Chapter 10
Analytical Approach to the Derivation of the Stress Field of a Cylindrical Shell with a Circular Hole under Axial Tension

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Abstract A new analytical approach to the stress field problem of the cylindrical shell with a circular cutout under axial tension is proposed. Classical models because of an expansion into small parameter have narrow range of applicability and almost do not differ from Kirsch case for plate. The new approach opens up opportunities for the analytical study of the stress field. The idea is to decompose each basis function into a Fourier series by separating the variables, which allows us to obtain an infinite system of algebraic equations for finding coefficients. One of the important steps of the study is that the authors were able to prove which of the equations of the system is a linear combination of several others. Excluding it made it possible to create a reduced system for finding unknown coefficients. The proposed approach does not impose mathematical restrictions on the values of the main parameter that characterizes the cylindrical shell.

Key words: Cylindrical Shell, Circular cutout, Elasticity theory

10.1 Introduction

In this paper[1], we propose a new analytical approach to the derivation of the stresses of a cylindrical shell with a circular hole under tension along forming axis. The state-

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[1] Dedicated to my teacher Prof. Peter Tovstik. Our last call was 13th of November 2020 when we discussed this article. Thank you that always believed in me, R.I.P.

H. Altenbach et al. (eds.), *Recent Approaches in the Theory of Plates and Plate-Like Structures*, Advanced Structured Materials 151,
https://doi.org/10.1007/978-3-030-87185-7_10

ment of the problem and the solution by the method of decomposition in the small parameter belongs to Lurie (1946). This parameter β characterizes the ratio of the radius of the hole, the thickness of the shell and the radius of curvature. Later, in the mid-1960s-70s, a surge of interest in this problem occurred not only among Soviet researchers, but also among foreign scientists who found an error in setting the boundary conditions at the boundary of a circular hole. Some of them reconsidered it by the same method (Houghton, 1961; Naghdi and Eringen, 1965; Pirogov and Iumatov, 1968; Murthy, 1969), others by numerical method of collocation (Eringen et al, 1965; Lekkerkerker, 1965; van Dyke, 1965). However, the proposed analytical approach was extremely cumbersome and worked for a very small range of values of the parameter β, which differed a little from the plane Kirsch problem, and the results obtained by the collocation method differed (Kashtanova et al, 2021). There were also attempts to solve this problem using the energy method (Pirogov and Iumatov, 1968; Adams, 1971) and the method of complex variables for arbitrary holes (Chekhov and Zakora, 1972; Hu et al, 1998). The resources of the considered methods have exhausted themselves without providing a convenient solution but no alternative methods have yet been proposed. Follow-up works rely on computer modelling, in particular, based on the finite element method (Yu et al, 2015; Chowdhury et al, 2016; Celebi et al, 2017; Storozhuk et al, 2018; Russo et al, 2019).

However, until now, the relevance and applicability of this problem remain high (Wu and Mu, 2003; Oterkus et al, 2007; Zhuang et al, 2015; Ray-Chaudhuri and Chawla, 2018), especially in the field of the aviation industry. And the analytical solution for the stress field in the hole area can give an impetus to the fundamental study of the issues of fracture and stability. This paper presents a new idea that makes it easy to find numerical values of stresses and opens up prospects for their analytical study. In this way, there are no mathematical restrictions on the values of the parameter β as it was before. In this paper, special attention is paid to the technique of solving the problem and a strict mathematical formulation.

10.2 Problem Formulation

We consider a cylindrical shell with a circular hole under tension p applied at infinity along forming axis x. The following symbols are used: parameter

$$\beta^2 = r_0^2 \frac{\sqrt{3(1-\nu^2)}}{4Rh},$$

where r_0 – the radius of the hole (without belittling the generality, we can take r_0 as a unit of measurement, i.e. $r_0 = 1$), R, h – the radius of curvature and the thickness of the shell, respectively, ν – Poisson's ratio. Parameter β is the main parameter responsible for the ratio of geometric parameters, including the curvature of the shell.

Note that the limiting case for $\beta \to 0$ leads us to the Kirsch problem. As it offered in Lurie (1946) we also introduce the function

$$\Phi = \frac{Eh}{8\beta^2 R} w - iU,$$

which depends on the deflection w and the stress function U. The relationship between the effort T and the function U is given as follows

$$\begin{pmatrix} T_x & T_{xy} \\ T_{xy} & T_y \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 U}{\partial y^2} & -\dfrac{\partial^2 U}{\partial x \partial y} \\ -\dfrac{\partial^2 U}{\partial x \partial y} & \dfrac{\partial^2 U}{\partial y^2}. \end{pmatrix}$$

The stress of the median surface of the thin shell is $\sigma = T/h$.

It is shown in Lurie (1946), Guz (1974), that the system of shell equilibrium equations reduces to the following equation

$$\Delta\Delta\Phi + 8i\beta^2 \frac{\partial^2 \Phi}{\partial x^2} = 0. \tag{10.1}$$

The full problem statement is to find a function that satisfies equation (10.1) and next boundary conditions

- at infinity

$$T_x = p, \quad T_{xy} = 0, \quad T_y = 0, \quad w = 0; \tag{10.2}$$

- at the boundary of a circular hole in the polar coordinate system (r, ϑ)

$$\begin{cases} T_{rr}|_{r=r_0} &= 0, \\ T_{r\vartheta}|_{r=r_0} &= 0, \\ M_r|_{r=r_0} &= 0, \\ Q_r|_{r=r_0} &= 0. \end{cases} \tag{10.3}$$

Here M_r is the moment, Q_r is the generalized boundary condition on a free edge (Lurie, 1946).

10.3 Solution

Despite the fact that the method for solving equation (10.1) is well known (Lurie, 1946; Naghdi and Eringen, 1965; Pirogov and Iumatov, 1968; Murthy, 1969; Guz, 1974; Kashtanova et al, 2021), some technical details were not given due attention. Consider two commuting linear operators

$$L_1 = \left(\Delta - 2i\alpha \frac{\partial}{\partial x}\right) \quad \text{and} \quad L_2 = \left(\Delta + 2i\alpha \frac{\partial}{\partial x}\right),$$

where $\alpha = (1+i)\beta$. Then Eq. (10.1) can be written as

$$L_1 L_2 \Phi = 0 \Leftrightarrow \Phi \in \mathrm{Ker} L_1 L_2.$$

That is, the problem is reduced to finding the kernel of the product $L_1 L_2$. From the fact that the operators commute, it follows that

$$\mathrm{Ker} L_1 + \mathrm{Ker} L_2 \subset \mathrm{Ker} L_1 L_2.$$

Finding the solutions of the equations $L_1 \Phi = 0$ and $L_2 \Phi = 0$ separately with the subsequent possibility of finding their sum greatly simplifies the solution of the original equation, since lowers its order. However, it is important to note that the sum of kernels $\mathrm{Ker} L_1 + \mathrm{Ker} L_2$ does not coincide with the set of all solutions of equation (10.1), which can lead to the loss of solutions. Therefore, this method can be used to prove the existence of a solution and find it constructively, but the study of uniqueness should be carried out separately.

By replacing and separating variables, it is easy to establish (Lurie, 1946; Guz, 1974) that the solutions of $L_{1,2} \Phi = 0$ are functions $\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta}$, where $n \in \mathbb{Z}_+$. The choice of Hankel functions in the construction of the solution is due to the fact that these are the only Bessel functions that tend to zero at an infinitely distant complex point (Watson, 1945):

$$\lim_{\rho \to +\infty} H_n^{(1)}(\rho e^{i\varphi}) = \lim_{\rho \to +\infty} H_n^{(2)}(\rho e^{-i\varphi}) = 0, \qquad \varphi \in [\varepsilon; \pi - \varepsilon].$$

Since $\alpha = (1+i)\beta$ has the argument $\pi/4 \in [\varepsilon; \pi-\varepsilon]$, functions $\mathrm{e}^{\pm i\alpha x} H_n^{(2)}(\alpha r) \mathrm{e}^{\pm i n \vartheta}$ obviously do not satisfy the boundary conditions, since the deflection $w \neq 0$ at infinity. At the same time, guided by Watson (1945), we can deduce that

$$|\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta}| \leq \frac{\tilde{C}}{\sqrt{r}} \mathrm{e}^{-\beta r (1 - |\cos\vartheta|)}.$$

Note that the first three boundary conditions of system (10.2) are set not with respect to the function U, but with respect to its second derivatives. Therefore, it is necessary to make sure that not only the potential, but also the stresses tend to zero at large r. This is true, since it follows from the recurrence relations for the Bessel functions (Watson, 1945) that

$$\frac{\partial}{\partial r} \left(\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta} \right)$$

is a linear combination of functions of the same type $\mathrm{e}^{\pm i\alpha x} H_m^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta}$. Moreover, calculations show that

$$\frac{1}{r} \frac{\partial}{\partial \vartheta} \left(\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta} \right)$$

is a linear combination of functions

$$\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta} \qquad \text{and} \qquad \frac{\mathrm{e}^{\pm i\alpha x} H_n^{(1)}(\alpha r) \mathrm{e}^{\pm i n \vartheta}}{r}.$$

Thus, the solution $e^{\pm i\alpha x}H_n^{(1)}(\alpha r)e^{\pm in\vartheta}$ satisfies all boundary conditions at infinity.

Using trigonometric form for $e^{\pm i\alpha x}H_n^{(1)}(\alpha r)e^{\pm in\vartheta}$ and taking into account the circular hole symmetry we get that the solution of the problem (10.1)-(10.3) is possible to find in following form for even and odd n (Lurie, 1946; Chowdhury et al, 2016):

$$\Phi = -i\frac{py^2}{2} + \sum_{n=0}^{\infty}(A_n + iB_n)\begin{bmatrix}\cos(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos(n\vartheta)\\ \sin(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos(n\vartheta)\end{bmatrix}. \quad (10.4)$$

The function Φ is the solution of the equation of mathematical physics (10.1), while Φ satisfies the boundary conditions (10.2). It remains only to find the coefficients A_n and B_n from the boundary conditions (10.3). Namely at this step the authors of previous works faced the greatest difficulties (for more information, see Kashtanova et al, 2021). Therefore, the main content part of the present paper is the method of searching for unknown coefficients.

10.4 New Approach

The main idea is to separate the variables r and ϑ in each basic function. Only in contrast to Lurie (1946), to achieve this goal, an expansion in the trigonometric Fourier series is proposed. The known Laurent series expansion $e^{\left(\frac{z}{2}(t-\frac{1}{t})\right)}$ (Watson, 1945) of

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} t^m J_m(z)$$

for $e^{i\alpha x}e^{in\vartheta}$ leads us to

$$\begin{aligned} e^{i\alpha x}e^{in\vartheta} &= e^{\frac{\beta(1+i)r}{2}\left(2i\frac{e^{i\vartheta}+e^{-i\vartheta}}{2}\right)}e^{in\vartheta} = e^{\frac{\beta(1+i)r}{2}\left(ie^{i\vartheta}-\frac{1}{ie^{i\vartheta}}\right)}e^{in\vartheta} \\ &= \left(\sum_{m=-\infty}^{\infty}\left(ie^{i\vartheta}\right)^m J_m((1+i)\beta r)\right)e^{in\vartheta} \\ &= \sum_{m=-\infty}^{\infty} i^m e^{i(n+m)\vartheta} J_m((1+i)\beta r) \\ &= \sum_{m=-\infty}^{\infty} i^{k-n} J_{k-n}((1+i)\beta r)e^{ik\vartheta} = \sum_{m=-\infty}^{\infty}(-i)^{k-n} J_{n-k}((1+i)\beta r)e^{ik\vartheta} \end{aligned}$$

Replacing n with $\breve{n}$ results in

$$e^{i\alpha x}e^{-in\vartheta} = \sum_{m=-\infty}^{\infty} i^{k+n} J_{k+n}((1+i)\beta r)e^{ik\vartheta}$$

If we add both equalities obtained, we get

$$e^{i\alpha x}\left(e^{in\vartheta}+e^{-in\vartheta}\right)=\sum_{m=-\infty}^{\infty}\left[(-i)^{k-n}J_{n-k}((1+i)\beta r)+i^{k+n}J_{k+n}((1+i)\beta r)\right]e^{ik\vartheta}$$

Now we can replace α by $\breve{}\alpha$ in the last formula and add both equalities for even n, and for odd n subtract the other from one:

$$\begin{aligned}\cos\alpha x\cdot\cos n\vartheta &= \frac{1}{4}\left(e^{i\alpha x}+e^{-i\alpha x}\right)\left(e^{in\vartheta}+e^{-in\vartheta}\right)\\ &= (-1)^{n/2}J_n((1+i)\beta r)\\ &+\sum_{l=1}^{\infty}(-1)^{l+(n/2)}(J_{n-2l}((1+i)\beta r)+J_{n+2l}((1+i)\beta r))\cos 2l\vartheta\\ \sin\alpha x\cdot\cos n\vartheta &= \frac{1}{4i}\left(e^{i\alpha x}-e^{-i\alpha x}\right)\left(e^{in\vartheta}+e^{-in\vartheta}\right)\\ &= (-1)^{(n-1)/2}J_n((1+i)\beta r)\\ &+\sum_{l=1}^{\infty}(-1)^{l+((n-1)/2)}(J_{n-2l}((1+i)\beta r)+J_{n+2l}((1+i)\beta r))\cos 2l\vartheta\end{aligned}$$

As a result, even and odd basis functions can be written in one general formula

$$\begin{aligned}\begin{bmatrix}\cos(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos n\vartheta\\ \sin(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos n\vartheta\end{bmatrix} &= f_n(r,\vartheta)=(-1)^{\left[\frac{n}{2}\right]}\frac{H_n^{(1)}((1+i)\beta r)}{H_n^{(1)}((1+i)\beta)}[J_n((1+i)\beta r)\\ &+\sum_{l=1}^{\infty}(-1)^l(J_{(n+2l)}((1+i)\beta r)+J_{(n-2l)}((1+i)\beta r))\cos 2l\vartheta],\end{aligned} \tag{10.5}$$

where $\left[\frac{n}{2}\right]$ is an integer part of the number. In the denominator a normalizing factor $H_n^{(1)}((1+i)\beta)$ is introduced. The latter is done so that the numerical values of the unknown coefficients have moderate values, with which it is convenient to work.

Further, for convenience, we introduce the notation for the Fourier coefficients in the trigonometric expansion of the basis function $g(r,n,l)$:

$$g(r,n,l)=(-1)^{\left[\frac{n}{2}\right]+l}\frac{H_n^{(1)}((1+i)\beta r)}{H_n^{(1)}((1+i)\beta)}(J_{n+2l}((1+i)\beta r)+J_{n-2l}((1+i)\beta r))$$

with $n=0,1,\ldots,\infty, l=0,1,\ldots,\infty$. Then (10.5) takes the form

$$f_n(r,\vartheta)=\frac{g(r,n,0)}{2}+\sum_{l=1}^{\infty}g(r,n,l)\cos 2l\vartheta. \tag{10.6}$$

Now solution (10.4) can be written as

$$\Phi(r,\vartheta)=-i\frac{py^2}{2}+\sum_{n=0}^{\infty}(a_n+ib_n)f_n(r,\vartheta) \tag{10.7}$$

That is convenient for substitution into the boundary conditions (10.3).

10.5 Boundary Conditions

First boundary condition $\sigma_{rr} = 0$ in polar coordinates

$$\mathcal{L}_1(U) = \mathcal{L}_1(-\mathrm{Im}\Phi) = 0, \quad \mathcal{L}_1 = \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

leads us to the equation

$$\frac{p}{2} + \frac{p}{2}\cos 2\vartheta - \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)\left(\frac{g'(r,n,0)}{2} + \sum_{l=1}^{\infty}(-4l^2 g(r,n,l) + g'(r,n,l))\cos 2l\vartheta\right) = 0. \tag{10.8}$$

The cosine coefficients give us the following system:

$$\begin{cases} \cos 0: & \frac{p}{2} - \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)\cdot\frac{g'(r,n,0)}{2} = 0 \\ \cos 2\vartheta: & \frac{p}{2} + \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)\cdot(4g(r,n,1) - g'(r,n,1)) = 0 \\ \cos 4\vartheta: & \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)\cdot(16g(r,n,2) - g'(r,n,2)) = 0 \\ & \dots \\ \cos 2l\vartheta: & \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)\cdot\left(4l^2 g(r,n,l) - g'(r,n,l)\right) = 0 \end{cases} \tag{10.9}$$

Second boundary condition $\sigma_{r\vartheta} = 0$

$$\mathcal{L}_2(U) = \mathcal{L}_2(-\mathrm{Im}\Phi) = 0, \quad \mathcal{L}_2 = \frac{1}{r^2}\frac{\partial}{\partial\vartheta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\vartheta}$$

give us

$$-\frac{p}{2}\sin 2\vartheta - \mathrm{Im}\sum_{n=0}^{\infty}(a_n + ib_n)2\sum_{l=1}^{\infty} l\cdot(g'(r,n,l) - g(r,n,l))\sin 2l\vartheta = 0. \tag{10.10}$$

The sine coefficients are:

$$\begin{cases} \sin 2\vartheta: & -\frac{p}{2} - \operatorname{Im}\sum_{n=0}^{\infty} 2(a_n+ib_n)\cdot(g'(r,n,1)-g(r,n,1)) = 0 \\ \sin 4\vartheta: & -\operatorname{Im}\sum_{n=0}^{\infty} 4(a_n+ib_n)\cdot(g'(r,n,2)-g(r,n,2)) = 0 \\ & \dots \\ \sin 2l\vartheta: & -\operatorname{Im}\sum_{n=0}^{\infty} 2l(a_n+ib_n)\cdot(g'(r,n,l)-g(r,n,l)) = 0 \end{cases} \tag{10.11}$$

From the third boundary condition $M_{rr} = 0$

$$\mathcal{L}_3(\operatorname{Re}\Phi) = 0, \quad \mathcal{L}_3 = -D\left(\frac{\partial^2}{\partial r^2} + \frac{\nu}{r}\frac{\partial}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right), \quad D = \frac{Eh^3}{12(1-\nu^2)},$$

where E – Young modulus and ν - Poisson ratio, we get

$$\operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)$$
$$\left(\frac{g''(r,n,0)}{2} + \nu\frac{g'(r,n,0)}{2} + \sum_{l=1}^{\infty}[g''(r,n,l)+\nu(g'(r,n,l)-4l^2 g(r,n,l))]\cos 2l\vartheta\right) = 0. \tag{10.12}$$

The cosine coefficients:

$$\begin{cases} \cos 0: & \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot\frac{\nu g'(r,n,0)+g''(r,n,0)}{2} = 0 \\ \cos 2\vartheta: & \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot(-4\nu g(r,n,1)+\nu g'(r,n,1)+g''(r,n,1)) = 0 \\ \cos 4\vartheta: & \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot(-16\nu g(r,n,2)+\nu g'(r,n,2)+g''(r,n,2)) = 0 \\ & \dots \\ \cos 2l\vartheta: & \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot\left(-4l^2\nu g(r,n,l)+\nu g'(r,n,l)+g''(r,n,l)\right) = 0 \end{cases} \tag{10.13}$$

From the fourth boundary condition $Q_r^* = 0$

$$\mathcal{L}_4(\operatorname{Re}\Phi) = 0, \quad \mathcal{L}_4 = -D\left(\frac{\partial}{\partial r}\Delta + (1-\nu)\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial \vartheta^2}\right), \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}$$

we find

$$\operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\left[\frac{-g'(r,n,0)+g''(r,n,0)+g'''(r,n,0)}{2}\right.$$
$$+\sum_{l=1}^{\infty}\left[4l^2(3-\nu)g(r,n,l)-(1+4l^2(2-\nu))g'(r,n,l)\right.$$
$$\left.+g''(r,n,l)+g'''(r,n,l)\right]\cos 2l\vartheta\Bigg]=0. \tag{10.14}$$

The cosine coefficients:

$$\begin{cases}
\cos 0: \quad \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot\left(\frac{-g'(r,n,0)+g''(r,n,0)+g'''(r,n,0)}{2}\right)=0\\
\cos 2\vartheta: \quad \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot(4(3-\nu)g(r,n,1)\\
\qquad -(9-4\nu)g'(r,n,1)+g''(r,n,1)+g'''(r,n,1))=0\\
\cos 4\vartheta: \quad \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot(16(3-\nu)g(r,n,2)\\
\qquad -(33-16\nu)g(r,n,2)+g''(r,n,2)+g'''(r,n,2))=0\\
\qquad \ldots\\
\cos 2l\vartheta: \operatorname{Re}\sum_{n=0}^{\infty}(a_n+ib_n)\cdot(4l^2(3-\nu)g(r,n,l)\\
\qquad -(1+4l^2(2-\nu))g'(r,n,l)+g''(r,n,l)+g'''(r,n,l))=0
\end{cases} \tag{10.15}$$

10.6 System Investigation

In the second pair of systems (10.13)–(10.15), the expressions under the sum sign can be multiplied by i. Then all the equations of the four systems will include only the imaginary part of the sum. From all the systems obtained, we compose a general linear system with an infinite number of unknowns and equations. Firstly, let us do some elementary transformations:

1. for $l > 1$, the equations of systems (10.9) and (10.11) can be written as follows:

$$\begin{cases}
4l^2\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,l)+(-1)\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,l)=0\\
2l\cdot\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,l)-2l\cdot\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,l)=0
\end{cases}$$

The result is a homogeneous system with respect to unknowns $\sum_{n=0}^{\infty}(a_n+ib_n)g(n,l)$ and $\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,l)$ with a determinant different from zero

$$\begin{vmatrix} 4l^2 & -1 \\ 2l & -2l \end{vmatrix} \neq 0,$$

and consequently, it has only a trivial solution

$$\begin{cases} \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,l)=0 \\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,l)=0 \end{cases} \tag{10.16}$$

2. for $l=1$ the same equations give an inhomogeneous system

$$\begin{cases} 4\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,1)+(-1)\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,1)=-\frac{p}{2} \\ 2\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,1)-2\operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,1)=\frac{p}{2} \end{cases}$$

Solving it with respect to unknowns, we obtain

$$\begin{cases} \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g(n,1)=-\frac{p}{4} \\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)g'(n,1)=-\frac{p}{2} \end{cases} \tag{10.17}$$

3. Let introduce the notation

$$\begin{aligned} t_3(n,l) &= i(-4l^2\nu g(n,l)+\nu g'(n,l)+g''(n,l)), \\ t_4(n,l) &= i(12l^2 g(n,l)-(1+\nu+4l^2(2-\nu))g'(n,l)+g'''(n,l)). \end{aligned}$$

In order to get rid of $g''(n,l)$ in the expression $t_4(n,l)$, we can subtract from the last equation of system (10.15) the last equation (10.13):

$$\begin{cases} \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)t_3(n,l)=0 \\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n+ib_n)t_4(n,l)=0 \end{cases} \tag{10.18}$$

It is important to note that the first equation of system (10.9), as proved by the authors, is a consequence of four equations: two equations of system (10.17) and two equations of system (10.18) at $l = 0$. The proof is based on the properties of the Bessel functions (Watson, 1945) and the idea is presented in the Appendix. The fact of linear dependence of the equations, but the lack of understanding of which ones perplexed the author of a previous work (Murthy, 1969; Pirogov and Iumatov, 1968; Naghdi and Eringen, 1965).

All equations of systems (10.16) - (10.18) can be written in matrix form (see Table 10.6). Thus, a linear system with an infinite number of equations and unknowns is obtained. In this case, the elements of the infinite matrix of the system, which differ significantly from zero, are located near the main diagonal. This is due to the values $J_k((1+i)\beta)$, on which all elements of the matrix of the system depend, become very small with increasing k, namely, next to the main diagonal there are elements whose index $k = n - 2l$ is close to zero. The submatrix composed of the first 4N rows and columns will have a nonzero determinant. The solving the system allows to find uniquely the coefficients for the first $2N$ basis functions. At the same time, as calculations show, with an increase in N, the first found coefficients practically do not change, and the coefficients at basis functions with large indexes tend to zero. This method has no mathematical restrictions on the values of the main parameter β. From the point of view of mechanics, this model is applicable for the range $0 \le \beta \le 3,5 - 4,5$ (Guz, 1974).

10.7 Results

The found coefficients a_n and b_n can be substituted into (10.4). Herewith, any finite partial sum

Table 10.1 System (10.16) - (10.18) in matrix form.

n	0		1		2		3			unknown	free terms
	Im	Re	Im	Re	Im	Re	Im	Re			
l											
0	$t_3(0,0)$	$t_3(0,0)$	$t_3(1,0)$	$t_3(1,0)$	$t_3(2,0)$	$t_3(2,0)$	$t_3(3,0)$	$t_3(3,0)$		a_0	0
0	$t_4(0,0)$	$t_4(0,0)$	$t_4(1,0)$	$t_4(1,0)$	$t_4(2,0)$	$t_4(2,0)$	$t_4(3,0)$	$t_4(3,0)$		b_0	0
1	$g(0,1)$	$g(0,1)$	$g(1,1)$	$g(1,1)$	$g(2,1)$	$g(2,1)$	$g(3,1)$	$g(3,1)$		a_1	$-\frac{p}{4}$
1	$g'(0,1)$	$g'(0,1)$	$g'(1,1)$	$g'(1,1)$	$g'(2,1)$	$g'(2,1)$	$g'(3,1)$	$g'(3,1)$	$\vdots$	b_1	$-\frac{p}{2}$
1	$t_3(0,1)$	$t_3(0,1)$	$t_3(1,1)$	$t_3(1,1)$	$t_3(2,1)$	$t_3(2,1)$	$t_3(3,1)$	$t_3(3,1)$		a_2	0
1	$t_4(0,1)$	$t_4(0,1)$	$t_4(1,1)$	$t_4(1,1)$	$t_4(2,1)$	$t_4(2,1)$	$t_4(3,1)$	$t_4(3,1)$		b_2	0
2	$g(0,2)$	$g(0,2)$	$g(1,2)$	$g(1,2)$	$g(2,2)$	$g(2,2)$	$g(3,2)$	$g(3,2)$		a_3	0
2	$g'(0,2)$	$g'(0,2)$	$g'(1,2)$	$g'(1,2)$	$g'(2,2)$	$g'(2,2)$	$g'(3,2)$	$g'(3,2)$		b_3	0

...

$$\Phi = -i\frac{py^2}{2} + \sum_{n=0}^{\infty} \frac{a_n + ib_n}{H_n^{(1)}[(1+i)\beta]} \begin{bmatrix} \cos(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos(n\vartheta) \\ \sin(\alpha x)\cdot H_n^{(1)}(\alpha r)\cdot\cos(n\vartheta) \end{bmatrix}.$$

is an exact solution of the mathematical physics equation in the domain (in contrast, for example, from the solution of this problem by the Ritz method) and satisfies the boundary conditions at infinity. As calculations show, the boundary conditions on the hole boundary are satisfied quite accurately for any $\beta \in (0;4]$ for 18 basis functions, the coefficients of which are found from the reduced system. E.g., the maximum deviation of the boundary conditions from zero for $\beta = 0.212$ is no more than 10^{-14}, and for $\beta = 4$ no more than $6\cdot 10^{-3}$. With increasing β, the maximum deviation increases: for greater accuracy, you can take 24 basis functions for large values of β, and then the deviation will be no more than $5\cdot 10^{-6}$. As β increases, the number of basis functions that significantly affect the response increases, i.e., the basis coefficients increase for large n. The results shown in the graph (Fig. 10.1a) completely coincide with the results obtained in van Dyke (1965) by the collocation method. In the works of different authors were different the results, and it remained unclear what results to rely on. Now it has been possible to find an analytical method that is easy to implement and gives reliable results and the possibility of further investigation of stresses.

Acknowledgements The reported study was funded by RFBR, project number 19-31-60008.

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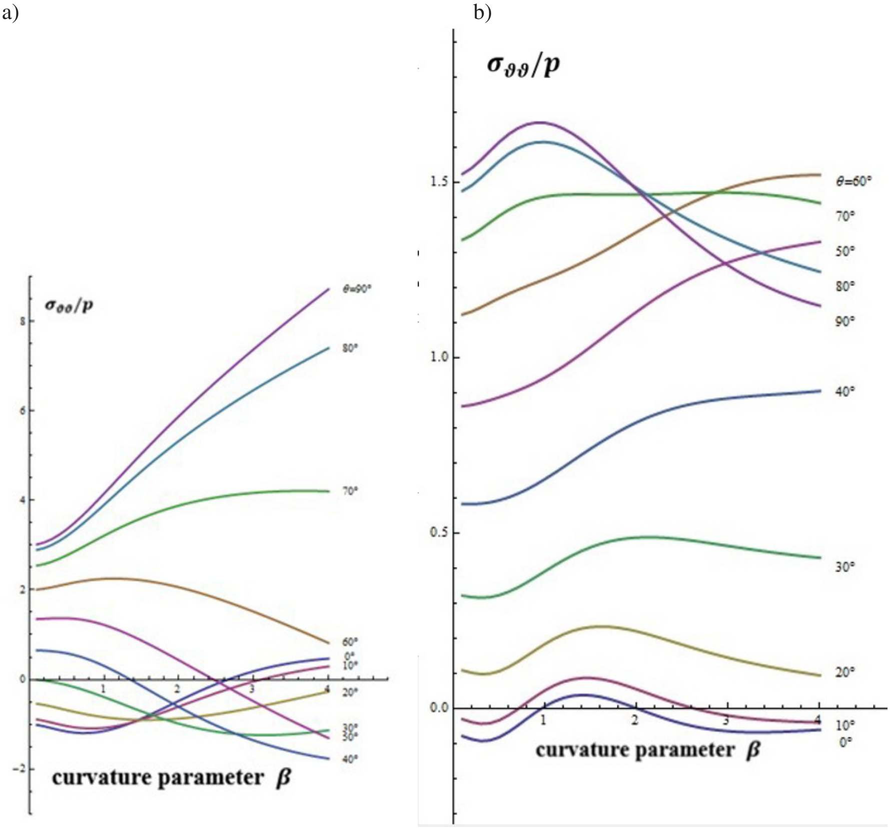

Fig. 10.1 Stress $\sigma_{\vartheta\vartheta}/p$ for all range of values $\beta \in (0;4]$ for $r = 1$ (left) – on the boarder and $r = 1.5$ (right).

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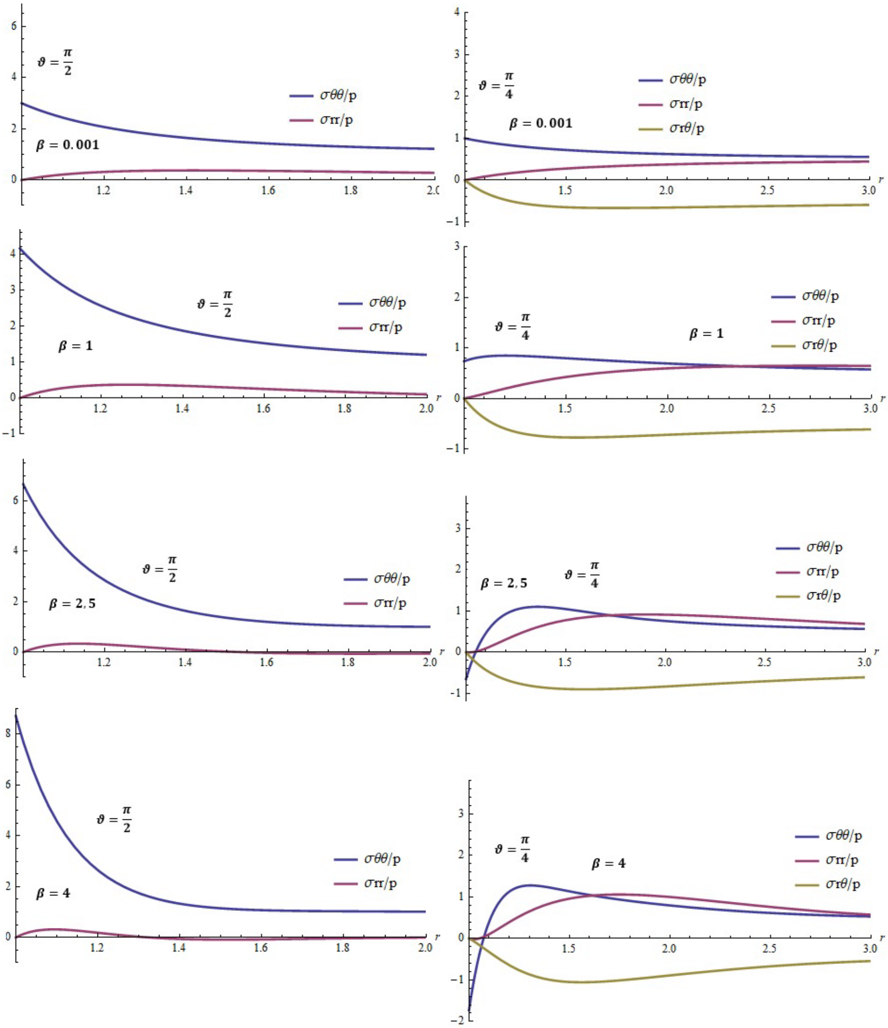

Fig. 10.2 Stresses for $\beta = 0,001; 1; 2,5; 4$ depending on r.

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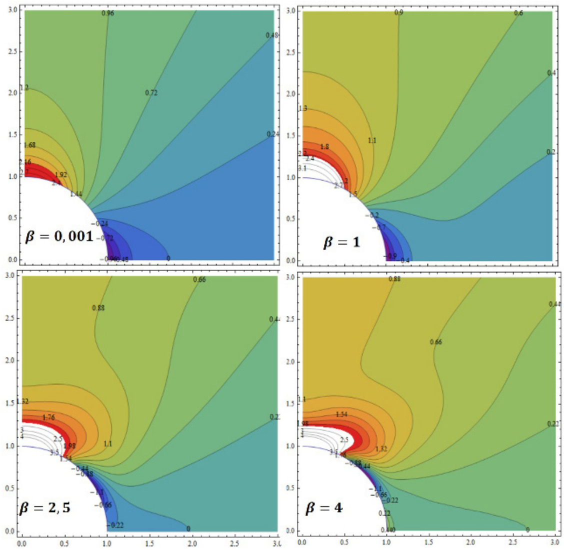

Fig. 10.3 Stress field for $\sigma_{\vartheta\vartheta}/p$.

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Zhuang L, Su B, Lin M, Liao Y, Peng Y, Zhou Y, Luo D (2015) Influence of the property of hole on stress concentration factor for isotropic plates. In: Araújo AL, Correia JR, Mota Soares CM (eds) 10th International Conference on Composite Science and Technology, Lisboa, pp 1–5

Appendix

Statement: the first equation in (10.9) is a consequence of four equations: two equations of system (10.17) and two equations of the system (10.18) for $l = 0$.

Proof. The following notation is introduced:

$$\text{I:}\quad (10.9)\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n + ib_n)g'(n,0) = p,$$

$$\text{II:}\quad (10.18)_1, l = 0\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n + ib_n)t_3(n,0) = 0,$$

$$\text{III:}\quad (10.18)_2, l = 0\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n + ib_n)t_4(n,0) = 0,$$

$$\text{IV:}\quad (10.17)_1\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n + ib_n)g(n,1) = -p/4,$$

$$\text{V:}\quad (10.17)_2\ \operatorname{Im}\sum_{n=0}^{\infty}(a_n + ib_n)g'(n,1) = -p/2.$$

The following fact is asserted:

$$4\beta^2[I+V+2IV] = II+III.$$

Equality for the right-hand sides is obvious. For the left-hand sides, we need to prove that $\forall n \in \mathbb{Z}_+$:

$$4\beta^2[g'(n,0)+g'(n,1)+2g(n,1)] = i(g''(n,0)+g'''(n,0)-g'(n,0)). \qquad (10.19)$$

Consider the linear differential operator

$$\mathcal{L}y = y'' + \frac{y'}{z} + \left(1-\frac{n^2}{z^2}\right)y.$$

Lemma 10.1. *Let be, $v \in \mathrm{Ker}\mathcal{L}$, i.e. u, v – Bessel function of index n, $G(z) = u(z)v(z)$. So*

$$G'''(z) + \frac{3}{z}G''(z) + \left(4-\frac{4n^2-1}{z^2}\right)G'(z) + \frac{4}{z}G(z) = 0.$$

The proof of the lemma is derived from the relations for the Bessel functions. If we apply the assertion of the lemma to

$$u(z) = (-1)^{\left[\frac{n}{2}\right]}\frac{2}{H_n^{(1)}((1+i)\beta)}H_n^{(1)}(z), \quad v(z) = J_n(z), \quad G(z) = u(z)v(z),$$

then we prove the Eq. (10.19).