3 Stochastic Comparisons

Abstract

In this chapter we use the representations obtained in the preceding chapter to stochastically compare the performance of different systems. We consider the main stochastic orders: the usual stochastic order, the hazard rate order, the mean residual life order, the reversed hazard rate order and the likelihood ratio order. We use different techniques depending on the assumptions made about the components. We consider systems with independent and identically distributed (IID) components, exchangeable (EXC) components, identically distributed (ID) components, independent (IND) components or dependent components. The dependence is modeled by using copulas (or joint reliability functions). This chapter is based on the review paper Navarro (2018b).

3.1 Main Stochastic Orders

First we give the definitions and the main properties of the stochastic orders considered here. Note that they can be used to compare both the system and the component lifetimes (i.e. non-negative random variables). For more properties and applications we refer the interested readers to Belzunce et al. (2016), Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

If X and Y are two random variables (representing the lifetimes of two different units or systems), there exist several ways to stochastically compare X and Y. The first option is to compare their means (or expected lifetimes) $\mu_X = E(X)$ and $\mu_Y = E(Y)$ (if they exist). Thus we write $X \leq_M Y$ (mean order) when $\mu_X \leq \mu_Y$.

The second main option is the (usual) stochastic order defined as follows.

Definition 3.1 X is said to be smaller than Y in the **stochastic order** (denoted by $X \leq_{ST} Y$ or by $F_X \leq_{ST} F_Y$) if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t, where $\bar{F}_X$ and $\bar{F}_Y$ are the reliability functions of X and Y, respectively.

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Note that here (and throughout the book) 'smaller than' means 'smaller than or equal to'. Also, if both $X \leq_{ST} Y$ and $X \geq_{ST} Y$ hold (i.e., $X =_{ST} Y$), then $\bar{F}_X(t) = \bar{F}_Y(t)$ for all t, that is, they have the same law (distribution). This order can also be called 'the reliability order' since $X \leq_{ST} Y$ means that the reliability of the units represented by Y is always equal to or greater than the reliability of the units represented by X.

The ST order $X \leq_{ST} Y$ is characterized by the following property:

$$E(g(X)) \leq E(g(Y)) \tag{3.1}$$

for any increasing function g such that these expectations exist. This property is sometimes used as a definition. Recall that, throughout the book, we use increasing (decreasing) in the weak sense, that is, a function g is increasing (decreasing) if $g(a) \leq g(b)$ ($\geq$) for all $a \leq b$. Therefore, the stochastic order can be seen as an extension of the expected value order for increasing functions. In particular, we have that $X \leq_{ST} Y$ implies $E(X) \leq E(Y)$ whenever both expectations exist. Also note that, from (2.3), if $X \leq_{ST} Y$ and $E(X) = E(Y)$ hold, then $X =_{ST} Y$.

Another characterization of this order is the following: $X \leq_{ST} Y$ if and only if there exist two random variables X^* and Y^* over the same probability space $(\Omega, S, \Pr)$ such that $X^* =_{ST} X$, $Y^* =_{ST} Y$ and $X^*(\omega) \leq Y^*(\omega)$ for all $\omega \in \Omega$ (see Shaked and Shanthikumar 2007, p. 5). However, note that if X and Y are defined over the same probability space Ω, $X \leq_{ST} Y$ does not necessarily imply that $X(\omega) \leq_{ST} Y(\omega)$ for all $\omega \in \Omega$. As an immediate consequence we have that if $X \leq_{ST} Y$, then $aX + b \leq_{ST} aY + b$ for all $a > 0$ and b. The ordering is reversed when $a < 0$.

Another option is to compare X and Y by comparing their respective aging functions. For example, the hazard rate order is defined as follows.

Definition 3.2 X is said to be smaller than Y in the **hazard (or failure) rate** order (denoted by $X \leq_{HR} Y$ or by $F_X \leq_{HR} F_Y$) if $\bar{F}_Y/\bar{F}_X$ is an increasing function (with the convention $a/0 = +\infty$ for all $a > 0$).

The HR order can be characterized in terms of the ST order by the following property:

$$X \leq_{HR} Y \Leftrightarrow (X - t|X > t) \leq_{ST} (Y - t|Y > t) \quad \text{for all } t. \tag{3.2}$$

Hence the HR order can be interpreted as follows: $X \leq_{HR} Y$ if and only if the residual lifetime of a used unit with age t from X is ST-smaller than the residual lifetime of a used unit with the same age t from Y for all t. Note that $X \leq_{HR} Y$ implies $X \leq_{ST} Y$.

If X and Y are two random variables with absolutely continuous (or discrete) distribution functions, then $X \leq_{HR} Y$ iff $h_X(t) \geq h_Y(t)$ for all t, where $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$ are the HR functions of X and Y, respectively.

Analogously, the reversed hazard rate order is defined as follows.

Definition 3.3 X is said to be smaller than Y in the **reversed hazard rate order** (denoted by $X \leq_{RHR} Y$ or by $F_X \leq_{RHR} F_Y$) if F_Y/F_X is an increasing function.

The RHR order can be characterized in terms of the ST order by the following property:

$$X \leq_{RHR} Y \Leftrightarrow (X|X \leq t) \leq_{ST} (Y|Y \leq t) \quad \text{for all } t$$

or equivalently, by

$$X \leq_{RHR} Y \Leftrightarrow (t - X|X \leq t) \geq_{ST} (t - Y|Y \leq t) \quad \text{for all } t. \tag{3.3}$$

From (3.3), the RHR order can be interpreted as follows: $X \leq_{RHR} Y$ holds if and only if the inactivity time of a unit which has failed before age t from X is ST-greater than the inactivity time of a unit which has failed before age t from Y for all t.

If X and Y are two random variables with absolutely continuous (or discrete) distribution functions, then $X \leq_{RHR} Y$ iff $\bar{h}_X(t) \leq \bar{h}_Y(t)$ for all t, where $\bar{h}_X = f_X/F_X$ and $\bar{h}_Y = f_Y/F_Y$ are the reverse hazard rate functions of X and Y, respectively.

It can be proved that the RHR order does not imply the HR order and that the HR order does not imply the RHR order. However, they are related by the following properties:

$$X \leq_{RHR} Y \Leftrightarrow -X \geq_{HR} -Y$$

and

$$X \leq_{HR} Y \Leftrightarrow -X \geq_{RHR} -Y$$

since $F_X(t) = \bar{F}_{-X}(-t)$, $f_X(t) = f_{-X}(-t)$, $h_X(t) = \bar{h}_{-X}(-t)$ and $\bar{h}_X(t) = h_{-X}(-t)$.

Next we give the definition of a stronger order also related with conditional expectations and aging properties, the likelihood ratio order.

Definition 3.4 If X and Y are two random variables with absolutely continuous (or discrete) distribution functions, X is said to be smaller than Y in the **likelihood ratio order** (denoted by $X \leq_{LR} Y$ or by $F_X \leq_{LR} F_Y$) if f_Y/f_X is increasing in the union of their supports, where f_X and f_Y are probability density (or probability mass) functions of X and Y, respectively.

Note that $X \leq_{LR} Y$ holds if and only if

$$f_X(y)f_Y(x) \leq f_X(x)f_Y(y)$$

for all $x < y$. The LR order can also be characterized by the following property:

$$X \leq_{LR} Y \Leftrightarrow (X|s < X \leq t) \leq_{ST} (Y|s < Y \leq t)$$

for all $s < t$ such that these conditional random variables exist (i.e., such that $F_X(s) < F_X(t)$ and $F_Y(s) < F_Y(t)$). This property can be used to give a general definition of the LR order. Hence the LR order can be interpreted as follows: $X \leq_{LR} Y$ if and only if when we know that a unit from X and another unit from Y have both failed in the interval $(s, t]$, the lifetime of the unit from X is ST-smaller than the lifetime of the unit from Y for all $s < t$. In particular, we obtain that the LR order implies both the HR and the RHR orders and, of course, the ST order. The LR order can also be characterized by the following property: $X \leq_{LR} Y$ holds iff $\eta_X \geq \eta_Y$, where $\eta_Z := -f'_Z/f_Z$ is known as the Glaser's eta function (see Glaser 1980).

The relationships between the preceding orders defined in terms of ST orderings of conditional random variables can be summarized as follows:

$$\begin{array}{ccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y \\ \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y \end{array}$$

where the reverse implications are not necessarily true.

We can complete the diagram above by including the orders based on conditional expectations given below. For a random variable Z, we define the upper end-point u_Z of its support as $u_Z := \sup\{x : F_Z(x) < 1\}$. Analogously, the lower end-point l_Z of its support is $l_Z := \inf\{x : F_Z(x) > 0\}$. Then the mean residual lifetime order is defined as follows.

Definition 3.5 X is said to be smaller than Y in the **mean residual life order** (denoted by $X \leq_{MRL} Y$ of by $F_X \leq_{MRL} F_Y$) if $u_X \leq u_Y$ and $m_X(t) \leq m_Y(t)$ for all $t < u_X$ for which that expectations exist, where $m_X(t) = E(X - t|X > t)$ and $m_Y(t) = E(Y - t|Y > t)$ are the MRL functions of X and Y, respectively.

Analogously, we can define the following orders based on conditional expectations.

Definition 3.6 X is said to be smaller than Y in the **mean inactivity time order** (denoted by $X \leq_{MIT} Y$) if $l_X \leq l_Y$ and $\bar{m}_X(t) \geq \bar{m}_Y(t)$ for all $t > l_Y$ for which that expectations exist, where $\bar{m}_X(t) = E(t - X|X \leq t)$ and $\bar{m}_Y(t) = E(t - Y|Y \leq t)$ are the MIT functions of X and Y, respectively.

Definition 3.7 X is said to be smaller than Y in the **doubly truncated mean order** (denoted by $X \leq_{DTM} Y$) if $m_X(s,t) \leq m_Y(s,t)$ for all $s < t$ for which that expectations exist, where $m_X(s,t) = E(X|s < X \leq t)$ and $m_Y(s,t) = E(Y|s < Y \leq t)$ are the DTM functions of X and Y, respectively.

The definitions and relationships between the orders defined in this section can be summarized in the diagram given in Table 3.1 that was obtained by Navarro et al. (1997). The first and last columns can be used as definitions for general distributions. The implications from the second column to the third column are consequences of the characterization of the ST order given in (3.1). The other implications can be obtained taking limits to ∞ or to $-\infty$. The reverse implications are not necessarily true.

Another option two compare two independent random variables X and Y defined over the same probability space is the following.

Definition 3.8 If X and Y are two independent random variables defined over the same probability space, X is said to be smaller than Y in **stochastic precedence** (denoted by X SP Y) if $\Pr(X \leq Y) \geq 1/2$.

Table 3.1 Relationships between the main stochastic orders. We use the notation $Z_t = (Z - t|Z > t)$, ${}_tZ = (t - Z|Z \le t)$ and ${}_sZ_t = (Z|s < Z \le t)$

$$
\begin{array}{ccccccc}
\bar F_X \le \bar F_Y & \Leftrightarrow & X \le_{ST} Y & \Rightarrow & X \le_M Y & \Leftrightarrow & E(X) \le E(Y) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
{}_tX \ge_{ST} {}_tY & \Leftrightarrow & X \le_{RHR} Y & \Rightarrow & X \le_{MIT} Y & \Leftrightarrow & E({}_tX) \ge E({}_tY) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
{}_sX_t \le_{ST} {}_sY_t & \Leftrightarrow & X \le_{LR} Y & \Rightarrow & X \le_{DTM} Y & \Leftrightarrow & E({}_sX_t) \le E({}_sY_t) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
X_t \le_{ST} Y_t & \Leftrightarrow & X \le_{HR} Y & \Rightarrow & X \le_{MRL} Y & \Leftrightarrow & E(X_t) \le E(Y_t) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\bar F_X \le \bar F_Y & \Leftrightarrow & X \le_{ST} Y & \Rightarrow & X \le_M Y & \Leftrightarrow & E(X) \le E(Y)
\end{array}
$$

It is an open problem to determine if stochastic precedence comparisons have the transitive property. They do not have it when X and Y are dependent. Hence we do not know if they define a proper order. For that reason we do not use the notation $X \le_{SP} Y$. Moreover, note that if both X SP Y and Y SP X hold, then we do not know if X and Y have the same law. However, stochastic precedence is a reasonable way to compare the lifetimes of two independent units or systems. Moreover, Arcones et al. (2002) prove that if X and Y are two independent random variables defined over the same probability space and $X \le_{ST} Y$ holds, then X SP Y. Hence stochastic precedence is a necessary condition for the ST order to hold. Stochastic precedence comparisons can be used as an alternative to the mean order when the ST order does not hold.

3.2 Systems with IID or EXC Components

First of all we prove that the k-out-of-n systems with IID components are LR-ordered (as expected).

Proposition 3.1 *If F is absolutely continuous, then*

$$X_{i:n} \le_{LR} X_{j:m}$$

for all $i \le j$ and $n - i \ge m - j$.

Proof From (2.12), we get

$$f_{i:n}(t) = i\binom{n}{i} f(t)F^{i-1}(t)\bar F^{n-i}(t)$$

and

$$f_{j:m}(t) = j\binom{m}{j} f(t)F^{j-1}(t)\bar F^{m-j}(t).$$

Hence

$$\frac{f_{j:m}(t)}{f_{i:n}(t)} = c\frac{F^{j-i}(t)}{\bar{F}^{n-i-m+j}(t)}$$

for a constant $c > 0$. As F is increasing and $\bar{F}$ is decreasing, this ratio is increaing in t under the stated assumptions and so the LR order holds. □

As a consequence, in the IID case, we have that

$$X_{i:m} \leq_{LR} X_{i:n} \leq_{LR} X_{j:n}$$

for all $i, j, n, m \in \mathbb{Z}$ such that $1 \leq i \leq j \leq n \leq m$. Note that $X_{i:n}$ is LR increasing in i and LR decreasing in n. In particular, the k-out-of-n systems (order statistics) are LR ordered in the IID case, that is,

$$X_{1:n} \leq_{LR} \cdots \leq_{LR} X_{n:n}. \tag{3.4}$$

As the LR order is the strongest one, then

$$X_{1:n} \leq_{ORD} \cdots \leq_{ORD} X_{n:n} \tag{3.5}$$

for $ORD = HR, RHR, ST, MRL, MIT, DTM$. This property also hold if F is not absolutely continuous. Actually,

$$X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{n:n} \tag{3.6}$$

holds in the general case since $X_{1:n} \leq \cdots \leq X_{n:n}$. In the general case we also have

$$X_{1:n} \leq_{ST} \cdots \leq_{ST} X_{1:1},$$

for the series systems,

$$X_{1:1} \leq_{ST} \cdots \leq_{ST} X_{n:n},$$

for the parallel systems and, in general, $X_{i:n} \leq_{ST} X_{j:m}$ whenever $i \leq j$ and $n - i \geq m - j$.

However, surprisingly, we will see that neither

$$X_{1:n} \leq_{HR} \cdots \leq_{HR} X_{n:n} \tag{3.7}$$

nor

$$X_{1:n} \leq_{MRL} \cdots \leq_{MRL} X_{n:n}. \tag{3.8}$$

hold in the general (or the EXC) case. This fact was first proved in Navarro and Shaked (2006).

Now we are ready to prove the first ordering results for systems with IID components based on Samaniego's signature representation. They were obtained in Kochar et al. (1999) and allows us to compare two systems just by comparing their respective signatures. Note that the signatures of order n can be considered as probability mass functions of discrete distributions over $\{1, \ldots, n\}$. Then they can be ordered by using the orders defined above.

Theorem 3.1 (Kochar et al. 1999) *Let T_1 and T_2 be the lifetimes of two coherent systems based on n IID components with a common continuous distribution function F. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be their respective signatures. Then the following properties hold:*

(i) *If* $\boldsymbol{s}_1 \le_{ST} \boldsymbol{s}_2$, *then* $T_1 \le_{ST} T_2$ *for all* F;
(ii) *If* $\boldsymbol{s}_1 \le_{HR} \boldsymbol{s}_2$, *then* $T_1 \le_{HR} T_2$ *for all* F;
(iii) *If* $\boldsymbol{s}_1 \le_{LR} \boldsymbol{s}_2$, *then* $T_1 \le_{LR} T_2$ *for all abs. cont. distribution functions* F.

The proof is obtained from Samaniego's representation (2.5), the ordering properties of the k-out-of-n systems in (3.5) for the IID case and the preservation ordering properties for mixtures of ordered distributions given in Theorems 1.A.6, 1.B.14 and 1.C.17 of Shaked and Shanthikumar (2007). Stochastic precedence comparisons were obtained in Theorem 5.6 of Samaniego (2007), p. 70.

These results can be extended to the EXC case by using the representations for coherent and semi-coherent systems obtained in the preceding chapter. These results were obtained in Navarro et al. (2008). Note that they also hold for systems with ID component lifetimes and a common DD survival copula due to Theorem 2.13.

Theorem 3.2 (Navarro et al. 2008) *Let* T_1 *and* T_2 *be the lifetimes of two semi-coherent (or coherent) systems with component lifetimes* $(X_1, \ldots, X_n)$ *having an exchangeable joint distribution function* $\boldsymbol{F}$, *and signatures of order n (signatures),* $\boldsymbol{s}_1^{(n)}$ *and* $\boldsymbol{s}_2^{(n)}$, *respectively. Then the following properties hold:*

(i) *If* $\boldsymbol{s}_1^{(n)} \le_{ST} \boldsymbol{s}_2^{(n)}$, *then* $T_1 \le_{ST} T_2$ *for all* $\boldsymbol{F}$;
(ii) *If* $\boldsymbol{s}_1^{(n)} \le_{HR} \boldsymbol{s}_2^{(n)}$, *then* $T_1 \le_{HR} T_2$ *for all* $\boldsymbol{F}$ *such that (3.7) holds;*
(iii) *If* $\boldsymbol{s}_1^{(n)} \le_{HR} \boldsymbol{s}_2^{(n)}$, *then* $T_1 \le_{MRL} T_2$ *for all* $\boldsymbol{F}$ *such that (3.8) holds;*
(iv) *If* $\boldsymbol{s}_1^{(n)} \le_{LR} \boldsymbol{s}_2^{(n)}$, *then* $T_1 \le_{LR} T_2$ *for all absolutely continuous or discrete joint distribution functions* $\boldsymbol{F}$ *such that (3.4) holds.*

As in the IID case, this theorem is an immediate consequence of the signature representation for the EXC case (2.27) and the mixture preservation properties obtained in Shaked and Shanthikumar (2007). However, in this case, we need to assume the respective ordering properties for the k-out-of-n systems (except in the case of the ST order where they are always true). Note that in (iii) we need the HR order for the signatures to get the MRL order for the system lifetimes when the k-out-of-n systems are MRL ordered. The MRL order for the signatures is not enough. Similar results holds for the MIT and RHR orders (see Navarro and Rubio 2011). Let us see an example.

Example 3.1 Let us consider the systems with lifetimes $T_1 = \min(X_1, \max(X_2, X_3))$ and $T_2 = \max(\min(X_1, X_2), \min(X_3, X_4))$. Note that they are of different orders (or that the first one is a semi-coherent system of order 4). So we need the signatures of order 4 to compare them. They are $\boldsymbol{s}_1^{(4)} = (1/4, 5/12, 1/3, 0)$ and $\boldsymbol{s}_2^{(4)} = \boldsymbol{s}_2 = (0, 2/3, 1/3, 0)$, respectively. We also have to assume that (X_1, X_2, X_3, X_4) has an EXC joint distribution $\boldsymbol{F}$ (or that they are IID or just ID with a DD survival copula).

To check the ST order we need to compute the reliability functions $\boldsymbol{S}_1^{(4)}$ and $\boldsymbol{S}_2^{(4)}$ of the respective signatures. They are given in the following table:

$\boldsymbol{s}_1^{(4)}$	1/4	5/12	1/3	0
$\boldsymbol{S}_1^{(4)}$	1	3/4	1/3	0
$\boldsymbol{s}_2^{(4)}$	0	2/3	1/3	0
$\boldsymbol{S}_2^{(4)}$	1	1	1/3	0

As $\boldsymbol{S}_1^{(4)} \le \boldsymbol{S}_2^{(4)}$, then $\boldsymbol{s}_1^{(4)} \le_{ST} \boldsymbol{s}_2^{(4)}$ holds. Therefore, from Theorem 3.2, (i), $T_1 \le_{ST} T_2$ holds for all EXC joint distributions $\boldsymbol{F}$. This includes the IID$\sim F$ case for any univariate distribution function F. Note that these systems cannot be ordered by using Theorem 3.1.

Analogously, to check the HR order, we need to compute the ratio of the reliability functions $\boldsymbol{S}_1^{(4)}$ and $\boldsymbol{S}_2^{(4)}$ of the respective signatures. They are given in the following table:

$\boldsymbol{S}_2^{(4)}$	1	1	1/3	0
$\boldsymbol{S}_1^{(4)}$	1	3/4	1/3	0
$\boldsymbol{S}_2^{(4)}/\boldsymbol{S}_1^{(4)}$	1	4/3	1	–

Hence $\boldsymbol{s}_1^{(4)}$ and $\boldsymbol{s}_2^{(4)}$ are not HR ordered. Therefore, we do not know if T_1 and T_2 are HR ordered for all EXC joint distributions $\boldsymbol{F}$ such that (3.7) holds (or all F in the IID case). Note that Theorems 3.1 and 3.2 just include sufficient conditions for this ordering.

Finally, if we want to get the LR order, we need to compute the ratio of the respective signatures $\boldsymbol{s}_1^{(4)}$ and $\boldsymbol{s}_2^{(4)}$. It is given in the following table:

$\boldsymbol{s}_2^{(4)}$	0	2/3	1/3	0
$\boldsymbol{s}_1^{(4)}$	1/4	5/12	1/3	0
$\boldsymbol{s}_2^{(4)}/\boldsymbol{s}_1^{(4)}$	0	8/5	1	–

As expected, $\boldsymbol{s}_1^{(4)}$ and $\boldsymbol{s}_2^{(4)}$ are not LR ordered (since they are not HR ordered). So we do not know what happen with the system lifetimes in the LR order. Note again that Theorem 3.2 just includes sufficient conditions.

To illustrate these theoretical results we consider the IID case with a standard exponential distribution. The system reliability functions are plotted in Fig. 3.1, left. As expected they are ordered. This property holds for any distribution function F. Even more, it holds for any joint EXC distribution function $\boldsymbol{F}$. The code in R to get this plot is the following:

```
R<-function(t) exp(-t)
s1<-c(1/4,5/12,1/3,0)
s2<-c(0,2/3,1/3,0)
R14<-function(t) (R(t))^4
R24<-function(t) 4*(R(t))^ 3-3*(R(t))^4
R34<-function(t) 6*(R(t))^2-8*(R(t))^3+3*(R(t))^ 4
```

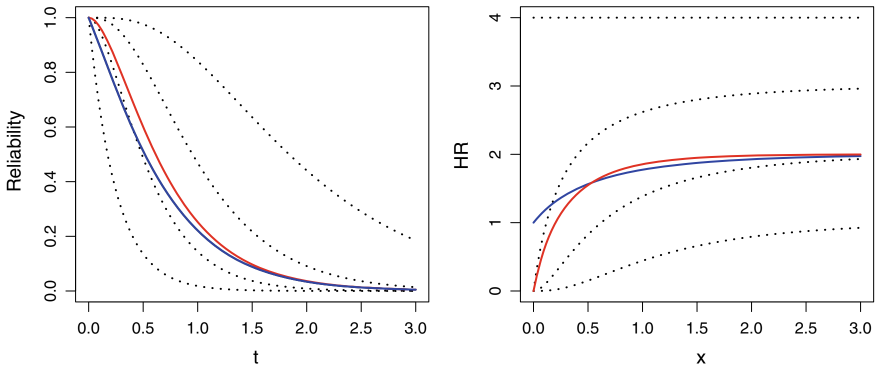

Fig. 3.1 Reliability functions (left) and hazard rate functions (right) for the systems T_1 (blue) and T_2 (red) in Example 3.1. The dotted lines correspond to the functions of the k-out-of-4 systems for $k = 1, 2, 3, 4$

```
R44<-function(t) 4*R(t)-6*(R(t))^2+4*(R(t))^ 3-1*(R(t))^4
R1<-function(t) {
  s1[1]*R14(t)+s1[2]*R24(t)+s1[3]*R34(t)+s1[4]*R44(t)
}
R2<-function(t) {
  s2[1]*R14(t)+s2[2]*R24(t)+s2[3]*R34(t)+s2[4]*R44(t)
}
curve(R14(x),xlab='t',ylab='Reliability',0,3,lty=3,lwd=2)
curve(R24(x),lty=3,add=T,lwd=2)
curve(R34(x),lty=3,add=T,lwd=2)
curve(R44(x),lty=3,add=T,lwd=2)
curve(R1(x),add=T,lwd=2)
curve(R2(x),add=T,col='red',lwd=2)
```

The system hazard rate functions are plotted in Fig. 3.1, right. In this case, they are not ordered. Thus, the second system is better when they are new but, from time $t = 0.5$ on (half a year if t is measured in years), the used systems with the first structure are a little bit better than that with the second. However, they have the same limiting behavior 2 when $t \to \infty$. Note that, in this example, the limiting behavior of the hazard rate function of the k-out-of-4 system is k for $k = 1, 2, 3, 4$ and that the common hazard rate of the components is $h(t) = 1$ for $t \geq 0$. The additional code to plot these hazard rate functions is the following:

```
f<-function(t) exp(-t)
f14<-function(t) f(t)*4*(R(t))^3
f24<-function(t) f(t)*(12*(R(t))^2-12*(R(t))^3)
f34<-function(t) f(t)*(12*R(t)-24*(R(t))^2+12*(R(t))^3)
f44<-function(t) f(t)*(4-12*R(t)+12*(R(t))^2-4*(R(t))^3)
```

```
f1<-function(t){
  s1[1]*f14(t)+s1[2]*f24(t)+s1[3]*f34(t)+s1[4]*f44(t)
}
f2<-function(t) {
  s2[1]*f14(t)+s2[2]*f24(t)+s2[3]*f34(t)+s2[4]*f44(t)
}
curve(f14(x)/R14(x),ylab='HR',0,3,lty=3,ylim=c(0,4),lwd=2)
curve(f24(x)/R24(x),lty=3,add=T,lwd=2)
curve(f34(x)/R34(x),lty=3,add=T,lwd=2)
curve(f44(x)/R44(x),lty=3,add=T,lwd=2)
curve(f1(x)/R1(x),add=T,col='blue',lwd=2)
curve(f2(x)/R2(x),add=T,col='red',lwd=2)
```
◀

Proceeding as in the preceding example we can obtain all the ordering properties for all the coherent systems with 1-4 components given in Table 2.1. They were obtained in Navarro et al. (2008) and are given in Figs. 3.2, 3.3 and 3.4. The systems with repeated signatures are not included in the graphs (since they are equal in law to other systems in the graphs). Note that in the EXC case, we need some extra-conditions for the HR, MRL and LR orders. We do not need them in the IID case. Also note that the graph for the ST and LR orders are symmetric, that is, $T_i \leq_{ORD} T_j$ iff the respective dual systems satisfy $T_j^D \leq_{ORD} T_i^D$. This is not the case for the HR and MRL orders. For the hazard rate order, we have $T_i \leq_{HR} T_j$ iff $T_j^D \leq_{RHR} T_i^D$ under the respective properties for the order statistics in the EXC case. A similar property holds for MRL and MIT orders.

As we have seen in the preceding example, when the signature ordering does not hold, we do not know if the systems are ordered since the theorems just contain sufficient conditions. In Navarro and Rubio (2011) it is proved that the conditions given in (i), (ii) and (iv) of the preceding theorem are actually necessary and sufficient

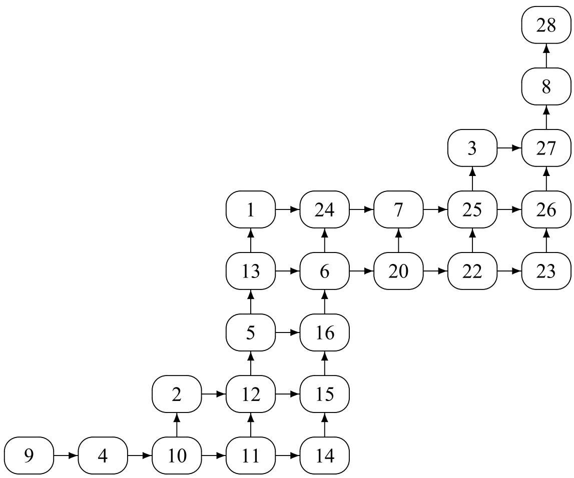

Fig. 3.2 ST orderings for the systems in Table 2.1 and an EXC $\boldsymbol{F}$. They also hold for the IID case

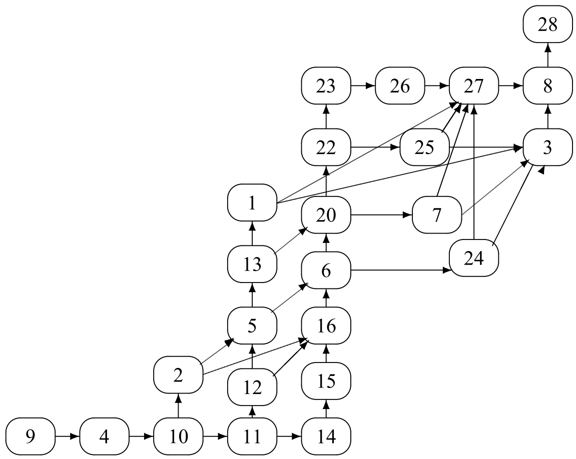

Fig. 3.3 HR (resp. MRL) orderings for the systems in Table 2.1 and an EXC $\boldsymbol{F}$ under (3.7) (resp. (3.8)). They also hold for the IID case (here both conditions hold and so it is better to get the HR order)

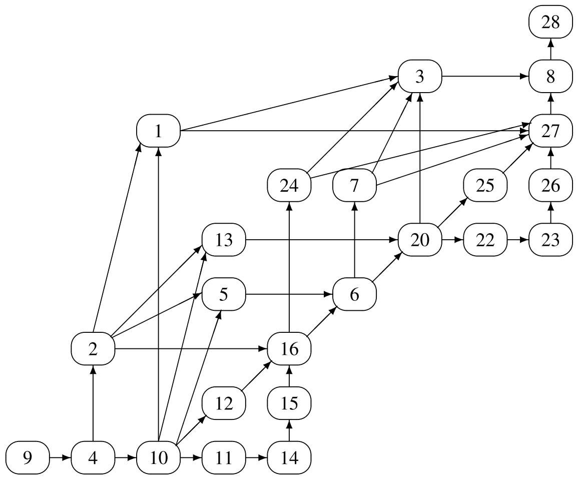

Fig. 3.4 LR orderings for the systems in Table 2.1 and and EXC $\boldsymbol{F}$ under (3.4). They also hold for the IID case for any absolute continuous distribution function F

conditions to have the ST, HR and LR orderings, respectively, for any exchangeable distribution function $\boldsymbol{F}$ under these conditions for the order statistics. This result can be stated as follows. This is not true for the MRL order and, as we will see later, it is not true for the IID case.

Theorem 3.3 (Navarro and Rubio, 2011) *Let T_1 and T_2 be the lifetimes of two semi-coherent (or coherent) systems with component lifetimes $(X_1, \dots, X_n)$ having an exchangeable joint distribution function $\boldsymbol{F}$, and signatures of order n (signatures), $\boldsymbol{s}_1^{(n)}$ and $\boldsymbol{s}_2^{(n)}$, respectively. Then the following properties hold:*

(i) $\boldsymbol{s}_1^{(n)} \le_{ST} \boldsymbol{s}_2^{(n)}$ *iff* $T_1 \le_{ST} T_2$ *for all* $\boldsymbol{F}$*;*
(ii) $\boldsymbol{s}_1^{(n)} \le_{HR} \boldsymbol{s}_2^{(n)}$ *iff* $T_1 \le_{HR} T_2$ *for all* $\boldsymbol{F}$ *such that (3.7) holds;*

(iii) $s_1^{(n)} \leq_{LR} s_2^{(n)}$ *iff* $T_1 \leq_{LR} T_2$ *for all absolutely continuous or discrete distribution functions* $\boldsymbol{F}$ *such that (3.4) holds.*

Proof The proofs of the "only if" parts are obtained from Theorem 3.2.

To prove the "if" part in (i), we assume that $T_1 \leq_{ST} T_2$ holds for all EXC $\boldsymbol{F}$. Then we consider a particular EXC $\boldsymbol{F}$ defined as follows. Let $(X_1, \dots, X_n)$ be a random vector defined as $X_i = \sigma(i)$ for $i = 1, \dots, n$, where $\sigma : [n] \to [n]$ is a randomly chosen permutation in the set of all the permutations of the set $[n]$. Clearly, the joint distribution $\boldsymbol{F}$ of $(X_1, \dots, X_n)$ is EXC since

$$(X_1, \dots, X_n) =_{ST} (X_{\tau(1)}, \dots, X_{\tau(n)})$$

holds for any permutation τ. Note that X_i has a uniform distribution on $[n]$ for all i. Moreover, as we choose a specific (common) permutation σ, the associated ordered data are $X_{i:n} = i$ for sure for all i. Therefore, $T_j = 1, \dots, n$ with probabilities $s_j^{(n)}$ for $j = 1, 2$. Hence, $T_1 \leq_{ST} T_2$ holds for this EXC distribution $\boldsymbol{F}$ which is equivalent to $s_1^{(n)} \leq_{ST} s_2^{(n)}$.

The proofs of the "if" parts of (ii) and (iii) are analogous taking into account that the specific EXC distribution defined above trivially satisfies (3.7) and (3.4) (since $X_{i:n} = i$ for sure for all i). □

This theorem assures that Figs. 3.2, 3.3 and 3.4 contain all the ordering properties for the EXC case. Navarro and Rubio (2011) noticed a surprising property: Some systems that cannot be ordered by using signatures of order n, can be ordered with signatures of order m for some $m > n$. This fact seems to be against the preceding theorem but this is not the case. Let us see an example that proves that Fig. 3.4 does not contain all the ordering properties for the IID case. We will see in the next section that the same happen for Fig. 3.3, providing a procedure to detect all the orderings for the IID case. We will also prove that Fig. 3.2 does contain all the ST orderings for the IID case.

Example 3.2 Let us consider the systems 5 and 24 from Table 2.1 with lifetimes $T_5 = \min(X_1, \max(X_2, X_3))$ and $T_{24} = \max(X_1, \min(X_2, X_3, X_4))$. Their signatures of order 4 are $s_5^{(4)} = (1/4, 5/12, 1/3, 0)$ and $s_{24}^{(4)} = (0, 1/2, 1/4, 1/4)$. The respective reliability vectors are:

$s_5^{(4)}$	1/4	5/12	1/3	0
$S_5^{(4)}$	1	3/4	1/3	0
$s_{24}^{(4)}$	0	1/2	1/4	1/4
$S_{24}^{(4)}$	1	1	1/2	1/4
$S_{24}^{(4)}/S_5^{(4)}$	1	4/3	3/2	$+\infty$

As $S_{24}^{(4)}/S_5^{(4)}$ is increasing, $s_5^{(4)} \leq_{HR} s_{24}^{(4)}$ holds. So we can connect these systems in Fig. 3.3 and their respective lifetimes satisfy $T_5 \leq_{HR} T_{24}$ for all EXC $\boldsymbol{F}$ satisfying (3.7). In particular this ordering holds for the IID case and the ST order holds for any EXC $\boldsymbol{F}$ (note that $s_5^{(4)} \leq_{ST} s_{24}^{(4)}$ holds).

However, to check the LR order we compute the following table:

$s_{24}^{(4)}$	0	1/2	1/4	1/4
$s_{5}^{(4)}$	1/4	5/12	1/3	0
$s_{24}^{(4)}/s_{5}^{(4)}$	0	6/5	3/4	$+\infty$

Therefore, $\boldsymbol{s}_5^{(4)}$ and $\boldsymbol{s}_{24}^{(4)}$ are not LR-ordered. So these systems are not LR ordered for all EXC $\boldsymbol{F}$ and so they are not connected in Fig. 3.4.

However, if we compute the respective signatures of order 5 from (2.29), we get

$$\boldsymbol{s}_5^{(5)} = \left(\frac{4}{5}\frac{1}{4}, \frac{1}{5}\frac{1}{4} + \frac{3}{5}\frac{5}{12}, \frac{2}{5}\frac{5}{12} + \frac{2}{5}\frac{1}{3}, \frac{3}{5}\frac{1}{3} + \frac{1}{5}0, \frac{4}{5}0\right) = \left(\frac{1}{5}, \frac{3}{10}, \frac{3}{10}, \frac{1}{5}, 0\right)$$

and

$$\boldsymbol{s}_{24}^{(5)} = \left(\frac{4}{5}0, \frac{1}{5}0 + \frac{3}{5}\frac{1}{2}, \frac{2}{5}\frac{1}{2} + \frac{2}{5}\frac{1}{4}, \frac{3}{5}\frac{1}{4} + \frac{1}{5}\frac{1}{4}, \frac{4}{5}\frac{1}{4}\right) = \left(0, \frac{3}{10}, \frac{3}{10}, \frac{1}{5}, 0\right).$$

Hence,

$s_{24}^{(5)}$	0	3/10	3/10	1/5	1/5
$s_{5}^{(5)}$	1/5	3/10	3/10	1/5	0
$s_{24}^{(5)}/s_{5}^{(5)}$	0	1	1	1	$+\infty$

Therefore $\boldsymbol{s}_5^{(5)} \leq_{LR} \boldsymbol{s}_{24}^{(5)}$ holds and, from Theorem 3.2, (iv), $T_5 \leq_{LR} T_{24}$ for any EXC joint distribution $\boldsymbol{F}$.

This property seems to contradict the property obtained with the signatures of order 4 taking into account that these properties are equivalent from Theorem 3.3, (iii). What is the explanation?

The answer is the following. Note that we have proved that $T_5 \leq_{LR} T_{24}$ for all EXC $\boldsymbol{F}$ of dimension 5. However, this is not true for all EXC $\boldsymbol{F}$ of dimension 4. In particular, this property fails for the distribution of dimension 4 constructed in the proof of Theorem 3.3 (since the systems' probability mass values $\boldsymbol{s}_5^{(4)}$ and $\boldsymbol{s}_{24}^{(4)}$ are not LR-ordered). This is due to the fact that this particular EXC distribution of dimension 4 cannot be extended (or included) in an exchangeable distribution of order 5. Note that we can affirm that $T_5 \leq_{LR} T_{24}$ holds for all EXC $\boldsymbol{F}$ of dimension 4 that can be extended (e.g. that are marginals) of EXC distributions of dimension 5. This is actually what happen in the IID case that can be extended to any dimension. So we can affirm that $T_5 \leq_{LR} T_{24}$ holds for the IID case and all distributions F (of dimension 1). Then note that we can connect these systems in the graph for the IID case. In the next section we will see how to complete the graphs for the IID case for all the orderings. ◀

3.3 Systems with ID Components

Recall that, from the preceding chapter, if the component lifetimes of a system are identically distributed with a common distribution F and a common reliability

$\bar{F} = 1 - F$, then the respective system's functions can be written as

$$F_T(t) = q(F(t)) \tag{3.9}$$

and

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)) \tag{3.10}$$

for all t, where q and $\bar{q}$ are two (univariate) distortion functions satisfying $\bar{q}(u) = 1 - q(1-u)$ for all $u \in [0, 1]$. These distortion functions are increasing and continuous and depend on the system structure (minimal path or cut sets) and on the dependence between the component lifetimes (copula or survival copula).

Hence, we can apply to systems with ID components all the ordering properties obtained for distorted distributions in Navarro et al. (2013, 2014) and Navarro and Gomis (2016). They are stated in the following proposition. We say that a function g is **bathtub (upside-down bathtub)** shaped if there exist $t_1 \le t_2$ such that $g(t)$ is decreasing (increasing) for $t \le t_1$, constant for $t \in [t_1, t_2]$, and increasing (decreasing) for $t \ge t_2$. In many applications, the hazard rate functions of the components are bathtub shaped.

Proposition 3.2 *If T_i has the reliability function $\bar{q}_i(\bar{F}(t))$ and the distribution function $q_i(F(t))$ for $i = 1, 2$, then the following properties hold:*

(i) $T_1 \le_{ST} T_2$ *for all F iff $\bar{q}_2 \ge \bar{q}_1$ (or $q_2 \le q_1$) in* $(0, 1)$*;*
(ii) $T_1 \le_{HR} T_2$ *for all F iff $\bar{q}_2/\bar{q}_1$ decreases in* $(0, 1)$*;*
(iii) $T_1 \le_{RHR} T_2$ *for all F iff q_2/q_1 increases in* $(0, 1)$*;*
(iv) $T_1 \le_{LR} T_2$ *for all absolutely continuous distribution functions F iff $\bar{q}_2'/\bar{q}_1'$ decreases (or q_2'/q_1' increases) in* $(0, 1)$*;*
(v) $T_1 \le_{MRL} T_2$ *for all F such that $E(T_1) \le E(T_2)$ if $\bar{q}_2/\bar{q}_1$ is bathtub in* $(0, 1)$.

Proof The proof (i) is immediate.

To prove (ii) we note that $T_1 \le_{HR} T_2$ holds iff

$$\frac{\bar{q}_2(\bar{F}(t))}{\bar{q}_1(\bar{F}(t))}$$

is increasing in t. Clearly, this property holds when $\bar{q}_2/\bar{q}_1$ decreases in $(0, 1)$ since $\bar{F}$ is decreasing. Conversely, if $T_1 \le_{HR} T_2$ holds for all F, then it holds for a continuous F (e.g. a standard exponential or a uniform distribution), and then $\bar{q}_2(u)/\bar{q}_1(u)$ is decreasing for $u \in (0, 1)$.

The proof of (iii) is similar to that of (ii).

To prove (iv) we recall that the respective PDF can be written as $f_i(t) = f(t)q_i'(F(t))$ for $i = 1, 2$, where $f = F'$ is the common baseline PDF. Hence, $T_1 \le_{LR} T_2$ holds iff the ratio

$$\frac{f_2(t)}{f_1(t)} = \frac{\bar{q}_2'(\bar{F}(t))}{\bar{q}_1'(\bar{F}(t))}$$

is increasing in t. Clearly, this property holds when $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$ since $\bar{F}$ is decreasing. Conversely, if $T_1 \leq_{LR} T_2$ holds for all F, then it holds for a continuous F and so $\bar{q}'_2/\bar{q}'_1$ decreases in $(0, 1)$. The proof for q'_2/q'_1 is similar.

Finally, to prove (v), we note that if $\bar{q}_2/\bar{q}_1$ is bathtub in $(0, 1)$, then the ratio

$$\frac{\bar{F}_2(t)}{\bar{F}_1(t)} = \frac{\bar{q}_2(\bar{F}(t))}{\bar{q}_1(\bar{F}(t))}$$

is bathtub in t. Hence, from the results given in Belzunce et al. (2013), $T_1 \leq_{MRL} T_2$ holds for all F such that $E(T_1) \leq E(T_2)$. □

Note that in all the orderings we have necessary and sufficient conditions except in (v) where we just have a sufficient condition and that there we need the additional condition $E(T_1) \leq E(T_2)$. Note that if $\bar{q}_2/\bar{q}_1$ is decreasing (or increasing), then we get the HR order from (ii) which is stronger than the MRL order. Moreover, we do not need the additional assumption $E(T_1) \leq E(T_2)$.

Clearly, these properties can be applied to compare systems with ID components having a common distribution function F by using the distortion representations obtained in Sect. 2.4. The result for the ST order can be stated as follows.

Proposition 3.3 *Let T_1 and T_2 be the lifetimes of two semi-coherent (or coherent) systems with ID component lifetimes having an common distribution function F, and distortion functions q_1 and q_2, respectively. Then the following properties are equivalent:*

(i) $\bar{q}_1 \leq \bar{q}_2$ *(or* $q_1 \geq q_2$*) in* $(0, 1)$*;*
(ii) $T_1 \leq_{ST} T_2$ *for all* F*;*
(iii) $T_1 \leq_{ST} T_2$ *for a continuous* F.

Proof From Proposition 3.2, (i), we have that (i) implies (ii).

Clearly, (ii) implies (iii).

Finally, if (iii) holds, then $T_1 \leq_{ST} T_2$ for a continuous F, that is, $\bar{F}_{T_1} \leq \bar{F}_{T_2}$. Hence, if $0 < u < 1$, then there exists t such that $\bar{F}(t) = u$ (since F is continuous). Therefore

$$\bar{q}_1(u) = \bar{q}_1(\bar{F}(t)) = \bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t) = \bar{q}_2(\bar{F}(t)) = \bar{q}_2(u)$$

for all $u \in (0, 1)$. □

For the HR order we have the following result.

Proposition 3.4 *Let T_1 and T_2 be the lifetimes of two semi-coherent (or coherent) systems with ID component lifetimes having an common distribution function F, and distortion functions q_1 and q_2, respectively. Then the following properties are equivalent:*

(i) $\bar{q}_2/\bar{q}_1$ *is decreasing in* $(0, 1)$*;*

(ii) $T_1 \leq_{HR} T_2$ *for all* F*;*
(iii) $T_1 \leq_{HR} T_2$ *for a continuous* F*.*

The proof is similar to that of the ST order. Note that in both cases, the systems may have different orders (i.e. numbers of components), different structures and different dependency relationships (copulas). The only requirement is that they have a common distribution function F. Also note that then we get distribution-free ordering results, that is, comparisons for any F. Similar results can be stated for the RHR and LR orders from Proposition 3.2. In the last case we need to assume that the respective distortion functions are differentiable. However, the result for the MRL ordering is different. It can be stated as follows.

Proposition 3.5 *Let* T_1 *and* T_2 *be the lifetimes of two semi-coherent (or coherent) systems with ID component lifetimes having an common distribution function* F*, and distortion functions* q_1 *and* q_2*, respectively. If* $\bar{q}_2/\bar{q}_1$ *is bathtub in* $(0, 1)$*, then* $T_1 \leq_{MRL} T_2$ *for all* F *such that* $E(T_1) \leq E(T_2)$.

The converse property does not hold (for strict bathtub shaped functions, that is, with both strict decreasing and strict increasing pieces). A counterexample can be seen in Navarro and Gomis (2016). Let us see how to apply the preceding results to systems with dependent ID components.

Example 3.3 Let us consider a series system and a parallel system with ID components having a common reliability $\bar{F}$ and a survival copula $\widehat{C}$. The reliability function of the series system $X_{1:2}$ can be written as

$$\bar{F}_{1:2}(t) = \Pr(X_{1:2} > t) = \Pr(X_1 > t, X_2 > t) = \widehat{C}(\bar{F}(t), \bar{F}(t)) = \bar{q}_{1:2}(\bar{F}(t)),$$

where $q_{1:2}(u) = \widehat{C}(u, u)$ is the diagonal section of the copula $\widehat{C}$.

Analogously, the reliability function of the parallel system $X_{2:2}$ is

$$\begin{aligned}\bar{F}_{2:2}(t) &= \Pr(\max(X_1, X_2) > t)\\ &= \Pr(X_1 > t) + \Pr(X_2 > t) - \Pr(X_1 > t, X_2 > t)\\ &= 2\bar{F}(t) - \widehat{C}(\bar{F}(t), \bar{F}(t))\\ &= \bar{q}_{2:2}(\bar{F}(t)),\end{aligned}$$

where $q_{2:2}(u) = 2u - \widehat{C}(u, u)$ for $u \in [0, 1]$.

Note that, in this case (and in the general case), we know that

$$X_{1:2} \leq_{ST} X_i \leq_{ST} X_{2:2}$$

holds for $i = 1, 2$, for all F and for all $\widehat{C}$.

From Proposition 3.4, $X_{1:2} \leq_{HR} X_i$ holds for all $\bar{F}$ iff the ratio

$$\frac{\bar{q}_{1:2}(u)}{\bar{q}_i(u)} = \frac{\widehat{C}(u, u)}{u}$$

is increasing in (0, 1). In a similar way, $X_i \leq_{HR} X_{2:2}$ holds for all F iff

$$\frac{\bar{q}_{2:2}(u)}{\bar{q}_i(u)} = \frac{2u - \widehat{C}(u,u)}{u}$$

decreases in (0, 1), that is, iff $\widehat{C}(u,u)/u$ is increasing in (0, 1). Analogously, it can also be proved that $X_{1:2} \leq_{HR} X_{2:2}$ holds for all F iff the same condition holds (i.e. $\widehat{C}(u,u)/u$ is increasing in (0, 1)). Therefore, in the ID case, these orderings are equivalent and they will just depend on the copula $\widehat{C}$ (they are distribution-free with respect to F).

Of course, if the components are IID, that is, $\widehat{C}(u,v) = uv$ for $u, v \in [0,1]$, then $\widehat{C}(u,u)/u = u$, which is increasing, and so

$$X_{1:2} \leq_{HR} X_i \leq_{HR} X_{2:2} \tag{3.11}$$

holds for $i = 1, 2$ and for all F. This is a well known property already obtained in the preceding section (by using the LR order).

Analogously, if we consider the following Clayton–Oakes copula

$$\widehat{C}(u,v) = \frac{uv}{u+v-uv}, \quad u, v \in [0,1], \tag{3.12}$$

which induces a positive dependence between the components, we get

$$\frac{\widehat{C}(u,u)}{u} = \frac{u^2}{2u^2 - u^3} = \frac{1}{2-u}$$

which is increasing in (0, 1). So (3.11) holds for all F and this copula. Of course, the same MRL orderings also hold for any F. However, there exist copulas such that this condition does not hold (see Example 4.1 in Navarro et al. 2018).

Let us study now the LR orderings. Thus, $X_{1:2} \leq_{LR} X_i$ holds for all F iff $\bar{q}'_{1:2}(u)/\bar{q}'_i(u) = \bar{q}'_{1:2}(u)$ is increasing in (0, 1), that is, when $\bar{q}_{1:2}(u)$ is convex in (0, 1). This is also the condition for the other LR orderings. In the IID case $\bar{q}_{1:2}(u) = u^2$ is convex in (0, 1). Thus we can prove again that

$$X_{1:2} \leq_{LR} X_i \leq_{LR} X_{2:2} \tag{3.13}$$

holds for any F in the IID case. For the copula (3.12), we note that

$$\bar{q}_{1:2}(u) = \widehat{C}(u,u) = \frac{u}{2-u}$$

is convex in (0, 1) and so (3.13) holds for any F.

To illustrate these theoretical results we consider a standard exponential distribution F, and then we plot in Fig. 3.5 the reliability functions (left) and the hazard rate functions (right) of these systems for the IID case (dashed lines) and the copula in (3.12) (continuous lines). The R-code to get these plots is the following:

```
# Reliability functions
#IID case:
R<-function(t) exp(-t)
qIID<-function(u) u^2
G12<-function(t) qIID(R(t))
```

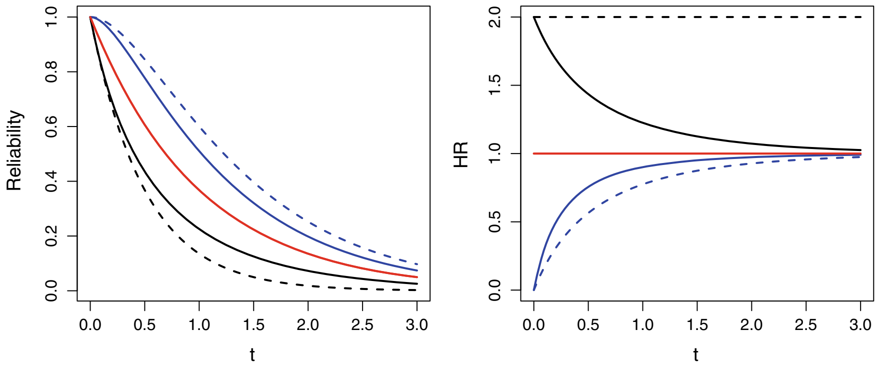

Fig. 3.5 Reliability (left) and hazard rate functions (right) for the series system $X_{1:2}$ (black), the components X_i (red) and the parallel system $X_{2:2}$ (blue) in Example 3.3 for the case of IID components (dashed lines) and dependent ID components (continuous lines) with the survival copula in (3.12)

```
G22<-function(t) 2*R(t)-G12(t)
curve(G12(x),xlab='t',ylab='Reliability',0,3,lty=2,lwd=2)
curve(G22(x),add=T,col='blue',lty=2,lwd=2)
curve(R(x),add=T,col='red',lwd=2)
#ID-C case:
C<-function(u,v) u*v/(u+v-u*v)
q<-function(u) C(u,u)
R12<-function(t) q(R(t))
R22<-function(t) 2*R(t)-R12(t)
curve(R12(x),xlab='t',add=T,lwd=2)
curve(R22(x),add=T,col='blue',lwd=2)
curve(R(x),add=T,col='red',lwd=2)
```

```
# Hazard rate functions
#IID case:
f<-function(t) exp(-t)
qpIID<-function(u) 2*u
g12<-function(t) f(t)*qpIID(R(t))
g22<-function(t) 2*f(t)-g12(t)
curve(g12(x)/G12(x),ylab='HR',0,3,ylim=c(0,2),lty=2,lwd=2)
curve(g22(x)/G22(x),add=T,col='blue',lty=2,lwd=2)
curve(f(x)/R(x),add=T,col='red',lwd=2)
#ID-C case:
qp<-function(u) 2/(2-u)^2
f12<-function(t) f(t)*qp(R(t))
f22<-function(t) 2*f(t)-f12(t)
```

```
curve(f12(x)/R12(x),add=T,lwd=2)
curve(f22(x)/R22(x),add=T,col='blue',lwd=2)
curve(f(x)/R(x),add=T,col='red',lwd=2)
```

Analogously, we can compare the systems obtained in the IID case with that obtained with the copula $\widehat{C}$. For example, $X_{1:2}^{IID} \le_{HR} X_{1:2}^{\widehat{C}}$ holds for all F since

$$\frac{\bar{q}_{1:2}^{\widehat{C}}(u)}{\bar{q}_{1:2}^{IID}(u)} = \frac{u/(2-u)}{u^2} = \frac{1}{2u-u^2}$$

is decreasing in $(0, 1)$. Analogously, we get that $X_{2:2}^{IID} \ge_{HR} X_{2:2}^{\widehat{C}}$ holds for all F since

$$\frac{\bar{q}_{2:2}^{\widehat{C}}(u)}{\bar{q}_{2:2}^{IID}(u)} = \frac{2u-u/(2-u)}{2u-u^2} = \frac{3-2u}{(2-u)^2}$$

is increasing in $(0, 1)$. Note that the series system improves with the positive dependency but that the parallel system get worse (see Fig. 3.5). ◀

As we have seen in the preceding example, the (distribution-free) ordering properties between two systems with ID components will just depend on the copula, that is, the dependence structure. So they can be related to well known positive/negative dependence properties. These relationships where studied in Navarro et al. (2018) and Navarro et al. (2021). For instance, the results obtained in the preceding example for series and parallel systems with two ID components can be stated as follows.

Proposition 3.6 *Let X_1 and X_2 be the lifetimes of two components having a common distribution function F and copula and survival copula C and $\widehat{C}$, respectively. Then the following properties are equivalent:*

(i) $X_{1:2} \le_{HR} X_1$ *for all* F*;*
(ii) $X_1 \le_{HR} X_{2.2}$ *for all* F*;*
(iii) $X_{1:2} \le_{HR} X_{2:2}$ *for all* F*;*
(iv) $\widehat{C}(u,u)/u$ *is increasing in* $(0, 1)$*;*
(v) $(1 - C(u,u))/(1-u)$ *is increasing in* $(0, 1)$.

Note that to prove (iv) (or (v)) we just need one of these orderings for a continuous distribution function F. Also note that in the ID case, as $\bar{F}_{2:2} = 2\bar{F} - \bar{F}_{1:2}$, then

$$\bar{F}(t) = \frac{1}{2}\bar{F}_{1:2}(t) + \frac{1}{2}\bar{F}_{2:2}(t)$$

for all t, that is, the common components' distribution is a uniform mixture of the distributions of the series and the parallel system. So the HR function of the components will be always between that of series and parallel systems (for any copula). This fact explains why the orderings stated in the preceding proposition are equivalent. For the LR order, we have the following result. The conditions for the RHR order can be seen in Theorem 4.2 of Navarro et al. (2018).

Proposition 3.7 *Let X_1 and X_2 be the lifetimes of two components having a common absolutely continuous distribution function F and copula and survival copula C and $\widehat{C}$, respectively. Then the following properties are equivalent:*

(i) $X_{1:2} \le_{LR} X_1$ *for all* F*;*
(ii) $X_1 \le_{LR} X_{2.2}$ *for all* F*;*
(iii) $X_{1:2} \le_{LR} X_{2:2}$ *for all* F*;*
(iv) $\widehat{C}(u,u)$ *is convex in* $(0,1)$.
(v) $C(u,u)$ *is convex in* $(0,1)$.

Analogously, the condition for the comparisons of the IID case with the DID case are the following. They were obtained in Proposition 17 of Navarro et al. (2021).

Proposition 3.8 *Let X_1 and X_2 be the lifetimes of two components having a common distribution function F and survival copula $\widehat{C}$. Let $\delta_{\widehat{C}}(u) = \widehat{C}(u,u)$ for $u \in [0,1]$. Let Y_1 and Y_2 be two IID lifetimes with distribution F.*

(i) $Y_{1:2} \le_{ST} X_{1:2}$ $(\ge_{ST})$ *for all* F *iff* $u^2 \le \delta_{\widehat{C}}(u)$ $(\ge)$ *for all* $u \in (0,1)$*;*
(ii) $Y_{1:2} \le_{HR} X_{1:2}$ $(\ge_{HR})$ *for all* F *iff* $\delta_{\widehat{C}}(u)/u^2$ *is decreasing (increasing) in* $(0,1)$*;*
(iii) $Y_{1:2} \le_{LR} X_{1:2}$ $(\ge_{LR})$ *for all abs. cont.* F *iff* $\delta'_{\widehat{C}}(u)/u$ *is decreasing (increasing) in* $(0,1)$*;*
(iv) $Y_{2:2} \ge_{ST} X_{2:2}$ $(\le_{ST})$ *for all* F *iff* $u^2 \le \delta_{\widehat{C}}(u)$ $(\ge)$ *for all* $u \in (0,1)$*;*
(v) $Y_{2:2} \ge_{HR} X_{2:2}$ $(\le_{HR})$ *for all* F *iff* $(2u - \delta_{\widehat{C}}(u))/(2u - u^2)$ *is increasing (decreasing) in* $(0,1)$*;*
(vi) $Y_{2:2} \ge_{LR} X_{2:2}$ $(\le_{LR})$ *for all abs. cont.* F *iff* $(2 - \delta'_{\widehat{C}}(u))/(1-u)$ *is increasing (decreasing) in* $(0,1)$.

Note that $Y_{1:2} \le_{ST} X_{1:2}$ $(\ge_{ST})$ holds iff $Y_{2:2} \ge_{ST} X_{2:2}$ $(\le_{ST})$. However, the other orderings are not equivalent. A random vector (X_1, X_2) is Positive (Negative) Quadrant Dependent, shortly written as PQD (NQD), if $\boldsymbol{F}(x,y) \ge F_1(x)F_2(y)$ $(\le)$ for all x, y, (see, e.g., Joe 1997). If F_1, F_2 are continuous, these (dependence) properties only depend on the copula.

Proposition 3.9 *Let X_1 and X_2 be the two random variables having distribution functions F_1 and F_2 and copula and survival copula C and $\widehat{C}$, respectively. Then the following properties are equivalent:*

(i) (X_1, X_2) *is PQD (NQD) for all* F_1, F_2*;*
(ii) (X_1, X_2) *is PQD (NQD) for two continuous distributions* F_1, F_2*;*
(iii) $C(u,v) \ge uv$ $(\le)$ *for all* $u, v \in [0,1]$*;*
(iv) $\widehat{C}(u,v) \ge uv$ $(\le)$ *for all* $u, v \in [0,1]$.

Note that, in the ID case, $Y_{1:2} \le_{ST} X_{1:2}$ $(\ge_{ST})$ and $Y_{2:2} \ge_{ST} X_{2:2}$ $(\le_{ST})$ hold for all F when (X_1, X_2) is PQD (NQD). Thus the series system is better under a

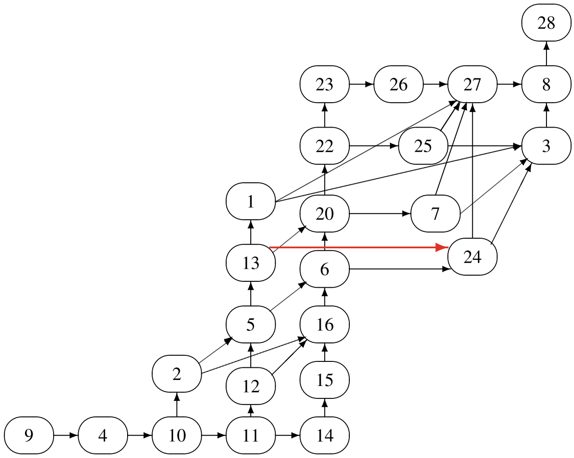

Fig. 3.6 All the HR orderings for the systems in Table 2.1 and IID components. The red arrow is an ordering that cannot be obtained by using signatures of order 4

positive dependence but the opposite holds for the parallel system (as we have seen in the preceding example). These orderings are reverted for the NQD condition. The other conditions can also be related with dependence properties (see next section). These are expectable properties (series systems improve under positive dependence since both component lifetimes are similar while parallel systems does so when they are different).

Note that the necessary and sufficient conditions obtained above can also be used to obtain all the distribution-free comparisons of coherent (or semi-coherent) systems with IID components. All the orderings for systems with 1-4 components (given in Table 2.1) were obtained in Navarro (2016). In some cases, these results improve the results obtained by using signatures (see the preceding section). For example, for the HR order we obtain the relationships given in Fig. 3.6. Note that we have a new ordering ($13 \to 24$) that cannot be obtained from signatures of order 4 (see Fig. 3.3). Analogously, for the LR order, we obtain the relationships given in Fig. 3.7. Note that we have three new orderings ($13 \to 24$, $5 \to 24$ and $13 \to 7$) that cannot be obtained from signatures of order 4 (see Fig. 3.4). For the ST order we obtain the same orderings given in the preceding section (see Fig. 3.2). However, for $n = 5$ and $n = 6$, there exist systems that can be ST-ordered with distortion functions but that cannot be ordered with signatures (see Rychlik et al. 2018). Moreover, note that the results based on distortions can also be used to check the ordering conditions for k-out-of-n systems needed in the ordering results based on signatures for the EXC case. For example, for $n = 3$, we can check if $X_{1:3} \leq_{HR} X_{2:3} \leq_{HR} X_{3:3}$ holds for a given copula C.

In other situations we may want to study if, for a fixed system (structure) and a fixed dependence (copula), an order is preserved. Thus, if the components $X_1, \ldots, X_n$ are ID$\sim F$ and $Y_1, \ldots, Y_n$ are ID$\sim G$, they share the same copula C and $F \leq_{ORD} G$ holds, we want to study if $T_1 \leq_{ORD} T_2$ holds (or holds under some conditions) for a given order ORD, where $T_1 = \phi(X_1, \ldots, X_n)$ and $T_2 = \phi(Y_1, \ldots, Y_n)$ are the lifetimes of two systems having the same structure.

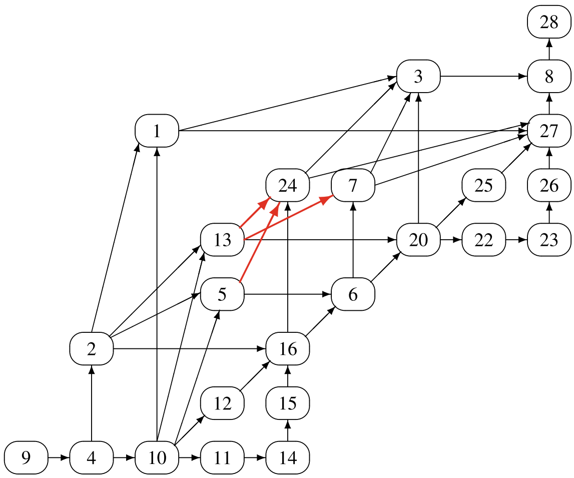

Fig. 3.7 All the LR orderings for the systems in Table 2.1 and IID components. The red arrows are three orderings that cannot be obtained by using signatures of order 4

To this end we can use the following ordering results for distorted distributions extracted from Navarro et al. (2013). The similar results for the non-ID case were obtained in Navarro et al. (2016).

Proposition 3.10 *Let X and Y be the two random variables having absolutely continuous distribution functions F_X and F_Y. Let T and S be two random variables having distribution functions $q(F_X)$ and $q(F_Y)$ for a distortion function q. Let $\bar{q}$ be the dual distortion function and let $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$, $\bar{\alpha}(u) = uq'(u)/q(u)$, and $\beta(u) = u\bar{q}''(u)/\bar{q}(u)$.*

(i) If $X \leq_{ST} Y$, then $T \leq_{ST} S$;
(ii) If $X \leq_{HR} Y$ and α is decreasing in $(0, 1)$, then $T \leq_{HR} S$;
(iii) If $X \leq_{RHR} Y$ and $\bar{\alpha}$ is increasing in $(0, 1)$, then $T \leq_{RHR} S$;
(iv) If $X \leq_{LR} Y$ and β is non-negative and decreasing in $(0, 1)$, then $T \leq_{LR} S$.

Proof The proof of (i) is immediate (since q and $\bar{q}$ are increasing functions).

To prove (ii), we assume $X \leq_{HR} Y$, that is, $h_X \geq h_Y$ holds for the respective hazard rate functions. Hence, $X \leq_{ST} Y$ also holds, that is, $\bar{F}_X \leq \bar{F}_Y$. Then we use (2.34) and that α is decreasing and non-negative to get

$$h_T(t) = \alpha(\bar{F}_X(t))h_X(t) \geq \alpha(\bar{F}_Y(t))h_Y(t) = h_S(t)$$

for all t, for the respective hazard rate functions of T and S. Then $T \leq_{HR} S$ holds.

The proof of (iii) is similar to the preceding one from (2.35).

Finally, to prove (iv), we note that $X \leq_{LR} Y$ implies $\eta_X \geq \eta_Y$ for the respective Glaser's eta functions defined in the first section of this chapter. Moreover, $X \leq_{LR} Y$ implies $X \leq_{HR} Y$ (i.e. $h_X \geq h_Y$) and $X \leq_{ST} Y$ (i.e. $\bar{F}_X \leq \bar{F}_Y$). Then we use (2.33) and that β is decreasing and non-negative to get

$$\eta_T(t) = \eta_X(t) + \beta(\bar{F}(t))h_X(t) \geq \eta_Y(t) + \beta(\bar{F}_Y(t))h_Y(t) = h_S(t)$$

for all t, for the respective Glaser's eta functions of T and S. Then $T \leq_{LR} S$ holds. □

Note that the ST-order is always preserved. However, we need some conditions for the preservations of the other orders. An alternative condition for the preservation of the HR is that the function $\bar{q}(uv)/\bar{q}(u)$ is increasing in $(0, 1)^2$. It can be proved that the HR order is preserved in k-out-of-n systems with IID components (i.e. α is decreasing for that systems). However, this is not the case for other coherent systems. Let us see an example.

Example 3.4 Let us consider the two systems with lifetimes T_1 and T_2, with a common structure $\phi(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3))$ and with IID components having distribution functions F and G, respectively. Then the common dual distortion function for these systems is

$$\bar{q}(u) = u + u^2 - u^3.$$

Hence,

$$\alpha(u) = u\frac{1 + 2u - 3u^2}{u + u^2 - u^3} = \frac{1 + 2u - 3u^2}{1 + u - u^2}.$$

By plotting α in $[0, 1]$, we see that it is non monotone (it first increases and then decreases). Therefore we do not know if the HR order is preserved. For example let us consider IID components having the reliability function

$$\bar{F}(t; a) = 1 - (1 - e^{-t})^a,\ t \geq 0 \tag{3.14}$$

for $a = 2, 5$. Then we plot in Fig. 3.8, left, the reliability functions of the components $\bar{F}(t) = \bar{F}(t; 2)$ (black dashed lines) and $\bar{G}(t) = \bar{F}(t; 5)$ (red dashed lines) and that of the respective systems (black and red continuous lines). As we can see, the components are ST ordered and this order is preserved in the systems (i.e. the system with the most reliable component, is more reliable than the other). In Fig. 3.8, right, we plot the hazard rate functions of the components (dashed lines) and the systems (continuous lines). As we can see, $F \leq_{HR} G$ holds. However, the hazard rate functions of the systems are not ordered. The code in R to get these plots is the following:

```
# Reliability functions:
R1<-function(t) 1-(1-exp(-t))^2
R2<-function(t) 1-(1-exp(-t))^5
q<-function(u) u+u^2-u^3
RT1<-function(t) q(R1(t))
RT2<-function(t) q(R2(t))
curve(RT1(x),xlab='t',ylab='Reliability',0,7,lwd=2)
curve(RT2(x),add=T,col='red',lwd=2)
curve(R1(x),add=T,lty=2,lwd=2)
curve(R2(x),add=T,col='red',lty=2,lwd=2)
```

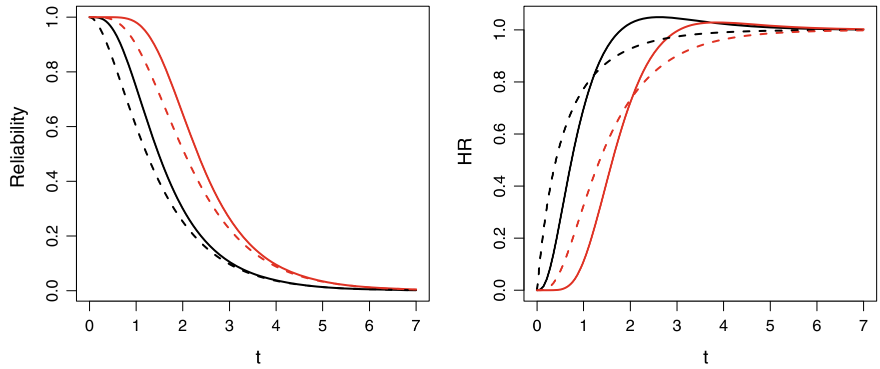

Fig. 3.8 Reliability functions (left) and hazard rate functions (right) for the components (dashed lines) and the systems (continuous lines) in Example 3.4 for the case of IID components with the reliability function in (3.14) and $a = 2$ (black) and $a = 5$ (red)

```
# Hazard rate functions:
f1<-function(t) 2*exp(-t)*(1-exp(-t))
f2<-function(t) 5*exp(-t)*(1-exp(-t))^4
qp<-function(u) 1+2*u-3*u^2
fT1<-function(t) f1(t)* qp(R1(t))
fT2<-function(t) f2(t)* qp(R2(t))
curve(fT1(x)/RT1(x),xlab='t',ylab='HR',0,7,lwd=2)
curve(fT2(x)/RT2(x),add=T,col='red',lwd=2)
curve(f1(x)/R1(x),add=T,lty=2,lwd=2)
curve(f2(x)/R2(x),add=T,col='red',lty=2,lwd=2)
```
◀

3.4 Systems with Non-ID Components

First, we recall that, from the representation results obtained in the preceding chapter, the system distribution function can be written (in the general case) as

$$F_T(t) = Q(F_1(t), \dots, F_n(t)),$$

and its reliability function as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)),$$

that is, they are generalized distorted distributions from the distributions of the component lifetimes. The explicit expression for the distortion functions Q and $\bar{Q}$ can be obtained from the minimal path (or cut) sets representation and the survival copula $\widehat{C}$ (or the copula C). So they only depend on the structure function and the dependence between the components (i.e. they do not depend on $F_1, \dots, F_n$).

If the components are independent (IND), then the function $\bar{Q}$ is a multinomial and it is known as the **reliability function of the structure** (see Barlow and Proschan 1975, p. 21). Actually, this multinomial is the one obtained in the pivotal decomposition (1.3) or in representation based on the Möbius transform (1.10) when these Boolean functions are extended to real numbers. In this case Q is also a multinomial.

In both cases we can use the following ordering results for generalized distorted distributions obtained in Navarro et al. (2016) (arbitrary components) and in Navarro and del Águila (2017) (ordered components). Note that we have necessary and sufficient conditions for the ST, HR and RHR orders. In Navarro et al. (2016) there are sufficient conditions for the LR order.

Theorem 3.4 *If T_i has the distribution function $Q_i(F_1,\dots,F_n)$ and the reliability function $\bar{Q}_i(\bar{F}_1,\dots,\bar{F}_n)$, for $i=1,2$, then the following properties hold:*

(i) $T_1 \le_{ST} T_2$ *for all* $F_1,\dots,F_n$ *iff* $\bar{Q}_1 \le \bar{Q}_2$ *(or* $Q_1 \ge Q_2$*) in* $(0,1)^n$;
(ii) $T_1 \le_{HR} T_2$ *for all* $F_1,\dots,F_n$ *iff* $\bar{Q}_2/\bar{Q}_1$ *is decreasing in* $(0,1)^n$;
(iii) $T_1 \le_{RHR} T_2$ *for all* $F_1,\dots,F_n$ *iff* Q_2/Q_1 *is increasing in* $(0,1)^n$.

Proof The proof of (i) is immediate.

To prove (ii) we note that $T_1 \le_{HR} T_2$ holds iff

$$\frac{\bar{F}_{T_2}(t)}{\bar{F}_{T_1}(t)} = \frac{\bar{Q}_2(\bar{F}_1(t),\dots,\bar{F}_n(t))}{\bar{Q}_1(\bar{F}_1(t),\dots,\bar{F}_n(t))} \tag{3.15}$$

is increasing in t.

If this ordering holds for all $F_1,\dots,F_n$ and we want to prove that $\bar{Q}_2/\bar{Q}_1$ is decreasing in u_1 for fixed $u_2,\dots,u_n \in (0,1)$, we choose distribution functions such that $\bar{F}_i(t) = u_i$ for $t \in (1,2)$ and $i = 2,\dots,n$ and $\bar{F}_1(t) = 1$ for $t \le 1$, $\bar{F}_1(t) = 2 - t$ for $t \in (1,2)$, and $\bar{F}_1(t) = 0$ for $t \ge 2$. Then, from (3.15), we have that $\bar{Q}_2(u_1,\dots,u_n)/\bar{Q}_1(u_1,\dots,u_n)$ is decreasing in u_1. We can prove that it is decreasing in the other variables in a similar way.

Conversely, if we assume that $\bar{Q}_2/\bar{Q}_1$ is decreasing in all its variables in $(0,1)^n$, as $\bar{F}_1,\dots,\bar{F}_n$ are decreasing, from (3.15), $\bar{F}_{T_2}(t)/\bar{F}_{T_1}(t)$ is increasing in t.

The proof of (iii) is similar to the proof of (ii). □

Theorem 3.5 *If T_i has the distribution function $Q_i(F_1,\dots,F_n)$ and the reliability function $\bar{Q}_i(\bar{F}_1,\dots,\bar{F}_n)$, for $i=1,2$, then the following properties hold:*

(i) $T_1 \le_{ST} T_2$ *for all* $F_1,\dots,F_n$ *such that* $F_1 \ge_{ST} \cdots \ge_{ST} F_n$ *iff* $\bar{Q}_1 \le \bar{Q}_2$ *in* $D = \{(u_1,\dots,u_n) \in [0,1]^n : u_1 \ge \cdots \ge u_n\}$;
(ii) $T_1 \le_{HR} T_2$ *for all* $F_1,\dots,F_n$ *such that* $F_1 \ge_{HR} \cdots \ge_{HR} F_n$ *iff the function*

$$\bar{H}(v_1,\dots,v_n) = \frac{\bar{Q}_2(v_1, v_1v_2,\dots,v_1\dots v_n)}{\bar{Q}_1(v_1, v_1v_2,\dots,v_1\dots v_n)} \tag{3.16}$$

is decreasing in $(0,1)^n$;

(iii) $T_1 \leq_{RHR} T_2$ *for all* $F_1, \ldots, F_n$ *such that* $F_1 \leq_{RHR} \cdots \leq_{RHR} F_n$ *iff the function*

$$H(v_1, \ldots, v_n) = \frac{Q_2(v_1, v_1 v_2, \ldots, v_1 \ldots v_n)}{Q_1(v_1, v_1 v_2, \ldots, v_1 \ldots v_n)} \quad (3.17)$$

is increasing in $(0, 1)^n$.

Proof The proof of (i) is immediate since $F_1 \geq_{ST} \cdots \geq_{ST} F_n$ implies $\bar{F}_1 \geq \cdots \geq \bar{F}_n$.

To prove (ii) we recall that $T_1 \leq_{HR} T_2$ holds iff the ratio in (3.15) is increasing in t.

If we want to prove that this ordering holds for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ when $\bar{H}$ is decreasing, we note $r_i = \bar{F}_i/\bar{F}_{i-1}$ is decreasing for $i = 2, \ldots, n$. Therefore, $r_i \in [0, 1]$ (since $r_i(0) = 1$). Moreover, $\bar{F}_1 \in [0, 1]$ and it is also decreasing. Hence,

$$\bar{H}(\bar{F}_1(t), r_2(t), \ldots, r_n(t)) = \frac{\bar{Q}_2(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))}{\bar{Q}_1(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))}$$

is increasing in t and so $T_1 \leq_{HR} T_2$ holds.

Conversely, let us assume that $T_1 \leq_{HR} T_2$ holds for all $F_1 \geq_{HR} \cdots \geq_{HR} F_n$. If we want to prove that $\bar{H}$ is decreasing in v_1 for fixed $v_2, \ldots, v_n \in (0, 1)$, we choose the following reliability functions:

$$\bar{F}_1(t) = \begin{cases} 1, \text{ for } 0 \leq t \leq 1 \\ 2 - t, \text{ for } 1 < t \leq 2 \\ 0, \text{ for } \quad t > 2 \end{cases}$$

and

$$\bar{F}_i(t) = \begin{cases} 1 - (1 - v_2 \ldots v_i)t, \text{ for } 0 \leq t \leq 1 \\ v_2 \ldots v_i(2 - t), \text{ for } 1 < t \leq 2 \\ 0, \text{ for } \quad t > 2 \end{cases}$$

for $i = 2, \ldots, n$. Hence

$$r_2(t) = \frac{\bar{F}_2(t)}{\bar{F}_1(t)} = \begin{cases} 1 - (1 - v_2)t, \text{ for } 0 \leq t \leq 1 \\ v_2, \text{ for } 1 < t \leq 2 \end{cases}$$

and

$$r_i(t) = \frac{\bar{F}_i(t)}{\bar{F}_{i-1}(t)} = \begin{cases} \frac{1-(1-v_2 \ldots v_i)t}{1-(1-v_2 \ldots v_{i-1})t}, \text{ for } 0 \leq t \leq 1 \\ v_i, \text{ for } 1 < t \leq 2 \end{cases}$$

for $i = 3, \ldots, n$, which are continuous and decreasing. Therefore, $F_1 \geq_{HR} \cdots \geq_{HR} F_n$ holds and from (3.15), we have that

$$\frac{\bar{Q}_2(\bar{F}_1(t), \ldots, \bar{F}_n(t))}{\bar{Q}_1(\bar{F}_1(t), \ldots, \bar{F}_n(t))} = \bar{H}(2 - t, v_2, \ldots, v_n)$$

is decreasing for $t \in (1, 2)$. So $\bar{H}(v_1, \ldots, v_n)$ is decreasing for $v_1 \in (0, 1)$, for all $v_2, \ldots, v_n \in (0, 1)$. We can prove that it is decreasing in the other variables in a similar way (see Navarro and del Águila 2017). For example, for the second variable, given $v_1, v_3, \ldots, v_n \in (0, 1)$, we can choose the following reliability functions:

$$\bar{F}_1(t) = \begin{cases} 1 - (1 - v_1)t, & \text{for } 0 \le t \le 1 \\ v_1, & \text{for } 1 < t \le 2 \\ v_1(3 - t), & \text{for } 2 < t \le 3 \\ 0, & \text{for } \quad t > 3 \end{cases}$$

$$\bar{F}_2(t) = \begin{cases} 1 - (1 - v_1)t, & \text{for } 0 \le t \le 1 \\ v_1(2 - t), & \text{for } 1 < t \le 2 \\ 0, & \text{for } \quad t > 2 \end{cases}$$

and

$$\bar{F}_i(t) = \begin{cases} v_i \ldots v_3(1 - (1 - v_1)t), & \text{for } 0 \le t \le 1 \\ v_i \ldots v_3 v_1(2 - t), & \text{for } 1 < t \le 2 \\ 0, & \text{for } \quad t > 2 \end{cases}$$

for $i = 3, \ldots, n$. Hence

$$r_2(t) = \frac{\bar{F}_2(t)}{\bar{F}_1(t)} = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 2 - t, & \text{for } 1 < t \le 2 \\ 0, & \text{for } 2 < t \le 3 \end{cases}$$

and

$$r_i(t) = \frac{\bar{F}_i(t)}{\bar{F}_{i-1}(t)} = \begin{cases} v_i, & \text{for } 0 \le t \le 1 \\ v_i, & \text{for } 1 < t \le 2 \end{cases}$$

for $i = 3, \ldots, n$, and the result holds as above.

The proof of (iii) is similar to the proof of (ii). □

Let us see an example which shows how to use the preceding theoretical results to compare systems.

Example 3.5 As in the preceding section, we can consider the series and parallel systems with lifetimes $X_{1:2}$ and $X_{2:2}$, respectively. Now we do not assume a common distribution for the component lifetimes X_1 and X_2. So they have arbitrary distribution functions F_1 and F_2, a copula C and a survival copula $\widehat{C}$. Remember that

$$X_{1:2} \le_{ST} X_i \le_{ST} X_{2:2}$$

holds for all F_1, F_2 and all C.

However, if we consider the hazard rate order, then

$$X_{1:2} \le_{HR} X_1$$

holds for all F_1, F_2 iff $\widehat{C}(u, v)/u$ is increasing in $(0, 1)^2$. Of course, this ordering holds for IND components since $\widehat{C}(u, v)/u = (uv)/u = v$ is increasing (a well known property). In this case, it can be proved that the hazard rate of the series system is $h_{1:2} = h_1 + h_2$, where h_i is the hazard rate of X_i. So $h_{1:2} \ge h_i$ for $i = 1, 2$.

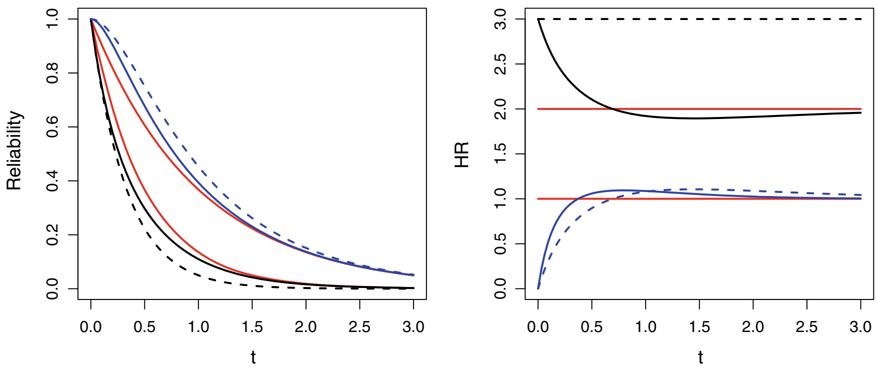

Fig. 3.9 Reliability (left) and hazard rate functions (right) for the series system $X_{1:2}$ (black), the components X_i (red) and the parallel system $X_{2:2}$ (blue) in Example 3.5 for the case of IND components (dashed lines) and dependent (continuous lines) components with the survival copula (3.12)

However, surprisingly, this property is not true when the components are dependent. Thus, if we consider the Clayton–Oakes survival copula (3.12), then

$$\frac{\widehat{C}(u,v)}{u} = \frac{v}{u+v-uv}$$

is decreasing in u and increasing in v. Therefore, for this copula,

$$X_{1:2} \leq_{HR} X_1$$

does not hold for all F_1, F_2. For example, if we consider two exponential distributions $F_i(t) = 1 - \exp(-it)$ for $t \geq 0$ and $i = 1, 2$, then we obtain the reliability (left) and hazard rate (right) functions plotted in Fig. 3.9 for IND components (dashed lines) and dependent components (continuous lines) with the survival copula in (3.12). Note that they are ST ordered in both cases (as expected), that $X_{1:2} \leq_{HR} X_1$ also holds in both cases, that $X_{1:2} \leq_{HR} X_2$ holds for the IND case but that it does not hold for this copula. Note that the used series systems with age t are going to be ST better (i.e., more reliable) than the used components X_2 with the same age t, for $t \geq 0.694$. Also note that they are equivalent when $t \to \infty$. However, the used series systems with age t are going to be ST worse than the used components X_1 with the same age t, for all t.

To explain these properties, we can use Theorem 3.5, (ii) to obtain that $X_{1:2} \leq_{HR} X_1$ holds for all $F_1 \geq_{HR} F_2$ iff the function

$$\bar{H}_1(v_1, v_2) = \frac{\bar{Q}_1(v_1, v_1 v_2)}{\bar{Q}_{1:2}(v_1, v_1 v_2)} = \frac{v_1}{\widehat{C}(v_1, v_1 v_2)}$$

is decreasing in $(0, 1)^2$. Analogously, $X_{1:2} \leq_{HR} X_2$ holds for all $F_1 \geq_{HR} F_2$ iff the function

$$\bar{H}_2(v_1, v_2) = \frac{\bar{Q}_2(v_1, v_1 v_2)}{\bar{Q}_{1:2}(v_1, v_1 v_2)} = \frac{v_1 v_2}{\widehat{C}(v_1, v_1 v_2)}$$

is decreasing in $(0, 1)^2$.

If the components are dependent with the survival copula in (3.12), then

$$\bar{H}_1(v_1, v_2) = \frac{v_1(v_1 + v_1v_2 - v_1^2v_2)}{v_1^2v_2} = \frac{1 + v_2 - v_1v_2}{v_2},$$

which is decreasing in $(0, 1)^2$, and

$$\bar{H}_2(v_1, v_2) = \frac{v_1v_2(v_1 + v_1v_2 - v_1^2v_2)}{v_1^2v_2} = 1 + v_2 - v_1v_2,$$

which is decreasing in v_1 but increasing in v_2. Hence, $X_{1:2} \leq_{HR} X_1$ holds for all $F_1 \geq_{HR} F_2$ (and this copula) but $X_{1:2} \leq_{HR} X_2$ does not hold for all $F_1 \geq_{HR} F_2$ (as we can see in Fig. 3.9). In this case, the series system is HR ordered with the best component (X_1) but not always with the worse one (X_2). If they are ID, both orderings hold (see Fig. 3.5 in the preceding section).

Let us study now the parallel system. For example, $X_1 \leq_{HR} X_{2:2}$ holds for all F_1, F_2 iff

$$\frac{u + v - \widehat{C}(u, v)}{u} = 1 + \frac{v - \widehat{C}(u, v)}{u}$$

is decreasing in $(0, 1)^2$. If the components are IND, then

$$\frac{v - \widehat{C}(u, v)}{u} = v\left(\frac{1}{u} - 1\right)$$

which is increasing in v and decreasing in u. So, surprisingly, this ordering does not hold for all F_1, F_2 even if the components are IND, as can be seen in Fig. 3.9, right, where $X_{2:2}$ (dashed blue line) and the best component X_1 (bottom red line) are not HR ordered. However, $X_{2:2}$ (dashed blue line) and the worse component X_2 (top red line) are HR ordered.

In this figure, the same holds for the Clayton–Oakes copula. As above we can use Theorem 3.5, (ii), to study if this is a general property. Thus $X_1 \leq_{HR} X_{2:2}$ holds for all $F_1 \geq_{HR} F_2$ iff the function

$$\bar{H}_3(v_1, v_2) = \frac{\bar{Q}_{2:2}(v_1, v_1v_2)}{\bar{Q}_1(v_1, v_1v_2)} = \frac{v_1 + v_1v_2 - \widehat{C}(v_1, v_1v_2)}{v_1}$$

is decreasing in $(0, 1)^2$. Analogously, $X_2 \leq_{HR} X_{2:2}$ holds for all $F_1 \geq_{HR} F_2$ iff the function

$$\bar{H}_4(v_1, v_2) = \frac{\bar{Q}_{2:2}(v_1, v_1v_2)}{\bar{Q}_2(v_1, v_1v_2)} = \frac{v_1 + v_1v_2 - \widehat{C}(v_1, v_1v_2)}{v_1v_2}$$

is decreasing in $(0, 1)^2$. If the components are IND, then

$$\bar{H}_3(v_1, v_2) = \frac{v_1 + v_1v_2 - v_1^2v_2}{v_1} = 1 + v_2 - v_1v_2$$

which is decreasing in v_1 and increasing in v_2. So X_1 and $X_{2:2}$ are not HR ordered in Fig. 3.9. However, if the components are IND, then

$$\bar{H}_4(v_1, v_2) = \frac{v_1 + v_1v_2 - v_1^2v_2}{v_1v_2} = \frac{1}{v_2} + 1 - v_1$$

which is decreasing in both v_1 and v_2. So X_2 and $X_{2:2}$ are HR ordered in Fig. 3.9. This is a general property for IND ordered components (the parallel system is HR better that the worse component). This property also holds for the chosen Clayton–Oakes copula since

$$\bar{H}_4(v_1, v_2) = 1 + \frac{1 - v_1 v_2}{v_2(1 + v_2 - v_1 v_2)}$$

is decreasing in both v_1 and v_2. However, X_1 and $X_{2:2}$ are not HR ordered as can be seen in Fig. 3.9, right. Also note that the series systems in both cases are HR ordered but that the parallel systems are not. The code in R to get these plots is the following:

```
#Reliability functions:
#IID case:
R1<-function(t) exp(-t)
R2<-function(t) exp(-2*t)
QIND<-function(u,v) u*v
G12<-function(t) QIND(R1(t),R2(t))
G22<-function(t) R1(t)+R2(t)-G12(t)
curve(G12(x),xlab='t',ylab='Reliability',0,3,lty=2,lwd=2)
curve(G22(x),add=T,col='blue',lty=2,lwd=2)
curve(R1(x),add=T,col='red',lty=2,lwd=2)
curve(R2(x),add=T,col='red',lty=2,lwd=2)
#Clayton
C<-function(u,v) u*v/(u+v-u*v)
R12<-function(t) C(R1(t),R2(t))
R22<-function(t) R1(t)+R2(t)-R12(t)
curve(R12(x),xlab='t',add=T,lwd=2)
curve(R22(x),add=T,col='blue',lwd=2)

#Hazard rate functions
#IND case
f1<-function(t) exp(-t)
f2<-function(t) 2*exp(-2*t)
Q1IID<-function(u,v) v #partial derivative 1
Q2IID<-function(u,v) u #partial derivative2
g12<-function(t) {
  f1(t)*Q1IID(R1(t),R2(t))+f2(t)*Q2IID(R1(t),R2(t))
}
g22<-function(t) f1(t)+f2(t)-g12(t)
curve(g12(x)/G12(x),0,3,ylab='HR',ylim=c(0,3),lty=2,lwd=2)
curve(g22(x)/G22(x),add=T,col='blue',lty=2,lwd=2)
curve(f1(x)/R1(x),add=T,col='red',lwd=2)
curve(f2(x)/R2(x),add=T,col='red',lwd=2)
#Clayton
C1<-function(u,v) v^2/(u+v-u*v)^2 #partial derivative 1
```

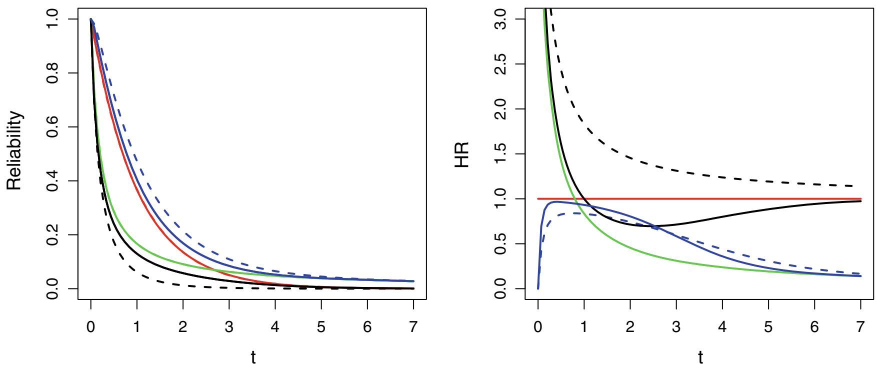

Fig. 3.10 Reliability (left) and hazard rate functions (right) for the series system $X_{1:2}$ (black), the parallel system $X_{2:2}$ (blue) and the components X_1 (Exponential, red) and X_2 (Pareto, green) in Example 3.5 for the case of IND components (dashed lines) and dependent components (continuous lines) with the survival copula (3.12)

```
C2<-function(u,v) u^2/(u+v-u*v)^2 #partial derivative 2
f12<-function(t) f1(t)*C1(R1(t),R2(t))+f2(t)*C2(R1(t),R2(t))
f22<-function(t) f1(t)+f2(t)-f12(t)
curve(f12(x)/R12(x),add=T,lwd=2)
curve(f22(x)/R22(x),add=T,col='blue',lwd=2)
```

We can modify this code to plot these functions for other marginals and/or other copulas. For example, if we consider the same exponential for X_1 but the Pareto distribution $F_2(t) = 1 - 1/(1 + 5t)$ for $t \geq 0$ for the second component lifetime X_2, then they are not ordered and we obtain the plot in Fig. 3.10 (for the same copula). Note that the series and parallel systems are HR ordered in the case of IND components (dashed lines) but that they are not ordered for the Clayton–Oakes copula (black and blue continuous lines). This is a really surprising property! ◀

As in the preceding section, we can obtain conditions for distribution-free ordering results based on properties of the copula and/or the survival copula. These conditions are related with negative dependence properties. These relationships were studied in Navarro et al. (2021). Let us see some examples. The proofs are straightforward.

To get these results we need the definitions of well-known dependence properties and how they can be stated in terms of copulas. A continuous random pair (X, Y) with copula C is said to be:

- **Positive (Negative) Quadrant Dependent**, shortly written as PQD (NQD), iff $\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y)$ for all x, y. If the marginal distributions are continuous, then the PQD (NQD) property is equivalent (see Proposition 3.9) to $C(u, v) \geq uv$ ($C(u, v) \leq uv$) in $[0, 1]^2$;

Table 3.2 Relationships among positive (left) and negative (right) dependence properties

$SI(Y\|X)$	$\Rightarrow$	$LTD(Y\|X)$	$SD(Y\|X)$	$\Rightarrow$	$LTI(Y\|X)$
$\Downarrow$		$\Downarrow$	$\Downarrow$		$\Downarrow$
$RTI(Y\|X)$	$\Rightarrow$	PQD	$RTD(Y\|X)$	$\Rightarrow$	NQD

- **Left Tail Decreasing (Increasing)** in X, shortly written as LTD$(Y|X)$ (LTI$(Y|X)$), if, and only if, $\Pr(Y \le y|X \le x)$ is decreasing (increasing) in x for all y or, equivalently, $C(u,v)/u$ is decreasing (increasing) in u for all v in $(0,1)^2$. The concepts LTD$(X|Y)$ and LTI$(X|Y)$ are defined in a similar way;
- **Right Tail Increasing (Decreasing)** in X, shortly written as RTI$(Y|X)$ (RTD$(Y|X)$), if, and only if, $\Pr(Y > y|X > x)$ is increasing (decreasing) in x for all y or, equivalently, $\widehat{C}(u,v)/u$ is decreasing (increasing) in u for all v in $(0,1)^2$;
- **Stochastically Increasing (Decreasing)** in X, shortly written as $SI(Y|X)$ ($SD(Y|X)$), if, and only if, $(Y|X=x)$ is ST-increasing (decreasing) in x.

We say that (X, Y) is LTD if it is both LTD$(Y|X)$ and LTD$(X|Y)$. The concepts LTI, RTI, RTD, SI and SD are defined similarly. The relationships among the above dependence properties are summarized in Table 3.2. Also note that the PQD (NQD) property implies that the Pearson correlation, Spearman correlation and Kendal tau coefficients are nonnegative (nonpositive), see Nelsen (2006). So all of them are positive (negative) dependence properties.

In the first proposition we compare the components with the series system.

Proposition 3.11 *Let X_1 and X_2 be component lifetimes with survival copula $\widehat{C}$ and distribution functions F_1 and F_2, respectively. Then the following statements are equivalent:*

(i) $X_{1:2} \le_{HR} X_1$ *holds for all* F_1 *and* F_2*;*
(ii) $\widehat{C}(u,v)/u$ *is increasing in* $u \in (0,1)$ *for every* $v \in (0,1)$*;*
(iii) $(v - 1 + C(1-u, 1-v))/u$ *is increasing in* $u \in (0,1)$ *for every* $v \in (0,1)$*;*
(iv) (X_1, X_2) *is* $RTD(X_2|X_1)$.

Note that we need a negative dependence property (RTD) to separate the series system from its components. For the RHR order we get the following conditions.

Proposition 3.12 *Let X_1 and X_2 be component lifetimes with copula C and with distribution functions F_1 and F_2, respectively. Then the following statements are equivalent:*

(i) $X_1 \le_{RHR} X_{2:2}$ *holds for all* F_1 *and* F_2*;*
(ii) $C(u,v)/u$ *is increasing in* $u \in (0,1)$ *for every* $v \in (0,1)$*;*
(iii) $(v - 1 + \widehat{C}(1-u, 1-v))/u$ *is increasing in* $u \in (0,1)$ *for every* $v \in (0,1)$.
(iv) (X_1, X_2) *is* $LTI(X_2|X_1)$.

Note that here we also need a negative dependence property and that the conditions are duals (by changing $\widehat{C}$ with C). To compare series and parallel systems we have the following condition.

Proposition 3.13 *Let X_1 and X_2 be component lifetimes with survival copula $\widehat{C}$ and with distribution functions F_1 and F_2, respectively. Then:*

(i) $X_{1:2} \leq_{HR} X_{2:2}$ holds for all F_1, F_2 iff $\widehat{C}(u, v)/(u + v)$ is increasing in $(0, 1)^2$;
(ii) $X_{1:2} \leq_{RHR} X_{2:2}$ holds for all F_1, F_2 iff $C(u, v)/(u+v)$ is increasing in $(0, 1)^2$.

For ordered components we have the following results.

Proposition 3.14 *Let X_1 and X_2 be component lifetimes. Then:*

(i) $X_{1:2} \leq_{HR} X_1$ holds for all $F_1 \geq_{HR} F_2$ iff $\widehat{C}(u, uv)/u$ is increasing in $(0, 1)^2$;
(ii) $X_{1:2} \leq_{HR} X_2$ holds for all $F_1 \geq_{HR} F_2$ iff $\widehat{C}(u, uv)/(uv)$ is increasing in $(0, 1)^2$;
(iii) $X_1 \leq_{HR} X_{2:2}$ holds for all $F_1 \geq_{HR} F_2$ iff $(uv - \widehat{C}(u, uv))/u$ is decreasing in $(0, 1)^2$;
(iv) $X_2 \leq_{HR} X_{2:2}$ holds for all $F_1 \geq_{HR} F_2$ iff $(u - \widehat{C}(u, uv))/(uv)$ is decreasing in $(0, 1)^2$;
(v) $X_{1:2} \leq_{HR} X_{2:2}$ holds for all $F_1 \geq_{HR} F_2$ iff $\widehat{C}(u, uv)/(u + uv)$ is increasing in $(0, 1)^2$.

Note that all the conditions for the survival copula $\widehat{C}$ in the preceding proposition can be seen as negative dependence properties.

Proposition 10 in Navarro et al. (2021) proves that, for any copula C, X_1 and $X_{2:2}$ are not HR ordered for all F_1, F_2. Note that $X_1 \leq_{HR} X_{2:2}$ holds for all F_1, F_2 iff

$$\frac{1 - C(1 - u, 1 - v)}{u}$$

is decreasing in $(0, 1)^2$. However, note that this ratio is always increasing in v. Hence, the results given in Example 3.5 for X_1 and $X_{2:2}$ are valid for any copula C (i.e. for some distribution functions F_1 and F_2 they are not HR ordered).

Analogously, it can be proved that X_1 and $X_{1:2}$ are not RHR ordered for all F_1, F_2. To get these orderings we need to assume ordered components (as stated in the preceding proposition).

As in the preceding section, we can compare systems with dependent and independent components. If X_1 and X_2 have a copula C and a survival copula $\widehat{C}$ and Y_1 and Y_2 are independent, X_1 and Y_1 have the common distribution function F_1 and X_2 and Y_2 have the common distribution function F_2, then we obtain the following results.

Proposition 3.15 *The following statements are equivalent:*

(i) $X_{1:2} \geq_{ST} Y_{1:2}$ *(respectively,* $\leq_{ST}$*) holds for all* F_1 *and* F_2*;*
(ii) $X_{2:2} \leq_{ST} Y_{2:2}$ *(respectively,* $\geq_{ST}$*) holds for all* F_1 *and* F_2*;*
(iii) $C(u, v) \geq uv$ *(respectively,* $C(u, v) \leq uv$*) in* $[0, 1]^2$*;*
(iv) $\widehat{C}(u, v) \geq uv$ *(respectively,* $\widehat{C}(u, v) \leq uv$*) in* $[0, 1]^2$*;*
(v) (X_1, X_2) *is PQD (respectively, NQD).*

Proposition 3.16 *The following statements are equivalent:*

(i) $X_{1:2} \geq_{HR} Y_{1:2}$ *(respectively,* $\leq_{HR}$*) holds for all* F_1 *and* F_2*;*
(ii) $\widehat{C}(u, v)/(uv)$ *is decreasing (respectively, increasing) in* $(0, 1)^2$*;*
(iii) (X_1, X_2) *is RTI (respectively, RTD).*

Proposition 3.17 *The following statements are equivalent:*

(i) $X_{2:2} \geq_{RHR} Y_{2:2}$ *(respectively,* $\leq_{RHR}$*) holds for all* F_1 *and* F_2*;*
(ii) $C(u, v)/(uv)$ *is increasing (respectively, decreasing) in* $(0, 1)^2$*;*
(iii) (X_1, X_2) *is LTI (respectively, LTD).*

As we have seen in Example 3.5, the comparison results for the general case can also be applied to systems with IND components. The results for all the semi-coherent systems with 1-3 components were obtained in Navarro and del Águila (2017). Their dual distortion functions are given in Table 3.3. All the ST and HR orderings for these

Table 3.3 Dual distortions functions of coherent systems with 1–3 independent components

N	$T = \psi(X_1, X_2, X_3)$	$\overline{Q}(u_1, u_2, u_3)$
1	$X_{1:3} = \min(X_1, X_2, X_3)$	$u_1u_2u_3$
2	$\min(X_2, X_3)$	u_2u_3
3	$\min(X_1, X_3)$	u_1u_3
4	$\min(X_1, X_2)$	u_1u_2
5	$\min(X_3, \max(X_1, X_2))$	$u_1u_3 + u_2u_3 - u_1u_2u_3$
6	$\min(X_2, \max(X_1, X_3))$	$u_1u_2 + u_2u_3 - u_1u_2u_3$
7	$\min(X_1, \max(X_2, X_3))$	$u_1u_2 + u_1u_3 - u_1u_2u_3$
8	X_3	u_3
9	X_2	u_2
10	X_1	u_1
11	$X_{2:3}$	$u_1u_2 + u_1u_3 + u_2u_3 - 2u_1u_2u_3$
12	$\max(X_3, \min(X_1, X_2))$	$u_3 + u_1u_2 - u_1u_2u_3$
13	$\max(X_2, \min(X_1, X_3))$	$u_2 + u_1u_3 - u_1u_2u_3$
14	$\max(X_1, \min(X_2, X_3))$	$u_1 + u_2u_3 - u_1u_2u_3$
15	$\max(X_2, X_3)$	$u_2 + u_3 - u_2u_3$
16	$\max(X_1, X_3)$	$u_1 + u_3 - u_1u_3$
17	$\max(X_1, X_2)$	$u_1 + u_2 - u_1u_2$
18	$X_{3:3} = \max(X_1, X_2, X_3)$	$u_1 + u_2 + u_3 - u_1u_2 - u_1u_3 - u_2u_3 + u_1u_2u_3$

Table 3.4 Relationships for the ST order between the coherent systems with independent components given in Table 3.3. The value 2 indicates that $T_i \leq_{ST} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{ST} T_j$ holds for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$. It also indicates that $T_i \leq_{ST} T_j$ does not hold for all F_1, F_2, F_3. The value 0 indicates that $T_i \leq_{ST} T_j$ does not hold for all $F_1 \geq_{ST} F_2 \geq_{ST} F_3$

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	2	2	2
3	0	2	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2
4	0	0	2	0	2	2	0	2	2	2	2	2	2	2	2	2	2
5	0	0	0	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6	0	0	0	0	2	1	0	2	1	2	2	2	2	2	2	2	2
7	0	0	0	0	0	2	0	0	2	2	2	2	2	2	2	2	2
8	0	0	0	0	0	0	2	1	1	0	2	1	1	2	2	1	2
9	0	0	0	0	0	0	0	2	1	0	0	2	1	2	1	2	2
10	0	0	0	0	0	0	0	0	2	0	0	0	2	0	2	2	2
11	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	1	1	2	2	1	2
13	0	0	0	0	0	0	0	0	0	0	0	2	1	2	1	2	2
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	2	2
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	2
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	2
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

systems are given in Tables 3.4 and 3.5. The value 2 indicates that the ordering holds for any components, the value 1 that it only holds for ordered components and the value 0 that it does not hold for ordered components. The relationships for the HR order and ordered components are summarized in the graph given in Fig. 3.11.

We conclude this section by showing how to proceed when the systems are built just by using two kind of components. Here we assume that T_1 and T_2 are the lifetimes of two coherent systems with components having one of the two (different) distribution functions F (type A) or G (type B). For example, we can consider the systems in Fig. 3.12.

Under this assumption, it is clear that the system reliability functions can be written as

$$\bar{F}_{T_i}(t) = \bar{Q}_i(\bar{F}, \bar{G})$$

for $i = 1, 2$, where $\bar{Q}_i : [0, 1]^2 \to [0, 1]$ are two (bivariate) distortion functions. They can be obtained from the general distortion functions obtained in Chap. 2 (by using minimal path or cut sets). Under some exchangeabillity assumptions between the components of the same type, these distortion functions can also be computed

Table 3.5 Relationships for the HR order between the coherent systems with independent components given in Table 3.3. The value 2 indicates that $T_i \leq_{HR} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column). The value 1 indicates that $T_i \leq_{HR} T_j$ holds for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$. It also indicates that $T_i \leq_{HR} T_j$ does not hold for all F_1, F_2, F_3. The value 0 means that $T_i \leq_{HR} T_j$ does not hold for all $F_1 \geq_{HR} F_2 \geq_{HR} F_3$

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	1	1	1
3	0	2	1	0	0	1	2	1	2	0	1	1	1	1	2	1	1
4	0	0	2	0	0	0	0	2	2	0	0	0	0	0	0	2	0
5	0	0	0	2	0	0	2	1	1	0	0	1	1	1	1	2	2
6	0	0	0	0	2	0	0	2	1	0	0	0	1	0	2	1	2
7	0	0	0	0	0	2	0	0	2	0	0	0	1	2	1	1	2
8	0	0	0	0	0	0	2	1	1	0	0	0	0	1	1	1	1
9	0	0	0	0	0	0	0	2	1	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	2	0	1	1	2	2	2	2
12	0	0	0	0	0	0	0	0	0	0	2	0	1	1	1	1	1
13	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	1
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

from the **survival signature** defined in Coolen and Coolen-Maturi (2012) (see also Samaniego and Navarro 2016).

For example, for the systems in Fig. 3.12, if we assume IND components, we have

$$\bar{Q}_1(u, v) = uv + v^2 - uv^2$$

and

$$\bar{Q}_2(u, v) = 2uv - uv^2$$

for $u, v \in [0, 1]^2$.

In this case $T_1 \leq_{ST} T_2$ (resp. $\geq_{ST}$) holds for all F, G iff $\bar{Q}_1 \leq \bar{Q}_2$ (resp. $\geq_{ST}$). If we define the **difference function**

$$\Delta(u, v) := \bar{Q}_2(u, v) - \bar{Q}_1(u, v),$$

this ordering holds for all F, G iff $\Delta(u, v) \geq 0$ (resp. ≤ 0) for all $u, v \in [0, 1]$.

However, in some cases, we need conditions between F and G to get this ordering. Thus, for the systems in Fig. 3.12, we obtain

$$\Delta(u, v) = 2uv - uv^2 - (uv + v^2 - uv^2) = uv - v^2.$$

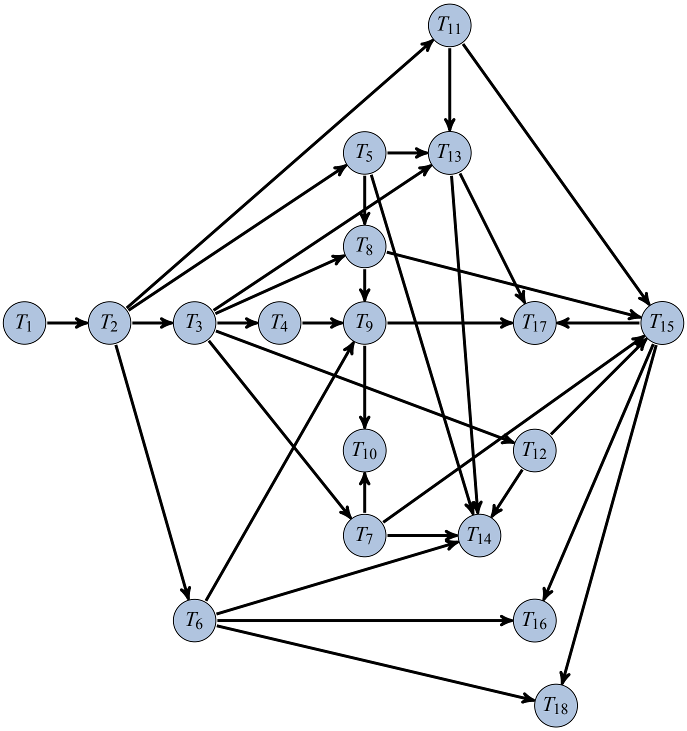

Fig. 3.11 Hazard rate ordering relationships between the coherent systems with 1–3 independent components given in Table 3.3 when $F_1 \geq_{HR} F_2 \geq_{HR} F_3$ holds

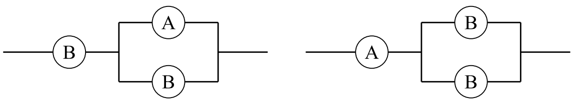

Fig. 3.12 Two coherent systems of order 3 with a similar structure built with components of type A and B.

Therefore, $\Delta(u, v) \geq 0$ (≤ 0) iff $u \geq v$ ($\leq$). Hence, $T_1 \leq_{ST} T_2$ (resp. $\geq_{ST}$) holds iff $\bar{F} \geq \bar{G}$ ($\leq$), that is, the best component should be placed at the first position (as expected). Note that they are not ST ordered when F and G are not ST ordered.

In other situations, the conditions to get this ordering can be more complicated. In this case we can proceed as follows. We plot the level curves (contour plot) of Δ in $[0, 1]^2$. In this plot we highlight the border line which leads to $\Delta = 0$ and we define the regions

$$\mathcal{R}_1 = \{(u, v) \in [0, 1] : \Delta(u, v) \leq 0\}$$

and

$$\mathcal{R}_2 = \{(u, v) \in [0, 1] : \Delta(u, v) \geq 0\}.$$

Then we add to this plot the parametric curve $(\bar{F}(t), \bar{G}(t))$ for $t \geq 0$. Note that this curve always starts at the point $(\bar{F}(0), \bar{G}(0)) = (1, 1)$ and finished at $(\bar{F}(\infty), \bar{G}(\infty)) = (0, 0)$. These plots were called **RR-plots** (Reliability-Reliability plots) in Samaniego and Navarro (2016). Thus, we have three options:

- If $(\bar{F}(t), \bar{G}(t)) \in \mathcal{R}_1$ for all $t \geq 0$, then $T_1 \geq_{ST} T_2$ for these F, G.
- If $(\bar{F}(t), \bar{G}(t)) \in \mathcal{R}_2$ for all $t \geq 0$, then $T_1 \leq_{ST} T_2$ for these F, G.
- In the other cases, T_1 and T_2 are not ST ordered for these F, G.

Let us see an example.

Example 3.6 Let us compare the first system T_1 in Fig. 3.12 with a 2-out-of-3 system $T_2 = X_{2:3}$ having two components of type A and one of type B. We assume that all the components are independent. Then

$$\bar{Q}_2(u, v) = 2uv + u^2 - 2u^2v$$

and

$$\Delta(u, v) = 2uv + u^2 - 2u^2v - (uv + v^2 - uv^2) = uv + u^2 - v^2 - 2u^2v + uv^2.$$

The level curves of Δ are plotted in Fig. 3.13, left. The regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are determined by the zero-level curve $\Delta = 0$ ($\mathcal{R}_1$, above, and $\mathcal{R}_2$, below). In the right plot we add several RR-plots. In the first one (blue line), we assume $\bar{G} = \bar{F}^2$. As the curve (RR-plot) belongs to the region $\mathcal{R}_2$, we have $T_1 \leq_{ST} T_2$. The same happen for the second example (red line), where we assume $\bar{G} = \bar{F}$. Note that we have this property $T_1 \leq_{ST} T_2$ for all $\bar{G} \leq \bar{F}$ (and also for some $\bar{G} \geq \bar{F}$). However, in the third case (green line), we assume $\bar{G}^2 = \bar{F}$ and the curve crosses both regions. Therefore, T_1 and T_2 are not ST-ordered. Finally, we choose $G = F^3$ (i.e. $\bar{G} = 1 - (1 - \bar{F})^3$) and then the curve (purple line) belongs to the region $\mathcal{R}_1$. So we have $T_1 \geq_{ST} T_2$. Note that the level curves can be used to determine approximately the difference

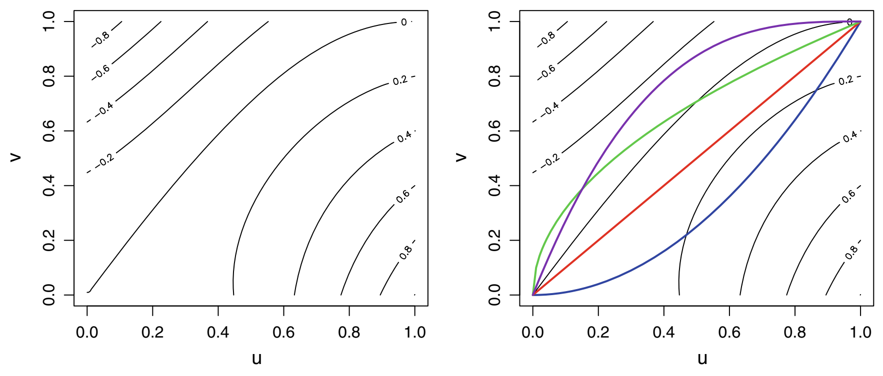

Fig. 3.13 Level curves of Δ for the systems in Example 3.6 and RR-plots (right) when we assume $\bar{G} = \bar{F}^2$ (blue line), $\bar{G} = \bar{F}$ (red line), $\bar{G}^2 = \bar{F}$ (green line) and $G = F^3$ (purple line)

between both system reliability functions. For example, if $F = G$ (red line), then $0 \le \bar{F}_2 - \bar{F}_1 \le 0.2$. The R code to get these plots is the following:

```
# RR-plots
Q1<-function(u,v) u*v+v^2-u*v^2
Q2<-function(u,v) 2*u*v+u^2-2*u^2*v
D<-function(u,v) Q2(u,v)-Q1(u,v)
x<-seq(0,1,0.01)
y<-seq(0,1,0.01)
z<-outer(x,y,D)
contour(x,y,z,xlab='u',ylab='v')
curve(x^ 2,add=T,col='blue',lwd=2)
curve(x+1-1,add=T,col='red',lwd=2)
curve(x^0.5,add=T,col='green',lwd=2)
curve(1-(1-x)^3,add=T,col='purple',lwd=2)
```
◀

3.5 A Parrondo Paradox in Reliability

The Parrondo's paradox shows how, in some games, a random strategy might be better than any deterministic strategy. Di Crescenzo (2007) noted that a similar paradox holds in reliability for series systems with independent heterogeneous components. The problem can be stated as follows.

Let $T = \min(X_1, X_2)$ be the lifetime of a series system with two independent components having reliability functions $\bar{F}_1$ and $\bar{F}_2$. We can assume that the components of type 1 are better than the others, that is, $\bar{F}_1 \ge \bar{F}_2$ (but we will see later that we do not need this assumption).

On the other hand, we can consider the series system with lifetime $S = \min(Y_1, Y_2)$, where Y_1 and Y_2 are IID with common reliability

$$\bar{G} = \frac{1}{2}\bar{F}_1 + \frac{1}{2}\bar{F}_2.$$

This system represents the case in which we choose the components randomly from a mixed population with a 50% of units of type 1 (with reliability $\bar{F}_1$) and a 50% of units of type 2 (with reliability $\bar{F}_2$), while in the first option we choose for sure one component of each type.

Which one is the best option? Does this property depend on $\bar{F}_1$ and $\bar{F}_2$? What is the best general option? Could this property be extended to other system structures? What happen if the components are dependent?

The respective system reliability functions in both options can be represented with distortions as

$$\bar{F}_T(t) = \Pr(X_1 > t, X_2 > t) = \bar{F}_1(t)\bar{F}_2(t) = \bar{Q}_T(\bar{F}_1(t), \bar{F}_2(t))$$

and

$$\bar{F}_S(t) = \Pr(Y_1 > t, Y_2 > t) = \bar{G}(t)\bar{G}(t) = \left(\frac{1}{2}\bar{F}_1(t) + \frac{1}{2}\bar{F}_2(t)\right)^2 = \bar{Q}_S(\bar{F}_1(t), \bar{F}_2(t))$$

with

$$\bar{Q}_T(u_1, u_2) = u_1 u_2$$

and

$$\bar{Q}_S(u_1, u_2) = \left(\frac{u_1 + u_2}{2}\right)^2$$

for $u_1, u_2 \in [0, 1]$. It is easy to prove that $\bar{Q}_T \leq \bar{Q}_S$ since

$$\sqrt{u_1 u_2} \leq \frac{u_1 + u_2}{2}$$

(the geometric mean is always less than the arithmetic mean), or just since

$$4u_1u_2 \leq u_1^2 + 2u_1u_2 + u_2^2$$

holds for all $u_1, u_2 \in [0, 1]$ because $0 \leq (u_1 - u_2)^2$. Note that we do not need the condition $u_1 \geq u_2$, that is, $\bar{F}_1 \geq \bar{F}_2$. They can be ordered in the reverse sense or even not ordered. In any case, the system with randomly chosen components is always ST better, that is, $T \leq_{ST} S$ for all $\bar{F}_1, \bar{F}_2$. So the Parrondo paradox holds!

The respective reliability functions for exponential components with means 5 and 1 can be seen in Fig. 3.14, left. In the right plot the first unit has a Weibull distribution with reliability $\bar{F}_1(t) = \exp(-t^4)$ for $t \geq 0$. The reliability functions of T and S are plotted in black and blue, respectively. Note that the first one is always worse than the second. As mentioned above, this property holds for all $\bar{F}_1, \bar{F}_2$. The red and orange plots correspond to the series systems obtained with just units of type 1 (red) or 2 (orange). Of course, in the left plot, the best option is the red curve, that is, the series system obtained with just the best units. In many situations this is not a

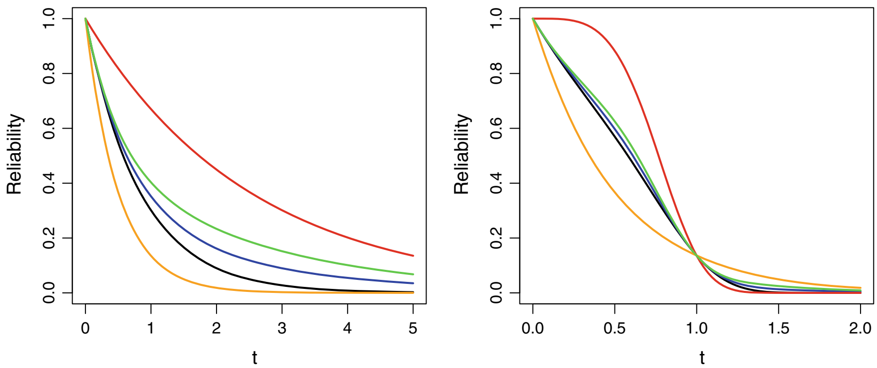

Fig. 3.14 Reliability functions for the series systems T (black) and S (blue) in Parrondo paradox for exponential (left) and Weibull (right) distributions. The red and orange plots correspond to the series systems obtained with just units of type 1 (red) or 2 (orange)

realistic option since we do not use the units of type 2, the bests units could be more expensive, we mighty not know which ones are the best units, or the units could be not ordered (as in the right plot). However, in the green plot we use an 50% of units of type 1 and 2 obtaining a better series system. How can we build this system? To answer this question we need some additional results.

First we are going to study when this "Parrondo paradox" holds. It is easy to see that it can be extended to series systems with n independent components. The explanation is simple since these systems are better when the units are similar (homogeneous). Hence, here the Parrondo paradox is not a paradox but an expectable property. This property is reverted for parallel systems since, in this case, the systems are better when the units are different (heterogeneous). What happen in other system structures? Do these properties hold when the components are dependent?

The answers to some of these questions were obtained in Navarro and Spizzichino (2010). They are based on the notions of Schur-concave and weakly Schur-concave functions defined as follows (see Durante and Papini 2007).

Definition 3.9 A function $g : \mathbb{R}^n \to \mathbb{R}$ is weakly Schur-concave (convex) if

$$g(u_1, \dots, u_n) \le g(\bar{u}, \dots, \bar{u}) \ (\ge)$$

for all $u_1, \dots, u_n$, where $\bar{u} = (u_1 + \cdots + u_n)/n$.

Definition 3.10 A function $g : \mathbb{R}^n \to \mathbb{R}$ is Schur-concave (convex) if

$$g(u_1, \dots, u_n) \le g(v_1, \dots, v_n) \ (\ge)$$

for all $u_1, \dots, u_n, v_1, \dots, v_n$ such that $u_1 + \cdots + u_n = v_1 + \cdots + v_n$ and such that

$$\sum_{i=1}^{j} u_{i:n} \le \sum_{i=1}^{j} v_{i:n}$$

for all $j = 1, \dots, n-1$, where $u_{i:n}$ and $v_{i:n}$ are the ordered values obtained from the respective vectors.

To explain the meaning of these properties let us consider $n = 2$. In both cases, we study the monotonicity of function $g(u_1, u_2)$ when we move the points in the line $u_1 + u_2 = c$. The function g is Schur-concave when it is increasing when the points move to the diagonal. Obviously, then the maximum value is obtained in the point at the diagonal $(\bar{u}, \bar{u})$, that is, then it is also weakly Schur-concave. For example, the function $g(u_1, u_2) = u_1 u_2$ is Schur-concave since if we assume $u_1 + u_2 = c$, then

$$g(u_1, u_2) = u_1 u_2 = u_1(c - u_1)$$

which is increasing for $u_1 \le c/2$ and decreasing for $u_1 \ge c/2$. Its maximum value is obtained when $u_1 = c/2$, that is, $u_1 = u_2$. The 3D plot and contour plot (level curves) can be seen in Fig. 3.15. Note that g increases when we move to the diagonal (mountain shape). Analogously, it can be proved that $g(u_1, u_2) = 1-(1-u_1)(1-u_2)$ is Schur-convex. The code for these plots is the following:

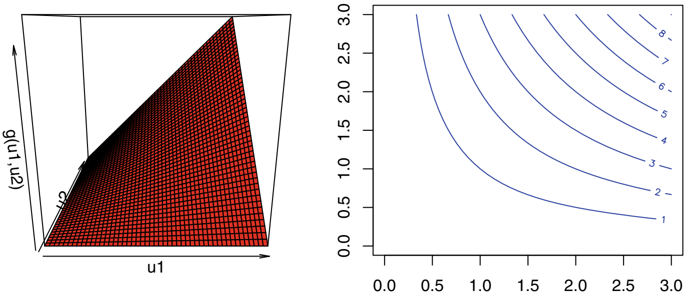

Fig. 3.15 Plot (left) and contour plot (right) for $g(u_1, u_2) = u_1 u_2$

```
#Schur-concave
g<-function(x,y) x*y
x<-seq(0,3,length=50)
y<-seq(0,3,length=50)
z<-outer(x,y,g)
persp(x,y,z,xlab='u1',ylab='u2',zlab='g(u1,u2)',col='red')
contour(x,y,z,col='blue')
```

Now we can state the following result.

Theorem 3.6 (Navarro and Spizzichino 2010) *Let $\bar{Q}$ be the dual distortion function of a system. Then the Parrondo paradox holds (is reverted) for this system if and only if $\bar{Q}$ is weakly Schur-concave (convex).*

Proof Note that to check the Parrondo paradox we have to compare

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t))$$

with

$$\bar{F}_S(t) = \bar{Q}(\bar{G}(t), \dots, \bar{G}(t))$$

where $\bar{G} = (\bar{F}_1 + \dots + \bar{F}_n)/n$. Hence, $\bar{F}_T \le \bar{F}_S$ holds if and only if $\bar{Q}$ is weakly Schur-concave. The property is reverted when $\bar{Q}$ is weakly Schur-convex. □

Of course, in particular, the Parrondo paradox holds (is reverted) when $\bar{Q}$ is Schur-concave (convex). For series systems with independent components, we have

$$\bar{Q}_{1:n}(u_1, \dots, u_n) = u_1 \dots u_n,$$

which is Schur-concave. So the Parrondo paradox holds for any $F_1, \dots, F_n$. If the components are dependent with a survival copula $\widehat{C}$, then

$$\bar{Q}_{1:n}(u_1, \dots, u_n) = \widehat{C}(u_1, \dots, u_n).$$

Hence, the Parrodo paradox holds if and only if $\widehat{C}$ is weakly Schur-concave. Many copulas are Schur-concave (see Nelsen 2006). For example, all the Archimedean copulas are Schur-concave. Do not exist strict Schur-convex copulas. There are some copulas that are at the same time Schur-convex and Schur-concave (i.e. they are Schur-constant). For them, both option coincide (i.e. $T =_{ST} S$).

However, Durante and Papini (2007) obtained a strict weakly Schur-convex copula. Hence, under this survival copula, the Parrondo paradox is reverted in this series system with dependent components. This is really a paradox since, in this case, it is better to have heterogeneous components in a series system!

These properties are reverted for parallel systems. If the components are independent, then their dual distortion function is

$$\bar{Q}_{n:n}(u_1,\ldots,u_n) = 1-(1-u_1)\ldots(1-u_n),$$

which is Schur-convex in $[0,1]^n$ and so the Parrondo paradox is reverted. If the components are dependent with a copula C, then

$$\bar{Q}_{n:n}(u_1,\ldots,u_n) = 1-C(1-u_1,\ldots,1-u_n)$$

and so the Parrondo paradox is reverted when C is weakly Schur-concave. So this property holds for many copulas. However, as stated above, it is not always true (which is also a paradox). For other system structures it is not easy to prove if $\bar{Q}$ is weakly Schur-concave/convex.

We can try to extend the Parrondo paradox to other (stronger) orders by using the comparison results obtained from distortions. For example, in the case of series systems with two independent components, to extend it to the HR order we haver to study the monotonicity of the ratio

$$\frac{\bar{Q}_S(u_1,u_2)}{\bar{Q}_T(u_1,u_2)} = \frac{(u_1+u_1)^2/4}{u_1u_2} = \frac{1}{2}+\frac{u_1}{4u_2}+\frac{u_2}{4u_1}.$$

It is easy to see that it is not monotone in $[0,1]^2$. So the Parrondo paradox cannot be extended to the HR order as can be seen in Fig. 3.16. Note that $T \le_{HR} S$ holds for two exponential distributions with mean 5 and 1(left) but that it does not hold when the first exponential is replaced with a Weibull (right) with hazard rate $h_1(t) = 4t^3$ for $t \ge 0$. Also note that, in both cases, the limiting value of h_S coincides with the one of the hazard rate of the series system obtained with the best components when $t \to \infty$. This is a well known property in mixture models where the leading term is determined by the best components since the worse components fail before (see Navarro and Hernández 2008a, and the references therein).

Let us come back now to the question of the green line in Fig. 3.14. To answer this question let us consider more general systems with randomized components. They were studied in Navarro et al. (2015). If we have two type of components with reliability function $\bar{F}_X$ and $\bar{F}_Y$ we can consider the deterministic system T_k which have k components from X and $n-k$ from Y. Its reliability function is

$$\bar{F}_{T_k}(t) = \bar{Q}(\underbrace{\bar{F}_X(t),\ldots,\bar{F}_X(t)}_{k\ times},\underbrace{\bar{F}_Y(t),\ldots,\bar{F}_Y(t)}_{n-k\ times})$$

for $k=0,\ldots,n$. Here $k=0$ means that we only use units from Y and $k=n$ that we just use units from X.

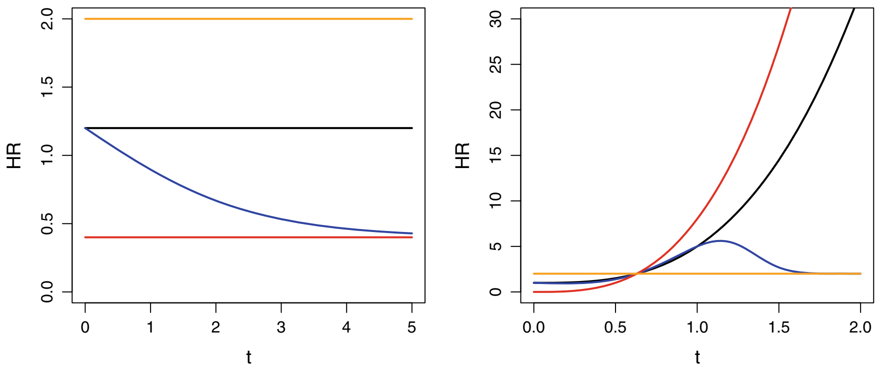

Fig. 3.16 Hazard rate functions for the series systems T (black) and S (blue) in Parrondo paradox for exponential (left) and Weibull (right) distributions. The red and orange plots correspond to the series systems obtained with just units of type 1 (red) or 2 (orange)

Then we can consider the randomized (mixed) system T_K which choose T_k when the random variable $K = k$, where K is a discrete random variable over the set $\{0, \ldots, n\}$.

Note that the systems T and S in the Parrondo paradox when $n = 2$ are obtained as $T = T_1$ (i.e. we choose one component of each type) and $S = T_K$ where $K = 0, 1, 2$ with probabilities 0.25, 0.5, 0.25, respectively. Note that $E(K) = 1$. Also note that T_1 can be obtained with the atom random variable K which takes the value 1 for sure.

Also note that the red and orange lines in Fig. 3.14 correspond to T_2 and T_0, respectively. As mentioned above, in some cases, these options are unrealistic because they only use units of one type. To have "fair" comparisons, we should impose $E(K) = 1$, that is, we use a 50% of units from each type. Under this condition, which one is the best option for K? The answer is given in the following result extracted from Navarro et al. (2015). There we use the convex (CX), increasing convex (ICX) and increasing concave (ICV) orders. Their definitions and main properties can be seen in Shaked and Shanthikumar (2007).

Proposition 3.18 *If the number k of components of type X is chosen randomly according to the random variables K_1 or K_2 and*

$$\varphi(k) = \bar{Q}(\underbrace{u, \ldots, u}_{k\ times}, \underbrace{v, \ldots, v}_{n-k\ times})$$

is convex (concave) in $\{0, 1, \ldots, n\}$ for all $u, v \in (0, 1)$, then:

(i) $K_1 \leq_{CX} K_2$ implies $T_{K_1} \leq_{ST} T_{K_2}$ $(\geq_{ST})$;
(ii) $X \geq_{ST} Y$ and $K_1 \leq_{ICX} K_2$ $(\leq_{ICV})$ imply $T_{K_1} \leq_{ST} T_{K_2}$ $(\geq_{ST})$.

This result says that if φ is convex, then the more convex K, the better. If the units from X are ST better than the ones from Y, then the convex ordering can be relaxed to the weakly ICX order. These properties are reverted when φ is concave.

In our series system with two independent components

$$\varphi(k) = u^k v^{n-k}$$

which is a convex function of k for any $u, v \in (0, 1)$ since

$$\varphi'(k) = \varphi(k) \log(u/v)$$

and

$$\varphi''(k) = \varphi(k)(\log(u/v))^2 = u^k v^{n-k}(\log(u/v))^2 \geq 0.$$

Hence, from (i) in the preceding proposition, when two randomized options are ordered in the convex order, the respective systems are ST ordered in the same sense. As mentioned above, the first option T is obtained with K_1 which takes the value 1 with probability 1 and the second S with K_2 with probabilities 0.25, 0.5, 0.25 for $k = 0, 1, 2$, respectively. Another reasonable assumption could be a uniform distribution, that is, $K_3 = 0, 1, 2$ with probability $1/3$. Finally, we could also consider K_4 with probabilities 0.5, 0, 0.5 for $k = 0, 1, 2$, respectively. In all these options we have $E(K_i) = 1$ for $i = 1, 2, 3, 4$ (i.e. they use the same number of components of each type). It can also be proved (e.g. by plotting their respective probability mass functions) that

$$K_1 \leq_{CX} K_2 \leq_{CX} K_3 \leq_{CX} K_4.$$

Therefore, from (i),

$$T_{K_1} \leq_{ST} T_{K_2} \leq_{ST} T_{K_3} \leq_{ST} T_{K_4}$$

for all $\bar{F}_X, \bar{F}_Y$. Actually, K_4 is the more convex option for K such that $E(K) = 1$. Hence it is always the best option in this system. It corresponds to the green line in Fig. 3.14 and it assumes that the series systems are built with two units of type X or with two units of type Y, randomly. This is the best option for our system and any $\bar{F}_X, \bar{F}_Y$, that is, the green line will be always above the other lines (reliabilities). Note that it is also a "random option" (so it can also be seen as a Parrondo paradox) and that we do not need $\bar{F}_X \geq \bar{F}_Y$. However, if this property holds, this best option could be unreasonable in practice since the 50% of the customers will have a very good system with two good units but the others will have a very bad system built with two bad units. In this case, what should be done in practice? The answer is not easy. Although this best option is very dispersed (we have very good and very bad systems), note that this what we do at home, for example, with the remote control when we have good and bad batteries (we put together the units of the same type). This is the best option for series systems with independent and heterogeneous components.

Problems

1. Compare two systems with IID components by using their signatures. Plot the respective functions to confirm (or reject) the comparisons obtained.
2. Check that an arrow in Fig. 3.2 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
3. Check that a no-arrow in Fig. 3.2 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.

4. Check that an arrow in Fig. 3.3 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
5. Check that a no-arrow in Fig. 3.3 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
6. Check that an arrow in Fig. 3.4 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
7. Check that a no-arrow in Fig. 3.4 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
8. Study the orderings for series and parallel systems with ID components for a given bivariate survival copula. Plot the respective functions to confirm (or reject) the comparisons obtained.
9. Study the orderings for series systems with ID components and two different bivariate survival copulas. Plot the respective functions to confirm (or reject) the comparisons obtained.
10. Study the orderings for two systems with ID components for a given trivariate survival copula. Plot the respective functions to confirm (or reject) the comparisons obtained.
11. Study the effect of the dependence parameter of a copula in the reliability of a system with ID components. Plot the respective functions to confirm (or reject) the comparisons obtained.
12. Study the orderings $X_{1:3} \leq_{HR} X_{2:3} \leq_{HR} X_{3:3}$ for ID components and a survival copula $\widehat{C}$.
13. Find an EXC copula for which $X_{1:2} \leq_{HR} X_{2:2}$ does not hold in the ID case. Plot the hazard rate functions to confirm that this comparison does not hold.
14. Check that a number in Table 3.4 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
15. Check that a number in Table 3.5 is correct. Plot the respective functions to confirm (or reject) the comparisons obtained.
16. Check that an arrow in Fig. 3.11 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
17. Check that a no-arrow in Fig. 3.11 is correct. Plot the respective functions to confirm (or reject) the comparison obtained.
18. Compare $X_{1:2}, X_1, X_2$ and $X_{2:2}$ for a fixed bivariate survival copula $\widehat{C}$ and arbitrary distributions F_1, F_2.
19. Compare two semi-coherent systems of order 3 for a fixed trivariate survival copula $\widehat{C}$ and arbitrary distributions F_1, F_2, F_3.
20. Compare two systems by using RR-plots.
21. Confirm the Parrondo paradox in series systems with independent components.
22. Confirm the Parrondo paradox in series systems with dependent components and an Archimedean copula.
23. Study the Parrondo paradox in a non-series system with independent components. Plot the respective reliability functions.
24. Prove that the Parrondo paradox holds for series systems with n independent components.
25. Prove that the Parrondo paradox is reverted for parallel systems with n independent components.