# Chapter 12
# Dimension Reduction in the Plate with Tunnel Cuts

Alexander G. Kolpakov and Sergei I. Rakin

**Abstract** We carry out dimension reduction in the homogenization theory 3D periodicity cell problem for the plate with a unidirectional system of channel cuts. We demonstrate that the original 3D problem may be reduced to several 2D problems. The main attention is paid to the solution near the top and the bottom surfaces of the plate Our numerical analysis indicates the existence of a new type of boundary layer at the upper and lower surfaces of the plate. We estimate the thickness of the found boundary layer. We also find a wrinkling effect on the top and bottom surfaces of the plate.

**Key words:** Plate with channel cuts, Dimension reduction, Top/bottom face boundary layers, Wrinkling effect

## 12.1 Introduction

The homogenization problem for the elastic bodies with holes/pores attracted the attention of numerous researchers. One can mention the pioneering paper of Cioranescu and Paulin (1979). Relevant references may be found in Cioranescu and Donato (1999); Cioranescu et al (2018) (mathematics) and Kalamkarov and Kolpakov (1997); Kolpakov and Kolpakov (2009) (applications to composite materials). The

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papers on the homogenization problem for plates with holes/pores/channels are not so numerous as the papers devoted to the homogenization problem for solids with holes/pores/channels.

The boundary layers in plates and shells were intensively discussed in literature after Rayleigh, Love, Lamb, and Basset (Sendeckyj, 1974). In the 1970s-1980s, boundary layers were intensively discussed for laminated composite materials (van Dyke, 1994; Pipes and Pagano, 1970), later - for fiber-reinforced composite materials see, e.g., Kalamkarov and Kolpakov (1997); Andrianov et al (2011). Numerous experimental, theoretical, and numerical results were reported. Note that the boundary layers in the plates and shells were associated exclusively with the transverse cut surface.

When considering the plates with channel cuts, we meet the new type of boundary layers associated with the top and the bottom surfaces of the plate. The boundary layers of this type never occur in the homogeneous or in the laminated plates.

## 12.2 Statement of the Problem

We consider a plate with a periodic system of cylindrical geometry channels. Suppose the cylinders are parallel to the $0x$–axis and form a periodic structure in the $0xz$-plane. The periodicity cell (PC) of such a structure is shown in Fig. 12.1a , and the cross-section is displayed in Figs. 12.1b. The choice of the length of the PC in Fig. 12.1a is voluntary.

Since the plate under consideration is invariant with respect to translation in the direction $0y$-axis, there is a reason to look for two-dimensional models to the plate. The dimension reduction procedures are known for the solids with periodic systems of fibers or holes (Grigolyuk et al, 1991; Grigolyuk and Fil'shtinskij, 1992; Lu, 1995; Mityushev and Rogozin, 2000; Drygaś et al, 2020). To the best knowledge of the authors, the first paper devoted to the dimension reduction in the bending problem for an elastic layer with tunnel cuts was Grigolyuk et al (1991). The mentioned paper was based on the double periodic function technique, thus treated the layer

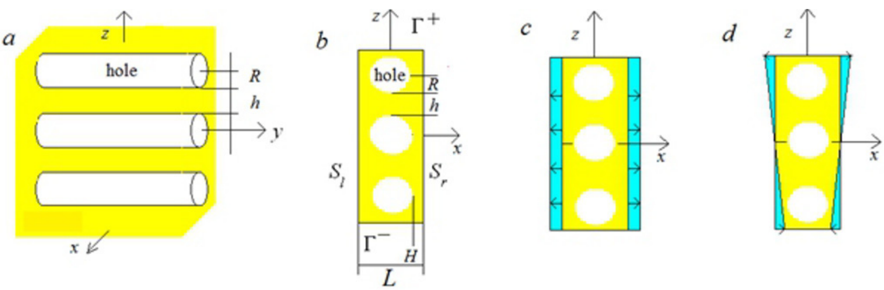


**Fig. 12.1** Periodicity cell (a) and the cross-section (b) of the plate with the channel cuts and the deformation modes of PC: (c) - tension, (d) - bending.

of "infinite" thickness. It means that Grigolyuk et al (1991) can be used to predict the stress-strain state (SSS) inside the plate, but not near-surface phenomena. But the plate has a finite thickness. The aim of this paper is the dimension reductions for plates of finite thickness. Our research indicates the existence of a new range of boundary layers - boundary layers at the upper and lower surfaces of the plate. We also find a wrinkling effect on the top and bottom surfaces of the plate. The results of Grigolyuk et al (1991) can be used to describe the stress-strain state (SSS) inside the plate, but not near-surface phenomena.

The starting point of our research is the periodicity cell problem (PCP) of the homogenization theory as applied to plates (Caillerie, 1984; Kohn and Vogelius, 1984), which has the following form:

$$\begin{aligned} &(a_{ijkl}N^{\mathrm{AB}\mu}_{k,l} + (-1)^\mu a_{ij\mathrm{AB}} z^\mu)_{,j} = 0 \quad \text{in} \quad P, \\ &(a_{ijkl}N^{\mathrm{AB}\mu}_{k,l} + (-1)^\mu a_{ij\mathrm{AB}} z^\mu) n_j = 0 \quad \text{on} \quad \Gamma \cup H, \\ &\mathbf{N}^{\mathrm{AB}\mu}(\mathbf{y}) \quad \text{periodic in} \quad x, y. \end{aligned} \tag{12.1}$$

with the superscript $\mu$ taking the values 0 or 1. In the plate PCP, the top $\Gamma^+$ and bottom $\Gamma^-$ surfaces are free. The PC may be subjected to in-plane ($\mu = 0$) or bending/torsion ($\mu = 1$) macroscopic deformation. These features distinguish the plates from the solids PCP. In the plate with channel cuts, the surfaces $H_i$ of the channels are also free. Denoted: $\Gamma = \Gamma^+ \cup \Gamma^-$ and $H = \cup_{i=1}^n$, where $n$ is the number of channels per one PC. The variables notation correspondence: $x \leftrightarrow 1, y \leftrightarrow 2, z \leftrightarrow 3$; the index $\mu = 0, 1$.

In the general case (Caillerie, 1984; Kohn and Vogelius, 1984), the local stresses in the PC are computed with the following formula:

$$\sigma_{ij} = a_{isjkl} N^{\mathrm{AB}\mu}_{k,j} + (-1)^\mu a_{ij\mathrm{AB}} z^\mu,$$

and the macroscopic stiffnesses of the plate are computed as

$$S^{\nu+\mu}_{\alpha\beta\mathrm{AB}} = \frac{1}{|\mathrm{Pr}P|} \int_P (a_{\alpha\beta kl} N^{\mathrm{AB}\nu}_{k,j} + (-1)^\nu z^\nu a_{\alpha\beta\mathrm{AB}})(-1)^\mu z^\mu \mathrm{d}x\mathrm{d}z,$$

where Pr$P$ is projection of the PC$P$ to the $0xy$-plane. The superscript $\nu$ can take the values 0 or 1.

The PC$P$ is a cylinder parallel to the $0y$-axis, see Fig. 12.1, and the elastic constants $a_{ijkl}$ are constants (we assume the plate is made of a homogeneous isotropic material). In this case, the solution to the problem (12.1) does not depend on the variable $y$ and has the form $\mathbf{N}^{\mathrm{AB}\mu} = \mathbf{N}^{\mathrm{AB}\mu}(x,z)$. Substituting $\mathbf{N}^{\mathrm{AB}\mu} = \mathbf{N}^{\mathrm{AB}\mu}(x,z)$ into (12.1), we arrive at the following 2D PCP:

$$\begin{cases} (a_{i\alpha k\beta} N^{\mathrm{AB}\mu}_{k,\beta} + (-1)^\mu a_{i\alpha\mathrm{AB}} z^\mu)_{,\alpha} = 0 \quad \text{in} \quad P, \\ (a_{i\alpha k\beta} N^{\mathrm{AB}\mu}_{k,\beta} + (-1)^\mu a_{i\alpha\mathrm{AB}} z^\mu) n_\alpha = 0 \quad \text{on} \quad \Gamma, \\ \mathbf{N}^{\mathrm{AB}\mu}(x,z) \quad \text{periodic in} \quad x. \end{cases} \tag{12.2}$$

Hereafter $\alpha,\beta = 1,3 : i,k = 1,2,3; \mathrm{A},\mathrm{B} = 1,1;2,2;1,2;2,1$. We use the same notation for the PC and its cross-sections, as well as for the boundaries of the plate and the boundaries of the channels. In (12.2)

$$a_{i\alpha k\beta}N^{\mathrm{AB}\mu}_{k,\beta}(\mathbf{y}) + (-1)^{\mu}a_{i\alpha\mathrm{AB}}z = a_{i\alpha\theta\beta}N^{\mathrm{AB}\mu}_{\theta,\beta}(x,z) + a_{i\alpha 2\beta}N^{\mathrm{AB}\mu}_{2,\beta}(x,z) + (-1)^{\mu}a_{i\alpha\mathrm{AB}}z^{\mu}. \tag{12.3}$$

Equation (12.3) makes it possible to decompose the boundary-value problem (12.1) into several 2D problems. The form of the 2D problems is determined by the index $i$ in (12.2). For this reason, we consider problem (12.2) for $i = 2$ and $i = \xi = 1,3 = x,z$, separately.

## 12.3 Problem 12.1 with Index $i = 2$

We assume the plate is made of homogeneous isotropic material. We will use the tensor notations $a_{ijkl}$ (it is convenient in our computations) for the elastic constants keeping into mind the relation of the elastic constants with Young's modulus $E$ and Poisson's ratio $\nu$ (Love, 2013)

$$a_{1111} = a_{3333} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \qquad a_{1133} = a_{3311} = a_{1122} = a_{3322} = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{12.4}$$

In the case under consideration, $a_{2\alpha\theta\beta} = 0, a_{2\alpha\mathrm{AB}} = 0$ (Love, 2013) and expression in (12.3) takes the form ($\alpha = 1,3$)

$$a_{2\alpha\theta\beta}N^{\mathrm{AB}\mu}_{\theta,\beta} + a_{2\alpha 2\beta}N^{\mathrm{AB}\mu}_{2,\beta} + (-1)^{\mu}a_{2\alpha\mathrm{AB}}z^{\mu} = a_{2\alpha 2\alpha}N^{\mathrm{AB}\mu}_{2,\alpha} + \begin{cases} (-1)^{\mu}a_{2121}z^{\mu} \\ \text{if} \quad \mathrm{AB} = 21,12 \\ 0 \quad \text{else} \end{cases} \tag{12.5}$$

By virtue of (12.5), the solution to (12.2) $N^{\mathrm{AB}\nu}_{2}(x,z) = 0$ if $\mathrm{AB} \neq 21$. Only the component $N^{21\nu}_{2}(x,z)$ is non-zero. It is the case of in-plane shift ($\mu = 0$) or torsion ($\mu = 1$). The in-plane shift is also called anti-plane deformation (Love, 2013).

The problem for $N^{21\nu}_{2}(x,z)$ takes the form

$$\begin{cases} (a_{2\alpha 2\alpha}N^{21\mu}_{2,\alpha} + (-1)^{\mu}a_{2121}z^{\mu}\delta_{\alpha 1})_{,\alpha} = 0 \quad \text{in} \quad P, \\ (a_{2\alpha 2\alpha}N^{21\mu}_{2,\alpha} + (-1)^{\mu}a_{2121}z^{\mu}\delta_{\alpha 1})n_{\alpha} = 0 \quad \text{on} \quad \Gamma \cup H, \\ N^{21\mu}_{2}(x,z) \quad \text{periodic in} \quad x. \end{cases} \tag{12.6}$$

The term $(-1)^{\mu}a_{2121}z^{\mu}\delta_{\alpha 1}$ in (12.6) may be eliminated. There exists a function $w(x,z)$, such that ($\nu = 0,1$)

$$a_{2\delta 2\delta}w_{,\delta} = (-1)^{\nu}a_{2121}z^{\nu}. \tag{12.7}$$

For $\delta = 2$ and $\delta = 3$, we obtain from (12.7) $a_{2121}w_{,1} = (-1)^{\nu}a_{2121}z^{\nu}$ and $a_{2323}w_{,3} = 0$. From these equalities, we obtain the following system of differential equations

$$w_{,1} = (-1)^{\nu}z^{\nu}, \qquad w_{,3} = 0. \tag{12.8}$$

### 12.3.1 In-plane Shift

For $\nu = 0$, the system (12.8) takes the form $w_{,1} = 1, w_{,3} = 0$. The solution to this system is $w(x,z) = x$. We introduce function $M(x,z) = N_2^{120}(x,z) + x$ and rewrite (12.6) in the form of the following boundary-value problem for the Laplace equation:

$$\begin{cases} \Delta M = 0 \quad \text{in} \quad P, \\ \dfrac{\partial M}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma \cup H, \\ M(x,z) - x \quad \text{periodic in} \quad x \in [-L, L]. \end{cases} \tag{12.9}$$

After some algebra we obtain the following formula for the computation of the local stresses:

$$\sigma_{ij} = a_{ij2\alpha}N^{120}_{2,\alpha} + a_{ij21} = a_{ij2\alpha}M_{,\alpha},$$

and the following formula for the computation of homogenized shift stiffness:

$$S^0_{2121} =< a_{212\alpha}N^{210}_{2,\alpha} + a_{2121} >=< a_{212\alpha}M_{,\alpha} > .$$

Hereafter

$$< \ldots >= \frac{1}{L}\int_P \ldots \mathrm{d}x\mathrm{d}z$$

means the "average value", where $L$ is the width of the 2D periodicity cell, Fig. 12.1b.

### 12.3.2 Torsion

For $\mu = 1$, (12.6) takes the form

$$\begin{cases} (a_{2\alpha 2\alpha}N^{211}_{2,\alpha} - a_{2121}z\delta_{\alpha 1})_{,\alpha} = 0 \quad \text{in} \quad P, \\ (a_{2\alpha 2\alpha}N^{211}_{2,\alpha} - a_{2121}z\delta_{\alpha 1})n_{\alpha} = 0 \quad \text{on} \quad \Gamma \cup H, \\ N_2^{211}(x,z) \quad \text{periodic in} \quad x. \end{cases} \tag{12.10}$$

For $\nu = 1$, the system (12.8) takes form $w_{,1} = -z, w_{,3} = 0$. It is a not integrable system of differential equations. For this system, the necessary integrability condition (Sedov, 1971) is not satisfied because $w_{,13} = -z_{,3} = -1 \neq, w_{,31} = 0$. As a result, it is impossible to eliminate the term $a_{2121}z\delta_{\alpha 1}$ in (12.10) in a simple way as above. The problem (12.10) may be written in a compact form in the following way. Introduce function $\varphi(x,z)$ as

$$\varphi_{,3} = a_{2121}(N^{211}_{2,1} - z), \qquad \varphi_{,1} = -a_{2323}N^{211}_{2,3} \tag{12.11}$$

The definition (12.11) uses the idea of the conjugate functions (Sedov, 1971). The existence of the function $\varphi(x,z)$ follows from the equality

$$\varphi_{,31} = (a_{2121}(x,z)(N^{211}_{2,1} - z))_{,1} + (a_{2323}(x,z)N^{211}_{2,3})_{,3} = 0,$$

which is the consequence of (12.10).

Express $\mathbf{N}^{211}_{2}(x,z)$ from (12.11)

$$N^{211}_{2,1} = \frac{1}{a_{2121}}\varphi_{,3} + z, \qquad N^{211}_{2,3} = -\frac{1}{a_{2121}}\varphi_{,1}. \tag{12.12}$$

Differentiation of (12.12) yields

$$0 = N^{211}_{2,13} - N^{211}_{2,31} = \left(\frac{1}{a_{2121}}\varphi_{,3} + z\right)_{,3} + \left(\frac{1}{a_{2121}}\varphi_{,1}\right)_{,1}.$$

Grouping the terms in the last equation, we arrive at the following Poisson equation:

$$\Delta\varphi = a_{2121}. \tag{12.13}$$

Consider the boundary conditions on the top and the bottom boundaries $\Gamma^+, \Gamma^-$ and the holes $H_i$ (12.6). With the use of the function $\varphi(x,z)$, these conditions can be written as follows:

$$(a_{2121}N^{21\nu}_{2,1} - a_{2121})n_1 + a_{2323}N^{21\nu}_{2,3}n_3 = \varphi_{,3}n_1 - \varphi_{,1}n_3 = \frac{\partial\varphi}{\partial s} = 0 \ \text{ on } \ \Gamma^+, \Gamma^- \ \text{ or } \ H_i, \tag{12.14}$$

where $\partial/\partial s$ is the derivative along the boundary $\Gamma^+, \Gamma^-$ or $H_i$. In view of (12.14), the function $\varphi(x,z)$ is constant on the top and bottom boundaries $\Gamma^+, \Gamma^-$ and $H_i$:

$$\varphi = \text{const} \quad \text{on} \quad \Gamma^+, \Gamma^-, H_i. \tag{12.15}$$

Without loss of generality, we can fix one constant. Let us assume that at the bottom boundary $\Gamma^-, \varphi(x,z) = 0$.

Integrating the first equation in (12.11) over $S_i$, see Fig. 12.1b, we can have

$$\varphi(h,-L) = \varphi(-h,-L) + \int_{-h}^{h} a_{2121}(N^{211}_{2,1} - z)\mathrm{d}z. \tag{12.16}$$

The asymmetric (out-of-plane) stiffness

$$S^1_{2121} = \frac{1}{L}\int\limits_P a_{2121}(N^{211}_{2,1} - z)\mathrm{d}x\mathrm{d}z.$$

Multiplying the differential equation in (12.10) by $x$ and integrating by parts, we have

$$\int\limits_P a_{2121}(N^{211}_{2,1} - z)\mathrm{d}x\mathrm{d}z = 2\frac{L}{2}\int\limits_{-h}^{h} a_{2121}(N^{211}_{2,1}\mathrm{d}z - z)\mathrm{d}x.$$

Then Eq. (12.16) becomes $\varphi(h,-L) = \varphi(-h,-L) + S_{2121}$. We have assumed that $\varphi(x,z) = 0$ on the bottom boundary $\Gamma^-$, thus, $\varphi(-h,-L) = 0$. Then $\varphi(h,-L) = S_{2121}$ and $\varphi(x,z) = S_{2121}$ on the top boundary $\Gamma^+$. As a result, we arrive at the following boundary value problem:

$$\begin{cases} \Delta\varphi = a_{2121} \quad \text{in} \quad P_0, \\ \varphi = 0 \quad \text{on} \quad \Gamma^-, \quad \varphi = S^1_{2121} \quad \text{on} \quad \Gamma^+, \varphi = C_1 \text{on} \quad H_i, \\ \varphi(x,z) \quad \text{periodic in} \quad x \in [-L,L]. \end{cases} \tag{12.17}$$

The local stresses are expressed in the form

$$\sigma_{ij} = a_{ij2\alpha}N^{211}_{2,\alpha} + a_{ij21}z = \frac{a_{ij21}}{a_{2121}}(\varphi_{,3} - \varphi_{,1}) \tag{12.18}$$

and the homogenized torsion stiffness is expressed in the form

$$S^2_{2121} = -< \varphi_{,3} - \varphi_{,1} > .$$

## 12.4 Problem 12.2 with Indices $i = \xi = 1,3 = x,z$. Deformation in the Direction Perpendicular to the Fibers

In this case, $a_{\xi\alpha\alpha2\beta}(\mathbf{y}) = 0$ and (12.3) takes the following form:

$$a_{i\alpha k\beta}N^{\mathrm{AB}\mu}_{k,\beta}(\mathbf{y}) + a_{\xi\alpha2\beta}z^\mu N^{\mathrm{AB}\mu}_{2,\beta}(\mathbf{y}) + (-1)^\mu a_{\xi\alpha\mathrm{AB}}z^\mu = a_{\xi\alpha\theta\beta}N^{\mathrm{AB}\mu}_{\theta,\beta}(\mathbf{y}) + (-1)^\mu a_{\xi\alpha\mathrm{AB}}z\mu.$$

Here AB = $11;22;12;21;\alpha,\beta,\theta,\xi = 1,3$. Then the PCP (12.2) takes the form

$$\begin{cases} (a_{\xi\alpha\theta\beta}N^{\mathrm{AB}\mu}_{\theta,\beta} + (-1)^\mu a_{\xi\alpha\mathrm{AB}}z^\mu)_{,\alpha} = 0 \quad \text{in} \quad P, \\ (a_{\xi\alpha\theta\beta}N^{\mathrm{AB}\mu}_{\theta,\beta} + (-1)^\mu a_{\xi\alpha\mathrm{AB}}z^\mu)n_\alpha = 0 \quad \text{on} \quad \Gamma \cup H, \\ (N^{\mathrm{AB}\mu}_1, N^{\mathrm{AB}\mu}_3)(x,z) \quad \text{periodic in} \quad x. \end{cases} \tag{12.19}$$

In the case under consideration elastic constants $a_{\xi\alpha12} = 0$ and $a_{\xi\alpha21} = 0$ for $i = \xi = 1,3$, then $(N^{21\mu}_1, N^{21\mu}_3 = N^{12\mu}_1, N^{12\mu}_3 = 0$. The problem is non-trivial only for $AB = 11;22$. Let us demonstrate the term $(-1)^\mu a_{\xi\alpha\mathrm{AB}}z^\mu$ in (12.19) may be represented in the form $(-1)^{\mu} a_{\xi\alpha\mathrm{AB}} e_{\theta\beta}^{\mathrm{AB}\mu}$ ($\delta = 2,3$) with the strains $e_{\theta\beta}^{\mathrm{AB}\mu} = v_{\theta\beta}^{\mathrm{AB}\mu}(\mu = 0,1)$:

$$a_{\xi\alpha\mathrm{AB}}(x,z)z^{\mu} = a_{\xi\alpha\theta\beta}(x,z)e_{\theta\beta}^{\mathrm{AB}\mu} \tag{12.20}$$

### 12.4.1 *Index* $AB = 22$*. Tension-compression and Bending Along the Fibers (in the* $0xz$*-plane)*

The typical overall deformations of the PC are shown in Figs. 12.1c, d. In the case under consideration, Eq. (12.20) takes the form $a_{\xi\alpha\theta\beta}e_{\theta\beta} = a_{\xi\alpha22}z^{\mu}$. Be written in the coordinate-wise form, it becomes

$$\begin{aligned}
&a_{1111}e_{11} + a_{1133}e_{33} = -a_{1122}z^{\mu},\\
&a_{3311}e_{11} + a_{3333}e_{33} = -a_{3322}z^{\mu},\\
&a_{1313}e_{13} = -a_{1322}z^{\mu} = 0, a_{3131}e_{31} = -a_{3122}z^{\mu} = 0.
\end{aligned} \tag{12.21}$$

Substituting into (12.21) the elastic constants (12.4), we obtain from the first two equations in the following system:

$$\begin{cases} (1-\nu)e_{11} + \nu e_{33} = -\nu(x,z)z^{\mu},\\ \nu e_{11} + (1-\nu)e_{33} = -\nu(x,z)z^{\mu}. \end{cases} \tag{12.22}$$

Solution to (12.22) is

$$e_{11} = e_{33} = -\nu z^{\nu}. \tag{12.23}$$

In addition, $e_{13} = e_{31} = 0$. Then

$$\frac{\partial v_1}{\partial x} = -\nu z^{\mu}, \quad \frac{\partial v_3}{\partial z} = -\nu z^{\mu}, \quad \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} = 0. \tag{12.24}$$

The solution to (12.24) may be obtained in the explicit form. For $\mu = 0$ from the first two equations in (12.24), we have $v_1 = -\nu x + f(z)$ and $v_3 = -\nu z + g(x)$. Substituting into the third equation in (12.24), we have $f'(z) + g'(x) = 0$, then $f(z) = 0$ and $g(x) = 0$.

For $\mu = 1$, we have from (12.24) $v_1 = -\nu zx + f(z)$ and $v_3 = -\frac{\nu}{2}z^2 + g(x)$. Substituting into the third equation in (12.24), we arrive at $-\nu x + f'(z) + g'(x) = 0$, and obtain $f'(z) = 0, g'(x) = \nu x$. Then $f(z) = 0$ and $g(x) = \frac{\nu}{2}x^2$. Finally, we have

$$v_1^{22\mu} = \begin{cases} -\nu x & \text{if} \quad \mu = 0,\\ -\nu zx & \text{if} \quad \mu = 1, \end{cases} \qquad v_3^{22\mu} = \begin{cases} -\nu z & \text{if} \quad \mu = 0,\\ -\frac{\nu}{2}z^2 + \frac{\nu}{2}x^2 & \text{if} \quad \mu = 1. \end{cases} \tag{12.25}$$

Introduce $(M_1^{\mathrm{AB}\mu}, M_3^{\mathrm{AB}\mu}), (N_1^{\mathrm{AB}\mu}, N_3^{\mathrm{AB}\mu}) + (v_1^{\mathrm{AB}\mu}, v_3^{\mathrm{AB}\mu})$. For $(M_1^{22\mu}, M_3^{22\mu})$, the third condition in (12.19) takes the form: $(M_1^{22\mu} - v_1^{22\mu}, M_3^{22\mu} - v_3^{22\mu})$ is periodic in $x$, or $[M_1^{22\mu}]_x = -\nu z^{\mu}[x]_x, [M_3^{22\mu}]_x = 0$ (the square brackets $[\ldots]_x$ mean the difference of the function on the opposite sides of the periodicity cell in the direction $Ox$). Here we use that

$$[v_1^{22\mu}]_x = \begin{cases} -\nu[x]_x & \text{if} \quad \mu = 0, \\ -\nu z[x]_x & \text{if} \quad \mu = 1, \end{cases} \qquad v_3^{22\mu} = \begin{cases} 0 & \text{if} \quad \mu = 0, \\ -\frac{\nu}{2}[x^2]_x = 0 & \text{if} \quad \mu = 1 \end{cases}$$

and

$$N_1^{220} = M_1^{220} + \nu x, N_3^{220} = M_3^{220} + \nu x, N_1^{221} = M_1^{221} + \nu z x, N_3^{2201} = M_3^{221} + \nu \frac{z^2}{2} - \nu \frac{x^2}{2}.$$

The problem (12.19) takes the form

$$\begin{cases} (a_{\xi\alpha\theta\beta} M_{\theta,\beta}^{22\nu})_{,\alpha} = 0 \quad \text{in} \quad P, \\ (a_{\xi\alpha\theta\beta} M_{\theta,\beta}^{22\nu} n_\alpha = 0 \quad \text{on} \quad \Gamma \cup H, \\ [M_1^{22\nu}]_x = -\nu z^\nu [x]_x, [M_3^{22\nu}]_x = 0. \end{cases} \tag{12.26}$$

The local stresses are computed with the formula

$$\sigma_{ij} = a_{ij21} M_{1,1}^{220} + a_{ij21}\nu + a_{ij23} M_{1,3}^{220} + a_{ij22} \tag{12.27}$$

for $\nu = 0$ - the tension along $0x$-axis; and with the formula

$$\sigma_{ij} = a_{ij21} M_{1,1}^{221} + a_{ij21}\nu z + a_{ij23} M_{1,3}^{221} + a_{ij23}\nu z + a_{ij22} z \tag{12.28}$$

for $\nu = 1$ - the bending in $0xz$-plane. The homogenized in-plane stiffnesses are computed with the formula

$$S_{ij22}^0 = < a_{ij21} M_{1,1}^{220} + a_{ij21}\nu + a_{ij23} M_{1,3}^{220} + a_{ij22} > \tag{12.29}$$

and the homogenized bending/torsion stiffnesses are computed with the formula

$$S_{ij22}^2 = < (a_{ij21} M_{1,1}^{221} + a_{ij21}\nu z + a_{ij23} M_{1,3}^{221} + a_{ij22}) z > .$$

### *12.4.2 Index* $AB = 11$. *Tension-compression and Bending Perpendicular to the Fibers (in the* $0yz$*-plane)*

In this case, Eq. (12.20) takes the form $a_{\xi\alpha\theta\beta} e_{\theta\beta} = a_{\xi\xi\alpha 11} z^\nu$ or, in the coordinate-wise form

$$\begin{aligned} &a_{1111} e_{11} + a_{1133} e_{33} = -a_{1111} z^\mu, \\ &a_{3311} e_{11} + a_{3333} e_{33} = -a_{3311} z^\mu, \\ &a_{1313} e_{13} = -a_{1311} z^\mu = 0, a_{3131} e_{31} = -a_{3111} z^\mu = 0. \end{aligned} \tag{12.30}$$

Writing in (12.30) the elastic tensor components in the terms of Young's modulus and Poisson ratio, see (12.4), we obtain from the first two equations in (12.30) the following system of equations:

$$\begin{aligned}(1-\nu)e_{11}+\nu e_{33} &= -(1-\nu)z^{\mu},\\ \nu e_{11}+(1-\nu)e_{33} &= -\nu z^{\mu}.\end{aligned}$$

The solution to this system is $e_{11} = -z\nu, e_{33} = 0$. Taking into account that $e_{13} = e_{31} = 0$, we arrive at

$$\frac{\partial v_1}{\partial x} = -z^{\mu}, \qquad \frac{\partial v_3}{\partial z} = 0, \qquad \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} = 0. \tag{12.31}$$

The problem (12.31) may be solved in the explicit form. For $\mu = 0$, from the first two equations in (12.22), we have $v_1 = -x + f(z)$ and $v_3 = g(x)$. Substituting into the third equation in (12.22), we arrive at $f'(z) + g'(x) = 0$, then $f(z) = 0$ and $g(x) = 0$. For $\mu = 1$, we have $v_1 = -zx + f(z)$ and $v_3 = g(x)$. Substituting into the third equation in (12.22), we arrive at $-x + f'(z) + g'(x) = 0$, and obtain $f'(z) = 0, g'(x) = \nu x$. Then $f(z) = 0$ and $g(x) = \dfrac{x^2}{2}$. Finally,

$$v_1^{11} = z^{\mu}x, \qquad v_3^{11} = \mu z^{\mu-1}\frac{x^2}{2} = \frac{x^2}{2}\begin{cases}0 & \text{if} \quad \mu = 0,\\ 1 & \text{if} \quad \mu = 1.\end{cases} \tag{12.32}$$

The third condition for $(M_1^{11\nu}, M_3^{11\nu})$ in (12.19) takes the form: $(M_1^{11\nu} - v_1^{11}, M_3^{11\nu} - v_2^{11},)$ periodic in $x$. With regard to (12.32), it can be written as $[M_1^{11\nu}]_x = -\nu z^{\mu}[x]_x$, $[M_1^{11\nu}]_x = 0$. Then (12.19) takes the form

$$\begin{cases}(a_{\xi\alpha\theta\beta}M^{11\nu}_{\theta,\beta})_{,\alpha} = 0 \quad \text{in} \quad P,\\ (a_{\xi\alpha\theta\beta}M^{11\nu}_{\theta,\beta}n_{\alpha} = 0 \quad \text{on} \quad \Gamma \cup H,\\ [M_1^{11\nu}]_x = -\nu z^{\nu}[x]_x, [M_3^{11\nu}]_x = 0.\end{cases} \tag{12.33}$$

The boundary displacements in (12.33) are similar to one displayed in Fig. 12.1d.

### 12.4.3 *Index* $AB = 12, 21$. *Shift/Torsion Perpendicular to the Fibers (in the* $0yz$*-plane)*

For $AB = 12$, Eq. (12.20) takes the form $a_{\xi\alpha\theta\beta}e_{\theta\beta} = a_{\xi\alpha 12}z^{\nu} = 0, \xi, \alpha = 1, 3$. Its solution is $e_{\theta\beta} = 0$. Then $v_1^{12} = v_3^{12} = 0$ and solution to (12.19) is $(M_1^{12\nu}, M_3^{12\nu}) = 0$. The non-trivial $M_2^{21\nu} \neq 0$ was discussed in Sect. 12.2.

## 12.5 Numerical Solutions

We present numerical solutions to several PCPs. In our computations Young's modulus $E_b = 2$ GPa and Poisson's ratio $\nu_b = 0.36$. The periodicity cell dimensions are

$h_1 = 1.1, h_2 = 2, h_3 = 1.1$, $h = 0.1$ and $2H = 0.1$. The radius of the fiber is 0.45. These values are indicated in the non-dimensional "fast" variables $\mathbf{y}$. The corresponding actual dimensional values are computed by multiplying by the characteristic size $\varepsilon$. The programs we developed by using the APDL programming language of the ANSYS FEM software (Thompson and Thompson, 2017). The finite elements PLANE183 are used for the fibers and the matrix, the characteristic size of the finite elements is 0.03. The total number of finite elements is about 11000.

### *12.5.1 The Boundary Layers*

The deformed PC and the local von Mises stress are displayed in Fig. 12.2. Figure 12.2a corresponds to the tension in $0x$-direction and Fig. 12.2b corresponds to the bending. The boundary layers at the top and the bottom surfaces of the PC are seen. The boundary layer thickness is less the thickness of one structural layer $2R + h$ (diameter of hole + surrounding material). In the core of plate the solution is periodic in the in-plane tension/shift modes. It the bending/torsion mode, the solution in the core of plate coincides with solution in the plate of “infinite” thickness Grigolyuk et al (1991).

If the plate is thick, these boundary layers do not influence the effective stiffness of the plate. But the boundary layers do influence the local SSS in the plate of any thickness. In particular, the boundary layers influence the strength of the plate. In the tension mode, the maximum von Mises stress $\sigma_{\mathrm{vM}} = 0.196\,10^9$ in the core of the plate occurs between the holes, see Fig. 12.2a. The maximum von Mises stress in the boundary layer is $\sigma_{\mathrm{vM}} = 0.252\,10^9$. The ratio of the maximuma is 1.29.

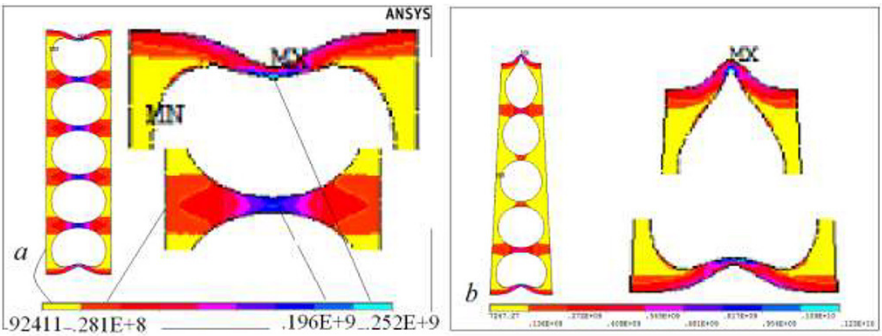


**Fig. 12.2** 5-hole PC and the top and the bottom surfaces of the PC (zoomed): a - tension and b - bending modes.

### 12.5.2 Wrinkling of the Top and Bottom Surfaces of the Plate

Figure 12.2 displays the top and bottom surfaces of the plate with channel cuts subjected to the overall tension - Fig. 12.2a, the overall bending - Fig. 12.2b. It is seen that the top and bottom surfaces are not flat in the case of the tension and not cylindrical in the case of bending. They are wavy. This is the wrinkling effect. The amplitude and period of the wrinkling are small (have the order of the PC dimension) but the corresponding change of the total length of the surfaces is not small. The wrinkling never occurs in the homogeneous or in the laminated plates. For the homogeneous or laminated plates, top and bottom surfaces are flat in the case of the tension and cylindrical in the case of bending.

## 12.6 The Macroscopic SSS of General Form

Solutions to a partial PCP corresponds to the basis macroscopic SSS: $e^{\nu}_{AB} = \delta_{AB}$, where $\delta_{AB}$ is Kronecker delta. For plate, we distinguish six basic macroscopic SSSs: two in-plane tensions and shift $e^{0}_{AB}$, and two bending and torsion $\rho_{AB}$. In accordance with the homogenization theory (Caillerie, 1984; Kohn and Vogelius, 1984), the local strains are computed as

$$e_{kl} = [\delta^{A}_{k}\delta^{B}_{l} + N^{AB0}_{k,l}(x,z)]e^{0}_{AB} + [-\delta^{A}_{k}\delta^{B}_{l}z + N^{AB1}_{k,l}(x,z)]\rho_{AB},$$

and the local stresses are computed as

$$\sigma_{ij} = [a_{ijAB}(x,z) + a_{ijAB}(x,z)N^{AB0}_{k,l}]e^{0}_{AB} + [-a_{ijAB}(x,z)z + a_{ijAB}(x,z)N^{AB1}_{k,l}]\rho_{AB}.$$

These formulas may be used for prospective analysis of the behavior of plates of unidirectional structures subjected to the macroscopic SSS $\{e^{0}_{AB}, e^{1}_{AB}\}$ of general form, for example, the investigation of the strength of such kind plates.

## 12.7 Conlusions

The original 3D PCP (12.1) is reduced to several 2D boundary-value problems. The boundary-value problems for Laplace (12.9) and Poisson (12.17) equations correspond to the anti-plane elasticity problems. The boundary-value problems (12.26) and (12.33) are the planar elasticity problems.

The obtained 2D problems may be analyzed numerically with high accuracy. Our numerical solutions demonstrate the existence of boundary layers near the top and the bottom surfaces of PC. The boundary layer thickness is less the thickness of one structural. The wrinkling effect takes place for the plates with a system of tunnel cuts.

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