Possibility Distributions Generated by Intuitionistic L-Fuzzy Sets

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Abstract. In this work, we bridge possibility theory with intuitionistic L-fuzzy sets, by identifying a special class of possibility distributions corresponding to intuitionistic L-fuzzy sets based on a complete residuated lattice with an involution. Moreover, taking the Łukasiewicz n-chains as structures of truth degrees, we propose an algorithm to compute the intuitionistic L-fuzzy set corresponding to a given possibility distribution, in case it exists.

Keywords: Possibility distributions · Intuitionistic L- fuzzy sets · Complete residuated lattices · Łukasiewicz n-chains

1 Introduction

Possibility distributions are the building blocks of *possibility theory* [15]. The concept of *possibility* was investigated by several scholars, expecially by Shackle [32], Lewis [27], Cohen [10], and Zadeh [33]. Moreover, possibility theory and its applications were widely explored by Dubois, Prade and colleagues in many works [12,13,16].

A possibility distribution π_x is a map associated to a variable x, from a universe U to a totally ordered scale L with a top and bottom, such as the unit interval $[0, 1]$. Depending on the interpretations, $\pi_x(u)$ estimates the degree of ease, the degree of unsurprizingness or of expectedness, the degree of acceptability or of preference related to the proposition "the value of x is u" [14]. Here, we focus on possibility distributions arising when a degree of plausibility needs to be assigned to an L-set as in the following example[1].

Suppose that $\mathcal{V}$ is a collection of features regarding a flat (for instance *small size* and *low price*). Then, each specific flat F is associated to an L-set $\omega_{\mathsf{F}} : \mathcal{V} \to \mathsf{L}$, where $\omega_{\mathsf{F}}(v)$ is the truth degree to which F has the attribute v of $\mathcal{V}$. Therefore, we could consider a possibility distribution π such that $\pi(\omega_{\mathsf{F}})$ expresses the degree of possibility that a given customer prefers a given flat F (described by ω_{F}) before he/she knows it in advance.

[1] L-sets were introduced by Goguen [21] as generalizations of fuzzy sets.

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Specifically, we deal with possibility distributions whose domain is composed of all L-sets on a given universe assuming that L is a complete residuated lattice with an involution [24]. Besides, they are interpreted as *preference functions*, thus standing for a counterpart to utility functions [11,17].

Mainly, we aim to discover the existing connections between this type of possibility distributions and intuitionistic L-fuzzy sets, a generalization of intuitionistic fuzzy sets, based on a lattice L instead of $[0,1]$ as the set of truth-values. To this purpose, we view intuitionistic L-fuzzy sets as generalizations of *orthopairs*, which are pairs of disjoint subsets of a universe used to model uncertainty [7]. Given a set of propositional variables $\mathcal{V}$, an orthopair (P, N) on $\mathcal{V}$ has an epistemic meaning: P is the set of variables *known to be true*, N is the set of variables *known to be false*, and $\mathcal{V} \setminus (P \cup N)$ is the set of *unknown variables* by a given agent. In [8], the authors provided the following correspondence between orthopairs and Boolean possibility distributions: An orthopair on $\mathcal{V}$ generates a Boolean possibility distribution whose domain Ω is made of all evaluation functions on $\mathcal{V}$. On the other hand, not all Boolean possibility distributions having Ω as domain are generated by an orthopair on $\mathcal{V}$. Consequently, orthopairs on $\mathcal{V}$ individuate a special class of Boolean possibility distributions on Ω. In this article, we intend to extend this correspondence by using fuzzy logic. More precisely, we identify intuitionistic L-fuzzy sets of a given universe $\mathcal{V}$ with particular possibility distributions, which assign a degree of L to each L-set of $\mathcal{V}$.

In providing theoretical results, we suppose that *complete residuated lattices with an involution* are our algebraic structures of truth values [19]. However, examples and algorithms are confined to finite substructures of the *standard Łukasiewicz MV-algebra* [6,28]. Our choice depends on that Łukasiewicz implication is usually used for fuzzy logic applications because it is the only plausible continuous implication operation on $[0,1]$ [31].

The article is organized as follows. The next section reviews some basic notions and results regarding residuated lattices and intuitionistic fuzzy sets. In Sect. 3, we firstly assign a special possibility distribution to each intuitionistic L-fuzzy set. Then, we prove that possibility distributions corresponding to intuitionistic L-fuzzy sets are normal. After that, confining to $IMTL$-algebras, we establish under what conditions a possibility distribution assumes value $\mathbf{0}$. In Sect. 4, we firstly show that there exist normal possibility distributions not generated by an intuitionistic L-fuzzy set. Then, we find the intuitionistic L-fuzzy set that generates a given possibility distribution, in case it exists. Moreover, in Subsect. 4.1, considering the Łukasiewicz n-chains as algebraic structures of truth degrees, we provide procedures to compute the possibility distribution generated by a given intuitionistic L-fuzzy set, and vice-versa, the intuitionistic L-fuzzy set generating a given possibility distribution. Finally, in the last section, we briefly discuss the potential developments of our results.

2 Preliminaries

This section describes some notations, preliminary notions and results, which will be used in this article.

2.1 Algebraic Structures of Truth Values

As basic structures of truth degrees, we choose complete residuated lattices, which are widely adopted for applications of fuzzy logic [22,23,26].

Definition 1 [25]. *A* residuated lattice *is an algebra* $\langle \mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \boldsymbol{0}, \boldsymbol{1}\rangle$*, where*

(i) $\langle \mathsf{L}, \wedge, \vee, \boldsymbol{0}, \boldsymbol{1}\rangle$ *is a bounded lattice;*
(ii) $\langle \mathsf{L}, \otimes, \boldsymbol{1}\rangle$ *is a commutative monoid, i.e.* $\otimes$ *is a binary operation that is commutative, associative, and* $a \otimes \boldsymbol{1} = a$ *for each* $a \in \mathsf{L}$*;*
(iii) $a \otimes b \leq c$ *if and only if* $a \leq b \rightarrow c$*, for each* $a, b, c \in \mathsf{L}$ *(adjunction property).*

A residuated lattice $(\mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$ is complete if its reduct $(\mathsf{L}, \wedge, \vee)$ is a complete lattice. In a residuated lattice, a unary operation named *negation* is derived as follows: $\neg x = x \rightarrow \mathbf{0}$, for each $x \in \mathsf{L}$. In this paper, we deal with residuated lattices where the negation is an *involution*, namely the so-called *double negation law* holds: $\neg\neg x = x$ for each $x \in \mathsf{L}$. The following proposition lists some properties satisfied by every residuated lattice with an involution.

Proposition 1. *Let* $\langle \mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \boldsymbol{0}, \boldsymbol{1}\rangle$ *be a residuated lattice, then the followings hold for each* $a, b, c \in \mathsf{L}$*:*

(i) $a \wedge b \leq a$ *and* $a \wedge b \leq b$*;*
(ii) if $a \leq b$ *and* $a \leq c$ *then* $a \leq b \wedge c$*;*
(iii) $a \vee b = \boldsymbol{1}$ *if and only if* $a = \boldsymbol{1}$ *or* $b = \boldsymbol{1}$*;*
(iv) $a \wedge b = \boldsymbol{1}$ *if and only if* $a = \boldsymbol{1}$ *and* $b = \boldsymbol{1}$*;*
(v) $a \wedge b = \boldsymbol{0}$ *if and only if* $a = \boldsymbol{0}$ *or* $b = \boldsymbol{0}$*;*
(vi) $a \otimes b = \boldsymbol{1}$ *if and only if* $a = \boldsymbol{1}$ *and* $b = \boldsymbol{1}$*;*
(vii) $a \rightarrow b = \boldsymbol{1}$ *if and only if* $a \leq b$*;*
(viii) if $\neg$ *is an involution, then* $a \rightarrow b = \neg b \rightarrow \neg a$*.*

Special residuated lattices with an involution are the so-called IMTL-algebras, which are the algebraic structures for monoidal t-norm based logic with an involutive negation.

Definition 2 [18]. *A residuated lattice with an involution* $\langle \mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \boldsymbol{0}, \boldsymbol{1}\rangle$ *is an* IMTL-algebra *if and only if it satisfies the* pre-linearity axiom*:*

$$(a \rightarrow b) \vee (b \rightarrow a) = \boldsymbol{1} \text{ for each } a, b \in \mathsf{L}. \tag{1}$$

In providing examples and algorithms, we must restrict to a class of finite substructures

$$\{\langle \text{Ł}_n, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle \text{ with } n \in \mathbb{N}\} \tag{2}$$

of the standard Łukasiewicz MV-algebra [6], where Ł_n is the *n-element Łukasiewicz chain* given by $\text{Ł}_n = \{{}^k/_n \mid 0 \leq k \leq n \text{ and } n \in \mathbb{Z}\}$, and the operations in (2) are defined as follows: let $a, b \in \text{Ł}_n$, then $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, $a \otimes b = \max(0, a + b - 1)$ (*Łukasiewicz conjunction*), and $a \rightarrow b = \min(1, 1 - a + b)$ (*Łukasiewicz implication*). Moreover, $\neg a = 1 - a$ for each $a \in \text{Ł}_n$. These structures also satisfy the *pre-linearity axiom* defined by (1). For convenience, we will indicate a residuated lattice $(\mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$ with its support L.

2.2 Intuitionistic Fuzzy Sets and Intuitionistic L-fuzzy Sets

Intuitionistic fuzzy sets (IF sets for short) were introduced by Atanassov in [1,2] to generalize the concept of *fuzzy sets* in order to explicitly take into account the *non-belongingness* to a set. More formally:

Definition 3. *Let X be a universe such that $X \neq \emptyset$. An* intuitionistic fuzzy set *A of X is defined as $A = \{(x, \mu(x), \nu(x)) \mid x \in X\}$, where the maps $\mu : X \to [0,1]$ and $\nu : X \to [0,1]$ satisfy the condition $\mu(x) + \nu(x) \leq 1$, for each $x \in X$.*

The values $\mu(x)$ and $\nu(x)$ are respectively called degree of membership and non-membership *of x to A, and $1 - (\mu(x) + \nu(x))$ is called* hesitation margin *of x to A.*

Let us observe that an IF set coincides with a fuzzy set when the hesitation margin of each element of the starting universe is equal to 0. In this work, we look at IF sets as generalizations of orthopairs by using fuzzy logic. Given a universe X, (P, N) is an *orthopair* on X if and only if $P, N \subseteq X$ and $P \cap N = \emptyset$ [7]. It is easy to understand that (P, N) can be identified with a particular intuitionistic fuzzy set $\{(x, \mu(x), \nu(x)) \mid x \in X\}$, where μ and ν coincides with the characteristic functions of P and N, respectively. That is, orthopairs coincide with the Boolean sub-collection of IF sets.

IF sets were extended to *intuitionistic L-fuzzy sets* (ILF sets for short) considering an appropriate lattice L instead of $[0,1]$ as the set of truth-values [3,4,20]. Our results are based on *intuitionistic L-fuzzy sets* valued on a complete residuated lattice satisfying the double negation law.

Definition 4. *Let $\langle \mathsf{L}, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ be a complete residuated lattice having an involution $\neg$, and let X be a non-empty set. An* intuitionistic L-fuzzy set *A of X is defined by $A = \{(x, \mu(x), \nu(x)) \mid x \in X\}$, where $\mu : X \to \mathsf{L}$ and $\nu : X \to \mathsf{L}$ satisfy the condition $\mu(x) \leq \neg\nu(x)$, for each $x \in X$[2].*

The components of an intuitionistic L-fuzzy set (μ, ν) of X satisfy the identity $\mu(x) \otimes \nu(x) = 0$ for each $x \in X$. Thus, if $\langle [0,1], \wedge, \vee, \otimes, \to, 0, 1\rangle$ is the standard Łukasiewicz MV-algebra, they represent *contrary properties* [5].

For convenience, in the sequel, we briefly write (μ, ν) to denote the intuitionistic L-set $\{(\mu(x), \nu(x)) \mid x \in X\}$ when X is clear from the context.

3 From Intuitionistic L-Fuzzy Sets to Possibility Distributions

In this section, we firstly assign a particular possibility distribution to each intuitionistic L-fuzzy set. Then, we prove that possibility distributions corresponding to ILF sets are normal. After that, confining to complete $IMTL$-algebras, we establish under what conditions a possibility distribution assumes value $\mathbf{0}$.

[2] We notice that, as in Definition 3, μ and ν have a symmetrical role, in the sense that $\mu(x) \leq \neg\nu(x)$ is equivalent to $\nu(x) \leq \neg\mu(x)$.

3.1 Possibility Distributions

A *possibility distribution* is a map from a universe X to a totally ordered scale L equipped with a top, a bottom, and an order-reversing map (such as the unit interval $[0,1]$ with the function assigning $1-\lambda$ to each $\lambda \in [0,1]$). The universe of discourse can be an attribute domain, a set of interpretation of a propositional language, etc. In this work, we focus on possibility distributions having the following form

$$\pi : \mathsf{L}^{\mathcal{V}} \to \mathsf{L}, \tag{3}$$

where L is a complete residuated lattice with an involution, and $\mathsf{L}^{\mathcal{V}}$ is the set of all L-sets of a non-empty universe $\mathcal{V}$, i.e., $\mathsf{L}^{\mathcal{V}} = \{\omega \mid \omega : \mathcal{V} \to \mathsf{L}\}$. Of course, since we choose complete residuated lattices with an involution as algebraic structures of truth degrees, our results also hold for the standard definition of possibility distribution, where L is a totally ordered scale. We use the symbol Π to denote the set of all possibility distributions given by (3), i.e., $\Pi = \{\pi \mid \pi : \mathsf{L}^{\mathcal{V}} \to \mathsf{L}\}$.

In possibility theory, a very important role is played by *normal possibility distributions* [15].

Definition 5. *A possibility distribution $\pi \in \Pi$ is* normal *if and only if there exists $\omega \in \mathsf{L}^{\mathcal{V}}$ such that $\pi(\omega) = \mathbf{1}$. Moreover, given $\pi \in \Pi$, we put $\mathcal{K}(\pi) = \{\omega \in \mathsf{L}^{\mathcal{V}} \mid \pi(\omega) = \mathbf{1}\}$, and we call $\mathcal{K}(\pi)$ the* kernel *of π.*

3.2 Possibility Distributions Generated by Intuitionistic L-fuzzy Sets

Every intuitionistic L-fuzzy set (μ, ν) determines a possibility distribution $\pi_{(\mu,\nu)} \in \Pi$.

Definition 6. *Let $\omega \in \mathsf{L}^{\mathcal{V}}$, then*

$$\pi_{(\mu,\nu)}(\omega) = \bigwedge_{v \in \mathcal{V}} (\mu(v) \to \omega(v)) \otimes (\nu(v) \to \neg\omega(v)). \tag{4}$$

We call $\pi_{(\mu,\nu)}$ the possibility distribution generated by *(μ, ν).*

Let us point out that our possibility distributions play a different role from those based on rough set theory [9,29,30]. Indeed, a possibility distribution, defined by (4), is viewed as a preference function that arises by aggregating the mappings of an intuitionistic L-fuzzy set, which are interpreted as preference functions too[3]. This interpretation can be better understood from the following illustrative example, where a possibility distribution is generated by an intuitionistic L-set in a concrete situation.

[3] Additionally, given a intuitionistic L-fuzzy set (μ, ν), the value $\pi_{(\mu,\nu)}(\omega)$ can be also understood as an answer to a bipolar fuzzy query given by (μ, ν), where μ and ν respectively express positive and negative elastic constraints.

Example 1. Imagine that a real estate agent wants to discover the degree of possibility to which a given client C prefers a given flat F that he/she does not know in advance, starting from a pair of specific preference functions, expressed by the client on a set of features concerning apartments.

Then, let $\mathcal{V} = \{v_1, \ldots, v_{10}\}$ be a collection of features regarding flats (for instance *small size* and *low price*), and let $Ł_5 = \{0, 0.25, 0.5, 0.75, 1\}$ be the 5-element Łukasiewicz chain (see Subsect. 2.1). We suppose that

– each flat F is described by an $Ł_5$-set $\omega_{\mathsf{F}} : \mathcal{V} \rightarrow Ł_5$, where $\omega_{\mathsf{F}}(v)$ is the truth degree to which F has the attribute v of $\mathcal{V}$;
– the preferences of a given customer C on the attributes of $\mathcal{V}$ are described by an intuitionistic $Ł_5$-fuzzy set $(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})$ of $\mathcal{V}$. This means that given $v \in \mathcal{V}$, C prefers apartments having the attribute v with a degree at least $\mu_{\mathsf{C}}(v)$ and at most $\neg\nu_{\mathsf{C}}(v)$ (i.e., C prefers apartments that do not have v with a degree at least $\nu(v)$) in the scale $Ł_5$. For example, if v is the attribute *small size*, then $\mu(v) = 0.5$ and $\nu(v) = 0.25$ respectively mean that the customer prefers flats being *small* at least 0.5 and not more than 0.75 in the scale $Ł_5$, since $\neg\nu(v) = 0.75$.
  Let us notice that μ and ν are also fuzzy constraints: given $v \in \mathcal{V}$, $\mu(v)$ and $\neg\nu(v)$ represent degrees of priority, namely a suitable flat must have the attribute v with a degree between $\mu(v)$ and $\neg\nu(v)$, according to the preference of C. Moreover, we say that μ and ν are respectively positive and negative preference functions because their interpretation is based on the customer preferences about the presence or the absence of certain properties in its ideal apartment.

Hence, $\pi_{(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})} : Ł_5^{\mathcal{V}} \rightarrow Ł_5$ given by (4), is a new preference function, where $\pi_{(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})}(\omega_{\mathsf{F}})$ is the degree of possibility that customer C prefers apartment ω_{F}, and it is computed by aggregating μ_{C} and ν_{C} that capture the preferences expressed by C on the attributes of $\mathcal{V}$. In other words, $\pi_{(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})}$ is a possibility distribution prescribing to what extent a flat is judged to be suitable for C according to the constraints given by $(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})$. For example, let F_i and F_j be flats represented by ω_{F_i} and ω_{F_j}, and let C be a customer whose preferences are represented by $(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})$ (see Table 1). By Eq. 4, $\pi_{(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})}(\omega_{\mathsf{F}_i}) = 1$ and $\pi_{(\mu_{\mathsf{C}}, \nu_{\mathsf{C}})}(\omega_{\mathsf{F}_j}) = 0.25$. This

Table 1. Values assumed by $\omega_{\mathsf{F}_i}, \omega_{\mathsf{F}_j}, \mu_{\mathsf{C}}$ and ν_{C} on $\mathcal{V}$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
ω_{F_i}	0.5	0.5	0.75	1	0	0.5	0.25	0.5	0.5	0.25
ω_{F_j}	0.75	0.5	0.5	0.25	0.25	0.5	0.75	1	0	0.25

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
μ_{C}	0.25	0.25	0.75	1	0	0	0	0.25	0.25	0.25
ν_{C}	0.5	0.5	0.25	0	0.25	0.5	0.75	0.25	0	0.5

means that we can believe that C prefers ω_{F_i} more than ω_{F_j}. Consequently, the real estate agent could propose ω_{F_i} to C directly and exclude ω_{F_j}.

In the following example, we find the possibility distribution generated by a given ILF set.

Example 2. Let $\mathcal{V} = \{a, b\}$, and let $Ł_5 = \{0, 0.25, 0.5, 0.75, 1\}$ be the 5-element Łukasiewicz chain (see Subsect. 2.1). Then, Π is composed of the $Ł_5$-sets $\omega_1, \ldots, \omega_{25} : \{a, b\} \to Ł_5$ defined by Table 2. We consider the ILF set (μ, ν) given by $\{(a, 0.25, 0.25), (b, 0.5, 0.5)\}$ (i.e. $\mu(a) = 0.25$, $\mu(b) = 0.5$, $\nu(a) = 0.25$, and $\nu(b) = 0.5$).

Then, by (4), the possibility distribution generated by (μ, ν) is given by

$$\pi_{(\mu,\nu)}(\omega_i) = \begin{cases} 1 & \text{if } i \in \{1, 2, 3\}, \\ 0.5 & \text{if } i \in \{4, 8, 9, 12, 13, 16, 17, 20, 21, 25\}, \\ 0.75 & \text{otherwise.} \end{cases} \tag{5}$$

Table 2. Values assumed by $\omega_1, \ldots, \omega_{25}$ on $\{a, b\}$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}
a	0.25	0.5	0.75	0	0	0	0	0	0.25	0.25	0.25	0.25	0.5
b	0.5	0.5	0.5	0	0.25	0.5	0.75	1	0	0.25	0.75	1	0

	ω_{14}	ω_{15}	ω_{16}	ω_{17}	ω_{18}	ω_{19}	ω_{20}	ω_{21}	ω_{22}	ω_{23}	ω_{24}	ω_{25}
a	0.5	0.5	0.5	0.75	0.75	0.75	0.75	1	1	1	1	1
b	0.25	0.75	1	0	0.25	0.75	1	0	0.25	0.5	0.75	1

Remark 1. When $\mathsf{L} = \{\mathbf{0}, \mathbf{1}\}$ and $\mathcal{V}$ is a set of propositional variables, Eq. 4 provides the following correspondence between Boolean possibility distributions and orthopairs, which has already been shown in [8]. Given an ILF set (μ, ν), then $\mu : \mathcal{V} \to \{\mathbf{0}, \mathbf{1}\}$ and $\nu : \mathcal{V} \to \{\mathbf{0}, \mathbf{1}\}$ are respectively the characteristic functions of the sets O_μ and O_ν that form an orthopair on $\mathcal{V}$. Furthermore, $\{\mathbf{0}, \mathbf{1}\}^{\mathcal{V}}$ consists of all Boolean evaluation functions on $\mathcal{V}$. Hence, it is easy to check that given $\pi \in \Pi$ and $\omega \in \{\mathbf{0}, \mathbf{1}\}^{\mathcal{V}}$, $\pi(\omega) = \mathbf{1}$ (according to Eq. 4) if and only if ω is a model of the propositional formula $\phi_\mu \wedge \phi_\nu$ such that

$$\phi_\mu := \begin{cases} \bigwedge_{v \in O_\mu} v & \text{if } O_\mu \neq \emptyset \\ \top & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_\nu := \begin{cases} \bigwedge_{v \in O_\nu} \neg v & \text{if } O_\nu \neq \emptyset \\ \top & \text{otherwise} \end{cases}$$

where $\wedge, \neg$, and $\top$ are respectively interpreted with the conjunction, the negation, and the top of a Boolean algebra.

An intuitionistic L-fuzzy set (μ, ν) determines also a collection of L-sets $\mathcal{I}_{(\mu,\nu)}$:

$$\mathcal{I}_{(\mu,\nu)} = \{\omega : \mathcal{V} \to \mathsf{L} \text{ such that } \mu(v) \leq \omega(v) \leq \neg\nu(v) \text{ for each } v \in \mathcal{V}\}. \quad (6)$$

Remark 2. $\mathcal{I}_{(\mu,\nu)}$ is a non-empty set since $\mu \in \mathcal{I}_{(\mu,\nu)}$.

The following theorem states that $\mathcal{I}_{(\mu,\nu)}$ coincides with the kernel of $\pi_{(\mu,\nu)}$, and so with $\mathcal{K}(\pi_{(\mu,\nu)}) = \{\omega \in \mathsf{L}^{\mathcal{V}} \mid \pi_{(\mu,\nu)}(\omega) = \mathbf{1}\}$.

Theorem 1. *Let (μ, ν) be an intuitionistic L-fuzzy set, and let $\omega \in L^{\mathcal{V}}$. Then, $\pi_{(\mu,\nu)}(\omega) = \mathbf{1}$ if and only if $\omega \in \mathcal{I}_{(\mu,\nu)}$.*

Proof. Let $\omega \in \mathsf{L}^{\mathcal{V}}$ such that $\pi_{(\mu,\nu)}(\omega) = \mathbf{1}$. Then, by (4), $\bigwedge_{v\in\mathcal{V}}(\mu(v) \to \omega(v)) \otimes (\nu(v) \to \neg\omega(v)) = 1$, for each $v \in \mathcal{V}$.

Using Proposition 1 (items (iii), (vi), and (vii)), we have that $\mu(v) \leq \omega(v)$ and $\nu(v) \leq \neg\omega(v)$, for each $v \in \mathcal{V}$.

Moreover, by Proposition 1(viii), $\nu(v) \leq \neg\omega(v)$ implies $\neg\neg\omega(v) \leq \neg\nu(v)$ for each $v \in \mathcal{V}$, and since $\neg$ is an involution, we finally get $\omega(v) \leq \neg\nu(v)$ for each $v \in \mathcal{V}$. Hence, $\mu(v) \leq \omega(v) \leq \neg\nu(v)$ for each $v \in \mathcal{V}$, and so, we can conclude that ω belongs to $\mathcal{I}_{(\mu,\nu)}$ (see (6)).

Analogously, we can prove that if $\omega \in \mathcal{I}_{(\mu,\nu)}$ then $\pi_{(\mu,\nu)}(\omega) = \mathbf{1}$.

Example 3. Consider Example 2, then $\mathcal{I}_{(\mu,\nu)} = \{\omega_1, \omega_2, \omega_3\}$, which is also equal to $\mathcal{K}(\pi_{(\mu,\nu)})$.

Therefore, as an immediate consequence of Theorem 1 and Remark 2, we have that possibility distributions generated by an intuitionistic L-set are always normal.

Corollary 1. *Let $\pi \in \Pi$. If π is generated by an intuitionistic L-fuzzy set, then π is normal.*

3.3 Possibility Distributions Generated by Intuitionistic L-fuzzy Sets Based on an IMTL-algebra

In this subsection, confining to complete IMTL-algebras, we discover when a possibility distribution (generated by an ILF set) assumes value $\mathbf{0}$.

At first, let us prove the following lemma.

Lemma 1. *Let $\langle L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ be a complete IMTL-algebra, and let (μ, ν) be an intuitionistic L-fuzzy set. Then, $(\mu(v) \to \omega(v)) \vee (\nu(v) \to \neg\omega(v)) = \mathbf{1}$, for each $v \in \mathcal{V}$.*

Proof. Let $v \in \mathcal{V}$ such that $\mu(v) \to \omega(v) \neq \mathbf{1}$. Since the pre-linearity axiom holds, we get $\omega(v) \to \mu(v) = \mathbf{1}$. By Proposition 1(vii), $\omega(v) \leq \mu(v)$. Moreover, by Definition 4, $\mu(v) \leq \neg\nu(v)$. Hence, $\omega(v) \leq \neg\nu(v)$ that is equivalent to $\omega(v) \to \neg\nu(v) = \mathbf{1}$. Finally, by Proposition 1 (viii), we get $\nu(v) \to \neg\omega(v) = \mathbf{1}$.

Analogously, let $v \in \mathcal{V}$, if $\nu(v) \to \neg\omega(v) \neq \mathbf{1}$, then we can prove that $\mu(v) \to \omega(v) = \mathbf{1}$.

By Proposition 1(iv), we conclude that $(\mu(v) \to \omega(v)) \vee (\nu(v) \to \neg\omega(v)) = \mathbf{1}$.

A possibility distribution generated by an ILF set (valued on a complete IMTL-algebra) is equal to $\mathbf{0}$ only in some particular cases. More precisely, the next theorem holds.

Theorem 2. *Let $\langle \mathsf{L}, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$ be a complete IMTL-algebra, let (μ, ν) be an intuitionistic L-fuzzy set, and let $\omega \in \mathsf{L}^{\mathcal{V}}$. Then, $\pi_{(\mu,\nu)}(\omega) = \mathbf{0}$ if and only if there exists $v \in \mathcal{V}$ such that $\mu(v) = \mathbf{1}$ and $\omega(v) = \mathbf{0}$, or $\mu(v) = \mathbf{0}$ and $\omega(v) = \mathbf{1}$.*

Proof. (⇐) This implication is trivial.
(⇒) Let $\omega \in \mathsf{L}^{\mathcal{V}}$ such that $\pi_{(\mu,\nu)}(\omega) = \mathbf{0}$. Then, by Proposition 1 (v), there exists $v \in \mathcal{V}$ such that

$$(\mu(v) \rightarrow \omega(v)) \otimes (\nu(v) \rightarrow \neg\omega(v)) = \mathbf{0}. \tag{7}$$

By Lemma 1,

$$\mu(v) \rightarrow \omega(v) = \mathbf{1} \text{ or } \nu(v) \rightarrow \neg\omega(v) = \mathbf{1}. \tag{8}$$

Eventually, by Definition 1 ($a \otimes \mathbf{1} = a$ for each $a \in \mathsf{L}$), Eqs. (7) and (8) imply that $\mu(v) = \mathbf{1}$ and $\omega(v) = \mathbf{0}$, or $\mu(v) = \mathbf{0}$ and $\omega(v) = \mathbf{1}$.

Example 4. Consider Example 2. Then, $\pi_{(\mu,\nu)}(\omega) \neq 0$ for each $\omega \in \text{Ł}_5^{\{a,b\}}$. In fact, $\mu(a), \mu(b) \notin \{0, 1\}$.

4 From Possibility Distributions to Intuitionistic Fuzzy Sets

This section mainly aims to find the intuitionistic L-fuzzy set that generates a given possibility distribution $\pi : \mathsf{L}^{\mathcal{V}} \rightarrow \mathsf{L}$ by means of Eq. 4.

Let us recall that Eq. 4 leads to define a normal possibility distribution for each intuitionistic L-set. On the other hand, it is not always possible to do the opposite. Namely, there exist normal possibility distributions from $\mathsf{L}^{\mathcal{V}}$ to L that do not correspond to any intuitionistic L-fuzzy set by means of Eq. 4. The following is an example.

Example 5. Consider Example 2, we can prove that no intuitionistic Ł_5-fuzzy set generates the possibility distribution $\pi : \text{Ł}_5^{\{a,b\}} \rightarrow \text{Ł}_5$ defined by the following formula: let $\omega_i \in \text{Ł}_5^{\{a,b\}}$,

$$\pi(\omega_i) = \begin{cases} 1 & \text{if } i \in \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Since the pre-linearity axiom holds in $\langle \text{Ł}_5, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, we can apply Theorem 2. Consequently, in case π is generated by an intuitionistic Ł_5-fuzzy set, it must be $\omega_i(a) \in \{0, 1\}$ or $\omega_i(b) \in \{0, 1\}$, for each $i \in \{4, \ldots, 25\}$. But, this contradicts Table 2, where $\omega_i(a)$ and $\omega_i(b)$ do not belong to $\{0, 1\}$ for each $i \in \{10, 11, 14, 15, 18, 19\}$.

In the sequel, we write Π^* to indicate the set of all possibility distributions of Π that are generated by an intuitionistic L-fuzzy set.

Now, we want to establish when a given possibility distribution belongs to Π^*. In order to do this, we firstly associate an intuitionistic L-fuzzy set to every possibility distribution starting from its kernel.

Definition 7. *Given $\pi \in \Pi$ and $v \in \mathcal{V}$, then $\mu_\pi, \nu_\pi : \mathcal{V} \to \mathsf{L}$ are defined as follows:*

$$\mu_\pi(v) = \bigwedge_{\omega \in \mathcal{K}(\pi)} \omega(v) \quad \text{and} \quad \nu_\pi(v) = \bigwedge_{\omega \in \mathcal{K}(\pi)} \neg\omega(v). \tag{10}$$

We can prove that the functions given by (10) form an intuitionistic L-fuzzy set.

Proposition 2. *Let $\pi \in \Pi$, then (μ_π, ν_π) is an intuitionistic L-fuzzy set.*

Proof. By Proposition 1(i), we get

$$\bigwedge_{\omega \in \mathcal{K}(\pi)} \omega(v) \leq \omega(v) \quad \text{and} \quad \bigwedge_{\omega \in \mathcal{K}(\pi)} \neg\omega(v) \leq \neg\omega(v), \quad \text{for each } v \in \mathcal{V}. \tag{11}$$

Moreover, by Proposition 1 (vii, viii),

$$\bigwedge_{\omega \in \mathcal{K}(\pi)} \neg\omega(v) \leq \neg\omega(v), \text{ implies that } \omega(v) \leq \neg \bigwedge_{\omega \in \mathcal{K}(\pi)} \neg\omega(v) \quad \text{for each } v \in \mathcal{V}. \tag{12}$$

Hence, by (10), we can conclude that $\mu_\pi(v) \leq \neg\nu_\pi(v)$, for each $v \in \mathcal{V}$.

Since (μ_π, ν_π) is an intuitionistic L-fuzzy set, it generates a new possibility distribution (by means of Eq. 4) that we denote with π^*. In general, π^* does not coincide with π. For example, it is easy to verify that if π is given by (5) then π^* is given by (9). Consequently, $\pi^* \neq \pi$. Of course, $\pi = \pi^*$ implies that $\pi \in \Pi^*$, and more precisely that π is generated by the intuitionistic L-fuzzy set (μ_π, ν_π). Furthermore, the following theorem shows that (μ_π, ν_π) is the only intuitionistic L-fuzzy set that can generate π.

Theorem 3. *Let $\pi \in \Pi$. If $\pi \in \Pi^*$, then π is generated by (μ_π, ν_π).*

Proof. Let $\pi \in \Pi^*$, there exists (μ, ν) that generates π. So, we want to prove that $(\mu, \nu) = (\mu_\pi, \nu_\pi)$.

First of all, we show that $\mu = \mu_\pi$. Let $v \in \mathcal{V}$. Then, by Theorem 1, $\mu(v) \leq \omega(v)$ for each $\omega \in \mathcal{K}(\pi)$. Consequently, by Proposition 1(ii), $\mu(v) \leq \bigwedge_{\omega \in \mathcal{K}(\pi)} \omega(v)$. Namely, $\mu(v) \leq \mu_\pi(v)$.

Moreover, if $\mu_\pi(v) < \mu(v)$, then there exists $\omega \in \mathcal{K}(\pi)$ such that $\omega(v) < \mu(v)$, but it contradicts Theorem 1. Then, $\mu_\pi(v) \leq \mu(v)$.

Analogously, we can prove that $\nu = \nu_\pi$.

Example 6. Consider the possibility distribution $\pi_{(\mu,\nu)}$ given by (5). For convenience, we indicate $\pi_{(\mu,\nu)}$ with π. Example 2 shows that π is generated by (μ,ν), which is $\{(a, 0.25, 0, 25), (b, 0.5, 0.5)\}$. Consequently, $\pi \in \Pi^*$. Moreover, Theorem 3 assures us that $(\mu,\nu) = (\mu_\pi, \nu_\pi)$. Indeed, $\mu_{\pi_{(\mu,\nu)}}(a) = \omega_1(a) \wedge \omega_2(a) \wedge \omega_3(a) = 0.25 \wedge 0.5 \wedge 0.75 = 0.25$, $\mu_{\pi_{(\mu,\nu)}}(b) = \omega_1(b) \wedge \omega_2(b) \wedge \omega_3(b) = 0.5 \wedge 0.5 \wedge 0.5 = 0.5$, $\nu_{\pi_{(\mu,\nu)}}(a) = \neg\omega_1(a) \wedge \neg\omega_2(a) \wedge \neg\omega_3(a) = 0.75 \wedge 0.5 \wedge 0.25 = 0.25$, and $\nu_{\pi_{(\mu,\nu)}}(b) = \neg\omega_1(b) \wedge \neg\omega_2(b) \wedge \neg\omega_3(b) = 0.5 \wedge 0.5 \wedge 0.5 = 0.5$.

Using Theorem 3, we provide a necessary and sufficient condition for a possibility distribution to be generated by an ILF set.

Corollary 2. *Let $\pi \in \Pi$. Then, $\pi \in \Pi^*$ if and only if $\pi = \pi^*$, namely*

$$\pi(\omega) = \bigwedge_{v \in \mathcal{V}} (\mu_\pi(v) \to \omega(v)) \otimes (\nu_\pi(v) \to \neg\omega(v)), \quad \textit{for each } \omega \in \mathsf{L}^{\mathcal{V}}.$$

The following proposition will be used in the next subsection. It shows that the kernel of π^* always includes that of π.

Proposition 3. *Let $\pi \in \Pi$. Then, $\mathcal{K}(\pi) \subseteq \mathcal{K}(\pi^*)$.*

Proof. Let $\omega \in \mathcal{K}(\pi)$. Then, by (11) and (12), we get $\mu_\pi(v) \leq \omega(v) \leq \neg\nu_\pi(v)$ for each $v \in \mathcal{V}$. Thus, by Theorem 1, $\omega \in \mathcal{K}(\pi^*)$.

4.1 An Algorithm to Find the Intuitionistic L-fuzzy Set Generating a Given Possibility Distribution

In this subsection, assuming that our structures of truth degrees are the Łukasiewicz n-chains defined by (2), we propose three algorithms to achieve the following goals.

(i) Compute the intuitionistic $\mathsf{Ł}_n$-fuzzy set corresponding to a given possibility distribution by means of (10).
(ii) Find the values assumed by the possibility distribution generated by a given intuitionistic $\mathsf{Ł}_n$-fuzzy set.
(iii) Establish whether or not a given possibility distribution is generated by an intuitionistic $\mathsf{Ł}_n$-fuzzy set.

Firstly, we propose the procedure INT-L-SET (see Algorithm 1) based on Eq. 10. Its input consists of a finite set $\mathcal{V}$, a positive integer n (to determine the corresponding Łukasiewicz n-chain), and a possibility distribution π from $\mathsf{Ł}_n^{\mathcal{V}}$ to $\mathsf{Ł}_n$. Its output is a pair (μ,ν) of mappings from $\mathcal{V}$ to $\mathsf{Ł}_n$. By Proposition 2, (μ,ν)

is an intuitionistic $Ł_n$-fuzzy set, and by Theorem 3, if $\pi \in \Pi^*$ then it generates π.

Algorithm 1: The algorithm for finding the intuitionistic $Ł_n$-fuzzy set corresponding to a given possibility distribution by means of (10).

```
procedure INT-L-SET (V, n, π)
foreach v ∈ V do
    μ(v), ν(v) → 1;
    foreach ω ∈ Ł_n^V such that π(ω) = 1 do
        μ(v) ← min{μ(v), ω(v)};
        ν(v) ← min{ν(v), 1 − ω(v)};
return (μ, ν);
end procedure
```

The next procedure (see Algorithm 2) is constructed by using the following proposition, where Eq. 4 is rewritten for all possibility distributions generated by an intuitionistic $Ł_n$-fuzzy set[4].

Proposition 4. *Let* π *be a possibility distribution generated by an intuitionistic* $Ł_n$*-fuzzy set* (μ, ν)*, and let* $\omega \in Ł_n^{\mathcal{V}}$*. Then,* $\pi(\omega) = \bigwedge_{v \in \mathcal{V}} \alpha_\omega(v)$*, where*

$$\alpha_\omega(v) = \begin{cases} \mu(v) \to \omega(v) & \text{if } \omega(v) \le \mu(v), \\ \nu(v) \to \neg\omega(v) & \text{if } \omega(v) \ge \neg\nu(v), \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

Proof. Let $v \in \mathcal{V}$ such that $\omega(v) \le \mu(v)$. By Definition 4, $\mu(v) \le \neg\nu(v)$. Hence, $\omega(v) \le \neg\nu(v)$. By Proposition 1(vii,viii), $\nu(v) \le \neg\omega(v)$, and so $\nu(v) \to \neg\omega(v) = 1$. Finally, using Eq. 4, $\alpha_\omega(v) = (\mu(v) \to \omega(v)) \otimes 1$, hence $\alpha_\omega(v) = (\mu(v) \to \omega(v))$ from Definition 1(ii).

Analogously, given $v \in \mathcal{V}$ such that $\omega(v) > \mu(v)$, we can prove that $\alpha_\omega(v)$ is given by (13).

Proposition 4 leads to the procedure VALUE (just apply the Łukasiewicz operations to (13)) taking as input a finite set $\mathcal{V}$, a function ω from $\mathcal{V}$ to $Ł_n$, and an intuitionistic $Ł_n$-fuzzy set (μ, ν), and producing as output the value $\pi_{(\mu,\nu)}(\omega)$.

[4] More in general, Proposition 4 holds when we consider complete residuated lattices with an involution and $[0, 1]$ as support.

Algorithm 2: The algorithm to find the values assumed by the possibility distribution generated by the intuitionistic $Ł_n$-fuzzy set (μ, ν).

```
procedure VALUE (𝒱, ω, (μ, ν))
m ← 1;
foreach v ∈ 𝒱 do
    if ω(v) < μ(v) then
        α_v(ω) ← 1 − μ(v) + ω(v);
    else
        if ω(v) > 1 − ν(v) then
            α_v(ω) ← 2 − ω(v) + ν(v);
        else
            α_v(ω) ← 1;
    m ← min{m, α_v(ω)};
return m;
end procedure
```

Finally, we present the procedure DISTRIBUTION to establish whether or not a given possibility distribution $\pi : Ł_n^{\mathcal{V}} \mapsto Ł_n$ is generated by the intuitionistic $Ł_n$-fuzzy set (μ_π, ν_π) (see Algorithm 3). In detail, firstly, the intuitionistic $Ł_n$-fuzzy set (μ_π, ν_π) is computed by INT-L-SET. Then, using the procedure VALUE, it is checked whether or not $\pi = \pi^*$, where π^* is the possibility distribution generated by (μ_π, ν_π). Eventually, if $\pi = \pi^*$, then π is generated by (μ_π, ν_π). Otherwise, the answer is that $\pi \notin \Pi^*$ (from Theorem 3 and Corollary 2). Moreover, by Proposition 3, we know that $\pi(\omega) = \pi^*(\omega)$ for each $\omega \in \mathcal{K}(\pi)$. Hence, we must apply the procedure VALUE only for each $\omega \in Ł_n^{\mathcal{V}} \setminus \mathcal{K}(\pi)$.

Algorithm 3: The algorithm to establish whether or not a given possibility distribution is generated by an intuitionistic $Ł_n$-fuzzy set.

```
procedure DISTRIBUTION (𝒱, n, π)
i ← 0;
(μ, ν) ← INT-L-SET(𝒱, n, π);
foreach ω ∈ Ł_n  such that  π(ω) ≠ 1 do
    if π(ω) ≠ VALUE(𝒱, ω, (μ, ν)) then
        i ← 1;
        break;
if i = 1 then
    print π is not generated by an intuitionistic Ł_n-fuzzy set;
else
    print π is generated by the intuitionistic Ł_n-fuzzy set (μ, ν);
return;
end procedure
```

5 Conclusions and Future Directions

In this article, we identified each intuitionistic L-fuzzy set with a special normal possibility distribution. On the other hand, we showed that not all normal possibility distributions can be identified with an intuitionistic L-fuzzy set.

In the future, we intend to explore the connection between possibility theory and intuitionistic L-fuzzy sets in more detail. As an example, we would like to discover other properties (in addition to normality) characterizing possibility distributions generated by ILF sets. Also, we could associate a collection of ILF sets to each possibility distribution, and hence, generalize by using fuzzy logic, the correspondence between Boolean possibility distributions and sets of orthopairs [8].

On a longer term, the link between the two theories could be used in applications, by applying techniques developed for IFS to Possibility Theory and, whenever possible, the other way round.

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