



Deciding FO-definability of Regular Languages

Agi Kurucz¹, Vladislav Ryzhikov^{2(✉)}, Yury Savateev^{2,3},
and Michael Zakharyashev^{2,3}

¹ King's College London, London, UK
agi.kurucz@kcl.ac.uk

² Birkbeck, University of London, London, UK
{vlad,yury,michael}@dcs.bbk.ac.uk

³ HSE University, Moscow, Russia

Abstract. We prove that, similarly to known PSPACE-completeness of recognising FO($<$)-definability of the language $L(\mathfrak{A})$ of a DFA \mathfrak{A} , deciding both FO($<$, \equiv)- and FO($<$, MOD)-definability (corresponding to circuit complexity in AC^0 and ACC^0) are PSPACE-complete. We obtain these results by first showing that known algebraic characterisations of FO-definability of $L(\mathfrak{A})$ can be captured by ‘localisable’ properties of the transition monoid of \mathfrak{A} . Using our criterion, we then generalise the known proof of PSPACE-hardness of FO($<$)-definability, and establish the upper bounds not only for arbitrary DFAs but also for 2NFAs.

1 Introduction

This paper gives answers to some open questions related to finite automata, logic and circuit complexity. Research in this area goes back (at least) to the early 1960s when Büchi [8], Elgot [12] and Trakhtenbrot [28] showed that MSO($<$) (monadic second-order) sentences over finite strict linear orders define exactly the class of regular languages.

FO($<$)-definable regular languages were proven to be the same as star-free languages [19], and their algebraic characterisation as languages with aperiodic syntactic monoids was obtained in [23]. Algebraic characterisations of FO-definability in other signatures, and circuit and descriptive complexity of regular languages were investigated in [3, 4, 26], which established an $AC^0/ACC^0/NC^1$ trichotomy. In particular, the regular languages decidable in AC^0 are definable by FO($<$, \equiv)-sentences with unary predicates $x \equiv 0 \pmod{n}$; those in ACC^0 are definable by FO($<$, MOD)-sentences with quantifiers $\exists^n x \psi(x)$ checking whether the number of positions satisfying ψ is divisible by n ; and all regular languages are definable in FO(RPR) with relational primitive recursion [11]; see Table 1.

The problem of deciding whether the language of a given DFA \mathfrak{A} is FO($<$)-definable is known to be PSPACE-complete [7, 10, 25] (which is also a special case of general results on finite monoids [5, 13]). As shown in [4], the algebraic criteria of Table 1 yield algorithms deciding whether a given regular language is in

Table 1. Definability, algebraic characterisations and circuit complexity of a regular language \mathbf{L} , where $M(\mathbf{L})$ is the syntactic monoid and $\eta_{\mathbf{L}}$ the syntactic morphism of \mathbf{L} .

Definability of \mathbf{L}	Algebraic characterisation of \mathbf{L}	Circuit complexity
FO($<$)	$M(\mathbf{L})$ is aperiodic	in AC^0
FO($<, \equiv$)	$\eta_{\mathbf{L}}$ is quasi-aperiodic	
FO($<, \text{MOD}$)	All groups in $M(\mathbf{L})$ are solvable	in ACC^0
FO(RPR)	Arbitrary $M(\mathbf{L})$	in NC^1
Not in FO($<, \text{MOD}$)	$M(\mathbf{L})$ contains an unsolvable group	NC^1 -hard

AC^0 and FO($<, \equiv$)-definable, or in ACC^0 and FO($<, \text{MOD}$)-definable, or NC^1 -complete and is not FO($<, \text{MOD}$)-definable (unless $\text{ACC}^0 = \text{NC}^1$). However, these ‘brute force’ algorithms are not optimal, requiring the generation of the whole transition monoid of \mathfrak{A} , which can be of exponential size [14]. As far as we know, the precise complexity of these decision problems has remained open.

Our interest in the exact complexity of these problems is motivated by recent advances in ontology-based data access (OBDA) with linear time temporal logic *LTL* [1, 2]. The classical (atemporal) OBDA paradigm [20, 30] relies on a reduction of answering a query mediated by an ontology under the open-world semantics to evaluating a database query in a standard language such as SQL or its extension—that is, essentially, an extension of first-order logic—under the closed-world semantic. In the context of temporal OBDA, answering *LTL* ontology-mediated queries is equivalent to deciding certain regular languages given by an NFA or 2NFA of (possibly) exponential size, which gives rise to the circuit complexity and FO-definability problems for those languages. For further details the reader is referred to [22], which relies on the results we obtain below.

Our contribution in this paper is as follows. Let \mathcal{L} be one of the languages FO($<, \equiv$) or FO($<, \text{MOD}$). First, using the algebraic characterisation results of [3, 4, 26], we obtain criteria for the \mathcal{L} -definability of the language $\mathbf{L}(\mathfrak{A})$ of any given DFA \mathfrak{A} in terms of a limited part of the transition monoid of \mathfrak{A} (Theorem 1). Then, by using our criteria and generalising the construction of [10], we show that deciding \mathcal{L} -definability of $\mathbf{L}(\mathfrak{A})$ for any minimal DFA \mathfrak{A} is PSPACE-hard (Theorem 2). Finally, we apply our criteria to give a PSPACE-algorithm deciding \mathcal{L} -definability of $\mathbf{L}(\mathfrak{A})$ for not only any DFA but any 2NFA \mathfrak{A} (Theorem 3).

2 Preliminaries

We begin by briefly reminding the reader of the basic algebraic and automata-theoretic notions required in the remainder of the paper.

2.1 Monoids and Groups

A *semigroup* is a structure $\mathfrak{S} = (S, \cdot)$ where \cdot is an associative binary operation. Given $s, s' \in S$ and $n > 0$, we write s^n for $s \cdot \dots \cdot s$ n -times, and often write ss'

for $s \cdot s'$. An element s in a semigroup \mathfrak{S} is *idempotent* if $s^2 = s$. An element e in \mathfrak{S} is an *identity* if $e \cdot x = x \cdot e = x$ for all $x \in S$. (It is easy to see that such an e is unique, if exists.) The identity element is clearly idempotent. A *monoid* is a semigroup with an identity element. For any element s in a monoid, we set $s^0 = e$. A monoid $\mathfrak{S} = (S, \cdot)$ is a *group* if, for any $x \in S$, there is $x^- \in S$ —the *inverse of x* —such that $x \cdot x^- = x^- \cdot x = e$ (every element of a group has a unique inverse). A group is *trivial* if it has one element, and *nontrivial* otherwise.

Given two groups $\mathfrak{G} = (G, \cdot)$ and $\mathfrak{G}' = (G', \cdot')$, a map $h: G \rightarrow G'$ is a *group homomorphism from \mathfrak{G} to \mathfrak{G}'* if $h(g_1 \cdot g_2) = h(g_1) \cdot' h(g_2)$ for all $g_1, g_2 \in G$. (It is easy to see that any group homomorphism maps the identity of \mathfrak{G} to the identity of \mathfrak{G}' and preserves the inverses. The set $\{h(g) \mid g \in G\}$ is closed under \cdot' , and so is a group, the *image of \mathfrak{G} under h* .) \mathfrak{G} is a *subgroup of \mathfrak{G}'* if $G \subseteq G'$ and the identity map id_G is a group homomorphism. Given $X \subseteq G$, the *subgroup of \mathfrak{G} generated by X* is the smallest subgroup of \mathfrak{G} containing X . The *order $o_{\mathfrak{G}}(g)$* of an element g in \mathfrak{G} is the smallest positive number n with $g^n = e$, which always exists. Clearly, $o_{\mathfrak{G}}(g) = o_{\mathfrak{G}}(g^-)$ and, if $g^k = e$ then $o_{\mathfrak{G}}(g)$ divides k . Also,

$$\text{if } g \text{ is a nonidentity element in a group } \mathfrak{G}, \text{ then } g^k \neq g^{k+1} \text{ for any } k. \quad (1)$$

A semigroup $\mathfrak{S}' = (S', \cdot')$ is a *subsemigroup* of a semigroup $\mathfrak{S} = (S, \cdot)$ if $S' \subseteq S$ and \cdot' is the restriction of \cdot to S' . Given a monoid $\mathbf{M} = (M, \cdot)$ and a set $S \subseteq M$, we say that S *contains the group $\mathfrak{G} = (G, \cdot)$* , if $G \subseteq S$ and \mathfrak{G} is a subsemigroup of \mathbf{M} . Note that we do **not** require the identity of \mathbf{M} to be in \mathfrak{G} , even if it is in S . If $S = M$, we also say that \mathbf{M} *contains the group \mathfrak{G}* , or \mathfrak{G} *is in \mathbf{M}* . We call a monoid \mathbf{M} *aperiodic* if it does not contain any nontrivial groups.

Let $\mathfrak{S} = (S, \cdot)$ be a finite semigroup and $s \in S$. By the pigeonhole principle, there exist $i, j \geq 1$ such that $i + j \leq |S| + 1$ and $s^i = s^{i+j}$. Take the minimal such numbers, that is, let $i_s, j_s \geq 1$ be such that $i_s + j_s \leq |S| + 1$ and $s^{i_s} = s^{i_s+j_s}$ but $s^{i_s}, s^{i_s+1}, \dots, s^{i_s+j_s-1}$ are all different. Then clearly $\mathfrak{G}_s = (G_s, \cdot)$, where $G_s = \{s^{i_s}, s^{i_s+1}, \dots, s^{i_s+j_s-1}\}$, is a subsemigroup of \mathfrak{S} . It is easy to see that there is $m \geq 1$ with $i_s \leq m \cdot j_s < i_s + j_s \leq |S| + 1$, and so $s^{m \cdot j_s}$ is idempotent. Thus, for every element s in a semigroup \mathfrak{S} , we have the following:

$$\text{there is } n \geq 1 \text{ such that } s^n \text{ is idempotent;} \quad (2)$$

$$\mathfrak{G}_s \text{ is a group in } \mathfrak{S} \text{ (isomorphic to the cyclic group } \mathbb{Z}_{j_s}\text{);} \quad (3)$$

$$\mathfrak{G}_s \text{ is nontrivial iff } s^n \neq s^{n+1} \text{ for any } n. \quad (4)$$

Let $\delta: Q \rightarrow Q$ be a function on a finite set $Q \neq \emptyset$. For any $p \in Q$, the subset $\{\delta^k(p) \mid k < \omega\}$ with the obvious multiplication is a semigroup, and so we have:

$$\text{for every } p \in Q, \text{ there is } n_p \geq 1 \text{ such that } \delta^{n_p}(\delta^{n_p}(p)) = \delta^{n_p}(p); \quad (5)$$

$$\text{there exist } q \in Q \text{ and } n \geq 1 \text{ such that } q = \delta^n(q); \quad (6)$$

for every $q \in Q$, if $q = \delta^k(q)$ for some $k \geq 1$,

$$\text{then there is } n, 1 \leq n \leq |Q|, \text{ with } q = \delta^n(q). \quad (7)$$

For a definition of *solvable* and *unsolvable* groups the reader is referred to [21]. Here, we only need the fact that any homomorphic image of a solvable group

is solvable and the Kaplan–Levy criterion [16] (generalising Thompson’s [27, Cor.3]) according to which a finite group \mathfrak{G} is unsolvable iff it contains three elements a, b, c , such that $o_{\mathfrak{G}}(a) = 2$, $o_{\mathfrak{G}}(b)$ is an odd prime, $o_{\mathfrak{G}}(c) > 1$ and coprime to both 2 and $o_{\mathfrak{G}}(b)$, and abc is the identity element of \mathfrak{G} .

A one-to-one and onto function on a finite set S is called a *permutation on S* . The *order of a permutation δ* is its order in the group of all permutations on S (whose operation is composition, and its identity element is the identity permutation id_S). We use the standard cycle notation for permutations.

Suppose \mathfrak{G} is a monoid of $Q \rightarrow Q$ functions, for some finite set $Q \neq \emptyset$. Let $S = \{q \in Q \mid e_{\mathfrak{G}}(q) = q\}$, where $e_{\mathfrak{G}}$ the identity element in \mathfrak{G} . For every function δ in \mathfrak{G} , let $\delta|_S$ denote the restriction of δ to S . Then we have the following:

$$\mathfrak{G} \text{ is a group iff } \delta|_S \text{ is a permutation on } S, \text{ for every } \delta \text{ in } \mathfrak{G}; \tag{8}$$

$$\text{if } \mathfrak{G} \text{ is a group and } \delta \text{ is a nonidentity element in it, then } \delta|_S \neq \text{id}_S \text{ and} \\ \text{the order of the permutation } \delta|_S \text{ divides } o_{\mathfrak{G}}(\delta). \tag{9}$$

2.2 Automata: DFAs, NFAs, 2NFAs

A *two-way nondeterministic finite automaton* is a quintuple $\mathfrak{A} = (Q, \Sigma, \delta, Q_0, F)$ that consists of an alphabet Σ , a finite set Q of states with a subset $Q_0 \neq \emptyset$ of initial states and a subset F of accepting states, and a transition function $\delta: Q \times \Sigma \rightarrow 2^{Q \times \{-1,0,1\}}$ indicating the next state and whether the head should move left (-1), right (1), or stay put. If $Q_0 = \{q_0\}$ and $|\delta(q, a)| = 1$, for all $q \in Q$ and $a \in \Sigma$, then \mathfrak{A} is *deterministic*, in which case we write $\mathfrak{A} = (Q, \Sigma, \delta, q_0, F)$. If $\delta(q, a) \subseteq Q \times \{1\}$, for all $q \in Q$ and $a \in \Sigma$, then \mathfrak{A} is a *one-way* automaton, and we write $\delta: Q \times \Sigma \rightarrow 2^Q$. As usual, DFA and NFA refer to one-way deterministic and non-deterministic finite automata, respectively, while 2DFA and 2NFA to the corresponding two-way automata. Given a 2NFA \mathfrak{A} , we write $q \rightarrow_{a,d} q'$ if $(q', d) \in \delta(q, a)$; given an NFA \mathfrak{A} , we write $q \rightarrow_a q'$ if $q' \in \delta(q, a)$. A *run* of a 2NFA \mathfrak{A} is a word in $(Q \times \mathbb{N})^*$. A run $(q_0, i_0), \dots, (q_m, i_m)$ is a *run of \mathfrak{A} on a word $w = a_0 \dots a_n \in \Sigma^*$* if $q_0 \in Q_0, i_0 = 0$ and there exist $d_0, \dots, d_{m-1} \in \{-1, 0, 1\}$ such that $q_j \rightarrow_{a_j, d_j} q_{j+1}$ and $i_{j+1} = i_j + d_j$ for all $j, 0 \leq j < m$. The run is *accepting* if $q_m \in F, i_m = n + 1$. \mathfrak{A} *accepts* $w \in \Sigma^*$ if there is an accepting run of \mathfrak{A} on w ; the language $L(\mathfrak{A})$ of \mathfrak{A} is the set of all words accepted by \mathfrak{A} .

Given an NFA \mathfrak{A} , states $q, q' \in Q$, and $w = a_0 \dots a_n \in \Sigma^*$, we write $q \rightarrow_w q'$ if either $w = \varepsilon$ and $q' = q$ or there is a run of \mathfrak{A} on w that starts with $(q_0, 0)$ and ends with $(q', n + 1)$. We say that a state $q \in Q$ is *reachable* if $q' \rightarrow_w q$, for some $q' \in Q_0$ and $w \in \Sigma^*$.

Given a DFA $\mathfrak{A} = (Q, \Sigma, \delta, q_0, F)$ and a word $w \in \Sigma^*$, we define a function $\delta_w: Q \rightarrow Q$ by taking $\delta_w(q) = q'$ iff $q \rightarrow_w q'$. We also define an equivalence relation \sim on the set $Q^r \subseteq Q$ of reachable states by taking $q \sim q'$ iff, for every $w \in \Sigma^*$, we have $\delta_w(q) \in F$ just in case $\delta_w(q') \in F$. We denote the \sim -class of q by q/\sim , and let $X/\sim = \{q/\sim \mid q \in X\}$ for any $X \subseteq Q^r$. Define $\tilde{\delta}_w: Q^r/\sim \rightarrow Q^r/\sim$ by taking $\tilde{\delta}_w(q/\sim) = \delta_w(q)/\sim$. Then $(Q^r/\sim, \Sigma, \tilde{\delta}, q_0/\sim, (F \cap Q^r)/\sim)$ is the *minimal DFA* whose language coincides with the language of \mathfrak{A} . Given a regular language L , we denote by \mathfrak{A}_L the minimal DFA whose language is L .

The *transition monoid* of a DFA \mathfrak{A} is $M(\mathfrak{A}) = (\{\delta_w \mid w \in \Sigma^*\}, \cdot)$ with $\delta_v \cdot \delta_w = \delta_{vw}$, for any v, w . The *syntactic monoid* $M(\mathbf{L})$ of \mathbf{L} is the transition monoid $M(\mathfrak{A}_{\mathbf{L}})$ of $\mathfrak{A}_{\mathbf{L}}$. The *syntactic morphism* of \mathbf{L} is the map $\eta_{\mathbf{L}}$ from Σ^* to the domain of $M(\mathbf{L})$ defined by $\eta_{\mathbf{L}}(w) = \tilde{\delta}_w$. We call $\eta_{\mathbf{L}}$ *quasi-aperiodic* if $\eta_{\mathbf{L}}(\Sigma^t)$ is aperiodic for every $t < \omega$.

Suppose $\mathcal{L} \in \{\text{FO}(<), \text{FO}(<, \equiv), \text{FO}(<, \text{MOD})\}$. A language \mathbf{L} over an alphabet Σ is \mathcal{L} -*definable* if there is an \mathcal{L} -sentence φ in the signature Σ , whose symbols are treated as unary predicates, such that, for any $w \in \Sigma^*$, we have $w = a_0 \dots a_n \in \mathbf{L}$ iff $\mathfrak{S}_w \models \varphi$, where \mathfrak{S}_w is an FO-structure with domain $\{0, \dots, n\}$ ordered by $<$, in which $\mathfrak{S}_w \models a(i)$ iff $a = a_i$, for $0 \leq i \leq n$.

Table 1 summarises the known results that connect definability of a regular language \mathbf{L} with properties of the syntactic monoid $M(\mathbf{L})$ and syntactic morphism $\eta_{\mathbf{L}}$ (see [4] for details) and with its circuit complexity under a reasonable binary encoding of \mathbf{L} 's alphabet (see, e.g., [7, Lemma 2.1]) and the assumption that $\text{ACC}^0 \neq \text{NC}^1$. We also remind the reader that a regular language is $\text{FO}(<)$ -definable iff it is star-free [26], and that $\text{AC}^0 \subsetneq \text{ACC}^0 \subseteq \text{NC}^1$ [15, 26].

3 Criteria of \mathcal{L} -definability

In this section, we show that the algebraic characterisations of FO-definability of $\mathbf{L}(\mathfrak{A})$ given in Table 1 can be captured by ‘localisable’ properties of the transition monoid of \mathfrak{A} , for any given DFA \mathfrak{A} . Note that Theorem 1 (i) was already observed in [25] and used in proving that $\text{FO}(<)$ -definability of $\mathbf{L}(\mathfrak{A})$ is PSPACE-complete [7, 10, 25]; while criteria (ii) and (iii) seem to be new.

Theorem 1. *For any DFA $\mathfrak{A} = (Q, \Sigma, \delta, q_0, F)$, the following criteria hold:*

- (i) $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<)$ -definable iff \mathfrak{A} contains a nontrivial cycle, that is, there exist a word $u \in \Sigma^*$, a state $q \in Q^r$, and a number $k \leq |Q|$ such that $q \not\sim \delta_u(q)$ and $q = \delta_{u^k}(q)$;
- (ii) $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<, \equiv)$ -definable iff there are words $u, v \in \Sigma^*$, a state $q \in Q^r$, and a number $k \leq |Q|$ such that $q \not\sim \delta_u(q)$, $q = \delta_{u^k}(q)$, $|v| = |u|$, and $\delta_{u^i}(q) = \delta_{u^i v}(q)$, for every $i < k$;
- (iii) $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<, \text{MOD})$ -definable iff there exist words $u, v \in \Sigma^*$, a state $q \in Q^r$ and numbers $k, l \leq |Q|$ such that k is an odd prime, $l > 1$ and coprime to both 2 and k , $q \not\sim \delta_u(q)$, $q \not\sim \delta_v(q)$, $q \not\sim \delta_{uv}(q)$ and, for all $x \in \{u, v\}^*$, we have $\delta_x(q) \sim \delta_{xu^2}(q) \sim \delta_{xv^k}(q) \sim \delta_{x(uv)^l}(q)$.

Proof. Throughout, we use the algebraic criteria of Table 1 for $\mathbf{L} = \mathbf{L}(\mathfrak{A})$. Thus, $M(\mathbf{L})$ is the transition monoid of the minimal DFA $\mathfrak{A}_{\mathbf{L}(\mathfrak{A})}$, whose transition function we denote by $\tilde{\delta}$.

(i) (\Rightarrow) Suppose \mathfrak{G} is a nontrivial group in $M(\mathfrak{A}_{\mathbf{L}(\mathfrak{A})})$. Let $u \in \Sigma^*$ be such that $\tilde{\delta}_u$ is a nonidentity element in \mathfrak{G} . We claim that there is $p \in Q^r$ such that $\tilde{\delta}_{u^n}(p/\sim) \neq \tilde{\delta}_{u^{n+1}}(p/\sim)$ for any $n > 0$. Indeed, otherwise for every $p \in Q^r$ there is $n_p > 0$ with $\tilde{\delta}_{u^{n_p}}(p/\sim) = \tilde{\delta}_{u^{n_p+1}}(p/\sim)$. Let $n = \max\{n_p \mid p \in Q^r\}$. Then $\tilde{\delta}_{u^n} = \tilde{\delta}_{u^{n+1}}$, contrary to (1).

By (5), there is $m \geq 1$ with $\tilde{\delta}_{u^{2m}}(p/\sim) = \tilde{\delta}_{u^m}(p/\sim)$. Let $s/\sim = \tilde{\delta}_{u^m}(p/\sim)$. Then $s/\sim = \tilde{\delta}_{u^m}(s/\sim)$, and so the restriction of δ_{u^m} to the subset s/\sim of Q^r is an $s/\sim \rightarrow s/\sim$ function. By (6), there exist $q \in s/\sim$ and $n \geq 1$ such that $(\delta_{u^m})^n(q) = q$. Thus, $\delta_{u^{mn}}(q) = q$, and so by (7), there is $k \leq |Q|$ with $\delta_{u^k}(q) = q$. As $s/\sim \neq \tilde{\delta}_u(s/\sim)$, we also have $q \not\sim \delta_u(q)$, as required.

(i) (\Leftarrow) Suppose the condition holds for \mathfrak{A} . Then there are $u \in \Sigma^*$, $q \in Q^r/\sim$, and $k < \omega$ such that $q \neq \tilde{\delta}_u(q)$ and $q = \tilde{\delta}_{u^k}(q)$. Then $\tilde{\delta}_{u^n} \neq \tilde{\delta}_{u^{n+1}}$ for any $n > 0$. Indeed, otherwise we would have some $n > 0$ with $\tilde{\delta}_{u^n}(q) = \tilde{\delta}_{u^{n+1}}(q)$. Let i, j be such that $n = i \cdot k + j$ and $j < k$. Then

$$q = \tilde{\delta}_{u^k}(q) = \tilde{\delta}_{u^{(i+1)k}}(q) = \tilde{\delta}_{u^n u^{k-j}}(q) = \tilde{\delta}_{u^{n+1} u^{k-j}}(q) = \tilde{\delta}_{u^{(i+1)k} u}(q) = \tilde{\delta}_u(q).$$

So, by (3) and (4), $\mathfrak{G}_{\tilde{\delta}_u}$ is a nontrivial group in $M(\mathfrak{A}_{L(\mathfrak{A})})$.

(ii) (\Rightarrow) Let \mathfrak{G} be a nontrivial group in $\eta_L(\Sigma^t)$, for some $t < \omega$, and let $u \in \Sigma^t$ be such that $\tilde{\delta}_u$ is a nonidentity element in \mathfrak{G} . As shown in the proof of (i) (\Rightarrow), there exist $s \in Q^r$ and $m \geq 1$ such that $s/\sim \neq \tilde{\delta}_u(s/\sim)$ and $s/\sim = \tilde{\delta}_{u^m}(s/\sim)$. Now let $v \in \Sigma^t$ be such that $\tilde{\delta}_v$ is the identity element in \mathfrak{G} , and consider δ_v . By (2), there is $\ell \geq 1$ such that δ_{v^ℓ} is idempotent. Then $\delta_{v^{2\ell-1}v^{2\ell}} = \delta_{v^{2\ell-1}}$. Thus, if we let $\bar{u} = uv^{2\ell-1}$ and $\bar{v} = v^{2\ell}$, then $|\bar{u}| = |\bar{v}|$ and $\delta_{\bar{u}^i} = \delta_{\bar{u}^i \bar{v}}$ for any $i < \omega$. Also, $\tilde{\delta}_{\bar{u}^i} = \tilde{\delta}_{\bar{u}^i}$ for every $i \geq 1$, and so the restriction of $\delta_{\bar{u}^m}$ to s/\sim is an $s/\sim \rightarrow s/\sim$ function. By (6), there exist $q \in s/\sim$ and $n \geq 1$ such that $(\delta_{\bar{u}^m})^n(q) = q$. Thus, $\delta_{\bar{u}^{mn}}(q) = q$, and so by (7), there is some $k \leq |Q|$ with $\delta_{\bar{u}^k}(q) = q$. As $s/\sim \neq \tilde{\delta}_{\bar{u}}(s/\sim) = \tilde{\delta}_{\bar{u}}(s/\sim)$, we also have $q \not\sim \delta_{\bar{u}}(q)$, as required.

(ii) (\Leftarrow) If the condition holds for \mathfrak{A} , then there exist $u, v \in \Sigma^*$, $q \in Q^r/\sim$, and $k < \omega$ such that $q \neq \tilde{\delta}_u(q)$, $q = \tilde{\delta}_{u^k}(q)$, $|v| = |u|$, and $\tilde{\delta}_{u^i}(q) = \tilde{\delta}_{u^i v}(q)$, for every $i < k$. As $M(\mathfrak{A}_{L(\mathfrak{A})})$ is finite, it has finitely many subsets. So there exist $i, j \geq 1$ such that $\eta_L(\Sigma^{i|u|}) = \eta_L(\Sigma^{(i+j)|u|})$. Let z be a multiple of j with $i \leq z < i + j$. Then $\eta_L(\Sigma^{z|u|}) = \eta_L(\Sigma^{z|u|}^2)$, and so $\eta_L(\Sigma^{z|u|})$ is closed under the composition of functions (that is, the semigroup operation of $M(\mathfrak{A}_{L(\mathfrak{A})})$). Let $w = uv^{z-1}$ and consider the group $\mathfrak{G}_{\tilde{\delta}_w}$ (defined above (2)–(4)). Then $G_{\tilde{\delta}_w} \subseteq \eta_L(\Sigma^{z|u|})$. We claim that $\mathfrak{G}_{\tilde{\delta}_w}$ is nontrivial. Indeed, we have $\tilde{\delta}_w(q) = \tilde{\delta}_{uv^{z-1}}(q) = \tilde{\delta}_u(q) \neq q$. On the other hand, $\tilde{\delta}_{w^k}(q) = \tilde{\delta}_{u^k}(q) = q$. By the proof of (i) (\Leftarrow), $\mathfrak{G}_{\tilde{\delta}_w}$ is nontrivial.

(iii) (\Rightarrow) Suppose \mathfrak{G} is an unsolvable group in $M(\mathfrak{A}_{L(\mathfrak{A})})$. By the Kaplan–Levy criterion, \mathfrak{G} contains three functions a, b, c such that $o_{\mathfrak{G}}(a) = 2$, $o_{\mathfrak{G}}(b)$ is an odd prime, $o_{\mathfrak{G}}(c) > 1$ and coprime to both 2 and $o_{\mathfrak{G}}(b)$, and $c \circ b \circ a = e_{\mathfrak{G}}$ for the identity element $e_{\mathfrak{G}}$ of \mathfrak{G} . Let $u, v \in \Sigma^*$ be such that $a = \tilde{\delta}_u$, $b = \tilde{\delta}_v$ and $c = (\tilde{\delta}_{uv})^-$, and let $k = o_{\mathfrak{G}}(\tilde{\delta}_v)$ and $r = o_{\mathfrak{G}}(c) = o_{\mathfrak{G}}(\tilde{\delta}_{uv})$. Then $r > 1$ and coprime to both 2 and k . Let $S = \{p \in Q^r/\sim \mid e_{\mathfrak{G}}(p) = p\}$. As $\tilde{\delta}_x$ is \mathfrak{G} for every $x \in \{u, v\}^*$, we have $e_{\mathfrak{G}} \circ \tilde{\delta}_x = \tilde{\delta}_x$. Thus,

$$\begin{aligned} \tilde{\delta}_{xu^2}(q) &= \tilde{\delta}_{u^2}(\tilde{\delta}_x(q)) = e_{\mathfrak{G}}(\tilde{\delta}_x(q)) = (e_{\mathfrak{G}} \circ \tilde{\delta}_x)(q) = \tilde{\delta}_x(q), \quad \text{and} \\ \tilde{\delta}_{xv^k}(q) &= \tilde{\delta}_{v^k}(\tilde{\delta}_x(q)) = e_{\mathfrak{G}}(\tilde{\delta}_x(q)) = (e_{\mathfrak{G}} \circ \tilde{\delta}_x)(q) = \tilde{\delta}_x(q), \quad \text{for every } q \in S. \end{aligned}$$

Then, by (8), each of $\tilde{\delta}_u|_S$, $\tilde{\delta}_v|_S$ and $\tilde{\delta}_{uv}|_S$ is a permutation on S . By (9), the order of $\tilde{\delta}_u|_S$ is 2, the order of $\tilde{\delta}_v|_S$ is k , and the order l of $\tilde{\delta}_{uv}|_S$ is a > 1 divisor

of r , and so it is coprime to both 2 and k . Also, we have $k, l \leq |S| \leq |Q|$. Further, for every x , if q is in S then $\tilde{\delta}_x(q) \in S$ as well. So we have

$$\tilde{\delta}_{x(uv)^l}(q) = \tilde{\delta}_{(uv)^l}(\tilde{\delta}_x(q)) = (\tilde{\delta}_{uv} \upharpoonright_S)^l(\tilde{\delta}_x(q)) = \text{id}_S(\tilde{\delta}_x(q)) = \tilde{\delta}_x(q), \quad \text{for all } q \in S.$$

It remains to show that there is $q \in S$ with $q \neq \tilde{\delta}_u(q)$, $q \neq \tilde{\delta}_v(q)$, and $q \neq \tilde{\delta}_{uv}(q)$. Recall that the length of any cycle in a permutation divides its order. First, we show there is $q \in S$ with $q \neq \tilde{\delta}_u(q)$ and $q \neq \tilde{\delta}_v(q)$. Indeed, as $\tilde{\delta}_u \upharpoonright_S \neq \text{id}_S$, there is $q \in S$ such that $\tilde{\delta}_u(q) = q' \neq q$. As the order of $\tilde{\delta}_u \upharpoonright_S$ is 2, $\tilde{\delta}_u(q') = q$. If both $\tilde{\delta}_v(q) = q$ and $\tilde{\delta}_v(q') = q'$ were the case, then $\tilde{\delta}_{uv}(q) = q'$ and $\tilde{\delta}_{uv}(q') = q$ would hold, and so (qq') would be a cycle in $\tilde{\delta}_{uv} \upharpoonright_S$, contrary to l being coprime to 2. So take some $q \in S$ with $\tilde{\delta}_u(q) = q' \neq q$ and $\tilde{\delta}_v(q) \neq q$. If $\tilde{\delta}_v(q') \neq q$ then $\tilde{\delta}_{uv}(q) \neq q$, and so q is a good choice. Suppose $\tilde{\delta}_v(q') = q$, and let $q'' = \tilde{\delta}_v(q)$. Then $q'' \neq q'$, as k is odd. Thus, $\tilde{\delta}_{uv}(q') \neq q'$, and so q' is a good choice.

(iii) (\Leftarrow) Suppose $u, v \in \Sigma^*$, $q \in Q^r$, and $k, l < \omega$ are satisfying the conditions. For every $x \in \{u, v\}^*$, we define an equivalence relation \approx_x on Q^r / \sim by taking $p \approx_x p'$ iff $\tilde{\delta}_x(p) = \tilde{\delta}_x(p')$. Then we clearly have that $\approx_x \subseteq \approx_{xy}$, for all $x, y \in \{u, v\}^*$. As Q is finite, there is $z \in \{u, v\}^*$ such that $\approx_z \subseteq \approx_{zy}$ for all $y \in \{u, v\}^*$. Take such a z . By (2), $\tilde{\delta}_z^n$ is idempotent for some $n \geq 1$. We let $w = z^n$. Then $\tilde{\delta}_w$ is idempotent and we also have that

$$\approx_w = \approx_{wy} \quad \text{for all } y \in \{u, v\}^*. \quad (10)$$

Let $G_{\{u,v\}} = \{\tilde{\delta}_{wxw} \mid x \in \{u, v\}^*\}$. Then $G_{\{u,v\}}$ is closed under composition. Let $\mathfrak{G}_{\{u,v\}}$ be the subsemigroup of $M(\mathfrak{A}_L(\mathfrak{A}))$ with universe $G_{\{u,v\}}$. Then $\tilde{\delta}_w = \tilde{\delta}_{w\varepsilon w}$ is an identity element in $\mathfrak{G}_{\{u,v\}}$. Let $S = \{p \in Q^r / \sim \mid \tilde{\delta}_w(p) = p\}$. We show that

$$\text{for every } \tilde{\delta} \text{ in } \mathfrak{G}_{\{u,v\}}, \tilde{\delta} \upharpoonright_S \text{ is a permutation on } S, \quad (11)$$

and so $\mathfrak{G}_{\{u,v\}}$ is a group by (8). Indeed, take some $x \in \{u, v\}^*$. As $\tilde{\delta}_w(\tilde{\delta}_{wxw}(p)) = \tilde{\delta}_{wxww}(p) = \tilde{\delta}_{wxw}(p)$ for any $p \in Q^r / \sim$, $\tilde{\delta}_{wxw} \upharpoonright_S$ is an $S \rightarrow S$ function. Also, if $p, p' \in S$ and $\tilde{\delta}_{wxw}(p) = \tilde{\delta}_{wxw}(p')$ then $p \approx_{wxw} p'$. Thus, by (10), $p \approx_w p'$, that is, $p = \tilde{\delta}_w(p) = \tilde{\delta}_w(p') = p'$, proving (11).

We show that $\mathfrak{G}_{\{u,v\}}$ is unsolvable by finding an unsolvable homomorphic image of it. Let $R = \{p \in Q^r / \sim \mid p = \tilde{\delta}_x(q) \text{ for some } x \in \{u, v\}^*\}$. We claim that, for every $\tilde{\delta}$ in $\mathfrak{G}_{\{u,v\}}$, $\tilde{\delta} \upharpoonright_R$ is a permutation on R , and so the function h mapping every $\tilde{\delta}$ to $\tilde{\delta} \upharpoonright_R$ is a group homomorphism from $\mathfrak{G}_{\{u,v\}}$ to the group of all permutations on R . Indeed, by (11), it is enough to show that $R \subseteq S$. Let $\bar{w} = \bar{z}_m \dots \bar{z}_1$, where $w = z_1 \dots z_m$ for some $z_i \in \{u, v\}$, $\bar{u} = u$ and $\bar{v} = v^{k-1}$. Since $\tilde{\delta}_x(q) = \tilde{\delta}_{x(u)^2}(q) = \tilde{\delta}_{(v)^k}(q)$ for all $x \in \{u, v\}^*$, we obtain that

$$\begin{aligned} \tilde{\delta}_{y\bar{w}\bar{w}}(q) &= \tilde{\delta}_{\bar{z}_{m-1} \dots \bar{z}_1}(\tilde{\delta}_{y\bar{z}_1 \dots \bar{z}_m \bar{z}_m}(q)) = \tilde{\delta}_{\bar{z}_{m-1} \dots \bar{z}_1}(\tilde{\delta}_{y\bar{z}_1 \dots \bar{z}_{m-1}}(q)) = \dots \\ &\dots = \tilde{\delta}_{\bar{z}_1}(\tilde{\delta}_{y\bar{z}_1}(q)) = \tilde{\delta}_{x\bar{z}_1}(q) = \tilde{\delta}_y(q), \quad \text{for all } y \in \{u, v\}^*. \end{aligned} \quad (12)$$

Now suppose $p \in R$, that is, $p = \tilde{\delta}_x(q)$ for some $x \in \{u, v\}^*$. Then, by (12),

$$\tilde{\delta}_w(p) = \tilde{\delta}_w(\tilde{\delta}_x(q)) = \tilde{\delta}_{xw}(q) = \tilde{\delta}_{xw\bar{w}\bar{w}}(q) = \tilde{\delta}_{xw\bar{w}}(q) = \tilde{\delta}_x(q) = p,$$

and so $p \in S$, as required.

Now let \mathfrak{G} be the image of $\mathfrak{G}_{\{u,v\}}$ under h . We prove that \mathfrak{G} is unsolvable by finding three elements a, b, c in it such that $o_{\mathfrak{G}}(a) = 2$, $o_{\mathfrak{G}}(b) = k$, $o_{\mathfrak{G}}(c)$ is coprime to both 2 and $o_{\mathfrak{G}}(b)$, and $c \circ b \circ a = \text{id}_R$ (the identity element of \mathfrak{G}). So let $a = h(\tilde{\delta}_{wuw})$, $b = h(\tilde{\delta}_{wvw})$, and $c = h(\tilde{\delta}_{wuvv})^-$. Observe that, for every $x \in \{u, v\}^*$, $h(\tilde{\delta}_{wxw}) = \tilde{\delta}_x \upharpoonright_R$, and so $c \circ b \circ a = \text{id}_R$. Also, for any $\tilde{\delta}_x(q) \in R$, $a^2(\tilde{\delta}_x(q)) = (\tilde{\delta}_u \upharpoonright_R)^2(\tilde{\delta}_x(q)) = \tilde{\delta}_{xu^2}(q) = \tilde{\delta}_x(q)$ by our assumption, so $a^2 = \text{id}_R$. On the other hand, $q \in R$ as $\tilde{\delta}_\varepsilon(q) = q$, and $\text{id}_R(q) = q \neq \tilde{\delta}_u(q)$ by assumption, so $a \neq \text{id}_R$. As $o_{\mathfrak{G}}(a)$ divides 2, $o_{\mathfrak{G}}(a) = 2$ follows. Similarly, we can show that $o_{\mathfrak{G}}(b) = k$ (using that $\tilde{\delta}_{xv^k}(q) = \tilde{\delta}_x(q)$ for every $x \in \{u, v\}^*$, and $u \neq \tilde{\delta}_v(q)$). Finally (using that $\tilde{\delta}_{x(uv)^l}(q) = \tilde{\delta}_x(q)$ for every $x \in \{u, v\}^*$, and $u \neq \tilde{\delta}_{uv}(q)$), we obtain that $h(\tilde{\delta}_{wuvv})^l = \text{id}_R$ and $h(\tilde{\delta}_{wuvv}) \neq \text{id}_R$. Therefore, it follows that $o_{\mathfrak{G}}(c) = o_{\mathfrak{G}}(h(\tilde{\delta}_{wuvv})^-) = o_{\mathfrak{G}}(h(\tilde{\delta}_{wuvv})) > 1$ and divides l , and so coprime to both 2 and k , as required.

4 Deciding FO-definability: PSPACE-hardness

Kozen [18] showed that deciding whether the intersection of the languages recognised by a set of given deterministic DFAs is non-empty is PSPACE-complete. By carefully analysing Kozen’s lower bound proof and using the criterion of Theorem 1 (i), Cho and Huynh [10] established that deciding FO($<$)-definability of $L(\mathfrak{A})$ is PSPACE-hard, for any given minimal DFA \mathfrak{A} . We generalise their construction and use the criteria in Theorem 1 (ii)–(iii) to cover FO($<$, \equiv)- and FO($<$, MOD)-definability as well.

Theorem 2. *For any $\mathcal{L} \in \{\text{FO}(<), \text{FO}(<, \equiv), \text{FO}(<, \text{MOD})\}$, deciding \mathcal{L} -definability of the language $L(\mathfrak{A})$ of a given minimal DFA \mathfrak{A} is PSPACE-hard.*

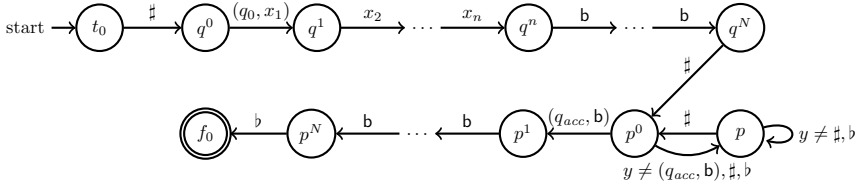
Proof. Let M be a deterministic Turing machine that decides a language using at most $N = P_M(n)$ tape cells on any input of size n , for some polynomial P_M . Given such an M and an input \mathbf{x} , our aim is to define three minimal DFAs whose languages are, respectively, FO($<$)-, FO($<$, \equiv)-, and FO($<$, MOD)-definable iff M rejects \mathbf{x} , and whose sizes are polynomial in N and the size $|M|$ of M .

Suppose $M = (Q, \Gamma, \gamma, \mathbf{b}, q_0, q_{acc})$ with a set Q of states, tape alphabet Γ with \mathbf{b} for blank, transition function γ , initial state q_0 and accepting state q_{acc} . Without loss of generality we assume that M erases the tape before accepting, its head is at the left-most cell in an accepting configuration, and if M does not accept the input, it runs forever. Given an input word $\mathbf{x} = x_1 \dots x_n$ over Γ , we represent configurations \mathbf{c} of the computation of M on \mathbf{x} by the N -long word written on the tape (with sufficiently many blanks at the end) in which the symbol y in the active cell is replaced by the pair (q, y) for the current state q . The accepting computation of M on \mathbf{x} is encoded by a word $\# \mathbf{c}_1 \# \mathbf{c}_2 \# \dots \# \mathbf{c}_{k-1} \# \mathbf{c}_k \mathbf{b}$ over the alphabet $\Sigma = \Gamma \cup (Q \times \Gamma) \cup \{\#, \mathbf{b}\}$, with $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k$ being the subsequent configurations. In particular, \mathbf{c}_1 is the initial configuration on \mathbf{x} (so it is of the form $(q_0, x_1)x_2 \dots x_n \mathbf{b} \dots \mathbf{b}$), and \mathbf{c}_k is the accepting configuration (so it is of the form $(q_{acc}, \mathbf{b})\mathbf{b} \dots \mathbf{b}$). As usual for this representation of computations, we may

regard γ as a partial function from $(\Gamma \cup (Q \times \Gamma) \cup \{\#\})^3$ to $\Gamma \cup (Q \times \Gamma)$ with $\gamma(\sigma_{i-1}^j, \sigma_i^j, \sigma_{i+1}^j) = \sigma_i^{j+1}$ for each $j < k$, where σ_i^j is the i th symbol of \mathbf{c}^j .

Let $p_{M,x} = p$ be the first prime such that $p \geq N + 2$ and $p \not\equiv \pm 1 \pmod{10}$. By [6, Corollary 1.6], p is polynomial in N . Our first aim is to construct a $p + 1$ -long sequence \mathfrak{A}_i of disjoint minimal DFAs over Σ . Each \mathfrak{A}_i has size polynomial in N and $|M|$, and it checks certain properties of an accepting computation on \mathbf{x} such that M accepts \mathbf{x} iff the intersection of the $L(\mathfrak{A}_i)$ is not empty and consists of the single word encoding the accepting computation on \mathbf{x} .

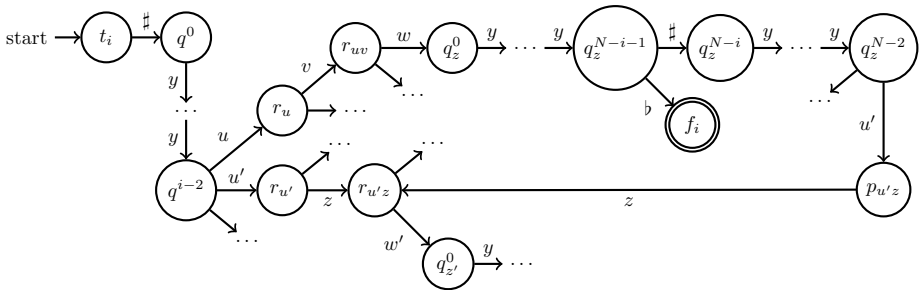
We define each \mathfrak{A}_i as an NFA, and assume that it can be turned to a DFA by adding a ‘trash state’ tr_i looping on itself with every $\sigma \in \Sigma$, and adding the missing transitions leading to tr_i . The DFA \mathfrak{A}_0 checks that an input starts with the initial configuration on \mathbf{x} and ends with the accepting configuration:



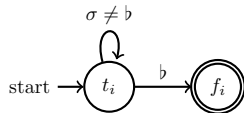
When $1 \leq i \leq N$, the DFA \mathfrak{A}_i checks, for all $j < k$, whether the i th symbol of \mathbf{c}^j changes ‘according to γ ’ in passing to \mathbf{c}^{j+1} . The non-trash part of its transition function δ^i is as follows, for $1 < i < N$. (For $i = 1$ and $i = N$ some adjustments are needed.) For all $u, u', v, w, w', y, z \in \Gamma \cup (Q \times \Gamma)$,

$$\begin{aligned} \delta_{\#}^i(t_i) &= q^0, & \delta_u^i(q^j) &= q^{j+1}, \text{ for } j = 0, \dots, i - 3, & \delta_u^i(q^{i-2}) &= r_u, & \delta_v^i(r_u) &= r_{uv}, \\ \delta_w^i(r_{uv}) &= q_{\gamma(u,v,w)}^0, & \delta_y^i(q_z^j) &= q_z^{j+1}, \text{ for } j = 0, \dots, N - 3, j \neq N - i - 1, \\ \delta_{\#}^i(q_z^{N-i-1}) &= q_z^{N-i}, & \delta_b^i(q_z^{N-i-1}) &= f_i, & \delta_{u'}^i(q_z^{N-2}) &= p_{u'z}, & \delta_z^i(p_{u'z}) &= r_{u'z}, \end{aligned}$$

see below, where $z = \gamma(u, v, w)$ and $z' = \gamma(u', z, w')$:



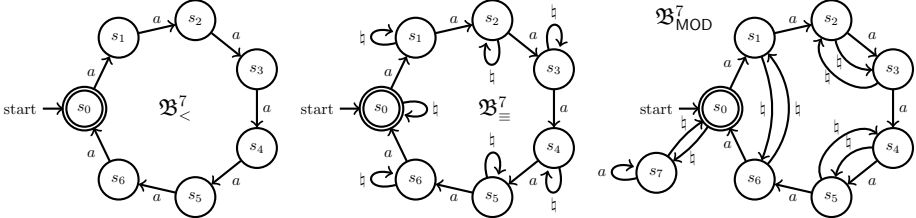
Finally, if $N + 1 \leq i \leq p$ then \mathfrak{A}_i accepts all words over Σ with a single occurrence of b , which is the input’s last character:



Note that $\mathfrak{A}_{p-1} = \mathfrak{A}_p$ as $p \geq N + 2$. It is not hard to check that each of the \mathfrak{A}_i is a minimal DFA that does not contain nontrivial cycles and the following holds:

Lemma 1. *M accepts x iff $\bigcap_{i=0}^p L(\mathfrak{A}_i) \neq \emptyset$, in which case this language consists of a single word that encodes the accepting computation of M on x .*

Next, we require three sequences of DFAs $\mathfrak{B}_{<}^p$, \mathfrak{B}_{\equiv}^p and $\mathfrak{B}_{\text{MOD}}^p$, where $p > 5$ is a prime number with $p \not\equiv \pm 1 \pmod{10}$; see the picture below for $p = 7$.



In general, the first sequence is $\mathfrak{B}_{<}^p = (\{s_i \mid i < p\}, \{a\}, \delta^{\mathfrak{B}_{<}^p}, s_0, \{s_0\})$, where $\delta_a^{\mathfrak{B}_{<}^p}(s_i) = s_j$ if $i, j < p$ and $j \equiv i + 1 \pmod{p}$. Then $L(\mathfrak{B}_{<}^p)$ comprises all words of the form $(a^p)^*$, $\mathfrak{B}_{<}^p$ is the minimal DFA for $L(\mathfrak{B}_{<}^p)$, and the syntactic monoid $M(\mathfrak{B}_{<}^p)$ is the cyclic group of order p (generated by the permutation $\delta_a^{\mathfrak{B}_{<}^p}$).

The second sequence is $\mathfrak{B}_{\equiv}^p = (\{s_i \mid i < p\}, \{a, \natural\}, \delta^{\mathfrak{B}_{\equiv}^p}, s_0, \{s_0\})$, where $\delta_{\natural}^{\mathfrak{B}_{\equiv}^p}(s_i) = s_i$ and $\delta_a^{\mathfrak{B}_{\equiv}^p}(s_i) = s_j$ if $i, j < p$ and $j \equiv i + 1 \pmod{p}$. One can check that $L(\mathfrak{B}_{\equiv}^p)$ comprises all words of a 's and \natural 's where the number of a 's is divisible by p , \mathfrak{B}_{\equiv}^p is the minimal DFA for this language, and $M(\mathfrak{B}_{\equiv}^p)$ is also the cyclic group of order p (generated by the permutation $\delta_a^{\mathfrak{B}_{\equiv}^p}$).

The third sequence is $\mathfrak{B}_{\text{MOD}}^p = (\{s_i \mid i \leq p\}, \{a, \natural\}, \delta^{\mathfrak{B}_{\text{MOD}}^p}, s_0, \{s_0\})$, where

- $\delta_a^{\mathfrak{B}_{\text{MOD}}^p}(s_p) = s_p$, and $\delta_a^{\mathfrak{B}_{\text{MOD}}^p}(s_i) = s_j$ whenever $i, j < p$ and $j \equiv i + 1 \pmod{p}$;
- $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}(s_0) = s_p$, $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}(s_p) = s_0$, and $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}(s_i) = s_j$ whenever $1 \leq i, j < p$ and $i \cdot j \equiv p - 1 \pmod{p}$, that is, $j = -1/i$ in the finite field \mathbb{F}_p .

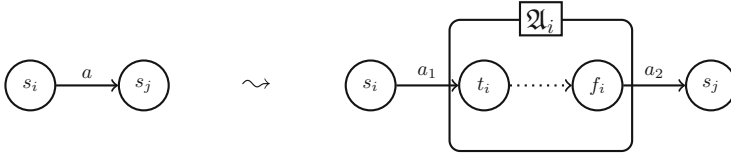
One can check that $\mathfrak{B}_{\text{MOD}}^p$ is the minimal DFA for its language, and the syntactic monoid $M(\mathfrak{B}_{\text{MOD}}^p)$ is the permutation group generated by $\delta_a^{\mathfrak{B}_{\text{MOD}}^p}$ and $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}$.

Lemma 2. *For any prime $p > 5$ with $p \not\equiv \pm 1 \pmod{10}$, the group $M(\mathfrak{B}_{\text{MOD}}^p)$ is unsolvable, but all of its proper subgroups are solvable.*

Proof. One can check that the order of the permutation $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}$ is 2, that of $\delta_a^{\mathfrak{B}_{\text{MOD}}^p}$ is p , while the order of the inverse of $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}$ is the same as the order of $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}$, which is 3. So $M(\mathfrak{B}_{\text{MOD}}^p)$ is unsolvable, for any prime p , by the Kaplan–Levy criterion. To prove that all proper subgroups of $M(\mathfrak{B}_{\text{MOD}}^p)$ are solvable, we show that $M(\mathfrak{B}_{\text{MOD}}^p)$ is a subgroup of the *projective special linear group* $\text{PSL}_2(p)$. If p is a prime with $p > 5$ and $p \not\equiv \pm 1 \pmod{10}$, then all proper subgroups of $\text{PSL}_2(p)$ are solvable; see, e.g., [17, Theorem 2.1]. (So $M(\mathfrak{B}_{\text{MOD}}^p)$ is in fact isomorphic to the unsolvable group $\text{PSL}_2(p)$.) Consider the set $P = \{0, 1, \dots, p - 1, \infty\}$ of all points of the projective line over the field \mathbb{F}_p . By identifying s_i with i for $i < p$,

and s_p with ∞ , we may regard the elements of $M(\mathfrak{B}_{\text{MOD}}^p)$ as $P \rightarrow P$ functions. The group $\text{PSL}_2(p)$ consists of all $P \rightarrow P$ functions of the form $i \mapsto \frac{w \cdot i + x}{y \cdot i + z}$, where $w \cdot z - x \cdot y = 1$, with the field arithmetic of \mathbb{F}_p extended by $i + \infty = \infty$ for any $i \in P$, $0 \cdot \infty = 1$ and $i \cdot \infty = \infty$ for $i \neq 0$. One can check that the two generators of $M(\mathfrak{B}_{\text{MOD}}^p)$ are in $\text{PSL}_2(p)$: take $w = 1, x = 1, y = 0, z = 1$ for $\delta_a^{\mathfrak{B}_{\text{MOD}}^p}$, and $w = 0, x = 1, y = p - 1, z = 0$ for $\delta_{\natural}^{\mathfrak{B}_{\text{MOD}}^p}$.

Finally, we define three automata $\mathfrak{A}_{<}, \mathfrak{A}_{\equiv}, \mathfrak{A}_{\text{MOD}}$ over the same tape alphabet $\Sigma_+ = \Sigma \cup \{a_1, a_2, \natural\}$, where a_1, a_2 are fresh symbols. We take, respectively, $\mathfrak{B}_{<}^p, \mathfrak{B}_{\equiv}^p, \mathfrak{B}_{\text{MOD}}^p$ and replace each transition $s_i \xrightarrow{a} s_j$ in them by a fresh copy of \mathfrak{A}_i , for $i \leq p$, as shown in the picture below.



We make $\mathfrak{A}_{<}, \mathfrak{A}_{\equiv}, \mathfrak{A}_{\text{MOD}}$ deterministic by adding a trash state tr looping on itself with every $y \in \Sigma_+$, and adding the missing transitions leading to tr . It follows that $\mathfrak{A}_{<}, \mathfrak{A}_{\equiv}$, and $\mathfrak{A}_{\text{MOD}}$ are minimal DFAs of size polynomial in $N, |M|$.

Lemma 3. (i) $L(\mathfrak{A}_{<})$ is FO($<$)-definable iff $\bigcap_{i=0}^p L(\mathfrak{A}_i) = \emptyset$.
 (ii) $L(\mathfrak{A}_{\equiv})$ is FO($<, \equiv$)-definable iff $\bigcap_{i=0}^p L(\mathfrak{A}_i) = \emptyset$.
 (iii) $L(\mathfrak{A}_{\text{MOD}})$ is FO($<, \text{MOD}$)-definable iff $\bigcap_{i=0}^p L(\mathfrak{A}_i) = \emptyset$.

Proof. As $\mathfrak{A}_{<}, \mathfrak{A}_{\equiv}, \mathfrak{A}_{\text{MOD}}$ are minimal, we can replace \sim by $=$ in the conditions of Theorem 1. For the (\Rightarrow) directions, given some $w \in \bigcap_{i=0}^p L(\mathfrak{A}_i)$, in each case we show how to satisfy the corresponding condition of Theorem 1: (i) take $u = a_1 w a_2, q = s_0$, and $k = p$; (ii) take $u = a_1 w a_2, v = \natural^{|u|}, q = s_0$, and $k = p$; (iii) take $u = \natural, v = a_1 w a_2, q = s_0, k = p$ and $l = 3$.

(\Leftarrow) We show that the corresponding condition of Theorem 1 implies non-emptiness of $\bigcap_{i=0}^p L(\mathfrak{A}_i)$. To this end, we define a $\Sigma_+^* \rightarrow \{a, \natural\}^*$ homomorphism by taking $h(\natural) = \natural, h(a_1) = a$, and $h(b) = \varepsilon$ for all other $b \in \Sigma_+$.

(i) and (ii): Let $\circ \in \{<, \equiv\}$ and suppose q is a state in \mathfrak{A}_0^p and $u' \in \Sigma_+^*$ such that $q \neq \delta_{u'}^{\mathfrak{A}_0^p}(q)$ and $q = \delta_{(u')^k}^{\mathfrak{A}_0^p}(q)$ for some k . Let $S = \{s_0, s_1, \dots, s_{p-1}\}$. We claim that there exist $s \in S$ and $u \in \Sigma_+^*$ such that

$$s \neq \delta_u^{\mathfrak{A}_0^p}(s), \quad (13)$$

$$\delta_x^{\mathfrak{A}_0^p}(s) \in S, \quad \text{for every } x \in \{u\}^*. \quad (14)$$

Indeed, observe that none of the states along the cyclic $q \xrightarrow{(u')^k} q$ path Π in \mathfrak{A}_0^p is tr . So there is some state along Π that is in S , as otherwise one of the \mathfrak{A}_i would contain a nontrivial cycle. Therefore, u' must be of the form $w \natural^n a_1 w'$ for some $w \in \Sigma^*, n < \omega$ and $w' \in \Sigma_+^*$. It is easy to see that $s = \delta_{(u')^{k-1} w}^{\mathfrak{A}_0^p}(q)$ and $u = \natural^n a_1 w' w$ is as required in (13) and (14).

As $M(\mathfrak{B}_0^p)$ is a finite group, the set $\{\delta_{h(x)}^{\mathfrak{B}_0^p} \mid x \in \{u\}^*\}$ forms a subgroup \mathfrak{G} in it (the subgroup generated by $\delta_{h(u)}^{\mathfrak{B}_0^p}$). We show that \mathfrak{G} is nontrivial by finding a nontrivial homomorphic image of it. To this end, (14) implies that, for every $x \in \{u\}^*$, the restriction $\delta_x^{\mathfrak{A}_0^p} \upharpoonright_{S'}$ of $\delta_x^{\mathfrak{A}_0^p}$ to the set $S' = \{\delta_y^{\mathfrak{A}_0^p}(s) \mid y \in \{u\}^*\}$ is an $S' \rightarrow S'$ function and $\delta_x^{\mathfrak{A}_0^p} \upharpoonright_{S'} = \delta_{h(x)}^{\mathfrak{B}_0^p} \upharpoonright_{S'}$. As $M(\mathfrak{B}_0^p)$ is a group of permutations on a set containing S' , $\delta_{h(x)}^{\mathfrak{B}_0^p} \upharpoonright_{S'}$ is a permutation of S' , for every $x \in \{u\}^*$. Thus, $\{\delta_{h(x)}^{\mathfrak{B}_0^p} \upharpoonright_{S'} \mid x \in \{u\}^*\}$ is a homomorphic image of \mathfrak{G} that is nontrivial by (13).

As \mathfrak{G} is a nontrivial subgroup of the cyclic group $M(\mathfrak{B}_0^p)$ of order p and p is a prime, $\mathfrak{G} = M(\mathfrak{B}_0^p)$. Then there is $x \in \{u\}^*$ with $\delta_{h(x)}^{\mathfrak{B}_0^p} = \delta_a^{\mathfrak{B}_0^p}$ (a permutation containing the p -cycle $(s_0 s_1 \dots s_{p-1})$ ‘around’ all elements of S), and so $S' = S$ and $x = \natural^n a_1 w a_2 w'$ for some $n < \omega$, $w \in \Sigma^*$, and $w' \in \Sigma_+^*$. As $n = 0$ when $\circ = <$ and $\delta_{\natural^n}^{\mathfrak{A}_0^p}(s)$ for every $s \in S$, $S' = S$ implies that $w \in \bigcap_{i=0}^{p-1} \mathbf{L}(\mathfrak{A}_i) = \bigcap_{i=0}^p \mathbf{L}(\mathfrak{A}_i)$.

(iii) Suppose q is a state in $\mathfrak{A}_{\text{MOD}}^p$ and $u', v' \in \Sigma_+^*$ such that $q \neq \delta_{u'}^{\mathfrak{A}_{\text{MOD}}^p}(q)$, $q \neq \delta_{v'}^{\mathfrak{A}_{\text{MOD}}^p}(q)$, $q \neq \delta_{u'v'}^{\mathfrak{A}_{\text{MOD}}^p}(q)$, and $\delta_x^{\mathfrak{A}_{\text{MOD}}^p}(q) = \delta_{x(u')^2}^{\mathfrak{A}_{\text{MOD}}^p}(q) = \delta_{x(v')^k}^{\mathfrak{A}_{\text{MOD}}^p}(q) = \delta_{x(u'v')^l}^{\mathfrak{A}_{\text{MOD}}^p}(q)$ for some odd prime k and number l that is coprime to both 2 and k . Take $S = \{s_0, s_1, \dots, s_p\}$. We claim that there exist $s \in S$ and $u, v \in \Sigma_+^*$ such that

$$s \neq \delta_u^{\mathfrak{A}_{\text{MOD}}^p}(s), \quad s \neq \delta_v^{\mathfrak{A}_{\text{MOD}}^p}(s), \quad s \neq \delta_{uv}^{\mathfrak{A}_{\text{MOD}}^p}(s), \tag{15}$$

$$\delta_x^{\mathfrak{A}_{\text{MOD}}^p}(s) \in S, \quad \text{for every } x \in \{u, v\}^*, \tag{16}$$

$$\delta_x^{\mathfrak{A}_{\text{MOD}}^p}(s) = \delta_{xu^2}^{\mathfrak{A}_{\text{MOD}}^p}(s) = \delta_{xv^k}^{\mathfrak{A}_{\text{MOD}}^p}(s) = \delta_{x(uv)^l}^{\mathfrak{A}_{\text{MOD}}^p}(s), \quad \text{for every } x \in \{u, v\}^*. \tag{17}$$

Indeed, by an argument similar to the one in the proof of (i) and (ii) above, we must have $u' = w_u \natural^n a_1 w'_u$ and $v' = w_v \natural^m a_1 w'_v$ for some $w_u, w_v \in \Sigma^*$, $n, m < \omega$ and $w'_u, w'_v \in \Sigma_+^*$. For every $x \in \{u, v\}^*$, as both $\delta_{xu^2}^{\mathfrak{A}_{\text{MOD}}^p}(q)$ and $\delta_{xw_v}^{\mathfrak{A}_{\text{MOD}}^p}(q)$ are in S , they must be the same state. Using this it is not hard to see that $s = \delta_{u'w_u}^{\mathfrak{A}_{\text{MOD}}^p}(q)$, $u = \natural^n a_1 w'_u w_u$ and $v = \natural^m a_1 w'_v w_v$ are as required in (15)–(17).

As $M(\mathfrak{B}_{\text{MOD}}^p)$ is a finite group, the set $\{\delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p} \mid x \in \{u, v\}^*\}$ forms a subgroup \mathfrak{G} in it (the subgroup generated by $\delta_{h(u)}^{\mathfrak{B}_{\text{MOD}}^p}$ and $\delta_{h(v)}^{\mathfrak{B}_{\text{MOD}}^p}$). We show that \mathfrak{G} is unsolvable by finding an unsolvable homomorphic image of it. To this end, we let $S' = \{\delta_y^{\mathfrak{A}_{\text{MOD}}^p}(s) \mid y \in \{u, v\}^*\}$. Then (16) implies that $S' \subseteq S$ and

$$\delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p}(s') = \delta_x^{\mathfrak{A}_{\text{MOD}}^p}(s') \in S', \quad \text{for all } s' \in S \text{ and } x \in \{u, v\}^*, \tag{18}$$

and so the restriction $\delta_x^{\mathfrak{A}_{\text{MOD}}^p} \upharpoonright_{S'}$ of $\delta_x^{\mathfrak{A}_{\text{MOD}}^p}$ to S' is an $S' \rightarrow S'$ function and $\delta_x^{\mathfrak{A}_{\text{MOD}}^p} \upharpoonright_{S'} = \delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'}$. As $M(\mathfrak{B}_{\text{MOD}}^p)$ is a group of permutations on a set containing S' , $\delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'}$ is a permutation of S' , for any $x \in \{u, v\}^*$. So $\{\delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'} \mid x \in \{u, v\}^*\}$ is a homomorphic image of \mathfrak{G} that is unsolvable by the Kaplan–Levy criterion: By (15), (17), and 2 and k being primes, the order of the permutation $\delta_{h(u)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'}$ is 2, the order of $\delta_{h(v)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'}$ is k , and the order of $\delta_{h(uv)}^{\mathfrak{B}_{\text{MOD}}^p} \upharpoonright_{S'}$ (which is the

same as the order of its inverse) is a > 1 divisor of l , and so coprime to both 2 and k .

As \mathfrak{G} is an unsolvable subgroup of $M(\mathfrak{B}_{\text{MOD}}^p)$, it follows from Lemma 2 that $\mathfrak{G} = M(\mathfrak{B}_{\text{MOD}}^p)$, and so $\{u, v\}^* \not\subseteq \mathfrak{h}^*$. We claim that $S' = S$ also follows. Indeed, let $x \in \{u, v\}^*$ be such that $\delta_{h(x)}^{\mathfrak{B}_{\text{MOD}}^p} = \delta_a^{\mathfrak{B}_{\text{MOD}}^p}$. As $|S'| \geq 2$ by (15), $s \in \{s_0, \dots, s_{p-1}\}$ must hold, and so $\{s_0, \dots, s_{p-1}\} \subseteq S'$ follows by (18). As there is $y \in \{u, v\}^*$ with $\delta_{h(y)}^{\mathfrak{B}_{\text{MOD}}^p} = \delta_{\mathfrak{h}}^{\mathfrak{B}_{\text{MOD}}^p}$, $s_p \in S'$ also follows by (18). Finally, as $\{u, v\}^* \not\subseteq \mathfrak{h}^*$, there is $x \in \{u, v\}^*$ of the form $\mathfrak{h}^n a_1 w a_2 w'$, for some $n < \omega$, $w \in \Sigma$ and $w' \in \Sigma_+^*$. As $S' = S$, $\delta_x^{\mathfrak{B}_{\text{MOD}}^p}(s_i) \in S$ for every $i \leq p$, and so $w \in \bigcap_{i=0}^p L(\mathfrak{A}_i)$.

Theorem 2 clearly follows from Lemmas 1 and 3.

5 Deciding \mathcal{L} -definability of 2NFAs in PSPACE

Using the criterion Theorem 1 (i), Stern [25] showed that deciding whether the language of any given DFA is FO($<$)-definable can be done in PSPACE. In this section, we also use the criteria of Theorem 1 to provide PSPACE-algorithms deciding whether the language of any given 2NFA is \mathcal{L} -definable, whenever $\mathcal{L} \in \{\text{FO}(<), \text{FO}(<, \equiv), \text{FO}(<, \text{MOD})\}$. Let $\mathfrak{A} = (Q, \Sigma, \delta, Q_0, F)$ be a 2NFA. Following [9], we first construct a(n exponential size) DFA \mathfrak{A}' such that $L(\mathfrak{A}) = L(\mathfrak{A}')$. To this end, for any $w \in \Sigma^+$, we introduce four binary relations $\mathfrak{b}_{lr}(w)$, $\mathfrak{b}_{rl}(w)$, $\mathfrak{b}_{rr}(w)$, and $\mathfrak{b}_{ll}(w)$ on Q describing the *left-to-right*, *right-to-left*, *right-to-right*, and *left-to-left behaviour* of \mathfrak{A} on w . Namely,

- $(q, q') \in \mathfrak{b}_{lr}(w)$ if there is a run of \mathfrak{A} on w from $(q, 0)$ to $(q', |w|)$;
- $(q, q') \in \mathfrak{b}_{rr}(w)$ if there is a run of \mathfrak{A} on w from $(q, |w| - 1)$ to $(q', |w|)$;
- $(q, q') \in \mathfrak{b}_{rl}(w)$ if, for some $a \in \Sigma$, there is a run on aw from $(q, |aw| - 1)$ to $(q', 0)$ such that no $(q'', 0)$ occurs in it before $(q', 0)$;
- $(q, q') \in \mathfrak{b}_{ll}(w)$ if, for some $a \in \Sigma$, there is a run on aw from $(q, 1)$ to $(q', 0)$ such that no $(q'', 0)$ occurs in it before $(q', 0)$.

For $w = \varepsilon$ (the empty word), we define the $\mathfrak{b}_{ij}(w)$ as the identity relation on Q . Let $\mathfrak{b} = (\mathfrak{b}_{lr}, \mathfrak{b}_{rl}, \mathfrak{b}_{rr}, \mathfrak{b}_{ll})$, where the \mathfrak{b}_{ij} are the behaviours of \mathfrak{A} on some $w \in \Sigma^*$, in which case we can also write $\mathfrak{b}(w)$, and let $\mathfrak{b}' = \mathfrak{b}(w')$, for some $w' \in \Sigma^*$. We define the composition $\mathfrak{b} \cdot \mathfrak{b}' = \mathfrak{b}''$ with components \mathfrak{b}''_{ij} as follows. Let X and Y be the transitive closure of $\mathfrak{b}'_{ll} \circ \mathfrak{b}_{rr}$ and $\mathfrak{b}_{rr} \circ \mathfrak{b}'_{ll}$, respectively. Then we set:

$$\begin{aligned} \mathfrak{b}''_{lr} &= \mathfrak{b}_{lr} \circ \mathfrak{b}'_{lr} \cup \mathfrak{b}_{lr} \circ X \circ \mathfrak{b}'_{lr}, & \mathfrak{b}''_{rl} &= \mathfrak{b}'_{rl} \circ \mathfrak{b}_{rl} \cup \mathfrak{b}'_{rl} \circ Y \circ \mathfrak{b}_{rl}, \\ \mathfrak{b}''_{rr} &= \mathfrak{b}'_{rr} \cup \mathfrak{b}'_{rl} \circ Y \circ \mathfrak{b}_{rr} \circ \mathfrak{b}'_{lr}, & \mathfrak{b}''_{ll} &= \mathfrak{b}_{ll} \cup \mathfrak{b}_{lr} \circ X \circ \mathfrak{b}'_{ll} \circ \mathfrak{b}_{rl}. \end{aligned}$$

One can check that $\mathfrak{b}'' = \mathfrak{b}(ww')$. Define a DFA $\mathfrak{A}'' = (Q', \Sigma, \delta', q'_0, F')$ by taking

$$\begin{aligned} Q' &= \{(B_{lr}, B_{rr}) \mid B_{lr} \subseteq Q_0 \times Q, B_{rr} \subseteq Q \times Q\}, & q'_0 &= (\{(q, q) \mid q \in Q_0\}, \emptyset), \\ F' &= \{(B_{lr}, B_{rr}) \mid (q_0, q) \in B_{lr}, \text{ for some } q_0 \in Q_0 \text{ and } q \in F\}, \\ \delta'_a((B_{lr}, B_{rr})) &= (B'_{lr}, B'_{rr}), \text{ with } B'_{lr} = B_{lr} \circ X(a) \circ \mathfrak{b}_{lr}(a), \\ & & B'_{rr} &= B_{rr} \cup \mathfrak{b}_{rl}(a) \circ Y(a) \circ \mathfrak{b}_{lr}(a), \end{aligned}$$

where $X(a)$ and $Y(a)$ are the reflexive and transitive closures of $\mathbf{b}_{ll}(a) \circ B_{rr}$ and $B_{rr} \circ \mathbf{b}_{ll}(a)$, respectively. It is not hard to see that, for any $w \in \Sigma^*$,

$$\begin{aligned} \delta'_w((B_{lr}, B_{rr})) = (B'_{lr}, B'_{rr}) \text{ iff } B'_{lr} = B_{lr} \circ X(w) \circ \mathbf{b}_{lr}(w) \text{ and} \\ B'_{rr} = B_{rr} \cup \mathbf{b}_{rl}(w) \circ Y(w) \circ \mathbf{b}_{lr}(w), \end{aligned} \quad (19)$$

where $X(w)$ and $Y(w)$ are the reflexive and transitive closures of $\mathbf{b}_{ll}(w) \circ B_{rr}$ and $B_{rr} \circ \mathbf{b}_{ll}(w)$, respectively. Also, one can show in a way similar to [24, 29] that

$$\mathbf{L}(\mathfrak{A}) = \mathbf{L}(\mathfrak{A}'). \quad (20)$$

Next, we show that, even if the size of \mathfrak{A}' is exponential in \mathfrak{A} , we can still use Theorem 1 to decide \mathcal{L} -definability of $\mathbf{L}(\mathfrak{A})$ in PSPACE:

Theorem 3. *For $\mathcal{L} \in \{\text{FO}(<), \text{FO}(<, \equiv), \text{FO}(<, \text{MOD})\}$, deciding \mathcal{L} -definability of $\mathbf{L}(\mathfrak{A})$, for any 2NFA \mathfrak{A} , is in PSPACE.*

Proof. Let \mathfrak{A}' be the DFA defined above for the given 2NFA \mathfrak{A} . By Theorem 1 (i) and (20), $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<)$ -definable iff there exist a word $u \in \Sigma^*$, a reachable state $q \in Q'$, and a number $k \leq |Q'|$ such that $q \not\sim \delta'_u(q)$ and $q = \delta'_{u^k}(q)$. We guess the required k in binary, q and a quadruple $\mathbf{b}(u)$ of binary relations on Q . Clearly, they all can be stored in polynomial space in $|\mathfrak{A}|$. To check that our guesses are correct, we first check that $\mathbf{b}(u)$ indeed corresponds to some $u \in \Sigma^*$. This is done by guessing a sequence $\mathbf{b}_0, \dots, \mathbf{b}_n$ of distinct quadruples of binary relations on Q such that $\mathbf{b}_0 = \mathbf{b}(u_0)$ and $\mathbf{b}_{i+1} = \mathbf{b}_i \cdot \mathbf{b}(u_{i+1})$, for some $u_0, \dots, u_n \in \Sigma$. (Any sequence with a subsequence starting after \mathbf{b}_i and ending with \mathbf{b}_{i+m} , for some i and m such that $\mathbf{b}_i = \mathbf{b}_{i+m}$, is equivalent, in the context of this proof, to the sequence with such a subsequence removed.) Thus, we can assume that $n \leq 2^{O(|Q|)}$, and so n can be guessed in binary and stored in PSPACE. So, the stage of our algorithm checking that $\mathbf{b}(u)$ corresponds to some $u \in \Sigma^*$ makes n iterations and continues to the next stage if $\mathbf{b}_n = \mathbf{b}(u)$ or terminates with an answer **no** otherwise. Now, using $\mathbf{b}(u)$, we compute $\mathbf{b}(u^k)$ by means of a sequence $\mathbf{b}_0, \dots, \mathbf{b}_k$, where $\mathbf{b}_0 = \mathbf{b}(u)$ and $\mathbf{b}_{i+1} = \mathbf{b}_i \cdot \mathbf{b}(u)$. With $\mathbf{b}(u)$ ($\mathbf{b}(u^k)$), we compute $\delta'_u(q)$ (respectively, $\delta'_{u^k}(q)$) in PSPACE using (19). If $\delta'_{u^k}(q) \neq q$, the algorithm terminates with an answer **no**. Otherwise, in the final stage of the algorithm, we check that $\delta'_u(q) \not\sim q$. This is done by guessing $v \in \Sigma^*$ such that $\delta'_v(q) = q_1$, $\delta'_v(\delta'_u(q)) = q_2$, and $q_1 \in F'$ iff $q_2 \notin F'$. We guess such a v (if exists) in the form of $\mathbf{b}(v)$ using an algorithm analogous to that for guessing u above.

By Theorem 1 (ii) and (20), $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<, \equiv)$ -definable iff there exist words $u, v \in \Sigma^*$, a reachable state $q \in Q'$, and a number $k \leq |Q'|$ such that $q \not\sim \delta'_u(q)$, $q = \delta'_{u^k}(q)$, $|v| = |u|$, and $\delta'_{u^i}(q) = \delta'_{v^i}(q)$, for all $i < k$. We outline how to modify the algorithm for $\text{FO}(<)$ above to check $\text{FO}(<, \equiv)$ -definability. First, we need to guess and check v in the form of $\mathbf{b}(v)$ in parallel with guessing and checking u in the form of $\mathbf{b}(u)$, making sure that $|v| = |u|$. For that, we guess a sequence of distinct pairs $(\mathbf{b}_0, \mathbf{b}'_0), \dots, (\mathbf{b}_n, \mathbf{b}'_n)$ such that the \mathbf{b}_i are as above, $\mathbf{b}'_0 = \mathbf{b}(v_0)$ and $\mathbf{b}'_{i+1} = \mathbf{b}'_i \cdot \mathbf{b}(v_{i+1})$, for some $v_0, \dots, v_n \in \Sigma$. (Any such sequence with a subsequence starting after $(\mathbf{b}_i, \mathbf{b}'_i)$ and ending with $(\mathbf{b}_{i+m}, \mathbf{b}'_{i+m})$, for some

i and m such that $(\mathbf{b}_i, \mathbf{b}'_i) = (\mathbf{b}_{i+m}, \mathbf{b}'_{i+m})$, is equivalent to the sequence with that subsequence removed.) So $n \leq 2^{O(|Q|)}$. For each $i < k$, we can then compute $\delta'_{u^i}(q)$ and $\delta'_{u^i v}(q)$, using (19), and check whether they are equal.

Finally, by Theorem 1 (iii) and (20), $\mathbf{L}(\mathfrak{A})$ is not $\text{FO}(<, \text{MOD})$ -definable iff there exist $u, v \in \Sigma^*$, a reachable state $q \in Q'$ and $k, l \leq |Q'|$ such that k is an odd prime, $l > 1$ and coprime to both 2 and k , $q \not\sim \delta'_u(q)$, $q \not\sim \delta'_v(q)$, $q \not\sim \delta'_{uv}(q)$, and $\delta'_x(q) \sim \delta'_{xu^2}(q) \sim \delta'_{xv^k}(q) \sim \delta'_{x(uv)^l}(q)$, for all $x \in \{u, v\}^*$. We start by guessing $u, v \in \Sigma^*$ in the form of $\mathbf{b}(u)$ and $\mathbf{b}(v)$, respectively. Also, we guess k and l in binary and check that k is an odd prime and l is coprime to both 2 and k . By (19), δ'_x is determined by $\mathbf{b}(x)$, for any $x \in \{u, v\}^*$. Thus, we can proceed as follows to verify that u, v, k and l are as required. We perform the following steps, for *each* quadruple \mathbf{b} of binary relations on Q . First, we check whether $\mathbf{b} = \mathbf{b}(x)$, for some $x \in \{u, v\}^*$ (we discuss the algorithm for this below). If this is not the case, we construct the *next* quadruple \mathbf{b}' and process it as this \mathbf{b} . If it is the case, we compute all the states $\delta'_x(q)$, $\delta'_{xu^2}(q)$, $\delta'_{xv^k}(q)$, $\delta'_{x(uv)^l}(q)$, $\delta'_u(q)$, $\delta'_v(q)$, $\delta'_{uv}(q)$, and check their required (non)equivalences w.r.t. \sim , using the same method as for checking $\delta'_u(q) \not\sim q$ above. If they do not hold as required, our algorithm terminates with an answer *no*. Otherwise, we construct the *next* quadruple \mathbf{b}' and process it as this \mathbf{b} . When all possible quadruples \mathbf{b} of binary relations of Q have been processed, the algorithm terminates with an answer *yes*.

Now, to check that a given quadruple \mathbf{b} is equal to $\mathbf{b}(x)$, for some $x \in \{u, v\}^*$, we simply guess a sequence $\mathbf{b}_0, \dots, \mathbf{b}_n$ of quadruples of binary relations on Q such that $\mathbf{b}_0 = \mathbf{b}(w_0)$, $\mathbf{b}_n = \mathbf{b}$ and $\mathbf{b}_{i+1} = \mathbf{b}_i \cdot \mathbf{b}(w_{i+1})$, where $w_i \in \{u, v\}^*$. It follows from the argument above that it is enough to consider $n \leq 2^{O(|Q|)}$.

6 Further Research

The results obtained in this paper have been used for deciding the rewritability type of ontology-mediated queries (OMQs) given in linear temporal logic *LTL* [22]. As mentioned in the introduction, *LTL* OMQs can be simulated by automata. In the worst case, the automata are of exponential size, and deciding FO-rewritability of some OMQs may become EXPSpace-complete. On the other hand, there are natural and practically important fragments of *LTL* with automata of special forms whose FO-rewritability can be decided in PSPACE, Π_2^P or CONP. However, it remains to be seen whether the corresponding algorithms, even in the simplest case of $\text{FO}(<)$ -definability, are efficient enough for applications in temporal OBDA. Note that the problems considered in this paper are also relevant to the optimisation problem for recursive SQL queries.

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