Deficiencies of Holomorphic Curves for Linear Systems in Projective Manifolds

Yoshihiro Aihara

Abstract In this note we shall give theorems on deficiencies of holomorphic curves $f : X \to M$, where X is a finite sheeted analytic covering space over $\mathbf{C}$ and M is a projective manifold. We first give an inequality of second main theorem type and a defect relation for f that generalizes the results in Aihara (Tohoku Math J 58:287–315, 2012). By making use of this defect relation, we give theorems on the structure of the set of deficient divisors of f. We also discuss methods for constructing holomorphic curves with deficient divisors.

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1 Introduction

The aim of this note is twofold. The first is to give generalizaition of the structure theorem for the set of deficient divisors in [1]. Let M be a projective algebraic manifold and $L \to M$ an ample line bundle. We denote by $|L|$ the complete linear system of L and let $\Lambda \subseteq |L|$ be a linear system. In the previous paper [1], we studied properties of the deficiencies of a holomorphic curve $f : \mathbf{C} \to M$ as functions on linear systems and gave the structure theorem for the set

$$\mathcal{D}_f = \{D \in \Lambda\, ;\ \delta_f(D) > \delta_f(B_\Lambda)\}$$

of deficient divisors. For definitions, see Sect. 2. In the proof of the structure theorem for $\mathcal{D}_f$, we used an inequality of the second main theorem type and a defect relation for f and Λ. In this note, we will generalize these to the case where holomorphic

Y. Aihara (✉)
Fukushima University, Fukushima, Japan
e-mail: aihara@educ.fukushima-u.ac.jp

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curves defined on finite sheeted analytic covering spaces over $\mathbf{C}$. The second is to give methods for constructing holomorphic curves with deficient divisors. Details will be published elsewhere.

2 Preliminaries

We recall some known facts on Nevanlinna theory for holomorphic curves. For details, see [5] and [6].

Let $\varpi : X \to \mathbf{C}$ be a finite analytic (ramified) covering space over $\mathbf{C}$ and let s_0 be its sheet number, that is, X is a one dimensional complex analytic space and $\varpi : X \to \mathbf{C}$ is a proper surjective holomorphic mapping with discrete fibers. Let z be the natural coordinate in $\mathbf{C}$, and set

$$X(r) = \varpi^{-1}(\{z \in \mathbf{C};\ |z| < r\}) \quad \text{and} \quad C(r) = \varpi^{-1}(\{z \in \mathbf{C};\ |z| = r\}).$$

For a (1,1)-current φ of order zero on X we set

$$N(r, \varphi) = \frac{1}{s_0}\int_1^r \langle \varphi, \chi_{X(t)} \rangle \frac{dt}{t},$$

where $\chi_{X(r)}$ denotes the characteristic function of $X(r)$.

Let M be a compact complex manifold and let $L \to M$ be a line bundle over M. We denote by $\Gamma(M, L)$ the space of all holomorphic sections of $L \to M$ and by $|L| = \mathbf{P}(\Gamma(M, L))$ the complete linear system of L. Denote by $\|\cdot\|$ a hermitian fiber metric in L and by ω its Chern form. Let $f : X \to M$ be a holomorphic curve. We set

$$T_f(r, L) = N(r, f^*\omega)$$

and call it the characteristic function of f with respect to L. If

$$\liminf_{r \to +\infty} \frac{T_f(r, L)}{\log r} = +\infty,$$

then f is said to be *transcendental*. We define the order ρ_f of $f : X \to M$ by

$$\rho_f = \limsup_{r \to +\infty} \frac{\log T_f(r, L)}{\log r}.$$

We notice that the definition of ρ_f is independent of a choice of positive line bundles $L \to M$. Let $D = (\sigma) \in |L|$ with $||\sigma|| \leq 1$ on M. Assume that $f(X)$ is not contained in Supp D. We define the proximity function of D by

$$m_f(r,\ D) = \frac{1}{s_0} \int_{C(r)} \log \left(\frac{1}{||\sigma(f(z))||} \right) \frac{d\theta}{2\pi}.$$

Then we have the following first main theorem for holomorphic curves $X \to M$.

Theorem 2.1 (First Main Theorem) *Let $L \to M$ be a line bundle over M and $f : X \to M$ a non-constant holomorphic curve. Then*

$$T_f(r,\ L) = N(r,\ f^* D) + m_f(r,\ D) + O(1)$$

for $D \in |L|$ with $f(X) \not\subseteq$ Supp D, where $O(1)$ stands for a bounded term as $r \to +\infty$.

Let f and D be as above. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r,\ D)}{T_f(r,\ L)}.$$

It is clear that $0 \leq \delta_f(D) \leq 1$. Then we have a defect function δ_f defined on $|L|$. If $\delta_f(D) > 0$, then D is called *a deficient divisor in the sense of Nevanlinna.*

Next, we recall some basic facts in value distribution theory for coherent ideal sheaves (cf. [6, Chapter 2]). Let $f : X \to M$ be a holomorphic curve and $\mathcal{I}$ a coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ of M. Let $\mathcal{U} = \{U_j\}$ be a finite open covering of M with a partition of unity $\{\eta_j\}$ subordinate to $\mathcal{U}$. We can assume that there exist finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$ such that every stalk $\mathcal{I}_p$ over $p \in U_j$ is generated by germs $(\sigma_{j1})_p, \ldots, (\sigma_{jl_j})_p$. Set

$$d_{\mathcal{I}}(p) = \left(\sum_j \eta_j(p) \sum_{k=1}^{l_j} \left| \sigma_{jk}(p) \right|^2 \right)^{1/2}.$$

We may assume that $d_{\mathcal{I}}(p) \leq 1$ for all $p \in M$. Set

$$\phi_{\mathcal{I}}(p) = -\log d_{\mathcal{I}}(p)$$

and call it the proximity potential for $\mathcal{I}$. It is easy to verify that $\phi_{\mathcal{I}}$ is well-defined up to addition by a bounded continuous function on M. We now define the proximity function $m_f(r, \mathcal{I})$ of f for $\mathcal{I}$, or equivalently, for the complex analytic subspace (may be non-reduced)

$$Y = (\text{Supp}\,(\mathcal{O}_M/\mathcal{I}),\ \mathcal{O}_M/\mathcal{I})$$

by

$$m_f(r, \mathcal{I}) = \frac{1}{s_0} \int_{C(r)} \phi_{\mathcal{I}}(f(z)) \frac{d\theta}{2\pi},$$

provided that $f(X)$ is not contained in $\operatorname{Supp} Y$. For $z_0 \in f^{-1}(\operatorname{Supp} Y)$, we can choose an open neighborhood U of z_0 and a positive integer ν such that

$$f^*\mathcal{I} = ((z - z_0)^\nu) \quad \text{on } U.$$

Then we see

$$\log d_{\mathcal{I}}(f(z)) = \nu \log |z - z_0| + h_U(z) \quad \text{for } z \in U,$$

where h_U is a C^∞-function on U. Thus we have the counting function $N(r, f^*\mathcal{I})$ as above. Moreover, we set

$$\omega_{\mathcal{I},f} = -dd^c h_U \quad \text{on } U,$$

where $d^c = (\sqrt{-1}/4\pi)(\overline{\partial} - \partial)$. We obtain a well-defined smooth $(1, 1)$-form $\omega_{\mathcal{I},f}$ on X. Define the characteristic function $T_f(r, \mathcal{I})$ of f for $\mathcal{I}$ by

$$T_f(r, \mathcal{I}) = \frac{1}{s_0} \int_1^r \frac{dt}{t} \int_{X(t)} \omega_{\mathcal{I},f}.$$

We have the first main theorem in value distribution theory for coherent ideal sheaves:

Theorem 2.2 (First Main Theorem) *Let $f : X \to M$ and $\mathcal{I}$ be as above. Then*

$$T_f(r, \mathcal{I}) = N(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) + O(1).$$

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a subspace with $\dim W \geq 2$. Set $\Lambda = \mathbf{P}(W)$. The base locus $\operatorname{Bs} \Lambda$ of Λ is defined by

$$\operatorname{Bs} \Lambda = \bigcap_{D \in \Lambda} \operatorname{Supp} D.$$

We define a coherent ideal sheaf $\mathcal{I}_0$ in the following way. For each $p \in M$, the stalk $\mathcal{I}_{0,p}$ is generated by all germs $(\sigma)_p$ for $\sigma \in W$. Then $\mathcal{I}_0$ defines the base locus of Λ as a complex analytic subspace B_Λ, that is,

$$B_\Lambda = (\operatorname{Supp} (\mathcal{O}_M/\mathcal{I}_0), \mathcal{O}_M/\mathcal{I}_0).$$

Hence $\mathrm{Bs}\,\Lambda = \mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_0)$. We define the deficiency of B_Λ for f by

$$\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r,\ \mathcal{I}_0)}{T_f(r,\ L)}.$$

Set

$$\mathcal{D}_f = \{D \in \Lambda;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

We call $\mathcal{D}_f$ *the set of deficient divisors* in Λ.

By making use of the generalized Crofton's formula due to R. Kobayashi ([6, Theorem 2.4.12]), we have the following proposition ([1, Proposition 4.1]).

Proposition 2.3 *The set $\mathcal{D}_f$ is a null set in the sense of the Lebesgue measure on Λ. In particular $\delta_f(D) = \delta_f(B_\Lambda)$ for almost all $D \in \Lambda$.*

This proposition plays an important role in what follows.

3 Inequality of the Second Main Theorem Type

We will give an inequality of the second main theorem type for a holomorphic curve $f : X \to M$ that generalizes Theorem 3.1 in [1]. For simplicity, we assume that f is of finite type. Let $W \subseteq \Gamma(M,\ L)$ be a linear subspace with $\dim W = l_0 + 1 \geq 2$ and set $\Lambda = \mathbf{P}(W)$. We call Λ a linear system included in $|L|$. Let $D_1, \ldots, D_q$ be divisors in Λ such that $D_j = (\sigma_j)$ for $\sigma_j \in W$. We first give a definition of *subgeneral position*. Set $Q = \{1, \ldots, q\}$ and take a basis $\{\psi_0, \ldots, \psi_{l_0}\}$ of W. We write

$$\sigma_j = \sum_{k=0}^{l_0} c_{jk}\psi_k \quad (c_{jk} \in \mathbf{C})$$

for each $j \in Q$. For a subset $R \subseteq Q$, we define a matrix A_R by $A_R = (c_{jk})_{j\in R, 0\leq k\leq l_0}$.

Definition 3.1 Let $N \geq l_0$ and $q \geq N + 1$. We say that $D_1, \ldots, D_q$ are in N-subgeneral position in Λ if

$$\mathrm{rank}\, A_R = l_0 + 1 \quad \text{for every subset } R \subseteq Q \text{ with } \sharp R = N + 1.$$

If they are in l_0-subgeneral position, we simply say that they are in general position.

Note that the above definition is different than the usual one (cf. [6, p. 114])

Let $\Phi_\Lambda : M \to \mathbf{P}(W^*)$ be a natural meromorphic mapping, where W^* is the dual of W. Then we have the linearly non-degenerate holomorphic curve

$$F_\Lambda = \Phi_\Lambda \circ f : X \to \mathbf{P}(W^*).$$

We let $W(F_\Lambda)$ denote the Wronskian of F_Λ.

Definition 3.2 If $\rho_f < +\infty$, then f is said to be of finite type.

Set

$$\kappa(X, \Lambda; N) = 2N - l_0 + 1 + (s_0 - 1)l_0(2N - l_0 + 1).$$

By making use of the methods in [1] and [4], we have an inequality of the second main theorem type as follows.

Theorem 3.3 *Let $f : X \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Let $D_1, \dots, D_q \in \Lambda$ be divisors in N-subgeneral position with $q > \kappa(X, \Lambda; N)$. Assume that f is of finite type. Then*

$$(q - \kappa(X, \Lambda; N))\,(T_f(r,\ L) - m_f(r,\ \mathcal{I}_0)) \le \sum_{j=1}^{q} N(r,\ f^*D_j) + E_f(r)$$

as $r \to +\infty$, where

$$E_f(r) = -\kappa(X, \Lambda; N)N(r,\ f^*\mathcal{I}_0) - \binom{N}{l_0} N(r,\ W(F_\Lambda)_0) + o(T_f(r,\ L)).$$

In order to get a defect relation from Theorem 3.3, we define a constant $\eta_f(B_\Lambda)$ by

$$\eta_f(B_\Lambda) = \liminf_{r \to +\infty} \frac{E_f(r)}{T_f(r, L)}.$$

It is clear that $\eta_f(B_\Lambda) \le 0$. Now, by Theorem 3.3, we have a defect relation.

Theorem 3.4 *Let Λ, f and $D_1, \dots, D_q$ be as in Theorem 3. Then*

$$\sum_{j=1}^{q}(\delta_f(D_j) - \delta_f(B_\Lambda)) \le (1 - \delta_f(B_\Lambda))\kappa(X, \Lambda) + \eta_f(B_\Lambda).$$

4 Structure Theorems for the Set of Deficient Divisors

In this section we give theorems on the structure of the set of deficient divisors. Let $L \to M$ be an ample line bundle and $f : \mathbf{C} \to M$ a transcendental holomorphic curve of finite type. Let $\Lambda \subseteq |L|$ be a linear system. Let

$$\mathcal{D}_f = \{D \in \Lambda\,;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

By making use of the above defect relation, we have the structure theorem for the set $\mathcal{D}_f$ (see [1, §5]).

Theorem 4.1 *The set $\mathcal{D}_f$ of deficient divisors is a union of at most countably many linear systems included in Λ. The set of values of deficiency of f is at most a countable subset $\{e_i\}$ of $[0, 1]$. For each e_i, there exist linear systems $\Lambda_1(e_i), \dots, \Lambda_s(e_i)$ included in Λ such that $e_i = \delta_f(B_{\Lambda_j(e_i)})$ for $j = 1, \dots, s$.*

By Theorem 5, there exists a family $\{\Lambda_j\}$ of at most countably many linear systems in Λ such that $\mathcal{D}_f = \bigcup_j \Lambda_j$. Define $\mathcal{L}_f = \{\Lambda_j\} \cup \{\Lambda\}$. We call $\mathcal{L}_f$ *the fundamental family of linear systems for* f. Then we have the following.

Proposition 4.2 *If $\delta_f(D) > \delta_f(B_\Lambda)$ for a divisor D in Λ, then there exists a linear system $\Lambda(D) \in \mathcal{L}_f$ such that*

$$\delta_f(D) = \delta_f(B_{\Lambda(D)}).$$

5 Methods for Constructing Holomorphic Curves with Deficiencies

In this section we consider the case where $M = \mathbf{P}^n(\mathbf{C})$ and $L = \mathcal{O}_{\mathbf{P}^n}(d)$. The existence of $f : X \to \mathbf{P}^n(\mathbf{C})$ with $\mathcal{D} \neq \emptyset$ is a delicate matter. In fact, S. Mori [3] showed that a family

$$\{f \in \mathrm{Hol}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}); \delta_f(H) = 0 \text{ for all } H \in |\mathcal{O}_{\mathbf{P}^n}(1)|\}$$

of holomprphic curves is dense in $\mathrm{Hol}(\mathbf{C}, \mathbf{P}^n(\mathbf{C}))$ with respect to a certain kind of topology. However, for any $\Lambda \subseteq |\mathcal{O}_{\mathbf{P}^n}(d)|$, there exists an algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ with $\mathcal{D}_f \neq \emptyset$. In fact, we have the following theorem [2, Theorem 3.2].

Theorem 5.1 *Let $D \in |\mathcal{O}_{\mathbf{P}^n}(d)|$. There exists a constant $\lambda(D)$ with $0 < \lambda(D) \leq d$ depending only on D that satisfies the following property: Let α be a positive real constant such that*

$$0 < \alpha \leq \frac{\lambda(D)}{d}.$$

Then there exists an algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *such that*

$$\delta_f(D) = \alpha.$$

We will generalize the above theorem for holomorphic curves defined on X.

Theorem 5.2 *Let* $D \in \Lambda \subseteq |\mathcal{O}_{\mathbf{P}^n}(d)|$. *Then there exists a finite sheeted analytic covering space* $\varpi : X \to \mathbf{C}$ *and an algebraically non-degenerate transcendental holomorphic curve* $f : X \to \mathbf{C}$ *with* $\rho_f = 0$ *such that* $\mathcal{D}_f \neq \emptyset$. *Furthermore, there exists a family* $\{\Lambda_j\}$ *of finitely many linear systems such that*

$$\mathcal{D}_f = \bigcup_j \Lambda_j.$$

The set of values of δ_f *is a finite set* $\{e_j\}$ *with*

$$\delta_f(B_{\Lambda_j}) \leq e_j \leq \frac{\mu(\Lambda_j)}{d}.$$

Here $\mu(\Lambda_j)$ *are constants depending only on* Λ_j *with* $0 < \mu(\Lambda_j) \leq d$.

Remark 5.3 In the case where X is an affine algebraic variety, there always exists an algebraically non-degenerate transcendental holomorphic curve that satisfies the above propeties.

The proofs of the above theorems are based on Valiron's theorem on algebroid functions of order zero (see [7]). Hence the resulting holomorphic curves are of order zero.

In the case where $d = 1$ and $X = \mathbf{C}$, we can construct holomorphic curves with $\mathcal{D}_f \neq \emptyset$ by another way (cf. [1, §6]). By using exponential curves

$$f(z) = (\exp a_0 z, \ldots, \exp a_n z) \quad (a_0, \ldots, a_n \in \mathbf{C}),$$

we can construct holomorphic curves $\mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ with $\mathcal{D}_f \neq \emptyset$. We denote by $\mathcal{C}_f$ the circumference of the convex polygon spanned by the set $\{a_0, \ldots, a_n\}$. If the convex polygon reduces to the segment with the end points with a_j and a_k, then we see $\mathcal{C}_f = 2|a_j - a_k|$. Let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ defined by

$$H : L(z) = \sum_{j=0}^{n} \alpha_j \zeta_j = 0 \quad (\alpha_0, \ldots, \alpha_n \in \mathbf{C}),$$

where $\zeta = (\zeta_0 : \ldots : \zeta_n)$ is a homogeneous coordinate system in $\mathbf{P}^n(\mathbf{C})$. We define the set J_H of index by $J_H = \{j\,;\ \alpha_j \neq 0\}$. Let $\mathcal{C}_f(H)$ be the circumference of the convex polygon spanned by the set $\{a_j\,;\ j \in J_H\}$. Then we have the following lemma.

Lemma 5.4 *Let* f *and* H *be as above. Then*

$$T_f(r, \mathcal{O}_{\mathbf{P}^n}(1)) = \frac{\mathcal{C}_f}{2\pi} r + O(1).$$

and the deficiency of f *for* H *is given by*

$$\delta_f(H) = 1 - \frac{\mathcal{C}_f(H)}{\mathcal{C}_f}.$$

Furthermore, the constant $\mathcal{C}_f(H)$ *depends only on* f *and* J_H*.*

By making use of this lemma, we have the following theorem.

Theorem 5.5 *Let* $\Lambda \subseteq |\mathcal{O}_{\mathbf{P}^n}(1)|$*. Then there is a transcendental holomorphic curve* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *non-degenerate with respect to* Λ *such that the set of values of* δ_f *is a finite set* $\{e_j\}$ *with* $0 < e_j < 1$*. Furthermore, there are finitely many linear systems* $\{\Lambda_j\}$ *included in* Λ *such that*

$$\delta_f(H) = e_j \quad \textit{for all} \quad H \in \Lambda_j \setminus (\bigcup_k \Lambda_{jk}),$$

where $\{\Lambda_{jk}\}$ *are linear systems included in* Λ_j*.*

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