Optimal Harvesting in Age- and Size-Structured Population Models

N. Hritonenko, M. C. A. Leite, and Y. Yatsenko

Abstract This paper briefly surveys age- and size- structured linear and nonlinear population models and reveals their practical meaning. It focuses on optimal control problems with two types of harvesting and their impact on the sustainable harvesting.

Keywords Age- and size-structured population models · Harvesting rate and effort · Two-dimensional optimal controls · PDE · State constraints

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1 Introduction

Various mathematical models have been suggested to capture population dynamics. PDE models are commonly used when age or size of individuals is relevant. They have been constantly extended to address practical needs, restrictions, and quotas. Sustainable harvesting is an important task in agriculture, aquaculture, forestry, fishery, and other applications. The paper aims to show relations between age- and size- structured models (Sect. 2).

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Harvesting in populations often depends on age or size of individuals. Such harvesting models have been offered and analyzed since 1980s, see surveys in [1, 13, 19], and references therein. Two types of harvesting control, harvesting rate and effort, and their practical relevance are analyzed in Sect. 3.

Finding effective harvesting regime in age- or size-structured population involves the optimal control of PDEs with two-dimensional controls and state constraints [1–3, 5, 9–17, 20, 23]. Analyzing optimal harvesting in linear and nonlinear population models, we explore the bang-bang form of the optimal harvesting and present applied interpretation of model assumptions and outcomes in Sect. 4.

2 Age- and Size-Dependent Population Models

The mathematical description of age- and size-structured populations has a long history, though the choice of the best model is still not always clear. Moreover, the relation between age and size in most biological populations is rather weak.

Let us briefly discuss connections and differences between two well-known linear PDE models.

The Lotka-McKendrick model describes the dynamics of an age-structured population under abundant resources [2, 10, 11, 13, 18, 19]

$$\frac{\partial x(t,a)}{\partial t} + \frac{\partial x(t,a)}{\partial a} = -\mu(t,a)x(t,a), \tag{2.1}$$

$$x(t,0) = p(t), \qquad x(0,a) = x_0(a), \qquad a \in [0,A), \quad t \in [0,\infty). \tag{2.2}$$

The function $x(t,a)$ represents the population density of individuals of age a at time t in the sense that the population size is given by $N(t) = \int_0^A x(t,a)da$ at time t, where A is the population maximum age. Then $\frac{\partial x(t,a)}{\partial a}$ is an "aging" term, $x_0(a)$ represents the population density at the initial moment, and $\mu(t,a)$ is the age-specific mortality rate. The influx of new individuals is determined by a fertility equation if the natural reproduction is allowed or by the density $p(t)$ of introduced new species in a fully managed population, in which all young individuals (trees, fish, etc.) are introduced from outside.

The related size-structured population model [7, 15] can be represented by

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial (g(t,l)x(t,l))}{\partial l} = -\mu(t,l)x(t,l), \tag{2.3}$$

$$g(t,l_0)x(t,l_0) = p(t),\ x(0,l) = x_0(l),\ l \in [l_0, l_{max}],\ t \in [0,\infty), \tag{2.4}$$

where $x(t,l),\ \mu(t,l),\ p(t)$, and $x_0(l)$ have the same meaning as in the model (2.1)–(2.2) but with respect to the size l that can be viewed as an individual's length, diameter, weight, volume, or other physiological quantity. The total population size at any time t in (2.3)–(2.4) is $N(t) = \int_{l_0}^{l_{max}} x(t,l)dl$ and $\dfrac{\partial(g(t,l)x(t,l))}{\partial l}$ is a "growth" term. The growth rate $g(t,l)$ can be interpreted by $\int_{l_1}^{l_2} \frac{1}{g(t,l)} dl$ as the time during which an individual grows from size l_1 to l_2, with $l_0 \le l_1 \le l_2 \le l_{max}$.

Equations (2.1)–(2.2) and (2.3)–(2.4) differ only by the presence of the growth rate $g(t,l)$. To justify g in the initial condition (2.4), let us consider a relation among the functions $x(t,l)$, $p(t)$, and $g(t,l)$ in a neighborhood of the initial time 0. The size of new individuals changes as $\Delta l \approx g(t,l)\Delta t$ during a small time period Δt, while the age changes as $\Delta a = \Delta t$. Then, $\Delta x \approx x(t,l_0)\Delta l \approx x(t,l_0)g(t,l_0)\Delta t$. Furthermore, new individuals of size l_0 brought into the population during Δt change their density by $\Delta x \approx p(t)\Delta t$ during Δt. Equating the last two statements we obtain (2.4). Similar reasoning explains the appearance of g in the second term of (2.3).

We consider examples of optimal control in age- and size-structured population models with relevant applied interpretation in the following sections.

3 Harvesting Control

Rational harvesting in populations is one of the most common applied problems in forestry, fishery, and agriculture [1–7, 9, 13, 16, 17, 19–22]. Depending on practical needs, harvesting objectives vary significantly and may include recommending the maximum sustainable yield or profit, minimizing environmental damage, preventing population extinction, and other requirements. Dependence of a harvesting control h on time, age, or size, or other factors greatly affects the investigation techniques. Let us consider two types of a two-dimensional harvesting control $h(t,a)$, the harvesting effort and rate.

Harvesting in fishery is often directly proportional to the density or abundance of a fish population. It can be measured by the overall effort of catching fish and the size of a stock.

Then the harvesting control $h(t,a) = u(t,a)x(t,a)$, where $u(t,a)$ is the *harvesting effort* that measures the expense made at time t to harvest fish of age a from all the available fish, and (2.1) can be extended to

$$\frac{\partial x(t,a)}{\partial t} + \frac{\partial x(t,a)}{\partial a} = -\mu(a)x(t,a) - u(t,a)x(t,a). \tag{3.1}$$

The model (3.1), (2.2) reflects the *catch-per-unit-effort* hypothesis and assumes the cost of harvesting to be proportional to the effort. In the case of a sole owner of a fish resource or in forestry, a certain portion of a population is harvested at a constant cost. Then the harvesting control $h(t,a) = u(t,a)$, where $u(t,a)$ is the *harvesting* *rate* or *intensity* at age a and time t, and the evolution equation (2.1) of a stock becomes

$$\frac{\partial x(t,a)}{\partial t} + \frac{\partial x(t,a)}{\partial a} = -\mu(a)x(t,a) - u(t,a). \tag{3.2}$$

The choice of a harvesting control is determined by the nature and objectives of a practical problem. The harvesting rate is commonly used in economics [11, 16], farming (agriculture, aquaculture), while the harvesting effort is preferred in modeling of wild populations, mostly forestry [6, 13, 16, 17, 23] and fishing. The Gordon-Schaefer, Beverton-Holt, and other bioeconomic models of open-access commercial fishing use the effort as an endogenous control.

4 Optimization in Age-, Size-Structured Models

Harvesting problems have been intensively investigated in [1–3, 8–17], and others. The corresponding optimal control problems (OCPs) are constructed to find the most effective harvesting policy. Formulation of a detailed objective function with meaningful applied interpretation often poses more challenges than the choice of a model. Investigation steps in OCPs usually include extremum conditions, the existence and uniqueness of solutions, qualitative and quantitative analyses of optimal trajectories, bang-bang solutions, and sustainable development of a system. Their outcomes are important in development of rational harvesting strategy. We will focus on OCPs with two-dimensional controls implemented in the Lotka-McKendrick model and its nonlinear modifications and analyze different bang-bang solutions depending on what harvesting control, rate or effort, has been chosen.

4.1 Optimization in Age-Structured Models

Let us consider the OCP of maximizing the harvesting profit in (3.1), (2.2).

$$\max_{u,p,x} I = \max_{u,p,x} \int_0^T e^{-rt} \left[\int_0^A c(t,a)u(t,a)x(t,a)da - b(t)p(t) \right] dt, \tag{4.1}$$

$$0 \le u(t,a) \le u_{max}, \qquad 0 \le p(t) \le p_{max}, \quad x(t,a) \ge 0, \tag{4.2}$$

where, $u \in L^\infty([0,T) \times [0,A)), p \in L^\infty[0,T)$, and $x_0 \in L^\infty[0,A), x \in C([0,T), L^\infty[0,A))$. In this OCP $c(t,a)$ is the unit price of the harvesting output, $b(t)$ is the price of introduced individuals, and e^{-rt} is the discounting factor. The functional (4.1) describes the present value of the profit as the difference of harvesting revenue and operating costs to bring new individuals.

An important feature of harvesting optimization models is the occurrence of bang-bang solutions. A bang-bang optimal control is a solution u that switches between the boundaries 0 and u_{max} of the constraint-inequality $0 \leq u \leq u_{max}$. The strong bang-bang principle defines conditions under which the optimal control takes only boundary values, while the weak bang-bang principle allows u to take interior values.

The *strong bang-bang principle* [10] for the OCP (4.1), (3.1), (4.2), (2.2), shows that the optimal harvesting effort $u(t, a)$ is of the form

$$u^\star(t,a) = \begin{cases} 0, & 0 \leq a < a^\star(t), \\ u_{max}, & a^\star(t) \leq a \leq A, \end{cases} \tag{4.3}$$

if

$$\frac{\partial c}{\partial a} > 0. \tag{4.4}$$

The *strong bang-bang principle* also occurs in the OCP (4.2), (4.1), (2.2) in the following nonlinear Gurtin-McCamy model

$$\frac{\partial x(t,a)}{\partial t} + \frac{\partial x(t,a)}{\partial a} = -\mu(E(t),a)x(t,a) - u(t,a)x(t,a), \tag{4.5}$$

where $E(t)$ reflects the environmental impact and intensity of the intra-specific competition. In this model the key assumptions are

$$\frac{\partial c}{\partial a} > 0, \quad \frac{\partial \mu}{\partial E} \geq 0, \quad \frac{\partial \mu}{\partial a} \geq 0. \tag{4.6}$$

The dependence of the mortality rate $\mu(E, a)$ on $E(t)$ reflects the intra-species competition. This is a common example of the *non-local nonlinearity* that represents non-local effects in the population. Although conditions (4.6) are stronger than condition (4.4), they are still realistic and reflect the increase of profit with individual's age and increase of the mortality caused by both, age and intra-species competition.

The age-structured population models with controlled harvesting rate are more natural in real applications, especially in forestry, though they lead to additional mathematical challenges caused by the active state constraint $x \geq 0$. The optimal control in the Lotka-McKendrik model (3.2) and (2.2) with two-dimensional optimal harvesting rate $u(t, a)$ and the objective

$$\max_{u,p,x} I = \max_{u,p,x} \int_0^T e^{-rt} \left[\int_0^A c(t,a)u(t,a)da - b(t)p(t) \right] dt \tag{4.7}$$

takes the *bang-bang form*

$$u^{\star}(t,a) = \begin{cases} 0, & 0 \le a < a^{\star}(t), \\ u_{max}, & a^{\star}(t) \le a \le a_e,(t), \\ 0, & a_e(t) < a \le A, \end{cases} \quad (4.8)$$

under restrictions

$$u_{max} >> 1 \quad \text{and} \quad \int_0^A c(t,a)e^{-\int_0^a \mu(t,\xi)d\xi} da > b(t), \quad t \in [0,\infty), \quad (4.9)$$

where $0 \le a^{\star}(t) < a_e(t) < A$, and the endogenous age $a_e(t)$ is determined from the condition $x(t, a_e(t)) = 0$ for each t.

The bang-bang (4.8) in the model with harvesting intensity is qualitatively different from the corresponding one (4.3) in the models with controlled effort. It states that the optimal strategy is to harvest individuals older than $a^{\star}(t)$ but younger than $a_e(t)$, that is, before they reach the maximum age A. This policy is more realistic than (4.3). For instance, in forestry, old trees, that do not have any market value, provide nutrition for younger trees when they die.

4.2 Optimization in Size-Structured Models

The existence of bang-bang solutions has been proven for some OCPs in size-structured models. Let us consider the following OCP in a Gurtin-McCamy model of forest management with a two-dimensional harvesting effort $u(t,l)$ that aims to find the functions $x(t,l)$, $u(t,l)$, $E(t)$, $p(t)$ for $t \in [0,\infty)$, $l \in [l_0, l_m]$, that maximize

$$\max_{u,p,x,E} J = \int_0^{\infty} e^{-rt} \left\{ \int_{l_0}^{l_m} c(t,l)u(t,l)x(t,l)dl - k(t)p(t) \right\} dt, \quad (4.10)$$

subject to the following constraints:

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial (g(E(t),l)x(t,l))}{\partial l} = -\mu(E(t),l)x(t,l)) - u(t,l)x(t,l), \quad (4.11)$$

$$E(t) = \chi \int_{l_0}^{l_m} l^2 x(t,l)dl, \quad g(E(t),l_0), x(t,l_0) = p(t), \quad (4.12)$$

$$0 \le u(t,l) \le u_{max}, \quad x(0,l) = x_0(l), \quad l \in [l_0, l_m], \quad t \in [0,\infty). \quad (4.13)$$

A forest is a renewable resource, which provides timber, offers recreation facilities, mitigates climate change, and improves air quality. Human intervention, natural disturbances, and climate change may cause irreversible and unfavorable changes in the forest dynamics. Therefore, modeling in forestry is vital to understand the dynamics of its development and predict negative consequences of climate change and human interaction.

The size of trees better suits biological and economic needs than their age. In applications to forestry, parameters and state variables of the model (4.10)–(4.13) are taken as follows: l is the tree diameter at breast height, $x(t,l)$ is the density of size-structured trees at time t, $E(t)$ reflects the environmental impact and intra-specific competition, $p(t)$ is the flux of young trees planted, and χ is a parameter related to a specific type of tree. The objective is to maximize the profit from logging. The OCP (4.10)–(4.13) introduced in [14] is a simplified version of a larger problem that maximizes profit from timber production and carbon sequestration studied in [9, 15–17]. The problem is of great importance in countries where forestry farms are compensated for keeping trees that sequester carbon from the atmosphere and store it in their trunks. Calibration of obtained theoretical outcomes from data on a Pinus Sylvestris forest in Catalonia, Spain, show good applicability of the model. It is shown in [9] and references therein that effects of climate change on a Pinus forest development can be captured by the growth rate

$$g(E,l) = (l_m - l)\hat{g}(E), \quad \hat{g}(E) = (\beta_0 - \beta_1)E, \quad \beta_0 > 0, \tag{4.14}$$

where β_0 and β_1 are parameters related to different climate change scenarios.

Assuming the growth rate (4.14) and

$$\frac{\partial c(t,l)}{\partial l} > 0, \qquad \frac{\partial \mu(E,l)}{\partial E} \geq 0, \qquad \frac{\partial \mu(E,l)}{\partial l} \geq 0, \tag{4.15}$$

the optimal control $u^\star(t,l)$ in OCP (4.10)–(4.13) has the following form

$$u^\star(t,l) = \begin{cases} 0, & l_0 \leq l < l^\star(t), \\ u_{max}, & l^\star(t) \leq l \leq l_m, \end{cases} \tag{4.16}$$

with, at most, one switching size $l^\star(t),\ l_0 \leq l^\star(t) \leq l_m,\ t \in [0,\infty)$.

Conditions (4.15) are similar to conditions (4.6) and have a similar applied interpretation. The bang-bang solution (4.16) is similar to (4.3) and justifies advantages of selective harvesting over clear cutting for sustainable forest development, which is in agreement with reality. Even countries that have a mandatory clear-cutting regime in forestry tend to change the law.

The optimal control for the linear size-structured model with controlled harvesting rate and natural reproduction was proven to be bang-bang in [11]. The nonlinear case remains an open question.

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