Compactness of Localization Operators on Modulation Spaces of ω-Tempered Distributions

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Abstract We give sufficient conditions for compactness of localization operators on modulation spaces $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$ of ω-tempered distributions whose short-time Fourier transform is in the weighted mixed space $L^{p,q}_{m_\lambda}$ for $m_\lambda(x) = e^{\lambda\omega(x)}$.

In this paper we study some properties of localization operators, which are pseudo-differential operators of time-frequency analysis suitable for applications to the reconstruction of signals, because they allow to recover a filtered version of the original signal. To introduce the problem, let us recall the *translation* and *modulation* operators

$$T_x f(y) = f(y-x), \quad M_\xi f(y) = e^{iy\cdot\xi} f(y), \qquad x, y \in \mathbb{R}^d,$$

and, for a window function $\psi \in L^2(\mathbb{R}^d)$, the *short-time Fourier transform* (briefly STFT) of a function $f \in L^2(\mathbb{R}^d)$

$$V_\psi f(z) = \langle f, M_\xi T_x \psi\rangle = \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)}e^{-iy\cdot\xi}\,dy, \qquad z = (x,\xi) \in \mathbb{R}^{2d}.$$

With respect to the inversion formula for the STFT (see [13, Cor. 3.2.3])

$$f = \frac{1}{(2\pi)^d\langle\gamma,\psi\rangle}\int_{\mathbb{R}^{2d}} V_\psi f(x,\xi) M_\xi T_x \gamma\,dx d\xi,$$

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which gives a reconstruction of the signal f, the localization operator, as defined in (0.2), modifies $V_\psi f(x,\xi)$ by multiplying it by a suitable $a(x,\xi)$ before reconstructing the signal, so that a filtered version of the original signal f is recovered.

Another important operator in time-frequency analysis that we shall need in the following is the *cross-Wigner transform* defined, for $f, g \in L^2(\mathbb{R}^d)$, by

$$\mathrm{Wig}(f,g)(x,\xi) = \int_{\mathbb{R}^d} f\big(x+\frac{t}{2}\big)\overline{g\big(x-\frac{t}{2}\big)}e^{-i\xi\cdot t}\,dt \qquad x,\xi\in\mathbb{R}^d.$$

The *Wigner transform* of f is then defined by $\mathrm{Wig}\, f := \mathrm{Wig}(f,f)$.

The above Fourier integral operators, with standard generalizations to more general spaces of functions or distributions, have been largely investigated in time-frequency analysis. In particular, results about boundedness or compactness related to the subject of this paper can be found, for instance, in [1, 7, 10–12, 16, 17].

Inspired by Cordero and Gröchenig [7] and Fernández and Galbis [10], our aim in this paper is to study boundedness of localization operators on modulation spaces in the setting of ω-tempered distributions, for a weight functions ω defined as below:

Definition 0.1 A *non-quasianalytic subadditive weight function* is a continuous increasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

(α) $\omega(t_1+t_2) \le \omega(t_1)+\omega(t_2), \qquad \forall t_1,t_2 \ge 0;$
(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt < +\infty;$
(γ) $\exists A\in\mathbb{R}, B>0$ s.t $\omega(t) \ge A + B\log(1+t), \qquad \forall t\ge 0;$
(δ) $\varphi_\omega(t) := \omega(e^t)$ is convex.

We then consider $\omega(\xi) := \omega(|\xi|)$ for $\xi \in \mathbb{C}^d$.

Definition 0.2 The space $\mathcal{S}_\omega(\mathbb{R}^d)$ is defined as the set of all $u \in L^1(\mathbb{R}^d)$ such that $u, \hat{u} \in C^\infty(\mathbb{R}^d)$ and

(i) $\forall\lambda>0, \alpha\in\mathbb{N}_0^d$: $\sup_{x\in\mathbb{R}^d} e^{\lambda\omega(x)}|D^\alpha u(x)| < +\infty,$
(ii) $\forall\lambda>0, \alpha\in\mathbb{N}_0^d$: $\sup_{\xi\in\mathbb{R}^d} e^{\lambda\omega(\xi)}|D^\alpha \hat{u}(\xi)| < +\infty,$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

Note that for $\omega(t) = \log(1+t)$ we obtain the classical Schwartz class $\mathcal{S}(\mathbb{R}^d)$, while in general $\mathcal{S}_\omega(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$. For more details about the spaces $\mathcal{S}_\omega(\mathbb{R}^d)$ we refer to [3–6]. In particular, we can define on $\mathcal{S}_\omega(\mathbb{R}^d)$ different equivalent systems of seminorms that make $\mathcal{S}_\omega(\mathbb{R}^d)$ a Fréchet nuclear space. It is also an algebra under multiplication and convolution.

The corresponding strong dual space is denoted by $\mathcal{S}'_\omega(\mathbb{R}^d)$ and its elements are called *ω-tempered distributions*. Moreover, $\mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{S}'_\omega(\mathbb{R}^d)$ and the Fourier transform, the short-time Fourier transform and the Wigner transform are continuous from $\mathcal{S}_\omega(\mathbb{R}^d)$ to $\mathcal{S}_\omega(\mathbb{R}^d)$ and from $\mathcal{S}'_\omega(\mathbb{R}^d)$ to $\mathcal{S}'_\omega(\mathbb{R}^d)$.

The "right" function spaces in time-frequency analysis to work with the STFT are the so-called *modulation spaces*, introduced by H. Feichtinger in [9]. In this context, we consider the weight $m_\lambda(z) := e^{\lambda\omega(z)}$, for $\lambda \in \mathbb{R}$, and define $L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})$ as the space of measurable functions f on $\mathbb{R}^{2d}$ such that

$$\|f\|_{L^{p,q}_{m_\lambda}} := \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\xi)|^p m_\lambda(x,\xi)^p \, dx \right)^{\frac{q}{p}} d\xi \Bigg)^{\frac{1}{q}} < +\infty,$$

for $1 \le p, q < +\infty$, with standard changes if p (or q) is $+\infty$. We define then, for $1 \le p, q \le +\infty$, the modulation space

$$\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d) := \{f \in \mathcal{S}'_\omega(\mathbb{R}^d) : V_\varphi f \in L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})\},$$

which is independent of the window function $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and is a Banach space with norm $\|f\|_{\mathbf{M}^{p,q}_{m_\lambda}} := \|V_\varphi f\|_{L^{p,q}_{m_\lambda}}$ (see [4]). Moreover, for $1 \le p, q < +\infty$, the space $\mathcal{S}_\omega(\mathbb{R}^d)$ is a dense subspace of $\mathbf{M}^{p,q}_{m_\lambda}$ by Boiti et al. [4, Prop. 3.9]. We shall denote $\mathbf{M}^p_{m_\lambda}(\mathbb{R}^d) = \mathbf{M}^{p,p}_{m_\lambda}(\mathbb{R}^d)$ and $\mathbf{M}^{p,q}(\mathbb{R}^d) = \mathbf{M}^{p,q}_{m_0}(\mathbb{R}^d)$.

As in [13, Thm. 12.2.2] if $p_1 \le p_2$, $q_1 \le q_2$, and $\lambda \le \mu$ then $\mathbf{M}^{p_1,q_1}_{m_\mu} \subseteq \mathbf{M}^{p_2,q_2}_{m_\lambda}$ with continuous inclusion (see [8, Lemma 2.3.16]). Set

$$m_{\lambda,1}(x) := m_\lambda(x,0), \quad m_{\lambda,2}(\xi) := m_\lambda(0,\xi),$$
$$v_\lambda(z) = e^{|\lambda|\omega(z)}, \quad v_{\lambda,1}(x) := v_\lambda(x,0), \quad v_{\lambda,2}(\xi) := v_\lambda(0,\xi),$$

and prove the following generalization of [7, Prop. 2.4]:

Proposition 0.3 *Let* $1 \le p, q, r, t, t' \le +\infty$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ *and* $\frac{1}{t} + \frac{1}{t'} = 1$. *Then, for all* $\lambda, \mu \in \mathbb{R}$ *and* $1 \le s \le +\infty$*,*

$$\mathbf{M}^{p,st}_{m_{\lambda,1}\otimes m_{\mu,2}}(\mathbb{R}^d) * \mathbf{M}^{q,st'}_{v_{\lambda,1}\otimes v_{\lambda,2}m_{-\mu,2}}(\mathbb{R}^d) \hookrightarrow \mathbf{M}^{r,s}_{m_\lambda}(\mathbb{R}^d)$$

and
$$\|f * g\|_{\mathbf{M}^{r,s}_{m_\lambda}} \le \|f\|_{\mathbf{M}^{p,st}_{m_{\lambda,1}\otimes m_{\mu,2}}} \|g\|_{\mathbf{M}^{q,st'}_{v_{\lambda,1}\otimes v_{\lambda,2}m_{-\mu,2}}}. \tag{0.1}$$

Proof For the Gaussian function $g_0(x) = e^{-\pi|x|^2} \in \mathcal{S}_\omega(\mathbb{R}^d)$ consider on $\mathbf{M}^{r,s}_{m_\lambda}$ the modulation norm with respect to the window function $g(x) := g_0 * g_0(x) = 2^{-d/2}e^{-\frac{\pi}{2}|x|^2} \in \mathcal{S}_\omega(\mathbb{R}^d)$. Since $m_\lambda(x,\xi) \le m_\lambda(x,0)v_\lambda(0,\xi)$ and $\overline{g_0(-x)} = g_0(x)$, by Gröchenig [13, Lemma 3.1.1], Young and Hölder inequalities:

$$\|f * h\|_{\mathbf{M}^{r,s}_{m_\lambda}} = \|V_g(f * h)\|_{L^{r,s}_{m_\lambda}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g(f * h)|^r m^r_\lambda(x,\xi) \, dx \right)^{\frac{s}{r}} d\xi \right)^{\frac{1}{s}}$$

$$\le \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(f * M_\xi g_0) * (h * M_\xi g_0)(x)|^r m_\lambda(x,0)^r \, dx \right)^{\frac{s}{r}} v^s_\lambda(0,\xi) \, d\xi \right)^{\frac{1}{s}}$$

$$= \left(\int_{\mathbb{R}^d} \|(f * M_\xi g_0) * (h * M_\xi g_0)\|^s_{L^r_{m_{\lambda,1}}} v^s_\lambda(0,\xi)\, d\xi \right)^{\frac{1}{s}}$$
$$\le \left(\int_{\mathbb{R}^d} \|f * M_\xi g_0\|^s_{L^p_{m_{\lambda,1}}} \|h * M_\xi g_0\|^s_{L^q_{v_{\lambda,1}}} v^s_\lambda(0,\xi)\, d\xi \right)^{\frac{1}{s}}$$
$$= \left(\int_{\mathbb{R}^d} \|V_{g_0} f\|^s_{L^p_{m_{\lambda,1}}} m^s_\mu(0,\xi) \|V_{g_0} h\|^s_{L^q_{v_{\lambda,1}}} m^s_{-\mu}(0,\xi) v^s_\lambda(0,\xi)\, d\xi \right)^{\frac{1}{s}}$$
$$\le \|f\|_{\mathbf{M}^{p,st}_{m_{\lambda,1}\otimes m_{\mu,2}}} \|h\|_{\mathbf{M}^{q,st'}_{v_{\lambda,1}\otimes v_{\lambda,2} m_{-\mu,2}}}.$$

□

Given two window functions $\psi, \gamma \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and a symbol $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$, the corresponding *localization operator* $L^a_{\psi,\gamma}$ is defined, for $f \in \mathcal{S}_\omega(\mathbb{R}^d)$, by

$$L^a_{\psi,\gamma} f = V^*_\gamma(a \cdot V_\psi f) = \int_{\mathbb{R}^{2d}} a(x,\xi) V_\psi f(x,\xi) M_\xi T_x \gamma\, dx d\xi, \tag{0.2}$$

where V^*_γ is the adjoint of V_γ. As in [2, Lemma 2.4] we have that $L^a_{\psi,\gamma}$ is a Weyl operator L^{a^w} with symbol $a^w = a * \mathrm{Wig}(\gamma, \psi)$:

$$L^{a^w} f := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \hat{a}^w(\xi, u) e^{-i\xi\cdot u} T_{-u} M_\xi f\, du d\xi. \tag{0.3}$$

Moreover, if $f, g \in \mathcal{S}_\omega(\mathbb{R}^d)$ then by definition of adjoint operator we can write

$$\langle L^a_{\psi,\gamma} f, g\rangle = \langle a \cdot V_\psi f, V_\gamma g\rangle = \langle a, \overline{V_\psi f} V_\gamma g\rangle,$$

and, similarly as in [13, Thm. 14.5.2] (see also [8, Teo. 2.3.21]), we have, for $a^w \in \mathbf{M}^{\infty,1}_{m_\mu}(\mathbb{R}^{2d})$ with $\mu \ge 0$,

$$\|L^{a^w} f\|_{\mathbf{M}^{p,q}_{m_\lambda}} = \|L^a_{\psi,\gamma} f\|_{\mathbf{M}^{p,q}_{m_\lambda}} \le \|a^w\|_{\mathbf{M}^{\infty,1}_{m_\mu}} \|f\|_{\mathbf{M}^{p,q}_{m_\lambda}}, \tag{0.4}$$

for all $f \in \mathbf{M}^{p,q}_{m_\lambda}$ and $\lambda \in \mathbb{R}$.

Theorem 0.4 *Let $\psi, \gamma \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and $a \in \mathbf{M}^\infty_{m_\lambda}(\mathbb{R}^{2d})$ for some $\lambda \ge 0$. Then $L^a_{\psi,\gamma}$ is bounded from $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$ to $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$, for $1 \le p, q < +\infty$, and*

$$\|L^a_{\psi,\gamma}\|_{op} \le \|a\|_{\mathbf{M}^\infty_{m_{-\lambda,2}}} \|\psi\|_{\mathbf{M}^1_{v_\lambda}} \|\gamma\|_{\mathbf{M}^p_{m_\lambda}}.$$

Proof By definition $V_\psi : \mathbf{M}^{p,q}_{m_\lambda} \to L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})$ and, by Boiti et al. [4, Prop. 3.7], $V^*_\gamma : L^{p,q}_{m_\lambda}(\mathbb{R}^{2d}) \to \mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$. Let $f \in \mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$. To prove that $L^a_{\psi,\gamma} f = V^*_\gamma(a \cdot V_\psi f) \in \mathbf{M}^{p,q}_{m_\lambda}$, it is then enough to show that $a \cdot V_\psi f \in L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})$. By the inversion formula [4, Prop. 3.7], given two window functions $\Phi, \Psi \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ with $\langle \Phi, \Psi\rangle \neq 0$, we have, for $z = (z_1, z_2) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |a(x,\xi)|^p |V_\psi f(x,\xi)|^p e^{p\lambda\omega(x,\xi)}\,dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}}\\
&\leq \frac{1}{(2\pi)^d}\frac{1}{|\langle\Phi,\Psi\rangle|}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{4d}} |V_\Psi a(z)|^p |M_{z_2}T_{z_1}\Phi(x,\xi)|^p dz\right)\right.\right.\\
&\left.\left.\cdot |V_\psi f(x,\xi)|^p e^{p\lambda\omega(x,\xi)} dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}}\\
&\leq \frac{1}{(2\pi)^d}\frac{1}{|\langle\Phi,\Psi\rangle|}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{4d}} \left(|V_\Psi a(z)|e^{\lambda\omega(z)}\right)^p |M_{z_2}T_{z_1}\Phi(x,\xi)|^p dz\right)\right.\right.\\
&\left.\left.\cdot |V_\psi f(x,\xi)|^p e^{p\lambda\omega(x,\xi)} dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}}\\
&\leq C\|V_\Psi a\|_{L^\infty_{m_\lambda}} \cdot \|V_\psi f\|_{L^{p,q}_{m_\lambda}} = C\|a\|_{\mathbf{M}^\infty_{m_\lambda}} \cdot \|f\|_{\mathbf{M}^{p,q}_{m_\lambda}},
\end{aligned}$$

for some $C > 0$. Therefore $a \cdot V_\psi f \in L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})$ and $L^a_{\psi,\gamma} f \in \mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$.

To prove that $L^a_{\psi.\gamma}$ is bounded, consider $g \in \mathcal{S}_\omega(\mathbb{R}^d)$ and set $\Psi = \mathrm{Wig}(g, g) \in \mathcal{S}_\omega(\mathbb{R}^{2d})$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2d}$, we set $\tilde{\xi} = (\xi_2, -\xi_1)$. By Cordero and Gröchenig [7, Lemma 2.2]

$$\begin{aligned}
\|\,\mathrm{Wig}(\gamma,\psi)\|_{\mathbf{M}^{1,p}_{m_{\lambda,2}}} &= \|V_\Psi\,\mathrm{Wig}(\gamma,\psi)\|_{L^{1,p}_{m_{\lambda,2}}}\\
&= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} \left|V_g\psi\Big(z+\frac{\tilde{\xi}}{2}\Big)V_g\gamma\Big(z-\frac{\tilde{\xi}}{2}\Big)\right| dz\right)^p m^p_{\lambda,2}(\xi)\,d\xi\right)^{\frac{1}{p}}.
\end{aligned}$$

By the change of variables $z + \frac{\tilde{\xi}}{2} = \tilde{z}$ and [4, formula (3.12)] we obtain (cf. also [7, Prop. 2.5]):

$$\begin{aligned}
\|\,\mathrm{Wig}(\gamma,\psi)\|_{\mathbf{M}^{1,p}_{m_{\lambda,2}}} &= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} |V_g\psi(\tilde{z})||V_g\gamma(\tilde{z}-\tilde{\xi})|\,d\tilde{z}\right)^p m^p_{\lambda,2}(\xi)\,d\xi\right)^{\frac{1}{p}}.\\
&= \left(\int_{\mathbb{R}^{2d}} (|V_g\psi(\tilde{z})| * |V_g\gamma(-\tilde{z})|)^p(\tilde{\xi})\,m^p_{\lambda,2}(\tilde{\xi})\,d\tilde{\xi}\right)^{\frac{1}{p}}\\
&\leq \|V_g\psi\|_{L^1_{v_\lambda}}\|V_g\gamma\|_{L^p_{m_\lambda}} = \|\psi\|_{\mathbf{M}^1_{v_\lambda}}\|\gamma\|_{\mathbf{M}^p_{m_\lambda}}.
\end{aligned} \tag{0.5}$$

Therefore $\mathrm{Wig}(\gamma, \psi) \in \mathbf{M}^1_{m_{\lambda,2}}(\mathbb{R}^{2d})$ and hence, from Proposition 0.3 (with $p = t = r = +\infty$, $q = s = t' = 1$, $\lambda = 0$ and $\mu = -\lambda$), we have that $\mathbf{M}^\infty_{m_{-\lambda,2}} * \mathbf{M}^1_{m_{\lambda,2}} \subseteq \mathbf{M}^{\infty,1}$, so that $a^w = a * \mathrm{Wig}(\gamma, \psi) \in \mathbf{M}^{\infty,1}$ and by (0.4) with $\mu = 0$

$$\|L^a_{\psi,\gamma}\|_{op} \le \|a^w\|_{\mathbf{M}^{\infty,1}}.$$

From (0.1) and (0.5) we finally have

$$\begin{aligned}\|L^a_{\psi,\gamma}\|_{op} &\le \|a * \mathrm{Wig}(\gamma, \psi)\|_{\mathbf{M}^{\infty,1}} \le \|a\|_{\mathbf{M}^\infty_{m_{-\lambda,2}}} \|\mathrm{Wig}(\gamma, \psi)\|_{\mathbf{M}^1_{m_{\lambda,2}}} \\ &\le \|a\|_{\mathbf{M}^\infty_{m_{-\lambda,2}}} \|\psi\|_{\mathbf{M}^1_{v_\lambda}} \|\gamma\|_{\mathbf{M}^p_{m_\lambda}}.\end{aligned}$$

□

A boundedness result analogous to that of Theorem 0.4 is proved, with different techniques, in [16] under further restrictions on the symbol $a(x, \xi)$ and without estimates on the norm of $L^a_{\psi,\gamma}$.

Set now

$$\mathbf{M}^{0,1}_{m_\lambda}(\mathbb{R}^d) = \{f \in \mathbf{M}^{\infty,1}_{m_\lambda}(\mathbb{R}^d) : \lim_{|x|\to\infty} \|V_g f(x, .)\|_{L^1_{m_\lambda}} e^{\lambda\omega(x)} = 0\}$$

and prove the following compactness result (cf. also [1, Prop. 2.3] and [12, Thm. 3.22]):

Theorem 0.5 *If $a^w \in \mathbf{M}^{0,1}_{m_\lambda}(\mathbb{R}^{2d})$ for some $\lambda \ge 0$, then L^{a^w} is a compact mapping of $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$ into itself, for $1 \le p, q < +\infty$.*

Proof The operator L^{a^w} maps $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$ into itself by (0.4). To prove that L^{a^w} is compact we first assume $a^w \in \mathcal{S}_\omega(\mathbb{R}^{2d})$. From (0.3)

$$\begin{aligned}L^{a^w} f(y) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \hat{a}^w(\xi, u) e^{-i\xi\cdot u} e^{i\xi\cdot(y+u)} f(y+u)\, du\, d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \hat{a}^w(\xi, x-y) e^{i\xi\cdot y} f(x)\, dx\, d\xi \\ &= \int_{\mathbb{R}^d} k(x, y) f(x)\, dx, \end{aligned} \tag{0.6}$$

with kernel $k(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{a}^w(\xi, x-y) e^{i\xi\cdot y} d\xi$. Note that $k(x, y) \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ because it is the inverse Fourier transform (with respect to the first variable) of the translation (with respect to the second variable) of $\hat{a}^w \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.

Now, let $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\alpha_0, \beta_0 > 0$ such that $\{\phi_{jl}\}_{j,l\in\mathbb{Z}^d} = \{M_{\beta_0 l}T_{\alpha_0 j}\phi\}_{j,l\in\mathbb{Z}^d}$ is a tight Gabor frame for $L^2(\mathbb{R}^d)$ (see [13, Def. 5.1.1] for the definition). Then $\{\Phi_{jlmn}\}_{j,l,m,n\in\mathbb{Z}^d} = \{\phi_{jl}(x)\phi_{mn}(y)\}_{j,l,m,n\in\mathbb{Z}^d}$ is a tight Gabor frame for $L^2(\mathbb{R}^{2d})$. Since $k \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ we have that $\langle k, \Phi_{jlmn}\rangle = V_\phi k(\alpha_0 j, \alpha_0 m, \beta_0 l, \beta_0 n) \in \ell^1$ and (see [4, Lemma 3.15])

$$k = \sum_{j,l,m,n\in\mathbb{Z}^d} \langle k, \Phi_{jlmn}\rangle \Phi_{jlmn}.$$

Therefore from (0.6)

$$L^{a^w} f = \sum_{j,l,m,n\in\mathbb{Z}^d} \langle k, \Phi_{jlmn}\rangle\langle \phi_{jl}, f\rangle\phi_{mn},$$

with $\langle k, \Phi_{jlmn}\rangle \in \ell^1$, $(\phi_{jl})_{j,l\in\mathbb{Z}^d}$ equicontinuous in $\mathbf{M}^{p',q'}_{m_{-\lambda}} = (\mathbf{M}^{p,q}_{m_\lambda})^*$ and $(\phi_{mn})_{m,n\in\mathbb{Z}^d}$ bounded in $\bigcup_{n\in\mathbb{N}} n\{f \in \mathbf{M}^{p,q}_{m_\lambda} : \|f\|_{\mathbf{M}^{p,q}_{m_\lambda}} < 1\}$, so that L^{a^w} is a nuclear operator from $\mathbf{M}^{p,q}_{m_\lambda}$ to $\mathbf{M}^{p,q}_{m_\lambda}$ (see [15, §17.3]). From [15, §17.3, Cor. 4] we thus have that L^{a^w} is compact.

Let us finally consider the general case $a \in \mathbf{M}^{0,1}_{m_\lambda}(\mathbb{R}^{2d})$. By Boiti et al. [4, Prop. 3.9] there exist $a_n \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ converging to a in $\mathbf{M}^{\infty,1}_{m_\lambda}$ and hence, by (0.4)

$$\|L^{a^w} - L^{a_n^w}\|_{\mathbf{M}^{p,q}_{m_\lambda}\to\mathbf{M}^{p,q}_{m_\lambda}} \le \|a - a_n\|_{\mathbf{M}^{\infty,1}_{m_\lambda}} \to 0.$$

Since the set of compact operators is closed we have that L^{a^w} is compact on $\mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$. □

We have the following generalization of [10, Lemma 3.4] and [11, Prop. 5.2]:

Lemma 0.6 *Let $g_0 \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $a \in \mathbf{M}^\infty_{m_\lambda}(\mathbb{R}^d)$, with $\lambda \ge 0$, such that*

$$\lim_{|x|\to+\infty} \sup_{|\xi|\le R} |V_{g_0}a(x,\xi)|e^{\lambda\omega(x,\xi)} = 0, \qquad \forall R > 0. \tag{0.7}$$

Then $a * H \in \mathbf{M}^{0,1}_{m_\lambda}(\mathbb{R}^d)$ for any $H \in \mathcal{S}_\omega(\mathbb{R}^d)$.

Proof The case $\lambda = 0$ has been proved in [10, Lemma 3.4]. Let $\lambda > 0$. Since $g_0 \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $H \in \mathcal{S}_\omega(\mathbb{R}^d)$, by Gröchenig and Zimmermann [14, Thm. 2.7] we have that $V_{g_0}H \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ and hence, for a fixed $\ell > 0$ (to be chosen later depending on λ), there exists $c_\lambda > 0$ such that

$$|V_{g_0}H(x,\xi)| \le c_\lambda e^{-3\ell\lambda\omega(x)}e^{-3\ell\lambda\omega(\xi)}, \qquad \forall x,\xi \in \mathbb{R}^d.$$

Now, as in the proof of Proposition 0.3, for $g = g_0 * g_0$, we have that $|V_g(a * H)(\cdot,\xi)| = |V_{g_0}a(\cdot,\xi) * V_{g_0}H(\cdot,\xi)|$. Since ω is increasing and subadditive we have

$$\begin{aligned}
&|V_g(a*H)(x,\xi)| \le \int_{\mathbb{R}^d} |V_{g_0}a(x-y,\xi)||V_{g_0}H(y,\xi)|dy\\
&\le c_\lambda e^{-3\ell\lambda\omega(\xi)}\int_{\mathbb{R}^d}|V_{g_0}a(x-y,\xi)|e^{-3\ell\lambda\omega(y)}dy\\
&= c_\lambda e^{-3\ell\lambda\omega(\xi)}\int_{\mathbb{R}^d}|V_{g_0}a(x-y,\xi)|e^{-3\ell\lambda\omega(y)}\, e^{\lambda\omega(x-y,\xi)}e^{-\lambda\omega(x-y,\xi)}dy\\
&\le c_\lambda e^{-3\ell\lambda\omega(\xi)}e^{-\lambda\omega(x)}\int_{\mathbb{R}^d}|V_{g_0}a(x-y,\xi)|e^{\lambda\omega(x-y,\xi)}e^{-(3\ell-1)\lambda\omega(y)}dy.
\end{aligned}$$

Since $a \in \mathbf{M}^\infty_{m_\lambda}(\mathbb{R}^d)$ we have that

$$\begin{aligned}
&e^{\lambda\omega(x)+2\ell\lambda\omega(\xi)}|V_g(a*H)(x,\xi)|\\
&\le\ c_\lambda e^{-\ell\lambda\omega(\xi)}\int_{\mathbb{R}^d}|V_g a(x-y,\xi)|e^{\lambda\omega(x-y,\xi)}e^{-(3\ell-1)\lambda\omega(y)}dy \qquad (0.8)\\
&\le\ c_\lambda e^{-\ell\lambda\omega(\xi)}\|a\|_{\mathbf{M}^\infty_{m_\lambda}}\int_{\mathbb{R}^d}e^{-(3\ell-1)\lambda\omega(y)}dy < +\infty, \qquad (0.9)
\end{aligned}$$

if $\ell > \frac{1}{3} + \frac{d}{3B\lambda}$, where B is the constant of condition (γ) in Definition 0.1. Since $\lim_{|\xi|\to+\infty}\omega(\xi) = +\infty$, from (0.9) we have that for all $\varepsilon > 0$ there exists $R_1 > 0$ such that

$$e^{\lambda\omega(x)+2\ell\lambda\omega(\xi)}|V_g(a*H)(x,\xi)| < \varepsilon, \quad \forall x,\xi \in \mathbb{R}^d, \quad |\xi| \ge R_1. \qquad (0.10)$$

We now choose $\delta > 0$ small enough so that

$$\delta\left(1 + c_\lambda\int_{\mathbb{R}^d}e^{-(3\ell-1)\lambda\omega(y)}\right)dy \le \varepsilon. \qquad (0.11)$$

From the hypothesis (0.7) we can choose $R_2 > 0$ sufficiently large so that

$$\sup_{|\xi|\le R_1}|V_{g_0}a(x,\xi)|e^{\lambda\omega(x,\xi)} < \delta, \quad |x| \ge R_2, \qquad (0.12)$$

$$\int_{|y|>R_2}e^{-(3\ell-1)\lambda\omega(y)}dy < \frac{\delta}{c_\lambda e^{-\ell\lambda\omega(\xi)}\|a\|_{\mathbf{M}^\infty_{m_\lambda}}}, \qquad |\xi| \le R_1. \qquad (0.13)$$

Therefore for $|x| \geq 2R_2$, $|y| \leq R_2$ (so that $|x-y| \geq R_2$) and $|\xi| \leq R_1$, by (0.8), (0.9), (0.13), (0.12) and (0.11):

$$\begin{aligned}
& e^{\lambda\omega(x)+2\ell\lambda\omega(\xi)}|V_g(a*H)(x,\xi)| \\
\leq\ & c_\lambda e^{-\ell\lambda\omega(\xi)}\|a\|_{\mathbf{M}^\infty_{m_\lambda}} \int_{|y|>R_2} e^{-(3\ell-1)\lambda\omega(y)}dy \\
& + c_\lambda e^{-\ell\lambda\omega(\xi)} \int_{|y|\leq R_2} |V_{g_0}a(x-y,\xi)|e^{\lambda\omega(x-y,\xi)}e^{-(3\ell-1)\lambda\omega(y)}dy \\
<\ & \delta + c_\lambda\delta \int_{\mathbb{R}^d} e^{-(3\ell-1)\lambda\omega(y)}dy \leq \varepsilon.
\end{aligned}$$

The above estimate, together with (0.10), gives

$$e^{\lambda\omega(x)} \int_{\mathbb{R}^d} |V_g(a*H)(x,\xi)|e^{\lambda\omega(\xi)}d\xi \leq \varepsilon \int_{\mathbb{R}^d} e^{-(2\ell-1)\lambda\omega(\xi)}d\xi, \qquad |x| \geq 2R_2.$$

Choosing now $\ell > \frac{1}{2} + \frac{d}{2B\lambda} > \frac{1}{3} + \frac{d}{3B\lambda}$ so that $e^{-(2\ell-1)\lambda\omega(\xi)} \in L^1(\mathbb{R}^d)$, we finally obtain

$$\lim_{|x|\to\infty} e^{\lambda\omega(x)}\|V_g(a*H)(x,.)\|_{L^1_{m_\lambda}} = 0.$$

□

Theorem 0.7 *Let* $\psi, \gamma \in \mathcal{S}_\omega(\mathbb{R}^d)$, $g_0 \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ *and* $a \in \mathbf{M}^\infty_{m_\lambda}(\mathbb{R}^{2d})$ *satisfying* (0.7)*, for some* $\lambda \geq 0$*. Then* $L^a_{\psi,\gamma} : \mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d) \to \mathbf{M}^{p,q}_{m_\lambda}(\mathbb{R}^d)$ *is compact, for* $1 \leq p, q < +\infty$.

Proof Set $H := W(\gamma,\psi) \in \mathcal{S}_\omega(\mathbb{R}^{2d})$. Since $a \in \mathbf{M}^\infty_{m_\lambda}(\mathbb{R}^{2d})$, by Lemma 0.6 we have that $a^w = a * H \in \mathbf{M}^{0,1}_{m_\lambda}(\mathbb{R}^{2d})$ and hence $L^a_{\psi,\gamma} = L^{a^w}$ is compact by Theorem 0.5. □

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