Redundancy and Repair Properties

5

Abstract

The term "redundancy" refers to the way a system can work even when some components have failed. All the coherent systems except the series systems have redundancy mechanisms in their structure functions. Moreover, sometimes, we may try to improve the reliability of a given system by adding some redundant components at different critical points. Other popular redundancy options are to add standby components in the system to replace the failed components or to repair these failed components. The main questions analyzed in this chapter are: What is the reliability of the (new) redundant system? What are the best points in the structure to add the redundant components? Which one is the best redundancy option? We also study some component importance indices that can be used to determine the best replacement options.

5.1 Redundancy Options

There are several redundancy options. Not all of them are available in practice for all the systems. Thus, we cannot use the same options for a plane or a rocket, that the ones used for ships or cars. For example, in the first cases we cannot wait for the system failure to apply the redundancy options (repairs).

In this introductory book we just analyze the most popular ones. There are two main options called "hot" and "cold" redundancies.

In the first case (**hot redundancy**), one "spare" is added to a component in the system with a given structure (which improves the behavior at this point). Both units work at the same time. The same can be done in other components as well. The most popular option is to add a new (similar) independent unit in parallel to a given component. In this case, the life length of the resulting structure at the ith position is $Y_i = \max(X_i, X'_i)$, where X_i is the lifetime of the original unit and X'_i the one of the associated spare. For example, if we consider a series system with two components,

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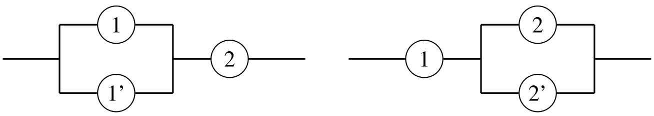

Fig. 5.1 A series system with two components and hot redundancies at positions 1 (left) and 2 (right)

it can be improved by adding a redundant component in parallel at positions 1 or 2 (see Fig. 5.1). Which one is the best option? To answer this question we need to know the characteristics of the units and the spares. Thus, we may assume that the spares have the same distributions as the units, that is, if $F_1, \ldots, F_n$ are the distribution functions of the components, we can assume that the spare at the ith position has distribution F_i. This is a reasonable assumption when the components are different. Another option is to assume that the components are similar and that a spare with distribution G can be added at any point. If the components are identically distributed and we assume $G = F_1 = \cdots = F_n$ both options coincide.

In the second case (**cold redundancy**), the spares are in standby and they replace the components when they fail. Here we also have several options in practice. For example, the standby units might be placed at fixed positions. Thus, if a plain has four engines (two in each wing), it could fly just with two (one in each wing), working the others just in case of the respective failures. Note that in a hot redundancy, the four engines are working from the beginning while in a cold redundancy the two engines in each wing work consecutively (one after the other). Which one is the best option? In other options, we might have just a spare that can be placed at any position in the system. Thus the spare wheel in a car (or a truck), can be placed at any position in case of failure.

In both options, we can consider different assumptions for the spares as well. As above, we can assume that the spares have the same distributions as the original components when they are new (because they are not working). This option is called **perfect repair** since it is equivalent to complete a perfect repair of that unit (a quite unrealistic situation in some systems). Another popular option is to assume that the spares have the same distributions of the original units but that they have the same age as the failed units. This situation is also unrealistic but it is stochastically equivalent to repair the unit to be as it was just before its failure. So it is called **minimal repair** and, in this way, in some situations, it is more realistic than the perfect repair considered above (which it is not a repair but a replacement with a new unit). In both cases, the lifetime of the mechanism at the ith position is $Y_i = X_i + X'_i$. In a perfect repair, we can assume that X_i and X'_i are independent and then the distribution of Y_i is the convolution of F_i and F'_i (see below). However, in a minimal repair, they are dependent since the distribution of X'_i depends on the age $t = X_i$ of the failed component (see below).

Finally, we note that the redundancies can be applied at different levels. If they are applied as considered above, we say that they are redundancies at the **components' level**. However, if they are applied to the entire system, then we say that they are

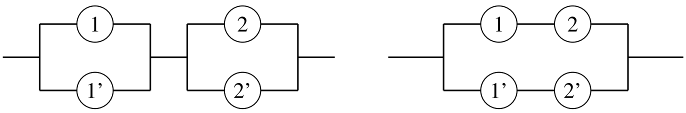

Fig. 5.2 A series system with hot redundancies at the components' level (left) and at the system's level (right)

redundancies at the **system's level**. Even more, if the system is composed of different modules with several units inside each module, the redundancy could be also applied at the **modules' level** (see, e.g., Torrado et al. 2021). For example, if we consider again a series system with two components, then we could add two spares at the components' level obtaining the system lifetime

$$T_c = \min(\max(X_1, X_1'), \max(X_2, X_2')).$$

or at the system's level obtaining

$$T_s = \max(\min(X_1, X_1'), \min(X_2, X_2')).$$

The different options can be seen in Fig. 5.2. Which one is the best option?

Many of these replacement options can be represented in a unified way by using distortions. The definition (extracted from Navarro and Fernández-Martínez 2021) is the following.

Definition 5.1 We say that $\bar{q} : [0, 1] \to [0, 1]$ is a **redundancy dual distortion function** if $\bar{q}$ is continuous, increasing and satisfies $\bar{q}(0) = 0$, $\bar{q}(1) = 1$, and $\bar{q}(u) \geq u$ for all $u \in [0, 1]$.

The purpose is to represent the reliability of the resulting mechanism at the ith position with $\bar{q}(\bar{F}_i)$ where $\bar{F}_i$ is the reliability of the original ith component. Thus, the meaning of the new condition $\bar{q}(u) \geq u$ for all $u \in [0, 1]$ is that the redundancy mechanism improves (in the stochastic order) the original one. Other additional conditions will be considered later.

Let us see some mechanisms that can be represented in this way. The first one is a hot spare connected in parallel. As mentioned above, the resulting structure at the ith position is $Y_i = \max(X_i, X_i')$ and its reliability $\bar{F}_{Y_i}(t) = \Pr(Y_i > t)$ is

$$\bar{F}_{Y_i}(t) = \Pr(\max(X_i, X_i') > t) = \Pr(X_i > t) + \Pr(X_i' > t) - \Pr(X_i > t, X_i') > t)$$

for all t. If we assume that X_i and X_i' are IID with a common reliability $\bar{F}_i$, then

$$\bar{F}_{Y_i}(t) = 2\bar{F}_i(t) - \bar{F}_i^2(t) = \bar{q}_{2:2}(\bar{F}_i(t)),$$

where $\bar{q}_{2:2}(u) = 2u - u^2$ is a distortion function satisfying $\bar{q}_{2:2}(u) \geq u$ for all $u \in [0, 1]$ (since $X_{2:2} \geq X_1$).

We can consider several changes in this model. For example, we could consider that the spare has a different (usually worse) reliability with a proportional hazard rate, i.e., $\Pr(X_i' > t) = \bar{F}_i^{\theta}(t)$ for $\theta > 0$, then

$$\bar{F}_{Y_i}(t) = \bar{F}_i(t) + \bar{F}_i^{\theta}(t) - \bar{F}_i^{1+\theta}(t) = \bar{q}_{\theta}(\bar{F}_i(t)),$$

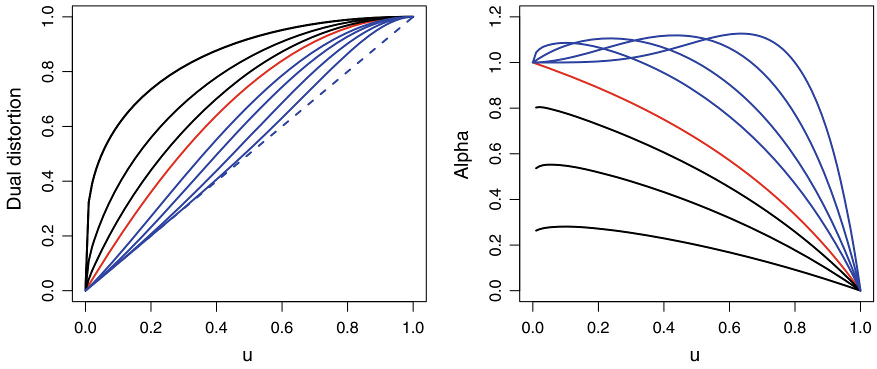

Fig. 5.3 Plots of $\bar{q}_\theta$ (left) and α_θ (right) for an independent hot redundant component with proportional hazard rate with $\theta = 0.25, 0.5, 0.75$ (black), $\theta = 1$ (red, IID case) and $\theta = 1.5, 2, 3, 5$ (blue)

where $\bar{q}_\theta(u) = u + u^\theta - u^{1+\theta}$ is a distortion function satisfying $\bar{q}_\theta(u) \geq u$ for all $u \in [0, 1]$ (since $\max(X_i, X_i') \geq X_i$). The hot redundant component is worse (better) than the original component when $\theta > 1$ $(0 < \theta < 1)$. The different distortion functions can be seen in Fig. 5.3, left. Note that the IID case is obtained when $\theta = 1$ (red curve) and that they are ST ordered. As it is a distortion, its hazard rate can be written as

$$h_\theta(t) = \alpha_\theta(\bar{F}(t))h(t),$$

where $h = f/\bar{F}$ is the hazard rate of $\bar{F}$ and

$$\alpha_\theta(u) = \frac{1 + \theta u^{\theta-1} - (1+\theta)u^\theta}{1 + u^{\theta-1} - u^\theta}$$

for $u \in [0, 1]$. The plots of α_θ can be seen in Fig. 5.3, right. As they are ordered for $0 < \theta < 1$, the respective repairs are hazard rate ordered. This is not the case for $\theta > 1$ (i.e. when the spare is worse than the original unit).

Another variation is to assume that X_i and X_i' are DID, that is, they are dependent and identically distributed. This is a reasonable assumption since they share the same environment. As in the preceding chapters, we can model this dependency through a survival copula $\widehat{C}$ which satisfies

$$\Pr(X_i > x, X_i' > y) = \widehat{C}(\bar{F}_i(x), \bar{F}_i(y))$$

for all x, y. Hence

$$\bar{F}_{Y_i}(t) = 2\bar{F}_i(t) - \widehat{C}(\bar{F}_i(t), \bar{F}_i(t)) = \bar{q}(\bar{F}_i(t)),$$

where $\bar{q}(u) = 2u - \widehat{C}(u, u)$ is a distortion function (which depends on $\hat{C}$) satisfying $\bar{q}(u) \geq u$ for all $u \in [0, 1]$ (since $\max(X_i, X_i') \geq X_i$).

Other interesting variations are to add $m - 1$ IID spares in parallel, which leads to the distortion function $\bar{q}_{m:m}(u) = 1 - (1-u)^m \geq u$ (since $X_{m:m} \geq X_1$), or to add them with any other system structure with distortion $\bar{q}$ satisfying $\bar{q}(u) \geq u$ for all $u \in [0, 1]$.

Some cold redundancies can also be represented in this way (i.e. as distortions). If the lifetime of the resulting mechanism is $\widetilde{Y}_i = X_i + X_i'$, then its reliability is

$$\bar{F}_{\widetilde{Y}_i}(t) = \Pr(X_i + X_i' > t) = \bar{F}_i(t) + \int_0^t \Pr(X_i' > t - x | X_i = x) f_i(x) dx \quad (5.1)$$

for all $t \geq 0$, where $f_i = -\bar{F}_i'$ is the PDF of X_i. If X_i and X_i' are IID (perfect repair), then the reliability function of $\widehat{Y}_i = X_i + X_i'$ is

$$\bar{F}_{\widehat{Y}_i}(t) = \bar{F}_i(t) + \int_0^t \bar{F}_i(t-x) f_i(x) dx$$

which is the well know formula for the reliability function of a convolution. It is represented as $\bar{F}_{\widehat{Y}_i} = \bar{F}_i * \bar{F}_i$. In some models, this reliability can also be represented as a distortion (e.g. with exponential distributions). The same happen if they are dependent (see Navarro and Sarabia 2020).

However, if we consider a **minimal repair** (MR), that is,

$$\Pr(X_i' > y | X_i = x) = \frac{\bar{F}_i(x+y)}{\bar{F}_i(x)}$$

for all $x, y \geq 0$, then from (5.1), the reliability function of $\widetilde{Y}_i = X_i + X_i'$ is

$$\bar{F}_{\widetilde{Y}_i}(t) = \bar{F}_i(t) + \int_0^t \frac{\bar{F}_i(t)}{\bar{F}_i(x)} f_i(x) dx = \bar{F}_i(t) - \bar{F}_i(t) \log \bar{F}_i(t) = \bar{q}_{MR}(\bar{F}_i(t)) \quad (5.2)$$

for all $t \geq 0$, where

$$\bar{q}_{MR}(u) = u - u \log(u)$$

is a distortion function satisfying $\bar{q}_{MR}(u) \geq u$ for all $u \in [0, 1]$ (since $X_i + X_i' \geq X_i$). This model is also known as the **relevation transform** and it was introduced in formula (3.1) of Krakowski (1973) with the notation $\bar{F}_i \# \bar{F}_i$. In this model we can also consider some variations. For example we can consider m minimal repairs obtaining

$$\bar{q}_m(u) = u \sum_{i=0}^{m} \frac{1}{i!} (-\log(u))^i \quad (5.3)$$

with $\bar{q}_m(u) \geq u$ for all $u \in [0, 1]$ (since $X_i + X_i' + \cdots \geq X_i$).

We can also consider **imperfect repairs** (IR) with

$$\Pr(X_i' > y | X_i = x) = \frac{\bar{F}_i^\theta(x+y)}{\bar{F}_i^\theta(x)}$$

for all $x, y \geq 0$ and $\theta > 1$ (the spare is worse than the original component). This option leads to

$$\bar{F}_{Y_i}(t) = \bar{F}_i(t) + \int_0^t \frac{\bar{F}_i^\theta(t)}{\bar{F}_i^\theta(x)} f_i(x) dx = \bar{F}_i(t) - \frac{1}{1-\theta} \bar{F}_i^\theta(t) \left[1 - \bar{F}_i^{1-\theta}(t)\right] = \bar{q}_\theta^{IR}(\bar{F}_i(t))$$

for all $t \geq 0$, where

$$\bar{q}_\theta^{IR}(u) = \frac{\theta}{\theta - 1} u - \frac{1}{\theta - 1} u^\theta$$

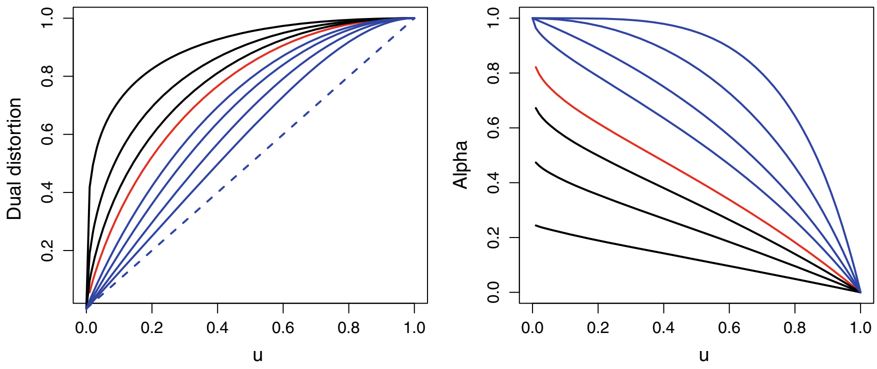

Fig. 5.4 Plots of $\bar{q}_\theta^{IR}$ (left) and α_θ^{IR} (right) for an imperfect repair with $\theta = 0.25, 0.5, 0.75$ (black), $\theta = 1$ (red, minimal repair) and $\theta = 1.5, 2, 3, 5$ (blue)

is a distortion function satisfying $\bar{q}_\theta^{IR}(u) \geq u$ for all $u \in [0, 1]$ (since $X_i + X_i' \geq X_i$). The case $0 < \theta < 1$ can be considered as well (although it could be unrealistic in some situations). Note that we obtain a negative mixture of $\bar{F}_i$ and $\bar{F}_i^\theta$. The plots of $\bar{q}_\theta^{IR}$ can be seen in Fig. 5.4, left. The case $\theta \to 1$ (red curve) represents the minimal repair case. As they are ordered, the respective repairs are ST ordered.

Its hazard rate can be written as

$$h_\theta^{IR}(t) = \alpha_\theta^{IR}(\bar{F}(t))h(t),$$

where $h = f/\bar{F}$ is the hazard rate of $\bar{F}$ and

$$\alpha_\theta^{IR}(u) = \frac{u(\bar{q}_\theta^{IR})'(u)}{\bar{q}_\theta^{IR}(u)} = \theta\frac{1 - u^{\theta-1}}{\theta - u^{\theta-1}}$$

for $u \in [0, 1]$. The plots of α_θ^{IR} can be seen in Fig. 5.4, right. As they are ordered, the respective repairs are hazard rate ordered.

We conclude this section by comparing the three main replacement options. Of course, if $Y_i = \max(X_i, X_i')$ (hot spare parallel), then $Y_i \leq X_i + X_i'$ and so, in particular, when they are independent $\bar{F}_{Y_i} \leq \bar{F}_i * \bar{F}_i$ (perfect repair or convolution). Under minimal repair $\bar{F}_{Y_i} \leq \bar{F}_i \# \bar{F}_i$ holds since

$$\bar{q}_{2:2}(u) = 2u - u^2 \leq \bar{q}_{MR}(u) = u - u\log(u)$$

for all $u \in [0, 1]$. Even more, as $\bar{q}_{MR}'/\bar{q}_{2:2}'$ is decreasing, then $Y_i \leq_{LR} \widetilde{Y}_i$ for all F, where $\widetilde{Y}_i$ represents the total lifetime from the beginning under a minimal repair. To compare $\widetilde{Y}_i$ (minimal repair) and $\widehat{Y}_i = X_i + X_i'$ (perfect repair or convolution) when X_i and X_i' are IID we have the following result.

Proposition 5.1 *With the notation introduced above, if F_i is NBU (NWU), then $\widetilde{Y}_i \leq_{ST} \widehat{Y}_i$ ($\geq_{ST}$).*

Proof Recall that NBU means that $\bar{F}_i(x)\bar{F}_i(y) \geq \bar{F}_i(x+y)$ for all $x, y \geq 0$. Hence, from (5.2), we get

$$\bar{F}_{\widetilde{Y}_i}(t) = \bar{F}_i(t) + \int_0^t \frac{\bar{F}_i(t)}{\bar{F}_i(x)} f_i(x)dx \leq \bar{F}_i(t) + \int_0^t \bar{F}_i(t-x) f_i(x)dx = \bar{F}_{\widehat{Y}_i}(t)$$

since $\bar{F}_i(t-x)\bar{F}_i(x) \geq \bar{F}_i(t)$ for all $0 \leq x \leq t$. The inequality is reversed for NWU distributions. □

Note that, for the "natural" aging property (NBU), the perfect repair is better than the minimal repair (as expected) and we can write

$$Y_i \leq_{ST} \widetilde{Y}_i \leq_{ST} \widehat{Y}_i.$$

For the dual class (NWU) we get

$$Y_i \leq_{ST} \widehat{Y}_i \leq_{ST} \widetilde{Y}_i.$$

Of course, for the exponential distribution (which is both NBU and NWU), we have $\widetilde{Y}_i =_{ST} \widehat{Y}_i$, that is, minimal and perfect repairs coincide.

5.2 Systems with ID Components

In this case we can compare different repair policies by using the ordering results for distorted distributions obtained in Chap. 3. Recall that, in the general case, the system reliability can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)),$$

where $\bar{Q}$ is a distortion function. If we assume that the components are ID, that is, $\bar{F}_1 = \cdots = \bar{F}_n = \bar{F}$ say, then this representation can be reduced to

$$\bar{F}_T(t) = \bar{Q}(\bar{F}(t), \ldots, \bar{F}(t)) = \bar{q}(\bar{F}(t)),$$

where $\bar{q}(u) = \bar{Q}(u, \ldots, u)$ is a distortion function.

If we apply a redundancy policy $\mathbf{r} = (r_1, \ldots, r_n)$ where the redundancy applied to the ith components is represented by $\bar{q}_{r_i}$, then the reliability function of the lifetime $T_{\mathbf{r}}$ of the resulting system can be written as

$$\bar{F}_{\mathbf{r}}(t) = \bar{Q}(\bar{q}_{r_1}(\bar{F}(t)), \ldots, \bar{q}_{r_n}(\bar{F}(t))) = \bar{q}_{\mathbf{r}}(\bar{F}(t)),$$

where

$$\bar{q}_{\mathbf{r}}(u) = \bar{Q}(\bar{q}_{r_1}(u), \ldots, \bar{q}_{r_n}(u))$$

for $u \in [0, 1]$. Note that if we do not apply redundancy to the ith component, then $\bar{q}_{r_i}(u) = u$. Of course, we always get $T \leq_{ST} T_{\mathbf{r}}$ since $\bar{Q}$ is increasing and we assume $\bar{q}_{r_i}(u) \geq u$ for $i = 1, \ldots, n$.

If we have another redundancy policy $\mathbf{s} = (s_1, \ldots, s_n)$, then the reliability function of the resulting system can be represented in a similar way with another distortion function $\bar{q}_{\mathbf{s}}$. Hence $T_{\mathbf{r}}$ and $T_{\mathbf{s}}$ can be compared just by comparing their respective distortion functions using Proposition 3.2.

The comparisons under minimal repairs were studied in Arriaza et al. (2018). Here $\mathbf{r} = (r_1, \ldots, r_n)$ means that r_i minimal repairs are applied to the ith component, with $r_i \geq 0$ for $i = 1, \ldots, n$. With this notation we can obtain the following classic result for series systems with IID components that can be traced back to Shaked and Shanthikumar (1992), Result 2.4(s) (see also Theorem 4 in Li and Ding 2010). Obviously, in the case of series systems with IID components, the repair strategy given by the vector $\mathbf{r} = (r_1, \ldots, r_n)$ is the same as that of $\mathbf{r}' = (r_{\pi(1)}, \ldots, r_{\pi(n)})$ for any permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$. So, without loss of generality, we can assume for this system that $r_1 \geq \ldots \geq r_n$. Moreover, we have $\bar{Q}(u_1, \ldots, u_n) = u_1 \ldots u_n$ and so

$$\bar{q}_{\mathbf{r}}(u) = \bar{q}_{r_1}(u) \ldots \bar{q}_{r_n}(u)$$

for $u \in [0, 1]$, where these distortions functions are defined as in (5.3). Hence we have the following theorem.

Theorem 5.1 (Shaked and Shanthikumar 1992) *Consider a series system with n IID components with a common reliability function $\bar{F}$. Suppose that we have available $m \in \mathbb{Z}_+$ minimal repairs that can be freely allocated to any component. Let $p, s \in \mathbb{Z}_+$ be the unique integer numbers such that $m = pn + s$ and $0 \leq s < n$. Then, the optimal allocation strategy, in terms of the usual stochastic order, is given by the vector*

$$\mathbf{r}^{\star} = (\overbrace{p+1,\ p+1, \ldots,\ p+1}^{s}, \overbrace{p,\ p, \ldots,\ p}^{n-s}).$$

As expected, the best option is to distribute all the available repairs as much as possible between the components. An alternative proof to that given in Shaked and Shanthikumar (1992) is provided in Arriaza et al. (2018). It is interesting to note here that if the optimal allocation strategy cannot be applied due to some other external constraint, then using the sequence $\{\mathbf{r}_i\}_{i \in \{1,\ldots,\upsilon\}}$ defined in this proof we always have available the second best choice as optimal strategy, and so on (or a path to improve the initial strategy $\mathbf{r}$). Also note that as a consequence of the proof, the worst option is always $(m, 0, \ldots, 0)$, i.e., to assign all the repairs to a fixed component.

We can also compare repairs in any other system structures. Let us see an example extracted from Arriaza et al. (2018). Additional results for minimal repairs can be seen in Navarro et al. (2019). Similar results can be obtained for other repair options based on distortions.

Example 5.1 Consider a 2-out-of-3 system with IID components with a common reliability function $\bar{F}$. Assume a fixed number $m = 7$ of available minimal repairs. Let us study all the possible ST comparisons of lifetimes $T_{\mathbf{r}}$ obtained from the repair policies $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_+$ with $r_1 \geq r_2 \geq r_3$ and $r_1 + r_2 + r_3 = 7$. Note that in this case they are also equivalent under permutations in $\mathbf{r}$. Firstly, given $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_+$ and assuming that the component lifetimes are independent, we obtain that the reliability function of the system lifetime $T_{\mathbf{r}}$ associated to $\mathbf{r}$ is

$$\bar{F}_{\mathbf{r}}(t) = \bar{Q}(\bar{F}_{(r_1)}(t), \bar{F}_{(r_2)}(t), \bar{F}_{(r_3)}(t)) = \bar{q}_{\mathbf{r}}(\bar{F}(t)),$$

where $\bar{Q}(u, v, w) = uv + uw + vw - 2uvw$,

$$\bar{q}_{\mathbf{r}}(u) = \bar{q}_{r_1}(u)\bar{q}_{r_2}(u) + \bar{q}_{r_1}(u)\bar{q}_{r_3}(u) + \bar{q}_{r_2}(u)\bar{q}_{r_3}(u) - 2\bar{q}_{r_1}(u)\bar{q}_{r_2}(u)\bar{q}_{r_3}(u)$$

and $\bar{q}_{r_i}$ is the distortion function given in (5.3) for $i = 1, 2, 3$. Then, we have that

$$T_{\mathbf{r}_1} \leq_{st} T_{\mathbf{r}_2} \text{ for all } \bar{F} \Leftrightarrow \bar{q}_{\mathbf{r}_1}(u) \leq \bar{q}_{\mathbf{r}_2}(u) \text{ for all } u \in (0, 1).$$

Therefore, if we want to compare two strategies $\mathbf{r}_1$ and $\mathbf{r}_2$, we just need to plot both functions, $\bar{q}_{\mathbf{r}_1}$ and $\bar{q}_{\mathbf{r}_2}$ on the interval [0, 1]. For instance, in this way we can confirm that $T_{\mathbf{r}_1} \leq_{ST} T_{\mathbf{r}_2}$ for all reliability functions $\bar{F}$ when $\mathbf{r}_1 = (7, 0, 0)$ and $\mathbf{r}_2 = (6, 1, 0)$. We will write $\mathbf{r}_1 \to \mathbf{r}_2$ to denote that the strategy $\mathbf{r}_1$ is better than $\mathbf{r}_2$ or, in other words, $T_{\mathbf{r}_2} \leq_* T_{\mathbf{r}_1}$ holds for a given order $\leq_*$.

Following the previous procedure, we obtain the graphs given in Fig. 5.5 with all the relationships for the comparisons of the repair strategies in the HR order (left) and in the ST order (right). The strategies that are not connected in the graph represent lifetimes of systems that are not comparable in the usual stochastic order (respectively, in the hazard rate order). In this case an optimal allocation strategy does not exist in terms of the usual stochastic order. Note that, a priori, all the minimal path sets of the 2-out-of-3 system are equally important due to the structure of the system. Note that the replacement policy represented by the vector $\mathbf{r}^\star = (4, 3, 0)$ (which applies all the repairs to the components in the first path set) is ordered with a larger number of alternatives (see Fig. 5.5). However, $\mathbf{r}^\star$ is not stochastically ordered neither with (3, 3, 1) nor with (3, 2, 2). Similar comments hold for the HR order.

5.3 Systems with Non-ID Components

As in the preceding section, we know that the reliability function of the system lifetime T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

for all t, where $\bar{Q}$ is a distortion function. Hence, if we apply a redundancy with distortion $\bar{q}(u) \geq u$ to the ith component, the reliability function of the resulting system lifetime T_i is

$$\bar{F}_{T_i}(t) = \bar{Q}_i(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

with

$$\bar{Q}_i(u_1, \ldots, u_n) = \bar{Q}(u_1, \ldots, u_{i-1}, \bar{q}(u_i), u_{i+1} \ldots, u_n)$$

for $0 \leq u_i \leq 1$ and $i = 1, \ldots, n$. Note that we are assuming a common redundancy mechanism (distortion) for all the components.

Of course, then we have $T \leq_{ST} T_i$ for $i = 1, \ldots, n$. However, we want to compare T_i and T_j to determine where the redundant component should be placed.

In the first result, extracted from Navarro and Fernández-Martínez (2021), we analyze series systems with independent components. In this case, we just compare T_1 and T_2.

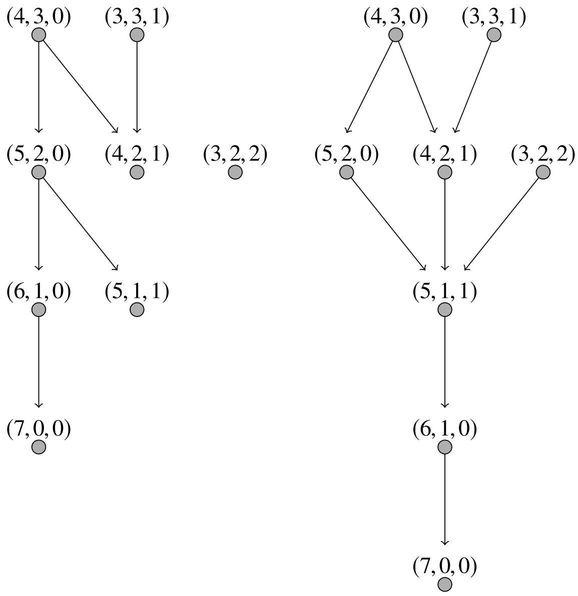

Fig. 5.5 Relationships among all possible lifetimes $T_{\mathbf{r}}$ after seven minimal repairs by using the hazard rate order (left) and the usual stochastic order (right) for the 2-out-of-3 system considered in Example 5.1

Proposition 5.2 *Let* $T = \min(X_1, \ldots, X_n)$ *with independent components.*

(i) *If* $X_1 \geq_{ST} X_2$ *and*

$$\frac{\bar{q}(u)}{u} \text{ is decreasing in } (0,1), \tag{5.4}$$

then $T_1 \leq_{ST} T_2$ *for all* $F_3, \ldots, F_n$.

(ii) *If* $X_1 \geq_{HR} X_2$ *and*

$$\frac{\bar{q}(uv)}{v\bar{q}(u)} \text{ is decreasing in } (0,1)^2, \tag{5.5}$$

then $T_1 \leq_{HR} T_2$ *for all* $F_3, \ldots, F_n$.

(iii) *The condition (5.4) holds iff* $T \leq_{HR} T_i$ *for all* $F_1, \ldots, F_n$ *and* $i = 1, \ldots, n$.

Proof (i) The condition $T_1 \leq_{ST} T_2$ holds iff

$$\bar{Q}_1(u_1, \ldots, u_n) = \bar{q}(u_1)u_2 \ldots u_n \leq \bar{u}_1 q(u_2) \ldots u_n = \bar{Q}_2(u_1, \ldots, u_n)$$

which is equivalent to

$$\bar{q}(u_1)u_2 \leq \bar{u}_1 q(u_2).$$

As we assume $\bar{F}_1 \geq \bar{F}_2$ and (5.4), we get

$$\frac{\bar{q}(\bar{F}_1(t))}{\bar{F}_1(t)} \leq \frac{\bar{q}(\bar{F}_2(t))}{\bar{F}_2(t)}$$

and so $T_1 \leq_{ST} T_2$ for all $F_3, \dots, F_n$.

(ii) The condition $T_1 \leq_{HR} T_2$ holds if and only if

$$\frac{\bar{Q}_2(\bar{F}_1(t), \dots, \bar{F}_n(t))}{\bar{Q}_1(\bar{F}_1(t), \dots, \bar{F}_n(t))} \text{ is increasing in } t,$$

which is equivalent to

$$\frac{\bar{F}_1(t)\bar{q}(\bar{F}_2(t))}{\bar{F}_2(t)\bar{q}(\bar{F}_1(t))} \text{ is increasing in } t$$

As we assume $X_1 \geq_{HR} X_2$, $g(t) = \bar{F}_2(t)/\bar{F}_1(t)$ is decreasing in t. Hence $g(t) \in [0, 1]$. Then, by applying (5.5) to $u = \bar{F}_1(t)$ and $v = g(t)$, we get that

$$\frac{\bar{F}_1(t)\bar{q}(\bar{F}_2(t))}{\bar{F}_2(t)\bar{q}(\bar{F}_1(t))}$$

is increasing in t and so $T_1 \leq_{HR} T_2$ holds for all $F_3, \dots, F_n$.

(iii) The condition $T \leq_{HR} T_i$ holds if and only if

$$\frac{\bar{Q}_i(\bar{F}_1(t), \dots, \bar{F}_n(t))}{\bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t))} \text{ is increasing in } t,$$

which is equivalent to

$$\frac{\bar{q}(\bar{F}_i(t))}{\bar{F}_i(t)} \text{ is increasing in } t$$

for all $\bar{F}_i$. This property is equivalent to (5.4). □

Note that (i) means that, under condition (5.4), the redundant component should be applied to the strongest components (in the ST order). To extend this property to the HR order we need the stronger condition (5.5). The meaning of (5.4) can be seen in (iii). It is equivalent to the condition: T_i is HR better than T and to the same ordering property for the original component X_i and the resulting redundancy mechanism Y_i.

The property (5.4) is satisfied for the usual redundancy mechanism. For example, for a hot IID spare added in parallel we have

$$\frac{\bar{q}_{2:2}(u)}{u} = \frac{2u - u^2}{u} = 2 - u$$

which is decreasing. The same happen for m independent spares added in parallel.

For a cold standby unit with minimal repair we have

$$\frac{\bar{q}_{MR}(u)}{u} = \frac{u - u \log u}{u} = 1 - \log u$$

that is also decreasing. Hence (5.4) holds. The same happen for m minimal repairs.

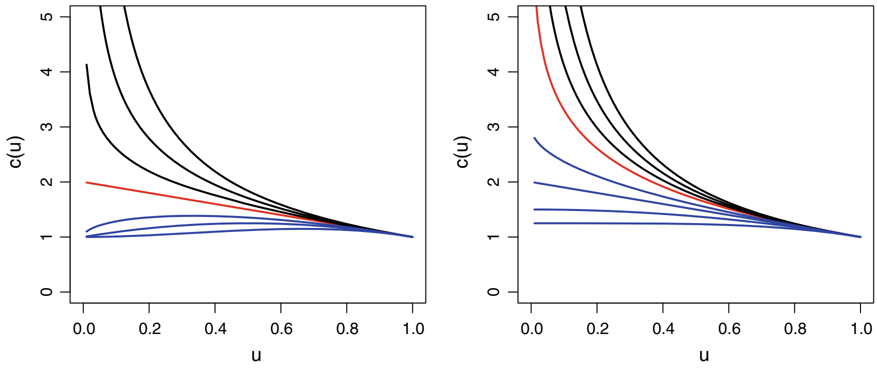

Fig. 5.6 Plots of $c(u) = \bar{q}_\theta(u)/u$ for a hot spare in parallel with reliability $\bar{F}^\theta$ (left) and an imperfect repair (right) with $\theta = 0.25, 0.5, 0.75$ (black), $\theta = 1$ (red, minimal repair) and $\theta = 1.5, 2, 3, 5$ (blue)

However, (5.4) is not always true. For example, if we add a spare in parallel with reliability $\bar{F}^\theta$, we obtain the plots in Fig. 5.6 (left) for $c(u) = \bar{q}_\theta(u)/u$. There we can see that function c is decreasing for $0 < \theta \leq 1$ but that it is not monotone for $\theta > 1$ (since $c(0) = c(1) = 1$).

However, (5.4) holds for imperfect repairs since

$$c(u) = \frac{\bar{q}_\theta(u)}{u} = \frac{\theta - u^{\theta-1}}{\theta - 1}$$

is decreasing in u for all $\theta > 0$ (see Fig. 5.6, right). As mentioned above, it is also decreasing for a minimal repair (red curve).

The condition (5.5) is not so common. For example, it fails in active redundancies since

$$\frac{\bar{q}(uv)}{v\bar{q}(u)} = \frac{2uv - u^2v^2}{2uv - u^2v} = \frac{2 - uv}{2 - u}$$

is increasing in u and decreasing in v in the set $(0, 1)^2$. The same happen for minimal repairs since

$$\frac{\bar{q}(uv)}{v\bar{q}(u)} = \frac{uv - uv\log(uv)}{uv - uv\log(u)} = 1 + \frac{-\log(v)}{1 - \log(u)}$$

is increasing in u and decreasing in v in the set $(0, 1)^2$.

Similar (reverse) results can be obtained for parallel systems with independent components. For example, if $X_1 \geq_{ST} X_2$ and $q(u)/u$ is increasing, then $T_1 \geq_{ST} T_2$ for all $F_3, \ldots, F_n$, that is, in this system, it is better to reinforce the strongest component (as expected). For other system structures the answer is not so clear, see Navarro and Fernández-Martínez (2021). The same happen if we consider dependent components. In these cases they can be compared by using distortions.

We conclude this section by establishing comparisons between redundancies at components' or system's levels. The BP (Barlow and Proschan) principle for active redundancies in parallel is established in the following theorem. It was given in Theorem 2.4 of Barlow and Proschan (1975), p. 8, (see also Samaniego 2007, p. 17).

Theorem 5.2 (BP-principle) *If we consider active redundancies added in parallel and the component and spares lifetimes have the same joint distribution, then the system with redundancy at components' level is always ST better than the system with redundancy at system's level.*

Proof We provide the proof for an active redundancy. The proof for m active redundancies is similar. If $X_1, \dots, X_n$ are the components' lifetimes and $X'_1, \dots, X'_n$ are the spares' lifetimes. We assume that in both redundancy options $(X_1, \dots, X_n, X'_1, \dots, X'_n)$ has the same joint distribution or, equivalently, that both systems are built with the same components.

Let $P_1, \dots, P_r$ be the minimal path sets of the original system. Then the minimal path sets of the system with redundancy at system's level are $P_1, \dots, P_r, P'_1, \dots, P'_r$, where P'_i is the set with the spares of the components in the set P_i. It is easy to see that all these sets are also path sets of the system with redundancy at components' level. Hence, if we assume that they have the same components, the system with redundancy at components' level works whenever the system with redundancy at system's level does so. Hence, their lifetimes are ordered for sure and so we have the ST order when the components have the same joint distribution (see Theorem 1.A.1 in Shaked and Shanthikumar 2007, p. 5). □

We must say that the assumption about a common joint distribution for the components and spares is quite unrealistic when the components are dependent (since the spares are placed at different positions). However, it holds when the components and spares are independent. Let us see an example.

Example 5.2 Let us consider the system with lifetime

$$T = \max(X_1, \min(X_2, X_3))$$

and independent components. Its dual distortion function is

$$\bar{Q}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3$$

for $u_1, u_2, u_3 \in [0, 1]$.

The lifetime of the system with redundancy at the components' level is

$$T_1 = \max(\max(X_1, X'_1), \min(\max(X_2, X'_2), \max(X_3, X'_3)),$$

where X'_1, X'_2, X'_3 represent the lifetimes of the spares. If we assume that the components and the spares are independent and that $X_i =_{ST} X'_i$ for $i = 1, 2, 3$, then the dual distortion function of T_1 is

$$\bar{Q}_1(u_1, u_2, u_3) = \bar{q}_{2:2}(u_1) + \bar{q}_{2:2}(u_2)\bar{q}_{2:2}(u_3) - \bar{q}_{2:2}(u_1)\bar{q}_{2:2}(u_2)\bar{q}_{2:2}(u_3)$$

for $u_1, u_2, u_3 \in [0, 1]$, where $\bar{q}_{2:2}(u) = 2u - u^2$ for $u \in [0, 1]$.

Analogously, the lifetime of the system with redundancy at the system's level is

$$T_2 = \max(\max(X_1, \min(X_2, X_3)), \max(X'_1, \min(X'_2, X'_3)))$$

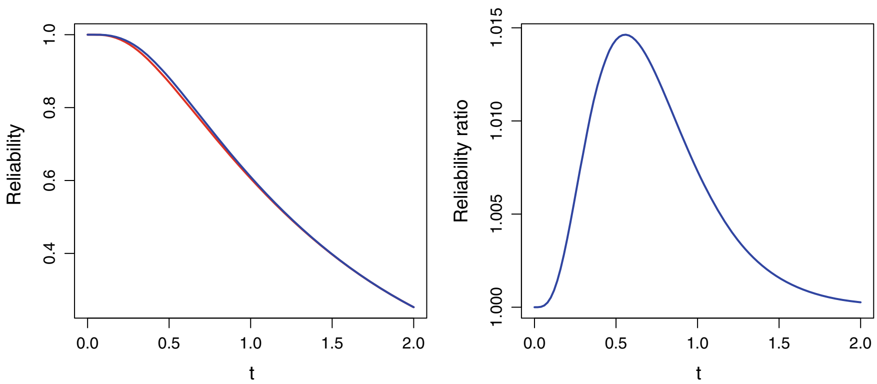

Fig. 5.7 Plots of the reliability functions (left) and its ratio (right) for the systems in Example 5.2 with redundancies at components' level (blue) and at system's level (red)

that is, in this case we have two independent copies of the system connected in parallel. If we assume the same joint distribution for components and spares as in the preceding case, then its dual distortion function is

$$\bar{Q}_2(u_1, u_2, u_3) = \bar{q}_{2:2}(\bar{Q}(u_1, u_2, u_3))$$

for $u_1, u_2, u_3 \in [0, 1]$.

Hence, from the preceding theorem (BP-principle), we have $T_1 \geq_{ST} T_2$ for all F_1, F_2, F_3. The respective reliability functions can be seen in Fig. 5.7 (left) for exponential components with hazard rates 1, 2, 3, respectively. Note that the reliabilities are very similar. The ratio in the right plot shows that this property cannot be extended to the hazard rate order.

5.4 Importance Indices

There exist several importance indices for the components in a system, especially in the case of independent components, see for example Barlow and Proschan (1975) and Kuo and Zhu (2012). Some of them only depend on the structure of the system, while others also depend on the components' distributions.

For example, the structural importance of the ith component is defined (see Barlow and Proschan 1975, p. 13) as

$$n_\phi(i) = \frac{1}{2^{n-1}} \sum_{x_j=0,1,\, j\neq i} [\phi(x_1, \dots, 1, \dots, x_n) - \phi(x_1, \dots, 0, \dots, x_n)],$$

where the ones and zeros are placed at the ith positions. This measure takes into account how many times the ith component is crucial for the system. If we consider a 2-out-of-3 system, then $n_\phi(i) = 1/2$ for $i = 1, 2, 3$ while if $\phi(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$, then $n_\phi(1) = 3/4$ and $n_\phi(i) = 1/4$ for $i = 2, 3$. The main advantage is that these indices can always be compared

Another popular index is the Barlow and Proschan (BP) importance measure defined as

$$m(i) = \Pr(T = X_i).$$

This index depends on the components' distributions. If we assume IID components with a common continuous distribution (no ties), then this index only depends on the structure and so it can be written as $m_\phi(i)$. For example, in a 2-out-of-3 system, $m_\phi(i) = 1/3$ for $i = 1, 2, 3$ while if $\phi(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$, then $m_\phi(1) = 4/6$ and $m_\phi(i) = 1/6$ for $i = 2, 3$. Note that in this index, with no ties, we have $\sum_{i=1}^n m(i) = 1$.

In the case of independent components, another popular index based on the reliability function of the structure $\bar{Q}_\Pi$ is

$$I_\phi(i) = \partial_i \bar{Q}_\Pi(u_1, \ldots, u_n),$$

where remember that $\bar{Q}_\Pi$ is also the dual distortion function based on the product copula (or the function obtained with the pivotal decomposition). It is known as the Birnbaum (B) importance measure (see Birnbaum 1969) and it can also be written as

$$I_\phi(i) = \bar{Q}(u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n) - \bar{Q}(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n)$$

or as

$$I_\phi(i) = E(\phi(X_1, \ldots, 1, \ldots, X_n) - \phi(X_1, \ldots, 0, \ldots, X_n)),$$

where $X_1, \ldots, X_n$ are IID with $\Pr(X_i = 1) = u_i$ and $\Pr(X_i = 0) = 1 - u_i$ for $i = 1, \ldots, n$ (see Barlow and Proschan 1975, p. 22). The main disadvantage is that this index is not a number but a function of $u_1, \ldots, u_n$. So the indices for the different components cannot be compared.

For example, for a 2-out-of-3 system, we get

$$I_\phi(1) = u_2 + u_3 - 2u_2u_3,$$

$$I_\phi(2) = u_1 + u_3 - 2u_1u_3$$

and

$$I_\phi(3) = u_1 + u_2 - 2u_1u_2.$$

However, if we consider the system $\phi(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$, then

$$I_\phi(1) = u_2 + u_3 - 2u_2u_3,$$

$$I_\phi(2) = u_1 - u_1u_3$$

and

$$I_\phi(3) = u_1 - u_1u_2.$$

If the components are IID, we can assume $u = u_1 = u_2 = u_3$ and then

$$I_\phi(i) = 2u - 2u^2,\ i = 1, 2, 3$$

in the 2-out-of-3 system (i.e. all the components have the same importance) while

$$I_\phi(1) = 2u - u^2 \geq u - u^2 = I_\phi(i),\ i = 2, 3$$

in the other system (the first component is more important than the others). These are expectable properties.

If we consider dependent components, the index should also take into account the dependence structure. However, it should not depend on the components' distributions. It should also be used to determine the best replacement positions. Actually, we may want to place the best components or the redundancies at the most critical (important) positions.

In this case (dependent components) the three equivalent expressions considered above for $I_\phi(i)$ lead to different options. The most useful one in practice is

$$I_{\phi,C}(i) = \partial_i \bar{Q}_C(u_1, \ldots, u_n), \tag{5.6}$$

where $\bar{Q}_C$ is the dual distortion function of the system when the dependence is determined by the copula C. Note that it also depends on the system structure ϕ. To simplify the notation we will just write I_i instead of $I_{\phi,C}(i)$. The meaning is clear the most important components are those which lead to a higher increment in the system reliability function (when they are improved).

Again the indices are functions of $u_1, \ldots, u_n$. However, as above, we could consider ID components and then they are just functions of $u = u_1 = \cdots = u_n$ with $I_i(u) := I_i(u, \ldots, u)$.

This index was analyzed in Miziuła and Navarro (2019) for dependent components proving that

$$m(i) = \Pr(T = X_i) = \int_0^\infty I_i(\bar{F}_1(t), \ldots, \bar{F}_n(t)) dF_i(t)$$

for $i = 1, \ldots, n$. In particular, if the components are ID, then

$$m(i) = \Pr(T = X_i) = \int_0^1 I_i(u) du. \tag{5.7}$$

In this case, $\Pr(T = X_i)$ does not depend on $F = F_1 = \cdots = F_n$ and, if $I_i(u) \leq I_j(u)$ for all $u \in [0, 1]$, then $m(i) \leq m(j)$ for all F.

This index can also be used to determine the best replacement position. The result extracted from Theorem 2.4 of Navarro et al. (2020) can be stated as follows. Its proof can be seen there.

Theorem 5.3 *If $I_1(u_1, \ldots, u_n) \leq I_2(u_1, \ldots, u_n)$ for all $u_1, \ldots, u_n$, then $T_1 \leq_{ST} T_2$, where T_i is the system obtained by applying a redundancy with dual distortion $\bar{q}$ to the ith component for $i = 1, 2$.*

The good point of the preceding theorem is that it holds for arbitrary redundancies satisfying $\bar{q}(u) \geq u$ for $u \in [0, 1]$. It can also be applied to mixed systems. However, the condition $I_1 \leq I_2$ assumed there is too strong. So some weaker conditions that lead to the similar result were analyzed in Navarro et al. (2020). Other conditions for specific redundancies (active redundancy in parallel or minimal repair) are analyzed as well.

Problems

1. Prove equation (5.3).
2. Compare two repair policies in a system with ID components.
3. Check an arrow in Fig. 5.5.
4. Study if Theorem 5.1 can be extended to hot redundancies of independent components added in parallel (Indication: Try to prove it first for $n = 2$).
5. Study the redundancy policies considered in Example 5.1 but using hot independent spares connected in parallel.
6. Compare a redundancy at different positions in a system with IID components.
7. Compare a redundancy at different positions in a system with independent components.
8. Compare a redundancy at different positions in a system with dependent components.
9. Confirm the BP-principle in a system with independent components.
10. Compute the BP and B importance measures in a system with IID components and confirm that (5.7) holds.
11. Compute the BP and B importance measures in a system with DID components and confirm that (5.7) holds.