Chapter 8
Hierarchical Models of Conduction of Heat in Continua Contained in Prismatic Shell-like Domains

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Abstract We construct hierarchical models for the heat conduction in standard and prismatic shell-like and rod-like 3D domains with non-Lipschitz boundary, in general.

Key words: Hierarchical models, Heat conduction equations, Prismatic shell-like and rod-like 3D domains with non-Lipschitz boundaries

8.1 Introduction

If the quantities, causing deformation and temperature, vary sufficiently slowly from zero to their finite values and remain in such a state, then we have a steady process, i.e., static process as $t \to \infty$. Therefore, displacements and temperature become independent of time and are functions only of the state. Thus, in the equation of conduction of Heat disappear derivatives with respect to time, in particular deformation tensor velocity $\dot{\varepsilon}_{ij}(x,t) \equiv 0$. So, the governing system of thermoelasticity will be split and after solving the independent BVPs for temperature change θ and substituting the found temperature change into governing system of thermoelasticity we arrive at independent BVP of elasticity with the additional (caused by temperature) member. In the theory of temperature stresses, which studies influence of heating the body surfaces and heat sources on the stress state of body it is assumed that the influence of $\dot{\varepsilon}_{kk}$ involved in the equation of heat conduction on body deformation is negligible (see Nowacki, 1975, pp. 90, 92, 93, 764).

Thus, for the above-mentioned and for analogous cases it is important to have hierarchical models separately for the heat conduction in standard and prismatic

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H. Altenbach et al. (eds.), *Recent Approaches in the Theory of Plates and Plate-Like Structures*, Advanced Structured Materials 151,
https://doi.org/10.1007/978-3-030-87185-7_8

shell-like and rod-like 3D domains with non-Lipschitz boundaries, in general, occupied by a continuum. In the present paper our purpose is to construct hierarchical models for heat conduction in prismatic shell-like 3D domains Ω with non-Lipschitz boundaries, in general. To this and we use I. Vekua's dimension redaction method (Vekua, 1955, 1965, 1973, 1985). We have a definite experience of application of this method, we have constructed hierarchical models: for micropolar elastic cusped prismatic shells (Jaiani, 2016b), elastic prismatic shells with microtemperatures (Jaiani, 2015), piezoelectric viscoelastic Kelvin-Voigt prismatic shells with voids (Jaiani, 2018b) for prismatic shells with mixed conditions on face surfaces (Jaiani, 2016c), layered prismatic shells (Jaiani, 2016a). The above-mentioned hierarchical models we easily reformulate from elastic to thermoelastic if in the constitutive relations, namely, in expression of stress tensor, to the right-hand side we add

$$-\gamma T\delta_{ij}, \qquad \gamma = \frac{\alpha E}{1-2\nu}$$

with the linear thermal definition coefficient α.

Now, within the framework of the last hierarchical models we may consider the states described at the beginning of the present section and handle them with the way indicated there.

8.2 Governing System of Conduction of Heat

The conservation of energy equation has the form (see, e.g., Dautray and Lions (1990, Chapter 1, Section 2, Subsection 6, Point 6.3 General Equations of Classical Thermoelasticity) and also Nowacki (1975, Chapter 3, Section 3.3; Section 3.4, Point 4))

$$\rho\theta\frac{ds}{dt} + \mathrm{div}\mathbf{q} = f, \qquad \theta := T(\mathbf{x},t) - T_0, \quad \text{in } \Omega, \tag{8.1}$$

provided the intrinsic energy is zero, where T_0 is the absolute temperature in a natural state $t = t_0$, T is the absolute temperature at the moment t, s is the specific entropy, $\mathbf{q}(\mathbf{x},t)$ is the heat flux vector (with components q_i in the considered reference frame, heat is crossing a unit element of fictitious surface $\partial\Omega$ passing through $\mathbf{x}$ and perpendicular to a unit outward normal $\mathbf{n}$. The passage being made in the sense and direction of the vector $\mathbf{q}$); here, it is the question of heat transmitted by conduction of the interior of Ω, $f(\mathbf{x},t)$ is density per unit volume defining a rate of heat supplied by external elements in the medium under consideration, e.i., so called "source" function is supposed to be given and is in fact zero in a certain number of applications. Fourier's law in the isotropic case looks like (see, e.g., Dautray and Lions (1990, Chapter 1, Section 2, Subsection 6, Point 6.3 General Equations of Classical Thermoelasticity)

$$\mathbf{q} = -k\mathrm{grad}\theta, \tag{8.2}$$

where k is the thermal conduction coefficient. In the steady case, from (8.1) we get

$$\mathrm{div}\mathbf{q} = f. \tag{8.3}$$

Now, about boundary conditions (BC):

(i) if the temperature $\bar{T}$ is prescribed on a part at the boundary $\partial\omega$, then we have

$$\theta = \bar{T} - T_0; \tag{8.4}$$

(ii) if the flux of heat across a part of the boundary is imposed, then we have BC of the type

$$-q_i n_i = \bar{q}_n \text{ given,} \tag{8.5}$$

which because of (8.2) becomes

$$\frac{\partial\theta}{\partial n} = \bar{g} \text{ given.} \tag{8.6}$$

Let the body occupy a prismatic $3D$ domain Ω with a non-Lipschitz boundary, in general, and the upper and lower face surfaces of the prismatic $3D$ domain be given by $x_3 = \overset{(+)}{h}(x_1, x_2)$ and $x_3 = \overset{(-)}{h}(x_1, x_2)$, respectively. Let further

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \omega,$$

denote the thickness of the domain occupied by the body, ω is a projection of the $3D$ domain on the plane $x_3 = 0$, a part of the boundary $\partial\omega$ is called a cusped edge if $2h = 0$ there (see also the beginning of Sect. 3 of Jaiani, 2018b).

$$2\widetilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2), \ (x_1, x_2) \in \omega.$$

Substituting (8.2) into (8.3) we obtain the heat equation

$$-(k(x_1, x_2)\theta_{,j})_{,j} = f \tag{8.7}$$

in the steady case.

8.3 Mathematical Moments

For the convenience of the reader we repeat the revelent material from Sect. 10 of Jaiani (2018b). Let $f(x_1, x_2, x_3)$ be a given function in $\overline{\Omega}$ having integrable partial derivatives, let f_r be its r-th order moment defined as follows

$$f_r(x_1, x_2) := \int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} f(x_1, x_2, x_3) P_r(a x_3 - b) dx_3,$$

where (see the end of Sect. 2 and the beginning of Sect. 3 of Jaiani, 2018b)

$$a(x_1,x_2) := \frac{1}{h(x_1,x_2)}, \quad b(x_1,x_2) := \frac{\widetilde{h}(x_1,x_2)}{h(x_1,x_2)},$$

$$2h(x_1,x_2) = \overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2) > 0,$$

$$2\widetilde{h}(x_1,x_2) = \overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2) > 0,$$

and

$$P_r(\tau) = \frac{1}{2^r r!}\frac{d^r(\tau^2-1)^r}{d\tau^r}, \quad r = 0,1,\cdots,$$

are the r-th order Legendre polynomials with the orhogonality property

$$\int_{-1}^{+1} P_m(\tau)P_n(\tau)d\tau = \frac{2}{2m+1}\delta_{mn}.$$

From here, substituting

$$\tau = ax_3 - b = \frac{2}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)}x_3 - \frac{\overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2)}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)},$$

we have

$$\Big(m+\frac{1}{2}\Big)a\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_m(ax_3-b)P_n(ax_3-b)dx_3 = \delta_{mn}.$$

Using the well-known formulas of integration by parts (with respect to x_3) and differentiation with respect to a parameter of integrals depending on parameters (x_α), taking into account $P_r(1) = 1$, $P_r(-1) = (-1)^r$, we deduce

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = -a\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r'(ax_3-b)f\,dx_3 + \overset{(+)}{f} - (-1)^r\overset{(-)}{f}, \qquad (8.8)$$

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 = f_{r,\alpha} - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha}$$

$$-\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r'(ax_3-b)(a_{,\alpha}x_3-b_{,\alpha})f\,dx_3, \quad \alpha=1,2, \tag{8.9}$$

where superscript prime means differentiation with respect to the argument ax_3-b, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, $\overset{(\pm)}{f} := f[x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)]$. Applying the following relations from the theory of the Legendre polynomials (see e.g. Jaiani, 2018a, pp. 338-339)

$$P_r'(\tau)=\sum_{s=0}^{r}(2s+1)\frac{1-(-1)^{r+s}}{2}P_s(\tau)\ {}^{1},$$

$$\tau P_r'(\tau)=rP_r(\tau)+P_{r-1}'(\tau)=rP_r(\tau)+\sum_{s=0}^{r-1}(2s+1)\frac{1+(-1)^{r+s}}{2}P_s(\tau)\ {}^{2}, \tag{8.10}$$

and, in view of $\frac{a_{,\alpha}}{a}=(\ln a)_{,\alpha}=-\frac{h_{,\alpha}}{h}$, $\frac{a_{,\alpha}}{a}b=\widetilde{h}a_{,\alpha}$, $b_{,\alpha}=(\widetilde{h}a)_{,\alpha}$, it is easily seen that

$$P_r'(ax_3-b)(a_{,\alpha}x_3-b_{,\alpha})=\frac{a_{,\alpha}}{a}(ax_3-b)P_r'(ax_3-b)+(\frac{a_{,\alpha}}{a}b-b_{,\alpha})P_r'(ax_3-b)$$

$$=-h_{,\alpha}h^{-1}(ax_3-b)P_r'(ax_3-b)-\widetilde{h}_{,\alpha}h^{-1}P_r'(ax_3-b)$$

$$=-\overset{r}{a}_{\alpha\underline{r}}P_r(ax_3-b)-\sum_{s=0}^{r-1}\overset{r}{a}_{\alpha s}P_s(ax_3-b)\ {}^{3}, \tag{8.11}$$

where

---

[1] On the top of the symbol $\sum$ both $r-1$ and r are true since the last term equals zero.

[2] On the top of the symbol $\sum$ both $r-2$ and $r-1$ are true since the last term equals zero.

[3] The following relations are valid

$$\sum_{s=0}^{r-1}(2s+1)\left[\frac{h_{,\alpha}+(-1)^{r+s}h_{,\alpha}}{2h}+\frac{\tilde{h}_{,\alpha}-(-1)^{r+s}\tilde{h}_{,\alpha}}{2h}\right]P_s(ax_3-b)$$

$$=\sum_{s=0}^{r-1}\frac{(2s+1)}{2h}\left(\frac{\overset{(+)}{h}_{,\alpha}-\overset{(-)}{h}_{,\alpha}+\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right.$$

$$\left.+\frac{\overset{(+)}{h}_{,\alpha}+\overset{(-)}{h}_{,\alpha}-\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right)P_s(ax_3-b)$$

$$=\sum_{s=0}^{r-1}(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}P_s(ax_3-b)$$

because of $h=\frac{\overset{(+)}{h}-\overset{(-)}{h}}{2}$, $\widetilde{h}=\frac{\overset{(+)}{h}+\overset{(-)}{h}}{2}$.

$$\overset{r}{a}_{\alpha \underline{r}} := r\frac{h_{,\alpha}}{h}, \;\; \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \;\; s \neq r. \tag{8.12}$$

Now, bearing in mind (8.11) and (8.10), from (8.9) and (8.8) we have

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3$$

$$= f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha}, \;\; \alpha = 1,2, \tag{8.13}$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s + \overset{(+)}{f} - (-1)^r \overset{(-)}{f}, \tag{8.14}$$

respectively. Here

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}, \tag{8.15}$$

clearly,

$$\overset{r}{a}_{3r} = 0. \tag{8.16}$$

Let

$$f(x_1,x_2,x_3) = \sum_{r=0}^{\infty} a\Big(r+\frac{1}{2}\Big) f_r(x_1,x_2) P_r(ax_3-b), \tag{8.17}$$

then

$$\overset{(\pm)}{f} := f(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)) = \sum_{s=0}^{\infty} a\Big(s+\frac{1}{2}\Big) f_s (\pm 1)^s$$

$$= \sum_{s=0}^{\infty} \frac{(\pm 1)^s (2s+1)}{2h} f_s, \;\; i = \overline{1,3}, \tag{8.18}$$

whence

$$\overset{(+)}{f} - (-1)^r \overset{(-)}{f} = -\sum_{s=0}^{\infty} \overset{r}{a}_{3s} f_s, \;\; i = \overline{1,3}, \tag{8.19}$$

$$\overset{(+)}{f}\overset{(+)}{h}_{,\alpha} - (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha} = \sum_{s=0}^{\infty} \overset{r}{a}{}^{*}_{\alpha s} f_s, \;\; i = \overline{1,3}, \;\; \alpha = 1,2, \tag{8.20}$$

where

$$\overset{r}{a}{}^{*}_{\alpha s} = \overset{r}{a}_{\alpha s}, \quad s \neq r, \quad \overset{r}{a}{}^{*}_{\alpha \underline{r}} = (2r+1)\frac{h_{,\alpha}}{h}. \tag{8.21}$$

Substituting (8.20) and (8.19) into (8.13) and (8.14), respectively, we get

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}{}^{*}_{\alpha s} f_s$$

$$= f_{r,\alpha} + \sum_{s=r}^{\infty} \overset{r}{b}_{\alpha s} f_s, \tag{8.22}$$

where

$$\overset{r}{b}_{js} := -\overset{r}{a}_{js}, \ s > r; \ \overset{r}{b}_{js} = 0, \ s < r; \tag{8.23}$$

$$\overset{r}{b}_{\alpha \underline{r}} := \overset{r}{a}_{\alpha \underline{r}} - \overset{r}{a}{}^{*}_{\alpha \underline{r}} = -(r+1)\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \ \overset{r}{b}_{3\underline{r}} = -\overset{r}{a}_{3\underline{r}} = 0, \tag{8.24}$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,3}\, dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}_{3s} f_s$$

$$= -\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s} f_s = \sum_{s=r+1}^{\infty} \overset{r}{b}_{3s} f_s, \tag{8.25}$$

respectively.

If $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are known (prescribed), then from (8.13) and (8.14), correspondingly, we obtain

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s$$

$$+ \overset{(+)}{f}\overset{(+)}{n}_{\alpha}\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} + (-1)^r \overset{(-)}{f}\overset{(-)}{n}_{\alpha}\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2} \tag{8.26}$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s$$

$$+\overset{(+)}{f}\overset{(+)}{n}_3\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2}+(-1)^r\overset{(-)}{f}\overset{(-)}{n}_3\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2}, \qquad (8.27)$$

since

$$\overset{(\pm)}{n}_\alpha = \frac{\mp\overset{(\pm)}{h}_{,\alpha}}{\sqrt{1+(\overset{(\pm)}{h}_{,1})^2+(\overset{(\pm)}{h}_{,2})^2}}, \quad \overset{(\pm)}{n}_3 = \frac{\pm 1}{\sqrt{1+(\overset{(\pm)}{h}_{,1})^2+(\overset{(\pm)}{h}_{,2})^2}}.$$

8.4 Construction of Hierarchical Models

To this end, applying Vekua's dimension reduction method (Vekua, 1955, 1965, 1973, 1985), we multiply (8.1), (8.2), (8.4), and (8.6) by $P_r(ax_3-b)$ and then integrate within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$. Using formulas (8.6), (8.7), (8.15), and (8.18), we assume the heat flux vector normal component $q(\mathbf{x},t,\mathbf{n})$ to be prescribed on the face surfaces, while on the lateral boundary of the body we assume to be hold either BC (8.4) or BC (8.6). Besides, we consider $\rho=\rho(x_1,x_2)$ and by calculations for temperature change θ on the face surfaces we employ (8.19), (8.20).

Thus, in the steady case: from (8.3), by virtue of (8.26), (8.27), we have

$$\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} q_{k,k}P_r(ax_3-b)dx_3 = \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} q_{\gamma,\gamma}P_r(ax_3-b)dx_3 + \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} v_{3,3}P_r(ax_3-b)dx_3$$

$$= q_{\gamma r,\gamma} + \sum_{s=0}^{r}\overset{r}{a}_{\gamma s}q_{\gamma s} - \overset{(+)}{q}_\gamma\overset{(+)}{h}_{,\gamma} + (-1)^r\overset{(-)}{q}_\gamma\overset{(-)}{h}_{,\gamma} + \sum_{s=0}^{r}\overset{r}{a}_{3s}q_{3s} + \overset{(+)}{q}_3 - (-1)^r\overset{(-)}{q}_3$$

$$= q_{\gamma r,\gamma} + \sum_{s=0}^{r}\overset{r}{a}_{\gamma s}q_{\gamma s} + q_{\overset{(+)}{n}}\sqrt{1+\overset{(+)}{h}_{,\gamma}\overset{(+)}{h}_{,\gamma}} + (-1)^r q_{\overset{(-)}{n}}\sqrt{1+\overset{(-)}{h}_{,\gamma}\overset{(-)}{h}_{,\gamma}} = f_r, \qquad (8.28)$$

$$r=0,1,2,...,$$

because of

$$\overset{(\pm)}{n}_\gamma = \frac{\mp\overset{(\pm)}{h}_{,\gamma}}{\sqrt{\overset{(\pm)}{h}_{,\alpha}\overset{(\pm)}{h}_{,\alpha}+1}}, \quad \overset{(\pm)}{n}_3 = \frac{\pm 1}{\sqrt{\overset{(\pm)}{h}_{,\alpha}\overset{(\pm)}{h}_{,\alpha}+1}}$$

from (8.2), provided $k=k(x_1,x_2)$, by virtue of (8.22)-(8.24), we get

$$
\begin{aligned}
q_{\gamma r} &= k(x_1,x_2)\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \theta_{,\gamma}(x_1,x_2,x_3)P_r(ax_3-b)dx_3 \\
&= k(x_1,x_2)\Big[\theta_{r,\gamma}+\sum_{s=0}^{r}\overset{r}{a}_{\gamma s}\theta_s-\overset{(+)}{\theta}\overset{(+)}{h}_{,\gamma}+(-1)^r\overset{(-)}{\theta}\overset{(-)}{h}_{,\gamma}\Big] \\
&= k(x_1,x_2)\Big[\theta_{r,\gamma}+\sum_{s=0}^{r}\overset{r}{a}_{\gamma s}\theta_s-\sum_{s=0}^{r}\overset{r}{a}{}^{*}_{\gamma s}\theta_s\Big] \\
&= k(x_1,x_2)\Big[\theta_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\theta_s\Big] \\
&= k(x_1,x_2)h^{r+1}(\bar{\theta}_r)_{,\gamma}+\sum_{s=r+1}^{\infty}\overset{r}{b}_{\gamma s}h^{s+1}\tilde{\theta}_s, \\
&\qquad \gamma=1,2,\quad r=0,1,2,...,
\end{aligned}
\tag{8.29}
$$

because of $\theta_{r,\gamma}+\overset{r}{b}_{\gamma r}\theta_r=h^{r+1}(\tilde{\theta}_r)_{,\gamma}$, $\tilde{\theta}_r := \frac{\theta_r}{h^{r+1}}$

$$
\begin{aligned}
q_{3r}(x_1,x_2) &= k(x_1,x_2)\Big[\sum_{s=0}^{r}\overset{r}{a}_{3s}q_{3s}+\overset{(+)}{\theta}+(-1)^r\overset{(-)}{\theta}\Big] \\
&= k(x_1,x_2)\sum_{s=r}^{\infty}\overset{r}{b}_{3s}\theta_s,\quad r=0,1,2,....
\end{aligned}
\tag{8.30}
$$

If we multiply by h^r the last equality in (8.28), the obtained equation

$$
h^r q_{\gamma r,\gamma}+h^r\sum_{s=0}^{r}\overset{r}{a}_{\gamma s}q_{\gamma s}+h^r\overset{r}{Q}=h^r f_r,\quad r=0,1,..., \tag{8.31}
$$

where

$$
\overset{r}{Q} := q_{\underset{n}{(+)}}\sqrt{1+\overset{(+)}{h}_{,\gamma}\overset{(+)}{h}_{,\gamma}}+(-1)^r q_{\underset{n}{(-)}}\sqrt{1+\overset{(-)}{h}_{,\gamma}\overset{(-)}{h}_{,\gamma}} \tag{8.32}
$$

we can rewrite as

$$
(h^r q_{\gamma r},_{\gamma}+h^r\sum_{s=0}^{r-1}\overset{r}{a}_{\gamma s}q_{\gamma s}+h^r\overset{r}{Q}=h^r f_r,\quad r=0,1,..., \tag{8.33}
$$

because of

$$
h^r q_{\gamma r},_{\gamma}+h^r\overset{r}{a}_{\gamma r}q_{\gamma r}=h^r q_{\gamma r,\gamma}+h^r r\frac{h_{,\gamma}}{h}=(h^r q_{\gamma r})_{,\gamma}. \tag{8.34}
$$

Now, considering weighted moments

$$\tilde{q}_{jr} := \frac{q_{jr}}{h^{r+1}}, \quad j = \overline{1,3}, \quad \tilde{\theta}_r := \frac{\theta_r}{h^{r+1}} \tag{8.35}$$

from (8.33) we get the following equations

$$(h^{2r+1}\tilde{q}_{\gamma r})_{,\gamma} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{\gamma s} h^{s+1} \tilde{q}_{\gamma s} + h^r \overset{r}{Q} = h^r f_r, \tag{8.36}$$
$$\gamma = 1,2, \quad r = 0,1,...,$$

with respect to weighted moments $\tilde{q}_{\gamma r}$, inserting (8.29) into (8.33) we derive hit equation in terms of weighted moments of θ:

$$\left[k(x_1,x_2)h^{2r+1}(x_1,x_2)\tilde{\theta}_{r,\gamma}\right]_{,\gamma} + \left[k(x_1,x_2)\sum_{s=r+1}^{\infty} \overset{r}{b}_{\gamma s} h^{s+1}\tilde{\theta}_s\right]_{,\gamma} = f_r, \tag{8.37}$$
$$r = 0,1,2,....$$

In other words we have rewritten heat equation (8.7) in terms of moments $\tilde{\theta}_s$, $s = r$, $r+1,....$ If we neglect moments of order $r > N$, we get Nth order approximation, i.e., Nth hierarchical model of heat transfer with the following BCs in moments

$$\theta_r = \bar{\theta}_r, \quad r = 0,1,2,...,N, \qquad q_{nr} = \bar{q}_{nr}, \quad r = 0,1,2,...,N, \tag{8.38}$$

where $\bar{\theta}_r$, $\bar{q}_{nr}$ we calculate from prescribed $\bar{\theta}$ and $\bar{q}_n$ after multiplying them by $P_r(ax_3 - b)$ and then integrating within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$. In the case of cusped edges they should be calculated as limits from the inside of domain.

The last, according to (8.29) may be rewritten as weighted Neumann BC

$$kh^{r+1}\frac{\partial\bar{\theta}_r}{\partial n} = \bar{g}_r, \quad r = 0,1,...,N, \tag{8.39}$$

Concentrated at point and at cusped edge (line) heat flux we define similar to definition of concentrated at point and at cusped edge (line) force (see Jaiani, 2008)

8.5 The $N = 0$ Approximation

In this case from (8.37)-(8.39) we get the following two BVPs:
Find $\theta_0 \in C^2$, satisfying equation

$$\left[k(x_1,x_2)h(x_1,x_2)\tilde{\theta}_{0,\gamma}\right]_{,\gamma} = f_0(x_1,x_2) \tag{8.40}$$

under either BC

$$\theta_0 = \bar{\theta}_0 \tag{8.41}$$

or BC

$$q_{n0} = \bar{q}_{n0}, \ \text{i.e.,} \ kh\frac{\partial \tilde{\theta}_0}{\partial n} = \bar{g}_0 \tag{8.42}$$

with prescribed $\bar{\theta}_0$ and $\bar{g}_0$.

8.6 Case of Cusped Bodies

Let now, in the $N = 0$ approximation (model) consider the body Ω with the half-thickness

$$h = h_0 x_2^\kappa, \quad h_0, \kappa = \text{const} > 0, \tag{8.43}$$

whose projection ω on plane $x_3 = 0$ is a strip

$$\{(x,y) : -\infty < x < +\infty, \ 0 < y < L, \ L = \text{const} > 0\},$$

Eq. (8.40) will get the form:

$$x_2^\kappa (k\tilde{\theta}_{0,1})_{,1} + (kx_2^\kappa \tilde{\theta}_{0,2})_{,2} = \frac{f_0(x_1,x_2)}{h_0}.$$

First we assume $k = k_0 = \text{const}$, then Eq. (8.40) looks like the following singular differential equation

$$u_{,11} + u_{,22} + \frac{\kappa}{x_2} u_{,2} = \frac{x_2^{-\kappa}}{kh_0} f_0(x_1,x_2), \tag{8.44}$$

i.e.,

$$x_2 \Delta u + \kappa u_{,2} = \frac{x_2^{1-\kappa}}{k_0 h_0} f_0(x_1,x_2). \tag{8.45}$$

Let us consider the rectangular part of the cusped strip bounded by lines $x = a$, $x = b$, $a < b$. From the main theorem (Jaiani, 1995) it immediately follows:

Theorem 8.1. *If $f_0(x_1,x_2) \equiv 0$, then for $\kappa < 1$ the Dirichlet Problem is well-posed, i.e. the weighted temperature $\tilde{\theta}_0$ should be prescribed on the whole boundary $\partial\omega$, while for $\kappa \geq 1$ the Keldysh Problem is well-posed, i.e., on the three non-cusped edges of the rectangular boundary weighted temperature $\tilde{\theta}_0$ should be prescribed but cusped edge $y = 0$, $a < x < b$ should be left without BC, provided solution $\tilde{\theta}_0$ is bounded.*

Now, we consider particular case when $\tilde{\theta}_0 = \tilde{\theta}_0(x_2)$, $f_0 = f_0(x_2)$, $k = k(x_2)$ and $k(x_2)h(x_2) > 0$ as $x_2 \in]0, L]$, $k(0)h(0) = 0$, then the general solution of equation (8.40), which takes the form of the following degenerate partial differential equation

$$\left[k(x_2)h(x_2)\tilde{\theta}_{0,2}(x_2)\right]_{,2} = f_0(x_2), \tag{8.46}$$

has the form

$$\tilde{\theta}_0(x_2) = c_1 \int_L^{x_2} \frac{d\tau}{k(\tau)h(\tau)} + \int_L^{x_2} \frac{d\tau}{k(\tau)h(\tau)} \int_L^{\tau} f_0(t)dt + c_2. \tag{8.47}$$

Whence, the Dirichlet problem is well-posed, i.e., the weighted temperature should be prescribed on both the edges $y = 0$ and $y = L$ if and only if

$$\int_0^{x_2^0} \frac{d\tau}{k(\tau)h(\tau)} < \infty, \tag{8.48}$$

for Keldysh type problem we have the condition

$$\int_0^{x_2^0} \frac{d\tau}{k(\tau)h(\tau)} = +\infty, \tag{8.49}$$

therefore, only at the edge $y = L$ should be prescribed the weighted temperature and the edge $y = 0$ should be freed from BC, provided we are looking for bounded solutions, i.e. we have the Keldysh type BVP. Moreover, both the BVP we solve in the explicit form under BCs

$$\tilde{\theta}_0(0) = \bar{\bar{\theta}}_0, \tag{8.50}$$

$$\tilde{\theta}_0(L) = \bar{\bar{\theta}}_L, \tag{8.51}$$

in the case of the Dirichlet type BVP and under BC (8.51) in the case of the Keldysh type BVP. The unique solutions have the form (8.47), where

$$c_2 = \bar{\bar{\theta}}_0, \tag{8.52}$$
$$c_1 = \Big[\int_L^0 \frac{d\tau}{k(\tau)h(\tau)} \Big]^{-1} \Big[\bar{\bar{\theta}}_0 - \bar{\bar{\theta}}_L - \int_L^0 \frac{d\tau}{k(\tau)h(\tau)} \int_L^{\tau} \frac{dt}{k(t)h(t)} \Big]$$

for the Dirichlet type BVP and with $c_1 = 0$ and (8.52) for the Keldysh type BVP (clearly in the particular case (8.43), when $k(\tau) = k_0 \neq 0$ we again obtain the condition $\kappa < 1$ for the Dirichlet Problem and the condition $\kappa \geq 1$ for the Keldysh problem).

The mixed BVP under BC (8.51) and the weighted Neumann condition (8.42) has a unique explicit solution (8.47), where

$$c_1 = \bar{g}_0 - \int_L^0 f_0(t)dt - \bar{\bar{\theta}}_L,$$

$$c_2 = \bar{\bar{\theta}}_L.$$

Indeed, from (8.42), bearing in mind (8.49) we obtain

$$\bar{g}_0 = c_1 + \int_L^0 f_0(t)dt + \bar{\tilde{\theta}}_L.$$

8.7 Conclusions

Differential hierarchical models for the heat condition equation in prismatic shell-like domains non-Lipschits, in general, are constructed and the peculiarities of setting of boundary conditions in the case of cusped domains are discussed. These results allow to investigate well-posedness of boundary value problems for thermoelastic bodies with non-Lipschits boundaries, in general when deformation and temperature vary sufficiently slowly and the governing system of thermoelasticity will be split into two independent BVPs for temperature and the deformed state of the body.

The peculiarities of nonclassical setting of BCs when either the thickness, or thermal conduction coefficient, or both ones vanish at the edge of prismatic shells are discussed, criteria of setting the Dirichlet and Keldysh type BVPs are established. Some concrete BVPs are solved in the explicit form.

In the $N = 0$ approximation a mixed BVP, when at non-cusped edge the weighted temperature and at cusped edge the concentrated at edge heat flux are prescribed, is solved.

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