A Note on Supersymmetry and Stochastic Differential Equations

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Abstract We obtain a dimensional reduction result for the law of a class of stochastic differential equations using a supersymmetric representation first introduced by Parisi and Sourlas.

Keywords Invariant measures of sdes · Dimensional reduction · Supersymmetric field theories

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1 Introduction

In this paper we want to exploit a supersymmetric representation of scalar stochastic differential equations (SDEs) with additive noise and nonlinear drift V' in order to prove the well known relation between the invariant law of these SDEs and the Gibbs measure $e^{-2V(x)}\mathrm{d}x$.

The supersymmetric representation of SDEs or more generally SPDEs was first noted by Parisi and Sourlas [15, 16] and it is well known and used in the physics literature (see, e.g. [21]) where the relation between supersymmetry, SDEs and Gibbs measures (called dimensional reduction) was formally established [7, 10]. In the case of elliptic SPDEs these formal arguments have been rigorously exploited and proved [3, 14] and applied to the stochastic quantization program of quantum field theory [2, 11]. In the present paper we want to propose a similar rigorous version of dimensional reduction for one dimensional SDEs. The proof proposed here follows more closely the methods used for dimensional reduction of elliptic equations

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used in [3] (see also [14]) rather than the formal proofs of the physics literature (see, e.g. [7, 10]). The dimensional reduction of parabolic and elliptic stochastic differential equations with additive noise and gradient type non-linearity is an example of a more general phenomenon involving a supersymmetric representation of generic stochastic differential equations with Gaussian white noise. Although there are some formal arguments for proving this (conjectured) relation between dimensional reduction, supersymmetry and generic stochastic differential equations (see, e.g., [21] Chap. 15), outside the elliptic case with additive noise cited above and the standard stochastic differential equation with additive noise treated here, to the best of our knowledge, there is no proof in the general setting.

We describe in more details the result proved in this paper. Here we consider the following SDE

$$\partial_t \phi(t) + m^2 \phi(t) + f(t) V'(\phi(t)) = \xi(t), \qquad t \in \mathbb{R}, \tag{1}$$

where $m > 0$, $f : \mathbb{R} \to \mathbb{R}_+$ is a compactly supported positive even smooth function such that $f(0) = 1$, $V : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function with all derivatives bounded and ξ is a Gaussian white noise on $\mathbb{R}$. Equation (1) has a unique solution $\phi : \mathbb{R} \to \mathbb{R}$ which coincides for sufficiently negative times with the Ornstein–Uhlenbeck process $\varphi = \mathcal{G} * \xi$ where

$$\mathcal{G}(t) = e^{-m^2 t} \mathbb{I}_{t>0}.$$

This solution satisfies the integral equation

$$\phi(t) + \mathcal{G} * (f V'(\phi))(t) = \varphi(t), \qquad t \in \mathbb{R}, \tag{2}$$

and moreover its law is invariant under the inversion $t \mapsto -t$ of the time variable.

The aim of this note is to prove the following theorem.

Theorem 1 *For any bounded measurable function $F : \mathbb{R} \to \mathbb{R}$ we have*

$$\mathbb{E}\left[F(\phi(0)) e^{-2\int_{-\infty}^{0} f'(t) V(\phi(t)) \mathrm{d}t}\right] = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} F(x) e^{-m^2 x^2 - 2V(x)} \mathrm{d}x$$

where

$$\mathcal{Z} = \frac{\mathbb{E}\left[e^{-2\int_{-\infty}^{0} f'(t) V(\phi(t)) \mathrm{d}t}\right]}{\int_{\mathbb{R}} e^{-m^2 x^2 - 2V(x)} \mathrm{d}x}.$$

Proof Let μ_φ be the law of the Gaussian field $\varphi = \mathcal{G} * \xi$ on the space $C(\mathbb{R}; \mathbb{R})$ endowed with the topology of uniform convergence on bounded intervals. Girsanov theorem implies that for any measurable bounded function $F : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}\left[F(\phi(0)) e^{-2\int_{-\infty}^{0} f'(t) V(\phi(t)) \mathrm{d}t}\right] = \int F(\varphi(0)) \exp(S(\varphi)) \mu_\varphi(\mathrm{d}\varphi). \tag{3}$$

with

$$S(\varphi) = \int_{-\infty}^{0} \left[\frac{f(t)}{2} V''(\varphi(t)) - \frac{1}{2}(f(t)V'(\varphi(t)))^2 - 2f'(t)V(\phi(t)) \right] \quad \mathrm{d}t + \\ - \int_{-\infty}^{0} f(t)V'(\varphi(t)) \circ \mathrm{d}B(t).$$

Here $(B(t))_{t\in\mathbb{R}}$ is the double sided Brownian motion (adapted with respect to φ) such that $\partial_t B = \xi = (\partial_t + m^2)\varphi$ and $\circ\mathrm{d}B$ denotes the corresponding Stratonovich integral.

Parisi and Sourlas [16] observed long ago that the r.h.s. of Eq. (3) admits a representation using a Gaussian super-field Φ defined on the superspace $(t, \theta, \bar{\theta})$, where $t \in \mathbb{R}$ is the usual time variable and $\theta, \bar{\theta}$ are two Grassmann variables playing the role of additional "fermionic" spatial coordinates (see Sect. 2 for the necessary notions and notations). For the moment let us simply remark that Φ can be rigorously constructed as a random field on a non-commutative probability space with expectation denoted by $\langle\cdot\rangle$ in such a way that the expectation of polynomials in Φ can be reduced, via an analog of Wick's theorem, to linear combinations of products of covariances. If the covariance of the super-field has the form

$$\langle \Phi(t, \theta, \bar{\theta})\Phi(s, \theta', \bar{\theta}')\rangle = \frac{1}{2m^2}\mathcal{G}(|t-s|) + \mathcal{G}(t-s)(\theta'-\theta)\bar{\theta}' - \mathcal{G}(s-t)(\theta'-\theta)\bar{\theta}, \tag{4}$$

then we will prove in Theorem 8 below that the following representation formula holds

$$\int F(\varphi(0)) \exp(S(\varphi))\mu_{\varphi}(\mathrm{d}\varphi) = \left\langle F(\Phi(0)) \exp\left(\int_{-\infty}^{0} f(t+2\theta\bar{\theta})V(\Phi(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta} \right) \right\rangle. \tag{5}$$

Note that on the l.h.s. we have usual (commutative) probabilistic objects while the r.h.s. is expressed in the language of non-commutative probability.

The interest of this reformulation lies in the fact that on the superspace $(t, \theta, \bar{\theta})$ one can define supersymmetric transformations which preserve the quantity $t + 2\theta\bar{\theta}$. Integrals of supersymmetric quantities satisfy well known localization (also called dimensional reduction) formulas [5, 6, 12, 14, 17] which express integrals over the superspace as evaluations in zero, more precisely if $F = f(t + 2\theta\bar{\theta}) \in \mathcal{S}(\mathfrak{S})$ is a supersymmetric function and $T \in \mathcal{S}'(\mathfrak{S})$ is a supersymmetric distribution we have that

$$\int_{-\infty}^{K} T(t, \theta, \bar{\theta}) \cdot F(t, \theta, \bar{\theta})\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta} = -2T_{\emptyset}(K)F_{\emptyset}(K)$$

for any $K \in \mathbb{R}$ (see Theorem 9 for a precise statement).

We cannot apply Theorem 9 directly to expression (5) since the superfield Φ is not supersymmetric. On the other hand the correlation function (4) is supersymmetric with respect to $(s, \theta', \bar{\theta}')$ when $t \geqslant s$ and with respect to $(t, \theta, \bar{\theta})$ when $t \leqslant s$. This property and the Markovianity of the kernel $\mathcal{G}$, namely

$$\mathcal{G}(t-s)\mathcal{G}(s-t)=0$$

when $s \neq t$, allows us to prove a localization property for the *expectation of super-symmetric linear functionals of* Φ (see Theorem 14), namely we prove that

$$\left\langle F(\Phi(0)) \exp\left(\int_{-\infty}^{0} f(t+2\theta\bar{\theta}) V(\Phi(t,\theta,\bar{\theta})) \mathrm{d}t \mathrm{d}\theta \mathrm{d}\bar{\theta}\right)\right\rangle = \langle F(\Phi(0)) \exp[-2\, V(\Phi(0))]\rangle.$$

Since $\Phi(0)$ is distributed as a Gaussian with mean 0 and variance $2m^{-2}$ this implies the claim. □

The rest of the paper contains details on the definition of the super-fields and the proofs of the intermediate results.

2 Super-Geometry and Gaussian Super-Fields

2.1 Some Notions of Super-Geometry

We denote by $\mathfrak{G}$ an infinite dimensional Grassmannian algebra generated by an enumerable number of free generators $\{1, \theta_1, \theta_2, \dots, \theta_n, \dots\}$. By this we mean that any element of $\Theta \in \mathfrak{G}$ can be written in a unique way using a finite number of sum and products between the generators θ_i. The product between the θ_i is anti-commuting which means that $\theta_i\theta_j = -\theta_j\theta_i$, for all $i, j \in \mathbb{N}$, and they commute with 1. We call $\mathfrak{G}_0 = \mathrm{span}\{1\}, \mathfrak{G}_1 = \mathrm{span}\{\theta_1, \theta_2, \dots, \theta_n, \dots\}$ and with $\mathfrak{G}_k = \mathrm{span}\{\theta_{i_1} \cdots \theta_{i_k} | \theta_{i_j} \in \mathfrak{G}_1\}$.

If $\theta_1, \dots, \theta_h \in \mathfrak{G}_1$ we denote by $\mathfrak{G}(\theta_1, \dots, \theta_h)$ the finite dimensional Grassmannian sub-algebra of $\mathfrak{G}$ generated by $\{1, \theta_1, \dots, \theta_h\}$, and we denote by $\mathfrak{O}_h$ the universal Grassmannian algebra generated by h elements. We suppose that there is an order between the generators of $\mathfrak{O}_h$. Once we fixed an order between $\theta_1, \dots, \theta_h$ there is a natural isomorphism between $\mathfrak{O}_h$ and $\mathfrak{G}(\theta_1, \dots, \theta_h)$.

We can define a notion of smooth function $F : \mathbb{R}^n \times \mathfrak{G}_1^h \to \mathfrak{G}$. Let $\tilde{F}$ be a smooth function from $\mathbb{R}^n$ taking values in $\mathfrak{O}_h$ which means an object of the form

$$\tilde{F}(x) = \widetilde{F}_{\emptyset}(x)1 + \sum_{i=1}^{h} \tilde{F}_i(x)\mathfrak{t}_i + \sum_{1 \leqslant i<j \leqslant h} \tilde{F}_{i,j}(x)\mathfrak{t}_i\mathfrak{t}_j + \cdots + \tilde{F}_{1,2,\dots,h}(x)\mathfrak{t}_1 \cdots \mathfrak{t}_h,$$

where $\mathfrak{t}_1, \dots, \mathfrak{t}_h$ are a set of generators of $\mathfrak{O}_h$ and $x \in \mathbb{R}^n$. We define F associated with $\tilde{F}$ in the following way: F associates to $(x, \theta_1, \dots, \theta_h) \in \mathbb{R}^n \times \mathfrak{G}_1^h$ the element $\tilde{F}(x) \in \mathfrak{G}(\theta_1, \dots, \theta_h)$ where we make the identification of $\mathfrak{G}(\theta_1, \dots, \theta_h)$ with $\mathfrak{O}_h$, i.e.

$$F(x,\theta_1,\ldots,\theta_h)=\widetilde{F_\emptyset}(x)1+\sum_{i=1}^{h}\tilde{F}_i(x)\theta_i+\sum_{1\leqslant i<j\leqslant h}\tilde{F}_{i,j}(x)\theta_i\theta_j+\cdots+\tilde{F}_{1,2,\ldots,h}(x)\theta_1\cdots\theta_h. \tag{6}$$

Hereafter we use the notation $F_\emptyset, F_{\theta_1},\ldots$ for denoting $F_\emptyset=\tilde{F}_\emptyset, F_{\theta_1}=\tilde{F}_1,\ldots$ .We say that F is a Schwartz function if $F_\emptyset, F_i,\ldots$ are Schwartz functions. We denote by $\mathcal{S}(\mathfrak{S}^h)$ the set of Schwartz functions with h anti-commuting variables.

If $H:\mathbb{R}\to\mathbb{R}$ is a smooth function we can define the composition $H\circ F$ in the following way

$$\begin{aligned}H\circ F(x,\theta_1,\ldots,\theta_h)=&\,H(F_\emptyset(x))1+H'(F_\emptyset(x))(F(x,\theta_1,\ldots,\theta_h)-F_\emptyset(x)1)\\&+\tfrac{1}{2}H''(F_\emptyset(x))(F(x,\theta_1,\ldots,\theta_h)-F_\emptyset(x)1)^2\\&\cdots+\tfrac{1}{h!}H^{(h)}(F_\emptyset(x))(F(x,\theta_1,\ldots,\theta_h)-F_\emptyset(x)1)^h.\end{aligned}$$

On $\mathfrak{S}$ it is possible to define a notion of integral called Berezin integral, in the following way $\int\theta\mathrm{d}\theta=1$, $\int\bar{\theta}\mathrm{d}\theta=0$ if $\bar{\theta}\in\mathfrak{S}_1$ and $\bar{\theta}\neq\theta$, $\int\Theta\theta\mathrm{d}\theta=\Theta$ where $\Theta=\theta_1\cdots\theta_h\in\mathfrak{S}_h$ and $\theta_i\neq\theta$ and $\int\cdot\mathrm{d}\theta$ is linear in its argument. The integral $\int\Theta\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_h$ is defined as $\int\left(\cdots\left(\int\left(\int\Theta\mathrm{d}\theta_1\right)\mathrm{d}\theta_2\right)\cdots\right)\mathrm{d}\theta_h$.

If F is a smooth function, defined on $\mathbb{R}^n\times\mathfrak{S}_1^h$, we can define the integral of F with respect to $\mathrm{d}x\mathrm{d}\theta_1\cdots d\theta_h$ in the following way $\int F(x,\theta_1,\ldots,\theta_h)\mathrm{d}x\mathrm{d}\theta_1\cdots\mathrm{d}\theta_h$ first applying the integral $\int\cdot\mathrm{d}x$ to the set of functions $F_\emptyset, F_i,\ldots$ (which are the component of the function F by Eq. (6)) obtaining an element of $\mathfrak{S}(\theta_1,\ldots,\theta_h)$ and then applying the Berezin integral to this result. Using this notion of integral and the induced duality between smooth functions, it is possible to define the notion of tempered distribution $T\in\mathcal{S}'(\mathbb{R}^n\times\mathfrak{S}^h)$. The distribution T is an object of the form

$$T(x,\theta_1,\ldots,\theta_k)=T_\emptyset(x)1+\sum_{i=1}^{h}T_{\theta_i}(x)\theta_i+\cdots+T_{\theta_1\ldots\theta_h}(x)\theta_1\cdots\theta_h$$

where $T_\emptyset(x)$, $T_{\theta_i}(x)$, ..., $T_{\theta_1\ldots\theta_h}(x)$ are Schwartz distributions on $\mathbb{R}^n$.

2.2 Construction of the Super-Field

Following the analogous construction in [3, 14] the super-field Φ is defined as

$$\Phi(t,\theta,\bar{\theta})=\varphi(t)+\bar{\psi}(t)\theta+\psi(t)\bar{\theta}+\omega(t)\theta\bar{\theta},$$

where $t\in\mathbb{R}$, and $\varphi,\psi,\bar{\psi},\omega$ are complex Gaussian fields realized as functional from $\mathcal{S}(\mathbb{R})$ into the set of operators $\mathcal{O}(\mathfrak{H})$ on a complex vector space $\mathfrak{H}$ with a fixed state Ω, and $\theta,\bar{\theta}$ are any pair of anti-commuting variables $\theta,\bar{\theta}\in\mathfrak{S}$ commuting with the operators ω,φ and anti-commuting with the operators $\psi,\bar{\psi}$. Hereafter we shall use the notation denote by $\langle a\rangle_\Omega=\langle\Omega,a(\Omega)\rangle_{\mathfrak{H}}$ for any $a\in\mathcal{O}(\mathfrak{H})$, where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the scalar product in $\mathcal{H}$. For a background on superfield, supermanifolds and Berezin integral see, e.g., [4, 8] and furthere referefnces in [2, 3].

The Gaussian fields $\varphi, \psi, \bar{\psi}, \omega$ mush be realized as operators defined from $\mathcal{S}(\mathbb{R})$ taking values in $\mathcal{O}(\mathfrak{H})$, for a suitable Hilbert space $\mathfrak{H}$ with a state $\Omega \in \mathfrak{H}$ such that the condition (4) holds. Making a formal computation we obtain that

$$\langle \Phi(t,\theta,\bar{\theta})\Phi(s,\theta',\bar{\theta}')\rangle_\Omega = \langle \varphi(t)\varphi(s)\rangle_\Omega - \langle \bar{\psi}(t)\psi(s)\rangle_\Omega \theta\bar{\theta}' - \langle \psi(t)\bar{\psi}(s)\rangle_\Omega \bar{\theta}\theta' + \\ + \langle \varphi(t)\omega(s)\rangle_\Omega \theta'\bar{\theta}' + \langle \omega(t)\varphi(s)\rangle_\Omega \theta\bar{\theta} + \langle \omega(t)\omega(s)\rangle_\Omega \theta\bar{\theta}\theta'\bar{\theta}'$$

from which we get

$$\langle \varphi(t)\varphi(s)\rangle_\Omega = \tfrac{1}{2m^2}\mathcal{G}(|t-s|)\ \langle \bar{\psi}(t)\psi(s)\rangle_\Omega = \mathcal{G}(t-s)\ \langle \varphi(t)\omega(s)\rangle_\Omega = \mathcal{G}(t-s) \\ \langle \omega(t)\omega(s)\rangle_\Omega = 0. \tag{7}$$

Using the commutation relations

$$\{\varphi(t),\varphi(s)\}_+ = 0\ \{\varphi(t),\omega(s)\}_+ = 0\ \{\omega(t),\omega(s)\}_+ = 0 \tag{8}$$

$$\{\bar{\psi}(t),\psi(s)\}_- = \{\psi(t),\psi(s)\}_- = \{\bar{\psi}(t),\bar{\psi}(s)\}_- = 0 \\ \{\varphi(t)\psi(s)\}_+ = \{\varphi(t)\bar{\psi}(s)\}_+ = \{\omega(t),\psi(s)\}_+ = \{\omega(t),\bar{\psi}(s)\}_+ = 0 \tag{9}$$

where $\{K_1, K_2\}_+ = K_1K_2 - K_2K_1$ and $\{K_1, K_2\}_- = K_1K_2 + K_2K_1$ (where K_1, $K_2 \in \mathcal{B}(\mathfrak{H})$) are the commutator and the anti-commutator of closed operators having a non void common core. By Wick's theorem (see, e.g. [9] Chap. 3 Sect. 8) the expectation of arbitrary polynomials in $\varphi, \psi, \bar{\psi}, \omega$ is completely determined.

The bosonic field φ is a standard (real and commutative) Gaussian field with covariance $\mathcal{G}(|t-s|)$. Also ω is a standard (complex and commutative) Gaussian field of the form

$$\omega(t) = \xi(t) + i\eta(t),$$

where $\xi = (\partial_t + m^2)\varphi$ and η is a Gaussian white noise with Cameron-Martin space $L^2(\mathbb{R})$ independent of φ. We can realize the Gaussian field φ, ω as (unbounded) operators defined on a Hilbert space $\mathfrak{H}_{',!}$ and with a state $\Omega_{\varphi,\omega}$. We can take $\mathfrak{H}_{\varphi,\omega} = L^2(\mu_{\varphi,\omega})$ where $\mu_{\varphi,\omega}$ is the law of (φ,ω) on $C(\mathbb{R}) \times \mathcal{S}'_{\mathbb{C}}(\mathbb{R})$ and $\Omega_{\varphi,\omega} = 1$.

The fermionic fields $\psi, \bar{\psi}$ are build as follows (for a different construction of fermionic fields see also [1]). Let a, b and a^*, b^* be two creation and annihilation operators defined as bounded functionals on $\mathcal{S}(\mathbb{R})$ taking values in $\mathcal{B}(\mathfrak{H}_{\psi,\bar{\psi}})$ (where $\mathfrak{H}_{\psi,\bar{\psi}}$ is a suitable Hilbert space with a fixed state $\Omega_{\psi,\bar{\psi}}$) such that

$$\{a(f), a(g)\}_- = \{b(f), b(g)\}_- = 0 \\ \{a(f), b(g)\}_- = \{a^*(f), b(g)\}_- = 0 \\ \{a^*(g), a(f)\}_- = \{b^*(g), b(f)\}_- = \left(\int_{\mathbb{R}} f(t)g(t)\mathrm{d}t\right) I_{\mathfrak{H}_{\psi,\bar{\psi}}},$$

for any $f, g \in \mathcal{S}(\mathbb{R})$, and such that

$$\langle a(f)K\rangle_{\Omega_{\psi,\bar{\psi}}} = \langle Ka^*(f)\rangle_{\Omega_{\psi,\bar{\psi}}} = \langle b(f)K\rangle_{\Omega_{\psi,\bar{\psi}}} = \langle Kb^*(f)\rangle_{\Omega_{\psi,\bar{\psi}}} = 0,$$

where K is any bounded operator $K \in \mathcal{B}(\mathfrak{H}_{\psi,\bar{\psi}})$. We define $\mathcal{U} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$\mathcal{U}(f)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-i\xi t}}{i\xi + m^2} \hat{f}(\xi) \mathrm{d}\xi.$$

We then write

$$\psi(f) = a^*(\mathcal{U}^*(f)) + b(f), \quad \bar{\psi}(f) = b^*(\mathcal{U}(f)) - a(f),$$

where $\mathcal{U}^*$ is the adjoint of $\mathcal{U}$ in $L^2(\mathbb{R})$ with respect to the Lebesgue measure on $\mathbb{R}$. In this way, we have

$$\{\bar{\psi}(t), \psi(s)\}_- = \{\psi(t), \psi(s)\}_- = \{\bar{\psi}(t), \bar{\psi}(s)\}_- = 0,$$

and also

$$\begin{aligned}\langle \bar{\psi}(f)\psi(g)\rangle_{\Omega_{\psi,\bar{\psi}}} &= \langle b^*(f)a^*(g)\rangle_{\Omega_{\psi,\bar{\psi}}} + \langle b^*(f)b(g)\rangle_{\Omega_{\psi,\bar{\psi}}} - \langle a(f)a^*(g)\rangle_{\Omega_{\psi,\bar{\psi}}} + \\ &\quad -\langle a(f)b(g)\rangle_{\Omega_{\psi,\bar{\psi}}} = \int_{\mathbb{R}} \mathcal{U}(f)(t)g(t)\mathrm{d}t \\ &= \int_{\mathbb{R}} g(t) \int_{-\infty}^t e^{-m^2(t-s)} f(s)\mathrm{d}s\mathrm{d}t = \int_{\mathbb{R}^2} g(t)\mathcal{G}(t-s)f(s)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

In other words we have $\langle \bar{\psi}(t)\psi(s)\rangle_{\Omega_{\psi,\bar{\psi}}} = \mathcal{G}(t-s)$ as required (for a more detailed proof see, e.g., [1, 15, 16, 21]). We can define the operators $\varphi, \psi, \bar{\psi}, \omega$ as acting on a unique (quantum) probability space, by taking

$$\mathfrak{H} = \mathfrak{H}_{\varphi,\omega} \otimes \mathfrak{H}_{\psi,\bar{\psi}} \qquad \Omega = \Omega_{\varphi,\omega} \otimes \Omega_{\psi,\bar{\psi}}.$$

In order to realize the field Φ in a rigorous way we consider a complex sub-algebra $\mathfrak{A} \subset \mathcal{O}(\mathfrak{H})$ such that $\varphi, \psi, \bar{\psi}, \omega$ take values in $\mathfrak{A}$ and for any smooth function $V : \mathbb{R} \to \mathbb{R}$ we have $V(\varphi(g)) \in \mathfrak{A}$, where g is any function in $\mathcal{S}(\mathbb{R})$. This sub-algebra $\mathfrak{A}$ is generated (from an algebraic point of view) by operators of the form $V(\varphi(g))$, $\omega(g)$, $\psi(g)$, $\bar{\psi}(g)$ and $I_{\mathfrak{H}}$. We consider the vector space $\mathcal{A} = \mathfrak{A} \times \mathfrak{S}$. There are two preferred hyperplanes $\mathcal{A}_{\mathfrak{A}}$ and $\mathcal{A}_{\mathfrak{S}}$ defined by

$$\mathcal{A}_{\mathfrak{A}} = \{(a, 1_{\mathfrak{S}})|a \in \mathfrak{A}\}, \quad \mathcal{A}_{\mathfrak{S}} = \{(I_{\mathfrak{H}}, \theta)|\theta \in \mathfrak{S}\},$$

with the natural immersions $i_{\mathfrak{A}} : \mathfrak{A} \to \mathcal{A}$ and $i_{\mathfrak{S}} : \mathfrak{S} \to \mathcal{A}$ defined by $i_{\mathfrak{A}}(a) = (a, 1_{\mathfrak{S}})$ and $i_{\mathfrak{S}}(\theta) = (I_{\mathfrak{H}}, \theta)$ (we note that $\mathcal{A}_{\mathfrak{A}} := i_{\mathfrak{A}}(\mathfrak{A})$ and $\mathcal{A}_{\mathfrak{S}} := i_{\mathfrak{S}}(\mathfrak{S})$). It is clear that $\mathcal{A}_{\mathfrak{A}}, \mathcal{A}_{\mathfrak{S}}$ generates the whole $\mathcal{A}$. On $\mathcal{A}$ we define the following product $\cdot$, in such a way that the maps $i_{\mathfrak{A}}$ and $i_{\mathfrak{S}}$ respect the product (i.e. $i_{\mathfrak{A}}(ab) = i_{\mathfrak{A}}(a) \cdot i_{\mathfrak{A}}(b)$ and $i_{\mathfrak{S}}(\theta_1\theta_2) = i_{\mathfrak{S}}(\theta_1) \cdot i_{\mathfrak{S}}(\theta_2)$) and such that

$$(V(\varphi(g)), 1_{\mathfrak{S}}) \cdot (I_{\mathfrak{H}}, \theta) = (I_{\mathfrak{H}}, \theta) \cdot (V(\varphi(g)), 1_{\mathfrak{S}}) = (V(\varphi(g)), \theta)$$

$$(\omega(g), 1_{\mathfrak{S}}) \cdot (I_{\mathfrak{H}}, \theta) = (I_{\mathfrak{H}}, \theta) \cdot (\omega(g), 1_{\mathfrak{S}}) = (\omega(g), \theta)$$

$$(\psi(g), 1_{\mathfrak{S}}) \cdot (I_{\mathfrak{H}}, \theta) = -(I_{\mathfrak{H}}, \theta) \cdot (\psi(g), 1_{\mathfrak{S}}) = (\psi(g), \theta)$$

$$(\bar{\psi}(g), 1_{\mathfrak{S}}) \cdot (I_{\mathfrak{H}}, \theta) = -(I_{\mathfrak{H}}, \theta) \cdot (\bar{\psi}(g), 1_{\mathfrak{S}}) = (\bar{\psi}(g), \theta)$$

where $g \in \mathcal{S}(\mathbb{R})$ and $\theta \in \mathfrak{S}_1$ (not in $\mathfrak{S}$). The product $\cdot$ can be uniquely extended (in a associative way) on $\mathcal{A}$ since $\mathcal{A}_{\mathfrak{A}}, \mathcal{A}_{\mathfrak{S}}$ generates the whole $\mathcal{A}$, operators of the form $V(\varphi(g))$, $\omega(g)$, $\psi(g)$, $\bar{\psi}(g)$ generates the whole $\mathfrak{A}$ and $\mathfrak{S}_1$ generates the whole $\mathfrak{S}$. Hereafter we will omit to explicitly write the product $\cdot$ if this omission does not cause any confusion.

On $\mathcal{A}$ we can define a linear operator $\langle\cdot\rangle : \mathcal{A} \to \mathcal{A}_{\mathfrak{S}} \simeq \mathfrak{S}$ by

$$\langle(a, \theta)\rangle = \langle a\rangle_{\Omega}(I_{\mathfrak{H}}, \theta).$$

Furthermore for any $\theta_1, \ldots, \theta_n \in \mathfrak{S}_1$ we define the linear operator $\int \cdot \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n : \mathcal{A} \to \mathcal{A}_{\mathfrak{A}}$ such $\int \cdot \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n|_{\mathcal{A}_{\mathfrak{S}}}$ is the usual Berezin integral induced by the identification $\mathcal{A}_{\mathfrak{S}} \simeq \mathfrak{S}$

$$\int (a, \theta)\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n = \left(\int \theta \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n\right)(a, 1_{\mathfrak{S}}).$$

Hereafter we identify the space $\mathfrak{A}$ and $\mathfrak{S}$ with $\mathcal{A}_{\mathfrak{A}}$ and $\mathcal{A}_{\mathfrak{S}}$ respectively, and we write instead of $(a, 1_{\mathfrak{S}})$, $(I_{\mathfrak{H}}, \theta)$, $(I_{\mathfrak{H}}, 1_{\mathfrak{S}})$ simply a, θ and 1 respectively (in this way we take also the tacit identification of $\mathrm{span}\{1_{\mathfrak{S}}\} = \mathfrak{S}_0$ with $\mathbb{R}$). Furthermore we identify $\varphi, \psi, \bar{\psi}, \omega$ with $i_{\mathfrak{A}} \circ \varphi, i_{\mathfrak{A}} \circ \psi, i_{\mathfrak{A}} \circ \bar{\psi}, i_{\mathfrak{A}} \circ \omega$.

Remark 2 Since $\psi, \bar{\psi}$ are "independent" with respect to φ and ω (since they can be realized on a space of the form $\mathfrak{H} = \mathfrak{H}_{\varphi,\omega} \otimes \mathfrak{H}_{\psi,\bar{\psi}}$) the expectation only with respect to the fields $\psi, \bar{\psi}$ is well defined, namely there exixts an operator $\langle\cdot\rangle_{\psi,\bar{\psi}} : \mathcal{O}_{\mathfrak{H}} \to \mathcal{O}_{',!}$ such that

$$\begin{aligned}\langle V(\varphi(t_1), \ldots, \varphi(t_k))\psi(t_1')\bar{\psi}(t_1'') \cdots \psi(t_{k'}')\bar{\psi}(t_{k'}'')\rangle_{\psi,\bar{\psi}} \\ = V(\varphi(t_1), \ldots, \varphi(t_k))\langle\psi(t_1')\bar{\psi}(t_1'') \cdots \psi(t_{k'}')\bar{\psi}(t_{k'}'')\rangle.\end{aligned}$$

This operator $\langle\cdot\rangle_{\psi,\bar{\psi}}$ extends to $\mathcal{A}$ in the same way in which the operator $\langle\cdot\rangle$ is extended on the whole $\mathcal{A}$.

2.3 Relation with SDEs

In this section we want to use the super-field Φ for representing the solution to the SDE (1) through the integral (3).

First of all we have to define the notion of composition of the super-field Φ with smooth functions. Consider the smooth function $H : \mathbb{R} \to \mathbb{R}$ growing at most exponentially at infinity. We can formally expand H in Taylor series and using the properties of $\theta, \bar{\theta}$ we obtain

$$\begin{aligned} H(\Phi(t,\theta,\bar{\theta})) = & H(\varphi(t)) + H'(\varphi(t))\bar{\psi}(t)\theta + H'(\varphi(t))\psi(t)\bar{\theta} + \\ & +(H'(\varphi(t))\omega(t) + H''(\varphi(t))\psi(t)\bar{\psi}(t))\theta\bar{\theta}. \end{aligned}$$

Unfortunately the products $H'(\varphi(t))\omega(t)$ and $H''(\varphi(t))\psi(t)\bar{\psi}(t)$ are ill defined since the factors are not regular enough. For this reason we consider a symmetric mollifier $\rho : \mathbb{R} \to \mathbb{R}_+$ (with $\rho(t) = \rho(-t)$) and the field $\Phi_\epsilon = \rho_\epsilon * \Phi$, where $\rho_\epsilon(t) = \epsilon^{-1}\rho(t\epsilon^{-1})$. If G is a super-function, $F : \mathbb{R} \to \mathbb{R}$ is a smooth function and $\mathbb{K}$ is an entire function we define

$$\begin{aligned} & \left\langle F(\varphi(0))\mathbb{K}\left(\int G(t,\theta,\bar{\theta})H(\Phi(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)\right\rangle \\ & \quad := \lim_{\epsilon\to 0}\left\langle F(\varphi_\varepsilon(0))\mathbb{K}\left(\int G(t,\theta,\bar{\theta})H(\Phi_\epsilon(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)\right\rangle. \end{aligned} \quad (10)$$

We want to prove that the previous expression is well defined and does not depend on ρ.

Remark 3 It is important to note that the expression (10) does not depend on ρ only if ρ is reflection symmetric (i.e. $\rho(t) = \rho(-t)$). If we choose a different ρ (such that for example $\int_{-\infty}^{0} \rho \mathrm{d}t \neq \int_0^{+\infty} \rho \mathrm{d}t$) we will obtain a different limit in (10). This is due to the fact that the products $H'(\varphi(t))\omega(t)$ and $H''(\varphi(t))\psi(t)\bar{\psi}(t)$ are ill defined and it is analogous to the possibility to obtain Itô or Stratonovich integral in stochastic calculus considering different approximations of the stochastic integral.

Lemma 4 *Let $F_1, \ldots, F_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be smooth functions with compact support in the first variable and growing at most exponentially at infinity in the second variable then we have*

$$\lim_{\epsilon\to 0}\left\langle \prod_{i=1}^{n}\int F_i(t,\varphi_\epsilon(t))\bar{\psi}_\epsilon(t)\psi_\epsilon(t)\mathrm{d}t\right\rangle_{\psi,\bar{\psi}} = \int \prod_{i=1}^{n} F_i(t_i,\varphi(t))\mathfrak{G}_n(t_1,\ldots,t_n)\mathrm{d}t_1\ldots\mathrm{d}t_n. \quad (11)$$

in $L^p(\mu_\varphi)$ for all $1 \leq p < +\infty$. Here $\mathfrak{G}_n(t_1,\ldots,t_n) = \det((G_{i,j})_{i,j=1,\ldots,n})$ with $G_{i,j} = \mathcal{G}(t_j - t_i)$ if $i \neq j$ and $G_{i,i} = 1/2$.

Proof It is simple to see that $\lim_{\epsilon\to 0}\langle\bar{\psi}_\epsilon(t_1)\psi_\epsilon(t_2)\rangle_{\psi,\bar{\psi}} = \mathcal{G}(t_1 - t_2)$ when $t_1 \neq t_2$ and $\lim_{\epsilon\to 0}\langle\bar{\psi}_\epsilon(t)\psi_\epsilon(t)\rangle_{\psi,\bar{\psi}} = \frac{1}{2}$ (this is due to the fact that $\rho(t) = \rho(-t)$). Since $\langle\bar{\psi}_\epsilon(t_1)\psi_\epsilon(t_1)\cdots\bar{\psi}_\epsilon(t_2)\psi_\epsilon(t_n)\rangle_{\psi,\bar{\psi}}$ is uniformly bounded in t and ϵ and $F_i(t,\varphi_\epsilon(t))$ is uniformly bounded in $L^p(\mu_\varphi)$ in t and ϵ the claim follows. □

Remark 5 Since only one between $\mathcal{G}(t-s)$ and $\mathcal{G}(s-t)$ is non zero if $F_1 = F_2 = \cdots = F_n$ then we get

$$\lim_{\epsilon\to 0}\left\langle\left(\int F_1(t,\varphi_\epsilon(t))\bar{\psi}_\epsilon(t)\psi_\epsilon(t)\mathrm{d}t\right)^n\right\rangle_{\psi,\bar{\psi}} = \left(\frac{1}{2}\int F_1(t,\varphi(t))\mathrm{d}t\right)^n.$$

Lemma 6 *Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be smooth functions with compact support in the first variable and growing at most exponentially at infinity in the second variable then we have in $L^p(\mu_{\varphi,\omega})$, for all $1 \leq p < +\infty$,*

$$\lim_{\epsilon \to 0} \int F(t, \varphi_\epsilon(t))\omega_\epsilon(t) = \int F(t, \varphi(t)) \circ \mathrm{d}B(t) + i \int F(t, \varphi(t))\mathrm{d}W(t)$$

where the first one is Stratonovich integral and the second one is Itô integral with respect to (double sided) Brownian motions $(B(t), W(t))_{t\in\mathbb{R}}$ such that $\partial_t B(t) = \xi(t)$ and $\partial_t W(t) = \eta(t)$ with $B_0 = W_0 = 0$.

Proof This is the Wong–Zakai theorem [13, 18–20]. □

Theorem 7 *When $\mathbb{K}$ is a polynomial and H grows at most exponentially at infinity, or $\mathbb{K}$ is entire and H is bounded with first and second derivative bounded, the limit (10) is well defined and does not depend on the symmetric mollifier ρ.*

Proof When $\mathbb{K}$ is a polynomial the thesis follows directly from Lemmas 4 and 6. If $\mathbb{K}$ is an entire function and H is a bounded function with first and second derivatives bounded it is possible to exchange the limit in ϵ with the power series, since

$$\left\langle \left| \left\langle \left(\int G(t, \theta, \bar{\theta}) H(\Phi_\epsilon(t, \theta, \bar{\theta})) \right)^k \mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta} \right\rangle_{\psi,\bar{\psi}} \right|^p \right\rangle$$

is uniformly bounded in ϵ for any $p \geq 1$. □

Theorem 8 *Suppose that $G(t, \theta, \bar{\theta}) = G_\emptyset(t) + G_{\theta\bar{\theta}}(t)\theta\bar{\theta}$ and that H is bounded with the first and second derivatives bounded then*

$$\begin{aligned}\left\langle F(\varphi(0)) \exp\left(\int G(t, \theta, \bar{\theta}) H(\Phi(t, \theta, \bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta} \right) \right\rangle = \\ = \int F(\varphi(0)) \exp\left(\frac{1}{2} \int G_\emptyset(t) H''(\varphi(t))\mathrm{d}t - \int G_\emptyset(t) H'(\varphi(t)) \circ \mathrm{d}\xi(t) + \right. \\ \left. - \frac{1}{2} \int (G_\emptyset(t) H'(\varphi(t)))^2 \mathrm{d}t - \int G_{\theta\bar{\theta}}(t) H(\varphi(t))\mathrm{d}t \right) \mu_\varphi(\mathrm{d}\varphi).\end{aligned} \tag{12}$$

Proof The proof follows from Theorem 7, the multiplicative property of exponentials, Remark 5, and the fact that the Fourier transform of a process integrated with respect to an independent Gaussian white noise can be computed explicitly and in this case gives the factor $\exp(-\frac{1}{2} \int (G_\emptyset(t) H'(\varphi(t)))^2 \mathrm{d}t)$. □

3 Supersymmetry and the Supersymmetric Field

3.1 The Supersymmetry

On $C^\infty(\mathbb{R} \times \mathfrak{S}_1^2)$ one can introduce the (graded) derivations

$$Q := 2\theta\partial_t + \partial_{\bar{\theta}}, \qquad \bar{Q} := 2\bar{\theta}\partial_t - \partial_\theta,$$

which are such that

$$Q(t + 2\theta\bar{\theta}) = 0 = \bar{Q}(t + 2\theta\bar{\theta}),$$

namely they annihilate the function $t + 2\theta\bar{\theta}$ defined on $\mathbb{R} \times \mathfrak{S}_1^2$. Moreover if $QF = \bar{Q}F = 0$, for F in $C^\infty(\mathbb{R} \times \mathfrak{S}_1^2)$, then we must have

$$0 = QF(x, \theta, \bar{\theta}) = 2\partial_t f_\emptyset(t)\theta + f_{\bar{\theta}}(t) + \partial_t f_{\bar{\theta}}(x)\theta\bar{\theta} - f_{\theta\bar{\theta}}(t)\theta$$

$$0 = \bar{Q}F(x, \theta, \bar{\theta}) = 2\partial_t f_\emptyset(t)\bar{\theta} + f_\theta(t) - \partial_t f_\theta(x)\theta\bar{\theta} - f_{\theta\bar{\theta}}(t)\bar{\theta}$$

and therefore

$$\partial_t f_\emptyset(t) = \frac{1}{2} f_{\theta\bar{\theta}}(t) \qquad \text{and} \qquad f_\theta(t) = f_{\bar{\theta}}(t) = 0.$$

This means that there exists an $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$f(t + 2\theta\bar{\theta}) = f(t) + 2f'(t)\theta\bar{\theta} = f_\emptyset(t) + f_{\theta\bar{\theta}}(t)\theta\bar{\theta} = F(t, \theta, \bar{\theta}).$$

Namely any function satisfying these two equations can be written in the form

$$F(t, \theta, \bar{\theta}) = f(t + 2\theta\bar{\theta}).$$

Suppose that $t > 0$, if we introduce the linear transformations

$$\tau(b, \bar{b}) \begin{pmatrix} t \\ \theta \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} t + 2\bar{b}\theta\rho + 2b\bar{\theta}\rho \\ \theta - b\rho \\ \bar{\theta} + \bar{b}\rho \end{pmatrix} \in \mathfrak{S}(\theta, \bar{\theta}, \rho)$$

for $b, \bar{b} \in \mathbb{R}$ and where $\rho \in \mathfrak{S}_1$ is a new odd variable different from $\theta, \bar{\theta}$, then we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}a}\right|_{a=0} \tau(ab, a\bar{b})F(t, \theta, \bar{\theta}) = \left.\frac{\mathrm{d}}{\mathrm{d}a}\right|_{a=0} F(\tau(ab, a\bar{b})(t, \theta, \bar{\theta})) = (b \cdot \bar{Q} + \bar{b} \cdot Q)F(t, \theta, \bar{\theta})$$

so $\tau(b, \bar{b}) = \exp(b \cdot \bar{Q} + \bar{b} \cdot Q)$ and $\tau(ab, a\bar{b})\tau(cb, c\bar{b}) = \tau((a + c)b, (a + c)\bar{b})$.

In particular $F \in C^\infty(\mathbb{R} \times \mathfrak{S}^2)$ is supersymmetric if and only if for any $b, \bar{b} \in \mathbb{R}$ we have $\tau(b, \bar{b})F = F$.

By duality the operators Q, $\bar{Q}$ and $\tau(b, \bar{b})$ also act on the space $\mathcal{S}'(\mathfrak{S})$ and we say that the distribution $T \in \mathcal{S}'(\mathfrak{S})$ is supersymmetric if it is invariant with respect to rotations in space and $QT = \bar{Q}T = 0$. For supersymmetric functions and distributions the following fundamental theorem holds.

Theorem 9 *Let $F \in \mathcal{S}(\mathfrak{S})$ and $T \in \mathcal{S}'(\mathfrak{S})$ such that $T_\emptyset$ is a continuous function. If both F and T are supersymmetric. Then for any $K \in \mathbb{R}$ we have the reduction formula*

$$\int_{-\infty}^{K} T(t, \theta, \bar{\theta}) \cdot F(t, \theta, \bar{\theta}) \mathrm{d}t \mathrm{d}\theta \mathrm{d}\bar{\theta} = -2 T_\emptyset(K) F_\emptyset(K). \tag{13}$$

Proof The proof can be found in [14], Lemma 4.5 for $\mathbb{R}^2$ and in [17] for the case of a general super-manifold. Here we give the proof only for the case where T is a super-function. In this case we have that $T(t, \theta, \bar{\theta}) = T_\emptyset(t) + 2T'_\emptyset(t)\theta\bar{\theta}$ and $F(t, \theta, \bar{\theta}) = F_\emptyset(t) + 2F'_\emptyset(t)\theta\bar{\theta}$, from which we have

$$\begin{aligned} T(t, \theta, \bar{\theta}) \cdot F(t, \theta, \bar{\theta}) &= T_\emptyset(t) F_\emptyset(t) + 2(T'_\emptyset(t) F_\emptyset(t) + T_\emptyset(t) F'_\emptyset(t))\theta\bar{\theta} \\ &= T_\emptyset(t) F_\emptyset(t) + 2\partial_t(T_\emptyset F_\emptyset)(t)\theta\bar{\theta}. \end{aligned}$$

By definition of Berezin integral we have

$$\begin{aligned} \int_{-\infty}^{K} T(t, \theta, \bar{\theta}) \cdot F(t, \theta, \bar{\theta}) \mathrm{d}x \mathrm{d}\theta \mathrm{d}\bar{\theta} &= -2 \int_{-\infty}^{K} \partial_t (T_\emptyset F_\emptyset)(t) \mathrm{d}t \\ &= -2 T_\emptyset(K) F_\emptyset(K). \end{aligned}$$

□

Remark 10 In Theorem 9 we can assume that $F = F_\emptyset(t) + F_{\theta\bar{\theta}}(t)\theta\bar{\theta}$ and $T(t, \theta, \bar{\theta}) = T_\emptyset(t) + T_{\theta\bar{\theta}}(t)\theta\bar{\theta}$ where $F_{\theta\bar{\theta}}(t) = 2F'_\emptyset(t)$ and $T_{\theta\bar{\theta}}(t) = 2T'_\emptyset(t)$ only for $t \leqslant K$. In this way we can consider supersymmetric functions only on the set $(-\infty, K]$.

3.2 Localization of Supersymmetric Averages

Remark 11 We note that the correlation function

$$C^\Phi(t, s, \theta, \bar{\theta}) := \langle \varphi(t) \Phi(s, \theta, \bar{\theta}) \rangle = \frac{1}{2m^2} \mathcal{G}(t-s) + \mathcal{G}(t-s)\theta\bar{\theta}$$

is a supersymmetric function when $t \geqslant s$.

Lemma 12 *Let $g(t)$ be smooth function with compact support, $t \in \mathbb{R}$, let P be a polynomial. Then for $t_1 > t_2 > \cdots > t_k$ and $M = (m_1, \ldots, m_k) \in \mathbb{N}^k$ we have*

$$\mathcal{H}^{M,G}_{\ell,P}(t_1, \ldots, t_k) =$$
$$= \left\langle \prod_{j=1}^{k} \varphi(t_j)^{m_j} \int_{-\infty}^{t_k} \int_{-\infty}^{\tau_1} \cdots \int_{-\infty}^{\tau_\ell} \prod_{i=1}^{\ell} g\left(\tau_i + 2\theta_i\bar{\theta}_i\right) P(\Phi(\tau_i, \theta_i, \bar{\theta}_i)) \mathrm{d}\tau_i \mathrm{d}\theta_i \mathrm{d}\bar{\theta}_i \right\rangle =$$
$$= \frac{(-2\, g(t_k))^\ell}{\ell!} \left\langle \prod_{j=1}^{k} \varphi(t_j)^{m_j} P(\varphi(t_k))^\ell \right\rangle.$$

Proof We prove the lemma by induction on ℓ and for simplicity we assume that $P(x) = x^n$, the general case being a straightforward generalization. Since the proof is essentially of combinatorial nature in the following we consider some ill defined objects like the products $\varphi(t)\omega(s)$ or $\psi(t)\bar{\psi}(s)$. This fact does not change the main idea of the proof since all the expectations with respect to the previous products are defined using the symmetric regularization proposed in Lemma 4 and Lemma 6, i.e. all the following computations can be made rigorous replacing φ, ω, ψ and $\bar{\psi}$ by the regularized Gaussian fields $\varphi_\epsilon, \omega_\epsilon, \psi_\epsilon$ and $\bar{\psi}_\epsilon$ (as defined in Lemma 4 and Lemma 6) and then taking the limit as $\epsilon \to 0$. The main difference between the proof below and the one involving the regularized fields is that in the regularized case we have also to consider the contractions of the form $\omega_\varepsilon(t)\varphi_\varepsilon(s)$ and $\psi_\varepsilon(t)\bar{\psi}_\varepsilon(s)$ when $s < t$ and $|s - t| < \varepsilon$. Since the contributions of this kind of terms are proportional to the support of the mollifier ρ_ε, they go to zero as $\varepsilon \to 0$. Let

$$Y^M(t_1, \ldots, t_k) := \prod_{j=1}^{k} \varphi(t_j)^{m_j}$$

We have

$$\mathcal{H}^{M,G}_{1,x^n}(t_1, \ldots, t_k) = \left\langle Y^M(t_1, \ldots, t_k) \int_{-\infty}^{t_k} g(\tau + 2\theta\bar{\theta})(\Phi(\tau, \theta, \bar{\theta}))^n \mathrm{d}\tau \mathrm{d}\theta \mathrm{d}\bar{\theta} \right\rangle =$$
$$= \int_{-\infty}^{t_k} g(\tau + 2\theta\bar{\theta}) \langle Y^M(t_1, \ldots, t_k)(\Phi(\tau, \theta, \bar{\theta}))^n \rangle \mathrm{d}\tau \mathrm{d}\theta \mathrm{d}\bar{\theta}.$$

Since Φ and φ are Gaussian fields, by Wick theorem and by Remark 11, we have that $\langle Y^M(t_1, \ldots, t_k)(\Phi(\tau, \theta, \bar{\theta}))^n \rangle$ is supersymmetric in $(\tau, \theta, \bar{\theta})$ when $\tau \leqslant t_k$. Moreover, given that $G = g(t + 2\theta\bar{\theta})$ is a supersymmetric function by Remark 10, we have the thesis.

Suppose now that the lemma holds for $\ell - 1 \in \mathbb{N}$, then letting

$$H(\tau_1) := \int_{-\infty}^{\tau_1} \cdots \int_{-\infty}^{\tau_\ell} \prod_{i=2}^{\ell} g(\tau_i + 2\theta_i\bar{\theta}_i)(\Phi(\tau_i, \theta_i, \bar{\theta}_i))^n \mathrm{d}\tau_i \mathrm{d}\theta_i \mathrm{d}\bar{\theta}_i,$$

we have

$$
\begin{aligned}
&\mathcal{H}^{M,g}_{\ell,x^n}(t_1,\ldots,t_k) = \\
&\quad = \left\langle Y^M(t_1,\ldots,t_k)\int_{-\infty}^{t_k}\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_\ell}\prod_{i=1}^{\ell} g\left(\tau_i+2\theta_i\bar{\theta}_i\right)(\Phi(\tau_i,\theta_i,\bar{\theta}_i))^n \mathrm{d}\tau_i\mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\right\rangle \\
&\quad = \int_{-\infty}^{t_k} g(\tau_1+2\theta_1\bar{\theta}_1)\langle Y^M(t_1,\ldots,t_k)\Phi(\tau_1,\theta_1,\bar{\theta}_1)^n H(\tau_1)\rangle \mathrm{d}\tau_1\mathrm{d}\theta_1\mathrm{d}\bar{\theta}_1 \\
&\quad = \int_{-\infty}^{t_k} g'(\tau_1)\mathcal{H}^{(M,n),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1)\mathrm{d}\tau_1 - n\int_{-\infty}^{t_k}\langle Y^M(t_1,\ldots,t_k)\varphi(\tau_1)^{n-1}\omega(\tau_1)H(\tau_1)\rangle\cdot \\
&\qquad \cdot g(\tau_1)\mathrm{d}\tau_1 - n(n-1)\int_{-\infty}^{t_k}\langle Y^M(t_1,\ldots,t_k)\varphi(\tau_1)^{n-2}\psi(\tau_1)\bar{\psi}(\tau_1)H(\tau_1)\rangle g(\tau_1)\mathrm{d}\tau_1.
\end{aligned}
$$

Here $(M,n)=(m_1,\ldots,m_k,n)$. By the induction hypothesis the first term in the sum is exactly

$$
\int_{-\infty}^{t_k} g'(\tau_1)\mathcal{H}^{(M,n),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1)\mathrm{d}\tau_1 = \int_{-\infty}^{t_k} g'(\tau_1)\frac{(2\,g(\tau_1))^{\ell-1}}{(\ell-1)!}\langle\varphi(\tau_1)^{\ell n}Y^M(t_1,\ldots,t_k)\rangle\mathrm{d}\tau_1.
$$

For the second term we note that

$$
\begin{aligned}
&\left\langle\varphi(\tau_1)^{n-1}\omega(\tau_1)Y^M(t_1,\ldots,t_k)\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_\ell}\prod_{i=2}^{\ell} g(\tau_i+2\theta_i\bar{\theta}_i)(\Phi(\tau_i,\theta_i,\bar{\theta}_i))^n\mathrm{d}\tau_i\mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\right\rangle = \\
&= \sum_{j=1}^{k} m_j\langle\omega(\tau_1)\varphi(t_j)\rangle\mathcal{H}^{(M-1_j,n-1),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1) + (n-1)\langle\varphi(\tau_1)\omega(\tau_1)\rangle\mathcal{H}^{(M,n-2),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1)
\end{aligned}
$$

where $1_j=(0,\ldots,1,0,\ldots,0)\in\mathbb{N}^k$ with 1 in the j-th position and where we used Wick's theorem and the fact that

$$
\langle\varphi(\tau_1)\omega(\tau_1)\rangle = \frac{1}{2} \quad \text{and} \quad \left\langle\omega(\tau_1)\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_\ell}\prod_{i=2}^{\ell} g(\tau_i+2\theta_i\bar{\theta}_i)(\Phi(\tau_i,\theta_i,\bar{\theta}_i))^n\mathrm{d}\tau_i\mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\right\rangle = 0.
$$

Furthermore for the third term we have

$$
\begin{aligned}
&\left\langle\varphi(\tau_1)^{n-2}\psi(\tau_1)\bar{\psi}(\tau_1)\prod_{j=1}^{k}\varphi(t_j)^{m_j}\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_\ell}\prod_{i=2}^{\ell} g(\tau_i+2\theta_i\bar{\theta}_i)(\Phi(\tau_i,\theta_i,\bar{\theta}_i))^n\mathrm{d}\tau_i\mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\right\rangle = \\
&\qquad = \langle\psi(\tau_1)\bar{\psi}(\tau_1)\rangle\mathcal{H}^{(M,n-2),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1).
\end{aligned}
$$

In this way we obtain that

$$
\begin{aligned}
\mathcal{H}^{M,g}_{\ell,x^n}(t_1,\ldots,t_k) &= (-1)^{\ell-1}2^{\ell-1}\int_{-\infty}^{t_k} g'(\tau_1)\frac{(g(\tau_1))^{\ell-1}}{(\ell-1)!}\langle\varphi(\tau_1)^{\ell n}Y^M(t_1,\ldots,t_k)\rangle\mathrm{d}\tau_1 + \\
&\quad - \sum_{j=1}^{k} m_j\langle\omega(\tau_1)\varphi(t_j)\rangle\cdot\mathcal{H}^{(M-1_j,n-1),g}_{\ell-1,x^n}(t_1,\ldots,t_k,\tau_1).
\end{aligned}
$$

Here we use the fact that $\langle \varphi(\tau_1)\omega(\tau_1)\rangle = -\langle \psi(\tau_1)\bar{\psi}(\tau_1)\rangle = \frac{1}{2}$. Noting that

$$\langle \varphi^{\ell n-2}(\tau)\psi(\tau)\bar{\psi}(\tau)Y^M(t_1,\ldots,t_k)\rangle + \langle \varphi(\tau)\omega(\tau)\rangle\langle \varphi^{\ell n-2}(\tau)Y^M(t_1,\ldots,t_k)\rangle = 0$$

we obtain

$$\mathcal{H}^{M,g}_{\ell,x^n}(t_1,\ldots,t_k) = (-2)^{\ell-1}\left\langle Y^M(t_1,\ldots,t_k)\int_{-\infty}^{t_k}\frac{(g(\tau+2\theta\bar{\theta}))^{\ell}}{\ell!}\Phi^{n\ell}(\tau,\theta,\bar{\theta})\mathrm{d}\tau\mathrm{d}\theta\mathrm{d}\bar{\theta}\right\rangle =$$

$$= \frac{(-2)^{\ell-1}}{\ell!}\mathcal{H}^{M,g^{\ell}}_{1,x^{n\ell}}(t_1,\ldots,t_k)$$

Finally, the thesis follows from the induction hypothesis for $\mathcal{H}^{M,g^{\ell}}_{1,x^{n\ell}}(t_1,\ldots,t_k)$. □

Corollary 13 *Let G be a supersymmetric function with compact support, then we have*

$$\left\langle \varphi(0)^m\left(\int_{-\infty}^{0}G(t,\theta,\bar{\theta})P(\Phi(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)^k\right\rangle = (-2G_{\emptyset}(0))^k\langle\varphi(0)^m P(\varphi(0))^k\rangle. \tag{14}$$

Proof Using the symmetry of the l.h.s. of (14) with respect to the exchanges $(\tau_i,\theta_i,\bar{\theta}_i) \longleftrightarrow (\tau_j,\theta_j,\bar{\theta}_j)$ we have that

$$\left\langle \varphi(0)^m\left(\int_{-\infty}^{0}G(t,\theta,\bar{\theta})P(\Phi(\tau,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)^k\right\rangle =$$

$$= k!\left\langle \varphi(0)^m\int_{-\infty}^{0}\int_{-\infty}^{\tau_1}\cdots\int_{-\infty}^{\tau_{k-1}}\prod_{i=1}^{k}G(\tau_i,\theta_i,\bar{\theta}_i)P(\Phi(\tau_i,\theta_i,\bar{\theta}_i))\mathrm{d}\tau_i\mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\right\rangle.$$

Then the claim follows directly from Lemma 12 taking $g = G_{\emptyset}$. □

Theorem 14 *Let F be a smooth bounded function, let G be a supersymmetric function with compact support, let H be a bounded function with all the derivatives bounded and let $\mathbb{K}$ be an entire function, then we have*

$$\left\langle F(\varphi(0))\mathbb{K}\left(\int_{-\infty}^{0}G(t,\theta,\bar{\theta})H(\Phi(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)\right\rangle = \langle F(\varphi(0))\cdot\mathbb{K}(-2G_{\emptyset}(0)\cdot\varphi(0))\rangle.$$

Proof Using the density of polynomials in the set of smooth functions with respect to the topology given by the one of the Sobolev space with respect to the Gaussian law of $\varphi(t)$, Corollary 13 implies that for any $k \in \mathbb{N}$ and F, G, H satisfying the hypothesis of the theorem

$$\left\langle F(\varphi(0))\left(\int_{-\infty}^{0}G(t,\theta,\bar{\theta})H(\Phi(t,\theta,\bar{\theta}))\mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)^k\right\rangle = \langle F(\varphi(0))[-2G_{\emptyset}(0)H(\varphi(0))]^k\rangle.$$

Expanding $\mathbb{K}$ in power series, exploiting the fact that

$$\left\langle \left| \left\langle \left(\int G(t,\theta,\bar{\theta}) H(\Phi(t,\theta,\bar{\theta})) \mathrm{d}t\mathrm{d}\theta\mathrm{d}\bar{\theta} \right)^k \right\rangle_{\psi,\bar{\psi}} \right|^p \right\rangle$$

is uniformly bounded when H is bounded, for any $p \geq 1$, we can exchange the series with the expectation $\langle \cdot \rangle$, and obtain in this way the thesis. □

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