Generalized Solutions to Equations for Probabilistic Characteristics of Levy Processes

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Abstract We study relations between stochastic differential equations with inhomogeneities reflecting continuous and discontinuous random perturbations and equations for probabilistic characteristics of processes specified by these stochastic equations. The application of two approaches: based on the Ito formula and on limit relations for process increments, allowed to obtain direct and backward integro-differential equations for various probabilistic characteristics and justify them in distribution spaces.

Keywords Stochastic equation · Ito formula · Generalized transition density · Levy process · Kolmogorov equation

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1 Introduction

A wide class of processes arising in various fields of natural science, economics and social phenomena, mathematically can be described using differential equations with random perturbations, stochastic differential equations (SDEs). The best-studied class of SDEs is one with random perturbations in the form of Wiener processes. Solutions of such equations (normal diffusion processes), due to continuity of Wiener process trajectories also have continuous trajectories. In addition, normal diffusion processes have the following characteristic property: the variance of the process deviation over time Δt is proportional to Δt. Therefore, modeling within the framework of diffusion-type equations is not suitable for describing processes with jumps and ones with variance proportional to $\Delta t^{\mu}, \mu \neq 1$. The

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behavioral features unusual for normal diffusion processes, can be modeled using Levy processes and more general Levy type processes.

Both in applications and in fundamental science, researchers are often interested not in processes themselves but their characteristics; therefore, the relationship between SDEs and equations of probabilistic characteristics of processes described by SDEs is one of the main directions of stochastic analysis. Most investigated remains the connection for diffusion processes and corresponding partial differential equations for their probabilistic characteristics.

In the paper, we study Levy type stochastic equations and obtain equations for probabilistic characteristics, which in the case are integro-differential (pseudo-differential), in contrast to partial differential equations of parabolic type corresponding to diffusion processes. For this purpose we distinguish two approaches:

– the approach based on the general Ito formula (see, e.g. [1, 2]) allowing to obtain functions of studied processes, which are averaged, and as a result we get the integro-differential equations for probabilistic characteristics.
– the approach allowing to obtain equations for probabilistic characteristics based on the existence of three limits for the random process under study: the limits of the quotient of dividing the local first and second moments by $\Delta t \to 0$ (conditions (3.1)–(3.2)) and the limit (3.3) characterizing the absence of the continuity property of Levy type processes (see, e.g. [3]).

There are deep, not always obvious connections between these approaches, and not all of them, despite many works devoted to the indicated issues, worked out in the desired completeness. In the paper we can see that in the both approaches twice differentiable functions appear in equations for probabilistic characteristics and we show that these functions can be used as test functions. We pay the important attention to substantiation of the resulting direct and backward integro-differential equations, in the general case not having classical solutions, in distribution (generalized functions) spaces.

2 Direct and Backward Equations for Probabilistic Characteristics Based on the Ito Formula

Let a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be given. We consider a random process $X = \{X(t), t \geq 0\}$, arising under the influence of continuous and discontinuous random disturbances. In general, this is a Levy type process defined by the stochastic equation

$$X(t) - x = \int_0^t a(X(s-))ds + \int_0^t b(X(s-))dW(s) + \int_0^t \int_{|q|\geq 1} K(X(s-), q)N(ds, dq) + \int_0^t \int_{|q|<1} F(X(s-), q)\widetilde{N}(ds, dq), \quad t \in [0; T]. \tag{2.1}$$

Here $W = \{W(t), t \geq 0\}$ is a standard Wiener process, $N(t, A)$ for any bounded from below set A is a Poisson random measure on $(\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}), \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\}))$, and $\widetilde{N}(t, A) := N(t, A) - t\nu(A)$ is a martingale-valued (compensated) Poisson random measure on this space. By the definition, the random variable $N(t, \cdot)(\omega)$ for $\omega \in \Omega$ and $t \geq 0$ is a counting measure on $\mathcal{B}(\mathbb{R} \setminus \{0\})$, and $N = \{N(t, A),\ t \geq 0\}$ is a Poisson process with intensity $\lambda = \nu(A) := \mathbf{E}[N(1, A)]$. We suppose the following conditions on coefficients supplying existence of a solution to (2.1): functions $a(\cdot)$, $b(\cdot)$, $F(\cdot, q)$ satisfy the Lipschitz and sub-linear growth conditions and $K(\cdot, q)$ is continuous.

Let $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$ and $\{X(t), t \geq 0\}$ be a Levy type process defined by (2.1) with $\mathtt{a}(\cdot) = a(X(\cdot))$, $\mathtt{b}(\cdot)$, $\mathtt{F}(\cdot, q)$ square integrable a.s. Then with probability 1 the following equality, the Ito formula for Levy type processes, holds (see, e.g. [2, c. 278]):

$$
\begin{aligned}
f(t, X(t)) - f(0, X(0)) &= \int_0^t f'_s(s, X(s-))ds + \int_0^t \mathtt{a}(s) f'_x(s, X(s-))ds \\
&+ \int_0^t \mathtt{b}(s) f'_x(s, X(s-))dW(s) + \frac{1}{2}\int_0^t \mathtt{b}^2(s) f''_{xx}(s, X(s-))ds \\
&+ \int_0^t \int_{|q|\geq 1} [f(s, X(s-) + \mathtt{K}(s, q)) - f(s, X(s-))]N(ds, dq) \\
&+ \int_0^t \int_{|q|<1} [f(s, X(s-) + \mathtt{F}(s, q)) - f(s, X(s-))]\widetilde{N}(ds, dq) \\
&+ \int_0^t \int_{|q|<1} [f(s, X(s-)+F(s, q)) - f(s, X(s-)) - F(s, q) f'_x(s, X(s-))]\nu(dq)ds.
\end{aligned} \tag{2.2}
$$

We start by deriving a direct (forward) equation for transition probability $P(\tau, y; t, A)$, the probability of transition from y at time τ to values on A at time t. To this end, using (2.2) first we obtain the equation for the process $\{f(X(t)), t \geq 0\}$, where X is a solution to (2.1) and $f \in C^2(\mathbb{R})$, then we apply the expectation. Using that integrals over W and over compensated Poisson process $\widetilde{N}$ are martingales, hence its expectations are zero, and changing the order of integration in other terms, we obtain

$$
\begin{aligned}
\mathbf{E}[f(X(t))] - f(x) &= \int_0^t \mathbf{E}[a(X(s-)) f'(X(s-)) + \frac{1}{2} b^2(X(s-)) f''(X(s-))]ds \\
&+ \int_0^t \int_{|q|\geq 1} \mathbf{E}[f(X(s-) + K(X(s-), q)) - f(X(s-))]\nu(dq)ds
\end{aligned}
$$

$$+\int_0^t \int_{|q|<1} \mathbf{E}[f(X(s-)+F(X(s-),q))-f(X(s-))$$
$$-F(X(s-),q)f'(X(s-))]\nu(dq)ds.$$

Further, since Levy type processes possess the property $P(X(s-)=X(s))=1$ for any $s>0$, we obtain the equality

$$\int_{\mathbb{R}} f(y)P(0,x;t,dy)-f(x)=\int_0^t \int_{\mathbb{R}}[a(z)f'(z)+\frac{1}{2}b^2(z)f''(z)]P(0,x;s,dz)ds$$

$$+\int_0^t \int_{|q|\geq 1} \int_{\mathbb{R}}[f(z+K(z,q))-f(z)]P(0,x;s,dz)\nu(dq)ds$$

$$+\int_0^t \int_{|q|<1} \int_{\mathbb{R}}[f(z+F(z,q))-f(z)-F(z,q)f'(z)]P(0,x;s,dz)\nu(dq)ds.$$

The right-hand side of the resulting equality are integrals with a variable upper limit, then after differentiating both sides of this equality with respect to parameter t we obtain the direct equation for transition probability:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(y)P(0,x;t,dy)=\int_{\mathbb{R}}[a(y)f'(y)+\frac{1}{2}b^2(y)f''(y)]P(0,x;t,dy)$$
$$+\int_{|q|\geq 1} \int_{\mathbb{R}}[f(y+K(y,q))-f(y)]P(0,x;t,dy)\nu(dq) \tag{2.3}$$
$$+\int_{|q|<1} \int_{\mathbb{R}}[f(y+F(y,q))-f(y)-F(y,q)f'(y)]P(0,x;t,dy)\nu(dq).$$

We show that (2.3) is correct in a space of distributions, where functions f will play the role of test functions for functionals determined by the transition probability. To do this, we consider $\Phi = C_c^2(\mathbb{R})$, the linear space of compactly supported twice continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ and Φ', the space of linear continuous functionals on Φ. Since integral $\int_{\mathbb{R}} f(y)P(0,x;t,dy)$ exists for any $f\in C_c(\mathbb{R})$ functional $p(0,x;t,\cdot)$ is well defined as follows:

$$\int_{\mathbb{R}} f(y)P(0,x;t,dy) =: \langle f(y), p(0,x;t,y)\rangle, \quad f\in C_c(\mathbb{R}). \tag{2.4}$$

In particular for $f\in\Phi$, we call $p(0,x;t,\cdot)$ the generalized transition probability density of X. If the transition probability has a classical density, then $p(0,x;t,\cdot)$ is a regular generalized function and (2.4) turns into the equality for integrals.

Having defined the functional $p(0,x;t,\cdot)$ on Φ, we pass to the formalization of Eq. (2.3) in Φ' and begin with "differential" terms of the equation. Since there exists $\frac{\partial}{\partial t}\int_{\mathbb{R}} f(y)P(0,x;t,dy)$, for $p(0,x;t,\cdot)$ exists a derivative with respect to t:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} f(y)P(0,x;t,dy) = \frac{\partial}{\partial t}\langle f(y), p(0,x;t,y)\rangle = \langle f(y), \frac{\partial}{\partial t}p(0,x;t,y)\rangle.$$

Next, we consider integral $\int_{\mathbb{R}}[a(y)f'(y)+\frac{1}{2}b^2(y)f''(y)]P(0,x;t,dy)$. By virtue of conditions on coefficients of (2.1) supplying existence of its solution, functions a,b satisfy the Lipschitz condition. It follows that products af' and b^2f'' define continuous functions with compact supports. Then the integral exists and is equal to

$$\langle f(y), -\frac{\partial}{\partial y}\left(a(y)p(0,x;t,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\Big(b^2(y)p(0,x;t,y)\Big)\rangle.$$

Next, we go on to formalize "integral" terms, integrals with respect to q included in the direct equation. Consider integral $\int_{|q|\geq 1}\int_{\mathbb{R}}[f(y+K(y,q)) - f(y)]P(0,x;t,dy)\nu(dq)$. Since, by conditions on coefficients, $K(\cdot,q)$, $|q|\geq 1$, is continuous, then $f(\cdot+K(\cdot,q))\in C_c(\mathbb{R})$ and the integral is equal to

$$\langle f(y), \int_{|q|\geq 1}\left(p(0,x;t,y-K(y,q)) - p(0,x;t,y)\right)\nu(dq)\rangle.$$

Finally, consider the last term in the right-hand side of (2.3). By virtue of the conditions imposed on F we obtain $f(\cdot+F(\cdot,q))$ and $F(\cdot,q)f'(\cdot)$ belong to $C_c(\mathbb{R})$. Then the term is equal to:

$$\langle f(y), \int_{|q|<1}\Big(p(0,x;t,y-F(y,q)) - p(0,x;t,y) + \frac{\partial}{\partial y}\left(F(y,q)p(0,x;t,y)\right)\Big)\nu(dq)\rangle.$$

Thus, it is shown that if coefficient a,b,K,F of (2.1) satisfy conditions supplying existence of its solution, then the direct equation for the generalized transition probability density of X is correct on functions $f\in\Phi$:

$$\langle f(y), \frac{\partial}{\partial t}p(0,x;t,y)\rangle = \langle f(y), -\frac{\partial}{\partial y}\left(a(y)p(0,x;t,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\Big(b^2(y)p(0,x;t,y)\Big)\rangle$$
$$+\langle f(y), \int_{|q|\geq 1}\left(p(0,x;t,y-K(y,q)) - p(0,x;t,y)\right)\nu(dq)\rangle \tag{2.5}$$

$$+\langle f(y), \int_{|q|<1} \Big(p(0,x;t,y-F(y,q)) - p(0,x;t,y)$$

$$+ \frac{\partial}{\partial y}\,(F(y,q)p(0,x;t,y))\Big)\,\nu(dq)\rangle.$$

By (2.4), the correctness of the direct equation for the generalized density on Φ leads to the correctness of (2.3) for the transition probability.

Now, briefly, due to the size restriction of the paper, we show that for the important in applications probabilistic characteristic

$$g(t,x) := \mathbf{E}^{t,x}[h(X(T))] = \int_{\mathbb{R}} h(y)P(t,x;T,dy),\ t\in[0;T],\quad h\in C_b(\mathbb{R}) \tag{2.6}$$

the backward equation is correct on Φ under some additional conditions on coefficients of (2.1).

At the first stage, we assume the existence of continuous partial derivatives $g'_t,\ g'_x,\ g''_{xx}$ and write the equation for $\{g(t,X(t)),\, t\in[0;T]\}$ using the Ito formula. By the Markov property of X, we have

$$\mathbf{E}[g(t,X(t))] = \mathbf{E}[\mathbf{E}^{t,X(t)}[h(X(T))]]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)P(t,x;T,dy)P(0,\xi;t,dx) = \int_{\mathbb{R}} h(y)P(0,\xi;T,dy) = \mathbf{E}[g(0,X(0))].$$

Therefore, the expectation of the right-hand side of the equation written for $\{g(t,X(t))\}$ on the basis of formula (2.2) is zero. Using the Fubini stochastic theorem we can move the expectation under the integral sign, then using the martingale property of W and $\widetilde{N}$ we obtain expectation of integrals over W and $\widetilde{N}$ are equal to zero.

If the evolution of X started at the moment $t\in[0;T]$ from the point $X(t)=x\in\mathbb{R}$, then the resulting equality leads to the backward equation for g:

$$-g'_t(t,x) = a(x)g'_x(t,x) + \frac{1}{2}b^2(x)g''_{xx}(t,x) + \int_{|q|\ge 1}\Big[g\Big(t,x+K(x,q)\Big) - g(t,x)\Big]\nu(dq)$$

$$+ \int_{|q|<1}\Big[g\Big(t,x+F(x,q)\Big) - g(t,x) - F(x,q)g'_x(t,x)\Big]\,\nu(dq),\quad t\in[0;T]. \tag{2.7}$$

Now we pass to the second stage, the study of the correctness of (2.7) in spaces of generalized functions. On the previous stage (2.7) was obtained under the assumption that there exist continuous partial derivatives $g'_t,\ g'_x,\ g''_{xx}$. The existence of derivatives $g'_x,\ g''_{xx}$ and corresponding derivatives $p'_x(t,x;T,\cdot),\ p''_{xx}(t,x;T,\cdot)$, as we can see from formulas (3.1)–(3.2) in the next section, is closely related to existence of derivatives of coefficients. In the general case, even provided that the coefficients of (2.7) are twice continuously differentiable, $g'_t,\ g'_x,\ g''_{xx}$ may not exist. More accurately: the following conditions guarantee the existence of the derivatives of g: coefficients $a(x), b(x),\ F(x,q), |q| < 1,\ K(x,q), |q| \geq 1$ are twice continuously differentiable with respect to x and their derivatives satisfy Lipschitz and sub-linear growth conditions, function h is twice continuously differentiable and its derivatives are bounded [4, 5]. Thus, we formalize the backward equation for g with $h \in C_b(\mathbb{R})$ on the space of test functions Φ under the conditions on $a(\cdot), b(\cdot),\ F(\cdot,q),\ K(\cdot,q)$ to be twice continuously differentiable on $\mathbb{R}$. Indeed, for $f \in \Phi$ and such $a,\ b,\ K,\ F$ the following equalities are correct:

$$\langle f(x),\ a(x)g'_x(t,x)\rangle = -\langle (a(x)f(x))',\, g(t,x)\rangle,$$

$$\langle f(x),\ b^2(x)g''_{xx}(t,x)\rangle = \langle (b^2(x)f(x))'',\ g(t,x)\rangle,$$

$$\langle f(x),\ F(x,q)g'_x(t,x)\rangle = -\langle (F(x,q)f(x))',\ g(t,x)\rangle,$$

$$\langle f(x),\ g(t,x+F(x,q))\rangle = \langle f(x-F(x,q)),\ g(t,x)\rangle,\ |q| < 1,$$

$$\langle f(x),\ g(t,x+K(x,q))\rangle = \langle f(x-K(x,q)),\ g(t,x)\rangle,\ |q| \geq 1,\ t \in [0;T].$$

Under the indicated conditions on coefficients, these equalities justify the correctness of (2.7) on $f \in \Phi$. It is not difficult to show that under these conditions a backward equation for the transition density can be obtained on Φ as well.

Remark It is important to note that although we justified the direct and backward equations in distribution spaces, the Cauchy problems for them: the problem with an initial condition for the direct equation and with final condition $g(T,x) = h(T)$ for the backward one, are well-posed from the point of view of the theory of ill-posed problems. This is the fundamental difference between the considered finite-dimensional problems and the infinite-dimensional ones, where ill-posedness can arise due to generators that do not generate semigroups of class C_0 [6].

3 The Approach via Limit Relations

This approach goes back to the ideas of A.N. Kolmogorov (see, e.g. [7]) for diffusion processes and is based on three limit values (3.1)–(3.3).

Let $p(t,x;T,y)$ be the transition probability density of a process X and let for any $\varepsilon > 0$ there exist finite limits

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|z-x|<\varepsilon} (z-x)p(t,x;t+\Delta t,z)dz = a(t,x) + O(\varepsilon), \tag{3.1}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|z-x|<\varepsilon}(z-x)^2 p(t,x;t+\Delta t,z)dz = b(t,x)+O(\varepsilon), \quad (3.2)$$

$$\lim_{\Delta t\to 0}\frac{p(t,x;t+\Delta t,z)}{\Delta t} = G(t,x;z), \qquad |z-x|>\varepsilon, \quad (3.3)$$

uniform with respect to x, z and t, and with respect to x and t in (3.2). Then for any $f\in C^2(\mathbb{R})$, transition density $p(t,x;T,y)$, $0\le t\le T<\infty$, satisfies the direct equation [3, pp. 51, 56]:

$$\frac{\partial}{\partial T}\int_{\mathbb{R}} f(y)p(t,x;T,y)dy = \int_{\mathbb{R}}\left[a(t,y)\frac{\partial f(y)}{\partial y}p(t,x;T,y) + \frac{1}{2}b(t,y)\frac{\partial^2}{\partial y^2}p(t,x;T,y)\right]dy$$

$$+\int_{\mathbb{R}} f(y)dy\int_{\mathbb{R}\setminus 0}dz\,[G(T,z;y)p(t,z;T,y)-G(T,y;z)p(t,x;T,y)]$$

and, under the assumption that there exist $p'_t,\ p'_x,\ p''_{xx}$, the backward equation:

$$-p'_t(t,x;T,y) = a(t,x)p'_x(t,x;T,y)+\frac{1}{2}b(t,x)p''_{xx}(t,x;T,y)$$

$$+\int_{\mathbb{R}\setminus 0}(p(t,z;T,y)-p(t,x;T,y))G(t,x;z)dz.$$

Note that in the case of diffusion processes, the third limit is zero, in general case this limit describes the discontinuity of X.

As an example of using the approach, we obtain direct and backward equations for the transition density of $X=\{X(t)=at+bW(t)+cN(t)\}$, where $W=\{W(t),\ t\ge 0\}$ is the standard Wiener process, $N=\{N(t), t\ge 0\}$ is the Poisson process with intensity λ, and a, b, c are constants. We assume that W and N are set independently of each other. Since the density of W is determined by the equality

$$p_W(t,x;T,y)=\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-x)^2}{2(T-t)}},$$

density of process $\{at+bW(t)\}$ has the form:

$$p_{a,b,W}(t,x;T,y)=\frac{1}{b\sqrt{2\pi(T-t)}}e^{-\frac{(y-x-a(T-t))^2}{2b^2(T-t)}}.$$

Knowing the law of distribution of a Poisson process, we write out the density of process cN

$$p_{cN}(t, x; T, y) = \sum_{k=0}^{[\frac{y-x}{c}]} \frac{(\lambda(T-t))^k}{k!} \delta(y - x - ck) e^{-\lambda(T-t)}.$$

As a result, we obtain the transition density of X as the convolution of densities of three independent processes:

$$p(t, x; T, y) = \frac{e^{-\lambda(T-t)}}{b\sqrt{2\pi(T-t)}} \sum_{k=0}^{\infty} \frac{(\lambda(T-t))^k}{k!} e^{-\frac{(y-x-ck-a(T-t))^2}{2b^2(T-t)}}.$$

Now we calculate limits (3.1)–(3.2):

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|z-x|<\varepsilon} (z-x) p(t, x; t+\Delta t, z) dz = \left[\begin{array}{ll} a, & \varepsilon \le c, \\ a + c\lambda, & \varepsilon > c \end{array} \right.$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|z-x|<\varepsilon} (z-x)^2 p(t, x; t+\Delta t, z) dz = \left[\begin{array}{ll} b^2, & \varepsilon \le c \\ b^2 + c^2\lambda, & \varepsilon > c \end{array} \right.$$

and function $G(t, x; z)$:

$$G(t, x; z) = \lim_{\Delta t \to 0} \frac{p(t, x; t+\Delta t, z)}{\Delta t} = \left[\begin{array}{ll} \lambda\delta(z - x - c), & \varepsilon \le c, \\ 0, & \varepsilon > c. \end{array} \right.$$

The found limit values allow us to write the direct and backward equations for the density of X on Φ:

$$\begin{aligned} \langle f(y), p'_T(t, x; T, y) \rangle = \langle f(y), -a p'_y(t, x; T, y) + \frac{b^2}{2} p''_{yy}(t, x; T, y) \\ + \lambda(p(t, x; T, y - c) - p(t, x; T, y)) \rangle. \end{aligned}$$

$$\begin{aligned} \langle f(x), -p'_t(t, x; T, y) \rangle = \langle f(x), a p'_x(t, x; T, y) + \frac{b^2}{2} p''_{xx}(t, x; T, y) \\ + \lambda(p(t, x + c; T, y) - p(t, x; T, y)) \rangle. \end{aligned}$$

In the considered example, since a and b are constants, we need not additional conditions on coefficients in the backward equation.

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