Chapter 15
Asymptotic Analysis of Buckling of Layered Rectangular Plates Accounting for Boundary Conditions and Edge Effects Induced by Shears

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Abstract Based on the equivalent single layer theory for laminated shells, buckling of layered rectangular plate under uniaxial compression with different variant of boundary conditions is studied. Equations in terms of the displacement, shear and stress functions, which take into account transverse shears inside the plate and near the edges with and without diaphragms, are used as the governing ones. Using the asymptotic approach, the buckling modes are constructed in the form of a superposition of the outer solution and the edge effect integrals induced by shears in the vicinity of the edges with or without diaphragms. Closed form relations for the critical buckling force accounting for shears are obtained for different variants of boundary conditions. It is detected that within one group of boundary conditions, the critical buckling forces can differ significantly depending on whether the edge is supplied with the diaphragm or not.

Key words: layered rectangular plate, shears, uniaxial compression, buckling, asymptotic approach, edge diaphragm, edge effects

15.1 Introduction

Buckling of thin plates is one of the extensive problems in the theory of thin-walled structures subjected to loading, which includes problems on buckling of single layer isotropic plates with various boundary conditions and under different schemes of loading, of composite and laminated plates based on different kinematic hypotheses, and others. Apparently, the first study on the stability of thin single layer isotropic rectangular plates in the framework of the classical Kirchhoff theory, was carried out by Bryan (1890). Applying the energy method, he obtained a simple formula for

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the critical force resulting in buckling of an uniaxially loaded rectangular plate with simply supported edges. Soon, following the Euler approach, Timoshenko (1907, 1910) and Reissner (1909) independently considered similar problems for a plate in which two edges loaded by a compressive force are simply supported, and the other ones have arbitrary boundary conditions. Later, Bubnov (1912) solved the problem on buckling of a rectangular plate, in which a pair of opposite sides is loaded by forces linearly varying along these sides. There was also considered a rectangular simply supported plate under the action of shear stresses uniformly distributed along the contour of the plate.

The above closed form classical solutions together with many others (e.g., see Timoshenko, 1936; Donell, 1976; Alfutov, 2000), obtained for isotropic plates obeying Kirchhoff's hypotheses, became a benchmark for subsequent investigations on buckling of layered plates. Solutions of problems on buckling of laminated plates with different boundary conditions and under various scheme of loading, using the equivalent single layer (ESL) theories based on the Kirchhoff assumptions, can be found in Ashton and Witney (1970); Reddy (2004). Obviously, similar solutions, ignoring shears and the transverse normal stresses, are not sufficiently accurate for laminated plates (Khdeir, 1989a,b). Therefore, the next contribution to the theory of buckling of layered plates was the use of the first-order shear deformation theory (FSDT) first proposed by Reissner (1945, 1952); Mindlin (1951) and then improved by other researches, see, e.g., the review papers Altenbach (1998); Qatu et al (2010). The series of studies based on this approach (see, among many others, Khdeir et al, 1987; Reddy and Khdeir, 1989; Nosier and Reddy, 1992) showed that taking transverse shears into account may give a large correction to the critical buckling force estimated within the classical shells theory for layered plates. The main drawback of the FSDT is that it does not allow satisfying the traction-free boundary conditions at the top and bottom surfaces of a laminated plate and so, it requires to introduce the shear correction factors (Mindlin, 1951). The next step in the development of more accurate approaches for modeling mechanical behavior of laminated plates is associated with the higher-order shear deformation theories (HSDT)s. They are based on quadratic, cubic and higher-order expansions at least of the in-plane displacements as functions of the transverse coordinate and comply with the traction-free boundary conditions on the face planes of a laminated plate, see relatively early papers Whitney and Sun (1973, 1974); Reddy (1984); Librescu et al (1987); Grigoliuk and Kulikov (1988), and also some more recent (Swaminathan and Ragounadin, 2004; Tovstik and Tovstik, 2007; Aydogdu, 2009; Amabili, 2015; Shi et al, 2018, to name a few). Using the developed HSDTs, many buckling problems of rectangular laminated plates were analyzed for various boundary conditions and schemes of loading, taking into account anisotropy of layers composing the plate. A lot of examples on buckling of laminated plates with symmetric and antisymmetric, cross- and angle-ply orientation of fibres can be found in Reddy (2004). An extensive literature on early studies of buckling of laminated plates can be also found there.

Due to the widespread use of composite plates and shells in engineering practice, the number of papers devoted to the buckling of laminated and functionally graded material plates, based on using HSDTs and high accurate layer-wise theories accounting the zig-zage effects, has increased dramatically, e.g., see the review article by Swaminathana et al (2015). These studies carried out, as a rule, using numerical methods, are characterized by high accuracy, but at the same time they are numerically highly-cost and do not allow obtaining closed form solutions and simple assessments for the critical buckling loads. In particular, a review of the available literature indicates the absence of any research of the transverse shear effects near the plate edges, induced by the edge diaphragms, on a value of the critical buckling load. At the same time, it should be noted that the asymptotic analysis of free vibration of a laminated cylindrical shell performed in Mikhasev and Botogova (2017); Mikhasev and Altenbach (2019c) displayed the strong dependence of the lowest eigenfrequencies on whether the shell edge is supplied with a diaphragm or not.

Motivated by the outcomes of Mikhasev and Botogova (2017); Mikhasev and Altenbach (2019c), we aim to investigate the influence of the transverse shears near edges on the value of the critical buckling force for rectangular plates which are pliable to shears. As the model we will use the ESL theory developed by Grigoliuk and Kulikov (1988) which is based on the generalized kinematical hypotheses of Timoshenko for the in-plane displacements and the parabolic distribution of transverse shear stresses through the plate thickness. This model complies with the traction-free boundary conditions on the top and bottom surfaces of a laminated plate and was verified by finite element simulation (Mikhasev and Altenbach, 2019a). In the framework of this theory, the full system of differential equations w.r.t. five unknowns is readily simplified and reduced to three equations for the displacement, stress and shear functions. These equations have more higher order than similar equations like Timoshenko-Reissner and, in a particular case, completely coincide with equations derived by Tovstik and Tovstik (2017a,b) from the 3D theory of elasticity. The higher order of this equations allows differing boundary conditions belonging to the same group (e.g., the clamped support group) depending on whether an edge has a diaphragm or not.

The asymptotic solutions of equations governing buckling of a rectangular laminated plate with various boundary conditions are constructed in the form of a superposition of the outer solution and the edge effect integrals induced by shears in the vicinity of an edge with or without a diaphragm. The corrections to classical relations for the critical buckling forces are derived.

15.2 Governing Equations

Consider a rectangular laminated plate with the sides a, b consisting of N transversally isotropic elastic layers. Each layer is characterized by the thickens h_k, Young's modulus E_k, the shear modulus G_k and Poisson's ratio ν_k, where $k = 1, 2, \dots, N$. The plate is referred to an orthogonal Cartesian coordinate system $Ox_1x_2x_3$ with the original plane Ox_1x_2 coinciding with the middle surface of any layer.

Let the plate be loaded with edges forces which generally generate the stress resultants $T^{\circ}_{11}, T^{\circ}_{22}, T^{\circ}_{12}$ in the original plane Ox_1x_2, where $T^{\circ}_{11}, T^{\circ}_{22}$ are the membrane forces acting in the x_1- and x_2- directions, respectively, and T°_{22} is the membrane shear force. Then the governing equations describing buckling of the plate based on the ESL theory by Grigoliuk-Kulikov can be written as follows:

$$D\left(1-\frac{\theta h^2}{\beta}\varDelta\right)\varDelta^2\chi-\left(T^{\circ}_{11}\frac{\partial^2}{\partial x_1^2}+2T^{\circ}_{12}\frac{\partial^2}{\partial x_1\partial x_2}+T^{\circ}_{22}\frac{\partial^2}{\partial x_2^2}\right)w=0, \tag{15.1}$$

$$w=\left(1-\frac{h^2}{\beta}\varDelta\right)\chi,\quad \frac{1-\nu}{2}\frac{h^2}{\beta}\varDelta\phi=\phi. \tag{15.2}$$

Here $\varDelta$ is the Laplace operator, $h=\sum\limits_{k=1}^{N}h_k$ is the total plate thickness, w is the normal displacement, χ and ϕ are the displacement and shear functions, respectively, E, D and ν are the reduced Young's modulus, bending stiffness and Poisson's ratio, respectively, and θ,β are the reduced shear parameters determined by equations:

$$D=\frac{Eh^3}{12(1-\nu^2)}\eta_3,\quad E=\frac{1-\nu^2}{h}\sum_{k=1}^{N}\frac{E_kh_k}{1-\nu_k^2},\quad \nu=\sum_{k=1}^{N}\frac{E_kh_k\nu_k}{1-\nu_k^2}\left(\sum_{k=1}^{N}\frac{E_kh_k}{1-\nu_k^2}\right)^{-1},$$

$$\theta=1-\frac{\eta_2^2}{\eta_1\eta_3},\quad \beta=\frac{12(1-\nu^2)}{Eh\eta_1}\left\{\frac{\left[\sum\limits_{k=1}^{N}\left(r_k-\frac{r_{k0}^2}{r_{kk}}\right)\right]^2}{\sum\limits_{k=1}^{N}\left(r_k-\frac{r_{k0}^2}{r_{kk}}\right)G_k^{-1}}+\sum_{k=1}^{N}\frac{r_{k0}^2}{r_{kk}}G_k\right\}. \tag{15.3}$$

Parameters η_i, r_k, r_{kn}, where $i=1,2,3; n=0,k$, appearing in (15.3) are introduced by the following relations:

$$r_k=\int_{z_{k-1}}^{z_k}f_0^2(z)\mathrm{d}z,\quad r_{kn}=\int_{z_{k-1}}^{z_k}f_k(z)f_n(z)\mathrm{d}z,\quad \eta_1=\sum_{k=1}^{N}\xi_k^{-1}\pi_{1k}\gamma_k-3c_{12}^2,$$
$$\eta_2=\sum_{k=1}^{N}\xi_k^{-1}\pi_{2k}\gamma_k-3c_{12}c_{13},\quad \eta_3=4\sum_{k=1}^{N}\left(\xi_k^2+3\zeta_{k-1}\zeta_k\right)\gamma_k-3c_{13}^2, \tag{15.4}$$

where

$$\gamma_k=\frac{E_kh_k}{1-\nu_k^2}\left(\sum_{k=1}^{N}\frac{E_kh_k}{1-\nu_k^2}\right)^{-1} \tag{15.5}$$

is the in-plane reduced stiffness of the k-th lamina, and

$$
\begin{aligned}
&c_{12} = \sum_{k=1}^{N} \xi_k^{-1} \pi_{3k} \gamma_k, && c_{13} = \sum_{k=1}^{N} (\zeta_{k-1} + \zeta_k) \gamma_k, \\
&\frac{1}{12} h^3 \pi_{1k} = \int\limits_{z_{k-1}}^{z_k} g^2(x_3) \mathrm{d}x_3, && \frac{1}{12} h^3 \pi_{2k} = \int\limits_{z_{k-1}}^{z_k} x_3 \, g(x_3) \mathrm{d}x_3, \\
&\frac{1}{2} h^2 \pi_{3k} = \int\limits_{z_{k-1}}^{z_k} g(x_3) \mathrm{d}x_3, && \eta_1 = \sum_{k=1}^{N} \xi_k^{-1} \pi_{1k} \gamma_k - 3c_{12}^2, \\
&\eta_2 = \sum_{k=1}^{N} \xi_k^{-1} \pi_{2k} \gamma_k - 3c_{12} c_{13}, && \eta_3 = 4 \sum_{k=1}^{N} \left(\xi_k^2 + 3\zeta_{k-1} \zeta_k \right) \gamma_k - 3c_{13}^2, \\
&h\xi_k = h_k, \quad h\zeta_n = z_n \, (n = 0, k).
\end{aligned}
\tag{15.6}
$$

Functions f_0, f_k, g are taken in the polynomial form:

$$
\begin{aligned}
&f_0(x_3) = \frac{1}{h^2}(x_3 - z_0)(z_N - x_3) \quad \text{for} \quad x_3 \in [z_0, z_N], \\
&f_k(x_3) = \frac{1}{h_k^2}(x_3 - z_{k-1})(z_k - x_3) \quad \text{for} \quad x_3 \in [z_{k-1}, z_k], \\
&f_k(x_3) = 0 \quad \text{for} \quad x_3 \notin [z_{k-1}, z_k], \quad g(x_3) = \int\limits_0^{x_3} f_0(z) \mathrm{d}z.
\end{aligned}
\tag{15.7}
$$

In Eqs. (15.4), (15.6), (15.7), $x_3 = z_k$ is the coordinate of the upper bound of the k^{th} layer, and $x_3 = z_0$ is the coordinate of the bottom face.

The dimensionless parameter θ depends on a number of layers and thickness of each lamina. For instance, for a single layer isotropic plate, $\theta = 1/85$. The estimates of this parameter for layered plates and panels depending on a number of layers and their mechanical properties can be found in Mikhasev et al (2019); Mikhasev and Altenbach (2019d). If $\theta = 0$, then Eq. (15.1) together with the first equation from (15.2) degenerates into the fourth-order differential one which coincides with the equation like Timoshenko-Reissner obtained by Tovstik and Tovstik (2017a,b); Morozov et al (2016a,b) for plates inhomogeneous in the thickness direction. However, the shear parameter β is calculated in other way. We note that the parameter $G = E\eta_1\beta/[12(1-\nu^2)]$ can be here treated as the effective (or reduced) shear modulus for laminated plate (Mikhasev and Altenbach, 2019b; Mikhasev and Tovstik, 2020).

We consider two groups of boundary conditions, the simple support group, and the clamped support group, which will be denoted by the letters S and C, respectively. Each of these groups consists of two variants boundary conditions which differ in the presence or absence of a diaphragm that prevents shears in the edge plane. To distinguish these condition in the framework of one fixed group, we will use the signs $^+$ and $^-$ for the edges with and without a diaphragm, respectively:

- S^+ – conditions,

$$\chi = \Delta\chi = \Delta^2\chi = \frac{\partial \phi}{\partial x_i} = 0; \tag{15.8}$$

- S^- – conditions,

$$\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = \frac{\partial^2}{\partial x_i^2}\left(1 - \frac{h^2}{\beta}\Delta\right) = 0,$$

$$\left(\frac{\partial^2}{\partial x_i^2} + \nu\frac{\partial^2}{\partial x_j^2}\right)\chi - (1-\nu)\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0, \tag{15.9}$$

$$2\frac{\partial^2 \chi}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial x_j^2} = 0;$$

- C^- – conditions,

$$\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = \frac{\partial \chi}{\partial x_i} = \frac{\partial}{\partial x_i}(\Delta\chi) = \phi = 0; \tag{15.10}$$

- C^+ – conditions,

$$\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = \frac{\partial}{\partial x_i}\left(1 - \frac{h^2}{\beta}\Delta\right)\chi = 0,$$
$$\frac{\partial \chi}{\partial x_i} - \frac{\partial \phi}{\partial x_j} = \frac{\partial \chi}{\partial x_j} + \frac{\partial \phi}{\partial x_i} = 0 \tag{15.11}$$

for $x_i = 0,\ x_i^*$, where $i, j = 1, 2;\ i \neq j$, and $x_1^* = a,\ x_2^* = b$.

There are 9 essentially different combinations of boundary conditions. In what follows, we consider only the following variants:

$$S^{\pm}S^{\pm}S^{+}S^{+}, \quad S^{\pm}C^{\pm}S^{+}S^{+}, \quad C^{\pm}C^{\pm}S^{+}S^{+},$$

where the first pair of letters denotes the boundary conditions at the edges $x_1 = 0$, $x_1 = a$, while the second one corresponds to conditions for $x_2 = 0$, $x_2 = b$. For instance, $S^+S^+S^+S^+$ stands for the plate with simply supported edges supplied with the diaphragms, while the combination $C^-C^-S^+S^+$ denotes a clamped support of the edges $x_1 = 0$, $x_1 = a$ without diaphragms.

As a rule, the plate loading is assumed to be one-parametric, so that

$$T_{ij}^{\circ} = -\lambda \frac{D}{a^2} t_{ij}^{\circ}, \tag{15.12}$$

where t_{ij}° is the dimensionless counterpart of T_{ij}°, and λ is the dimensionless load parameter. It is important that at least one of the parameters t_{ij}° be positive, that corresponds to the plate compression. The problem is to find the minimum positive value of a parameter λ for which the governing Eqs. (15.1), (15.2) with some specified variant of boundary conditions (15.8)-(15.11) have a non-trivial solution. Here, the case $t^\circ_{12} \neq 0$ is not considered.

15.3 Simply Supported Plate with the Edge Diaphragms

Consider the simplest case denoted as $S^+S^+S^+S^+$ when all edges are simply supported and supplied with the diaphragms. The corresponding boundary conditions are given by Eqs. (15.8). This case is probably the only one that allows us to construct a solution in the explicit simple form and analyse the effect of shears on the critical load. Without loss of generality, we assume that $t^\circ_{22} = 1$, and $t^\circ_{11} = t_1$ is any constant.

In this case the unique solution of the last equation from (15.2) satisfying the boundary conditions (15.8) is the trivial function, $\phi = 0$, while the displacement function can be represented as

$$\chi = c_0 \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b}, \tag{15.13}$$

where n, m are natural numbers, and c_0 is a nonzero constant. The substitution of (15.13) into Eqs. (15.1), (15.2) results in the simple formula for the eigenvalue

$$\lambda = \frac{\pi^2}{l^2}\frac{(n^2 + l^2m^2)^2[1 + \theta K(n^2 + l^2m^2)]}{(n^2 + t_1 l^2m^2)[1 + K(n^2 + l^2m^2)]}, \tag{15.14}$$

where $l = b/a$, and

$$K = \frac{\pi^2 h^2}{\beta b^2} \tag{15.15}$$

is the dimensionless shear parameter. The required critical value

$$\lambda^* = \min_{n,m} \lambda(n,m) = \lambda(n^*, m^*) \tag{15.16}$$

is the function of θ and K, where the dimensionless shear parameter K depends on the reduced shear modulus G, see Eq. (15.3). Because θ is a small value, the shear parameter K is the main one affecting the critical buckling force.

If all edges are uniformly loaded ($t_1 = 1$), then

$$n^* = m^* = 1, \quad \lambda^* = \lambda^*_{cl}\frac{1 + \theta K(1 + l^2)}{1 + K(1 + l^2)}, \tag{15.17}$$

where the eigenvalue

$$\lambda^*_{cl} = \frac{\pi^2(1 + l^2)}{l^2} \tag{15.18}$$

corresponds to the classical value of the buckling force for a single layer isotropic plate (Alfutov, 2000).

Equation (15.17) shows that an increase in the shear parameter K (i.e., a decrease in the effective shear modulus G) leads to a decrease in the critical buckling force for a multilayer plate pliable to transverse shears. Because $\phi = 0$, then the edge effect in the case under consideration is absent. In other words, the presence of the diaphragms at simply supported edges does not generates shears localized near these edges.

15.4 Buckling Modes Accounting for the Edge Effects

In this section we consider the plates with boundary conditions belonging to the group $Y^{\pm}Z^{\pm}S^{+}S^{+}$, where Y and Z denote either S or C conditions. In what follows, we assume that $t^{\circ}_{22} = 1$ and $t^{\circ}_{11} = 0$, i.e., the plate is compressed only in the x_2 – direction. For these combinations of boundary conditions the general solution of Eqs. (15.1), (15.2) can be represented in the form:

$$\chi = X(x)\sin\delta_n y, \quad \phi = \Phi(x)\cos\delta_n y, \tag{15.19}$$

where

$$x = \frac{x_1}{a}, \quad y = \frac{x_2}{a}, \quad \delta = \frac{\pi n}{l}. \tag{15.20}$$

Then Eqs. (15.1), (15.2) can be rewritten as follows:

$$(1 - \theta\kappa\Delta_1)\Delta_1^2 X - \lambda\delta^2(1 - \kappa\Delta_1)X = 0, \tag{15.21}$$

$$\frac{1-\nu}{2}\kappa\Delta_1\Phi - \Phi = 0, \tag{15.22}$$

where $\Delta_1 = \frac{d^2}{dx^2} - \delta^2$ is the differential operator, and $\kappa = \frac{h^2}{\beta a^2}$ is the dimensionless shear parameter.

Accounting for (15.19), the boundary conditions for an unloaded edge ($x = 0$ or $x = 1$) read:

- S^{+} – conditions,

$$X = 0, \quad \left(\frac{d^2}{dx^2} - \delta^2\right)X = 0, \quad \left(\frac{d^2}{dx^2} - \delta^2\right)^2 X = 0, \quad \frac{d\Phi}{dx} = 0. \tag{15.23}$$

- S^{-} – conditions,

$$\begin{gathered}(1 - \kappa\Delta_1)X = 0, \quad \frac{d^2}{dx^2}(1 - \kappa\Delta_1)X = 0,\\ \left(\frac{d^2}{dx^2} - \nu\delta^2\right)X + (1-\nu)\delta\frac{d\Phi}{dx} = 0,\\ 2\delta\frac{dX}{dx} + \frac{d^2\Phi}{dx^2} + \delta^2\Phi = 0;\end{gathered} \tag{15.24}$$

- C^- – conditions,

$$(1-\kappa\Delta_1)X = 0, \quad \frac{dX}{dx} = 0,$$
$$\Delta_1 \frac{dX}{dx} = 0, \quad \Phi = 0; \tag{15.25}$$

- C^+ – conditions,

$$(1-\kappa\Delta_1)X = 0, \quad \frac{d}{dx}(1-\kappa\Delta_1)X = 0,$$
$$\frac{dX}{dx} + \delta\Phi = 0, \quad \delta X + \frac{d\Phi}{dx} = 0. \tag{15.26}$$

Depending on the orders of the shear parameters κ and θ, there are the following two distinctive cases:

- Case (A) $\kappa = \varepsilon^2$ is a small parameter, and $\theta = O(1)$ as $\varepsilon \to 0$;
- Case (B) θ is a small parameter, and $\kappa = O(1)$ as $\theta \to 0$.

Case (A) corresponds to very thin plates with the reduced Young's and shear moduli of the same order ($E \sim G$), and case (B) is related to thin plates for which $G \ll E$.

15.4.1 Layered Plates with the Reduced Young's and Shear Moduli of the same Order

Consider case (A). Regardless of the type of boundary conditions, the general solution of Eq. (15.22) is the function

$$\Phi = \varepsilon^{\iota_1} c_1 \mathrm{e}^{-\frac{1}{\varepsilon}\sqrt{\frac{2+\varepsilon^2\delta^2(1-\nu)}{1-\nu}}x} + \varepsilon^{\iota_2} c_2 \mathrm{e}^{\frac{1}{\varepsilon}\sqrt{\frac{2+\varepsilon^2\delta^2(1-\nu)}{1-\nu}}(x-1)}, \tag{15.27}$$

where ι_i are the indices of intensity of the function Φ, and c_1, c_2 are constants of the order $O(1)$ to be determined from appropriate boundary conditions.

Consider Eq. (15.21). Its solution can be constructed in the form of the superposition of a solution, $X^{(m)}$, valid in the plate interior (the so-called "outer solution"), with a pair of boundary layers, $X_1^{(e)}$ and $X_2^{(e)}$, fading off away from the left and from the right plate ends, respectively:

$$X = X^{(m)}(x,\varepsilon) + \varepsilon^{\alpha_1} X_1^{(e)}(x,\varepsilon) + \varepsilon^{\alpha_2} X_2^{(e)}(x,\varepsilon), \tag{15.28}$$

where α_1, α_2 are indices of intensity of the edge effect integrals. We assume also that the following order relations hold:

$$\frac{\partial X^{(m)}}{\partial x} \sim X^{(m)}, \quad \frac{\partial X_i^{(e)}}{\partial x} \sim \varepsilon^{-\varsigma_i} X_i^{(m)} \quad \text{as} \quad \varepsilon \to 0. \tag{15.29}$$

The positive parameters ς_i are named the indices of variation of the edge effect integrals. The indices α_1, α_2 depends on the boundary conditions and should be specified for each edge.

To derive an edge effect equation describing behaviour of the solution in the neighbourhood of the left and right ends, we scale in the vicinity of both edges. For instance, for the left edge we assume $x = \varepsilon^{\varsigma_1}\zeta$ and compare the main term in Eq. (15.21) containing the six-order derivative with others. As a result, we obtain $\varsigma_i = 1$ for both ends, and the edge effect equation reads

$$\theta\frac{d^6X_i^{(e)}}{d\zeta^6} - \left(1+3\varepsilon^2\theta\delta^2\right)\frac{d^4X_i^{(e)}}{d\zeta^4} + \varepsilon^2\delta^2\left(2+3\varepsilon^2\theta\delta^2-\varepsilon^2\lambda\right)\frac{d^2X_i^{(e)}}{d\zeta^2} \\ -\varepsilon^4\delta^2\left[\delta^2+\varepsilon^2\theta\delta^4-\lambda\left(1+\varepsilon^2\delta^2\right)\right]X_i^{(e)} = 0. \tag{15.30}$$

Its solution is sought in the form of asymptotic series

$$X_i^{(e)} = \sum_{j=1}^{\infty}\varepsilon^j\chi_{ij}^{(e)}(\zeta).$$

Here we give only the leading terms of these expansions, returning to the original argument x:

$$X_1^{(e)} = a_1\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}} + O\left(\varepsilon\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}}\right), \quad X_2^{(e)} = a_2\mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}} + O\left(\varepsilon\mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}}\right), \tag{15.31}$$

where a_i are constants to be determined from the boundary conditions.

The outer solution $X^{(m)}$ as well as the eigenvalue λ are also sought in the form of series

$$X^{(m)} = \chi_0(x) + \varepsilon\chi_1(x) + \ldots, \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots \tag{15.32}$$

Substituting (15.32) into Eq. (15.21) and grouping coefficients of the same powers of ε leads to the sequence of differential equations:

$$\sum_{j=0}^{k}\mathbf{L}_j\chi_{k-j} = 0, \tag{15.33}$$

where

$$\mathbf{L}_0 = \frac{d^4}{dx^4} - 2\delta^2\frac{d^2}{dx^2} + \delta^2\left(\delta^2-\lambda_0\right), \quad \mathbf{L}_1 = -\lambda_1\delta^2, \ldots \tag{15.34}$$

To specify the boundary conditions for the functions $\chi_k(x)$, we substitute expansions (15.28), (15.31), (15.32) into appropriate conditions from (15.23)-(15.26), equate coefficients at the same powers of ε and impose the following requirements:

- in the leading approximation ($k = 0$), the boundary conditions for $\chi_0(x)$ should be homogeneous;
- the leading approximation generates equations coupling constants c_i with the function $\chi_0'(x)$ evaluated at the boundaries;

- the kth-order ($k \geq 1$) approximation results in the inhomogeneous boundary conditions for $\chi_k(x)$ and relations for a_i as well.

We note that indices α_i, ι_i depend on the type of boundary conditions and can be different on the left and right edges.

15.4.1.1 Plate with S⁻S⁻S⁺S⁺ - Boundary Conditions

Let the unloaded edges be simply supported and not supplied with diaphragms. The corresponding boundary conditions are given by relations (15.24). Here, we obtain $\alpha_i = 3$, $\iota_i = 2$ for $i = 1,2$.

In the leading approximation, one obtains the homogeneous boundary conditions,

$$\chi_0(0) = \chi_0(1) = 0, \quad \chi_0''(0) = \chi_0''(1) = 0, \tag{15.35}$$

and the pair of relations:

$$2\delta\chi_0'(0) + \frac{2}{1-\nu}c_1 = 0, \quad 2\delta\chi_0'(1) + \frac{2}{1-\nu}c_2 = 0. \tag{15.36}$$

The first-order approximation leads to the inhomogeneous boundary conditions,

$$\chi_1(0) = \chi_1(1) = 0, \tag{15.37}$$

$$\chi_1''(0) - \frac{\theta - 1}{\theta^2}a_1 = 0, \quad \chi_1''(1) - \frac{\theta - 1}{\theta^2}a_2 = 0, \tag{15.38}$$

and generates two the equations coupling constants a_i, c_i with the function $\chi_1''(x)$ evaluated at $x = 0$ and $x = 1$:

$$\begin{aligned} \chi_1''(0) + \frac{a_1}{\theta} - \delta(1-\nu)\frac{\sqrt{2}}{\sqrt{1-\nu}}c_1 = 0, \\ \chi_1''(1) + \frac{a_2}{\theta} + \delta(1-\nu)\frac{\sqrt{2}}{\sqrt{1-\nu}}c_2 = 0. \end{aligned} \tag{15.39}$$

Interrupting the process of finding the boundary conditions for functions $\chi_k(x)$, we consider the boundary-value problems arising in the first two approximations.

In the leading approximation ($k = 0$), one has the homogeneous differential equation $\mathbf{L}_0\chi_0 = 0$ with the homogeneous boundary conditions (15.35). The solution of this classical boundary-value problem is the eigenfunction $\chi_0 = c_0 \sin \pi m x$ with the associated eigenvalue

$$\lambda_0 = \lambda_0(n,m) = \frac{\left[(\pi m)^2 + \delta^2\right]^2}{\delta^2}. \tag{15.40}$$

The critical buckling force is evaluated as

$$\lambda_0^* = \min_{n,m} = \lambda(n^*, m^*) = \lambda(n^*, 1) = \frac{\pi^2\left(l^2 + n^{*2}\right)^2}{l^2 n^{*2}}, \tag{15.41}$$

where $n^* = 1$, if $l = b/a \leq 1$, and $n^* = \mathrm{In}(l)$ for $l > 1$. Here $\mathrm{In}(z)$ stands for the integer part of a number z. Then $\chi_0^* = c_0 \sin \pi x$ is the associated eigenfunction. Below, the asterisk $*$ is omitted for all parameters and the eigenfunction.

Now we can calculate

$$c_1 = -\pi\delta(1-\nu)c_0, \quad c_2 = \pi\delta(1-\nu)c_0, \tag{15.42}$$

where c_0 is an arbitrary constant that remains undefined in the framework of the linear problem. Here and below, the parameter $\delta = \delta_n$ is calculated at $n = n^*$.

Subtracting Eqs. (15.39) from Eqs. (15.38) and accounting for (15.42), we obtain the relations for constants

$$a_1 = a_2 = \sqrt{2}\pi\delta^2(1-\nu)^{3/2}\theta^2 c_0. \tag{15.43}$$

Then the pair of boundary conditions (15.39) can be rewritten as

$$\chi_1''(0) = \chi_1''(1) = \kappa c_0, \tag{15.44}$$

where

$$\kappa = \sqrt{2}\pi\delta^2(1-\nu)^{3/2}(1-\theta)c_0. \tag{15.45}$$

Consider the inhomogeneous differential equation (15.33) in the first-order approximation:

$$\frac{d^4\chi_1}{dx^4} - 2\delta^2\frac{d^2\chi_1}{dx^2} + \delta^2\left(\delta^2 - \lambda_0\right)\chi_1 = \lambda_1\delta^2\chi_0. \tag{15.46}$$

We note that the operator $\mathbf{L}_0$ is self-conjugated. Therefore, regardless of the type of boundary conditions imposed on the function $\chi_1(x)$, the condition for the existence of a solution to Eq. (15.46) will be as follows:

$$\begin{aligned}
&\chi_1'''(1)\chi_0(1) - \chi_1'''(0)\chi_0(0) - \chi_1''(1)\chi_0'(1) + \chi_1''(0)\chi_0'(0) \\
&+\chi_1'(1)\chi_0''(1) - \chi_1'(0)\chi_0''(0) - \chi_1(1)\chi_0'''(1) + \chi_1(0)\chi_0'''(0) \\
&-2\delta^2\left[\chi_1'(1)\chi_0(1) - \chi_1'(0)\chi_0(0) - \chi_1(1)\chi_0'(1) + \chi_1(0)\chi_0'(0)\right] \\
&-\lambda_1\delta^2\int_0^1 \chi_0^2(x)dx = 0.
\end{aligned} \tag{15.47}$$

Returning to the case of $S^-S^-S^+S^+$ – boundary conditions specified by relations (15.35), (15.37) and (15.44), we arrive at the parameter correcting the eigenvalue:

$$\lambda_1 = \frac{4\kappa\pi}{\delta^2}. \tag{15.48}$$

Then an approximate relations for the required eigenvalue and eigenmode can be written as follows:

$$\lambda = \frac{\left(\pi^2+\delta^2\right)^2}{\delta^2}\left[1+\varepsilon\Lambda_1+O\left(\varepsilon^2\right)\right],$$
$$\chi \approx c_0 \sin\delta_n y\left[\sin\pi x+\varepsilon\chi_1(x)+\varepsilon^3\sqrt{2}\pi\delta^2\theta^2(1-\nu)^{3/2}\left(\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}}+\mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}}\right)\right], \tag{15.49}$$

where

$$\Lambda_1 = \frac{4\sqrt{2}\pi^2(1-\nu)^{3/2}\delta^2(1-\theta)}{(\pi^2+\delta^2)^2} > 0, \tag{15.50}$$

and $\chi_1(x)$ is the partial solution of Eq. (15.46) with the boundary conditions (15.37), (15.44).

We note that although the correction of the edge effect integrals to the eigenmode is of the order $\varepsilon^3\left(\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}},\ \mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}}\right)$, the error of relation (15.49) for χ has the order $O\left(\varepsilon^2\right)$.

We compare eigenvalue (15.49) with the analogous value given by relations (15.14), (15.16), corresponding to the simply supported plate with diaphragms at all edges. Note that $m^* = 1$ for $t_1 = 0$ in (15.14), (15.16). Since Eqs. (15.49), (15.49) are asymptotic, we expand formula (15.14) also into the series in a small parameter ε keeping in mind that $K = \varepsilon^2\pi^2 l^{-2}$:

$$\lambda = \frac{\left(\pi^2+\delta^2\right)^2}{\delta^2}\left[1+\varepsilon^2\Lambda_2+O(\varepsilon^4)\right], \quad \Lambda_2 = (1-\theta)\left(\pi^2+\delta^2\right) > 0. \tag{15.51}$$

It can be seen that in the leading approximation the classical eigenvalues λ_0^* evaluated by Eqs. (15.49) and (15.51), which ignore the shear effects in a plate, are the same. The effect of shears on the buckling force turns to be different in plates with and without diaphragms. In the plate with simply supported edges with the diaphragms, the edge effects induced by shears are absent, and shears, taking place in the interior region of the plate, leads to a minor reduction of the buckling force with respect to the classical value λ_0^*, the normalized correction being a value of the order $O(\varepsilon^2)$.

Conversely, if there are no diaphragms at the simply supported edges, then near these edges shears occur, which lead to edge effects in the buckling form and increase the critical force with a normalized correction of the order $O(\varepsilon)$. It is interesting to note that similar reinforcing effect of the edge shears takes place in a cylindrical shell without diaphragms at the simply supported edges when the shell is under external pressure (Mikhasev and Botogova, 2017).

15.4.1.2 Plate with C±C±S+S+ - Boundary Conditions

Consider a plate with the clamped unloaded edges without diaphragms ($C^-C^-S^+S^+$ – conditions). The corresponding boundary conditions are given by relations (15.25). In this case $\alpha_1 = \alpha_2 = 3$, and $c_1 = c_2 = 0$ so that $\Phi = 0$.

In the leading approximation, the boundary conditions read

$$\chi_0(0) = \chi_0(1) = 0, \quad \chi_0'(0) = \chi_0'(1) = 0. \tag{15.52}$$

In addition, one obtains constants

$$a_1 = -a_2 = \theta^{3/2}\chi_0'''(0). \tag{15.53}$$

Consider the homogeneous differential equation $\mathbf{L}_0\chi_0 = 0$ with the boundary conditions (15.52). This boundary-value problem has a straightforward exact solution

$$\chi_0(x,\delta) = c_1 e^{-\alpha x} + c_2 e^{\alpha(x-1)} + c_3 \sin\gamma x + c_4 \cos\gamma x, \tag{15.54}$$

where c_j are constants determined from conditions (15.52), and

$$\alpha = \sqrt{\delta\lambda_0^{1/2} + \delta^2}, \quad \gamma = \sqrt{\delta\lambda_0^{1/2} - \delta^2}, \quad \delta^2 < \lambda_0. \tag{15.55}$$

Let $\lambda(\delta)$ be the minimum positive eigenvalue for a fixed δ. The required eigenvalue λ_0^* corresponding to the plate buckling is determined as follows:

$$\lambda_0^* = \min_n \lambda_0(\delta(n)) = \lambda_0(\delta(n^*)) = \lambda_0(\delta^*).$$

The procedure to determine $n^*, \delta^*, \lambda_0^*$ will be described below (for different variants of boundary conditions).

In the fist-order approximation, one has the inhomogeneous differential equation (15.46) with the homogeneous boundary conditions (15.52) for χ_1. This inhomogeneous boundary-value problem implies $\lambda_1 = 0$, and the eigenfunction χ_1 is given with accuracy up to a constant by Eq. (15.54).

The second-order approximation, taking into account (15.53), generates the inhomogeneous boundary-value problem:

$$\mathbf{L}_0\chi_2 = \mathbf{N}\chi_0 + \lambda_2\delta^2\chi_0, \tag{15.56}$$

$$\begin{gathered} \chi_2(0) = \chi_0''(0), \quad \chi_2(1) = \chi_0''(1), \\ \chi_2'(0) = \frac{a_1}{\sqrt{\theta}} = \theta\chi_0'''(0), \quad \chi_2'(1) = -\frac{a_2}{\sqrt{\theta}} = \theta\chi_0'''(1), \end{gathered} \tag{15.57}$$

where $\mathbf{N}$ is the differential operator introduced as follows:

$$\mathbf{N} = \theta\frac{\mathrm{d}^6}{\mathrm{d}x^6} - 3\theta\delta^2\frac{\mathrm{d}^4}{\mathrm{d}x^4} + \delta^2\left(3\theta\delta^2 - \lambda_0\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \delta^4\left(\lambda_0 - \theta\delta^2\right).$$

The condition for existence of a solution of this problem results in the correction for the eigenvalue λ_0^*:

$$\lambda_2 = \left\{(\theta-1)[\chi_0''(1)\chi_0'''(1) - \chi_0''(0)\chi_0'''(0)] - Z_N\right\} Z_1^{-1}, \tag{15.58}$$

where

$$Z_N = \int_0^1 \chi_0(x)\mathbf{N}\chi_0(x)dx, \quad Z_1 = \delta^2 \int_0^1 \chi_0^2(x)dx. \tag{15.59}$$

We note that the correction λ_2 is evaluated for $\lambda_0 = \lambda_0^*, \delta = \delta^*$.

Breaking the process of constructing the buckling mode, we write the approximate formulas for the critical load parameter λ^* and the corresponding bucking mode:

$$\begin{gathered}\lambda^* = \lambda_0^*\left[1 + \varepsilon^2 \Lambda_2 + O\left(\varepsilon^3\right)\right], \\ \chi \approx \sin\delta_n y\left\{\chi_0(x) + \varepsilon^2\chi_2(x) - \varepsilon^3\theta^{3/2}\left[\chi_0'''(0)\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}} + \chi_0'''(1)\mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}}\right]\right\},\end{gathered} \tag{15.60}$$

where $\Lambda_2 = \lambda_2/\lambda_0^*$.

Now, let the unloaded edges be clamped and supplied with diaphragms ($C^+C^+S^+S^+$ – conditions). This case is not much different from the previous one (for $C^-C^-S^+S^+$– conditions, see Eqs. (15.26)). Here, $\alpha_1 = \alpha_2 = 3$, and the boundary-value problems arising in the first three approximations are the same, so that all equations from (15.52) to (15.60) are valid. The only difference is that the function Φ defined by Eq. (15.27) is nonzero here. The asymptotic analysis of the boundary-value problems (15.33), (15.26) implies $\iota_1 = \iota_2 = 3$ and constants

$$c_1 = \sqrt{\frac{1-\nu}{2}}\delta\chi_0''(0), \quad c_2 = -\sqrt{\frac{1-\nu}{2}}\delta\chi_0''(1). \tag{15.61}$$

Thus, the presence of diaphragms on the clamped edges does not influence on the critical parameter λ^*, see Eq. (15.60), found from the first three approximations. An additional correction for the eigenvalue λ^* can be determined from considering the highest approximations.

15.4.1.3 Plate with S$^\pm$C$^-$S$^+$S$^+$ - Boundary Conditions

Let the left edge be simply supported and the right one be clamped, with both edges free of diaphragms ($S^-C^-S^+S^+$ – conditions, see Eqs. (15.24), (15.25)). In this case $\alpha_1 = \alpha_2 = 3$, $\iota_1 = 2$, $c_2 = 0$. In the leading approximation, one has the homogeneous boundary conditions

$$\chi_0(0) = \chi_0''(0) = 0, \quad \chi_0(1) = \chi_0'(1) = 0. \tag{15.62}$$

The first approximation implies

$$c_1 = -\delta(1-\nu)\chi_0'(0), \quad a_1 = -\sqrt{2}(1-\nu)^{3/2}\theta^2\delta^2\chi_0'(0), \; a_2 = -\theta^{3/2}\chi_0'''(1), \quad (15.63)$$

and the boundary conditions for $\chi_1(x)$ become as follows:

$$\chi_1(0) = 0, \quad \chi_1''(0) = -\sqrt{2}(1-\nu)^{3/2}(1-\theta)\delta^2\chi_0'(0), \quad \chi_1(1) = \chi_1'(1) = 0. \quad (15.64)$$

Condition (15.47) for the existence of a solution of the inhomogeneous boundary-value problem (15.33), (15.64) at $k = 1$ results in the following correction

$$\lambda_1 = -\frac{\sqrt{2}(1-\nu)^{3/2}(1-\theta)[\chi_0'(0)]^2}{\int_0^1 \chi_0^2(x)dx}. \quad (15.65)$$

Hence, we arrive at the following relations for the critical value of the load parameter and associated buckling mode:

$$\begin{aligned} \lambda^* &= \lambda_0^*\left[1 + \varepsilon\Lambda_1 + O\left(\varepsilon^2\right)\right], \\ \chi &\approx \sin\delta_n y\{\chi_0(x) + \varepsilon\chi_1(x) \\ &-\varepsilon^3\left[\sqrt{2}(1-\nu)^{3/2}\theta^2\delta^2\chi_0'(0)e^{-\frac{x}{\varepsilon}\sqrt{\theta}} + \theta^{3/2}\chi_0'''(1)e^{\frac{x-1}{\varepsilon\sqrt{\theta}}}\right]\}, \end{aligned} \quad (15.66)$$

where $\Lambda_1 = \lambda_1/\lambda_0^*$.

Now we consider the variant of $S^+C^-S^+S^+$ – conditions, see Eqs. (15.23) and (15.25). In contrast to the previous case, here the left simply supported edge is supplied with the diaphragm. The asymptotic analysis of the sequence of Eqs. (15.33) with corresponding boundary conditions implies $\alpha_1 = 4, \alpha_2 = 3$, $\Phi = 0$.

The homogeneous boundary conditions for the leading approximation are as in the previous case and specified by Eqs. (15.62). Considering the first-order approximation gives $\lambda_1 = 0, \chi_1 = 0$, and the second-order approximation implies parameters

$$a_1 = -\theta^2\chi_0^{IV}(0), \quad a_2 = -\theta^{3/2}\chi_0'''(1), \quad (15.67)$$

and the boundary conditions for χ_2:

$$\begin{aligned} &\chi_2(0) = 0, \quad \chi_2''(0) = \theta\chi_0^{IV}(0), \\ &\chi_2(1) = \chi_0''(1), \quad \chi_2'(1) = \theta\chi_0'''(1). \end{aligned} \quad (15.68)$$

A solution of the inhomogeneous boundary-value problem (15.56), (15.68) exists if and only if

$$\lambda_2 = \frac{\theta\left[\chi_0^{IV}(0)\chi_0'(0) + \chi_0'''(1)\chi_0''^{(1)}\right] - \chi_0''(1)\chi_0'''(1) - \int\limits_0^1 \chi_0 \mathbf{N}\chi_0 dx}{\delta^2 \int\limits_0^1 \chi_0^2(x)dx}. \tag{15.69}$$

Thus, for the $S^-C^-S^+S^+$ – boundary conditions the critical value of the load parameter is calculated by the first relation from (15.60), the associated buckling mode being the function

$$\chi \approx \sin\delta_n y\left[\chi_0(x) + \varepsilon^2\chi_2(x) - \varepsilon^3\theta^{3/2}\chi_0'''(1)\mathrm{e}^{\frac{x-1}{\varepsilon\sqrt{\theta}}} - \varepsilon^4\theta^2\chi_0^{IV}(0)\mathrm{e}^{-\frac{x}{\varepsilon\sqrt{\theta}}}\right]. \tag{15.70}$$

15.4.2 Layered Plates with Small Reduced Shear Modulus

Consider case (B), where θ is a small parameter, and $\kappa = O(1)$ as $\theta \to 0$. The procedure of constructing the asymptotic solution remains the same with the following modifications. A solution of Eq. (15.21) is sought in the form:

$$X = X^{(m)}(x,\theta) + \theta^{\alpha_1}X_1^{(e)}(x,\theta) + \theta^{\alpha_2}X_2^{(e)}(x,\theta), \tag{15.71}$$

where

$$X^{(m)} = \chi_0(x) + \theta^{1/2}\chi_1(x) + \theta\chi_2(x) + \ldots, \quad \lambda = \lambda_0 + \theta^{1/2}\lambda_1 + \theta\lambda_2 + \ldots \tag{15.72}$$

and

$$X_1^{(e)} = a_1\mathrm{e}^{-\frac{x}{\sqrt{\theta}}} + O\left(\theta^{1/2}\mathrm{e}^{-\frac{x}{\sqrt{\theta}}}\right), \quad X_2^{(e)} = a_2\mathrm{e}^{\frac{x-1}{\sqrt{\theta}}} + O\left(\theta^{1/2}\mathrm{e}^{\frac{x-1}{\sqrt{\theta}}}\right), \tag{15.73}$$

The differential equation of the leading approximation reads

$$\mathbf{L}_0\chi_0 \equiv \frac{d^4\chi_0}{dx^4} - (2\delta^2 - \lambda_0\delta^2\kappa)\frac{d^2\chi_0}{dx^2} + [\delta^4 - \lambda_0\delta^2(1+\kappa\delta^2)]\chi_0 = 0. \tag{15.74}$$

Its general solution is defined by Eq. (15.54), where

$$\begin{aligned}\alpha &= \frac{\sqrt{2}}{2}\sqrt{\delta^2(2-\lambda_0\kappa) + \sqrt{\delta^2\lambda_0(4+\lambda_0\kappa^2\delta^2)}},\\ \gamma &= \frac{\sqrt{2}}{2}\sqrt{-\delta^2(2-\lambda_0\kappa) + \sqrt{\delta^2\lambda_0(4+\lambda_0\kappa^2\delta^2)}},\end{aligned} \tag{15.75}$$

and $\lambda_0 > \frac{\delta^2}{1+\kappa\delta^2}$.

Here we restrict ourselves to the consideration of the $C^-C^-S^+S^+$ – boundary conditions. In this case, $\Phi = 0$, $\alpha_1 = \alpha_2 = 3/2$, and the boundary conditions in the leading approximation read

$$\begin{aligned} &\kappa\chi_0''(0)-(1+\kappa\delta^2)\chi_0(0)=0, \quad \chi_0'(0)=0,\\ &\kappa\chi_0''(1)-(1+\kappa\delta^2)\chi_0(0)=0, \quad \chi_0'(1)=0 \end{aligned} \tag{15.76}$$

In what follows, λ_0 is the minimum positive eigenvalue of the boundary-value problem (15.74), (15.76).

The next approximation yields the inhomogeneous boundary-value problem

$$\begin{aligned} &\mathbf{L}_0\chi_1(x)=\lambda_1\delta^2\left[(1+\kappa\delta^2)\chi_0(x)-\kappa\chi_0''(x)\right],\\ &\kappa\chi_1''(0)-(1+\kappa\delta^2)\chi_1(0)=-\kappa\chi_0'''(0), \quad \chi_1'(0)=0,\\ &\kappa\chi_1''(1)-(1+\kappa\delta^2)\chi_1(0)=\kappa\chi_0'''(1), \quad \chi_1'(1)=0 \end{aligned} \tag{15.77}$$

and the pair of constants, $a_1=\chi_0'''(0), \quad a_2=-\chi_0'''(1)$ for the edge effect integrals (15.73).

We note that the boundary-value problem (15.74), (15.76) is not self-conjugated. Let the function $\chi_*(x)$ be a solution of the homogeneous Eq. (15.74) with the following conjugated boundary conditions:

$$\begin{aligned} &\kappa\chi_*'''(0)-(\kappa\delta^2-\lambda_0\kappa^2\delta^2-1)\chi_*'(0)=0, \chi_*(0)=0,\\ &\kappa\chi_*'''(1)-(\kappa\delta^2-\lambda_0\kappa^2\delta^2-1)\chi_*'(1)=0, \chi_*(1)=0. \end{aligned} \tag{15.78}$$

Then the comparability conditions for the inhomogeneous boundary-value problem (15.77) results in the following correction for the eigenvalue λ_0:

$$\lambda_1=-\frac{\chi_0'''(0)\chi_*'(0)+\chi_0'''(1)\chi_*'(1)}{\delta^2(1+\kappa\delta^2)\int\limits_0^1\chi_0(x)\chi_*(x)dx-\delta^2\kappa\int\limits_0^1\chi_0''(x)\chi_*(x)dx}. \tag{15.79}$$

15.5 Analysis of Influence of Boundary Conditions and Edge Effects on Critical Force

At first, we consider case (A). In the leading approximation ($k=0$), the homogeneous boundary-value problems represented by Eq. (15.33) and by corresponding boundary conditions coincide with similar classical problems for single layer plates when shear effect is ignored. The careful analysis of the influence of boundary conditions on buckling of single layer isotropic rectangular plates can be found in Alfutov (2000). In particular, diagrams of the critical compressive force versus the side ratio $l=b/a$ are presented for all possible variants of boundary conditions. Here, we give similar plots of the load parameter λ_0 as of the function of a parameter $\delta = \pi n/l$. The minimum positive eigenvalue $\lambda_0(\delta)$ versus a fixed parameter δ is depicted in Fig. 15.1 for the three distinctive variants of boundary conditions, $S^{\pm}S^{\pm}S^{+}S^{+}$, $S^{\pm}C^{\pm}S^{+}S^{+}$ and $C^{\pm}C^{\pm}S^{+}S^{+}$ – conditions (as a reminder, the value of λ_0 does not depend on whether the edges are equipped with diaphragms or not).

Let δ_m be a value at which the function $\lambda_0(\delta)$ takes a minimum value λ_m. Here, $\delta_m = \pi$, $\lambda_m \approx 39.478$ for $S^{\pm}S^{\pm}$ – boundary conditions at the unloaded edges, $\delta_m \approx 3.95, \lambda_m \approx 53.392$ for $S^{\pm}C^{\pm}$ – conditions and $\delta_m \approx 4.75, \lambda_m \approx 68.800$ for $C^{\pm}C^{\pm}$ – conditions. The required eigenvalue λ_0^* corresponding to the plate buckling strongly depends on the sides ratio $l = b/a$ and is determined as follows. If $l < \pi/\delta_m$, then $n^* = 1$, $\delta^* = \pi n^*/l = \pi/l$, and for $l \geq \pi/\delta_m$, one obtains $n^* = \mathrm{In}(\delta_m l/\pi)$, where the sign $\mathrm{In}(z)$ as above denotes the integer part of a number z. In both cases, $\lambda_0^* = \lambda_0(\delta^*)$.

The correction $\varepsilon^k \lambda_k$, taking into account shears, strongly depends on the type of boundary conditions. For the plates with $S^{-}S^{-}$, $S^{-}C^{-}$ – conditions at the unloaded edges $x = 0, 1$, we obtain the correction $\varepsilon\lambda_1$ of order $O(\varepsilon)$, while for the plates with $S^{+}S^{+}, C^{\pm}C^{\pm}, S^{+}C^{-}$ – conditions this correction becomes smaller and is a value of order $O(\varepsilon^2)$. Other words, if even one simply supported edge is free of a diaphragm, then the effect of shears on the critical buckling force increases.

A sign of the correction as well as its value depend on the ratio $\delta = b/a$ and the shear parameter θ. We remind that a parameter θ is the function of many magnitudes such as a number of layer, thickness and Young's modulus of each layer. A parameter θ is generally small (Mikhasev et al, 2019). In Figs. 15.2 - 15.4, the relative corrections $\Lambda_1 = \lambda_1/\lambda_0$ and $\Lambda_2 = \lambda_2/\lambda_0$ are depicted as functions of δ for $\theta = 0.01, 0.1, 0.5, 0.8$.

It is seen that for $S^{-}S^{-}$ – conditions at the unloaded edges, the correction Λ_1 is always positive. When θ is infinitely small, then the correction is maximum for any δ. Note that $\theta = 0$ corresponds to the Timoshenko-Reissner model (Tovstik and Tovstik, 2017a) when the edge effects are ignored and only the transverse shears inside the plate are taken into account. Hence, accounting for shears near the simply supported edges without diaphragms reduces the positive correction and, as consequence, the critical buckling force. For any fixed θ, the correction Λ_1 reaches the

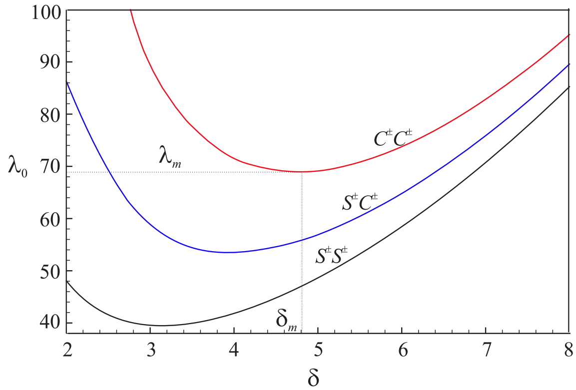

Fig. 15.1 The first positive eigenvalue $\lambda_0(\delta)$ vs. a parameter δ for $S^{\pm}S^{\pm}$, $S^{\pm}C^{\pm}$, $C^{\pm}C^{\pm}$ – boundary conditions at the unloaded edges. Case (A).

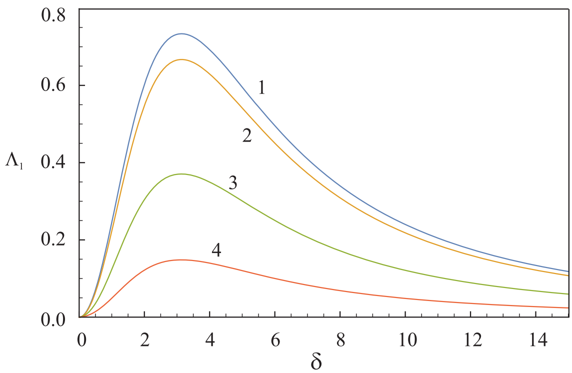

Fig. 15.2 The relative correction $\Lambda_1 = \lambda_1/\lambda_0$, taking into account shears, vs. parameter δ at various $\theta = 0.01, 0.1, 0.5, 0.8$ (curves 1, 2, 3, 4, respectively) for plates with S^-S^- – boundary conditions at the unloaded edges.

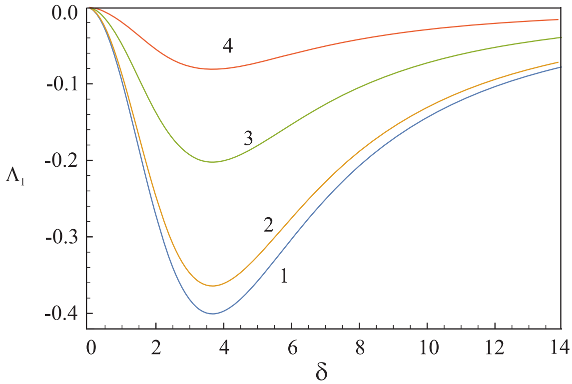

Fig. 15.3 The relative correction $\Lambda_1 = \lambda_1/\lambda_0$, taking into account shears, v.s parameter δ at various $\theta = 0.01, 0.1, 0.5, 0.8$ (curves 1, 2, 3, 4, respectively) for plates with S^-C^- – boundary conditions at the unloaded edges.

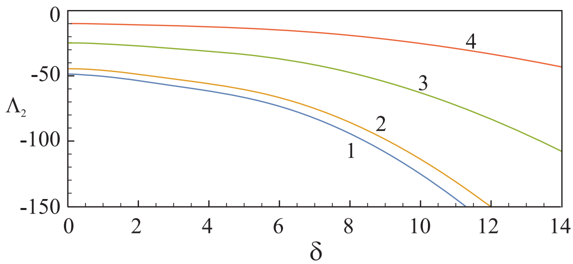

Fig. 15.4 The relative correction $\Lambda_2 = \lambda_2/\lambda_0$, taking into account shears, vs. parameter δ at various $\theta = 0.01, 0.1, 0.5, 0.8$ (curves 1, 2, 3, 4, respectively) for plates with $C^{\pm}C^{\pm}$ – boundary conditions at the unloaded edges.

maximum value at $\delta = \pi$. When $\delta \to 0$ or $\delta \to \infty$ (that corresponds to the degeneration of a plate into an infinitely narrow stripe or beam, respectively), the correction Λ_1 vanishes.

For S^-C^-– boundary conditions at the unloaded edges, the relative correction Λ_1 becomes negative. The maximum absolute value of Λ_1 is achieved at $\delta \approx 3.95$ for any θ; the smaller the parameter δ, the greater the value of $|\Lambda_1|$.

Finally, for $C^{\pm}C^{\pm}$ – conditions, the correction is always negative for any δ and its absolute value increases together with δ. We note that relations (15.58), (15.60) for $C^{\pm}C^{\pm}$ – conditions as well as Eq. (15.69) for S^+C^- – conditions are asymptotically correct if $\varepsilon^2\Lambda_2 \ll 1$.

Now we analyze case (B) for C^-C^- – conditions at the unloaded edges. Here, θ is assumed to be a small parameter, while κ is a finite value of the order $O(1)$ as $\theta \to 0$. In this case, in contrast to case (A), the eigenvalue λ_0 evaluated in the leading approximation takes into account the transverse shear inside the plate. In Fig. 15.5, the first positive eigenvalue λ_0 of the boundary-value problem (15.74), (15.76) is shown as the function of a parameter δ for different values of the shear parameter κ. The minimum value λ_m of the function $\lambda_0(\delta)$ and the associated argument δ_m are shown in Table 15.1 for various κ. We note the following limit relation $\lim\limits_{\kappa \to 0} \lambda_m = 68.80$, where $\lambda_m = 68.80$ corresponds to the $C^{\pm}C^{\pm}$ – boundary conditions for case (A), see Fig. 15.1. The required critical buckling force λ_0^* is evaluated in accordance with the rule described above for case (A).

In Fig. 15.6, the relative corrections Λ_1 are depicted as functions of δ for different κ. In contrast to case (A) considered for $C^{\pm}C^{\pm}$ – boundary conditions (see Eq. (15.58) and Fig. 15.4), the positive correction $\varepsilon\Lambda_1$ takes into account only the edge effect integrals induced by the transverse shears in the neighborhood of the clamped edges $x = 0, x = 1$ without diaphragms. Thus, accounting for shears in the vicinity of the clamped edges results in the increase of the critical buckling force evaluated in the framework of the Timoshenko-Reissner model. It can be seen that the correction Λ_1 falls down when the shear parameter κ decreases. For each fixed κ, there exists such value of δ for which this correction takes the maximum value. It is also interesting to note that the correction Λ_1 becomes weakly dependent on the shear parameter for large δ (when a plate is degenerated into a beam) and vanishes as $\delta \to \infty$ for any κ.

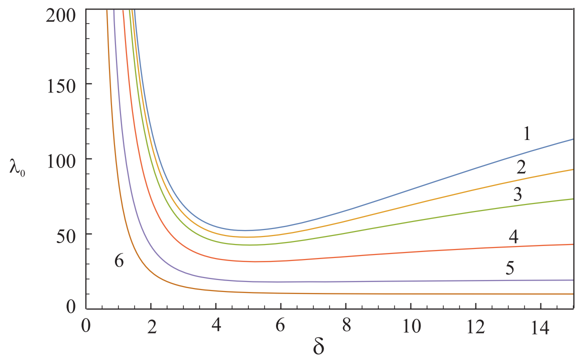

Fig. 15.5 The first positive eigenvalue $\lambda_0(\delta)$ vs. a parameter δ for C^-C^- – boundary conditions at the unloaded edges for different values of the shear parameter κ = 0.005, 0.07, 0.01, 0.02, 0.05, 0.1 (curves 1, 2, 3, 4, 5, 6, respectively). Case (B).

Table 15.1 Parameters λ_m, δ_m for different values of the shear parameter κ.

κ	0.005	0.007	0.01	0.02	0.05	0.1
δ_m	4.92	4.96	5.02	5.21	6.25	15.00
λ_m	52.21	47.81	42.56	31.54	18.00	10.01

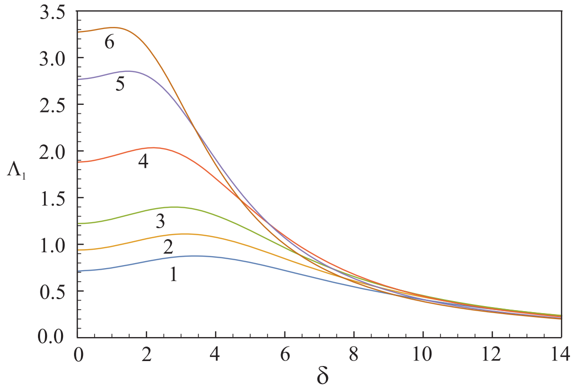

Fig. 15.6 The relative correction $\Lambda_1 = \lambda_1/\lambda_0$, taking into account shears in the vicinity of the unloaded edges with C^-C^- – boundary conditions, vs. a parameter δ for different values of the shear parameter κ = 0.005, 0.07, 0.01, 0.02, 0.05, 0.1 (curves 1, 2, 3, 4, 5, 6, respectively). Case (B).

15.6 Conclusions

Based on the ESL theory for laminated shells, buckling of layered rectangular plates uniaxially compressed by in-plane forces was studied. The loaded edges were assumed to be simply supported and supplied with diaphragms while for other edges two groups of boundary conditions, the clamped and simple support groups, with or without diaphragm(s) were considered. The solutions of governing equations were constructed in the form of a superposition of the outer solution and the edge effect integrals accounting shears in the neighbourhood of the unloaded edges. It was found out that the effect of boundary conditions on the critical buckling load depends on whether one of the unloaded edges is equipped with the diaphragm or not. In particular, if there are no diaphragms at all unloaded simply supported edges, then a correction to the classical buckling force turns out to be an order of magnitude higher than for a plate equipped with a diaphragm at least on one of the unloaded simply supported edges.

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