Chapter 3
Bending Stiffness of Multilayer Plates with Alternating Soft and Hard Layers

Alexander K. Belyaev, Nikita F. Morozov, Petr E. Tovstik, and Tatiana P. Tovstik

Abstract The bending stiffness of a multilayer plate with alternating soft and hard layers is considered under the assumption that the deformation wavelength is substantially greater than the plate thickness. We discuss the approximate methods for determining the shear compliance required for replacing a multilayer plate with an equivalent single-layer Timoshenko – Reissner plate. A comparison is made with the exact solution of the three-dimensional problem of the theory of elasticity. The dependence of shear compliance on the ratio of Young's moduli of layers and on their location is investigated.

Key words: Plate vibrations and buckling, Multilayer plate, Long-wave deformation, Generalized Timoshenko – Reissner model, Transverse shear stiffness

3.1 Introduction

The approach to plate theory based on the hypothesis of a straight non-deformable normal, which was proposed and developed by Kirchhoff (1876) and then applied and improved for shells by Love (1927), is the main two-dimensional model of the theory of thin plates and shells. The range of applicability of this model is limited to single-layer plates made of a homogeneous isotropic material. However for anisotropic plates with low shear stiffness, for plates with oblique anisotropy, for

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multilayer plates with alternating soft and hard layers, the Kirchhoff – Love (KL) model leads to significant inaccuracies, that means, it becomes necessary to use refined models.

A more complex model was proposed in the first half of the 20th century by Timoshenko (1921); Reissner (2021). The Timoshenko – Reissner (TR) theory takes into account rotations of the mid surface normals, that is, includes the effect of transverse shear deformation. In this case, the plate can be considered as a material plane, the elements of which have translational and rotational degrees of freedom. In the limiting case, when the shear stiffness is equal to infinity, the TR model turns to the KL model.

The TR accounting for the transverse shear leads to a significant refinement of the results compared to the KL model for anisotropic plates with low transverse shear stiffness and for multilayer plates with alternating soft and hard layers. For multilayer plates, an equivalent single-layer TR plate made of a homogeneous material is introduced in Tovstik and Tovstik (2017a,b), which models a multilayer plate form a perspective of deflections, vibrations and buckling. The equivalent bending stiffness can be found using the same formulas as in the KL model, but determining the shear stiffness presents certain difficulties and is discussed in detail in what follows. In this paper, to determine this rigidity, we use an asymptotic expansion of the solution of a three-dimensional problem in a series in powers of a small dimensionless thickness (Tovstik and Tovstik, 2014; Morozov et al, 2016). Other methods for determining the shear stiffness are also discussed in Hill (1965); Grigolyuk and Kulikov (1988). These methods are discussed using the example of the problem of free vibrations of a multilayer plate with transversely isotropic layers. A comparison is made with the exact solution of the three-dimensional problem. The dependence of the shear compliance, bending stiffness, vibration frequency and buckling of a multilayer plate on the ratio of Young's moduli of layers and on the arrangement of layers is investigated.

3.2 Free Vibration and Bending of Multilayer Plate

Let us first consider free bending vibrations of a transversely isotropic homogeneous plate with the deflection $w(x,y,t) = w_0 \sin px \sin qy \sin \omega t$. This deflection is typical for vibrating infinite plate, as well as vibration of a rectangular simply supported plate. In the latter case $p = p_m = m\pi/L_x,\ q = q_n = n\pi/L_y,\quad m,n = 1,2,\ldots,$ where $L_x,\ L_y$ stand for length of the corresponding size. For the TR model, the vibration frequency ω is related to the dimensionless frequency parameter

$$\lambda = \frac{\rho h^2 \omega^2}{E_0}$$

and given by the equations (Tovstik and Tovstik, 2017a,b)

$$\lambda = \lambda^{TR} = \frac{\lambda^{KL}}{1+g}, \quad \lambda^{KL} = D\mu^4, \tag{3.1}$$

where $E_0 = E/(1-\nu^2)$, $\mu = rh = 2\pi h/L$, $r^2 = p^2 + q^2$. Here ρ is the material mass density, h is the plate thickness, $L = (L_x^{-2} + L_y^{-2})^{-\frac{1}{2}}$ is a typical wave length, E is the Young modulus, ν is the Poisson ratio, μ is a small parameter proportional to the ratio of the plate thickness to a typical wave length, $D = 1/12$ is a dimensionless parameter of the bending stiffness, $g = (E_0\mu^2)/(10G_{13})$ is a parameter of influence of transverse shear, G_{13} is the transverse shear modulus. For isotropic layers $G_{13} = E/(2(1+\nu))$, while for transversally isotropic layers G_{13} is an independent parameter. For thin plates ($\mu \ll 1$) when $E/G_{13} \sim 1$ the term g in (1) can be neglected whereas for $G_{13} \ll E$ the shear correction factor becomes considerable. When $g = 0$, i.e. when the shear is not taken into account, Eq. (3.1) yields the KL formula $\lambda^{KL} = D\mu^4$.

For KL model, the static deflection $w(x,y) = w_0 \sin px \sin qy$ of the plate under the normal load $f(x,y,t) = f_0 \sin px \sin qy$ is given by the equation

$$\frac{Dr^4 w_0}{1+g} = f_0, \tag{3.2}$$

cf. Tovstik and Tovstik (2017a,b), where the bending stiffness D and the shear compliance g are given by Eq. (3.1). The objective of the present work is to develop Eqs. (3.1) and (3.2) for multilayer plates.

3.3 Asymptotic Integration of Three-dimensional Equations

For a multilayer plate, the elastic moduli and the mass density become piecewise constant functions of the transverse coordinate z, $0 \le z \le h$. This parametrization is more convenient than the usual $-h/2 \le z \le h/2$, since for coordinates used here $z \ne 0$ at the neutral layer. We introduce dimensionless variables (with$\hat{}$)

$$\{u_1, u_2, w, z\} = h\{\hat{u}_1, \hat{u}_2, \hat{w}, \hat{z}\}, \quad \{x, y\} = L\{\hat{x}, \hat{y}, \}$$
$$\{E, E_0, G_{13}, \sigma_{ij}\} = E_*\{\hat{E}, \hat{E}_0, \hat{G}_{13}, \hat{\sigma}_{ij}\}, \quad i, j = 1, 2, 3,$$

where u_1, u_2 are tangential displacements along the x, y axes, σ_{ij} are stress tensor components, E_* - thickness-average Young's modulus. Here and after we omit mark$\hat{}$.

For a multilayer plate, the exact expression for the frequency parameter λ can be found from a three-dimensional boundary value problem, which for transverse vibrations in dimensionless variables is reduced to the form (Tovstik and Tovstik, 2014; Morozov et al, 2016; Tovstik and Tovstik, 2017a,b):

$$
\begin{aligned}
&\frac{dw}{dz} = -\mu^2 c_\nu(z)u + \mu^4 c_3(z)\sigma_{33}, && \frac{du}{dz} = w + \mu^2 c_g(z)\sigma, \\
&\frac{d\sigma}{dz} = E_0(z)u + \mu^2 c_\nu(z)\sigma_{33} - \mu^2\lambda\rho(z)u, && \frac{d\sigma_{33}}{dz} = -\sigma - \lambda\rho(z)w, \\
&\sigma = \sigma_{33} = 0, \quad z = 0, 1.
\end{aligned}
\tag{3.3}
$$

Here the displacements and the z coordinate are related to the plate thickness, whereas the stresses and the elastic moduli are related to Young's modulus of the rigid layer, and the densities are related to the density of the rigid layer. Instead of tangential movements u_1, u_2 and stresses σ_{13}, σ_{23} we introduce auxiliary unknowns quantities $u = \mu(pu_1 + qu_2)$, $\sigma = \mu^3(p\sigma_{13} + q\sigma_{23})$ and

$$
c_\nu = \frac{\nu}{1-\nu}, \quad c_3 = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}, \quad c_g = \frac{1}{G_{13}}.
$$

By asymptotic expansion of the solution to the boundary value problem (3.3) in powers of the small parameter μ in Timoshenko (1921); Reissner (2021); Tovstik and Tovstik (2017b), an expression for the frequency parameter λ was obtained in a form similar to (3.1):

$$
\lambda = \frac{\rho_* h^2\omega^2}{E_*} = \frac{\lambda^{KL}}{1+g}, \quad \lambda^{KL} = D_*\mu^4, \quad D_* = \frac{1}{E_*}\int_0^1 E_0(z)(z-a)^2 dz, \tag{3.4}
$$

where

$$
\begin{aligned}
&g = g^a + O(\mu^4), \qquad g^a = \mu^2(A_g + A_\nu + J + J_\nu), \\
&\{E_*, \rho_*\} = \int_0^1 \{E_0(z), \rho(z)\} dz, \qquad a = \frac{1}{E_*}\int_0^1 E_0(z) z dz, \\
&A_g = \frac{1}{E_* D_*}\int_0^1 \frac{\left(\int_0^z E_0(z_1)(z_1 - a)dz_1\right)^2}{G_{13}(z)} dz.
\end{aligned}
\tag{3.5}
$$

Here E_*, ρ_* is the thickness average tensile stiffness and density, D_* is the bending stiffness parameter and a is the coordinate of the neutral layer. The second order terms g^a take into account the lateral shear compliance (A_g), the Poisson tension of a normal fiber (A_ν), the inertia of rotational motion (J) and the inertia of the Poisson extension (J_ν). The quantities A_ν, J and J_ν are not shown here, cf. Tovstik and Tovstik (2017a,b).

For the problem of statics (3.2), the formula for deflection, accurate to terms of the second order of smallness, has the form:

$$
w^{TR} = w^{KL}(1+g), \quad w^{KL} = \frac{f_0}{Dr^4}, \quad g = g^a = \mu^2\left(A_g + A_\nu\right). \tag{3.6}
$$

3.4 The Transverse Shear Stiffness

Calculations for multilayer plates by Eqs. (3.4)-(3.6) are associated with the calculation of iterated integrals of piecewise constant functions, and hence are rather cumbersome. That is why we will consider the possibility of simplifying them in what follows. Consider a plate with alternating isotropic hard and soft layers and denote by η the ratio of Young's moduli of hard and soft layers. If the parameter η increases, then the moduli G_{13} of the soft layers of the plate decrease and, by virtue of formula (3.4), the coefficient A_g also increases, while the remaining second-order coefficients A_v, J, J_v remain significantly smaller than A_g.

Consider, for example, a three-layer plate with the layer thicknesses $h_1 = 0.3$, $h_2 = 0.6$, $h_3 = 0.1$. Respectively, Young's moduli of hard and soft layers are equal to $E_1 = E_3 = 1$, $E_2 = 1/\eta$, Poisson's ratios $\nu_1 = \nu_3 = 0.3$, $\nu_2 = 0.35$. For a number of values of η, the coefficients of the second order of smallness are given in Table 3.1. We put approximately $g^a = \mu^2 A_g$, thus returning to the TR model, which takes into account only the shear and ignores the other terms of the second order of smallness.

Calculations in Tovstik and Tovstik (2017a,b) showed that at $\eta \leq 1000$, $\mu = 0.1$ the error of Eq. (3.4) for $g = g^a = \mu^2 A_g$ does not exceed 4%. In what follows, the error of this replacement is discussed in more detail, cf. Hill (1965).

3.5 The Exact Value of the Shear Stiffness

By virtue of Eq. (3.5), we have the estimate

$$g^a = \mu^2 A_g = O(\mu^2 \eta). \tag{3.7}$$

For very large η, i.e., for a large ratio of the stiffness of the layers we have $g^a > 1$, Eq. (3.4) for $g = g^a$ becomes inaccurate and it is necessary to find the exact value $g = g^e$ for which Eq. (3.4) gives the exact value $\lambda = \lambda^e$. In order to find it we put $c_v = c_3 = 0$ in system (3.3) and omit the term $\mu^2 \lambda \rho(z) u$ in the third equation. We obtain $w = 1$ and the auxiliary boundary-value problem

$$\frac{du}{dz} = w + \mu^2 c_g(z)\sigma, \quad \frac{d\sigma}{dz} = E_0(z)u, \quad \sigma(0) = \sigma(1) = 0. \tag{3.8}$$

Table 3.1 Terms of the second order of smallness.

η	A_g	A_v	J	J_v	a	D_*
1	0.299	0.0928	0.1150	0.0308	0.502	0.0824
10	1.461	0.0875	0.1114	0.0081	0.384	0.1202
100	12.921	0.0844	0.1149	0.0026	0.354	0.1253
1000	127.515	0.0840	0.1154	0.0019	0.350	0.1259

After solving it, from the compatibility condition of the fourth equation (3.3), we find $\lambda = -\int_0^1 \sigma(z)dz$ from Eq. (3.3) we have

$$g^e = \frac{1}{\mu^2}\left(\frac{D_*}{\lambda} - 1\right). \tag{3.9}$$

Some examples of calculation are provided in Sect. 3.8.

Equation (3.9) is obtained from consideration of free vibrations. Calculation of the same value g^e from the statics problem is more difficult, because with an exact statement, the deflection depends on the distribution of the load over the thickness and the types of the layer, cf. Tovstik and Tovstik (2017a).

3.6 About the TR Model for a Homogeneous Transversally Isotropic Plate

According to the TR model, the frequency parameter λ for a homogeneous transversely isotropic plate is determined by Eq. (3.4), in which $g = g_0 = \dfrac{q}{10}$, $q = \dfrac{\mu^2 E_0}{G_{13}}$. Let us estimate the accuracy of this formula for $g_0 > 1$. For a homogeneous plate, problem (3.6) has a closed form solution

$$\sigma = \frac{G}{\mu^2}\left(\frac{\cosh\left(\sqrt{q}(z-0.5)\right)}{\frac{\sqrt{q}}{2}} - 1\right)$$

and Eq. (3.9) yields

$$g^e = \frac{q}{12\left(2\cosh\left(\sqrt{q}/2\right)/\sqrt{q} - 1\right)} - 1. \tag{3.10}$$

Calculations using Eq. (3.10) gave the following results:

$q/10 =$	0.1	1	10	100	1000
$g^e =$	0.0999	0.989	9.42	88.0	849

from which it follows that with an increase in q, the exact value of g^e deviates downward from the value $q/10$, recommended by the TR model.

3.7 Other Ways to Calculate the Shear Parameter g

In the classical paper by Hill (1965) two models, Voigt and Reuss, for estimating the transverse shear modulus of a composite material are given $G_V = \gamma_1 G_1 + \gamma_2 G_2$

and $G_R = \left(\frac{\gamma_1}{G_1} + \frac{\gamma_2}{G_2}\right)^{-1}$. The second formula for the plate with N layers takes the form of the sum of shear compliance of the layers

$$g = \sum_{n=1}^{N} \frac{\gamma_n}{G_n}, \tag{3.11}$$

where γ_n do not depend in transverse shear moduli of the layers G_n. G_n are the independent coefficients, the formulas for which are not given here. Note that Eq. (3.11) for A_g is reduced to (11) after calculating the integrals.

The monograph by Grigolyuk and Kulikov (1988) (GK) proposed an algorithm for taking into account the transverse shear effect for multilayer plates and shells. It is expedient to return to this algorithm, because the recent works (Mikhasev and Altenbach, 2019; Morozov et al, 2020), as well as a number of other works, reported application of this algorithm for solving some particular problems. This algorithm is based on the hypothesis of distribution of the transverse shear deformations over the plate thickness. According to Grigolyuk and Kulikov (1988), the formula for g can be written as:

$$\left(\left(\sum_{n=1}^{N} \frac{\alpha_n}{G_n}\right)^{-1} + \sum_{n=1}^{N} \beta_n G_n\right)^{-1}, \tag{3.12}$$

where α_n and β_n are G_n - independent coefficients. The explicit form of the formula for g is given in Grigolyuk and Kulikov (1988); Mikhasev and Altenbach (2019); Morozov et al (2020). Calculations have shown that the GK algorithm can be used only for plates with a small ratio η of Young's moduli of layers which is also discussed in Grigolyuk and Kulikov (1988). With an increase in η, the error of Eq. (3.10) for $\Delta(\eta)$ grows rapidly. For example, for the plate considered in Table 3.1, the error $\Delta(1) = 1.2\%$, $\Delta(10) = 42\%$ where at $\eta = 100$, the value of g given by Eq. (3.10) is 10 times greater than the exact value. Apparently, the hypotheses underlying the GK model and violating the continuity of shear stresses at the layer boundary need to be corrected.

3.8 Numerical Results. Three-layer Plate Symmetrical in Thickness

The shear parameter g and the associated vibration frequency ω depend on many parameters. A number of special cases are considered below.

Consider a plate with the parameters $h_1 = h_3 = 0.3$, $h_2 = 0.4$, $E_1 = E_3 = 1$, $E_2 = 1/\eta$, $\nu_1 = \nu_2 = \nu_3 = 0.3$, $\rho_1 = \rho_3 = 1$, $\rho_2 = 1/\eta$. There are two free parameters left: the thickness parameter μ and the Young's modulus ratio η. As follows from estimate 3.7, the allowance for the transverse shear is associated with the value $\mu^2\eta$; therefore, we introduce the combined parameter $p = \mu^2\eta$ and carry out the calculations at a fixed value of the parameter $\mu = 0.1$. Table 3.2 shows for a number of p

values: approximate values of the shear parameter $g^a = \mu^2 A_g$ found by asymptotic formula (3.4), and the exact values of g^e found by Eq. (3.7); the exact values of λ^e of the frequency parameter λ obtained by solving the three-dimensional problem (3.2). The remaining values of the parameter λ are approximate. They are obtained by formula (3), and the values of λ^{ap}, λ^{TR}, λ^{KL} are calculated for $g = g^e$, for $g = g^a = \mu^2 A_g$ and for $g = 0$, respectively. The λ^{TR} value corresponds to the TR model with allowance for the shear according to the approximate model (3.4). The λ^{KL} value corresponds to the KL model, which does not take into account the transverse shear.

Comparison of columns 3–4 and 5–8 allows us to judge the areas of applicability of the approximate models. The KL model is applicable only at $\eta \le 10$ (or at $p \le 0.1$). The asymptotic approach of the second order of accuracy which leads to the values of g^a and λ^{TR} is certainly applicable for $\eta \le 100$ and gives a noticeable error for $100 < \eta \le 1000$. In this case, parameter g^a exceeds the exact value g^e. Using the g^e value gives fairly accurate results over the entire considered range of $\eta \le 10000$, as evidenced by the comparison of columns 5 and 6 (when calculating λ^{ap} only the shear is accurately taken into account while the other second-order effects are ignored).

Calculations were also carried out at $\mu = 0.316$ and at $\mu = 0.0316$ however, the numerical results are not presented, because they differ from those given in Table 3.2 by less than 1% (except for the parameter η which is 10 times less or more, respectively).

3.9 Three-layer Plate Asymmetric in Thickness

Consider a three-layer plate with a constant thickness of the soft layer $h_2 = 0.4$ and with variable thicknesses of the hard layers $0 < h_1 \le 0.3$, $h_3 = 0.6 - h_1$. The rest of the parameters are the same as in Sect. 3.4. When $h_1 = 0.3$ the plate is symmetrical in thickness, and the difference $0.3 - h_1$ serves as a measure of asymmetry. Let us discuss function $\lambda^e(\eta, h_1)$. From the results of Table 3.3, it follows that at $1 < \eta \le 100$ the frequency decreases with an increase in the asymmetry of the plate (with a

Table 3.2 Shear and frequency parameters depending on p at $\mu = 0.1$.

1	2	3	4	5	6	7	8
p	η	g^a	g^e	λ^e	λ^{ap}	λ^{TR}	λ^{KL}
0.01	1	0.00286	0.00286	0.0913	0.0913	0.0913	0.0916
0.1	10	0.0174	0.0174	0.1321	0.1325	0.1325	0.1348
1	100	0.163	0.161	0.1222	0.1223	0.1224	0.1420
10	1000	1.62	1.47	0.0578	0.0578	0.0545	0.1432
100	10000	16.2	8.1	0.0157	0.0157	0.0083	0.1432

Table 3.3 The frequency parameter λ^e versus parameters η and h_1 at $\mu = 0.1$.

h_1	$\eta = 1$	10	100	1000	10000
0.3	0.0913	0.1321	0.1222	0.0578	0.0157
0.2	0.0913	0.1231	0.1141	0.0583	0.0183
0.1	0.0913	0.0953	0.0876	0.0575	0.0263
0.05	0.0913	0.0740	0.0652	0.0524	0.0320

decrease in the thickness h_1). At higher η, with increasing asymmetry, the frequency increases and reaches a maximum at a certain value of h_1 and then decreases. For a fixed value of h_1 and with increasing η, the frequency first increases and, upon reaching the maximum, decreases. At $h_1 = 0.05$, the maximum is reached at $\eta = 1.15$ and is not shown in Table 3.3.

3.10 Multilayer Plate

Consider a multilayer plate with an odd number $2n+1$ of alternating hard and soft layers of the same thickness h_1 and h_2 with parameters $\mu = 0.1$, $\eta = 100$. Let ξ denote the fraction of the volume occupied by soft layers. Table 3.4 shows the values of the frequency parameter λ^e depending on the parameters n and ξ. It follows from the results presented in the table that the frequency decreases with an increase in the number of layers, approaching the limit corresponding to a transversely isotropic homogeneous plate. The last row of Table 3.4 was constructed according to Eq. (3.8) with $q = (1-\xi+\eta\xi)(\eta(1-\xi)+\xi)$, whence it follows that function $g^e(\xi)$ is even with respect to the point $\xi = 0.5$. The latter circumstance is associated with the peculiarity of specifying the density of the soft layers $\rho_2 = 1/\eta$.

With a small number of layers, the frequency increases with the fraction ξ of the soft material, whereas this regularity is violated with a large number of layers.

Table 3.4 Parameter λ^e depending on n and ξ at $\mu = 0.1,\ \eta = 100$.

n	$\xi = 0.1$	0.3	0.5	0.7	0.9
3	0.0962	0.1115	0.1354	0.1712	0.2190
5	0.0946	0.1024	0.1153	0.1338	0.1567
11	0.0915	0.0941	0.0991	0.1070	0.1169
21	0.0904	0.0904	0.0926	0.0970	0.1032
101	0.0893	0.0871	0.0868	0.0884	0.0919
∞	0.0891	0.0863	0.0854	0.0863	0.0891

3.11 Buckling of a Multilayer Plate Under Uniform Compression

Consider a multilayer simply supported rectangular plate with sides $L_x,\ L_y$ which is uniformly compressed by tangential strain e. The following initial conditions accepted in the plane of the plate

$$T = T_1 = T_2 = eE_{1*}, \qquad E_{1*} = \int_0^1 E_1(z)dz, \qquad E_1 = \frac{E}{1-\nu}, \tag{3.13}$$

which upon the buckling $w(x,y) = w_0 \sin px \sin qy$ generate a load $f(x,y) = T\Delta w$ where Δ denotes the Laplace operator. Equation (3.2) in which D and g are calculated by Eqs. (3.4) and (3.5) after separation of variables $f_0 = eE_* r^2 w$ yields the critical deformation

$$e = \frac{D_* \mu^2}{(1+g)E_{1*}}, \qquad \mu = rh, \qquad r^2 = \frac{\pi^2}{L_x^2} + \frac{\pi^2}{L_y^2}. \tag{3.14}$$

To estimate the error in Eq. (3.14), let us turn to the exact system (3.3). The last two equations of take the form

$$\frac{d\sigma}{dz} = E_0(z)u + \mu^2 c_\nu(z)\sigma_{33}, \qquad \frac{d\sigma_{33}}{dz} = -\sigma - e\mu^2 E_1(z)w. \tag{3.15}$$

Let us consider the compression buckling of a plate asymmetric in thickness with the parameters $E_1 = E_3 = 1,\ E_2 = 1/\eta,\ h_1 = 0.1,\ h_2 = 0.6,\ h_3 = 0.3,\ \nu_1 = \nu_2 = \nu_3 = 0.3$. As in Table 3.2, parameter η will vary within wide limits $0 \le \eta \le 10000$. The calculations were carried out for the relative thickness $\mu = 0.1$. As in the vibration problem, the result depends on the combined parameter $p = \mu^2\eta$, so it can be used for other values of μ.

Table 3.5 shows the exact values of e_0^e found when integrating system (3.3) taking into account replacement (3.15); the values of e_0^{TR} found by Eq. (3.14) at $g = g^a$, along with the values of e_0^{KL} corresponding to the KL model and found by the same Eq. (3.14) at $g = 0$ (without taking into account the effect of transverse shear). As in the case of vibration, the KL model gives acceptable results only at $p \le 0.1$ whereas the TR model using the second-order accuracy parameter $g = g^a$ gives good results

Table 3.5 Critical deformation $e = 10^{-3}e_0$ versus p for $\mu = 0.1$.

p	η	g^a	e_0^e	e_0^{TR}	e_0^{KL}
0.01	1	0.00286	0.639	0.639	0.641
0.1	10	0.0141	0.914	0.914	0.926
1	100	0.124	0.858	0.857	0.960
10	1000	1.23	0.448	0.434	0.967
100	10000	12.3	0.111	0.073	0.967

at $p \le 1$ and acceptable results at $p \le 10$. For $p > 10$, the exact value $g = g^e$ should be used, the calculation of which is reduced to solving a simpler boundary value problem (3.8), otherwise it is necessary to solve the complete problem (3.3).

Note that the parameters g^a and g^e depend on the parameters of multilayer plate, but they are the same for the problems of vibration, statics, and buckling since boundary value problem (3.8) does not change when calculating g^e.

3.12 Discussion

The frequency of bending vibrations of a multilayer plate was found to be calculated by Eq. (3.4) corresponding to the TR model, in which the denominator $1+g$ takes into account the effect of transverse shear. A combined parameter $p = \mu^2\eta$ is introduced, which determines the range of applicability of various approaches in calculating g (μ is a small parameter of thickness and η is the ratio of Young's moduli of layers). When $p \le 1$ for a homogeneous plate $g = \frac{E_0\mu^2}{10G_{13}}$, and for a multilayer plate $g = g^a = \mu^2 A_g$, see (3.5). If $p > 1$ these formulas become inaccurate. For a homogeneous plate, g is calculated using the explicit formula (3.10). This gives an estimate of the error of the TR model for $g = \frac{E_0\mu^2}{10G_{13}}$. For a multilayer plate, the value $g = g^e$ is calculated by Eq. (3.9). The use of this value of g gives fairly accurate results in the entire considered range of parameters $0.001 \le \mu \le 0.3,\ 1 \le \eta \le 10000$ which is confirmed by comparison with the exact solution of the three-dimensional problem (3.3). A number of particular problems have been solved. For a three-layer plate, the influence of the location of the soft layer on the vibration frequencies is investigated. A multilayer plate with a constant fraction ξ of the volume occupied by soft layers is considered, and the influence of parameter ξ and the number of layers on the vibration frequency is investigated.

The results obtained for the factor $1+g$ which takes into account the effect of transverse shear are also applicable without changes for the static problem of deflection of a multilayer plate under the action of a static harmonic load of the form $f = f_0 \sin px \sin qy$. These results are also used to solve the buckling problem for a multilayer plate under uniform compression in its plane. Equation (3.14) for critical deformation is a generalization of Eq. (3.4). In this case, the range of applicability of the approximate KL and TR models turns out to be the same as in the vibrational case.

For multilayer transversely isotropic plates, the presented results can be considered final. In Tovstik (2019), an asymptotic approximation of the second order of accuracy was constructed for a plate inhomogeneous in thickness with anisotropy of general form (with 21 elastic moduli), which leads to a rather cumbersome model that requires simplifications and a corresponding analysis of the error. In particular, a multilayer plate with orthotropic layers generally does not have a neutral layer. That is, the longitudinal and bending deformations are not separated and the calculation becomes more complicated. Only partial results have been obtained in Belyaev et al (2019)[1] and the problem remains to be tackled.

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[1] Petr E. Tovstik wrote several papers together with the other authors - this is the last one of this series.