^n|} \int_\Omega M\left(x, c_4 |\boldsymbol{\alpha}^n| \nabla \mathbf{w}^i\right) \, \mathrm{d}x \\ &\le k \max_{i=1,\dots k} \int_\Omega M\left(x, c_4 |\boldsymbol{\alpha}^n| \nabla \mathbf{w}^i\right) \, \mathrm{d}x, \end{aligned}$$

which is finite as $\{\boldsymbol{\alpha}^n\}_{n=1}^\infty$ is bounded. From these estimates one deduces the uniform integrability of $\mathbf{A}(\cdot, \nabla\omega(\boldsymbol{\alpha}^n))$. As $\mathbf{A}(\cdot, \nabla\omega(\boldsymbol{\alpha})) \in L^1(\Omega; \mathbb{R}^{d\times N})$ and $\nabla \mathbf{w}^i \in L^\infty\left(\Omega; \mathbb{R}^{d\times N}\right)$, we conclude that h_j^n is uniformly integrable and thus the Vitali theorem provides that $\mathbf{s}$ is continuous, indeed

$$|\mathbf{s}(\boldsymbol{\alpha}^n) - \mathbf{s}(\boldsymbol{\alpha})| \le k \max_{j=1,\dots,k} \int_\Omega |h_j^n| \, \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Now, we show that $\mathbf{s}$ satisfies (8.6) – the assumption of Lemma 8.53. Employing $(A2e)$, the Fenchel–Young inequality and Lemma 3.1.14 (ii) we deduce

$$\begin{aligned} \mathbf{s}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} &= \int_\Omega \mathbf{A}(x, \nabla\omega(\boldsymbol{\alpha})) \cdot \nabla\omega(\boldsymbol{\alpha}) - \mathbf{F} \cdot \nabla\omega(\boldsymbol{\alpha}) \, \mathrm{d}x \\ &\ge \tfrac{1}{2} \int_\Omega M(x, c_1 \nabla\omega(\boldsymbol{\alpha})) \, \mathrm{d}x - M^*\left(x, \tfrac{2}{c_1}\mathbf{F}\right) \, \mathrm{d}x \\ &\ge \tfrac{1}{2}(c_1 \|\nabla\omega(\boldsymbol{\alpha})\|_{L_M(\Omega)} - 1) - \int_\Omega M^*\left(x, \tfrac{2}{c_1}\mathbf{F}\right) \, \mathrm{d}x. \end{aligned} \tag{4.11}$$

Let us show that

$$\|\nabla\omega(\boldsymbol{\alpha})\|_{L_M(\Omega)} \to \infty \text{ as } |\boldsymbol{\alpha}| \to \infty. \tag{4.12}$$

We observe that $\boldsymbol{\alpha} \mapsto \|\nabla\omega(\boldsymbol{\alpha})\|_{L_M(\Omega)}$ is a continuous function, in particular it is continuous on the unit sphere S_1 in $\mathbb{R}^k$, which is compact. Thus the minimum of $\|\nabla\omega(\boldsymbol{\alpha})\|_{L_M(\Omega)}$ on S_1 is attained at some $\boldsymbol{\beta} \in S_1$. We intend to show that

$$\|\nabla\omega(\boldsymbol{\beta})\|_{L_M(\Omega)} > 0, \tag{4.13}$$

and to this end we first assume the contrary that $\|\nabla\omega(\boldsymbol{\beta})\|_{L_M(\Omega)} = 0$. This assumption also implies that $\left\|\sum_{i=1}^k \beta_i^k \nabla \mathbf{w}^i\right\|_{L^1(\Omega)} = 0$ and by the Poincaré inequality $\|\sum_{i=1}^k \beta_i^k \mathbf{w}^i\|_{L^1(\Omega)} = 0$. Hence necessarily $\sum_{i=1}^k \beta_i^k \mathbf{w}^i = 0$ a.e. in Ω, which implies, since $\{\mathbf{w}^i\}_{i=1}^k$ are linearly independent, that $\beta_i^k = 0$ for each $i = 1, \dots, k$, which is a contradiction and thus (4.13) holds. We have

$$\|\nabla\omega(\boldsymbol{\alpha})\|_{L_M(\Omega)} = |\boldsymbol{\alpha}| \left\|\nabla\omega\left(\frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}\right)\right\|_{L_M(\Omega)} \geq |\boldsymbol{\alpha}| \|\nabla\omega(\boldsymbol{\beta})\|_{L_M(\Omega)}$$

and thus (4.12) follows easily. For R large enough we obtain that $\mathbf{s}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq 0$ for $|\boldsymbol{\alpha}| = R$. Consequently, by Lemma 8.53 there is an $\boldsymbol{\alpha} \in \mathbb{R}^k$ satisfying (4.8).

Uniform estimates. We show uniform estimates for $\{\mathbf{u}^k\}_{k=1}^\infty$ and $\{\mathbf{A}\left(x, \nabla\mathbf{u}^k\right)\}_{k=1}^\infty$. Multiplying (4.8) by α_i^k and summing over $i = 1, \dots, k$ yields

$$\int_\Omega \mathbf{A}\left(x, \nabla\mathbf{u}^k\right) \cdot \nabla\mathbf{u}^k \, \mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla\mathbf{u}^k \, \mathrm{d}x. \tag{4.14}$$

Using $(A2e)$ we obtain an estimate

$$\tfrac{1}{2} \int_\Omega M(x, c_1\nabla\mathbf{u}^k) \, \mathrm{d}x \leq \tfrac{1}{2} \int_\Omega \mathbf{A}\left(x, \nabla\mathbf{u}^k\right) \cdot \nabla\mathbf{u}^k \, \mathrm{d}x. \tag{4.15}$$

Lemma 2.1.23 (i) and the Fenchel–Young inequality allow us to infer that

$$\begin{aligned} \int_\Omega \tfrac{4}{c_1}\mathbf{F} \cdot \tfrac{c_1}{4}\nabla\mathbf{u}^k \, \mathrm{d}x &\leq \int_\Omega M\left(x, \tfrac{c_1}{4}\nabla\mathbf{u}^k\right) \mathrm{d}x + \int_\Omega M^*\left(x, \tfrac{4}{c_1}\mathbf{F}\right) \mathrm{d}x \\ &\leq \tfrac{1}{4} \int_\Omega M\left(x, c_1\nabla\mathbf{u}^k\right) \mathrm{d}x + \int_\Omega M^*\left(x, \tfrac{4}{c_1}\mathbf{F}\right) \mathrm{d}x. \end{aligned} \tag{4.16}$$

And thus

$$\tfrac{1}{4} \int_\Omega M\left(x, c_1\nabla\mathbf{u}^k\right) \mathrm{d}x + \tfrac{1}{2} \int_\Omega \mathbf{A}\left(x, \nabla\mathbf{u}^k\right) \cdot \nabla\mathbf{u}^k \, \mathrm{d}x \leq \int_\Omega M^*\left(x, \tfrac{4}{c_1}\mathbf{F}\right) \mathrm{d}x.$$

Since the right-hand side of the latter inequality is finite as $\mathbf{F} \in E_{M^*}(\Omega; \mathbb{R}^{d\times N})$, we infer the existence of $\mathbf{u} \in W_0^1 L_M(\Omega; \mathbb{R}^N)$ such that

$$\nabla\mathbf{u}^k \overset{*}{\rightharpoonup} \nabla\mathbf{u} \text{ in } L_M(\Omega; \mathbb{R}^{d\times N}), \tag{4.17}$$

as $k \to \infty$. Lemma 3.8.2 provides that $\mathbf{A}(\cdot, \nabla\mathbf{u}^k)$ is bounded in $L_{M^*}(\Omega; \mathbb{R}^{d\times N})$ and since M^* satisfies the Δ_2-condition it holds that there exists an $\bar{\mathbf{A}} \in E_{M^*}(\Omega; \mathbb{R}^{d\times N})$ such that

$$\mathbf{A}(\cdot, \nabla\mathbf{u}^k) \overset{*}{\rightharpoonup} \bar{\mathbf{A}} \text{ in } E_{M^*}(\Omega; \mathbb{R}^{d\times N}). \tag{4.18}$$

Characterization of the limit. We identify the limit function $\bar{\mathbf{A}}$. Again the assumptions on the right-hand side (i) and (ii) require a twofold argumentation. Firstly employing the convergence (4.18) in (4.8) we have

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla\mathbf{w}^i \, \mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla\mathbf{w}^i \, \mathrm{d}x \tag{4.19}$$

for each $i = 1, \dots, k$. Multiplying by α_i^k and summing over $i = 1, \dots, k$ we get

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u}^k \, dx = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}^k \, dx. \tag{4.20}$$

Since $\bar{\mathbf{A}} \in E_{M^*}(\Omega; \mathbb{R}^{d\times N})$, we obtain using the convergences (4.17) and (4.18)

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u} \, dx = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u} \, dx. \tag{4.21}$$

Moreover, the application of (4.17) in (4.14) yields

$$\lim_{k\to\infty} \int_\Omega \mathbf{A}\left(x, \nabla \mathbf{u}^k\right) \cdot \nabla \mathbf{u}^k \, dx = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u} \, dx. \tag{4.22}$$

Let us choose an arbitrary $\mathbf{W} \in L^\infty(\Omega; \mathbb{R}^{d\times N})$. The monotonicity of $\mathbf{A}$ combined with (4.14) yields

$$\begin{aligned} 0 &\le \int_\Omega \left(\mathbf{A}\left(x, \nabla \mathbf{u}^k\right) - \mathbf{A}(x, \mathbf{W})\right) \cdot (\nabla \mathbf{u}^k - \mathbf{W}) \, dx \\ &= \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}^k - \mathbf{A}(x, \nabla \mathbf{u}^k) \cdot \mathbf{W} - \mathbf{A}(x, \mathbf{W}) \cdot (\nabla \mathbf{u}^{n,k} - \mathbf{W}) \, dx. \end{aligned}$$

We employ (4.17) and (4.18) to perform the limit passage $k \to \infty$ in the latter inequality and use (4.21) to obtain

$$0 \le \int_\Omega (\bar{\mathbf{A}} - \mathbf{A}(x, \mathbf{W})) \cdot (\nabla \mathbf{u} - \mathbf{W}) \, dx. \tag{4.23}$$

The proof is completed using the monotonicity trick described in Section 4.1.2.

Uniqueness of solutions. We show the uniqueness of a weak solution. Supposing that $\mathbf{u}_1, \mathbf{u}_2$ are weak solutions satisfying (4.7), we subtract the weak formulations corresponding to $\mathbf{u}_1$ and $\mathbf{u}_2$ to obtain

$$\int_\Omega (\mathbf{A}(x, \mathbf{u}_1) - \mathbf{A}(x, \mathbf{u}_2)) \cdot \nabla \boldsymbol{\varphi} \, dx = 0 \; \forall \boldsymbol{\varphi} \in W_0^1 L^M(\Omega; \mathbb{R}^N). \tag{4.24}$$

As we have $\mathbf{u}_1 - \mathbf{u}_2 \in W_0^1 L^M(\Omega; \mathbb{R}^N)$, we set $\boldsymbol{\varphi} := \mathbf{u}_1 - \mathbf{u}_2$ in (4.24) to get

$$\int_\Omega (\mathbf{A}(x, \mathbf{u}_1) - \mathbf{A}(x, \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) \, dx = 0.$$

Then $(A3e^*)$ implies $\nabla(\mathbf{u}_1 - \mathbf{u}_2) = 0$ a.e. in Ω. Regarding the zero trace of $\mathbf{u}_1 - \mathbf{u}_2$ on $\partial\Omega$ we conclude $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Ω. □

The second statement concerns the case when M satisfies the Δ_2-condition. A naive idea to show existence of solutions in this case would be to follow the lines of the proof of Theorem 4.1.2. This however breaks down at the limit passage from (4.20) to (4.21). We only know that $\mathbf{A}(\cdot, \nabla \mathbf{u}) \in L_{M^*}(\Omega; \mathbb{R}^{d\times N})$, and $L_{M^*}(\Omega; \mathbb{R}^{d\times N})$ is not the predual space to $L_M(\Omega; \mathbb{R}^{d\times N})$ anymore. Therefore we cannot use weak-$*$

convergence arguments in $L_M(\Omega)$. For that reason the approach is different here – first a weak solution of the dual problem to (4.28) will be found, and then we deduce the existence of a weak solution to the original problem. Before starting this procedure we need to specify how the *dual problem* is understood here. For the construction we will use an inverse operator to $\mathbf{A}$, which we shall denote by $\mathbf{B}$, i.e.

$$\mathbf{A}(x, \mathbf{B}(\xi)) = \xi \text{ for a.a } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{d\times N}.$$

Indeed, as $\mathbf{A}$ is strictly monotone, it is a homeomorphism on $\mathbb{R}^{d\times N}$ therefore the inverse operator exists and is also strictly monotone. Before passing to the existence theorem let us concentrate on the growth conditions for $\mathbf{B}$ that follow from $(A2e)$. Choosing $\xi = \mathbf{B}(\eta)$ for an arbitrary $\eta \in \mathbb{R}^{d\times N}$ we immediately obtain

$$M(x, c_1\mathbf{B}(x,\eta)) \le \mathbf{B}(x,\eta)\cdot\eta, \tag{4.25}$$

$$c_2 M^*(x, c_3\eta) \le M(x, c_4\mathbf{B}(\eta)). \tag{4.26}$$

Again, the overall impediment lies in the possibly that $c_1 \neq c_4$. We can however manage in a simpler way here than in the case of the operator $\mathbf{A}$. In particular there is no need to formulate an analogue to Lemma 3.8.2, as having the advantage of the Δ_2-condition for M we can proceed with a simple argument.

As we understand well the behavior of an inverse operator $\mathbf{B}$, we may now formulate an existence theorem in the case when M satisfies the Δ_2-condition. Note that the statement here is not fully analogous, in particular solutions are elements of a larger space, i.e. $W_0^1 L_M(\Omega;\mathbb{R}^d) \subset V_0^M(\Omega)$

Theorem 4.1.3 *Let $N \ge 1$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Assume that an operator* $\mathbf{A}$ *satisfies (A1e), (A2e), (A3e*) and $M : \Omega\times\mathbb{R}^{d\times N} \to [0,\infty)$ is an N-function such that it satisfies the Δ_2-condition. Assume that $\mathbf{f} = \operatorname{div}\mathbf{F}$ and $\mathbf{F} \in E_{M^*}(\Omega;\mathbb{R}^{d\times N})$. Then there exists a unique weak solution to problem* (4.1)*, which is a function*

$$\mathbf{u} \in V_0^M(\Omega) \tag{4.27}$$

such that

$$\int_\Omega \mathbf{A}(x, \nabla\mathbf{u}(x))\cdot\nabla\boldsymbol{\varphi}(x)\,\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\nabla\boldsymbol{\varphi}(x)\,\mathrm{d}x \tag{4.28}$$

is satisfied for all $\boldsymbol{\varphi} \in V_0^M(\Omega)$.

Proof. We use the above defined inverse operator $\mathbf{B}$ to formulate the dual problem to (4.28), i.e. we will search for a function $\mathbf{T} \in L_{M^*}^{div}(\Omega;\mathbb{R}^{d\times N})$ satisfying

$$\int_\Omega \mathbf{B}(x, \mathbf{T}(x) + \mathbf{F}(x))\cdot\psi(x)\,\mathrm{d}x = 0 \text{ for all } \psi \in E_{M^*}^{\mathrm{div}}(\Omega;\mathbb{R}^{d\times N}). \tag{4.29}$$

Still before concentrating on (4.29), we will discuss why the solvability of (4.29) directly implies the statement of the lemma. We want to conclude that once we know that (4.29) holds, then there exists a $\mathbf{u} \in V_0^M(\Omega)$ such that $\nabla\mathbf{u} = \mathbf{B}(\cdot, \mathbf{T}+\mathbf{F})$ and $\mathbf{u}$ satisfies (4.28). We define an extension

$$\tilde{\mathbf{B}}(x) := \begin{cases} \mathbf{B}(x, \mathbf{T}(x) + \mathbf{F}(x)) & x \in \Omega \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then (4.29) can be equivalently written as

$$\int_{\mathbb{R}^N} \tilde{\mathbf{B}}(x) \cdot \psi(x)\, \mathrm{d}x = 0 \tag{4.30}$$

for all $\psi \in C^\infty(\mathbb{R}^N; \mathbb{R}^{d\times N})$ with $\operatorname{div}\psi = 0$ in $\mathbb{R}^N$.

The de Rham theorem (see Theorem 8.45) yields the existence of a distribution $\mathbf{p}$ such that

$$\tilde{\mathbf{B}} = \nabla \mathbf{p}.$$

Then $\nabla\mathbf{p} = 0$ outside of Ω, thus $\mathbf{p}$ is equal to some constant $\bar{\mathbf{p}}$ in $\mathbb{R}^N \setminus \Omega$. We consider then $\mathbf{u} := \mathbf{p} - \bar{\mathbf{p}}$ and observe that obviously $\nabla\mathbf{u} = \nabla\mathbf{p}$, and thus

$$\nabla\mathbf{u} \in L^1(\Omega) \quad \text{and} \quad \mathbf{u} = 0 \ \text{ in } \ \mathbb{R}^N \setminus \Omega.$$

These two conditions imply that $\mathbf{u} = 0$ on $\partial\Omega$ in the sense of trace. And thus we can use the Poincaré inequality

$$\|\mathbf{u}\|_{L^1(\Omega)} \le c\|\nabla\mathbf{u}\|_{L^1(\Omega)} = c\|\tilde{\mathbf{B}}\|_{L^1(\Omega)}. \tag{4.31}$$

The above inequality allows us to conclude that $\mathbf{u} \in W_0^{1,1}(\Omega)$ and as $\tilde{\mathbf{B}} \in E_M(\Omega; \mathbb{R}^{d\times N})$, thus also $\nabla\mathbf{u} \in E_M(\Omega; \mathbb{R}^{d\times N})$, i.e., $\mathbf{u} \in V_0^M(\Omega)$. By the definition of $\mathbf{B}$ we obtain

$$\int_\Omega \mathbf{A}(x, \nabla\mathbf{u}) \cdot \nabla\boldsymbol{\varphi}\, \mathrm{d}x = \int_\Omega \mathbf{A}(x, \mathbf{B}(\mathbf{T}+\mathbf{F})) \cdot \nabla\boldsymbol{\varphi}\, \mathrm{d}x = \int_\Omega (\mathbf{T}+\mathbf{F}) \cdot \nabla\boldsymbol{\varphi}\, \mathrm{d}x$$

for all $\boldsymbol{\varphi} \in V_0^M$. Once we show that for all such $\boldsymbol{\varphi}$

$$\int_\Omega \mathbf{T} \cdot \nabla\boldsymbol{\varphi} = 0,$$

it immediately follows that $\mathbf{u}$ satisfies (4.28). Indeed as $\mathbf{T} \in L^{\mathrm{div}}_{M^*}\left(\Omega; \mathbb{R}^{d\times N}\right)$, it can be approximated in the weak-$*$ topology by a sequence of divergence-free smooth functions. And since $\nabla\boldsymbol{\varphi}$ is an element of $L_M(\Omega; \mathbb{R}^{d\times N})$, which is a separable space as M satisfies the Δ_2-condition, then the weak-$*$ convergence argument is justified.

Hence, let us focus on (4.29). We observe that the space $E^{\mathrm{div}}_{M^*}(\Omega; \mathbb{R}^{d\times N})$ is separable since it is a closed subspace of the separable space $E_{M^*}\left(\Omega; \mathbb{R}^{d\times N}\right)$. Thus there is a linearly independent subset $\{\mathbf{W}^i\}_{i=1}^\infty$ of $E^{\mathrm{div}}_{M^*}\left(\Omega; \mathbb{R}^{d\times N}\right)$ such that

$$\overline{\operatorname{span}\{\mathbf{W}^i\}_{i=1}^\infty}^{\|\cdot\|_{L_M}} = E^{\mathrm{div}}_{M^*}\left(\Omega; \mathbb{R}^{d\times N}\right)$$

and the $\mathbf{W}^i$ are smooth, divergence-free functions for all $i \in \mathbb{N}$.

Existence of solutions to the approximate problem. We will construct Galerkin approximations to (4.29). Define $\mathbf{T}^k := \sum_{i=1}^k \alpha_i^k \mathbf{W}^i$ for $k \in \mathbb{N}$, where $\alpha_i^k \in \mathbb{R}$ are chosen in such a way that

$$\int_\Omega \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right) \cdot \mathbf{W}^j \, \mathrm{d}x = 0 \tag{4.32}$$

for all $j = 1, \ldots, k$.

Let us show the existence of $\boldsymbol{\alpha} = (\alpha_1^k, \ldots, \alpha_k^k) \in \mathbb{R}^k$ satisfying (4.32). We want to apply Lemma 8.53 on a mapping $\mathbf{s} : \mathbb{R}^k \to \mathbb{R}^k$ defined as

$$s_j(\boldsymbol{\alpha}) = \int_\Omega \mathbf{B}(x, \mathbf{T}^k + \mathbf{F}) \cdot \mathbf{W}^j \, \mathrm{d}x, \quad j = 1, \ldots, k.$$

First, we show that $\mathbf{s}$ is continuous. We define $\mathcal{W}(\boldsymbol{\alpha}) := \sum_{i=1}^k \alpha_i^k \mathbf{W}^i$. Let us suppose that $\boldsymbol{\alpha}^n \to \boldsymbol{\alpha}$ in $\mathbb{R}^k$. We observe that for

$$h_j^n := (\mathbf{B}(x, \mathcal{W}(\boldsymbol{\alpha}^n) + \mathbf{F}) - \mathbf{B}(x, \mathcal{W}(\boldsymbol{\alpha}) + \mathbf{F})) \cdot \mathbf{W}^j$$

we have $h_j^n \to 0$ as $n \to \infty$ a.e. in Ω. From (4.40) one concludes the uniform integrability of $\mathbf{B}(x, \mathcal{W}(\boldsymbol{\alpha}^n))$. We also have

$$\mathbf{B}(x, \mathcal{W}(\boldsymbol{\alpha}) + \mathbf{F}) \in L^1\left(\Omega; \mathbb{R}^{d\times N}\right) \text{ and } \mathbf{W}^j \in L^\infty\left(\Omega; \mathbb{R}^{d\times N}\right)$$

and therefore $|h_j^n|$ is uniformly integrable. Consequently, $\mathbf{s}$ is continuous since by the Vitali convergence theorem (Theorem 8.23) we get

$$|\mathbf{s}(\boldsymbol{\alpha}^n) - \mathbf{s}(\boldsymbol{\alpha})| \le k \max_{j=1,\ldots,k} \int_\Omega |h_j^n| \, \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Next, we verify that $\mathbf{s}$ satisfies (8.6) – the assumption of Lemma 8.53. We show that

$$\|\mathcal{W}(\boldsymbol{\alpha}) + \mathbf{F}\|_{L_{M^*}(\Omega)} \to \infty \text{ as } |\boldsymbol{\alpha}| \to \infty. \tag{4.33}$$

We observe that $\min_{|\boldsymbol{\alpha}|=1} \|\mathcal{W}(\boldsymbol{\alpha})\|_{L_{M^*}(\Omega)} > 0$, which follows from the fact that $\{\mathbf{W}^i\}_{i=1}^k$ are linearly independent. Since $\mathbf{F} \in E_{M^*}\left(\Omega; \mathbb{R}^{d\times N}\right)$, we find $R_0 > 0$ such that $\frac{\|\mathbf{F}\|_{L_{M^*}}}{|\boldsymbol{\alpha}|} \le \frac{1}{2} \min_{|\boldsymbol{\beta}|=1} \|\mathcal{W}(\boldsymbol{\beta})\|_{L_{M^*}(\Omega)}$ for all $\boldsymbol{\alpha} \in \mathbb{R}^k$ with $|\boldsymbol{\alpha}| \ge R_0$. Considering such $\boldsymbol{\alpha}$ we get by the triangle inequality

$$\begin{aligned}
\|\mathcal{W}(\boldsymbol{\alpha}) + \mathbf{F}\|_{L_{M^*}(\Omega)} &\ge \|\mathcal{W}(\boldsymbol{\alpha})\|_{L_{M^*}(\Omega)} - \|\mathbf{F}\|_{L_{M^*}(\Omega)} \\
&\ge |\boldsymbol{\alpha}| \left(\left\| \mathcal{W}\left(\frac{\alpha}{|\boldsymbol{\alpha}|}\right) \right\|_{L_{M^*}(\Omega)} - \frac{\|\mathbf{F}\|_{L_{M^*}(\Omega)}}{|\boldsymbol{\alpha}|} \right) \\
&\ge \frac{1}{2} |\boldsymbol{\alpha}| \min_{|\boldsymbol{\beta}|=1} \|\mathcal{W}(\boldsymbol{\beta})\|_{L_{M^*}(\Omega)}.
\end{aligned}$$

Hence (4.33) follows. By (4.25) and the Fenchel–Young inequality we have

$$
\begin{aligned}
\mathbf{s}(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha} &= \int_\Omega \mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right)\cdot\mathcal{W}(\boldsymbol{\alpha})\,\mathrm{d}x\\
&= \int_\Omega \mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right)\cdot(\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})\,\mathrm{d}x - \int_\Omega \mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right)\cdot\mathbf{F}\,\mathrm{d}x\\
&\geq \int_\Omega M(x,c_1\mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right))\,\mathrm{d}x - \tfrac{1}{2}\int_\Omega M(x,c_1\mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right))\,\mathrm{d}x\\
&\quad - \int_\Omega M^*(x,\tfrac{2}{c_1}\mathbf{F})\,\mathrm{d}x\\
&= \tfrac{1}{2}\int_\Omega M(x,c_1\mathbf{B}\left(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\right)\,\mathrm{d}x - \int_\Omega M^*(x,\tfrac{2}{c_1}\mathbf{F})\,\mathrm{d}x =: I^1.
\end{aligned}
$$

Since M satisfies the Δ_2-condition,

$$
M(x,c_1\mathbf{B}(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})) \geq \tfrac{1}{c_{\Delta_2}}M(x,2c_1\mathbf{B}(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})) - \tfrac{1}{c_{\Delta_2}}h(x)
$$

for some constant $c_{\Delta_2} > 0$ and integrable function h. Let us now choose $k \in \mathbb{N}$ sufficiently large such that

$$
2^k c_1 > c_4 \tag{4.34}
$$

and observe that then

$$
M(x,c_1\mathbf{B}(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})) \geq \left(\tfrac{1}{c_{\Delta_2}}\right)^k M(x,2^k c_1\mathbf{B}(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})) - \sum_{i=1}^{k}\left(\tfrac{1}{c_{\Delta_2}}\right)^i h(x). \tag{4.35}
$$

Using (4.34)–(4.35), Lemma 2.1.23 (ii) and (4.26) we continue the estimate

$$
\begin{aligned}
I^1 &\geq \tfrac{1}{2(c_{\Delta_2})^k}\int_\Omega M(x,c_4\mathbf{B}(x,\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F})\,\mathrm{d}x - \int_\Omega\sum_{i=1}^{k}\left(\tfrac{1}{c_{\Delta_2}}\right)^i h(x)\,\mathrm{d}x\\
&\quad - \int_\Omega M^*(x,\tfrac{2}{c_1}\mathbf{F})\,\mathrm{d}x\\
&\geq \tfrac{c_2}{2(c_{\Delta_2})^k}\int_\Omega M^*(x,c_3(\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}))\,\mathrm{d}x - \int_\Omega\sum_{i=1}^{k}\left(\tfrac{1}{c_{\Delta_2}}\right)^i h(x)\,\mathrm{d}x\\
&\quad - \int_\Omega M^*(x,\tfrac{2}{c_1}\mathbf{F})\,\mathrm{d}x\\
&\geq \tfrac{c_2}{2(c_{\Delta_2})^k}(c_3\|\mathcal{W}(\boldsymbol{\alpha})+\mathbf{F}\|_{L_{M^*}(\Omega)}-1) - \int_\Omega\sum_{i=1}^{k}\left(\tfrac{1}{c_{\Delta_2}}\right)^i h(x)\,\mathrm{d}x\\
&\quad - \int_\Omega M^*(x,\tfrac{2}{c_1}\mathbf{F})\,\mathrm{d}x.
\end{aligned} \tag{4.36}
$$

The last inequality follows from Lemma 3.1.14 (ii). Then using (4.33) we find $R \geq R_0$ such that $\mathbf{s}(\boldsymbol{\alpha})\cdot\boldsymbol{\alpha} \geq 0$ for all $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = R$. Thus according to Lemma 8.53 we have the existence of $\boldsymbol{\alpha}$ satisfying (4.32).

Uniform estimates. Multiplying (4.32) by α_j^k and summing over $j = 1, \ldots, k$ yields

$$\int_\Omega \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right) \cdot \mathbf{T}^k \, \mathrm{d}x = 0. \tag{4.37}$$

Applying (4.25) and (4.37) we obtain an estimate

$$\begin{aligned}\int_\Omega M(x, c_1 \mathbf{B}(x, \mathbf{T}^k + \mathbf{F})) &\le \int_\Omega \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right) \cdot (\mathbf{T}^k + \mathbf{F}) \, \mathrm{d}x \\ &= \int_\Omega \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right) \cdot \mathbf{F} \, \mathrm{d}x,\end{aligned} \tag{4.38}$$

whereas the right-hand side can be estimated with the help of Lemma 2.1.23 (i) and the Fenchel–Young inequality

$$\begin{aligned}&\int_\Omega \tfrac{c_1}{2} \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right) \cdot \tfrac{2}{c_1} \mathbf{F} \, \mathrm{d}x \\ &\quad \le \int_\Omega \tfrac{1}{2} M\left(x, c_1 \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right)\right) \mathrm{d}x + \int_\Omega M^*\left(x, \tfrac{2}{c_1} \mathbf{F}\right) \mathrm{d}x.\end{aligned} \tag{4.39}$$

And thus

$$\tfrac{1}{2} \int_\Omega M\left(x, c_1 \mathbf{B}\left(x, \mathbf{T}^k + \mathbf{F}\right)\right) \mathrm{d}x \le \int_\Omega M^*\left(x, \tfrac{2}{c_1} \mathbf{F}\right) \mathrm{d}x, \tag{4.40}$$

where the right-hand side is bounded since $\mathbf{F} \in E_{M^*}(\Omega; \mathbb{R}^{d \times N})$.

Hereafter we follow estimates (4.35) and (4.36), again observing here the utility of the Δ_2-condition assumed for M. Thereby, with the same notation as above, we infer that

$$\frac{c_2}{2(c_{\Delta_2})^k} \int_\Omega M^*\left(x, c_3(\mathbf{T}^k + \mathbf{F})\right) \mathrm{d}x \le \int_\Omega \sum_{i=1}^k \left(\frac{1}{c_{\Delta_2}}\right)^i h(x) \, \mathrm{d}x + \int_\Omega M^*\left(x, \tfrac{2}{c_1} \mathbf{F}\right) \mathrm{d}x. \tag{4.41}$$

Since the right-hand sides of inequalities (4.40) and (4.41) are finite, we infer the existence of $\mathbf{T} \in L^{\mathrm{div}}_{M^*}(\Omega; \mathbb{R}^N)$ and $\bar{\mathbf{B}} \in E_M(\Omega; \mathbb{R}^{d \times N})$ such that (note here that for $(4.42)_2$ we use the fact that M satisfies the Δ_2-condition)

$$\begin{aligned}\mathbf{T}^k + \mathbf{F} &\overset{*}{\rightharpoonup} \mathbf{T} + \mathbf{F} \text{ in } L_{M^*}(\Omega; \mathbb{R}^{d \times N}), \\ \mathbf{B}(\cdot, \mathbf{T}^k + \mathbf{F}) &\overset{*}{\rightharpoonup} \bar{\mathbf{B}} \qquad \text{in } E_M(\Omega; \mathbb{R}^{d \times N}).\end{aligned} \tag{4.42}$$

Employing the convergence $(4.42)_2$ in (4.37) we have for all $i \in \mathbb{N}$

$$\int_\Omega \bar{\mathbf{B}} \cdot \mathbf{W}^i \, \mathrm{d}x = 0. \tag{4.43}$$

Consequently, since $\{\mathbf{W}^i\}_{i=1}^\infty$ forms a basis we also have for all $\mathbf{W} \in E^{\mathrm{div}}_{M^*}\left(\Omega; \mathbb{R}^{d \times N}\right)$

$$\int_\Omega \bar{\mathbf{B}} \cdot \mathbf{W} \, \mathrm{d}x = 0. \tag{4.44}$$

Thus to prove (4.29), it remains to identify $\bar{\mathbf{W}}$.

Multiplying the i-th equation in (4.43) by α_i^k and summing the result over $i = 1, \ldots, k$ yields

$$\int_\Omega \bar{\mathbf{B}} \cdot (\mathbf{T}^k + \mathbf{F}) \, \mathrm{d}x = \int_\Omega \bar{\mathbf{B}} \cdot \mathbf{F} \, \mathrm{d}x.$$

We apply the convergence $(4.42)_1$, which is possible since $\bar{\mathbf{B}} \in L_M(\Omega; \mathbb{R}^{d\times N}) = E_M(\Omega; \mathbb{R}^{d\times N})$ as M is assumed to satisfy the Δ_2-condition, to obtain

$$\int_\Omega \bar{\mathbf{B}} \cdot \mathbf{T} \, \mathrm{d}x = 0. \tag{4.45}$$

Let us identify $\bar{\mathbf{B}}$ with the help of the variant of Minty's trick for nonseparable and nonreflexive function spaces. First, using the monotonicity of $\mathbf{B}$ and (4.37) we get

$$\begin{aligned} 0 &\le \int_\Omega \Big(\mathbf{B}(x, \mathbf{T}^k + \mathbf{F}) - \mathbf{B}(x, \mathbf{W})\Big) \cdot (\mathbf{T}^k + \mathbf{F} - \mathbf{W}) \, \mathrm{d}x \\ &= \int_\Omega \mathbf{B}(x, \mathbf{T}^k + \mathbf{F}) \cdot (\mathbf{F} - \mathbf{W}) - \mathbf{B}(x, \mathbf{W}) \cdot (\mathbf{T}^k + \mathbf{F} - \mathbf{W}) \, \mathrm{d}x \end{aligned}$$

for an arbitrary but fixed $\mathbf{W} \in L^\infty(\Omega; \mathbb{R}^{d\times N})$. Then performing the limit passage $k \to \infty$ in the latter inequality and using (4.42) and (4.45) we arrive at

$$0 \le \int_\Omega \bar{\mathbf{B}} \cdot (\mathbf{F} - \mathbf{W}) - \mathbf{B}(x, \mathbf{W}) \cdot (\mathbf{T} + \mathbf{F} - \mathbf{W}) \, \mathrm{d}x = \int_\Omega \big(\bar{\mathbf{B}} - \mathbf{B}(x, \mathbf{W})\big) \cdot (\mathbf{T} + \mathbf{F} - \mathbf{W}) \, \mathrm{d}x, \tag{4.46}$$

which corresponds to (4.5) in Theorem 4.1.1. Therefore $\bar{\mathbf{B}} = \mathbf{B}(\mathbf{T} + \mathbf{F})$ a.e. in Ω.

Step 6: One easily obtains uniqueness of a weak solution. Supposing that $\mathbf{u}_1, \mathbf{u}_2$ are different weak solutions of (4.28), we get after testing the difference of weak formulations for $\mathbf{u}_1$ and $\mathbf{u}_2$ by the difference $\mathbf{u}_1 - \mathbf{u}_2$, which is a proper test function in (4.28) as $\mathbf{u}_1, \mathbf{u}_2 \in V_0^M(\Omega)$, that

$$\int_\Omega (\mathbf{A}(x, \nabla \mathbf{u}_1) - \mathbf{A}(x, \nabla \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) \, \mathrm{d}x = 0.$$

Since $\mathbf{A}$ is strictly monotone, we have $\nabla(\mathbf{u}_1 - \mathbf{u}_2) = 0$ a.e. in Ω and the zero trace of $\mathbf{u}_1 - \mathbf{u}_2$ on $\partial\Omega$ implies $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Ω. □

4.1.4 Elliptic problems via the modular density approach

This section presents the method of showing the existence of weak solutions in the case when there is no information on the growth of an N-function or its conjugate in the second variable, but under control on an N-function implying that the smooth functions are dense in the modular topology in the space where the solution is expected. Density in the Sobolev-type space is elaborated in Section 3.7 under conditions *(Me)* or *(Me)$_p$* that prescribe balance of the modulus of continuity of M with respect to the first variable with its growth with respect to the second one. To provide the existence of weak solutions in this case, we use the result of [263] based on the ideas of Gossez [173], as is done in [178] – via a regularized problem. The result of [263] provides the existence to a problem in the isotropic Orlicz–Sobolev setting (with the modular function depending on the norm of the gradient of solution only). To avoid introducing overwhelming notation, we recall here only a direct consequence of results of [263, Section 5] important for the case considered here.

Corollary 4.1.4 *Let Ω be a bounded open domain in $\mathbb{R}^N$ and $\overline{m} : [0,\infty) \to [0,\infty)$ be a homogeneous and isotropic N-function. We consider a Carathéodory function $\mathbf{a} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$, which is strictly monotone, i.e.*

$$\Big(\mathbf{a}(x,\xi_1)-\mathbf{a}(x,\xi_2)\Big)\cdot(\xi_1-\xi_2) > 0 \quad \textit{for a.a. } x\in\Omega \textit{ and all } \xi_1\neq\xi_2\in\mathbb{R}^N$$

and satisfies growth and coercivity conditions: for some $C_0, C_1, C_2 > 0$, a.a. $x \in \Omega$, and all $\xi \in \mathbb{R}^N$

$$C_0\,\overline{m}(|\xi|) \le \mathbf{a}(x,\xi)\cdot\xi \qquad \textit{and} \qquad |\mathbf{a}(x,\xi)| \le C_1(\overline{m}^*)^{-1}\big(\overline{m}(C_2|\xi|)\big). \tag{4.47}$$

Then the problem

$$\begin{cases} -\mathrm{div}\,\mathbf{a}(x,\nabla v) = g \in L^\infty(\Omega) & \text{in } \Omega,\\ v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.48}$$

has at least one weak solution $v \in W^1_0 L_{\overline{m}}(\Omega)$.

Proof. To apply [263, Theorem 5.1], one has to ensure that the operator is pseudomonotone, which we get via [263, Theorem 4.3]. It suffices to verify the assumptions [263, $(A_1)-(A_3)$] therein. Note that [263, (A_1)] requires $\mathbf{a}$ to be a Carathéodory function, [263, (A_2)] coincides precisely with (4.47), whereas [263, (A_3)] is just the monotonicity. □

The following theorem, only for simplicity, is formulated in the scalar case and under assumption $(A2e)^*$. These various simplifications in formulation, that seem not to be optimal, on one hand improve readability. But on the other hand, they serve to present the existence result for a problem with bounded data in such a way that is useful in the next chapter on renormalized solutions as an approximation for a problem with merely integrable data.

Theorem 4.1.5 *Suppose $\mathbf{a} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies assumptions (A1e), $(A2e)^*$ and (A3e) with an N-function $M : \Omega\times\mathbb{R}^N \to [0,\infty)$, and $g \in L^\infty(\Omega)$. Moreover, assume that M satisfies assumption (Me) or (Me)$_p$. Then there exists a weak solution to the problem*

$$\begin{cases} -\mathrm{div}\,\mathbf{a}(x,\nabla u) = g & \text{in} \quad \Omega,\\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Namely, there exists a $u \in W_0^{1,1}(\Omega)$ such that $\nabla u \in L_M(\Omega;\mathbb{R}^N)$, for which the following formulation

$$\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla\varphi \, \mathrm{d}x = \int_\Omega g\varphi \, \mathrm{d}x \tag{4.49}$$

holds for all $\varphi \in C_c^\infty(\Omega)$*. Moreover,* $\mathbf{a}(\cdot,\nabla u) \in L_{M^*}(\Omega;\mathbb{R}^N)$.

The proof is given at the end of this section.

Remark 4.1.6. The proof heavily relies on density results from Section 3.7. Thus, following Remark 3.7.11 in that section, we also notice here that instead of assuming *(Me)* or *(Me)$_p$*, we could assume $(Me)^*$, which is less restrictive in the case $p < N$ and $N > 1$, see Remark 3.7.11 for detailed formulation.

Remark 4.1.7. In fact, Theorem 4.1.5 can be formulated under the assumption that there exists an $F : \Omega \to \mathbb{R}^N$ such that $g = \operatorname{div} F$ and $F \in E_{M^*}(\Omega)$, which is less restrictive than assuming that $g \in L^\infty(\Omega)$. Indeed, observe that if Ω is bounded, then $g \in L^\infty(\Omega)$ implies that $g \in L^p(\Omega)$ for some $p > N$. Assume for the moment that $\int_\Omega g \, \mathrm{d}x = 0$. Then by Lemma 8.57 there exists an F such that

$$\|F\|_{W^{1,p}(\Omega)} \le c\|g\|_{L^p(\Omega)}$$

and $\operatorname{div} F = g$. Since $W^{1,p}(\Omega) \subset L^\infty(\Omega)$, then $F \in L^\infty(\Omega)$ and thus also $F \in E_{M^*}(\Omega)$. Notice that if $\int_\Omega g \, \mathrm{d}x \neq 0$ then we can rewrite the problem as follows

$$\operatorname{div} F = g - g_\Omega + g_\Omega$$

with $g_\Omega = \fint_\Omega g(x) \, \mathrm{d}x$ and then divide it into solving two problems

$$\operatorname{div} F_1 = g - g_\Omega \quad \text{and} \quad \operatorname{div} F_2 = g_\Omega$$

with $F = F_1 + F_2$. For the first one we apply the procedure previously described, as the integral of the right-hand side already vanishes. The second problem can be solved immediately. Since g_Ω is a constant, then $F(x) = g_\Omega x_1$ is an example of a solution.

To prove Theorem 4.1.5 we consider the regularized problem posed in an isotropic space. Let $m : \mathbb{R}^N \to \mathbb{R}$ be an isotropic function, i.e. $m(\xi) = \overline{m}(|\xi|)$ with some

$$\overline{m} : [0,\infty) \to [0,\infty),$$

that grows significantly faster than M (see Definition 3.2.3). Note that since Ω has finite measure and m grows significantly faster than M, by Proposition 3.2.4 we have that

$$L_m(\Omega) \subset E_M(\Omega).$$

Recall that we use the notation ∇ for a gradient with respect to the spatial variable. Let us also introduce the notation $\nabla_\xi := \nabla_\xi$. Using it

$$\nabla_\xi m(\xi) = \nabla_\xi \overline{m}(|\xi|) = \xi \overline{m}'(|\xi|)/|\xi|.$$

Observe that due to Remark 2.1.33 this gives equality in the Fenchel–Young inequality in the following way

$$\nabla_\xi m(\xi)\cdot\xi = \overline{m}(|\xi|)+\overline{m}^*(|\nabla_\xi m(\xi)|). \tag{4.50}$$

Taking an arbitrary N-function $m(\xi)=\overline{m}(|\xi|)$ which grows significantly faster than M we observe that m is strictly monotone as a gradient of a strictly convex function, i.e. for all $\xi,\eta\in\mathbb{R}^N$ it holds that

$$(\nabla_\xi m(\xi)-\nabla_\xi m(\eta))\cdot(\xi-\eta)>0. \tag{4.51}$$

The following proposition yields the existence of solutions to a regularized problem. Note that the constructed solution is in a classical isotropic Orlicz space $W^1L_{\overline{m}}(\Omega)$, see (3.54).

Proposition 4.1.8 *Assume a vector field* $\mathbf{a}:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ *satisfies assumptions (A1e), (A2e)** *and (A3e) with an N-function* $M:\Omega\times\mathbb{R}^N\to[0,\infty)$, *and* $\overline{m}:[0,\infty)\to[0,\infty)$ *is an isotropic N-function such that* $m(\xi)=\overline{m}(|\xi|)$, *where* $m\in C^1(\mathbb{R}^N)$ *satisfies* (4.51) *and grows significantly faster than* M. *Moreover, let*

$$M(x,\xi)\le\overline{m}(|\xi|)\quad \textit{for every } \xi\in\mathbb{R}^N \textit{ and a.a. } x\in\Omega,$$

and $g\in L^\infty(\Omega)$. *We define a regularized operator as*

$$\mathbf{a}_\theta(x,\xi):=\mathbf{a}(x,\xi)+\theta\nabla_\xi m(\xi)\qquad \textit{for a.a. } x\in\Omega \textit{ and all } \xi\in\mathbb{R}^N. \tag{4.52}$$

Then for every $\theta\in(0,1]$ *there exists a weak solution to the problem*

$$\begin{cases}-\operatorname{div}\mathbf{a}_\theta(x,\nabla u^\theta)=g & \textit{in } \Omega,\\ u^\theta(x)=0 & \textit{on } \partial\Omega.\end{cases} \tag{4.53}$$

Namely, there exists a $u^\theta\in W_0^{1,1}(\Omega)\cap W^1L_{\overline{m}}(\Omega)$ *such that*

$$\int_\Omega \mathbf{a}_\theta(x,\nabla u^\theta)\cdot\nabla\varphi\,\mathrm{d}x=\int_\Omega g\varphi\,\mathrm{d}x\qquad \textit{for all } \varphi\in C_c^\infty(\Omega). \tag{4.54}$$

Proof. We apply Corollary 4.1.4. By the comments before the statement, it suffices to derive (4.47) from *(A2e)**. Indeed, we will show

$$|\mathbf{a}_\theta(x,\xi)|\le\tfrac{2}{c_\mathbf{a}}(\overline{m}^*)^{-1}\left(\overline{m}\left(\left|\tfrac{2}{c_\mathbf{a}}\xi\right|\right)\right). \tag{4.55}$$

Using the Fenchel–Young inequality (2.33), the convexity of $\overline{m}^*$, and since $c_\mathbf{a},\theta\in(0,1]$, we have

$$\begin{aligned}\mathbf{a}_\theta(x,\xi)\cdot\xi&\le\overline{m}\left(\left|\tfrac{2}{c_\mathbf{a}}\xi\right|\right)+\overline{m}^*\left(\left|\tfrac{c_\mathbf{a}}{2}\mathbf{a}_\theta(x,\xi)\right|\right)\\ &\le\overline{m}\left(\left|\tfrac{2}{c_\mathbf{a}}\xi\right|\right)+c_\mathbf{a}\overline{m}^*\left(\left|\tfrac{1}{2}\mathbf{a}_\theta(x,\xi)\right|\right).\end{aligned}$$

On the other hand, Lemma 2.1.37 implies that $\overline{m}^*(x,\xi)\le M^*(|\xi|)$ for every $\xi\in\mathbb{R}^N$ and a.a. $x\in\Omega$. Therefore, when we take into account *(A2e)*, (4.50), the convexity of $\overline{m}^*$, and drop positive terms we obtain

$$
\begin{aligned}
\mathbf{a}_\theta(x,\xi)\cdot\xi &\geq M(x,\xi)+\nabla_\xi m(\xi)\cdot\xi\\
&\geq c_{\mathbf{a}}M^*(x,\mathbf{a}(x,\xi))+\theta\overline{m}(|\xi|)+\theta\overline{m}^*\left(\left|\nabla_\xi m(\xi)\right|\right)\\
&\geq 2c_{\mathbf{a}}\left(\tfrac{1}{2}\overline{m}^*(|\mathbf{a}(x,\xi)|)+\tfrac{1}{2}\overline{m}^*\left(\left|\theta\nabla_\xi m(\xi)\right|\right)\right)\\
&\geq 2c_{\mathbf{a}}\overline{m}^*\left(\tfrac{1}{2}\left|\mathbf{a}_\theta(x,\xi)\right|\right).
\end{aligned}
$$

Merging both of the above observations, we get

$$
c_{\mathbf{a}}\overline{m}^*\left(\tfrac{1}{2}\left|\mathbf{a}_\theta(x,\xi)\right|\right)\leq\overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right).
$$

By convexity of $\overline{m}^*$ we get (4.55) and therefore due to Corollary 4.1.4 we arrive at the claim. □

Below we prove Theorem 4.1.5. The general idea to get the existence of weak solutions to the problem with bounded data (4.49) is to apply Proposition 4.1.8 and let $\theta\to 0$.

Proof (of Theorem 4.1.5). We prove a priori estimates, interpret them as inferring certain types of convergence, and conclude the proof using the monotonicity argument.

A priori estimates. We fix $\varphi\in W_0^1L_{\overline{m}}(\Omega)$. According to the classical isotropic version of the approximation theorem due to Gossez (Theorem 8.35) we consider an approximate sequence $\{\varphi_\delta\}_\delta\subset C_c^\infty(\Omega)$ such that

$$
\varphi_\delta\xrightarrow[\delta\to 0]{\overline{m}}\varphi\ \text{ modularly in the isotropic Orlicz space }\ W^1L_{\overline{m}}(\Omega).
$$

Using the estimates on the growth of $\mathbf{a}_\theta$, (4.54), and Lemma 3.4.7 we get

$$
\int_\Omega\mathbf{a}_\theta(x,\nabla u^\theta)\cdot\nabla\varphi\,\mathrm{d}x=\lim_{\delta\to 0}\int_\Omega\mathbf{a}_\theta(x,\nabla u^\theta)\cdot\nabla\varphi_\delta\,\mathrm{d}x=\lim_{\delta\to 0}\int_\Omega g\,\varphi_\delta\,\mathrm{d}x=\int_\Omega g\,\varphi\,\mathrm{d}x.
$$

Thus, we can use $u^\theta\in W_0^1L_{\overline{m}}(\Omega)$ as a test function in the weak formulation (4.54) to obtain

$$
\int_\Omega\left(\mathbf{a}(x,\nabla u^\theta)+\theta\nabla_\xi m(\nabla u^\theta)\right)\cdot\nabla u^\theta\,\mathrm{d}x=\int_\Omega g\,u^\theta\,\mathrm{d}x. \tag{4.56}
$$

By *(A2e)* and (4.50) we get

$$
\int_\Omega M(x,\nabla u^\theta)+\int_\Omega\theta m(\nabla u^\theta)+\theta m^*(\nabla_\xi m(\nabla u^\theta))\,\mathrm{d}x\leq\int_\Omega gu^\theta\,\mathrm{d}x. \tag{4.57}
$$

To estimate the right-hand side we are going to apply the Fenchel–Young inequality (2.33) and the modular Poincaré inequality (Theorem 9.3) with constants $c_P^1,c_P^2>0$. For this let us consider an N-function $p:[0,\infty)\to[0,\infty)$ given by

$$
p(s)=\tfrac{1}{2c_P^2}m_1(s)
$$

with m_1 being a minorant of M from the definition of an N-function. Then on the right-hand side of (4.57) we have

$$\begin{aligned}\int_\Omega g u^\theta \,\mathrm{d}x\,\mathrm{d}t &\le \int_\Omega p^*\left(\tfrac{|g|}{c_P^1}\right) \mathrm{d}x + \int_\Omega p(c_P^1|u^\theta|)\,\mathrm{d}x\\ &\le \int_\Omega p^*\left(\tfrac{|g|}{c_P^1}\right) \mathrm{d}x + c_P^2 \int_\Omega p(|\nabla u^\theta|)\,\mathrm{d}x\\ &\le \int_\Omega p^*\left(\tfrac{|g|}{c_P^1}\right) \mathrm{d}x + \frac{1}{2}\int_\Omega M(x,\nabla u^\theta)\,\mathrm{d}x.\end{aligned}$$

Consequently, we infer that (4.57) implies

$$\frac{1}{2}\int_\Omega M(x,\nabla u_n^\theta) + \int_\Omega \theta m(\nabla u_n^\theta) + \theta m^*(\nabla_\xi m(\nabla u_n^\theta))\,\mathrm{d}x \le |\Omega| p^*(\|g\|_{L^\infty}/c_P^1) = C_g.$$

Note that the right-hand side above is bounded since $g \in L^\infty(\Omega)$. This observation implies the following a priori estimates

$$\begin{aligned}\frac{1}{2}\int_\Omega M(x,\nabla u^\theta)\,\mathrm{d}x &\le C_g,\\ \int_\Omega \theta m^*(\nabla_\xi m(|\nabla u^\theta|))\,\mathrm{d}x &\le C_g.\end{aligned} \tag{4.58}$$

Moreover, according to *(A2e)** we have

$$c_{\mathbf{a}} \int_\Omega M^*(x,\mathbf{a}(x,\nabla u^\theta))\,\mathrm{d}x \le C_g. \tag{4.59}$$

Existence of weak limits. The Banach–Alaoglu Theorem (Theorem 8.31) states that $\{\nabla u^\theta\}_{\theta\in(0,1)}$ is weakly-$*$ compact in L_M. The Dunford–Pettis Theorem (Theorem 8.21), and the fact that M is an N-function (Definition 2.2.2) imply that $\{u^\theta\}_{\theta\in(0,1)}$ is equiintegrable in $W_0^{1,1}(\Omega)$. Therefore, there exists a subsequence of $\theta \to 0$, such that

$$u^\theta \rightharpoonup u \quad \text{weakly in} \quad W^{1,1}(\Omega), \tag{4.60}$$

$$\nabla u^\theta \overset{*}{\rightharpoonup} \nabla u \quad \text{weakly-}* \text{ in} \quad L_M(\Omega;\mathbb{R}^N), \tag{4.61}$$

with some $u \in W_0^{1,1}(\Omega)$ with $\nabla u \in L_M(\Omega;\mathbb{R}^N)$ and, due to (4.59), there exists an $\boldsymbol{\alpha} \in L_{M^*}(\Omega;\mathbb{R}^N)$ such that

$$\mathbf{a}(\cdot,\nabla u^\theta) \overset{*}{\rightharpoonup} \boldsymbol{\alpha} \quad \text{weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^N). \tag{4.62}$$

Identification of the limit $\boldsymbol{\alpha}$. To use the monotonicity trick, we need to show that

$$\limsup_{\theta\to 0} \int_\Omega \mathbf{a}(x,\nabla u^\theta)\cdot\nabla u^\theta\,\mathrm{d}x \le \int_\Omega \boldsymbol{\alpha}\cdot\nabla u\,\mathrm{d}x. \tag{4.63}$$

Recall the weak formulation of the regularized problem (4.54). We will motivate that on the left-hand side the term corresponding to the second part of $\mathbf{a}_\theta$, see (4.52), vanishes. Namely we will show that

$$\theta|\nabla_\xi m(\nabla u^\theta)| \xrightarrow[\theta\to 0]{} 0 \qquad \text{in } L^1(\Omega) \tag{4.64}$$

by the Vitali convergence theorem (Theorem 8.23). For its application we need to infer uniform integrability and convergence in measure to 0.

Since m^* is an N-function, for $\theta \in (0,1)$ we have $m^*(\theta s) \le \theta m^*(s)$. This together with the $L^1(\Omega)$-bound (4.58) for $\theta m^*(\nabla_\xi m(\nabla u^\theta))$, which is uniform with respect to θ, we get an $L^1(\Omega)$-bound for $\{m^*(\theta\nabla_\xi m(\nabla u^\theta))\}_{\theta\in(0,1)}$. Therefore, de la Vallée Poussin's theorem (Theorem 3.4.2) implies the uniform integrability of $\{\theta\nabla_\xi m(\nabla u^\theta)\}_{\theta\in(0,1)}$.

To show convergence in measure to 0 in (4.64), we suppose the opposite, i.e. that there exist $\gamma_1, \gamma_2 > 0$ such that

$$\liminf_{\theta\to 0} |\{x:\ \theta|\nabla_\xi m(\nabla u^\theta)| > \gamma_1\}| > \gamma_2.$$

On the set $\{x:\ \theta|\nabla_\xi m(\nabla u^\theta)| > \gamma_1\}$ we have

$$\theta m^*\left(\frac{\gamma_1}{\theta}\right) \le \theta m^*\left(|\nabla_\xi m(\nabla u^\theta)|\right).$$

Note however that since m^* is an N-function we have

$$\theta m^*\left(\frac{\gamma_1}{\theta}\right) = \gamma_1 \frac{m^*\left(\frac{\gamma_1}{\theta}\right)}{\frac{\gamma_1}{\theta}} \xrightarrow[\theta\to 0]{} \infty. \tag{4.65}$$

On the other hand,

$$\gamma_2 \theta m^*\left(\frac{\gamma_1}{\theta}\right) \le \int_\Omega \theta m^*\left(|\nabla_\xi m(\nabla u^\theta)|\right)\,\mathrm{d}x < C_g, \tag{4.66}$$

where the last inequality is a consequence of (4.58). The convergence (4.65) contradicts the estimate (4.66) and, in turn, we can apply Vitali's convergence theorem (Theorem 8.23) to justify the convergence in (4.64).

Therefore, we can pass to the limit in the weak formulation of the regularized problem (4.54). Because of (4.62) we obtain

$$\int_\Omega \boldsymbol{\alpha}\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega g\varphi\,\mathrm{d}x. \tag{4.67}$$

When we get rid of the nonnegative term $\int_\Omega \theta\nabla_\xi m(\nabla u^\theta)\cdot\nabla u^\theta\,\mathrm{d}x$ in (4.56) and then pass to the limit as $\theta \searrow 0$, we get (4.63).

To prove

$$\mathbf{a}(x,\nabla u) = \boldsymbol{\alpha} \qquad \text{a.e.} \quad \text{in} \quad \Omega, \tag{4.68}$$

we notice that (4.67), monotonicity assumption *(A3e)*, and (4.63) imply

$$\int_{\Omega}(\mathbf{a}(x,\eta)-\boldsymbol{\alpha})\cdot(\eta-\nabla u)\,\mathrm{d}x\geq 0. \tag{4.69}$$

Therefore, we are in a position to apply the monotonicity trick (Theorem 4.1.1) with $\boldsymbol{\mathcal{A}}=\boldsymbol{\alpha}$ and $\boldsymbol{\xi}=\nabla u$ to get (4.68).

Conclusion. We pass to the limit in the weak formulation of the bounded regularized problem (4.54) due to (4.60), (4.61), (4.62), and (4.68), obtaining the existence of $u\in V_0^M(\Omega)$ and satisfying (4.49), which ends the proof. □

4.2 Parabolic equation

We study the problem

$$\begin{cases}\partial_t u-\operatorname{div}\mathbf{a}(t,x,\nabla u)=g(t,x) & \text{in } \Omega_T,\\ u(t,x)=0 & \text{on } \partial\Omega,\\ u(0,x)=u_0(x) & \text{in } \Omega,\end{cases} \tag{4.70}$$

where $[0,T)$ is a finite interval, Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $\Omega_T=(0,T)\times\Omega$, $N>1$, $g\in L^\infty(\Omega_T)$, $u_0\in L^\infty(\Omega)$, and $\mathbf{a}:[0,T)\times\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is controlled by an anisotropic modular function M inhomogeneous in space and time, that is

$$M:[0,T)\times\Omega\times\mathbb{R}^N\to[0,\infty).$$

The function spaces where the solutions are expected are defined and discussed in Section 3.6. Let us recall only

$$\begin{aligned}V_T^M(\Omega)&:=\{u\in L^1(0,T;W_0^{1,1}(\Omega)):\ \nabla u\in L_M(\Omega_T;\mathbb{R}^N)\},\\ V_T^{M,\infty}(\Omega)&:=\{u\in L^\infty(0,T;L^2(\Omega))\cap L^1(0,T;W_0^{1,1}(\Omega)):\ \nabla u\in L_M(\Omega_T;\mathbb{R}^N)\}\\ &=V_T^M(\Omega)\cap L^\infty(0,T;L^2(\Omega)).\end{aligned}$$

The problem is considered in spaces where the modular function M is regular enough so that the Lavrentiev phenomenon does not occur.

4.2.1 Assumptions on the operator

We consider (4.70) with $\mathbf{a}$ having growth and coercivity described by means of an N-function $M:[0,T)\times\Omega\times\mathbb{R}^N\to[0,\infty)$. An N-function is defined in Definition 2.2.2, while its conjugate M^* in Definition 2.1.28. We assume that $\mathbf{a}$ satisfies the following conditions.

(A1p) $\mathbf{a}:\Omega_T\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function;

(A2p) There exists an N-function $M:[0,T)\times\Omega\times\mathbb{R}^N\to[0,\infty)$ and a constant $c_{\mathbf{a}}\in(0,1]$ such that for all $\xi\in\mathbb{R}^N$ we have

$$c_{\mathbf{a}}M^*(t,x,\mathbf{a}(t,x,\xi)) \le M(t,x,\xi) \qquad \text{and} \qquad M(t,x,\xi) \le \mathbf{a}(t,x,\xi)\cdot\xi;$$

(A3p) For all $\xi,\eta \in \mathbb{R}^N$ and a.a. $(t,x) \in \Omega_T$ we have

$$(\mathbf{a}(t,x,\xi)-\mathbf{a}(t,x,\eta))\cdot(\xi-\eta) \ge 0.$$

Remark 4.2.1. Conditions *(A1p)–(A3p)* correspond to the assumptions for the elliptic problem *(A1e)–(A3e)* in Section 4.1.1 and their meaning can be discussed in precisely the same way. We stress that they extend classical growth and coercivity conditions and keep anisotropy. Recall that the meaning of condition *(A2e)* is discussed in detail in relation to other conditions appearing in the literature in Section 3.8.2. For clarity of presentation we provide the analysis with one constant $c_{\mathbf{a}}$ only, but by the ideas of Lemma 3.8.2 one can carry out the analysis under parabolic counterparts of (3.88) and (3.89) with arbitrary $c_1,c_2,c_3,c_4 > 0$.

4.2.2 *Approximation in space*

In this section we concentrate on the first approximation result, called here 'Approximation in space' to distinguish it from the more delicate goal of Section 5.3.2, which will be needed for the existence of renormalized solutions. As in the elliptic case, we study spaces equipped with M with general growth and later with at least a power-type growth. To describe its local behavior we make use of

$$M_{I,Q}(\xi) := \operatorname*{ess\,inf}_{\substack{t\in I\cap[0,T]\\ x\in Q\cap\overline{\Omega}}} M(t,x,\xi) \tag{4.71}$$

defined for some interval $I \subset [0,\infty)$ and cube $Q \subset \mathbb{R}^N$ and recall that $(M_{I,Q})^{**}(\xi) = ((M_{I,Q}(\xi))^*)^*$ stands for the second conjugate, see (2.36). Recall that the second conjugate of a function is its greatest convex minorant (Corollary 2.1.42). This part is a refinement of approximation results of [80] applied therein as well as in [81].

In correspondence to related assumptions provided in Sections 3.7.1 and 3.7.2 in order to study elliptic problems, we investigate parabolic problems under condition *(Mp)* or *(Mp)$_p$* defined below.

4.2.2.1 Parabolic condition I (general growth)

Anisotropic case
For an N-function $M : [0,T)\times\Omega\times\mathbb{R}^N \to [0,\infty)$ we consider the following assumption.

(Mp) There exists a function $\Theta : [0,\infty)^2 \to [0,\infty)$ nondecreasing with respect to each of the variables, such that

$$\limsup_{\delta\to 0^+} \Theta(\delta,\delta^{-N}) < \infty, \tag{4.72}$$

which expresses the relation between $M(t,x,\xi)$ and $M_{I,Q}(\xi)$. Namely, we assume that there exists a $\delta_0 > 0$, such that for every interval $I \subset \mathbb{R}$ such that $|I| < \delta < \delta_0$, and every cube $Q \subset \mathbb{R}^N$ with $\operatorname{diam} Q < 4\delta\sqrt{N}$

$$\frac{M(t,x,\xi)}{(M_{I,Q})^{**}(\xi)} \le \Theta(\delta,|\xi|) \tag{4.73}$$

for a.e. $t \in I$, a.e. $x \in Q \cap \Omega$, and for all $\xi \in \mathbb{R}^N : |\xi| > 1$, where $(M_{I,Q})^{**}$ is the second conjugate to the infimum from (4.71), which by Corollary 2.1.42 coincides with its greatest convex minorant.

Isotropic case
For an N-function $M : [0,T)\times\Omega\times[0,\infty) \to [0,\infty)$ we consider the following assumption.

(Mp^i) There exists a function $\Theta^i : [0,\infty)^2 \to [0,\infty)$ nondecreasing with respect to each of the variables, such that

$$\limsup_{\delta\to 0^+} \Theta^i(\delta,\delta^{-N}) < \infty, \tag{4.74}$$

and for a.a. $t,r \in [0,T]$ and $x,y \in \overline{\Omega}$ we have

$$\frac{M(t,x,s)}{M(r,y,s)} \le \Theta^i\left(|t-r| + c_{sp}|x-y|, s\right). \tag{4.75}$$

Let us pass to a wide range of examples within our setting. Their proofs follow the same lines as in Example 4.2.2.

Example 4.2.2. We have the following examples of pairs M and Θ satisfying *(Mp)* and thus being admissible in our results on the density of smooth functions which will appear later.

- **Orlicz.** If $M(t,x,\xi) = M(\xi)$, i.e. it is independent of t and x, then it satisfies *(Mp)* with $\Theta(\tau,s) \equiv 1$. The fully anisotropic case is included.
- **Variable exponent.** Suppose that $M_v(t,x,s) = |s|^{p(t,x)}$, $1 < p_- \le p(\cdot) \le p_+(\cdot) < p_+ < \infty$, satisfies *(Mp)* with $\Theta(\tau,s) = \max\{s^{\omega(\tau)}, s^{-\omega(\tau)}\}$, where $\omega(\tau) = c/(\log(1/\tau))$ is the modulus of continuity of p, see (3.58). This is ensured when p is log-Hölder continuous.
- **Borderline double-phase.** When $M = |\xi|^p + a(t,x)|\xi|^p \log(\mathrm{e}+|\xi|)$ with $1 < p < \infty$ and possibly touching zero weight $a : \Omega_T \to [0,\infty)$, we can take Θ as in (3.59). Then *(Mp)* is satisfied for a log-Hölder continuous a, cf. [191, 24].
- **Orlicz double-phase.** Suppose $M(t,x,\xi) = M_1(\xi) + a(t,x)M_2(\xi)$ where M_1, M_2 are (possibly anisotropic) homogeneous N-functions (without prescribed growth) such that $M_1(\xi) \le M_2(\xi)$ for $\xi \in \mathbb{R}^N$ with $|\xi| > 1$, the function $a : \Omega_T \to [0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a. Then we can consider Θ as in (3.60) and *(Mp)* is satisfied if

$$\limsup_{\delta\to 0}\omega_a(\delta)\frac{\overline{M}_2(\delta^{-N})}{\underline{M}_1(\delta^{-N})}<\infty,$$

where $\underline{M}_1(s) := \inf_{\xi:\,|\xi|=s} M_1(\xi)$ and $\overline{M}_2(s) := \sup_{\xi:\,|\xi|=s} M_2(\xi)$.

- **Musielak–Orlicz.** If M has the form

$$M(t,x,\xi)=\sum_{j=1}^{K}k_j(t,x)M_j(\xi)+M_0(t,x,|\xi|),\quad K\in\mathbb{N},$$

where M_0 satisfies *(Mp^i)*, all M_j for $j=1,\dots,K$ are N-functions and all k_j are positive and satisfy $\frac{k_j(t,x)}{k_j(s,y)}\le C_j\Theta_j(|t-s|+c_{sp}|x-y|)$ with $C_j>0$ and $\Theta_j:[0,\infty)\to[0,\infty)$ and $\Theta_j\in L^\infty$ for $j=1,\dots,K$, then we can choose

$$\Theta(r,s)=\sum_{j=1}^{K}\Theta_j(r)+\Theta_0(r,s)\qquad\text{with}\qquad\limsup_{\delta\to 0^+}\Theta(\delta,\delta^{-N})<\infty.$$

4.2.2.2 Parabolic condition II (at least power-type growth)

We concentrate here on modular functions dependent on the time and space variables that have at least power-type growth, i.e. if

$$M(t,x,s)\ge c|s|^p\qquad\text{with } p>1\text{ and } c>0.$$

Let us recall that $M_{I,Q}$ is defined in (4.71). We consider the following balance conditions.

Anisotropic case
For an N-function $M:[0,T)\times\Omega\times\mathbb{R}^N\to[0,\infty)$ we consider the following assumption.

(Mp)$_p$ For $\xi\in\mathbb{R}^N$ such that $|\xi|>1$,

$$M(t,x,\xi)\ge c_{gr}|\xi|^p\qquad\text{with}\quad p>1\text{ and } c_{gr}>0\tag{4.76}$$

and there exists a function $\Theta_p:[0,\infty)^2\to[0,\infty)$ nondecreasing with respect to each of the variables, such that (4.73) holds with

$$\limsup_{\delta\to 0^+}\Theta_p(\delta,\delta^{-\frac{N}{p}})<\infty.\tag{4.77}$$

Isotropic case
For an N-function $M:[0,T)\times\Omega\times[0,\infty)\to[0,\infty)$ we consider the following assumption.

(Mp^i)$_p$ There exists a function $\Theta^i_p:[0,\infty)^2\to[0,\infty)$, nondecreasing with respect to each of the variables, satisfying

$$\limsup_{\delta\to 0^+} \Theta^i_p(\delta, \delta^{-N/p}) < \infty, \tag{4.78}$$

such that for all $s > 1$ and for a.a. $t, r \in [0,T]$ and $x, y \in \overline{\Omega}$,

$$\begin{cases} M(t,x,s) \geq c_{gr}\, s^p \qquad \text{with } \; p > 1 \text{ and } \; c_{gr} > 0, \\ \frac{M(t,x,s)}{M(r,y,s)} \leq \Theta^i_p\left(|t-r| + c_{sp}|x-y|, s\right). \end{cases} \tag{4.79}$$

Let us pass to a wide range of examples within our setting.

Example 4.2.3. We have the following examples of M satisfying *(Mp)*$_p$ and thus being admissible in our results on the density of smooth functions.

- **Double phase.** Suppose $M = |\xi|^p + a(t,x)|\xi|^q$ with $1 < p, q < \infty$ and a function $a : \Omega_T \to [0,\infty)$ is such that $a \in C^{0,\alpha}(\Omega_T)$ and possibly touching zero. We can take Θ as in (3.62) and *(Mp)*$_p$ is satisfied if $q/p \leq 1 + \alpha/N$.
  The range of parameters is sharp for regularity of minimizers [98].
- **Variable exponent double phase.** Suppose $M(t,x,\xi) = |\xi|^{p(t,x)} + a(t,x)|\xi|^{q(t,x)}$ with $p, q : \Omega_T \to (1,\infty)$ such that $1 < p_- \leq p(t,x) \leq q(t,x) \leq q_+ < \infty$ and a function $a : \Omega_T \to [0,\infty)$ is such that $a \in C^{0,\alpha}(\Omega_T)$ and possibly touching zero. We can take Θ as in (3.64) and *(Mp)*$_p$ is satisfied if

$$\sup_{(t,x)\in\Omega_T} \big(q(t,x) - p(t,x)\big) \leq \frac{\alpha p_-}{N}.$$

- **Orlicz double phase.** Suppose $M(t,x,\xi) = M_1(\xi) + a(t,x)M_2(\xi)$, where M_1, M_2 are (possibly anisotropic) homogeneous N-functions (without prescribed growth) such that $|\xi|^p \leq M_1(\xi) \leq M_2(\xi)$ for ξ such that $|\xi| > 1$, and moreover the function $a : \Omega_T \to [0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a. Then we can take Θ as in (3.60) and *(Mp)*$_p$ is satisfied if

$$\limsup_{\delta\to 0} \omega_a(\delta) \frac{\overline{M}_2(\delta^{-N/p})}{\underline{M}_1(\delta^{-N/p})} < \infty,$$

  where $\underline{M}_1(s) := \inf_{\xi:\,|\xi|=s} M_1(\xi)$ and $\overline{M}_2(s) := \sup_{\xi:\,|\xi|=s} M_2(\xi)$.

4.2.2.3 Between isotropic and anisotropic conditions in the parabolic case

Exactly the same method as in the proof of the elliptic analogue (Theorem 3.7.4) leads to the simplification of anisotropic condition *(Mp)* (resp. *(Mp)*$_p$) to its isotropic counterpart, namely *(Mp^i)* (resp. *(Mp^i)*$_p$).

Theorem 4.2.4 *Isotropic conditions are sufficient to get their anisotropic versions. That is, if M satisfies (Mp^i), then M satisfies (Mp); and if M satisfies (Mp^i)$_p$, then M satisfies (Mp)$_p$.*

This theorem is a direct consequence of the following geometrical observation, which can be proved by the same arguments as Proposition 3.7.5. Note that the corresponding statement for anisotropic M is false in general, cf. Remark 3.7.6.

Proposition 4.2.5 *Let Ω be an open subset of $\mathbb{R}^N$, $M_{I,Q}$ be defined by* (4.71)*, and an N-function M satisfy (Mp^i) or (Mp^i)$_p$. Let $\varepsilon > 0$ be an arbitrary (small) number. Then, for a.a. $t,r \in [0,T]$ and a.a. $x,y \in \Omega$, such that $|t-r| + c_{sp}|x-y|^N \le \varepsilon/2$ we have*

$$\frac{M(t,y,s)}{(M_{I,Q})^{**}(s)} \le 4(\Theta^i(\varepsilon,s))^2. \tag{4.80}$$

4.2.2.4 Approximation in space

The approximation in space follows the scheme of [79] modified by ideas of [7]. Unlike [304] we do not require M^* to satisfy a balance condition of the type *(Mp)* or *(Mp)$_p$*. Recall that *(Mp)* is given in Section 4.2.2.1, whereas *(Mp)$_p$* in Section 4.2.2.2.

Theorem 4.2.6 (Approximation in space) *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ and an N-function $M : [0,T) \times \Omega \to [0,\infty)$ satisfies condition (Mp) or (Mp)$_p$. Then for any $\varphi \in V_T^{M,\infty}(\Omega)$ there exists a sequence*

$$\{\varphi_\delta\}_{\delta>0} \subset L^\infty(0,T; C_c^\infty(\Omega)),$$

such that for $\delta \to 0$

$$\varphi_\delta \to \varphi \ \textit{ strongly in } L^1(\Omega_T) \quad \textit{and} \quad \nabla\varphi_\delta \xrightarrow{M} \nabla\varphi \ \textit{ modularly in } L_M(\Omega_T;\mathbb{R}^N).$$

Moreover, there exists a $c = c(\Omega) > 0$, such that $\|\varphi_\delta\|_{L^\infty(\Omega)} \le c\|\varphi\|_{L^\infty(\Omega)}$.

By virtue of Theorem 4.2.4, the above result has a simple isotropic version.

Theorem 4.2.7 (Approximation in space – Isotropic case) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and an N-function $M : [0,T) \times \Omega \to [0,\infty)$ satisfy condition (Mp^i) or (Mp^i)$_p$. Then for any $\varphi \in V_T^{M,\infty}(\Omega)$ there exists a sequence*

$$\{\varphi_\delta\}_{\delta>0} \subset L^\infty(0,T; C_c^\infty(\Omega))$$

such that for $\delta \to 0$

$$\varphi_\delta \to \varphi \ \textit{ strongly in } L^1(\Omega_T) \quad \textit{and} \quad \nabla\varphi_\delta \xrightarrow{M} \nabla\varphi \ \textit{ modularly in } L_M(\Omega_T;\mathbb{R}^N).$$

Moreover, there exists a $c = c(\Omega) > 0$ such that $\|\varphi_\delta\|_{L^\infty(\Omega)} \le c\|\varphi\|_{L^\infty(\Omega)}$.

To deal with the approximation in space we construct an approximate sequence based on the convolution, then we provide a uniform estimate on a star-shaped domain to be in a position to prove Theorem 4.2.6.

Set $\kappa_\delta := 1 - 2\delta/R$ as before in (3.71). For a measurable function

$$\xi : [0,T) \times \mathbb{R}^N \to \mathbb{R}^N, \qquad \text{such that} \qquad \operatorname{supp}\xi \subset [0,T] \times \Omega,$$

let us define

$$S_\delta \xi(t,x) = \int_\Omega \rho_\delta(x-y)\xi(t,y/\kappa_\delta)\,\mathrm{d}y, \tag{4.81}$$

where $\rho_\delta(x) = \rho(x/\delta)/\delta^N$ is a standard regularizing kernel on $\mathbb{R}^N$ (i.e. $\rho \in C^\infty(\mathbb{R}^N)$, $\operatorname{supp}\rho \subset\subset B(0,1)$ and $\int_\Omega \rho(x)dx = 1$, $\rho(x) = \rho(-x)$), such that $0 \le \rho \le 1$. For sufficiently small $\delta > 0$ of course $S_\delta \xi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$. By the very definition S_δ preserves the L^∞-norm.

Proposition 4.2.8 *Suppose $M : [0,T)\times\Omega \to [0,\infty)$ is an N-function satisfying condition (Mp) or $(Mp)_p$, and Ω is a bounded star-shaped domain with respect to a ball B_R for some $R > 0$. If S_δ is given by* (4.81)*, then there exist constants $C, \delta_1 > 0$ independent of δ such that for all $\delta < \delta_1$ and all $\xi \in \mathcal{L}_M(\Omega_T;\mathbb{R}^N)$ we have*

$$\begin{aligned}\int_{\Omega_T} M(t,x,S_\delta\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\{m_1(|\xi|)\le 1\}} & m_2(|\xi(t,x)|)\,\mathrm{d}x\,\mathrm{d}t \\ &+ C\int_{\Omega_T} M(t,x,\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.82}$$

where m_1, m_2 are the minorant and majorant, respectively, of an N-function (see Definition 2.2.2).

Proof. We present the proof only in the case when Ω is a star-shaped domain with respect to a ball centered at the origin. For the general case one should change variables moving the center of B_R to the origin, then proceed with the proof as below, and then reverse the change of variables.

Fix an arbitrary $\xi \in L_M(\Omega_T;\mathbb{R}^N)$. We note that under assumption *(Mp)* without loss of generality it can be assumed that

$$\|\xi\|_{L^\infty(0,T;L^1(\Omega))} \le \tfrac{1}{2^N}. \tag{4.83}$$

On the other hand, if *$(Mp)_p$* is in power, we may assume that $\|\xi\|_{L^\infty(0,T;L^p(\Omega))} \le \widetilde{c}$ with absolute constant $\widetilde{c} > 0$ (we will choose it soon). We notice that

$$\begin{aligned}\int_{\Omega_T} M(t,x,S_\delta\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t &\le \int_{\{M(\cdot,\cdot,S_\delta\xi)\le 1\}} M(t,x,S_\delta\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\{M(\cdot,\cdot,S_\delta\xi)\ge 1\}} M(t,x,S_\delta\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{\{m_1(|S_\delta\xi(\cdot)|)\le 1\}} m_2(|S_\delta\xi(t,x)|)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\{M(\cdot,\cdot,S_\delta\xi(\cdot))\ge 1\}} M(t,x,S_\delta\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t \\ &=: \mathsf{I}_\delta + \mathsf{J}_\delta.\end{aligned}$$

To deal with I_δ we notice that $\{m_1(|S_\delta\xi(\cdot)|) \le 1\} = \{m_2(|S_\delta\xi(\cdot)|) \le c\}$ for $c = m_2 \circ m_1^{-1}(1)$ and we have the following pointwise estimate

$$m_2(|S_\delta\xi(\cdot)|)\mathbb{1}_{\{m_1(|S_\delta\xi(\cdot)|)\le 1\}}(\cdot) \le c.$$

Hence, by Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\limsup_{\delta\searrow 0} \mathsf{I}_\delta &= \limsup_{\delta\searrow 0} \int_{\{m_1(|S_\delta\xi|)\le 1\}} m_2(|S_\delta\xi(t,x)|)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\{m_1(|\xi|)\le 1\}} m_2(|\xi(t,x)|)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{4.84}$$

Thus, we concentrate now on J_δ. For $0<\delta<R/2$ it holds that

$$\overline{\kappa_\delta\Omega+\delta B(0,1)}\subset\Omega,$$

see (3.75). Further we consider only δ sufficiently small that $S_\delta\xi\in L^\infty(0,T;C_c^\infty(\Omega))$.

We split the domain Ω into small cubes and the time interval into small pieces with uniformly controlled size. Then we will use assumption *(Mp)* (or *(Mp)$_p$*) on each small time-space cube separately. We fix $0<\delta<R/4$ and define families of sets $\{Q_j^\delta\}_{j=1}^{N_\delta}$ and $\{I_i^\delta\}_{i=1}^{N_\delta^T}$ having the following properties. By $\{I_i^\delta\}_{i=1}^{N_\delta^T}$ we denote a finite family of closed subintervals $I_i^\delta\subset[0,T]$ of length not greater than δ, such that

$$I_i^\delta=[t_i^\delta,t_{i+1}^\delta) \qquad \text{and} \qquad [0,T]=\bigcup_{i=1}^{N_\delta^T} I_i^\delta.$$

Let $\{Q_j^\delta\}_{j=1}^{N_\delta}$ be a family of N-dimensional cubes covering the set Ω, having edges of length 2δ, and such that

$$\operatorname{int}Q_j^\delta\cap\operatorname{int}Q_i^\delta=\emptyset \quad \text{for } i\neq j \text{ and } \quad \Omega\subset\bigcup_{j=1}^{N_\delta}Q_j^\delta.$$

With each cube Q_j^δ we associate the cube $\widetilde{Q}_j^\delta$ centered at the same point q_j and with parallel corresponding edges of length 4δ.

By condition *(Mp)* or *(Mp)$_p$*, the relation between $M(t,x,\xi)$ and

$$M_{i,j}^\delta(\xi):=\operatorname{ess\,inf}_{t\in I_i^\delta,\,x\in\widetilde{Q}_j^\delta\cap\Omega}M(t,x,\xi)\tag{4.85}$$

can be expressed as

$$\frac{M(t,x,\xi)}{(M_{i,j}^\delta)^{**}(\xi)}\le\Theta(\delta,|\xi|),\tag{4.86}$$

holding for a.e. $(t,x)\in I_i^\delta\times Q_j^\delta$ and all $\xi\in\mathbb{R}^N$ with $|\xi|>1$, where by $(M_{i,j}^\delta)^{**}(\xi)=((M_{i,j}^\delta(\xi))^*)^*$ we denote the second conjugate of $M_{i,j}^\delta$. Recall that by Corollary 2.1.42 it coincides with the greatest convex minorant of $M_{i,j}^\delta$.

We split the domain into small pieces and, since $M(t,x,\xi_\delta(t,x))>0$ in $\{M(\cdot,\cdot,\xi_\delta)\ge 1\}$, we may multiply and divide by $(M_{i,j}^\delta)^{**}$ to write the left-hand side of (4.82) in the following way

$$
\begin{aligned}
\mathsf{J}_\delta &= \sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{Q_j^\delta\cap\Omega} M(t,x,S_\delta(\xi(t,x)))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\delta)\geq 1\}}\,\mathrm{d}x\,\mathrm{d}t\\
&= \sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{Q_j^\delta\cap\Omega} \frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))}(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\delta)\geq 1\}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.87}
$$

We shall now show that

$$
\frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))} \leq C \tag{4.88}
$$

for sufficiently small $\delta > 0$, $x \in Q_j^\delta \cap \Omega$, $t \in I_i^\delta \cap [0,T]$, with C independent of δ, t, x, i, j and ξ. To get it we fix an arbitrary cube and subinterval and take $(t,x) \in I_i^\delta \times Q_j^\delta$. For sufficiently small δ, due to (4.86), we obtain

$$
\frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))} \leq \Theta(\delta, |S_\delta(\xi(t,x))|). \tag{4.89}
$$

To estimate the right-hand side of (4.89) we make use of the definition of S_δ given in (4.81). In fact, for any $(t,x) \in \Omega_T$ and each $\delta > 0$ we have $\rho_\delta(x-y) \leq 1/\delta^N$. Having (4.83), we observe that

$$
|S_\delta\xi(t,x)| \leq \frac{1}{\delta^N}\int_\Omega |\xi(t,y/\kappa_\delta)|\;\mathrm{d}y \leq \frac{\kappa_\delta^N}{\delta^N}\|\xi\|_{L^\infty(0,T;L^1(\Omega))} \leq \delta^{-N}. \tag{4.90}
$$

Note that in the case of *(Mp)$_p$* we just estimate $|S_\delta\xi(t,x)| \leq \delta^{-N/p}$ using the Hölder inequality. Indeed,

$$
\begin{aligned}
|S_\delta\xi(t,x)| &\leq \left(\int_\Omega |\xi(t,y/\kappa_\delta)|^p\;\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_\Omega \rho_\delta^{\frac{p}{p-1}}\;\mathrm{d}y\right)^{\frac{p-1}{p}} \leq \frac{1}{\widetilde{c}\delta^{N/p}}\|\xi\|_{L^\infty(0,T;L^p(\Omega))}\\
&\leq \delta^{-\frac{N}{p}},
\end{aligned}
$$

where we chose $\widetilde{c}$ for the second inequality to hold. The last estimate is true, as we used $\widetilde{c}$ as the normalization constant.

We apply these estimates in (4.89). When we recall (4.72) (resp. (4.77) in the case of (4.76)), we obtain for all $\delta < \delta_1$ with some $\delta_1 > 0$ the estimates

$$
\frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))} \leq \Theta(\delta,\delta^{-N}) < C,
$$

$$
\left(\text{resp. } \frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))} \leq \Theta(\delta,\delta^{-N/p}) < C\right),
$$

which completes the proof of (4.88).

Let us go back to (4.87). We apply (4.88) to obtain

$$\begin{aligned}
\mathsf{J}_\delta &= \\
&\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{Q_j^\delta\cap\Omega}\frac{M(t,x,S_\delta(\xi(t,x)))}{(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))}(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\delta)\geq 1\}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{Q_j^\delta\cap\Omega}(M_{i,j}^\delta)^{**}(S_\delta(\xi(t,x)))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\delta)\geq 1\}}\,\mathrm{d}x\,\mathrm{d}t =: \mathsf{J}_\delta^1.
\end{aligned}$$

In order to apply Jensen's inequality we carefully change the domain of integration by writing an indicator of relevant cubes. Let us recall that $\widetilde{Q}_j^\delta$ is an expansion of the cube Q_j^δ with the same center. We have

$$\begin{aligned}
\mathsf{J}_\delta^1 &\leq C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{Q_j^\delta\cap\Omega}(M_{i,j}^\delta)^{**}\left(\int_{B(0,\delta)}\rho_\delta(y)\xi\left(t,\frac{x-y}{\kappa_\delta}\right)\mathrm{d}y\right)\mathbb{1}_{Q_j^\delta\cap\Omega}(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\mathbb{R}^N}(M_{i,j}^\delta)^{**}\left(\int_{B(0,\delta)}\rho_\delta(y)\xi\left(t,\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{Q_j^\delta\cap\Omega}(x)\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \\
&\leq C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\mathbb{R}^N}(M_{i,j}^\delta)^{**}\left(\int_{\mathbb{R}^N}\rho_\delta(y)\xi\left(t,\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{\widetilde{Q}_j^\delta\cap\Omega}(x-y)\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \\
&=: \mathsf{J}_\delta^2.
\end{aligned}$$

Further, the right-hand side above can be estimated with the use of Jensen's inequality by the following quantity

$$\begin{aligned}
\mathsf{J}_\delta^2 \leq\ & C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\delta(y)(M_{i,j}^\delta)^{**}\left(\xi\left(t,\frac{x-y}{\kappa_\delta}\right)\mathbb{1}_{\widetilde{Q}_j^\delta\cap\Omega}(x-y)\right)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\|\rho_\delta\|_{L^1(B(0,\delta);\mathbb{R}^N)}\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\mathbb{R}^N}(M_{i,j}^\delta)^{**}\left(\xi\left(t,\frac{z}{\kappa_\delta}\right)\mathbb{1}_{\widetilde{Q}_j^\delta\cap\Omega}(z)\right)\mathrm{d}z\,\mathrm{d}t \\
&\leq C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\widetilde{Q}_j^\delta\cap\Omega}(M_{i,j}^\delta)^{**}\left(\xi\left(t,\frac{z}{\kappa_\delta}\right)\right)\mathrm{d}z\,\mathrm{d}t =: \mathsf{J}_\delta^3.
\end{aligned}$$

We applied above Young's convolution inequality (Lemma 8.26), the uniform boundedness of ρ_δ, and once again the fact that $(M_{i,j}^\delta)^{**}(\xi) = 0$ if and only if $\xi = 0$. Since $(M_{i,j}^\delta)^{**}$ is the (greatest convex) minorant of $M_{i,j}^\delta$ (see Corollary 2.1.42), we know that for every $t \in I_i^\delta$, $w \in \widetilde{Q}_j^\delta \cap \Omega$, and every vector $\xi \in \mathbb{R}^N$ it holds that

$$(M_{i,j}^\delta)^{**}(\xi) \leq M_{i,j}^\delta(\xi) \leq M(t,w,\xi).$$

Thus we can estimate

$$\begin{aligned}\mathsf{J}_\delta^3 &\le C\sum_{i=1}^{N_\delta^T}\sum_{j=1}^{N_\delta}\int_{I_i^\delta}\int_{\widetilde{Q}_j^\delta\cap\Omega} M_{i,j}^\delta\left(\xi\left(t,\frac{z}{\kappa_\delta}\right)\right)\mathrm{d}z\,\mathrm{d}t\\ &\le C\sum_{j=1}^{N_\delta}\int_0^T\int_{\widetilde{Q}_j^\delta\cap\Omega} M\left(t,\frac{z}{\kappa_\delta},\xi\left(t,\frac{z}{\kappa_\delta}\right)\right)\mathrm{d}z\,\mathrm{d}t =: \mathsf{J}_\delta^4,\end{aligned}$$

where we used that $(M_j^\delta)^{**}$ is (the greatest convex) minorant of M_j^δ (Corollary 2.1.42). To estimate it further we substitute $x := z/\kappa_\delta$ and observe that as in (3.83) we have

$$\kappa_\delta\widetilde{Q}_j^\delta \subset Q_j^{c_\Omega\delta},$$

for $c_\Omega = 4(1+\mathrm{diam}\Omega/R)$ since $\delta < R/4$. Therefore we infer that

$$\mathsf{J}_\delta^4 \le C\sum_{j=1}^{N_\delta}\int_0^T\int_{Q_j^{c_\Omega\delta}} M(t,x,\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t \le C(N)\int_{\Omega_T} M(t,x,\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

The last inequality above takes into account the measure of a finite number of repeating parts of cubes. Summing up the estimates we conclude (4.82). □

Now we are in a position to prove the approximation-in-space result.

Proof (of Theorem 4.2.6). Since Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, then by Lemma 8.2 the set $\overline{\Omega}$ can be covered by a finite family of sets $\{G_i\}_{i\in I}$ such that each

$$\Omega_i := \Omega\cap G_i$$

is a star-shaped domain with respect to the balls $\{B^i\}_{i\in I}$, respectively. Then

$$\Omega = \bigcup_{i\in I}\Omega_i.$$

Let us introduce a partition of unity θ_i, i.e.

$$0\le\theta_i\le 1,\quad \theta_i\in C_c^\infty(G_i),\quad \sum_{i\in I}\theta_i(x) = 1 \text{ for } x\in\Omega,$$

which exists due to Lemma 8.3.

We fix an arbitrary $\varphi\in V_T^{M,\infty}(\Omega)$. We will prove that there exists a constant $\lambda > 0$ such that

$$\lim_{\delta\to 0^+}\int_{\Omega_T} M\left(t,x,\frac{\nabla S_\delta(\varphi)-\nabla\varphi}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t = 0,$$

for S_δ given by (4.81). Let

$$A_i := (0,T)\times\Omega_i.$$

Since

$$\int_{\Omega_T} M\left(t,x,\frac{\nabla S_\delta(\varphi)-\nabla\varphi}{\lambda}\right)\,\mathrm{d}x\,\mathrm{d}t \le \sum_{i\in I}\frac{\lambda^i}{\lambda}\int_{A_i} M\left(t,x,\frac{\nabla S_\delta(\varphi)-\nabla\varphi}{\lambda^i}\right)\,\mathrm{d}x\,\mathrm{d}t$$

for some $\lambda_i > 0$ such that $\lambda = \sum_i \lambda^i$, and there is a finite number of A_is on the right-hand side, we prove the convergence to zero of each of them.

Let us choose a family of measurable sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} E_n = A_i$ and a simple vector-valued function

$$E^n(t,x) = \sum_{j=0}^{n} \mathbb{1}_{E_j}(t,x)\eta_j, \tag{4.91}$$

where $\{\eta_j\}_{j=0}^n$ is a family of vectors, such that $\{E^n\}_{n\in\mathbb{N}}$ converges modularly in L_M to $\nabla(\theta_i\varphi)$ with λ_3^i (cf. Definition 3.4.3) whose existence is ensured by Theorem 3.4.11. Since

$$\nabla S_\delta(\theta_i\varphi)-\nabla(\theta_i\varphi) = (\nabla S_\delta(\theta_i\varphi)-S_\delta E^n)+(S_\delta E^n - E^n)+(E^n-\nabla(\theta_i\varphi)),$$

by Jensen's inequality we can estimate

$$\begin{aligned}
&\int_{A_i} M\left(t,x,\frac{\nabla S_\delta(\theta_i\varphi)-\nabla(\theta_i\varphi)}{\lambda^i}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{\lambda_1^i}{\lambda^i}\int_{A_i} M\left(t,x,\frac{\nabla S_\delta(\theta_i\varphi)-S_\delta E^n}{\lambda_1^i}\right)\,\mathrm{d}x\,\mathrm{d}t + \frac{\lambda_2^i}{\lambda^i}\int_{A_i} M\left(t,x,\frac{S_\delta E^n - E^n}{\lambda_2^i}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{\lambda_3^i}{\lambda^i}\int_{A_i} M\left(t,x,\frac{E^n-\nabla(\theta_i\varphi)}{\lambda_3^i}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&= L_1^{n,\delta}+L_2^{n,\delta}+L_3^n,
\end{aligned}\tag{4.92}$$

where $\lambda^i = \sum_{j=1}^3 \lambda_j^i$, $\lambda_j^i > 0$. We have λ_3^i fixed already. We take $\lambda_1^i = \lambda_3^i$ and leave λ_2^i to be chosen in a moment.

We recall that $\varphi \in V_T^{M,\infty}(\Omega)$, so for each $i \in I$ we have

$$\theta_i \cdot \varphi \in L^\infty(A_i)\cap L^\infty(0,T;L^2(\Omega_i))\cap L^1(0,T;W_0^{1,1}(\Omega_i))$$

and

$$\nabla(\theta_i\varphi) \;=\; \varphi\nabla\theta_i + \theta_i\nabla\varphi \;\in\; L_M(\Omega_T;\mathbb{R}^N).$$

Furthermore,

$$\sum_{i\in I}\nabla(\theta_i\varphi) = \nabla\varphi.$$

Since

$$L_1^{n,\delta} = \frac{\lambda_1^i}{\lambda^i} \int_{A_i} M\left(t, x, S_\delta\left(\frac{E^n - \nabla(\theta_i \varphi)}{\lambda_1^i}\right)\right) \mathrm{d}x\, \mathrm{d}t,$$

Proposition 4.2.8 implies that we can estimate

$$\begin{aligned} 0 \le L_1^{n,\delta} \le & \int_{\left\{ m_1\left(\frac{|E^n - \nabla(\theta_i\varphi)|}{\lambda_1^i}\right) \le 1 \right\}} m_2\left(\frac{|E^n - \nabla(\theta_i\varphi)|}{\lambda_1^i}\right) \mathrm{d}x\, \mathrm{d}t \\ & + C \int_{A_i} M\left(t, x, \frac{E^n - \nabla\varphi}{\lambda_3^i}\right) \mathrm{d}x\, \mathrm{d}t =: K^n, \end{aligned}$$

where $\lim_{n\to\infty} K^n = 0$. Consequently,

$$\lim_{l\to\infty} \lim_{\delta\to 0^+} L_1^{n,\delta} = 0.$$

It suffices to prove the convergence of $L_2^{n,\delta}$. Let us recall that E^n is given by (4.91). By Jensen's inequality and Fubini's theorem we get that

$$\begin{aligned} \frac{\lambda^i}{\lambda_2^i} L_2^{n,\delta} &= \int_{A_i} M\left(t, x, \frac{E^n(t,x) - S_\delta E^n(t,x)}{\lambda_2^i}\right) \mathrm{d}x\, \mathrm{d}t \\ &= \int_{A_i} M\left(t, x, \frac{1}{\lambda_2^i} \int_{B(0,\delta)} \varrho_\delta(y) \cdot \right. \\ &\qquad \left. \cdot \sum_{j=0}^{n} \left[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}\left(t, \frac{x-y}{\kappa_\delta}\right)\eta_j\left(t, \frac{x-y}{\kappa_\delta}\right)\right] \mathrm{d}y \right) \mathrm{d}x\, \mathrm{d}t \qquad (4.93) \\ &\le \int_{B(0,\delta)} \varrho_\delta(y) \left(\int_{A_i} M\left(t, x, \frac{1}{\lambda_2^i} \cdot \right.\right. \\ &\qquad \left.\left. \cdot \sum_{j=0}^{n} \left[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}\left(t, \frac{x-y}{\kappa_\delta}\right)\eta_j\left(t, \frac{x-y}{\kappa_\delta}\right)\right]\right) \mathrm{d}x \right) \mathrm{d}y\, \mathrm{d}t. \end{aligned}$$

As the shift operator is continuous in L^1, we have pointwise convergence

$$\sum_{j=0}^{n} \left[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}\left(t, \frac{x-y}{\kappa_\delta}\right)\eta_j\left(t, \frac{x-y}{\kappa_\delta}\right)\right] \xrightarrow{\delta\to 0} 0.$$

Moreover, for arbitrary fixed $\lambda_2^i > 0$ we have

$$M\left(t,x,\frac{1}{\lambda_2^i}\sum_{j=0}^n\left[\mathbb{1}_{E_j}(t,x)\eta_j(t,x)-\mathbb{1}_{E_j}\left(t,\frac{x-y}{\kappa_\delta}\right)\eta_j\left(t,\frac{x-y}{\kappa_\delta}\right)\right]\right)$$
$$\leq \sup_{\zeta\in\mathbb{R}^N:\ |\zeta|=1} M\left(t,x,\frac{2}{\lambda_2^i}\sum_{j=0}^n\|\eta_j\|_{L^\infty(E_j)}\zeta\right)<\infty.$$

The Lebesgue dominated convergence theorem justifies convergence to zero of the right-hand side of (4.93) for any $\lambda_2^i>0$.

Altogether we have proved that the right-hand side of (4.92) vanishes in the limit, which completes the proof of modular convergence of the approximate sequence. The modular convergence of gradients implies their strong L^1-convergence and the Poincaré inequality gives the claim. □

Remark 4.2.9. To avoid decomposition into star-shaped domains, in the isotropic setting one can apply the shrinking approach. For this one should consider a mapping that shrinks the area near the boundary into the interior of the domain. Its construction for a bounded domain with C^2-boundary is presented in [304]. Note that this mapping and the inferred further approximation theorem are applied therein in the proof of existence of weak solutions to a bounded data problem in a space changing with time via the Galerkin method.

4.2.3 Integration by parts formula

In the classical setting one is equipped with a factorization of the norm resulting from the fact that

$$L^p([0,T)\times\Omega)=L^p([0,T);L^p(\Omega)),$$

which in our case is excluded. The Musielak–Orlicz version of the right-hand side, when M depends on time, has no meaning. Actually already the general growth of M makes this Bochner-type factorization of the norm impossible, see the comments in the Introduction. To bypass the typical use of such a factorization, we shall need to prove the so-called *integration by parts formula*. We prove it under assumptions *(Mp)* from Section 4.2.2.1 or *(Mp)$_p$* from Section 4.2.2.2. Note that the proof involves various approximation results. The proof also holds, with minor modifications, in the case when M does not depend on t and $M,M^*\in\Delta_2$, using approximation coming from Mazur's lemma.

Theorem 4.2.10 (Integration by parts formula) *Suppose $M:\Omega_T\times\mathbb{R}^N\to[0,\infty)$ is an N-function satisfying (Mp) or (Mp)$_p$ and that $u:\Omega_T\to\mathbb{R}$ is a measurable function such that for every $k\geq 0$, $T_k(u)\in V_T^M(\Omega)$, $u(t,x)\in L^\infty([0,T];L^1(\Omega))$. Assume that there exist $\boldsymbol{\mathcal{A}}\in L_{M^*}(\Omega_T;\mathbb{R}^N)$, $F\in L^1(\Omega_T)$, and $u_0\in L^1(\Omega)$ such that $u_0(x):=u(0,x)$, with which for all $\varphi\in C_c^\infty([0,T)\times\Omega)$ it holds that*

$$-\int_{\Omega_T}(u-u_0)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\boldsymbol{\mathcal{A}}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}F\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{4.94}$$

Then

$$-\int_{\Omega_T}\left(\int_{u_0}^{u} h(\sigma)\,\mathrm{d}\sigma\right)\partial_t\xi\ \ \mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathcal{A}\cdot\nabla(h(u)\xi)\ \ \mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}Fh(u)\xi\ \ \mathrm{d}x\,\mathrm{d}t$$

holds for all $h\in W^{1,\infty}(\mathbb{R})$ *such that* $\operatorname{supp}(h')$ *is compact and all* $\xi\in V_T^{M,\infty}(\Omega)$ *such that* $\partial_t\xi\in L^\infty(\Omega_T)$ *and* $\operatorname{supp}\xi(\cdot,x)\subset[0,T)$ *for a.e.* $x\in\Omega$.
In particular, the formula holds for $\xi\in C_c^\infty([0,T)\times\overline{\Omega})$.

Proof. Fix an arbitrary $h\in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}(h')$ is compact. Let us recall that $(\cdot)_+,(\cdot)_-$ denote the positive and negative parts of the argument, respectively. Note that $h_1,h_2:\mathbb{R}\to\mathbb{R}$ given by

$$h_1(t)=\int_{-\infty}^{t}(h')_+(s)\,\mathrm{d}s\quad\text{and}\quad h_2(t)=\int_{-\infty}^{t}(h')_-(s)\,\mathrm{d}s$$

are compactly supported Lipschitz continuous functions. Furthermore, h_1 is non-decreasing, h_2 is non-increasing, and $h=h_1+h_2$. Since there exists a $k>0$ such that $\operatorname{supp}(h')\subset[-k,k]$, we can write

$$h(u)=h(T_k(u))=h_1(T_k(u))+h_2(T_k(u)).$$

Of course, $h_1(T_k(u)),h_2(T_k(u))\in L^\infty(\Omega_T)$ and

$$\nabla(h_1(T_k(u))),\nabla(h_2(T_k(u)))\in L_M(\Omega_T;\mathbb{R}^N).$$

By Theorem 4.2.6 there exists a modularly converging sequence $\{\nabla(T_k(u))_\varepsilon\}_\varepsilon$. Then due to modular convergence and Theorem 3.4.4 we get the uniform integrability of

$$\left\{M\left(x,\frac{h_1'((T_k(u))_\varepsilon)\nabla(T_k(u))_\varepsilon}{\lambda}\right)\right\}_\varepsilon\qquad\text{for some }\ \lambda>0.$$

We start with the proof for nonnegative ξ, which we extend in the following way

$$\xi(t,x)=\begin{cases}\xi(-t,x), & t<0,\\ \xi(t,x), & t\in[0,T],\\ 0, & t>T.\end{cases}\tag{4.95}$$

Further we extend $u(t,x)=u_0(x)$ for $t<0$ and we fix $d>0$. We set

$$\zeta=h_1(T_k(u))\xi\tag{4.96}$$

and its right and left Steklov averages $\zeta_d,\widetilde{\zeta}_d(t,x):\Omega_T\to\mathbb{R}$ given by

$$\begin{aligned}\zeta_d(t,x):=&\frac{1}{d}\int_t^{t+d}\zeta(\sigma,x)\,\mathrm{d}\sigma,\\ \widetilde{\zeta}_d(t,x):=&\frac{1}{d}\int_{t-d}^{t}\zeta(\sigma,x)\,\mathrm{d}\sigma.\end{aligned}\tag{4.97}$$

Note that due to the same reasoning as for $h(u)$, also

$$\zeta_d, \widetilde{\zeta}_d(t,x) \in V_T^{M,\infty}(\Omega).$$

Furthermore, $\partial_t \zeta_d, \partial_t \widetilde{\zeta}_d(t,x) \in L^\infty(\Omega_T)$. Then $\zeta_d(T,x) = \widetilde{\zeta}_d(0,x) = 0$ for all $x \in \Omega$ and $d > 0$. By Theorem 4.2.6 we have approximate sequences $\{\zeta_d\}_\varepsilon, \{\widetilde{\zeta}_d\}_\varepsilon \in C_c^\infty(0,T;C_c^\infty(\Omega))$ that can be used as test functions in (4.94) to get

$$\begin{aligned}\int_{\Omega_T} \boldsymbol{\mathcal{A}} \cdot \nabla((\zeta_d)_\varepsilon)\, \mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F(\zeta_d)_\varepsilon\, \mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} (u(t,x) - u_0(x))\, \partial_t((\zeta_d)_\varepsilon)\, \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.98}$$

Since modular convergence entails weak convergence and

$$\{(\zeta_d)_\varepsilon\}_\varepsilon \text{ is uniformly bounded in } L^\infty,$$

Lebesgue's dominated convergence theorem enables us to pass to the limit as $\varepsilon \longrightarrow 0$. If we set $\zeta(t,x) = 0$ for $t > T$, we obtain

$$\begin{aligned}\int_{\Omega_T} \boldsymbol{\mathcal{A}} \cdot \nabla\zeta_d\, \mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\zeta_d\, \mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} (u(t,x) - u_0(x))\, \frac{1}{d}\, (\zeta(t+d,x) - \zeta(t,x))\, \mathrm{d}x\,\mathrm{d}t \\ = \frac{1}{d}\,(J_1 + J_2 + J_3),\end{aligned} \tag{4.99}$$

where

$$\begin{aligned} J_1 &:= \int_0^T \int_\Omega \zeta(t+d,x)u(t,x)\, \mathrm{d}x\,\mathrm{d}t = \int_d^T \int_\Omega \zeta(t,x)u(t-d,x)\, \mathrm{d}x\,\mathrm{d}t, \\ J_2 &:= -\int_0^T \int_\Omega \zeta(t,x)u(t,x)\, \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.100}$$

$$\begin{aligned} J_3 &:= -\int_0^T \int_\Omega \zeta(t+d,x)u_0(x)\, \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \zeta(t,x)u_0(x)\, \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^d \int_\Omega \zeta(t,x)u(t-d,x)\, \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.101}$$

Using (4.100) and (4.101) in (4.99) we get

$$\begin{aligned}\int_{\Omega_T} \boldsymbol{\mathcal{A}} \cdot \nabla\zeta_d\, \mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\zeta_d\, \mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} \frac{1}{d}\zeta(t,x)\,(u(t-d,x) - u(t,x))\, \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.102}$$

Since for any $s_1, s_2 \in \mathbb{R}$ it holds that

$$\int_{s_1}^{s_2} h_1(T_k(\sigma))\,\mathrm{d}\sigma \geq h_1(T_k(s_1))(s_2 - s_1), \tag{4.103}$$

we infer that

$$\frac{1}{d}\int_{\Omega_T} \zeta(t,x)\,(u(t-d,x) - u(t,x))\ \mathrm{d}x\,\mathrm{d}t \\ \leq \frac{1}{d}\int_{\Omega_T} \xi(t,x) \int_{u(t,x)}^{u(t-d,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\ \ \mathrm{d}x\,\mathrm{d}t.$$

Applying this in (4.102), by the same reasoning as in (4.101), we get

$$\begin{aligned} &\int_{\Omega_T} \mathcal{A} \cdot \nabla \zeta_d\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\zeta_d\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \frac{1}{d}\int_{\Omega_T} \xi(t,x) \left(\int_{u(t,x)}^{u(t-d,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma \right) \mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{d}\int_{\Omega_T} (\xi(t+d,x) - \xi(t,x)) \left(\int_{u(0,x)}^{u(t-d,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma \right) \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.104}$$

Taking into account the definition of ζ_d (4.97) and passing to a subsequence if necessary, we have

$$\zeta_d \xrightarrow[d\searrow 0]{*} \xi h_1(T_k(u)) \text{ weakly-}* \text{ in } L^\infty(\Omega_T).$$

Since $\nabla\zeta_d = \big((\nabla\xi)h_1(T_k(u))\big)_d + \big(\xi\nabla(h_1(T_k(u)))\big)_d$ and

$$\big((\nabla\xi)h_1(T_k(u))\big)_d \xrightarrow[d\searrow 0]{*} (\nabla\xi)h_1(T_k(u)) \quad \text{weakly-}* \text{ in } L^\infty(\Omega_T;\mathbb{R}^N),$$

by Jensen's inequality in (4.97) we get

$$\nabla\zeta_d \xrightarrow[d\searrow 0]{M} \nabla(\xi h_1(T_k(u))) \quad \text{modularly in } L_M(\Omega_T,\mathbb{R}^N).$$

Since additionally we have

$$\zeta_d \xrightarrow[d\searrow 0]{*} \xi h_1(T_k(u)) \quad \text{weakly-}* \text{ in } L^\infty(\Omega_T),$$

we can pass to the limit in (4.104) and get

$$\int_{\Omega_T} \mathcal{A} \cdot \nabla(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \int_{\Omega_T} \partial_t\xi \int_{u_0}^{u(t,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\ \ \mathrm{d}x\,\mathrm{d}t. \tag{4.105}$$

Since $T_k(u_0) \in L^\infty(\Omega)$, it can be approximated by a sequence $\{u_0^n\}_n \subset C_c^\infty(\Omega)$ such that

$$T_k(u_0^n) \xrightarrow[n\to\infty]{} T_k(u_0) \quad \text{strongly in } L^1(\Omega) \text{ and a.e. in } \Omega.$$

Let us recall that for $t < 0$ and all $x \in \Omega$ we put $u(t,x) = u_0(x)$, we consider nonnegative $\xi \in C_c^\infty([0,T)\times\Omega)$ extended as in (4.95), and the sequence $\{(\widetilde{\zeta}_d)_\varepsilon\} \subset C_c^\infty([0,T)\times\Omega)$ of smooth functions approximates $\widetilde{\zeta}_d$ (given by (4.97)) in the modular topology, i.e.

$$(\widetilde{\zeta}_d)_\varepsilon \xrightarrow[\varepsilon\to 0]{M} \widetilde{\zeta}_d.$$

Therefore, $(\widetilde{\zeta}_d)_\varepsilon$ can be used as a test function in (4.94) and using the arguments as in (4.98), we pass to the limit as $\varepsilon \to 0$, inferring

$$\begin{aligned}\int_{\Omega_T} \mathcal{A}\cdot\nabla\widetilde{\zeta}_d \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\widetilde{\zeta}_d \,\mathrm{d}x\,\mathrm{d}t &\\ = \int_{\Omega_T} \frac{1}{d}\left(\zeta(t,x)-\zeta(t-d,x)\right)\left(u(t,x)-u_0(x)\right)\,\mathrm{d}x\,\mathrm{d}t&\\ = \frac{1}{d}\left(K_1+K_2+K_3\right),&\end{aligned}$$

where

$$\begin{aligned} K_1 &:= \int_0^T \int_\Omega \zeta(t,x)u(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_d^{T+d}\int_\Omega \zeta(t-d,x)u(t-d,x)\,\mathrm{d}x\,\mathrm{d}t,\\ K_2 &:= -\int_0^T\int_\Omega \zeta(t-d,x)u(t,x)\,\mathrm{d}x\,\mathrm{d}t, \qquad (4.106)\\ K_3 &:= -\int_0^T\int_\Omega \zeta(t,x)u_0(x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \zeta(t-d,x)u_0(x)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^d\int_\Omega \zeta(t-d,x)u_0(x)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.107)\end{aligned}$$

Summing this up, by (4.106) and (4.107), we have

$$\int_{\Omega_T} \mathcal{A}\cdot\nabla\widetilde{\zeta}_d\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\widetilde{\zeta}_d\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{d}\left(L_1+L_2\right), \qquad (4.108)$$

with

$$\begin{aligned} L_1 &:= \int_d^T\int_\Omega \zeta(t-d,x)(u(t-d,x)-u(t,x))\,\mathrm{d}x\,\mathrm{d}t,\\ L_2 &:= \int_0^d\int_\Omega h_1(T_k(u_0^n))\xi(u(t-d,x)-u(t,x))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^d\int_\Omega (h_1(T_k(u_0))-h_1(T_k(u_0^n)))\xi(u(t-d,x)-u(t,x))\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for sufficiently small d, because $\xi(\cdot,x)$ has a compact support in $[0,T)$ almost everywhere in Ω. Having (4.103), we infer that for a.e. $(t,x) \in (d,T)\times\Omega$,

$$\int_{u(t-d,x)}^{u(t,x)} -(h_1(T_k(\sigma)))\,\mathrm{d}\sigma \le -(u(t,x)-u(t-d,x))h_1(T_k(u(t-d,x))), \quad (4.109)$$

and a.e. in $(t,x) \in (0,d)\times\Omega$ we have

$$\int_{u(t-d,x)}^{u(t,x)} -(h_1(T_k(\sigma)))\,\mathrm{d}\sigma \le -(u(t,x)-u_0)h_1(T_k(u_0)). \quad (4.110)$$

Combining (4.108), (4.109), and (4.110), we get

$$\begin{aligned}
&\int_{\Omega_T} \mathcal{A}\cdot\nabla\widetilde{\zeta}_d \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F\widetilde{\zeta}_d \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \ge \frac{1}{d}\int_{\Omega_T} \xi(t,x)\left(\int_{u(t,x)}^{u(t-d,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad + \int_0^d\int_\Omega (h_1(T_k(u_0)) - h_1(T_k(u_0^n)))\xi(u_0-u(t,x))\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \ge \int_{\Omega_T} \frac{\xi(t-d,x)-\xi(t,x)}{d}\left(\int_{u(t,x)}^{u(t-d,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad - \int_\Omega |h_1(T_k(u_0^n)) - h_1(T_k(u_0))|\,|\xi|(|u_0|+|u(t,x)|)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We pass to the limit as $d \searrow 0$ and then $n\to\infty$ on the left-hand side above, as in (4.105) by the Lebesgue dominated convergence theorem. We obtain

$$\begin{aligned}
&\int_{\Omega_T} \mathcal{A}\cdot\nabla(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \ge \int_{\Omega_T} \partial_t\xi \int_{u_0}^{u(t,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\;\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \quad (4.111)$$

Using estimates from above (4.105) and from below (4.111) we infer that

$$\begin{aligned}
&\int_{\Omega_T} \mathcal{A}\cdot\nabla(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} F(h_1(T_k(u))\xi)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad = \int_{\Omega_T} \partial_t\xi \int_{u_0}^{u(t,x)} h_1(T_k(\sigma))\,\mathrm{d}\sigma\;\mathrm{d}x\,\mathrm{d}t
\end{aligned} \quad (4.112)$$

holds for all nondecreasing and Lipschitz $h_1:\mathbb{R}\to\mathbb{R}$ and for all nonnegative ξ.

We can replace $h_1(T_k(u))$ by $-h_2(T_k(u))$ in (4.112) and in turn we can also replace it by $h(T_k(u)) = h(u)$. We have $\xi = \xi_+ + \xi_-$, where $\xi_+,\xi_- \in V_T^{M,\infty}(\Omega)$, which leads to the desired conclusion. □

4.2.4 The monotonicity trick in the parabolic case

In order to identify some limits we shall use the following parabolic monotonicity trick used in several variants in [79, 81, 183, 188, 303, 326].

Theorem 4.2.11 (Monotonicity trick in the parabolic case) *Suppose that* Ω *is a bounded open domain in* $\mathbb{R}^N$, $[0,T]$ *is a bounded interval, and* $\mathbf{a}$ *satisfies conditions (A1p)–(A2p) with an N-function* $M : \Omega \times \mathbb{R}^N \to [0,\infty)$. *Assume further that there exist*

$$\boldsymbol{\mathcal{A}} \in L_{M^*}(\Omega_T;\mathbb{R}^N) \quad \textit{and} \quad \boldsymbol{\xi} \in L_M(\Omega_T;\mathbb{R}^N),$$

such that

$$\int_{\Omega_T} \big(\boldsymbol{\mathcal{A}} - \mathbf{a}(t,x,\boldsymbol{\eta})\big) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta})\, \mathrm{d}x\, \mathrm{d}t \geq 0 \quad \textit{for all } \boldsymbol{\eta} \in \mathbb{R}^N.$$

Then

$$\mathbf{a}(t,x,\boldsymbol{\xi}) = \boldsymbol{\mathcal{A}} \qquad \textit{a.e. in } \Omega_T.$$

Proof. The proof follows precisely the same lines as the proof of its elliptic version, namely Theorem 4.1.1. Indeed, time-dependence either of the operator, or the modular function, does not interfere with the method. It suffices to use now *(A1p)–(A2p)* instead of the earlier *(A1e)–(A2e)*. □

4.2.5 Bounded-data parabolic problems

We apply the result of [130] providing the existence of a solution to a problem in the isotropic Orlicz–Sobolev setting (with the modular function depending on the norm of the gradient of the solution only). To avoid introducing overwhelming notation, we give here only a direct simplification of [130, Theorem 2] to our situation. It reads as follows.

Corollary 4.2.12 *Let* Ω *be a bounded open domain in* $\mathbb{R}^N$ *and* $\overline{m} : [0,\infty) \to [0,\infty)$. *We consider a Carathéodory function* $\mathbf{a} : [0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, *which is strictly monotone, i.e. for a.a.* $(t,x) \in \Omega_T$ *and all* $\xi_1 \neq \xi_2 \in \mathbb{R}^N$

$$\Big(\mathbf{a}(t,x,\xi_1) - \mathbf{a}(t,x,\xi_2)\Big) \cdot (\xi_1 - \xi_2) > 0.$$

Moreover, we assume that $\mathbf{a}$ *satisfies the following growth and coercivity conditions: for some* $c_0, c_1, c_2 > 0$, *for a.a.* $x \in \Omega$, *and for all* $\xi \in \mathbb{R}^N$ *we have*

$$c_0\, \overline{m}(|\xi|) \leq \mathbf{a}(t,x,\xi) \cdot \xi \qquad \textit{and} \qquad |\mathbf{a}(t,x,\xi)| \leq c_1 (\overline{m}^*)^{-1}\big(\overline{m}(c_2|\xi|)\big). \tag{4.113}$$

Then the problem

$$\begin{cases} \partial_t v - \operatorname{div} \mathbf{a}(t,x,\nabla v) = g \in L^\infty(\Omega_T) & \text{in } \Omega_T, \\ v(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0,\cdot) = v_0(\cdot) \in L^2(\Omega) & \text{in } \Omega \end{cases} \tag{4.114}$$

has at least one weak solution $v \in W^1L_{\overline{m}}(\Omega_T) \cap C([0,T];L^2(\Omega))$.

Moreover, the energy equality

$$\frac{1}{2}\int_\Omega (v(\tau,x))^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega (v_0(x))^2\,\mathrm{d}x + \int_{\Omega_\tau} \mathbf{a}(t,x,\nabla v)\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_\tau} gv\,\mathrm{d}x\,\mathrm{d}t \tag{4.115}$$

is satisfied for all $\tau \in [0,T]$.

The application of the above result gives the following proposition yielding the existence of solutions to a regularized problem. For this we will employ

$$m : \mathbb{R}^N \to [0,\infty) \text{ such that } m(\xi) = \overline{m}(|\xi|). \tag{4.116}$$

Recall that we use the notation ∇ to denote the gradient with respect to the spatial variable x, while to denote the gradient with respect to ξ we write ∇_ξ. Using this notation

$$\nabla_\xi \overline{m}(|\xi|) = \xi \overline{m}'(|\xi|)/|\xi|.$$

According to Remark 2.1.33 this gives equality in the Fenchel–Young inequality in the following way

$$\nabla_\xi m(\xi)\cdot\xi = \overline{m}(|\xi|) + \overline{m}^*(|\nabla_\xi m(\xi)|). \tag{4.117}$$

Since we take an N-function m which grows significantly faster than M we observe that m is strictly monotone as a gradient of a strictly convex function, i.e.

$$(\nabla_\xi m(\xi) - \nabla_\xi m(\eta))\cdot(\xi-\eta) > 0 \quad \text{for all } \xi,\eta \in \mathbb{R}^N. \tag{4.118}$$

We have the following result on the family of regularized problems.

Proposition 4.2.13 *Let an N-function* $M : [0,T)\times\Omega\times\mathbb{R}^N \to [0,\infty)$ *satisfy assumption (Mp) or* $(Mp)_p$ *and a function* $\mathbf{a}$ *satisfy assumptions (A1p)–(A3p). Assume that* $m : \mathbb{R}^N \to [0,\infty)$ *is an isotropic N-function such that* $m \in C^1(\mathbb{R}^N)$ *and it grows significantly faster than M (see Definition 3.2.3) and*

$$M(\xi) \le m(\xi) \quad \textit{for all } \xi \in \mathbb{R}^N. \tag{4.119}$$

We consider a regularized operator given by

$$\mathbf{a}_\theta(t,x,\xi) := \mathbf{a}(t,x,\xi) + \theta\nabla_\xi m(\xi) \quad \textit{for all } (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^N. \tag{4.120}$$

Let $f \in L^\infty(\Omega_T)$ *and* $u_0 \in L^2(\Omega)$. *Then for every* $\theta \in (0,1]$ *there exists a weak solution to the problem*

$$\begin{cases} \partial_t u^\theta - \operatorname{div}\mathbf{a}_\theta(t,x,\nabla u^\theta) = f & \text{in } \Omega_T, \\ u^\theta(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u^\theta(0,\cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{4.121}$$

Namely, there exists a $u^\theta \in C([0,T];L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ *with* $\nabla u^\theta \in L_m(\Omega_T;\mathbb{R}^N)$, *such that*

$$-\int_{\Omega_T} u^\theta \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega u^\theta(T)\varphi(T) \, \mathrm{d}x - \int_\Omega u^\theta(0)\varphi(0) \, \mathrm{d}x$$
$$+ \int_{\Omega_T} \mathbf{a}_\theta(t,x,\nabla u^\theta) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} f\varphi \, \mathrm{d}x \, \mathrm{d}t \quad (4.122)$$

holds for $\varphi \in C^\infty([0,T];C_c^\infty(\Omega))$.

Furthermore,

- *the family* $\{u^\theta\}_\theta$ *is uniformly bounded in* $L^\infty(0,T;L^2(\Omega))$,
- *the family* $\{\nabla u^\theta\}_\theta$ *is uniformly integrable in* $L_M(\Omega_T;\mathbb{R}^N)$,
- *the family* $\{\mathbf{a}(t,x,\nabla u^\theta)\}_\theta$ *is uniformly bounded in* $L_{M^*}(\Omega_T;\mathbb{R}^N)$,
- *the family* $\{\theta m^*(\nabla_\xi m(|\nabla u^\theta|)\}_\theta$ *is uniformly bounded in* $L^1(\Omega_T)$.

Moreover, the following energy equality

$$\frac{1}{2}\int_\Omega (u^\theta(\tau,x))^2 \, \mathrm{d}x - \frac{1}{2}\int_\Omega (u_0(x))^2 \, \mathrm{d}x + \int_{\Omega_\tau} \mathbf{a}_\theta(t,x,\nabla u^\theta) \cdot \nabla u^\theta \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_{\Omega_\tau} f u^\theta \, \mathrm{d}x \, \mathrm{d}t \quad (4.123)$$

is satisfied for all $\tau \in [0,T]$.

Proof. To get existence we apply Corollary 4.2.12, whereas a priori estimates result from the analysis of the structure of regularization. Recall that $m(\xi) = \overline{m}(|\xi|)$, see (4.116).

Existence. To apply Corollary 4.2.12 we shall show (4.113). The coercivity condition results directly from *(A2p)*. The bound on growth follows from the Fenchel–Young inequality (2.33), (4.117), and *(A3p)*

$$c_{\mathbf{a}}\overline{m}^*\left(\tfrac{1}{2}\,|\mathbf{a}_\theta(x,\xi)|\right) \le \overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)$$

and further, by convexity of $\overline{m}^*$,

$$|\mathbf{a}_\theta(x,\xi)| \le 2(\overline{m}^*)^{-1}\left(\tfrac{1}{c_{\mathbf{a}}}\overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)\right) \le \tfrac{2}{c_{\mathbf{a}}}(\overline{m}^*)^{-1}\left(\overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)\right).$$

Therefore, by Corollary 4.2.12 (coming from [130, Theorem 2]) it suffices to show

$$\overline{m}^*(|c_1\mathbf{a}_\theta(t,x,\xi)|) \le \overline{m}(|c_2\xi|),$$

which follows from equality (4.117) in the Fenchel–Young inequality (2.33) and *(A3p)* where $c_{\mathbf{a}}, \theta \in (0,1]$. We have

$$
\begin{aligned}
\mathbf{a}_\theta(t,x,\xi)\cdot\xi &\le \overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)+\overline{m}^*\left(\left|\tfrac{c_{\mathbf{a}}}{2}\mathbf{a}_\theta(t,x,\xi)\right|\right)\\
&\le \overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)+c_{\mathbf{a}}\overline{m}^*\left(\left|\tfrac{1}{2}\mathbf{a}_\theta(t,x,\xi)\right|\right),
\end{aligned}
$$

but on the other hand

$$
\begin{aligned}
\mathbf{a}_\theta(t,x,\xi)\cdot\xi &\ge M(t,x,\xi)+\theta\overline{m}(|\xi|)+\theta\overline{m}^*\left(\left|\nabla_\xi m(\xi)\right|\right)\\
&\ge c_{\mathbf{a}}M^*(t,x,\mathbf{a}(t,x,\xi))+0+\theta\overline{m}^*\left(\left|\nabla_\xi m(\xi)\right|\right)\\
&\ge 2c_{\mathbf{a}}\left[\tfrac{1}{2}\overline{m}^*\left(|\mathbf{a}(t,x,\xi)|\right)+\tfrac{1}{2}\overline{m}^*\left(\left|\theta\nabla_\xi m(\xi)\right|\right)\right]\\
&\ge 2c_{\mathbf{a}}\overline{m}^*\left(\tfrac{1}{2}\left|\mathbf{a}_\theta(t,x,\xi)\right|\right),
\end{aligned}
$$

where we used Jensen's inequality and (4.120).

Therefore, we get $c_{\mathbf{a}}\overline{m}^*\left(\frac{1}{2}\left|\mathbf{a}_\theta(t,x,\xi)\right|\right)\le\overline{m}\left(\left|\frac{2}{c_{\mathbf{a}}}\xi\right|\right)$. Then by the strict monotonicity of $\overline{m}^*$, the estimate

$$
|\mathbf{a}_\theta(t,x,\xi)|\le\tfrac{2}{c_{\mathbf{a}}}(\overline{m}^*)^{-1}\left(\overline{m}\left(\left|\tfrac{2}{c_{\mathbf{a}}}\xi\right|\right)\right)
$$

and Corollary 4.2.12 gives the claim, i.e. the existence of a solution

$$
u^\theta\in C([0,T];L^2(\Omega))\cap L^1(0,T;W_0^{1,1}(\Omega))\quad\text{with}\quad\nabla u^\theta\in L_m(\Omega_T;\mathbb{R}^N).
$$

Now, we shall show uniform boundedness of $\{u^\theta\}_\theta$.

A priori estimates. By the energy equality (4.123), *(A2p)*, and (4.117) we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega(u^\theta(\tau))^2\,\mathrm{d}x+\int_{\Omega_\tau}M(t,x,\nabla u^\theta)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\Omega_\tau}\theta m(\nabla u^\theta)+\theta m^*(\nabla_\xi m(\nabla u^\theta))\,\mathrm{d}x\,\mathrm{d}t\\
&\le\int_{\Omega_\tau}fu^\theta\,\mathrm{d}x\,\mathrm{d}t+\tfrac{1}{2}\int_\Omega(u_0)^2\,\mathrm{d}x.
\end{aligned}
\tag{4.124}
$$

We estimate further the right-hand side using the Fenchel–Young inequality (2.33) and the modular Poincaré inequality (Theorem 9.3). For this let us consider any homogeneous and isotropic N-function $b:[0,\infty)\to[0,\infty)$ such that

$$
b(s)\le\tfrac{1}{2c_P^2}m_1(s),
$$

where m_1 is the minorant of M from the definition of an N-function, c_P^2 is the constant from the modular Poincaré inequality for b. Then on the right-hand side of (4.124) we have

$$\begin{aligned}\int_{\Omega_\tau} f u^\theta \,\mathrm{d}x\,\mathrm{d}t &\le \int_{\Omega_\tau} b^*(\|f\|_{L^\infty(\Omega)}/c_1^P)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau} b(c_1^P|u^\theta|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le |\Omega_T| b^*(\|f\|_{L^\infty(\Omega)}/c_1^P) + c_P^2 \int_{\Omega_\tau} b(|\nabla u^\theta|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le c(\Omega,T,f,N) + \frac{1}{2}\int_{\Omega_\tau} M(t,x,\nabla u^\theta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Consequently, we infer from (4.124) that

$$\begin{aligned}\frac{1}{2}\int_\Omega (u^\theta(\tau))^2\,\mathrm{d}x &+ \int_{\Omega_\tau} \frac{1}{2} M(t,x,\nabla u^\theta)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\Omega_\tau} \theta m(\nabla u^\theta) + \theta m^*(\nabla_\xi m(\nabla u^\theta))\,\mathrm{d}x\,\mathrm{d}t\\ &\le c(\Omega,T,f,N) + \frac{1}{2}\|u_0\|_{L^2(\Omega)} = c(\Omega,T,f,N,u_0) = \widetilde{C}.\end{aligned}$$

When we take into account that τ is arbitrary, this observation leads to a priori estimates

$$\sup_{\tau\in[0,T]} \|u^\theta(\tau)\|^2_{L^2(\Omega)} \le \widetilde{C}, \tag{4.125}$$

$$\int_{\Omega_T} M(t,x,\nabla u^\theta)\,\mathrm{d}x\,\mathrm{d}t \le 2\widetilde{C}, \tag{4.126}$$

$$\int_{\Omega_T} \theta m^*(\nabla_\xi m(\nabla u^\theta))\,\mathrm{d}x\,\mathrm{d}t \le \widetilde{C}. \tag{4.127}$$

Moreover, *(A2p)* implies then

$$c_{\mathbf{a}} \int_{\Omega_T} M^*(t,x,\mathbf{a}(t,x,\nabla u^\theta))\,\mathrm{d}x\,\mathrm{d}t \le 2\widetilde{C}. \tag{4.128}$$

And thus, the uniform boundednesses of the claim follow. □

Let us prepare some easy observations that will turn out to be instrumental in letting $\theta \to 0$ in (4.122).

Lemma 4.2.14 *Under assumptions of Proposition 4.2.13, for any $\varphi : \Omega_T \to \mathbb{R}^d$ such that $\varphi \in L^\infty(\Omega_T;\mathbb{R}^d)$, we have*

$$\lim_{\theta\to 0} \int_{\Omega_T} \theta \nabla_\xi m(\nabla u^\theta)\varphi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Proof. We motivate it by the Vitali convergence theorem (Theorem 8.23). For its application we need to infer uniform integrability and convergence in measure to 0. We point out that since m^* is an N-function, for $\theta \in (0,1)$ we have $m^*(\theta\cdot) \le \theta m^*(\cdot)$. This together with the $L^1(\Omega_T)$-bound (4.127) for $\theta m^*(\nabla_\xi m(\nabla u^\theta))$, which is uniform with respect to θ, we get an $L^1(\Omega_T)$-bound for $\{m^*(\theta\nabla_\xi m(\nabla u^\theta))\}_\theta$. Therefore, Theorem 3.4.2 implies the uniform integrability of $\{\theta\nabla_\xi m(\nabla u^\theta)\}_\theta$. In order to show convergence in measure to 0 in (4.136), we suppose the opposite, i.e. that there exist $c_1, c_2 > 0$ such that

$$\inf_{\theta} |\{(t,x) :\ \theta|\nabla_\xi m(\nabla u^\theta)| > c_1\}| > c_2.$$

In fact, on the set $\{(t,x) :\ \theta|\nabla_\xi m(\nabla u^\theta)| > c_1\}$ we have

$$\theta\overline{m}^*\left(\frac{c_1}{\theta}\right) \le \theta\overline{m}^*\left(|\nabla_\xi m(\nabla u^\theta)|\right).$$

By the fact that m^* is an N-function we also have

$$\theta\overline{m}^*\left(\frac{c_1}{\theta}\right) = c_1 \frac{\overline{m}^*\left(\frac{c_1}{\theta}\right)}{\frac{c_1}{\theta}} \xrightarrow[\theta\to 0]{} \infty.$$

On the other hand,

$$c_2\theta\overline{m}^*\left(\frac{c_1}{\theta}\right) \le \int_{\Omega_T} \theta m^*\left(\nabla_\xi m(\nabla u^\theta)\right)\,\mathrm{d}x\,\mathrm{d}t < \widetilde{C},$$

where the last inequality is a consequence of the a priori estimate (4.127). Since the last two displays yield a contradiction, we can apply the Vitali convergence theorem (Theorem 8.23) to justify the convergence in the claim. □

The next result deals with the time derivative of u^θ and will be used to deduce the pointwise convergence.

Lemma 4.2.15 *Under the assumptions of Proposition 4.2.13, for every $\theta > 0$, we have $\partial_t u^\theta \in \left(W^1E_m(\Omega_T)\right)^*$ and for every $\varphi \in W^1E_m(\Omega_T)$ we have the following inequality*

$$\int_{\Omega_T} \partial_t u^\theta\,\varphi\,\mathrm{d}x\,\mathrm{d}t \le C\|\varphi\|_{W^1L_m(\Omega_T)}, \tag{4.129}$$

where the constant C is independent of θ.

Proof. First, let $\varphi \in C_0^\infty((0,T)\times\Omega)$. By the weak formulation of (4.121) we have

$$-\int_{\Omega_T} u^\theta(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_{\Omega_T} \mathbf{a}(t,x,\nabla u^\theta)\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x$$
$$+\int_{\Omega_T} \theta\nabla_\xi m(|\nabla u^\theta|)\cdot\nabla\varphi\,\mathrm{d}t\,\mathrm{d}x = \int_{\Omega_T} f(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

We can estimate the left-hand side using Hölder's inequality (Lemma 3.1.15) to get

$$\left|\int_{\Omega_T} u^\theta(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| \le 2\left\|\mathbf{a}(t,x,\nabla u^\theta)\right\|_{L_{m^*}(\Omega_T;\mathbb{R}^N)}\left\|\nabla\varphi\right\|_{L_m(\Omega_T;\mathbb{R}^N)}$$
$$+2\theta\left\|\nabla_\xi m(|\nabla u^\theta|)\right\|_{L_{m^*}(\Omega_T;\mathbb{R}^N)}\left\|\nabla\varphi\right\|_{L_m(\Omega_T;\mathbb{R}^N)}$$
$$+2\,|\Omega_T|\,m^*\left(\|f\|_\infty\right)\left\|\varphi\right\|_{L_m(\Omega_T;\mathbb{R}^N)}.$$

Since $M(t,x,\xi) \le m(\xi)$ by (4.119), Lemma 2.1.37 implies that $m^*(\xi) \le M^*(t,x,\xi)$ and hence we have

$$\left\|\mathbf{a}(t,x,\nabla u^\theta)\right\|_{L_{m^*}(\Omega_T;\mathbb{R}^N)} \le \left\|\mathbf{a}(t,x,\nabla u^\theta)\right\|_{L_{M^*}(\Omega_T;\mathbb{R}^N)}.$$

Therefore, we can use uniform estimates from Proposition 4.2.13 and the modular Poincaré inequality (Theorem 9.3) to get (4.129) for $\varphi \in C_0^\infty((0,T)\times\Omega)$. The general case follows by the density of $C_0^\infty((0,T)\times\Omega)$ in $W_0^1E_m(\Omega_T)$, cf. [175]. □

Lemma 4.2.16 *Under assumptions of Proposition 4.2.13, the sequence* $\{u^\theta\}_{\theta\in(0,1]}$ *is relatively compact in* $L^1(0,T;L^1(\Omega))$*. In particular, it has a subsequence converging a.e. in* Ω_T*.*

Proof. We apply the Aubin–Lions lemma (Theorem 8.50) with

$$X_0 = W_0^{1,1}(\Omega),\;\; X = L^1(\Omega),\;\; \text{and}\;\; X_1 = W^{-2,r}(\Omega)\;\; \text{for some } r > N.$$

Then $W_0^{2,r}(\Omega)$ is continuously embedded in $C^1(\Omega)$, cf. [171, Corollary 7.11]. By the Rellich–Kondrachov theorem (Theorem 8.48), X_0 is compactly embedded in X. If $f \in L^1(\Omega)$ and $\varphi \in W_0^{2,r}(\Omega)$, then

$$\left|\int_\Omega f\varphi\,\mathrm{d}x\right| \le \|f\|_{L^1}\|\varphi\|_{L^\infty} \le C\|f\|_{L^1}\|\varphi\|_{W^{2,r}},$$

for some constant $C > 0$, therefore X is continuously embedded in X_1.

By Proposition 4.2.13 the sequence $\{u^\theta\}_\theta$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{\nabla u^\theta\}_\theta$ is uniformly bounded in $L_{M^*}(\Omega_T)$. In particular,

$$\left\{u^\theta\right\}_{\theta\in(0,1]} \text{ is uniformly bounded in } L^1(0,T;W_0^{1,1}(\Omega)) \text{ and } L^2(0,T;L^1(\Omega)).$$

Let $\varphi \in L^\infty(0,T;W_0^{2,r}(\Omega))$ with $\|\varphi\|_{L^\infty(0,T;W_0^{2,r}(\Omega))} \le 1$. Notice that $\left\{|\partial_t u^\theta\varphi|\right\}_\theta$ is bounded in L^1 uniformly with respect to φ and $\theta \in (0,1]$. Indeed, by the choice of r, there exists a constant $C > 0$ such that $|\varphi| \le C$ and $|\nabla\varphi| \le C$, so in particular, $\varphi \in W_0^1E_m(\Omega_T)$ and if suffices to apply Lemma 4.2.15. This shows that

$$\left\{\partial_t u^\theta\right\}_\theta \text{ is uniformly bounded in } L^1(0,T;W^{-2,r}(\Omega)),$$

thus the Aubin–Lions lemma (Theorem 8.50) implies that

$$\left\{u^\theta\right\}_{\theta\in[0,1)}$$

is relatively compact in $L^1(0,T;L^1(\Omega))$. □

We prove the existence of a weak solution for a non-regularized problem with bounded data by passing to the limit as $\theta \to 0$ in the regularized problem (4.121). Note that to get weak solutions, we exploit the *integration by parts formula* from Theorem 4.2.10.

Theorem 4.2.17 (Existence of weak solutions to a parabolic problem) *Suppose* $[0,T)$ *is a finite interval,* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^N$*,* $N > 1$*,* $g \in L^\infty(\Omega_T)$*,* $u_0 \in L^\infty(\Omega)$*, and the function* **a** *satisfies assumptions (A1p)–(A3p) with an N-function* $M : [0,T)\times\Omega\times\mathbb{R}^N \to [0,\infty)$*. Assume further that* M *satisfies the condition (Mp) or (Mp)*$_p$*. Then there exists a weak solution to the problem*

$$\begin{cases} \partial_t u - \operatorname{div} \mathbf{a}(t,x,\nabla u) = g & \text{in } \Omega_T, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{4.130}$$

Namely, there exists a $u \in V_T^M(\Omega)$ *such that for any* $\varphi \in C_c^\infty([0,T)\times\Omega)$

$$-\int_{\Omega_T} u\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u(0)\varphi(0)\,\mathrm{d}x + \int_{\Omega_T} \mathbf{a}(t,x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} g\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{4.131}$$

Proof. We provide the proof in the case of M satisfying the condition *(Mp)* or *(Mp)$_p$* via an approximation coming from Theorem 4.2.6 and Proposition 5.3.9.

We apply Proposition 4.2.13 and let $\theta \to 0$. Uniform estimates provided therein imply that there exist a subsequence of $\theta \to 0$ and a function $u^\theta \in V_T^M(\Omega)$ such that

$$u^\theta \overset{*}{\rightharpoonup} u \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.132}$$

$$\nabla u^\theta \overset{*}{\rightharpoonup} \nabla u \quad \text{weakly-}* \text{ in } L_M(\Omega_T;\mathbb{R}^N), \tag{4.133}$$

with some $u \in V_T^{M,\infty}(\Omega)$ and there exists an $\boldsymbol{\alpha} \in L_{M^*}(\Omega_T;\mathbb{R}^N)$ such that

$$\mathbf{a}(\cdot,\cdot,\nabla u^\theta) \overset{*}{\rightharpoonup} \boldsymbol{\alpha} \quad \text{weakly-}* \text{ in } L_{M^*}(\Omega_T;\mathbb{R}^N). \tag{4.134}$$

Identification of the limit $\boldsymbol{\alpha}$. Uniform estimates. We need to show

$$\limsup_{\theta\to 0} \int_{\Omega_T} \mathbf{a}(t,x,\nabla u^\theta)\cdot\nabla u^\theta \,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} \boldsymbol{\alpha}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t. \tag{4.135}$$

The aim now is to pass to the limit as $\theta \searrow 0$ in the regularized problem (4.122) and (4.123). In the first term on the left-hand side therein, due to Lemma 4.2.16, up to a subsequence we have

$$\lim_{\theta\searrow 0} \int_{\Omega_T} u^\theta\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, we need to motivate that on the left-hand side one of the terms vanishes. Note that by Lemma 4.2.14

$$\theta|\nabla_\xi m(\nabla u^\theta)| \xrightarrow[\theta\to 0]{} 0 \qquad \text{in } L^1(\Omega_T). \tag{4.136}$$

Hence, we can pass to the limit in the weak formulation of the regularized problem (4.122). By (4.134) we infer

$$-\int_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_0\varphi(0)\,\mathrm{d}x + \int_{\Omega_T} \boldsymbol{\alpha}\cdot\nabla\varphi\,\mathrm{d}x = \int_{\Omega_T} g\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{4.137}$$

for all $\varphi \in C_c^\infty([0;T)\times\Omega)$.

We apply the integration by parts formula from Theorem 4.2.10 applied to (4.137) with $\boldsymbol{\mathcal{A}} = \boldsymbol{\alpha}$, $F = g$, and $h(\cdot) = T_k(\cdot)$, i.e. we have

$$-\int_{\Omega_T}\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\partial_t\xi\,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T}\boldsymbol{\alpha}\cdot\nabla(T_k(u)\xi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} gT_k(u)\xi\,\mathrm{d}x\,\mathrm{d}t,$$

for every $\xi \in C_c^\infty([0,T)\times\overline{\Omega})$. Let the two-parameter family of functions $\vartheta^{\tau,r} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\vartheta^{\tau,r}(t) := \left(\omega_r * \mathbb{1}_{[0,\tau)}\right)(t),$$

where ω_r is a standard regularizing kernel, that is $\omega_r \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\omega_r \subset (-r,r)$. Note that $\operatorname{supp}\vartheta^{\tau,r} = [-r,\tau+r)$. In particular, for every τ there exists an r_τ such that for all $r < r_\tau$ we have $\vartheta^{\tau,r} \in C_c^\infty([0,T))$. By taking $\xi(t,x) = \vartheta^{\tau,r}(t)$ in the integration by parts formula above, we get

$$\begin{aligned}
-\int_{\Omega_T}&\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\partial_t\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{\Omega_T}\boldsymbol{\alpha}\cdot\nabla(T_k(u))\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} gT_k(u)\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.138}$$

On the right-hand side we integrate by parts obtaining

$$-\int_{\Omega_T}\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\partial_t\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\partial_t\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t.$$

Then we pass to the limit as $r \to 0$, apply Fubini's theorem, and integrate over the time variable

$$\begin{aligned}
\lim_{r\to 0}\int_{\Omega_T}\partial_t\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_\tau}\partial_t\left(\int_{u_0}^{u(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega}\left(\int_{u_0}^{u(\tau,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x.
\end{aligned}$$

Passing to the limit as $r \to 0$ in (4.138) and using the content of the last display we get the following for a.e. $\tau \in [0,T)$

$$\begin{aligned}
\int_{\Omega}&\left(\int_0^{u(\tau,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_0(x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x \\
&= -\int_{\Omega_\tau}\boldsymbol{\alpha}\cdot\nabla T_k(u)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau} gT_k(u)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By applying the Lebesgue monotone convergence theorem for $k \to \infty$ we obtain

$$\frac{1}{2}\|u(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2 = -\int_{\Omega_\tau}\boldsymbol{\alpha}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau} gu\,\mathrm{d}x\,\mathrm{d}t.$$

Notice that by considering the energy equality (4.123) in the first term on the left-hand side, taking into account the weak lower semi-continuity of the L^2-norm, and (4.132) we see that

$$\|u^\theta(\tau)\|^2_{L^2(\Omega)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\tau-\epsilon}^{\tau}\|u^\theta(s)\|^2_{L^2(\Omega)}ds \geq \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\tau-\epsilon}^{\tau}\|u(s)\|^2_{L^2(\Omega)}ds = \|u(\tau)\|^2_{L^2(\Omega)}.$$

After dropping the nonnegative term $\int_{\Omega_T}\theta\nabla_\xi m(\nabla u^\theta)\cdot\nabla u^\theta\,\mathrm{d}x\,\mathrm{d}t$ in (4.123), the passage to the limit as $\theta\searrow 0$ is justified, so we get

$$\frac{1}{2}\|u(\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \limsup_{\theta\searrow 0}\int_{\Omega_T}\mathbf{a}(t,x,\nabla u^\theta)\cdot\nabla u^\theta\,\mathrm{d}x\,\mathrm{d}t \leq \int_{\Omega_T} gu\,\mathrm{d}x\,\mathrm{d}t. \tag{4.139}$$

Thus, (4.135) follows.

Identification of the limit $\boldsymbol{\alpha}$. Conclusion by the monotonicity trick. Let us concentrate on proving that

$$\mathbf{a}(t,x,\nabla u) = \boldsymbol{\alpha} \qquad \text{a.e.} \quad \text{in} \quad \Omega_T. \tag{4.140}$$

Since $\mathbf{a}$ is monotone by *(A3p)*, we have that

$$(\mathbf{a}(t,x,\nabla u^\theta) - \mathbf{a}(t,x,\eta))\cdot(\nabla u^\theta - \eta) \geq 0$$

a.e. in Ω_T, and for any $\eta\in L^\infty(\Omega_T;\mathbb{R}^N)\subset E_M(\Omega_T;\mathbb{R}^N)$. By *(A2p)* we see that $\mathbf{a}(\cdot,\cdot,\eta)\in E_{M^*}(\Omega_T,\mathbb{R}^N)$. Moreover, having (4.133), (4.134), and (4.135) we pass to the limit as $\theta\searrow 0$ to conclude that

$$\int_{\Omega_T}(\boldsymbol{\alpha} - \mathbf{a}(t,x,\eta))\cdot(\nabla u - \eta)\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{4.141}$$

Then Theorem 4.2.11 with

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\alpha} \qquad \text{and} \qquad \boldsymbol{\xi} = \nabla u$$

implies (4.140).

Conclusion of the proof of Theorem 4.2.17. We can pass to the limit in the weak formulation of the bounded regularized problem (4.122), because of (4.132), (4.133), (4.134), and (4.140). In turn we get the existence of $u\in V_T^M(\Omega)$ satisfying

$$-\int_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u(0)\varphi(0)\,\mathrm{d}x + \int_{\Omega_T}\mathbf{a}(t,x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} g\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi\in C_c^\infty([0;T)\times\Omega)$, i.e. (4.131), which ends the proof of existence. □

Remark 4.2.18. The result presented in Theorem 4.2.17 has been generalized in two directions:

(i) It is possible to avoid mollification of a test function in time, inasmuch as the equation itself can provide information on the regularity of the time derivative u_t, i.e. $u_t \in L^1(0,T;X)$ with X being some negative Sobolev space. There are definite advantages of this approach, as the assumption of regularity of an N-function with respect to time is not needed anymore.
(ii) Similarly as in the elliptic case, see Remark 3.7.11, we can change the exponent under the arguments of the function Θ in conditions (Mp) and $(Mp)_p$. Again in the case when $p < N$ and $N > 1$ such a condition is less restrictive.

For both of these generalizations of Theorem 4.2.17 we refer to Theorem 1.23 in [63], where conditions (Mp) and $(Mp)_p$ are replaced with the following condition.

$(Mp)^*$ There exists a function $\Theta : (0,T)\times[0,1]\times[0,\infty) \to [0,\infty)$ which is nondecreasing with respect to the second and the third variable, and moreover there exist $\xi_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that for every cube $Q \subset \mathbb{R}^d$ with edge $\delta \in (0,\delta_0)$ and all $\xi \in \mathbb{R}^d$ with $|\xi| > \xi_0$ we have

$$\frac{M(t,x,\xi)}{M_Q^{**}(t,\xi)} \le \Theta(t,\delta,|\xi|), \tag{4.142}$$

where M_Q^{**} is the second convex conjugate to M_Q.

The corresponding isotropic conditions (Mp^i) and $(Mp^i)_p$ are replaced with the following condition.

$(Mp^i)^*$ There exists a function $\Theta : (0,T)\times[0,1]\times[0,\infty) \to [0,\infty)$ which is nondecreasing with respect to the second and third variable such that

$$\limsup_{\delta\to 0^+} \Theta(t,\delta,C\delta^{-1}) \text{ is bounded uniformly in time } t \in (0,T) \tag{4.143}$$

and

$$\frac{M(t,x,r)}{M(t,y,r)} \le \Theta(t,|x-y|,r).$$

Remark 4.2.19. Similarly as in case of elliptic problems, weak solutions to parabolic equations have been considered under assumption $M^* \in \Delta_2$ and M independent of time, see Lemma 4.3 in [188].

Chapter 5
Renormalized Solutions

5.1 Problems With Irregular Data

We concentrate here on second-order elliptic and parabolic partial differential equations of a simple structure, but with merely integrable data. Irregular data influence the choice of notion of solution, whereas the general growth of the operator complicates the approximate procedure used in order to obtain the solution.

5.1.1 Consequences of mere integrability of data

Suppose **a** has growth described by the use of an N-function as in Section 3.8.2 and applied as in the study of weak solutions in Chapter 4. To study the problem

$$-\operatorname{div}\mathbf{a}(x,\nabla u)=f\in L^1(\Omega)\quad\text{and}\quad \partial_t u-\operatorname{div}\mathbf{a}(t,x,\nabla u)=f\in L^1(\Omega_T),\tag{5.1}$$

a special notion of solution has to be employed. To explain why, let us consider the classical Poisson equation on bounded $\Omega\subset\mathbb{R}^N$, namely

$$\begin{cases}-\Delta u=f & \text{in }\ \Omega,\\ \quad u=0 & \text{on }\ \partial\Omega.\end{cases}\tag{5.2}$$

If the boundary $\partial\Omega$ is smooth enough, a solution to the above problem can be expressed by means of the Green function G via the formula

$$u(x)=\int_\Omega f(y)G(x,y)\,\mathrm{d}y.\tag{5.3}$$

When the problem is posed on the unit ball and $N>2$, i.e. $\Omega=B(0,1)$, we make use of the fundamental solution

$$\Gamma(z)=c(N)|z|^{2-N}$$

and get that the Green function is given by

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5_5

$$G(x,y) = \Gamma(y-x) - \Gamma\Big(|x|\big(y - \tfrac{x}{|x|^2}\big)\Big),$$

so apparently u obtained by (5.3) solves (5.2) in the distributional sense, but it does *not* belong to the natural energy space $W^{1,2}(\Omega)$, when f is merely integrable. Thus, it is not a weak solution and there is no weak solution to this problem. To admit arbitrary $f \in L^1(\Omega)$ one needs to consider a generalized notion of solution. The easy way would be to analyze distributional solutions, but they can be deprived of basic good properties. In particular, we cannot ensure uniqueness. The classical example of non-uniqueness of distributional solutions comes from Serrin [292]. He shows that a linear homogeneous equation of the type $\operatorname{div}(A(x)Du) = 0$ defined on a ball, with a strongly elliptic and bounded, measurable matrix $A(x)$, has (at least) two distributional solutions. One of them belongs to the natural energy space $W^{1,2}(B(0,1))$, whereas the second one does not and is called a pathological solution.

The point is then to distinguish the solutions having a proper interpretation and exclude the wild ones. An interesting special notion of solution, besides its existence, is that it has to satisfy reasonable, say physical, conditions that ensure uniqueness. To relax the classical requirement for a solution to (5.2) to belong to $W^{1,2}(\Omega)$, we will expect to control the energy of our solutions by conditions of the form

$$\int_{\{l \le |u| < l+1\}} |\nabla u|^2 \, \mathrm{d}x \xrightarrow[l\to\infty]{} 0.$$

The problem with uniqueness appearing in the linear equation is obviously shared by the p-harmonic problem

$$-\Delta_p u = f \in L^1(\Omega),$$

as well as its anisotropic, Orlicz, and Musielak–Orlicz generalizations described in Section 3.8. Indeed, when on the right-hand side the data is merely integrable, the weak formulations of $(5.1)_1$, i.e.

$$\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x,$$

cannot be expected to hold for every

$$\varphi \in V_0^M(\Omega) = \{\phi \in W_0^{1,1}(\Omega) : \ \nabla\phi \in L_M(\Omega;\mathbb{R}^N)\},$$

and in the parabolic case, the weak formulation of $(5.1)_2$ reads

$$-\int_{\Omega_T} (u-u_0)\partial_t\varphi \, \mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathbf{a}(t,x,\nabla u)\cdot\nabla\varphi \, \mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\,\varphi \, \mathrm{d}x\,\mathrm{d}t,$$

which *cannot* hold for every

$$\varphi \in V_T^M(\Omega) = \{\phi \in L^1(0,T;W_0^{1,1}(\Omega)) : \ \nabla\phi \in L_M(\Omega_T;\mathbb{R}^N)\}.$$

Instead, we expect control on radiation of energy

$$\int_{\{l\le|u|<l+1\}} \mathbf{a}(x,\nabla u)\cdot\nabla u \;\mathrm{d}x \xrightarrow[l\to\infty]{} 0 \tag{5.4}$$

or

$$\int_{\{l\le|u|<l+1\}} \mathbf{a}(t,x,\nabla u)\cdot\nabla u \;\mathrm{d}x\,\mathrm{d}t \xrightarrow[l\to\infty]{} 0. \tag{5.5}$$

5.1.2 Various notions of solutions

We can study weak solutions to (5.1), when the datum f belongs to the dual space which we expect the solution to belong to. There are a few already classical notions of solutions introduced in order to consider less regular data. DiPerna and Lions introduced the notion of *renormalized solutions* in [120] in their investigation on the Boltzmann equation. For further foundations of the theory we refer to Boccardo, Giachetti, Diaz, and Murat [47] and Murat [260]. Other seminal ideas for problems with data below duality come from Boccardo and Gallouët [43, 44], where the so-called *solutions obtained as a limit of approximation*, SOLA for short, are considered. Finally, the *entropy solutions* are considered starting from cornerstones laid by Benilan, Boccardo, Gallouët, Gariepy, Pierre, and Vazquéz [31], Boccardo, Gallouët, and Orsina [46], and Dall'Aglio [103].

Recently attention has been paid to the notion of approximable solutions, somehow merging the ideas of SOLA and entropy solutions, see [95] and also [9, 78, 85, 84]. Some of the mentioned results are relevant also in the context of measure data problems. Let us refer to e.g. [44, 46, 102, 101, 124, 132, 261, 73, 78, 77, 85, 84] for elliptic results and e.g. [39, 58, 273, 274] for parabolic results. The uniqueness in the case of arbitrary measure data is a long-standing open problem. Namely, sharp conditions for a measure to ensure uniqueness are not known. Nevertheless, below we restrict ourselves to the L^1-data equations avoiding this challenge.

An interesting feature is that the mentioned, distinct kinds of notion of solutions can coincide. See [216] for a result for elliptic problems involving nonnegative measure datum and the p-Laplace operator, [127] for the equivalence between entropy and renormalized solutions to parabolic problems with polynomial growth, [334, 335] for the corresponding results in the variable exponent and the Orlicz settings, [73] for the equivalence between SOLA and renormalized solutions in the reflexive Musielak–Orlicz case, and [233] for the equivalence between entropy and renormalized solutions to L^1-data problems in the non-reflexive Musielak–Orlicz spaces. It would be interesting to find the regime where the notions of solutions do not coincide.

Renormalized solutions. In the elliptic setting the foundations of the study of renormalized solutions, providing results for operators with polynomial growth, were laid by [47, 103, 260]. In the parabolic setting, renormalized solutions were studied first in [36, 37, 40, 41, 42] and further in [38, 126, 127, 273, 274]. These studies were continued under weaker assumptions on the data [39, 58, 102].

For recent existence results for elliptic problems we refer to [6, 8, 27, 32, 125, 146, 186, 187, 193, 230, 239, 78]. In [146, 193, 239, 73] isotropic, separable and reflexive Musielak–Orlicz spaces are employed, [27] concerns anisotropic variable exponent spaces, [125] studies separable, but not reflexive Musielak–Orlicz spaces, while [230] anisotropic, but separable and reflexive Orlicz spaces. Renormalized solutions to elliptic problems in Orlicz spaces are explored in [6, 8, 32], while in Musielak–Orlicz spaces in [179, 186, 187, 109, 73, 233].

As for parabolic problems in the variable exponent setting we refer to [30, 234, 334] and for the model of thermoviscoelasticity to [69]. For very recent results on entropy and renormalized solutions, we refer also to [69, 154, 242, 335]. This issue in parabolic problems in non-reflexive Orlicz–Sobolev spaces is studied in [189, 242, 282, 335], while in the inhomogeneous and non-reflexive Musielak–Orlicz spaces in [188] under certain growth conditions on the modular function and in [79, 81] under regularity restrictions.

5.1.3 Comments on the scheme of the proof of existence

We want to present the analysis on (5.1), developing the study of [179, 81]. In fact, we provide the existence and uniqueness of renormalized solutions to (5.1) under assumptions described in Section 3.8.2, but when no growth restrictions of doubling type are imposed on the anisotropic modular function M and when the operator $\mathbf{a}$ is weakly monotone. Notice that our research includes the fully anisotropic Orlicz setting under no growth conditions of doubling type, since then the regularity assumption is trivially satisfied. What is more, in order to obtain existence, this assumption can be simply skipped not only in the Orlicz case, but also in reflexive spaces, that is, among others, in variable exponent, weighted Sobolev and double phase spaces, no matter how irregular the exponent or the weights are. The lack of precise control on the growth of the leading part of the operator, together with the low integrability of the right-hand side triggers noticeable difficulties in the study of convergence of approximation. An additional consequence of resigning from imposing the Δ_2-condition on the conjugate of the modular function is that it complicates the meaning of the dual pairing, see Chapter 4. Since $(L_M)^* \neq L_{M^*}$ (see Theorem 3.5.3), we need the modular approximation result of Theorem 3.7.7 (for the elliptic case) or Theorems 4.2.6 and 5.3.12 (for the parabolic case) from the very beginning – in order to get a priori estimates.

Let us summarize briefly the scheme of the proof of existence, which is the same in the elliptic and the parabolic case. Initially we show the existence of weak solutions to the regularized problem with bounded data and then, using the Browder–Minty monotonicity trick, the existence of weak solutions to problems involving the original (non-regularized) operator, still with bounded data. Passing to L^1-data problems we establish a priori estimates and the radiation control condition relating to (5.4)–(5.5) (see later *(R3e)*, resp. *(R3p)*), but for u_s – solutions to problems with truncated data. From these results we infer certain types of convergence of symmetric truncations of a solution to problems with truncated data, i.e. $\{T_k(u_s)\}_{s>0}$. The next step is to identify $\mathbf{a}(x,\nabla T_k(u))$ as the weak-$*$ limit in L_{M^*} of $\{\mathbf{a}(x,\nabla T_k(u_s))\}_{s>0}$ and use the monotonicity trick. Finally we conclude the proof of existence of renormalized solutions motivating the weak L^1-convergence of $\{\mathbf{a}(x,\nabla T_k(u_s))\cdot\nabla T_k(u_s)\}_{s>0}$ via the Young measures. In the end we provide uniqueness as a result of the method of test functions.

Besides obvious technical complications, the main difference between the elliptic and the parabolic approach is that in the parabolic case we exploit the notion of a renormalized solution to get the existence of a weak solution to a regularized problem with truncated data.

No density property is necessary in the reflexive spaces. The proofs are formulated in the case when the modular function is regular enough to ensure the absence of Lavrentiev's phenomenon. Nonetheless, in the elliptic case as well as in the parabolic case when $M = M(x,\xi)$, for existence we can simply bypass this restriction provided are dealing with reflexive spaces, i.e. whenever both $M, M^* \in \Delta_2$. This is justified since the method keeps all of the limits in the strong closure of the smooth functions and Mazur's lemma (Theorem 8.32) ensures the existence of a strongly converging finite affine combination of the weakly converging sequence. Note that all spaces L_M equipped with doubling M from Example 2.3.2 are reflexive, including variable exponent spaces without regularity assumptions and double-phase spaces without a closeness condition.

Let us notice further that the regularity condition is necessary only in the approximation. The entire proof of existence and uniqueness in the elliptic case as well as in the parabolic case when $M = M(x,\xi)$ works assuming only that M is an N-function. For full generality in our study of parabolic problems, when $M = M(t,x,\xi)$, we employ a far more delicate approximation which holds under the same balance condition.

5.2 Renormalized Solutions to Elliptic Problems

5.2.1 Formulation of the problem

Recall that the truncation T_k is defined in (3.55).

Definition 5.2.1 (Renormalized solutions to an elliptic equation). We call a function u a *renormalized solution* to (4.1) if it satisfies the following conditions.

(R1e) $u:\Omega\to\mathbb{R}$ is measurable and for each $k>0$

$$T_k(u)\in V_0^M(\Omega) \qquad \text{and} \qquad \mathbf{a}(x,\nabla T_k(u))\in L_{M^*}(\Omega;\mathbb{R}^N).$$

(R2e) For every compactly supported $h\in W^{1,\infty}(\mathbb{R})$ and all $\varphi\in V_0^M(\Omega)\cap L^\infty(\Omega)$ we have

$$\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla(h(u)\varphi)\;\mathrm{d}x = \int_\Omega f h(u)\varphi\;\mathrm{d}x.$$

(R3e) $\int_{\{l<|u|<l+1\}} \mathbf{a}(x,\nabla u)\cdot\nabla u \,\mathrm{d}x \to 0$ as $l \to \infty$.

Remark 5.2.2. Let us note that condition *(R3e)* restricts the energy of admissible solutions to those having the expected meaning as described in Section 5.1.1. This condition will be a key tool to obtain uniqueness of renormalized solutions to L^1-data problems.

We prove the existence of renormalized solutions to the general elliptic equation (4.1) with merely integrable data. Recall that assumptions on the operator *(A1e)–(A3e)* are given in Section 4.1.1 (their generalization is discussed in Section 3.8.2), moreover for modular density of smooth functions via Theorem 3.7.7 we need *(Me)* given in Section 3.7.1 or *(Me)*$_p$ from Section 3.7.2.

Theorem 5.2.3 (Existence of renormalized solutions) *Suppose* $\Omega \subset \mathbb{R}^N$, $N > 1$, $f \in L^1(\Omega)$, *and a function* $\mathbf{a}: \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ *satisfy assumptions (A1e)–(A3e) with an N-function* $M: \Omega\times\mathbb{R}^N \to [0,\infty)$. *Assume further that at least one of the following assumptions holds:*

(i) *M satisfies the condition (Me) or (Me)$_p$;*
(ii) $M, M^* \in \Delta_2$.

Then there exists a renormalized weak solution to the problem (4.1)*, i.e. a function u satisfying (R1e)–(R3e) of Definition 5.2.1.*

Remark 5.2.4. Similarly as in the case of weak solutions, renormalized solutions to elliptic equations have been considered under the assumption $M^* \in \Delta_2$, see Theorem 2.6 in [109].

Remark 5.2.5 (Uniqueness of renormalized solutions). We show also that in the case of (i), if the operator is strictly monotone then the renormalized solution from Theorem 5.2.3 is unique. Uniqueness holds true also for the case (ii) among the solutions obtained as a limit of the approximation we construct.

The proof is presented only for M satisfying *(i)*. As explained in the introduction to this chapter, for existence no regularity of M is necessary for modular approximation in our proof in the reflexive spaces (like all variable exponent spaces with $1 < p_- \le p(\cdot) \le p_+ < \infty$, double-phase spaces with $1 < p,q < \infty$ and bounded weight, mixed spaces, involving more phases, milder or rapid transition between them etc.), as well as in the general classical Orlicz setting, including fully anisotropic spaces.

Remark 5.2.6 (Skipping *(Me) / (Me)*$_p$ **– reflexive case).** Theorem 5.2.3 provides existence results when $M, M^* \in \Delta_2$, that is, for example, in the following cases.

- When $M(x,\xi) = |\xi|^p$, with $1 < p < \infty$, in classical Sobolev spaces for the p-Laplace problem $-\Delta_p u = f \in L^1(\Omega)$, as well as for

$$-\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = f(x) \in L^1(\Omega)$$

  with measurable a such that $0 < a_- \le a(\cdot) \le a_+ < \infty$.

- When $M(x,\xi) = |\xi|^{p(x)}$, with $1 < p_- \le p \le p_+ < \infty$ in variable exponent spaces; for
$$-\mathrm{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big) = f(x) \in L^1(\Omega)$$
with measurable a such that $0 < a_- \le a(\cdot) \le a_+ < \infty$ and $1 < p_- \le p \le p_+ < \infty$.
- When $M(x,\xi) = |\xi|^p + a(t,x)|\xi|^q$, with $1 < p,q < \infty$ and $a : \Omega_T \to [0,\infty)$ being a bounded and measurable function possibly touching zero (no matter how irregular it is) in double phase spaces; for
$$-\mathrm{div}\Big(b(x)\big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big)\Big) = f(x) \in L^1(\Omega)$$
with measurable b such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.
- When $M(x,\xi) = |\xi|^{p(x)} + a(x)|\xi|^{q(x)}$, with $1 < c_1 \le p(x),q(x) \le c_2 < \infty$ and the function $a \in L^\infty(\Omega)$ nonnegative a.e. in Ω in variable exponent double-phase spaces; for
$$-\mathrm{div}\Big(b(x)\big(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\big)\Big) = f(x) \in L^1(\Omega)$$
with measurable b such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.
- When $M(x,\xi) = M_1(\xi) + a(x)M_2(\xi)$, where M_1, M_2 are (possibly anisotropic) homogeneous N-functions, such that $M_1, M_2, M_1^*, M_2^* \in \Delta_2$, and moreover the function $a \in L^\infty(\Omega)$ is nonnegative a.e. in Ω in Orlicz double phase spaces; for
$$-\mathrm{div}\left(b(x)\left(\tfrac{M_1(\nabla u)}{|\nabla u|^2}\cdot\nabla u + a(x)\tfrac{M_2(\nabla u)}{|\nabla u|^2}\cdot\nabla u\right)\right) = f(x) \in L^1(\Omega)$$
with measurable b such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.

Remark 5.2.7 (Skipping ***(Me) / (Me)$_p$*** **– Orlicz case).** In the pure Orlicz case, i.e. when
$$M(x,\xi) = M(\xi),$$
the balance conditions do not carry any information. Therefore, as a direct consequence of Theorem 5.2.3 we get the existence of unique renormalized solutions to the elliptic problem (4.1) under conditions therein in an anisotropic Orlicz space without growth restrictions of doubling type. This includes the case of $L\log^\alpha L$-spaces for $\alpha \ge 0$, when $M(x,\xi) = |\xi|\log^\alpha(1+|\xi|)$ and
$$-\mathrm{div}\Big(a(x)\tfrac{\log^\alpha(\mathrm{e}+|\nabla u|)}{|\nabla u|}\nabla u\Big) = f(x) \in L^1(\Omega)$$
with measurable a such that $0 < a_- \le a(\cdot) \le a_+ < \infty$. Note that in this case M grows essentially slower than a power function of any power larger that 1.

To give examples in nonreflexive Musielak–Orlicz spaces we shall relax the growth restrictions. According to Examples 3.7.2 and 3.7.3 we infer the existence of renormalized solutions in the following cases.

Example 5.2.8 (Orlicz double phase space). When $M(x,\xi) = M_1(\xi) + a(x)M_2(\xi)$, where M_1, M_2 are (possibly anisotropic) homogeneous N-functions (without prescribed growth) such that $M_1(\xi) \le M_2(\xi)$ for ξ such that $|\xi| > 1$, and moreover the function $a : \Omega \longrightarrow [0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a, we infer existence and uniqueness for solutions to the problem

$$-\mathrm{div}\left(b(x)\left(\frac{M_1(\nabla u)}{|\nabla u|^2}\cdot\nabla u + a(x)\frac{M_2(\nabla u)}{|\nabla u|^2}\cdot\nabla u\right)\right) = f(x) \in L^1(\Omega)$$

with measurable b such that $0 < b_- \le b(\cdot) \le b_+ < \infty$, provided

$$\limsup_{\delta\to 0}\omega_a(\delta)\frac{\overline{M}_2(\delta^{-N})}{\underline{M}_1(\delta^{-N})} < \infty,$$

where $\underline{M}_1(s) := \inf_{\xi:\,|\xi|=s} M_1(\xi)$ and $\overline{M}_2(s) := \sup_{\xi:\,|\xi|=s} M_2(\xi)$, or – when M_1 has at least power p growth – provided

$$\limsup_{\delta\to 0}\omega_a(\delta)\frac{\overline{M}_2(\delta^{-N/p})}{\underline{M}_1(\delta^{-N/p})} < \infty.$$

One can easily modify this example to get its variable exponent-type version or to involve more than two phases. Also other choices of M coming from Examples 3.7.2 and 3.7.3 generate a wide range of examples.

5.2.2 Existence and uniqueness

From now on in order to ensure approximation properties of our space by Theorem 3.7.7 we assume that M satisfies either *(Me)* (see Section 3.7.1) or *(Me)$_p$* (see Section 3.7.2), as it is explained in Section 5.1.3 how to construct, in the reflexive case, an approximation to our solution that has the same properties. We are now in position to present the proof of existence and uniqueness.

Proof (of Theorem 5.2.3). We start with the existence of a solution to a regularized problem, then we show a priori estimates, the radiation-control condition for the solutions to the regularized problem, and finally we concentrate on the most challenging part – passing to the limit. Lastly, we describe the comparison principle which implies the uniqueness of solutions.

Step 1. Problems with truncated data

The existence of a solution to the problem with truncated data

$$\begin{cases} -\mathrm{div}\,\mathbf{a}(x,\nabla u_s) = T_s(f) & \text{in} \quad \Omega, \\ u_s(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.6}$$

for $s > 0$ is a direct consequence of Theorem 4.1.5 with $g = T_s(f)$, where T stands for the symmetric truncation at the level s which is defined in (3.55).

Step 2. A priori estimates

In order to get uniform integrability of the sequences $\{\mathbf{a}(x,\nabla T_k(u_s))\}_{s>0}$ and $\{\nabla T_k(u_s)\}_{s>0}$ we need to obtain the following a priori estimates. For a weak solution u_s to (5.6), $s > 0$ and $f \in L^1(\Omega)$, we have the following estimates for any $k > 0$

$$\int_\Omega M\big(x, \nabla T_k(u_s)\big)\,\mathrm{d}x \le 2k\|f\|_{L^1(\Omega)}, \tag{5.7}$$

$$\int_\Omega M^*\Big(x, \mathbf{a}\big(x, \nabla T_k(u_s)\big)\Big)\,\mathrm{d}x \le \tfrac{2k}{c_{\mathbf{a}}}\|f\|_{L^1(\Omega)}. \tag{5.8}$$

Indeed, observe that due to assumption *(A2e)* we have

$$\begin{aligned} c_{\mathbf{a}}\int_\Omega M^*\Big(x, \mathbf{a}\big(x, \nabla T_k(u_s)\big)\Big)\,\mathrm{d}x &\le \int_\Omega M\big(x, \nabla T_k(u_s)\big)\,\mathrm{d}x \\ &\le \int_\Omega \mathbf{a}\big(x, \nabla T_k(u_s)\big)\cdot \nabla T_k(u_s)\,\mathrm{d}x. \end{aligned}$$

Let us consider $\{(T_k(u_s))_\delta\}_\delta \subset C_c^\infty(\Omega)$ – a sequence approximating $T_k(u_s)$ in the modular topology from Theorem 3.7.7. Then we have

$$\begin{aligned} \int_\Omega \mathbf{a}\big(x, \nabla T_k(u_s)\big)\cdot \nabla T_k(u_s)\,\mathrm{d}x &= \lim_{\delta\to 0}\int_\Omega \mathbf{a}\big(x, \nabla T_k(u_s)\big)\cdot \nabla (T_k(u_s))_\delta\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_\Omega T_s(f)(T_k(u_s))_\delta\,\mathrm{d}x = \int_\Omega T_s(f)T_k(u_s)\,\mathrm{d}x. \end{aligned}$$

Combining these observations we infer

$$c_{\mathbf{a}}\int_\Omega M^*\Big(x, \mathbf{a}\big(x, \nabla T_k(u_s)\big)\Big)\,\mathrm{d}x \le \int_\Omega M\big(x, \nabla T_k(u_s)\big)\,\mathrm{d}x \le 2\|f\|_{L^1(\Omega)}$$

and thus (5.7) and (5.8) follow.

Step 3. Controlled radiation

In this step we show that for any weak solution u_s to (5.6) ($s > 0$ and $f \in L^1(\Omega)$), there exists a $\gamma : [0,\infty) \to [0,\infty)$ independent of l, s such that $\lim_{r\to 0}\gamma(r) = 0$ and for every $l > 0$

$$\int_{\{l<|u_s|<l+1\}} \mathbf{a}(x, \nabla u_s)\cdot \nabla u_s\,\mathrm{d}x \le \gamma\left(\frac{l}{m_1(c_1 l)}\right) \qquad \text{for some } \; c_1 = c_1(\Omega) > 0. \tag{5.9}$$

Recall that m_1 is the minorant of M from the definition of an N-function.

We notice that the meaning of truncations (see (3.55) for the definition) implies

$$\begin{aligned} \int_{\{l<|u_s|<l+1\}} \mathbf{a}(x, \nabla u_s)\cdot \nabla u_s\,\mathrm{d}x &= \int_{\{l<|u_s|<l+1\}} \mathbf{a}(x, \nabla T_{l+1}(u_s))\cdot \nabla T_{l+1}(u_s)\,\mathrm{d}x \\ &= \int_\Omega \mathbf{a}(x, \nabla u_s)\cdot \nabla (T_{l+1}(u_s) - T_l(u_s))\,\mathrm{d}x. \end{aligned} \tag{5.10}$$

We cannot directly use the weak formulation here, because $(T_{l+1}(u_s) - T_l(u_s))$ is not admissible as a test function. We have to consider $\big\{(T_{l+1}(u_s) - T_l(u_s))_\delta\big\}_\delta$ – a sequence of smooth functions approximating the function $(T_{l+1}(u_s) - T_l(u_s))$ in the modular topology, which exists due to Theorem 3.7.7. Using elements of this approximate sequence as test functions in (5.6) we get

$$\begin{aligned}\int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla(T_{l+1}(u_s)-T_l(u_s))\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla(T_{l+1}(u_s)-T_l(u_s))_\delta\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_\Omega T_s(f)(T_{l+1}(u_s)-T_l(u_s))_\delta\,\mathrm{d}x \\ &= \int_\Omega T_s(f)(T_{l+1}(u_s)-T_l(u_s))\,\mathrm{d}x \\ &\le \int_{\{|u_s|\ge l\}}|f|\,\mathrm{d}x.\end{aligned} \tag{5.11}$$

Our aim is now to estimate the right-hand side above. With this aim we firstly find control over the size of a domain of integration. We note that for m_1 we have

$$|\{|u_s|\ge l\}| = |\{|T_l(u_s)|=l\}| = |\{|T_l(u_s)|\ge l\}| = |\{m_1(c_1|T_l(u_s)|)\ge m_1(c_1 l)\}|.$$

Moreover, for $l>0$ we apply the Chebyshev inequality (Theorem 8.28) and the Poincaré inequality (Theorem 9.3) involving m_1 – a convex minorant of M from the definition of an N-function, to get

$$\begin{aligned}|\{|u_s|\ge l\}| &\le \int_\Omega \frac{m_1(c_1|T_l(u_s)|)}{m_1(c_1 l)}\,\mathrm{d}x \\ &\le \frac{c_2}{m_1(c_1 l)}\int_\Omega m_1(|\nabla T_l(u_s)|)\,\mathrm{d}x.\end{aligned} \tag{5.12}$$

Since m_1 is a minorant of M and using the a priori estimate (5.7) we continue the above estimates as follows

$$|\{|u_s|\ge l\}| \le \frac{c_2}{m_1(c_1 l)}\int_\Omega M(x,\nabla T_l(u_s))\,\mathrm{d}x \le c(M,N,\Omega)\|f\|_{L^1(\Omega)}\frac{l}{m_1(c_1 l)}. \tag{5.13}$$

The right-hand side above vanishes when $l\to\infty$, because m_1 is assumed to be superlinear at infinity. Consequently, there exists a $\gamma:[0,\infty)\longrightarrow[0,\infty)$ independent of l,s, for which $\lim_{r\to 0}\gamma(r)=0$ and

$$\int_E |f|\,\mathrm{d}x \le \gamma(|E|).$$

In particular, due to (5.12) and (5.13) we may write

$$\int_{\{|u_s|\ge l\}}|f|\,\mathrm{d}x \le \gamma\left(\frac{l}{m_1(c_1 l)}\right). \tag{5.14}$$

Altogether we conclude (5.9), because due to (5.10), (5.11), and (5.14) we have

$$\int_{\{l<|u_s|<l+1\}} \mathbf{a}(x,\nabla u_s)\cdot\nabla u_s \,\mathrm{d}x \le \int_{\{|u_s|\ge l\}} |f|\,\mathrm{d}x \le \gamma\left(\frac{l}{m_1(c_1 l)}\right).$$

Step 4. Convergence of truncations

We characterize various limits involving u_s and its gradient. The aim of this step is to show that some subsequence of $\{u_s\}_{s>0}$ has a limit $u:\Omega\to\mathbb{R}$ in the sense that

$$u_s \xrightarrow[s\to\infty]{} u \quad \text{a.e. in } \Omega, \tag{5.15}$$

such that $T_k(u)\in V_0^M(\Omega)$ for every $k>0$ and it holds that

$$\big|\{|u|>l\}\big| \xrightarrow[l\to\infty]{} 0, \tag{5.16}$$

and such that for each $k\in\mathbb{N}$ and $s\to\infty$

$$T_k(u_s)\to T_k(u) \qquad \text{strongly in } L^1(\Omega), \tag{5.17}$$

$$\nabla T_k(u_s)\rightharpoonup \nabla T_k(u) \qquad \text{weakly in } L^1(\Omega), \tag{5.18}$$

$$\nabla T_k(u_s)\xrightharpoonup{*} \nabla T_k(u) \qquad \text{weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^N), \tag{5.19}$$

$$\mathbf{a}(x,\nabla T_k(u_s))\xrightharpoonup{*} \mathbf{a}(x,\nabla T_k(u)) \qquad \text{weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^N). \tag{5.20}$$

Fix an arbitrary $k\in\mathbb{N}$. The proved a priori estimate (5.7) reads

$$\int_\Omega M\big(x,\nabla T_k(u_s)\big)\,\mathrm{d}x \le ck\|f\|_{L^1(\Omega)}$$

and the Banach–Alaoglu theorem (Theorem 8.31) implies further that $\{T_k(u_s)\}_{s>0}$ is weakly-$*$ compact in L_M. The Dunford–Pettis theorem (Theorem 8.21) and the fact that M is an N-function (according to Definition 2.2.2) imply that for each k

$$\text{the sequence } \{T_k(u_s)\}_{s>0} \text{ is bounded in } W_0^{1,1}(\Omega).$$

Since Ω is bounded, for fixed $k\in\mathbb{N}$ convergence in (5.17) results from uniform integrability in $L^1(\Omega)$ of bounded functions $T_k(u_s)$ obtained due to the Rellich–Kondrachov theorem (Theorem 8.48) for $W^{1,1}(\Omega)$. Hence, there exists a function u such that

$$T_k(u_s) \xrightarrow[s\to\infty]{} T_k(u) \quad \text{strongly in } L^1(\Omega),$$

$$\nabla T_k(u_s) \xrightharpoonup[s\to\infty]{} \nabla T_k(u) \quad \text{weakly in } L^1(\Omega;\mathbb{R}^N).$$

Consequently, up to a subsequence, we have $u_s\to u$ in measure and (5.15). By the recalled a priori estimate and the Dunford–Pettis theorem (Theorem 8.21) we infer that, up to a subsequence, we have

$$\nabla T_k(u_s) \xrightharpoonup[s\to\infty]{*} \nabla T_k(u) \quad \text{weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^N),$$

in particular implying (5.18) and (5.19). Meanwhile, since the last term on the right-hand side of (5.13) converges to zero (because m_1 is superlinear at infinity) and $u_s \to u$ in measure, we deduce (5.16).

Let us concentrate on (5.20). For every k we define

$$\mathcal{A}_{s,k} = \mathbf{a}(x, \nabla T_k(u_s(x))).$$

By the same arguments as above, from the second a priori estimate (5.8) we deduce that up to a subsequence there exists an $\mathcal{A}_k \in L_{M^*}(\Omega;\mathbb{R}^N)$ such that

$$\mathcal{A}_{s,k} \overset{*}{\rightharpoonup} \mathcal{A}_k \qquad \text{weakly-}* \text{ in } L_{M^*}(\Omega;\mathbb{R}^N). \tag{5.21}$$

Identification of the limit of $\mathbf{a}(x, \nabla T_k(u_s(x)))$. Our aim is now to show that in (5.21) the limit has the form

$$\mathcal{A}_k(x) = \mathbf{a}(x, \nabla T_k(u)) \quad \text{a.e. in } \Omega. \tag{5.22}$$

In order to apply the monotonicity trick in the identification of the limit, we need to show that

$$\int_\Omega (\mathcal{A}_k - \mathbf{a}(x,\eta)) \cdot (\nabla T_k(u) - \eta)\, \mathrm{d}x \geq 0. \tag{5.23}$$

The main step to get it is to prove that

$$\limsup_{s\to\infty} \int_\Omega \mathcal{A}_{s,k} \cdot \nabla T_k(u_s)\, \mathrm{d}x = \int_\Omega \mathcal{A}_k \cdot \nabla T_k(u)\, \mathrm{d}x. \tag{5.24}$$

We take an auxiliary function $\psi_l : \mathbb{R} \to [0,1]$ given by

$$\psi_l(r) = \min\{(l+1-|r|)_+, 1\} \tag{5.25}$$

and an approximate sequence $\{\nabla(T_k(u))_\delta\}_\delta$ of smooth functions such that

$$\nabla(T_k(u))_\delta \xrightarrow[\delta\to 0]{M} \nabla T_k(u) \quad \text{modularly in } L_M(\Omega;\mathbb{R}^N),$$

which exists due to Theorem 3.7.7. We shall show first that

$$\lim_{\delta\to 0} \limsup_{s\to\infty} \int_\Omega \mathcal{A}_{s,k} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta\right]\, \mathrm{d}x = 0. \tag{5.26}$$

Notice that due to *(A2e)* one has that $\mathbf{a}(x,0) = 0$, therefore for $l \geq k$ we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}_{s,k} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta\right] \psi_l(u_s)\, \mathrm{d}x \\
&\quad = \int_\Omega \mathcal{A}_{s,k} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta\right]\, \mathrm{d}x \\
&\qquad + \int_{\{|u_s|>l\}} \mathbf{a}(x,0) \cdot \nabla \left[0 - (T_k(u))_\delta\right] (\psi_l(u_s) - 1)\, \mathrm{d}x
\end{aligned}$$

$$= \int_\Omega \boldsymbol{\mathcal{A}}_{s,k} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta \right] \, \mathrm{d}x$$

and thus (5.26) is equivalent to

$$\lim_{l\to\infty} \lim_{\delta\to 0} \limsup_{s\to\infty} \int_\Omega \boldsymbol{\mathcal{A}}_{s,k} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta \right] \psi_l(u_s) \, \mathrm{d}x = 0. \tag{5.27}$$

Actually, it suffices to show that

$$\lim_{l\to\infty} \lim_{\delta\to 0} \limsup_{s\to\infty} \int_\Omega \boldsymbol{\mathcal{A}}_{s,l+1} \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta \right] \psi_l(u_s) \, \mathrm{d}x = 0. \tag{5.28}$$

Indeed, having (5.28) and $\mathbf{a}(x,0) = 0$, the equality (5.27) will be proved when the following expression is shown to tend to 0 as $s \to \infty$ and $\delta \to 0$ (still $k \le l$)

$$\begin{aligned} II &= \int_\Omega (\boldsymbol{\mathcal{A}}_{s,k} - \boldsymbol{\mathcal{A}}_{s,l+1}) \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta \right] \psi_l(u_s) \, \mathrm{d}x \\ &= \int_\Omega (\boldsymbol{\mathcal{A}}_{s,l+1} - \mathbf{a}(x,0)) \cdot \nabla (T_k(u))_\delta \mathbb{1}_{\{k<|u_s|\}} \psi_l(u_s) \, \mathrm{d}x \\ &= \int_\Omega \boldsymbol{\mathcal{A}}_{s,l+1} \cdot \nabla (T_k(u))_\delta \mathbb{1}_{\{k<|u_s|\}} \psi_l(u_s) \, \mathrm{d}x. \end{aligned}$$

We need to justify that

$$\begin{aligned} \lim_{\delta\to 0} \limsup_{s\to\infty} |II| &\le \lim_{\delta\to 0} \limsup_{s\to\infty} \int_\Omega |\boldsymbol{\mathcal{A}}_{s,l+1}| \mathbb{1}_{\{k<|u_s|\}} \psi_l(u_s) |\nabla (T_k(u))_\delta| \, \mathrm{d}x \\ &\le \lim_{\delta\to 0} \int_\Omega |\boldsymbol{\mathcal{A}}_{l+1}| \mathbb{1}_{\{k<|u|\}} \psi_l(u) |\nabla (T_k(u))_\delta| \, \mathrm{d}x \qquad (5.29) \\ &= \int_\Omega |\boldsymbol{\mathcal{A}}_{l+1}| \mathbb{1}_{\{k<|u|\}} \psi_l(u) |\nabla T_k(u)| \, \mathrm{d}x = 0. \end{aligned}$$

For the limit as $s \to \infty$ we will use Lemma 8.22 with

$$w^s = |\boldsymbol{\mathcal{A}}_{s,l+1}| \cdot |\nabla (T_k(u))_\delta| \xrightharpoonup[s\to\infty]{} |\boldsymbol{\mathcal{A}}_{l+1}| \cdot |\nabla (T_k(u))_\delta| = w \qquad \text{in } L^1(\Omega)$$

and $v^s = \mathbb{1}_{\{t<|u_s|\}}$. The convergence $w^s \rightharpoonup w$ in $L^1(\Omega)$ is a consequence of (5.21), whereas Lemma 8.24 implies that $v^s \to v = \mathbb{1}_{\{t<|u|\}}$ a.e. in Ω. The limit as $\delta \to 0$ results from the modular convergence in (5.29). By modular convergence and Theorem 3.4.4, the sequence

$$\left\{ M\left(x, \frac{\nabla (T_k(u))_\delta}{\lambda}\right) \right\}_\delta \qquad \text{is uniformly bounded in } L^1(\Omega) \text{ for some } \lambda$$

and, consequently, by Theorem 3.4.2 the sequence $\{\nabla (T_k(u))_\delta\}_\delta$ is uniformly integrable. By the Vitali convergence theorem (Theorem 3.4.4) we can pass to the limit as in (5.29). The last equality therein follows from the definition of truncation, because

$$T_k(u) \mathbb{1}_{\{k<|u|\}} = 0.$$

Thus we get (5.29) and, consequently, (5.27) and (5.26) hold.

To get (5.28) we test (5.6) by $\varphi_{\bar{\delta}}$, approximating modularly

$$\varphi = \psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta)$$

(cf. Theorem 3.7.7), where ψ_l is given by (5.25) and passing to the limit as $\bar{\delta} \to 0$ we get

$$\int_\Omega \mathbf{a}(x, \nabla u_s) \cdot \nabla \left[\psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta)\right] \, \mathrm{d}x \\ = \int_\Omega T_s(f)\psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta) \, \mathrm{d}x. \tag{5.30}$$

We observe that the right-hand side of (5.30) tends to zero, that is

$$\lim_{l\to\infty} \lim_{\delta\to 0} \lim_{s\to\infty} \int_\Omega T_s(f)\psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta) \, \mathrm{d}x = 0.$$

Indeed, the convergence a.e. is ensured by (5.15) and to apply the Lebesgue dominated convergence theorem we note that

$$\begin{aligned}
&\lim_{\delta\to 0} \lim_{s\to\infty} \left| \int_\Omega T_s(f)\psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta) \, \mathrm{d}x \right| \\
&\quad \le \lim_{\delta\to 0} \lim_{s\to\infty} \int_\Omega |T_s(f)|\psi_l(u_s) \cdot |T_k(u_s) - T_k(u)| \, \mathrm{d}x \\
&\qquad + \lim_{\delta\to 0} \lim_{s\to\infty} \int_\Omega |T_s(f)|\psi_l(u_s) \cdot |T_k(u) - (T_k(u))_\delta| \, \mathrm{d}x \\
&\quad \le \lim_{\delta\to 0} \lim_{s\to\infty} \int_\Omega 2k|f| \, \mathrm{d}x + \lim_{\delta\to 0} \lim_{s\to\infty} \int_\Omega |f| \cdot |T_k(u) - (T_k(u))_\delta| \, \mathrm{d}x \\
&\quad = 2k\|f\|_{L^1(\Omega)} + \lim_{\delta\to 0} \int_\Omega |f| \cdot |T_k(u) - (T_k(u))_\delta| \, \mathrm{d}x,
\end{aligned}$$

where according to Theorem 3.7.7 we have $|(T_k(u))_\delta| < ck$ and thus

$$|T_k(u) - (T_k(u))_\delta| < (1+c)k. \tag{5.31}$$

Let us now concentrate on the left-hand side of (5.30) and write

$$\begin{aligned}
&\int_\Omega \mathbf{a}(x, \nabla u_s) \cdot \nabla \left[\psi_l(u_s)(T_k(u_s) - (T_k(u))_\delta)\right] \, \mathrm{d}x \\
&\quad = \int_\Omega \mathbf{a}(x, \nabla u_s) \cdot \nabla \psi_l(u_s) \left[T_k(u_s) - (T_k(u))_\delta\right] \, \mathrm{d}x \\
&\qquad + \int_\Omega \mathbf{a}(x, \nabla u_s) \cdot \nabla \left[T_k(u_s) - (T_k(u))_\delta\right] \psi_l(u_s) \, \mathrm{d}x \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{5.32}$$

By the Cauchy–Schwarz inequality and (5.31) we can estimate

$$
\begin{aligned}
&\lim_{l\to\infty}\left(\lim_{\delta\to 0}\limsup_{s\to\infty}|I_1|\right)\\
&\quad\le \lim_{l\to\infty}\left(\lim_{\delta\to 0}\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}}|\mathbf{a}(x,\nabla u_s)\cdot\nabla u_s|\,|T_k(u_s)-(T_k(u))_\delta|\,\mathrm{d}x\right)\\
&\quad\le c\,k\lim_{l\to\infty}\left(\lim_{\delta\to 0}\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}}|\mathbf{a}(x,\nabla u_s)\cdot\nabla u_s|\,\mathrm{d}x\right)\\
&\quad= c\,k\lim_{l\to\infty}\left(\limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}}\mathbf{a}(x,\nabla u_s)\cdot\nabla u_s\,\mathrm{d}x\right)\\
&\quad\le c\,k\lim_{l\to\infty}\gamma\left(\frac{l}{m_1(c_1 l)}\right)=0,
\end{aligned}
$$

where the last line follows due to (5.9). To complete the argument justifying the convergence of the left-hand side of (5.30), we notice that for I_2 from (5.32) it holds that

$$
\begin{aligned}
&\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty} I_2\\
&=\lim_{l\to\infty}\lim_{\delta\to 0}\limsup_{s\to\infty}\int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla\left[T_k(u_s)-(T_k(u))_\delta\right]\psi_l(u_s)\,\mathrm{d}x=0.
\end{aligned}
\tag{5.33}
$$

Then taking into account the above limits, (5.33) is equivalent to (5.28). Therefore, (5.27) follows and, consequently, we have also (5.26). Due to (5.21), for fixed δ,

$$
\lim_{s\to\infty}\int_\Omega \boldsymbol{\mathcal{A}}_{s,k}\cdot\nabla(T_k(u))_\delta\,\mathrm{d}x=\int_\Omega \boldsymbol{\mathcal{A}}_k\cdot\nabla(T_k(u))_\delta\,\mathrm{d}x. \tag{5.34}
$$

Then (5.26) together with (5.34) imply

$$
\begin{aligned}
\limsup_{s\to\infty}\int_\Omega \boldsymbol{\mathcal{A}}_{s,k}\cdot\nabla T_k(u_s)\,\mathrm{d}x &= \lim_{\delta\to 0}\int_\Omega \boldsymbol{\mathcal{A}}_k\cdot\nabla(T_k(u))_\delta\,\mathrm{d}x\\
&=\int_\Omega \boldsymbol{\mathcal{A}}_k\cdot\nabla T_k(u)\,\mathrm{d}x,
\end{aligned}
\tag{5.35}
$$

where the last equality is obtained in the same way as (5.29). Finally, (5.24) also follows.

We are about to complete the proof of identification of the limit of $\{\boldsymbol{\mathcal{A}}_{s,k}\}_{s>0}$ by the monotonicity trick of Theorem 4.1.1. By the monotonicity of $\mathbf{a}$ from *(A3e)* we have

$$
\int_\Omega \boldsymbol{\mathcal{A}}_{s,k}\cdot\eta\;\mathrm{d}x+\int_\Omega \mathbf{a}(x,\eta)\cdot(\nabla T_k(u_s)-\eta)\,\mathrm{d}x\le\int_\Omega \boldsymbol{\mathcal{A}}_{s,k}\cdot\nabla T_k(u_s)\,\mathrm{d}x
$$

for any $\eta\in\mathbb{R}^N$. Taking the upper limit as $s\to\infty$ above (due to (5.24), (5.21), and (5.19)) we infer that

$$
\int_\Omega \boldsymbol{\mathcal{A}}_k\cdot\eta\;\mathrm{d}x+\int_\Omega \mathbf{a}(x,\eta)\cdot(\nabla T_k(u)-\eta)\,\mathrm{d}x\le\int_\Omega \boldsymbol{\mathcal{A}}_k\cdot\nabla T_k(u)\,\mathrm{d}x,
$$

which is equivalent to (5.23). We are in a position to apply Theorem 4.1.1 with

$$\mathcal{A} = \mathcal{A}_k \qquad \text{and} \qquad \xi = \nabla(T_k(u))$$

to conclude (5.22).

Step 5. Renormalized solutions

The aim of this step is to complete the proof of Theorem 5.2.3, that is, the existence of renormalized solutions. More precisely, we show that u obtained as a limit in the previous step is in fact a renormalized solution according to Definition 5.2.1.

Condition *(R1e)***.**

Note that (5.19) and (5.20) imply that u satisfies *(R1e)*.

Condition *(R2e)***.**

Since $T_k(u) \in V_0^M(\Omega) \cap L^\infty(\Omega)$, Theorem 3.7.7 ensures that there exists a sequence $\{u_r\}_{r>0} \subset C_c^\infty(\Omega)$ indexed by $r \to \infty$, such that

$$\begin{aligned} u_r &\to u && \text{a.e. in } \Omega,\\ \nabla T_k(u_r) &\overset{*}{\rightharpoonup} \nabla T_k(u) && \text{weakly-}* \text{ in } L_M(\Omega;\mathbb{R}^N),\\ \nabla h(u_r) &\rightharpoonup \nabla h(u) && \text{weakly in } L_M(\Omega;\mathbb{R}^N), \end{aligned}$$

with arbitrary $h \in C_c^1(\mathbb{R})$. We fix such h. Then we test (5.6) by $\psi_l(u_s)h(u_r)\phi$ with a hat function ψ_l defined in (5.25) and $\phi \in W_0^{1,\infty}(\Omega)$. We get

$$L_{s,r,l} := \int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla[\psi_l(u_s)h(u_r)\phi]\,\mathrm{d}x = \int_\Omega T_s(f)\psi_l(u_s)h(u_r)\phi\,\mathrm{d}x =: R_{s,r,l}.$$

Let us justify passing to the limit on both sides of the last display. Due to the Lebesgue dominated convergence theorem it holds that

$$\lim_{l\to\infty}\lim_{r\to\infty}\limsup_{s\to\infty} R_{s,r,l} = \int_\Omega f h(u)\phi\,\mathrm{d}x.$$

Let us concentrate on the left-hand side by writing

$$\begin{aligned} L_{s,r,l} &= \int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla\psi_l(u_s)\,h(u_r)\phi\,\mathrm{d}x + \int_\Omega \mathbf{a}(x,\nabla u_s)\cdot\nabla[h(u_r)\phi]\psi_l(u_s)\,\mathrm{d}x\\ &=: L^1_{s,r,l} + L^2_{s,r,l}, \end{aligned}$$

where

$$\begin{aligned} &\lim_{l\to\infty}\lim_{r\to\infty}\limsup_{s\to\infty}|L^1_{s,r,l}|\\ &\qquad \le \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\lim_{l\to\infty}\lim_{r\to\infty}\left(\sup_s \int_{\{l<|u_s|<l+1\}} \mathcal{A}_{s,l+1}(x)\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x\right)\\ &\qquad = 0 \end{aligned}$$

due to the radiation control condition (5.9). As for $L^2_{s,r,l}$ we notice that, up to a subsequence, it holds that

$$\mathcal{A}_{s,l+1} \xrightarrow[s\to\infty]{} \mathbf{a}(x,\nabla T_{l+1}(u)) \quad \text{weakly in } L^1(\Omega).$$

Indeed, the a priori estimate (5.8) and Theorem 3.4.2 give uniform integrability. Then, taking into account weak-$*$ convergence (5.20), the Dunford–Pettis theorem (Theorem 8.21) ensures weak L^1-convergence up to a subsequence. Moreover, since

$$|\psi_l(u_s)| \le 1,$$
$$\nabla(h(u_r)\phi) \in L^\infty(\Omega;\mathbb{R}^N)$$

and for $s \to \infty$

$$\psi_l(u_s) \to \psi_l(u) \quad \text{a.e. in } \Omega,$$

the sequence

$$\Big\{\mathbf{a}(x,\nabla u_s)\psi_l(u_s)\nabla[h(u_r)\phi]\Big\}_{s>0} \quad \text{is uniformly integrable in } L^1(\Omega).$$

As a consequence of Chacon's biting lemma (Theorem 8.38) we notice that

$$\limsup_{r\to\infty}\limsup_{s\to\infty}\int_\Omega \mathcal{A}_{s,l+1}\cdot\nabla h(u_r)\psi_l(u_s)\,\mathrm{d}x = \int_\Omega \mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla[h(u)\phi]\psi_l(u)\,\mathrm{d}x.$$

Since $\operatorname{supp} h(u) \subset [-K,K]$ for some $K\in\mathbb{N}$ and we can consider only $l > K+1$, we infer that

$$\begin{aligned}\lim_{l\to\infty}&\limsup_{r\to\infty}\limsup_{s\to\infty} L^2_{s,r,l}\\ &= \lim_{l\to\infty}\int_\Omega \mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla[h(u)\phi]\psi_l(u)\,\mathrm{d}x\\ &= \int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla[h(u)\phi]\,\mathrm{d}x,\end{aligned}$$

and our solution u satisfies condition *(R2e)*.

Condition ***(R3e)*****.**
We have to show that

$$\int_{\{l<|u|<l+1\}} \mathbf{a}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x = \int_{\{l<|u|<l+1\}} \mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u)\,\mathrm{d}x \xrightarrow[l\to\infty]{} 0.$$

We start by showing that

$$\mathcal{A}_{s,l+1}\cdot\nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{} \mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u) \quad \text{weakly in } L^1(\Omega). \tag{5.36}$$

We will apply Chacon's biting lemma (Theorem 8.38) and the Young measures (Theorem 8.41). First we observe that the sequence

$$\left\{\left(\boldsymbol{\mathcal{A}}_{s,l+1}-\mathbf{a}(x,\nabla T_{l+1}(u))\right)\cdot\left(\nabla T_{l+1}(u_s)-\nabla T_{l+1}(u)\right)\right\}_{s>0}$$

is uniformly bounded in $L^1(\Omega)$. Indeed, we might write

$$\int_\Omega\left(\boldsymbol{\mathcal{A}}_{s,l+1}-\mathbf{a}(x,\nabla T_{l+1}(u))\right)\cdot\left(\nabla T_{l+1}(u_s)-\nabla T_{l+1}(u)\right)\,\mathrm{d}x\le IV_1+IV_2+IV_3+IV_4,$$

where

$$\begin{aligned}
IV_1&:=\int_\Omega\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x,\\
IV_2&:=\int_\Omega\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u)\,\mathrm{d}x,\\
IV_3&:=\int_\Omega\mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u_s)\,\mathrm{d}x,\\
IV_4&:=\int_\Omega\mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u)\,\mathrm{d}x,
\end{aligned}$$

where each of the terms can be estimated with the use of the Fenchel–Young inequality (Lemma 2.1.32) and the a priori estimate (5.7) in the following way

$$\begin{aligned}
IV_1&\le\int_\Omega M^*(x,\mathbf{a}(x,\nabla T_{l+1}(u_s)))+M(x,\nabla T_{l+1}(u_s))\,\mathrm{d}x\\
&\le 2(l+1)\|f\|_{L^1(\Omega)}(1+1/c_{\mathbf{a}}),
\end{aligned}$$

which yields uniform boundedness in s. In turn, $IV_1+IV_2+IV_3+IV_4$ is uniformly bounded. Then the monotonicity of $\mathbf{a}(x,\cdot)$ and Chacon's biting lemma (Theorem 8.38) give, up to a subsequence, convergence in the sense of biting (Definition 8.36) of the product

$$\begin{aligned}
0&\le\left(\boldsymbol{\mathcal{A}}_{s,l+1}-\mathbf{a}(x,\nabla T_{l+1}(u))\right)\cdot\left(\nabla T_{l+1}(u_s)-\nabla T_{l+1}(u)\right)\\
&\xrightarrow[s\to\infty]{b}\int_{\mathbb{R}^N}\left(\mathbf{a}(x,\lambda)-\mathbf{a}(x,\nabla T_{l+1}(u))\right)\cdot\left(\lambda-\nabla T_{l+1}(u)\right)\mathrm{d}\nu_x(\lambda),
\end{aligned}\tag{5.37}$$

where ν_x denotes the Young measure generated by the sequence $\{\nabla T_{l+1}(u_s)\}_{s>0}$.

Since due to (5.18) we have weak convergence $\nabla T_{l+1}(u_s)\rightharpoonup\nabla T_{l+1}(u)$ in $L^1(\Omega)$ for $s\to\infty$, we have that

$$\int_{\mathbb{R}^N}\lambda\,\mathrm{d}\nu_x(\lambda)=\nabla T_{l+1}(u)\qquad\text{for a.e. } x\in\Omega.$$

Then

$$\int_{\mathbb{R}^N}\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\left(\lambda-\nabla T_{l+1}(u)\right)\mathrm{d}\nu_x(\lambda)=0$$

and the limit in (5.37) is equal for a.e. $x\in\Omega$ to

$$\int_{\mathbb{R}^N} \Big(\mathbf{a}(x,\lambda) - \mathbf{a}(x,\nabla T_{l+1}(u))\Big) \cdot \big(\lambda - \nabla T_{l+1}(u)\big)\, \mathrm{d}\nu_x(\lambda) \\ = \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\cdot\lambda\,\mathrm{d}\nu_x(\lambda) - \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\cdot\nabla T_{l+1}(u)\,\mathrm{d}\nu_x(\lambda). \tag{5.38}$$

Uniform boundedness of the sequence $\{\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u_s)\}_{s>0}$ in $L^1(\Omega)$ resulting from (5.9) enables us to apply once again the Chacon's biting lemma (Theorem 8.38) to obtain

$$\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{b} \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\cdot\lambda\,\mathrm{d}\nu_x(\lambda).$$

Moreover, assumption *(A2e)* implies $\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u_s)\ge 0$. Therefore, due to (5.38) and (5.37), we have

$$\limsup_{s\to\infty} \mathbf{a}(x,\nabla T_{l+1}(u_s))\cdot\nabla T_{l+1}(u_s) \ge \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\cdot\lambda\,\mathrm{d}\nu_x(\lambda).$$

Taking into account that in (5.35) we characterize the above limit, we can put

$$\boldsymbol{\mathcal{A}}_k = \mathbf{a}(x,\nabla T_{l+1}(u)) = \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\,\mathrm{d}\nu_x(\lambda),$$

and the above expression implies

$$\nabla T_{l+1}(u)\int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\,\mathrm{d}\nu_x(\lambda) \ge \int_{\mathbb{R}^N} \mathbf{a}(x,\lambda)\cdot\lambda\,\mathrm{d}\nu_x(\lambda).$$

When we apply this together with (5.38), we infer that the limit in (5.37) is non-positive and

$$\Big(\boldsymbol{\mathcal{A}}_{s,l+1} - \mathbf{a}(x,\nabla T_{l+1}(u))\Big)\cdot\big(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\big) \xrightarrow[s\to\infty]{b} 0.$$

Since $\mathbf{a}(x,\nabla T_{l+1}(u))\in L_{M^*}(\Omega;\mathbb{R}^N)$, we can choose an ascending family of shrinking sets E_j^{l+1}, i.e. such that

$$\left|E_j^{l+1}\right| \to 0 \quad \text{for} \quad j\to\infty$$

and

$$\mathbf{a}\big(x,\nabla T_{l+1}(u)\big)\in L^\infty\Big(\Omega\setminus E_j^{l+1}\Big).$$

From (5.19) we have $\nabla T_{l+1}(u_s)\rightharpoonup\nabla T_{l+1}(u)$ weakly in $L_M(\Omega,\mathbb{R}^N)$ as $s\to\infty$. Therefore, we get

$$\mathbf{a}(x,\nabla T_{l+1}(u))\cdot\big(\nabla T_{l+1}(u_s) - \nabla T_{l+1}(u)\big) \xrightarrow[s\to\infty]{b} 0$$

and similarly we conclude that

$$\boldsymbol{\mathcal{A}}_{s,l+1}\cdot\nabla T_{l+1}(u) \xrightarrow[s\to\infty]{b} \mathbf{a}(x,\nabla T_{l+1}(u))\cdot\nabla T_{l+1}(u).$$

Summing it up we get

$$\mathcal{A}_{s,l+1} \cdot \nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{b} \mathbf{a}(x, \nabla T_{l+1}(u)) \cdot \nabla T_{l+1}(u).$$

In the end, Chacon's biting lemma (Theorem 8.38) together with (5.35) and (5.20) results in (5.36). We have now the main ingredient of the proof of *(R3e)*.

Note that by (3.56) for any $l \in \mathbb{N}$ we have

$$\nabla u_s = 0 \quad \text{a.e. in} \quad \{x \in \Omega : \ |u_s| \in \{l, l+1\}\}.$$

Then (5.9) implies

$$\lim_{l\to\infty} \limsup_{s\to\infty} \int_{\{l-1<|u_s|<l+2\}} \mathbf{a}(x, \nabla u_s) \cdot \nabla u_s \, \mathrm{d}x = 0.$$

For the function $g_l : \mathbb{R} \to \mathbb{R}$ defined by

$$g_l(r) = \begin{cases} 1 & \text{if } \ l \le |r| \le l+1, \\ 0 & \text{if } \ |r| < l-1 \text{ or } |r| > l+2, \\ \text{is affine otherwise,} \end{cases}$$

we have

$$\int_{\{l-1<|u|<l+2\}} \mathbf{a}(x, \nabla u) \cdot \nabla u \, \mathrm{d}x \le \int_\Omega g_l(u) \, \mathbf{a}(x, \nabla T_{l+2}(u)) \cdot \nabla T_{l+2}(u) \, \mathrm{d}x. \tag{5.39}$$

Let us recall that condition *(A2e)* implies that $\mathbf{a}(x,\xi) \cdot \xi \ge 0$. Thus, due to (5.39), we may write

$$\begin{aligned} 0 &\le \lim_{l\to\infty} \int_{\{l-1<|u|<l+2\}} \mathbf{a}(x, \nabla u) \cdot \nabla u \, \mathrm{d}x \\ &\le \lim_{l\to\infty} \int_\Omega g_l(u) \, \mathbf{a}(x, \nabla T_{l+2}(u)) \cdot \nabla T_{l+2}(u) \, \mathrm{d}x =: V. \end{aligned}$$

By (5.36) we have that

$$\mathbf{a}(x, \nabla T_{l+1}(u_s)) \cdot \nabla T_{l+1}(u_s) \xrightarrow[s\to\infty]{} \mathbf{a}(x, \nabla T_{l+1}(u)) \cdot \nabla T_{l+1}(u) \text{ weakly in } \ L^1(\Omega),$$

whereas g_l is a continuous and bounded function, so

$$\begin{aligned} V &= \lim_{l\to\infty} \int_\Omega g_l(u) \, \mathbf{a}(x, \nabla T_{l+2}(u)) \cdot \nabla T_{l+2}(u) \, \mathrm{d}x \\ &= \lim_{l\to\infty} \lim_{s\to\infty} \int_\Omega g_l(u) \, \mathbf{a}(x, \nabla T_{l+2}(u_s)) \cdot \nabla T_{l+2}(u_s) \, \mathrm{d}x \\ &\le \lim_{l\to\infty} \limsup_{s\to\infty} \int_{\{l-1<|u_s|<l+2\}} \mathbf{a}(x, \nabla u_s) \cdot \nabla u_s \, \mathrm{d}x, \end{aligned}$$

where the last line is a direct consequence of the definition of g_l. As declared in (5.9), there exists a function γ independent of l and s such that $\lim_{r\to\infty}\gamma(r)=0$ and

$$\int_{\{l<|u_s|<l+1\}} \mathbf{a}(x,\nabla u_s)\cdot\nabla u_s \,\mathrm{d}x \le \gamma\left(\tfrac{l}{m_1(c_1 l)}\right).$$

Here $c_1=c_1(\Omega)$ and m_1 is a superlinear function being a minorant of M (from the definition of an N-function). Therefore, we get that

$$0\le V\le \limsup_{s\to\infty}\int_{\{l<|u_s|<l+1\}} \mathbf{a}(x,\nabla u_s)\cdot\nabla u_s \,\mathrm{d}x \le \gamma\left(\tfrac{l}{m_1(c_1 l)}\right) \xrightarrow[l\to\infty]{} 0.$$

When we let $s,l\to\infty$, the integral in the last display vanishes. Hence, u satisfies condition *(R3e)*.

Summing up, u is a renormalized solution.

Uniqueness. Now we consider renormalized solutions v^i, $i=1,2$, to (4.1) with a strictly monotone operator constructed as above for the same datum $f\in L^1(\Omega)$. We show that then $v^1=v^2$ a.e. in Ω.

In both renormalized formulations, for v^1 and v^2 the choice of $h=\psi_l$ defined in (5.25) is admissible. Moreover, we can take $\varphi=T_k(T_{l+1}v_1-T_{l+1}v_2)$, because by Theorem 3.7.7 (requiring *(Me)* or *(Me)$_p$*), we simply have $\varphi\in L^\infty(\Omega)\cap V_0^M(\Omega)$. Testing the renormalized formulations for v^1 and v^2 against this choice of h and φ and then subtracting the second of them from the first one we obtain

$$I_1-I_2+I_3+I_4-I_5=I_6,$$

where

$$\begin{aligned}
I_1 &:= \int_\Omega \psi_l'(v^1)\big(\mathbf{a}(x,\nabla v^1)\cdot\nabla T_{l+1}v^1\big)T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x,\\
I_2 &:= \int_\Omega \psi_l'(v^2)\big(\mathbf{a}(x,\nabla v^2)\cdot\nabla T_{l+1}v^2\big)T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x,\\
I_3 &:= \int_\Omega (\mathbf{a}(x,\nabla v^1)-\mathbf{a}(x,\nabla v^2))\cdot\nabla T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x,\\
I_4 &:= \int_\Omega (1-\psi_l(v^2))\,\mathbf{a}(x,\nabla v^2)\cdot\nabla T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x,\\
I_5 &:= \int_\Omega (1-\psi_l(v^1))\,\mathbf{a}(x,\nabla v^1)\cdot\nabla T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x,\\
I_6 &:= \int_\Omega f(\psi_l(v^1)-\psi_l(v^2))T_k(T_{l+1}v^1-T_{l+1}v^2)\,\mathrm{d}x.
\end{aligned}$$

We want to pass to the limit as $l\to\infty$. On the right-hand side we have an integrable function integrated over a shrinking set, so the Lebesgue dominated convergence theorem implies that $\lim_{l\to\infty}I_6=0$. On the left-hand side, we pass to the limit in I_1 and I_2 using the radiation control condition *(R3e)*. Clearly,

$$|I_1| \le k \int_{\{l \le |v^1| \le l+1\}} \mathbf{a}(x, \nabla v^1) \cdot \nabla v^1 \,\mathrm{d}x,$$

by *(R3e)* and from the fact that the measure of the sets $\{l \le |v^1| \le l+1\}$ tends to zero as $l \to \infty$, we infer $\lim_{l\to\infty} |I_1| = 0$. By the same arguments $\lim_{l\to\infty} |I_2| = 0$.

Now we pass to the limit in I_4 and I_5. As the argument is similar for both terms we show it only for I_4. We have

$$\begin{aligned} |I_4| &\le \int_{\{|v^2| \ge l,\ 0<|T_{l+1}v^1 - T_{l+1}v^2|<k\}} \left|\mathbf{a}(x, \nabla v^2) \cdot \nabla T_{l+1} v^1 + \mathbf{a}(x, \nabla v^2) \cdot \nabla T_{l+1} v^2\right| \mathrm{d}x \\ &\le \int_{\{l \le |v^2| \le l+k+1\}} \mathbf{a}(x, \nabla v^2) \cdot \nabla T_{l+1} v^2 \,\mathrm{d}x \\ &\quad + \int_{\{l \le |v^2| \le l+k+1,\ l-k \le |v^1| \le l+1\}} |\mathbf{a}(x, \nabla v^2) \cdot \nabla T_{l+1} v^1| \,\mathrm{d}x. \end{aligned}$$

The first integral in the last display tends to zero by *(R3e)*, whereas to deal with the second one we observe that

$$\begin{aligned} &\int_{\{l \le |v^2| \le l+k+1,\ l-k \le |v^1| \le l+1\}} |\mathbf{a}(x, \nabla v^2) \cdot \nabla T_{l+1} v^1| \,\mathrm{d}x \\ &\quad \le \int_{\{l \le |v^2| \le l+k+1\}} M^*(x, \mathbf{a}(x, \nabla v^2))\, dx + \int_{\{l-k \le |v^1| \le l+1\}} M(x, \nabla v^1) \,\mathrm{d}x \\ &\quad \le \frac{1}{c_{\mathbf{a}}} \int_{\{l \le |v^2| \le l+k+1\}} \mathbf{a}(x, \nabla v^2) \cdot \nabla v^2 \, dx + \int_{\{l-k \le |v^1| \le l+1\}} \mathbf{a}(x, \nabla v^1) \cdot \nabla v^1 \,\mathrm{d}x, \end{aligned}$$

and all terms on the rightmost-side converge to zero either by *(R3e)*. We deal with I_3. Let us fix an arbitrary $l_0 > 0$ and consider $l+1 \ge l_0$. The following holds

$$\begin{aligned} I_3 &= \int_{\{0<|T_{l+1}v^1 - T_{l+1}v^2|<k\}} (\mathbf{a}(x, \nabla v^1) - \mathbf{a}(x, \nabla v^2)) \cdot \nabla (T_{l+1} v^1 - T_{l+1} v^2) \,\mathrm{d}x \\ &\ge \int_{\{0<|T_{l+1}v^1 - T_{l+1}v^2|<k,\ |v^1| \le l_0,\ |v^2| \le l_0\}} (\mathbf{a}(x, \nabla v^1) - \mathbf{a}(x, \nabla v^2)) \cdot \nabla (T_{l+1} v^1 - T_{l+1} v^2) \,\mathrm{d}x \\ &= \int_{\{0<|v^1 - v^2|<k,\ |v^1| \le l_0,\ |v^2| \le l_0\}} (\mathbf{a}(x, \nabla v^1) - \mathbf{a}(x, \nabla v^2)) \cdot \nabla (T_{l+1} v^1 - T_{l+1} v^2) \,\mathrm{d}x. \end{aligned}$$

As we know that $\lim_{l\to\infty} I_3 = 0$ it follows that

$$0 = \int_{\{0<|v^1 - v^2|<k,\ |v^1| \le l_0,\ |v^2| \le l_0\}} (\mathbf{a}(x, \nabla v^1) - \mathbf{a}(x, \nabla v^2)) \cdot \nabla (v^1 - v^2) \,\mathrm{d}x,$$

which means, by the strict monotonicity of $\mathbf{a}$,

$$\left| \left\{ 0 < |v^1 - v^2| < k,\ |v^1| \le l_0,\ |v^2| \le l_0 \right\} \right| = 0.$$

As k and l_0 are arbitrary, we deduce that $v^1 = v^2$ a.e. in Ω. □

Let us recall that the condition $M, M^* \in \Delta_2$ is equivalent to reflexivity of the involved Musielak–Orlicz space, see Remark 3.3.3. In this case we can bypass the balance assumptions *(Me)* and *(Me)$_p$*.

Remark 5.2.9 (Uniqueness in the reflexive case). When both $M, M^* \in \Delta_2$, we can get the same conclusion on uniqueness, but for those solutions that are obtained by the construction. As above in both renormalized formulations, for v^1 and v^2 the choice of $h = \psi_l$ defined in (5.25) is admissible. Moreover, we can take $\varphi = T_k(T_{l+1}v_1 - T_{l+1}v_2)$, because we assume that v^1 and v^2 are obtained as the modular limits of solutions to approximate problems, and of course the gradients of their truncations on level k are uniformly bounded in $L_M(\Omega;\mathbb{R}^N)$ (and the weak-$*$ topology of $L_M(\Omega;\mathbb{R}^N)$ on bounded sets is metrizable as this space has a separable predual space). We can use the diagonal argument to obtain a sequence of functions belonging to $W^{1,\infty}(\Omega)$ which converges to $T_k(T_{l+1}v^1 - T_{l+1}v^2)$. The remaining arguments do not need any modification.

5.2.3 Exercises

There are various directions in which the problem treated in Theorem 5.2.3 can be developed.

- To cover more general conditions ensuring the density of the smooth functions, one can refine the result of Theorem 3.7.7. The possible ways are indicated in Remarks 3.7.11 and 3.7.13.
- One can relax the requirement on the growth condition. In [186] the existence of renormalized solutions is provided under the restriction $M^* \in \Delta_2$, but not $M \in \Delta_2$. One may think about the continuation of ideas of Theorem 4.1.3 to prove the existence of renormalized solutions imposing $M \in \Delta_2$, but not $M^* \in \Delta_2$.
- Other notions of very weak solutions can be studied under various regimes. In particular it would be interesting to verify under what assumptions the notions of SOLA, entropy solutions, renormalized solutions or generalized superharmonic functions essentially differ from each other.
- One can study what kind of lower-order terms can be incorporated into the equation or what kind of structural conditions need to be imposed on the operator if $\mathbf{a} = \mathbf{a}(x, u, \nabla u)$, see [186, 78, 47]. Since the related problem for differential inclusions is also likely to attract attention [109], these modifications can also be considered there.
- The question of how to consider more general data is open, see e.g. [73, 83, 9]. In particular, there is an open problem for measure data equations involving nonlinear operators (even of power growth), namely what is the optimal assumption on a measure datum ensuring uniqueness of a very weak solution?
- Precise regularity of solutions to measure data nonstandard growth elliptic equations and their gradients is known only in some special cases [9, 45, 95, 72, 74, 45, 77].

5.3 Renormalized Solutions to Parabolic Problems

5.3.1 Formulation of the problem

Definition 5.3.1 (Renormalized solutions to a parabolic equation). We call a function u a *renormalized solution* to (4.70) if it satisfies the following conditions.

(R1p) $u : \Omega_T \to \mathbb{R}$ is a measurable function and for each $k > 0$

$$T_k(u) \in V_T^M(\Omega) \qquad \text{and} \qquad \mathbf{a}(\cdot,\cdot,\nabla T_k(u)) \in L_{M^*}(\Omega_T;\mathbb{R}^N).$$

(R2p) For every compactly supported $h \in W^{1,\infty}(\mathbb{R})$ and all $\varphi \in V_T^{M,\infty}(\Omega)$ such that $\partial_t\varphi \in L^\infty(\Omega_T)$ and $\varphi(\cdot,x)$ has a compact support in $[0,T)$ for a.e. $x \in \Omega$, we have

$$-\int_{\Omega_T}\left(\int_{u_0(x)}^{u(t,x)} h(\sigma)\,\mathrm{d}\sigma\right)\partial_t\varphi \;\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathbf{a}(t,x,\nabla u)\cdot\nabla(h(u)\varphi)\;\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f h(u)\varphi\;\,\mathrm{d}x\,\mathrm{d}t.$$

(R3p) $\displaystyle\int_{\{l<|u|<l+1\}} \mathbf{a}(t,x,\nabla u)\cdot\nabla u\;\,\mathrm{d}x\,\mathrm{d}t \to 0 \;\text{ as }\; l \to \infty.$

Remark 5.3.2. Condition *(R3p)* is the one ensuring the comparison principle and, consequently, also uniqueness.

We prove the existence of unique renormalized solutions to the general elliptic equation (4.70) under the assumptions *(A1p)–(A3p)* on the operator from Section 4.2.1. Note that *(Mp)* or *(Mp)$_p$* are given in Sections 4.2.2.1 and 4.2.2.2, respectively. They ensure approximation properties of the space via theorems of Section 4.2.2.

Theorem 5.3.3 (Existence and uniqueness of renormalized solutions) *Suppose that* $[0,T]$ *is a finite interval,* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^N$*,* $N > 1$*,* $f \in L^1(\Omega_T)$*,* $u_0 \in L^1(\Omega)$*, and a function* $\mathbf{a}$ *satisfy assumptions (A1p)–(A3p) with an N-function* $M : [0,T]\times\Omega\times\mathbb{R}^N \to [0,\infty)$ *satisfying (Mp) or (Mp)$_p$. Then there exists a unique renormalized solution to the problem* (4.70)*. Namely, there exists a unique function u which satisfies (R1p)–(R3p) of Definition 5.3.1.*

Remark 5.3.4. Similarly as in the case of weak solutions, the renormalized solutions to parabolic equations have been considered under the assumption that $M^* \in \Delta_2$ and M is independent of time, see Theorem 1.1 in [188].

Note that in the general classical Orlicz setting (including fully anisotropic spaces) we can skip assumption *(Mp) / (Mp)$_p$*.

Remark 5.3.5 (Skipping *(Mp) / (Mp)$_p$* – Orlicz case). In the pure Orlicz case, i.e. when

$$M(t,x,\xi) = M(\xi),$$

the balance conditions do not carry any information. Therefore, as a direct consequence of Theorem 5.2.3 we get the existence of renormalized solutions to the parabolic problem (4.70) under conditions therein in an anisotropic Orlicz space without growth restrictions. Namely, whenever $\Omega \subset \mathbb{R}^N$, $N > 1$, $T < \infty$, $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, and the function $\mathbf{a}$ satisfies assumptions *(A1p)–(A3p)* with a homogeneous N-function M. This analysis covers the classical power-growth problems, i.e. when $M(x,\xi) = |\xi|^p$ with $1 < p < \infty$ in Sobolev spaces (when $\nabla u \in L^1(0,T;W_0^{1,p}(\Omega))$ for the p-Laplace problem $\partial_t u - \Delta_p u = f$, we study

$$\partial_t u - \operatorname{div}\big(b(t,x)|\nabla u|^{p-2}\nabla u\big) = f(t,x) \in L^1(\Omega_T)$$

with bounded $b : \Omega_T \to [0,\infty)$ such that $0 < b_- \le b(\cdot) \le b_+ < \infty$. On the other hand, it also covers the case of $M(x,\xi) = |\xi|\log^\alpha(1+|\xi|)$, $\alpha \ge 0$, and consequently problems posed in $L\log^\alpha L$ spaces, e.g.

$$\partial_t u - \operatorname{div}\Big(b(t,x)\tfrac{\log^\alpha(\mathrm{e}+|\nabla u|)}{|\nabla u|}\nabla u\Big) = f(t,x) \in L^1(\Omega_T)$$

with bounded and measurable $b : \Omega_T \to [0,\infty)$ such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.

We infer the existence of renormalized solutions in the following cases.

Example 5.3.6 (Problems under condition *(Mp)*).

- When $M(x,\xi) = |\xi|^{p(t,x)}$ in variable exponent spaces with log-Hölder continuous $p : \Omega_T \to (1,\infty)$ such that $1 < p_- \le p(\cdot) \le p_+ < \infty$, we study

$$\partial_t u - \operatorname{div}\big(b(t,x)|\nabla u|^{p(t,x)-2}\nabla u\big) = f(t,x) \in L^1(\Omega_T)$$

  with bounded and measurable $b : \Omega_T \to [0,\infty)$ such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.
- When $M(x,\xi) = |\xi|^p + a(t,x)|\xi|^p\log(\mathrm{e}+|\xi|)$ in double phase spaces with mild transition, with $1 < p < \infty$ and with a log-Hölder and possibly touching zero weight $a : \Omega_T \to [0,\infty)$, we study

$$\partial_t u - \operatorname{div}\Big(b(t,x)\big(1+a(t,x)\log(\mathrm{e}+|\nabla u|)\big)|\nabla u|^{p-2}\nabla u\Big) = f(t,x) \in L^1(\Omega_T)$$

  where $b : \Omega_T \to [0,\infty)$ is bounded, measurable, and such that $0 < b_- \le b(\cdot) \le b_+ < \infty$.

Example 5.3.7 (Problems under condition *(Mp)$_p$*).

- When $M(x,\xi) = |\xi|^p + a(t,x)|\xi|^q$ in double phase spaces, with $1 < p,q < \infty$ and a function $a : \Omega_T \to [0,\infty)$ being such that $a \in C^{0,\alpha}(\Omega_T)$ and possibly touching zero; we study

$$\partial_t u - \operatorname{div}\Big(b(t,x)\big(|\nabla u|^{p-2}\nabla u + a(t,x)(|\nabla u|^{q-2}\nabla u\big)\Big) = f(t,x) \in L^1(\Omega_T),$$

  where $b : \Omega_T \to [0,\infty)$ is bounded, measurable, and such that $0 < b_- \le b(\cdot) \le b_+ < \infty$ and $\frac{q}{p} \le 1 + \frac{\alpha}{N}$.

- When $M(t,x,\xi) = |\xi|^{p(t,x)} + a(t,x)|\xi|^{q(t,x)}$ in variable exponent double-phase spaces with log-Hölder $p,q : \Omega_T \to (1,\infty)$ such that $1 < p_- < p(t,x) < q(t,x) \leq c < \infty$ and a function $a : \Omega_T \to [0,\infty)$ being such that $a \in C^{0,\alpha}(\Omega_T)$ and possibly touching zero; we study

$$\partial_t u - \operatorname{div}\Big(b(t,x)\big(|\nabla u|^{p(t,x)-2}\nabla u + a(t,x)(|\nabla u|^{q(t,x)-2}\nabla u\big)\Big) = f(t,x) \in L^1(\Omega_T),$$

where $b : \Omega_T \to [0,\infty)$ is bounded, measurable, and such that $0 < b_- \leq b(\cdot) \leq b_+ < \infty$ $\sup_{(t,x)\in\Omega_T} \big(q(t,x) - p(t,x)\big) \leq \alpha \frac{p_-}{N}$.

Example 5.3.8 (Orlicz double phase space). When

$$M(t,x,\xi) = M_1(\xi) + a(t,x)M_2(\xi),$$

where M_1, M_2 are (possibly anisotropic) homogeneous N-functions without prescribed growth such that $M_1(\xi) \leq M_2(\xi)$ for ξ such that $|\xi| > 1$, and moreover the function $a : \Omega_T \to [0,\infty)$ is bounded and has a modulus of continuity denoted by ω_a, we infer existence and uniqueness for solutions to the problem

$$u_t - \operatorname{div}\left(b(x)\left(\frac{M_1(\nabla u)}{|\nabla u|^2} \cdot \nabla u + a(x)\frac{M_2(\nabla u)}{|\nabla u|^2} \cdot \nabla u\right)\right) = f(x) \in L^1(\Omega_T)$$

with measurable b such that $0 < b_- \leq b(\cdot) \leq b_+ < \infty$, provided

$$\limsup_{\delta\to 0} \omega_a(\delta) \frac{\overline{M_2}(\delta^{-N})}{\underline{M}_1(\delta^{-N})} < \infty,$$

where $\underline{M}_1(s) := \inf_{\xi:\,|\xi|=s} M_1(\xi)$ and $\overline{M_2}(s) := \sup_{\xi:\,|\xi|=s} M_2(\xi)$, or – when M_1 has at least power growth – provided

$$\limsup_{\delta\to 0} \omega_a(\delta) \frac{\overline{M_2}(\delta^{-N/p})}{\underline{M}_1(\delta^{-N/p})} < \infty.$$

One can easily modify this example to get its variable exponent-type version or to involve more than two phases. Other choices of M coming from Examples 4.2.2 and 4.2.3 generate a wide range of examples.

5.3.2 Approximation in time

Unlike the proof of existence of weak solutions, we need two more subtle approximation results, which are called ‘Approximation in time’ to distinguish from ‘Approximation in space’ from Section 3.7. The first one in fact states that under our regime right and left Steklov averages of a function converge modularly to this function. This part was not needed in [79], due to the lack of time-dependence of M. Indeed, therein the following approximation result follows directly from Jensen’s inequality. Here we need to carefully examine the uniform estimate and convergence. Recall that *(Mp)* is given in Section 4.2.2.1, whereas *(Mp)$_p$* in Section 4.2.2.2.

Proposition 5.3.9 (Approximation in time I) *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N>1$, an N-function $M:[0,T]\times\Omega\times\mathbb{R}^N\to[0,\infty)$ satisfies condition (Mp) or $(Mp)_p$, and $\varphi\in V_T^{M,\infty}(\Omega)$. Consider linear mappings $\varphi\mapsto\varphi_d$ and $\varphi\mapsto\widetilde{\varphi}_d$, given by*

$$\varphi_d(t,x):=\frac{1}{d}\int_t^{t+d}\varphi(\sigma,x)\,\mathrm{d}\sigma \qquad \text{and} \qquad \widetilde{\varphi}_d(t,x):=\frac{1}{d}\int_{t-d}^{t}\varphi(\sigma,x)\,\mathrm{d}\sigma. \tag{5.40}$$

For $d\to 0$, both $\varphi_d\to\varphi$ and $\widetilde{\varphi}_d\to\varphi$ converge strongly in $W^{1,1}_{loc}(\Omega)$. Moreover,

$$\nabla(\varphi_d)\xrightarrow[d\to 0]{}\nabla\varphi \quad \text{and} \quad \nabla(\widetilde{\varphi}_d)\xrightarrow[d\to 0]{}\nabla\varphi \quad \text{modularly in } L_M(\Omega_T;\mathbb{R}^N).$$

Furthermore, $\|\varphi_d\|_{L^\infty(\Omega_T)}\le\|\varphi\|_{L^\infty(\Omega_T)}$ and $\|\widetilde{\varphi}_d\|_{L^\infty(\Omega_T)}\le\|\varphi\|_{L^\infty(\Omega_T)}$.

Before the proof, we need to provide the following uniform estimate.

Lemma 5.3.10 *Suppose an N-function M satisfies assumptions (Mp) or $(Mp)_p$. Consider the linear mapping $\varphi\mapsto\widetilde{\varphi}_d$ given by* (5.40). *Then, there exists a constant $C>0$, independent of d, such that for all sufficiently small $d>0$ and every $\eta\in V_T^{M,\infty}(\Omega)$ it holds that*

$$\begin{aligned}\int_{\Omega_T}M(t,x,\widetilde{\eta}_d(t,x))\,\mathrm{d}x\,\mathrm{d}t\le&\int_{\{m_1(|\eta(\cdot,\cdot)|)\le 1\}}m_2(|\eta(t,x)|)\,\mathrm{d}x\,\mathrm{d}t\\&+C\int_{\Omega_T}M\left(t,x,\eta(t,x)\right)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{5.41}$$

Proof. Fix arbitrary $\eta\in V_T^{M,\infty}(\Omega)$ and small $d>0$. The proof is similar to that of Proposition 3.7.10. First we notice that

$$\begin{aligned}&\int_{\Omega_T}M(t,x,\widetilde{\eta}_d(t,x))\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le\int_{\{M(\cdot,\cdot,\widetilde{\eta}_d)\le 1\}}M(t,x,\widetilde{\eta}_d(t,x))\,\mathrm{d}x\,\mathrm{d}t+\int_{\{M(\cdot,\cdot,\widetilde{\eta}_d)\ge 1\}}M(t,x,\widetilde{\eta}_d(t,x))\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le\int_{\{m_1(|\widetilde{\eta}_d|)\le 1\}}m_2(|\widetilde{\eta}_d(t,x)|)\,\mathrm{d}x\,\mathrm{d}t+\int_{\{M(\cdot,\cdot,\widetilde{\eta}_d)\ge 1\}}M(t,x,\widetilde{\eta}_d(t,x))\,\mathrm{d}x\,\mathrm{d}t\\&\quad=:\mathsf{I}_d+\mathsf{J}_d.\end{aligned}$$

To deal with I_d we notice that $\{m_1(|\widetilde{\eta}_d(\cdot)|)\le 1\}=\{m_2(|\widetilde{\eta}_d(\cdot)|)\le c\}$ for $c=m_2\circ m_1^{-1}(1)$ and we have the following pointwise estimate

$$m_2(|\widetilde{\eta}_d(\cdot)|)\mathbb{1}_{\{m_1(|\widetilde{\eta}_d(\cdot)|)\le 1\}}(\cdot)\le c.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\limsup_{d\searrow 0}\mathsf{I}_d=\limsup_{d\searrow 0}\int_{\{m_1(|\widetilde{\eta}_d|)\le 1\}}m_2(|\widetilde{\eta}_d(x)|)\,\mathrm{d}x\,\mathrm{d}t=\int_{\{m_1(|\eta|)\le 1\}}m_2(|\eta|)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.42}$$

Thus, we concentrate now on J_d. In order to make use of balance condition *(Mp)* or *(Mp)$_p$* we need to find a proper division of the interval $[0,T]$ such that we are able to estimate the infimum of M over a small sub-interval by an infimum over a cylinder. Within this proof we understand that M is extended by 0 outside $[0,T]$.

We define intervals $I_i^d = [t_i^d, t_{i+1}^d)$, for $i = 1,\ldots,N_d^T$, such that $|I_i^d| < d$ and

$$\widetilde{I}_i^d := [t_i^d - d, t_{i+1}^d) \cap [0,T]. \tag{5.43}$$

Of course

$$|\widetilde{I}_i^d| \le 2|I_i^d| < 2d.$$

We employ

$$M_{i,d}(x,\xi) := \inf_{t\in\widetilde{I}_i^d} M(t,x,\xi)$$

and its second conjugate $(M_{i,d})^{**}(x,\xi) = ((M_{i,d}(x,\xi))^*)^*$. Recall also the notation for the infimum over a cylinder $M_{i,j}^\delta$ introduced in (4.85). For $y > 0$ we denote by $\lceil y\rceil$ the smallest natural number larger than or equal to y. Since for every $i = 1,\ldots,N_d^T$ it holds that

$$\widetilde{I}_i^d \subset I_{\lceil i/2\rceil}^{2d},$$

for a.e. $x \in \widetilde{Q}_j^d$, $j = 1,\ldots,N_d$, and $i = 1,\ldots,N_d^T$, we have

$$M_{\lceil i/2\rceil,j}^{2d}(\xi) \le M_{i,d}(x,\xi).$$

Therefore, for a.a. (t,x) in $I_{\lceil i/2\rceil}^{2d} \times \widetilde{Q}_j^d$ we have

$$\frac{M(t,x,\xi)}{(M_{i,d})^{**}(x,\xi)} \le \frac{M(t,x,\xi)}{(M_{\lceil i/2\rceil,j}^{2d})^{**}(\xi)}.$$

The right-hand side above can be further estimated by the use of our balance assumption. We get

$$\frac{M(t,x,\xi)}{(M_{i,d})^{**}(x,\xi)} \le \Theta(2d,|\xi|). \tag{5.44}$$

We have $\eta \in V_T^{M,\infty}(\Omega)$, small $d > 0$, and $\widetilde{\eta}_d$ given by (5.40). Our goal is to obtain (5.41). Let us notice that when we split the time interval we may write

$$\begin{aligned}\mathsf{J}_d &= \sum_{i=1}^{N_d^T} \int_\Omega \int_{I_i^d} M(t,x,\widetilde{\eta}_d(t,x))\mathbb{1}_{\{M(\cdot,\widetilde{\eta}_d)\ge 1\}}\,\mathrm{d}t\,\mathrm{d}x\\ &= \sum_{i=1}^{N_d^T} \int_\Omega \int_{I_i^d} \frac{M(t,x,\widetilde{\eta}_d(t,x))}{(M_{i,d})^{**}(x,\widetilde{\eta}_d(t,x))}(M_{i,d})^{**}(x,\widetilde{\eta}_d(t,x))\mathbb{1}_{\{M(\cdot,\widetilde{\eta}_d)\ge 1\}}\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{5.45}$$

We used above that $M(t,x,\xi) = 0$ whenever $\xi = 0$. Now we need to estimate the fraction in the last integral. For any $x \in \Omega$ we choose Q_j^d including x. Then, by (5.44) we infer that for arbitrary $t \in I_i^d$,

$$\frac{M(t,x,\widetilde{\eta}_d(t,x))}{(M_{i,d})^{**}(x,\widetilde{\eta}_d(t,x))} \le \Theta\left(2d, |\widetilde{\eta}_d(t,x)|\right). \tag{5.46}$$

Our aim now is to estimate the quantity from (5.46) by a constant independent of x,t,i,j, and d. Since without loss of generality it can be assumed that

$$\|\eta\|_{L^\infty(0,T;L^\infty(\Omega))} \le 1,$$

we have

$$|\widetilde{\eta}_d(t,x)| \le \frac{1}{d}\int_{t-d}^{t} |\eta(s,x)|\, ds \le |\Omega|\, \|\eta\|_{L^\infty(0,T;L^\infty(\Omega))} \le c(\Omega). \tag{5.47}$$

By monotonicity of Θ, we have

$$\Theta(2d, |\widetilde{\eta}_d(t,x)|) \le \Theta(2d, c(\Omega)),$$

which by assumption *(Mp)* (or *(Mp)$_p$*) can be estimated further by a uniform constant c. Thus, the right-hand side of (5.46) can be estimated by the same c. By applying it in (5.45) we get that

$$\mathsf{J}_d \le c\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}(M_{i,d})^{**}\left(x, \int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)\eta(t-\sigma,x)\,\mathrm{d}\sigma\right)\mathrm{d}t\;\mathrm{d}x =: \mathsf{J}_d^1.$$

By extending the domain of integration, we notice that

$$\begin{aligned}
\mathsf{J}_d^1 &= c\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}(M_{i,d})^{**}\left(x, \int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)\mathbb{1}_{I_i^d}(t)\eta(t-\sigma,x)\,\mathrm{d}\sigma\right)\mathrm{d}t\;\mathrm{d}x\\
&\le c\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}(M_{i,d})^{**}\left(x, \int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)\mathbb{1}_{\widetilde{I}_i^d}(t-\sigma)\eta(t-\sigma,x)\,\mathrm{d}\sigma\right)\mathrm{d}t\;\mathrm{d}x =: \mathsf{J}_d^2.
\end{aligned}$$

Continuing the estimates with the use of Jensen's inequality and the fact that the second conjugate is the greatest convex minorant (Corollary 2.1.42), we get

$$\begin{aligned}
\mathsf{J}_d^2 &\le c\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}\int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)(M_{i,d})^{**}\left(x, \mathbb{1}_{\widetilde{I}_i^d}(t-\sigma)\eta(t-\sigma,x)\right)\mathrm{d}\sigma\;\mathrm{d}t\;\mathrm{d}x\\
&\le c\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}\int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)M\left(t-\sigma,x,\eta(t-\sigma,x)\right)\mathrm{d}\sigma\;\mathrm{d}t\;\mathrm{d}x =: \mathsf{J}_d^3.
\end{aligned}$$

We can compute the sum above and apply Young's convolution inequality (Lemma 8.26) to obtain

$$
\begin{aligned}
\mathsf{J}^3_d &\le c\int_\Omega\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(\sigma)M(t-\sigma,x,\eta(t-\sigma,x))\,\mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}x\\
&\le c\int_\Omega\frac{1}{d}\|\mathbb{1}_{[0,d)}(\cdot)\|_{L^1(\mathbb{R})}\cdot\|M(\cdot,x,\eta(\cdot,x))\|_{L^1(\mathbb{R})}\,\mathrm{d}x\\
&\le C\|M(\cdot,\cdot,\eta(\cdot,\cdot))\|_{L^1(\Omega_T)}.
\end{aligned}
$$

To sum up, we have shown that

$$
\mathsf{J}_d \le \mathsf{J}^1_d \le \mathsf{J}^2_d \le \mathsf{J}^3_d \le C\int_{\Omega_T} M(t,x,\eta(t,x))\,\mathrm{d}x\,\mathrm{d}t
$$

which in the view of (4.84) concludes the proof. □

Remark 5.3.11. Minor modifications lead to the same result for $\varphi \mapsto \varphi_d$. In the case of $\varphi \mapsto \varphi_d$ in (5.43) we should extend the interval to the right, namely we should consider $(t^d_i, t^d_{i+1}+d]\cap[0,T]$.

We are in position to prove the approximation in time of the regularizations defined in (5.40).

Proof (of Proposition 5.3.9). We show the modular convergence $\nabla(\widetilde{\varphi}_d) \to \nabla\varphi$ only, because for the justification of modular convergence $\nabla(\varphi_d) \to \nabla\varphi$ one uses precisely the same reasoning. Then main tool is Lemma 5.3.10. From the definition of this regularization we directly infer that

$$
\varphi_d \in W^{1,\infty}(0,T;V_T^{M,\infty}(\Omega)) \quad \text{and} \quad \nabla(\widetilde{\varphi}_d) = \widetilde{(\nabla\varphi)}_d.
$$

It suffices now to prove the modular convergence

$$
\nabla(\varphi_d) \xrightarrow[d\to 0]{M} \nabla\varphi \quad \text{in} \quad L_M(\Omega_T;\mathbb{R}^N).
$$

We will construct our approximation using simple functions that are dense in L_M in the modular topology, see Theorem 3.4.11. We take a family of measurable sets $\{\tilde{E}_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}}\tilde{E}_n = \Omega_T$ and a sequence of simple vector-valued functions $\{\tilde{E}^n\}_{n\in\mathbb{N}}$ given by

$$
\tilde{E}^n(t,x) = \sum_{j=0}^{n}\mathbb{1}_{\tilde{E}_j}(t,x)\widetilde{\eta}_j,
$$

where $\{\widetilde{\eta}_j\}_{j=0}^n$ is a family of vectors, such that $\{\tilde{E}^n\}_{n\in\mathbb{N}}$ converges modularly to $\nabla\varphi$ with $\tilde{\lambda}_3$ as $n\to\infty$ (cf. Definition 3.4.3). We write a telescopic sum

$$
\nabla(\widetilde{\varphi}_d) - \nabla\varphi = \big(\nabla(\widetilde{\varphi}_d) - (\tilde{E}^n)_d\big) + ((\tilde{E}^n)_d - \tilde{E}^n) + (\tilde{E}^n - \nabla\varphi).
$$

It is enough to prove the convergence of each of these terms. Indeed, by Jensen's inequality we have

$$\begin{aligned}\int_{\Omega_T} M\left(t,x,\frac{\nabla(\widetilde{\varphi}_d)-\nabla\varphi}{\tilde\lambda}\right)\,\mathrm{d}x\,\mathrm{d}t \le{}& \frac{\tilde\lambda_1}{\tilde\lambda}\int_{\Omega_T} M\left(t,x,\frac{\nabla(\widetilde{\varphi}_d)-(\tilde E^n)_d}{\tilde\lambda_1}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{\tilde\lambda_2}{\tilde\lambda}\int_{\Omega_T} M\left(t,x,\frac{(\tilde E^n)_d-\tilde E^n}{\tilde\lambda_2}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{\tilde\lambda_3}{\tilde\lambda}\int_{\Omega_T} M\left(t,x,\frac{\tilde E^n-\nabla\varphi}{\tilde\lambda_3}\right)\,\mathrm{d}x\,\mathrm{d}t\\
={}& L_1^{n,d}+L_2^{n,d}+L_3^n,\end{aligned}$$

where $\tilde\lambda=\sum_{i=1}^3\tilde\lambda_i$, $\tilde\lambda_i>0$. We have $\tilde\lambda_3$ fixed already for L_3^n to be small. We take $\tilde\lambda_1=\tilde\lambda_3$, and $\tilde\lambda_2$ will be chosen soon. Modular convergence will follow provided we can pass to the limit with all of the terms in the right-hand side tending to zero.

In order to pass to the limit as $d\to 0$, we can estimate

$$\begin{aligned}0\le L_1^{n,d}\le{}& \int_{\left\{m_1\left(\frac{|\tilde E^n-\sum_{i\in I}\nabla(\theta_i\varphi)|}{\tilde\lambda_1}\right)\le 1\right\}} m_2\left(\frac{|\tilde E^n-\sum_{i\in I}\nabla(\theta_i\varphi)|}{\tilde\lambda_1}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+C\int_{\Omega_T} M\left(t,x,\frac{\tilde E^n-\nabla\varphi}{\tilde\lambda_3}\right)\,\mathrm{d}x\,\mathrm{d}t =: K^n,\end{aligned}$$

where $\lim_{n\to\infty}K^n=0$. Consequently,

$$\lim_{n\to\infty}\limsup_{d\to 0} L_1^{n,d}=0.$$

In the case of $L_2^{n,d}$, by Jensen's inequality and then Fubini's theorem we obtain

$$\begin{aligned}\frac{\tilde\lambda}{\tilde\lambda_2}L_2^{n,d}={}&\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d} M\Bigg(t,x,\\
&\frac{1}{\tilde\lambda_2}\int_{\mathbb{R}}\frac{1}{d}\mathbb{1}_{[0,d)}(s)\sum_{j=0}^{n}[\mathbb{1}_{\tilde E_j}(t,x)\widetilde{\eta}_j(t,x)-\mathbb{1}_{\tilde E_j}(s-t,x)\widetilde{\eta}_j(s-t,x)]\,\mathrm{d}s\Bigg)\,\mathrm{d}t\,\mathrm{d}x\\
\le{}&\sum_{i=1}^{N_d^T}\int_\Omega\int_{I_i^d}\frac{1}{d}\mathbb{1}_{[0,d)}(s)M\Bigg(t,x,\\
&\frac{1}{\tilde\lambda_2}\sum_{j=0}^{n}[\mathbb{1}_{\tilde E_j}(t,x)\widetilde{\eta}_j(t,x)-\mathbb{1}_{\tilde E_j}(s-t,x)\widetilde{\eta}_j(s-t,x)]\Bigg)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\
\le{}&\int_{\Omega_i}\sum_{i=1}^{N_d^T}\int_{I_i^d} M\left(t,x,\frac{1}{\tilde\lambda_2}\sum_{j=0}^{n}[\mathbb{1}_{\tilde E_j}(t,x)\widetilde{\eta}_j(t,x)-\mathbb{1}_{\tilde E_j}(s-t,x)\widetilde{\eta}_j(s-t,x)]\right)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{5.48}$$

Since the shift operator is continuous in L^1, we have pointwise convergence

$$\sum_{j=0}^{n}[\mathbb{1}_{\tilde{E}_j}(t,x)\widetilde{\eta}_j(t,x)-\mathbb{1}_{\tilde{E}_j}(s-t,x)\widetilde{\eta}_j(s-t,x)]\xrightarrow[d\to 0]{}0,$$

because $s-t<d$. Moreover, for arbitrary $\tilde{\lambda}_2>0$ one can estimate

$$M\left(t,x,\frac{1}{\tilde{\lambda}_2}\sum_{j=0}^{n}[\mathbb{1}_{\tilde{E}_j}(t,x)\widetilde{\eta}_j(t,x)-\mathbb{1}_{\tilde{E}_j}(s-t,x)\widetilde{\eta}_j(s-t,x)]\right)$$
$$\leq \sup_{\zeta\in\mathbb{R}^N:\ |\zeta|=1} M\left(t,x,\frac{1}{2\tilde{\lambda}_2}\sum_{j=0}^{n}\|\widetilde{\eta}_j\|_{L^\infty(\tilde{E}_j)}\zeta\right)<\infty.$$

Thus, the Lebesgue dominated convergence theorem justifies convergence to zero of the right-hand side of (5.48).

We have proved that $L_1^{n,d}+L_2^{n,d}+L_3^n$ converges to zero, which completes the proof of modular convergence of the approximate sequence. The modular convergence of gradients implies their strong L^1-convergence and the Poincaré inequality ends the proof. Of course, by (5.40) the L^∞-norm is preserved too. □

One more precise approximation result is needed. It has to converge modularly, commute with the space gradient, and have properly convergent time derivatives. When the modular function is time-dependent we cannot use the Landes regularization coming from [228], as was done in [188, 79]. The reason is that in this case the Landes regularization no longer maps L_M into itself. Moreover, we shall need a few more delicate properties here. Nonetheless, a careful merging of the ideas of Landes on the small but not uniformly controlled time intervals enables us to prove the following result. We essentially need a balance condition in the proof, but it turns out to be less demanding than *(Mp)* and *(Mp)*$_p$, which are imposed anyway for other approximation theorems applied in the proof of existence of weak and renormalized solutions. This result was proved for the first time in [81].

Theorem 5.3.12 (Approximation in time II) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, an N-function $M:[0,T]\times\Omega\times\mathbb{R}^N\to[0,\infty)$ satisfy condition (Mp) or (Mp)$_p$, $\varphi\in V_T^M(\Omega)$, and $\varphi_0\in L^\infty(\Omega)$. Then there exist sequences*

$$\{\varphi_\mu\}_{\mu>2},\{\varphi_\mu^\bullet\}_{\mu>2}\subset V_T^M(\Omega),\quad \{(\nabla\varphi)_\mu^\bullet\}_{\mu>2}\subset L_M(\Omega_T;\mathbb{R}^N)$$

such that

(i) *for every μ and a.e. $x\in\Omega$ the function $\varphi_\mu(\cdot,x)$ is in $C^\infty([0,T))$ and satisfies*

$$\begin{cases}\partial_t\varphi_\mu &=\mu(\varphi-\varphi_\mu)\ \ a.e.\ in\ \Omega_T,\\ \varphi_\mu(0,x)&=\varphi_0(x)\quad a.e.\ in\ \Omega,\end{cases}\tag{5.49}$$

(ii) *for every μ we have $\varphi_\mu^\bullet(0,x)=\varphi_0(x)(1-\mathrm{e}^{-\log^2\mu})$,*
(iii) $(\nabla\varphi)_\mu^\bullet=\nabla(\varphi_\mu^\bullet)$,
(iv) $\varphi_\mu^\bullet\xrightarrow[\mu\to\infty]{}\varphi$ *strongly in $L^1(\Omega_T)$ and* $(\nabla\varphi)_\mu^\bullet\xrightarrow[\mu\to\infty]{M}\nabla\varphi$ *modularly in $L_M(\Omega_T;\mathbb{R}^N)$.*

(v) *If additionally* $\varphi \in L^\infty(\Omega_T)$*, then* $\|\varphi^\bullet_\mu\|_{L^\infty(\Omega_T)} \le \|\varphi\|_{L^\infty(\Omega_T)}$*, for every* μ *and a.e.* $x \in \Omega$ *the function* $\varphi^\bullet_\mu(\cdot,x)$ *belongs to* $W^{1,\infty}([0,T])$ *and furthermore*

$$\lim_{\mu\to\infty} \|\varphi_\mu - \varphi^\bullet_\mu\|_{L^\infty(\Omega_T)} = 0 \qquad \textit{and} \qquad \lim_{\mu\to\infty} \|\partial_t(\varphi_\mu - \varphi^\bullet_\mu)\|_{L^\infty(\Omega_T)} = 0.$$

The approximate sequence is constructed as a truncated convolution with particular kernel. For a measurable function $\xi : \mathbb{R}\times\Omega \to \mathbb{R}^N$ and

$$\varrho_\mu(s) = \mu \mathrm{e}^{-\mu s} \mathbb{1}_{[0,\infty)}(s), \qquad \mu > 2,$$

the regularized function $\xi_\mu : \mathbb{R}\times\Omega \to \mathbb{R}$ is defined by

$$\xi_\mu(t,x) := (\varrho_\mu * \xi)(t,x),$$

where $*$ stands for the convolution in the time variable. Then

$$\xi_\mu(t,x) = \mu \int_{-\infty}^{t} \mathrm{e}^{\mu(s-t)} \xi(s,x)\,\mathrm{d}s. \tag{5.50}$$

Define further

$$\xi^\bullet_\mu(t,x) = \mu \int_{t-\varepsilon(\mu)}^{t} \mathrm{e}^{\mu(s-t)} \xi(s,x)\,\mathrm{d}s \qquad \text{with} \qquad \varepsilon(\mu) = \frac{\log^2 \mu}{\mu}. \tag{5.51}$$

We provide a uniform estimate in the following lemma and then conclude the proof of Theorem 5.3.12.

Lemma 5.3.13 *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^N$*. Suppose an N-function* $M : [0,T]\times\Omega\times\mathbb{R}^N \to [0,\infty)$ *satisfies assumptions (Mp) or* $(Mp)_p$*. We extend an arbitrary* $\xi \in V_T^{M,\infty}(\Omega)$ *by* $\xi(0,x)$ *on* $(-\infty,0)$ *and by* 0 *on* (T,∞)*. If* $\xi^\bullet_\mu$ *is given by* (5.51)*, then there exist constants* $C_1, C_2 > 0$ *independent of* μ*, such that for all* $\mu > 2$ *and every* $\xi \in V_T^{M,\infty}(\Omega)$ *we have*

$$\int_{\Omega_T} M(t,x,\xi^\bullet_\mu(t,x))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\{m_1(|\xi|)\le 1\}} m_2(|\xi|)\,\mathrm{d}x\,\mathrm{d}t + C_1 \int_{\Omega_T} M(t,x,C_2\xi(t,x))\,\mathrm{d}x\,\mathrm{d}t. \tag{5.52}$$

Proof. The beginning of the proof is very similar to the proofs of Proposition 3.7.10 and Lemma 5.3.10, where we give more comments on the method. We need, however, to split the time interval in a more delicate way. First we notice that

$$\int_{\Omega_T} M(t,x,\xi^\bullet_\mu(t,x))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\{M(\cdot,\cdot,\xi^\bullet_\mu)\le 1\}} M(t,x,\xi^\bullet_\mu(t,x))\,\mathrm{d}x\,\mathrm{d}t + \int_{\{M(\cdot,\cdot,\xi^\bullet_\mu)\ge 1\}} M(t,x,\xi^\bullet_\mu(t,x))\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_{\{m_1(|\xi_\mu^\bullet|)\leq 1\}} m_2(|\xi_\mu^\bullet(t,x)|)\,\mathrm{d}x\,\mathrm{d}t + \int_{\{M(\cdot,\cdot,\xi_\mu^\bullet)\geq 1\}} M(t,x,\xi_\mu^\bullet(t,x))\,\mathrm{d}x\,\mathrm{d}t$$
$$=: \mathsf{I}_\mu + \mathsf{J}_\mu.$$

To deal with I_μ we notice that $\{m_1(|\xi_\mu^\bullet(\cdot)|)\leq 1\} = \{m_2(|\xi_\mu^\bullet(\cdot)|)\leq c\}$ for $c = m_2\circ m_1^{-1}(1)$ and we have the following pointwise estimate

$$m_2(|\xi_\mu^\bullet(\cdot)|)\mathbb{1}_{\{m_1(|\xi_\mu^\bullet(\cdot)|)\leq 1\}}(\cdot) \leq c.$$

Hence, Lebesgue's dominated convergence theorem enables us to justify that

$$\limsup_{\mu\nearrow\infty} \mathsf{I}_\mu = \limsup_{\mu\nearrow 0} \int_{\{m_1(|\xi_\mu^\bullet|)\leq 1\}} m_2(|\xi_\mu^\bullet(x)|)\,\mathrm{d}x\,\mathrm{d}t = \int_{\{m_1(|\xi|)\leq 1\}} m_2(|\xi|)\,\mathrm{d}x\,\mathrm{d}t.$$

We concentrate now on J_μ. We fix an arbitrary parameter $\mu > 2$ and consider families $\{I_i^{\frac{1}{\mu}}\}_{i\in\mathrm{I}}$ and $\{J_i^{\frac{1}{\mu}}\}_{i\in\mathrm{I}}$ of time intervals

$$I_i^{\frac{1}{\mu}} = \left[t_i^{\frac{1}{\mu}}, t_{i+1}^{\frac{1}{\mu}}\right)$$

covering $[0,T]$ and such that $|I_i^{\frac{1}{\mu}}| = |I_k^{\frac{1}{\mu}}| = \frac{1}{\mu}$ for every $i,k > 1$ and

$$J_i^{\frac{1}{\mu}} := \left[t_i^{\frac{1}{\mu}}, t_{i+1}^{\frac{1}{\mu}} + \varepsilon(\mu)\right) \qquad \text{and} \qquad |J_i^{\frac{1}{\mu}}| < \frac{1}{\mu} + \varepsilon(\mu) =: \nu(\mu),$$

where $\varepsilon(\mu)$ is given by (5.51). Let us stress that the $J_i^{\frac{1}{\mu}}$ are shrinking, that is

$$\lim_{\mu\to\infty} |J_i^{\frac{1}{\mu}}| = \lim_{\mu\to\infty} \nu(\mu) = 0.$$

We consider infima over small time sub-intervals

$$M_{i,\frac{1}{\mu}}(x,\eta) = \inf\left\{M(t,x,\eta):\ \ t\in J_i^{\frac{1}{\mu}}\cap[0,T]\right\},$$

and their second conjugates $(M_{i,\frac{1}{\mu}})^{**}$ coinciding with the greatest convex minorant (see Theorem 2.1.41). Since $M(t,x,\xi) > 0$ in $\{M(\cdot,\cdot,\xi_\mu^\bullet)\leq 1\}$, we have

$$\begin{aligned}\mathsf{J}_\mu &= \sum_{i=1}^{N_{1/\mu}^t} \int_\Omega \int_{I_i^{\frac{1}{\mu}}} M(t,x,\xi_\mu^\bullet(t,x))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\mu^\bullet)\leq 1\}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ &= \sum_{i=1}^{N_{1/\mu}^t} \int_\Omega \int_{I_i^{\frac{1}{\mu}}} \frac{M(t,x,\xi_\mu^\bullet(t,x))}{(M_{i,\frac{1}{\mu}})^{**}(x,\xi_\mu^\bullet(t,x))}(M_{i,\frac{1}{\mu}})^{**}(x,\xi_\mu^\bullet(t,x))\mathbb{1}_{\{M(\cdot,\cdot,\xi_\mu^\bullet)\leq 1\}}(t,x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{5.53}$$

To estimate the fraction in the last integral by a constant independent of i and μ, we need to employ the balance condition *(Mp)* or *(Mp)$_p$*. For this we note that for every $i \in \{1,\dots,N^t_{1/\mu}\}$ there exists a $k \in \{1,\dots,N^t_{\nu(\mu)}\}$ such that we have

$$J_i^{\frac{1}{\mu}} \subset \widehat{I}_i^{\nu(\mu)}$$

with $\widehat{I}_i^{\nu(\mu)}$ having the same properties as $I_i^{\frac{1}{\mu}}$ described above, but with length $\delta = \nu(\mu)$ and coming from some other division of $[0,T]$ chosen for each i. Conditions *(Mp)* or *(Mp)$_p$* are formulated with the use of an infimum over a cylinder, so we need to construct proper cylinders. Let $\{Q_j^\delta\}_{j=1}^{N_\delta}$ be a family of N-dimensional cubes covering the set Ω, which consists of closed cubes of edge length 2δ, such that

$$\operatorname{int} Q_j^\delta \cap \operatorname{int} Q_i^\delta = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \Omega \subset \bigcup_{j=1}^{N_\delta} Q_j^\delta.$$

With any cube Q_j^δ we associate a cube $\widetilde{Q}_j^\delta$ centered at the same point and with parallel corresponding edges of length 4δ. Define

$$\widehat{M}_{k,j}^{\nu(\mu)}(\xi) := \inf\{M(t,x,\eta) : \ t \in \widehat{I}_k^{\nu(\mu)} \cap [0,T],\ x \in \widetilde{Q}_j^{\nu(\mu)}\}.$$

Now we are in position to compare the infimum over a relevant cylinder with an infimum over a short interval. Since for a.e. $x \in \widetilde{Q}_j^{\nu(\mu)}$, $j = 1,\dots,N_{1/\mu}$, and $i = 1,\dots,N^t_{1/\mu}$ it holds that

$$\widehat{M}_{k,j}^{\nu(\mu)}(\xi) \le M_{i,\frac{1}{\mu}}(x,\xi),$$

hence we can estimate

$$\frac{M(t,x,\eta)}{(M_{i,\frac{1}{\mu}})^{**}(x,\eta)} \le \frac{M(t,x,\eta)}{(\widehat{M}_{k,j}^{\nu(\mu)})^{**}(\eta)} \le \Theta\big(\nu(\mu),|\eta|\big). \tag{5.54}$$

For every $x \in \Omega$ we can choose the cube including it and having the properties needed for the above estimate. The outcome of (5.54) is uniform with respect to $(t,x) \in \left(\widehat{I}_k^{\nu(\mu)} \cap [0,T]\right) \cap \widetilde{Q}_j^{\nu(\mu)}$.

As without loss of generality it can be assumed that $\|\xi\|_{L^\infty(0,T;L^1(\Omega))} \le 1$, we have

$$\begin{aligned} |\xi_\mu^\bullet(t,x)| &\le \mu \int_{t-\varepsilon(\mu)}^t \mathrm{e}^{\mu(s-t)}\, |\xi(s,x)|\, \mathrm{d}s \\ &\le c(\Omega)\mu\|\xi\|_{L^\infty(0,T;L^1(\Omega))} \le c(\Omega)\mu. \end{aligned} \tag{5.55}$$

Then for every $x \in \Omega$ we can choose a cube $\widetilde{Q}_j^{\nu(\mu)}$ including x. By (5.54) and *(Mp)* given in Section 4.2.2.1 (resp. *(Mp)$_p$* from Section 4.2.2.2), we observe that for arbitrary $(t,x) \in \left([0,T] \cap I_i^{\frac{1}{\mu}}\right) \times \left(\Omega \cap \widetilde{Q}_j^{\nu(\mu)}\right)$ we have

$$\frac{M(t,x,\xi^\bullet_\mu(t,x))}{(M_{i,\frac{1}{\mu}})^{**}(x,\xi^\bullet_\mu(t,x))} \le \Theta\Big(\nu(\mu),|\xi^\bullet_\mu(t,x)|\Big) \le \Theta\Big(\nu(\mu),c(\Omega)\mu\Big), \tag{5.56}$$

where the last inequality is justified by (5.55) and the monotonicity of Θ with respect to the second variable. This bound is uniform with respect to (t,x). Since Θ is nondecreasing with respect to the first variable we see that

$$\limsup_{\mu\to\infty}\Theta(\nu(\mu),c(\Omega)\mu) \le \limsup_{\delta\to 0}\Theta\Big((1+\log^2(c(\Omega)\delta^N))c(\Omega)\delta^N,\delta^{-N}\Big).$$

For all $\delta < \delta_0(N)$ we have

$$\Theta\Big((1+\log^2(c(\Omega)\delta^N))c(\Omega)\delta^N,\delta^{-N}\Big) \le \Theta(\delta,\delta^{-N}) < c < \infty,$$

where the last estimate holds due to *(Mp)*. In the case of *(Mp)$_p$* by the same arguments we get that

$$\limsup_{\mu\to\infty}\Theta_p(\nu(\mu),c(\Omega)\mu) < c < \infty.$$

In both cases we can estimate the right-hand side of (5.56) over a cube $\widetilde{Q}_j^{\nu(\mu)}$ by a constant c not depending on μ. In turn, in (5.53) we obtain

$$\mathsf{J}_\mu \le c\sum_{i=1}^{N^t_{1/\mu}}\int_\Omega\int_{I_i^{\frac{1}{\mu}}}(M_{i,\frac{1}{\mu}})^{**}(x,\xi^\bullet_\mu(t,x))\,\mathrm{d}t\,\mathrm{d}x. \tag{5.57}$$

We go back to (5.53). By Jensen's inequality with an intrinsic constant

$$c_J(\mu) = 1/(1-\mathrm{e}^{-\mu\varepsilon(\mu)}) \le 1/(1-\mathrm{e}^{-1}) = c_J(1) =: C_2 < 1,$$

we obtain

$$\begin{aligned}\mathsf{J}_\mu &\le c\sum_{i=1}^{N^t_\mu}\int_\Omega\int_{I_i^{\frac{1}{\mu}}}(M_{i,\frac{1}{\mu}})^{**}\left(x,\mu\int_{t-s-\varepsilon(\mu)}^{t-s}\mathrm{e}^{\mu(s-t)}\xi(s,x)\,\mathrm{d}s\right)\mathrm{d}t\,\mathrm{d}x\\ &\le c\sum_{i=1}^{N^t_\mu}\int_\Omega\int_{I_i^{\frac{1}{\mu}}}\mu\int_{t-s-\varepsilon(\mu)}^{t-s}\mathrm{e}^{\mu(t-s)}(M_{i,\frac{1}{\mu}})^{**}(x,c_J(\mu)\xi(s,x))\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x =: \mathsf{J}^1_\mu.\end{aligned}$$

We continue our estimations using the fact that the second conjugate is the greatest convex minorant (Corollary 2.1.42) and Young's convolution inequality (Lemma 8.26). We get

$$\begin{aligned}\mathsf{J}^1_\mu &= c\sum_{i=1}^{N^t_\mu}\int_\Omega\int_{I_i^{\frac{1}{\mu}}}\mu\int_{t-s-\varepsilon(\mu)}^{t-s}\mathrm{e}^{\mu(t-s)}M(t,x,c_J(1)\xi(t,x))\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\ &\le c\sum_{i=1}^{N^t_\mu}\int_\Omega\|\mu\mathrm{e}^{\mu\cdot}\|_{L^1(-\varepsilon(\mu),0)}\cdot\|M(\cdot,x,C_2\xi(\cdot,x))\|_{L^1\left(I_i^{1/\mu}\cap[0,T]\right)}\,\mathrm{d}x\end{aligned}$$

$$\leq C_1 \| M(\cdot,\cdot,C_2\xi) \|_{L^1(\Omega_T)},$$

where for the last inequality we estimate $\|\mu e^{\mu\cdot}\|_{L^1(-\varepsilon(\mu),0)} \leq 1$ and compute the sum. Summing the last two displays gives the claim. □

Let us present a proof of Theorem 5.3.12.

Proof. We extend $\varphi \in V_T^{M,\infty}(\Omega)$ by $\varphi(0,x) = \varphi_0(x)$ on $(-\infty,0)$ and by 0 on (T,∞). We shall prove that the sequences we are looking for are: $\{\varphi_\mu\}_{\mu>2}$ coming from (5.50) and $\{\varphi_\mu^\bullet\}_{\mu>2}$ coming from (5.51).

Properties (i), (ii), (iii) can be proved by simple computations. Indeed, (i) follows since we have

$$\begin{aligned}\partial_t \varphi_\mu(t,x) &= \partial_t \left(\mu \int_{-\infty}^t e^{\mu(s-t)} \xi(s,x)\, ds \right) \\ &= \mu \left(\varphi(t,x) - \mu \int_{-\infty}^t e^{\mu(s-t)} \varphi(s,x)\, ds \right) = \mu \left(\varphi(t,x) - \varphi_\mu(t,x) \right).\end{aligned}$$

We justify (ii) immediately from (5.51) while we notice that

$$\begin{aligned}\varphi_\mu^\bullet(0,x) &= \mu \int_{0-\varepsilon(\mu)}^0 e^{\mu(s-0)} \xi(s,x)\, ds \\ &= \varphi(0,x) e^{\mu s}\Big|_{-\varepsilon(\mu)}^0 = \varphi_0(x) \left(1 - e^{-\log^2 \mu} \right).\end{aligned}$$

As for (iii), we see that

$$\begin{aligned}(\nabla\varphi)_\mu^\bullet(t,x) &= \mu \int_{t-\varepsilon(\mu)}^t e^{\mu(s-t)} \nabla\varphi(s,x)\, ds \\ &= \nabla \left(\mu \int_{t-\varepsilon(\mu)}^t e^{\mu(s-t)} \varphi(s,x)\, ds \right) = \nabla \left(\varphi_\mu^\bullet(t,x) \right).\end{aligned}$$

Let us concentrate now on showing (iv), i.e. the modular convergence

$$\nabla(\varphi_\mu^\bullet) \xrightarrow[\mu\to\infty]{M} \nabla\varphi \quad \text{in} \quad L_M(\Omega_T;\mathbb{R}^N),$$

which suffices for strong convergence $\varphi_\mu^\bullet \xrightarrow[\mu\to\infty]{} \varphi$ in $L^1(\Omega_T)$ due to the Rellich–Kondrachov theorem (Theorem 8.48) for $W^{1,1}(\Omega)$. As in the previous proofs of approximation properties of the space we base our argument on the modular density of simple functions from Theorem 3.4.11. We start by constructing a simple function which is close to φ. We take a family of measurable sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} E_n = \Omega_T$ and vector-valued simple functions

$$E^n(t,x) = \sum_{j=0}^n \mathbb{1}_{E_j}(t,x)\eta_j,$$

with a family of vectors $\{\eta_j\}_{j=0}^n$, such that $\{E^n\}_{n\in\mathbb{N}}$ converges modularly to $\nabla\varphi$ with λ_4 (cf. Definition 3.4.3). Its existence is justified by Theorem 3.4.11. Let us write a telescopic sum

$$\nabla(\varphi_\mu^\bullet) - \nabla\varphi = \left(\nabla(\varphi_\mu^\bullet) - (E^n)_\mu^\bullet\right) + ((E^n)_\mu^\bullet - E^n) + (E^n - \nabla\varphi).$$

Then Jensen's inequality implies

$$\begin{aligned}\int_{\Omega_T} M\left(t,x,\frac{\nabla(\varphi_\mu^\bullet)-\nabla\varphi}{\lambda}\right)\,\mathrm{d}x\,\mathrm{d}t &\le \frac{\lambda_1}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{\nabla(\varphi_\mu^\bullet)-(E^n)_\mu^\bullet}{\lambda_1}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac{\lambda_2}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{(E^n)_\mu^\bullet-E^n}{\lambda_2}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac{\lambda_3}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{E^n-\nabla\varphi}{\lambda_3}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &= L_1^{n,\mu}+L_2^{n,\mu}+L_3^{\mu},\end{aligned}$$

where $\lambda=\sum_{i=1}^3\lambda_i$, $\lambda_i>0$. We have λ_3 fixed already for the last term to be small. Let us take $\lambda_1=\lambda_3/C_2$ and leave for a moment the choice of λ_2. We will justify convergence to zero of each of the terms on the right-hand side of the last display.

Due to Lemma 5.3.13 we can estimate

$$\begin{aligned}0\le L_1^{n,\mu}\le &\int_{\left\{m_1\left(\frac{|E^n-\nabla\varphi|}{\lambda_1}\right)\le 1\right\}} m_2\left(\frac{|E^n-\nabla\varphi|}{\lambda_1}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &+C\int_{\Omega_T} M\left(t,x,\frac{E^n-\nabla\varphi}{\lambda_3}\right)\,\mathrm{d}x\,\mathrm{d}t =: K^n,\end{aligned}$$

where $\lim_{n\to\infty}K^n=0$. Consequently,

$$\lim_{n\to\infty}\limsup_{\mu\to\infty} L_1^{n,\mu}=0.$$

As for $L_2^{n,\mu}$, Jensen's inequality and then Fubini's theorem lead to

$$\begin{aligned}\frac{\lambda}{\lambda_2}L_2^{n,\mu} &= C\sum_{i=1}^{N_\mu^T}\int_\Omega\int_{I_i^{\frac{1}{\mu}}} M\Bigg(t,x,\frac{1}{\lambda_2}\int_{\mathbb{R}}\mu e^{\mu(s)}\mathbb{1}_{(-\infty,0]}(s)\,\cdot\\ &\qquad\cdot\sum_{j=0}^n[\mathbb{1}_{E_j}(t,x)\eta_j(t,x)-\mathbb{1}_{E_j}(s-t,x)\eta_j(s-t,x)]\,\mathrm{d}s\Bigg)\,\mathrm{d}t\,\mathrm{d}x\\ &\le C\sum_{i=1}^{N_\mu^T}\int_\Omega\int_{I_i^{\frac{1}{\mu}}}\mu e^{\mu(s)}\mathbb{1}_{(-\infty,0]}(s)M\Bigg(t,x,\frac{1}{\lambda_2}\,\cdot\\ &\qquad\cdot\sum_{j=0}^n[\mathbb{1}_{E_j}(t,x)\eta_j(t,x)-\mathbb{1}_{E_j}(s-t,x)\eta_j(s-t,x)]\Bigg)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

$$\leq C \int_{\Omega_i} \sum_{i=1}^{N_\mu^T} \int_{I_i^{\frac{1}{\mu}}} M\left(t,x,\frac{1}{\lambda_2}\sum_{j=0}^{n}[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}(s-t,x)\eta_j(s-t,x)]\right) \mathrm{d}t\, \mathrm{d}x. \tag{5.58}$$

Since the shift operator is continuous in L^1 we have the pointwise convergence

$$\sum_{j=0}^{n}[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}(s-t,x)\eta_j(s-t,x)] \xrightarrow[\mu\to\infty]{} 0,$$

because $s-t < 1/\mu$. For arbitrary fixed $\lambda_2 > 0$ we have

$$\begin{aligned} M&\left(t,x,\tfrac{1}{\lambda_2}\sum_{j=0}^{n}[\mathbb{1}_{E_j}(t,x)\eta_j(t,x) - \mathbb{1}_{E_j}(s-t,x)\eta_j(s-t,x)]\right) \\ &\leq \sup_{\zeta\in\mathbb{R}^N:\ |\zeta|=1} M\left(t,x,\tfrac{1}{\lambda_2}\sum_{j=0}^{n}\|\eta_j\|_{L^\infty(E_j)}\zeta\right) < \infty \end{aligned}$$

and by the Lebesgue dominated convergence theorem, the right-hand side of (5.58) converges to zero. Thus, we have the convergence of $\{L_1^{n,\mu} + L_2^{n,\mu} + L_3^{\mu}\}_{n\in\mathbb{N},\mu>2}$ to zero for $n \to \infty$ and $\mu \to \infty$, which completes the proof of modular convergence of the approximating sequence.

To complete the proof it suffices to show (v). The L^∞-norm is preserved directly from the formula (5.51). Let us note that

$$\varphi_\mu(t,x) - \varphi_\mu^\bullet(t,x) = \mu \int_{-\infty}^{t-\varepsilon(\mu)} \mathrm{e}^{\mu(s-t)}\varphi(s,x)\,\mathrm{d}s.$$

Since we assume $\varphi \in L^\infty(\Omega_T)$, it follows that

$$\|\varphi_\mu - \varphi_\mu^\bullet\|_{L^\infty(\Omega_T)} \leq \|\varphi\|_{L^\infty(\Omega_T)}\mathrm{e}^{-\mu\varepsilon(\mu)} = \|\varphi\|_{L^\infty(\Omega_T)}\mathrm{e}^{-\log^2\mu} \xrightarrow[\mu\to\infty]{} 0.$$

One may justify that for every μ and a.e. $x \in \Omega$ the function $\varphi_\mu^\bullet(\cdot,x)$ is in $W^{1,\infty}([0,T))$ using Young's inequality for a convolution with a measure (Lemma 8.27). Indeed, for every μ function $\partial_t\varphi_\mu^\bullet(\cdot,x)$ has bounded total variation, because its accumulation points have finite mass. Moreover, direct computation shows that

$$\begin{aligned} \|\partial_t\left(\varphi_\mu(\cdot,x) - \varphi_\mu^\bullet(\cdot,x)\right)\|_{L^\infty(0,T)} &\leq 2\|\varphi\|_{L^\infty(\Omega_T)}\mu\mathrm{e}^{-\mu\varepsilon(\mu)} \\ &= 2\|\varphi\|_{L^\infty(\Omega_T)}\mathrm{e}^{\log\mu(1-\log\mu)} \xrightarrow[\mu\to\infty]{} 0 \end{aligned}$$

uniformly for $x \in \Omega$, which completes the proof. □

5.3.3 The comparison principle

The comparison principle we provide below is the consequence of the choice of a proper family of test functions. Note that the proof essentially uses the decay condition *(R3p)*.

Theorem 5.3.14 *Suppose* **a** *satisfies (A1p)–(A3p) with an N-function* $M : [0,T]\times\Omega\times\mathbb{R}^N \to [0,\infty)$ *satisfying (Mp) or* $(Mp)_p$. *Let* v^i, $i = 1,2$, *be renormalized solutions to*

$$\begin{cases} \partial_t v^i - \operatorname{div} \mathbf{a}(t,x,\nabla v^i) = f^i \in L^1(\Omega_T), \\ v^i(0,x) = v^i_0(x) \in L^1(\Omega), \end{cases}$$

then for a.e. $\tau \in (0,T)$ *it holds that*

$$\begin{aligned} &\int_\Omega \big(v^1(\tau,x) - v^2(\tau,x)\big)_+ \,\mathrm{d}x \\ &\qquad \le \int_{\Omega_T} \big(f^1(t,x) - f^2(t,x)\big)\mathrm{sgn}_0^+(v^1(\tau,x) - v^2(\tau,x))\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega \big(v^1_0(x) - v^2_0(x)\big)\mathrm{sgn}_0^+(v^1(\tau,x) - v^2(\tau,x))\,\mathrm{d}x, \end{aligned} \tag{5.59}$$

where sgn_0^+ *denotes the positive part of the signum function.*

If additionally $f^1 \le f^2$ *a.e. in* Ω_T *and* $v^1_0 \le v^2_0$ *in* Ω, *then* $v^1 \le v^2$ *a.e. in* Ω_T.

Proof. Let us define a two-parameter family of functions $\beta^{\tau,r} : \mathbb{R} \to \mathbb{R}$ by

$$\beta^{\tau,r}(s) := \begin{cases} 1 & \text{for } s \in [0,\tau], \\ \frac{-s+\tau+r}{r} & \text{for } s \in [\tau,\tau+r], \\ 0 & \text{for } s \in [\tau+r,T] \end{cases}$$

with arbitrary $\tau \in (0,T)$ and sufficiently small $r > 0$, such that $\tau + r < T$, a one-parameter family of functions $H_\delta : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta(s) := \begin{cases} 0, & s \le 0, \\ s/\delta, & s \in (0,\delta), \\ 1, & s \ge \delta, \end{cases}$$

with $\delta \in (0,1)$ and sets

$$Q_T^\delta := \{(t,x) : 0 < T_{l+1}(v^1) - T_{l+1}(v^2) < \delta\},$$

$$Q_T^{\delta+} := \{(t,x) : T_{l+1}(v^1) - T_{l+1}(v^2) \ge \delta\}.$$

We use *(R2p)* from the definition of a renormalized solution (Definition 5.3.1) with

$$h(v^1) = \psi_l(v^1), \qquad \varphi = H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)$$

and

$$h(v^2) = \psi_l(v^2), \qquad \varphi = H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t),$$

where ψ_l is a hat-function from (5.25). In turn for $i = 1,2$ we get the equalities

$$-\int_{\Omega_T} \left(\int_{v_0^i(x)}^{v^i(t,x)} \psi_l(\sigma)\,\mathrm{d}\sigma \right) \partial_t \varphi \;\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathbf{a}(t,x,\nabla v^i)\cdot \nabla(\psi_l(v^i)\varphi)\;\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f^i \psi_l(v^i)\varphi \;\mathrm{d}x\,\mathrm{d}t.$$

Note that this choice is admissible due to Theorem 4.2.6 and Lemma 3.4.7. When we subtract the equality for $i = 2$ from the one for $i = 2$, we obtain the equation for the difference of these problems, reading

$$D_1^{\delta,r,l,\tau} + D_2^{\delta,r,l,\tau} + D_3^{\delta,r,l,\tau} + D_4^{\delta,r,l,\tau} + D_5^{\delta,r,l,\tau} = D_R^{\delta,r,l,\tau},$$

where

$$D_1^{\delta,r,l,\tau} := -\int_{\Omega_T} \left(\int_{v_0^1}^{v_0^2} \psi_l(\sigma)\,\mathrm{d}\sigma + \int_{v^2}^{v^1} \psi_l(\sigma)\,\mathrm{d}\sigma \right)\cdot$$
$$\cdot\, \partial_t (H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2)))\beta^{\tau,r}(t)\;\mathrm{d}x\,\mathrm{d}t,$$
$$D_2^{\delta,r,l,\tau} := \int_\Omega \frac{1}{r}\int_\tau^{\tau+r} \left(\int_{v_0^1}^{v_0^2} \psi_l(\sigma)\,\mathrm{d}\sigma + \int_{v^2}^{v^1} \psi_l(\sigma)\,\mathrm{d}\sigma \right) H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2))\,\mathrm{d}t\;\mathrm{d}x,$$
$$D_3^{\delta,r,l,\tau} := \int_{Q_T^\delta} \frac{\psi_l(v^1) - \psi_l(v^2)}{\delta} (\mathbf{a}(t,x,\nabla v^1)\cdot \nabla((T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t))\;\mathrm{d}x\,\mathrm{d}t,$$
$$D_4^{\delta,r,l,\tau} := \int_{Q_T^\delta} \frac{\psi_l(v^2)}{\delta} (\mathbf{a}(t,x,\nabla v^1) - \mathbf{a}(t,x,\nabla v^2))\cdot \nabla((T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t))\;\mathrm{d}x\,\mathrm{d}t,$$
$$D_5^{\delta,r,l,\tau} := \int_{Q_T^\delta \cup Q_T^{\delta+}} (\mathbf{a}(t,x,\nabla v^1)\cdot \nabla v^1 \psi_l'(v^1) - \mathbf{a}(t,x,\nabla v^2)\cdot \nabla v^2 \psi_l'(v^2))\cdot$$
$$\cdot\, H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)\;\mathrm{d}x\,\mathrm{d}t,$$
$$D_R^{\delta,r,l,\tau} := \int_{Q_T^\delta \cup Q_T^{\delta+}} (f^1\psi_l(v^1) - f^2\psi_l(v^2)) H_\delta(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)\;\mathrm{d}x\,\mathrm{d}t.$$

Our aim is now to get rid of the limits of $D_1^{\delta,r,l,\tau}$, $D_3^{\delta,r,l,\tau}, D_4^{\delta,r,l,\tau}, D_5^{\delta,r,l,\tau}$ and obtain the final estimates from limits of $D_2^{\delta,r,l,\tau}$ and $D_R^{\delta,r,l,\tau}$.

By properties of $\beta^{\tau,r}$ we can estimate

$$|D_1^{\delta,r,l,\tau}| \le \int_{Q_T^\delta} \left| \partial_t \left(\int_{v_0^1}^{v_0^2} \psi_l(\sigma)\,\mathrm{d}\sigma + \int_{v^2}^{v^1} \psi_l(\sigma)\,\mathrm{d}\sigma \right)\right| \cdot$$
$$\cdot \left| \frac{1}{\delta}(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)\right| \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T^\delta} \left| \left(\int_{v_0^1}^{v_0^2} \psi_l(\sigma)\,\mathrm{d}\sigma + \int_{v^2}^{v^1} \psi_l(\sigma)\,\mathrm{d}\sigma \right)\right| \cdot$$
$$\cdot \left| \frac{1}{\delta}(T_{l+1}(v^1) - T_{l+1}(v^2))\partial_t \beta^{\tau,r}(t)\right| \mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_{Q_T^\delta} \left| \left(\psi_l(\partial_t(T_{l+1}(v^1)) - \psi_l(\partial_t T_{l+1}(v^2)) \right) \frac{1}{\delta} \cdot \delta \right| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q_T^\delta} \left| \left(T_{l+1}(v_0^1) - T_{l+1}(v_0^2) + T_{l+1}(v^1) - T_{l+1}(v^2) \right) \frac{1}{\delta} \cdot \delta \right| \,\mathrm{d}x\,\mathrm{d}t \\
&\le (2+4(l+1))|Q_T^\delta|.
\end{aligned}
$$

Therefore, by Lebesgue's dominated convergence theorem we can pass to the limit $D_1^{\delta,r,l,\tau} \to 0$ as $\delta \to 0$. To justify the convergence of $D_2^{\delta,r,l,\tau}$, $D_5^{\delta,r,l,\tau}$, and $D_R^{\delta,r,l,\tau}$ for $\delta \to 0$, it also suffices to apply the dominated convergence theorem. On the other hand, $D_3^{\delta,r,l,\tau} \ge 0$ can be justified by the monotonicity of truncations, whilst the monotonicity of $\mathbf{a}$ implies $D_4^{\delta,r,l,\tau} \ge 0$. To sum up, by dropping nonnegative terms on the left-hand side, passing to the limit as $\delta \to 0$ in the remaining terms, and by setting

$$
D_2^{r,l,\tau} := \lim_{\delta \to 0} D_2^{\delta,r,l,\tau} \qquad \text{and} \qquad D_5^{r,l,\tau} := \lim_{\delta \to 0} D_5^{\delta,r,l,\tau},
$$

we obtain

$$
\begin{aligned}
D_2^{r,l,\tau} + D_5^{r,l,\tau} &= \lim_{\delta \to 0} D_2^{\delta,r,l,\tau} + \lim_{\delta \to 0} D_5^{\delta,r,l,\tau} \\
&= \int_\Omega \frac{1}{r} \int_\tau^{\tau+r} \left(\int_{v_0^1}^{v_0^2} \psi_l(\sigma)\,\mathrm{d}\sigma + \int_{v^2}^{v^1} \psi_l(\sigma)\,\mathrm{d}\sigma \right) \operatorname{sgn}_0^+(T_{l+1}(v^1) - T_{l+1}(v^2))\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{\Omega_T} (\mathbf{a}(t,x,\nabla v^1) \cdot \nabla v^1 \psi_l'(v^1) - \mathbf{a}(t,x,\nabla v^2) \cdot \nabla v^2 \psi_l'(v^2)) \cdot \\
&\qquad\qquad \cdot \operatorname{sgn}_0^+(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{\Omega_T} (f^1\psi_l(v^1) - f^2\psi_l(v^2)) \operatorname{sgn}_0^+(T_{l+1}(v^1) - T_{l+1}(v^2))\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\delta \to 0} D_R^{\delta,r,l,\tau}.
\end{aligned}
$$

Due to (5.66) and the uniform boundedness of

$$
\{\operatorname{sgn}_0^+(T_{l+1}(v^1) - T_{l+1}(v^2))\beta^{\tau,r}(t)\}_{l>0}
$$

we get that $\lim_{l\to\infty} D_5^{r,l,\tau} = 0$. Lebesgue's monotone convergence theorem enables us to pass to the limit as $l \to \infty$ also in $D_2^{r,l,\tau}$ and $D_5^{r,l,\tau}$. Consequently, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{r} \int_\tau^{\tau+r} \left(v_0^2 - v_0^1 + v^1 - v^2 \right) \operatorname{sgn}_0^+(v^1 - v^2)\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad\qquad\qquad \le \int_{\Omega_T} \left(f^1 - f^2 \right) \operatorname{sgn}_0^+(v^1 - v^2)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since a.e. $\tau \in [0,T)$ is a Lebesgue point of the integrand on the left-hand side, we can pass to the limit as $r \to 0$. By rearranging terms we conclude (5.59).

To motivate the final conclusion for $f^1 \le f^2$ a.e. in Ω_T and $v_0^1 \le v_0^2$ in Ω, note that in (5.59) the left-hand side is nonnegative and the right-hand side is nonpositive.

Hence,

$$\mathrm{sgn}_0^+(v^1(\tau,x) - v^2(\tau,x)) = 0 \quad \text{for a.e.} \quad x \in \Omega_\tau \ \text{ and a.e. } \ \tau \in (0,T)$$

and consequently $v^1 \le v^2$ a.e. in Ω_T. □

5.3.4 Existence and uniqueness

The proof of Theorem 5.3.3 is obtained in several steps. Recall that the existence of solutions to bounded data problems has already been covered in Theorem 4.2.17. We start with a priori estimates for truncations of a sequence of solutions to bounded data problems. The use of the truncation method at this stage is already classical and dates back to the pioneering papers of [43, 47, 31].

From now on, to use the approximation results described in Theorems 4.2.6 and Theorem 5.3.12, we assume that M satisfies a balance condition, that is, *(Mp)* from Section 4.2.2.1 or *(Mp)$_p$* from Section 4.2.2.2 is in power.

Step 1. Problem with truncated data

The existence of weak solutions to a problem under our regime and with bounded data g and u_0 is provided in Theorem 4.2.17. Therefore, the problem

$$\begin{cases} \partial_t u_n - \operatorname{div} \mathbf{a}(t,x,\nabla u_n) = T_n(f) & \text{in } \Omega_T, \\ u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0,\cdot) = u_{0,n}(\cdot) = T_n(u_0) & \text{in } \Omega \end{cases} \tag{5.60}$$

for every $n \in \mathbb{N}$ is a direct consequence of Theorem 4.2.17 with $g = T_n(f)$, where T_n stands for the symmetric truncations at the level n which is defined in (3.55). Namely, for every n there exists

$$u_n \in V_T^M(\Omega) = \{u \in L^1(0,T;W_0^{1,1}(\Omega)) : \ \nabla u \in L_M(\Omega_T;\mathbb{R}^N)\},$$

such that for any $\varphi \in C_c^\infty([0,T)\times\Omega)$ it holds that

$$-\int_{\Omega_T} u_n \partial_t \varphi \ \mathrm{d}x\,\mathrm{d}t - \int_\Omega u_n(0)\varphi(0) \ \mathrm{d}x + \int_{\Omega_T} \mathbf{a}(t,x,\nabla u_n)\cdot\nabla\varphi \ \mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} T_n(f)\varphi \ \mathrm{d}x\,\mathrm{d}t. \tag{5.61}$$

Step 2. A priori estimates

Our aim is to apply the integration by parts formula from Theorem 4.2.10 to a weak solution u_n to (5.60) and a particular choice of $\mathcal{A}$, F, h and ξ therein. Let us prepare for this. We already know that

$$T_k(u_n) \in V_T^{M,\infty}(\Omega) = V_T^M(\Omega) \cap L^\infty([0,T];L^1(\Omega)).$$

Let the two-parameter family of functions $\vartheta^{\tau,r} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\vartheta^{\tau,r}(t) := \left(\omega_r * \mathbb{1}_{[0,\tau)}\right)(t), \tag{5.62}$$

where $r > 0$, $\tau \in (0,T)$, and ω_r is a standard regularizing kernel, that is $\omega_r \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\omega_r \subset (-r,r)$. Note that

$$\operatorname{supp}\vartheta^{\tau,r} = [-r,\tau+r)$$

and for every τ there exists an r_τ such that for all $r < r_\tau$ we have

$$\vartheta^{\tau,r}|_{[0,T)} \in C_c^\infty([0,T)).$$

When we apply Theorem 4.2.10 with

$$\mathcal{A} = \mathbf{a}(t,x,\nabla u_n), \quad F = T_n(f), \quad h(\cdot) = T_k(\cdot), \quad \text{and} \quad \xi(t,x) = \vartheta^{\tau,r}(t),$$

we have

$$-\int_{\Omega_T}\left(\int_{u_{0,n}(x)}^{u_n(t,x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\partial_t(\vartheta^{\tau,r})\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\mathbf{a}(t,x,\nabla u_n)\cdot\nabla(T_k(u_n)\vartheta^{\tau,r})\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} T_n(f)T_k(u_n)\vartheta^{\tau,r}\,\mathrm{d}x\,\mathrm{d}t.$$

We can pass to the limit as $r \to 0$, for a.e. $\tau \in [0,T]$ to infer

$$\int_\Omega\left(\int_0^{u_n(\tau,x)} T_k(\sigma)\,\mathrm{d}\sigma - \int_0^{u_{0,n}(x)} T_k(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x + \int_{\Omega_\tau}\mathbf{a}(t,x,\nabla u_n)\cdot\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_\tau} T_n(f)T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t,$$

and in turn

$$\tfrac{1}{2}\|T_k\left(u_n(\tau)\right)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|T_k\left(u_{0,n}\right)\|^2_{L^2(\Omega)} + \int_{\Omega_\tau}\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_\tau} T_n(f)T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

The coercivity condition *(A2p)* results in

$$c_{\mathbf{a}}M^*(t,x,\nabla T_k(u_n)) \le M(t,x,\nabla T_k(u_n)) \le \mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n). \tag{5.63}$$

Combining the last two displays we get

$$\frac{1}{2}\|T_k\left(u_n(\tau)\right)\|^2_{L^2(\Omega)} - \frac{1}{2}\|T_k\left(u_{0,n}\right)\|^2_{L^2(\Omega)} + \int_{\Omega_\tau} M(t,x,\nabla T_k(u_n))\,\mathrm{d}x\,\mathrm{d}t$$
$$\le k\|f\|_{L^1(\Omega_T)}.$$

Noticing that

$$\|T_k(u_{0,n})\|^2_{L^2(\Omega)} \le k\|u_{0,n}\|_{L^1(\Omega)} = k\|T_n(u_0)\|_{L^1(\Omega)}$$

and that $\tau \in (0,T)$ is arbitrary, by fixing

$$w_2(k,f,u_0) := k\left(\|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|_{L^1(\Omega)}\right)$$

and making use of (5.63) again, we get a priori estimates of the following form

$$\begin{aligned}\int_{\Omega_\tau} M(t,x,\nabla T_k(u_n))\,\mathrm{d}x\,\mathrm{d}t &\le w_2(k,f,u_0),\\ c_{\mathbf{a}}\int_{\Omega_\tau} M^*(t,x,\mathbf{a}(t,x,\nabla T_k(u_n)))\,\mathrm{d}x\,\mathrm{d}t &\le w_2(k,f,u_0).\end{aligned} \tag{5.64}$$

Step 3. Decay condition

We prove that if u_n is a weak solution to (5.60), $n > 0$, then it holds that

$$\lim_{l\to\infty} |\{|u_n| > l\}| = 0 \tag{5.65}$$

and

$$\lim_{l\to\infty}\limsup_{n\to\infty}\int_{\{l<|u_n|<l+1\}} \mathbf{a}(t,x,\nabla u_n)\cdot\nabla u_n \;\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.66}$$

We concentrate first on (5.65). To this end we consider m_1 – the minorant of M from the definition of an N-function and $c_P^1, c_P^2 > 0$ being constants from the Poincaré inequality (Theorem 9.3) involving m_1. We have

$$|\{|u_n| \ge l\}| = |\{|T_l(u_n)| = l\}| = |\{|T_l(u_n)| \ge l\}| = |\{m_1(c_P^1|T_l(u_n)|) \ge m_1(c_P^1 l)\}|,$$

so for $l > 1$ by Chebyshev's inequality (Theorem 8.28) and the Poincaré inequality (Theorem 9.3), we have

$$|\{|u_n| \ge l\}| \le \int_{\Omega_T} \frac{m_1(c_P^1|T_l(u_n)|)}{m_1(c_P^1 l)}\,\mathrm{d}x\,\mathrm{d}t \le \frac{c(N,\Omega,T)}{m_1(c_P^1 l)}\int_{\Omega_T} c_P^2 m_1(|\nabla T_l(u_n)|)\,\mathrm{d}x\,\mathrm{d}t$$

Since m_1 is a minorant of M and by the a priori estimate (5.64) we continue the above estimates to notice that

$$\begin{aligned}|\{|u_n| \ge l\}| &\le \frac{c(N,\Omega,T)}{m_1(c_P^1 l)}\int_{\Omega_T} M(t,x,\nabla T_l(u_n))\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(M,N,\Omega,T)\left(\|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|_{L^1(\Omega)}\right)\frac{l}{m_1(c_P^1 l)}.\end{aligned}$$

As we know that $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, dependence on data will now be hidden in a constant. The function m_1 is superlinear at infinity, so we can conclude that

$$|\{|u_n| \geq l\}| \leq C(f, u_0, M, N, \Omega, T) \frac{l}{m_1(c_P^1 l)} \xrightarrow[l \to \infty]{} 0. \tag{5.67}$$

In turn, this proves (5.65).

To prove (5.66), we consider a family of nonincreasing functions $\phi_r \in C_c^\infty([0,T))$, such that

$$\phi_r(t) := \begin{cases} 1 \text{ for } t \in [0, T-2r], \\ 0 \text{ for } t \in [T-r, T], \end{cases} \quad \text{and} \quad G_l(s) := T_{l+1}(s) - T_l(s).$$

Since $u_n \in V_T^M(\Omega)$ is a weak solution to (5.60), we can use

$$\varphi(t,x) = G_l(u_n(t,x))\phi_r(t)$$

as a test function in (5.61) and get

$$\int_{\Omega_T} (\partial_t u_n) G_l(u_n)\phi_r \, \mathrm{d}x\,\mathrm{d}t + \int_{\{l<|u_n|<l+1\}} \mathbf{a}(t,x,\nabla u_n)\cdot \nabla u_n \, \phi_r \, \mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f G_l(u_n)\phi_r \, \mathrm{d}x\,\mathrm{d}t. \tag{5.68}$$

On the left-hand side above a direct computation shows that

$$\int_0^T (\partial_t u_n) G_l(u_n)\phi_r \,\mathrm{d}t = \int_0^T \partial_t \left(\int_0^{u_n} G_l(s)\,\mathrm{d}s \right) \phi_r \,\mathrm{d}t$$
$$= -\phi_r(0) \int_0^{u_{0,n}} G_l(s)\,\mathrm{d}s - \int_0^T \int_0^{u_n} G_l(s)\,\mathrm{d}s\, \partial_t \phi_r \,\mathrm{d}t,$$

where

$$\int_0^{u_n} G_l(s)\,\mathrm{d}s \geq 0 \quad \text{and} \quad \partial_t \phi_r \leq 0.$$

Therefore, we have

$$-\int_0^T \int_0^{u_n} G_l(s)\,\mathrm{d}s\, \partial_t \phi_r \,\mathrm{d}t \geq 0$$

and in turn from (5.68) it follows that

$$\int_{\{l<|u_n|<l+1\}} \mathbf{a}(t,x,\nabla u_n)\cdot \nabla u_n \, \phi_r \, \mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_{\Omega_T} f G_l(u_n)\phi_r \, \mathrm{d}x\,\mathrm{d}t + \int_\Omega \phi_r(0) \int_0^{u_{0,n}} G_l(s)\,\mathrm{d}s \, \mathrm{d}x.$$

It suffices now to show that the right-hand side above tends to zero as $l \to \infty$. Note that

$$\int_\Omega \int_0^{|u_{0,n}|} |G_l(s)|\, \mathrm{d}s\, \mathrm{d}x \le \int_\Omega \int_0^{|u_{0,n}|} \mathbb{1}_{\{s>l\}}\, \mathrm{d}s\, \mathrm{d}x$$
$$= \int_{\{||u_{0,n}|-l|>0\}} \big(|u_{0,n}|-l\big)\, \mathrm{d}x \xrightarrow[l\to\infty]{} 0$$

and

$$\int_{\Omega_T} f G_l(u_n)\phi_r\, \mathrm{d}x\, \mathrm{d}t \le \int_{\{|u_n|>l\}} |f|\, \mathrm{d}x\, \mathrm{d}t \xrightarrow[l\to\infty]{} 0,$$

where the convergence is justified by the fact that f is integrable and the domain of integration shrinks, see (5.67). Consequently, (5.66) follows.

Step 4. Convergence of the truncations $T_k(u_n)$

Our aim is now to interpret the a priori estimates (5.64) as inferring various properties of limits of a weak solution $u_n \in V_T^{M,\infty}(\Omega)$ to the problem (5.60) for arbitrary $k > 0$. Namely, we show that there exists a measurable function u such that $T_k(u) \in V_T^M(\Omega)$ and

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in } L^1(0,T;W_0^{1,1}(\Omega)), \tag{5.69}$$

$$T_k(u_n) \overset{*}{\rightharpoonup} T_k(u) \qquad \text{weakly-}* \text{ in } L^\infty(\Omega_T), \tag{5.70}$$

$$\nabla T_k(u_n) \overset{*}{\rightharpoonup} \nabla T_k(u) \qquad \text{weakly-}* \text{ in } L_M(\Omega_T;\mathbb{R}^N), \tag{5.71}$$

$$\mathbf{a}(t,x,\nabla T_k(u_n)) \overset{*}{\rightharpoonup} \boldsymbol{\mathcal{A}}_k \qquad \text{weakly-}* \text{ in } L_{M^*}(\Omega_T;\mathbb{R}^N), \tag{5.72}$$

for some $\boldsymbol{\mathcal{A}}_k \in L_{M^*}(\Omega_T;\mathbb{R}^N)$.

In fact, the weak lower semi-continuity of a convex functional together with the above a priori estimates (5.64), the Dunford–Pettis theorem (Theorem 8.21), and the Banach–Alaoglu theorem (Theorem 8.31) imply the existence of u such that $T_k u \in V_T^M(\Omega)$ for every $k > 0$ and (5.69), (5.70), (5.71) hold. By the same reasoning there exists an $\boldsymbol{\mathcal{A}}_k$ such that (5.72) holds. Similar arguments are presented in detail in Step 4 of the proof of Theorem 5.2.3.

Step 5. Almost everywhere limit

This step is devoted to showing that if u_n is a weak solution to (5.60), then for the function u obtained in the limit in the previous step, we have $T_k u \in V_T^M(\Omega)$ for every $k > 0$,

$$u_n \to u \quad \text{ a.e. in } \Omega_T, \tag{5.73}$$

and

$$\lim_{l\to\infty} |\{|u| > l\}| = 0. \tag{5.74}$$

We make use of the notion of renormalized solutions already in this step. In fact, to prove (5.74) we apply the comparison principle of Theorem 5.3.14. We can do this since by Theorem 4.2.10 any weak solution u_n is a renormalized solution. Let us denote asymmetric truncations by

$$T^{k,l}(f)(x) = \begin{cases} -k & \text{if } f \le -k, \\ f & \text{if } -k < f < l, \\ l & \text{if } f \ge l \end{cases}$$

and by $u^{a,b}$ – a weak solution to a problem with truncated data

$$u_t - \operatorname{div} \mathbf{a}(t,x,\nabla u) = T^{a,b}(f), \qquad u(0,x) = T^{a,b}(u_0),$$

which exists due to Theorem 4.2.17. By the comparison principle (Theorem 5.3.14) we get that for $0 < l < l'$ and $0 < k < k'$ it holds that

$$u^{k',l} \le u^{k,l} \le u^{k,l'} \tag{5.75}$$

for a.e. $(t,x) \in \Omega_T$. The sequence $\{u^{k,l}\}_{l>0}$ is monotone, so $\lim_{l\to\infty} u^{k,l}$ exists a.e. in Ω_T and we denote it by $u^{k,\infty}$. Having (5.75) we infer that also

$$u^{k',\infty} \le u^{k,\infty} \quad \text{a.e. in } \Omega_T.$$

Altogether, the following limit exists

$$u^{\infty,\infty} = \lim_{k\to\infty} u^{k,\infty} \quad \text{a.e. in } \Omega_T.$$

Consequently, due to the uniqueness of the limit, we get the convergence (5.73). Then condition (5.74) is a direct consequence of (5.65).

Step 6. Identification of the limit of $\{\mathbf{a}(t,x,\nabla T_k(u_n))\}_{n\in\mathbb{N}}$

As in the elliptic case, in this step we employ the monotonicity trick to identify the limit (5.72). Nonetheless, this is a particularly interesting step, because it is essentially more complex than the case of M constant in time, e.g. [79, 188]. As a matter of fact, the classical tool of Landes regularization cannot be applied anymore and we need to apply a very subtle approximation provided by Theorem 5.3.12. For this result we need *(Mp)* or *(Mp)$_p$* from Section 4.2.2.1 or 4.2.2.2, respectively.

The aim now is to show that

$$\mathbf{a}(t,x,\nabla T_k(u_n)) \xrightharpoonup{*} \mathbf{a}(t,x,\nabla T_k(u)) \quad \text{weakly-}* \text{ in } L_{M^*}(\Omega_T;\mathbb{R}^N), \tag{5.76}$$

by proving that in (5.72) for fixed k we have

$$\mathcal{A}_k = \mathbf{a}(t,x,\nabla T_k(u)). \tag{5.77}$$

Let us fix an arbitrary nonnegative $w \in C_c^\infty([0,T))$. In order to use the monotonicity trick of Theorem 4.2.11 we show that

$$\limsup_{n\to\infty} \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n)) \cdot \nabla(T_k(u_n))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} w\,\mathcal{A}_k \cdot \nabla(T_k(u))\,\mathrm{d}x\,\mathrm{d}t. \tag{5.78}$$

We take the approximate sequence $\{(T_k(u))^\bullet_\mu\}_{\mu>2}$ from Theorem 5.3.12. It satisfies

$$(\nabla T_k(u))_\mu^\bullet \xrightarrow{M} \nabla T_k(u) \quad \text{modularly in } L_M(\Omega_T;\mathbb{R}^N)$$

when $\mu \to \infty$. We define $\psi_l : \mathbb{R} \to \mathbb{R}$ by

$$\psi_l(s) := \min\{(l+1-|s|)_+, 1\}. \tag{5.79}$$

We use the integration by parts formula (Theorem 4.2.10) on (5.60) with

$$\mathcal{A} = \mathbf{a}(t,x,\nabla u_n) \in L_M(\Omega_T;\mathbb{R}^N) \qquad \text{and} \qquad F = T_n f \in L^1(\Omega_T)$$

twice: for the first time with

$$h(\cdot) = \psi_l(\cdot)T_k(\cdot) \qquad \text{and} \qquad \xi = w$$

and for the second time with

$$h(\cdot) = \psi_l(\cdot) \qquad \text{and} \qquad \xi = w(T_k(u))_\mu^\bullet.$$

By subtracting the second from the first we get

$$I_1^{n,\mu,l} + I_2^{n,\mu,l} + I_3^{n,\mu,l} = I_4^{n,\mu,l}, \tag{5.80}$$

where

$$\begin{aligned}
I_1^{n,\mu,l} &:= -\int_{\Omega_T} \partial_t w \Big(\int_{u_{0,n}}^{u_n} \psi_l(s) T_k(s)\,\mathrm{d}s \Big)\, \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T} \partial_t (w (T_k(u))_\mu^\bullet) \Big(\int_{u_{0,n}}^{u_n} \psi_l(s)\,\mathrm{d}s \Big)\, \mathrm{d}x\,\mathrm{d}t, \\
I_2^{n,\mu,l} &:= \int_{\Omega_T} w \psi_l(u_n)\, \mathbf{a}(t,x,\nabla u_n) \cdot \nabla (T_k(u_n) - (T_k(u))_\mu^\bullet)\, \mathrm{d}x\,\mathrm{d}t, \\
I_3^{n,\mu,l} &:= \int_{\Omega_T} w \psi_l'(u_n) (T_k(u_n) - (T_k(u))_\mu^\bullet)\, \mathbf{a}(t,x,\nabla u_n) \cdot \nabla u_n \, \mathrm{d}x\,\mathrm{d}t, \\
I_4^{n,\mu,l} &:= \int_{\Omega_T} w T_n f \psi_l(u_n) (T_k(u_n) - (T_k(u))_\mu^\bullet)\, \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We need to justify that we can pass to the limit as $n \to \infty$, then $\mu \to \infty$, and finally with $l \to \infty$. Roughly speaking to prove that the limit of $I_2^{n,\mu,l}$ is nonpositive, we show that the limit of $I_1^{n,\mu,l}$ is nonnegative, while $I_3^{n,\mu,l}$ and $I_4^{n,\mu,l}$ tend to zero.

Limit of $I_1^{n,\mu,l}$. To prove that

$$\limsup_{l\to\infty} \limsup_{\mu\to\infty} \limsup_{n\to\infty} I_1^{n,\mu,l} \geq 0 \tag{5.81}$$

by the properties of the approximation, i.e. due to (5.49), we write

$$I_1^{n,\mu,l} = I_{1,1}^{n,\mu,l} + I_{1,2}^{n,\mu,l} + I_{1,3}^{n,\mu,l},$$

where

$$I_{1,1}^{n,\mu,l} := -\int_{\Omega_T} \partial_t w \Big(\int_{u_{0,n}}^{u_n} \psi_l(s) T_k(s)\, \mathrm{d}s \Big)\; \mathrm{d}x\, \mathrm{d}t,$$
$$I_{1,2}^{n,\mu,l} := \int_{\Omega_T} (\partial_t w)\; (T_k(u))_\mu^\bullet \left(\int_{u_{0,n}}^{u_n} \psi_l(s)\, \mathrm{d}s \right)\; \mathrm{d}x\, \mathrm{d}t,$$
$$I_{1,3}^{n,\mu,l} := \int_{\Omega_T} w\, \partial_t \big((T_k(u))_\mu^\bullet \big) \left(\int_{u_{0,n}}^{u_n} \psi_l(s)\, \mathrm{d}s \right)\; \mathrm{d}x\, \mathrm{d}t.$$

Since $s \mapsto \psi_l(s) T_k(s)$ has a compact support, by (5.73) we have the convergence $u_n \to u$ a.e. in Ω_T, and by continuity of the integral, we can justify passing to the limit as $n \to \infty$ in $I_{1,1}^{n,\mu,l}$ to get

$$\lim_{n\to\infty} I_{1,1}^{n,\mu,l} = -\int_{\Omega_T} \partial_t w \int_{u_0}^{u} \psi_l(s) T_k(s)\, \mathrm{d}s\; \mathrm{d}x\, \mathrm{d}t =: I_{1,1}^l.$$

Furthermore, by integration by parts we have

$$\begin{aligned} \int_0^w \psi_l(s) T_k(s)\, \mathrm{d}s &= \int_0^w \int_0^{T_k(s)} \psi_l(s)\, \mathrm{d}\sigma\; \mathrm{d}s \\ &= T_k(w) \int_0^w \psi_l(s)\; \mathrm{d}s - \int_0^{T_k(w)} \int_0^{\sigma} \psi_l(s)\; \mathrm{d}s\, \mathrm{d}\sigma, \end{aligned} \tag{5.82}$$

so we can write

$$\begin{aligned} I_{1,1}^l = &-\int_{\Omega_T} \partial_t w \left(T_k(u) \int_0^u \psi_l(s)\, \mathrm{d}s - \int_0^{T_k(u)} \int_0^{\sigma} \psi_l(s)\, \mathrm{d}s\, \mathrm{d}\sigma \right) \mathrm{d}x\, \mathrm{d}t \\ &+ \int_\Omega w(0) \left(\int_0^{T_k(u_0)} \int_0^{\sigma} \psi_l(s)\, \mathrm{d}s\, \mathrm{d}\sigma - T_k(u_0) \int_0^{u_0} \psi_l(s)\, \mathrm{d}s \right) \mathrm{d}x. \end{aligned}$$

In the case of $I_{1,2}^{n,\mu,l}$, having the pointwise convergence of the integrand when $n \to \infty$ and boundedness of all of the involved terms, Lemma 8.22 justifies passing to the limit. By the properties of the approximation (Theorem 5.3.12 (iv) and (v) we pass to the limit as $\mu \to \infty$ and obtain

$$\lim_{\mu\to\infty} \lim_{n\to\infty} I_{1,2}^{n,\mu,l} = \int_{\Omega_T} (\partial_t w)\, T_k(u) \left(\int_0^u \psi_l(s)\, \mathrm{d}s - \int_0^{u_0} \psi_l(s)\, \mathrm{d}s \right) \mathrm{d}x\, \mathrm{d}t =: I_{1,2}^l.$$

In the case of $I_{1,3}^{n,\mu,l}$ we can let $n \to \infty$ by the same arguments as for $I_{1,2}^{n,\mu,l}$ and get

$$\begin{aligned} \lim_{n\to\infty} I_{1,3}^{n,\mu,l} &= \int_{\Omega_T} w\, \partial_t \big((T_k(u))_\mu \big) \left(\int_{u_0}^{u} \psi_l(s)\, \mathrm{d}s \right) \mathrm{d}x\, \mathrm{d}t \\ &\quad - \int_{\Omega_T} w \left(\partial_t \big((T_k(u))_\mu - (T_k(u))_\mu^\bullet \big) \right) \left(\int_{u_0}^{u} \psi_l(s)\, \mathrm{d}s \right) \mathrm{d}x\, \mathrm{d}t \\ &= I_{1,3,1}^{\mu,l} + I_{1,3,2}^{\mu,l} + I_{1,3,3}^{\mu,l} + I_{1,3,4}^{\mu,l} + I_{1,3,5}^{\mu,l}, \end{aligned}$$

with

$$
\begin{aligned}
I^{\mu,l}_{1,3,1} &:= \int_{\Omega_T} w\,\partial_t\big((T_k(u))_\mu\big)\Big(\int_{T_k(u)}^{u} \psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t,\\
I^{\mu,l}_{1,3,2} &:= \int_{\Omega_T} w\,\partial_t\big((T_k(u))_\mu\big)\Big(\int_{(T_k(u))_\mu}^{T_k(u)} \psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t,\\
I^{\mu,l}_{1,3,3} &:= \int_{\Omega_T} w\,\partial_t\big((T_k(u))_\mu\big)\Big(\int_{0}^{(T_k(u))_\mu} \psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t,\\
I^{\mu,l}_{1,3,4} &:= -\int_{\Omega_T} w\,\partial_t\big((T_k(u))_\mu\big)\Big(\int_{0}^{u_0} \psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t,\\
I^{\mu,l}_{1,3,5} &:= -\int_{\Omega_T} w\,\Big(\partial_t\big((T_k(u))_\mu-(T_k(u))^{\bullet}_\mu\big)\Big)\Big(\int_{u_0}^{u} \psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where we want to pass to the limit as $\mu\to\infty$. Note that due to Theorem 5.3.12 (v) we directly infer that $\lim_{\mu\to\infty} I^{\mu,l}_{1,3,5}=0$.

The convergence of $I^{\mu,l}_{1,3,3}$ can be justified by integration by parts and continuity of the integral since

$$
\begin{aligned}
\lim_{\mu\to\infty} I^{\mu,l}_{1,3,3} = &-\int_{\Omega_T} (\partial_t w)\Big(\int_0^{T_k(u)}\int_0^{\sigma} \psi_l(s)\,\mathrm{d}s\,\mathrm{d}\sigma\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{\Omega} w(0)\Big(\int_0^{T_k(u_0)}\int_0^{\sigma} \psi_l(s)\,\mathrm{d}s\,\mathrm{d}\sigma\Big)\,\mathrm{d}x\,\mathrm{d}t\\
=:&\ I^{l}_{1,3,3}.
\end{aligned}
$$

As for $I^{\mu,l}_{1,3,4}$ we integrate by parts with respect to the time variable and pass to the limit due to Lemma 8.22 to get

$$
\begin{aligned}
\lim_{\mu\to\infty} I^{\mu,l}_{1,3,4} &= \lim_{\mu\to\infty}\int_{\Omega}\int_0^T (\partial_t w)(T_k(u))_\mu\,\mathrm{d}t\Big(\int_0^{u_0}\psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\\
&\quad+\int_{\Omega} w(0)T_k(u_0)\Big(\int_0^{u_0}\psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\Omega_T}(\partial_t w)T_k(u)\Big(\int_0^{u_0}\psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{\Omega} w(0)T_k(u_0)\Big(\int_0^{u_0}\psi_l(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=: I^{l}_{1,3,4}.
\end{aligned}
$$

By summing all the terms and the formula from (5.82), after passing to the limit as $\mu\to\infty$, we get

$$
I^l_{1,1}+I^l_{1,2}+I^l_{1,3,3}+I^l_{1,3,4}=0. \tag{5.83}
$$

Notice that in order to get (5.81) it suffices now to prove that

$$\limsup_{l\to\infty}\limsup_{\mu\to\infty}\left(I^{\mu,l}_{1,3,1}+I^{\mu,l}_{1,3,2}\right)\geq 0. \tag{5.84}$$

We will do this by applying the property of the approximation (5.49). Let us notice first that

$$\begin{aligned} I^{\mu,l}_{1,3,1} &= \int_{\Omega_T} w\,\mu\big(T_k(u)-(T_k(u))_\mu\big)\left(\int_{T_k(u)}^{u}\psi_l(s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\{u\leq -k\}} w\,\mu\big(-k-(T_k(u))_\mu\big)\left(\int_{-k}^{u}\psi_l(s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{\{u\geq k\}} w\,\mu\big(k-(T_k(u))_\mu\big)\left(\int_{k}^{u}\psi_l(s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\geq 0, \end{aligned}$$

where on the set $\{|u|\geq k\}$ the most internal integral vanishes and each of the remaining terms is nonnegative. Moreover, again due to (5.49), we get

$$I^{\mu,l}_{1,3,2}=\int_{\Omega_T} w\,\mu\big(T_k(u)-(T_k(u))_\mu\big)\left(\int_{(T_k(u))_\mu}^{T_k(u)}\psi_l(s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\geq 0,$$

which is justified by monotonicity of truncation.

Thus we conclude (5.84) and consequently (5.81).

Limit of $I_3^{n,\mu,l}$. The coercivity condition *(A2p)* implies nonnegativeness of $\mathbf{a}(t,x,\nabla u_n)\cdot\nabla u_n$, so the radiation control property (5.66) is equivalent to

$$\lim_{l\to\infty}\limsup_{n\to\infty}\int_{\{l<|u_n|<l+1\}}|\mathbf{a}(t,x,\nabla u_n)\cdot\nabla u_n|\,\mathrm{d}x\,\mathrm{d}t=0.$$

In such a case

$$\begin{aligned} |I_3^{n,\mu,l}| &= \left|\int_{\Omega_T} w\psi_l'(u_n)\Big(T_k(u_n)-(T_k(u))^{\bullet}_{\mu}\Big)\mathbf{a}(t,x,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leq 2k||w||_{L^\infty(\mathbb{R})}\int_{\{l<|u_n|<l+1\}}|\mathbf{a}(t,x,\nabla u_n)\cdot\nabla u_n|\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which is independent of μ, so directly we have that

$$\lim_{l\to\infty}\lim_{\mu\to\infty}\limsup_{n\to\infty} I_3^{n,\mu,l}=0.$$

Limit of $I_4^{n,\mu,l}$. We apply the Lebesgue dominated convergence theorem to justify the limit as $n\to\infty$. Indeed, we have the continuity of the integrand, (5.73), i.e. $u_n\to u$ a.e. in Ω_T. Having convergence

$$(T_k(u))^{\bullet}_{\mu}\xrightarrow[\mu\to\infty]{} T_k(u)\qquad\text{a.e. in }\ \Omega_T$$

due to Theorem 5.3.12 and boundedness in L^1 of the rest terms means we can apply the Lebesgue dominated convergence theorem and pass to the limit as $\mu\to\infty$ to get

$$\lim_{l\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty} I_4^{n,\mu,l} = 0.$$

Conclusion via the monotonicity trick. Since we have (5.80), by passing there to the limit, since $I_3^{n,\mu,l}$ and $I_4^{n,\mu,l}$ tend to zero, we get

$$\begin{aligned}0 &= \limsup_{l\to\infty}\limsup_{\delta\to 0}\limsup_{\mu\to\infty}\limsup_{n\to\infty} I_1^{n,\mu,l}\\ &\quad+\limsup_{l\to\infty}\limsup_{\delta\to 0}\limsup_{\mu\to\infty}\limsup_{n\to\infty}\Big(\\ &\qquad\int_{\Omega_T} w\psi_l(u_n)\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-(T_k(u))_\mu^\bullet))\,\mathrm{d}x\,\mathrm{d}t\Big).\end{aligned}$$

If we also take into account (5.81) the above line becomes

$$\begin{aligned}&\limsup_{l\to\infty}\limsup_{\delta\to 0}\limsup_{\mu\to\infty}\limsup_{n\to\infty}\Big(\\ &\qquad\int_{\Omega_T} w\psi_l(u_n)\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-(T_k(u))_\mu^\bullet))\,\mathrm{d}x\,\mathrm{d}t\Big)\le 0.\end{aligned}$$

By the coercivity assumption *(A2p)* we have $\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)))\ge 0$ and $\mathbf{a}(t,x,0)=0$, so since $w,\psi_l\ge 0$ and by (5.79) for sufficiently large l,μ,n, we infer

$$\int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla((T_k(u))_\mu^\bullet)\,\mathrm{d}x\,\mathrm{d}t.$$

Let us concentrate on the right-hand side above. Recall that $\nabla(T_k(u))_\mu^\bullet\in L_M(\Omega_T;\mathbb{R}^N)$ and that (5.72) holds, so for sufficiently large μ

$$\limsup_{n\to\infty}\int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla(T_k(u_n))\,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} w\,\boldsymbol{\mathcal{A}}_k\cdot\nabla((T_k(u))_\mu^\bullet)\,\mathrm{d}x\,\mathrm{d}t.$$

We recall again that $\nabla((T_k(u))_\mu^\bullet)\xrightarrow{M}\nabla T_k(u)$ modularly in $L_M(\Omega_T;\mathbb{R}^N)$ as $\mu\to\infty$ (by Theorem 5.3.12), so by Corollary 3.4.7

$$\lim_{\mu\to 0}\int_{\Omega_T} w\,\boldsymbol{\mathcal{A}}_k\cdot\nabla((T_k(u))_\mu^\bullet)\,\mathrm{d}x = \int_{\Omega_T} w\,\boldsymbol{\mathcal{A}}_k\cdot\nabla T_k(u)\,\mathrm{d}x.$$

Consequently, we obtain (5.78). Following the monotonicity argument of Theorem 4.2.11, as in the proof of Theorem 4.2.17, we prove (5.77). In fact, the monotonicity assumption of *(A3p)* implies

$$\Big(\mathbf{a}(t,x,\nabla(T_k(u_n))-\mathbf{a}(t,x,\eta)\Big)\cdot(\nabla(T_k(u_n))-\eta)\ge 0$$

a.e. in Ω_T for any $\eta\in L^\infty(\Omega_T;\mathbb{R}^N)\subset E_M(\Omega_T;\mathbb{R}^N)$. Due to Theorem 3.5.3

$$\mathbf{a}(\cdot,\cdot,\eta)\in L_{M^*}(\Omega_T,\mathbb{R}^N)=(E_M(\Omega_T,\mathbb{R}^N))^*,$$

so we can pass to the limit as $n \to \infty$ and take into account (5.78) to conclude that

$$\int_{\Omega_T} w(\boldsymbol{\mathcal{A}}_k - \mathbf{a}(t,x,\eta)) \cdot (\nabla(T_k(u)) - \eta)\, \mathrm{d}x\,\mathrm{d}t \geq 0. \tag{5.85}$$

Then Theorem 4.2.11 with

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}_k \qquad \text{and} \qquad \boldsymbol{\xi} = \nabla(T_k(u))$$

implies (5.77), which completes the proof of this step.

Step 7. Renormalized solutions

We are in a position to complete the proof of existence by showing that u obtained as a limit in the previous steps is a renormalized solution. This part follows the ideas of [79, 188]. We need to apply here the integration by parts formula, so indeed approximation from [80] is used and, consequently, we require condition *(Mp)* or *(Mp)*$_p$ from Section 4.2.2.1 or 4.2.2.2, respectively, to use the approximation theorems from Section 4.2.2.

We aim to show that the limit function u from the previous steps is the unique renormalized solution we are looking for.

Condition *(R1p)*.
Due to Step 4 and (5.76) if u_n solves (5.60) its limit u satisfies condition *(R1p)*.

Condition *(R3p)***.**
The aim now is to prove the key convergence for condition *(R3p)*, namely

$$\mathbf{a}(t,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n) \xrightarrow[n\to\infty]{} \mathbf{a}(t,x,\nabla T_k(u)) \cdot \nabla T_k(u) \ \text{ weakly in } L^1(\Omega_T). \tag{5.86}$$

The arguments follow the same arguments as in the proof of Theorem 5.2.3. The above display is a parabolic version of (5.36). We motivate the convergence in both cases by an argument based on Chacon's biting lemma (Theorem 8.37) and the Young measures (Theorem 8.41). We take a nonnegative $w \in C_c^\infty([0,T))$. Let us observe that the sequence

$$\{b_n\}_{n\in\mathbb{N}} := \Big\{ w\Big(\mathbf{a}(t,x,\nabla T_k(u_n)) - \mathbf{a}(t,x,\nabla T_k(u))\Big) \cdot \big(\nabla T_k(u_n) - \nabla T_k(u_n)\big) \Big\}_{n\in\mathbb{N}}$$

is uniformly bounded in $L^1(\Omega_T)$ due to the a priori estimate (5.64) and the Fenchel–Young inequality (2.33). Indeed, we might write

$$\int_{\Omega_T} b_n \,\mathrm{d}x\,\mathrm{d}t \leq J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 := \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_2 := \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_3 := \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u))\cdot\nabla T_k(u_n)\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_4 := \int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u))\cdot\nabla T_k(u)\,\mathrm{d}x\,\mathrm{d}t,$$

and each of the terms can be estimated in the following way

$$\begin{aligned} J_1 &\le \|w\|_{L^\infty}\left(\int_\Omega M^*(t,x,\mathbf{a}(t,x,\nabla T_k(u_n))) + M(t,x,\nabla T_k(u_n))\right)\mathrm{d}x \\ &\le c(f,u_0,c_{\mathbf{a}},w,M)k, \end{aligned}$$

which yields uniform boundedness in n. In turn, $\|b_n\|_{L^1(\Omega_T)} \le J_1+J_2+J_3+J_4$, that is, $\{b_n\}_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega_T)$. The monotonicity of $\mathbf{a}(t,x,\cdot)$ from *(A3p)* ensures that $b_n \ge 0$. Therefore, Theorem 8.41 combined with Theorem 8.37 give, up to a subsequence, convergence

$$\begin{aligned} 0 &\le w\Big(\mathbf{a}(t,x,\nabla T_k(u_n)) - \mathbf{a}(t,x,\nabla T_k(u))\Big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\big) \\ &\xrightarrow[n\to\infty]{b} w\int_{\mathbb{R}^{N+1}}\Big(\mathbf{a}(t,x,\lambda)-\mathbf{a}(t,x,\nabla T_k(u))\Big)\cdot\big(\lambda-\nabla T_k(u)\big)\,\mathrm{d}\nu_{t,x}(\lambda), \end{aligned} \tag{5.87}$$

where $\nu_{t,x}$ denotes the Young measure generated by the sequence $\{\nabla T_k(u_n)\}_{n\in\mathbb{N}}$. Since (5.69) supplies us with weak convergence $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $L^1(\Omega_T;\mathbb{R}^N)$, we infer that

$$\int_{\mathbb{R}^{N+1}} \lambda\, d\nu_{t,x}(\lambda) = \nabla T_k(u) \quad \text{for a.e. } t\in(0,T) \text{ and a.e. } x\in\Omega.$$

In turn

$$\int_{\mathbb{R}^{N+1}} \mathbf{a}(t,x,\nabla T_k(u))\cdot\big(\lambda-\nabla T_k(u)\big)\,\mathrm{d}\nu_{t,x}(\lambda) = 0$$

and, consequently, the limit in (5.87) for a.e. $t\in(0,T)$ and a.e. $x\in\Omega$ satisfy

$$\begin{aligned} w&\int_{\mathbb{R}^{N+1}}\Big(\mathbf{a}(t,x,\lambda)-\mathbf{a}(t,x,\nabla T_k(u))\Big)\cdot\big(\lambda-\nabla T_k(u)\big)\,\mathrm{d}\nu_{t,x}(\lambda) \\ &= w\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\cdot\lambda\,\mathrm{d}\nu_{t,x}(\lambda) - w\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\cdot\nabla T_k(u)\,\mathrm{d}\nu_{t,x}(\lambda). \end{aligned} \tag{5.88}$$

The uniform boundedness of the sequence

$$\big\{w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\big\}_{n\in\mathbb{N}} \quad \text{in } L^1(\Omega_T)$$

coming from the Fenchel–Young inequality and a priori estimates (5.64) (as in the case of J_1 above) enables us to apply once again Theorem 8.41 combined with Theorem 8.37 to obtain

$$w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\xrightarrow[n\to\infty]{b} w\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\cdot\lambda\,\mathrm{d}\nu_{t,x}(\lambda).$$

Since assumption *(A2p)* implies $\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\geq 0$ by (5.88) and (5.87), we have

$$\int_{\Omega_T} w\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\cdot\lambda\,\mathrm{d}\nu_{t,x}(\lambda)\;\,\mathrm{d}x\,\mathrm{d}t \leq \limsup_{n\to\infty}\int_{\Omega_T} w\,\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\;\,\mathrm{d}x\,\mathrm{d}t.$$

We already have (5.76), so we can put

$$\mathcal{A}_k=\mathbf{a}(t,x,\nabla T_k(u))=\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\,\mathrm{d}\nu_{t,x}(\lambda),$$

and consequently the above expression implies

$$\int_{\Omega_T} w\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\cdot\lambda\,\mathrm{d}\nu_{t,x}(\lambda)\;\mathrm{d}x\,\mathrm{d}t\leq\int_{\Omega_T} w\,\nabla T_k(u)\cdot\int_{\mathbb{R}^{N+1}}\mathbf{a}(t,x,\lambda)\,\mathrm{d}\nu_{t,x}(\lambda)\;\mathrm{d}x\,\mathrm{d}t.$$

We apply it together with (5.88) to get that the limit in (5.87) is non-positive and that

$$w\Big(\mathbf{a}(t,x,\nabla T_k(u_n))-\mathbf{a}(t,x,\nabla T_k(u))\Big)\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\xrightarrow[n\to\infty]{b}0$$

with arbitrary nonnegative $w\in C_c^\infty([0,T))$. When we take into account that $\mathbf{a}(t,x,\nabla T_k(u))\in L_{M^*}(\Omega_T;\mathbb{R}^N)$, we can choose an ascending family of sets E_j^k, such that

$$\big|E_j^k\big|\to 0\;\text{ for }\; j\to\infty\qquad\text{and}\qquad \mathbf{a}(t,x,\nabla T_k(u))\in L^\infty(\Omega_T\setminus E_j^k;\mathbb{R}^N).$$

Recall that $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $L^1(\Omega_T;\mathbb{R}^N)$, so

$$\mathbf{a}(t,x,\nabla T_k(u))\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\xrightarrow[n\to\infty]{b}0$$

and similarly we infer that

$$\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u)\xrightarrow[n\to\infty]{b}\mathbf{a}(t,x,\nabla T_k(u))\cdot\nabla T_k(u).$$

Summing up, we get

$$\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\xrightarrow[n\to\infty]{b}\mathbf{a}(t,x,\nabla T_k(u))\cdot\nabla T_k(u).$$

The coercivity assumption *(A2p)* ensures that both the right and the left-hand sides are nonnegative. Recall that Theorem 8.38 together with (5.78) and (5.72) results in (5.86).

Notice that $\nabla u_n = 0$ a.e. in $\{(t,x) : \ |u_n(t,x)| \in \{l, l+1\}\}$, so by (5.66) we get

$$\lim_{l\to\infty} \sup_{n>0} \int_{\{l-1<|u_n|<l+2\}} \mathbf{a}(t,x,\nabla u_n)\cdot \nabla u_n \ \mathrm{d}x = 0.$$

Let us define $g^l : \mathbb{R} \to \mathbb{R}$ by

$$g^l(s) = \begin{cases} 1 & \text{if } \ l \le |s| \le l+1, \\ 0 & \text{if } \ |s| < l-1 \text{ or } |s| > l+2, \\ \text{is affine otherwise.} \end{cases}$$

Therefore

$$\int_{\{l<|u|<l+1\}} \mathbf{a}(t,x,\nabla u)\cdot \nabla u \ \mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} g^l(u)\,\mathbf{a}(t,x,\nabla T_{l+2}(u))\cdot \nabla T_{l+2}(u)\ \mathrm{d}x\,\mathrm{d}t. \tag{5.89}$$

Since (5.73) (i.e. $u_n \to u$ a.e. in Ω_T) and due to (5.74) we have

$$\lim_{l\to\infty} |\{|u| > l\}| = 0.$$

By *(A3p)* we have

$$\mathbf{a}(t,x,\xi)\cdot \xi \ge 0,$$

so we can estimate the limit of the right-hand side of (5.89) in the following way

$$\begin{aligned} 0 &\le \lim_{l\to\infty} \int_{\{l-1<|u|<l+2\}} \mathbf{a}(t,x,\nabla u)\cdot \nabla u \ \mathrm{d}x\,\mathrm{d}t \\ &\le \lim_{l\to\infty} \int_{\Omega} g^l(u)\ \mathbf{a}(t,x,\nabla T_{l+2}(u))\cdot \nabla T_{l+2}(u)\ \mathrm{d}x\,\mathrm{d}t =: L. \end{aligned}$$

Having weak convergence of (5.86) and recalling that the function g^l is continuous and bounded, we infer that

$$L = \lim_{l\to\infty} \lim_{n\to\infty} \int_{\Omega} g^l(u_n)\ \mathbf{a}(t,x,\nabla T_{l+2}(u_n))\cdot \nabla T_{l+2}(u_n)\ \mathrm{d}x\,\mathrm{d}t,$$

which can be estimated from above due to the definition of g_l as follows

$$L \le \lim_{l\to\infty} \lim_{n\to\infty} \int_{\{l-1<|u_n|<l+2\}} \mathbf{a}(t,x,\nabla T_{l+2}(u_n))\cdot \nabla T_{l+2}(u_n)\ \mathrm{d}x\,\mathrm{d}t = 0,$$

where the last equality comes from (5.66). In turn u satisfies condition *(R3p)*.

Condition ***(R2p)*****.**
We apply the integration by parts formula (Theorem 4.2.10) for (5.60), with arbitrary $h \in C_c^1(\mathbb{R})$ and $\xi \in C_c^\infty([0,T)\times\Omega)$, obtaining

$$-\int_{\Omega_T}\left(\int_{u_{0,n}}^{u_n} h(\sigma)\,\mathrm{d}\sigma\right)\partial_t\xi\ \,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mathbf{a}(t,x,\nabla u_n)\cdot\nabla(h(u_n)\xi)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega_T} T_n(f)h(u_n)\xi\,\mathrm{d}x\,\mathrm{d}t. \quad (5.90)$$

It suffices to justify passing to the limit as $n\to\infty$ above for fixed $R>0$ such that $\operatorname{supp} h\subset[-R,R]$. The right-hand side converges to the desired limit due to the Lebesgue dominated convergence theorem since $T_n f\to f$ in $L^1(\Omega_T)$ and $\{h(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded.

To deal with the limit on the left-hand side we notice that

$$\lim_{n\to\infty}-\int_{\Omega_T}\left(\int_{u_{0,n}}^{u_n} h(\sigma)\,\mathrm{d}\sigma\right)\partial_t\xi\ \,\mathrm{d}x\,\mathrm{d}t=-\int_{\Omega_T}\left(\int_{u_0}^{u} h(\sigma)\,\mathrm{d}\sigma\right)\partial_t\xi\,\mathrm{d}x\,\mathrm{d}t,$$

where the equality is justified by the continuity of the integral. As for the second expression on the left-hand side of (5.90), since $\operatorname{supp} h\subset[-R,R]$, we write

$$\begin{aligned}&\int_{\Omega_T}\mathbf{a}(t,x,\nabla u_n)\cdot\nabla(h(u_n)\xi)\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{\Omega_T} h'(T_R(u_n))\,\mathbf{a}(t,x,\nabla T_R(u_n))\cdot\nabla T_R(u_n)\,\xi\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+\int_{\Omega_T} h(T_R(u_n))\,\mathbf{a}(t,x,\nabla T_R(u_n))\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t\\&=: III_1^n+III_2^n.\end{aligned}$$

Recall (5.86), that is, the weak convergence of $\{\mathbf{a}(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\}_{n\in\mathbb{N}}$ in $L^1(\Omega_T)$. By (5.73) we see that $h'(u_n)\xi\to h'(u)\xi$ a.e. in Ω_T and

$$\|h'(u_n)\xi\|_{L^\infty(\Omega_T)}\le\|h'(u_n)\|_{L^\infty(\Omega_T)}\|\xi\|_{L^\infty(\Omega_T)},$$

so we can pass to the limit as $n\to\infty$ in III_1^n. In order to justify the case of III_2^n we recall that (5.76) implies the weak convergence

$$\mathbf{a}(t,x,\nabla T_R(u_n))\xrightarrow[n\to\infty]{}\mathbf{a}(t,x,\nabla T_R(u))\quad\text{in }\ L^1(\Omega_T).$$

Furthermore, $\{h(T_R(u_n))\}_{n\in\mathbb{N}}$ converges a.e. in Ω_T to $h(T_R(u))$ and is uniformly bounded in $L^\infty(\Omega_T)$, therefore we can pass to the limit as $n\to\infty$. Putting this together we conclude that

$$\begin{aligned}\lim_{n\to\infty}(III_1^n+III_2^n)&=\int_{\Omega_T} h'(T_R(u))\,\mathbf{a}(t,x,\nabla T_R(u))\cdot\nabla T_R(u)\,\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_{\Omega_T} h(T_R(u))\,\mathbf{a}(t,x,\nabla T_R(u))\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore, all the expressions of (5.90) converge to the desired limits in *(R2p)*.

We already proved that u satisfies *(R1p)*, *(R2p)*, and *(R3p)*, hence it is a renormalized solution to (4.70).

Uniqueness. Uniqueness is a direct consequence of the comparison principle (Theorem 5.3.14). Assume there exist two renormalized solutions v^1 and v^2 to the problem (4.70) for given data f and v_0. To show that they are equal a.e. in Ω_T it suffices to apply (5.59), which implies that for a.e. $\tau \in (0,T)$ we have

$$\int_\Omega \big(v^1(\tau,x) - v^2(\tau,x)\big)\mathrm{sgn}_0^+(v^1(\tau,x) - v^2(\tau,x))\,\mathrm{d}x \le 0$$

as well as

$$\int_\Omega \big(v^2(\tau,x) - v^1(\tau,x)\big)\mathrm{sgn}_0^+(v^2(\tau,x) - v^1(\tau,x))\,\mathrm{d}x \le 0.$$

Consequently, $v^1(\tau,x) = v^2(\tau,x)$ for a.a. $(\tau,x) \in (0,T)\times\Omega$.

5.3.5 Exercises

The problem treated in Theorem 5.3.3 can be developed in various directions.

- To cover more general conditions ensuring the density of smooth functions, one can refine the results of Theorems 8.35, 4.2.6, and 5.3.12. The possible ways are indicated in Remarks 3.7.11 and 3.7.13. It is possible to lower the regularity imposed on M with respect to the time variable. Note that the existence of weak solutions is actually proved under almost no restriction with respect to the time variable [63]. Nonetheless, proving Step 6 under more general conditions will be highly challenging.
- The growth conditions can be relaxed. In [188] the existence of renormalized solutions is provided for M independent of the time variable and under the restriction $M^* \in \Delta_2$, but not $M \in \Delta_2$. It would be interesting to extend it to $M = M(t,x,\xi)$. On the other hand, one may think about extending the ideas of Theorem 4.1.3 to prove the existence of renormalized solutions imposing $M \in \Delta_2$, but not $M^* \in \Delta_2$.
- Other notions of very weak solutions can be studied under various regimes. In particular it would be interesting to verify under what assumptions the notions essentially differ from each other.
- One can study which kinds of lower-order terms can be incorporated into the equation or what kind of structural conditions need to be imposed on the operator if $\mathbf{a} = \mathbf{a}(t,x,u,\nabla u)$, see [188].
- Following the pioneering contribution for differential inclusions [109] it might be interesting to study its parabolic version under various regimes mentioned above (and with lower-order terms included).
- One may be interested in related problem with problems with data more general than merely integrable, see e.g. [273, 274]. In particular, there is an open problem

for measure data equations involving nonlinear operators (even of power growth), namely what the optimal assumption on a measure datum ensuring uniqueness of a very weak solution is, cf. [46].

- The regularity theory of solutions to measure data parabolic equations and their gradients with such a general growth is essentially an open field.

Chapter 6
Homogenization of Elliptic Boundary Value Problems

The main concern of this chapter is a homogenization process for families of strongly nonlinear elliptic problems with a homogeneous Dirichlet boundary condition. The spatial inhomogeneity in the study of homogenization is motivated by the phenomenon of the creation of a porous structure under the influence of an electric field. An example of this phenomenon is the formation of such structures in metal oxides, such as aluminium and titanium, in the process of anodization. Experiments reveal that in the growing oxide layer spatially irregular pores are formed. This is due to the dependence of oxide conductivity on the electric field, cf. [288]. A benefit of the anodization process is that an oxide film increases resistance to corrosion and wear, and provides better adhesion for paint primers and glues than the bare metal itself, thus the study of anodization has attracted a great deal of attention, see [204, 200] among many other references. The mathematical interpretation of homogenization is simply that it is an averaging of PDEs with oscillating coefficients.

The growth and the coercivity of an elliptic operator is assumed to be prescribed by an inhomogeneous anisotropic N-function. The dependence of an N-function on a spatial variable has a significant impact on the problem as it means the homogenization process will change the underlying function spaces and the nonlinear elliptic operator at each step. For this reason the presented results are not just a generalization of homogenization of nonlinear elliptic systems in the standard L^p-setting, but are qualitatively different.

6.1 Formulation of the Homogenization Problem

Homogenization is an approach to studying problems involving operators with rapidly oscillating coefficients. Translating this to physical language it is a way to study heterogeneous materials, i.e. such that some microstructure is present. The length scale of oscillations in the coefficients is significantly smaller than the size of the domain.

The fundamental lecture of Tartar [310] and also consequent works [290, 269] can be recognized as the origin of the study of homogenization of elliptic equations.

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
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These results quickly became of the highest interest among the properties of elliptic systems with periodic structure. The homogenization process was also the starting point for the development of the two-scale convergence technique, introduced by Allaire [11] and later generalized for different operators in [341]. This approach made it possible to find a homogenized equation and prove convergence in a single process.

Let now $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with $N \geq 2$. The parameter ε is a positive number which is considered to be small in comparison to the size of the domain Ω. Given $\mathbf{F}: \Omega \to \mathbb{R}^{d\times N}$ and a nonlinear operator $\mathbf{A}: \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ we study elliptic systems of the form

$$\begin{aligned} \operatorname{div} \mathbf{A}\left(\tfrac{x}{\varepsilon}, \nabla \mathbf{u}^{\varepsilon}\right) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u}^{\varepsilon} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{6.1}$$

where $\mathbf{u}^{\varepsilon}: \Omega \to \mathbb{R}^d$ is an unknown. As the length scale of oscillating coefficients is visibly smaller than the size of the domain, studying such an equation would be too complex, and thus in the homogenization process we let $\varepsilon \to 0$ in (6.1) and expect to show that $\mathbf{u}^{\varepsilon} \to \mathbf{u}$, where the limit $\mathbf{u}$ solves the following nonlinear elliptic problem with an operator independent of a spatial variable, i.e.,

$$\begin{aligned} \operatorname{div} \hat{\mathbf{A}}(\nabla \mathbf{u}) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{6.2}$$

and the operator $\hat{\mathbf{A}}$ is defined as

$$\hat{\mathbf{A}}(\boldsymbol{\xi}) := \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \mathbf{W}(y))\, \mathrm{d}y. \tag{6.3}$$

Here by Y we mean a periodicity cell of a fixed size, for simplicity chosen as the unit cube, $Y := (0,1)^N$ and $\mathbf{W}$ solves the cell problem, i.e., $\mathbf{W} := \nabla \mathbf{w}$ with Y-periodic $\mathbf{w}: \mathbb{R}^N \to \mathbb{R}^N$ solving

$$\operatorname{div} \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}(y)) = 0 \text{ in } Y. \tag{6.4}$$

Note that now $\boldsymbol{\xi}$ plays the role of a parameter and y is a variable. Summarizing, we try to understand how the microscopic properties of a material influence its macroscopic behavior. A good understanding of the above formulated cell problem is a starting point for further analysis, and thus the presentation will start by collecting its properties, see Section 6.5

We start by formulating the assumptions on the operator $\mathbf{A}$. They partially correspond to analogous conditions appearing in preceding chapters in view of growth conditions prescribed by an N-function. For simplicity we follow the generality of growth and coercivity conditions introduced in the original papers on the topic [59, 60], however the whole analysis could be conducted under the most general conditions $(A2e)$. In addition, $\mathbf{A}$ is periodic in the first variable:

(A1) $\mathbf{A}$ is a Carathéodory mapping.
(A2) $\mathbf{A}$ is Y-periodic, i.e. periodic in each argument y_i, $i = 1, \ldots, N$, with period 1.

(A3) There exists an N-function $M\colon \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ and a constant $c>0$ such that for a.a. $y \in Y$ and all $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$

$$\mathbf{A}(y,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geq c(M(y,\boldsymbol{\xi})+M^*(y,\mathbf{A}(y,\boldsymbol{\xi}))).$$

(A4) For all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times N}$ such that $\boldsymbol{\xi} \neq \boldsymbol{\eta}$ and a.a. $y \in Y$, we have

$$(\mathbf{A}(y,\boldsymbol{\xi})-\mathbf{A}(y,\boldsymbol{\eta}))\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})>0.$$

Before formulating the theorem, which is the main content of the current chapter, we briefly describe how the study of homogenization of elliptic equations has developed. The first results go back to the works of Oleĭnik and Zhikov [269] and Allaire [11]. However, the setting of non-standard growth conditions of the operator $\mathbf{A}$, which is of particular interest to us, appeared in [341]. The authors considered the growth prescribed by means of a variable exponent $p(x)$, so the corresponding function spaces were varying with respect to $\varepsilon \to 0$ in the homogenization process. Notice that in the $L^{p(x)}$ setting they required that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, so the corresponding functions spaces were reflexive and separable as well. The first attempt to deal with an N-function not satisfying the Δ_2-condition was in [59], where for an operator $\mathbf{A}$ satisfying (A1)–(A4) and an N-function M the limit $\varepsilon \to 0$ was successfully established, provided that M is log-Hölder continuous with respect to the first variable.

Later, the same authors showed that for discontinuous functions M one can obtain a fairly complete theory provided that M or M^* satisfy Δ_2-condition, but without any assumption on the continuity with respect to the spatial variable, see [60]. This result, supplemented with numerous details to ease the understanding of proof steps, is presented in the current chapter. It is a particularly interesting case as it makes it possible to model discontinuity of conductivities from one phase to the other.

6.2 Definitions, Main Result and the Strategy

In this section we introduce definitions of solutions to the original problem and discuss the limit problem, and then explain that the difficulty in the homogenization process turns upon the possibility of varying function spaces. For simplicity we define $M^\varepsilon(x,\boldsymbol{\xi}) = M\left(\frac{x}{\varepsilon},\boldsymbol{\xi}\right)$ for fixed ε. Throughout the chapter we present a twin-track reasoning – the cases of the Δ_2-condition for an N-function M and the Δ_2-condition for its conjugate M^* need to be treated differently and thus the definitions and theorems have two variants.

Definition 6.2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, the operator $\mathbf{A}$ satisfy (A1)–(A4), an N-function M be Y-periodic in the first variable and $\mathbf{F} \in L^\infty(\Omega;\mathbb{R}^{d\times N})$.

(i) If the conjugate N-function M^* satisfies the Δ_2-condition, then we call $\mathbf{u}^\varepsilon$ a solution to problem (6.1) if for fixed $\varepsilon \in (0,1)$

$$\mathbf{u}^{\varepsilon} \in W_0^1 L_{M^{\varepsilon}}\left(\Omega;\mathbb{R}^N\right)$$

and

$$\int_{\Omega} \mathbf{A}\left(\tfrac{x}{\varepsilon}, \nabla \mathbf{u}^{\varepsilon}(x)\right) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x = \int_{\Omega} \mathbf{F}(x) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x \tag{6.5}$$

is satisfied for all $\boldsymbol{\varphi} \in W_0^1 L_{M^{\varepsilon}}\left(\Omega;\mathbb{R}^N\right)$.

(ii) If the N-function M satisfies the Δ_2-condition, then we call $\mathbf{u}^{\varepsilon}$ a solution to problem (6.1) if for fixed $\varepsilon \in (0,1)$

$$\mathbf{u}^{\varepsilon} \in V_0^{M^{\varepsilon}}$$

and

$$\int_{\Omega} \mathbf{A}\left(\tfrac{x}{\varepsilon}, \nabla \mathbf{u}^{\varepsilon}(x)\right) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x = \int_{\Omega} \mathbf{F}(x) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x \tag{6.6}$$

is satisfied for all $\boldsymbol{\varphi} \in V_0^{M^{\varepsilon}}$.

As announced earlier, we formulate below the homogenization result. We allow ourselves some imprecision here when mentioning the notion of solution to the limit problem (6.2), which has not yet been precisely defined. However, this definition will appear after a careful analysis of the growth conditions, see Definition 6.6.8. Note that, even though we know the conditions that are satisfied by an operator $\mathbf{A}$, the growth conditions that are satisfied by $\hat{\mathbf{A}}$ need to be deduced.

Theorem 6.2.2 *Let* $\mathbf{A}$ *satisfy (A1)–(A4), an N-function M be Y-periodic and let at least one of the following conditions hold:*

1. *M satisfies the Δ_2-condition,*
2. *M^*, the convex conjugate N-function to M, satisfies the Δ_2-condition.*

Furthermore, assume that

$$\mathbf{F} \in L^{\infty}(\Omega;\mathbb{R}^{d\times N}) \tag{6.7}$$

and for any $\varepsilon > 0$ let $\mathbf{u}^{\varepsilon}$ be a unique weak solution to the problem (6.1) *according to Definition 6.2.1. Then, as $\varepsilon \to 0$,*

$$\mathbf{u}^{\varepsilon} \rightharpoonup \mathbf{u} \text{ in } W_0^{1,1}(\Omega;\mathbb{R}^d),$$

where $\mathbf{u}$ *is a unique weak solution to* (6.2)*, provided that either the considered problem is scalar, i.e., $d = 1$, or the embeddings*

$$W_0^1 L_{m_1}(\Omega) \hookrightarrow L_{m_2}(\Omega) \text{ and } W^1 L_{m_1}(Y) \hookrightarrow L_{m_2}(Y) \tag{6.8}$$

hold true.

In the above theorem, as well as in the further parts of the chapter, we use Musielak–Orlicz (or Musielak–Orlicz–Sobolev) spaces generated by Young functions m_1, m_2 coming from condition (2.37). To stay consistent with the definition of an N-function we should understand these functions as follows: $m_i(z,\xi) := m_i(|\xi|)$. Note that we used the same letters for a Young function and for the N-function for simplicity of notation.

Growth and coercivity conditions naturally lead to a priori estimates implying that

$$\int_\Omega M\left(\tfrac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon\right)\,\mathrm{d}x \quad\text{and}\quad \int_\Omega M^*\left(\tfrac{x}{\varepsilon},\mathbf{A}\left(\tfrac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon\right)\right)\,\mathrm{d}x$$

are bounded. Note however that this bound is not uniform in ε, which means that by passing to the limit as $\varepsilon\to 0$ the function spaces L_{M^ε} would change in each step. In such case a passage to the limit is not possible. For that reason we shall benefit from the uniform bound we have on an N-function, cf. Definition 2.2.2, by two Young functions m_1 and m_2. Thus $\nabla\mathbf{u}^\varepsilon$ is uniformly bounded in $L_{m_1}(\Omega)$ and $\mathbf{A}\left(\frac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon\right)$ in $L_{m_2^*}(\Omega)$, which are already objects that allow for uniform estimates and compactness conclusions. We will see, however, that there are no definite disadvantages to working in less general spaces generated by these Young functions. It immediately appears questionable whether the product of these limits has a chance to be well-defined, as the spaces $L_{m_1}(\Omega)$ and $L_{m_2^*}(\Omega)$ are not associate spaces. However, this problem will be solved after identifying new N-functions that on one hand prescribe growth and coercivity conditions of $\hat{\mathbf{A}}(\nabla\mathbf{u})$, and on the other hand define associate spaces that allow the above mentioned limits to have a well-defined product. As expected, the limiting operator $\hat{\mathbf{A}}(\nabla\mathbf{u})$ does not depend on x.

Before starting the proof of Theorem 6.2.2, we first assemble some facts: The existence of solutions to problems (6.1) and (6.2) need to be established. We can already do this for problem (6.1), but to handle problem (6.2) special attention needs to be directed to a cell problem. In particular, in the first step the growth conditions of $\hat{\mathbf{A}}$ need to be found, i.e. an appropriate N-function need to be identified, which is the main content of Section 6.5.

Before however answering all the existence questions, we firstly collect the tools needed for the passage to the limit as $\varepsilon\to 0$ (see Section 6.4). This exposition concentrates on isotropic homogeneous spaces for the reasons mentioned above, i.e. the passage to the limit is performed exclusively in such spaces.

6.3 The Functional Setting

In accordance with the notation that has been used so far, $\Omega\subset\mathbb{R}^N$ is a bounded domain and $Y:=(0,1)^N$, which we endow with the Luxemburg norm

$$\|\mathbf{v}\|_{L_M}=\|\mathbf{v}\|_{E_M}:=\inf\left\{\lambda>0:\int_\Omega\int_Y M\left(y,\frac{\mathbf{v}(x,y)}{\lambda}\right)\,\mathrm{d}y\,\mathrm{d}x\le 1\right\}.$$

From here on in this chapter we assume that whenever a function depends on a variable from Y, it is always Y-periodic, even if the Y-periodicity is not mentioned.

The spaces of periodic functions were introduced in Section 3.6. For our considerations we introduce the following closed subspaces of the spaces $E_M^{per}(Y;\mathbb{R}^{d\times N})$ and $E_{M^*}^{per}(Y;\mathbb{R}^{d\times N})$, as well as their annihilators

$$
\begin{aligned}
G &:= \{\nabla \mathbf{w} : \mathbf{w} \in W^1_{per}E_M(Y;\mathbb{R}^d)\},\\
G^{\perp} &:= \{\mathbf{W} \in L_{M^*}(Y;\mathbb{R}^{d\times N}) : \int_Y \mathbf{W}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0 \text{ for all } \mathbf{V} \in G\},\\
D &:= E^{per,\mathrm{div}}_{M^*}(Y;\mathbb{R}^{d\times N}),\\
D^{\perp} &:= \{\mathbf{W} \in L_M(Y;\mathbb{R}^{d\times N}) : \int_Y \mathbf{W}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0 \text{ for all } \mathbf{V} \in D\}.
\end{aligned}
$$

We note that

$$D^{\perp} = \{\nabla \mathbf{v} : \mathbf{v} \in V^M_{per}\}. \tag{6.9}$$

By $G^{\perp\perp}$ and $D^{\perp\perp}$ we understand the second annihilators. To prescribe their basic characterization we already need to distinguish two cases according to whether M or M^* satisfy the Δ_2-condition. If M^* satisfies the Δ_2-condition, then $(L_{M^*})^* = L_M$ and we can observe that

$$G^{\perp\perp} = \{\mathbf{W} \in L_M(Y;\mathbb{R}^{d\times N}) : \int_Y \mathbf{W}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0 \ \text{ for all } \mathbf{V} \in G^{\perp}(Y)\}. \tag{6.10}$$

Similarly, if M satisfies the Δ_2-condition, then $(L_M)^* = L_{M^*}$ and we obtain

$$D^{\perp\perp} = \{\mathbf{W} \in L_{M^*}(Y;\mathbb{R}^{d\times N}) : \int_Y \mathbf{W}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0 \ \text{ for all } \mathbf{V} \in D^{\perp}\}. \tag{6.11}$$

6.4 Homogenization Tools in the Setting of Musielak–Orlicz Spaces

The method of periodic unfolding is one of the tools used in homogenization problems. It has its origins in the L^p setting in the works [320]. It essentially relies on two ideas: firstly one doubles the dimension by introducing the unfolding operator S_ε. This step allows one to use standard weak or strong convergence results in L^p instead of the tools of two-scale convergence. Indeed, this procedure allows us to associate to a function in $L^p(\Omega)$ a function $v(S_\varepsilon)$, which is an element of $L^p(\Omega\times Y)$, and it turns out that two-scale convergence of a sequence in L^p is equivalent to the weak convergence in $L^p(\Omega\times Y)$ of the unfolded sequence. We recall here that a sequence of functions v^ε in $L^p(\Omega)$, $p \in (1,\infty)$, is said to two-scale converge to a limit $v^0(y,x) \in L^p(\Omega\times Y)$ if for any function $\varphi(x,y) \in C^\infty_c(\Omega, C^\infty_{per}(Y))$ it holds that

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon(x)\varphi(x,\tfrac{x}{\varepsilon})\,\mathrm{d}x = \int_\Omega\int_Y v^0(y,x)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.12}$$

Here Y is assumed to be a unit cube, otherwise the right-hand side would be divided by the measure of the periodicity cell $|Y|$. Thus one shows that two-scale convergence of a sequence v^ε in $L^p(\Omega)$ is equivalent to the weak convergence of $v^\varepsilon(S_\varepsilon)$ in $L^p(\Omega\times Y)$, see [96, Proposition 2.14].

The current setting of Musielak–Orlicz spaces, because of their non-reflexivity, only provides conclusions on the weak-$*$ compactness of bounded sets, thus for our purposes we need to use the condition of convergence of the unfolded sequence $v^\varepsilon \circ S_\varepsilon$ as a definition of two-scale convergence, additionally underlining precisely what type of convergence we have in mind. In the forthcoming lemma the relations to the standard definition of two-scale convergence (6.12) are discussed.

The second equally important element of the periodic unfolding method is separating the characteristic scales, which means that every function is decomposed into two parts.

For a more rigorous presentation of these ideas we define functions $n : \mathbb{R} \to \mathbb{Z}$

$$n(t) := \max\{n \in \mathbb{Z} : n \le t\},\ \forall t \in \mathbb{R}, \tag{6.13}$$

and

$$\lfloor x \rfloor := (n(x_1), \dots, n(x_d)),\ \forall x \in \mathbb{R}^N. \tag{6.14}$$

Set

$$r(x) := x - \lfloor x \rfloor. \tag{6.15}$$

Then obviously for any $x \in \mathbb{R}^N$, $\varepsilon > 0$, we have a two-scale decomposition

$$x = \varepsilon \left(\left\lfloor \tfrac{x}{\varepsilon} \right\rfloor + r\left(\tfrac{x}{\varepsilon}\right)\right), \tag{6.16}$$

where r is a remainder function. Then we define for any $\varepsilon > 0$ a two-scale composition function $S_\varepsilon : Y \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$S_\varepsilon(y,x) := \varepsilon \left(\left\lfloor \tfrac{x}{\varepsilon} \right\rfloor + y\right). \tag{6.17}$$

It follows immediately that

$$S_\varepsilon(y,x) \to x \text{ uniformly in } Y \times \mathbb{R}^N \text{ as } \varepsilon \to 0 \tag{6.18}$$

since $S_\varepsilon(y,x) = x + \varepsilon\left(y - r\left(\tfrac{x}{\varepsilon}\right)\right)$.

Definition 6.4.1. Assume $m : [0,\infty) \to [0,\infty)$ is a Young function. We say that a sequence of functions $\{v^\varepsilon\}_{\varepsilon>0} \subset L_m(\mathbb{R}^N)$

(i) converges to v^0 *weakly-∗ two-scale* in $L_m(\mathbb{R}^N \times Y)$, written $v^\varepsilon \xrightharpoonup{2-s}{}^{*} v^0$, if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 weakly-∗ in $L_m(\mathbb{R}^N \times Y)$,

(ii) converges to v^0 *strongly two-scale* in $E_m(\mathbb{R}^N \times Y)$, written $v^\varepsilon \xrightarrow{2-s} v^0$, if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 strongly in $E_m(\mathbb{R}^N \times Y)$.

We define two-scale convergence in $L_m(\Omega \times Y)$ as two-scale convergence in $L_m(\mathbb{R}^N \times Y)$ for functions extended by zero to $\mathbb{R}^N \setminus \Omega$.

In the following lemma we point out the relation of the above definition of two-scale convergence with the standard one recalled in (6.12). The proof of this lemma appears at the end of this section, since it uses some convergence properties that are proved afterwards (Lemma 6.4.4 part (iii)).

Lemma 6.4.2 *If $v^\varepsilon \xrightharpoonup{2-s}{}^{*} v^0$ in $L_m(\Omega \times Y)$, then for any $\psi \in C_c^\infty\left(\Omega; C_{per}^\infty(Y)\right)$*

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x)\psi\left(\tfrac{x}{\varepsilon}, x\right)\, \mathrm{d}x = \int_\Omega \int_Y v^0(y,x)\psi(y,x)\, \mathrm{d}y\, \mathrm{d}x. \tag{6.19}$$

The following two lemmas provide essential aspects of two-scale convergence, which appear to be a core of the proof of the main result. The first one we recall with its short proof from [320].

Lemma 6.4.3 (Lemma 1.1, [320]) *Let g be measurable with respect to a σ-algebra generated by the product of the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}^N$ and the σ-algebra of all Borel measurable subsets of Y. Assume in addition that $g \in L^1(\mathbb{R}^N; L^\infty(Y))$ and extend it by Y-periodicity to $\mathbb{R}^N$ for a.a. $x \in \mathbb{R}^N$. Then, for any $\varepsilon > 0$, the function $(y,x) \mapsto g(y, S_\varepsilon(y,x))$ is integrable and*

$$\int_{\mathbb{R}^N} g\left(\tfrac{x}{\varepsilon}, x\right)\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_Y g(y, S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. Firstly we observe that the functions $x \mapsto g(\frac{x}{\varepsilon}, x)$ as well as $x \mapsto g(y, S_\varepsilon(y,x))$ are measurable. From the definition (6.17) we conclude that the mapping $(y,x) \mapsto (y, S_\varepsilon(y,x))$ is piecewise constant with respect to x and affine with respect to y. Observe that since $\mathbb{R}^N = \bigcup_{m\in\mathbb{Z}^N}(\varepsilon m + \varepsilon Y)$ and $\lfloor\frac{x}{\varepsilon}\rfloor = m$ for any $x \in \varepsilon m + \varepsilon Y$ the following holds

$$\begin{aligned}\int_{\mathbb{R}^N} g\left(\tfrac{x}{\varepsilon}, x\right)\,\mathrm{d}x &= \sum_{m\in\mathbb{Z}^N}\int_{\varepsilon m+\varepsilon Y} g\left(\tfrac{x}{\varepsilon}, x\right)\,\mathrm{d}x = \varepsilon^N \sum_{m\in\mathbb{Z}^N}\int_Y g\left(y, \varepsilon(m+y)\right)\,\mathrm{d}y \\ &= \sum_{m\in\mathbb{Z}^N}\int_{\varepsilon m+\varepsilon Y}\mathrm{d}x \int_Y g\left(y, \varepsilon\left(\left\lfloor\tfrac{x}{\varepsilon}\right\rfloor + y\right)\right)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\mathrm{d}x\int_Y g(y, S_\varepsilon(y,x))\,\mathrm{d}y.\end{aligned} \tag{6.20}$$

□

Lemma 6.4.4 *Assume that $m : [0,\infty) \to [0,\infty)$ is a Young function.*

(i) *Let $\{v^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L_m(\Omega)$. Then there is a $v^0 \in L_m(\Omega\times Y)$ such that, passing to a subsequence if necessary,*

$$v^\varepsilon \xrightarrow{2-s}{}^{*} v^0 \text{ in } L_m(\Omega\times Y)$$

as $\varepsilon \to 0$.

(ii) *If $v^\varepsilon \xrightarrow{2-s}{}^{*} v^0$ in $L_m(\Omega\times Y)$ then*

$$v^\varepsilon \overset{*}{\rightharpoonup} \int_Y v^0(y,\cdot)\,\mathrm{d}y \text{ in } L_m(\Omega).$$

(iii) *If $v^\varepsilon \xrightarrow{2-s}{}^{*} v^0$ in $L_m(\Omega\times Y)$ and $w^\varepsilon \xrightarrow{2-s} w^0$ in $E_{m^*}(\Omega\times Y)$, then*

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon(x)w^\varepsilon(x)\,\mathrm{d}x = \int_\Omega\int_Y v^0(y,x)w^0(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

(iv) *Let $v^\varepsilon \overset{*}{\rightharpoonup} v^0$ in $W_0^1 L_m(\Omega)$. Then*

$$v^\varepsilon \xrightarrow{2-s}{}^{*} v^0 \textit{ in } L_m(\Omega\times Y)$$

and there is a $\mathbf{v} \in L_m(\Omega\times Y;\mathbb{R}^d)$ *such that*

$$\nabla v^\varepsilon \xrightarrow{2-s}{}^{*} \nabla v^0 + \mathbf{v} \textit{ in } L_m(\Omega\times Y;\mathbb{R}^d)$$

as $\varepsilon \to 0$, *and*

$$\int_Y \mathbf{v}(y,x)\cdot\psi(y)\,\mathrm{d}y = 0$$

for a.a. $x\in\Omega$ *and any* $\psi \in C^\infty_{per}(Y;\mathbb{R}^d), \operatorname{div}\psi = 0$ *in* Y.

(v) *Let* $\Phi : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ *be a Carathéodory function,* $\Phi(y,\cdot)$ *be convex for almost all* $y\in Y$ *and* $\Phi(\cdot,\boldsymbol{\xi})$ *be* Y*-periodic for any* $\boldsymbol{\xi}\in\mathbb{R}^{d\times N}$*. Moreover, let* $\Phi \geq 0$ *and* $\Phi(\cdot,0)=0$*. If*

$$\mathbf{U}^\varepsilon \xrightarrow{2-s}{}^{*} \mathbf{U} \text{ in } L_m(\Omega\times Y;\mathbb{R}^{d\times N})$$

then

$$\liminf_{\varepsilon\to 0}\int_\Omega \Phi\left(\tfrac{x}{\varepsilon},\mathbf{U}^\varepsilon(x)\right)\,\mathrm{d}x \geq \int_\Omega\int_Y \Phi(y,\mathbf{U}(y,x))\,\mathrm{d}y\,\mathrm{d}x.$$

Proof. In order to show (i), we first apply Lemma 6.4.3 to a function $g = m\left(\frac{|v^\varepsilon|}{\lambda}\right)$, which is independent of y, where $\{v^\varepsilon\}_{\varepsilon>0}$ is an arbitrary bounded sequence in $L_m(\Omega)$. Consequently this implies

$$c \geq \int_\Omega m\left(\frac{|v^\varepsilon(x)|}{\lambda}\right)\,\mathrm{d}x = \int_\Omega\int_Y m\left(\frac{|v^\varepsilon(S_\varepsilon(y,x))|}{\lambda}\right)\,\mathrm{d}y\,\mathrm{d}x$$

for some $\lambda>0$. This boundedness yields that there exists a subsequence, also denoted by $\{v^\varepsilon\circ S_\varepsilon\}_{\varepsilon>0}$, and a limit function $v^0\in L_m(\Omega\times Y)$ such that

$$v^\varepsilon\circ S_\varepsilon \overset{*}{\rightharpoonup} v^0 \text{ in } L_m(\Omega\times Y)$$

as $\varepsilon\to 0$ by the Banach–Alaoglu theorem (Theorem 8.31). We recall that $L_m(\Omega\times Y) = (E_{m^*}(\Omega\times Y))^*$ and $E_{m^*}(\Omega\times Y)$ is a separable space. Assertion (i) then obviously follows by the definition of weak-$*$ two-scale convergence.

Point (ii) is an immediate consequence of the definition of the weak-$*$ two-scale convergence in $L_m(\Omega\times Y)$ once we use test functions, which are independent of the y-variable.

To prove assertion (iii) one again applies Lemma 6.4.3, this time to the function $g = vw$. With the help of Hölder's inequality (cf. Lemma 3.1.15) it is easy to show the integrability of g an hence that the assumptions of the lemma are satisfied. Thus

$$\begin{aligned}\int_\Omega v^\varepsilon(x)w^\varepsilon(x)\,\mathrm{d}x &= \int_\Omega\int_Y v(y,S_\varepsilon(y,x))w(y,S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_\Omega\int_Y v(y,S_\varepsilon(y,x))(w(y,S_\varepsilon(y,x))-w^0(y,x))\,\mathrm{d}y\,\mathrm{d}x \qquad (6.21)\\ &+ \int_\Omega\int_Y v(y,S_\varepsilon(y,x))w^0(y,x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

The first term on the right-hand side converges to zero since w^ε two-scale converges strongly in $E_{m^*}(\Omega)$ and $v^\varepsilon \circ S_\varepsilon$ is bounded in $L_m(\Omega\times Y)$. The second term converges, immediately from the definition of the weak-$*$ two-scale convergence in $L_m(\Omega\times Y)$, to $\int_\Omega \int_Y v^0 w^0 \,\mathrm{d}y\,\mathrm{d}x$. Observe that w^0, as an element of $E_{m^*}(\Omega\times Y)$, and since $(E_{m^*}(\Omega\times Y))^* = L_m(\Omega\times Y)$, is a proper test function.

In order to show (iv) we observe first that $\{v^\varepsilon\}_{\varepsilon>0}$ is bounded in $L_m(\Omega)$. Thus by (i) there is a $v^0 \in L_m(\Omega\times Y)$ and a subsequence, labelled the same, such that $v^\varepsilon \xrightharpoonup{2-s}{}^* v^0$ in $L_m(\Omega\times Y)$. Then (iii) implies for all $\varphi \in C_c^\infty(\Omega, C_{per}^\infty(Y)^N)$ that

$$\begin{aligned}
0 &= -\lim_{\varepsilon\to 0} \varepsilon \int_\Omega \nabla v^\varepsilon(x)\cdot\varphi\left(\tfrac{x}{\varepsilon},x\right)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\varepsilon\int_\Omega v^\varepsilon(x)\mathrm{div}\left[\varphi\left(\tfrac{x}{\varepsilon},x\right)\right]\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_\Omega \varepsilon v^\varepsilon(x)\mathrm{div}_x\varphi\left(\tfrac{x}{\varepsilon},x\right) + v^\varepsilon(x)\mathrm{div}_y\varphi\left(\tfrac{x}{\varepsilon},x\right)\,\mathrm{d}x \\
&= \int_\Omega\int_Y v^0(y,x)\mathrm{div}_y\varphi(y,x)\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

which implies that v^0 is independent of y. If v is a weak-$*$ limit in $L_m(\Omega)$ of v^ε, then $v = \int_Y v^0$ by (ii). We thus see that for any weakly-$*$ two-scale convergent subsequence of $\{v^\varepsilon\}_{\varepsilon>0}$ the limit is v. Hence v is the weak-$*$ two-scale limit of the entire sequence $\{v^\varepsilon\}_{\varepsilon>0}$. Applying (i) to the sequence $\{\nabla v^\varepsilon\}_{\varepsilon>0}$ we infer the existence of $\mathbf{w} \in L_m(\Omega\times Y;\mathbb{R}^d)$ such that

$$\nabla v^\varepsilon \xrightharpoonup{2-s}{}^* \mathbf{w} \text{ in } L_m(\Omega\times Y;\mathbb{R}^d)$$

as $\varepsilon \to 0$. Let us choose $z \in C_c^\infty(\Omega)$ and $\psi \in C_{per}^\infty(Y;\mathbb{R}^d)$ with $\mathrm{div}_y\psi = 0$ in Y. Then, the continuity of ψ implies that the sequence $\psi^\varepsilon(x) := \psi\left(\frac{x}{\varepsilon}\right)$ two-scale converges in $E_{m^*}(\Omega\times Y)$, i.e. $\psi^\varepsilon \xrightarrow{2-s} \psi$ in $E_{m^*}(\Omega\times Y)$ as $\varepsilon \to 0$. It then follows from (iii) that

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla v^\varepsilon(x)\cdot z(x)\psi\left(\tfrac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Omega\int_Y \mathbf{w}(y,x)\cdot z(x)\psi(y)\,\mathrm{d}y\,\mathrm{d}x$$

whereas the integration by parts yields

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega \nabla v^\varepsilon(x)\cdot z(x)\psi\left(\tfrac{x}{\varepsilon}\right)\,\mathrm{d}x &= -\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon(x)\nabla z(x)\cdot\psi\left(\tfrac{x}{\varepsilon}\right)\,\mathrm{d}x \\
&= -\int_\Omega\int_Y v^0(x)\nabla z(x)\cdot\psi(y)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_Y \nabla v^0(x)\cdot z(x)\psi(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Hence, choosing the function $\mathbf{v} := \mathbf{w} - \nabla v^0$, we observe that it has all the properties required in the assertion (iv).

We complete the proof by showing that assertion (v) holds. Note that $\mathbf{U}^\varepsilon \xrightharpoonup{2-s}{}^* \mathbf{U}$ in $L_m(\Omega\times Y;\mathbb{R}^{d\times N})$ implies $\mathbf{U}^\varepsilon \rightharpoonup \mathbf{U}$ in $L^1(\Omega\times Y;\mathbb{R}^{d\times N})$. Thus it follows from Lemma 6.4.3 and a standard weak lower semicontinuity property that for $\mathbf{U}^\varepsilon, \mathbf{U}$ extended by zero in $\mathbb{R}^N \setminus \Omega$

$$\liminf_{\varepsilon\to 0}\int_\Omega \Phi\left(\tfrac{x}{\varepsilon},\mathbf{U}^\varepsilon(x)\right)\,\mathrm{d}x = \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\Phi\left(y,\mathbf{U}^\varepsilon(S_\varepsilon(y,x))\right)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geq \int_{\Omega\times Y}\Phi(y,\mathbf{U}(y,x))\,\mathrm{d}x\,\mathrm{d}y.$$

Hence we conclude (vi). □

Proof (of Lemma 6.4.2). First we will observe that the following fact holds. Let $\psi : Y\times\Omega \to \mathbb{R}$ be an Y-periodic smooth function. Obviously $\psi \in E_{m^*}(\Omega\times Y)$. Consider a sequence $\psi^\varepsilon(x) := \psi\left(\tfrac{x}{\varepsilon},x\right)$. Then $\psi^\varepsilon \xrightarrow{2-s} \psi$ in $E_{m^*}(\Omega\times Y)$ as $\varepsilon\to 0$.

To show that assertion (6.19) holds, we fix a weakly-$*$ two-scale convergent sequence $\{v^\varepsilon\}_{\varepsilon>0}\subset L_m(\Omega)$ with a limit $v^0\in L_m(\Omega\times Y)$ and $\psi\in C_c^\infty(\Omega;C_{per}^\infty(Y))$. Then we have $v^\varepsilon(x)\psi(x,y)\in L^1(\mathbb{R}^N;L^\infty(Y))$ provided that we set $v^\varepsilon = 0$ in $\mathbb{R}^N\setminus\Omega$, $\psi = 0$ in $(\mathbb{R}^N\setminus\Omega)\times Y$. Therefore by Lemma 6.4.3 we get

$$\int_\Omega v^\varepsilon(x)\psi\left(\tfrac{x}{\varepsilon},x\right)\,\mathrm{d}x = \int_\Omega\int_Y v^\varepsilon(S_\varepsilon(y,x))\psi(y,S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x.$$

Combining this with the convergence results $v^\varepsilon \xrightarrow{2-s}{}^{*}\, v^0$ in $L_m(\Omega\times Y)$ and $\psi\left(\tfrac{x}{\varepsilon},x\right)\xrightarrow{2-s}\psi(y,x)$ in $E_{m^*}(\Omega\times Y)$ as $\varepsilon\to 0$, we infer

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon(x)\psi\left(\tfrac{x}{\varepsilon},x\right)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_\Omega\int_Y v^\varepsilon(S_\varepsilon(y,x))\psi\left(y,S_\varepsilon(y,x)\right)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_\Omega\int_Y v^0(y,x)\psi(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

with the help of assertion (iii) of Lemma 6.4.3. □

6.5 Properties of the Cell Problem

Our aim is to investigate the properties of the homogenized operator $\hat{\mathbf{A}}$, defined by (6.3). However before discussing its properties it is necessary to know that the problem of defining $\hat{\mathbf{A}}$, which is problem (6.4), that we also call the cell problem, is solvable. First, we give a definition of weak solutions to (6.4), and then we prove the existence result.

Definition 6.5.1. Let $\mathbf{A}$ satisfy (A1)–(A4) and an N-function M be Y-periodic.

1. If M satisfies the Δ_2-condition, then for arbitrary $\boldsymbol{\xi}\in\mathbb{R}^{d\times N}$ we say that $\mathbf{w}_{\boldsymbol{\xi}}$ is a weak solution to (6.4) if
$$\mathbf{w}_{\boldsymbol{\xi}}\in W^1_{per}L_M(Y;\mathbb{R}^d)$$
and
$$\int_Y \mathbf{A}\left(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\boldsymbol{\xi}}(y)\right)\cdot\nabla\boldsymbol{\varphi}(y)\,\mathrm{d}y = 0 \qquad \text{for all } \boldsymbol{\varphi}\in W^1_{per}L_M\left(Y;\mathbb{R}^d\right). \tag{6.22}$$

2. If M^*, the convex conjugate N-function to M, satisfies the Δ_2-condition, then for arbitrary $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ we say that $\mathbf{w}_{\boldsymbol{\xi}}$ is a weak solution to (6.4) if

$$\mathbf{w}_{\boldsymbol{\xi}} \in V_{per}^{M}$$

and

$$\int_Y \mathbf{A}\left(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)\right) \cdot \nabla \boldsymbol{\varphi}(y)\, \mathrm{d}y = 0 \qquad \text{for all } \boldsymbol{\varphi} \in V_{per}^{M}. \tag{6.23}$$

Theorem 6.5.2 *Let* $\mathbf{A}$ *satisfy (A1)–(A4), an N-function M be Y-periodic, and let at least one of the following conditions hold:*

1. *M satisfies the Δ_2-condition,*
2. *M^*, the convex conjugate N-function to M, satisfies the Δ_2-condition.*

Then for arbitrary $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$*, the problem* (6.4) *admits a unique weak solution. Moreover,*

$$\boldsymbol{\xi}^j \to \boldsymbol{\xi} \text{ in } \mathbb{R}^{d\times N} \text{ implies } \mathbf{A}(\cdot, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}) \overset{*}{\rightharpoonup} \mathbf{A}(\cdot, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}) \text{ in } L_{M^*}(Y; \mathbb{R}^{d\times N}), \tag{6.24}$$

where $\mathbf{w}_{\boldsymbol{\xi}^j}$ *is a solution of the cell problem corresponding to* $\boldsymbol{\xi}^j$ *and* $\mathbf{w}_{\boldsymbol{\xi}}$ *to* $\boldsymbol{\xi}$*, respectively.*

Proof. The existence and uniqueness of solution $\mathbf{w}_{\boldsymbol{\xi}}$ can be obtained by a straightforward modification of Theorem 4.1.3 and Theorem 4.1.2, respectively.

Thus in the remaining part of the proof we concentrate on showing (6.24). Assume that $\{\boldsymbol{\xi}^j\}_{j=1}^{\infty}$ is such that $\boldsymbol{\xi}^j \to \boldsymbol{\xi}$ in $\mathbb{R}^{d\times N}$ as $j \to \infty$. By $\mathbf{w}_{\boldsymbol{\xi}^j}$ we mean a weak solution (according to Definition 6.5.1) to the problem

$$\operatorname{div} \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) = 0 \text{ in } Y. \tag{6.25}$$

The first part of the theorem provides that there exists a weak solution $\mathbf{w}_{\boldsymbol{\xi}^j}$, which is an admissible test function in (6.22), (6.23) respectively, with $\boldsymbol{\xi}^j$ instead of $\boldsymbol{\xi}$, and thus we obtain

$$\int_Y \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) \cdot \nabla \mathbf{w}_{\boldsymbol{\xi}^j}\, \mathrm{d}y = 0. \tag{6.26}$$

Using the assumption (A3), (6.26) and the Fenchel–Young inequality in consequent steps of the estimate below we infer that

$$\begin{aligned}
c \int_Y & M^*(y, \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y))) + M(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j})\, \mathrm{d}y \\
&\le \int_Y \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) \cdot (\boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j})\, \mathrm{d}y \\
&= \int_Y \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) \cdot \boldsymbol{\xi}^j\, \mathrm{d}y \\
&\le \tfrac{c}{2} \int_Y M^*(y, \mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)))\, \mathrm{d}y + \int_Y M\left(y, \tfrac{2}{c}\boldsymbol{\xi}^j\right)\, \mathrm{d}y.
\end{aligned}$$

By the definition of an N-function there exists a Young function m_2 such that

$$\int_Y M\left(y, \tfrac{2}{c}\boldsymbol{\xi}^j\right) \, \mathrm{d}y \le \int_Y m_2\left(|\tfrac{2}{c}\boldsymbol{\xi}^j|\right) \, \mathrm{d}y$$

and the integral on the right-hand side is finite as $\{\boldsymbol{\xi}^j\}_{j=1}^{\infty}$ is bounded. Thus we conclude that the sequence $\{\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j})\}_{j=1}^{\infty}$ is uniformly bounded in $L_{M^*}(Y; \mathbb{R}^{d\times N})$, so there exists a $\mathbf{Z} \in L_{M^*}(Y; \mathbb{R}^{d\times N})$ such that, passing to a subsequence if necessary,

$$\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) \overset{*}{\rightharpoonup} \mathbf{Z} \text{ in } L_{M^*}(Y; \mathbb{R}^{d\times N}) \tag{6.27}$$

as $\varepsilon \to 0$. In the next step we shall show that $\mathbf{Z} = \mathbf{A}(\cdot, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}})$ for almost all $y \in Y$, where $\mathbf{w}_{\boldsymbol{\xi}}$ is a weak solution corresponding to $\boldsymbol{\xi}$, which exists according to the first part of the theorem. The monotonicity of $\mathbf{A}$ implies that

$$\begin{aligned}
&\int_Y \left|(\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y) - \boldsymbol{\xi} - \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))\right| \mathrm{d}y \\
&\quad = \int_Y (\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y) - \boldsymbol{\xi} - \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)) \, \mathrm{d}y
\end{aligned}$$

Since $\mathbf{w}_{\boldsymbol{\xi}}$ and $\mathbf{w}_{\boldsymbol{\xi}^j}$ are weak solutions, they may be used as test functions in (6.4) and (6.25) respectively. Consequently the above expression can be rewritten as follows

$$\begin{aligned}
&\int_Y (\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y) - \boldsymbol{\xi} - \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)) \, \mathrm{d}y \\
&\quad = \int_Y (\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j - \boldsymbol{\xi}) \, \mathrm{d}y.
\end{aligned} \tag{6.28}$$

Since we showed that $\{\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j})\}_{j=1}^{\infty}$ is uniformly bounded in $L_{M^*}(Y; \mathbb{R}^{d\times N})$, and the same is true for the term $\mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))$, we conclude that the terms $\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y))$ and $\mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))$ are uniformly bounded in $L^1(Y; \mathbb{R}^{d\times N})$ and thus the last term in (6.28) can be easily estimated

$$\int_Y (\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j - \boldsymbol{\xi}) \, \mathrm{d}y \le C|\boldsymbol{\xi}^j - \boldsymbol{\xi}|.$$

Thus letting $j \to \infty$ allows us to conclude that the left-hand side converges to zero. Recalling the identity (6.28) we infer that the sequence

$$\{(\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y)) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi}^j + \nabla \mathbf{w}_{\boldsymbol{\xi}^j}(y) - \boldsymbol{\xi} - \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))\}_{j\in\mathbb{N}} \tag{6.29}$$

converges in $L^1(Y)$ to zero, and therefore also is weakly precompact in $L^1(Y)$. Thus the limit can be characterized with the help of Young measures (see Theorem 8.41) as follows

$$\int_{\mathbb{R}^d} (\mathbf{A}(y, \boldsymbol{\xi} + \boldsymbol{\lambda}) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y))) \cdot (\boldsymbol{\xi} + \boldsymbol{\lambda} - \boldsymbol{\xi} - \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)) \, \mathrm{d}\nu_y(\boldsymbol{\lambda}). \tag{6.30}$$

In view of the strict monotonicity of $\mathbf{A}$, see (A4), we find that the Young measure ν_y has to concentrated in one point, i.e. $\nu_y = \delta_{\{\nabla \mathbf{w}_\xi(y)\}}$. As the Young measure is a Dirac measure, the sequence generating the Young measure converges almost everywhere in Y. Thus we obtain by letting $j \to \infty$ that

$$\nabla \mathbf{w}_{\xi^j} \to \nabla \mathbf{w} \qquad \text{a.e. in } Y.$$

The continuity of $\mathbf{A}$ with respect to the second variable also implies that also

$$\mathbf{A}(y, \xi + \nabla \mathbf{w}_{\xi^j}) \to \mathbf{A}(y, \nabla \mathbf{w}) \qquad \text{a.e. in } Y$$

and since this limit coincides with a weak-$*$ limit (6.27), it holds that $\mathbf{Z}(y) = \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}(y))$ for a.a. $y \in Y$. The uniqueness of this solution implies that not only a subsequence selected from $\{\mathbf{A}(y, \boldsymbol{\xi}^j + \nabla \mathbf{w}_{\xi^j}(y))\}_{j=1}^{\infty}$ converges weakly-$*$ to $\mathbf{A}(\cdot, \boldsymbol{\xi} + \nabla \mathbf{w})$ in $L_{M^*}(Y; \mathbb{R}^{d\times N})$ but the whole sequence converges to the same limit, which completes the proof of (6.24). □

6.6 The Homogenized Operator and the Limit Problem

This section concentrates on the properties of the homogenized operator. The discussion is split into two cases: either M or M^* satisfy the Δ_2-condition. We first prepare the tools for specifying the growth properties of $\hat{\mathbf{A}}$ in the case when M^* satisfies the Δ_2-condition. For this purpose we define the functional $f : \mathbb{R}^{d\times N} \to [0, \infty)$ as

$$f(\boldsymbol{\xi}) = \inf_{\mathbf{W} \in G} \int_Y M(y, \boldsymbol{\xi} + \mathbf{W}(y))\, \mathrm{d}y \tag{6.31}$$

and discuss its properties.

Lemma 6.6.1 *Let an N-function M be Y-periodic and satisfy the stability condition* (2.37) *with functions m_1, m_2. Then the functional f defined by* (6.31) *is also an N-function and the corresponding condition is satisfied with the same functions m_1, m_2, i.e.*

$$m_1(|\boldsymbol{\xi}|) \le f(\boldsymbol{\xi}) \le m_2(|\boldsymbol{\xi}|) \tag{6.32}$$

for a.a. $\xi \in \mathbb{R}^{d\times N}$.

Proof. We start by showing that the first inequality in (6.32) holds. Obviously, an average over Y of the gradient of an Y-periodic function vanishes. Thus, using the estimate (2.37) for M and Jensen's inequality we have

$$f(\boldsymbol{\xi}) \ge \inf_{\mathbf{W} \in G} \int_Y m_1(|\boldsymbol{\xi} + \mathbf{W}(y)|)\, \mathrm{d}y \ge \inf_{\mathbf{W} \in G} m_1\left(\left|\boldsymbol{\xi} + \int_Y \mathbf{W}(y)\, \mathrm{d}y\right|\right) \ge m_1(|\boldsymbol{\xi}|).$$

On the other hand, since G is a subspace of $E_M^{per}(Y; \mathbb{R}^{d\times N})$, we have $\mathbf{0} \in G$ and thus the upper bound on M implies that $f(\boldsymbol{\xi}) \le m_2(|\boldsymbol{\xi}|)$. Having (6.32) we immediately conclude that $f(0) = 0$. Obviously, since M is even in the second argument and G is a subspace of $E_M^{per}(Y;\mathbb{R}^{d\times N})$ we conclude that $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$.

In order to show the convexity of f we take $\lambda \in (0,1)$, $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d\times N}$ and $\mathbf{W}_1, \mathbf{W}_2 \in G$. Again the fact that G is a subspace of $E_M^{per}(Y;\mathbb{R}^{d\times N})$ and the convexity of M yields

$$f(\lambda\boldsymbol{\xi}_1 + (1-\lambda)\boldsymbol{\xi}_2) \le \lambda \int_Y M(y, \boldsymbol{\xi}_1 + \mathbf{W}_1(y))\,\mathrm{d}y + (1-\lambda)\int_Y M(y, \boldsymbol{\xi}_2 + \mathbf{W}_2(y))\,\mathrm{d}y.$$

By taking an infimum over $\mathbf{W}_1$ and $\mathbf{W}_2$ we immediately arrive at the definition of convexity. □

Next we recall a general functional analytic fact that will be used later for a characterization of a conjugate function to f.

Lemma 6.6.2 *Let X be a Banach space, V be a subspace of X, and g be a closed, convex functional on X that is continuous at some $x \in V$. Then*

$$\inf_{x\in V}\{g(x) - \langle \eta, x\rangle\} + \inf_{\xi\in V^\perp} g^*(\eta+\xi) = 0 \tag{6.33}$$

for all $\eta \in X^*$.

Proof. Directly using the definition of a convex conjugate we have

$$\forall \xi \in X^* (g-\eta)^*(\xi) = \sup_{x\in X}\{\langle \eta+\xi, x\rangle - g(x)\} = g^*(\eta+\xi). \tag{6.34}$$

By Theorem 8.34 for a closed, convex functional A that is continuous at some $x \in V$ it holds that

$$\inf_{x\in V} A(x) + \inf_{x^*\in V^\perp} A^*(x^*) = 0.$$

Let us then choose $A(x) := (g-\eta)(x)$, which indeed is closed, convex and continuous at some $x \in V$, and use the expression for A^* established by (6.34) in the above equality to conclude (6.33). □

Lemma 6.6.3 *Let an N-function M be Y-periodic and f be defined by* (6.31). *Then the conjugate N-function f^* to f is given by*

$$f^*(\boldsymbol{\xi}) = \inf_{\substack{\mathbf{W}^*\in G^\perp,\\ \int_Y \mathbf{W}^*(y)\,\mathrm{d}y = \boldsymbol{\xi}}} \int_Y M^*(y, \mathbf{W}^*(y))\,\mathrm{d}y. \tag{6.35}$$

Proof. We define a functional $\mathcal{F} : L_M(Y;\mathbb{R}^{d\times N}) \to \mathbb{R}$ as

$$\mathcal{F}(\mathbf{w}) = \int_Y M(y, \mathbf{w}(y))\,\mathrm{d}y$$

and rewrite the definition of the conjugate function f^* as follows

$$f^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ \boldsymbol{\xi}\cdot\boldsymbol{\eta} - \inf_{\mathbf{W}\in G} \mathcal{F}(\boldsymbol{\eta}+\mathbf{W}) \right\}. \tag{6.36}$$

Using again that an average over Y of the gradient of a Y-periodic function vanishes we obtain

$$\begin{aligned} f^*(\boldsymbol{\xi}) &= \sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ -\inf_{\mathbf{W}\in G} \left\{ \mathcal{F}(\boldsymbol{\eta}+\mathbf{W}) - \int_Y \boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\,\mathrm{d}y \right\}\right\} \\ &= -\inf_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ \inf_{\mathbf{W}\in G} \left\{ \mathcal{F}(\boldsymbol{\eta}+\mathbf{W}) - \int_Y \boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\,\mathrm{d}y \right\}\right\} \\ &= -\inf_{\mathbf{V}\in\mathbb{R}^{d\times N}\oplus G} \left\{ \mathcal{F}(\mathbf{V}) - \int_Y \boldsymbol{\xi}\cdot\mathbf{V}(y)\,\mathrm{d}y \right\}. \end{aligned} \tag{6.37}$$

In the first step we show that for all $\boldsymbol{\xi}\in\mathbb{R}^{d\times N}$

$$\inf_{\mathbf{V}\in\mathbb{R}^{d\times N}\oplus G} \left\{ \mathcal{F}(\mathbf{V}) - \int_Y \boldsymbol{\xi}\cdot\mathbf{V}(y)\,\mathrm{d}y \right\} + \inf_{\mathbf{W}^*\in(\mathbb{R}^{d\times N}\oplus G)^\perp} \mathcal{F}^*(\boldsymbol{\xi}+\mathbf{W}^*) = 0. \tag{6.38}$$

This statement follows from Lemma 6.6.2 applied to a functional $\mathcal{F}$. To check whether the assumptions are satisfied, we first show that $\mathcal{F}$ is closed, or equivalently, that if $\mathbf{w}_{\xi^j}\to\mathbf{W}$ in $L_M(Y;\mathbb{R}^{d\times N})$ then

$$\liminf_{\varepsilon\to 0}\mathcal{F}(\mathbf{W}^k)\geq\mathcal{F}(\mathbf{W}). \tag{6.39}$$

Obviously $\mathbf{W}^k\to\mathbf{W}$ in $L_M(Y;\mathbb{R}^{d\times N})$ implies $\mathbf{W}^k\to\mathbf{W}$ in $L^1(Y;\mathbb{R}^{d\times N})$. Thus (6.39) follows as integral functionals with a Carathéodory integrand are lower semicontinuous, see [172, Theorem 4.2].

Directly from the definition and due to the convexity of M for $\|\mathbf{v}\|_{L_M}\leq 1$ it follows that

$$\mathcal{F}(\mathbf{v})\leq\|\mathbf{v}\|_{L_M}, \tag{6.40}$$

which immediately implies that $\mathcal{F}$ is continuous at $\mathbf{0}\in G$, which allows us to conclude by Lemma 6.6.2 that (6.38) holds.

Finally, we recall the definition of the conjugate functional $\mathcal{F}^*: L_{M^*}(Y)\to\mathbb{R}$

$$\mathcal{F}^*(\mathbf{v}^*) := \sup_{\mathbf{v}\in L_M(Y)} \left(\int_Y \mathbf{v}\cdot\mathbf{v}^*\,\mathrm{d}x - \mathcal{F}(\mathbf{v}) \right)$$

and it is not difficult to observe by using the Young inequality that

$$\mathcal{F}^*(\mathbf{v}^*) = \int_Y M^*(x,\mathbf{v}^*(x))\,\mathrm{d}x. \tag{6.41}$$

Having this characterization, (6.37) and (6.38) provide that

$$f^*(\boldsymbol{\xi}) = \inf_{\mathbf{W}^*\in(\mathbb{R}^{d\times N}\oplus G)^\perp} \int_Y M^*(y,\mathbf{W}^*(y)+\boldsymbol{\xi})\,\mathrm{d}y \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^{d\times N}.$$

Finally, to conclude (6.35) we need to show that

$$\left(\mathbb{R}^{d\times N}\oplus G\right)^{\perp}=\left\{\mathbf{W}^*\in G^{\perp}:\int_Y \mathbf{W}^*(y)\,\mathrm{d}y=0\right\}=:(G^{\perp})_0.$$

Obviously $(G^{\perp})_0\subset\left(\mathbb{R}^{d\times N}\oplus G\right)^{\perp}$. In order to get the opposite inclusion, we choose $\mathbf{W}^*\in(\mathbb{R}^{d\times N}\oplus G)^{\perp}$. By the definition of the annihilator

$$\int_Y \mathbf{W}^*\cdot(\boldsymbol{\eta}+\mathbf{W})\,\mathrm{d}y=0$$

for any $\boldsymbol{\eta}\in\mathbb{R}^{d\times N}$ and $\mathbf{W}\in G$. Setting $\mathbf{W}=0$ and $\boldsymbol{\eta}=\int_Y\mathbf{W}^*$ we get the condition $\int_Y\mathbf{W}^*=0$, whereas $\mathbf{W}^*\in G^{\perp}$ follows by setting $\boldsymbol{\eta}=0$. □

Finally, we state the key property of f provided that M^* satisfies the Δ_2-condition. Note that this lemma will play an essential role in the homogenization process.

Lemma 6.6.4 *Let an N-function M be Y-periodic, M^* satisfy the Δ_2-condition and f be defined by* (6.31). *Then f can be alternatively expressed as*

$$f(\boldsymbol{\xi})=\inf_{\mathbf{W}\in G^{\perp\perp}(Y)}\int_Y M(y,\boldsymbol{\xi}+\mathbf{W}(y))\,\mathrm{d}y. \tag{6.42}$$

Proof. According to Lemma 6.6.3 the conjugate function f^* is expressed by the formula (6.35). In the first step we compute f^{**}, which is the second conjugate of f. Defining the functional $\mathcal{G}$ as

$$\mathcal{G}(\mathbf{W})=\int_Y M^*(y,\mathbf{W}(y))\,\mathrm{d}y$$

one can show that $\mathcal{G}$ is closed, continuous at $\mathbf{0}\in G^{\perp}$ and the fact that

$$\mathcal{G}^*(\mathbf{W}^*)=\int_Y M(y,\mathbf{W}^*(y))\,\mathrm{d}y$$

along similar lines as the analogous facts for the functional $\mathcal{F}$ in the proof of Lemma 6.6.3. Then we compute

$$\begin{aligned}
f^{**}(\boldsymbol{\xi})&=\sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}}\left\{\boldsymbol{\xi}\cdot\boldsymbol{\eta}-\inf_{\mathbf{W}\in G_0^{\perp}}\mathcal{G}(\boldsymbol{\eta}+\mathbf{W})\right\}\\
&=\sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}}\left\{-\inf_{\mathbf{W}\in G_0^{\perp}}\left\{\mathcal{G}(\boldsymbol{\eta}+\mathbf{W})-\int_Y\boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\,\mathrm{d}y\right\}\right\}\\
&=-\inf_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}}\left\{\inf_{\mathbf{W}\in G_0^{\perp}}\left\{\mathcal{G}(\boldsymbol{\eta}+\mathbf{W})-\int_Y\boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\,\mathrm{d}y\right\}\right\}\\
&=-\inf_{\mathbf{V}\in\mathbb{R}^{d\times N}\oplus G_0^{\perp}}\left\{\mathcal{G}(\mathbf{V})-\int_Y\boldsymbol{\xi}\cdot\mathbf{V}(y)\,\mathrm{d}y\right\}\\
&=-\inf_{\mathbf{V}\in G^{\perp}}\left\{\mathcal{G}(\mathbf{V})-\int_Y\boldsymbol{\xi}\cdot\mathbf{V}(y)\,\mathrm{d}y\right\}=\inf_{\mathbf{U}\in G^{\perp\perp}(Y)}\mathcal{G}^*(\boldsymbol{\xi}+\mathbf{U})
\end{aligned}$$

$$= \inf_{\mathbf{U}\in G^{\perp\perp}(Y)} \int_Y M(y, \boldsymbol{\xi} + \mathbf{U}(y))\, \mathrm{d}y,$$

where the last equality follows by Lemma 6.6.2. We immediately conclude (6.42) since $f = f^{**}$ as f is convex and lower semicontinuous. □

For the case when M satisfies the Δ_2-condition we introduce a functional h^* : $\mathbb{R}^{d\times N} \to [0,\infty)$ as

$$h^*(\boldsymbol{\xi}) = \inf_{\mathbf{W}\in D_0} \int_Y M^*(y, \boldsymbol{\xi} + \mathbf{W}(y))\, \mathrm{d}y, \tag{6.43}$$

where

$$D_0 := \{\mathbf{W} \in D : \int_Y \mathbf{W}(y)\, \mathrm{d}y = 0\}. \tag{6.44}$$

The properties of h^* are summarized in the ensuing lemma. Since the proof is analogous to the proof of the corresponding properties of f in Lemma 6.6.1, it is omitted.

Lemma 6.6.5 *Let M be a Y-periodic N-function satisfying the stability condition* (2.37) *with functions m_1, m_2 and let M^* be its conjugate function. Then the functional h^* defined by* (6.43) *is also an N-function and the corresponding condition is satisfied with the conjugate functions m_1^*, m_2^*, i.e.,*

$$m_2^*(|\boldsymbol{\xi}|) \le h^*(\boldsymbol{\xi}) \le m_1^*(|\boldsymbol{\xi}|) \tag{6.45}$$

for a.a. $\xi \in \mathbb{R}^{d\times N}$.

Next, we present a characterization of h^* and its conjugate function in the case when M satisfies the Δ_2-condition.

Lemma 6.6.6 *Let an N-function M be Y-periodic and satisfy the Δ_2-condition and h^* be defined by* (6.43). *Then*

$$h^{**}(\boldsymbol{\xi}) := (h^*)^*(\boldsymbol{\xi}) = \inf_{\mathbf{V}\in D^\perp} \int_Y M(y, \boldsymbol{\xi} + \mathbf{V}(y))\, \mathrm{d}y \tag{6.46}$$

and in addition h^* can be equivalently expressed as

$$h^*(\boldsymbol{\xi}) = \inf_{\mathbf{W}\in D_0^{\perp\perp}} \int_Y M^*(y, \boldsymbol{\xi} + \mathbf{W}(y))\, \mathrm{d}y, \tag{6.47}$$

where $D_0^{\perp\perp} := \{\mathbf{W} \in D^{\perp\perp} : \int_Y \mathbf{W}(y)\, \mathrm{d}y = 0\}$.

Proof. Statement (6.46) is shown in the same way as (6.35) and thus we skip the proof.

To prove (6.47) we compute $h^{***} := (h^{**})^*$. Since M satisfies the Δ_2-condition, we have

$$D^\perp = \{\mathbf{V} \in E_M^{per,\mathrm{div}}(Y; \mathbb{R}^{d\times N}) : \int_Y \mathbf{V}(y)\cdot \mathbf{W}(y)\, \mathrm{d}y = 0 \text{ for all } \mathbf{W} \in D\}$$

and $(\mathbb{R}^{d\times N} \oplus D^{\perp})^{\perp} = D_0^{\perp\perp}$. Accordingly, we obtain by Lemma 6.6.2

$$h^{***}(\boldsymbol{\xi}) = -\inf_{\mathbf{V}\in\mathbb{R}^{d\times N}\oplus D^{\perp}} \int_Y M(y,\mathbf{V}(y)) - \boldsymbol{\xi}\cdot\mathbf{V}(y)\,\mathrm{d}y = \inf_{\mathbf{W}\in D_0^{\perp\perp}} \int_Y M^*(y,\boldsymbol{\xi}+\mathbf{W}(y))\,\mathrm{d}y.$$

As h^* is convex and continuous, the latter identity implies (6.47). □

The N-functions f and f^*, h^* and h^{**} respectively, were introduced in order to indicate the growth and coercivity properties of the operator $\hat{\mathbf{A}}$ as it is stated, among other properties of $\hat{\mathbf{A}}$, in the following lemma.

Lemma 6.6.7 *Let the operator* $\mathbf{A}$ *satisfy (A1)–(A4) and the N-function M be Y-periodic. Then:*

$(\hat{A}1)$ $\hat{\mathbf{A}}$ *is continuous on* $\mathbb{R}^{d\times N}$.
$(\hat{A}2)$ *There is a constant* $c > 0$ *such that for all* $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$

$$\hat{\mathbf{A}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geq c(f(\boldsymbol{\xi}) + f^*(\hat{\mathbf{A}}(\boldsymbol{\xi}))) \text{ provided that } M^* \text{ satisfies } \Delta_2\text{–condition},$$
$$\hat{\mathbf{A}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geq c(h^{**}(\boldsymbol{\xi}) + h^*(\hat{\mathbf{A}}(\boldsymbol{\xi}))) \text{ provided that } M \text{ satisfies } \Delta_2\text{–condition} \tag{6.48}$$

$(\hat{A}3)$ *For all* $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times N}$, $\boldsymbol{\xi} \neq \boldsymbol{\eta}$,

$$(\hat{\mathbf{A}}(\boldsymbol{\xi}) - \hat{\mathbf{A}}(\boldsymbol{\eta}))\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) > 0.$$

Proof. To show $(\hat{A}1)$ we consider $\{\boldsymbol{\xi}^j\}_{j=1}^{\infty}$ such that $\boldsymbol{\xi}^j \to \boldsymbol{\xi}$ in $\mathbb{R}^{d\times N}$ as $j\to\infty$. Moreover, let $\{\mathbf{w}_{\boldsymbol{\xi}^j}\}_{j=1}^{\infty}$ be a sequence of weak solutions of the cell problem (6.4) corresponding to $\boldsymbol{\xi}^j$ and $\mathbf{w}$ be a weak solution of the cell problem (6.4) corresponding to $\boldsymbol{\xi}$. Existence and uniqueness of these solutions is provided by Lemma 6.5.2 Then it holds for an arbitrary but fixed $\boldsymbol{\eta} \in \mathbb{R}^{d\times N}$ that

$$(\hat{\mathbf{A}}(\boldsymbol{\xi}^j) - \hat{\mathbf{A}}(\boldsymbol{\xi}))\cdot\boldsymbol{\eta} = \int_Y (\mathbf{A}(y,\boldsymbol{\xi}^j + \nabla\mathbf{w}_{\boldsymbol{\xi}^j}) - \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}))\cdot\boldsymbol{\eta}\,\mathrm{d}y \to 0$$

as $j\to\infty$ by (6.24). Since $\mathbb{R}^{d\times N}$ is finite-dimensional, we conclude $(\hat{A}1)$ from the latter convergence.

Let $\mathbf{w}_{\boldsymbol{\xi}}$ still be a solution of the cell problem corresponding to $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$. If M^* satisfies the Δ_2-condition, we know that $\mathbf{w}_{\boldsymbol{\xi}} \in W^1_{per}L_M(Y;\mathbb{R}^d)$ and if M satisfies the Δ_2-condition then $\mathbf{w}_{\boldsymbol{\xi}} \in V^M_{per}$. In both cases we know that the following identity is satisfied

$$\int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\boldsymbol{\xi}})\cdot\nabla\mathbf{w}_{\boldsymbol{\xi}}\,\mathrm{d}y = 0. \tag{6.49}$$

Then, using (6.49) and the growth conditions of $\mathbf{A}$ it directly follows that

$$
\begin{aligned}
\hat{\mathbf{A}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} &= \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y\cdot\boldsymbol{\xi} \\
&= \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y\cdot(\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y \qquad (6.50)\\
&\geq c\int_Y M(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y)) + M^*(y,\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y.
\end{aligned}
$$

The above estimate will lead us to the proof of growth and coercivity conditions of $\mathbf{A}$.

First, we deal with the case when M^* satisfies the Δ_2-condition. The estimate (6.50) cannot be immediately applied, since $\mathbf{w}_{\xi}$ is not necessarily an element of $W^1_{per}E_M(Y;\mathbb{R}^d)$, and thus $\nabla\mathbf{w}_{\xi}$ may not be an element of G, which excludes the possibility of using the direct formula for f, i.e. (6.31). Our strategy is to show that $\nabla\mathbf{w}_{\xi}\in G^{\perp\perp}$ and use Lemma 6.6.4. Choosing an arbitrary $\mathbf{V}\in G^{\perp}\subset E_{M^*}(Y;\mathbb{R}^{d\times N})$ and taking into account that $\mathbf{w}_{\xi}\in W^1_{per}L_M(Y;\mathbb{R}^d)$ is a weak-$*$ limit of a sequence $\{\mathbf{w}_{\xi^j}\}_{j=1}^{\infty}\subset W^1_{per}E_M(Y;\mathbb{R}^d)$ we obtain

$$
\int_Y \nabla\mathbf{w}_{\xi}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = \lim_{\varepsilon\to 0}\int_Y \nabla\mathbf{w}_{\xi^j}(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0.
$$

Thus $\nabla\mathbf{w}_{\xi}\in G^{\perp\perp}$, and we can use Lemma 6.6.4, which provides a characterization of f to infer

$$
\int_Y M(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y \geq f(\boldsymbol{\xi}). \qquad (6.51)
$$

As $\mathbf{w}_{\xi}$ is a solution of the cell problem and $G\subset\{\nabla\mathbf{v}:\mathbf{v}\in W^1_{per}L_M(Y;\mathbb{R}^d)\}$, the weak formulation (6.22) immediately implies that $\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w})\in G^{\perp}$. Thus we deduce the next estimate

$$
\int_Y M^*(y,\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y))\,\mathrm{d}y \geq f^*(\hat{\mathbf{A}}(\boldsymbol{\xi})). \qquad (6.52)
$$

Finally, estimate $(6.48)_1$ is obtained as a consequence of (6.50), (6.51) and (6.52).

In a similar fashion we will show the estimates for the case when M satisfies the Δ_2-condition. Since D is defined as a closure of smooth periodic divergence-free functions, it directly follows that

$$
\nabla\mathbf{w}_{\xi}\in D^{\perp},
$$

which, by (6.46), implies

$$
\int_Y M(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi}(y))\,\mathrm{d}y \geq h^{**}(\boldsymbol{\xi}). \qquad (6.53)
$$

Next, since $\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi})\in L_{M^*}(Y;\mathbb{R}^{d\times N})$ and by the assumption that M satisfies the Δ_2-condition the observation (6.9) is available, we conclude using the weak formulation (6.23) that

$$
\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}_{\xi})\in D^{\perp\perp}.
$$

Consequently, Lemma 6.6.6 yields

$$\int_Y M^*(y, \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}_{\boldsymbol{\xi}}(y)))\, \mathrm{d}y \geq h^*(\hat{\mathbf{A}}(\boldsymbol{\xi})). \tag{6.54}$$

Estimate $(6.48)_2$ then follows from (6.50), (6.54) and (6.53).

In order to show ($\hat{\mathbf{A}}$3) we fix $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d\times N}, \boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$. Let then $\mathbf{w}_{\xi_1}$ and $\mathbf{w}_{\xi_2}$ be corresponding weak solutions of the cell problem (6.4), which gives

$$\int_Y \mathbf{A}(y, \boldsymbol{\xi}_i + \nabla \mathbf{w}_{\xi_i}(y)) \cdot \nabla \mathbf{w}_{\xi_j}(y)\, \mathrm{d}y = 0 \text{ for } i, j = 1, 2 \tag{6.55}$$

in the same way as for (6.26). Then using (6.55) and the strict monotonicity of $\mathbf{A}$, we deduce

$$\begin{aligned}(\hat{\mathbf{A}}(\boldsymbol{\xi}_1) - \hat{\mathbf{A}}(\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) &= \int_Y (\mathbf{A}(y, \boldsymbol{\xi}_1 + \nabla \mathbf{w}_1) - \mathbf{A}(y, \boldsymbol{\xi}_2 + \nabla \mathbf{w}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\, \mathrm{d}y \\ &= \int_Y (\mathbf{A}(y, \boldsymbol{\xi}_1 + \nabla \mathbf{w}_1) - \mathbf{A}(y, \boldsymbol{\xi}_2 + \nabla \mathbf{w}_2)) \cdot (\boldsymbol{\xi}_1 + \nabla \mathbf{w}_1 - \boldsymbol{\xi}_2 - \nabla \mathbf{w}_2)\, \mathrm{d}y > 0\end{aligned}$$

and this completes the proof of ($\hat{A}$3). □

Now that we have the growth conditions of $\hat{\mathbf{A}}$ we are ready to complete the argument by formulating a definition of a solution to the homogenized problem (6.2)

Definition 6.6.8. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, the operator $\hat{\mathbf{A}}$ satisfy ($\hat{A}$1)–($\hat{A}$3), an N-function f be given by (6.31) and an N-function h^* be given by (6.43). Moreover, let $\mathbf{F} \in L^\infty(\Omega; \mathbb{R}^{d\times N})$.

(i) If the conjugate N-function M^* satisfies the Δ_2-condition, then we call $\mathbf{u}$ a solution to problem (6.2) if

$$\mathbf{u} \in W_0^1 L_f \left(\Omega; \mathbb{R}^d\right)$$

and

$$\int_\Omega \hat{\mathbf{A}}\left(\nabla \mathbf{u}(x)\right) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x = \int_\Omega \mathbf{F}(x) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x \tag{6.56}$$

is satisfied for all $\boldsymbol{\varphi} \in W_0^1 L_f\left(\Omega; \mathbb{R}^d\right)$.

(ii) If the N-function M satisfies the Δ_2-condition, then we call $\mathbf{u}$ a solution to problem (6.2) if

$$\mathbf{u} \in V_0^{h^{**}}$$

and

$$\int_\Omega \hat{\mathbf{A}}\left(\nabla \mathbf{u}(x)\right) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x = \int_\Omega \mathbf{F}(x) \cdot \nabla \boldsymbol{\varphi}(x)\, \mathrm{d}x \tag{6.57}$$

is satisfied for all $\boldsymbol{\varphi} \in V_0^{h^{**}}$.

Note that solutions to (6.2) will be constructed with the help of Theorem 6.2.2. Since $\bar{\mathbf{A}}$ is strictly monotone, this solution will be unique.

6.7 Existence of Solutions for a Fixed ε

In this short section we formulate a theorem concerning the existence and uniqueness of a solution to problem (6.1) for an arbitrary but fixed $\varepsilon > 0$. This fact follows from the existence theory presented in Chapter 4. Recall that a notion of solution to (6.1) is given by Definition 6.2.1.

Theorem 6.7.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, the operator $\mathbf{A}$ satisfy (A1)–(A4), and let at least one of the following conditions hold:*

1. *M satisfies the Δ_2-condition,*
2. *M^*, the convex conjugate N-function to M, satisfies the Δ_2-condition.*

Then for a fixed $\varepsilon \in (0,1)$ there exists a unique weak solution to (6.1).

Next we state an estimate that is uniform with respect to ε. For brevity we will use the notation

$$\mathbf{A}^\varepsilon(x) := \mathbf{A}(\tfrac{x}{\varepsilon}, \nabla \mathbf{u}^\varepsilon(x)).$$

Lemma 6.7.2 *Let the assumptions of Theorem 6.7.1 be satisfied and $\mathbf{u}^\varepsilon$ be a weak solution to* (6.1). *Then we have*

$$\sup_{0<\varepsilon<1} \int_\Omega M^\varepsilon\left(x, \nabla \mathbf{u}^\varepsilon(x)\right) + (M^\varepsilon)^*\left(x, \mathbf{A}^\varepsilon(x)\right)\, \mathrm{d}x \le c < \infty, \tag{6.58}$$

where $\{\mathbf{A}^\varepsilon\}_{\varepsilon>0}$ is bounded in $L_{m_2^*}(\Omega; \mathbb{R}^{d\times N})$ and $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ is bounded in $V_0^{m_1}$.

Proof. Choosing $\boldsymbol{\varphi} = \mathbf{u}^\varepsilon$ as a test function in (6.5) and (6.6) depending on whether M^* or M satisfy the Δ_2-condition, we obtain the following integral identity

$$\int_\Omega \mathbf{A}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon \, \mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}^\varepsilon \, \mathrm{d}x. \tag{6.59}$$

Observe that, by the Fenchel–Young inequality, the constant c appearing in the growth and coercivity condition $(A1)$ is less than or equal to one. Using again the Fenchel–Young inequality and Lemma 2.1.23 part (i) gives

$$c\int_\Omega M^\varepsilon(x, \nabla \mathbf{u}^\varepsilon) + (M^\varepsilon)^*(x, \mathbf{A}^\varepsilon)\, \mathrm{d}x \le \int_\Omega (M^\varepsilon)^*\left(x, \tfrac{2}{c}\mathbf{F}\right) + \tfrac{c}{2} M^\varepsilon(x, \nabla \mathbf{u}^\varepsilon)\, \mathrm{d}x.$$

Consequently, using the uniform estimates of N-functions we obtain

$$\begin{aligned} c\int_\Omega \tfrac{1}{2} m_1(|\nabla \mathbf{u}^\varepsilon|) + m_2^*(|\mathbf{A}^\varepsilon|)\, \mathrm{d}x &\le c\int_\Omega \tfrac{1}{2} M^\varepsilon\left(x, \nabla \mathbf{u}^\varepsilon\right) + (M^\varepsilon)^*\left(x, \mathbf{A}^\varepsilon\right)\, \mathrm{d}x \\ &\le \int_\Omega m_1^*\left(\tfrac{2}{c}|\mathbf{F}|\right)\, \mathrm{d}x. \end{aligned} \tag{6.60}$$

Due to (6.7) the integral on the right-hand side is finite and thus (6.58) follows. By the Poincaré inequality (Theorem 9.3), which holds in homogeneous isotropic spaces, the bounds for $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ and $\{\mathbf{A}^\varepsilon\}_{\varepsilon>0}$ also follow from (6.60). □

6.8 Limit Passage to the Homogenized Problem

The current section uses all the tools established so far in this chapter and presents the key step of the limit passage from problem (6.1) to (6.2). We present the proof of Theorem 6.2.2. The proof is divided into two parts. The first part prepares the necessary technical facts, while the second one concentrates on the limit of the nonlinearity. Note that within these two parts we repeatedly need to consider the cases of M or M^* satisfying the Δ_2-condition separately.

Proof (of Theorem 6.2.2). Before presenting the rigorous proof, let us provide a short itinerary of the whole strategy. First we derive uniform bounds for $\mathbf{u}^\varepsilon$ and $\mathbf{A}(\frac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon)$. These estimates allow us to conclude that $\{\nabla\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ to some $\nabla\mathbf{u}$ in $L_{m_1}(\Omega;\mathbb{R}^{d\times N})$ and $\{\mathbf{A}^\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ to a limit $\bar{\mathbf{A}}\in L_{m_2^*}(\Omega;\mathbb{R}^{d\times N})$. Then we show that the sequence $\{\nabla\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ two-scale to $\nabla\mathbf{u}+\mathbf{U}$ in $L_{m_1}(\Omega\times Y;\mathbb{R}^{d\times N})$ and $\{\mathbf{A}^\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ two-scale to $\mathbf{A}^0$ in $L_{m_2^*}(\Omega\times Y;\mathbb{R}^{d\times N})$. Consequently, we apply the weak-$*$ two-scale semicontinuity of convex functionals to improve the regularity of limit functions, i.e., we obtain $\nabla\mathbf{u}\in L_f(\Omega;\mathbb{R}^{d\times N})$ and $\bar{\mathbf{A}}=\int_Y\mathbf{A}^0\in L_{f^*}(\Omega;\mathbb{R}^{d\times N})$. This ensures that $\int_\Omega\bar{\mathbf{A}}\cdot\nabla\mathbf{u}\,\mathrm{d}x$ is meaningful. We conclude with a variant of the Minty trick for nonreflexive function spaces to identify the limit $\bar{\mathbf{A}}$.

Part 1 (Technical facts). For any $\varepsilon>0$ let $\mathbf{u}^\varepsilon$ be a unique weak solution to the problem (6.1) according to Definition 6.2.1. In the sequel, when letting $\varepsilon\to 0$ we may pass to a subsequence if needed, not necessarily stressing this fact. The uniform estimates, which were proved in Lemma 6.7.2, imply the following convergences

$$\begin{aligned}\mathbf{u}^\varepsilon &\overset{*}{\rightharpoonup}\mathbf{u} &&\text{in } L_{m_1}(\Omega;\mathbb{R}^{d}),\\ \nabla\mathbf{u}^\varepsilon &\overset{*}{\rightharpoonup}\nabla\mathbf{u} &&\text{in } L_{m_1}(\Omega;\mathbb{R}^{d\times N}),\\ \mathbf{A}^\varepsilon &\overset{*}{\rightharpoonup}\bar{\mathbf{A}} &&\text{in } L_{m_2^*}(\Omega;\mathbb{R}^{d\times N}).\end{aligned}\tag{6.61}$$

As a consequence of $(6.61)_{1,2}$ and Lemma 6.4.4 (iv) we obtain the existence of a function $\mathbf{U}\in L_{m_1}(\Omega\times Y;\mathbb{R}^{d\times N})$ such that

$$\begin{aligned}\mathbf{u}^\varepsilon &\xrightarrow{2-s}{}^{*}\,\mathbf{u} &&\text{in } L_{m_1}(\Omega\times Y;\mathbb{R}^{d}),\\ \nabla\mathbf{u}^\varepsilon &\xrightarrow{2-s}{}^{*}\,\nabla\mathbf{u}+\mathbf{U} &&\text{in } L_{m_1}(\Omega\times Y;\mathbb{R}^{d\times N}),\end{aligned}\tag{6.62}$$

where $\mathbf{U}$ satisfies

$$\int_Y\mathbf{U}(y,x)\cdot\psi(y)\,\mathrm{d}y=0\qquad\forall\psi\in C^\infty_{per}(Y;\mathbb{R}^{d\times N}).\tag{6.63}$$

Lemma 6.4.4 (vi) and Lemma 6.7.2 imply the existence of a function $\mathbf{A}^0\in L_{m_2^*}(\Omega\times Y;\mathbb{R}^{d\times N})$ such that

$$\mathbf{A}^\varepsilon\xrightarrow{2-s}{}^{*}\,\mathbf{A}^0\text{ in }L_{m_2^*}(\Omega\times Y;\mathbb{R}^{d\times N}).\tag{6.64}$$

Using convergence results (6.62) and (6.64), from the weak lower semicontinuity proved in part (v) of Lemma 6.4.4 we infer for $\Phi = M$ and $\Phi = M^*$ respectively that

$$\begin{aligned}\int_\Omega\int_Y M(y,\nabla\mathbf{u}+\mathbf{U}) + M^*(y,\mathbf{A}^0)\,\mathrm{d}y\,\mathrm{d}x \\ \le \liminf_{\varepsilon\to 0}\int_\Omega\int_Y M\left(\tfrac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon\right) + M^*\left(\tfrac{x}{\varepsilon},\mathbf{A}^\varepsilon\right)\,\mathrm{d}y\,\mathrm{d}x < \infty.\end{aligned}\tag{6.65}$$

Next, by Lemma 6.4.3 and Lemma 6.7.2 we get

$$\sup_{\varepsilon>0}\int_\Omega\int_Y M(y,\nabla\mathbf{u}^\varepsilon(S_\varepsilon(y,x)))\,\mathrm{d}y\,\mathrm{d}x = \sup_{\varepsilon>0}\int_\Omega M\left(\tfrac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon(x)\right)\,\mathrm{d}x < \infty,$$

$$\begin{aligned}\sup_{\varepsilon>0}\int_\Omega\int_Y M^*(y,\mathbf{A}(y,\nabla\mathbf{u}^\varepsilon(S_\varepsilon(y,x)))\,\mathrm{d}y\,\mathrm{d}x = \sup_{\varepsilon>0}\int_\Omega M^*\left(\tfrac{x}{\varepsilon},\mathbf{A}\left(\tfrac{x}{\varepsilon},\nabla\mathbf{u}^\varepsilon(x)\right)\right)\,\mathrm{d}x \\ < \infty.\end{aligned}$$

Accordingly, there exist functions $\mathbf{V}\in L_M(\Omega\times Y;\mathbb{R}^{d\times N})$ and $\tilde{\mathbf{A}}\in L_{M^*}(\Omega\times Y;\mathbb{R}^{d\times N})$ such that as $\varepsilon\longrightarrow 0$ we conclude

$$\begin{aligned}\nabla\mathbf{u}^\varepsilon\circ S_\varepsilon &\overset{*}{\rightharpoonup}\mathbf{V} \text{ in } L_M(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}^\varepsilon\circ S_\varepsilon &\overset{*}{\rightharpoonup}\tilde{\mathbf{A}} \text{ in } L_{M^*}(\Omega\times Y;\mathbb{R}^{d\times N}).\end{aligned}\tag{6.66}$$

Hence in view of (6.65) we infer using (6.62) and (6.64) that $\mathbf{V} = \nabla\mathbf{u}+\mathbf{U}$, $\tilde{\mathbf{A}} = \mathbf{A}^0$, i.e., we have concluded

$$\begin{aligned}(\nabla\mathbf{u}^\varepsilon)\circ S_\varepsilon &\overset{*}{\rightharpoonup}\nabla\mathbf{u}+\mathbf{U} \text{ in } L_M(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}^\varepsilon\circ S_\varepsilon &\overset{*}{\rightharpoonup}\mathbf{A}^0 \qquad \text{ in } L_{M^*}(\Omega\times Y;\mathbb{R}^{d\times N}).\end{aligned}\tag{6.67}$$

By Lemma 6.4.4 (ii) we get that the limit functions $\bar{\mathbf{A}}$ and $\mathbf{A}^0$ are related via

$$\bar{\mathbf{A}} = \int_Y \mathbf{A}^0\,\mathrm{d}y.\tag{6.68}$$

Until now it has not been necessary to distinguish between the cases of M^* or M satisfying the Δ_2-condition. However we now need to start treating them separately.

Case 1: Assume that M^* satisfies the Δ_2-condition. Within each case we again need to distinguish between the scalar case $d = 1$ and the accomplishment of the embedding $W_0^1L_{m_1}\hookrightarrow L_{m_2}$. Let us first deal with the case $d = 1$. To emphasize that the scalar case is being considered we shall use the simple notation u for the solution.

Firstly we concentrate on the vector field $\mathbf{U}$, in particular on showing that it is an element of $G^{\perp\perp}$. We recall that the truncation operator T_ℓ was introduced in (3.55). Lemma 3.7.9 collects some of its properties. Note that $\|T_\ell u^\varepsilon\|_{L^\infty}\le h$ is uniformly (in ε) bounded and thus, up to a subsequence, it converges weakly-$*$ in $L^\infty(\Omega\times Y)$. The limit function is identified using the Lebesgue dominated convergence theorem from $(6.61)_{1,2}$ together with the compact embedding of $W^{1,1}(\Omega)$ to $L^1(\Omega)$. Along the same lines as the proof of convergence of (6.67), we argue that for any $\ell > 0$

$$\begin{aligned} T_\ell u^\varepsilon \circ S_\varepsilon &\overset{*}{\rightharpoonup} T_\ell u && \text{in } L^\infty(\Omega\times Y), \\ \nabla T_\ell u^\varepsilon \circ S_\varepsilon &\overset{*}{\rightharpoonup} \nabla T_\ell u + \mathbf{U}^\ell && \text{in } L_M(\Omega\times Y;\mathbb{R}^N) \end{aligned} \tag{6.69}$$

holds as $\varepsilon \to 0$. The usefulness of these convergences will become apparent below.

Now, choose an arbitrary, but fixed $\varphi \in C_c^\infty(\Omega)$ and $\mathbf{V} \in G^\perp$, which without loss of generality may be assumed to satisfy $\int_Y \mathbf{V} = 0$. Then, as $\operatorname{div}\mathbf{V} = 0$ a.e. in Y we obtain for an arbitrary but fixed $\ell > 0$ using Lemma 6.4.3 twice and integrating by parts

$$\begin{aligned} \int_\Omega\int_Y \nabla T_\ell u^\varepsilon(S_\varepsilon(y,x))\cdot\mathbf{V}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x &= \int_\Omega \nabla T_\ell u^\varepsilon(x)\cdot\mathbf{V}(\tfrac{x}{\varepsilon})\varphi(x)\,\mathrm{d}x \\ &= -\int_\Omega T_\ell u^\varepsilon(x)\mathbf{V}(\tfrac{x}{\varepsilon})\cdot\nabla\varphi(x)\,\mathrm{d}x \\ &= -\int_\Omega\int_Y T_\ell u^\varepsilon(S_\varepsilon(y,x))\cdot\mathbf{V}(y)\cdot\nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Performing the passage to the limit as $\varepsilon \to 0$ in the latter identity with the help of (6.69) yields for an arbitrary but fixed $\ell > 0$

$$\begin{aligned} &\int_\Omega\int_Y (\nabla T_\ell u(x) + \mathbf{U}^\ell(y,x))\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad = -\int_\Omega\int_Y T_\ell u(x)\mathbf{V}(y)\nabla\varphi(x)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{6.70}$$

Observe that the right-hand side can be equivalently written

$$-\int_\Omega\int_Y T_\ell u(x)\mathbf{V}(y)\nabla\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = -\left(\int_Y \mathbf{V}(y)\,\mathrm{d}y\right)\cdot\int_\Omega T_\ell u(x)\nabla\varphi(x)\,\mathrm{d}x = 0,$$

whereas the conclusion that it vanishes is a consequence of $\int_Y \mathbf{V}\,\mathrm{d}y = 0$. On the other hand, the left-hand side satisfies

$$\int_\Omega\int_Y \mathbf{U}^\ell(y,x)\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_Y (\nabla T_\ell u(x) + \mathbf{U}^\ell(y,x))\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x,$$

which again is an obvious consequence of the fact that $\int_Y \mathbf{V}\,\mathrm{d}y = 0$ and the remaining terms do not depend on y. Consequently

$$\int_\Omega\int_Y \mathbf{U}^\ell(y,x)\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = 0,$$

which means that $\mathbf{U}^\ell \in G^{\perp\perp}$ for any $\ell > 0$. Define $\mathbf{W}^\ell = \nabla T_\ell u + \mathbf{U}^\ell$ and $\mathbf{W} = \nabla u + \mathbf{U}$. Using Lemma 6.4.4 (v) we infer

$$\begin{aligned} \int_\Omega\int_Y M(y,\mathbf{W}^\ell(y,x))\,\mathrm{d}y\,\mathrm{d}x &\le \liminf_{\varepsilon\to 0}\int_\Omega M\left(\tfrac{x}{\varepsilon},\nabla T_\ell u^\varepsilon(x)\right)\,\mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0}\int_\Omega M\left(\tfrac{x}{\varepsilon},\nabla u^\varepsilon(x)\right)\,\mathrm{d}x < \infty. \end{aligned}$$

The above uniform estimate implies that there exist a subsequence, labeled the same, and $\tilde{\mathbf{W}} \in L_M(\Omega\times Y;\mathbb{R}^N)$ such that $\mathbf{W}^\ell \overset{*}{\rightharpoonup} \tilde{\mathbf{W}}$ in $L_M(\Omega\times Y;\mathbb{R}^N)$. On the other hand we obtain due to the weak lower semicontinuity of the L^1-norm that

$$\begin{aligned}\int_\Omega\int_Y |\mathbf{W}(y,x)-\mathbf{W}^\ell(y,x)|\,\mathrm{d}y\,\mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0}\int_\Omega\int_Y |\nabla u^\varepsilon(S_\varepsilon(y,x)) - \nabla T_\ell u^\varepsilon(S_\varepsilon(y,x))|\,\mathrm{d}y\,\mathrm{d}x \\ &= \liminf_{\varepsilon\to 0}\int_{\{|u^\varepsilon(S_\varepsilon(y,x))|>\ell\}} |\nabla u^\varepsilon(S_\varepsilon(y,x))|\,\mathrm{d}y\,\mathrm{d}x \\ &\le c\mu(|\{|u^\varepsilon(S_\varepsilon(y,x))|>\ell\}|),\end{aligned}$$

where μ is continuous at 0 and $\mu(0)=0$. Thus we conclude from the uniform bound on $\{u^\varepsilon\circ S_\varepsilon\}_{\varepsilon>0}$, which follows from Lemma 6.7.2, that $\mathbf{W}^\ell\to\mathbf{W}$ in $L^1(\Omega\times Y;\mathbb{R}^N)$. Consequently, as the L^1- and L_M-limit coincide, we have $\tilde{\mathbf{W}}=\mathbf{W}=\nabla u+\mathbf{U}$ a.e. in $\Omega\times Y$, which along with $\nabla T_h u \xrightarrow{M} \nabla u$ in $L_M(\Omega)$ implies $\mathbf{U}^\ell \xrightarrow{M} \mathbf{U}$ in $L_M(\Omega\times Y;\mathbb{R}^N)$. With the help of Corollary 3.4.7, we obtain

$$\int_\Omega\int_Y \mathbf{U}(y,x)\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = \lim_{\ell\to\infty}\int_\Omega\int_Y \mathbf{U}^\ell(y,x)\cdot\mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = 0,$$

from which it follows that

$$\mathbf{U}(\cdot,x)\in G^{\perp\perp}. \tag{6.71}$$

For an arbitrary $\nabla\mathbf{v}=\mathbf{V}\in G$ and $\varphi\in C_c^\infty(\Omega)$ we observe that Lemma 3.7.9 implies

$$\int_\Omega\int_Y \mathbf{A}^0(y,x)\cdot\nabla\mathbf{v}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = \lim_{\ell\to\infty}\int_\Omega\int_Y \mathbf{A}^0(y,x)\cdot\nabla T_\ell\mathbf{v}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x. \tag{6.72}$$

From the convergence $(6.66)_2$ and Lemma 6.4.3 we conclude further that

$$\begin{aligned}\int_\Omega\int_Y \mathbf{A}^0(y,x)\cdot\nabla T_\ell\mathbf{v}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_\Omega\int_Y \mathbf{A}^\varepsilon(y,\nabla\mathbf{u}(S_\varepsilon(y,x)))\cdot\nabla T_\ell\mathbf{v}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}^\varepsilon(x)\cdot\nabla T_\ell\mathbf{v}(\tfrac{x}{y})\varphi(x)\,\mathrm{d}y\,\mathrm{d}x =: I_1^{\varepsilon,\ell}.\end{aligned} \tag{6.73}$$

Before transforming the term $I_1^{\varepsilon,\ell}$ notice that obviously

$$\nabla_x\left[T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right] = T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\otimes\nabla\varphi(x)+\tfrac{1}{\varepsilon}\nabla T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x) \tag{6.74}$$

holds. Using (6.74), the weak formulation (6.5) and Lemma 6.4.3 gives

$$
\begin{aligned}
&\lim_{\ell\to\infty} I_1^{\varepsilon,\ell}\\
&= \lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon\int_\Omega \mathbf{A}^\varepsilon(x)\cdot\nabla_x\left(T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right)\,\mathrm{d}x - \varepsilon\int_\Omega \mathbf{A}^\varepsilon(x)\cdot T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\otimes\nabla\varphi(x)\,\mathrm{d}x\\
&= \lim_{\ell\to\infty}\lim_{\varepsilon\to 0} \varepsilon\int_\Omega \mathbf{F}(x)\cdot\nabla_x\left(T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right)\,\mathrm{d}x - \varepsilon\int_\Omega \mathbf{A}^\varepsilon(x)\cdot T_\ell\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\otimes\nabla\varphi(x)\,\mathrm{d}x\\
&= \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_\Omega\int_Y \mathbf{F}(S_\varepsilon(y,x))\cdot\nabla_y\left(T_\ell\mathbf{v}(y)\varphi(S_\varepsilon(y,x))\right)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad -\varepsilon\int_\Omega\int_Y \mathbf{A}^\varepsilon(S_\varepsilon(y,x))\cdot T_\ell\mathbf{v}(y)\otimes\nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\
&= \lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\int_\Omega\int_Y \mathbf{F}(S_\varepsilon(y,x))\cdot\nabla_y T_\ell\mathbf{v}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad +\varepsilon\int_\Omega\int_Y \mathbf{F}(S_\varepsilon(y,x))\cdot T_\ell\mathbf{v}(y)\otimes\nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad -\varepsilon\int_\Omega\int_Y \mathbf{A}^\varepsilon(S_\varepsilon(y,x))\cdot T_\ell\mathbf{v}(y)\otimes\nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\
&= \lim_{\ell\to\infty}\int_\Omega \mathbf{F}(x)\varphi(x)\,\mathrm{d}x\cdot\int_Y\nabla_y T_\ell\mathbf{v}(y)\,\mathrm{d}y = 0,
\end{aligned}
\tag{6.75}
$$

where we also used the fact that $T_\ell\mathbf{v}$ is Y-periodic. Summarizing the steps (6.72)–(6.75) we conclude that

$$\int_\Omega\int_Y \mathbf{A}^0(y,x)\cdot\nabla\mathbf{v}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = 0$$

and thus we have shown that

$$\mathbf{A}^0(\cdot,x)\in G^\perp. \tag{6.76}$$

Now, we consider the case $d>1$ and in addition assume that $W_0^1L_{m_1}\hookrightarrow L_{m_2}$. We will show that under such assumptions it is also possible to show that (6.71) and (6.76) hold.

With this aim we choose an arbitrary but fixed $\varphi\in C_c^\infty(\Omega)$, $\mathbf{V}\in G^\perp$ and without loss of generality assume that $\int_Y\mathbf{V}=0$. Then as $\operatorname{div}\mathbf{V}=0$ a.e. in Y we obtain, using integration by parts and Lemma 6.4.3 twice

$$
\begin{aligned}
&\int_\Omega\int_Y \nabla\mathbf{u}^\varepsilon(S_\varepsilon(y,x))\cdot\mathbf{V}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \nabla\mathbf{u}^\varepsilon(x)\cdot\mathbf{V}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\,\mathrm{d}x\\
&= -\int_\Omega \mathbf{V}\left(\tfrac{x}{\varepsilon}\right)\cdot\mathbf{u}^\varepsilon(x)\otimes\nabla\varphi(x)\,\mathrm{d}x = -\int_\Omega\int_Y \mathbf{V}(y)\cdot\mathbf{u}^\varepsilon(S_\varepsilon(y,x))\otimes\nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}x.
\end{aligned}
$$

Since we assumed an appropriate embedding, the integral on the right-hand side is finite. We then pass to the limit as $\varepsilon\to 0$ and use the convergence $(6.66)_1$ to the left-hand side and the convergence $(6.62)_1$ to the right-hand side of the latter identity to infer that

$$\int_\Omega \int_Y \mathbf{U}(y,x)\cdot \mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \int_Y (\nabla \mathbf{u}(x)+\mathbf{U}(y,x))\cdot \mathbf{V}(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_Y \mathbf{V}(y)\,\mathrm{d}y \cdot \int_\Omega \mathbf{u}(x)\otimes \nabla\varphi(x)\,\mathrm{d}y\,\mathrm{d}x = 0,$$

which implies (6.71).

In order to show (6.76) we choose an arbitrary but fixed $\varphi \in C_c^\infty(\Omega)$, and $\nabla \mathbf{v} \in G$, to obtain

$$\begin{aligned}\int_\Omega \mathbf{A}^\varepsilon(x)\cdot \nabla \mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\,\mathrm{d}x =& \varepsilon \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \nabla_x \mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\,\mathrm{d}x\\ &= \varepsilon \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \nabla_y \left(\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right)\\ &- \varepsilon \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\otimes \nabla\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Again we use the embedding $W_0^1 L_{m_1} \hookrightarrow L_{m_2}$ to argue that the second integral on the right-hand side is well defined. In a similar fashion as in the scalar case, by using Lemma 6.4.3 and the weak formulation (6.5) we infer

$$\begin{aligned}&\int_\Omega \int_Y \mathbf{A}^\varepsilon(S_\varepsilon(y,x))\cdot \nabla \mathbf{v}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \nabla \mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\,\mathrm{d}x\\ &= \varepsilon \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \nabla_x \left(\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right)\,\mathrm{d}x - \varepsilon \int_\Omega \mathbf{A}^\varepsilon(x)\cdot \mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\otimes \nabla\varphi(x)\,\mathrm{d}x\\ &= \varepsilon \int_\Omega \mathbf{F}(x)\cdot \nabla_x \left(\mathbf{v}\left(\tfrac{x}{\varepsilon}\right)\varphi(x)\right) - \varepsilon \int_\Omega \int_Y \mathbf{A}^\varepsilon(S_\varepsilon(y,x))\cdot \mathbf{v}(y)\otimes \nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_\Omega \int_Y \mathbf{F}(S_\varepsilon(y,x))\cdot \nabla_y \mathbf{v}(y)\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad + \varepsilon \int_\Omega \int_Y \mathbf{F}(S_\varepsilon(y,x))\cdot \mathbf{v}(y)\otimes \nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad - \varepsilon \int_\Omega \int_Y \mathbf{A}^\varepsilon(S_\varepsilon(y,x))\cdot \mathbf{v}(y)\otimes \nabla\varphi(S_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\\ &=: I^{\varepsilon,1} + I^{\varepsilon,2} + I^{\varepsilon,3}.\end{aligned} \tag{6.77}$$

Letting $\varepsilon \to 0$ in the latter we will now concentrate on showing that all the terms on the right-hand side vanish. Lemma 6.7.2 and the embedding $W_0^1 L_{m_1} \hookrightarrow L_{m_2}$ and the definition of an N-function, particularly condition (2.37), imply that

$$\begin{aligned}&\lim_{\varepsilon\to 0} I^{\varepsilon,1} = \int_\Omega \mathbf{F}(x)\varphi(x)\,\mathrm{d}x \cdot \int_Y \nabla \mathbf{v}(y)\,\mathrm{d}y = 0,\\ &\lim_{\varepsilon\to 0} I^{\varepsilon,2} \le c \limsup_{\varepsilon\to 0} \varepsilon \|\mathbf{F}\|_{L^\infty(\Omega)} \|\nabla \mathbf{v}\|_{L_M(Y)} \|\nabla\varphi\|_{L^\infty(\Omega)} = 0,\\ &\lim_{\varepsilon\to 0} I^{\varepsilon,3} \le \limsup_{\varepsilon\to 0} \varepsilon \|\mathbf{A}^\varepsilon\|_{L_{m_2^*}(\Omega)} \|\nabla \mathbf{v}\|_{L_M(Y)} \|\nabla\varphi\|_{L^\infty(\Omega)} = 0.\end{aligned}$$

Thus passing to the limit on the left-hand side of (6.77) using the convergence described by $(6.67)_2$ we get

$$\int_\Omega \int_Y \mathbf{A}^0(y,x) \cdot \nabla \mathbf{v}(y) \varphi(x) \, \mathrm{d}y \, \mathrm{d}x = 0,$$

which implies (6.76).

Finally, we observe that independently, whether we consider a scalar equation or a system ($d = 1$ or $d > 1$) Lemma 6.6.4 and (6.65) imply that

$$\mathbf{u} \in V_0^f. \tag{6.78}$$

Using the expression for f^*, (6.76) and (6.65) we obtain that

$$\bar{\mathbf{A}} \in L_{f^*}(\Omega; \mathbb{R}^{d\times N}). \tag{6.79}$$

Case 2: Assume that M satisfies the Δ_2-condition. We only sketch the argument here. Instead of showing (6.71) we conclude that

$$\mathbf{U}(\cdot, x) \in D^\perp. \tag{6.80}$$

Indeed, we fix $\mathbf{V} \in D$, $\varphi \in C_c^\infty(\Omega)$ and proceed analogously to the proof of (6.71). Taking into account (6.9) we fix $\nabla \mathbf{v} \in D^\perp$, $\varphi \in C_c^\infty(\Omega)$ and repeating the proof of (6.76) we obtain

$$\mathbf{A}^0(\cdot, x) \in D^{\perp\perp}. \tag{6.81}$$

Analogously to (6.78) one obtains

$$\mathbf{u} \in V_0^{h^{**}}, \tag{6.82}$$

when employing Lemma 6.6.6 and (6.65). Using the expression for h^*, (6.76) and (6.65) we obtain

$$\bar{\mathbf{A}} \in L_{h^*}(\Omega; \mathbb{R}^{d\times N}). \tag{6.83}$$

We are able to complete **Part 1** of the proof with the observation that, both in *Case 1* and *Case 2*, the function $\bar{\mathbf{A}}$ satisfies

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \boldsymbol{\varphi} = \int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \tag{6.84}$$

for all $\boldsymbol{\varphi} \in C_c^\infty\left(\Omega; \mathbb{R}^d\right)$. The above identity is obtained by performing a passage to the limit as $\varepsilon \to 0$ in (6.5) for smooth compactly supported test functions using the convergence $(6.61)_2$.

We are now ready to pass to the second part of the proof and properly identify the limit in (6.84).

Part 2 (Identification of $\bar{\mathbf{A}}$). The proof is divided into five steps.
Step 1: First we show that the following identity

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon\,\mathrm{d}x=\int_\Omega \bar{\mathbf{A}}\cdot\nabla\mathbf{u}\,\mathrm{d}x \tag{6.85}$$

holds. Considering identity (6.5) with $\boldsymbol{\varphi}=\mathbf{u}^\varepsilon$ we conclude with the help of convergence $(6.61)_2$ that

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_\Omega \mathbf{F}\cdot\nabla\mathbf{u}^\varepsilon\,\mathrm{d}x=\int_\Omega \mathbf{F}\cdot\nabla\mathbf{u}\,\mathrm{d}x. \tag{6.86}$$

Case 1: Assume that M^* satisfies the Δ_2-condition. Moreover, assume first that $d=1$. Since Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, then by Lemma 8.2 the set $\overline{\Omega}$ can be covered by a finite family of sets $\{G_i\}_{i\in I}$ such that each $\Omega_i=\Omega\cap G_i$ is a star-shaped domain with respect to balls $\{B^i\}_{i\in I}$, respectively. Then

$$\Omega=\bigcup_{i\in I}\Omega_i.$$

Let us introduce a partition of unity θ_i, i.e.

$$0\le\theta_i\le 1,\quad \theta_i\in C_c^\infty(G_i),\quad \sum_{i\in I}\theta_i(x)=1\ \text{ for }\ x\in\Omega,$$

which exists due to Lemma 8.3. For each $\ell\in\mathbb{N}$ consider the truncation $T_\ell u$ and its decomposition in the form

$$T_\ell u(x)=\sum_{i=1}^K T_\ell u(x)\theta_i(x),\ x\in\Omega.$$

As $\nabla(T_j u\theta_i)=\nabla T_\ell u\theta_i+T_\ell u\nabla\theta_i\in L_f(\Omega_i;\mathbb{R}^d)$ and $\mathrm{supp}(u\theta_i)\subset\Omega_i$ for each $i,\ell\in\mathbb{N}$, we can apply Theorem 3.7.7 to construct an approximating sequence $\{v^n_{\ell,i}\}\subset C_c^\infty(\Omega)$ such that

$$\nabla v^n_{\ell,i}\xrightarrow{f}\nabla(T_\ell u\theta_i)\text{ modularly in }L_f(\Omega_i)\text{ for each }i=1,\dots,K.$$

Observe that since f is independent of x, then the assumption (Me) of Theorem 3.7.7 is trivially satisfied. We define $v^n_\ell:=\sum_{i=1}^K v^n_{\ell,i}$ and obtain from (6.84) using Lemmas 3.7.9 and Corollary 3.4.7

$$\int_\Omega\bar{\mathbf{A}}\cdot\nabla u\,\mathrm{d}x=\lim_{\ell\to\infty}\lim_{n\to\infty}\int_\Omega\bar{\mathbf{A}}\cdot\nabla v^n_\ell\,\mathrm{d}x=\lim_{\ell\to\infty}\lim_{n\to\infty}\int_\Omega\mathbf{F}\cdot\nabla v^n_\ell\,\mathrm{d}x=\int_\Omega\mathbf{F}\cdot\nabla u\,\mathrm{d}x.$$

Hence (6.85) follows from (6.86) and the latter identity.

Next, we consider the case when $k\ge 1$ and the embedding $W^1_0L_{m_1}\hookrightarrow L_{m_2}$ holds. Although in the vector case the usefulness of the truncation method fails, we consider the following decomposition of $\mathbf{u}$

$$\mathbf{u}(x) = \sum_{i=1}^{K} \mathbf{u}(x)\theta_i(x),\ x \in \Omega,$$

where $\{\theta_i\}_{i=1}^K$ is the partition of unity introduced above. In view of the assumed embedding we conclude that $\nabla(\mathbf{u}\theta_i) \in L_f(\Omega_i;\mathbb{R}^{d\times N})$ and moreover $\mathrm{supp}(\mathbf{u}\theta_i) \subset \Omega_i$ for each $i = 1,\ldots,K$. Using again Theorem 3.7.7 we find sequences $\{\mathbf{u}_i^n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega;\mathbb{R}^{d\times N})$ such that

$$\nabla \mathbf{u}_i^n \xrightarrow{f} \nabla(\mathbf{u}\theta_i) \text{ modularly in } L_f(\Omega;\mathbb{R}^{d\times N}).$$

For each $n \in \mathbb{N}$ define $\mathbf{u}^n =: \sum_{i=1}^K \mathbf{u}_i^n$ and observe that by Lemma 3.4.6 we get

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u}\, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u}^n\, \mathrm{d}x = \lim_{n\to\infty} \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}^n\, \mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}\, \mathrm{d}x,$$

which implies (6.85) along with (6.86) also for the case $d > 1$. This completes the reasoning of Step 1 for the case of M^* satisfying the Δ_2-condition both in the scalar and the vector case.

Case 2: Assume that M satisfies the Δ_2-condition. We note that if M satisfies the Δ_2-condition we can proceed analogously using (6.80)–(6.83) and the approximation by smooth compactly supported functions in the modular topology of gradients in $L_{h^{**}}(\Omega;\mathbb{R}^d)$, $L_{h^{**}}(\Omega;\mathbb{R}^{d\times N})$ respectively.

Step 2: We devote this step to showing that the following inequality

$$\int_\Omega \int_Y (\mathbf{A}^0(y,x) - \mathbf{A}(y,\mathbf{V}(x,y))) \cdot (\nabla \mathbf{u}(x) + \mathbf{U}(y,x) - \mathbf{V}(x,y))\, \mathrm{d}y\, \mathrm{d}x \geq 0 \tag{6.87}$$

holds for all $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y;\mathbb{R}^{d\times N}))$. For $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y;\mathbb{R}^{d\times N}))$ we have that

$$\mathbf{A}(\cdot,\mathbf{V}) \in L^\infty(\Omega\times Y;\mathbb{R}^{d\times N}). \tag{6.88}$$

To show that (6.88) indeed holds observe that conditions of type (A3), see formulation (3.90) for a general statement, imply conditions (3.89), see Proposition 3.8.1. Moreover, since M and M^* are N-functions, we have

$$c_1 m_1^*(c_3\mathbf{A}(\cdot,\mathbf{V})) \leq c_2 M^*(\cdot, c_3\mathbf{A}(\cdot,\mathbf{V})) \leq M(\cdot, c_4\mathbf{V}) \leq m_2(c_4\mathbf{V}). \tag{6.89}$$

These estimates yield

$$\|\mathbf{A}(\cdot,\mathbf{V})\|_{L^\infty(\Omega\times Y)} \leq \tfrac{1}{c_3}(m_1^*)^{-1}\left(\tfrac{1}{c_2} m_2(c_4\|\mathbf{v}\|_{L^\infty(\Omega\times Y)})\right). \tag{6.90}$$

Define the sequences

$$\mathbf{V}^\varepsilon(x) := \mathbf{V}(\tfrac{x}{\varepsilon},x) \text{ and } \tilde{\mathbf{A}}^\varepsilon(x) := \mathbf{A}(\tfrac{x}{\varepsilon},\mathbf{V}^\varepsilon).$$

Due to the appropriate embeddings

$$L^\infty(\Omega\times Y;\mathbb{R}^{d\times N}) \subset E_{m_1^*}(\Omega\times Y;\mathbb{R}^{d\times N}) \subset E_{m_2^*}(\Omega\times Y;\mathbb{R}^{d\times N})$$

we obtain, when letting $\varepsilon \to 0$,

$$\begin{aligned} &\mathbf{V}^\varepsilon \xrightarrow{2-s} \mathbf{V} && \text{in } E_{m_i}(\Omega\times Y;\mathbb{R}^{d\times N}), \\ &\tilde{\mathbf{A}}^\varepsilon \xrightarrow{2-s} \mathbf{A}(\cdot,\mathbf{V}(\cdot,\cdot)) && \text{in } E_{m_i^*}(\Omega\times Y;\mathbb{R}^{d\times N}), i=1,2, \end{aligned} \tag{6.91}$$

and consequently

$$\lim_{\varepsilon\to 0}\int_\Omega \tilde{\mathbf{A}}^\varepsilon(x)\cdot\mathbf{V}^\varepsilon(x)\,\mathrm{d}x = \int_\Omega\int_Y \mathbf{A}(y,\mathbf{V}(y,x))\cdot\mathbf{V}(y,x)\,\mathrm{d}y\,\mathrm{d}x. \tag{6.92}$$

By the monotonicity condition (A3) we have

$$\int_\Omega (\mathbf{A}^\varepsilon(x)-\tilde{\mathbf{A}}^\varepsilon(x))\cdot(\nabla\mathbf{u}^\varepsilon(x)-\mathbf{V}^\varepsilon(x))\,\mathrm{d}x \geq 0. \tag{6.93}$$

The above inequality will lead us to (6.87). Let $\varepsilon \to 0$ in (6.93). We will use **Step 1**, namely identity (6.85), together with the characterization (6.68) to infer that

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{A}^\varepsilon(x)\cdot\nabla\mathbf{u}^\varepsilon(x)\,\mathrm{d}x = \int_\Omega\int_Y \mathbf{A}^0\cdot\nabla\mathbf{u}\,\mathrm{d}y\,\mathrm{d}x. \tag{6.94}$$

As we have already proved (6.71) and (6.76), it immediately follows that

$$\int_\Omega\int_Y \mathbf{A}^0\cdot\nabla\mathbf{u} = \int_\Omega\int_Y \mathbf{A}^0\cdot(\nabla\mathbf{u}+\mathbf{U}). \tag{6.95}$$

Passing to the limit in the remaining terms of (6.93) easily follows with the help of (6.67), (6.91) together with Lemma 6.4.4 (v) and (6.92). Thus the proof of this part is complete.

Step 3: Our next step is to sharpen the preceding observation and to show that (6.87) holds not only for $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y;\mathbb{R}^{d\times N}))$ but also for $\mathbf{V} \in L^\infty(\Omega\times Y;\mathbb{R}^{d\times N})$. For this purpose we take an arbitrary function $\mathbf{V} \in L^\infty(\Omega\times Y;\mathbb{R}^{d\times N})$ and consider a sequence $\{K^m\}_{m\in\mathbb{N}}$ of compact subsets of Ω such that

$$K^1 \subset K^2 \subset \ldots\Omega \text{ and } \bigcup_{m=1}^\infty K^m = \Omega.$$

Define $\mathbf{V}^m := \mathbf{V}\mathbb{1}_{K^m}$ and observe that obviously $\mathbf{V}^m$ are bounded in $L^\infty(\Omega\times Y;\mathbb{R}^{d\times N})$ for every $m \in \mathbb{N}$, thus there exists a positive constant c such that

$$\|\mathbf{A}(\cdot,\mathbf{V}^m)\|_{L^\infty(\Omega\times Y)} \leq c \quad \text{for all } m\in\mathbb{N}, \tag{6.96}$$

which follows by arguments analogous to those used in the proof of (6.88). Using 2.37 and (6.96) gives

$$\int_\Omega\int_Y M(y,\mathbf{V}^m)+M^*(y,\mathbf{A}(y,\mathbf{V}^m))\,\mathrm{d}y\,\mathrm{d}x \le \int_\Omega\int_Y m_2(|\mathbf{V}^m|)+m_1^*(|\mathbf{A}(y,\mathbf{V}^m)|)\,\mathrm{d}y\,\mathrm{d}x$$
$$\le \int_\Omega\int_Y m_2(\|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)})+m_1^*(\|\mathbf{A}(\cdot,\mathbf{V}^m)\|_{L^\infty(\Omega\times Y)}) \le c.$$

Boundedness of the modulars allows us to conclude with the help of Lemma 3.4.2 that $\{\mathbf{V}^m\}_{m=1}^\infty$ and $\{\mathbf{A}(\cdot,\mathbf{V}^m)\}_{m=1}^\infty$ are uniformly integrable. The uniform integrability together with the convergence in measure of these sequences, which can be easily shown, by Theorem 3.4.4 give as $m\to\infty$

$$\begin{aligned}\mathbf{V}^m &\xrightarrow{M} \mathbf{V} \text{ in } L_M(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}(\cdot,\mathbf{V}^m) &\xrightarrow{M^*} \mathbf{A}(\cdot,\mathbf{V}) \text{ in } L_{M^*}(\Omega\times Y;\mathbb{R}^{d\times N}).\end{aligned} \tag{6.97}$$

Let us consider a standard mollifier $\omega\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. Since $\mathbf{V}^m$ is supported in $K^m\subset\Omega$ for all m, we can find for every m a sequence $\delta^n\to 0$ as $n\to\infty$ such that, defining $\mathbf{V}^{m,n} := \mathbf{V}^m * \omega^n$, where $\omega^n(z) = (\delta^n)^{-2N}\omega\left(\frac{z}{\delta^n}\right)$. This procedure provides both smoothness and compact support of $\mathbf{V}^{m,n}$, i.e. we have $\mathbf{V}^{m,n}\in C_c^\infty(\Omega;C_{per}^\infty(Y))^{d\times N}$. Obviously $\|\mathbf{V}^{m,n}\|_{L^\infty(\Omega\times Y)} \le \|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)}$.

In the same manner as we showed the convergences (6.97), now we conclude that for every m

$$\begin{aligned}\mathbf{V}^{m,n} &\xrightarrow{M} \mathbf{V}^m \text{ in } L_M(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}(\cdot,\mathbf{V}^{m,n}) &\xrightarrow{M^*} \mathbf{A}(\cdot,\mathbf{V}^m) \text{ in } L_{M^*}(\Omega\times Y;\mathbb{R}^{d\times N})\end{aligned} \tag{6.98}$$

as $n\to\infty$. Finally, using (6.97), (6.98) and Lemma 3.4.6 we infer from Step 2 that

$$\begin{aligned}0 &\le \lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega\int_Y (\mathbf{A}^0-\mathbf{A}(y,\mathbf{V}^{m,n}))\cdot(\nabla\mathbf{u}+\mathbf{U}-\mathbf{V}^{m,n})\\ &= \int_\Omega\int_Y (\mathbf{A}^0-\mathbf{A}(y,\mathbf{V}))\cdot(\nabla\mathbf{u}+\mathbf{U}-\mathbf{V}).\end{aligned}$$

Step 4: The final step in establishing the limit of the nonlinear term is to use the monotonicity trick, which was described in detail in Chapter 4. This step follows the same idea, however for completeness of our argument in this case we explain this step with great care. For $\ell>0$ define

$$\mathcal{S}_\ell = \{(x,y)\in\Omega\times Y : |\nabla\mathbf{u}(x)+\mathbf{U}(y,x)|\le\ell\}$$

and let $\mathbb{1}_\ell$ be the characteristic function of $\mathcal{S}_\ell$. As we showed in **Step 3** that (6.87) holds for $\mathbf{V}\in L^\infty(\Omega\times Y;\mathbb{R}^{d\times N})$, we will now choose

$$\mathbf{V} = (\nabla\mathbf{u}+\mathbf{U})\mathbb{1}_j+\lambda\bar{\mathbf{V}}\mathbb{1}_i,$$

where $0<i<j$ and $\lambda\in(0,1)$ and $\bar{\mathbf{V}}\in L^\infty(\Omega\times Y;\mathbb{R}^{d\times N})$. Thus (6.87) can be rewritten as follows

$$
\begin{aligned}
0 \le & \int_\Omega \int_Y \mathbf{A}^0 \cdot (\nabla \mathbf{u} + \mathbf{U} - (\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j) \, \mathrm{d}y \, \mathrm{d}x \\
& - \int_\Omega \int_Y \mathbf{A}(y, (\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j + \lambda \bar{\mathbf{V}} \mathbb{1}_i)) \cdot (\nabla \mathbf{u} + \mathbf{U} - (\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j) \\
& + \lambda \int_\Omega \int_Y (\mathbf{A}(y, (\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j + \lambda \bar{\mathbf{V}} \mathbb{1}_i) - \mathbf{A}^0) \cdot \bar{\mathbf{V}} \mathbb{1}_i \, \mathrm{d}y \, \mathrm{d}x.
\end{aligned} \tag{6.99}
$$

The first term on the right-hand side is equal to the integral

$$
\int_{\Omega \times Y \setminus \mathcal{S}_j} \mathbf{A}^0 \cdot (\nabla \mathbf{u} + \mathbf{U} - (\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j) \, \mathrm{d}y \, \mathrm{d}x
$$

and it vanishes when passing to the limit as $j \to \infty$ by the Lebesgue dominated convergence theorem and the fact that $|\Omega \times Y \setminus \mathcal{S}_j| \to 0$ as $j \to \infty$. Observe that

$$
(\nabla \mathbf{u} + \mathbf{U}) \mathbb{1}_j + \lambda \bar{\mathbf{V}} \mathbb{1}_i = 0 \text{ in } \mathcal{S}_j,
$$

thus the second integral in (6.99) also vanishes. We divide the resulting inequality by λ and conclude the following

$$
\int_{\mathcal{S}_i} (\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}}) - \mathbf{A}^0) \cdot \bar{\mathbf{V}} \, \mathrm{d}y \, \mathrm{d}x \ge 0. \tag{6.100}
$$

Since M^* is an N-function, by (2.37) we obtain

$$
\begin{aligned}
\int_{\mathcal{S}_i} M^*(y, \mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}})) \, \mathrm{d}y \, \mathrm{d}x &\le \int_{\mathcal{S}_i} m_1^*(|\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}})|) \, \mathrm{d}y \, \mathrm{d}x \\
\le |\mathcal{S}_i| m_1^*(\|\mathbf{A}(\cdot, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}})\|_{L^\infty(\mathcal{S}_i)}) \le c.
\end{aligned} \tag{6.101}
$$

To pass to the limit as $\lambda \to 0$ in (6.100) we need to estimate

$$
\|\mathbf{A}(\cdot, \nabla \mathbf{u} + \mathbf{U} + \lambda \mathbf{V})\|_{L^\infty(\mathcal{S}_i)}
$$

uniformly with respect to λ. For this purpose we proceed in a similar way as in (6.96) since

$$
\|\nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}}\|_{L^\infty(\mathcal{S}_i)} \le \|\nabla \mathbf{u} + \mathbf{U}\|_{L^\infty(\mathcal{S}_i)} + \|\bar{\mathbf{V}}\|_{L^\infty(\Omega \times Y)} \le i + \|\bar{\mathbf{V}}\|_{L^\infty(\Omega \times Y)}.
$$

As

$$
\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}}) \to \mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U}) \text{ a.e. in } \mathcal{S}_i
$$

and $\{\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \bar{\mathbf{V}})\}_{\lambda \in (0,1)}$ is uniformly integrable on $\mathcal{S}_i$ due to (6.101) and Lemma 3.4.2, again by the Vitali theorem we conclude

$$
\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U} + \lambda \mathbf{V}) \to \mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U}) \text{ in } L^1(\mathcal{S}_i)
$$

as $\lambda \to 0_+$. Thus passing to the limit in (6.100) we arrive at

$$
\int_{\mathcal{S}_i} (\mathbf{A}(y, \nabla \mathbf{u} + \mathbf{U}) - \mathbf{A}^0) \cdot \mathbf{V} \, \mathrm{d}y \, \mathrm{d}x \ge 0.
$$

Now, the appropriate choice of the function $\bar{\mathbf{V}}$ leads to the conclusion. Indeed, choosing

$$\bar{\mathbf{V}} = -\frac{\mathbf{A}^0-\mathbf{A}(y,\nabla\mathbf{u}+\mathbf{U})}{|\mathbf{A}^0-\mathbf{A}(y,\nabla\mathbf{u}+\mathbf{U})|+1}$$

yields

$$\mathbf{A}^0(y,x) = \mathbf{A}(y,\nabla\mathbf{u}(x)+\mathbf{U}(y,x)) \text{ for a.a. } (x,y)\in\mathcal{S}_i. \tag{6.102}$$

Recall that i was arbitrary and $|\Omega\times Y\setminus\mathcal{S}_i|\to 0$ as $i\to\infty$, thus the above relation holds a.e. in the whole $\Omega\times Y$. Moreover, due to the properties (6.71) and (6.76) we obtain that $\mathbf{U}(\cdot,x)$ is equal to the gradient of a weak solution of the cell problem (6.4) corresponding to $\boldsymbol{\xi} = \nabla\mathbf{u}(x)$. Finally, we get by (6.68) and (6.3) that

$$\bar{\mathbf{A}}(x) = \int_Y \mathbf{A}^0(y,x)\,\mathrm{d}y = \int_Y \mathbf{A}(y,\nabla\mathbf{u}(x)+\mathbf{U}(y,x))\,\mathrm{d}y = \hat{\mathbf{A}}(\nabla\mathbf{u}(x)), \tag{6.103}$$

and we obtained the desired characterization.

Step 5: Since it has already been established that (6.2) possesses a unique solution $\mathbf{u}$ and we can extract from any subsequence of $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ a subsequence that converges to $\mathbf{u}$ weakly-$*$ in $W_0^1L_{m_1}(\Omega;\mathbb{R}^N)$, and consequently also weakly in $W_0^{1,1}(\Omega;\mathbb{R}^d)$, the whole sequence $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ converges to $\mathbf{u}$ weakly-$*$ in $W_0^1L_{m_1}(\Omega;\mathbb{R}^d)$, and weakly in $W_0^{1,1}(\Omega;\mathbb{R}^d)$, respectively. □

Chapter 7
Non-Newtonian Fluids

7.1 Introducing the Problem

In this chapter we concentrate on a large class of problems arising from the dynamics of incompressible non-Newtonian fluids with nonstandard rheology. By non-Newtonian fluids we mean here fluids which do not satisfy Newton's law of viscosity, i.e. viscosity is constant and independent of stress. In the case of non-Newtonian fluids viscosity may change under various stimuli like shear rate, or a magnetic or electric field. When subjected to such a force, the fluid may become more liquid or solid, and becomes runnier or more solid when shaken. Because of this property of changeable viscosity these fluids have numerous industrial, military and natural science applications.

To be more precise, in various models described by systems of PDEs the rheology – behavior of the medium – is reflected by the constitutive relation between the viscous stress tensor $\mathbf{S}$ and the shear stress $\mathbf{D}\mathbf{u}$, which is the symmetric part of the velocity gradient, sometimes called the deformation tensor ($\mathbf{u}$ denotes the velocity field of the fluid). In particular, we will investigate fluids for which the relation between the viscous stress tensor $\mathbf{S}$ and the shear stress $\mathbf{D}\mathbf{u}$ is nonlinear and we concentrate on the case when this relation may be anisotropic, inhomogeneous and not necessarily of polynomial type.

Among the various types of non-Newtonian fluids, we can distinguish shear thickening and shear thinning fluids, and magneto- and electro-rheologial fluids.

For shear thickening fluids (STF), also called dilatant fluids, viscosity increases with increased stress, e.g. oobleck (corn starch suspended in water) and nanosilica with polyethylene glycol. This type of fluid has an interesting military application. An STF fluid behaves as a liquid until another object strikes it with high kinetic energy. In this case the fluid increases its viscosity in milliseconds and behaves almost like a solid. Moreover this process is completely reversible, which makes such a fluid an ideal component in the production of fabrics and materials for armor, for military, medical and sports purposes. The obtained material has high elasticity combined with protection against needles, knives and bullets [108, 129, 203, 231].

There is also wide range of fluids with the shear thinning property. In this case the fluid viscosity decreases with an increase of stress, e.g. nail polish, ketchup, latex

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paint, ice, blood. We mention here two constitutive relations: the Prandtl–Eyring model, cf. [140], where the stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \eta_0 \frac{\operatorname{arc\,sinh}(\lambda|\mathbf{Du}|)}{\lambda|\mathbf{Du}|}\mathbf{Du}$$

and the modified Powell–Eyring model, cf. [275],

$$\mathbf{S} = \eta_\infty \mathbf{Du} + (\eta_0 - \eta_\infty)\frac{\ln(1+\lambda|\mathbf{Du}|)}{(\lambda|\mathbf{Du}|)^m}\mathbf{Du}, \tag{7.1}$$

where η_∞, η_0, λ, m are material constants. Both above models are broadly used in geophysics, engineering and medical applications, e.g. for the modeling of glacier ice, cf. [209], blood flow, cf. [284, 285] and many other contexts, cf. [67, 271, 295, 330].

We also would like to cover the case of constitutive relations which may depend on spatial variables and directions of $\mathbf{Du}$. Anisotropic and inhomogeneous effects are in particular present in the case of electrorheological and magnetorheological fluids. Electrorheological (ER) fluids are suspensions of extremely fine non-conducting but electrically active particles in an electrically insulating fluid. The viscosity of these fluids changes reversibly when an electric field is applied. A magnetorheological fluid is a mixture of magnetic particles suspended in a carrier fluid, usually a type of oil. When such a fluid is subjected to a magnetic field, the fluid significantly increases its viscosity, sometimes to the point of becoming a viscoelastic solid. In both cases the effect of the increase of viscosity is caused by the fact that particles suspended in the fluid form a particular structure depending on the applied field, e.g. a column-like structure may be formed. In such case the viscosity of the fluid may depend on the spatial point in the domain, directions of lines of the electric/magnetic field and various directions of the shear rate.

In general the constitutive relation may depend also on density distribution, temperature, electric field, or spatial variables. This chapter is directed towards existence and properties of weak solutions to the systems of equations describing the motion of non-Newtonian fluids.

In particular in this chapter we focus on incompressible flow. Then the considered model takes the form of the following system of equations:

$$\begin{aligned} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S}(x,\mathbf{Du}) + \nabla\pi = \mathbf{f} \quad &\text{in } (0,T)\times\Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad &\text{in } (0,T)\times\Omega, \end{aligned} \tag{7.2}$$

where $\mathbf{u}$ denotes the velocity field of a fluid; π is a pressure; Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary; $(0,T)$ with $T<\infty$ is a finite time interval; $\mathbf{f}$ is a given body force; and

$$\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$$

stands for the symmetric part of the gradient of the velocity field. The first equation describes the balance of momentum and the last one stands for an incompressibility condition. We supplement the above system with no-slip boundary conditions for velocity (or a zero Dirichlet boundary condition), namely $\mathbf{u} = 0$ on $\partial\Omega$, for all $t \in (0,T)$.

In this chapter we consider various generalizations of (7.2). Depending on the particular problem considered here the viscosity of the fluid is not only allowed to depend on $\mathbf{Du}$, but it may depend on a spatial variable x, density distribution or temperature of the fluid.

Since our aim is to study various phenomena of non-Newtonian fluids, we consider a general form of the stress tensor $\mathbf{S}$. Our analysis includes power-law and Carreau-type models which are quite popular in rheology, chemical engineering and colloidal mechanics. In particular, we formulate the growth conditions of the stress tensor in the following way: we assume that there exist an N-function M and its conjugate M^*, and a constant $c \in (0,1]$ such that

$$\mathbf{S}(x,\mathbf{Du}) : \mathbf{Du} \geq c\big(M(x,\mathbf{Du}) + M^*(x,\mathbf{S}(x,\mathbf{Du}))\big) \quad \text{for a.a. } x \in \Omega \tag{7.3}$$

and $\mathbf{S}$ is monotone, i.e.

$$(\mathbf{S}(x,\boldsymbol{\xi}_1) - \mathbf{S}(x,\boldsymbol{\xi}_2)) : (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq 0 \quad \text{for a.a. } x \in \Omega \text{ and all } \boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2 \in \mathbb{R}^{N\times N}.$$

For a discussion on the form of growth conditions for the higher order terms we refer the reader to Section 3.8.2. Let us emphasize that in our considerations, as in previous chapters, it is important that we do not assume that $\mathbf{S}$ has only polynomial-structure, which may not suffice to describe the nonstandard behavior of the fluid. Hence the N-function defining a functional space does not satisfy the Δ_2-condition and is possibly inhomogeneous and anisotropic.

Since we allow $\mathbf{S}$ to depend on the spatial variable x, the N-function also depends on $x \in \Omega$. This corresponds to the possible inhomogeneity of the medium as in the case of electrorheological or magnetorheological fluids. The dependence of $\mathbf{S}$, and consequently of M, on a whole tensor results from the fact that the viscosity may differ in different directions of the symmetric part of the velocity gradient $\mathbf{Du}$ and the growth condition for the stress tensor may be dependent on the whole tensor $\mathbf{Du}$, not only on $|\mathbf{Du}|$. In our considerations the general growth of $\mathbf{S}$ is provided by quite general properties of the N-function M defining an anisotropic Musielak–Orlicz space L_M. As we do not want to be restricted here by no-faster than polynomial growth on $\mathbf{S}$ or by the doubling condition (to cover the case of significantly shear thickening fluids) we do not assume that the Δ_2-condition is satisfied by M.

In the majority of publications concerning non-Newtonian fluids a p-structure for $\mathbf{S}$ is assumed and then typically the stress tensor takes the following form

$$\mathbf{S} = \mu(\kappa + |\mathbf{Du}|)^{p-2}\mathbf{Du}$$

$$\text{or } \mathbf{S} = \mu(\kappa + |\mathbf{Du}|^2)^{(p-2)/2}\mathbf{Du},$$

for constants $\kappa > 0$ and $\mu > 0$. Then the growth and coercivity conditions for the stress tensor take the following form of polynomial type:

$$\begin{aligned} |\mathbf{S}(x,\mathbf{Du})| &\leq c(1 + |\mathbf{Du}|^2)^{(p-2)/2}|\mathbf{Du}|, \\ \mathbf{S}(x,\mathbf{Du}) : \mathbf{Du} &\geq c(1 + |\mathbf{Du}|^2)^{(p-2)/2}|\mathbf{Du}|^2, \end{aligned} \tag{7.4}$$

see e.g. [158, 160, 244]. Unfortunately such a formulation is not adequate for fluids that rapidly and significantly change their viscosity, i.e. when the growth of the stress tensor may be much faster than polynomial, or very slow, faster than linear, but not comparable with any polynomial. Moreover conditions (7.4) do not allow the situation when the stress tensor differs in various directions of the shear stress or is inhomogeneous in spatial variables.

On the other hand one might be interested in studying the constitutive relation for fluids with dependence on an external field. Let us mention here electrorheological fluids. Mathematical models of such fluids were considered by Rajagopal and Růžička in [280]. The authors derive governing equations for the motion of electrorheological fluids, where the complex interactions between the thermo-mechanical and electromagnetic fields are taken into account (see also [279]). In the case of such fluids, from the representation theorem it follows that the most general form of the stress tensor $\mathbf{S}$ (cf. [287]) is given by

$$\mathbf{S} = \alpha_1 \mathrm{e}\otimes\mathrm{e} + \alpha_2\mathbf{D} + \alpha_3\mathbf{D}^2 + \alpha_4(\mathbf{D}\mathrm{e}\otimes\mathrm{e} + \mathrm{e}\otimes\mathbf{D}\mathrm{e}) + \alpha_5(\mathbf{D}^2\mathrm{e}\otimes\mathrm{e} + \mathrm{e}\otimes\mathbf{D}^2\mathrm{e})$$

where α_i, $i = 1,\dots,5$, may be functions of the invariants

$$|\mathrm{e}|^2,\ \mathrm{tr}\,\mathbf{D}^2,\ \mathrm{tr}\,\mathbf{D}^3,\ \mathrm{tr}\,(\mathbf{D}\mathrm{e}\otimes\mathrm{e}),\ \mathrm{tr}\,(\mathbf{D}^2\mathrm{e}\otimes\mathrm{e}).$$

It can be shown that for $i = 1, 3, 5$, $\alpha_i = 0$ the stress tensor in the form

$$\mathbf{S} = |\mathrm{tr}\,\mathbf{D}^2|^3\mathbf{D} + |\mathrm{tr}\,(\mathbf{D}^2\mathrm{e}\otimes\mathrm{e})|^6(\mathbf{D}\mathrm{e}\otimes\mathrm{e} + \mathrm{e}\otimes\mathbf{D}\mathrm{e}) \tag{7.5}$$

is thermodynamically admissible (i.e. $\mathbf{S}:\mathbf{D} \geq 0$), satisfies a principle of material frame-indifference and is monotone. However, for $\mathrm{e} = (1,0,0)$ it can be calculated that the isotropic growth conditions $|\mathbf{S}(\mathbf{D},\mathrm{e})| \leq c(1+|\mathbf{D}|)^{p-1}$, and $\mathbf{S}(\mathbf{D},\mathrm{e}):\mathbf{D} \geq c|\mathbf{D}|^p$ are not satisfied, since the tensor $\mathbf{S}$ possesses growth of different powers in various directions of $\mathbf{D}$. In particular, this means that one may need to consider $\mathbf{S}$ possessing growth of different powers in various directions of $\mathbf{Du}$.

As mentioned above we also would like to cover the case of constitutive relations which may depend on spatial variables. As an example we can again recall the class of electrorheological fluids. Such fluids can be seen as suspensions of extremely fine non-conducting particles in an electrically insulating liquid. This case was considered, for example, in [287], where in order to describe the behavior of the fluid an isotropic but inhomogeneous N-function of the form $M(x,z) = |z|^{p(x)}$ with $1 < p_- \leq p(x) \leq p^+ < \infty$ was used.

Let us now briefly recall some related results concerning the theory of existence of solutions to systems describing non-Newtonian fluids. The analysis of the time-dependent flow of homogeneous (density was assumed to be constant) non-Newtonian fluids of power-law type was initiated in [224, 225], where the global existence of weak solutions for the exponent $p \geq 1 + (2N)/(N+2)$ (N stands for space dimension) was proved under Dirichlet boundary conditions. Later the steady flow was investigated in [159], where the existence of weak solutions was obtained for a constant exponent lower than above, $p > \frac{2N}{N+2}$, $N \geq 2$, by Lipschitz truncation methods.

In [287] generalized Lebesgue spaces $L^{p(\cdot)}$ were used to analyze the flow of an electrorheological fluid. In this work the exponent $p(\cdot)$ is such that $1 < p_0 \leq p(x) \leq p_\infty < \infty$, where $p \in C^1(\Omega)$ is a function of an electric field E, i.e. $p = p(|E|^2)$, and $p_0 > \frac{3N}{N+2}$ in the case of steady flow, where $N \geq 2$ (the space dimension). The Δ_2-condition is then satisfied and consequently the space is reflexive and separable, which is not the case for our considerations. Later in [116] the above result was improved, by allowing a smaller lower bound on the exponent, using Lipschitz truncation methods in the $L^{p(x)}$-setting for $\mathbf{S}$, where $\frac{2N}{N+2} < p(\cdot) < \infty$ was log-Hölder continuous and $\mathbf{S}$ was strictly monotone.

In [325] the author showed the existence of weak solutions to the unsteady motion of an incompressible homogenous power-law fluid with shear rate dependent viscosity with $p > 2(N+1)/(N+2)$ without strong restrictions on the shape and size of Ω. The author applied an L^∞-test function and a local pressure method. Finally the existence of global in time weak solutions with Dirichlet boundary conditions for $p > (2N)/(N+2)$ was achieved in [118] by Lipschitz truncation and local pressure methods.

Most of the available results concerning heterogeneous (without the assumption that density is constant) incompressible fluids again deal with the polynomial dependence between $\mathbf{S}$ and $|\mathbf{Du}|$. In this case the system (7.2) needs to be supplemented with a balance of mass (continuity equation):

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,$$

where ρ denotes the density of the fluid. The analysis of heterogeneous Newtonian ($p = 2$ in standard growth condition (7.4)) fluids was given in [15] in the seventies. Next in [237] the concept of renormalized solutions was presented, which made it possible to obtain convergence and continuity properties of the density.

The first result for unsteady flow of heterogeneous non-Newtonian fluids goes back to [152], where the existence of Dirichlet weak solutions was obtained for $p \geq 12/5$ if $N = 3$, and for $p \geq 2$ if $N = 2$. Later in [177] the existence of space-periodic weak solutions for $p \geq 2$ was shown and regularity properties of weak solutions were found for $p \geq 20/9$ if $N = 3$ and $p \geq 2$ if $N = 2$. In [160] the existence of a weak solution was shown for generalized Newtonian fluids of power-law type for $p > (3N+2)/(N+2)$. The authors also needed the existence of the potential of $\mathbf{S}$.

Next if we additionally want to consider heat effects and include the temperature as a changing unknown, one can add to the system (7.2) balance of thermal energy (see Section 7.2). The most closely related result concerning heterogeneous, incompressible and heat-conducting non-Newtonian fluids, but of growth conditions of polynomial type for $p \geq 11/5$ with $N = 3$, can be found in [158].

Non-Newtonian fluids in the framework of anisotropic Musielak–Orlicz spaces have been studied using a variety of approaches. Considerations on the existence of weak solutions in the case of homogeneous, incompressible non-Newtonian fluids can be found in [180], where $\mathbf{S}$ was assumed to be strictly monotone. The authors used Young measure techniques in place of monotonicity methods. The additional assumption of strict monotonicity makes it possible to conclude that the measure-valued solution is of the form of a Dirac measure and then the system has weak solution. As mentioned in previous chapters, the monotonicity method for non-reflexive anisotropic Musielak–Orlicz spaces was used in [183, 326, 328], allowing only the monotonicity of $\mathbf{S}$ to be assumed.

In this chapter we investigate the following problems:

- The flow of heterogeneous heat-conducting fluids which depends also on density and thermal energy. This means that the above system (7.2) needs to be supplemented with two equations: balance of mass (continuity equation) and balance of thermal energy. In this case the stress tensor depends also on density and temperature. Here we are restricted by the condition that M grows essentially faster than $|\cdot|^p$ with $p \geq 11/5$ for three-dimensional space, which allows us to consider only shear thickening fluids (STF).
- The flow of incompressible non-Newtonian fluids described by a generalized Stokes system which allows us to consider shear thinning fluids, since the convective term can be omitted in this model. In this case we allow M and its conjugate M^* not to satisfy the Δ_2-condition.
- The system describing fluid-structure interaction where the motion of rigid bodies immersed in the non-Newtonian fluid is taken into account. Here the important issue is local reconstruction of the pressure function, which is neglected in the weak formulation for the above two problems due to the incompressibility condition.

The setting considered in this chapter needs tools which generalize results not only of classical Lebesgue and Sobolev spaces (related to power-law fluids), but also in variable exponent, anisotropic and classical Orlicz spaces, which has already been emphasized in previous chapters.

Let us now comment on the content of particular sections of this chapter.

In Section 7.2 our aim is to show the existence of weak solutions to the system consisting of balance of mass, momentum and thermal energy, see (7.7)–(7.12). Since we do not assume here that M satisfies the Δ_2-condition, such a formulation allows us to capture shear thickening fluids, even very rapidly thickening, e.g. exponential growth. The proof is based on the construction of a proper approximation and showing that the approximation converges to a weak solution, i.e. showing sequential stability.

In order to show the convergence in a nonlinear viscous term we apply monotonicity methods adapted to the case of anisotropic Musielak–Orlicz spaces developed in [326, 328, 183], see also [263] and our presentation in Section 4.1.2. Also here we deal with the lack of a classical integration by parts formula and in the present section we recall the formula obtained in [328] by adaptation of arguments from [183] and [158], see also Theorem 4.2.10 in Chapter 4. Let us emphasize that we assume here that M^* satisfies the Δ_2-condition, therefore $L_{M^*} = E_{M^*}$, which facilities the analysis since it allows us to use the weak-$*$ convergence of the approximation for symmetric gradients of solutions.

One of the essential difficulties we have to face here in order show the weak sequential stability concerns the right-hand side of the energy equation, which reads as follows

$$\partial_t(\varrho\theta) + \operatorname{div}(\varrho\mathbf{u}\theta) - \operatorname{div}\mathbf{q}(\varrho,\theta,\nabla\theta) = \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : \mathbf{Du},$$

where θ denotes the absolute temperature and $\mathbf{q}$ is a heat flux function. Setting $\mathbf{S}^n := \mathbf{S}(\cdot,\varrho^n,\theta^n,\mathbf{Du}^n)$ we need to show that $\mathbf{S}^n : \mathbf{Du}^n \rightharpoonup \mathbf{S}(\cdot,\varrho,\theta,\mathbf{Du}) : \mathbf{Du}$ weakly in $L^1(\Omega_T)$, where $\{\varrho^n\}_{n=1}^\infty$, $\{\mathbf{u}^n\}_{n=1}^\infty$, $\{\theta^n\}_{n=1}^\infty$ are approximation sequences of ϱ, $\mathbf{u}$, θ and $\{\mathbf{S}^n\}_{n=1}^\infty \subset L_{M^*}(\Omega_T)$, $\{\mathbf{Du}^n\}_{n=1}^\infty \subset L_M(\Omega_T)$. Note that when working with reflexive spaces (such as L^p) the monotonicity is a sufficient argument to conclude from $(\mathbf{S}^n - \mathbf{S}) : (\mathbf{Du}^n - \mathbf{Du}) \rightharpoonup 0$ in L^1 that $\mathbf{S}^n : \mathbf{Du}^n \rightharpoonup \mathbf{S} : \mathbf{Du}$ weakly in L^1. However, once the space is not reflexive, as is the case for our L_M-space, then the convergence may fail if one is not able to provide modular convergence of sequences $\mathbf{S}^n$ and $\mathbf{Du}^n$ in proper spaces.

For the current problem we use the Chacon Biting Lemma (Theorem 8.38) and Young measures (Theorem 8.41) to show that the product of our two sequences converges weakly in L^1 and consequently to provide the sequential stability of the right-hand side of the energy equation. Similar arguments in the framework of anisotropic Musielak–Orlicz spaces were used in [188] for parabolic equations and later also in [218] for the thermo-visco-elasticity model and in Section 5.3 (Step 7). See also the works of Chlebicka et al. recalled in Chapter 5.

The results and methodology of Section 7.2 are mainly based on [249] and the references therein, see also [180, 183, 326, 328, 329].

In Section 7.3 we consider incompressible non-Newtonian fluids described by a generalized Stokes system (see (7.134)–(7.135)). In (7.2) we skip the convective term $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$. Moreover, the N-function M and its conjugate in condition (7.3) are homogeneous (do not depend on x and t) but still anisotropic. In order to show weak sequential stability, when the convective term is present, the common assumption providing compactness in this term in the case of power-law fluids satisfying (7.4) is that $p \geq \frac{3N+2}{N+2}$. Also in Section 7.2 we assume that the growth of M is at least as $|\cdot|^p$ with $p \geq \frac{11}{5}$, since we consider the problem in three-dimensional space. In the case of a generalized Stokes system we are allowed to skip these assumptions and moreover we do not assume that M or M^* needs to satisfy the Δ_2-condition. With such conditions on $\mathbf{S}$, namely with the lack of the lower bound on M of polynomial type, we can capture a wide class of models and it opens the possibility of including flows with different behavior, in particular shear thinning fluids.

Here we are particularly interested in a rheology close to linear in at least one direction. Note that $\mathbf{S}$ can be of the form given by (7.1) with $\eta_\infty = 0$ and $m = 1$. However for the case of non-star-shaped domains we need to assume some conditions on the upper growth of M, but this does not conflict with the goal of describing a close to linear rheology.

To motivate the simplified model without the convective term let us mention two situations. If the flow is assumed to be slow, then the term $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$ can be assumed to be very small and therefore neglected, hence the whole system (7.2) reduces to a generalized Stokes system. Another situation is the case of simple flows, namely a Poisseuille type flow, between two fixed parallel plates, which is driven by a constant pressure gradient (e.g. see [205]). Also when regarding blood flow (shear thinning) it seems to be important to consider simple flows, as the geometry of vessels can be simplified to a flow in a pipe. An analysis of both models in the steady case (also without convective term) through the variational approach was undertaken by Fuchs and Seregin in [163, 164].

Let us observe that as soon as $M(x,\cdot) \geq c|\cdot|^p$ with $p > \frac{11}{5}$ for a.a. $x \in \Omega \subset \mathbb{R}^3$, solutions are bounded in an appropriate Sobolev space $W^{1,p}(\Omega;\mathbb{R}^3)$ which is compactly embedded in $L^2(\Omega;\mathbb{R}^3)$. If $M^* \in \Delta_2$ we gain that $L_{M^*}(\Omega_T;\mathbb{R}^{N\times N}_{\rm sym}) = E_{M^*}(\Omega_T;\mathbb{R}^{N\times N}_{\rm sym})$ is a separable space. The above two assumptions are used in Section 7.2. The naturally arising question which we answer in Section 7.3 is whether the existence of solutions can still be proved after relaxing the assumptions on M and M^*. The preliminary studies in this direction were done for an abstract parabolic equation, cf. [182]. Also the convergence of a full discretization of a quasilinear parabolic equation can be found in [134]. In Section 7.3 we give a proof based on showing that weak-$*$ and modular limits for symmetric gradients coincide.

In the above considerations help comes from the generalization of the Korn–Sobolev inequality, stated for homogeneous and isotropic N-function m:

$$\|m(|\mathbf{u}|)\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C_N \|m(|\mathbf{Du}|)\|_{L^1(\Omega)},$$

see Theorem 9.4. In general one of the most important tools in the existence theory for problems in fluid mechanics is a Korn type inequality, which allows us to provide an estimate of the gradient via a symmetric gradient in appropriate norms. There are numerous classical results such as Poincaré, Sobolev, Korn inequalities which have been generalized from Lebesgue and Sobolev spaces to Orlicz and Musielak–Orlicz spaces. Let us recall here results of Cianchi on the Sobolev inequality, see [91, 92] and the results concerning the embedding of a particular type of Orlicz–Sobolev space, namely $BLD(\Omega) := \{\mathbf{u} \in L^1(\Omega;\mathbb{R}^N) \ : \ |\mathbf{Du}| \in L_m(\Omega)\}$ where $L_m(\Omega)$ is defined by the function $m(\xi) = \xi \ln(\xi+1), \xi \in \mathbb{R}_+$, given in [162].

The generalization of the Korn inequality, namely

$$\int_\Omega m(|\nabla \mathbf{u}|)\,\mathrm{d}x \leq c \int_\Omega m(|\mathbf{Du}|)\,\mathrm{d}x,$$

is valid for the case of homogeneous and isotropic N-functions m and its conjugate m^* satisfying the Δ_2-condition, see e.g. [161]. In [94] the author exhibits balance conditions between used N-functions for a Korn type inequality to hold. However, this is not the case for our considerations in Section 7.3 and therefore we need to generalize Strauss' result, cf. [300], that

$$\|\mathbf{u}\|_{L^{\frac{N}{N-1}}(\Omega)} \leq c\|\mathbf{Du}\|_{L^1(\Omega)} \quad \text{with some } c > 0$$

to the case of appropriate homogeneous and isotropic N-functions.

In Section 7.4 we provide the decomposition and local estimates for the pressure function. This part is stated in anisotropic Musielak–Orlicz spaces. Then we show how this method can be used to investigate the existence of weak solutions for the non-stationary flow of incompressible non-Newtonian fluids. However this particular result is stated in the isotropic and homogeneous Orlicz space setting where the governing N-function does not have to satisfy the Δ_2-condition. Therefore we can cover the case of shear thickening fluids. The considerations of Sections 7.4 are based on [327, 329].

The problem of existence and construction of a pressure function arises in the theory of non-stationary incompressible non-Newtonian fluids. Since in the analysis of the mechanics of incompressible fluids one mainly considers divergence-free test functions in the definition of a weak solution, the pressure usually does not appear, see the definitions of weak solutions in Section 7.2 and 7.3. However the pressure function π can be identified a posteriori in some cases, see e.g. [158]. For the power law fluids it can be shown that there exists a π of the form

$$\pi = \pi_{\text{reg}} + \partial_t \pi_{\text{harm}},$$

where some $\pi_{\text{reg}} \in L^q(I; L^q(\Omega))$ for q, $1 < q < \infty$ and $\pi_{\text{harm}} \in L^2(I; L^2(\Omega))$, where I stands for the time interval and $\Omega \subset \mathbb{R}^3$ is a spatial domain occupied by the fluid. However, as the time derivative $\partial_t \pi_{\text{harm}}$ is present here, we do not know if π is an integrable function on the time space cylinder, or even if it is a measurable function in $(t,x) \in \Omega_T$. Furthermore, if the pressure is introduced by the De Rham theorem (see Theorem 8.46), we still do not know what the best function space is where the pressure function exists. In Section 7.4 we extend the method of local pressure to the case of Musielak–Orlicz spaces. The concept of local pressure was introduced by J. Wolf in [325] in order to obtain an existence result for the non-stationary motion of a non-Newtonian fluid with shear rate dependent viscosity of a power-law type where no restriction on shape or size of the spatial domain was an issue. The local pressure estimates are based on variational methods. Here and in [150, 325] the pressure is decomposed into a measurable function π_{reg} and the singular part $\partial_t \pi_{\text{harm}}$, where π_{harm} is harmonic with respect to a space variable. In [325] the author provides optimal a priori estimates for the components π_{reg} and π_{harm}, which are achieved mainly by L^q-estimates for weak solutions to the Laplace equation. Later in [150] the authors employed different methods to derive estimates for the pressure components π_{reg} and π_{harm}. Their construction is based on the Riesz transform, which seems to be more suitable for application to problems associated with fluids of a power-law type. Such methodology allows the regular part π_{reg} to share the same regularity properties (integrability) as the nonlinear viscous part in the momentum equation for the velocity field of the fluid (in the case of power-law fluids).

Our construction of the local pressure is based on the Riesz transform as in [150], but we state the problem in the more general setting of Musielak–Orlicz spaces. Note that the Riesz transform in general cannot be well defined as an operator from one Orlicz space to the same one. If M and M^* do not satisfy the Δ_2-condition it can turn out that it is continuous from one Orlicz space to another larger one. Moreover, in general $L_M(0,T; L_M(\Omega)) \neq L_M((0,T) \times \Omega)$. Consequently we are not able to show, for the time being, that π_{reg} is in the same space as the viscous term (with its generality), but possibly in a weaker/larger space, see Theorem 7.4.1. Therefore the method of local pressure seems to be more delicate than in the classical L^p-setting.

In order to investigate the existence of weak solutions to the motion of one or several rigid bodies in a non-Newtonian fluid with the above nonstandard (not necessarily polynomial) rheology we use the concept of weak solutions based on the Eulerian reference system and on a class of test functions which depend on the position of the rigid bodies. This idea was introduced in [212], see also [110, 111, 166, 167, 202, 289, 291]. In order to prove the existence of weak solutions to the problem one needs to construct a proper approximation based on penalization/replacement of rigid bodies by a fluid of very high viscosity proportional to $\frac{1}{\varepsilon}$, $\varepsilon \to 0$. Then the monotonicity argument has to be localized to the 'fluid' part of the time-space cylinder. We cannot test the momentum equation by functions with non-zero support on regions which contain rigid bodies, since we can control neither the penalizing term $\mu_\varepsilon \mathbf{S}(\mathbf{Du}_\varepsilon)$ nor $\mu_\varepsilon \mathbf{Du}_\varepsilon$. At this stage of our investigation, the problem has to be localized in the fluid part separately from the rigid bodies. This requires the investigation of the pressure function locally in the fluid part of the time space cylinder Ω_T. If the tensor $\mathbf{S}$ satisfies the same conditions as (7.3) with a homogeneous and isotropic N-function m which does not satisfy the Δ_2-condition, then the regularity of π_{reg} can be lower than the regularity of the viscous term, which in fact makes the problem even more delicate for power-law fluids.

7.2 Heat-Conducting Non-Newtonian Fluids

In this section we show the existence of weak solutions for unsteady flow of non-Newtonian, incompressible, heterogeneous (here this means that density is not assumed to be constant), heat-conducting fluids with generalized form of the stress tensor without restriction on its upper growth of polynomial type or without assuming the Δ_2-condition for the governing N-function. Let us emphasize that we do not assume any smallness condition on the initial data in order to obtain long-time existence. As in the previous chapter, monotonicity methods, integration by parts adapted to nonreflexive spaces and Young measure techniques are crucial to the proof. This section is based on [180, 183, 249, 328].

7.2.1 A few words about notation

Let us recall that Ω stands for a bounded domain in $\mathbb{R}^N$, $(0,T)$ is a time interval and $\Omega_T := (0,T)\times\Omega$.

Let us introduce some functions spaces which will be used within this chapter. Let $\mathcal{V}$ be the set of all smooth compactly supported functions on Ω which are divergence-free

$$\mathcal{V}(\Omega) := \{\varphi \in C_c^\infty(\Omega) \,:\, \operatorname{div}\varphi = 0\}$$

and the related spaces:

$$\begin{aligned} L^2_{\text{div}}(\Omega) &:= \text{ the closure of } \mathcal{V} \text{ with respect to the } \|\cdot\|_{L^2}\text{-norm} \\ W^{1,p}_{0,\text{div}}(\Omega) &:= \text{ the closure of } \mathcal{V} \text{ with respect to the } \|\nabla(\cdot)\|_{L^p}\text{-norm.} \end{aligned} \tag{7.6}$$

Let $W^{-1,p'}(\Omega;\mathbb{R}^N) = (W^{1,p}_0(\Omega;\mathbb{R}^N))^*$, $W^{-1,p'}_{\text{div}}(\Omega;\mathbb{R}^N) = (W^{1,p}_{0,\text{div}}(\Omega;\mathbb{R}^N))^*$. By p' we mean the conjugate exponent to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finally, we recall that the Nikolskii space $N^{\alpha,p}(0,T;X)$ corresponding to the Banach space X and the exponents $\alpha \in (0,1)$ and $p \in [1,\infty]$ is given by

$$N^{\alpha,p}(0,T;X) := \{f \in L^p(0,T;X) \ : \ \sup_{0<h<T} h^{-\alpha} \|\tau_h f - f\|_{L^p(0,T-h;X)} < \infty\},$$

where $\tau_h f(t) = f(t+h)$ for a.a. $t \in [0,T-h]$.

7.2.2 *Existence of weak solutions. Formulation of the problem*

The mathematical model of the flow of an incompressible, heterogeneous (density dependent), non-Newtonian, heat-conducting fluid can be described in terms of the mass density of the fluid $\varrho : \Omega_T \to \mathbb{R}$, the velocity field $\mathbf{u} : \Omega_T \to \mathbb{R}^3$, and the absolute temperature $\theta : \Omega_T \to \mathbb{R}$. The motion can be governed by the following system of equations consisting of balance of mass (continuity equation)

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in} \quad \Omega_T, \tag{7.7}$$

balance of momentum (momentum equation)

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) + \nabla \pi = \varrho \mathbf{f} \quad \text{in} \quad \Omega_T, \tag{7.8}$$

and balance of thermal energy (thermal energy equation)

$$\partial_t(\varrho\theta) + \operatorname{div}(\varrho \mathbf{u}\theta) - \operatorname{div} \mathbf{q}(\varrho,\theta,\nabla\theta) = \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : \mathbf{Du} \quad \text{in} \quad \Omega_T. \tag{7.9}$$

The last equation is in fact a balance of internal energy. One can find a discussion about the possible choices of the last equation in [64], see also [149]. Since we are considering incompressible fluids we set

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega_T. \tag{7.10}$$

We supplement the above system with initial data

$$\varrho(0,x) = \varrho_0(x), \quad \mathbf{u}(0,x) = \mathbf{u}_0, \quad \theta(0,x) = \theta_0(x) \quad \text{in} \quad \Omega, \tag{7.11}$$

and with a zero Dirichlet boundary condition for the velocity field, and no-heat flux through the boundary

$$\mathbf{u}(t,x) = 0, \quad \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on} \quad [0,T] \times \partial\Omega. \tag{7.12}$$

In the above $\pi : \Omega_T \to \mathbb{R}$ is a pressure function, $\mathbf{S}$ – a stress tensor, $\mathbf{q}$ – a thermal flux vector, and $\mathbf{f} : \Omega_T \to \mathbb{R}^3$ – a given outer force. The set $\Omega \subset \mathbb{R}^3$ is a bounded domain with a regular boundary $\partial\Omega$ (of class, say $C^{2,\nu}$, $\nu > 0$, taken for convenience). We consider the above system on the time-space cylinder $\Omega_T = (0,T) \times \Omega$ where $T \in (0,+\infty)$ is given. The tensor $\mathbf{Du} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ stands for a symmetric part of the velocity gradient.

For the above system we assume that the initial density ϱ and temperature θ satisfy

$$\varrho(0,\cdot)=\varrho_0\in L^\infty(\Omega) \quad \text{and} \quad 0<\varrho_*\le\varrho_0(x)\le\varrho^*<+\infty \quad \text{for a.a. } x\in\Omega, \tag{7.13}$$

$$\theta_0\in L^1(\Omega) \quad \text{and} \quad 0<\theta_*\le\theta_0(x) \quad \text{for a.a. } x\in\Omega, \tag{7.14}$$

where $\varrho_*,\ \varrho^*,\ \theta_*$ are constants.

We formulate the growth conditions of the stress tensor with the help of an anisotropic and inhomogeneous N-function $M:\Omega\times\mathbb{R}^{3\times3}\to[0,\infty)$. We assume here that the stress tensor $\mathbf{S}:\Omega\times[0,\infty)\times[0,\infty)\times\mathbb{R}^{3\times3}_{\text{sym}}\to\mathbb{R}^{3\times3}_{\text{sym}}$ satisfies:

(S1h) $\mathbf{S}(x,\varrho,\theta,\mathbf{K})$ is a Carathéodory function (i.e., measurable function of x for all $\varrho,\theta>0$ and $\mathbf{K}\in\mathbb{R}^{3\times3}_{\text{sym}}$ and continuous function of θ,ϱ and $\mathbf{K}$ for a.a. $x\in\Omega$) and $\mathbf{S}(x,\varrho,\theta,\mathbf{0})=\mathbf{0}$.

(S2h) There exist a positive constant $c_c\in(0,1)$, an N-function M and its conjugate M^*, $M,\ M^*:\Omega\times\mathbb{R}^{3\times3}\to[0,\infty)$ such that for all $\mathbf{K}\in\mathbb{R}^{3\times3}_{\text{sym}}$, $\theta,\varrho>0$ and a.a. $t,\ x\in\Omega_T$ the following growth and coercivity condition

$$\mathbf{S}(x,\varrho,\theta,\mathbf{K}):\mathbf{K}\ge c_c\big(M(x,\mathbf{K})+M^*(x,\mathbf{S}(x,\varrho,\theta,\mathbf{K}))\big) \tag{7.15}$$

holds.

(S3h) $\mathbf{S}$ is monotone, that is,

$$\big(\mathbf{S}(x,\varrho,\theta,\mathbf{K}_1)-\mathbf{S}(x,\varrho,\theta,\mathbf{K}_2)\big):\big(\mathbf{K}_1-\mathbf{K}_2\big)\ge0,$$

for all $\mathbf{K}_1,\mathbf{K}_2\in\mathbb{R}^{3\times3}_{\text{sym}}$, $\varrho>0$, $\theta>0$ and a.a. $x\in\Omega$.

Let us emphasize that the stress tensor $\mathbf{S}$ may depend here not only on a shear stress but also on a fluid density and a fluid temperature.

The heat flux $\mathbf{q}$ in our model takes a quite common form. Let us remark that we do not concentrate here on choosing the most optimal form for the heat flux. Similarly as in [158], we expect that $\mathbf{q}(\varrho,\theta,\nabla\theta)$ behaves as

$$\kappa(\varrho)\theta^\beta\nabla\theta=\kappa(\varrho)\tfrac{1}{\beta+1}\nabla\theta^{\beta+1} \quad \text{for } \beta\in\mathbb{R}$$

such that $\kappa(\varrho)$ satisfies

$$0\le\kappa_*\le\kappa(\varrho)\le\kappa^*<\infty,$$

where κ_*,κ^* are some fixed constants. In particular, we require that $\mathbf{q}:[0,\infty)\times[0,\infty)\times\mathbb{R}^3\to\mathbb{R}^3$ satisfies

$$\mathbf{q}(\varrho,\theta,\nabla\theta)=\kappa_0(\varrho,\theta)\nabla\theta \quad \text{with } \kappa_0\in C([0,\infty)\times[0,\infty)) \tag{7.16}$$

and for all $\theta,\varrho>0,\ \nabla\theta\in\mathbb{R}^3$

$$\begin{aligned}\mathbf{q}(\varrho,\theta,\nabla\theta)\cdot\nabla\theta&\ge\kappa_*\theta^\beta|\nabla\theta|^2 \quad \text{with } \beta\in\mathbb{R} \text{ and } \kappa_*>0,\\ |\mathbf{q}(\varrho,\theta,\nabla\theta)|&\le\kappa^*\theta^\beta|\nabla\theta| \quad \text{with} \quad \kappa^*>0.\end{aligned} \tag{7.17}$$

Let us start with a definition of a weak solution to the system (7.7)–(7.12).

Definition 7.2.1. Let Ω be bounded domain in $\mathbb{R}^3$, let $(0,T)$ be finite time interval, and let $p > 1$. Let ϱ_0 satisfy (7.13), $\mathbf{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$, θ_0 satisfy (7.14) and $\mathbf{f} \in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$. We call a triple $(\varrho, \mathbf{u}, \theta)$ a *weak solution* to (7.7)–(7.12) if:

- the continuity equation is satisfied in a weak sense, namely

$$\int_0^T \langle \partial_t \varrho, z\rangle \,\mathrm{d}t - \int_0^T \int_\Omega \varrho \mathbf{u} \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{7.18}$$

  for all $z \in L^r(0,T;W^{1,r}(\Omega))$ with $r = 5p/(5p-3)$, i.e.

$$\int_{s_1}^{s_2} \int_\Omega (\varrho \partial_t z + (\varrho \mathbf{u}) \cdot \nabla z) \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (\varrho z(s_2) - \varrho z(s_1)) \,\mathrm{d}x \tag{7.19}$$

  for all z smooth and $s_1, s_2 \in [0,T]$, $s_1 < s_2$.
- the momentum equation is satisfied in a weak sense, namely

$$\begin{aligned}\int_0^T \int_\Omega (-\varrho \mathbf{u} \cdot \partial_t \varphi - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi + \mathbf{S}(x, \varrho, \theta, \mathbf{D}\mathbf{u}) : \mathbf{D}\varphi) \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_\Omega \varrho \mathbf{f} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0) \,\mathrm{d}x\end{aligned} \tag{7.20}$$

  for all $\varphi \in C_c^\infty((-\infty,T);\mathcal{V})$.
- the thermal energy equation is satisfied in a weak sense, namely

$$\begin{aligned}\int_0^T \langle \partial_t(\varrho\theta), h\rangle \,\mathrm{d}t + \int_0^T \int_\Omega (-\varrho\theta \mathbf{u} \cdot \nabla h + \mathbf{q}(\varrho, \theta, \nabla\theta) \cdot \nabla h) \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_\Omega (\mathbf{S}(x, \varrho, \theta, \mathbf{D}\mathbf{u}) : \mathbf{D}\mathbf{u})\, h \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{7.21}$$

  for all $h = L^\infty(0,T;W^{1,q}(\Omega))$ with q sufficiently large, where the duality paring above is between $(W^{1,q}(\Omega))^*$ and $W^{1,q}(\Omega)$.

Theorem 7.2.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,\nu}$ boundary, where $\nu \in (0,1)$ and let $(0,T)$ be finite time interval. Let $M : \Omega \times \mathbb{R}^{3\times3} \to [0,\infty)$ be an N-function satisfying for some $\underline{c} > 0$, $\widetilde{C} \geq 0$ and for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$*

$$M(x,\boldsymbol{\xi}) \geq \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C} \quad \text{with } p \geq \tfrac{11}{5}. \tag{7.22}$$

Let us assume that the conjugate to M function

$$M^* \text{ satisfies the } \Delta_2\text{-condition.} \tag{7.23}$$

Let $\mathbf{S}$ satisfy conditions (S1h)–(S3h). Moreover let $\mathbf{q}$ satisfy (7.16), (7.17) *with*

$$\beta > -\min\left\{\tfrac{2}{3}, \tfrac{3p-5}{3p-3}\right\}.$$

Let $\mathbf{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$ *and* $\varrho_0 \in L^\infty(\Omega)$ *with* $0 < \varrho_* \le \varrho_0(x) \le \varrho^* < +\infty$ *for a.a.* $x \in \Omega$. *Let* $\theta_0 \in L^1(\Omega)$, $0 < \theta_* \le \theta_0$ *for a.a.* $x \in \Omega$ *and let* $\mathbf{f} \in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$. *Then there exists a weak solution to* (7.7)–(7.12) *in the sense of Definition 7.2.1.*

Moreover

- $0 < \varrho_* \le \varrho(t,x) \le \varrho^*$ *for a.a.* $(t,x) \in \Omega_T$,
- $\varrho \in C([0,T];L^q(\Omega))$ *for arbitrary* $q \in [1,\infty)$,
- $\partial_t \varrho \in L^{5p/3}(0,T;(W^{1,5p/(5p-3)}(\Omega))^*)$,
- $\mathbf{u} \in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)) \cap L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)) \cap N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$,
- $\mathbf{Du} \in L_M(\Omega_T;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *and* $(\varrho\mathbf{u},\psi) \in C([0,T])$ *for all* $\psi \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$,
- $\theta \in L^\infty(0,T;L^1(\Omega))$ *and* $\theta \ge \theta_* > 0$ *for a.a.* $(t,x) \in \Omega_T$,
- $\theta^{\frac{\beta-\lambda+1}{2}} \in L^2(0,T;W^{1,2}(\Omega))$ *for all* $\lambda \in (0,1)$,
- $\mathbf{q} \in L^m(0,T;L^m(\Omega))$ *for* $m \in \left[1, \frac{3\beta+5}{3\beta+4}\right)$,
- $\partial_t(\varrho\theta) \in L^1(0,T;(W^{1,q}(\Omega))^*)$ *with q sufficiently large.*

Moreover, the initial conditions are achieved in the following way

$$\begin{aligned} &\lim_{t\to 0^+}\left(\|\varrho(t)-\varrho_0\|_{L^q(\Omega)} + \|\mathbf{u}(t)-\mathbf{u}_0\|^2_{L^2(\Omega)}\right) = 0 \quad \text{for some } q \in [1,\infty),\\ &\lim_{t\to 0^+}\int_\Omega \varrho\theta(t)h\,\mathrm{d}x = \int_\Omega \varrho_0\theta_0 h\,\mathrm{d}x \quad \text{for all } h \in L^\infty(\Omega). \end{aligned} \tag{7.24}$$

Here we have restricted ourselves to a flow in a domain of space dimension $N = 3$, just for the brevity of the presentation. The existence result can be extended to the case of arbitrary $N \ge 2$ and $p \ge \frac{3N+2}{N+2}$.

Let us remark that the assumption (7.22) on the exponent $p \ge \frac{11}{5}$ restricts our consideration to the case of shear thickening fluids. Since in our approach we use as a test function an approximation of the solution in the space where we a priori expect the solution will be, in order to proceed with the convergence in the convective term the restriction (7.22) is crucial. If one is able to use a method based on Lipschitz truncation, we expect this could be relaxed to the condition $p > \frac{6}{5}$ for dimension $N = 3$, see [116].

7.2.3 The proof of existence of weak solutions

The proof of Theorem 7.2.2 is provided in steps.

Step 1. The n**-approximate problem.**

In order to prove Theorem 7.2.2 we start by constructing n-approximate solutions. Let

$$\begin{gathered} \{\omega_i\}_{i=1}^\infty \text{ be an orthonormal basis of } W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)\\ \text{such that } \{\omega_i\}_{i=1}^\infty \subset W^{1,2p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3), \end{gathered} \tag{7.25}$$

where elements of the basis are constructed with the help of eigenfunctions of the problem

$$((\omega_i, \varphi))_s = \lambda_i \int_\Omega \omega_i \cdot \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in V_s,$$

where $((\cdot,\cdot))_s$ denotes the scalar product in V_s defined by

$$V_s := \text{ the closure of } \mathcal{V} \text{ with respect to the } W^{s,2}(\Omega)\text{-norm for } s > 3. \tag{7.26}$$

The existence of the above basis is provided by Lemma 8.55. By the Sobolev embedding Theorem 8.47 we have that

$$W^{s-1,2}(\Omega) \subset L^\infty(\Omega). \tag{7.27}$$

Then we consider the n-approximate velocity $\mathbf{u}^n \in C([0,T]; W^{1,2p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))$ of the following form

$$\mathbf{u}^n := \sum_{i=1}^n \alpha_i^n(t)\omega^i \qquad \text{for } i = 1,2,\dots, \tag{7.28}$$

where $\alpha_i^n \in C([0,T])$. The condition $\operatorname{div}\mathbf{u}^n = 0$ is fulfilled automatically, since $\mathbf{u}^n$ is a linear combination of divergence-free functions. The n-approximate solution, namely the triple $(\varrho^n, \mathbf{u}^n, \theta^n)$, satisfies

$$\int_0^T \langle \partial_t \varrho^n, z\rangle \, \mathrm{d}t - \int_0^T \int_\Omega \varrho^n \mathbf{u}^n \cdot \nabla z \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{7.29}$$

for all $z \in L^r(0,T; W^{1,r}(\Omega))$ with $r = 5p/(5p-3)$, and

$$0 < \varrho_* \le \varrho^n(t,x) \le \varrho^* < +\infty \quad \text{for a.a. } (t,x) \in \Omega_T, \tag{7.30}$$

$$\theta^n \in L^\infty(0,T;L^2(\Omega)) \cup L^s(0,T;W^{1,s}(\Omega)) \qquad \text{with } s = \min\left\{2, \frac{5\beta+10}{\beta+5}\right\},$$
$$\text{and } \theta^n \ge \theta_* \text{ in } \Omega_T, \tag{7.31}$$

$$\begin{aligned} \langle \partial_t(\varrho^n\mathbf{u}^n), \omega_i\rangle + \int_\Omega (-\varrho^n \mathbf{u}^n \otimes \mathbf{u}^n : \nabla\omega_i + \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{D}\mathbf{u}^n) : \mathbf{D}\omega_i) \, \mathrm{d}x \\ = \int_\Omega \varrho^n \mathbf{f}^n \cdot \omega_i \, \mathrm{d}x \end{aligned} \tag{7.32}$$

for all $i = 1,\dots,n$ and a.a. $t \in [0,T]$, and

$$\begin{aligned} \int_0^T \langle \partial_t(\theta^n\varrho^n), h\rangle \, \mathrm{d}t + \int_0^T \int_\Omega (-\theta^n \varrho^n \mathbf{u}^n \cdot \nabla h + \kappa_0(\varrho^n\theta^n)\nabla\theta^n \cdot \nabla h) \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{D}\mathbf{u}^n) : \mathbf{D}\mathbf{u}^n h \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{7.33}$$

for all $h \in L^\infty(0,T;W^{1,q}(\Omega))$ for large enough q. For the initial data we set

$$\varrho^n(0,\cdot) = \varrho_0, \quad \mathbf{u}^n(0,\cdot) = P^n\mathbf{u}_0, \quad \theta^n(0,\cdot) = \theta_0^n, \tag{7.34}$$

where P^n denotes the orthogonal projection of $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$ onto the linear hull of $\{\omega_i\}_{i=1}^n$ and

$$P^n \mathbf{u}_0 \to \mathbf{u}_0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3) \quad \text{as } n \to \infty, \tag{7.35}$$

and θ_0^n is a smooth regularization of θ_0 such that

$$\theta_0^n \to \theta_0 \quad \text{strongly in } L^1(\Omega).$$

Moreover, in (7.32) $\{\mathbf{f}^n\}_n$ stands for a standard smooth regularization of $\mathbf{f}$ (or regular enough approximation to provide the existence of an approximate solution) such that

$$\mathbf{f}^n \to \mathbf{f} \quad \text{strongly in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3)) \quad \text{as } n \to \infty. \tag{7.36}$$

Let us remark that we can understand (7.29) in the following sense: for $z \in C^\infty([0,T]\times\Omega)$

$$\int_{t_1}^{t_2} (\varrho^n \partial_t z + \varrho^n \mathbf{u}^n \cdot \nabla z)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varrho^n z(t_2)\,\mathrm{d}x - \int_\Omega \varrho^n z(t_1)\,\mathrm{d}x \tag{7.37}$$

for a.a. $t_1,\ t_2$ such that $0 \le t_1 \le t_2 \le T$. In a similar way we can rewrite (7.32).

The existence of a triple $(\varrho^n, \mathbf{u}^n, \theta^n)$ being a solution to (7.28)–(7.33) with initial data (7.34) can be proved by a two-step approximation: regularization of the continuity equation and a finite-dimensional approximation of the temperature function. It is quite technical but most of the difficulties are not directly related to growth conditions (7.15) or the Musielak–Orlicz setting, since due to (7.27) for any fixed n both $\mathbf{S}^n$ and $\mathbf{D}\mathbf{u}^n$ are in $L^\infty(\Omega_T;\mathbb{R}^{3\times 3})$. Therefore it is enough to adapt the proof given for the power-law type fluid in [158, Section 6]. For details, see Section 8.3.

Step 2. Uniform estimates for ρ^n and $\mathbf{u}^n$.

Let us denote the sequence of solutions to the n-approximate problem (7.29)–(7.33) by $\{(\varrho^n, \mathbf{u}^n, \theta^n)\}_n$ with $n = 1,2,\dots$. Now we concentrate on providing estimates which are uniform with respect to n which in later steps will allow us to pass to the limit as $n \to \infty$.

Let us multiply (7.32) by α_i^n, take a sum over $i = 1,\dots,n$, and use (7.29) with $z = \frac{1}{2}|\mathbf{u}^n|^2$. This leads us to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho^n |\mathbf{u}^n|^2\,\mathrm{d}x + \int_\Omega \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{D}\mathbf{u}^n) : \mathbf{D}\mathbf{u}^n\,\mathrm{d}x = \int_\Omega \varrho^n \mathbf{f}^n \cdot \mathbf{u}^n\,\mathrm{d}x. \tag{7.38}$$

By the Hölder, the Poincaré, the Korn (Lemma 8.54) and the Young inequalities, assumption (7.22) and inequality (7.30) we can estimate the right-hand side of (7.38) in the following way

$$\left|\int_\Omega \varrho^n \mathbf{f}^n \cdot \mathbf{u}^n\,\mathrm{d}x\right| \le C(\Omega, c_c, \underline{c}, \varrho^*, p)\|\mathbf{f}^n\|^{p'}_{L^{p'}(\Omega)} + \frac{c_c}{2}\int_\Omega M(x,\mathbf{D}\mathbf{u}^n)\,\mathrm{d}x + C(\Omega, c_c, \widetilde{C}), \tag{7.39}$$

where the constant c_c corresponds to (7.15) and $\widetilde{C}$, $\underline{c}$ to (7.22). Next let us integrate (7.38) over the time interval $(0,s_0)$. The use of estimates (7.39), (7.30), the coercivity conditions (S2h) on $\mathbf{S}$, uniform continuity of P^n with respect to n and (7.36) give the uniform with respect to n estimates

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho^n(s_0)|\mathbf{u}^n(s_0)|^2\,\mathrm{d}x\\&+\int_0^{s_0}\int_\Omega\left\{\frac{c_c}{2}M(x,\mathbf{Du}^n)+c_cM^*(x,\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n))\right\}\mathrm{d}x\,\mathrm{d}t\\&\le C(\Omega,c_c,\underline{c},\varrho^*,p,\widetilde{C},\|\mathbf{f}\|_{L^{p'}(0,T;L^{p'}(\Omega))})+\frac{1}{2}\varrho^*\|\mathbf{u}_0\|^2_{L^2(\Omega)}.\end{aligned}\tag{7.40}$$

Here C is a nonnegative constant independent of n, but it depends on the given data.

As (7.40) holds, from the condition (7.22) we infer that $\{\mathbf{Du}^n\}_{n=1}^\infty$ is uniformly bounded in the space $L^p(\Omega_T;\mathbb{R}^{3\times3})$, i.e.

$$\int_0^T \|\mathbf{Du}^n\|^p_{L^p(\Omega)}\,\mathrm{d}t\le C.\tag{7.41}$$

By the Korn inequality (Lemma 8.54) we obtain

$$\int_0^T \|\nabla\mathbf{u}^n\|^p_{L^p(\Omega)}\,\mathrm{d}t\le C.\tag{7.42}$$

From estimate (7.40) it is straightforward to show that

$$\|\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n\|_{L^1(\Omega_T)}\le C,\tag{7.43}$$

$$\|\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n)\|_{L^1(\Omega_T)}\le C.\tag{7.44}$$

Moreover, the sequence $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n)\}_{n=1}^\infty$ is uniformly bounded in the generalized Musielak–Orlicz class $\mathcal{L}_{M^*}(\Omega_T;\mathbb{R}^{3\times3})$.

According to (7.40) and (7.30) we have

$$\sup_{t\in[0,T]}\|\mathbf{u}^n(t)\|^2_{L^2(\Omega)}\le C\quad\text{and}\quad\sup_{t\in[0,T]}\|\varrho^n(t)|\mathbf{u}^n(t)|^2\|_{L^1(\Omega)}\le C,\tag{7.45}$$

where C is a positive constant dependent on the size of data, but independent of n. By (7.42), the zero boundary condition for the velocity field, and the Poincaré inequality we deduce that

$$\|\mathbf{u}^n\|_{L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))}\le C.$$

Due to the classical Sobolev inequality (see Theorem 8.47) also

$$\|\mathbf{u}^n\|_{L^p(0,T;L^{3p/(3-p)}(\Omega;\mathbb{R}^3))}\le C.$$

Then a classical interpolation between spaces $L^\infty(0,T;L^2)$ and $L^p(0,T;L^{3p/(3-p)})$ (see Lemma 8.56) gives

$$\int_0^T \|\mathbf{u}^n\|_{L^r(\Omega)}^r \,\mathrm{d}t \le C \quad \text{for } 1 \le r \le 5p/3. \tag{7.46}$$

Let us remark here that the above particular argument deals with the case $p < 3$. The case $p \ge 3$ can be treated more easily, due to L^∞ embedding (see Theorem 8.47). Hence by the L^∞ bound on density (7.30) and by (7.46) we infer

$$\int_0^T \|\varrho^n \mathbf{u}^n\|_{L^{5p/3}(\Omega)}^{5p/3} \,\mathrm{d}t \le C. \tag{7.47}$$

Using again (7.30) and bounds on velocity field (7.42) and (7.46), by the Hölder inequality we are led to

$$\begin{aligned}\int_0^T \int_\Omega |\varrho^n \mathbf{u}^n \otimes \mathbf{u}^n : \nabla \mathbf{u}^n| \,\mathrm{d}x\,\mathrm{d}t &\le C\varrho^* \|\mathbf{u}^n\|_{L^{5p/3}(0,T;L^{5p/3}(\Omega))}^2 \|\nabla \mathbf{u}^n\|\|_{L^p(0,T;L^p(\Omega))} \\ &\le C \quad \Longleftrightarrow \quad p \ge \frac{11}{5}.\end{aligned}$$

Here the restriction for the exponent p stated in (7.22) is crucial. Next by (7.47) and (7.30) we obtain from the continuity equation (7.29) that

$$\int_0^T \|\partial_t \varrho^n\|_{(W^{1,5p/(5p-3)}(\Omega))^*}^{5p/3} \,\mathrm{d}t \le C. \tag{7.48}$$

Step 3. Uniform estimates for $\mathbf{u}^n$ in Nikolskii space.

Now let us show that the sequence $\{\mathbf{u}^n\}_{n=1}^\infty$ is uniformly bounded with respect to n in the Nikolskii space $N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$, that is,

$$\sup_{0<\delta<T} \delta^{-\frac{1}{2}} \left(\int_0^{T-\delta} \|\mathbf{u}^n(s+\delta) - \mathbf{u}^n(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \right)^{\frac{1}{2}} < C. \tag{7.49}$$

The proof of this fact is based on an argument from [15, Chapter 3, Lemma 1.2] with modifications concerning a change of L^2-structure to L^p-structure and due to the nonlinear term controlled by the coercivity condition (7.15).

Let us fix δ and s, $0 < \delta < T$, $0 \le s \le T - \delta$. Next we test the momentum equation (7.32) at time t by $\mathbf{u}^n(s+\delta) - \mathbf{u}^n(s)$ and integrate the equation over the time interval $(s, s+\delta)$ with respect to time t. Applying the integration by parts formula with respect to time, the continuity equality (7.29) and the following obvious identity

$$\begin{aligned}&\varrho^n(s+\delta)\mathbf{u}^n(s+\delta) - \varrho(s)\mathbf{u}^n(s) \\ &\qquad = \varrho^n(s+\delta)[\mathbf{u}^n(s+\delta) - \mathbf{u}^n(s)] + [\varrho^n(s+\delta) - \varrho^n(s)]\mathbf{u}^n(s)\end{aligned}$$

we get

$$
\begin{aligned}
&\int_\Omega \varrho^n(s+\delta)|\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)|^2\,\mathrm{d}x\\
&+\int_\Omega [\varrho^n(s+\delta)-\varrho^n(s)]\mathbf{u}^n(s)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\\
&+\int_s^{s+\delta}\int_\Omega \operatorname{div}(\varrho^n(t)\mathbf{u}^n(t))\mathbf{u}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_s^{s+\delta}\int_\Omega \varrho^n(t)[\nabla\mathbf{u}^n(t)]\mathbf{u}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_s^{s+\delta}\int_\Omega \mathbf{S}(x,\varrho^n(t),\theta^n(t),\mathbf{D}\mathbf{u}^n(t)):\mathbf{D}[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{f}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{7.50}
$$

Now, let us test the continuity equation (7.29) at time t by $\mathbf{u}^n(s)\cdot(\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s))$ and integrate the equation over the time interval $(s,s+\delta)$ with respect to t to find that

$$
\begin{aligned}
&\int_\Omega [\varrho^n(s+\delta)-\varrho^n(s)]\mathbf{u}^n(s)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\\
&=-\int_s^{s+\delta}\int_\Omega \operatorname{div}(\varrho^n(t)\mathbf{u}^n(t))\mathbf{u}^n(s)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using the above relation in (7.50), since

$$
\begin{aligned}
&\int_\Omega \operatorname{div}(\varrho^n\mathbf{u}^n)\mathbf{u}^n(s)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\\
&\quad=-\int_\Omega \varrho^n(t)[\nabla\mathbf{u}^n(s)]\mathbf{u}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\\
&\qquad-\int_\Omega \varrho^n(t)\mathbf{u}^n(s)\otimes\mathbf{u}^n(t)\cdot\nabla[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x,
\end{aligned}
\tag{7.51}
$$

and by (7.30) we have that

$$
\begin{aligned}
&\|\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}x\\
&\le\frac{1}{\varrho_*}\Big\{\Big|-\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{u}^n(s)\otimes\mathbf{u}^n(t)\cdot\nabla[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{u}^n(t)\otimes\mathbf{u}^n(t)\cdot\nabla[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_s^{s+\delta}\int_\Omega \varrho^n(t)[\nabla\mathbf{u}^n(s)]\mathbf{u}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_s^{s+\delta}\int_\Omega \mathbf{S}(x,\varrho^n(t),\theta^n(t),\mathbf{D}\mathbf{u}^n(t)):\mathbf{D}[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{f}^n(t)\cdot[\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)]\,\mathrm{d}x\,\mathrm{d}t\Big|\Big\}=:\frac{1}{\varrho_*}\left|\sum_{k=1}^{10}I_k\right|.
\end{aligned}
\tag{7.52}
$$

Next we integrate (7.52) over $(0,T-\delta)$ with respect to time s. Our aim now is to show that for any of the ten addends $I_k(s)$, $k=1,2,\dots,10$, on the right-hand side of (7.52), the following estimates hold true

$$\int_0^{T-\delta} I_k(s)\,\mathrm{d}s \le \lambda_k\delta \quad \text{for } k=1,2,\dots,10, \tag{7.53}$$

where the constants λ_k are independent of δ and n. In order to deal with the first six integrals let us set, for the moment, $q:=5p/3$. Using the L^∞-bound on density (7.30), the Hölder inequality, and the Fenchel–Young (Lemma 2.1.32) we infer for one of representative terms that

$$\begin{aligned}
&\left|\int_0^{T-\delta} I_1(s)\,\mathrm{d}s\right| = \left|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{u}^n(s)\otimes\mathbf{u}^n(t)\cdot\nabla\mathbf{u}^n(s+\delta)\,\mathrm{d}x\,\mathrm{d}t\mathrm{d}s\right|\\
\le&\varrho^*\int_0^{T-\delta}\int_s^{s+\delta} \|\mathbf{u}^n(s)\|_{L^q(\Omega)}\|\mathbf{u}^n(t)\|_{L^q(\Omega)}\|\nabla\mathbf{u}^n(s+\delta)\|_{L^p(\Omega)}\,\mathrm{d}t\mathrm{d}s\\
\le&\delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\mathbf{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\left|\frac{1}{\delta}\int_s^{s+\delta}\|\mathbf{u}^n(t)\|_{L^q(\Omega)}\,\mathrm{d}t\right|^q\right.\\
&\left.+\frac{1}{p}\|\nabla\mathbf{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}\mathrm{d}s =: J.
\end{aligned}$$

Then Jensen's inequality and the following obvious relation

$$\int_0^{T-\delta}\frac{1}{\delta}\int_s^{s+\delta} a(t)\,\mathrm{d}t\,\mathrm{d}s \le \int_0^T a(s)\,\mathrm{d}s \qquad \text{for } a(t)\ge 0$$

gives us that

$$\begin{aligned}
J \le&\delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\mathbf{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\frac{1}{\delta}\int_s^{s+\delta}\|\mathbf{u}^n(t)\|^q_{L^q(\Omega)}\,\mathrm{d}t\right.\\
&\left.+\frac{1}{p}\|\nabla\mathbf{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}\mathrm{d}s\\
\le&\delta\varrho^*\left(\frac{1}{q}\|\mathbf{u}^n(s)\|^q_{L^q(0,T;L^q(\Omega))}+\frac{1}{q}\|\mathbf{u}^n(s)\|^q_{L^q(0,T;L^q(\Omega))}+\frac{1}{p}\|\nabla\mathbf{u}^n\|^p_{L^p(0,T:L^p(\Omega))}\right).
\end{aligned}$$

In order to estimate the right-hand side of the above with $q=5p/3$ we use estimates (7.42) and (7.46). Then we find that

$$\left|\int_0^{T-\delta} I_1(s)\,\mathrm{d}s\right| \le \lambda_1\delta,$$

where λ_1 is independent of δ and n. In a similar way we treat I_k with $k=2,\dots,6$.

Now we concentrate on the nonlinear viscous term. By Fubini's theorem, the Fenchel–Young inequality (see Lemma 2.1.32) and Jensen's inequality (see Corollary 2.1.24) we get the following estimates

$$\begin{aligned}
&\left|\int_0^{T-\delta} I_7(s)\,\mathrm{d}s\right| \\
=&\left|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \mathbf{S}(x,\varrho^n(t),\theta^n(t),\mathbf{Du}^n(t)) : \mathbf{Du}^n(s+\delta)\,\mathrm{d}x\,\mathrm{d}t\mathrm{d}s\right| \\
=&\delta\int_0^{T-\delta}\int_\Omega\left|\frac{1}{\delta}\int_s^{s+\delta}\mathbf{S}(x,\varrho^n(t),\theta^n(t),\mathbf{Du}^n(t))\,\mathrm{d}t\cdot\mathbf{Du}^n(s+\delta)\right|\,\mathrm{d}x\,\mathrm{d}s \\
\le&\delta\int_0^{T-\delta}\int_\Omega\Big\{M^*\left(x,\frac{1}{\delta}\int_s^{s+\delta}\mathbf{S}(t,x,\varrho^n(t),\mathbf{Du}^n(t))\,\mathrm{d}t\right) \\
&+M\left(x,\mathbf{Du}^n(s+\delta)\right)\Big\}\,\mathrm{d}x\mathrm{d}s \\
\le&\delta\int_\Omega\int_0^{T-\delta}\Big\{\frac{1}{\delta}\int_s^{s+\delta}M^*\left(x,\mathbf{S}(x,\varrho^n(t),\theta^n(t),\mathbf{Du}^n(t))\right)\,\mathrm{d}t \\
&+M\left(x,\mathbf{Du}^n(s+\delta)\right)\Big\}\,\mathrm{d}s\,\mathrm{d}x \\
\le&\int_\Omega\left\{\int_0^T M^*\left(x,\mathbf{S}(x,\varrho^n(s),\theta^n(s),\mathbf{Du}^n(s))\right)\,\mathrm{d}s+\int_0^{T-\delta}M\left(x,\mathbf{Du}^n(s+\delta)\right)\,\mathrm{d}s\right\}\,\mathrm{d}x \\
\le&\lambda_7\delta,
\end{aligned}$$

where λ_7 is uniform with respect to n. For the last inequality (7.40) is applied. In a similar way we treat I_8.

Due to (7.36) and (7.46) we obtain

$$\begin{aligned}
\left|\int_0^{T-\delta} I_9(s)\,\mathrm{d}s\right| &= \left|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \varrho^n(t)\mathbf{f}^n(t)\cdot\mathbf{u}^n(s+\delta)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\right| \\
&\le \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{p'}\left|\frac{1}{\delta}\int_s^{s+\delta}\|\mathbf{f}^n(t)\|_{L^{p'}(\Omega)}\,\mathrm{d}t\right|^{p'}+\|\mathbf{u}^n(s+\delta)\|_{L^p(\Omega)}\right\}\,\mathrm{d}s \\
&\le \delta\varrho^*\left(\frac{1}{p'}\|\mathbf{f}^n\|^{p'}_{L^{p'}(0,T;L^{p'}(\Omega))}+\frac{1}{p}\|\mathbf{u}^n\|^p_{L^p(0,T;L^p(\Omega))}\right)\le\lambda_9\delta.
\end{aligned}$$

In the same way we proceed with I_{10}. Summarizing all of the above estimates we infer that (7.53) holds. As we already know that $\{\mathbf{u}^n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$ (see (7.45)), we get that

$$\int_0^{T-\delta}\|\mathbf{u}^n(s+\delta)-\mathbf{u}^n(s)\|^2_{L^2(\Omega)}ds\le\lambda\delta, \tag{7.54}$$

where λ is independent of n and δ. Consequently $\{\mathbf{u}^n\}_{n=1}^\infty$ is uniformly bounded in Nikolskii space $N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$. In particular, (7.49) holds true for all $n\in\mathbb{N}$ and

$$\mathbf{u}\in N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)).$$

Step 4. Uniform estimates for θ^n.

Now let us concentrate on the energy equation and estimates on the temperature function. We notice that taking $h=1$ in (7.33), by the Fenchel–Young inequality (see

Lemma 2.1.32), (7.40) and (7.30) we obtain the following

$$\sup_{t\in[0,T]} ||\varrho^n\theta^n||_{L^1(\Omega)} \le C \quad \text{and} \quad \sup_{t\in[0,T]} ||\theta^n||_{L^1(\Omega)} \le C. \tag{7.55}$$

Taking $h = -(\theta^n)^{-\lambda}$ with $\lambda \in (0,1)$ in (7.33), as $\theta^n \ge \theta_*$ (see (8.63)), we have that

$$||(\theta^n)^{-\lambda}||_{L^\infty(\Omega_T)} \le C.$$

Therefore we find that

$$\int_0^T ||(\theta^n)^{\frac{\beta-\lambda-1}{2}}\nabla\theta^n||^2_{L^2(\Omega)}\,\mathrm{d}t = C_1\int_0^T ||\nabla[(\theta^n)^{\frac{\beta-\lambda+1}{2}}]||^2_{L^2(\Omega)}\,\mathrm{d}t \le C_2, \tag{7.56}$$

which allows us to infer by (7.56) that

$$\begin{aligned}
&\int_0^T ||(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \\
&\quad\le \int_0^T ||(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T ||\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad\le \int_0^T ||(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{L^2(\Omega)}\,\mathrm{d}t + C_2 \\
&\quad\le C_3\Big[\int_0^T ||(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{L^1(\Omega)}\,\mathrm{d}t + \int_0^T ||\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}}||^2_{L^2(\Omega)}\,\mathrm{d}t\Big] + C_2 \le C_4.
\end{aligned} \tag{7.57}$$

Since $W^{1,2}(\Omega) \subset L^6(\Omega)$, we obtain

$$\int_0^T ||(\theta^n)||^{\beta-\lambda+1}_{L^{3(\beta-\lambda+1)}(\Omega)}\,\mathrm{d}t \le C. \tag{7.58}$$

By the interpolation argument, (7.55), and (7.58) we conclude that (here we use the restriction that $\beta > -\frac{2}{3}$)

$$\int_0^T ||(\theta^n)||^s_{L^s(\Omega)}\,\mathrm{d}t \le C \quad \text{for all } s \in \Big[1, \tfrac{5}{3}+\beta\Big). \tag{7.59}$$

According to the assumption on heat flux (7.17) we have

$$\begin{aligned}
\int_{\Omega_T} |\mathbf{q}^n|^m\,\mathrm{d}x\,\mathrm{d}t &\le \int_{\Omega_T} \left|\kappa^*(\theta^n)^\beta|\nabla\theta^n|\right|^m\,\mathrm{d}x\,\mathrm{d}t \\
&\le c(\kappa^*,\lambda,\beta,m)\int_{\Omega_T} |\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}}|^m(\theta^n)^{\beta m-\frac{\beta-\lambda-1}{2}m}\,\mathrm{d}x\,\mathrm{d}t \\
&\le c\left(\int_{\Omega_T} |\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{m}{2}}\left((\theta^n)^{\frac{m(\beta+\lambda+1)}{2-m}}\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{m}{2}}.
\end{aligned} \tag{7.60}$$

By estimates (7.57) and (7.59) we find that

$$\int_{\Omega_T} |\kappa_0 \nabla\theta^n|^m \, \mathrm{d}x\,\mathrm{d}t \le C \quad \text{for all} \quad m \in \left[1, \frac{5+3\beta}{4+3\beta}\right). \tag{7.61}$$

Notice that $\mathbf{q}^n \in L^m$ if and only if $\frac{m(\beta+\lambda+1)}{2-m} < \frac{5}{3}+\beta$, which gives the restriction on m in Theorem 7.2.2.

Finally we aim to estimate the last term in the energy equation. Due to the Hölder inequality, the Sobolev embedding, the interpolation argument, and the above considerations, in particular by (7.55), (7.59), (7.56), we infer that

$$\begin{aligned}
&\int_0^T \|\theta^n \varrho^n \mathbf{u}^n\|^\gamma_{L^\gamma(\Omega)} \,\mathrm{d}t \le \varrho^* \int_0^T \|\mathbf{u}^n\|^\gamma_{L^{\frac{3p}{3-p}}(\Omega)} \|\theta^n\|^\gamma_{L^{\frac{3p\gamma}{(3+\gamma)p-3\gamma}}(\Omega)} \,\mathrm{d}t \\
&\le C_1 \int_0^T \|\mathbf{u}^n\|^\gamma_{W^{1,p}(\Omega)} \|\theta^n\|^{(1-\alpha)\gamma}_{L^1(\Omega)} \|\theta^n\|^{\alpha\gamma}_{L^{3(\beta-\lambda+1)}(\Omega)} \,\mathrm{d}t \\
&\underset{(7.55),(7.59)}{\le} C_1 \int_0^T \|\mathbf{u}^n\|^\gamma_{W^{1,p}(\Omega)} \|\theta^n\|^{\alpha\gamma}_{L^{3(\beta-\lambda+1)}(\Omega)} \,\mathrm{d}t \\
&\le C_1 \left[\int_0^T \|\theta^n\|^{\beta-\lambda+1}_{L^{3(\beta-\lambda+1)}(\Omega)} \,\mathrm{d}t\right]^{\frac{\gamma\alpha}{\beta-\lambda+1}} \left[\int_0^T \|\mathbf{u}^n\|^{\frac{(\beta-\lambda+1)\gamma}{\beta-\lambda+1-\alpha\gamma}}_{W^{1,p}(\Omega)} \,\mathrm{d}t\right]^{\frac{\beta-\lambda+1-\alpha\gamma}{\beta-\lambda+1}} \underset{(7.56)}{\le} C_2.
\end{aligned}$$

In the above the parameter $\alpha \in [0,1]$ is chosen such that

$$\frac{(3+\gamma)p-3\gamma}{3p\gamma} = \frac{1-\alpha}{1} + \frac{\alpha}{3(\beta-\lambda+1)}. \tag{7.62}$$

Notice also that the last inequality in (7.63) gives constraints combining values of β, α, λ, p and γ, i.e.

$$\frac{(\beta-\lambda+1)\gamma}{\beta-\lambda+1-\alpha\gamma} = p.$$

Using formula (7.62) we claim that $\gamma > 1$ if $\beta > -\frac{3p-5}{3p-3}$, which is the restriction required in Theorem 7.2.2. Summarizing we obtain that for $p < 3$ and appropriate β there exists a $\gamma > 1$ such that

$$\|\varrho^n \mathbf{u}^n \theta^n\|_{L^1(0,T;L^\gamma(\Omega))} < C. \tag{7.63}$$

Let us remark that the above holds also for $p \ge 3$ due to embedding results for $W^{1,p}$, see Theorem 8.47. In this case $\gamma > 1$ if $\beta > -\frac{2}{3}$.

Finally, by balance of thermal energy we find that

$$\|\partial_t(\varrho^n\theta^n)\|_{L^1(0,T;(W^{1,s}(\Omega))^*)} < C \quad \text{for } s \text{ sufficiently large.} \tag{7.64}$$

More precisely

$$\|\partial_t(\varrho^n\theta^n)\|_{L^1(0,T;(W^{1,s}(\Omega))^*)} = \sup_{\|h\|_{W^{1,s}(\Omega)} \le 1} |\langle \partial_t(\varrho^n\theta^n)(\tau), h\rangle| \le g^n(\tau), \tag{7.65}$$

where $\|g^n(\tau)\|_{L^1(0,T)} \le C < \infty$. Recall the thermal energy equation (7.33) which holds for a.a. $\tau \in (0,T)$ and estimates on terms appearing in (7.33), namely (7.43), (7.61), (7.63). Then integration of (7.65) over $(0,T)$ with respect to τ allows us to infer that (7.64) holds.

Step 5. Weak convergence of $(\varrho^n, \mathbf{u}^n, \theta^n)$ as $n \to \infty$.

The uniform estimates obtained in the above section together with the Banach–Alaoglu theorem, see Theorem 8.31, provide the existence of subsequences selected from $\{\varrho^n\}_{n=1}^\infty$, $\{\mathbf{u}^n\}_{n=1}^\infty$, $\{\theta^n\}_{n=1}^\infty$ such that for $n \to \infty$

$$\varrho^n \rightharpoonup \varrho \quad \text{weakly in } L^q(\Omega_T) \text{ for any } q \in [1,\infty) \text{ and weakly-}* \text{ in } L^\infty(\Omega_T), \tag{7.66}$$

$$\partial_t \varrho^n \rightharpoonup \partial_t \varrho \quad \text{weakly in } L^{5p/3}(0,T;(W^{1,5p/(5p-3)}(\Omega))^*), \tag{7.67}$$

$$\begin{aligned} \mathbf{u}^n \rightharpoonup \mathbf{u} \quad & \text{weakly in } L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)) \text{ and } L^{5p/3}(\Omega_T;\mathbb{R}^3) \\ & \text{and weakly-}* \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)), \end{aligned} \tag{7.68}$$

$$\theta^n \rightharpoonup \theta \quad \text{weakly in } L^q(\Omega_T) \text{ for any } q \in [1,5/3+\beta). \tag{7.69}$$

Moreover, due to (7.47), (7.57), there exist $\overline{\varrho\mathbf{u}} \in L^{5p/3}(\Omega_T;\mathbb{R}^3)$ and also $\overline{\theta^\alpha} \in L^2(0,T;W^{1,2}(\Omega))$ such that

$$\varrho^n\mathbf{u}^n \rightharpoonup \overline{\varrho\mathbf{u}} \quad \text{weakly in } L^{5p/3}(\Omega_T;\mathbb{R}^3), \tag{7.70}$$

$$(\theta^n)^\alpha \rightharpoonup \overline{(\theta)^\alpha} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \text{ for } \alpha \in (0,(\beta+1)/2). \tag{7.71}$$

Let us clarify here that when in the above we have an overlined $\bar{\cdot}$ object, we mean that there exists a limit of a proper subsequence. The other issue is to identify it and to be able to 'erase' the bar. Additionally, as E_M and E_{M^*} are separable spaces (see Theorem 3.4.14) and $(E_M)^* = L_{M^*}$, $(E_{M^*})^* = L_M$ (see Theorem 3.5.3 and Theorem 2.1.41), the following holds

$$\mathbf{Du}^n \overset{*}{\rightharpoonup} \mathbf{Du} \quad \text{weakly-}* \text{ in } L_M(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}}), \tag{7.72}$$

$$\mathbf{S}(\cdot,\varrho^n,\theta^n,\mathbf{Du}^n) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \quad \text{weakly-}* \text{ in } L_{M^*}(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}}), \tag{7.73}$$

where

$$\overline{\mathbf{S}} \in \mathcal{L}_{M^*}(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \text{ and } \mathbf{Du} \in \mathcal{L}_M(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \tag{7.74}$$

which results from the convexity of M, M^*. Due to Theorem 3.4.2 we conclude the uniform integrability of $\{\mathbf{S}(\cdot,\varrho^n,\theta^n,\mathbf{Du}^n)\}_{n=1}^\infty$ in L^1. Thus by the Dunford–Pettis theorem (Theorem 8.21 there exists a tensor $\overline{\mathbf{S}} \in L^1(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ such that

$$\mathbf{S}(\cdot,\varrho^n,\theta^n,\mathbf{Du}^n) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^1(\Omega_T;\mathbb{R}^{3\times3}). \tag{7.75}$$

Step 6. Strong convergence of $(\varrho^n, \mathbf{u}^n, \theta^n)$ as $n \to \infty$.

Our aim now is to prove the strong convergence of the triple $(\varrho^n, \mathbf{u}^n, \theta^n)$ using the Aubins–Lions arguments (Theorem 8.50) and the Div-Curl lemma (Lemma 8.52).

Let us start with the strong convergence of the velocity field. By (7.68), (7.54), and due to an Aubin–Lions type argument (Theorem 8.49) we find that

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } L^2(\Omega_T;\mathbb{R}^3).$$

Then by (7.46) and an interpolation argument we also get that

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } L^q(\Omega_T;\mathbb{R}^3) \text{ with } q \in \left[1, \tfrac{5p}{3}\right). \tag{7.76}$$

Using the L^∞-bound on density (7.30) and (7.48), (7.67), together with the Aubin–Lions argument (Theorem 8.51) we obtain that

$$\varrho^n \to \varrho \quad \text{strongly in } C([0,T];(W^{1,5p/(5p-3)}(\Omega))^*).$$

The standard concept of the renormalized solutions of DiPerna and Lions for continuity equations (see Proposition 8.63) leads to

$$\varrho^n \to \varrho \text{ strongly in } C([0,T];L^q(\Omega)) \text{ for all } q \in [1,\infty) \text{ and a.e. in } \Omega_T. \tag{7.77}$$

Notice that (7.77) provides the continuity of density to initial condition stated as the very first part of (7.24), namely that

$$\lim_{t\to 0} \|\varrho(t) - \varrho_0\|_{L^q(\Omega)} = 0 \quad \text{for all } q \in [1,\infty).$$

Our next aim is to show that

$$\varrho^n\mathbf{u}^n \rightharpoonup \varrho\mathbf{u} \quad \text{weakly in } L^q(\Omega_T;\mathbb{R}^3) \text{ for all } q \in \left[1, \tfrac{5p}{3}\right]. \tag{7.78}$$

Indeed, the above strong convergence (7.77) implies that

$$\varrho^n \to \varrho \text{ strongly in } L^{\frac{5p}{3}+\gamma}(0,T;L^{\frac{5p}{3}+\gamma}(\Omega;\mathbb{R}^3)),$$

where $\gamma \in [0,\infty)$. This together with convergence of the velocity field (7.68) implies that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \varrho^n\mathbf{u}^n \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \varrho\mathbf{u} \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for every $\varphi \in (L^{\frac{5p}{3}+\varepsilon}(0,T;L^{\frac{5p}{3}+\varepsilon}(\Omega;\mathbb{R}^3)))^*$, where $\varepsilon(\gamma) \in [0, \frac{5p}{3})$. Therefore (7.70) implies that (7.78) holds.

Now employing strong and weak convergence of the velocity sequence (7.76) and (7.68), the L^∞-bound on the density sequence, and strong convergence of the density sequence (7.77) we obtain

$$\varrho^n\mathbf{u}^n \otimes \mathbf{u}^n \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^\gamma(0,T;L^\gamma(\Omega;\mathbb{R}^{3\times 3}))$$

for γ sufficiently large, i.e. $\frac{1}{q} + \frac{6}{5p} < \frac{1}{\gamma}$ with arbitrary q. A density argument together with (7.68) ensures that for $p \geq \frac{11}{5}$ we have

$$\varrho^n\mathbf{u}^n \otimes \mathbf{u}^n \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^{3\times 3})). \tag{7.79}$$

Now we are going to show the convergence of $\{\varrho^n\theta^n\mathbf{u}^n\}_{n=1}^\infty$ using the Div-Curl lemma (Lemma 8.52). For this reason we set

$$\tilde{\mathbf{a}}^n = (\varrho^n\theta^n, \varrho^n\theta^n u_1^n + \kappa_0\partial_{x_1}\theta^n, \varrho^n\theta^n u_2^n + \kappa_0\partial_{x_2}\theta^n, \varrho^n\theta^n u_3^n + \kappa_0\partial_{x_3}\theta^n),$$

where $\mathbf{u}^n = (u_1^n, u_2^n, u_3^n)$ and

$$\tilde{\mathbf{b}}^n = ((\theta^n)^\alpha, 0, 0, 0) \quad \text{with } \alpha \in (0, (\beta+1)/2),$$

here α is rather small. By bounds (7.61), (7.63), (7.31) and convergence (7.77) we infer that

$$\tilde{\mathbf{a}}^n \rightharpoonup (\overline{\varrho^n\theta^n}, \overline{\varrho^n\theta^n u_1^n} + \overline{\kappa_0\nabla\theta^n}, \overline{\varrho^n\theta^n u_2^n} + \overline{\kappa_0\nabla\theta^n}, \overline{\varrho^n\theta^n u_3^n} + \overline{\kappa_0\nabla\theta^n}) \text{ in } L^s(\Omega_T)$$

for some $s > 1$ close to 1 and

$$\tilde{\mathbf{b}}^n \rightharpoonup (\overline{\theta^\alpha}, 0, 0, 0) \quad \text{weakly in } L^r(\Omega_T) \text{ for } r \text{ such that } \frac{1}{s} + \frac{1}{r} < 1$$

(which is possible for small α and due to condition (7.69)). The energy equation provides that

$$\mathrm{Div}_{t,x}\tilde{\mathbf{a}}^n = \partial_t(\varrho^n\theta^n) + \mathrm{div}\,(\varrho^n\theta^n\mathbf{u}^n + \kappa_0\nabla\theta^n) = \mathbf{S}(x, \varrho^n, \theta^n, \mathbf{Du}^n) : \mathbf{Du}^n.$$

Since (7.43) holds, we find that $\mathbf{S}(x, \varrho^n, \theta^n, \mathbf{Du}^n) : \mathbf{Du}^n \in L^1(\Omega_T) \subset\subset W^{-1,\hat{r}}(\Omega_T)$, where $\hat{r} \in (1, 4/3)$. On the other hand, by estimates (7.57) we infer that

$$\mathrm{Curl}_{t,x}\tilde{\mathbf{b}}^n = \begin{pmatrix} 0 & \nabla(\theta^n)^\alpha \\ -(\nabla(\theta^n)^\alpha)^T & \mathbf{O} \end{pmatrix} \in L^2(\Omega_T; \mathbb{R}^{4\times 4}) \subset\subset W^{-1,2}(\Omega_T; \mathbb{R}^{4\times 4}).$$

Then according to Lemma 8.52 we finally get that

$$\varrho^n(\theta^n)^{\alpha+1} \rightharpoonup \varrho\theta\overline{\theta^\alpha} \quad \text{weakly in } L^{1+\eta}(\Omega_T) \text{ for some } \eta > 0,$$

where η is chosen such that $\frac{1}{s} + \frac{1}{r} = \frac{1}{\eta+1} < 1$. Then by the simple manipulation $\varrho(\theta^n)^{\alpha+1} = ((\varrho - \varrho^n) + \varrho^n)(\theta^n)^{\alpha+1}$ the above combined with the estimate (7.59) on $\{\theta^n\}_n$ and strong convergence (7.77) imply that

$$\varrho(\theta^n)^{\alpha+1} \rightharpoonup \varrho\theta\overline{\theta^\alpha} \quad \text{weakly in } L^{1+\zeta}(\Omega_T) \text{ for some } \zeta > 0 \tag{7.80}$$

(here α is such that $\alpha + 1 < \frac{5}{3} + \beta$). Now our aim is to show that

$$\overline{\theta^\alpha} = \theta^\alpha \quad \text{a.e. in } \Omega_T. \tag{7.81}$$

To this end we employ the classical Browder and Minty trick (see e.g. [138]). Indeed, noticing that y^α for $y \in [0, \infty)$, $\alpha > 0$, is an increasing function we find that

$$0 \le \int_0^T \int_\Omega \varrho[(\theta^n)^\alpha - h^\alpha](\theta^n - h)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } h \in L^{1+\eta}(\Omega_T).$$

Next let us pass to the limit as $n \longrightarrow \infty$. By limits (7.69), (7.77), and (7.80) we obtain

$$0 \leq \int_0^T \int_\Omega \varrho[\overline{\theta^\alpha} - h^\alpha](\theta - h)\, \mathrm{d}x\, \mathrm{d}t \qquad \text{for all } h \in L^{1+\eta}(\Omega_T).$$

Setting $h = \theta - \lambda v$ for $\lambda > 0, v \in L^{1+\eta}(\Omega_T)$ and $h = \theta + \lambda v$, then passing to the limit as $\lambda \longrightarrow 0$ we conclude that

$$0 = \int_0^T \int_\Omega \varrho[\overline{\theta^\alpha} - \theta^\alpha] v\, \mathrm{d}x\, \mathrm{d}t \qquad \text{for all } v \in L^{1+\eta}(\Omega_T).$$

Therefore as $\varrho > \varrho_*$ we deduce that (7.81) holds. Hence by (7.80)

$$\varrho^{\frac{1}{1+\alpha}} \theta^n \rightharpoonup \varrho^{\frac{1}{1+\alpha}} \theta \qquad \text{weakly in } L^{1+\alpha}(\Omega_T)$$

and

$$\|\varrho^{\frac{1}{1+\alpha}} \theta^n\|_{L^{1+\alpha}(\Omega_T)} \longrightarrow \|\varrho^{\frac{1}{1+\alpha}} \theta\|_{L^{1+\alpha}(\Omega_T)}.$$

Therefore for a subsequence

$$\varrho^{\frac{1}{1+\alpha}} \theta^n \longrightarrow \varrho^{\frac{1}{1+\alpha}} \theta \qquad \text{strongly in } L^{1+\alpha}(\Omega_T).$$

Since ϱ is bounded from above and below, see (7.30), $\theta^n > \theta_*$ by (7.31), the weak limit (7.69) for the approximate temperature sequence together with the above strong convergence leads to

$$\theta^n \longrightarrow \theta \qquad \text{strongly in } L^q \text{ for all } q \in [1, 5/3 + \beta) \text{ and a.e. in } \Omega_T. \tag{7.82}$$

The above strong limit together with (7.71) imply that

$$(\theta^n)^\alpha \rightharpoonup \theta^\alpha \qquad \text{weakly in } L^2(0,T; W^{1,2}(\Omega)) \text{ for all } \alpha \in (0, (\beta+1)/2). \tag{7.83}$$

Due to the strong convergence of the approximate density sequence (7.77) and (7.82) we obtain that

$$\varrho^n \theta^n \longrightarrow \varrho\theta \qquad \text{strongly in } L^q(\Omega_T) \text{ for all } q \in [1, 5/3 + \beta). \tag{7.84}$$

By strong convergence of the approximate velocity sequence (7.76), and by (7.77), (7.82) we conclude that

$$\varrho^n \theta^n \mathbf{u}^n \longrightarrow \varrho\theta\mathbf{u} \qquad \text{strongly in } L^1(\Omega_T; \mathbb{R}^3). \tag{7.85}$$

Next, let us consider the convergence of the sequence $\{\mathbf{q}(\varrho^n, \theta^n, \nabla\theta^n)\}_n$. Due to (7.16) we obtain

$$\mathbf{q}(\varrho^n, \theta^n, \nabla\theta^n) = \kappa_0(\varrho^n, \theta^n)\nabla\theta^n = \frac{2}{\beta - \lambda + 1}(\theta^n)^{\frac{-\beta+\lambda+1}{2}} \kappa_0(\varrho^n, \theta^n)\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}}. \tag{7.86}$$

Inequality (7.59) can be used to ensure

$$\int_0^T \|(\theta^n)^{\frac{-\beta+\lambda+1}{2}}\kappa_0(\varrho^n,\theta^n)\|^{2r}_{L^{2r}(\Omega)}\,\mathrm{d}t \le \int_{\Omega_T}(\theta^n)^{r(\beta+\lambda+1)}\,\mathrm{d}x\,\mathrm{d}t \le C \tag{7.87}$$

for r such that $r(\beta+\lambda+1) = 5/3+\beta-\lambda$. Notice that $r > 1$ for λ small enough. Then by (7.87) we have that

$$\{(\theta^n)^{\frac{-\beta+\lambda+1}{2}}\kappa_0(\varrho^n,\theta^n)\}_n \text{ is uniformly integrable in } L^2(\Omega_T).$$

The almost everywhere convergence of $\{\varrho^n\}_{n=1}^\infty, \{\theta^n\}_{n=1}^\infty$ shown in (7.77) and (7.82) combined with the Vitali convergence theorem (Theorem 8.23) leads to

$$(\theta^n)^{\frac{-\beta+\lambda+1}{2}}\kappa_0(\varrho^n,\theta^n) \to \theta^{\frac{-\beta+\lambda+1}{2}}\kappa_0(\varrho,\theta) \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Moreover, according to (7.83) we obtain that

$$\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}} \rightharpoonup \nabla\theta^{\frac{\beta-\lambda+1}{2}} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$

which applied to (7.86) and by (7.61) give us that

$$\mathbf{q}(\varrho^n,\theta^n,\nabla\theta^n) \rightharpoonup \mathbf{q}(\varrho,\theta,\nabla\theta) \quad \text{weakly in } L^s(\Omega_T;\mathbb{R}^3) \text{ for all } s \in \left(1, \frac{5+3\beta}{4+3\beta}\right). \tag{7.88}$$

Step 7. Passing to the limit as $n \to \infty$ in the continuity and momentum equation.

Summarizing the arguments of the previous steps we are allowed to pass to the limit as $n \to \infty$ in the system (7.29)–(7.32).

With the limits (7.67), (7.77), (7.78) at hand we pass to the limit as $n \to \infty$ in (7.29) and we get that

$$\int_0^T \langle \partial_t\varrho, z\rangle\,\mathrm{d}t - \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{7.89}$$

for all $z \in L^r(0,T;W^{1,r}(\Omega))$ with $r = 5p/(5p-3)$.

Let us now collect convergences for all terms from the momentum equation, namely (7.36), (7.75), (7.79), (7.34), (7.35). Multiplying the approximate momentum equation (7.32) by $\psi \in C_c^\infty(-\infty,T)$, integrating the result over $(0,T)$ with respect to time we pass to the limit as $n \to \infty$ obtaining that

$$\begin{aligned}\int_0^T\int_\Omega \Big(-\varrho\mathbf{u}\cdot\partial_t\varphi - \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\varphi + \overline{\mathbf{S}} : \mathbf{D}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \varrho\mathbf{f}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0)\,\mathrm{d}x\end{aligned} \tag{7.90}$$

for all $\varphi \in C_c^\infty((-\infty,T);\mathcal{V})$. Here we used the fact that the family of functions of the form $\psi\omega_i$ where $\psi \in C_c^\infty(-\infty,T)$, ω_i, $i = 1,2,\ldots$, given by (7.25) is dense in

$C_c^\infty((-\infty,T);\mathcal{V})$. Then it remains only to characterize the nonlinear viscous term $\overline{\mathbf{S}}$. That is the aim of the forthcoming steps.

We have now collected most of the ingredients allowing us to pass to the limit in the balance of thermal energy (7.33). It remains only to show the convergence in the right-hand side of (7.33), which will be provided in the penultimate Step 11.

Step 8. Integration by parts.

Let us recall that the classical integration by parts formula does not hold for our considered problem, since Musielak–Orlicz spaces are not in general reflexive and smooth functions are not dense, if the Δ_2-condition is not satisfied. Also, in general there is no equivalence between the Bochner type space $L_M(0,T;L_M(\Omega))$ and $L_M(\Omega_T)$. This problem has already been investigated in previous chapters, see Section 4.2.3. However in order to give a complete treatment of the fluid flow problem, we also present it here, since some steps need to be treated differently.

Our goal now will be to show that if (7.90) is satisfied, then for a.a. s_0 and s such that $0 < s_0 \le s < T$ it holds that

$$\begin{aligned}\frac{1}{2}\int_\Omega \varrho(s,x)|\mathbf{u}(s,x)|^2\,\mathrm{d}x + \int_{s_0}^{s}\int_\Omega \overline{\mathbf{S}} : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{s_0}^{s}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \varrho(s_0,x)|\mathbf{u}(s_0,x)|^2\,\mathrm{d}x.\end{aligned} \tag{7.91}$$

The proof here is based on a proper choice of a test function in (7.90) and makes use of Steklov regularization with respect to the time variable. To this end let us introduce the following notation: for any function g (for which the integrals below make sense) and for $\lambda > 0$

$$\begin{aligned}(\tilde{\sigma}_\lambda^+ * g)(t,x) &:= \frac{1}{\lambda}\int_0^\lambda g(t+\tau,x)\,\mathrm{d}\tau,\\ (\tilde{\sigma}_\lambda^- * g)(t,x) &:= \frac{1}{\lambda}\int_{-\lambda}^0 g(t+\tau,x)\,\mathrm{d}\tau,\end{aligned} \tag{7.92}$$

where $*$ means convolution over the time variable, and let us set

$$\begin{aligned}D^{+\lambda}g &:= \frac{g(t+\lambda,x)-g(t,x)}{\lambda},\\ D^{-\lambda}g &:= \frac{g(t,x)-g(t-\lambda,x)}{\lambda}.\end{aligned}$$

We observe that

$$\partial_t(\tilde{\sigma}_\lambda^+ * g) = D^{+\lambda}g \qquad \text{and} \qquad \partial_t(\tilde{\sigma}_\lambda^- * g) = D^{-\lambda}g. \tag{7.93}$$

Taking $\lambda > 0$ and $0 < s_0 < s < T$ such that $\lambda \le \min\{s_0, T-s\}$ let us multiply the momentum equation (7.32) by

$$\tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \alpha_i^j(t))\, \mathbb{1}_{(s_0,s)}(t)).$$

Next we sum up over $i = 1,\dots,j$, where $j \le n$ and integrate this sum over the time interval $(0,T)$. Noticing that

$$\tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \mathbf{u}^j)\, \mathbb{1}_{(s_0,s)}) = \sum_{i=1}^{j} \tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \alpha_i^j(t))\, \mathbb{1}_{(s_0,s)})\omega^i(x)$$

let us define

$$\mathbf{u}^{\lambda,j} := \tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \mathbf{u}^j)\, \mathbb{1}_{(s_0,s)})$$

with $\lambda \le \min\{s_0, T-s\}$. As

$$\int_0^T \left\langle \partial_t(\varrho^n \mathbf{u}^n), \mathbf{u}^{\lambda,j} \right\rangle \mathrm{d}t = \int_0^T \left\langle \partial_t(\tilde{\sigma}_\lambda^- * (\varrho^n \mathbf{u}^n)), ((\tilde{\sigma}_\lambda^- * \mathbf{u}^j)\mathbb{1}_{(s_0,s)}) \right\rangle \mathrm{d}t,$$

and $j \le n$, we have

$$\begin{aligned} \int_{s_0}^s \left\langle (\partial_t(\tilde{\sigma}_\lambda^- * \varrho^n \mathbf{u}^n)), (\tilde{\sigma}_\lambda^- * \mathbf{u}^j) \right\rangle \mathrm{d}t &= \int_0^T \int_\Omega (\varrho^n \mathbf{u}^n \otimes \mathbf{u}^n) : \nabla \mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_0^T \int_\Omega \mathbf{S}^n : \mathbf{D}\mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho^n \mathbf{f}^n \cdot \mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.94}$$

Let us notice that for fixed λ and j we have that $\mathbf{u}^{\lambda,j} \in L^\infty(\Omega_T;\mathbb{R}^3)$ and $\mathbf{D}\mathbf{u}^{\lambda,j} \in L^\infty(\Omega_T;\mathbb{R}^{3\times3})$. Our aim is now to pass to the limit with n. For the first term on the left-hand side of (7.94) we use the fact that $\tilde{\sigma}_\lambda^- * \mathbf{u}^j$ is locally Lipschitz with respect to the time variable and (7.78) holds. For the terms on the left-hand side we use the weak limit in $L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^{3\times3}))$ for the convective term (7.79), the weak-$*$ convergence in $L_M(\Omega_T;\mathbb{R}^{3\times3})$ for the nonlinear viscous term (7.73), and the strong convergence in $L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ for the forcing term (7.36) with strong convergence for the approximate density sequence (7.77), respectively. Then as $n \to \infty$ we have

$$\begin{aligned} \int_{s_0}^s \left\langle (\partial_t(\tilde{\sigma}_\lambda^- * \varrho \mathbf{u})), (\tilde{\sigma}_\lambda^- * \mathbf{u}^j) \right\rangle \mathrm{d}t &= \int_0^T \int_\Omega (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}^{\lambda,j} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.95}$$

Our aim now is to replace in (7.94) $\mathbf{u}^{\lambda,j}$ by $\mathbf{u}^\lambda$ defined as follows

$$\mathbf{u}^\lambda := \tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \mathbf{u})\, \mathbb{1}_{(s_0,s)})$$

with $0 < \lambda < \min\{s_0, T-s\}$. For this purpose let us define the truncation operator $\mathbf{T}_m : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ such that

$$\mathbf{T}_m(\mathbf{K}) := \begin{cases} \mathbf{K} & \text{if } |\mathbf{K}| \le m, \\ m\frac{\mathbf{K}}{|\mathbf{K}|} & \text{if } |\mathbf{K}| > m. \end{cases}$$

Then observe the following identity

$$\begin{aligned}\int_{s_0}^{s}\left\langle(\partial_t(\tilde{\sigma}_\lambda^- *(\varrho\mathbf{u})),(\tilde{\sigma}_\lambda^- *\mathbf{u}^j)\right\rangle \mathrm{d}t &= \int_0^T\int_\Omega(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega(\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}}) : \mathbf{D}\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega\mathbf{T}_m(\overline{\mathbf{S}}) : \mathbf{D}\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{7.96}$$

Now let us focus on the right-hand side of (7.96) and investigate the first and the last term. Note that

$$\mathbf{u}^{\lambda,j} \rightharpoonup \mathbf{u}^\lambda \quad \text{weakly in } L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)) \text{ as } j\to\infty.$$

Since $p \ge \frac{11}{5}$ and ϱ is bounded, we infer that

$$\int_0^T\int_\Omega(\varrho\mathbf{u}\otimes\mathbf{u})\cdot\nabla\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega(\varrho\mathbf{u}\otimes\mathbf{u})\cdot\nabla\mathbf{u}^{\lambda}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } j\to\infty.$$

As $\mathbf{f}\in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ we treat the source term in the same way. Hence

$$\int_0^T\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}^{\lambda}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } j\to\infty.$$

Now we analyze the second term on the right-hand side of (7.96). Let us fix $k\in\mathbb{N}$. Due to the Fenchel–Young inequality (see Lemma 2.1.32), the convexity of M and as M^* satisfies the Δ_2-condition (see (7.23)) with some nonnegative integrable function $h:\Omega\to[0,\infty)$ (see (2.38)), we infer the following

$$\begin{aligned}\int_0^T\int_\Omega|(\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}}) : \mathbf{D}\mathbf{u}^{\lambda,j}|\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T\int_\Omega M^*(x,2^k(\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}}))\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega M(x,\frac{1}{2^k}\mathbf{D}\mathbf{u}^{\lambda,j})\,\mathrm{d}x\,\mathrm{d}t\\ &\le c_{\Delta_2}^k\int_0^T\int_\Omega M^*(x,\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}})\,\mathrm{d}x\,\mathrm{d}t\\ &+k\int_0^T\int_\Omega h(x)\mathbb{1}_{\{|\overline{\mathbf{S}}(t,x)|>m\}}\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{1}{2^k}\int_0^T\int_\Omega M(x,\mathbf{D}\mathbf{u}^{\lambda,j})\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{7.97}$$

By (7.40) and noticing that reasoning as in the proof of Lemma 3.4.8 holds also for Steklov regularization (7.92), we get that for each $0<\lambda\le\min\{s_0,T-s\}$ it holds that

$$\sup_{\lambda}\sup_{j\in\mathbb{N}}\int_0^T\int_\Omega M(x,\mathbf{Du}^{\lambda,j})\,\mathrm{d}x\,\mathrm{d}t<C,$$

where C is a nonnegative constant independent of j and λ. Consequently we infer that

$$\lim_{k\to\infty}\frac{1}{2^k}\sup_{\lambda}\sup_{j\in\mathbb{N}}\int_0^T\int_\Omega M(x,\mathbf{Du}^{\lambda,j})\,\mathrm{d}x\,\mathrm{d}t=0. \tag{7.98}$$

By the convexity and symmetry of M^*, and since $M^*(x,0)=0$ a.e. in Ω_T, for m large enough (here such that $2|1-(m/|\overline{\mathbf{S}}|)|<1$)) we infer the following

$$\begin{aligned}M^*(x,\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}})&=M^*(x,\overline{\mathbf{S}}-\mathbf{T}_m(\overline{\mathbf{S}}))\\&=M^*(x,0)\mathbb{1}_{\{|\overline{\mathbf{S}}|\le m\}}+M^*\left(x,\overline{\mathbf{S}}\left(1-\tfrac{m}{|\overline{\mathbf{S}}|}\right)\right)\mathbb{1}_{\{|\overline{\mathbf{S}}|>m\}}\\&\le M^*(x,\overline{\mathbf{S}}).\end{aligned}$$

As M^* satisfies the Δ_2-condition and $\overline{\mathbf{S}}\in\mathcal{L}_{M^*}(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}})$, the above inequality provides by the Lebesgue convergence theorem that

$$\int_{\Omega_T}M^*(x,\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}})\,\mathrm{d}x\,\mathrm{d}t\to0\quad\text{ as }m\to\infty.$$

Hence

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{s_0}^s\int_\Omega c^k_{\Delta_2}M^*(x,\mathbf{T}_m(\overline{\mathbf{S}})-\overline{\mathbf{S}})+kh(x)\mathbb{1}_{\{|\overline{\mathbf{S}}(t,x)|>m\}}\,\mathrm{d}x\,\mathrm{d}t=0. \tag{7.99}$$

Therefore we can pass to the limits in the second term on the right-hand side of (7.96) (together with (7.97)) consecutively as $j\to\infty$, $m\to\infty$ and $k\to\infty$.

Concerning the third term on the right-hand side of (7.96) – let us notice that

$$\mathbf{Du}^{\lambda,j}\rightharpoonup\mathbf{Du}^{\lambda}\quad\text{ weakly in }L^p(\Omega_T;\mathbb{R}^{3\times3})\text{ as }j\to\infty$$

Since $\mathbf{T}_m(\overline{\mathbf{S}})\to\overline{\mathbf{S}}$ a.e. in Ω_T as $m\to\infty$ and since $|\mathbf{T}_m(\overline{\mathbf{S}}):\mathbf{Du}^\lambda|\le|\overline{\mathbf{S}}:\mathbf{Du}^\lambda|$, by symmetry of the N-function, from the Fenchel–Young inequality we have an integrable majorant for the sequence $\{\mathbf{T}_m(\overline{\mathbf{S}}):\mathbf{Du}^\lambda\}_m$. Then by the Lebesgue convergence theorem

$$\lim_{m\to\infty}\lim_{j\to\infty}\int_0^T\int_\Omega\mathbf{T}_m(\overline{\mathbf{S}}):\mathbf{Du}^{\lambda,j}\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega\overline{\mathbf{S}}:\mathbf{Du}^\lambda\,\mathrm{d}x\,\mathrm{d}t.$$

Now let us concentrate on the left hand-side term of (7.94). Recall that $\varrho\mathbf{u}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, then $\tilde{\sigma}^-_\lambda*\varrho\mathbf{u}$ is a Lipschitz function with respect to the time variable, and therefore $\partial_t(\tilde{\sigma}^-_\lambda*\varrho\mathbf{u})\in L^\infty(0,T;L^2(\Omega))$. By (7.68) and letting $j\to\infty$ we get

$$\begin{aligned}L_\lambda&:=\int_{s_0}^s\int_\Omega(\partial_t(\tilde{\sigma}^-_\lambda*(\varrho\mathbf{u}))\cdot(\tilde{\sigma}^-_\lambda*\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{s_0}^s\int_\Omega\Big((\varrho\mathbf{u}\otimes\mathbf{u}):\nabla\mathbf{u}^\lambda-\overline{\mathbf{S}}:\mathbf{Du}^\lambda+\varrho\mathbf{f}\cdot\mathbf{u}^\lambda\Big)\,\mathrm{d}x\,\mathrm{d}t=:R_\lambda.\end{aligned} \tag{7.100}$$

Now our aim is to pass to the limit as $\lambda \to 0^+$. We infer from (7.93) that

$$L_\lambda = \int_{s_0}^{s} \int_\Omega (D^{-\lambda}(\varrho \mathbf{u})) \cdot (\tilde{\sigma}_\lambda^- * \mathbf{u}) \,\mathrm{d}x\,\mathrm{d}t. \tag{7.101}$$

Observe that due to (7.93) and the relation (satisfied in a weak sense)

$$D^{-\lambda}\varrho = -\mathrm{div}_x(\tilde{\sigma}_\lambda^- * (\varrho \mathbf{u})),$$

which holds since ϱ and $\mathbf{u}$ solve the continuity equation (7.89) in a weak sense, we have that

$$\begin{aligned} L_\lambda &= \int_{s_0}^{s} \int_\Omega (\varrho D^{-\lambda}\mathbf{u}) \cdot (\tilde{\sigma}_\lambda^- * \mathbf{u}) + ((D^{-\lambda}\varrho)\mathbf{u}(t-\lambda)) \cdot (\tilde{\sigma}_\lambda^- * \mathbf{u}) \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{s_0}^{s} \int_\Omega \varrho \frac{1}{2} \partial_t |\tilde{\sigma}_\lambda^- * \mathbf{u}|^2 + \left(\tilde{\sigma}_\lambda^- * (\varrho \mathbf{u})\right) \cdot \left(\nabla \left(\mathbf{u}(t-\lambda) \cdot (\tilde{\sigma}_\lambda^- * \mathbf{u})\right)\right) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.102}$$

Let us insert $z = \frac{1}{2}|\tilde{\sigma}_\lambda^- * \mathbf{u}|^2$ into the weak formulation of the continuity equation, which gives for all s_0, $s \in [0,T]$, $s_0 < s$, that

$$\int_{s_0}^{s} \int_\Omega (\varrho(\tau) \cdot \partial_t z(\tau) + \varrho(\tau)\mathbf{u}(\tau) \cdot \nabla z(\tau)) \,\mathrm{d}x\mathrm{d}\tau = \int_\Omega \varrho(s) \cdot z(s) - \varrho(s_0) \cdot z(s_0) \,\mathrm{d}x$$

(for all $z \in L^r(0,T;W^{1,r})$ with $r = 5p/(5p-3)$ and $\partial_t z \in L^{1+\delta}(0,T;L^{1+\delta}(\Omega))$). Hence we have

$$\begin{aligned} L_\lambda = &\int_\Omega \varrho(s) \cdot (\frac{1}{2}|\tilde{\sigma}_\lambda^- * \mathbf{u}(s)|^2) \,\mathrm{d}x - \int_\Omega \varrho(s_0) \cdot (\frac{1}{2}|\tilde{\sigma}_\lambda^- * \mathbf{u}(s_0)|^2) \,\mathrm{d}x \\ &- \int_{s_0}^{s} \int_\Omega (\varrho \mathbf{u}) \cdot (\frac{1}{2}\nabla|\tilde{\sigma}_\lambda^- * \mathbf{u}|^2) \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{s_0}^{s} \int_\Omega \left(\tilde{\sigma}_\lambda^- * (\varrho \mathbf{u})\right) \cdot \left(\nabla \left[\mathbf{u}(t-\lambda) \cdot (\tilde{\sigma}_\lambda^- * \mathbf{u})\right]\right) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.103}$$

Let us notice that

$$\begin{aligned} \tilde{\sigma}_\lambda^- * \mathbf{u} \to \mathbf{u} \quad &\text{strongly, locally in time, in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)) \\ &\text{and in } L^{5p/3}(0,T;L^{5p/3}(\Omega;\mathbb{R}^3)), \end{aligned}$$

$$\nabla(\tilde{\sigma}_\lambda^- * \mathbf{u}) \to \nabla \mathbf{u} \quad \text{strongly, locally in time, in } L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times 3})).$$

The same arguments work for translation $\tau_{-\lambda}\mathbf{u} = \mathbf{u}(t-\lambda)$. Then by the Hölder inequality, letting $\lambda \to 0^+$ in the above we get for almost all s_0 and s in $(0,T)$ the following

$$\begin{aligned}\lim_{\lambda\to 0^+} L_\lambda &= \frac{1}{2}\int_\Omega \varrho(s,x)|\mathbf{u}(s,x)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \varrho(s_0,x)|\mathbf{u}(s_0,x)|^2\,\mathrm{d}x\\ &\quad + \int_{s_0}^{s}\int_\Omega (\varrho\mathbf{u})\cdot\left(\frac{1}{2}\nabla|\mathbf{u}|^2\right)\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2}\int_\Omega \varrho(s,x)|\mathbf{u}(s,x)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \varrho(s_0,x)|\mathbf{u}(s_0,x)|^2\,\mathrm{d}x\\ &\quad + \int_{s_0}^{s}\int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.104}$$

Next let us concentrate on the right-hand side of (7.100) and pass to the limit as $\lambda \to 0$. First we analyze the convergence of the convective term

$$\int_{s_0}^{s}\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{u}^\lambda)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to (7.42), the sequence $\{\nabla\mathbf{u}^\lambda\}_\lambda = \{\nabla\left(\tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \mathbf{u})\mathbb{1}_{(s_0,s)})\right)\}_\lambda$ is uniformly bounded with respect to λ in $L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))$. Hence, for a subsequence if needed, we get that

$$\lim_{\lambda\to 0^+}\int_{s_0}^{s}\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{u}^\lambda)\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^{s}\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t. \tag{7.105}$$

As $\mathbf{f} \in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ and as ϱ is bounded by (7.30), in the same way we find that

$$\lim_{\lambda\to 0^+}\int_{s_0}^{s}\int_\Omega (\varrho\mathbf{f})\cdot\mathbf{u}^\lambda\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.106}$$

Let us concentrate now on the term

$$\int_0^T\int_\Omega \overline{\mathbf{S}} : (\tilde{\sigma}_\lambda^+ * ((\tilde{\sigma}_\lambda^- * \mathbf{Du})\mathbb{1}_{(s_0,s)}))\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^{s}\int_\Omega (\tilde{\sigma}_\lambda^- * \overline{\mathbf{S}}) : (\tilde{\sigma}_\lambda^- * \mathbf{Du})\,\mathrm{d}x\,\mathrm{d}t.$$

The sequences

$$\{\tilde{\sigma}_\lambda^- * \overline{\mathbf{S}}\}_\lambda \text{ and } \{\tilde{\sigma}_\lambda^- * \mathbf{Du}\}_\lambda \text{ converge in measure on } \Omega_T \tag{7.107}$$

due to Lemma 3.4.8, which holds also for Steklov regularization. Hence arguments similar to Lemma 3.4.8 with (7.74) imply that the sequences $\{\tilde{\sigma}_\lambda^- * \overline{\mathbf{S}}\}_\lambda$ and $\{\tilde{\sigma}_\lambda^- * \mathbf{Du}\}_\lambda$ are uniformly integrable, which together with (7.107) give

$$\begin{aligned}\tilde{\sigma}_\lambda^- * \mathbf{Du} &\xrightarrow{M} \mathbf{Du} \quad \text{modularly in } L_M(\Omega_T;\mathbb{R}^{3\times 3}_{\mathrm{sym}}),\\ \tilde{\sigma}_\lambda^- * \overline{\mathbf{S}} &\xrightarrow{M^*} \overline{\mathbf{S}} \quad \text{modularly in } L_{M^*}(\Omega_T;\mathbb{R}^{3\times 3}_{\mathrm{sym}}).\end{aligned} \tag{7.108}$$

Then Lemma 3.4.6 allows us to conclude

$$\lim_{\lambda\to 0^+}\int_{s_0}^{s}\int_\Omega (\tilde{\sigma}_\lambda^- * \overline{\mathbf{S}}) : (\tilde{\sigma}_\lambda^- * \mathbf{Du})\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \overline{\mathbf{S}} : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.109}$$

Summarizing the arguments for (7.104), (7.109) and (7.106) we are able to pass to the limit in (7.94) and we obtain (7.91).

Step 9. Continuity with respect to time in the weak topology and the initial condition.

Note that as $\varrho \in C([0,T],L^q(\Omega))$ for $q \in [1,\infty)$, $\varrho_* \le \varrho \le \varrho^*$, and $\mathbf{u} \in L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^3)$, we may conclude that $\varrho(\cdot)\mathbf{u}(\cdot)$ is continuous in time in the weak topology, namely for $s_1 \in (0,T)$ for all $\tilde{\varphi} \in L^2_{\text{div}}(\Omega)$,

$$\lim_{s_2\to s_1}\int_\Omega (\varrho(s_2)\mathbf{u}(s_2) - \varrho(s_1)\mathbf{u}(s_1))\cdot\tilde{\varphi}\,\mathrm{d}x = 0.$$

In particular we observe that

$$\lim_{s_1\to 0}\int_\Omega (\varrho(s_1)\mathbf{u}(s_1) - \varrho_0\mathbf{u}_0)\cdot\tilde{\varphi}\,\mathrm{d}x = 0 \quad \text{for all } \tilde{\varphi} \in L^2_{\text{div}}(\Omega;\mathbb{R}^3). \tag{7.110}$$

Then integrating (7.38) over the time interval $(0,s_1)$, using (7.30) and the fact that

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n) : \mathbf{Du}^n \ge 0 \quad \text{a.e. in } \Omega_T,$$

which holds because of the monotonicity and as $\mathbf{S}(\cdot,\cdot,\cdot,\mathbf{0}) = \mathbf{0}$, and taking the limit as $n \to \infty$ we obtain

$$\int_\Omega (\varrho(s_1)|\mathbf{u}(s_1)|^2 - \varrho(0)|\mathbf{u}(0)|^2\,\mathrm{d}x \le 2\varrho^*\int_0^{s_1}\int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.111}$$

If we employ the obvious identity

$$\begin{aligned}&\|\sqrt{\varrho(s_1)}(\mathbf{u}(s_1) - \mathbf{u}_0)\|^2_{L^2(\Omega)}\\ &\qquad = \int_\Omega \left(\varrho(s_1)|\mathbf{u}(s_1)|^2 - 2\varrho(s_1)\mathbf{u}(s_1)\cdot\mathbf{u}_0 + \varrho(s_1)|\mathbf{u}_0|^2\right)\,\mathrm{d}x\end{aligned}$$

then the second part of property (7.24) is an easy consequence of (7.111) and

$$\begin{aligned}&\|\sqrt{\varrho(s_1)}(\mathbf{u}(s_1) - \mathbf{u}_0)\|^2_{L^2(\Omega)}\\ &= \int \left(\varrho(s_1)|\mathbf{u}(s_1)|^2 - 2\varrho(s_1)\mathbf{u}(s_1)\cdot\mathbf{u}_0 + \varrho(s_1)\cdot|\mathbf{u}_0|^2\right)\,\mathrm{d}x\\ &= \int_\Omega \left(\varrho(s_1)|\mathbf{u}(s_1)|^2 - \varrho_0|\mathbf{u}_0|^2 - 2(\varrho(s_1)\mathbf{u}(s_1) - \varrho_0\mathbf{u}_0)\cdot\mathbf{u}_0 + (\varrho(s_1) - \varrho_0)|\mathbf{u}_0|^2\right)\,\mathrm{d}x\\ &\le 2\varrho^*\int_0^{s_1}\int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t - 2\int_\Omega (\varrho(s_1)\mathbf{u}(s_1) - \varrho_0\mathbf{u}_0)\cdot\mathbf{u}_0\,\mathrm{d}x + \int_\Omega (\varrho(s_1) - \varrho_0)|\mathbf{u}_0|^2\,\mathrm{d}x.\end{aligned} \tag{7.112}$$

Now let us observe that using in (7.89) a test function of the form $\mathbb{1}_{(t_1,t_2)}h$ with $h \in W^{1,r}(\Omega)$ with $r = 5p/(5p-3)$, partial integration with respect to time and density of $W^{1,r}$ in L^1 gives

$$\lim_{t_2\to t_1}\int_\Omega \varrho(t_2)h\,\mathrm{d}x = \int_\Omega \varrho(t_1)h\,\mathrm{d}x \quad \text{for all } h\in L^1(\Omega) \text{ and } t_1\in[0,T]. \tag{7.113}$$

Letting $s_1 \to 0^+$ in (7.112) using (7.110), (7.113) and as $\mathbf{f}\in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ and $\mathbf{u}\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^3))$, we conclude that

$$\lim_{s_1\to 0}\|\sqrt{\varrho(s_1)}(\mathbf{u}(s_1)-\mathbf{u}_0)\|^2_{L^2(\Omega)} = 0. \tag{7.114}$$

Hence this implies together with (7.30) the second part of (7.24). The above arguments and (7.112), (7.114) also imply the fact, which we will use later, that

$$\lim_{s_1\to 0}\int_\Omega \varrho(s_1)|\mathbf{u}(s_1)|^2\,\mathrm{d}x = \int_\Omega \varrho_0|\mathbf{u}_0|^2\,\mathrm{d}x. \tag{7.115}$$

Step 10. Monotonicity argument to show $\overline{\mathbf{S}} = \mathbf{S}(x,\varrho,\theta,\mathbf{Du})$.

Let us now concentrate on the weak limit $\overline{\mathbf{S}}$. We aim to show that

$$\overline{\mathbf{S}} = \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) \quad \text{a.e. in } \Omega_T. \tag{7.116}$$

As in the previous chapter the proof is based on the monotonicity method in nonreflexive anisotropic Musielak–Orlicz spaces. We follow here arguments analogous to the one from Section 4.1.2. However since it is slightly modified due to the dependence of $\mathbf{S}$ on density and temperature, we recall it here for the convenience of the reader.

Using the integration by parts formula, see (7.91), and letting $s_0\to 0$, see (7.115), we find that

$$\frac{1}{2}\int_\Omega \varrho(s,x)|\mathbf{u}(s,x)|^2\,\mathrm{d}x + \int_0^s\int_\Omega \overline{\mathbf{S}}:\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^s\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \varrho_0(x)|\mathbf{u}_0(x)|^2\,\mathrm{d}x.$$

Next let us integrate the equation (7.38) over the interval $(0,s)$, let $n\to\infty$ and compare the result with the above one. Thus we infer that

$$\limsup_{n\to\infty}\int_0^s\int_\Omega \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n\,\mathrm{d}x\,\mathrm{d}t \le \int_0^s\int_\Omega \overline{\mathbf{S}}:\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.117}$$

Let $\Omega_s = (0,s)\times\Omega$. Due to the monotonicity of $\mathbf{S}$ (see condition (S3h)) we get that

$$\int_{\Omega_s}(\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) - \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n)):(\mathbf{w}-\mathbf{Du}^n)\,\mathrm{d}x\,\mathrm{d}t \ge 0 \tag{7.118}$$

holds for all $\mathbf{w}\in L^\infty(\Omega_T;\mathbb{R}^{3\times 3})$.

Now let us show the following fact:

$$\mathbf{S}(\cdot,l,\vartheta,\mathbf{w}) \text{ is bounded for } \mathbf{w}\in L^\infty(\Omega_T;\mathbb{R}^{3\times3}_{\mathrm{sym}}) \text{ and for } l,\vartheta\in\mathbb{R}. \tag{7.119}$$

Indeed, this statement can be proved by contradiction. Suppose that $\mathbf{S}(x,l,s,\mathbf{w})$ is unbounded. Since M is nonnegative, by the coercivity condition (7.15), we have that

$$|\mathbf{w}| \geq \frac{M^*(x,\mathbf{S}(x,l,\vartheta,\mathbf{w}))}{|\mathbf{S}(x,l,\vartheta,\mathbf{w})|}.$$

Then the right-hand side tends to infinity as $|\mathbf{S}(x,l,\vartheta,\mathbf{w})| \to \infty$, since M^* is superlinear at infinity (see property 4. in Definition 2.2.2 of an N-function together with condition 3. of Definition 2.2.1), which contradicts that $\mathbf{w} \in L^\infty(\Omega_T;\mathbb{R}^{3\times 3})$. Therefore we find that $\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) \in L^\infty(\Omega_T;\mathbb{R}^{3\times 3})$.

Due to the continuity of $\mathbf{S}$ with respect to the second and the third argument (i.e. with respect to density and temperature) and a.e. convergence of the sequences $\{\varrho^n\}_{n=1}^\infty$, $\{\theta^n\}_{n=1}^\infty$ we have that

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) \to \mathbf{S}(x,\varrho,\theta,\mathbf{w}) \quad \text{a.e. in } \Omega_T.$$

Since $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w})\}_{n=1}^\infty \subset L^\infty(\Omega_s;\mathbb{R}^{3\times 3})$ we obtain uniform integrability in L^1 of the sequence $\{M^*(\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}))\}_{n=1}^\infty$. By Theorem 3.4.4

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) \xrightarrow{M^*} \mathbf{S}(x,\varrho,\theta,\mathbf{w}) \quad \text{modularly in } L_{M^*}(\Omega_T;\mathbb{R}^{3\times 3}).$$

Since M^* satisfies the Δ_2-condition, the modular and strong convergence in L_{M^*} coincide

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) \to \mathbf{S}(x,\varrho,\theta,\mathbf{w}) \quad \text{strongly in } L_{M^*}(\Omega_T;\mathbb{R}^{3\times 3}).$$

Therefore by the weak-$*$ convergence in $L_M(\Omega_T;\mathbb{R}^{3\times 3})$ of $\{\mathbf{Du}^n\}_n$ (see (7.72)) we find that

$$\lim_{n\to\infty} \int_{\Omega_s} \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{w}) : \mathbf{Du}^n \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_s} \mathbf{S}(x,\varrho,\theta,\mathbf{w}) : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.120}$$

Let us pass to the limit as $n \to \infty$ in (7.118). By weak-$*$ convergence in $L_{M^*}(\Omega_T;\mathbb{R}^{3\times 3})$ of the subsequence $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n)\}_n$ (see (7.73)) and by (7.117), (7.120) we obtain

$$\int_{\Omega_s} \overline{\mathbf{S}} : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t \geq \int_{\Omega_s} \overline{\mathbf{S}} : \mathbf{w}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_s} \mathbf{S}(x,\varrho,\theta,\mathbf{w}) : (\mathbf{Du}-\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t \tag{7.121}$$

and consequently

$$\int_{\Omega_s} (\mathbf{S}(x,\varrho,\theta,\mathbf{w}) - \overline{\mathbf{S}}) : (\mathbf{w}-\mathbf{Du})\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{7.122}$$

As in Section 4.1.2 we choose the function $\mathbf{w}$ by setting

$$\mathbf{w} = (\mathbf{Du})\mathbb{1}_{\Omega_s{}^i} + h\mathbf{v}\mathbb{1}_{\Omega_s{}^j},$$

with

$$\Omega_s{}^k = \{(t,x) \in \Omega_s : |\mathbf{Du}(t,x)| \le k \ \text{ a.e. in } \Omega_s\}$$

and where $k > 0$, $0 < j < i$, $h > 0$ and $\mathbf{v} \in L^\infty(\Omega_T;\mathbb{R}^{3\times 3})$ are arbitrary. Since $\mathbf{S}(x,\varrho,\theta,\mathbf{0}) = \mathbf{0}$, from (7.122) we infer

$$-\int_{\Omega_s\setminus\Omega_s{}^i}(\mathbf{S}(x,\varrho,\theta,\mathbf{0})-\overline{\mathbf{S}}):\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t + h\int_{\Omega_s{}^j}(\mathbf{S}(x,\varrho,\theta,\mathbf{Du}+h\mathbf{v})-\overline{\mathbf{S}}):\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \ge 0, \tag{7.123}$$

where obviously

$$\int_{\Omega_s\setminus\Omega_s{}^i}\overline{\mathbf{S}}:\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_s}(\overline{\mathbf{S}}:\mathbf{Du})\mathbb{1}_{\Omega_s\setminus\Omega_s{}^i}\,\mathrm{d}x\,\mathrm{d}t.$$

By (7.74) and by the Fenchel–Young inequality (see Lemma 2.1.32) we obtain that $\int_{\Omega_T}\overline{\mathbf{S}}:\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t < \infty$ and consequently

$$(\overline{\mathbf{S}}:\mathbf{Du})\mathbb{1}_{\Omega_s\setminus\Omega_s{}^i} \to 0 \quad \text{ a.e. in } \Omega_s \text{ for } i\to\infty.$$

The Lebesgue dominated convergence theorem gives

$$\lim_{i\to\infty}\int_{\Omega_s\setminus\Omega_s{}^i}\overline{\mathbf{S}}:\mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Let us pass to the limit as $i\to\infty$ in (7.123) and divide by h. Hence we have that

$$\int_{\Omega_s{}^j}(\mathbf{S}(x,\varrho,\theta,\mathbf{Du}+h\mathbf{v})-\overline{\mathbf{S}}):\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \ge 0.$$

As $\mathbf{Du}+h\mathbf{v}\to\mathbf{Du}$ a.e. in $\Omega_s{}^j$ when $h\to 0^+$ and as $\{\mathbf{S}(x,\varrho,\theta,\mathbf{Du}+h\mathbf{v})\}_{h>0} \subset L^\infty(\Omega_s{}^j;\mathbb{R}^{3\times 3})$, $|\Omega_s{}^j| < \infty$, the Vitali convergence theorem (Theorem (8.23)) yields

$$\mathbf{S}(x,\varrho,\theta,\mathbf{Du}+h\mathbf{v}) \to \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) \quad \text{strongly in} \quad L^1(\Omega_s{}^j;\mathbb{R}^{3\times 3}) \text{ as } h\to 0^+$$

and

$$\int_{\Omega_s{}^j}(\mathbf{S}(x,\varrho,\theta,\mathbf{Du}+h\mathbf{v})-\overline{\mathbf{S}}):\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_s{}^j}(\mathbf{S}(x,\varrho,\theta,\mathbf{Du})-\overline{\mathbf{S}}):\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \text{ as } h\to 0^+.$$

Therefore

$$\int_{\Omega_s{}^j}(\mathbf{S}(x,\varrho,\theta,\mathbf{Du})-\overline{\mathbf{S}}):\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \ge 0 \quad \text{ for all } \mathbf{v}\in L^\infty(\Omega_s;\mathbb{R}^{3\times 3}).$$

Let us chose $\mathbf{v}$ such that $\mathbf{v} = -\frac{\mathbf{S}(x,\varrho,\theta,\mathbf{Du})-\overline{\mathbf{S}}}{|\mathbf{S}(x,\varrho,\theta,\mathbf{Du})-\overline{\mathbf{S}}|}$ if $\mathbf{S}(x,\varrho,\theta,\mathbf{Du}) \ne \overline{\mathbf{S}}$ and $\mathbf{v} = 0$ if $\mathbf{S}(x,\varrho,\theta,\mathbf{Du}) = \overline{\mathbf{S}}$. Therefore, we find that

$$\int_{\Omega_s{}^j}|\mathbf{S}(x,\varrho,\theta,\mathbf{Du})-\overline{\mathbf{S}}|\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Hence $\mathbf{S}(x,\varrho,\theta,\mathbf{Du}) = \overline{\mathbf{S}}$ a.e. in $\Omega_s{}^j$ and as j is arbitrary it also holds a.e. in Ω_s for almost all s such that $0 < s < T$. Finally we conclude that (7.116) holds true and we are allowed to replace $\overline{\mathbf{S}}$ by $\mathbf{S}(x,\varrho,\theta,\mathbf{Du})$ in (7.90).

Step 11. Convergence of $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n\}_n$.

Note that (7.116), (7.117), the limits shown in the part concerning weak limits (7.66)–(7.75) and the part concerning strong convergence (7.76)–(7.88) allow us to pass to the limit as $n \to \infty$ in the approximate thermal energy equation (7.33), but instead of equality in the limit we still can only conclude inequality (due to (7.117)). Therefore we concentrate now on the convergence of the sequence $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n\}_{n=1}^\infty$ and our aim is to show that the following holds

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n \rightharpoonup \mathbf{S}(x,\varrho,\theta,\mathbf{Du}):\mathbf{Du} \quad \text{weakly in } L^1(\Omega_T). \tag{7.124}$$

The idea follows from [188] (later also used in [218]) and is based on the concept of biting convergence and the theory of Young measures. For the definition of the biting limit, see Definition 8.36. In particular, we apply here Lemma 8.39. The methodology can also be found in Section 5.3 Step 7. However, we recall here all the details for clarity of presentation so that new details concerning the dependence of $\mathbf{S}$ on temperature and density are not missed.

Let us set

$$\{a_n\}_{i=1}^\infty := \{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}^n\}_{i=1}^\infty .$$

Our aim now is to show that for $\{a_n\}_{i=1}^\infty$ the assumptions of Lemma 8.39 are fulfilled. As a consequence, this leads to the weak convergence of a_n in $L^1(\Omega_T)$.

Assumption (i) is fulfilled due to monotonicity condition (S3h) and as $\mathbf{S}(\cdot,\cdot,\cdot,0) = 0$, namely $a_n \geq 0$.

Next (iii) is a straightforward consequence of (7.117).

Finally we have to show (ii) – biting convergence of a_n to $a := \mathbf{S}(x,\varrho,\theta,\mathbf{Du}):\mathbf{Du}$.

By the monotonicity of $\mathbf{S}$ (see (S3h)) we have

$$0 \leq (\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n) - \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du})):(\mathbf{Du}^n - \mathbf{Du}). \tag{7.125}$$

By coercivity condition (S2h), the Fenchel–Young inequality (see Lemma 2.1.32) and convexity of the N-function M^* we infer that

$$cM(x,\mathbf{Du}) + \tfrac{2c-d}{2} M^*(x,\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du})) \leq M\left(x, \tfrac{2}{d}\mathbf{Du}\right)$$

with $d = \min\{c,1\}$. As $\mathbf{Du} \in L_M(\Omega_T;\mathbb{R}^{3\times3})$, we find that $\{\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du})\}_{n=1}^\infty$ is uniformly bounded in $L_{M^*}(\Omega_T;\mathbb{R}^{3\times3})$. Due to (7.40) and by the generalized Hölder inequality (see Lemma 3.1.15) the right-hand side of (7.125) is uniformly bounded in $L^1(\Omega_T)$. Therefore, by Theorem 8.37 combined with Theorem 8.41, there exists a Young measure $\mu_{t,x}(\cdot,\cdot,\cdot)$ satisfying up to a subsequence

$$\begin{aligned} 0 \leq &(\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n) - \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du})):(\mathbf{Du}^n - \mathbf{Du}) \\ &\xrightarrow{b} \int_{\mathbb{R}^2\times\mathbb{R}^{3\times3}} (\mathbf{S}(x,l,s,\lambda) - \mathbf{S}(x,l,s,\mathbf{Du}))):(\lambda - \mathbf{Du})\,\mathrm{d}\mu_{t,x}(s,l,\lambda) := L \end{aligned} \tag{7.126}$$

as $n \longrightarrow \infty$. Applying Lemma 8.44, by (7.77) and (7.82) we have in fact that $\mu_{t,x}(\cdot,\cdot,\cdot)$ can be rewritten in the form $\delta_{\varrho,\theta}(l,s)\otimes\nu_{t,x}(\lambda)$. This gives that

$$\begin{aligned} L &= \int_{\mathbb{R}^{3\times 3}} (\mathbf{S}(x,\varrho,\theta,\lambda) - \mathbf{S}(x,\varrho,\theta,\mathbf{Du})) : (\lambda - \mathbf{Du})\,\mathrm{d}\nu_{t,x}(\lambda) \\ &= \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\lambda) : (\lambda - \mathbf{Du})\,\mathrm{d}\nu_{t,x}(\lambda) - \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : (\lambda - \mathbf{Du})\,\mathrm{d}\nu_{t,x}(\lambda). \end{aligned} \tag{7.127}$$

Notice that $\mathbf{S}(x,\varrho,\theta,\mathbf{Du})$ is independent of λ and

$$\int_{\mathbb{R}^{3\times 3}} \lambda \mathrm{d}\nu_{t,x}(\lambda) = \mathbf{Du} \quad \text{for a.e. } (t,x) \in \Omega_T$$

by Theorem 8.41 and since $\mathbf{Du}^n \rightharpoonup \mathbf{Du}$ in $L^1(\Omega_T;\mathbb{R}^{3\times 3})$ (consequence of (7.72)). Then we have that

$$\begin{aligned} &\int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : (\lambda - \mathbf{Du})\,\mathrm{d}\nu_{t,x}(\lambda) \\ &\qquad = \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : \left(\int_{\mathbb{R}^{3\times 3}} \lambda\,\mathrm{d}\nu_{t,x}(\lambda) - \mathbf{Du} \right) = 0. \end{aligned} \tag{7.128}$$

As the second term of the right-hand side of (7.127) disappears, the biting limit of (7.126) becomes

$$L = \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\lambda) : (\lambda - \mathbf{Du})\,\mathrm{d}\nu_{t,x}(\lambda). \tag{7.129}$$

By the Fenchel–Young inequality (see Lemma 2.1.32) and (7.40), $\{a_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^1(\Omega_T)$. Therefore we obtain that

$$\begin{aligned} a_n = \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n) : \mathbf{Du}^n &\xrightarrow{b} \int_{\mathbb{R}^2\times\mathbb{R}^{3\times 3}} \mathbf{S}(x,l,s,\lambda) : \lambda\,\mathrm{d}\mu_{t,x}(l,s,\lambda) \\ &= \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\lambda) : \lambda\,\mathrm{d}\nu_{t,x}(\lambda). \end{aligned}$$

Then as $a_n \geq 0$ for $n = 1,\dots,\infty$, by Lemma 8.43 and due to (7.117), (7.116), we get that

$$\begin{aligned} \int_{\Omega_T} \mathbf{S}(x,\varrho,\theta,\mathbf{Du}) : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t &\geq \liminf_{n\to\infty} \int_{\Omega_T} \mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n) : \mathbf{Du}^n\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_{\Omega_T} \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,\varrho,\theta,\lambda) : \lambda\,\mathrm{d}\nu_{t,x}(\lambda)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.130}$$

On the other hand due to (7.75) and (7.116) we have that,

$$\mathbf{S}(x,\varrho,\theta,\mathbf{Du}) = \int_{\mathbb{R}^{3\times 3}} \mathbf{S}(x,l,s,\lambda)\,\mathrm{d}\nu_{t,x}(\lambda).$$

So by (7.129) and (7.130) the right-hand side (7.125) is non-positive, as is the right-hand side of (7.126). This implies that

$$(\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n)-\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du})):(\mathbf{Du}^n-\mathbf{Du})\xrightarrow{b}0. \tag{7.131}$$

In a similar way as (7.128) we find that

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}):(\mathbf{Du}^n-\mathbf{Du})\xrightarrow{b}0, \tag{7.132}$$

and one can obtain also that

$$\mathbf{S}(x,\varrho^n,\theta^n,\mathbf{Du}^n):\mathbf{Du}\xrightarrow{b}\mathbf{S}(x,\varrho,\theta,\mathbf{Du}):\mathbf{Du}. \tag{7.133}$$

Let us sum up (7.131)–(7.133). This implies that $a_n \xrightarrow{b} a$. Hence assumption (ii) of Lemma 8.39 is fulfilled and from its statement we conclude that (7.124) is shown.

Step 12. The limit in the thermal energy equation.

Finally let us recall the strong convergence in $L^1(\Omega_T;\mathbb{R}^3)$ of $\{\varrho^n\mathbf{u}^n\theta^n\}_n$ (see (7.85)), the weak convergence in $L^s(\Omega_T;\mathbb{R}^3)$ with proper s of $\{\mathbf{q}(\varrho^n,\theta^n,\nabla\theta^n)\}_n$ (see (7.88)), and the above weak convergence (7.124). Then letting $n\to\infty$ in (7.33) we obtain the thermal energy equation (7.21) from the Definition 7.2.1 of a weak solution. All that is left now is to establish the convergence of the first term on the left-hand side of (7.33) to the first term on the left-hand side of (7.21). Notice that from (7.33), (7.64), by (7.124), (7.85), (7.88) we have that

$$\int_0^T \langle z,h\rangle\,\mathrm{d}t := \lim_{n\to\infty}\int_0^T \langle\partial_t(\theta^n\varrho^n),h\rangle\,\mathrm{d}t$$

exists for all $h\in L^\infty(0,T;W^{1,q}(\Omega))$ with large q. On the other hand, by (7.84) we find that

$$\lim_{n\to\infty}\int_0^T \langle\partial_t(\theta^n\varrho^n),h\rangle\,\mathrm{d}t = \int_0^T \langle\partial_t(\varrho\theta),h\rangle\,\mathrm{d}t$$

for all $h\in C_c^\infty(0,T;W^{1,q}(\Omega))$. Hence $z=\partial_t(\varrho\theta)$.

To observe how the initial data is achieved, i.e. $(7.24)_3$, we take as a test function in (7.21) $\mathbb{1}_{[0,t]}h$ with $h\in W^{1,q}(\Omega)$.

The proof of Theorem 7.2.2 is completed.

7.3 A Generalized Stokes System

This section concerns a generalized Stokes system with the nonlinear viscous term having growth conditions prescribed by an N-function which places the problem of existence of weak solutions in homogeneous and anisotropic Orlicz spaces. Our main interest here is directed toward relaxing the growth assumptions on the N-function in comparison to those presented in Section 7.2. In particular, we want to capture the shear thinning fluids with rheology close to linear, namely to avoid an N-function M being supported from below by a polynomial of power larger then 2, see the condition (7.22). Here we consider the case of anisotropic but homogeneous functions. The main result of this section is the existence of weak solutions to the generalized Stokes system. Additionally, for the purpose of the existence proof, we need a version of the Korn–Sobolev inequality in the Orlicz setting (Theorem 9.4). This section is based on [180, 183, 184].

7.3.1 Formulation of the problem and the existence result

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with a sufficiently smooth boundary $\partial\Omega$ (say $C^{2+\nu}$ with $\nu > 0$), $(0,T)$ the time interval with $T < \infty$, $\Omega_T = (0,T)\times\Omega$, $\mathbf{u} : \Omega_T \to \mathbb{R}^N$ the velocity of a fluid, $\pi : \Omega_T \to \mathbb{R}$ the pressure function and $\mathbf{S} + \mathbf{I}\pi$ the Cauchy stress tensor. Here we do not consider density and temperature as unknowns. The flow is prescribed by the generalized incompressible Stokes system, which consist of balance of momentum, the condition of incompressibility, and initial data:

$$\partial_t \mathbf{u} - \operatorname{div} \mathbf{S}(t,x,\mathbf{Du}) + \nabla\pi = \mathbf{f} \quad \text{in } (0,T)\times\Omega, \tag{7.134}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0,T)\times\Omega, \tag{7.135}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{7.136}$$

$$\mathbf{u}(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{7.137}$$

For the viscous stress tensor $\mathbf{S}$ we assume that

(S1s) $\mathbf{S} : [0,T]\times\Omega\times\mathbb{R}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}^{N\times N}_{\mathrm{sym}}$ is a Carathéodory function (i.e., measurable with respect to t and x and continuous with respect to the last variable).

(S2s) There exists an anisotropic N-function $M : \mathbb{R}^{N\times N}_{\mathrm{sym}} \to [0,\infty)$ and a constant $c_c > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ the following growth and coercivity condition is satisfied

$$\mathbf{S}(t,x,\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c_c\big(M(\boldsymbol{\xi}) + M^*(\mathbf{S}(t,x,\boldsymbol{\xi}))\big). \tag{7.138}$$

(S3s) $\mathbf{S}$ is monotone, i.e. for all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ and for a.a. $(t,x) \in \Omega_T$

$$\big(\mathbf{S}(t,x,\boldsymbol{\xi}) - \mathbf{S}(t,x,\boldsymbol{\eta})\big) : \big(\boldsymbol{\xi} - \boldsymbol{\eta}\big) \ge 0.$$

We define the space of functions with symmetric gradient in $L_M(\Omega;\mathbb{R}^{N\times N})$, namely

$$BD_M(\Omega;\mathbb{R}^N) := \{\mathbf{u} \in L^1(\Omega;\mathbb{R}^N) \,:\, \mathbf{Du} \in L_M(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})\}.$$

The space $BD_M(\Omega)$ is a Banach space with the norm

$$\|\mathbf{u}\|_{BD_M(\Omega)} := \|\mathbf{u}\|_{L^1(\Omega)} + \|\mathbf{Du}\|_{L_M(\Omega)}$$

and it is a subspace of the space of bounded deformations $BD(\Omega)$, i.e.

$$BD(\Omega;\mathbb{R}^N) := \{\mathbf{u} \in L^1(\Omega;\mathbb{R}^N) \,:\, [\mathbf{Du}]_{i,j} \in \mathcal{M}(\Omega), \text{ for } i,j = 1,\dots,N\},$$

where $\mathcal{M}(\Omega)$ denotes the space of signed Radon measures with finite mass on Ω and

$$[\mathbf{Du}]_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

According to [313, Theorem 1.1.] there exists a unique continuous operator

$$\gamma_0 \text{ from } BD(\Omega;\mathbb{R}^N) \text{ onto } L^1(\partial\Omega;\mathbb{R}^N)$$

such that the generalized Green formula

$$2\int_\Omega \phi[\mathbf{Du}]_{i,j}\,\mathrm{d}x = -\int_\Omega \left(u_j \frac{\partial \phi}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_j}\right)\mathrm{d}x + \int_{\partial\Omega} \phi\left(\gamma_0(u_i)n_j + \gamma_0(u_j)n_i\right)\mathrm{d}\mathcal{H}^{d-1} \tag{7.139}$$

holds for every $\phi \in C^1(\overline{\Omega})$, where $\mathbf{n} = (n_1,\dots,n_N)$ is the unit outer normal vector on $\partial\Omega$ and $\gamma_0(u_i)$ is the i-th component of $\gamma_0(\mathbf{u})$ and $\mathcal{H}^{N-1}$ is the $(N-1)$-Hausdorff measure. Notice that such a γ_0 is a generalization of the trace operator in Sobolev spaces to the case of BD space. Moreover, if $\mathbf{u} \in C(\overline{\Omega};\mathbb{R}^N)$, then $\gamma_0(\mathbf{u}) = \mathbf{u}|_{\partial\Omega}$. Observe that the above coincides with the classical trace operator in classical Sobolev spaces, if $\mathbf{u} \in W_0^{1,1}(\Omega;\mathbb{R}^N)$.

With the above understanding of the trace in a generalized sense we define the subspace and the subset of $BD_M(\Omega;\mathbb{R}^N)$ as follows

$$BD_{M,0}(\Omega;\mathbb{R}^N) := \{\mathbf{u} \in BD_M(\Omega;\mathbb{R}^N) \,:\, \gamma_0(\mathbf{u}) = 0\},$$

$$\mathcal{BD}_{M,0}(\Omega;\mathbb{R}^N) := \{\mathbf{u} \in BD_M(\Omega;\mathbb{R}^N) \,:\, \mathbf{Du} \in \mathcal{L}_M(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}) \text{ and } \gamma_0(\mathbf{u}) = 0\}. \tag{7.140}$$

Moreover, let us define also

$$BD_M(\Omega_T;\mathbb{R}^N) := \{\mathbf{u} \in L^1(\Omega_T;\mathbb{R}^N) \,:\, \mathbf{Du} \in L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}})\}$$

and the related subspace

$$BD_{M,0}(\Omega_T;\mathbb{R}^N) := \{\mathbf{u} \in BD_M(\Omega_T;\mathbb{R}^N) \,:\, \gamma_0(\mathbf{u}) = 0\},$$

where γ_0 is understood as follows

$$\begin{aligned} 2\int_{\Omega_T} \phi[\mathbf{Du}]_{i,j}\,\mathrm{d}x\,\mathrm{d}t = &-\int_{\Omega_T} \left(u_j \frac{\partial \phi}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_j}\right)\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{(0,T)\times\partial\Omega} \phi\left(\gamma_0(u_i)n_j + \gamma_0(u_j)n_i\right)\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}t \end{aligned} \tag{7.141}$$

for all $\phi \in C^1(\overline{\Omega}_T)$ and $i,j = 1,\dots,N$. If $\mathbf{u} \in BD_M(\Omega_T;\mathbb{R}^N)$, then we have $\mathbf{u}(t,\cdot) \in BD_M(\Omega;\mathbb{R}^N)$ for a.a. $t \in (0,T)$. For such vector fields it is equivalent that $\mathbf{u} \in BD_{M,0}(\Omega_T;\mathbb{R}^N)$ and that $\mathbf{u}(t,\cdot) \in BD_{M,0}(\Omega;\mathbb{R}^N)$ for a.a. $t \in (0,T)$. The [313, Proposition 1.1.] gives us that there exists an extension operator from $BD(\Omega;\mathbb{R}^N)$ to

$BD(\mathbb{R}^N)$ and consequently we are able to extend the functions in $BD_{M,0}(\Omega_T;\mathbb{R}^N)$ by zero to functions in $BD_M([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$.

In the forthcoming part we will consider the closure of $C_c^\infty(\Omega;\mathbb{R}^N)$ with respect to two topologies, i.e.

(i) the modular topology of $L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}})$, which we denote by Y_0^M, namely

$$\begin{aligned} Y_0^M(\Omega_T;\mathbb{R}^N) := \{&\mathbf{u}\in L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^N)) \ : \ \mathbf{Du}\in L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}}),\\ &\exists\, \{\mathbf{u}^j\}_{j=1}^\infty \subset C_c^\infty((-\infty,T);\mathcal{V}) : \mathbf{u}^j \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^N))\\ &\text{and } \mathbf{Du}^j \xrightarrow{M} \mathbf{Du} \text{ modularly in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}})\}, \end{aligned} \tag{7.142}$$

(ii) the weak-$*$ topology of $L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}})$, which we denote by Z_0^M, namely

$$\begin{aligned} Z_0^M(\Omega_T;\mathbb{R}^N) := \{&\mathbf{u}\in L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^N)) \ : \ \mathbf{Du}\in L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}}),\\ &\exists\, \{\mathbf{u}^j\}_{j=1}^\infty \subset C_c^\infty((-\infty,T);\mathcal{V}) : \mathbf{u}^j \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^N))\\ &\text{and } \mathbf{Du}^j \overset{*}{\rightharpoonup} \mathbf{Du} \text{ weakly-}* \text{ in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\text{sym}})\}. \end{aligned} \tag{7.143}$$

Let us now formulate the result on existence of weak solutions to the initial-boundary value problem (7.134)–(7.137). We study the problem in two different types of domains:

- the domain Ω is star-shaped, an N-function is anisotropic, and we do not need any additional restriction on the growth of the N-function.
- the domain is arbitrary, with a sufficiently smooth boundary. For this case we recall the minorant and majorant of an N-function which are Young functions as described in Definition 2.2.2 $m_1, m_2 : [0,\infty) \to [0,\infty)$ satisfying

$$m_1(|\xi|) \le M(\xi) \le m_2(|\xi|). \tag{7.144}$$

  In this case the existence result is formulated under the control of the spread between m_1 and m_2.

Observe that one could choose here for m_1 and m_2

$$\begin{aligned} m_1(r) &:= \tilde{m}_1^{**}(r), \quad \text{where } \tilde{m}_1(r) := \min_{\boldsymbol{\xi}\in\mathbb{R}^{N\times N}_{\text{sym}}, |\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}),\\ m_2(r) &:= \max_{\boldsymbol{\xi}\in\mathbb{R}^{N\times N}_{\text{sym}}, |\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}), \end{aligned}$$

see Section 2.1.4.

Theorem 7.3.1 *Let Ω be a bounded domain in $\mathbb{R}^N$, $N\ge 2$, with smooth boundary. Let $M:\mathbb{R}^{N\times N}_{\text{sym}} \to [0,\infty)$ be a homogeneous, anisotropic N-function with minorant and majorant m_1, $m_2 : [0,\infty) \to [0,\infty)$ being the Young functions described in Definition 2.2.2. Let condition (D1) or (D2) be satisfied.*

(D1) Ω *is a bounded star-shaped domain,*
(D2) Ω *is a bounded non-star-shaped domain and*

$$m_2(r) \le c_m((m_1(r))^{\frac{N}{N-1}} + |r|^2 + 1) \quad \text{for all } r \in [0,\infty), \tag{7.145}$$

and

$$m_1 \text{ satisfies the } \Delta_2\text{-condition,} \tag{7.146}$$

Let $\mathbf{S}$ *satisfy conditions (S1s)–(S3s) and let* $\mathbf{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N)$ *and* $\mathbf{f} \in E_{m_1^*}(\Omega_T;\mathbb{R}^N)$ *be given. Then there exists a weak solution to the system* (7.134)–(7.137)*. Namely, there exists a* $\mathbf{u} \in Z_0^M(\Omega_T;\mathbb{R}^N)$ *such that*

$$\int_{\Omega_T} (-\mathbf{u}\cdot\partial_t\varphi + \mathbf{S}(t,x,\mathbf{D}\mathbf{u}) : \mathbf{D}\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \mathbf{f}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} \mathbf{u}_0\cdot\varphi(0)\,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(-\infty,T;\mathcal{V})$.

In order to prove the above result we will proceed as follows:

- First we show that the spaces Y_0^M and Z_0^M defined above coincide and explain how this fact is used in the integration by parts formula. To do so will we need the Korn–Sobolev-type inequality in Orlicz spaces shown in Theorem 9.4;
- Next we give a proof of Theorem 7.3.1 starting with the construction of a proper approximation and using the part mentioned above.

7.3.2 Domains and closures

In this subsection we study the issue of closures of smooth functions with respect to various topologies and the two spaces Y_0^M and Z_0^M defined in the beginning of the section by (7.142) and (7.143). Our aim is to show the equivalence between these two spaces. We start with the simpler case of star-shaped domains. Then we extend the result to arbitrary domains with regular boundary, where the set Ω is considered as a sum of star-shaped domains. In particular, for this case the Korn–Sobolev inequality (9.12) provides an essential estimate. A requirement which appears for non-star-shaped domains is the constraint on the spread between m_1 and m_2 and on the growth of m_1 – both are represented by assumption (D2) in Theorem 7.3.1.

In this following part we consider the issue of integration by parts, where the equivalence between the spaces Y_0^M and Z_0^M appears to be crucial.

Let us start with the case of star-shaped domains:

Lemma 7.1 (star-shaped domains). *Let* $M : \mathbb{R}^{N\times N}_{\mathrm{sym}} \to [0,\infty)$ *be a homogeneous and anisotropic N-function,* Ω *be a bounded star-shaped domain,* $(0,T)$ *be a finite time interval, and* $Y_0^M(\Omega_T;\mathbb{R}^N)$, $Z_0^M(\Omega_T;\mathbb{R}^N)$ *be the function spaces defined by (7.142) and (7.143) respectively. Then*

$$Y_0^M = Z_0^M.$$

Moreover, if $\mathbf{u} \in Y_0^M$, $\overline{\mathbf{S}} \in \mathcal{L}_{M^*}(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, $\mathbf{f} \in \mathcal{L}_{m_1^*}(\Omega_T;\mathbb{R}^N)$ *and*

$$-\int_0^T \int_\Omega \mathbf{u}\cdot\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \mathbf{f}\cdot\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \varphi \in C_c^\infty(\Omega_T), \tag{7.147}$$

then

$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^s \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\mathbf{u} \,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^s \int_\Omega \mathbf{f}\cdot\mathbf{u} \,\mathrm{d}x\,\mathrm{d}t$$

for a.a. s_0, s *such that* $0 < s_0 < s < T$.

Proof. **Part 1: ($Y_0^M = Z_0^M$).**
As the modular topology is stronger than weak-$*$, we have

$$Y_0^M(\Omega_T;\mathbb{R}^N) \subset Z_0^M(\Omega_T;\mathbb{R}^N).$$

Hence we concentrate on proving the opposite inclusion, i.e.

$$Z_0^M(\Omega_T;\mathbb{R}^N) \subset Y_0^M(\Omega_T;\mathbb{R}^N). \tag{7.148}$$

To this end we want to extend $\mathbf{u}$ by zero outside of Ω to the whole of $\mathbb{R}^N$ and then regularize it. In order to extend $\mathbf{u}$ we notice that

$$Z_0^M(\Omega_T;\mathbb{R}^N) \subset BD_{M,0}(\Omega_T;\mathbb{R}^N).$$

By definition each $\mathbf{u} \in Z_0^M(\Omega_T;\mathbb{R}^N)$ is an element of $BD_M(\Omega_T;\mathbb{R}^N)$. So now let us show that it vanishes on the boundary. Let us recall that $\mathbf{u}$ satisfies the formula (7.141). Let us take a sequence

$$\{\mathbf{u}^k\}_{k=1}^\infty := \text{compactly supported smooth functions}$$
$$\text{with the properties prescribed in the definition of the space } Z_0^M.$$

Inserting this sequence into (7.141) we obtain

$$2\int_{\Omega_T} \phi[\mathbf{D}\mathbf{u}^k]_{i,j} \,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T} \left(u_j^k \frac{\partial\phi}{\partial x_i} + u_i^k \frac{\partial\phi}{\partial x_j} \right) \mathrm{d}x\,\mathrm{d}t \tag{7.149}$$

for all $\phi \in C^1(\overline{\Omega}_T)$ and $i,j = 1,\ldots,N$. By the linearity of all terms we pass to the weak-$*$ limit in (7.149) and we conclude that the boundary term is zero.

Next we introduce $\mathbf{u}^\lambda$. Here the index λ over the function $\mathbf{v}$ denotes the following

$$\mathbf{v}^\lambda(t,x) := \mathbf{v}(t,\lambda(x-x_0)+x_0) \tag{7.150}$$

where x_0 is a vantage point of Ω and $\lambda \in (0,1)$.

Let

$$\varepsilon_\lambda = \frac{1}{2}\mathrm{dist}(\partial\Omega,\lambda\Omega), \text{ where } \lambda\Omega := \{y = \lambda(x-x_0)+x_0 \mid x \in \Omega\}.$$

Let us define then

$$\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x) := \sigma_\delta * ((\varrho_\varepsilon * \mathbf{u}^\lambda(t,x))\, \mathbb{1}_{(s_0,s)}), \tag{7.151}$$

where $\varrho_\varepsilon(x) = \frac{1}{\varepsilon^N}\varrho(\frac{x}{\varepsilon})$ is a standard regularizing kernel on $\mathbb{R}^N$ (i.e. $\varrho \in C^\infty(\mathbb{R}^N)$, ϱ has a compact support in $B(0,1)$ and $\int_{\mathbb{R}^N}\varrho(x)\,\mathrm{d}x = 1, \varrho(x) = \varrho(-x)$) and the convolution is with respect to the space variable x, $\varepsilon < \frac{\varepsilon_\lambda}{2}$ and $\sigma_\delta(t) = \frac{1}{\delta}\sigma(\frac{t}{\delta})$ is a regularizing kernel on $\mathbb{R}$ (i.e. $\sigma \in C^\infty(\mathbb{R})$, σ has a compact support in $B(0,1)$ and $\int_{\mathbb{R}}\sigma(\tau)\,\mathrm{d}\tau = 1, \sigma(t) = \sigma(-t)$) and the convolution is with respect to the time variable t with $\delta < \min\{s_0, T-s\}$. Notice that the approximation function $\mathbf{u}^{\delta,\lambda,\varepsilon}$ also has zero trace.

Let us pass to the limit as $\varepsilon \to 0$ as a first step. We have then that

$$\mathbf{Du}^{\delta,\lambda,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathbf{Du}^{\delta,\lambda} \quad \text{strongly in } L^1(\Omega_T;\mathbb{R}^{N\times N}).$$

For a.a. $t \in [0,T]$ the function $\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,\cdot) \in L^1(\Omega;\mathbb{R}^{N\times N})$ and

$$\varrho_\varepsilon * \mathbf{Du}^{\delta,\lambda}(t,\cdot) \xrightarrow[\varepsilon\to 0]{} \mathbf{Du}^{\delta,\lambda}(t,\cdot) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^{N\times N})$$

then

$$\varrho_\varepsilon * \mathbf{Du}^{\delta,\lambda} \xrightarrow[\varepsilon\to 0]{} \mathbf{Du}^{\delta,\lambda} \quad \text{in measure on the set } [0,T]\times\Omega.$$

By Lemma 3.4.8

$$\{M(\beta\mathbf{Du}^{\delta,\lambda,\varepsilon})\}_{\varepsilon>0} \text{ is uniformly integrable in } L^1$$

and then by Theorem 3.4.4 we infer that

$$\mathbf{Du}^{\delta,\lambda,\varepsilon} \xrightarrow[\varepsilon\to 0]{M} \mathbf{Du}^{\delta,\lambda} \text{ modularly in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}}).$$

Next, passing to the limit as $\lambda \to 1$ we obtain that

$$\mathbf{Du}^{\delta,\lambda} \xrightarrow[\lambda\to 1]{} \mathbf{Du}^{\delta} \quad \text{strongly in } L^1(\Omega_T;\mathbb{R}^{N\times N})$$

and again the above together with the uniform integrability of $\{M(\beta\mathbf{Du}^{\delta,\lambda})\}_\lambda$ gives

$$\mathbf{Du}^{\delta,\lambda} \xrightarrow[\lambda\to 1]{M} \mathbf{Du}^{\delta} \quad \text{modularly in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}}).$$

To pass to the limit as $\delta \to 0^+$ we use similar arguments as for convergence with $\varepsilon \to 0^+$. Finally we observe that $Y_0^M = Z_0^M$.

Part 2: The integration by parts formula.
Let us define now a new approximation sequence (denoted in the same way as the previous one)

$$\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x) := \sigma_\delta * ((\sigma_\delta * \varrho_\varepsilon * \mathbf{u}^\lambda(t,x))\, \mathbb{1}_{(s_0,s)}) \tag{7.152}$$

with $\varepsilon < \frac{\varepsilon_\lambda}{2}$ and $\sigma < \frac{1}{2}\min\{s_0, T-s\}$. We test each equation in (7.147) by $\mathbf{u}^{\delta,\lambda,\varepsilon}$, noting that it is a sufficiently regular and admissible test function, to get

$$\int_{s_0}^{s}\int_\Omega (\mathbf{u}*\sigma_\delta)\cdot\partial_t(\mathbf{u}^{\lambda,\varepsilon}*\sigma_\delta)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \overline{\mathbf{S}}:\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.153}$$

The left-hand side of (7.153) is equivalent to

$$\int_{s_0}^{s}\int_\Omega (\mathbf{u}*\sigma_\delta)\cdot(\mathbf{u}^{\lambda,\varepsilon}*\partial_t\sigma_\delta)\,\mathrm{d}x\,\mathrm{d}t$$

and to pass to the limit as $\varepsilon\longrightarrow 0$ and $\lambda\longrightarrow 1$ it is enough to observe that

$$\mathbf{u}^{\lambda,\varepsilon}\overset{*}{\rightharpoonup}\mathbf{u}\quad\text{weakly-}*\text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N)).$$

To handle the right-hand side of (7.153) we use the results shown in the first part of the proof. In order to prove the convergence of the term $\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t$ we apply Theorem 9.4 with an N-function m_1 and observe that

$$\left(\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{N}{N-1}}\,\mathrm{d}x\right)^{\frac{N-1}{N}} \le C_N\int_\Omega m_1(|\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x)$$

for a.a. $t\in[0,T]$. Consequently Hölder's inequality implies that

$$\int_0^T\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))\,\mathrm{d}x\,\mathrm{d}t \le C_{\Omega,N}\int_0^T\int_\Omega (m_1(|\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\,\mathrm{d}t.$$

By the definition of m_1 together with the above we obtain

$$\int_0^T\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))\,\mathrm{d}x\,\mathrm{d}t \le C_{\Omega,N}\int_0^T\int_\Omega M(\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\,\mathrm{d}t. \tag{7.154}$$

Relation (7.154) and the following modular convergences

$$\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}\xrightarrow[\varepsilon\to 0]{M}\mathbf{D}\mathbf{u}^{\delta,\lambda},\quad \mathbf{D}\mathbf{u}^{\delta,\lambda}\xrightarrow[\lambda\to 1]{M}\mathbf{D}\mathbf{u}^{\delta}\quad\text{modularly in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}})$$

imply that

$$\mathbf{u}^{\delta,\lambda,\varepsilon}\xrightarrow[\varepsilon\to 0]{m_1}\mathbf{u}^{\delta,\lambda},\quad \mathbf{u}^{\delta,\lambda}\xrightarrow[\lambda\to 1]{m_1}\mathbf{u}^{\delta}\quad\text{modularly in } L_{m_1}(\Omega_T;\mathbb{R}^N).$$

By Lemma 3.4.6 with N-functions m_1^* and m_1 we obtain

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_{\Omega_T}\mathbf{f}\cdot\mathbf{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\mathbf{f}\cdot\mathbf{u}^{\delta}\,\mathrm{d}x\,\mathrm{d}t.$$

Similarly by Lemma 3.4.6 with N-functions M and M^* we get that

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_{\Omega_T}\overline{\mathbf{S}}:\mathbf{D}\mathbf{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\overline{\mathbf{S}}:\mathbf{D}\mathbf{u}^{\delta}\,\mathrm{d}x\,\mathrm{d}t.$$

Note that for all $0 < s_0 < s < T$ it follows that

$$\int_{s_0}^{s} \int_\Omega (\sigma_\delta * \mathbf{u}) \cdot \partial_t (\sigma_\delta * \mathbf{u}) \, dx \, dt = \int_{s_0}^{s} \frac{1}{2} \frac{d}{dt} \|\sigma_\delta * \mathbf{u}\|^2_{L^2(\Omega)} \, dt$$
$$= \frac{1}{2} \|\sigma_\delta * \mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2} \|\sigma_\delta * \mathbf{u}(s_0)\|^2_{L^2(\Omega)}.$$

Passing to the limit as $\delta \to 0$ and obtain for almost all $0 < s_0 < s < T$ (namely for all Lebesgue points of the function $\mathbf{u}(t)$), the following identity

$$\lim_{\delta \to 0} \int_{s_0}^{s} \int_\Omega (\mathbf{u} * \sigma_\delta) \cdot \partial_t (\mathbf{u} * \sigma_\delta) \, dx \, dt = \frac{1}{2} \|\mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2} \|\mathbf{u}(s_0)\|^2_{L^2(\Omega)}. \quad (7.155)$$

Let us concentrate now on the term

$$\int_0^T \int_\Omega \overline{\mathbf{S}} : (\sigma_\delta * ((\sigma_\delta * \mathbf{Du}) \, \mathbb{1}_{(s_0,s)})) \, dx \, dt = \int_{s_0}^{s} \int_\Omega (\sigma_\delta * \overline{\mathbf{S}}) : (\sigma_\delta * \mathbf{Du}) \, dx \, dt.$$

We observe that

$$\sigma_\delta * \overline{\mathbf{S}} \to \overline{\mathbf{S}} \quad \text{in measure on } \Omega_T \text{ as } \delta \to 0$$

and

$$\sigma_\delta * \mathbf{Du} \to \mathbf{Du} \quad \text{in measure on } \Omega_T \text{ as } \delta \to 0.$$

Moreover, the assumptions $\mathbf{u} \in Y_0^M(\Omega_T; \mathbb{R}^N)$ and $\overline{\mathbf{S}} \in \mathcal{L}_{M^*}(\Omega_T; \mathbb{R}^{N \times N}_{\text{sym}})$ provide that

$$\int_0^T \int_\Omega M(\mathbf{Du}) \, dx \, dt < \infty \quad \text{and} \quad \int_0^T \int_\Omega M^*(\overline{\mathbf{S}}) \, dx \, dt < \infty.$$

Therefore, using the same method as above we conclude that the sequences

$$\{M^*(\sigma_\delta * \overline{\mathbf{S}})\}_\delta \text{ and } \{M(\sigma_\delta * \mathbf{Du})\}_\delta \text{ are uniformly integrable}$$

and by Theorem 3.4.4 we have

$$\sigma_\delta * \mathbf{Du} \xrightarrow[\delta \to 0]{M} \mathbf{Du} \quad \text{modularly in } L_M(\Omega_T; \mathbb{R}^{N \times N}_{\text{sym}}),$$
$$\sigma_\delta * \overline{\mathbf{S}} \xrightarrow[\delta \to 0]{M^*} \overline{\mathbf{S}} \quad \text{modularly in } L_{M^*}(\Omega_T; \mathbb{R}^{N \times N}_{\text{sym}}).$$

Next by Lemma 3.4.6 we have

$$\lim_{\delta \to 0} \int_{s_0}^{s} \int_\Omega (\sigma_\delta * \overline{\mathbf{S}}) : (\sigma_\delta * \mathbf{Du}) \, dx \, dt = \int_{s_0}^{s} \int_\Omega \overline{\mathbf{S}} : \mathbf{Du} \, dx \, dt. \quad (7.156)$$

We treat the source term in the same way, except instead of the N-function M we consider m_1. Hence we have

$$\int_0^T \int_\Omega \mathbf{f} \cdot (\sigma_\delta * ((\sigma_\delta * \mathbf{u})\, \mathbb{1}_{(s_0,s)}))\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^s \int_\Omega (\sigma_\delta * \mathbf{f}) \cdot (\sigma_\delta * \mathbf{u})\,\mathrm{d}x\,\mathrm{d}t.$$

Noticing that

$$\sigma^\delta * \mathbf{u} \xrightarrow[\delta\to 0]{m_1} \mathbf{u} \quad \text{modularly in } L_{m_1}(\Omega_T;\mathbb{R}^N),$$
$$\sigma^\delta * \mathbf{f} \xrightarrow[\delta\to 0]{m_1^*} \mathbf{f} \quad \text{modularly in } L_{m_1^*}(\Omega_T;\mathbb{R}^N)$$

we infer

$$\lim_{\delta\to 0} \int_{s_0}^s \int_\Omega (\sigma^\delta * \mathbf{f}) \cdot (\sigma^\delta * \mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^s \int_\Omega \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.157}$$

Summarizing (7.155), (7.156) and (7.157) we obtain after passing to the limit with ε, λ and δ in (7.153) that

$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^s \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^s \int_\Omega \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \tag{7.158}$$

for almost all $0 < s_0 < s < T$. □

Lemma 7.2 (Non-star-shaped domains with the control of anisotropy). *Let* $M : \mathbb{R}^{N\times N}_{\mathrm{sym}} \to [0,\infty)$ *be a homogeneous and anisotropic N-function such that*

$$m_2(r) \le c_m((m_1(r))^{\frac{N}{N-1}} + |r|^2 + 1) \quad \textit{for } r \in [0,\infty) \tag{7.159}$$

and let

$$m_1 \textit{ satisfy the } \Delta_2\textit{-condition.}$$

Let Ω *be a bounded domain with a sufficiently smooth boundary and* $(0,T)$ *be a finite time interval, and let* $Y^M_0(\Omega_T;\mathbb{R}^N)$, $Z^M_0(\Omega_T;\mathbb{R}^N)$ *be the function spaces defined by (7.142) and (7.143) respectively. Then*

$$Y^M_0 = Z^M_0.$$

Moreover, if $\mathbf{u} \in Y^M_0(\Omega_T;\mathbb{R}^N)$, $\overline{\mathbf{S}} \in \mathcal{L}_{M^*}(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, $\mathbf{f} \in \mathcal{L}_{m_1^*}(\Omega_T;\mathbb{R}^N)$ *and*

$$-\int_0^T \int_\Omega \mathbf{u} \cdot \partial_t \varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \mathbf{f} \cdot \varphi\,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \varphi \in C^\infty_c(\Omega_T), \tag{7.160}$$

then

$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^s \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^s \int_\Omega \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t$$

for a.a. s_0, s *such that* $0 < s_0 < s < T$.

Proof. Let us start by recalling the fact that for Lipschitz domains there exists a finite family of star-shaped domains

$$\{\Omega_i\}_{i\in J} \quad \text{such that } \Omega = \bigcup_{i\in J}\Omega_i, \tag{7.161}$$

see Lemma 8.2. Let us introduce the partition of unity

$$\theta_i \text{ with } 0 \le \theta_i \le 1, \theta_i \in C_c^\infty(\Omega_i), \operatorname{supp}\theta_i = \Omega_i, \sum_{i\in J}\theta_i(x) = 1 \text{ for } x \in \Omega.$$

Applying now Theorem 9.4 with the N-function (homogeneous and isotropic) m_1 we obtain

$$\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{N}{N-1}}\,\mathrm{d}x \le C_N\left(\int_\Omega m_1(|\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\right)^{\frac{N}{N-1}}$$

for a.a. $t \in [0,T]$, where $\mathbf{u}^{\delta,\lambda,\varepsilon}$ is defined as in (7.151). Consequently

$$\int_0^T\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{N}{N-1}}\,\mathrm{d}x\,\mathrm{d}t \le C_N\int_0^T\left(\int_\Omega (m_1(|\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\right)^{\frac{N}{N-1}}\mathrm{d}t.$$

By definition of m_1, see (7.144), and as $T < \infty$ we find that

$$\begin{aligned}\int_0^T\int_\Omega (m_1(|\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{N}{N-1}}\,\mathrm{d}x\,\mathrm{d}t &\le C_N\int_0^T\left(\int_\Omega M(\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\right)^{\frac{N}{N-1}}\mathrm{d}t\\ &\le C_{T,N}\sup_{t\in[0,T]}\left(\int_\Omega M(\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\right)^{\frac{N}{N-1}}.\end{aligned} \tag{7.162}$$

Next we show that the right-hand side of (7.162) is bounded for fixed δ. To this end we use Jensen's inequality, Fubini's theorem and the nonnegativity of M in the following way

$$\begin{aligned}\int_\Omega M(\mathbf{Du}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x &\le \int_\Omega\int_{B_\delta} M(\mathbf{Du}^{\lambda,\varepsilon}(t-\tau,x))\sigma_\delta(\tau)\,\mathrm{d}\tau\mathrm{d}x\\ &= \int_{B_\delta}\int_\Omega M(\mathbf{Du}^{\lambda,\varepsilon}(t-\tau,x))\sigma_\delta(\tau)\,\mathrm{d}x\mathrm{d}\tau\\ &\le \|\sigma_\delta\|_{L^\infty(B(0,\delta))}\|M(\mathbf{Du}^{\lambda,\varepsilon})\|_{L^1(B(t,\delta)\times\Omega)}\\ &\le \|\sigma_\delta\|_{L^\infty(B(0,\delta))}\|M(\mathbf{Du}^{\lambda,\varepsilon})\|_{L^1(\Omega_T)}.\end{aligned} \tag{7.163}$$

As $m_2(r) \le c_m((m_1(r))^{\frac{N}{N-1}} + |r|^2 + 1)$ and $\nabla\theta_i \in L^\infty(\Omega;\mathbb{R}^N)$ we get that

$$\begin{aligned}(\mathbf{D}(\mathbf{u}^{\delta,\lambda})\theta_i^\lambda)^\varepsilon + \frac{1}{2}(\mathbf{u}^\delta\otimes\nabla\theta_i)^{\lambda,\varepsilon} + \frac{1}{2}(\nabla\theta_i\otimes\mathbf{u}^\delta)^{\lambda,\varepsilon}&\\ = \mathbf{D}(\mathbf{u}^\delta\theta_i)^{\lambda,\varepsilon} \in L_M(\Omega_T^i;\mathbb{R}^{N\times N}_{\mathrm{sym}}),&\end{aligned}$$

where $\Omega_T^i = (0,T)\times\Omega^i$ with $\Omega^i = \operatorname{supp}\theta_i$.

Let us concentrate now on the function

$$\mathbf{u}^{\delta,\lambda,\varepsilon}(t,x) = \sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda,$$

where $\{\cdot\}^\lambda$ is defined by (7.150). Notice that $\mathbf{u}^{\delta,\lambda,\varepsilon}$ is in general not divergence-free. Therefore we introduce for a.a. $t \in (0,T)$ the function $\varphi^{\lambda,\varepsilon}(t,\cdot) \in L_{m_1^{\frac{N}{N-1}}}(\Omega;\mathbb{R}^N)$ which for a.a. $t \in (0,T)$ is a solution to the problem

$$\begin{aligned} \operatorname{div}\varphi^{\lambda,\varepsilon}(t,\cdot) &= \sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}(t,\cdot)\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i\right\}^\lambda \quad \text{in } \Omega \\ \varphi^{\lambda,\varepsilon}(t,\cdot) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{7.164}$$

The existence of such a $\varphi^{\lambda,\varepsilon}$ is provided by Proposition 8.60 applied to the above problem (7.164) with N-function $m_1^{\frac{N}{N-1}}$ that satisfies the Δ_2-condition. Note that the quasiconvexity condition in Proposition 8.60 is satisfied with $\gamma = \frac{N-1}{N}$. Then we can follow the case of star-shaped domains to complete the proof. The difference is that instead of the sequence defined by (7.151), in order to show the integration by parts formula, we consider

$$\psi^{\delta,\lambda,\varepsilon}(t,x) := \sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}(t,x)\, \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda - \varphi^{\lambda,\varepsilon}(t,x).$$

It remains to show that $\varphi^{\lambda,\varepsilon}$ vanishes in the limit as $\lambda \to 1$ and $\varepsilon \to 0$. To this end we notice that Proposition 8.60 implies the estimate

$$\begin{aligned} \int_\Omega m_1^{\frac{N}{N-1}}(|\mathbf{D}\varphi^{\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x &\le \int_\Omega m_1^{\frac{N}{N-1}}(|\nabla\varphi^{\lambda,\varepsilon}|)\,\mathrm{d}x \\ &\le c\int_\Omega m_1^{\frac{N}{N-1}}(|\sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i\right\}^\lambda|) \end{aligned} \tag{7.165}$$

for a.a. $t \in (0,T)$. Let us integrate (7.165) over the time interval $(0,T)$. Since for every $i \in J$, see (7.161), the sequence

$$\varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i\right\}^\lambda \xrightarrow{m_1^{\frac{N}{N-1}}} \left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i \quad \text{modularly in } L_{m_1^{\frac{N}{N-1}}}(\Omega_T)$$

as $\varepsilon \to 0$ and $\lambda \to 1$ and $\sum_{i\in J}\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i = 0$, we immediately conclude that

$$\sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\cdot\nabla\theta_i\right\}^\lambda \xrightarrow{m_1^{\frac{N}{N-1}}} 0 \quad \text{modularly in } L_{m_1^{\frac{N}{N-1}}}(\Omega_T) \tag{7.166}$$

as $\varepsilon \to 0$ and $\lambda \to 1$. Therefore

$$\mathbf{D}\varphi^{\lambda,\varepsilon} \xrightarrow{m_1^{\frac{N}{N-1}}} 0 \quad \text{modularly in } L_{m_1^{\frac{N}{N-1}}}(\Omega_T;\mathbb{R}^{N\times N}). \tag{7.167}$$

Next we use the same arguments as for the star-shaped domain case. However instead of the function defined by (7.152), we test the weak formulation (7.160) with

$$\zeta^{\delta,\lambda,\varepsilon}(t,x) := \sum_{i\in J} \varrho_\varepsilon * \left\{\sigma_\delta * \left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda - \sigma_\delta * \left(\sigma_\delta * \varphi^{\lambda,\varepsilon}(t,x)\, \mathbb{1}_{(s_0,s)}\right). \tag{7.168}$$

As a result we obtain the analogue of (7.153). Then in order to pass to the limit as $\lambda \to 1$ and $\varepsilon \to 0$ it remains to show that terms corresponding to the second part of the test function (7.168) vanish, i.e., the following three related limits hold

$$\lim_{\varepsilon\to 0,\lambda\to 1} \int_{s_0}^{s} \int_\Omega (\mathbf{u} * \sigma_\delta) \cdot \partial_t \left(\sigma_\delta * \varphi^{\lambda,\varepsilon}(t,x)\, \mathbb{1}_{(s_0,s)}\right) \mathrm{d}x\, \mathrm{d}t = 0, \tag{7.169}$$

$$\lim_{\varepsilon\to 0,\lambda\to 1} \int_0^T \int_\Omega \overline{\mathbf{S}} : \sigma_\delta * \left(\sigma_\delta * \mathbf{D}\varphi^{\lambda,\varepsilon}(t,x)\, \mathbb{1}_{(s_0,s)}\right) \mathrm{d}x\, \mathrm{d}t = 0, \tag{7.170}$$

$$\lim_{\varepsilon\to 0,\lambda\to 1} \int_0^T \int_\Omega \mathbf{f} \cdot \sigma_\delta * \left(\sigma_\delta * \varphi^{\lambda,\varepsilon}(t,x)\, \mathbb{1}_{(s_0,s)}\right) \mathrm{d}x\, \mathrm{d}t = 0. \tag{7.171}$$

To show (7.169) we apply Proposition 8.60 with the N-function $m = |\cdot|^2$ and the Poincaré inequality, which gives us

$$\begin{aligned} \|\varphi^{\lambda,\varepsilon}(t,\cdot)\|_{L^2(\Omega)} &\le c_1 \|\nabla\varphi^{\lambda,\varepsilon}(t,\cdot)\|_{L^2(\Omega)} \\ &\le c_2 \Big\|\sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}(t,\cdot)\mathbb{1}_{(s_0,s)}\right) \cdot \nabla\theta_i\right\}^\lambda \Big\|_{L^2(\Omega)} \end{aligned} \tag{7.172}$$

for a.a. $t \in (0,T)$. Since the term on the left-hand side of (7.169) is equivalent to

$$\int_{s_0}^{s} \int_\Omega (\mathbf{u} * \sigma_\delta) \cdot \left(\varphi^{\lambda,\varepsilon} * \partial_t \sigma_\delta\right) \mathrm{d}x\, \mathrm{d}t,$$

we pass to the limit using the fact that

$$\sum_{i\in J} \varrho_\varepsilon * \left\{\left(\sigma_\delta * \mathbf{u}\, \mathbb{1}_{(s_0,s)}\right) \cdot \nabla\theta_i\right\}^\lambda \overset{*}{\rightharpoonup} 0 \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)),$$

thus

$$\varphi^{\lambda,\varepsilon} \overset{*}{\rightharpoonup} 0 \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^N)) \text{ as } \varepsilon \to 0 \text{ and } \lambda \to 1.$$

Now let us concentrate on the limit (7.170). Due to (7.167), assumption (7.159), and estimate (7.172) we have that

$$\{M(\alpha\mathbf{D}\varphi^{\lambda,\varepsilon})\}_{\lambda,\varepsilon} \text{ is uniformly integrable with some } \alpha > 0. \tag{7.173}$$

Moreover, by Theorem 3.4.4 the convergence (7.167) implies

$$\mathbf{D}\varphi^{\lambda,\varepsilon} \to 0 \quad \text{in measure.}$$

Hence using again Theorem 3.4.4 with a function M we conclude that

$$\mathbf{D}\varphi^{\lambda,\varepsilon} \to 0 \quad \text{modularly in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}}) \text{ as } \varepsilon \to 0 \text{ and } \lambda \to 1.$$

Hence (7.170) is satisfied.

Finally, the limit in (7.171) is a consequence of (7.165), (7.166) and Hölder's inequality, which all imply that

$$\nabla\varphi^{\lambda,\varepsilon} \to 0 \quad \text{modularly in } L_{m_1}(\Omega_T;\mathbb{R}^{N\times N})$$

and since $\varphi = 0$ on $\partial\Omega$ we obtain

$$\varphi^{\lambda,\varepsilon} \to 0 \quad \text{modularly in } L_{m_1}(\Omega_T;\mathbb{R}^d) \text{ as } \varepsilon \to 0 \text{ and } \lambda \to 1.$$

Now to complete the proof of Lemma 7.2 we follow the case of star-shaped domains. □

7.3.3 The proof of existence

With tools from the previous section at hand let us concentrate now on completing the proof of Theorem 7.3.1

We start with the construction of Galerkin approximations to (7.134)–(7.137) using a basis $\{\omega_i\}_{i=1}^{\infty}$ consisting of eigenvectors of the Stokes operator. We define

$$\mathbf{u}^n := \sum_{i=1}^{n} \alpha_i^n(t)\omega_i,$$

where the $\alpha_i^n(t)$ solve the system

$$\begin{aligned} \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^n \cdot \omega_i \,\mathrm{d}x + \int_\Omega \mathbf{S}(t,x,\mathbf{D}\mathbf{u}^n) : \mathbf{D}\omega_i \,\mathrm{d}x &= \int_\Omega \mathbf{f}\cdot\omega_i \,\mathrm{d}x, \\ \mathbf{u}^k(0) &= P^n\mathbf{u}_0, \end{aligned} \tag{7.174}$$

where $i = 1,\ldots,n$ and

P^n denotes the orthogonal projection of $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N)$ on $\mathrm{conv}\{\omega_1,\ldots,\omega_n\}$.

Let us observe that the system (7.174) can be rewritten as a system of ordinary differential equations. We obtain local in time solvability – existence of $\alpha_i^n(t)$ – due to the Peano existence theorem for systems of ordinary differential equations. According to the uniform bounds on $\mathbf{u}^n$ presented in what follows, the existence of $\alpha_i^n(t)$ can be shown globally for any finite time. Here we skip the details, since one can adapt the arguments from [245, Section 5.2], see also [328, 134].

Multiplying each equation of (7.174) by $\alpha_i^n(t)$, summing over $i = 1, \dots, n$ we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}^n\|^2_{L^2(\Omega)} + \int_\Omega \mathbf{S}(t,x,\mathbf{Du}^n) : \mathbf{Du}^n \,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{u}^n \,\mathrm{d}x. \tag{7.175}$$

The Fenchel–Young inequality (see Lemma 2.1.32), Hölder's inequality, Theorem 9.4 and convexity of the N-function tell us that

$$\begin{aligned}\left|\int_\Omega \mathbf{f}\cdot\mathbf{u}^n \,\mathrm{d}x\right| &\le \int_\Omega \left|\frac{2\tilde{c}}{c}\mathbf{f}\cdot\frac{c}{2\tilde{c}}\mathbf{u}^n\right| \mathrm{d}x \\ &\le \int_\Omega m_1^*\left(\frac{2\tilde{c}}{c}|\mathbf{f}|\right)\mathrm{d}x + \int_\Omega m_1\left(\frac{c}{2\tilde{c}}|\mathbf{u}^n|\right)\mathrm{d}x \\ &\le \int_\Omega m_1^*\left(\frac{2\tilde{c}}{c}|\mathbf{f}|\right)\mathrm{d}x + |\Omega|^{\frac{1}{N}}\left(\int_\Omega m_1\left(\frac{c}{2\tilde{c}}|\mathbf{u}^n|\right)\mathrm{d}x\right)^{\frac{N-1}{N}} \\ &\le \int_\Omega m_1^*\left(\frac{2\tilde{c}}{c}|\mathbf{f}|\right)\mathrm{d}x + |\Omega|^{\frac{1}{N}}C_N\int_\Omega m_1\left(\frac{c}{2\tilde{c}}|\mathbf{Du}^n|\right)\mathrm{d}x \\ &\le \int_\Omega m_1^*\left(\frac{2\tilde{c}}{c}|\mathbf{f}|\right)\mathrm{d}x + \frac{c}{2}\int_\Omega M(\mathbf{Du}^n)\,\mathrm{d}x.\end{aligned} \tag{7.176}$$

In (7.176) we choose constants such that $\max(|\Omega|^{\frac{1}{N}}C_N, \frac{c}{2}) < \tilde{c} < \infty$, where C_N comes from Theorem 9.4. To explain the last inequality in (7.176) we use the relation between m_1 and M, the convexity of M, and that $M(0) = 0$ and $0 < c \le 1$ (which is an obvious consequence of combining (7.138) with the Fenchel–Young inequality).

Let us now integrate (7.175) over the time interval $(0,t)$ with $t \le T$. Using estimate (7.176) and the coercivity condition (S2s) on $\mathbf{S}$ we obtain the following estimates

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}^n(t)\|^2_{L^2(\Omega)} + \frac{c}{2}\int_0^t\int_\Omega M(\mathbf{Du}^n)\,\mathrm{d}x\,\mathrm{d}t + c\int_0^t\int_\Omega M^*(\mathbf{S}(t,x,\mathbf{Du}^n))\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_0^t\int_\Omega m_1^*\left(\frac{2\tilde{c}}{c}|\mathbf{f}|\right)\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega)},\end{aligned} \tag{7.177}$$

for all $t \in (0,T]$. Hence due to the Banach–Alaoglu theorem (see Theorem 8.31) there exists a subsequence such that

$$\mathbf{Du}^n \overset{*}{\rightharpoonup} \mathbf{Du} \quad \text{weakly-}* \text{ in } L_M(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}})$$

and

$$\mathbf{S}(\cdot,\cdot,\mathbf{Du}^n) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \quad \text{weakly-}* \text{ in } L_{M^*}(\Omega_T;\mathbb{R}^{N\times N}_{\mathrm{sym}}).$$

From (7.177) we conclude also that

$$\|\mathbf{u}^n\|_{L^\infty(0,T;L^2(\Omega))} \le C \tag{7.178}$$

and consequently we have at least for a subsequence

$$\mathbf{u}^n \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N)).$$

After passing to the limit as $n \longrightarrow \infty$ in (7.174) by the density arguments we obtain the following identity

$$-\int_{\Omega_T} \mathbf{u}\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \overline{\mathbf{S}} : \mathbf{D}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \mathbf{f}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} \mathbf{u}_0\cdot\varphi(0,x)\,\mathrm{d}x \quad (7.179)$$

for all $\varphi \in C_c^\infty((-\infty,T);\mathcal{V})$.

In the remaining part of the proof we will concentrate on the characterization of the limit $\overline{\mathbf{S}}$. Since the weak-$*$ and modular limits coincide here, Lemma 7.1 for star-shaped domains or Lemma 7.2 for non-star-shaped domains and the equality (7.179) provide

$$\frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^{s}\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \quad (7.180)$$

for a.a. $0 < s_0 < s < T$. To pass to the limit as $s_0 \to 0$ we need to establish the continuity of $\mathbf{u}$ with respect to time in the weak topology in $L^2(\Omega;\mathbb{R}^N)$. For this reason let us concentrate for a while on the sequence $\{\frac{\mathrm{d}\mathbf{u}^n}{\mathrm{d}t}\}$. Taking $\varphi \in L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^N))$, $\|\varphi\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega))} \le 1$, where $r > \frac{N}{2}+1$, we observe that

$$\left\langle \frac{\mathrm{d}\mathbf{u}^n}{\mathrm{d}t},\varphi\right\rangle = \left\langle \frac{\mathrm{d}\mathbf{u}^n}{\mathrm{d}t},P^k\varphi\right\rangle = -\int_\Omega \mathbf{S}(t,x,\mathbf{D}\mathbf{u}^n) : \mathbf{D}(P^n\varphi)\,\mathrm{d}x + \int_\Omega \mathbf{f}\cdot(P^n\varphi)\,\mathrm{d}x. \quad (7.181)$$

As $\|P^n\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)} \le \|\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)}$ and $W^{r-1,2}(\Omega) \subset L^\infty(\Omega)$ we get the following

$$\begin{aligned}
\left|\int_0^T\int_\Omega \mathbf{S}(t,x,\mathbf{D}\mathbf{u}^n) : \mathbf{D}(P^n\varphi)\,\mathrm{d}x\,\mathrm{d}t\right| &\le \int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)}\|\mathbf{D}(P^n\varphi)\|_{L^\infty(\Omega)}\,\mathrm{d}t\\
&\le c\int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)}\|P^n\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)}\,\mathrm{d}t\\
&\le c\int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)}\|\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)}\,\mathrm{d}t\\
&\le c\|\mathbf{S}(\cdot,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega_T)}\|\varphi\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega))}
\end{aligned} \quad (7.182)$$

and

$$\begin{aligned}
\left|\int_0^T\int_\Omega \mathbf{f}\cdot P^n\varphi\,\mathrm{d}x\,\mathrm{d}t\right| &\le \int_0^T \|\mathbf{f}\|_{L^1(\Omega)}\|P^n\varphi\|_{L^\infty(\Omega)}\,\mathrm{d}t\\
&\le c\int_0^T \|\mathbf{f}\|_{L^1(\Omega)}\|P^n\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)}\,\mathrm{d}t \le c\int_0^T \|\mathbf{f}\|_{L^1(\Omega)}\|\varphi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)}\,\mathrm{d}t\\
&\le c\|\mathbf{f}\|_{L^1(\Omega_T)}\|\varphi\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega))}.
\end{aligned} \quad (7.183)$$

By the assumptions on $\mathbf{f}$ and by (7.177) we have that $\{\mathbf{f}\}_n$ and $\{\mathbf{S}(\cdot,\cdot,\mathbf{D}\mathbf{u}^n)\}_N$ are bounded in L^1. Therefore we conclude that

$$\frac{\mathrm{d}\mathbf{u}^n}{\mathrm{d}t} \text{ is bounded in } L^1(0,T;(W^{r,2}_{0,\mathrm{div}})^*(\Omega)).$$

By (7.177) and the assumptions on $\mathbf{f}$ there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_{\Omega_T} \left[M^*(\mathbf{S}(t,x,\mathbf{D}\mathbf{u}^n)) + m_1^*(|\mathbf{f}|)\right] \mathrm{d}x\,\mathrm{d}t \le C.$$

Since $m_2^* \le M^*$ by Lemma 2.1.37 and by Jensen's inequality, we obtain

$$\sup_{k\in\mathbb{N}} |\Omega| \int_0^T \left[m_2^*(\|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)}) + m_1^*(\|\mathbf{f}\|_{L^1(\Omega)})\right] \mathrm{d}t < C.$$

Hence by Theorem 3.4.2 the sequence $\{\|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)}\}_n$ and $\|\mathbf{f}\|_{L^1(\Omega)}$ are uniformly integrable in $L^1(0,T)$. Then we notice that we can find a monotone, continuous function $L:\mathbb{R}_+ \to \mathbb{R}_+$, with $L(0) = 0$ which is independent of n and

$$\int_{s_1}^{s_2} \left(\|\mathbf{S}(t,\cdot,\mathbf{D}\mathbf{u}^n)\|_{L^1(\Omega)} + \|\mathbf{f}\|_{L^1(\Omega)}\right) \mathrm{d}t \le L(|s_1 - s_2|)$$

for any $s_1, s_2 \in [0,T]$. Consequently, estimates (7.182)–(7.183) and (7.181) provide that

$$\left|\int_{s_1}^{s_2} \left\langle \frac{\mathrm{d}\mathbf{u}^n}{\mathrm{d}t}, \varphi \right\rangle \mathrm{d}t\right| \le L(|s_1 - s_2|)$$

for all φ with $\operatorname{supp}\varphi \subset (s_1,s_2) \subset [0,T]$ and $\|\varphi\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega))} \le 1$. Note that

$$\begin{aligned}
&\|\mathbf{u}^n(s_1) - \mathbf{u}^n(s_2)\|_{(W^{r,2}_{0,\mathrm{div}}(\Omega))^*} \\
&= \sup_{\|\psi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)} \le 1} |\langle \mathbf{u}^n(s_1) - \mathbf{u}^n(s_2), \psi\rangle| = \sup_{\|\psi\|_{W^{r,2}_{0,\mathrm{div}}(\Omega)} \le 1} \left|\left\langle \int_{s_1}^{s_2} \frac{\mathrm{d}\mathbf{u}^n(t)}{\mathrm{d}t}, \psi \right\rangle\right| \\
&\le \sup\left\{ \int_0^T \left|\left\langle \frac{\mathrm{d}\mathbf{u}^n(\tau)}{\mathrm{d}\tau}, \varphi \right\rangle\right| \mathrm{d}t \;:\; \|\varphi\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}}(\Omega))} \le 1,\ \operatorname{supp}\varphi \subset (s_1,s_2) \right\}.
\end{aligned}$$

The above implies that

$$\sup_{n\in\mathbb{N}} \|\mathbf{u}^n(s_1) - \mathbf{u}^n(s_2)\|_{(W^{r,2}_{0,\mathrm{div}}(\Omega))^*} \le L(|s_1 - s_2|). \tag{7.184}$$

The estimate (7.184) ensures that the family of functions

$$\{\mathbf{u}^n : [0,T] \to (W^{r,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^N)^*\}_n \text{ is equicontinuous.}$$

By (7.178) and by the compact embedding $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N) \subset\subset (W^{r,2}_{0,\mathrm{div}})^*$ (as $r > \frac{N}{2} + 1$) we infer by the Arzelà–Ascoli theorem that the sequence

$$\{\mathbf{u}^n\}_{n=1}^\infty \text{ is relatively compact in } C([0,T];(W^{r,2}_{0,\mathrm{div}}(\Omega))^*).$$

Therefore $\mathbf{u} \in C([0,T];(W^{r,2}_{0,\mathrm{div}}(\Omega))^*)$ and

$$\mathbf{u}(s_0^i) \overset{i\to\infty}{\longrightarrow} \mathbf{u}(0) \quad \text{strongly in } (W^{r,2}_{0,\mathrm{div}}(\Omega))^*. \tag{7.185}$$

On the other hand $\mathbf{u} \in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N))$, and we can choose a sequence $\{s_0^i\}_i$, $s_0^i \to 0^+$ as $i \to \infty$, such that

$$\mathbf{u}(s_0^i) \rightharpoonup \mathbf{u}(0) \quad \text{weakly in } L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N).$$

The limit (7.185) coincides with the above weak limit in $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N)$ and therefore we infer that

$$\liminf_{i\to\infty} \|\mathbf{u}(s_0)\|_{L^2(\Omega)} \geq \|\mathbf{u}_0\|_{L^2(\Omega)}. \tag{7.186}$$

Let s be any Lebesgue point of $\mathbf{u}$. Integration of (7.175) over the time interval $(0,s)$ gives

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^s \int_\Omega & \mathbf{S}(t,x,\mathbf{Du}^n) : \mathbf{Du}^n \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^s \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega)} - \liminf_{n\to\infty} \frac{1}{2}\|\mathbf{u}^n(s)\|^2_{L^2(\Omega)} \\
&\leq \int_0^s \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} \\
&\leq \liminf_{i\to\infty} \left(\int_{s_0^i}^s \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}(s_0^i)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} \right) \\
&= \lim_{i\to\infty} \int_{s_0^i}^s \int_\Omega \overline{\mathbf{S}} : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t = \int_0^s \int_\Omega \overline{\mathbf{S}} : \mathbf{Du}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{7.187}$$

For the last two steps in the above we used (7.186) and (7.180). The monotonicity of $\mathbf{S}$ provides that

$$\int_0^s \int_\Omega (\mathbf{S}(t,x,\mathbf{v}) - \mathbf{S}(t,x,\mathbf{Du}^n)) : (\mathbf{v} - \mathbf{Du}^n)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{7.188}$$

holds for all $\mathbf{v} \in L^\infty(\Omega_T;\mathbb{R}^{N\times N})$. Using (7.187) and (7.188) we follow the same steps as in Section 4.1.2 or in Section 7.2.3 to show by the monotonicity trick

$$\overline{\mathbf{S}} = \mathbf{S}(t,x,\mathbf{Du}) \text{ a.e. in } \Omega_T.$$

This finishes the proof of Theorem 7.3.1.

7.4 Local Pressure and the Fluid-Structure Interaction Problem for Non-Newtonian Fluids

In this section we provide a decomposition and local estimates for the pressure function for the non-stationary flow of incompressible non-Newtonian fluids. We show also that this method can be applied to prove the existence of weak solutions to the problem of motion of one or several rigid bodies in a non-Newtonian incompressible fluid with growth conditions given by a homogenous and isotropic N-function Δ_2-condition. The below considerations are based on [327, 329]

7.4.1 Decomposition of the pressure function and local estimates

Let us begin by recalling some basic properties of particular homogenous and isotropic N-functions, which will justify the forthcoming assumptions in this and in the next section. Let $\beta, \gamma \in (0,\infty)$ and $\tau \in [0,\infty)$. Let us denote by $L\log^{\beta} L(\Omega)$ the Orlicz space associated with the N-function $m : [0,\infty) \longrightarrow [0,\infty)$, $m(\tau) = \tau(\log(\tau+1))^{\beta}$ and by $L_{\exp^{\gamma}}(\Omega)$ the Orlicz space associated with the N-function defined for $\tau > 1$ by $\widetilde{m}(\tau) = \exp(\tau^{\gamma})$. Note that $L\log^{\beta} L = E\log^{\beta} L(\Omega)$,

$$\left(E_{\exp^{\gamma}}(\Omega)\right)^{*} = L\log^{1/\gamma} L(\Omega) \quad \text{and} \quad \left(L\log^{\beta} L(\Omega)\right)^{*} = L_{\exp^{1/\beta}}(\Omega)$$

hold, see [221].

The following result concerning local reconstruction of the pressure function holds:

Theorem 7.4.1 *Let $B \subset \mathbb{R}^3$ be a bounded domain with a regular C^3 boundary ∂B and $I = (t_0, t_1)$ be a finite time interval. Let $\tilde{m}_1, \tilde{m}_2 : [0,\infty) \longrightarrow [0,\infty)$ be homogeneous and isotropic N-functions defined for $\beta > 0$ by*

$$\begin{aligned} \tilde{m}_1(\tau) &= \tau\log^{\beta+1}(\tau+1), \\ \tilde{m}_2(\tau) &= \tau\log^{\beta}(\tau+1). \end{aligned} \tag{7.189}$$

Let $M_3 : (I\times B)\times\mathbb{R}^{3\times3} \longrightarrow [0,\infty)$ be an N-function such that for some $c_1, c_2 > 0$

$$c_1\tilde{m}_1(|\xi|) \le M_3(t,x,\xi) < c_2|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^{3\times3}. \tag{7.190}$$

Assume that $\mathbf{U} \in L^{\infty}(I; L^2(B;\mathbb{R}^3))$, $\operatorname{div}\mathbf{U} = 0$ in a weak sense, namely

$$\int_I\int_B \mathbf{U}\cdot\nabla\psi \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \textit{for all } \psi \in C_c^{\infty}(I\times B),$$

and $\mathbf{T} \in L_{M_3}(I\times B;\mathbb{R}^{3\times3})$ satisfy the integral identity

$$\int_I\int_B \left(\mathbf{U}\cdot\partial_t\varphi + \mathbf{T} : \nabla\varphi\right)\mathrm{d}x\,\mathrm{d}t = 0 \tag{7.191}$$

for all $\varphi \in C_c^\infty(I\times B;\mathbb{R}^3)$, $\operatorname{div}\varphi = 0$.

Then there exist two functions, called regular and harmonic components, π_{reg}, π_{harm}, *such that*

$$\pi_{\text{reg}} \in L^1(I; L\log^\beta L(B)), \qquad \int_B \pi_{\text{harm}}(t,\cdot)\,\mathrm{d}x = 0 \quad \textit{for a.a. } t\in I,$$

$$\pi_{\text{harm}}(t,\cdot) \in (C_c^\infty(B))^*,\ \Delta\pi_{\text{harm}} = 0 \textit{ in } (C_c^\infty(I\times B))^*,$$

namely

$$\int_I\int_B \pi_{\text{harm}}\Delta\psi\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \textit{for all } \psi\in C_c^\infty(I\times B),$$

and π_{reg}, π_{harm} *satisfy*

$$\int_I\int_B \Big(\mathbf{U}\cdot\partial_t\varphi + \mathsf{T}:\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_B \Big(\pi_{\text{harm}}\partial_t \operatorname{div}\varphi + \pi_{\text{reg}}\operatorname{div}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \qquad (7.192)$$

for any $\varphi\in C_c^\infty(I\times B;\mathbb{R}^3)$. *Moreover,*

$$\|\pi_{\text{reg}}\|_{L_1(I;L\log^\beta L(B))} \le c(\tilde{m}_2)\|\mathsf{T}\|_{L_{M_3}(I\times B;\mathbb{R}^{3\times 3})} \qquad (7.193)$$

and

$$\pi_{\text{harm}}(t,\cdot)|_{B'} \in C^\infty(B'),\ \textit{where } B'\subset\subset B, \qquad (7.194)$$

$$\|\pi_{\text{harm}}\|_{L^\infty(I;L^1(B))} \le c(M_2,I,B)\Big(\|\mathsf{T}\|_{L_{M_3}(I\times B;\mathbb{R}^3)} + \|\mathbf{U}\|_{L^\infty(I;L^2(B;\mathbb{R}^3))}\Big). \qquad (7.195)$$

Proof. Let us start with the 'regular' component of the pressure π_{reg}, which we define as

$$\pi_{\text{reg}}(t,\cdot) := \mathcal{R}:\mathsf{T} = \sum_{i,j=1}^3 \mathcal{R}_{i,j}[T_{i,j}](t,\cdot) \text{ in } \mathbb{R}^3 \text{ for a.a. } t\in I,$$

where $\mathcal{R}$ denotes the 'double' Riesz transform (see (8.7)) and $\mathsf{T} = [T_{i,j}]_{i;j}$, $i=1,2,3$, $j=1,2,3$, has been extended by zero outside of B. Using Lemma 8.61 (see (8.9)) we find that the mappings

$$\mathcal{R}_{i,j}|_B : L\log^{\beta+1}L(B) \to L\log^\beta L(B) \quad \text{are bounded for } i,j=1,2,3.$$

Therefore we get (7.193) by the following

$$\begin{aligned}\|\pi_{\text{reg}}\|_{L^1(I;L\log^\beta L(B))} &= \|\mathcal{R}:\mathsf{T}\|_{L^1(I;L\log^\beta L(B))} \le c_1(\tilde{m}_2)\|\mathsf{T}\|_{L^1(I;L\log^{\beta+1}L(B))}\\ &\le c_2(\tilde{m}_2)\|\mathsf{T}\|_{L\log^{\beta+1}L(I\times B)} \le c_3(\tilde{m}_2)\|\mathsf{T}\|_{L_{M_3}(I\times B)},\end{aligned} \qquad (7.196)$$

where we use the fact that $L_{M_3}(I\times B;\mathbb{R}^{3\times 3}) \subset L^1(I;L_{M_3}(B;\mathbb{R}^{3\times 3}))$ and (7.190).

By definition of π_{reg} and the double Riesz transform, see (8.7),

$$\int_B \pi_{\text{reg}}\Delta\psi\,\mathrm{d}x = \int_B \mathsf{T}:\nabla^2\psi\,\mathrm{d}x \quad \text{for any } \psi\in C_c^\infty(B). \qquad (7.197)$$

Notice that by (7.191) we can redefine $\mathbf{U}$ with respect to time on a set of zero measure such that the mappings

$$t \mapsto \int_B \mathbf{U} \cdot \psi \, \mathrm{d}x \text{ are continuous on } \bar{I} \text{ for any } \psi \in C_c^\infty(B;\mathbb{R}^3) \text{ with } \operatorname{div}\psi = 0.$$

Then we find that the Helmholtz projection of $\mathbf{U}$ on the space of divergence-free functions is continuous in time with respect to the weak topology in $L^2(B;\mathbb{R}^3)$. Then considering our equation (7.191) with test function $\varphi(t,x) = \eta(t)\psi(x)$ such that $\eta \in C_c^\infty(I)$, $\psi \in C_c^\infty(B;\mathbb{R}^3)$, $\operatorname{div}\psi = 0$ we infer that

$$\int_I \Big[\int_B (\mathbf{U}(t,\cdot) - \mathbf{U}(t_0,\cdot)) \cdot \psi \, \mathrm{d}x \Big] \partial_t \eta \, \mathrm{d}t - \int_I \Big[\int_B \Big(\int_{t_0}^t \mathsf{T}(s,\cdot) \, \mathrm{d}s \Big) : \nabla\psi \, \mathrm{d}x \Big] \partial_t \eta \, \mathrm{d}t = 0.$$

By the above and Theorems 8.46 and 8.45, there exists a function (pressure function) $\pi = \pi(t,\cdot)$ such that

$$\int_B (\mathbf{U}(t,\cdot) - \mathbf{U}(t_0,\cdot)) \cdot \psi \, \mathrm{d}x - \int_B \Big(\int_{t_0}^t \mathsf{T}(s,\cdot) \, \mathrm{d}s \Big) : \nabla\psi \, \mathrm{d}x = \int_B \pi(t,\cdot) \operatorname{div}\psi \, \mathrm{d}x \tag{7.198}$$

for all $t \in I$ and all $\psi \in C_c^\infty(B;\mathbb{R}^3)$. Notice that the term on the right-hand side is measurable and integrable with respect to the time variable, since the left-hand side is measurable and integrable. Moreover, by Theorem 8.45 and 8.46 for a.a. $t \in I$

$$\int_B \pi(t,\cdot) \, \mathrm{d}x = 0 \quad \text{and} \quad \pi(t,\cdot) \in (C_c^\infty(B))^*. \tag{7.199}$$

Let us test (7.198) by $\partial_t \zeta$, $\zeta \in C_c^\infty(I)$, and integrate over the time interval I. Then setting $\varphi(t,x) = \zeta(t)\psi(x)$ we find that

$$\int_I \int_B \Big(\mathbf{U} \cdot \partial_t \varphi + \mathsf{T} : \nabla\varphi \Big) \mathrm{d}x \, \mathrm{d}t = \int_I \int_B \pi \partial_t \operatorname{div} \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{7.200}$$

for any $\varphi \in C_c^\infty(I \times B;\mathbb{R}^3)$.

Define now the harmonic part of the pressure function in the following way:

$$\pi_{\mathrm{harm}}(t,\cdot) = \pi(t,\cdot) + \Big(\int_{t_0}^t \Big[\pi_{\mathrm{reg}}(\tau,\cdot) - \frac{1}{|B|} \int_B \pi_{\mathrm{reg}}(\tau,\cdot) \, \mathrm{d}x \Big] \, \mathrm{d}\tau \Big). \tag{7.201}$$

Next our aim is to show that $p_{\mathrm{harm}}(t,\cdot)$ is, in fact, a harmonic function for $t \in I$. For this reason we take test functions of the form $\psi = \nabla\gamma$, with $\gamma \in C_c^\infty(B)$ in (7.198), thus

$$\int_B (\mathbf{U}(t,\cdot) - \mathbf{U}(t_0,\cdot)) \cdot \nabla\gamma \, \mathrm{d}x - \int_B \Big(\int_{t_0}^t \mathsf{T}(s,\cdot) \, \mathrm{d}s \Big) : \nabla^2\gamma \, \mathrm{d}x = \int_B \pi(t,\cdot) \Delta\gamma \, \mathrm{d}x$$

for all $t \in I$. The first term on the left-hand side disappears after integration by parts and as $\operatorname{div}\mathbf{U} = 0$. Due to (7.197) and (7.201) we infer that

$$\int_B \pi_{\text{harm}}(t,\cdot)\Delta\gamma \, dx = 0 \quad \text{for all } \gamma \in C_c^\infty(B). \tag{7.202}$$

Inserting (7.201) into (7.200) and using the integration by parts formula we obtain also that (7.192) holds true.

Finally the Weyl lemma (see Lemma 8.62 with $f = 0$) provides that the function p_{harm} is regular locally in B, namely $\pi_{\text{harm}} \in C^\infty(B')$, where $B' \subset\subset B$ and B' has a smooth boundary. In this way we obtain (7.194).

Next our aim is to show that (7.195) holds due to (7.201) and (7.198). For this purpose we will use a result involving the Bogovskii operator in L^∞-space, see Lemma 8.58. Let us use in (7.198) a test function ψ such that

$$\operatorname{div}\psi = \left(\operatorname{sgn} p - \frac{1}{|B|}\int_B \operatorname{sgn} p\right) \in L^\infty(B).$$

By Lemma 8.58 we find in fact that $\nabla\psi \in L_{\exp}(B)$. In particular, also $\psi \in L^2(B)$. Then by the Hölder inequality, the generalized Hölder inequality and since $L_{M_3} \subset L\log^{\beta+1} L$, we infer from (7.198) that

$$\operatorname*{ess\,sup}_{t\in I} \|p(t,\cdot)\|_{L^1(B)} \le c(B,M)\left\{\|\mathbf{U}\|_{L^\infty(I;L^2(B))} + \|\mathbf{T}\|_{L_{M_3}(I\times B)}\right\}. \tag{7.203}$$

Therefore (7.201) and (7.193) ensures (7.195). □

7.4.2 *Motion of rigid bodies in non-Newtonian fluid. An application of the method*

The method of local reconstruction of the pressure function can facilitate the mathematical analysis of the motion of one or several rigid bodies immersed in an incompressible viscous fluid which occupies a bounded domain $\Omega \subset \mathbb{R}^3$. Below we present just a draft of the proof and we emphasize how the reconstruction of the pressure function from the previous section can be used in order to show the existence of weak solutions to such problem. One can find details of this result and its proof in [150, 267] for the case of power-law fluids and for the more general case of isotropic Orlicz spaces in [329].

Let us start with the assumptions on the viscous stress tensor:

(S1b) the viscous stress tensor $\mathbf{S}$ depends on the symmetric part of the gradient of the velocity field $\mathbf{u}$, i.e.

$$\mathbf{S} : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}} \text{ and } \mathbf{S}(\mathbf{0}) = \mathbf{0},\ \mathbf{S} = \mathbf{S}(\mathbf{D}\mathbf{u}) \text{ is continuous.}$$

(S2b) there exist a positive constant c and N-functions $m, m^* : [0,\infty) \to [0,\infty)$ (m^* denotes the conjugate function to m) such that

$$\mathbf{S}(\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c\big(m(|\boldsymbol{\xi}|) + m^*(|\mathbf{S}(\boldsymbol{\xi})|)\big) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\text{sym}}. \tag{7.204}$$

(S3b) $\mathbf{S}$ is monotone, i.e.

$$\big(\mathbf{S}(\boldsymbol{\xi})-\mathbf{S}(\boldsymbol{\eta})\big) : (\boldsymbol{\xi}-\boldsymbol{\eta}) \geq 0 \text{ for all } \boldsymbol{\xi} \neq \boldsymbol{\eta},\ \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}. \tag{7.205}$$

For a discussion on the form of the growth conditions in the Orlicz (Musielak–Orlicz) setting, see Section 3.8.2.

Formulation of the problem

We formulate the following problem (see [150, 329]): let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$, occupied by an incompressible fluid containing rigid bodies. The initial position of the rigid bodies is determined through a family of domains

$$S_i \subset \Omega \subset \mathbb{R}^3,\ i = 1,\ldots,n,$$

which are diffeomorphic to a ball in $\mathbb{R}^3$. In order to avoid additional difficulties the boundaries of all rigid bodies are assumed to be sufficiently regular, namely there exists a

$$\delta_0 > 0 \tag{7.206}$$

such that for any $x \in \partial S_i$ there are two closed balls $B^{\mathrm{int}}_{\delta_0}$, $B^{\mathrm{ext}}_{\delta_0}$ of radius δ_0 such that

$$x \in B^{\mathrm{int}}_{\delta_0} \cap B^{\mathrm{ext}}_{\delta_0},\ B^{\mathrm{int}}_{\delta_0} \subset \overline{S}_i,\ B^{\mathrm{ext}}_{\delta_0} \subset \mathbb{R}^3 \setminus S_i.$$

We assume the same for the considered physical space $\Omega \subset \mathbb{R}^3$, namely for any $x \in \partial\Omega$ there are two closed balls $B^{\mathrm{int}}_{\delta_0}$, $B^{\mathrm{ext}}_{\delta_0}$ of radius δ_0 such that

$$x \in B^{\mathrm{int}}_{\delta_0} \cap B^{\mathrm{ext}}_{\delta_0},\ B^{\mathrm{int}}_{\delta_0} \subset \overline{\Omega},\ B^{\mathrm{ext}}_{\delta_0} \subset \mathbb{R}^3 \setminus \Omega.$$

We represent the motion of the rigid body S_i by the associated mapping η_i such that

$$\begin{gathered} \eta_i = \eta_i(t,x),\ t \in [0,T),\ x \in \mathbb{R}^3, \\ \eta_i(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an isometry for all } t \in [0,T) \\ \text{and } \eta_i(0,x) = x \text{ for all } x \in \mathbb{R}^3,\ \ i = 1,\ldots,n. \end{gathered}$$

Let us emphasize that the position of the rigid bodies is not known a priori for $t > 0$ and depend on the flow. In particular, the position of the body S_i at time $t \in [0,T)$ is represented by the following formula

$$S_i(t) = \eta_i(t,S_i),\ i = 1,\ldots,n. \tag{7.207}$$

Using the above terms we introduce the following domains:

$$\Omega^S_T \text{ is the rigid (solid) part of the time space cylinder,}$$

$$\Omega^S_T := \bigcup_{i=1,\ldots,n} \Big\{(t,x)\ :\ t \in [0,T],\ x \in \overline{S_i}(t)\Big\}$$

and

$$\Omega_T^f \text{ is the fluid part of the time space cylinder}$$

$$\Omega_T^f := \Omega_T \setminus \Omega_T^S.$$

Let us denote the velocity field of the system by $\mathbf{u} : \Omega_T \to \mathbb{R}^3$ and decompose it for the fluid and rigid parts as follows

$$\mathbf{u}_f = \mathbf{u} \text{ on } \Omega_T^f \quad \text{and} \quad \mathbf{u}_S = \mathbf{u} \text{ on } \Omega_T^S.$$

Let $x_i(t)$ denote the position of the center of mass of S_i at time t and

$$x_i(t) = \frac{1}{\bar{m}_i} \int_{\overline{S}_i(t)} \varrho_{S_i}(t,x) x \,\mathrm{d}x, \quad \text{where} \quad \bar{m}_i = \int_{\overline{S}_i(t)} \varrho_{S_i}(t,x)\,\mathrm{d}x.$$

Here $\bar{m}_i$ is the total mass of the i-th rigid body of mass density ϱ_{S_i}.

Since the mappings η_i are isometries, we can write

$$\eta_i(t,x) = x_i(t) + \mathbf{O}_i(t)x, \quad t \in [0,T),\ i = 1,2,\ldots,n,$$

where $\mathbf{O}_i(t)$ is a matrix satisfying $\mathbf{O}_i^T\mathbf{O}_i = \mathbf{Id}$. Notice that the motion η_i is absolutely continuous. We define the translation velocity $\mathbf{U}_i(t)$ and the angular velocity $\mathbf{Q}$ of the body by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = \mathbf{U}_i(t), \qquad \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{O}_i(t)\Big)\mathbf{O}_i^T(t) = \mathbf{Q}_i(t) \text{ a.a. on } (0,T). \tag{7.208}$$

Therefore, the solid velocity in the Eulerian coordinate system can be written as

$$\mathbf{u}_{S_i}(t,x) = \frac{\partial \eta}{\partial t}(t, \eta^{-1}(t,x)) = \mathbf{U}_i(t) + \mathbf{Q}_i(t)(x - x_i(t)).$$

The total force F_{S_i} acting on $\overline{S}_i$ consists of the body force and contact force, i.e.

$$F_{S_i}(t) = \int_{\partial\overline{S}_i(t)} \mathbb{T}\mathbf{n}\,\mathrm{d}\sigma + \int_{\overline{S}_i(t)} \varrho_{S_i}\mathbf{g}_{S_i}\,\mathrm{d}x,$$

where $\mathbf{n}$ is the unit outward normal vector and $\mathbb{T}$ denotes the Cauchy stress tensor,

$$\mathbb{T} = \mathbf{S}(\mathbf{D}\mathbf{u}_f) - \pi_f \mathbf{I}.$$

The expression $\mathbb{T}\mathbf{n}$ stands for the local force applied by the fluid on the surface ∂S_i (e.g. the buoyancy force). The term $\mathbf{g}_{S_i}$ denotes the specific body (volume) force (e.g. the gravitation force). Here $\overline{S}_i(t) = \eta(t, \overline{S}_i)$. Due to Newton's second law, we have

$$m_i \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{S}_i} \varrho_{S_i}\mathbf{u}_{S_i}\,\mathrm{d}x = \int_{\partial\overline{S}_i(t)} \mathbb{T}\mathbf{n}\,\mathrm{d}\sigma + \int_{\overline{S}_i(t)} \varrho_{S_i}\mathbf{g}_{S_i}\,\mathrm{d}x. \tag{7.209}$$

As the angular velocity $\mathbf{Q}_i$ is skew symmetric by (7.208), there exists a vector ω_i such that

$$\mathbf{Q}_i(t)(x-x_i) = \omega_i(t)\times(x-x_i).$$

Assuming continuity of the stress, the balance of linear and angular momentum for the body S_i can be seen as follows

$$\mathbf{J}_i \frac{\mathrm{d}}{\mathrm{d}t}\omega_i = \mathbf{J}_i\omega_i(t)\times\omega_i(t) + \int_{\partial\overline{S}_i(t)} (x-x_i)\times\mathbb{T}\mathbf{n}\,\mathrm{d}\sigma + \int_{\overline{S}_i(t)} \varrho_{S_i}(x-x_i)\times\mathbf{g}_{S_i}\,\mathrm{d}x, \tag{7.210}$$

where $\mathbf{J}_i$ is the inertial tensor defined through

$$\mathbf{J}_i\mathbf{a}\cdot\mathbf{b} = \int_{\overline{S}_i(t)} \varrho_{S_i}(\mathbf{a}\times(x-x_i))\cdot(\mathbf{b}\times(x-x_i))\,\mathrm{d}x.$$

Notice that the equations (7.209) and (7.210) determine the motion of the rigid body $\overline{S}_i$, $i=1,\dots,n$.

Here the state of the fluid is determined by

$$\varrho_f : \Omega_T^f \to \mathbb{R}, \text{ the density of the fluid,}$$

$$\text{and } \mathbf{u}_f : \Omega_T^f \to \mathbb{R}^3, \text{ the velocity field of the fluid,}$$

and is governed by the following system of equation on the set Ω_T^f consisting of the continuity equation, balance of momentum and condition for incompressibility:

$$\partial_t\varrho_f + \operatorname{div}(\varrho_f\mathbf{u}_f) = 0, \tag{7.211}$$

$$\partial_t(\varrho_f\mathbf{u}_f) + \operatorname{div}(\varrho_f\mathbf{u}_f\otimes\mathbf{u}_f) + \nabla\pi = \operatorname{div}(\mathbf{S}) + \varrho_f\mathbf{g}_f, \tag{7.212}$$

$$\operatorname{div}\mathbf{u}_f = 0, \tag{7.213}$$

where $\pi : \Omega_T^f \to \mathbb{R}$ is the pressure and $\mathbf{g}_f : \Omega_T^f \to \mathbb{R}^3$ is the volume force (e.g. gravity).

Concerning the boundary conditions we assume there is no slip on the boundary of the physical domain $\partial\Omega$ and on the boundary of each rigid body S_i $(i=1,\dots,n)$ is assumed to coincide with the velocity of the rigid object. This means that

$$\mathbf{u}_f(t,x) = 0 \text{ on } \partial\Omega \text{ and } \mathbf{u}_f(t,x) = \mathbf{u}_S(t,x) \text{ on } \partial S(t) \text{ for all } t\in[0,T].$$

To close the system we need to specify the relation between the velocity $\mathbf{u}$ and the motion of solids given by the isometries η_i. We say that the velocity field $\mathbf{u}$ is compatible with the family of motions $\{\eta_1,\dots,\eta_n\}$ if

$$\mathbf{u}(t,x) = \mathbf{u}_{S_i}(t,x) = \mathbf{U}_i(t) + \mathbf{Q}_i(t)(x-x_i(t)) \text{ for a.a. } x\in\overline{S}_i(t),\ i=1,\dots,n \tag{7.214}$$

for a.a. $t\in[0,T)$, where $\mathbf{u}^{S_i}$ is the solid velocity. More details concerning the formulation of this problem can be found, for example, in [148].

Let us emphasize that for this problem the concept of weak solutions is based on the Eulerian reference system and on a class of test functions which depend on the position of the rigid bodies.

To be more precise: when considering the mass density $\varrho=\varrho(t,x)$, the velocity field $\mathbf{u}=\mathbf{u}(t,x)$ at time $t\in(0,T)$ and the spatial position $x\in\Omega$, those functions, governed by system of the equations (7.209), (7.210) and (7.211)–(7.213), satisfy the following integral identities

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi\Big)\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \varrho_0 \varphi\,\mathrm{d}x \quad \text{for all } \varphi \in C^1([0,T)\times\Omega), \tag{7.215}$$

and

$$\begin{aligned}&\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \mathbf{D}\varphi - \mathbf{S} : \mathbf{D}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega \varrho \mathbf{g} \cdot \varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi\,\mathrm{d}x\end{aligned} \tag{7.216}$$

for any test function $\varphi \in C_c^1([0,T)\times\Omega;\mathbb{R}^3)$ associated with the position of rigid bodies, in other words with the rigid motion. Namely

$$\varphi(t,\cdot) \in [\mathcal{RM}](t), \tag{7.217}$$

where

$$\begin{aligned}[\mathcal{RM}](t) := \big\{\phi \in C_c^1(\Omega;\mathbb{R}^3) \,:\, \operatorname{div}\phi = 0 \text{ in } \Omega, \\ \mathbf{D}\phi \text{ has compact support on } \Omega \setminus \cup_{i=1}^n \overline{S}_i(t)\big\}.\end{aligned} \tag{7.218}$$

Here the viscous stress tensor $\mathbf{S}$ is assumed to satisfy conditions (S1b)–(S3b), $\mathbf{g}$ is a given potential driving force and ϱ_0, $\mathbf{u}_0$ stand for the initial distribution of the density and the velocity, respectively.

The tensor $\mathbf{Du} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is also called a deformation rate tensor, since $\mathbf{u}$ stands for a velocity field. The kernel of this tensor is a rigid vector field. This means that if we assume that S is a connected domain in $\mathbb{R}^3$ and $\mathbf{u} : S \longrightarrow \mathbb{R}^3$ is a velocity field, then

$$\mathbf{Du} = \mathbf{0} \text{ in } S \text{ if and only if the motion is rigid.}$$

That is, there exists a vector $\mathbf{a} \in \mathbb{R}^3$ and an antisymmetric tensor $\mathbf{A} \in \mathbb{R}^{3\times 3}$ such that $\mathbf{u}(x) = \mathbf{a} + \mathbf{A}x$ for $x \in S$. For a proof of this fact see, for instance, [312, Theorem 5.1]. In other words, rigid motion means that the distance between any pair of points is conserved. Therefore we are able to determine the position of rigid bodies using the condition that the deformation rate tensor vanishes in the domains corresponding to the bodies. In particular, this observation plays an essential role in the formulation of the problem, choice of test functions in (7.217), and for the strategy of the proof of the existence of weak solutions.

Let us now formulate the existence result, which can be found with a detailed proof in [329, Theorem 4.1]. Below we give an outline of the proof.

Theorem 7.4.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ and let the following assumptions be satisfied:*

- *Let the initial position of the rigid bodies be given by a family of open sets*

$$S_i \subset \Omega \subset \mathbb{R}^3,\ S_i \text{ diffeomorphic to a ball for } i = 1, \ldots, n,$$

  where both ∂S_i, $i = 1, \ldots, n$, and $\partial\Omega$ belong to the regularity class specified by (7.206).
- *Let* $\operatorname{dist}[\overline{S}_i, \overline{S}_j] > 0$ *for* $i \neq j$, $\operatorname{dist}[\overline{S}_i, \mathbb{R}^3 \setminus \Omega] > 0$ *for any* $i, j = 1, \ldots, n$.
- *Let the viscous stress tensor* $\mathbf{S}$ *satisfy assumptions (S1b)–(S3b).*
- *Let* $m : [0, \infty) \to [0, \infty)$ *be a homogenous and isotropic N-function satisfying the following:*

  – *for some positive constants c_1, c_2*

$$c_1 |\cdot|^p \leq M(|\cdot|) \leq c_2 \exp^{\frac{1}{\beta+1}}(|\cdot|) \quad \text{for } p \geq 4,\ \beta > 0, \tag{7.219}$$

  – $M(|\cdot|^{\frac{1}{4}})$ *is convex,*
  – *the conjugate function m^* to m satisfies the Δ_2-condition.*

- *Let* $\mathbf{g} = \operatorname{div} \mathbf{F}$, *where* $\mathbf{F} \in W^{1,\infty}(\Omega; \mathbb{R}^{3\times3})$, *be given.*
- *Let the initial distribution of the density be given by*

$$\varrho_0 = \begin{cases} \varrho_f = \text{const} > 0 \text{ in } \Omega \setminus \cup_{i=1}^n \overline{S}_i, \\ \\ \varrho_{S_i} \text{ on } S_i, \text{ where } \varrho_{S_i} \in L^\infty(\Omega),\ \operatorname{ess\,inf}_{S_i} \varrho_{S_i} > 0,\ i = 1, \ldots, n, \end{cases}$$

  while the initial velocity field $\mathbf{u}_0 \in L^2(\Omega; \mathbb{R}^3)$ *satisfies*

$$\operatorname{div} \mathbf{u}_0 = 0 \text{ in } \mathcal{D}'(\Omega),\ \mathbf{D}\mathbf{u}_0 = 0 \text{ in } \mathcal{D}'(S_i; \mathbb{R}^{3\times3}) \text{ for } i = 1, \ldots, n.$$

Then there exist a density function $\varrho \in C([0, T]; L^1(\Omega))$ *satisfying*

$$0 < \operatorname*{ess\,inf}_\Omega \varrho(t, \cdot) \leq \operatorname*{ess\,sup}_\Omega \varrho(t, \cdot) < \infty \text{ for all } t \in [0, T],$$

a family of isometries $\{\eta_i(t, \cdot)\}_{i=1}^n$, $\eta_i(0, x) = x$, $x \in \mathbb{R}^3$, *and a velocity field* $\mathbf{u}$ *satisfying*

$$\mathbf{u} \in L^\infty(0, T; L^2_{\text{div}}(\Omega; \mathbb{R}^3)) \cap L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^3)), \quad \mathbf{D}\mathbf{u} \in L_M(\Omega_T; \mathbb{R}^{3\times3}),$$

compatible with $\{\eta_i\}_{i=1}^n$ *in the sense specified by* (7.208), (7.214). *Moreover* ϱ, $\mathbf{u}$ *satisfy*

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi\Big) \,\mathrm{d}x \,\mathrm{d}t = -\int_\Omega \varrho_0 \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in C^1([0, T) \times \Omega),$$

for any test function $\varphi \in C_c^1([0, T) \times \Omega)$, *and*

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \mathbf{D}\varphi - \mathbf{S} : \mathbf{D}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^T \int_\Omega \varrho\mathbf{g}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi\,\mathrm{d}x$$

for any $\varphi \in C_c^1([0,T)\times\Omega;\mathbb{R}^3)$ *such that* $\varphi(t,\cdot) \in [\mathcal{RM}](t)$ *for all* $t \in [0,T)$.

A few words about the proof.

The proof of Theorem 7.4.2 starts by constructing a two-level approximation scheme as in [150, 267]. Therefore we introduce the following approximation scheme:

$$\begin{aligned} \partial_t\varrho + \mathrm{div}\,(\varrho[\mathbf{u}]_\delta) &= 0, \\ \partial_t(\varrho\mathbf{u}) + \mathrm{div}\,(\varrho\mathbf{u}\otimes[\mathbf{u}]_\delta) + \nabla\pi &= \mathrm{div}\,([\mu_\varepsilon]_\delta\mathbf{S}) - \chi_\varepsilon\mathbf{u} + \varrho\,\mathrm{div}\,\mathbf{F} \\ \partial_t\mu_\varepsilon + \mathrm{div}\,(\mu_\varepsilon[\mathbf{u}]_\delta) &= 0, \\ \mathrm{div}\,\mathbf{u} &= 0. \end{aligned} \tag{7.220}$$

As Ω is bounded, we can assume that $\Omega \subset [-L,L]^3$ for a certain $L > 0$ and study the system (7.220) on a spatial torus denoted by:

$$\mathcal{T} = [(-L,L)|_{\{-L,L\}}]^3.$$

Now all quantities are assumed to be spatially periodic with period $2L$, where we extend the initial velocity field $\mathbf{u}_0$ by 0 outside of Ω and the density by ϱ_f, the constant density of the fluid. We also extend the outer force in such a way that $\mathbf{F} \in W^{1,\infty}(\mathcal{T})$.

In the approximation (7.220) the rigid bodies are replaced by a fluid of high viscosity μ_ε, becoming singular as $\varepsilon \to 0$. In the fluid part (where is no rigid body) μ_ε stays equal to 1. More precisely, we prescribe the ε-dependent 'artificial viscosity' $\mu : (0,T)\times\mathcal{T} \to \mathbb{R}$ with initial data given by

$$\mu(0,\cdot) = \mu_{0,\varepsilon} = 1 + \tfrac{1}{\varepsilon}\sum_{i=1}^n \mu_{S_i}, \tag{7.221}$$

where

$$\mu_{S_i} \in C_c^\infty(S_i),\ \mu_{S_i}(t,x) = 0 \text{ whenever } \mathrm{dist}[x,\partial S_i(t)] < \delta,$$

$$\mu_{S_i}(t,x) > 0 \text{ for } x \in S_i(t),\ t \in [0,T),\ \mathrm{dist}[x,\partial S_i(t)] > \delta \quad \text{for } i = 1,\dots,n. \tag{7.222}$$

The 'viscosity' μ can be understood as the penalization introduced by Hoffmann and Starovoitov [202] and San Martin et al. [289], where the rigid bodies are replaced by the fluid of high viscosity becoming singular (unbounded) for $\varepsilon \to 0$.

Furthermore, we also penalize the region out of the set Ω and we take

$$\chi_\varepsilon = \tfrac{1}{\varepsilon}\chi,\ \chi \in C_c^\infty(\mathcal{T}),\ \chi > 0 \text{ on } \mathcal{T}\setminus\Omega,\ \chi = 0 \text{ in } \overline{\Omega}. \tag{7.223}$$

For simplicity we assume that the density of the fluid is constant. The extra parameter $\delta > 0$ is introduced to improve properties of the approximation and to keep the density constant in the approximate fluid region in order to construct the local pressure and

$[\ ]_\delta$ denotes spatial convolution with the standard regularizing kernel.

For fixed $\varepsilon > 0$ and $\delta > 0$, we can report an existence result that can be proved by means of the monotonicity argument for nonreflexive spaces as in the previous Section 7.2 (see also [328, 180, 183, 326]).

Let us denote by $\{\varrho_\varepsilon, \mu_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ the family of approximate solutions associated with the problem (7.220)–(7.223). For brevity of notation we omit the dependence of this sequence on δ.

The first step is to pass to the limit as $\varepsilon \to 0$ for fixed δ and identify the positions of the rigid bodies. Details of this procedure can be found in [150, 329]. Here we pass to the next difficulty, which is more specific to the Orlicz space setting we are working in.

The main problem, inherent to the theory of non-Newtonian fluids, is that we have to identify the nonlinear viscous term, when we want to pass to the limit as $\varepsilon \to 0$. This problem appears to be more delicate than in Section 7.2 and Section 7.3 since the monotonicity argument must be localized to the 'fluid' part of the time-space cylinder. We are not allowed here to test the momentum equation by functions with non-zero support on Ω_T^s, since we cannot control either the penalizing term $\mu_\varepsilon \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)$ or $\mu_\varepsilon \mathbf{D}\mathbf{u}_\varepsilon$. At this stage of our investigation, the problem of the existence of weak solutions, or rather passage with approximation parameter to the limit, have to be localized in the fluid part separately from the rigid bodies. However, this requires the investigation of the pressure function locally in the fluid part of the time space cylinder Ω_T, which does not vanish in the local weak form momentum equation of the approximation scheme.

In order to characterize the nonlinear term in the fluid part we consider the momentum equation of the approximate problem on the time interval $I \subset [0,T]$ and the spatial domain $B \subset \Omega$ such that $I \times B$ is in the 'fluid' part of the time space cylinder. Let us point out that according to a result of Starovoitov [299, Theorem 3.1], two rigid objects cannot collide. This follows from the fact that the considered fluid is incompressible and the velocity gradients are assumed to be bounded in the Lebesgue space L^p, with $p \geq 4$. Together with the regularity of the domain Ω and the rigid bodies S_i, this ensures that the fluid domain can be 'covered' by such a choice of small time-space cylinders, at least for fixed δ.

We can assume that $\varrho = \varrho_f = \text{const}$ in $I \times B$. In particular, we have for any $\varphi \in C_c^\infty(I \times B; \mathbb{R}^3)$, $\operatorname{div} \varphi = 0$ that

$$\int_I \int_B \varrho_f \mathbf{u}_\varepsilon \cdot \partial_t \varphi + (\varrho_f \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon) - \varrho_f \mathbf{F}) : \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = 0.$$

In the above formula we are separated from the rigid bodies and we can apply Theorem 7.4.1 with the N-function $M_3 = m^*$, the function

$$\mathbf{U} := \varrho_f \mathbf{u}, \quad \text{and} \quad \mathsf{T} := \varrho_f \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta - \mathsf{S}(\mathsf{D}\mathbf{u}_\varepsilon) - \varrho_f \mathsf{F}.$$

Let us remark that the assumption of the lower bound in Theorem 7.4.1 for an N-function m^*, i.e. $\tilde{m}_1(\tau) = \tau \log^{\beta+1}(\tau+1) \le M_3 = m^*(\tau)$ for $\tau \in \mathbb{R}_+$, $\beta > 0$, implies that we need to assume also that $m(\tau) \le c\big(\exp(\tau^{\frac{1}{\beta+1}}) - 1\big)$ for some positive constant c (see (7.219)).

According to Theorem 7.4.1, for any $\varepsilon > 0$, there exist two scalar functions $\pi^\varepsilon_{\text{reg}}$, $\pi^\varepsilon_{\text{harm}}$ such that

$$\pi^\varepsilon_{\text{reg}} \in L^1(I; L\log^\beta L(B)),\ \pi^\varepsilon_{\text{harm}} \in L^\infty(I; L^1(B)) \ \text{ are uniformly bounded for all } \varepsilon \tag{7.224}$$

and $\pi^\varepsilon_{\text{harm}}$ is a harmonic function with respect to x, i.e.

$$\Delta \pi^\varepsilon_{\text{harm}} = 0, \quad \int_B \pi^\varepsilon_{\text{harm}}(t,\cdot) = 0 \quad \text{for } t \in I.$$

Moreover, the following is satisfied

$$\begin{aligned} \int_0^T \int_\Omega \Big[& (\varrho_f \mathbf{u}_\varepsilon + \nabla \pi^\varepsilon_{\text{harm}}) \cdot \partial_t \varphi \\ & + (\varrho_f \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta - \mathsf{S}(\mathsf{D}\mathbf{u}_\varepsilon) - \varrho_f \mathsf{F} + \pi^\varepsilon_{\text{reg}} \mathsf{I}) : \nabla \varphi \Big] \,\mathrm{d}x\,\mathrm{d}t = 0 \end{aligned} \tag{7.225}$$

for any test function $\varphi \in C^\infty_c(I \times B; \mathbb{R}^3)$.

Standard estimates (see (7.194)) implies that

$$\pi^\varepsilon_{\text{harm}} \quad \text{is uniformly bounded in } L^\infty(I; W^{2,2}_{loc}(B)). \tag{7.226}$$

By similar arguments as in Section 7.2 we find that

$$\|\mathbf{u}_\varepsilon\|_{L^p(I, W^{1,p}(B;\mathbb{R}^3))} \le C.$$

Moreover, the equation (7.225) implies that

$$\|\partial_t(\varrho_f \mathbf{u}_\varepsilon + \nabla \pi^\varepsilon_{\text{harm}})\|_{L^1(I;(W^{s,2}_0(B))^*)} < C,$$

where $s > 5/3$ (then $W^{s-1,2}(B) \subset L^\infty(B)$). Hence the Lions–Aubin argument (see Theorem 8.50) gives us that

$$\varrho_f \mathbf{u}_\varepsilon + \nabla \pi^\varepsilon_{\text{harm}} \to \varrho_f \mathbf{u} + \nabla \pi_{\text{harm}} \text{ in } L^2(I; L^2(B'; \mathbb{R}^3)), \tag{7.227}$$

for arbitrary $B' \subset\subset B$ as $\varepsilon \to 0$ ($\varrho_f = \text{const in } B$).

By estimates similar to those in Section 7.2 one can show that the velocity field

$$\{\mathbf{u}_\varepsilon|_B\}_{\varepsilon>0} \text{ is precompact in } L^2(0,T; L^2(B; \mathbb{R}^3)).$$

Hence we infer that

$$\nabla\pi^{\varepsilon}_{\text{harm}} \to \nabla\pi_{\text{harm}} \text{ in } L^2(I;L^2(B';\mathbb{R}^3)). \tag{7.228}$$

The argument holds true for any compact $B' \subset B$. Therefore when passing to the limit as $\varepsilon \to 0$ in (7.225) we provide proper convergence in the first term.

Next we deduce, again similarly as in previous sections, that the sequence $\{\mathbf{S}(\mathbf{Du}_\varepsilon)|_{I\times B}\}_{\varepsilon>0}$ up to a subsequence satisfies

$$\|\mathbf{S}(\mathbf{Du}_\varepsilon)\|_{L_{m^*}(I\times B)} \le C,$$

$$\mathbf{S}(\mathbf{Du}_\varepsilon) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \text{ weakly-}* \text{ in } L_{m^*}(I\times B;\mathbb{R}^{3\times 3}_{\text{sym}}), \tag{7.229}$$

$$\text{and} \quad \mathbf{S}(\mathbf{Du}_\varepsilon) \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^1(I\times B;\mathbb{R}^{3\times 3}_{\text{sym}}).$$

Manipulations based on embedding theory and the growth assumption (7.219) on m provide that

$$\|\mathbf{T}^\varepsilon\|_{L_{m^*}(I\times B)} = \|(\varrho_f \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{Du}_\varepsilon) - \varrho_f \mathbf{F})|_{I\times B}\|_{L_{m^*}(I\times B)} \le C$$

uniformly with respect to ε. Therefore there exists some $\bar{\mathbf{T}} \in L_{m^*}(I\times B)$ such that up to a subsequence

$$\mathbf{T}^\varepsilon \overset{*}{\rightharpoonup} \bar{\mathbf{T}} \quad \text{weakly-}* \text{ in } L_{m^*}(I\times B;\mathbb{R}^{3\times 3})$$

and we infer that

$$\bar{\mathbf{T}} = (\varrho_f \mathbf{u} \otimes [\mathbf{u}]_\delta - \overline{\mathbf{S}} - \varrho_f \mathbf{F})|_{I\times B}.$$

As $\mathcal{R}_{i,j}$ given by (8.7) is a linear operator, using the properties of difference quotients, we provide that

$$\|\mathcal{R}_{i,j}[\phi]|_B\|_{W^{1,r}(B)} \le c\|\phi\|_{W^{1,r}(B)} \quad \text{for any } r \in (1,\infty),$$

for any function $\phi \in W^{1,r}(B)$ with compact support contained in an open set B, where on the left-hand side ϕ is extended by zero (preserving the norm). Therefore the functions $\mathcal{R}_{i,j}\partial_{x_k}\varphi_k$, $i, j, k = 1,2,3$, are sufficiently regular to allow us to get by (7.229) that

$$\int_I\int_B p^{\varepsilon}_{\text{reg}}\mathbf{I} : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I\int_B (\mathcal{R} : \mathbf{T}^\varepsilon)\mathbf{I} : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_I\int_B \sum_{i,j,k=1}^{3} T^{\varepsilon}_{i,j}\mathcal{R}_{j,i}[\partial_{x_k}\varphi_k]\,\mathrm{d}x\,\mathrm{d}t \to \int_I\int_B \sum_{i,j,k=1}^{3} T_{i,j}\mathcal{R}_{j,i}[\partial_{x_k}\varphi_k]\,\mathrm{d}x\,\mathrm{d}t \text{ as } \varepsilon \to 0 \tag{7.230}$$

for any test function $\varphi \in C^\infty_c(I\times B;\mathbb{R}^3)$.

Finally passing to the limit as $\varepsilon \to 0$ by precompactness of $\{\mathbf{u}_\varepsilon\}_\varepsilon$ in L^2 and by (7.225), (7.229) and (7.227), (7.230) we get

$$\int_I \int_B \Big[(\varrho_f \mathbf{u} + \nabla p_{\text{harm}}) \cdot \partial_t \varphi + (\varrho_f \mathbf{u} \otimes [\mathbf{u}]_\delta - \overline{\mathbf{S}}) : \nabla \varphi \Big] + \sum_{i,j,k=1}^{3} T_{i,j} \mathcal{R}_{j,i} \partial_{x_k} \varphi_k \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{7.231}$$

for any test function $\varphi \in C_c^\infty(I \times B; \mathbb{R}^3)$.

The next step is to use (7.225) and (7.231) with strong convergence (7.228) to characterize the nonlinear viscous term using monotonicity methods as in Section 4.1.2 or Section 7.2. But first we have to show that

$$\limsup_{\varepsilon \to 0} \int_I \int_B r \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \le \int_I \int_B r \mathbf{S}(\mathbf{D}\mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \quad \text{for any } r \in C_c^\infty(B),$$

which requires the use of the integration by parts formula. This has already been highlighted in Section 4.2 and Section 7.2, 7.3. To this end we take for any $s_0, s_1 \in I$ and sufficiently small h

$$\varphi = \sigma_h * (\mathbb{1}_{(s_0,s_1)} (\sigma_h * r(\varrho_f \mathbf{u}_\varepsilon + \nabla \pi^\varepsilon_{\text{harm}}))) \quad \text{with any } r \in C_c^\infty(B)$$

as a test function in (7.225). Here $*$ stands for convolution in the time variable with regularising kernel σ_h (i.e. $\sigma \in C^\infty(\mathbb{R})$, $\operatorname{supp} \sigma \in B_1(0)$, $\sigma(-t) = \sigma(t)$, $\int_{\mathbb{R}} \sigma(t) \, \mathrm{d}t = 1$, $\sigma_h(t) = \frac{1}{h} \sigma(\frac{1}{h})$). Then one proceed similarly as in Section 7.2.

Note that with the above information we are able to characterize the nonlinear term in a fluid region as in previous sections, which is essential for showing the existence of weak solutions. We leave the remaining details (which can be found in [329]) to the reader.

Part III
Auxiliaries

This part provides basic and classical results, and auxiliary facts, together with references to the literature.

Chapter 8
Basics

8.1 Measure Theory

Lemma 8.1 (Lemma II.1.3 in [168]) *Let Ω be locally Lipschitz. Then, there exist m locally Lipschitz bounded domains $G_1, \dots, G_m$ such that $\partial\Omega \subset \bigcup_{i=1}^m G_i$ and the domains $\Omega_i = \Omega \cap G_i$, $i = 1, \dots, m$, are (locally Lipschitz and) star-shaped with respect to a ball B_i with $B_i \subset \Omega_i$.*

Lemma 8.2 *Let Ω be locally Lipschitz. Then, there exist $m + r$ locally Lipschitz bounded domains $G_1, \dots, G_{m+r}$ such that $\Omega \subset \bigcup_{i=1}^{m+r} G_i$ and the domains $\Omega_i = \Omega \cap G_i$, $i = 1, \dots, m+r$, are (locally Lipschitz and) star-shaped with respect to a ball B_i with $B_i \subset \Omega_i$. Moreover, $\Omega = \bigcup_{i=1}^{m+r} \Omega_i$.*

Proof. By Lemma 8.1 the boundary $\partial\Omega$ can be covered by a finite family of sets $\{G_i\}_{i=1}^m$ and

$$\Omega_i = \Omega \cap G_i, \quad i = 1, \dots, m,$$

are star-shaped with respect to a ball B_i with $B_i \subset \Omega_i$. Note that $\Omega \setminus (\bigcup_{i=1}^m G_i)$ is a compact set, as is the boundary $\partial\Omega$, and $[\Omega \setminus (\bigcup_{i=1}^m G_i)] \cap \partial\Omega = \emptyset$, thus we can choose $\delta > 0$ small enough so that

$$\operatorname{dist}\left(\Omega \setminus \left(\bigcup_{i=1}^m G_i\right), \partial\Omega\right) > \delta > 0.$$

Again, since $\Omega \setminus (\bigcup_{i=1}^m G_i)$ is compact, one can choose its finite covering

$$\{B(x_j, \delta/2)\}_{j=1}^r.$$

Obviously, each such ball is contained in Ω, thus we can write

$$G_{i+m} = \Omega_{i+m} = B(x_i, \delta/2), \quad i = 1, \dots, r.$$

Then $\{G_i\}_{i=1}^{r+m}$ is a desired covering and $\{\Omega_i\}_{i=1}^{r+m}$ a family of star-shaped domains. □

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Lemma 8.3 (Proposition 2.3, Chapter 1 in [266]) *Let $\overline{\Omega} \subset \mathbb{R}^N$ be a compact set and let $G_1, \ldots G_{m+r}$ be an open covering of $\overline{\Omega}$. Then, there exist functions θ_i, $i = 1, \ldots, m+r$, satisfying the following properties*

(i) $0 \le \theta_i \le 1$ $i = 1, \ldots, m+r$
(ii) $\theta_i \in C_c^\infty(G_i)$, $i = 1, \ldots, m+r$
(iii) $\sum_{i=1}^{m+r} \theta_i(x) = 1$, *for all* $x \in \bar{\Omega}$.

The family $\{\theta_i\}_{i=1}^{m+r}$ *is referred to as a partition of unity in* $\overline{\Omega}$ *corresponding to the covering* $G_1, \ldots G_{m+r}$.

Our main reference for properties of measures is [139].

Definition 8.4 (Measure). A mapping $\mu : 2^X \to [0, \infty]$ is called a *measure* on X if $\mu(\emptyset) = 0$ and $\mu(A) \le \sum_{k=1}^\infty \mu(A_k)$ whenever $A \subset \bigcup_{k=1}^\infty A_k$.

Note that the mapping in this definition is often called an 'outer measure'.

Definition 8.5 (σ-algebra). A collection of subsets $S \subset 2^X$ is called a *σ-algebra* provided

(i) $\emptyset, X \in S$;
(ii) $A \in S$ implies $X \setminus A \in S$;
(iii) $A_k \in S$ for $k = 1, \ldots$ implies $\bigcup_{k=1}^\infty A_k \in S$.

Definition 8.6. The *Borel σ-algebra* of $\mathbb{R}^N$ is the smallest σ-algebra of $\mathbb{R}^N$ containing the open subsets of $\mathbb{R}^N$. Each element of the Borel σ-algebra is called a *Borel set*.

Definition 8.7.

- A measure μ on X is called *regular* if for each set $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$.
- A measure μ on $\mathbb{R}^N$ is called *Borel* if every Borel set is μ-measurable.
- A measure μ on $\mathbb{R}^N$ is called *Borel regular* if μ is Borel and for each set $A \subset \mathbb{R}^N$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.
- A measure μ on X is called a *Radon measure* if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^N$.

Definition 8.8. A function $f : \mathbb{R}^N \to \mathbb{R}$ is called *(Lebesgue) measurable* if for any open $U \subset \mathbb{R}$, $f^{-1}(U)$ is measurable. A function $f : \mathbb{R}^N \to \mathbb{R}^m$ is called *measurable* if each of its coordinates is measurable.

In this monograph mostly Lebesgue measurable functions are considered. Therefore, by *measurable* we shall mean *Lebesgue measurable*. The results below hold true for general μ, for proofs, see [139].

Theorem 8.9 (Theorem 6, Section 1.1 in [139]) *Measurable functions share the following basic properties.*

(i) *If* $f, g : \mathbb{R}^N \to \mathbb{R}$ *are measurable, then so are* $f + g$, fg, $\min\{f, g\}$, *and* $\max\{f, g\}$. *The function* f/g *is also measurable, provided* $g \neq 0$ *on* $\mathbb{R}^N$.
(ii) *If* $f_k : \mathbb{R}^N \to [-\infty, \infty]$ *are measurable for* $k = 1, 2, \ldots$, *then* $\inf_{k>1} f_k$, $\sup_{k>1} f_k$, $\liminf_{k\to\infty} f_k$, *and* $\limsup_{k\to\infty} f_k$ *are also measurable.*

Theorem 8.10 (Luzin, Theorem 2, Section 1.2 in [139]) *Let* $f : \mathbb{R}^N \to \mathbb{R}^m$ *be a measurable function an* $A \subset \mathbb{R}^N$ *a set of finite measure. Then for fixed* $\varepsilon > 0$ *there exists a compact set* $K \subset A$ *such that* $|A \setminus K| < \varepsilon$ *and* $f|_K$ *is continuous.*

Theorem 8.11 (Egorov, Theorem 3, Section 1.3 in [139]) *Let* $f_n : \mathbb{R}^N \to \mathbb{R}^m$ $(n = 1, 2, \ldots)$ *be a sequence of measurable functions. Assume that* $A \subset \mathbb{R}^N$ *is a measurable set of finite measure and* $f_n \to f$ *a.e. on* A. *Then for every* $\varepsilon > 0$ *there exists a measurable set* $B \subset A$ *such that* $|A \setminus B| < \varepsilon$ *and* $f_n \to f$ *uniformly on* B.

Theorem 8.12 (Scorza–Dragoni, Theorem 4.5 in [283]) *Suppose that* $A \subset \mathbb{R}^N$ *is a measurable set of finite measure and* $f : A \times \mathbb{R}^N \to \mathbb{R}$ *is a Carathéodory function such that for almost every fixed* $z \in A$ *the function* $f(z, \cdot)$ *is uniformly continuous. Then, there exists an increasing sequence of compact sets* $S_k \subset A$, $k \in \mathbb{N}$, *with* $|A \setminus S_k| \searrow 0$ *such that* $f|_{S_k \times \mathbb{R}^N}$ *is continuous.*

Definition 8.13 (Absolute continuity). A measure ν is said to be *absolutely continuous* with respect to μ (written $\nu \ll \mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \subset \mathbb{R}^N$.

Theorem 8.14 (Radon–Nikodym, Theorem 3.2.2 in [48]) *Let* ν, μ *be finite measures on* $\mathbb{R}^N$. *Then* $\nu \ll \mu$ *precisely when there exists a* μ*-measurable function* f *such that* ν *is given by* $\nu(A) = \int_A f \,\mathrm{d}\mu$ *for all* μ*-measurable sets* $A \subset \mathbb{R}^N$.

Theorem 8.15 (Riesz representation theorem, Theorem 1, Section 1.8 in [139]) *Let* $L : C_c(\mathbb{R}^N; \mathbb{R}^m) \to \mathbb{R}$ *be a linear functional satisfying*

$$\sup\{L(f) : \ f \in C_c(\mathbb{R}^N; \mathbb{R}^m), |f| \leq 1, \operatorname{supp} f \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^N$. *Then there exists a Radon measure* μ *on* $\mathbb{R}^N$ *and a* μ*-measurable function* $\sigma : \mathbb{R}^N \to \mathbb{R}^m$ *such that* $|\sigma(x)| = 1$ *for* μ*-a.e.* x *and* $L(f) = \int_{\mathbb{R}^N} f \cdot \sigma \,\mathrm{d}\mu$ *for all* $f \in C_c(\mathbb{R}^N; \mathbb{R}^m)$.

Definition 8.16 (Convergence in measure). Let $Z \subset \mathbb{R}^N$. We call a sequence $\{f_n\}_{n=1}^\infty$ of measurable functions $f_n : Z \to \mathbb{R}^d$ *convergent in measure* to a measurable function $f : Z \to \mathbb{R}^d$ if for any $\varepsilon > 0$ it holds that

$$\lim_{n\to\infty} \Big|\{z \in Z : \ |f_n(z) - f(z)| > \varepsilon\}\Big| = 0.$$

Definition 8.17 (Superlinear function). We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *superlinear at infinity* if

$$\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = \infty.$$

We say that a function $f:\mathbb{R}^d \to \mathbb{R}$ is *superlinear at the origin* if

$$\lim_{|\xi|\to 0} \frac{f(\xi)}{|\xi|} = 0.$$

Definition 8.18 (Uniform integrability). Let $Z \subset \mathbb{R}^N$. We call a sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(Z;\mathbb{R}^d)$ *uniformly integrable* if the following two conditions hold

(i) for any $\varepsilon > 0$ there exists a measurable set A with $|A| < \infty$ such that $\int_{Z\setminus A} |f_n|\, \mathrm{d}x < \varepsilon$ for every $n \in \mathbb{N}$;
(ii) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every measurable set E, if $|E| < \delta$, then $\int_E |f_n|\, \mathrm{d}z < \varepsilon$ for every $n \in \mathbb{N}$.

Note that the condition (i) is trivially satisfied when Z has finite measure (by taking $A = Z$).

We give below equivalent characterizations of uniform integrability. Condition (ii) is known as de la Valleé–Poussin's theorem. This fact is proved in [14] for a more general measure, but we restrict our attention to the most classical case of Lebesgue measure.

Lemma 8.19 (Proposition 1.27 in [14]) *Suppose $Z \subset \mathbb{R}^N$ is such that $|Z| < \infty$ and a sequence $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^1(Z;\mathbb{R}^d)$. Then the following conditions are equivalent:*

(i) *the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable;*
(ii) *it holds that*

$$\{f_n\} \subset \left\{ f \in L^1(Z;\mathbb{R}^d) : \int_Z \varphi(|f|)\, \mathrm{d}z \le 1 \right\}$$

for some increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$;
(iii) *it holds that*

$$\lim_{R\to\infty} \left(\sup_{n\in\mathbb{N}} \int_{\{z\in Z:\ |f_n(z)|\ge R\}} |f_n(z)|\, \mathrm{d}z \right) = 0.$$

Lemma 8.20 *If $Z \subset \mathbb{R}^N$ is such that $|Z| < \infty$, then a sequence $\{f_n\}_{n=1}^{\infty}$ bounded in $L^1(Z;\mathbb{R}^d)$ is uniformly integrable if and only if for every $\varepsilon > 0$ there exists an $R > 0$ such that*

$$\sup_{n\in\mathbb{N}} \int_Z (|f_n(z)| - R)_+\, \mathrm{d}z \le \varepsilon. \tag{8.1}$$

Proof. Suppose $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable and notice that

$$\int_Z (|f_n(z)| - R)_+\, \mathrm{d}x \le \int_{\{z\in Z:\ |f_n(z)|\ge R\}} |f_n(z)|\, \mathrm{d}z,$$

so (8.1) follows from Lemma 8.19.

On the other hand, if we assume (8.1), fix arbitrary $\varepsilon > 0$ and take $R > 0$, such that (ii) of Definition 8.18 holds true with $\varepsilon/2$. Then we choose $\delta < \varepsilon/(2R)$. We have

$$\sup_{n\in\mathbb{N}} \sup_{\substack{A\subset Z\\ |A|<\delta}} \int_A |f_n(z)|\,\mathrm{d}z = \sup_{n\in\mathbb{N}} \sup_{\substack{A\subset Z\\ |A|<\delta}} \int_A R+(|f_n(z)|-R)\,\mathrm{d}x$$
$$\leq \sup_{\substack{A\subset Z\\ |A|<\delta}} |A|R + \sup_{n\in\mathbb{N}} \int_Z (|f_n(z)|-R)_+\,\mathrm{d}z$$
$$\leq \delta R + \tfrac{\varepsilon}{2} \leq \varepsilon. \qquad \square$$

8.2 Functional Analysis

Theorem 8.21 (Dunford–Pettis, Theorem 4.30 in [57]) *A sequence $\{f_n\}_n$ is uniformly integrable in $L^1(Z)$ if and only if it is relatively compact in the weak topology.*

Lemma 8.22 *Suppose $g_n \xrightarrow[n\to\infty]{} g$ in $L^1(Z)$ and $f_n, f \in L^\infty(Z)$. Assume further that there exists a $C > 0$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty} < C$ and $f_n \xrightarrow[n\to\infty]{a.e.} f$. Then*

$$\|f_n g_n\|_{L^1(Z)} \xrightarrow[n\to\infty]{} \|fg\|_{L^1(Z)}.$$

Proof. By Theorem 8.21 the sequence $\{g_n\}$ is uniformly integrable. We fix an arbitrary $\varepsilon > 0$. According to Lemma 8.20, we choose $\delta > 0$ such that

$$\sup_{|A|<\delta} \sup_{n\in\mathbb{N}} \int_A |g_n|\,\mathrm{d}z \leq \frac{\varepsilon}{C}. \tag{8.2}$$

By Egorov's theorem (Theorem 8.11) there exists a measurable set B with $|Z\setminus B| < \delta$ such that $f_n \to f$ strongly in $L^\infty(B)$. Let us write

$$\int_Z f_n g_n\,\mathrm{d}z = \int_{Z\setminus B} f_n g_n\,\mathrm{d}z + \int_B f_n g_n\,\mathrm{d}z.$$

Given the assumed boundedness and convergence of $\{f_n\}$, we get that $\|f_n g_n\|_{L^1(B)} \to \|fg\|_{L^1(B)}$. On the other hand, we have

$$\int_{Z\setminus B} f_n g_n\,\mathrm{d}z = \int_{Z\setminus B} f_n(g_n-g)\,\mathrm{d}z + \int_{Z\setminus B} f_n g\,\mathrm{d}z,$$

where on the right-hand side due to (8.2) the first term can be estimated as follows

$$\left|\int_{Z\setminus B} f_n(g_n-g)\,\mathrm{d}z\right| \leq 2\varepsilon,$$

and the second one is convergent by the assumption to $\|fg\|_{L^1(B)}$. To conclude we combine the above remarks and recall that $\varepsilon > 0$ was arbitrarily small. $\square$

We make use of the following version of Vitali's convergence theorem resulting from [133, 5.6. Konvergenzsatz von Vitali p. 262]. In [133] one can find an extended version covering the case $0 \le p < \infty$ and possibly unbounded Z.

Theorem 8.23 (Vitali's convergence theorem) *If $1 \le p < \infty$, $f_n, f \in L^p(Z)$, $|Z| < \infty$, then the following conditions are equivalent:*

(i) $f_n \to f$ *in* $L^p(Z)$;
(ii) $f_n \to f$ *in measure and* $\{f_n\}$ *is uniformly integrable in* $L^p(Z)$.

Lemma 8.24 (Lemma 9.1 in [170]) *If $g_n : Z \to \mathbb{R}$ are measurable functions converging to g almost everywhere, then for each regular value t of the limit function g we have $\mathbb{1}_{\{t<|g_n|\}} \xrightarrow[n\to\infty]{} \mathbb{1}_{\{t<|g|\}}$ a.e. in Z.*

Lemma 8.25 (Lemma 9.4 in [57]) *Let $N \ge 2$ and let $f_1, f_2, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For $x \in \mathbb{R}^N$ and $1 \le i \le N$ set $x_i' = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$, i.e. x_i is omitted from the list. Then the function $f(x) := f_1(x_1') f_2(x_2') \cdot \cdots \cdot f_N(x_N')$, $x \in \mathbb{R}^N$, belongs to $L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} |f| \, \mathrm{d}x \le \prod_{i=1}^{N} \left(\int_{\mathbb{R}^{N-1}} |f_i|^{N-1} \, \mathrm{d}x_i' \right)^{\frac{1}{N-1}}.$$

Lemma 8.26 (Young inequality for a convolution of functions, Theorem 3.9.4 in [48]) *If $f, g \in L^1(\mathbb{R}^N)$, then $\|f * g\|_{L^1(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)} \|g\|_{L^1(\mathbb{R}^N)}$.*

The above inequality actually also holds for measures, see [48, Theorem 3.9.9] or [176, Theorem 1.2.13 and Example 1.2.14]. We will need it only in the following version.

Lemma 8.27 (Young inequality for a convolution with a measure) *If $f \in L^1(\mathbb{R}^N)$ and μ is a bounded Borel measure, then $\|f * \mu\|_{L^1(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)} |\mu|(\mathbb{R}^N)$.*

Theorem 8.28 (Chebyshev's inequality, Theorem 2.5.3 in [48]) *Suppose $Z \subset \mathbb{R}^N$ and $f : \mathbb{R}^N \to [-\infty, +\infty]$ is an integrable function, then for any real number $t > 0$ it holds that*

$$\left|\{z \in Z : \ f(z) \ge t\}\right| \le \frac{1}{t} \int_{\{z \in Z:\ f(z) \ge t\}} f(z) \, \mathrm{d}z.$$

Theorem 8.29 (Hahn–Banach extension theorem, Theorem 1.1. in [57]) *Let E be a vector space over $\mathbb{R}$ and $p : E \to \mathbb{R}$ be a function satisfying*

(i) $p(\lambda x) = \lambda p(x)$ *for every* $x \in E$ *and* $\lambda > 0$,
(ii) $p(x+y) \le p(x) + p(y)$ *for every* $x, y \in E$.

Assume further that $G \subset E$ is a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that $g(x) \le p(x)$ for every $x \in G$. Under these assumptions, there exists a linear functional f defined on all of E that extends g, i.e., $g(x) = f(x)$ for every $x \in G$ and such that $f(x) \le p(x)$ for every $x \in E$.

The following lemma is one of the components of the proof of the hyperplane separation theorem, cf. Theorem 1.6 in [57].

Theorem 8.30 (Lemma 1.3 in [57]) *Let E be a normed vector space, $C \subset E$ be a nonempty open convex set and let $x_0 \notin C$. Then there exists a linear functional $v \in E^*$ such that $v(x) < v(x_0)$ for each $x \in C$. In particular, the hyperplane $\{x \in E : v(x) = v(x_0)\}$ separates $\{x_0\}$ and C.*

Theorem 8.31 (Banach–Alaoglu, Corollary 3.30 in [57]) *Let E be a separable Banach space and let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E^*. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges in the weak-$*$ topology.*

Theorem 8.32 (Mazur's lemma, Corollary 3.8 in [57]) *Suppose $x_n \xrightharpoonup[n\to\infty]{} x$ in a Banach space E. Then, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset E$ of convex combinations, i.e.*

$$y_n = \sum_{i=n}^{N(n)} \alpha_{n,i} x_i, \qquad 0 \le \alpha_{n,i} \le 1, \qquad \sum_{i=n}^{N(n)} \alpha_{n,i} = 1,$$

converging strongly in E (that is, such that $\|y_n - x\|_X \to 0$ as $n \to \infty$).

Due to the nature of our problem, we often deal with sequences that converge weakly-$*$. In the case of reflexive spaces the weak and weak-$*$ topologies coincide, therefore we find the following corollary of Mazur's lemma useful.

Corollary 8.33 *Suppose $x_n \xrightharpoonup[n\to\infty]{*} x$ weakly-$*$ in a reflexive Banach space E. Then, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset E$ made up of convex combinations of the x_n's such that $y_n \to x$ in E.*

Theorem 8.34 (Duality theorem, Theorem 14.2 in [211]) *Let A be a closed convex function on X and let V be a subspace of X. Suppose that there is a point of V when A is continuous. Then the following relation holds*

$$\inf_{x\in V} A(x) + inf_{x\in V^\perp} A^*(x^*) = 0.$$

In the Orlicz setting we have the following result due to Gossez. Modular convergence is defined in Section 3.4.

Theorem 8.35 (Gossez's approximation, Theorem 4 in [175]) *Suppose $\Omega \subset \mathbb{R}^N$, $N \ge 1$ is a bounded Lipschitz domain, $m : [0,\infty) \to [0,\infty)$ is a homogeneous and isotropic N-function, and $u \in W_0^1 L_m(\Omega)$. Then there exists a sequence $\{u_\delta\}_\delta \in C_0^\infty(\Omega)$ such that*

$$u_\delta \xrightarrow[\delta\to\infty]{mod} u \quad in \quad W^1 L_m(\Omega).$$

Moreover, if $u \in L^\infty(\Omega)$, then $\|u_\delta\|_{L^\infty(\Omega)} \le c(\Omega)\|u\|_{L^\infty(\Omega)}$.

Definition 8.36 (Biting convergence). Let $f_n, f \in L^1(Z)$ for every $n \in \mathbb{N}$. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ *converges in the sense of biting* to f in $L^1(Z)$ (and denote it by $f_n \xrightarrow{b} f$), if there exists a sequence E_k of measurable subsets of Z such that $\lim_{k\to\infty} |E_k| = 0$ and for every k we have $f_n \rightharpoonup f$ in $L^1(Z \setminus E_k)$.

Theorem 8.37 (Chacon's biting lemma, Theorem 6.6 in [272]) *Suppose the sequence $\{f_n\}_n$ is uniformly bounded in $L^1(Z)$. Then there exists an $f \in L^1(Z)$ such that $f_n \xrightarrow{b} f$.*

A consequence of the above result is the following.

Theorem 8.38 (Lemma 6.9 in [272]) *Let $f_n \in L^1(Z)$ for every $n \in \mathbb{N}$, $f_n(z) \geq 0$ for every $n \in \mathbb{N}$ and a.e. x in Z. Moreover, suppose*

$$f_n \xrightarrow{b} f \quad \text{and} \quad \limsup_{n\to\infty} \int_Z f_n \,\mathrm{d}x \leq \int_Z f \,\mathrm{d}x.$$

Then $f_n \rightharpoonup f$ weakly in $L^1(Z)$ for $n \to \infty$.

We may reformulate Theorem 8.37 and Theorem 8.38 in the following way:

Lemma 8.39 *Let $\{a_n\}_{i=1}^{\infty}$ be a bounded sequence in $L^1(\Omega_T)$ and let $0 \leq a_0 \in L^1(\Omega_T)$. If assumptions*

(i) $a_n \geq -a_0$ *for all* $n = 1, 2, 3, \ldots,$
(ii) $a_n \xrightarrow{b} a$ *as* $n \to \infty$,
(iii) $\limsup_{n\to\infty} \int_{\Omega_T} a_n \,\mathrm{d}x\,\mathrm{d}t \leq \int_{\Omega_T} a \,\mathrm{d}x\,\mathrm{d}t$,

hold, then

$$a_n \rightharpoonup a \quad \text{weakly in } L^1(\Omega_T) \text{ as } n \to \infty.$$

For the proof, see also [272, 188].

Definition 8.40 (Tightness condition). Let $Z \subset \mathbb{R}^d$. A sequence $\{\xi_k\}_{k\in\mathbb{N}}$ of measurable functions $\xi_k : Z \to \mathbb{R}^N$ is said to satisfy the tightness condition if

$$\lim_{R\to\infty} \sup_{k\in\mathbb{N}} |\{z : \ |\xi_k(z)| \geq R\}| = 0.$$

Theorem 8.41 (Fundamental theorem for Young measures) *Let $Z \subset \mathbb{R}^N$ and $\xi_j : Z \to \mathbb{R}^d$ be a sequence of measurable functions. Then there exists a subsequence $\{\xi_{j,k}\}$ and a family of weakly-$*$ measurable maps $\nu_z : Z \to \mathcal{M}(\mathbb{R}^d)$, such that:*

(i) $\nu_z \geq 0$, $\|\nu_z\|_{\mathcal{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathrm{d}\nu_z \leq 1$ *for a.e.* $z \in Z$.
(ii) *For every* $f \in C_0(\mathbb{R}^d)$, *we have* $f(\xi_{j,k}) \overset{*}{\rightharpoonup} \bar{f}$ *weakly-$*$ in* $L^\infty(Z)$. *Moreover,*

$$\bar{f}(z) = \int_{\mathbb{R}^d} f(\lambda)\,\mathrm{d}\nu_z(\lambda).$$

(iii) *Let* $K \subset \mathbb{R}^d$ *be compact and* $\operatorname{dist}(\xi_{j,k}, K) \to 0$ *in measure, then* $\operatorname{supp} \nu_z \subset K$.
(iv) $\|\nu_z\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $z \in Z$ *if and only if the tightness condition is satisfied.*
(v) *If the tightness condition is satisfied,* $A \subset Z$ *is measurable,* $f \in C(\mathbb{R}^d)$, *and* $\{f(\xi_{j,k})\}$ *is relatively weakly compact in* $L^1(A)$, *then*

$$f(\xi_{j,k}) \rightharpoonup \bar{f} \quad \text{in } L^1(A) \quad \text{and} \quad \bar{f}(z) = \int_{\mathbb{R}^d} f(\lambda)\,\mathrm{d}\nu_z(\lambda).$$

The family of maps $\nu_z : Z \to \mathcal{M}(\mathbb{R}^d)$ is called the Young measure generated by $\{\xi_{j,k}\}$.

Remark 8.42. The notion of Young measures dates back to [331, 332, 333] whereas the proof of the above theorem can be found in [20], see also [18] and [217].

Lemma 8.43 (Corollary 3.3 in [259]) *Let $Z \subset \mathbb{R}^N$ be a measurable set of finite measure and let $z_k : Z \to \mathbb{R}^d$ be a sequence of measurable functions which generates the Young measure ν. Let $f : Z \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function and assume that the negative part $(f(x,\xi_k(x))_-$ is weakly relatively compact in $L^1(Z)$. Then*

$$\liminf_{k\to\infty} \int_Z f(z,\xi_k(z))\,\mathrm{d}z \geq \int_Z \int_{\mathbb{R}^d} f(z,\lambda)\,\mathrm{d}\nu_z\,\mathrm{d}x.$$

Lemma 8.44 (Corollary 3.4 in [259]) *Let $Z \subset \mathbb{R}^N$ be a measurable set of finite measure, let $u_j : Z \to \mathbb{R}^{d_1}$, $v_j : Z \to \mathbb{R}^{d_2}$ be measurable and suppose that $u_j \to u$ a.e. while v_j generates the Young measure ν. Then the sequence of pairs $(u_j, v_j) : Z \to \mathbb{R}^{d_1+d_2}$ generates the Young measure $z \mapsto \delta_{u(z)} \otimes \nu_z$. Here $\otimes$ denotes the tensor product of measures.*

Theorem 8.45 (De Rham, Proposition 1.1 in [311]) *Let $q \in (C_c^\infty(\Omega;\mathbb{R}^N))^*$, where Ω is an open subset of $\mathbb{R}^N$, be such that*

$$\langle q,\psi\rangle_{(C_c^\infty(\Omega))'\times C_c^\infty(\Omega)} = 0 \tag{8.3}$$

for all $\psi \in C_c^\infty(\Omega;\mathbb{R}^N)$ such that $\operatorname{div}\psi = 0$. Then there exists an $f \in (C_c^\infty(\Omega;\mathbb{R}^N))^*$ such that

$$q = \nabla f. \tag{8.4}$$

Theorem 8.46 (De Rham, Lemma 2.2.1 in [298]) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an arbitrary domain. Let $\Omega_0 \subset \Omega$ be a bounded nonempty subdomain such that $\overline{\Omega_0} \subset \Omega$, and let $1 < q < \infty$. Suppose $\mathbf{f} \in W_{loc}^{-1,q}(\Omega;\mathbb{R}^N)$ satisfies*

$$[\mathbf{f},\mathbf{v}] = 0 \quad \textit{for all } \mathbf{v} \in C_c^\infty(\Omega;\mathbb{R}^N),\ \operatorname{div}\mathbf{v} = 0.$$

Then there exists a unique $p \in L_{loc}^q(\Omega)$ satisfying $\nabla p = \mathbf{f}$ in the sense of distributions and

$$\int_{\Omega_0} p\,\mathrm{d}x = 0.$$

In the above $[\mathbf{f},\mathbf{v}]$ means the value of the functional $\mathbf{f}$ at $\mathbf{v}$.

Theorem 8.47 (Sobolev embedding, Theorem 1.20 in [286]) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded Lipschitz domain. The space $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ provided that the exponent p^* is defined as:*

(i) $p^* = \frac{Np}{N-p}$ *for* $p \in [1,N)$,
(ii) p^* *is arbitrarily large real if* $p = N$,
(iii) $p^* = +\infty$ *if* $p > N$.

Theorem 8.48 (Rellich–Kondrachov, Theorem 9.16 in [57]) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded Lipschitz domain, and let $1 \leq p \leq n$. If $p^* = \frac{np}{n-p}$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for every $1 \leq q < p^*$.*

Theorem 8.49 (Aubin–Lions I, Theorem 3 in [293]) *Let X_1, X_2 be Banach spaces, such that $X_1 \subset\subset X_2$. Let $F \subset L^p(0,T;X_2)$ where $1 \le p \le \infty$, and let F be bounded in $L^1_{loc}(0,T;X_1)$, and $\|\tau_h f - f\|_{L^p(0,T;X_2)} \to 0$ as $h \to 0$ uniformly for $f \in F$. Then F is relatively compact in $L^p(0,T;X_2)$ (and in $C(0,T;X_2)$ for $p = \infty$).*

Theorem 8.50 (Aubin–Lions II, Corollary 8 in [293]) *Let X_0, X and X_1 be Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1. Suppose that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^q(0,T;X_1)$ and $L^1(0,T;X_0)$. Moreover, assume that the sequence of distributional time derivatives $\{\partial_t f_n\}$ is bounded in $L^1(0,T;X_1)$. Then, $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^p(0,T;X)$ for any $1 \le p < q$.*

Theorem 8.51 (Aubin–Lions III, Corollary 4 in [293]) *Let X_0, X and X_1 be Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1. Let the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ be bounded in $L^p(0,T;X_0)$ where $1 \le p < \infty$ and let the sequence of distributional time derivatives $\{\partial_t f_n\}_{n\in\mathbb{N}}$ be bounded in $L^1(0,T;X_1)$. Then $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^p(0,T;X)$.*

Let $\{f_n\}_{n\in\mathbb{N}}$ be bounded in $L^\infty(0,T;X_0)$ and let $\{\partial_t f_n\}_{n\in\mathbb{N}}$ be bounded in $L^r(0,T;X_1)$, where $r > 1$. Then $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact in $C(0,T;X)$.

We recall below the Div-Curl lemma, which can be found in [309, 158].

Let us start with the notation: for $\mathbf{a} = (a_0, a_1, a_2, a_3)$

$$\mathrm{Div}_{t,x}\mathbf{a} := \partial_t a_0 + \sum_{i=1}^{3} \partial_{x_i} a_i \quad \text{and} \quad \mathrm{Curl}_{t,x}\mathbf{a} := \nabla_{t,x}\mathbf{a} - (\nabla_{t,x}\mathbf{a})^T. \tag{8.5}$$

Lemma 8.52 (Div-Curl lemma) *Let $\Omega_T = (0,T)\times\Omega \subset \mathbb{R}^4$ be a bounded set. Let $p,q,l,s \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{l}$ and the vector fields $\mathbf{a}^n, \mathbf{b}^n$ satisfy*

$$\mathbf{a}^n \rightharpoonup \mathbf{a} \quad \text{weakly in} \quad L^p(\Omega_T;\mathbb{R}^4) \quad \text{and} \quad \mathbf{b}^n \rightharpoonup \mathbf{b} \quad \text{weakly in} \quad L^q(\Omega_T;\mathbb{R}^4),$$

and $Div_{t,x}\mathbf{a}$ and $Curl_{t,x}\mathbf{b}$ are precompact in $W^{-1,s}(\Omega_T)$ and $W^{-1,s}(\Omega_T;\mathbb{R}^{4\times 4})$ respectively. Then

$$\mathbf{a}^n \cdot \mathbf{b}^n \rightharpoonup \mathbf{a}\cdot\mathbf{b} \quad \text{weakly in } L^l(\Omega_T),$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^4$.

Lemma 8.53 (Zeros of vector field, Section 9.1 in [138]) *Let $\mathbf{s} : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping and*

$$\mathbf{s}(x)\cdot x \ge 0 \text{ if } |x| = r \tag{8.6}$$

for some $r > 0$. Then there is a point x with $|x| \le r$ such that $\mathbf{s}(x) = 0$.

Lemma 8.54 (Korn's inequality, Theorem 1.10 in [245]) *Let $\Omega \subset \mathbb{R}^N$ with $N \ge 1$ be an open bounded set such that $\partial\Omega \in C^{0,1}$ (Lipschitz continuous). Let $1 < p < \infty$ and let $\mathbf{v} \in W^{1,p}_0(\Omega;\mathbb{R}^N)$. Then there exists a constant C_p depending on p and Ω such that the inequality*

$$C_p \|\mathbf{v}\|_{W^{1,p}(\Omega)} \le \|\mathbf{D}\mathbf{v}\|_{L^p(\Omega)}$$

holds, where $\mathbf{D}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v})$.

Lemma 8.55 (Theorem 4.11 in Chapter A.4 of [245]) *Let us recall that* V_s *is defined by* (7.26). *Let* $((\cdot,\cdot))_s$ *denotes the scalar product in* V_s*, while* $(\cdot,\cdot)$ *denotes the scalar product in* $L^2(\Omega;\mathbb{R}^N)$. *Then there exists a countable set* $\{\lambda_i\}_{i=1}^\infty$ *and a corresponding family of eigenvectors* $\{\omega^i\}_{i=1}^\infty$ *solving the problem*

$$((\omega_i,\varphi))_s = \lambda_i \int_\Omega \omega_i \cdot \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in V_s$$

such that $(\omega_i,\omega_j) = \delta_{i,j}$*, for all* $i,j \in \mathbb{N}$*, being the Kronecker delta,* $1 \le \lambda_1 \le \lambda_2 \le \cdots$ *and* $\lambda_i \to \infty$ *as* $i \to \infty$*, and* $\{\omega_i\}_{i=1}^\infty$ *forms a basis in* V_s. *Moreover, if* $L^2_{\mathrm{div},n}(\Omega;\mathbb{R}^N) := \mathrm{span}\{\omega_1,\ldots,\omega_n\}$ *(a linear hull) and* $P^n : V_s \to L^2_{\mathrm{div},n}(\Omega;\mathbb{R}^N)$ *is given by* $P^n(\mathbf{v}) := \sum_{i=1}^n (\mathbf{v},\omega_i)\omega_i$*, then we obtain that for* $\mathbf{v} \in V_s$

$$\|P^n(\mathbf{v})\|_{W^{s,2}(\Omega)} \le \|\mathbf{v}\|_{W^{s,2}(\Omega)}, \quad \|P^n(\mathbf{v})\|_{L^2(\Omega)} \le \|\mathbf{v}\|_{L^2(\Omega)}.$$

Lemma 8.56 (Interpolation between Bochner spaces, Proposition 1.41 in [286]) *Let* $\Omega \subset \mathbb{R}^N$ *be bounded domain and* I *be bounded time interval. Let*

$$f \in L^{p_1}(0,T;L^{q_1}(\Omega)) \cap L^{p_2}(0,T;L^{q_2}(\Omega)),$$

where $p_1,p_2,q_1,q_2 \in [1,\infty]$. *Let* $\lambda \in [0,1]$. *If*

$$\tfrac{1}{p} = \tfrac{\lambda}{p_1} + \tfrac{1-\lambda}{p_2} \quad \text{and} \quad \tfrac{1}{q} = \tfrac{\lambda}{q_1} + \tfrac{1-\lambda}{q_2},$$

then

$$\|f\|_{L^p(I;L^q(\Omega))} \le \|f\|^{\lambda}_{L^{p_1}(I;L^{q_1}(\Omega))}\|f\|^{1-\lambda}_{L^{p_2}(I;L^{q_2}(\Omega))}.$$

Let us recall some results concerning the Bogovskii operator.

Lemma 8.57 (Lemma II.2.1.1 in [298]) *Let* $\Omega \subset \mathbb{R}^N$*,* $N \ge 2$*, be a bounded Lipschitz domain. Let* $1 < q < \infty$. *Then we have for each* $g \in L^q(\Omega)$ *with* $\int_\Omega g \,\mathrm{d}x = 0$*, that there exists at least one* $\mathbf{v} \in W_0^{1,q}(\Omega;\mathbb{R}^N)$ *satisfying*

$$\mathrm{div}\,\mathbf{v} = g, \quad \|\nabla\mathbf{v}\|_{L^q(\Omega)} \le C(q,\Omega)\|g\|_{L^q(\Omega)}.$$

Lemma 8.58 (Theorem 5.2 in [322].) *Let* Ω *be bounded Lipschitz domain in* $\mathbb{R}^N$*,* $N \ge 2$. *Let* $m : [0,\infty) \to [0,\infty)$ *be an isotopic and homogeneous Young function of the form* $m(\tau) = \exp(\tau) - \tau - 1$. *Let* $\mathbf{v} : \Omega \to \mathbb{R}^N$*,* $f : \Omega \to \mathbb{R}$*,* $f \in L^\infty(\Omega)$ *and* $\int_\Omega f = 0$. *Then there exists at least one solution* $\mathbf{v} \in W_0^{1,1}(\Omega;\mathbb{R}^N)$ *satisfying* $\mathrm{div}\,\mathbf{v} = f$*,* $\mathbf{v}|_{\partial\Omega} = 0$. *Furthermore*

$$\|\mathbf{v}\|_{L_m(\Omega)} + \|\nabla\mathbf{v}\|_{L_m(\Omega)} \le c\|f\|_{L^\infty(\Omega)}$$

for some constant $c > 0$.

Definition 8.59 (Quasiconvexity). We say that a function $f : [0,\infty) \to \mathbb{R}$ is *quasiconvex* if for all $x,y \in [0,\infty)$ and $\lambda \in [0,1]$ one has that $f(\lambda x + (1-\lambda y) \le \max\{f(x),f(y)\}$.

Proposition 8.60 ([321], Section 6 in [117]) *Let Ω be a bounded domain with a Lipschitz boundary. Let $m : [0,\infty) \to [0,\infty)$ be an isotropic and homogeneous N-function satisfying the Δ_2-condition and such that m^γ is quasiconvex for some $\gamma \in (0,1)$. Then, for any $f \in L_m(\Omega)$ such that*

$$\int_\Omega f \, \mathrm{d}x = 0,$$

the problem of finding a vector field $\mathbf{v} : \Omega \to \mathbb{R}^N$ such that

$$\begin{aligned} \operatorname{div} \mathbf{v} &= f \quad \text{in } \Omega \\ \mathbf{v} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has at least one solution $\mathbf{v} \in L_m(\Omega;\mathbb{R}^N)$ and $\nabla \mathbf{v} \in L_m(\Omega;\mathbb{R}^{N\times N})$. Moreover, for some positive constant c

$$\int_\Omega m(|\nabla \mathbf{v}|) \, \mathrm{d}x \le c \int_\Omega m(|f|) \, \mathrm{d}x.$$

Now let us introduce the Riesz transform in an isotropic Orlicz space.

Let $\mathcal{R}_{i,j}$ stand for a 'double' Riesz transform of an integrable function g on $\mathbb{R}^3$, which can be given by a Fourier transform $\mathcal{F}$ as

$$\mathcal{R}_{i,j}[g] = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2}\right)\mathcal{F}[g] = \partial_{x_i}\partial_{x_j}\Delta^{-1} g, \quad i,j = 1,2,3, \tag{8.7}$$

where

$$\Delta^{-1} g(x) = \mathcal{F}^{-1}\left(\frac{-1}{|\xi|^2}\right)\mathcal{F}[g] = \int_{\mathbb{R}^3} \frac{g(y)}{|x-y|} \mathrm{d}y.$$

Lemma 8.61 ([135]) *Let Ω be a bounded domain, let $b : \mathbb{R}^3 \to \mathbb{R}$ be a multiplier, α be a multi-index such that $|\alpha| \le 2$, and*

$$|\xi|^{|\alpha|} |D^\alpha b(\xi)| \le C < \infty .$$

Then for any $\beta > 0$ there exists a constant $c(\beta)$ such that for all $g \in L \log^{\beta+1} L(\Omega)$

$$\|(\mathcal{F}^{-1} b \mathcal{F})[g]|_\Omega\|_{L \log^\beta L} \le c(\beta) \|g\|_{L \log^{\beta+1} L}, \tag{8.8}$$

where g is extended to be 0 on $\mathbb{R}^3 \setminus \Omega$. In particular, for any $\beta > 0$ and $g \in L \log^{\beta+1} L(\Omega)$

$$\|\mathcal{R}_{i,j}[g]|_\Omega\|_{L \log^\beta L} \le c(\beta) \|g\|_{L \log^{\beta+1} L}. \tag{8.9}$$

Lemma 8.62 (Weyl's lemma, [302]) *Let Ω be an open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. If $u \in \mathcal{D}'(\Omega;\mathbb{R})$ (the space of Schwartz distributions on Ω) satisfies $\Delta u = f$, $f \in C_c^\infty(\Omega;\mathbb{R})$ in the sense that*

$$\langle \Delta\psi, u\rangle = \langle \psi, f\rangle, \quad \psi \in C_c^\infty(\Omega;\mathbb{R}),$$

then $u \in C^\infty(\Omega;\mathbb{R})$.

8.3 Approximation

In this section we prove the existence of an approximate solution to the system describing the flow of a heat-conducting non-Newtonian fluid given by (7.7)–(7.9).

Before stating the main result about the whole approximation let us first recall a result of Lions concerning the approximation of the continuity equation.

Proposition 8.63 (DiPerna and Lions, [237]) *Let Ω be bounded domain in $\mathbb{R}^3$ with $C^{2,\nu}$ boundary, $\nu > 0$. Let $(0,T)$ be a finite time interval. Let sequences $\{\varrho^\lambda\}_\lambda$, $\{\mathbf{u}^\lambda\}_\lambda$, $\lambda > 0$ be given. Assume that*

$$\varrho^\lambda \in C([0,T];L^1(\Omega)), \quad 0 < \varrho_* \le \varrho^\lambda \le \varrho^* < \infty \text{ a.e. in } \Omega_T,$$

$$\varrho^\lambda(0,\cdot) = \varrho_0^\lambda \quad \text{and } \varrho_* \le \varrho^\lambda \le \varrho^*,$$

$$\varrho_0^\lambda \to \varrho_0 \quad \text{strongly in } L^1(\Omega) \text{ as } \lambda \to \infty,$$

$$\mathbf{u}^\lambda \in L^2(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)), \quad \mathbf{u}^\lambda|_{\partial\Omega} = 0, \quad \operatorname{div}\mathbf{u}^\lambda = 0 \text{ a.e. in } \Omega_T,$$

$$\|\mathbf{u}^\lambda\|_{L^2(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))} \le C,$$

$$\mathbf{u}^\lambda \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)) \text{ as } \lambda \to \infty.$$

Assume that ϱ^λ and $\mathbf{u}^\lambda$ for each fixed $\lambda > 0$ satisfy the following

$$\int_0^T \int_\Omega \varrho^\lambda \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega (\varrho^\lambda \mathbf{u}^\lambda)\cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \varrho_0^\lambda \varphi(0,\cdot)\,\mathrm{d}x$$

for all $\varphi \in C_c^\infty([0,T);C_c^\infty(\Omega))$.

Then

$$\rho^\lambda \to \rho \quad \text{strongly in } C(0,T;L^p(\Omega)) \text{ and a.e. in } \Omega_T \text{ for all } p \in [1,\infty) \text{ as } \lambda \to \infty.$$

In order to prove the above result one can follow pp. 43–46 in [237].

Below we show the existence of an approximate solution to (7.7)–(7.9), namely to a triple $(\varrho^n, \mathbf{u}^n, \theta^n)$ which is a solution to (7.28)–(7.33) with initial data (7.35). The construction of the proof contains a two-step approximation which is based on the standard methods of artificial viscosity and combines the continuous problem with two Ritz–Galerkin finite-dimensional systems. We adapt the proof given for the power-law type fluid in [158, Section 6]. Below we present most of the steps of the reasoning. Some details are skipped, however these can be found, for example, in [158, Section 6], [64], [328, Section 4.1].

Definition of an n-approximate problem.

Let us recall that here Ω is a bounded domain in $\mathbb{R}^3$ with $C^{2,\nu}$ boundary, $\nu > 0$, and $(0,T)$ is a finite time interval. Let $M : \Omega \times \mathbb{R}^{3\times 3} \to [0,\infty)$ be an N-function satisfying for some $\underline{c} > 0$, $\widetilde{C} \geq 0$ and for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$M(x,\boldsymbol{\xi}) \geq \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C} \quad \text{with } p \geq \tfrac{11}{5}.$$

Let $\mathbf{S}$ satisfy conditions (S1h)–(S3h) from Section 7.2.

Moreover, let $\mathbf{q} : [0,\infty) \times [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfy

$$\mathbf{q}(\varrho,\theta,\nabla\theta) = \kappa_0(\varrho,\theta)\nabla\theta \quad \text{with } \kappa_0 \in C([0,\infty)\times[0,\infty)) \tag{8.10}$$

and for all $\theta, \varrho > 0$, $\nabla\theta \in \mathbb{R}^3$

$$\begin{aligned} \mathbf{q}(\varrho,\theta,\nabla\theta)\cdot\nabla\theta &\geq \kappa_*\theta^\beta|\nabla\theta|^2 \quad \text{with } \beta \in \mathbb{R} \text{ and } \kappa_* > 0, \\ |\mathbf{q}(\varrho,\theta,\nabla\theta)| &\leq \kappa^*\theta^\beta|\nabla\theta| \qquad \text{with} \quad \kappa^* > 0. \end{aligned} \tag{8.11}$$

with

$$\beta > -\min\left\{\tfrac{2}{3}, \tfrac{3p-5}{3p-3}\right\}.$$

Let us recall here the n-approximate problem from Section 7.2.

Let

$$\{\omega_i\}_{i=1}^\infty \text{ be an orthonormal basis of } W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)$$
$$\text{such that } \{\omega_i\}_{i=1}^\infty \subset W^{1,2p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3),$$

introduced in Section 7.2 by (7.25).

Then we define the n-approximate velocity $\mathbf{u}^n \in C([0,T];W^{1,2p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))$ of the following form

$$\mathbf{u}^n := \sum_{i=1}^n \alpha_i^n(t)\omega^i \quad \text{for } i = 1,2,\ldots, \tag{8.12}$$

where $\alpha_i^n \in C([0,T])$. The n-approximate solution is defined such that the triple $(\varrho^n, \mathbf{u}^n, \theta^n)$ satisfies

$$\int_0^T \langle \partial_t \varrho^n, z\rangle \,\mathrm{d}t - \int_0^T \int_\Omega \varrho^n \mathbf{u}^n \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{8.13}$$

for all $z \in L^r(0,T;W^{1,r}(\Omega))$ with $r = 5p/(5p-3)$, and

$$0 < \varrho_* \leq \varrho^n(t,x) \leq \varrho^* < +\infty \quad \text{for a.a. } (t,x) \in \Omega_T, \tag{8.14}$$

$$\theta^n \in L^\infty(0,T;L^2(\Omega)) \cup L^s(0,T;W^{1,s}(\Omega)) \quad \text{with } s = \min\left\{2, \frac{5\beta+10}{\beta+5}\right\},$$

$$\text{and } \theta^n \geq \theta_* \text{ in } \Omega_T, \tag{8.15}$$

$$\begin{aligned}\langle \partial_t(\varrho^n \mathbf{u}^n), \omega_i \rangle + \int_\Omega (-\varrho^n \mathbf{u}^n \otimes \mathbf{u}^n : \nabla \omega_i + \mathbf{S}(x, \varrho^n, \theta^n, \mathbf{D}\mathbf{u}^n) : \mathbf{D}\omega_i) \, \mathrm{d}x \\ = \int_\Omega \varrho^n \mathbf{f}^n \cdot \omega_i \, \mathrm{d}x\end{aligned} \tag{8.16}$$

for all $i = 1, \dots, n$ and a.a. $t \in [0, T]$, and

$$\begin{aligned}\int_0^T \langle \partial_t(\theta^n \varrho^n), h \rangle \, \mathrm{d}t + \int_0^T \int_\Omega (-\theta^n \varrho^n \mathbf{u}^n \cdot \nabla h + \kappa_0(\varrho^n \theta^n) \nabla \theta^n \cdot \nabla h) \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \mathbf{S}(x, \varrho^n, \theta^n, \mathbf{D}\mathbf{u}^n) : \mathbf{D}\mathbf{u}^n h \, \mathrm{d}x \, \mathrm{d}t\end{aligned} \tag{8.17}$$

for all $h \in L^\infty(0, T; W^{1,q}(\Omega))$ for large enough q. For the initial data we set

$$\begin{aligned}&\varrho^n(0, \cdot) = \varrho_0 \in L^1(\Omega), \quad 0 < \varrho_* \leq \varrho_0^n \leq \varrho^* < \infty, \\ &\mathbf{u}^n(0, \cdot) = P^n \mathbf{u}_0, \quad \mathbf{u}_0^n \in L^2_{\mathrm{div}}(\Omega; \mathbb{R}^3), \\ &\theta^n(0, \cdot) = \theta_0^n \in C_c^\infty(\Omega), \quad 0 < \theta_* \leq \theta_0^n,\end{aligned} \tag{8.18}$$

where P^n denotes the orthogonal projection of $L^2_{\mathrm{div}}(\Omega; \mathbb{R}^3)$ onto the linear hull of $\{\omega_i\}_{i=1}^n$. Moreover,

$$\mathbf{f}^n \in C_c^\infty(\Omega_T; \mathbb{R}^3) \tag{8.19}$$

stands for a standard smooth regularization of $\mathbf{f} \in L^{p'}(0, T; L^{p'}(\Omega; \mathbb{R}^3)$ (or regular enough approximation to ensure the existence of an approximate solution).

Theorem 8.64 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,\nu}$ boundary, where $\nu \in (0, 1)$ and let $(0, T)$ be a finite time interval. Let $M : \Omega \times \mathbb{R}^{3\times3} \to [0, \infty)$ be an N-function satisfying for some $\underline{c} > 0$, $\widetilde{C} \geq 0$ and for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$*

$$M(x, \boldsymbol{\xi}) \geq \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C} \quad \textit{with } p \geq \tfrac{11}{5}. \tag{8.20}$$

Let $\mathbf{S}$ satisfy conditions (S1h)–(S3h) from Section 7.2. Moreover let $\mathbf{q}$ satisfy (8.10), (8.11) *with*

$$\beta > -\min\left\{\tfrac{2}{3}, \tfrac{3p-5}{3p-3}\right\}.$$

Let $\mathbf{u}_0^n$, ϱ_0^n, θ_0 satisfy (8.18). *Let $\mathbf{f}^n \in C_c^\infty(\Omega_T; \mathbb{R}^3)$.*

Then there exists a triple $(\varrho^n, \mathbf{u}^n, \theta^n)$ satisfying (8.12)–(8.17).

Proof. To define the new two-step approximation to the n-approximate problem let us introduce a basis spanning the space where we construct a k-approximation of θ^n. Namely

$$\{w_j\}_{j=1}^\infty \text{ a smooth basis of } W^{1,2}(\Omega) \text{ orthonormal in } L^2(\Omega).$$

Then we look for a triple $(\varrho^{n,k,\epsilon}, \mathbf{u}^{n,k,\epsilon}, \theta^{n,k,\epsilon})$ where $\mathbf{u}^{n,k,\epsilon}$ and $\theta^{n,k,\epsilon}$ are defined by

$$\mathbf{u}^{n,k,\epsilon} := \sum_{i=1}^{n} \alpha_i^{n,k,\epsilon}(t)\omega_i \quad \text{and} \quad \theta^{n,k,\epsilon} := \sum_{i=1}^{k} \nu_i^{n,k,\epsilon}(t) w_i, \tag{8.21}$$

where $\nu_i^{n,k,\epsilon} \in C([0,T])$, and $(\varrho^{n,k,\epsilon}, \mathbf{u}^{n,k,\epsilon}, \theta^{n,k,\epsilon})$ satisfies the following

$$\partial_t \varrho^{n,k,\epsilon} + \operatorname{div}(\varrho^{n,k,\epsilon}\mathbf{u}^{n,k,\epsilon}) - \epsilon\Delta\varrho^{n,k,\epsilon} = 0 \text{ in } \Omega_T, \tag{8.22}$$

$$\nabla\varrho^{n,k,\epsilon} \cdot \mathbf{n} = 0 \quad \text{on } [0,T]\times\partial\Omega, \quad \varrho^{n,k,\epsilon}(0,\cdot) = \varrho_0 \quad \text{in} \quad \Omega, \tag{8.23}$$

$$\begin{aligned}\int_\Omega \left(\varrho^{n,k,\epsilon}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{n,k,\epsilon}\cdot\omega_i + \varrho^{n,k,\epsilon}[\nabla\mathbf{u}^{n,k,\epsilon}]\mathbf{u}^{n,k,\epsilon}\cdot\omega_i + \tilde{\mathbf{S}}^{n,k,\epsilon} : \mathbf{D}\omega_i\right)\mathrm{d}x \\ - \int_\Omega \left(\epsilon\nabla\varrho^{n,k,\epsilon}\cdot[\nabla\mathbf{u}^{n,k,\epsilon}]\omega_i\right)\mathrm{d}x = \int_\Omega \varrho^{n,k,\epsilon}\mathbf{f}^n\cdot\omega_i\,\mathrm{d}x\end{aligned} \tag{8.24}$$

for all $i = 1, 2, \ldots, n$,

$$\mathbf{u}^{n,k,\epsilon}(0,\cdot) = \mathbf{u}_0^{n,k,\epsilon} = \sum_{i=1}^{n}\alpha_i^{k,\epsilon}(0)\omega_i = P^n\mathbf{u}_0 \quad \text{in } \Omega, \tag{8.25}$$

$$\begin{aligned}\int_\Omega \left(\varrho^{n,k,\epsilon}\frac{\mathrm{d}}{\mathrm{d}t}\theta^{n,k,\epsilon}w_j + \varrho^{n,k,\epsilon}[\nabla\theta^{n,k,\epsilon}]\cdot\mathbf{u}^{n,k,\epsilon}w_j + \tilde{\kappa}^{n,k,\epsilon}\nabla\theta^{n,k,\epsilon}\cdot\nabla w_j\right)\mathrm{d}x \\ - \int_\Omega \epsilon\nabla\varrho^{n,k,\epsilon}\cdot\nabla\theta^{n,k,\epsilon}w_j\,\mathrm{d}x = \int_\Omega \tilde{\mathbf{S}}^{n,k,\epsilon} : \mathbf{D}\mathbf{u}^{n,k,\epsilon}w_j\,\mathrm{d}x\end{aligned} \tag{8.26}$$

for all $j = 1, 2, \ldots, k$,

$$\theta^{n,k,\epsilon}(0,\cdot) = \theta_0^{n,k,\epsilon} = \sum_{j=1}^{k}\nu_j^{n,k,\epsilon}(0)w_j = P^k(\theta_0^n) \quad \text{in } \Omega, \tag{8.27}$$

where we set

$$\theta_{max}^{n,k,\epsilon} := \max\{\theta^{n,k,\epsilon}, \theta_*\}, \quad \tilde{\mathbf{S}}^{n,k,\epsilon} := \mathbf{S}(x, \varrho^{n,k,\epsilon}, \theta_{max}^{n,k,\epsilon}, \mathbf{D}\mathbf{u}^{k,\epsilon})$$
$$\text{and } \tilde{\kappa}^{n,k,\epsilon} := \kappa_0(\varrho^{n,k,\epsilon}, \theta_{max}^{n,k,\epsilon}).$$

Again P^n means projection of $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$ onto linear hull spanned by $\{\omega_i\}_{i=1}^n$ and P^k analogously projection of L^2 onto $\operatorname{span}\{w_j\}_{j=1}^k$.

Notice that the system (8.22)–(8.27) combines one continuous problem (8.22) with two Ritz–Galerkin finite-dimensional systems. As $\mathbf{u}^{n,k,\epsilon}$ is a linear combination of the first n basis functions, which are bounded, there exists exactly one weak solution $\varrho^{n,k,\epsilon}$ to (8.22), that by a classical weak minimum/maximum principle, see [223], satisfies

$$\varrho_* \le \varrho^{n,k,\epsilon} \le \varrho^* \quad \text{a.e. in } \Omega_T. \tag{8.28}$$

Then in order to solve the system (8.24)–(8.27) for fixed $k \in \mathbb{N}$, $\varepsilon > 0$, and $n \in \mathbb{N}$ one can apply Schauder's fixed point theorem and basic estimates. These will appear in the following part. Therefore we skip the details concerning the solvability of

(8.22)–(8.27). The analogous system but with $\epsilon = 0$ can be solved by a combination of characteristic methods, Schauder's fixed point theorem, and basic estimates. For a more detailed proof in the barotropic (without heat effects) case, see [328].

Passing to the limit as $\varepsilon \to 0$.

Let us provide uniform estimates for (8.22)–(8.26) for every $\varepsilon > 0$.

First, let us multiply the equation (8.22) by $\varrho^{n,k,\epsilon}$ and integrate over Ω_T. Since $\operatorname{div}\mathbf{u}^{n,k,\epsilon} = 0$, this leads to

$$\sup_{t\in[0,T]} \|\varrho^{n,k,\epsilon}(t)\|^2_{L^2(\Omega)} + 2\epsilon \int_0^T \|\nabla\varrho^{n,k,\epsilon}\|_2^2 \,\mathrm{d}t \le \|\varrho_0\|_2^2. \tag{8.29}$$

By taking the L^2-scalar product of (8.22) and a smooth z we obtain that

$$\langle \partial_t \varrho^{n,k,\epsilon}, z\rangle - \int_\Omega (\varrho^{n,k,\epsilon}\mathbf{u}^{n,k,\epsilon}) \cdot \nabla z \,\mathrm{d}x + \int_\Omega \epsilon \nabla\varrho^{n,k,\epsilon} \cdot \nabla z \,\mathrm{d}x = 0. \tag{8.30}$$

Next let us multiply the i-th equation in (8.24) by $\alpha^{n,k,\epsilon}$, take the sum over $i = 1, \dots, n$, and let us use (8.30) with $z = |\mathbf{u}^{n,k,\epsilon}|^2/2$ (a density argument is used here) integrated over $(0,t)$. Hence we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho^{n,k,\epsilon}|\mathbf{u}^{n,k,\epsilon}|^2 \,\mathrm{d}x + \int_\Omega \tilde{\mathbf{S}}^{n,k,\epsilon} : \mathbf{D}\mathbf{u}^{n,k,\epsilon} \,\mathrm{d}x = \int_\Omega \varrho^{n,k,\epsilon}\mathbf{f}^n \cdot \mathbf{u}^{n,k,\epsilon} \,\mathrm{d}x. \tag{8.31}$$

Using the Hölder, the Korn, and the Fenchel–Young inequality (see Lemma 2.1.32), (8.28), (8.20) we are able to estimate the right-hand side of (8.31) as follows

$$\begin{aligned} &\left|\int_0^t \int_\Omega \varrho^{n,k,\epsilon}\mathbf{f}^n \cdot \mathbf{u}^{n,k,\epsilon} \,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C\|\mathbf{f}^n\|_{L^{p'}(0,T;L^{p'}(\Omega))} + \int_0^t \frac{c_c}{2}\int_\Omega M(x, \mathbf{D}\mathbf{u}^{n,k,\epsilon}) \,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{8.32}$$

where $C = C(\Omega, \underline{c}, c_c\rho^*, p)$. Integrating (8.31) over time $(0,t)$, by (8.32) and assumption (7.15) we obtain that

$$\begin{aligned} &\left\|\sqrt{\varrho^{n,k,\epsilon}}\mathbf{u}^{n,k,\epsilon}(t)\right\|^2_{L^2(\Omega)} \\ &+ \int_0^t \int_\Omega M^*(x, \tilde{\mathbf{S}}^{n,k,\epsilon}) \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega M(x, \mathbf{D}\mathbf{u}^{n,k,\epsilon}) \,\mathrm{d}x\,\mathrm{d}\tau \le C, \end{aligned} \tag{8.33}$$

where $C = C(\varrho_0, \mathbf{u}_0, \mathbf{f}, \Omega, c_c, \underline{c}, \rho^*, p)$. Combining (8.33) with (7.22), together with the classical Korn inequality we have

$$\|\mathbf{u}^{n,k,\epsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}^{n,k,\epsilon}\|_{L^p(0,T;W^{1,p}(\Omega))} \le C.$$

Multiplying the j-th equation of (8.26) by $\nu_j^{n,k,\epsilon}$, taking the sum over $j = 1, \dots, k$, using the L^2-scalar product of (8.30) with $|\theta^{n,k,\epsilon}|^2/2$, and integrating the result over

$(0,t)$, using Gronwall's inequality we are led to

$$\|\theta^{n,k,\epsilon}\|_{L^\infty(0,T;L^2)} + \left\|\sqrt{\varrho^{n,k,\epsilon}}\,\theta^{n,k,\epsilon}\right\|_{L^\infty(0,T;L^2)} + \left\|\sqrt{\tilde{\kappa}^{n,k,\epsilon}}\,\nabla\theta^{n,k,\epsilon}\right\|_{L^2(0,T;L^2)}$$
$$\leq C\|\sqrt{\varrho_0}\theta_0\|^2_{L^2(\Omega)} + C\|\tilde{\mathbf{S}}^{n,k,\epsilon} : \mathbf{D}(\mathbf{u}^{n,k,\epsilon})\|_{L^2(0,T;L^2(\Omega))} \leq C(n).$$

In the above, for the last inequality we used (8.33) and the fact that $\mathbf{u}^{n,k,\epsilon}$ is a linear combination of the first n elements of the basis $\{\omega^i\}_{i=1}^\infty$ such that $\nabla\omega^i \in L^\infty(\Omega)$ (see (7.27)). By the same reasoning as for (7.119) we find that

$$\|\tilde{\mathbf{S}}^{n,k,\epsilon}\|_{L^\infty(\Omega_T)} + \|\mathbf{D}\mathbf{u}^{n,k,\epsilon}\|_{L^\infty(\Omega_T)} \leq C(n). \tag{8.34}$$

Let us recall that k and n are fixed. Let us multiply the i-th equation of (8.24) by $\frac{\mathrm{d}\alpha_i^{n,k,\epsilon}}{\mathrm{d}t}$ and the j-th equation of (8.26) by $\frac{\mathrm{d}\nu_j^{n,k,\epsilon}}{\mathrm{d}t}$. We conclude that

$$\int_0^T \left|\frac{\mathrm{d}\alpha_i^{n,k,\epsilon}}{\mathrm{d}t}\right| \mathrm{d}t \leq C(n) \quad \text{for } i = 1, \cdots, n,$$
$$\int_0^T \left|\frac{\mathrm{d}\nu_j^{n,k,\epsilon}}{\mathrm{d}t}\right| \mathrm{d}t \leq C(k) \quad \text{for } j = 1, \cdots, k,$$

in particular, we have that

$$\begin{aligned} \|\alpha_i^{n,k,\epsilon}\|_{W^{1,2}(0,T)} &\leq C(n) \quad \text{for } i = 1, \cdots, n, \\ \|\nu_j^{n,k,\epsilon}\|_{W^{1,2}(0,T)} &\leq C(k) \quad \text{for } j = 1, \cdots, k. \end{aligned} \tag{8.35}$$

Summarizing estimates (8.33)–(8.35) we can pass to the limit with ϵ, proving that

$$\varrho^{n,k,\epsilon} \overset{*}{\rightharpoonup} \varrho^{n,k} \quad \text{weakly-}* \text{ in } L^\infty(\Omega_T), \tag{8.36}$$

$$\alpha_i^{n,k,\epsilon} \rightharpoonup \alpha_i^{n,k} \quad \text{weakly in } W^{1,2}(0,T) \text{ and strongly in } C([0,T]) \text{ for } i = 1, \cdots, n, \tag{8.37}$$

$$\nu_j^{n,k,\epsilon} \rightharpoonup \nu_j^{n,k} \quad \text{weakly in } W^{1,2}(0,T) \text{ and strongly in } C([0,T]) \text{ for } j = 1, \cdots, k. \tag{8.38}$$

Consequently by (8.37)

$$\mathbf{u}^{n,k,\epsilon} \to \mathbf{u}^{n,k} \quad \text{strongly in } L^{2p}(0,T; W_n^{1,2p}(\Omega;\mathbb{R}^3)), \tag{8.39}$$

where $W_n^{1,2p}$ stands for $P^n(W^{1,2p})$.

From (8.36) and (8.39), by taking the limit as $\varepsilon \to 0$ in the weak formulation of (8.22) we find that $\varrho^{n,k}$ weakly satisfies the transport equation

$$\partial_t \varrho^{n,k} + \operatorname{div}(\varrho^{n,k}\mathbf{u}^{n,k}) = 0.$$

Notice that the last term on the left-hand side of (8.30) vanishes as $\varepsilon \to 0$ due to the bound (8.29). Then applying the DiPerna–Lions theory of the renormalized solutions to the transport equations (see [121, 238]) one infers that

$$\|\varrho^{n,k}(t)\|^2_{L^2(\Omega)} = \|\varrho_0\|^2_{L^2(\Omega)}. \tag{8.40}$$

By (8.29) and (8.36) we also find that

$$\|\varrho^{n,k}(t)\|^2_{L^2(\Omega)} \le \liminf_{\varepsilon\to 0} \|\varrho^{n,k,\varepsilon}(t)\|^2_{L^2(\Omega)} \le \limsup_{\varepsilon\to 0} \|\varrho^{n,k,\varepsilon}(t)\|^2_{L^2(\Omega)} \le \|\varrho_0\|^2_{L^2(\Omega)}. \tag{8.41}$$

Then (8.41) together with (8.40) implies that (possibly for a subsequence)

$$\varrho^{n,k,\epsilon} \to \varrho^{n,k} \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } \Omega_T. \tag{8.42}$$

Let us concentrate now on the nonlinear viscous term. It converges a.e. in Ω_T, since the arguments converge a.e., due to (8.38), (8.39). Then the uniform integrability of $\{\tilde{\mathbf{S}}^{n,k,\epsilon}\}_\varepsilon$ provided by (8.34) gives by the Vitali convergence theorem (Theorem (8.23)) that

$$\tilde{\mathbf{S}}^{n,k,\epsilon} \to \tilde{\mathbf{S}}^{n,k} \quad \text{strongly in } L^{p'}(0,T;L^{p'}(\Omega_T;\mathbb{R}^{3\times 3})). \tag{8.43}$$

Notice that by (8.29), (8.35) the last term on the left-hand side of (8.26) converges to zero as $\varepsilon \to 0$.

Consequently by (8.21), (8.36), (8.37), (8.38), (8.39), (8.42), (8.43), we obtain that the limit triple $(\varrho^{n,k}, \mathbf{u}^{n,k}, \theta^{n,k})$ solves

$$\int_0^T \langle \partial_t \varrho^n, z\rangle \,\mathrm{d}t - \int_0^T \int_\Omega \varrho^n \mathbf{u}^n \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{8.44}$$

for all $z \in L^s(0,T;W^{1,s}(\Omega))$ with arbitrary $s \in [1,\infty)$,

$$\varrho^{n,k}(0,\cdot) = \varrho_0 \text{ in } \Omega,$$

$$\int_\Omega \left(\varrho^{n,k} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}^{n,k} \cdot \omega_i + \varrho^{n,k}[\nabla \mathbf{u}^{n,k}]\mathbf{u}^{n,k} \cdot \omega_i + \tilde{\mathbf{S}}^{n,k} : \mathbf{D}\omega_i \right) \mathrm{d}x = \int_\Omega \varrho^{n,k}\mathbf{f}^n \cdot \omega_i \,\mathrm{d}x \quad \text{for all } i = 1,2,\dots,n, \tag{8.45}$$

$$\mathbf{u}^{n,k}(0,\cdot) = P^n \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\begin{aligned} &\int_\Omega \left(\varrho^{n,k} \frac{\mathrm{d}}{\mathrm{d}t}\theta^{n,k} w_j + \varrho^{n,k}[\nabla\theta^{n,k}] \cdot \mathbf{u}^{n,k} w_j + \tilde{\kappa}^{n,k} \nabla\theta^{n,k} \cdot \nabla w_j \right) \mathrm{d}x \\ &\quad = \int_\Omega \tilde{\mathbf{S}}^{n,k} : \mathbf{D}\mathbf{u}^{n,k} w_j \,\mathrm{d}x \quad \text{for all } j = 1,2,\dots,k, \end{aligned} \tag{8.46}$$

$$\theta^{n,k}(0,\cdot) = P^k(\theta_0^n) \quad \text{in } \Omega.$$

Here

$$\theta^{n,k}_{max} := \max\{\theta^{n,k}, \theta_*\}, \quad \tilde{\mathbf{S}}^{n,k} := \mathbf{S}(x, \varrho^{n,k}, \theta^{n,k}_{max}, \mathbf{D}\mathbf{u}^{n,k})$$
$$\text{and } \tilde{\kappa}^{n,k} := \kappa_0(\varrho^{n,k}, \theta^{n,k}_{max}).$$

Passing to the limit as $k \to \infty$.

The next step is to pass to the limit as $k \to \infty$. To this end let us collect estimates uniform with respect to k. From (8.28), (8.44) we find that

$$\varrho_* \le \varrho^{n,k} \le \varrho^* \quad \text{and} \quad \int_0^T \|\partial_t \varrho^{n,k}\|^{s/(s-1)}_{(W^{1,s})^*(\Omega)}\, \mathrm{d}t \le c \quad \text{with arbitrary } s \in (1, \infty). \tag{8.47}$$

Repeating the procedures as for (8.33)–(8.35) we obtain the following estimates

$$\left\| \sqrt{\varrho^{n,k}} \mathbf{u}^{n,k} \right\|_{L^\infty(0,T;L^2(\Omega))} \le C,$$

$$\|\mathbf{u}^{n,k}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}^{n,k}\|_{L^p(0,T;W^{1,p}(\Omega))} \le C,$$

$$\|\mathbf{D}\mathbf{u}^{n,k}\|_{L_M(\Omega_T)} + \|\tilde{\mathbf{S}}^{n,k}\|_{L_{M^*}(\Omega_T)} \le C,$$

$$\|\theta^{n,k}\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \sqrt{\varrho^{n,k}} \theta^{n,k} \right\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \sqrt{\tilde{\kappa}^{n,k}} \nabla \theta^{n,k} \right\|_{L^2(0,T;L^2(\Omega))} \le C(n),$$

$$\|\alpha^{n,k}_i\|_{W^{1,2}(0,T)} \le C(n) \text{ for } 1 = 1, \ldots, n, \tag{8.48}$$

where $\tilde{\kappa}^{n,k}(\theta^{n,k}) := \kappa_0(\varrho^{n,k}, \theta^{n,k}_{max})$. Similarly as for (8.34) we infer also that

$$\|\tilde{\mathbf{S}}^{n,k}\|_{L^\infty(\Omega_T)} + \|\mathbf{D}\mathbf{u}^{n,k}\|_{L^\infty(\Omega_T)} \le C(n). \tag{8.49}$$

Let us set

$$\bar{\kappa}(\theta^{n,k}) := (\theta^{n,k})^\beta \quad \text{if } \theta^{n,k} \ge \theta_*,$$
$$\bar{\kappa}(\theta^{n,k}) := \theta_*^\beta \quad \text{if } \theta^{n,k} < \theta_*.$$

Let $\bar{K}$ be a primitive function to $\sqrt{\bar{\kappa}^{n,k}}$, i.e.

$$\bar{K}(\theta^{n,k}) = \theta_*^{\frac{\beta}{2}} \theta^{n,k} \quad \text{for } \theta^{n,k} < \theta_*,$$

$$\bar{K}(\theta^{n,k}) = \tfrac{2}{\beta+2} (\theta^{n,k})^{\frac{\beta+2}{2}} + \tfrac{\beta}{\beta+2} \theta_*^{\frac{\beta+2}{2}} \quad \text{for } \theta^{n,k} \ge \theta_*.$$

Then we may infer that (the details here are skipped and can be found in [158, 64])

$$\|\bar{K}(\theta^{n,k})\|_{L^\infty(0,T;L^2(\Omega))} + \|\bar{K}(\theta^{n,k})\|_{L^2(0,T;W^{1,2}(\Omega))} \le C(n), \tag{8.50}$$

$$\|\tilde{\kappa}^{n,k}\nabla\theta^{n,k}\|_{L^{p_1}(0,T;L^{p_1}(\Omega))} \le C(n) \quad \text{with } p_1 = 2 \text{ for } \beta \le 0$$
$$\text{and } p_1 = (3\beta+10)/(3\beta+5) \text{ for } \beta > 0, \tag{8.51}$$

$$\left\| \frac{\tilde{\kappa}^{n,k}}{\sqrt{\bar{\kappa}(\theta^{n,k})}} \right\|_{L^{p_2}(\Omega_T)} \quad \text{with } p_2 = \infty \text{ for } \beta \le 0 \tag{8.52}$$
$$\text{and } p_2 = (6\beta+20)/(3\beta) \text{ for } \beta > 0,$$

$$\|\nabla\theta^{n,k}\|_{L^{p_3}(\Omega_T)} \le C(n) \text{ with } p_3 = (5\beta+10)/(\beta+5) \text{ for } \beta \le 0 \tag{8.53}$$
$$\text{and } p_3 = 2 \text{ for } \beta > 0.$$

Due to the above estimates, (8.46), and the continuity of the projection P^n we also obtain

$$\|\partial_t(\varrho^k\theta^{n,k})\|_{L^{p_4'}(0,T;W^{-1,p_4'}(\Omega))} \le C(n) \quad \text{where } p_4 = \min\{2,(3\beta+10)/(3\beta+5)\}. \tag{8.54}$$

In particular, due to (8.48)

$$\alpha^{n,k} \rightharpoonup \alpha^n \quad \text{weakly in } W^{1,2}(0,T) \text{ and strongly in } C([0,T]),$$

consequently

$$\mathbf{u}^{n,k} \to \mathbf{u}^n \text{ strongly in } L^{2p}(0,T;W_n^{1,2p}(\Omega;\mathbb{R}^3)) \text{ and a.e. in } \Omega_T. \tag{8.55}$$

Using the theory of renormalized solutions of DiPerna and Lions, see Proposition 8.63, we conclude that

$$\varrho^{n,k} \to \varrho^n \text{ strongly in } C(0,T;L^q(\Omega)) \text{ and a.e. in } \Omega_T \text{ for all } q \in [1,\infty). \tag{8.56}$$

By the uniform estimates on $\{\theta^{n,k}\}_k$, (8.56), we infer the following

$$\theta^{n,k} \rightharpoonup \theta^n \quad \text{weakly in } L^{p_3}(0,T;W^{1,p_3}(\Omega)) \text{ with } p_3 \text{ as in (8.53)}, \tag{8.57}$$

$$\varrho^{n,k}\theta^{n,k} \overset{*}{\rightharpoonup} \varrho^n\theta^n \text{ weakly-}* \text{ in } \left\{ z \in L^\infty(0,T;L^2(\Omega)),\ \partial_t z \in L^{p_4'}(0,T;W^{-1,p_4'}(\Omega)) \right\}, \tag{8.58}$$

which by the Aubin–Lions arguments, Theorem 8.50, implies that

$$\varrho^{n,k}\theta^{n,k} \to \varrho^n\theta^n \quad \text{strongly in } C(0,T;(W^{1,p_4})^*) \text{ with } p_4 \text{ as in (8.54)}. \tag{8.59}$$

Then due to (8.47), (8.56), (8.57), (8.59), by an interpolation argument, for a subsequence if necessary, we have that

$$\theta^{n,k} \to \theta^n \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{8.60}$$

By (8.49), (8.55), (8.56), (8.60), as for (8.43) we infer that

$$\tilde{\mathbf{S}}^{n,k} \to \tilde{\mathbf{S}}^n := \tilde{\mathbf{S}}(x,\varrho^n,\theta^n_{max},\mathbf{Du}^n) \quad \text{strongly in } L^{p'}(0,T;L^{p'}(\Omega_T;\mathbb{R}^{3\times 3})). \tag{8.61}$$

Let us emphasise that it is crucial here that n is fixed. Therefore there is no need to enter the Musielak–Orlicz space structure. However even that is still very straightforward in this case, since in (8.61) one may quite easily obtain modular convergence in $L_M(\Omega_T;\mathbb{R}^{3\times3})$. Finally, from a.e. convergence (8.56), (8.60), estimates (8.50) and (8.52) we infer that

$$\bar{K}(\theta^{n,k}) \rightharpoonup \bar{K}(\theta^n) \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

$$\frac{\tilde{\kappa}^{n,k}}{\sqrt{\bar{\kappa}(\theta^{n,k})}} \to \frac{\tilde{\kappa}^n}{\sqrt{\bar{\kappa}(\theta^n)}} \quad \text{strongly in } L^{\tilde{p}}(\Omega_T) \quad \text{with } \tilde{p} < p_2.$$

Therefore we infer that

$$\tilde{\kappa}^{n,k}\nabla\theta^{n,k} \rightharpoonup \tilde{\kappa}^n\nabla\theta^n \text{ weakly in } L^{p_4}(0,T;W^{1,p_4}(\Omega;\mathbb{R}^3)) \text{ with } p_4 \text{ as in (8.54)}, \tag{8.62}$$

where $\tilde{\kappa}^n := \kappa_0(\varrho^n,\theta^n_{\max})$ with $\theta^n_{max} = \max\{\theta^n,\theta_*\}$.

Summarizing (8.55)–(8.62) we pass to the limit in the system (8.44)–(8.46) obtaining the equations stated as in (8.13)–(8.17). In addition, from the minimum principle we get

$$0 < \theta_* \leq \theta^n \quad \text{a.a. } \in \Omega_T. \tag{8.63}$$

This implies that $\tilde{\mathbf{S}}^n = \mathbf{S}^n$ and $\tilde{\kappa}^n = \kappa^n$ a.e. in Ω_T, which finishes the proof of existence of solutions to the n-approximation (8.13)–(8.17). □

Chapter 9
Functional Inequalities

9.1 Sobolev-Type Embedding

Suppose $m : [0,\infty) \to [0,\infty)$ is a homogeneous and isotropic N-function. In the case of the Orlicz–Sobolev spaces there are known embedding results into some Orlicz space, namely

$$W_0^1 L_m(\Omega) \hookrightarrow L_{\tilde{m}}(\Omega),$$

with $\tilde{m}$ growing in a certain sense faster than m. In particular, there are known optimal embeddings, in the sense that there does not exist a bigger Orlicz space such that the embedding is still continuous. See [89] for the isotropic and [91] for the anisotropic case.

To avoid an excess of unnecessary complications we shall give details only in the isotropic setting, but there are also known anisotropic optimal embeddings [91]. The embedding of an isotropic Orlicz–Sobolev space into an optimal Orlicz space was proved by Cianchi [89]. Two cases are distinguished of a quickly or slowly growing modular function m, corresponding to the cases of a p-Laplacian with $p > N$ and $p \leq N$. For several purposes it is enough to use – let us roughly call it – a simple embedding capturing all types of growth of the modular function. The simple embedding, which yields that

$$W_0^1 L_m(\Omega) \hookrightarrow L_{m^{N'}}(\Omega)$$

with $m^{N'} = m^{\frac{N}{N-1}}$, is provided below, after we give the optimal embedding. It is obviously weaker than the optimal embedding, but it is easy to apply and sufficient for us since it captures a general N-function m independently of any growth conditions with one formulation. Let us stress that since the approximation results of Section 3.7 and their applications to PDEs require Ω to be a Lipschitz bounded domain, we present all of the following results on such domains. See e.g. [95] for an overview of the issue of the regularity of the boundary in relation to the embedding.

To recall the optimal embeddings we employ, we note that in [89] the Sobolev inequality is proved under the restriction

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5_9

$$\int_0 \left(\frac{t}{m(t)}\right)^{\frac{1}{N-1}} \mathrm{d}t < \infty \tag{9.1}$$

on the growth of m at the origin. As for the method it is only important to verify (9.1) for an arbitrary positive right limit, and we follow the customary notation to avoid prescribing its right limit. Nonetheless, the properties of L_m depend on the behavior of $m(s)$ for large values of s and (9.1) can be easily bypassed in applications. We define functions H_N and m_N by the following formulas

$$H_N(s) := \int_0^s \left(\frac{t}{m(t)}\right)^{\frac{1}{N-1}} \mathrm{d}t \qquad \text{and} \qquad m_N(t) := m(H_N^{-1}(t)). \tag{9.2}$$

When (9.1) is satisfied and the growth of an N-function m at infinity is slow, that is

$$\int^\infty \left(\frac{t}{m(t)}\right)^{\frac{1}{N-1}} \mathrm{d}t = \infty, \tag{9.3}$$

where again the left limit is omitted in the notation, then [89, Theorem 3] provides the following continuous embedding

$$W_0^1 L_m(\Omega) \hookrightarrow L_{m_N}(\Omega), \tag{9.4}$$

where m_N is given by (9.2). Otherwise, when the growth of B at infinity is fast, that is,

$$\int^\infty \left(\frac{t}{m(t)}\right)^{\frac{1}{N-1}} \mathrm{d}t < \infty, \tag{9.5}$$

then we have the following continuous embedding

$$W^1 L_m(\Omega) \hookrightarrow L^\infty(\Omega). \tag{9.6}$$

This result was first proved in [308], see also [88].

In the general case, independently of the growth conditions we provide the easy embedding

$$W_0^1 L_m(\Omega) \hookrightarrow L_{m^{N'}}(\Omega).$$

Theorem 9.1 (Modular Sobolev–Poincaré inequality) *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$, and $m : [0,\infty) \to [0,\infty)$ is an arbitrary homogeneous and isotropic N-function. Then there exist constants $c_1, c_2 > 0$ depending on Ω, such that for every $u \in W_0^1 L_m(\Omega)$ it holds that*

$$\left(\int_\Omega m^{N'}(c_1|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} \leq c_2 \int_\Omega m(|\nabla u|)\,\mathrm{d}x. \tag{9.7}$$

Proof. The proof consists of three steps starting with the case of smooth and compactly supported functions on a small cube including the origin, then turning to the

Orlicz–Sobolev space (still on the small cube) and by a scaling argument concluding the claim on a general set.

Step 1. We start with the proof for $u \in C_0^\infty(\Omega)$ with $\operatorname{supp} u \Subset [-1/4, 1/4]^N$. We extend u by 0 outside Ω and note that for every $j = 1, \ldots, N$ we have

$$|u(x)| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_j u(x)| dx_j.$$

As m is increasing, we can apply it to both sides above and get for any $j = 1, \ldots, N$

$$m(|u(x)|) \leq m\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_j u(x)| dx_j\right) \leq m\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla u(x)| \, \mathrm{d}x_j\right) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\nabla u(x)|\right) \, \mathrm{d}x_j.$$

Here we used Jensen's inequality.

When we multiply N copies of the above inequality for $j = 1, \ldots, N$, we obtain

$$m^N\left(|u(x)|\right) \leq \prod_{j=1}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\nabla u(x)|\right) \, \mathrm{d}x_j.$$

By raising both sides in the last display to the power $1/(N-1)$, then integrating over Ω, one gets

$$\begin{aligned} \int_\Omega m^{\frac{N}{N-1}}\left(|u(x)|\right) \mathrm{d}x &= \int_{Q^N} m^{\frac{N}{N-1}}\left(|u(x)|\right) \mathrm{d}x \\ &\leq \int_{Q^N} \prod_{j=1}^{N} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\nabla u(x)|\right) \, \mathrm{d}x_j\right)^{\frac{1}{N-1}} \mathrm{d}x =: I_1. \end{aligned}$$

We apply Lemma 8.25 and obtain

$$I_1 \leq \prod_{j=1}^{N} \left(\int_{Q^{N-1}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\nabla u(x)|\right) \, \mathrm{d}x_j\right)^{\frac{N-1}{N-1}} \mathrm{d}x'\right)^{\frac{1}{N-1}} =: I_2.$$

It suffices to note that

$$I_2 = \prod_{j=1}^{N} \left(\int_{Q^N} m\left(|\nabla u(x)|\right) \, \mathrm{d}x\right)^{\frac{1}{N-1}} = \left(\int_\Omega m\left(|\nabla u(x)|\right) \, \mathrm{d}x\right)^{\frac{N}{N-1}}.$$

Summing up the above estimates we end up with (9.7) for smooth functions with support included in a small cube.

Step 2. We pass to the Orlicz–Sobolev functions supported in small sets. Let $u \in W_0^1 L_m(\Omega)$. Then by Theorem 8.35 there exists a sequence $\{u_\delta\}_\delta \subset C_0^\infty(\Omega)$ such that

$$u_\delta \longrightarrow u \quad \text{modularly in } W^1 L_m(\Omega).$$

Since $\{u_\delta\}_\delta$ is a Cauchy sequence in the modular topology in $W^1L_m(\Omega)$ and the inequality obtained above holds for every u_δ, by (9.7) $\{u_\delta\}_\delta$ is also a Cauchy sequence in the modular topology in $L_{m^{N'}}(\Omega)$.

Due to the modular convergence we get $\nabla u_\delta \to \nabla u$ in measure. By Jensen's inequality (Corollary 2.1.24) and properties of modular convergence together with the Lebesgue dominated convergence theorem, we can pass to the limit as $\delta \to 0$ to get the final claim on the small set Ω including the origin.

Step 3. We are in position to prove the claim on an arbitrary bounded set Ω. If Ω includes the origin, it is contained in the cube of edge length $D = \operatorname{diam}\Omega$. Then $\widetilde{u}(x) = u(4Dx)$ has $\operatorname{supp}\widetilde{u} \subset \Omega_1 \subset \left[-\frac{1}{4}, \frac{1}{4}\right]^N$. We have

$$\left(\int_\Omega m^{N'}(|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} = \left((4D)^N \int_{\Omega_1} m^{N'}(|\widetilde{u}|)\right)^{\frac{1}{N'}} \mathrm{d}x \le (4D)^{\frac{N}{N'}} \int_{\Omega_1} m(|\nabla\widetilde{u}|)\,\mathrm{d}x$$
$$= \frac{1}{4D}\int_\Omega m(4D|\nabla u|)\,\mathrm{d}x.$$

Since the Lebesgue measure is translation-invariant, we have the estimate on an arbitrary domain. □

The result below can be obtained as a consequence of Theorem 9.1. Nonetheless, we include the following proof from [22] to highlight how substantially less technical the analysis is under the Δ_2-condition.

Theorem 9.2 (Modular Sobolev–Poincaré inequality) *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \ge 1$, and $m : [0,\infty) \to [0,\infty)$ is an arbitrary homogeneous and isotropic continuously differentiable N-function satisfying the Δ_2-condition. Then there exists a constant $C = C(N, |\Omega|, m) > 0$ such that for every $u \in W_0^{1,1}(\Omega)$ with $\nabla u \in L_m(\Omega)$*

$$\int_\Omega m^{N'}(|u|)\,\mathrm{d}x \le C\left(\int_\Omega m(|\nabla u|)\,\mathrm{d}x\right)^{N'}.$$

Proof. We start with the proof for fixed $u \in C_0^1(\Omega)$ and then conclude by the density argument. The classical Sobolev inequality gives

$$\left(\int_\Omega m^{N'}(|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} \le C\int_\Omega |\nabla(m(|u|)|\,\mathrm{d}x. \tag{9.8}$$

Since $m \in \Delta_2$, it satisfies

$$m'(t) \le c\frac{m(t)}{t} \tag{9.9}$$

where m' is the right-derivative of m. Moreover, due to Lemma 2.3.11, we get

$$m^*\left(\frac{m(t)}{t}\right) \le m(t). \tag{9.10}$$

Then using (9.9), the Fenchel–Young inequality (2.33), and (9.10) we arrive at

$$\begin{aligned}|\nabla(m(|u|)| = m'(|u|)|\nabla u| &\le c\tfrac{m(|u|)}{u}|\nabla u| \\ &\le \varepsilon m^*\left(\tfrac{m(|u|)}{|u|}\right) + cm(|\nabla u|) \le \varepsilon m(|u|) + cm(|\nabla u|).\end{aligned} \tag{9.11}$$

Summing up, we have

$$\left(\int_\Omega m^{N'}(|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} \le C\int_\Omega |\nabla(m(|\nabla u|)|\,\mathrm{d}x$$

$$\le C\varepsilon\int_\Omega m(|u|)\,\mathrm{d}x + Cc_\varepsilon\int_\Omega |\nabla(m(|\nabla u|)|\,\mathrm{d}x,$$

where according to the Hölder inequality we obtain

$$\left(\int_\Omega m^{N'}(|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} \le \varepsilon C|\Omega|^{\frac{1}{N}}\left(\int_\Omega m^{N'}(|u|)\,\mathrm{d}x\right)^{\frac{1}{N'}} + Cc_\varepsilon\int_\Omega |\nabla(m(|\nabla u|)|\,\mathrm{d}x.$$

Now we can choose ε small enough to absorb it into the right-hand side and obtain the claim for $u \in C_0^1(\Omega)$. Since due to Theorem 8.35 the space $V_0^m(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the modular topology, we infer the general claim by a standard approximation argument. □

A highly useful tool is the following modular Poincaré-type inequality in the Orlicz setting without growth restrictions. It is, via the Hölder inequality, a direct consequence of Theorem 9.1 (or Theorem 9.2 in the Δ_2-setting).

Theorem 9.3 (Modular Poincaré inequality) *Let* $m : [0,\infty) \to [0,\infty)$ *be an arbitrary homogeneous and isotropic N-function and* $\Omega \subset \mathbb{R}^N$ *be a bounded domain, then there exist* $c_{p_1}, c_{p_2} > 0$ *such that for every* $u \in W_0^1L_m(\Omega)$ *it holds that*

$$\int_\Omega m(c_{p_1}|u|)\,\mathrm{d}x \le c_{p_2}\int_\Omega m(|\nabla u|)\,\mathrm{d}x.$$

If additionally $m \in \Delta_2$*, there exists a* $c_{m_2} > 0$ *such that for every* $u \in W_0^1L_m(\Omega)$ *it holds that*

$$\int_\Omega m(|u|)\,\mathrm{d}x \le c_{m_2}\int_\Omega m(|\nabla u|)\,\mathrm{d}x.$$

For more general recent results on modular Poincaré inequalities, see [169].

9.2 The Korn Inequality

The following version of the Korn–Sobolev inequality for the case of isotropic Orlicz spaces holds:

Theorem 9.4 *Let* $m : [0,\infty) \to [0,\infty)$ *be a homogeneous and isotropic N-function and* Ω *be a bounded domain such that* $\overline{\Omega} \subset [-\frac{1}{4}, \frac{1}{4}]^N$*, and* $\mathbf{u} \in \mathcal{BD}_{m,0}(\Omega;\mathbb{R}^n)$ *(for the definition of* $\mathcal{BD}_{m,0}$*, see* (7.140)*). Then for some constant* $C_N > 0$ *depending on the space dimension N the following holds*

$$\|m(|\mathbf{u}|)\|_{L^{\frac{N}{N-1}}(\Omega)} \le C_N \|m(|\mathbf{Du}|)\|_{L^1(\Omega)}. \tag{9.12}$$

The proof below can be found in [185] and generalizes the result of [300].

Proof. As a first step let us show that (9.12) holds for $\mathbf{u} \in X(\Omega;\mathbb{R}^N)$, where

$$X(\Omega;\mathbb{R}^N) := \{\varphi \in C_c^1(\Omega;\mathbb{R}^N) \ : \ \int_\Omega m(|\mathbf{D}\varphi|)\,\mathrm{d}x < \infty\}$$

and next in the second step we will extend this result for $\mathbf{u} \in \mathcal{BD}_{m,0}(\Omega;\mathbb{R}^N)$.

Step 1.
Let us assume that $\mathbf{u} \in X(\Omega;\mathbb{R}^N)$ and $\operatorname{supp}\mathbf{u} \subset [-\frac{1}{4}, \frac{1}{4}]^N$. Let us denote $(1,1,\dots,1)$ by δ_N. According to the mean value theorem in the integral form (see e.g. [13]) we have that

$$u_i(x) = \int_{-\frac{1}{2}}^{0} \sum_{j=1}^{N} \partial_j u_i(x+s\delta_N)\,\mathrm{d}s = -\int_0^{\frac{1}{2}} \sum_{j=1}^{N} \partial_j u_i(x+s\delta_N)\,\mathrm{d}s$$

and

$$\sum_{i=1}^{N} u_i(x) = \int_{-\frac{1}{2}}^{0} \sum_{i,j=1}^{N} \partial_j u_i(x+s\delta_N)\,\mathrm{d}s = -\int_0^{\frac{1}{2}} \sum_{i,j=1}^{N} \partial_j u_i(x+s\delta_N)\,\mathrm{d}s.$$

Hence

$$\begin{aligned} 2\sum_{i=1}^{N} u_i(x) &= \int_{-\frac{1}{2}}^{0} \sum_{i,j=1}^{N} \left(\partial_j u_i(x+s\delta_N) + \partial_i u_j(x+s\delta_N)\right)\,\mathrm{d}s \\ &= -\int_0^{\frac{1}{2}} \sum_{i,j=1}^{N} \left(\partial_j u_i(x+s\delta_N) + \partial_i u_j(x+s\delta_N)\right)\,\mathrm{d}s \end{aligned}$$

and consequently it follows that

$$4|\sum_{i=1}^{N} u_i(x)| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{i,j=1}^{N} |\partial_j u_i(x+s\delta_N) + \partial_i u_j(x+s\delta_N)|\,\mathrm{d}s. \tag{9.13}$$

Applying the N-function $m : [0,\infty) \to [0,\infty)$ to (9.13), by convexity of m and Jensen's inequality, and using the fact that the support of $\mathbf{u}$ is in $[-\frac{1}{4}, \frac{1}{4}]^N$, we observe that

$$\begin{aligned} &\left(m\left(|\sum_{i=1}^{N} u_i(x)|\right)\right)^{\frac{1}{N-1}} \\ &\le \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1}^{N} |\partial_j u_i(x+s\delta_N) + \partial_i u_j(x+s\delta_N)|\right)\mathrm{d}s\right)^{\frac{1}{N-1}}. \end{aligned}$$

Let $e_k = (0,...,0,1,0,...,0)$ be a unit vector along the x_k-axis and $f_k = \delta_N - e_k = (1,...,1,0,1,...,1)$ for $k \in \{1,...,N-1\}$. Notice that

$$\sum_{i=1}^{N} u_i(x) = \int_{-\frac{1}{2}}^{0} \sum_{i,j=1,i\neq k,j\neq k}^{N} \partial_j u_i(x+sf_k)\,\mathrm{d}s + \int_{-\frac{1}{2}}^{0} \partial_k u_k(x+se_k)\,\mathrm{d}s$$
$$= -\int_{0}^{\frac{1}{2}} \sum_{i,j=1,i\neq k,j\neq k}^{N} \partial_j u_i(x+sf_k)\,\mathrm{d}s - \int_{0}^{\frac{1}{2}} \partial_k u_k(x+se_k)\,\mathrm{d}s.$$

Consequently

$$\left(m(|\sum_{i=1}^{N} u_i(x)|)\right)^{\frac{1}{N-1}} \leq$$
$$\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1,i\neq k,j\neq k}^{N} |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)| + \frac{1}{2}|\partial_k u_k(x+se_k)|\right)\mathrm{d}s\right]^{\frac{1}{N-1}}$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{N-1}} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{N} |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\right.\right.$$
$$\left.\left. + m\left(|\partial_k u_k(x+se_k)|\right)\right\}\mathrm{d}s\right]^{\frac{1}{N-1}}$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{N-1}} C\left\{\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{N} |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\mathrm{d}s\right]^{\frac{1}{N-1}}\right.$$
$$\left. + \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)|\right)\mathrm{d}s\right]^{\frac{1}{N-1}}\right\}. \tag{9.14}$$

Next, multiplying expression $\left(m(|\sum_{i=1}^{N} u_i(x)|)\right)^{\frac{1}{N-1}}$ by itself N times we infer that

$$\int_{\mathbb{R}^N} \left(m\left(\left|\sum_{i=1}^{N} u_i(x)\right|\right)\right)^{\frac{N}{N-1}} \mathrm{d}x_1 ... \mathrm{d}x_N$$
$$\leq C\int_{\mathbb{R}^N} \left[\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1}^{N} |\partial_j u_i(x+s\delta_N)+\partial_i u_j(x+s\delta_N)|\right)\mathrm{d}s\right)^{\frac{1}{N-1}}\right.$$
$$\prod_{k=1}^{N-1}\left[\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{N} |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\right)^{\frac{1}{N-1}}\right.$$

$$+\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)|\right)\,\mathrm{d}s\right)^{\frac{1}{N-1}}\Bigg]\Bigg]\,\mathrm{d}x_1...\mathrm{d}x_N \tag{9.15}$$

$$=C\sum_{\sigma}\int_{\mathbb{R}^N}\Bigg[\Bigg(\Bigg(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1}^{N}|\partial_j u_i(x+s\delta_N)+\partial_i u_j(x+s\delta_N)|\right)\mathrm{d}s\Bigg)^{\frac{1}{N-1}}$$

$$\prod_{k=1,k\in\sigma}^{N-1}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{N}|\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\right)^{\frac{1}{N-1}}$$

$$\prod_{k=1,k\notin\sigma}^{N-1}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)|\right)\,\mathrm{d}s\right)^{\frac{1}{N-1}}\Bigg]\,\mathrm{d}x_1...\mathrm{d}x_N$$

where σ runs over all possible subsets of $\{1,2,...,N-1\}$. As $\operatorname{supp}\mathbf{u}\subset[-\frac{1}{4},\frac{1}{4}]^N$, due to Fubini's theorem one can observe that

$$\int_{\mathbb{R}^N}\left[m\left(|\sum_{i=1}^{N}u_i|\right)\right]^{\frac{N}{N-1}}\mathrm{d}x_1...\mathrm{d}x_N$$

$$\le C\sum_{\sigma}\left[\,]\int_{\mathbb{R}^N} m\left(\frac{1}{4}\sum_{i,j=1}^{N}|\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x\right]^{\frac{1}{N-1}}$$

$$\prod_{k=1,k\in\sigma}^{N-1}\left[\int_{\mathbb{R}^N} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{N}|\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x\right]^{\frac{1}{N-1}} \tag{9.16}$$

$$\prod_{k=1,k\notin\sigma}^{N-1}\left[\int_{\mathbb{R}^N} m\left(|\partial_k u_k(x)|\right)\,\mathrm{d}x\right]^{\frac{1}{N-1}}.$$

In a similar way, by integration over lines $(1,-1,1,....,-1)$ etc., instead of these we can obtain the same bound for any $\|m(|\sum_{i=1}^{N}v_i(x)u_i|)\|_{L^{N/(N-1)}(\mathbb{R}^N)}^{N/(N-1)}$, where $v_i\in\{\pm1,0\}$. Next, let v_i vary by setting

$$v_i(x)=\operatorname{sgn}u_i(x),$$

then

$$\int_{\mathbb{R}^N}\left(m\left(\sum_{i=1}^{N}|u_i(x)|\right)\right)^{\frac{N}{N-1}}\mathrm{d}x_1...\mathrm{d}x_N\le\int_{\mathbb{R}^N}\left(m\left(\sum_{i=1}^{N}|v_i(x)u_i(x)|\right)\right)^{\frac{N}{N-1}}\mathrm{d}x_1...\mathrm{d}x_N$$

has the same bound (up to a constant 2^N). Indeed, let

$$\Upsilon=\{\gamma=(\gamma_1,\gamma_2,\gamma_3):\gamma_i\in\{-1,0,1\},i=1,2,3\},$$

$$A_\gamma=\{x\in\mathbb{R}^N:\operatorname{sgn}u_i(x)=v_i(x)=\gamma_i,i=1,2,3\}.$$

Estimates (9.14), (9.16) also hold if we integrate over any measurable subset of $\mathbb{R}^N$ instead of the whole of $\mathbb{R}^N$. Notice that $\{A_\gamma\}_\gamma$ is a division of $\mathbb{R}^N$ into measurable subsets. Observe that

$$|\sum_{i=1}^N v_i(x)u_i(x)| = \sum_{i=1}^N v_i(x)u_i(x) \geq 0.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N} |\sum_{i=1}^N v_i(x)u_i(x)|\, \mathrm{d}x &= \sum_{\gamma\in\Upsilon}\int_{A_\gamma} |\sum_{i=1}^N v_i(x)u_i(x)|\, \mathrm{d}x \\ &= \sum_{\gamma\in\Upsilon}\int_{A_\gamma} \sum_{i=1}^N |v_i(x)u_i(x)|\, \mathrm{d}x = I_1,\end{aligned}$$

where $v_i(x)$ is constant on any subset of the division $\{A_\gamma\}_\gamma$. Therefore all expressions in the above summation over γ are nonnegative and independent of $v_i(x)$. Then we infer

$$I_1 = \sum_{\gamma\in\Upsilon}\int_{A_\gamma} \sum_{i=1}^N |u_i(x)|\, \mathrm{d}x = \int_{\mathbb{R}^N} \sum_{i=1}^N |u_i(x)|\, \mathrm{d}x$$

but notice also that

$$I_1 \leq 2^N \int_{\mathbb{R}^N} m(|\sum_{i=1}^N v_i(x)u_i(x)|)\, \mathrm{d}x.$$

Hence we deduce that

$$\int_{\mathbb{R}^N} m(\sum_{i=1}^N |u_i(x)|)\, \mathrm{d}x \leq 2^N \int_{\mathbb{R}^N} m(|\sum_{i=1}^N v_i(x)u_i(x)|)\, \mathrm{d}x.$$

Finally, since the geometric mean of nonnegative numbers is no greater than the arithmetic mean, we can estimate the right-hand side of (9.16) as follows

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(m(|\sum_{i=1}^N u_i|)\right)^{\frac{N}{N-1}} \mathrm{d}x_1 \ldots \mathrm{d}x_N \\ &\leq C\sum_\sigma \left(\frac{1}{N}\right)^{\frac{N}{N-1}} \left[\int_{\mathbb{R}^N} m\left(\frac{1}{4}\sum_{i,j=1}^N |\partial_j u_i(x) + \partial_i u_j(x)|\right) \mathrm{d}x \right.\\ &+ \sum_{k=1,k\in\sigma}^{N-1} \int_{\mathbb{R}^N} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^N |\partial_j u_i(x) + \partial_i u_j(x)|\right) \mathrm{d}x \\ &\left. + \sum_{k=1,k\notin\sigma}^{N-1} \int_{\mathbb{R}^N} m\left(\frac{1}{2}|\partial_k u_k(x)|\right) \mathrm{d}x\right]^{\frac{N}{N-1}} = I_2.\end{aligned} \tag{9.17}$$

As m is convex and $m(0) = 0$, we obtain that

$$\begin{aligned} I_2 &\le C \sum_{\sigma} \left(\frac{1}{N}\right)^{\frac{N}{N-1}} \Big[\frac{1}{2}\int_{\mathbb{R}^N} m\left(\frac{1}{2}\sum_{i,j=1}^{N} |\partial_j u_i(x) + \partial_i u_j(x)|\right) \mathrm{d}x \\ &+ \sum_{k=1, k\in\sigma}^{N-1} \int_{\mathbb{R}^N} m\left(\frac{1}{2} \sum_{i,j=1, i\neq k, j\neq k}^{N-1} |\partial_j u_i(x) + \partial_i u_j(x)|\right) \mathrm{d}x \\ &+ \sum_{k=1, k\notin\sigma}^{N-1} \int_{\mathbb{R}^N} m\left(\frac{1}{2}|\partial_k u_k(x)|\right) \mathrm{d}x\Big]^{\frac{N}{N-1}} \\ &\le \Big[C(N) \int_{\mathbb{R}^N} m\left(\frac{1}{2}\sum_{i,j=1}^{N} |\partial_j u_i(x) + \partial_i u_j(x)|\right) \mathrm{d}x\Big]^{\frac{N}{N-1}}. \end{aligned} \tag{9.18}$$

Summarizing (9.17), (9.18) we infer that (9.12) holds for $\mathbf{u} \in X(\Omega; \mathbb{R}^n)$.

Step 2.

Let $\tilde{\Omega}$ be a bounded domain such that $[-\frac{1}{4}, \frac{1}{4}] \supset \tilde{\Omega} \supset \overline{\Omega}$ and let $C_c^\infty(\tilde{\Omega}; \mathbb{R}^N)$ be the set of smooth functions in $\mathbb{R}^N$ with support in $\tilde{\Omega}$. Step 1 ensures that $\mathbf{u} \in C_c^\infty(\tilde{\Omega}; \mathbb{R}^N)$ with $\operatorname{supp} \mathbf{u} \in [-\frac{1}{4}, \frac{1}{4}]^N$ satisfies

$$\|m(|\mathbf{u}|)\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \le C_N \|m(|\mathbf{Du}|)\|_{L^1(\mathbb{R}^N)}. \tag{9.19}$$

To show (9.19) for all $\mathbf{u} \in \mathcal{BD}_{m,0}(\Omega; \mathbb{R}^N)$, we extend $\mathbf{u}$ by zero outside of the set Ω. Notice that $\mathbf{u} \in \mathcal{BD}_{M,0}(\tilde{\Omega}; \mathbb{R}^N)$. Now let us construct a regularized sequence

$$\mathbf{u}^\varepsilon(x) := \varrho_\varepsilon * \mathbf{u}(x),$$

where $\varepsilon < \frac{1}{2}\operatorname{dist}(\partial\tilde{\Omega}, \Omega)$ and ϱ_ε is a standard regularizing kernel (nonnegative smooth function such that $\int_{\mathbb{R}^N} \varrho(x)\,\mathrm{d}x = 1$ and $\varrho_\varepsilon(x) = \frac{1}{\varepsilon}\varrho(\frac{1}{\varepsilon}x)$) and the convolution is with respect to the x-variable. Since $\mathbf{u}^\varepsilon$ is smooth and of compact support in $\tilde{\Omega}$, inequality (9.19) holds true for $\mathbf{u}^\varepsilon$. Passing to the limit as $\varepsilon \to 0$ we have that

$$\mathbf{u}^\varepsilon \to \mathbf{u}, \quad \mathbf{Du}^\varepsilon \to \mathbf{Du} \quad \text{ a.e. in } \mathbb{R}^N$$

and consequently by the continuity of an N-function m we have that

$$m(|\mathbf{u}^\varepsilon|) \to m(|\mathbf{u}|), \quad m(|\mathbf{Du}^\varepsilon|) \to m(|\mathbf{Du}|) \quad \text{ a.e. in } \mathbb{R}^N.$$

Moreover, due to Lemma 3.4.8 $\{m(\mathbf{u}^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable in L^1. Finally, by the Vitali convergence theorem (see Theorem 8.23) we obtain that

$$\begin{aligned} m(|\mathbf{u}^\varepsilon|) &\to m(|\mathbf{u}|) \quad \text{strongly in } L^1(\mathbb{R}^N), \\ m(|\mathbf{Du}^\varepsilon|) &\to m(|\mathbf{Du}|) \quad \text{strongly in } L^1(\mathbb{R}^N). \end{aligned}$$

The above gives that the limit $\mathbf{u}$ satisfies inequality (9.19). □

Remark 9.2.1. In the case when Ω is a bounded domain, which is not necessarily contained in $[-\frac{1}{4}, \frac{1}{4}]^N$, we can use a rescaling of the space variables. Then we get that

$$\|m(|\mathbf{u}|)\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C_N \|m(|C_r \mathbf{Du}|)\|_{L^1(\Omega)},$$

where C_r is a constant dependent on the Jacobian of the rescaling.

References

1. E. Acerbi, I. Fonseca, and G. Mingione. Existence and regularity for mixtures of micromagnetic materials. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 462(2072):2225–2243, 2006.
2. E. Acerbi and G. Mingione. Regularity results for a class of functionals with non-standard growth. *Arch. Ration. Mech. Anal.*, 156(2):121–140, 2001.
3. E. Acerbi and G. Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.
4. T. Adamowicz and P. Górka. The Liouville theorems for elliptic equations with nonstandard growth. *Commun. Pure Appl. Anal.*, 14(6):2377–2392, 2015.
5. R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
6. L. Aharouch, J. Bennouna, and A. Touzani. Existence of renormalized solution of some elliptic problems in Orlicz spaces. *Rev. Mat. Complut.*, 22(1):91–110, 2009.
7. Y. Ahmida, P. Gwiazda, I. Skrzypczak, and Youssfi A. Gossez's approximation theorems in Musielak–Orlicz–Sobolev spaces. *J. Funct. Anal.*, 275(9):2538–2571, 2018.
8. A. Aissaoui Fqayeh, A. Benkirane, M. El Moumni, and A. Youssfi. Existence of renormalized solutions for some strongly nonlinear elliptic equations in Orlicz spaces. *Georgian Math. J.*, 22(3):305–321, 2015.
9. A. Alberico, I. Chlebicka, A. Cianchi, and A. Zatorska-Goldstein. Fully anisotropic elliptic problems with minimally integrable data. *Calc. Var. PDEs*, 58:186, 2019.
10. A. Alberico, G. di Blasio, and F. Feo. A priori estimates for solutions to anisotropic elliptic problems via symmetrization. *Math. Nachr.*, 290(7):986–1003, 2017.
11. G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.
12. C. O. Alves and V. D. Radulescu. The Lane-Emden equation with variable double-phase and multiple regime. *Proc. Amer. Math. Soc.*, 148(7):2937–2952, 2020.
13. H. Amann and J. Escher. *Analysis. II.* Birkhäuser Verlag, Basel, 2008.
14. L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
15. S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov. *Boundary value problems in mechanics of nonhomogeneus fluids*, volume 22 of *Studies in Mathematics and its Applications (translated from Russian)*. North-Holland, Amsterdam, 1990.
16. A. Kh. Balci, L. Diening, and M. Surnachev. New examples on Lavrentiev gap using fractals. *Calc. Var. Partial Differential Equations*, 59(5):Paper No. 180, 34, 2020.
17. A. Kh. Balci and M. Surnachev. Lavrentiev gap for some classes of generalized Orlicz functions. *Nonlinear Anal.*, 207:112329, 2021.
18. E. J. Balder. A general approach to lower semicontinuity and lower closure in optimal control theory. *SIAM J. Control Optim.*, 22(4):570–598, 1984.

I. Chlebicka et al., *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-88856-5

19. J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.
20. J. M. Ball. A version of the fundamental theorem for Young measures. In *PDEs and continuum models of phase transitions (Nice, 1988)*, volume 344 of *Lecture Notes in Phys.*, pages 207–215. Springer, Berlin, 1989.
21. G. Barletta and A. Cianchi. Dirichlet problems for fully anisotropic elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 147(1):25–60, 2017.
22. P. Baroni. Riesz potential estimates for a general class of quasilinear equations. *Calc. Var. Partial Differential Equations*, 53(3-4):803–846, 2015.
23. P. Baroni and I. Chlebicka. Regularity of minimizers of functionals with mild double phase without assumption of log-Hölder continuity imposed on a weight. *preprint*, 2021.
24. P. Baroni, M. Colombo, and G. Mingione. Nonautonomous functionals, borderline cases and related function classes. *St. Petersburg Math. J.*, 27(3):347–379, 2016.
25. P. Baroni, M. Colombo, and G. Mingione. Regularity for general functionals with double phase. *Calc. Var. Partial Differential Equations*, 57(2):Art. 62, 48, 2018.
26. L. Beck and G. Mingione. Lipschitz bounds and non-uniform ellipticity. *Comm. Pure Appl. Math.*, 73(5):944–1034, 2020.
27. M. B. Benboubker, H. Chrayteh, H. Hjiaj, and C. Yazough. Existence of solutions in the sense of distributions of anisotropic nonlinear elliptic equations with variable exponent. *Topol. Methods Nonlinear Anal.*, 46(2):665–693, 2015.
28. M. Bendahmane, K. H. Karlsen, and M. Saad. Nonlinear anisotropic elliptic and parabolic equations with variable exponents and L^1 data. *Commun. Pure Appl. Anal.*, 12(3):1201–1220, 2013.
29. M. Bendahmane, M. Langlais, and M. Saad. On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease. *Nonlinear Anal.*, 54(4):617–636, 2003.
30. M. Bendahmane, P. Wittbold, and A. Zimmermann. Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data. *J. Differential Equations*, 249(6):1483–1515, 2010.
31. P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.-L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.
32. A. Benkirane and J. Bennouna. Existence of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms in Orlicz spaces. In *Partial differential equations*, volume 229 of *Lecture Notes in Pure and Appl. Math.*, pages 125–138. Dekker, New York, 2002.
33. A. Benkirane and A. Elmahi. An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces. *Nonlinear Anal.*, 36(1, Ser. A: Theory Methods):11–24, 1999.
34. C. Bennett and R. Sharpley. *Interpolation of Operators*. Academic Press, 1988.
35. A. Benyaiche, P. Harjulehto, P. Hästö, and A. Karppinen. The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth. *J. Differential Equations*, 275:790–814, 2021.
36. D. Blanchard. Renormalized solutions for parabolic problems with L^1 data. In *Free boundary problems, theory and applications (Zakopane, 1995)*, volume 363 of *Pitman Res. Notes Math. Ser.*, pages 177–185. Longman, Harlow, 1996.
37. D. Blanchard, F. Murat, and H. Redwane. Existence et unicité de la solution renormalisée d'un problème parabolique non linéaire assez général. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):575–580, 1999.
38. D. Blanchard, F. Murat, and H. Redwane. Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems. *J. Differential Equations*, 177(2):331–374, 2001.
39. D. Blanchard, F. Petitta, and H. Redwane. Renormalized solutions of nonlinear parabolic equations with diffuse measure data. *Manuscripta Math.*, 141(3-4):601–635, 2013.
40. D. Blanchard and H. Redwane. Solutions renormalisées d'équations paraboliques à deux non linéarités. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(8):831–835, 1994.
41. D. Blanchard and H. Redwane. Renormalized solutions for a class of nonlinear evolution problems. *J. Math. Pures Appl. (9)*, 77(2):117–151, 1998.

42. L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, 1997.
43. L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.
44. L. Boccardo and T. Gallouët. Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations*, 17(3-4):641–655, 1992.
45. L. Boccardo, T. Gallouët, and P. Marcellini. Anisotropic equations in L^1. *Differential Integral Equations*, 9(1):209–212, 1996.
46. L. Boccardo, T. Gallouët, and L. Orsina. Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(5):539–551, 1996.
47. L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat. Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *J. Differential Equations*, 106(2):215–237, 1993.
48. V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.
49. V. Bögelein, F. Duzaar, P. Marcellini, and C. Scheven. A variational approach to doubly nonlinear equations. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 29(4):739–772, 2018.
50. M. Bonforte, J. Dolbeault, M. Muratori, and B. Nazaret. Weighted fast diffusion equations (Part I): Sharp asymptotic rates without symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities. *Kinet. Relat. Models*, 10(1):33–59, 2017.
51. M. Bonforte and N. Simonov. Quantitative a priori estimates for fast diffusion equations with Caffarelli-Kohn-Nirenberg weights. Harnack inequalities and Hölder continuity. *Adv. Math.*, 345:1075–1161, 2019.
52. M. Borowski and I. Chlebicka. Modular density of smooth functions in inhomogeneous and fully anisotropic Musielak–Orlicz–Sobolev spaces. *preprint*, 2021.
53. P. Bousquet and L. Brasco. C^1 regularity of orthotropic p-harmonic functions in the plane. *Anal. PDE*, 11(4):813–854, 2018.
54. P. Bousquet, L. Brasco, C. Leone, and A. Verde. On the Lipschitz character of orthotropic p-harmonic functions. *Calc. Var. Partial Differential Equations*, 57(3):Art. 88, 33, 2018.
55. D. Breit and A. Cianchi. Negative Orlicz–Sobolev norms and strongly nonlinear systems in fluid mechanics. *J. Differential Equations*, 259(1):48–83, 2015.
56. D. Breit and A. Verde. Quasiconvex variational functionals in Orlicz–Sobolev spaces. *Ann. Mat. Pura Appl. (4)*, 192(2):255–271, 2013.
57. H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
58. M. Bulíček, L. Consiglieri, and J. Málek. On solvability of a non-linear heat equation with a non-integrable convective term and data involving measures. *Nonlinear Anal. Real World Appl.*, 12(1):571–591, 2011.
59. M. Bulíček, P. Gwiazda, M. Kalousek, and A. Świerczewska-Gwiazda. Homogenization of nonlinear elliptic systems in nonreflexive Musielak–Orlicz spaces. *Nonlinearity*, 32(3):1073–1110, 2019.
60. M. Bulíček, P. Gwiazda, M. Kalousek, and A. Świerczewska-Gwiazda. Existence and homogenization of nonlinear elliptic systems in nonreflexive spaces. *Nonlinear Anal.*, 194:111487, 34, 2020.
61. M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, 44(4):2756–2801, 2012.
62. M. Bulíček, P. Gwiazda, and J. Skrzeczkowski. An improved range of exponents for absence of Lavrentiev phenomenon for double phase functionals. In preparation.
63. M. Bulíček, P. Gwiazda, and J. Skrzeczkowski. Parabolic equations in Musielak–Orlicz spaces with discontinuous in time N-function. *J. Differential Equations*, 290:17–56, 2021.
64. M. Bulíček, J. Málek, and K. R. Rajagopal. Mathematical analysis of unsteady flows of fluids with pressure, shear-rate and temperature dependent material moduli, that slip at solid boundaries. *SIAM J. Math. Anal.*, 5(2):665–707, 2009.
65. S.-S. Byun and J. Oh. Regularity results for generalized double phase functionals. *Analysis & PDE*, 13:1269–1300, 2019.
66. P. Caldiroli and R. Musina. On a variational degenerate elliptic problem. *NoDEA Nonlinear Differential Equations Appl.*, 7(2):187–199, 2000.

67. O. N. Cavatorta and R. D. Tonini. Dimensionless velocity profiles and parameter maps for non-Newtonian fluids. *International Communications in Heat and Mass Transfer*, 14(4):359–369, 1987.
68. M. Cencelj, V. D. Radulescu, and D. D. Repovš. Double phase problems with variable growth. *Nonlinear Anal.*, 177(part A):270–287, 2018.
69. K. Chełmiński and S. Owczarek. Renormalised solutions in thermo-visco-plasticity for a Norton–Hoff type model. Part II: the limit case. *Nonlinear Anal. Real World Appl.*, 31:643–660, 2016.
70. Y. Chen, S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406 (electronic), 2006.
71. I. Chlebicka. A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces. *Nonl. Analysis*, 175:1 – 27, 2018.
72. I. Chlebicka. Gradient estimates for problems with Orlicz growth. *Nonlinear Anal.*, 194:111364, 32, 2020.
73. I. Chlebicka. Measure data elliptic problems with generalized Orlicz growth. *arXiv:2008.02495*, 2020.
74. I. Chlebicka. Regularizing effect of the lower-order term for elliptic problems with Orlicz growth. *Israel J. Math.*, 236(2):967–1000, 2020.
75. I. Chlebicka and C. De Filippis. Removable sets for nonuniformly elliptic problems. *Ann. Mat. Pura Appl. (4)*, 199(2):619–649, 2020.
76. I. Chlebicka, C. De Filippis, and L. Koch. Boundary regularity for manifold constrained $p(x)$-harmonic maps. *J. London Math. Soc.*, 2021.
77. I. Chlebicka, F. Giannetti, and Zatorska-Goldstein A. Wolff potentials and local behaviour of solutions to measure data elliptic problems with Orlicz growth. *arXiv:2006.02172*, 2020.
78. I. Chlebicka, F. Giannetti, and A. Zatorska-Goldstein. Elliptic problems with growth in nonreflexive Orlicz spaces and with measure or L^1 data. *J. Math. Anal. Appl.*, 479(1):185–213, 2019.
79. I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. Well-posedness of parabolic equations in the non-reflexive and anisotropic Musielak–Orlicz spaces in the class of renormalized solutions. *J. Differential Equations*, 265(11):5716–5766, 2018.
80. I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. Parabolic equations in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentiev's phenomenon. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(5):1431–1465, 2019.
81. I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. Renormalized solutions to parabolic equations in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentiev's phenomenon. *J. Differential Equations*, 267(2):1129–1166, 2019.
82. I. Chlebicka and A. Karppinen. Removable sets in elliptic equations with Musielak–Orlicz growth. *J. Math. Anal. Appl.*, 501(1):124073, 2021.
83. I. Chlebicka and P. Nayar. Essentially fully anisotropic Orlicz functions and uniqueness to measure data problem. *Math. Methods Appl. Sci.*, pages 1–25, 2021.
84. I. Chlebicka, Y. Youn, and A. Zatorska-Goldstein. Measure data systems with Orlicz growth. *arXiv:2106.11639*, 2021.
85. I. Chlebicka, Y. Youn, and A. Zatorska-Goldstein. Wolff potentials and measure data vectorial problems with Orlicz growth. *arXiv:2102.09313*, 2021.
86. I. Chlebicka and A. Zatorska-Goldstein. Existence to nonlinear parabolic problems with unbounded weights. *J. Evol. Equations*, 19:1–19, 2019.
87. I. Chlebicka and A. Zatorska-Goldstein. Generalized superharmonic functions with strongly nonlinear operator. *Potential Analysis*, 2021.
88. A. Cianchi. Continuity properties of functions from Orlicz–Sobolev spaces and embedding theorems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(3):575–608, 1996.
89. A. Cianchi. Boundedness of solutions to variational problems under general growth conditions. *Comm. Partial Differential Equations*, 22(9-10):1629–1646, 1997.
90. A. Cianchi. Some results in the theory of Orlicz spaces and applications to variational problems. In *Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998)*, pages 50–92. Acad. Sci. Czech Repub., Prague, 1999.
91. A. Cianchi. A fully anisotropic Sobolev inequality. *Pacific J. Math.*, 196(2):283–295, 2000.

92. A. Cianchi. Optimal Orlicz–Sobolev embeddings. *Rev. Mat. Iberoam.*, 20:427–474, 2004.
93. A. Cianchi. Symmetrization in anisotropic elliptic problems. *Comm. Partial Differential Equations*, 32(4-6):693–717, 2007.
94. A. Cianchi. Korn type inequalities in Orlicz spaces. *Journal of Functional Analysis*, 267(7):2313–2352, 2014.
95. A. Cianchi and V. Maz'ya. Quasilinear elliptic problems with general growth and merely integrable, or measure, data. *Nonlinear Anal.*, 164:189–215, 2017.
96. D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4):1585–1620, 2008.
97. M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. *Arch. Ration. Mech. Anal.*, 218(1):219–273, 2015.
98. M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.*, 215(2):443–496, 2015.
99. D. Cruz-Uribe and P. Hästö. Extrapolation and interpolation in generalized Orlicz spaces. *Trans. Amer. Math. Soc.*, 370(6):4323–4349, 2018.
100. D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.
101. G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Definition and existence of renormalized solutions of elliptic equations with general measure data. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):481–486, 1997.
102. G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.
103. A. Dall'Aglio. Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations. *Ann. Mat. Pura Appl. (4)*, 170:207–240, 1996.
104. L. D'Ambrosio. Liouville theorems for anisotropic quasilinear inequalities. *Nonlinear Anal.*, 70(8):2855–2869, 2009.
105. C. De Filippis. Partial regularity for manifold constrained $p(x)$-harmonic maps. *Calc. Var. Partial Differential Equations*, 58(2):Paper No. 47, 38, 2019.
106. C. De Filippis and G. Mingione. Manifold constrained non-uniformly elliptic problems. *J. Geom. Analysis*, 30:1661–1723, 2020.
107. C. De Filippis and G. Palatucci. Hölder regularity for nonlocal double phase equations. *J. Differential Equations*, 267(1):547–586, 2019.
108. M. Decker, C. Halbach, C. Nam, N. Wagner, and E. Wetzel. Stab resistance of shear thickening fluid (stf)-treated fabrics. *Composites science and technology*, 67(3):565–578, 2007.
109. A. Denkowska, P. Gwiazda, and P. Kalita. On renormalized solutions to elliptic inclusions with nonstandard growth. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 21, 52, 2021.
110. B. Desjardins and M. J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.*, 146:59–71, 1999.
111. B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction: Compressible and incompressible models. *Commun. Partial Differential Equations*, 25:1399–1413, 2000.
112. T. Di Marco and P. Marcellini. A-priori gradient bound for elliptic systems under either slow or fast growth conditions. *Calc. Var. Partial Differential Equations*, 59(4):Paper No. 120, 26, 2020.
113. L. Diening. Maximal function on Musielak–Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.
114. L. Diening and F. Ettwein. Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.*, 20(3):523–556, 2008.
115. L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.
116. L. Diening, J. Málek, and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM, Control, Optimisation and Calculus of Variations*, 14(2):211–232, 2008.

117. L. Diening, M. Růžička, and K. Schumacher. A decomposition technique for John domains. *Ann. Acad. Sci. Fenn. Math.*, 35:87–114, 2010.
118. L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions for unsteady motion of generalized Newtonian fluids. *Ann. Scuola Norm. Sup. Pisa.*, 9:1–46, 2010.
119. T. Dinu. Entire solutions of multivalued nonlinear Schrödinger equations in Sobolev spaces with variable exponent. *Nonlinear Anal.*, 65(7):1414–1424, 2006.
120. R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.
121. R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
122. T. Donaldson. Nonlinear elliptic boundary value problems in Orlicz–Sobolev spaces. *J. Differential Equations*, 10:507–528, 1971.
123. T. Donaldson. Inhomogeneous Orlicz–Sobolev spaces and nonlinear parabolic initial value problems. *J. Differential Equations*, 16:201–256, 1974.
124. G. Dong and X. Fang. Existence results for some nonlinear elliptic equations with measure data in Orlicz–Sobolev spaces. *Bound. Value Probl.*, pages 2015:18, 22, 2015.
125. G. Dong and X. Fang. Differential equations of divergence form in separable Musielak–Orlicz–Sobolev spaces. *Bound. Value Probl.*, pages 2016:106, 19, 2016.
126. J. Droniou, A. Porretta, and A. Prignet. Parabolic capacity and soft measures for nonlinear equations. *Potential Anal.*, 19(2):99–161, 2003.
127. J. Droniou and A. Prignet. Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):181–205, 2007.
128. S. Dudek and I. Skrzypczak. Liouville theorems for elliptic problems in variable exponent spaces. *Commun. Pure Appl. Anal.*, 16(2):513–532, 2017.
129. R. G. Egres Jr, Y. S. Lee, J. E. Kirkwood, K. M. Kirkwood, E. D. Wetzel, and N. J. Wagne. 'Liquid armour': Protective fabrics utilising shear thickening fluids. *Proceedings of the 4th International Conference of Safety and Protective Fabrics, Pittsburg, PA, October*, pages 26–27, 2004.
130. A. Elmahi and D. Meskine. Parabolic equations in Orlicz spaces. *J. London Math. Soc. (2)*, 72(2):410–428, 2005.
131. A. Elmahi and D. Meskine. Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces. *Nonlinear Anal.*, 60(1):1–35, 2005.
132. A. Elmahi and D. Meskine. Elliptic inequalities with lower order terms and L^1 data in Orlicz spaces. *J. Math. Anal. Appl.*, 328(2):1417–1434, 2007.
133. J. Elstrodt. *Maß- und Integrationstheorie*. Springer-Lehrbuch. [Springer Textbook]. Springer-Verlag, Berlin, fourth edition, 2005. Grundwissen Mathematik. [Basic Knowledge in Mathematics].
134. E. Emmrich and A. Wróblewska-Kamińska. Convergence of a full discretization of quasi-linear parabolic equations in isotropic and anisotropic Orlicz spaces. *SIAM J. Numer. Anal.*, 51(2):1163–1184, 2013.
135. R. Erban. On the existence of solutions to the Navier–Stokes equations of two-dimensional compressible flow. *Mathematical Methods in the Applied Sciences*, 26:489–517, 2003.
136. A. Esposito, F. Leonetti, and P. Vincenzo Petricca. Absence of Lavrentiev gap for non-autonomous functionals with (p,q)-growth. *Adv. Nonlinear Anal.*, 8(1):73–78, 2019.
137. L. Esposito, F. Leonetti, and G. Mingione. Sharp regularity for functionals with (p,q) growth. *J. Differential Equations*, 204(1):5–55, 2004.
138. L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
139. L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
140. H. J. Eyring. Viscosity, plasticity, and diffusion as examples of absolute reaction rates. *J. Chemical Physics*, 4:283–291, 1936.
141. E. B. Fabes, D. Jerison, and C. E. Kenig. Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure. *Ann. of Math. (2)*, 119(1):121–141, 1984.

142. E. B. Fabes, C. E. Kenig, and D. Jerison. Boundary behavior of solutions to degenerate elliptic equations. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 577–589. Wadsworth, Belmont, CA, 1983.
143. E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.
144. X. Fan. Boundary trace embedding theorems for variable exponent Sobolev spaces. *J. Math. Anal. Appl.*, 339(2):1395–1412, 2008.
145. X. Fan. Positive solutions to $p(x)$-Laplacian–Dirichlet problems with sign-changing nonlinearities. *Glasg. Math. J.*, 52(3):505–516, 2010.
146. X. Fan. Differential equations of divergence form in Musielak-Sobolev spaces and a sub-supersolution method. *J. Math. Anal. Appl.*, 386(2):593–604, 2012.
147. X. Fan and Q. Zhang. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.*, 52(8):1843–1852, 2003.
148. E. Feireisl. On the motion of rigid bodies in a viscous compressible fluid. *Arch. Rational Mech. Anal.*, 167:281–308, 2003.
149. E. Feireisl. *Dynamics of Viscous Compressible Fluids*. Oxford Lecture Series in Mathematics and Its Applications, 2004.
150. E. Feireisl, M. Hillairet, and Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian fluid. *Nonlinearity*, 21:2008, 1349-1366.
151. J. D. Fernandes, J. Groisman, and S. T. Melo. Harnack inequality for a class of degenerate elliptic operators. *Z. Anal. Anwendungen*, 22(1):129–146, 2003.
152. E. Fernández-Cara, F. Guillén-Gonzalez, and R. R. Ortega. Some theoretical results for viscoplastic and dilatant fluids with variable density. *Nonlinear Anal.*, 28(6):1079–1100, 1997.
153. A. Fiorenza and M. Krbec. Indices of Orlicz spaces and some applications. *Comment. Math. Univ. Carolin.*, 38(3):433–451, 1997.
154. J. Fischer. Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems. *Arch. Ration. Mech. Anal.*, 218(1):553–587, 2015.
155. I. Fonseca, J. Malý, and G. Mingione. Scalar minimizers with fractal singular sets. *Arch. Ration. Mech. Anal.*, 172(2):295–307, 2004.
156. I. Fragalà, F. Gazzola, and B. Kawohl. Existence and nonexistence results for anisotropic quasilinear elliptic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(5):715–734, 2004.
157. I. Fragalà, F. Gazzola, and G. Lieberman. Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains. *Discrete Contin. Dyn. Syst.*, suppl.:280–286, 2005.
158. J. Frehse, J. Málek, and M. Růžička. Large data existence results for unsteady flows of inhomogeneus heat-conducting incompressible fluids. *Communication in Partial Differential Equations*, 35(10):1891–1919, 2010.
159. J. Frehse, J. Málek, and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.*, 34(5):1064–1083, 2003.
160. J. Frehse and M. Růžička. Non-homogenous generalized Newtonian fluids. *Mathematische Zeitschrift*, 260:355–375, 2008.
161. M. Fuchs. Korn inequalities in Orlicz space. *Irish Math. Soc. Bulletin*, 65:5–9, 2010.
162. M. Fuchs and M. Bildhauer. Compact embeddings of the space of functions with bounded logarithmic deformation. *Preprint Universitaet des Saarlandes*, 276, 2010.
163. M. Fuchs and G. Seregin. Variational methods for fluids of Prandtl–Eyring type and plastic materials with logarithmic hardening. *Math. Methods Appl. Sci.*, 22:317–351, 1999.
164. M. Fuchs and G. Seregin. Variational methods for problems from plasticity theory and for generalized Newtonian fluids. *Ann. Univ. Sarav., Ser. Math.*, 10:1–283, 1999.
165. H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band 38.
166. G. P. Galdi. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Arch. Rat. Mech. Anal.*, 148:53–88, 1999.

167. G. P. Galdi. *On the motion of a rigid body in a viscous fluid: A mathematical analysis with applications*, volume I of *Handbook of Mathematical Fluid Dynamics*. Elsevier Sci., Amsterdam, 2002.
168. G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.
169. F. Giannetti and A. Fiorenza. Removability of zero modular capacity. *Rev. Mat. Complut.*, 2020.
170. F. Giannetti, T. Iwaniec, J. Onninen, and A. Verde. Estimates of Jacobians by subdeterminants. *J. Geom. Anal.*, 12(2):223–254, 2002.
171. D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
172. E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
173. J.-P. Gossez. Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. *Trans. Amer. Math. Soc.*, 190:163–205, 1974.
174. J.-P. Gossez. Orlicz–Sobolev spaces and nonlinear elliptic boundary value problems. In *Nonlinear analysis, function spaces and applications (Proc. Spring School, Horni Bradlo, 1978)*, pages 59–94. Teubner, Leipzig, 1979.
175. J.-P. Gossez. Some approximation properties in Orlicz–Sobolev spaces. *Studia Math.*, 74(1):17–24, 1982.
176. L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.
177. F. Gulilén-González. Density-dependent incompressible fluids with non-Newtonian viscosity. *Czechoslovak Math. J.*, 54(3):637–656, 2004.
178. P. Gwiazda, P. Minakowski, and A. Wróblewska-Kamińska. Elliptic problems in generalized Orlicz–Musielak spaces. *Cent. Eur. J. Math.*, 10(6):2019–2032, 2012.
179. P. Gwiazda, I. Skrzypczak, and A. Zatorska-Goldstein. Existence of renormalized solutions to elliptic equation in Musielak–Orlicz space. *J. Differential Equations*, 264(1):341–377, 2018.
180. P. Gwiazda and A. Świerczewska-Gwiazda. On non-Newtonian fluids with a property of rapid thickening under different stimulus. *Math. Models Methods Appl. Sci.*, 18(7):1073–1092, 2008.
181. P. Gwiazda and A. Świerczewska-Gwiazda. On steady non-Newtonian fluids with growth conditions in generalized Orlicz spaces. *Topol. Methods Nonlinear Anal.*, 32(1):103–113, 2008.
182. P. Gwiazda and A. Świerczewska-Gwiazda. Parabolic equations in anisotropic Orlicz spaces with general $\mathbb{N}$-functions. In *Parabolic problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 301–311. Birkhäuser/Springer Basel AG, Basel, 2011.
183. P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.*, 33(2):125–137, 2010.
184. P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Generalized Stokes system in Orlicz spaces. *Discrete Contin. Dyn. Syst.*, 32(6):2125–2146, 2012.
185. P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Generalized Stokes system in Orlicz spaces. *Discrete Contin. Dyn. Syst.*, 32(6):2125–2146, 2012.
186. P. Gwiazda, P. Wittbold, A. Wróblewska, and A. Zimmermann. Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces. *J. Differential Equations*, 253(2):635–666, 2012.
187. P. Gwiazda, P. Wittbold, A. Wróblewska-Kamińska, and A. Zimmermann. Corrigendum to 'Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces' [J. Differential Equations 253 (2) (2012) 635–666]. *J. Differential Equations*, 253(9):2734–2738, 2012.
188. P. Gwiazda, P. Wittbold, A. Wróblewska-Kamińska, and A. Zimmermann. Renormalized solutions to nonlinear parabolic problems in generalized Musielak–Orlicz spaces. *Nonlinear Anal.*, 129:1–36, 2015.

189. S. Hadj Nassar, H. Moussa, and M. Rhoudaf. Renormalized solution for a nonlinear parabolic problems with noncoercivity in divergence form in Orlicz spaces. *Appl. Math. Comput.*, 249:253–264, 2014.
190. P. Harjulehto and P. Hästö. The Riesz potential in generalized Orlicz spaces. *Forum Math.*, 29(1):229–244, 2017.
191. P. Harjulehto and P. Hästö. *Orlicz spaces and Generalized Orlicz Spaces*, volume 2236 of *Lecture Notes in Mathematics*. Springer, 2019.
192. P. Harjulehto, P. Hästö, and A. Karppinen. Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions. *Nonlinear Anal.*, 177(part B):543–552, 2018.
193. P. Harjulehto, P. Hästö, and R. Klén. Generalized Orlicz spaces and related PDE. *Nonlinear Anal.*, 143:155–173, 2016.
194. P. Harjulehto, P. Hästö, Ú. V. Lê, and M. Nuortio. Overview of differential equations with non-standard growth. *Nonlinear Anal.*, 72(12):4551–4574, 2010.
195. P. Harjulehto, P. Hästö, and M. Lee. Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22:549–582, 2021.
196. P. Harjulehto, P. Hästö, and O. Toivanen. Hölder regularity of quasiminimizers under generalized growth conditions. *Calc. Var. Partial Differential Equations*, 56(2):56:22, 2017.
197. P. Harjulehto and A. Karppinen. Sharp growth conditions for boundedness of maximal function in generalized Orlicz spaces. *arXiv:2103.13774*, 2021.
198. P. Hästö. The maximal operator on generalized Orlicz spaces. *J. Funct. Anal.*, 269(12):4038–4048, 2015.
199. P. Hästö and J. Ok. Higher integrability for parabolic systems with Orlicz growth. *J. Eur. Math. Soc. arXiv:1906.04639*, 2020.
200. K. R. Hebert and J. E. Houser. A model for coupled electrical migration and stress-driven transport in anodic oxide films. *Journal of the Electrochemical Society*, 156(8):C275–C281, 2009.
201. J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.
202. K.-H. Hoffmann and V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two dimensional case. *Adv. Math. Sci. Appl.*, 9:633–64, 1999.
203. J. M. Houghton, B. A. Schiffman, D. P. Kalman, E. D. Wetzel, and N. J. Wagner. Hypodermic Needle Puncture of Shear Thickening Fluid (STF)–Treated Fabrics. *Proceedings of SAMPE. Baltimore MD*, 2007.
204. J. E. Houser and K. R. Hebert. Modeling the potential distribution in porous anodic alumina films during steady-state growth. *Journal of the Electrochemical Society*, 153(12):B566–B573, 2006.
205. J. Hron, C. Le Roux, J. Málek, and K. Rajagopal. Flows of incompressible fluids subject to Navier's slip on the boundary. *Comput. Math. Appl.*, 56:2128–2143, 2008.
206. H. Hudzik. A generalization of Sobolev spaces. I. *Funct. Approximatio Comment. Math.*, 2:67–73, 1976.
207. H. Hudzik. A generalization of Sobolev spaces. II. *Funct. Approximatio Comment. Math.*, 3:77–85, 1976.
208. H. Hudzik and A. Kamińska. On uniformly convexifiable and B-convex Musielak–Orlicz spaces. *Comment. Math. Prace Mat.*, 25(1):59–75, 1985.
209. K. Hutter. Theoretical glaciology: material science of ice and the mechanics of glaciers and ice sheets. *Dordrecht, etc., D. Reidel Publishing Co.; Tokyo, Terra Scientific Publishing Co.*, 1983.
210. A. V. Ivanov. Second-order quasilinear degenerate and nonuniformly elliptic and parabolic equations. *Trudy Mat. Inst. Steklov.*, 160:285, 1982.
211. V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].
212. N. V. Judakov. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. *Dinamika Splošn. Sredy Vyp.*, 18:249–253, 1974.

213. A. Kałamajska, K. Pietruska-Pałuba, and I. Skrzypczak. Nonexistence results for differential inequalities involving A-Laplacian. *Adv. Differential Equations*, 17(3-4):307–336, 2012.
214. A. Kamińska. On some compactness criterion for Orlicz subspace $E_\Phi(\Omega)$. *Comment. Math. Prace Mat.*, 22(2):245–255, 1980/81.
215. A. Kamińska. Some convexity properties of Musielak–Orlicz spaces of Bochner type. In *Proceedings of the 13th winter school on abstract analysis (Srní, 1985)*, volume 10, pages 63–73 (1986), 1985.
216. T. Kilpeläinen, T. Kuusi, and A. Tuhola-Kujanpää. Superharmonic functions are locally renormalized solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(6):775–795, 2011.
217. D. Kinderlehrer and P. Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rational Mech. Anal.*, 115(4):329–365, 1991.
218. F. Z. Klawe. Thermo-visco-elasticity for models with growth conditions in Orlicz spaces. *Topol. Methods Nonlinear Anal.*, 47(2):457–497, 2016.
219. V. S. Klimov. Imbedding theorems and geometric inequalities. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(3):645–671, 710, 1976.
220. M. A. Krasnoselskii and J. B. Rutickii. *Convex functions and Orlicz spaces*. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.
221. A. Kufner, O. John, and S. Fučik. *Function spaces*. Noordhoff International Publishing. Prague: Publishing House of the Czechoslovak Academy of Sciences, 1977.
222. A. Kufner and B. Opic. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.*, 25(3):537–554, 1984.
223. O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
224. O. A. Ladyzhenskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Stek. Inst. Math.*, 102:95–118, 1967.
225. O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow 2nd ed.* Gordon and Breach, New York, 1969.
226. O. A. Ladyzhenskaya and N. N. Ural′ tseva. Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 23:677–703, 1970.
227. O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.
228. R. Landes. On the existence of weak solutions for quasilinear parabolic initial-boundary value problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 89(3-4):217–237, 1981.
229. M. Lavrentiev. Sur quelques problèmes du calcul des variations. *Ann. Mat. Pura Appl.*, 41:107–124, 1927.
230. V. K. Le. On second order elliptic equations and variational inequalities with anisotropic principal operators. *Topol. Methods Nonlinear Anal.*, 44(1):41–72, 2014.
231. Y. S. Lee, E. D. Wetzel, and N. J. Wagner. The ballistic impact characteristics of Kevlar woven fabrics impregnated with a colloidal shear thickening fluid. *Journal of Materials Science*, 38(13):2825–2833, 2004.
232. Jean Leray and Jacques-Louis Lions. Quelques résulatats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder. *Bull. Soc. Math. France*, 93:97–107, 1965.
233. Y. Li, F. Yao, and S. Zhou. Entropy and renormalized solutions to the general nonlinear elliptic equations in Musielak–Orlicz spaces. *Nonlinear Anal. Real World Appl.*, 61:103330, 2021.
234. Z. Li and W. Gao. Existence of renormalized solutions to a nonlinear parabolic equation in L^1 setting with nonstandard growth condition and gradient term. *Math. Methods Appl. Sci.*, 38(14):3043–3062, 2015.
235. G. M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations. *Comm. Partial Differential Equations*, 16(2-3):311–361, 1991.
236. G. M. Lieberman. Gradient estimates for anisotropic elliptic equations. *Adv. Differential Equations*, 10(7):767–812, 2005.

237. J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.
238. P. L. Lions. *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*. Oxford Lecture Series in Mathematics and its Applications, vol. 3. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1996.
239. D. Liu and P. Zhao. Solutions for a quasilinear elliptic equation in Musielak-Sobolev spaces. *Nonlinear Anal. Real World Appl.*, 26:315–329, 2015.
240. J. Liu. Positive solutions of the $p(x)$-Laplace equation with singular nonlinearity. *Nonlinear Anal.*, 72(12):4428–4437, 2010.
241. W. A. J. Luxemburg. *Banach function spaces*. Thesis, Technische Hogeschool te Delft, 1955.
242. M. Mabdaoui, H. Moussa, and M. Rhoudaf. Entropy solutions for a nonlinear parabolic problems with lower order term in Orlicz spaces. *Anal. Math. Phys.*, 7(1):47–76, 2017.
243. F.-Y. Maeda, Y. Sawano, and T. Shimomura. Some norm inequalities in Musielak–Orlicz spaces. *Ann. Acad. Sci. Fenn. Math.*, 41(2):721–744, 2016.
244. J. Málek, J. Nečas, and M. Růžička. On the non-Newtonian incompressible fluids. *Math. Models Methods Appl. Sci.*, 3:35–63, 1993.
245. J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and Measure-valued Solutions to Evolutionary PDEs*. Chapman and Hall, 1996.
246. P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions. *Arch. Rational Mech. Anal.*, 105(3):267–284, 1989.
247. P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations*, 90(1):1–30, 1991.
248. Paolo Marcellini. Growth conditions and regularity for weak solutions to nonlinear elliptic pdes. *J. Math. Anal. Appl.*, 501(1):124408, 32, 2021.
249. B. Matejczyk and A. Wróblewska-Kamińska. Unsteady flows of heat-conducting non-Newtonian fluids in generalised Orlicz spaces. *Nonlinearity*, 31(3):701–727, 2018.
250. V. G. Maz′ya. Weak solutions of the Dirichlet and Neumann problems. *Trudy Moskov. Mat. Obšč.*, 20:137–172, 1969.
251. O. Méndez and J. Lang. *Analysis on function spaces of Musielak–Orlicz type*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2019.
252. A. Menovschikov, A. Molchanova, and L. Scarpa. An extended variational theory for nonlinear evolution equations via modular spaces. *SIAM J. Math. Anal. (SIMA)*, 2021.
253. N. G. Meyers and J. Serrin. $H = W$. *Proc. Nat. Acad. Sci. U.S.A.*, 51:1055–1056, 1964.
254. M. Mihailescu and D. Stancu-Dumitru. On an eigenvalue problem involving the $p(x)$-Laplace operator plus a non-local term. *Differ. Equ. Appl.*, 1(3):367–378, 2009.
255. P. Mikkonen. On the Wolff potential and quasilinear elliptic equations involving measures. *Ann. Acad. Sci. Fenn. Math. Diss.*, 104:71, 1996.
256. G. Mingione and V. D. Radulescu. Recent developments in problems with nonstandard growth and nonuniform ellipticity. *J. Math. Anal. Appl.*, 2021.
257. Y. Mizuta, E. Nakai, T. Ohno, and T. Shimomura. Campanato–Morrey spaces for the double phase functionals with variable exponents. *Nonlinear Anal.*, 197:111827, 19, 2020.
258. B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.
259. S. Müller. *Variational models for microstructure and phase transitions*. M volume 1713 of Lecture Notes in Math.: Calculus of Variations and Geometric Evolution Problems. Springer-Verlag, Berlin Heidelberg, 1999.
260. F. Murat. Soluciones renormalizadas de edp elipticas no lineales. *Publ. Laboratoire d'Analyse Numérique, Univ. Paris 6, R 93023*, 1993.
261. F. Murat and A. Porretta. Stability properties, existence, and nonexistence of renormalized solutions for elliptic equations with measure data. *Comm. Partial Differential Equations*, 27(11-12):2267–2310, 2002.
262. J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.
263. V. Mustonen and M. Tienari. On monotone-like mappings in Orlicz–Sobolev spaces. *Math. Bohem.*, 124(2-3):255–271, 1999.
264. Vesa Mustonen and Matti Tienari. A remark on the Leray-Lions condition. *Nonlinear Anal.*, 49(7, Ser. A: Theory Methods):1015–1022, 2002.

265. H. Nakano. *Modulared Semi-Ordered Linear Spaces*. Maruzen Co., Ltd., Tokyo, 1950.
266. Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.
267. Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian and heat-conducting fluid. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 55(2):325–352, 2009.
268. H. T. Nguyen and D. Pączka. Weak and Young measure solutions for hyperbolic initial boundary value problems of elastodynamics in the Orlicz–Sobolev space setting. *SIAM J. Math. Anal.*, 48(2):1297–1331, 2016.
269. O. A. Oleĭnik and V. V. Zhikov. On the homogenization of elliptic operators with almost-periodic coefficients. In *Proceedings of the international conference on partial differential equations dedicated to Luigi Amerio on his 70th birthday (Milan/Como, 1982)*, volume 52, pages 149–166 (1985), 1982.
270. S. Pastukhova and V. Chiadó Piat. Homogenization of multivalued monotone operators with variable growth exponent. *Netw. Heter. Media*, 15(2):281–305, 2020.
271. M. Patel and M. G. Timol. Numerical treatment of Powell–Eyring fluid flow using method of satisfaction of asymptotic boundary conditions (msabc). *Appl. Numer. Math.*, 59(10):2584–2592, 2009.
272. P. Pedregal. *Parametrized measures and variational principles*. Birkhäuser, 2012.
273. F. Petitta. Renormalized solutions of nonlinear parabolic equations with general measure data. *Ann. Mat. Pura Appl. (4)*, 187(4):563–604, 2008.
274. F. Petitta, A. C. Ponce, and A. Porretta. Diffuse measures and nonlinear parabolic equations. *J. Evol. Equ.*, 11(4):861–905, 2011.
275. R. E. Powell and H. Eyring. Mechanisms for the relation theory of viscosity. *Nature*, 154:427–428, 1944.
276. P. Pucci and Q. Zhang. Existence of entire solutions for a class of variable exponent elliptic equations. *J. Differential Equations*, 257(5):1529–1566, 2014.
277. V. D. Radulescu. Nonlinear elliptic equations with variable exponent: old and new. *Nonlinear Anal.*, 121:336–369, 2015.
278. M. A. Ragusa and A. Tachikawa. Regularity for minimizers for functionals of double phase with variable exponents. *Adv. Nonlinear Anal.*, 9(1):710–728, 2020.
279. K. Rajagopal and M. Růžička. On the modeling of electrorheological materials. *Mech. Res. Commun.*, 23:401–407, 1996.
280. K.R. Rajagopal and M. Růžička. Mathematical modeling of electrorheological materials. *Continuum Mechanics and Thermodynamics*, 13(1):59–78, 2001.
281. M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.
282. H. Redwane. Existence results for a class of nonlinear parabolic equations in Orlicz spaces. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 2, 19, 2010.
283. F. Rindler. *Calculus of variations*. Universitext. Springer, Cham, 2018.
284. A. M. Robertson. Review of relevant continuum mechanics. *In* Hemodynamical flows *of* Oberwolfach Semin. *Birkhäuser, Basel.*, 37:1–62, 2008.
285. A. M. Robertson, A. Sequeira, and M. V. Kameneva. Hemorheology. *In* Hemodynamical flows *of* Oberwolfach Semin. *Birkhäuser, Basel,*, 37:63–120, 2008.
286. T. Roubiček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2005.
287. M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
288. C. Sample and A. A. Golovin. Formation of porous metal oxides in anodization process. *Phys. Rev. E*, 74:041606, 2006.
289. J. A. San Martin, V. Starovoitov, and M. Tucsnak. Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.*, 161:93–112, 2002.
290. E. Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin-New York, 1980.
291. D. Serre. Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Japan J. Appl. Math.*, 4(1):99–110, 1987.

292. J. Serrin. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 18:385–387, 1964.
293. J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.
294. I. B. Simonenko. Interpolation and extrapolation of linear operators in Orlicz spaces. *Mat. Sb. (N.S.)*, 63 (105):536–553, 1964.
295. V. Sirohi, M. G. Timol, and N. L. Kalthia. Powell-Eyring model flow near an accelerated plate. *Fluid Dynamics Research*, 2(3):193, 1987.
296. M. S. Skaff. Vector valued Orlicz spaces generalized N-functions. I. *Pacific J. Math.*, 28:193–206, 1969.
297. M. S. Skaff. Vector valued Orlicz spaces generalized N-functions. II. *Pacific J. Math.*, 28:413–430, 1969.
298. H. Sohr. *The Navier-Stokes equations*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2001. An elementary functional analytic approach, [2013 reprint of the 2001 original].
299. V. N. Starovoitov. Behavior of a rigid body in an incompressible viscous fluid near boundary. *In International Series of Numerical Mathematics*, 147:313–327, 2003.
300. M. J. Strauss. Variations of Korn's and Sobolev's inequalities. *Partial Diff. Equ., Berkeley 1971, Proc. Sympos. Pure Math.*, 23:207–214, 1973.
301. B. Stroffolini. Some remarks on the regularity of anisotropic variational problems. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 17:229–239, 1993.
302. D. W. Stroock. Weyl's lemma, one of many. *London Mathematical Society Lecture Note Series. Cambridge University Press.*, 354:164–173, 2008.
303. A. Świerczewska-Gwiazda. Anisotropic parabolic problems with slowly or rapidly growing terms. *Colloq. Math.*, 134(1):113–130, 2014.
304. A. Świerczewska-Gwiazda. Nonlinear parabolic problems in Musielak–Orlicz spaces. *Nonlinear Anal.*, 98:48–65, 2014.
305. A. Tachikawa. On the singular set of minimizers of $p(x)$-energies. *Calc. Var. Partial Differential Equations*, 50(1-2):145–169, 2014.
306. G. Talenti. Elliptic equations and rearrangements. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3(4):697–718, 1976.
307. G. Talenti. Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura Appl. (4)*, 120:160–184, 1979.
308. G. Talenti. An embedding theorem. In *Partial differential equations and the calculus of variations, Vol. II*, volume 2 of *Progr. Nonlinear Differential Equations Appl.*, pages 919–924. Birkhäuser Boston, Boston, MA, 1989.
309. L. Tartar. *Compensated compactness and applications to partial differential*, volume IV of *Nonlinear analysis and mechanics: Heriot-Watt Symposium*. Pitman, Boston, Mass., 1979.
310. L. Tartar. Cours Peccot au Collège de France. partially written by F. Murat in Séminaire d'Analyse Fonctionelle et Numérique de l'Université d'Alger, unpublished., 1979.
311. R. Temam. *Navier Stokes Equations. Theory and Numerical Analysis*. Noth-Holland Publishing Company, 1977.
312. R. Temam and A. Miranville. *Mathematical Modeling in Continuum Mechanics*. Cambridge University Press, 2005.
313. R. Temam and G. Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75:7–21, 1980/81.
314. Mario Troisi. Teoremi di inclusione per spazi di Sobolev non isotropi. *Ricerche Mat.*, 18:3–24, 1969.
315. N. S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.
316. N. S. Trudinger. An imbedding theorem for $H_0(G,\Omega)$ spaces. *Studia Math.*, 50:17–30, 1974.
317. B. O. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
318. J. Vétois. Existence and regularity for critical anisotropic equations with critical directions. *Adv. Differential Equations*, 16(1-2):61–83, 2011.
319. J. Vétois. Strong maximum principles for anisotropic elliptic and parabolic equations. *Adv. Nonlinear Stud.*, 12(1):101–114, 2012.

320. A. Visintin. Towards a two-scale calculus. *ESAIM Control Optim. Calc. Var.*, 12(3):371–397, 2006.
321. R. Vodák. The problem $\nabla \cdot \mathbf{v} = f$ and singular integrals on Orlicz spaces. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, 41:161–173, 2002.
322. R. Vodák. Behaviour of weak solutions of compressible Navier-Stokes equations for isothermal fluids with a nonlinear stress tensor. *Journal of Evolution Equations*, 4(2):213–247, 2004.
323. H. F. Weinberger. Symmetrization in uniformly elliptic problems. In *Studies in mathematical analysis and related topics*, pages 424–428. Stanford Univ. Press, Stanford, Calif., 1962.
324. P. Wittbold and A. Zimmermann. Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data. *Nonlinear Anal.*, 72(6):2990–3008, 2010.
325. J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mechanics*, 9(1):104–138, 2007.
326. A. Wróblewska. Steady flow of non-Newtonian fluids – monotonicity methods in generalized Orlicz spaces. *Nonlinear Anal.*, 72(11):4136–4147, 2010.
327. A. Wróblewska-Kamińska. Local pressure methods in Orlicz spaces for the motions of rigid bodies in an non-Newtonian fluid with general growth conditions. *Discrete and Continuous Dynamical Systems S*, 6(5):1417–1425, 2013.
328. A. Wróblewska-Kamińska. Unsteady flows of non-Newtonian fluids in generalized Orlicz spaces. *Discrete Contin. Dyn. Syst.*, 33(6):2565–2592, 2013.
329. A. Wróblewska-Kamińska. Existence result for the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces. *Nonlinearity*, 27:685–716, 2014.
330. H. Yoon and A. Ghajar. A note on the Powell-Eyring fluid model. *International Communications in Heat and Mass Transfer*, 14(4):381–390, 1987.
331. L. C. Young. Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *C.R. Soc. Sci. Lettres de Varsovie*, 30(III):212–234, 1937.
332. L. C. Young. Generalized surfaces in the calculus of variations. *Ann. of Math.*, 43:84–103, 1942.
333. L. C. Young. Generalized surfaces in the calculus of variations. II. *Ann. of Math.*, 43:530–544, 1942.
334. C. Zhang and S. Zhou. Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data. *J. Differential Equations*, 248(6):1376–1400, 2010.
335. C. Zhang and S. Zhou. The well-posedness of renormalized solutions for a non-uniformly parabolic equation. *Proc. Amer. Math. Soc.*, 145(6):2577–2589, 2017.
336. V. V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(4):675–710, 877, 1986.
337. V. V. Zhikov. On Lavrentiev's phenomenon. *Russian J. Math. Phys.*, 3(2):249–269, 1995.
338. V. V. Zhikov. Meyer-type estimates for solving the nonlinear Stokes system. *Differ. Uravn.*, 33(1):107–114, 143, 1997.
339. V. V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5(1):105–116 (1998), 1997.
340. V. V. Zhikov. On variational problems and nonlinear elliptic equations with nonstandard growth conditions. *J. Math. Sci. (N.Y.)*, 173(5):463–570, 2011. Problems in mathematical analysis. No. 54.
341. V. V. Zhikov and S. E. Pastukhova. Homogenization of monotone operators under conditions of coercitivity and growth of variable order. *Mat. Zametki*, 90(1):53–69, 2011.

List of Symbols

General notation

$\lvert\ \rvert$	Euclidean length of a vector $\xi \in \mathbb{R}^N$; for $\xi = (\xi_1, \dots, \xi_N)$ we have $\lvert\xi\rvert = \left(\sum_{i=1}^N \xi_i^2\right)^{\frac{1}{2}}$
$\cdot$	scalar product of two vectors, i.e. for $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n$ and $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$ we have $\xi \cdot \eta = \sum_{i=1}^N \xi_i \eta_i$
$:$	the Frobenius product of two second-order tensors; for $\boldsymbol{\xi} = [\xi_{i,j}]_{i=1,\dots,N,\, j=1,\dots,N} \in \mathbb{R}^{N\times N}$, $\boldsymbol{\eta} = [\eta_{i,j}]_{i=1,\dots,N,\, j=1,\dots,N} \in \mathbb{R}^{N\times N}$ we have $$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^N \xi_{i,j}\eta_{i,j}$$
$\otimes$	the tensor product of two vectors; for $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$, we have $\xi \otimes \eta := [\xi_i \eta_j]_{i=1,\dots,N,\, j=1,\dots,N}$, that is $$\xi \otimes \eta := \begin{pmatrix} \xi_1\eta_1 & \xi_1\eta_2 & \cdots & \xi_1\eta_N \\ \xi_2\eta_1 & \xi_2\eta_2 & \cdots & \xi_2\eta_N \\ \vdots & \vdots & \ddots & \vdots \\ \xi_N\eta_1 & \xi_N\eta_2 & \cdots & \xi_N\eta_N \end{pmatrix} \in \mathbb{R}^{N\times N}$$
$\langle\cdot,\cdot\rangle$	duality pairing, i.e. for $a \in X^*$ and $b \in X$, $\langle a, b\rangle$ is a duality pairing
A^T	the transpose of a square matrix $\mathsf{A} = \{a_{i,j}\}_{i,j}^N$, $\mathsf{A}^T = \{a_{j,i}\}$
p'	$p' = \frac{p}{p-1}$ – the Hölder conjugate to p; a number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$
T_k	symmetric truncation at level k; $T_k(f)(x) := \min\{\max\{-k, f(x)\}, k\}$; (3.55)
T_m	truncation operator applied to a square matrix; $\mathsf{T}_m(\mathsf{K}) = \mathsf{K}$ if $\lvert\mathsf{K}\rvert \le m$ and $\mathsf{T}_m(\mathsf{K}) = m(\mathsf{K}/\lvert\mathsf{K}\rvert)$ for $\lvert\mathsf{K}\rvert \le m$ where $\mathsf{K} \in \mathbb{R}^{N\times N}$
f_+	$(f(s))_+ := \max\{f(s), 0\}$ – the positive part of function f
f_-	$(f(s))_- := \min\{f(s), 0\}$ – the negative part of function f
sgn_0^+	the positive part of the signum function
$\mathbb{1}_A$	the indicator function of the set A

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$\rightharpoonup$	weak convergence; Section 3.4
$\overset{*}{\rightharpoonup}$	weak-$*$ convergence; Section 3.4
$\xrightarrow{M}$	modular convergence in Orlicz space (L_M or L_m); Definition 3.4.3
$\xrightarrow{mod}$	modular convergence in Orlicz–Sobolev space (V_M or W^1L_m); Definition 3.4.3, Theorems 3.7.7, 4.2.6, and 5.3.12
$\xrightarrow{b}$	biting convergence; Definition 8.36
∇u	the gradient of u with respect to the spatial variable
$\mathbf{D}\mathbf{u}$	the symmetric gradient with respect to the spatial variable, i.e. $\mathbf{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$
$\boldsymbol{\partial}, \boldsymbol{\partial}_\xi$	subdifferential or partial subdifferential; when with a function $M : Z \times \mathbb{R}^d \to [0,\infty)$ we associate a one-parameter family of functions $M_z : \mathbb{R}^d \to [0,\infty)$ by $M(z,\xi) = M_z(\xi)$, then $\boldsymbol{\partial} M_z(\xi) = \boldsymbol{\partial}_\xi M(z,\xi)$ is a subdifferential with respect to the variable ξ
epi	epi$f$ – epigraph of a function f; Definition 2.1.5
Im	Imf – image of a function f
lin	$\mathrm{lin}\{x_j\}_{j\in J}$ is the space of linear combinations of $\{x_j\}_{j\in J}$
arg min	$\arg\min_{x\in A} f(x) = \{x \in A : \ f(x) = \min_{y\in A} f(y)\}$
arg max	$\arg\max_{x\in A} f(x) = \{x \in A : \ f(x) = \max_{y\in A} f(y)\}$
$\mathcal{F}$	Fourier transform
$\mathcal{R}_{i,j}[g]$	double Riesz transform of an integrable function g on $\mathbb{R}^3$, Section 8.2

Sets

$\mathbb{R}^d$	d-dimensional real Euclidean space
$Z \subset \mathbb{R}^d$	bounded and connected subset of $\mathbb{R}^d$
$\mathbb{Q}^d \subset \mathbb{R}^d$	subset of $\mathbb{R}^d$ of vectors having rational coordinates
$\mathbb{R}^{N\times N}_{\mathrm{sym}}$	the space of $N \times N$ symmetric matrices with real coefficients
$\Omega \subset \mathbb{R}^N$	an open bounded set
Ω_t	$(0,t) \times \Omega$ for $t \in (0,T]$
Y	the unit cube $(0,1)^N$

***N*-functions**

M	an (anisotropic and inhomogeneous) N-function $M : Z \times \mathbb{R}^d \to [0,\infty)$, Definition 2.2.2
M^*	Young's conjugate function to $M : Z \times \mathbb{R}^d \to [0,\infty)$, Definition 2.1.28
$m, \bar{m}, \ldots$	isotropic and homogeneous N-functions $m, \bar{m}, \cdots : [0,\infty) \to [0,\infty)$
m_1 and m_2	minorant and majorant of M from definition of N-function
Δ_2	$M \in \Delta_2$ if M satisfies the Δ_2-condition, Definition 2.2.5
ρ_M	modular $\rho_M(\xi) := \int_Z M(z,\xi(z))\ \mathrm{d}z$, $Z \subset \mathbb{R}^d$

Spaces

$\mathcal{M}(\Omega)$	the space of signed Radon measures with finite mass in Ω
$\mathcal{H}^{N-1}(\partial\Omega)$	the space of $(N-1)$-Hausdorff measures on $\partial\Omega$
$L^p(\Omega)$	the classical Lebesgue spaces, $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^N$

$W^{1,p}(\Omega)$	the classical Sobolev spaces, $1 \le p \le \infty$, $\Omega \subset \mathbb{R}^N$
$C(\Omega)$	the space of functions continuous on Ω
$C^k(\Omega)$	the space of functions continuously differentiable on Ω up to the order $k \in \mathbb{N}$
$C_c^\infty(\Omega)$	the space of smooth functions $u : \mathbb{R}^N \to \mathbb{R}$ with compact support in $\Omega \subset \mathbb{R}^N$
$C_c^\infty(\Omega;\mathbb{R}^N)$	the space of smooth functions $\mathbf{v} : \mathbb{R}^N \to \mathbb{R}^N$ with compact support in $\Omega \subset \mathbb{R}^N$
$C^{0,\alpha}(\Omega)$	the space of functions $f : \Omega \to \mathbb{R}$ is called α-Hölder continuous, $\alpha \in (0,1]$, i.e. such that there exists $C_a > 0$ such that for every $x, y \in \Omega$ we have $\lvert f(x) - f(y)\rvert \le C_a \lvert x-y\rvert^\alpha$
$C^{k,\alpha}(\Omega)$	the space of $C^k(\Omega)$ functions such that k-derivative is α-Hölder continuous, $k \in \mathbb{N}$, $\alpha \in (0,1]$
$W^1L_m(\Omega)$	with m being a homogeneous and isotropic N-function is the classical Orlicz–Sobolev space; Appendix 9
$W_0^1L_m(\Omega)$	with m being a homogeneous and isotropic N-function is the closure of $C_c^\infty(\Omega)$ in $W^1L_m(\Omega)$ with respect to the topology $\sigma(L_m, E_{m^*})$; Section 3.6, Appendix 9
$W^{1,m}(\Omega)$	with doubling m being a homogeneous and isotropic N-function is the classical Orlicz–Sobolev space; i.e. if $m, m^* \in \Delta_2$, then $W^1L_m(\Omega) = W^1E_m(\Omega) = W^1\mathcal{L}_m(\Omega) =: W^{1,m}(\Omega)$
$\mathcal{L}_M(Z;\mathbb{R}^d)$	the generalized Musielak–Orlicz class; the set of all measurable functions $\xi : Z \to \mathbb{R}^d$ such that $\rho_M(\xi) < \infty$; Section 3.1
$L_M(Z;\mathbb{R}^d)$	the generalized Musielak–Orlicz space; it is the smallest linear space containing $\mathcal{L}_M(Z;\mathbb{R}^d)$; Section 3.1
$E_M(Z;\mathbb{R}^d)$	the largest linear space contained in $\mathcal{L}_M(Z;\mathbb{R}^d)$; Section 3.1
$L^p\log^\beta L(\Omega)$	special case of $L_M(\Omega,\mathbb{R}^d)$ with $M(x,\xi) = \lvert\xi\rvert^p\log^\beta(e+\lvert x\rvert)$
$E^p\log^\beta E(\Omega)$	special case of $E_M(\Omega,\mathbb{R}^d)$ with $M(x,\xi) = \lvert\xi\rvert^p\log^\beta(e+\lvert x\rvert)$
$L_{\exp^\gamma}(\Omega)$	special case of $L_M(\Omega,\mathbb{R}^d)$ with $M(x,\xi) = \exp(\lvert\xi\rvert^\gamma)$ for $\lvert\xi\rvert > 1$
$V_0^M(\Omega)$	the space of functions $u \in W_0^{1,1}(\Omega)$ such that $\nabla u \in L_M(\Omega;\mathbb{R}^N)$; Sections 3.6, 3.7, and 4.1
$\mathcal{T}V_0^M(\Omega)$	the space of measurable functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ it holds that $T_k(u) \in W_0^{1,1}(\Omega)$ and $\nabla T_k(u) \in L_M(\Omega;\mathbb{R}^N)$; Sections 3.6, 3.7, and 4.1
$V_T^M(\Omega)$	the space of functions $u \in L^1(0,T;W_0^{1,1}(\Omega))$ such that $\nabla u \in L_M(\Omega_T;\mathbb{R}^N)$; Sections 3.6, 4.2.2, and 4.2
$V_T^{M,\infty}(\Omega)$	the space of functions $u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ such that $\nabla u \in L_M(\Omega_T;\mathbb{R}^N)$; it coincides with $V_T^M(\Omega) \cap L^\infty(0,T;L^2(\Omega))$; Sections 3.6, 4.2.2, and 4.2
$BD_M(\Omega;\mathbb{R}^N)$	the space of functions $\mathbf{u} \in L^1(\Omega;\mathbb{R}^N)$ such that $\mathbf{Du} \in L_M(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$
$BD_{M,0}(\Omega;\mathbb{R}^N)$	the space of functions from $BD_M(\Omega;\mathbb{R}^N)$ with zero trace, see Section 7.3

$\mathcal{BD}_{M,0}(\Omega;\mathbb{R}^N)$	the space of functions from $BD_M(\Omega;\mathbb{R}^N)$ with zero trace and symmetric gradient in $\mathcal{L}(\Omega;\mathbb{R}^{N\times N}_{\text{sym}})$, see Section 7.3
$Y_0^M(\Omega_T;\mathbb{R}^N)$	the closure of C_c^∞ functions in modular topology for symmetric gradients, see Section 7.3
$Z_0^M(\Omega_T;\mathbb{R}^N)$	the closure of C_c^∞ functions weak-$*$ topology for symmetric gradients, see Section 7.3
$\mathcal{V}(\Omega)$	the set of $u \in C_c^\infty(\Omega)$ such that $\operatorname{div} u = 0$
$L^2_{\text{div}}(\Omega)$	the closure of $\mathcal{V}$ with respect to the L^2-norm
$W^{1,p}_{0,\text{div}}(\Omega)$	the closure of $\mathcal{V}$ with respect to the L^p-norm of the gradient
$N^{\alpha,p}(0,T;X)$	the Nikolskii space corresponding to the Banach space X and exponents $\alpha \in (0,1)$ and $p \in [1,\infty]$, see Section 7.2.1
$V_s(\Omega)$	the closure of $\mathcal{V}(\Omega)$ with respect to the $W^{s,2(\Omega)}$-norm
$W^{1,1}_{per}(Y;\mathbb{R}^N)$	the closure of the set of such $\mathbf{v} \in C^\infty_{per}(Y;\mathbb{R}^N)$ that $\int_Y \mathbf{v}\,\mathrm{d}x = 0$ in $W^{1,1}(\mathbb{R}^N;\mathbb{R}^N)$
$C^\infty_{per}(Y;\mathbb{R}^N)$	the set of $\mathbf{v} \in C^\infty(\mathbb{R}^N;\mathbb{R}^N)$ such that $\mathbf{v}$ is Y-periodic
$C^\infty_{per,\text{div}}(Y;\mathbb{R}^N)$	the set of $\mathbf{v} \in C^\infty_{per}(Y;\mathbb{R}^N)$ such that $\operatorname{div}\mathbf{v} = 0$ in Y
$C^\infty_{c,\text{div}}(\Omega;\mathbb{R}^N)$	the set of $\mathbf{v} \in C_c^\infty(\Omega;\mathbb{R}^N)$ such that $\operatorname{div}\mathbf{v} = 0$ in Ω
$\mathcal{D}'(\Omega;\mathbb{R})$	the space of Schwartz distributions on Ω
$BD(\Omega;\mathbb{R}^N)$	the space of bounded deformations, the space of $\mathbf{u} \in L^1(\Omega;\mathbb{R}^N)$ such that $[\mathbf{Du}]_{i,j} \in \mathcal{M}(\Omega)$ for $i,j = 1,\dots,N$

Glossary

doubling	we say that an N-function M is doubling, if $M, M^* \in \Delta_2$
superlinear	we say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is superlinear at infinity if $\lim_{\lvert\xi\rvert\to\infty} f(\xi)/\lvert\xi\rvert = \infty$ and superlinear in the origin if $\lim_{\lvert\xi\rvert\to 0} f(\xi)/\lvert\xi\rvert = 0$
proper	a function $f : X \to \overline{\mathbb{R}}$ is called proper if there exists an $x \in X$ such that $f(x) < \infty$ and for every x, $f(x) > -\infty$; in other words if the epigraph of f is non-empty and contains no vertical lines

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