1 Coherent System Structures

Abstract

In this chapter we study the basic properties of the main concept in the Reliability Theory: the coherent system structures. In the first section we give the formal definitions of coherent and semi-coherent (binary) system structures, providing several examples. We do not study non-coherent systems here. We refer the interested readers in that systems to Borgonovo (2010), Imakhlaf et al. (2017) and the references therein. The main properties of coherent systems are given in the second section, including several representations for the structure function of the system. Relationships with simple games, connectivity properties of networks and mixed systems are studied in the third section. The fourth section contains some results for multi-state systems with binary components. The components' importance indices are not studied here. Some of them are studied in Chap. 5. In a first reading, Sects. 1.3 and 1.4 can be skipped (if you want).

1.1 Coherent Structures

The systems are the main concepts in the Reliability Theory. They are "structures" built by using several components. The main assumption is that the state of the system only depends on the states of the components through a "structure function". In this section we assume that the system and the components only have two possible states, a functioning state represented by a 1 and a failure state represented by a 0. Then the formal (mathematical) definition of (binary) *system* can be stated as follows.

Definition 1.1 A (binary) **system** with (binary) components of order n is a Boolean structure function (map)

$$\phi : \{0, 1\}^n \to \{0, 1\},$$

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where $\phi(x_1, \ldots, x_n) \in \{0, 1\}$ represents the system's state that is completely determined by the components' states represented by $x_1, \ldots, x_n \in \{0, 1\}$.

To simplify, we just use the word "system" to represent a binary system with binary components. Here it is natural to assume some additional properties for the structure function ϕ. For example, we can expect that a system does not work when all the components fail or that the system works when all the components do so. Analogously, we may also assume that if a broken component is replaced by a functioning component (or it is repaired), then the system state cannot be worse. These assumptions lead to the concept of *semi-coherent systems*. If one (or more) of these properties fails, then we have a non-coherent system that are studied in the references mentioned above.

Definition 1.2 A **semi-coherent system** of order n is a system

$$\phi : \{0, 1\}^n \to \{0, 1\}$$

satisfying the following properties:

(i) ϕ is increasing;
(ii) $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$.

Throughout the book we use the words "increasing" and "decreasing" in a wide sense, that is, a function g is increasing (resp. decreasing) when

$$g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n) \quad (\geq)$$

for all $x_1 \leq y_1, \ldots, x_n \leq y_n$.

Semi-coherent systems may have "irrelevant" components, that is, components that do not affect the system. The formal definition is the following.

Definition 1.3 The ith component is **irrelevant** for the system ϕ if

$$\phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \phi(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \{0, 1\}$. If this is not the case, then it is a **relevant** component.

For example, the structure function $\phi(x_1, x_2) = x_1$ is a semi-coherent system of order 2 that represents the system formed just with the first component. Here the second component is irrelevant for the system since $\phi(x_1, 0) = \phi(x_1, 1)$ for all x_1. To avoid this problem we consider the concept of coherent system defined as follows. This is the main concept in the present book.

Definition 1.4 A **coherent system** of order n is a system

$$\phi : \{0, 1\}^n \to \{0, 1\}$$

satisfying the following properties:

(i) ϕ is increasing;
(ii) ϕ is strictly increasing in each variable in at least a point.

Clearly, the second condition can be replaced with: "All the components are relevant" and we have the following property.

Proposition 1.1 *All the coherent systems are also semi-coherent systems.*

Proof The condition (ii) in the preceding definition implies that, in particular, ϕ is strictly increasing in x_1 in at least a point, that is, there exist $x_2, \ldots, x_n \in \{0, 1\}$ such that

$$0 = \phi(0, x_2, \ldots, x_n) < \phi(1, x_2, \ldots, x_n) = 1.$$

Hence, from (i), we have

$$0 \leq \phi(0, \ldots, 0) \leq \phi(0, x_2, \ldots, x_n) = 0$$

and

$$1 = \phi(1, x_2, \ldots, x_n) \leq \phi(1, \ldots, 1) \leq 1.$$

Therefore, $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$. □

Note that some semi-coherent systems of order n can be considered as an extension of a coherent system in a dimension $k < n$. For example, the semi-coherent system in dimension 2 defined by $\phi(x_1, x_2) = x_1$ is an extension of the coherent system $\phi(x_1) = x_1$ in dimension 1.

Also note that, from a mathematical point of view, the coherent systems $\phi_1(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3))$ and $\phi_2(x_1, x_2, x_3) = \min(x_2, \max(x_1, x_3))$ are different. However, when we plot them they have a similar "structure" (see Fig. 1.1). This fact is important when we want to count all the coherent systems of a given dimension (see next section). To consider this fact we need the following definition.

Definition 1.5 We say that two systems ϕ_1 and ϕ_2 of order n are **equivalent under permutations** (shortly written as $\phi_1 \sim \phi_2$) if

$$\phi_1(x_1, \ldots, x_n) = \phi_2(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

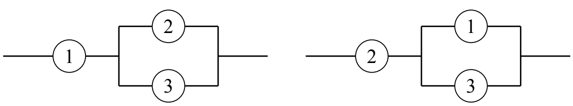

Fig. 1.1 Two coherent systems of order 3 with a similar structure

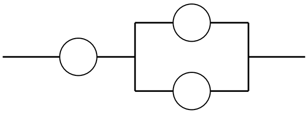

Fig. 1.2 A general structure for a coherent systems of order 3

The equivalence classes determined by this relationship can also be called "systems". For example, the systems given in Fig. 1.1 can be represented by the equivalence class represented by the system in Fig. 1.2.

A coherent (or semi-coherent) system can be determined by the sets of components that assure that the system works (resp. fails) when these components work (fail). The formal definition of such sets is the following.

Definition 1.6 A non-empty set $P \subseteq \{1, \ldots, n\}$ is a **path set** of a system ϕ if $\phi(x_1, \ldots, x_n) = 1$ when $x_i = 1$ for all $i \in P$. A non-empty set $C \subseteq \{1, \ldots, n\}$ is a **cut set** of ϕ if $\phi(x_1, \ldots, x_n) = 0$ when $x_i = 0$ for all $i \in C$. A path set P is a **minimal path set** if it does not contain other path sets. A cut set C is a **minimal cut set** if it does not contain other cut sets.

The sets of path and cut sets of a system ϕ are represented by $\mathcal{P}$ and $\mathcal{C}$. Then we have the following properties. To simplify, in the book, we use "iff" instead of "if and only if".

Proposition 1.2 *Let ϕ be a system. Then:*

(i) *ϕ is increasing iff $\mathcal{P}$ is closed under super-inclusions (i.e. if $P \in \mathcal{P}$ and $P \subseteq P^*$, then $P^* \in \mathcal{P}$).*
(ii) *ϕ is increasing iff $\mathcal{C}$ is closed under super-inclusions.*
(iii) *$\phi(0, \ldots, 0) = 0$ iff $\{1, \ldots, n\} \in \mathcal{C}$.*
(iv) *$\phi(1, \ldots, 1) = 1$ iff $\{1, \ldots, n\} \in \mathcal{P}$.*
(v) *ϕ is semi-coherent iff $\mathcal{P}$ is non-empty, closed under super-inclusions and does not contain the empty set.*
(vi) *ϕ is semi-coherent iff $\mathcal{C}$ is non-empty, closed under super-inclusions and does not contain the empty set.*

Note that, in semi-coherent systems, $\mathcal{P}$ and $\mathcal{C}$ have the same structural properties. To explain this fact we need another concept that can be stated as follows.

Definition 1.7 The **dual system** of a system ϕ is the system

$$\phi^D : \{0, 1\}^n \to \{0, 1\}$$

defined by $\phi^D(x_1, \ldots, x_n) := 1 - \phi(1 - x_1, \ldots, 1 - x_n)$ for all $x_1, \ldots, x_n \in \{0, 1\}$.

The following properties for the dual systems can be proved easily.

Proposition 1.3 *Let ϕ be a coherent (resp. semi-coherent) system and let ϕ^D be its dual system. Then:*

(i) *ϕ^D is a coherent (resp. semi-coherent) system.*
(ii) *A set is a path set of ϕ iff it is a cut set of ϕ^D.*
(iii) *A set is a cut set of ϕ iff it is a path set of ϕ^D.*
(iv) *A set is a minimal path set of ϕ iff it is a minimal cut set of ϕ^D.*
(v) *A set is a minimal cut set of ϕ iff it is a minimal path set of ϕ^D.*
(vi) $(\phi^D)^D = \phi$.

Let us see now several examples of coherent and semi-coherent systems. The main structures are series and parallel structures defined as follows.

Definition 1.8 The **series system** of order n is

$$\phi_{1:n}(x_1, \ldots, x_n) := \min(x_1, \ldots, x_n).$$

The **parallel system** of order n is

$$\phi_{n:n}(x_1, \ldots, x_n) := \max(x_1, \ldots, x_n).$$

The **series system** with components in the set P is

$$\phi_P(x_1, \ldots, x_n) := \min_{i \in P} x_i.$$

The **parallel system** with components in the set P is

$$\phi^P(x_1, \ldots, x_n) := \max_{i \in P} x_i.$$

The series and parallel systems of order n are coherent systems but that based on a set P are just semi-coherent systems. Of course, $\phi_{\{1,\ldots,n\}} = \phi_{1:n}$ and $\phi^{\{1,\ldots,n\}} = \phi_{n:n}$ and, in these cases, they are also coherent systems. Moreover, the dual system of ϕ_P is ϕ^P and vice versa.

Note that Boolean functions can be expressed in many different ways. The main options are to use min and max operators (as above) or to use polynomials (or multinomials). For example, the series system ϕ_P can also be written as

$$\phi_P(x_1, \ldots, x_n) = \prod_{i \in P} x_i.$$

Note that these options coincide when $x_i \in \{0, 1\}$ but that they are different when we extend these functions to other sets (see next chapter). Analogously, the parallel system ϕ^P can also be written as

$$\phi^P(x_1, \ldots, x_n) = \coprod_{i \in P} x_i$$

where the coproduct $\coprod$ is defined as

$$\coprod_{i \in P} x_i = 1 - \prod_{i \in P}(1 - x_i).$$

For example,

$$\phi_{2:2}(x_1, x_2) = \max(x_1, x_2) = x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

for all $x_1, x_2 \in \{0, 1\}$.

We will see in the next section that all the coherent (or semi-coherent) systems can be written by using series and parallel structures.

Other relevant structures are the *k-out-of-n systems* that work when at least k of their n components work. The explicit definition is the following.

Definition 1.9 The ***k*-out-of-*n* system** is defined by

$$\phi_{n-k+1:n}(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_1 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + \cdots + x_n < k \end{cases} \tag{1.1}$$

for $k = 1, \ldots, n$.

The minimal path sets of the k-out-of-n system are all the sets with exactly k components. So it has $\binom{n}{k}$ minimal path sets. Note that with this definition the 1-out-of-n system is the parallel system $\phi_{n:n}$ and the n-out-of-n system is the series system $\phi_{1:n}$. If $(x_{1:n}, \ldots, x_{n:n})$ represents the increasing ordered vector obtained from $(x_1, \ldots, x_n)$, then

$$\phi_{n-k+1:n}(x_1, \ldots, x_n) = x_{n-k+1:n}$$

for $k = 1, \ldots, n$. This notation is the same as that used to represent the *order statistics*, that is, the ordered data obtained from a sample (see, e.g., Arnold et al. 2008; David and Nagaraja 2003). For example, the 2-out-of-3 system is

$$\phi_{2:3}(x_1, x_2, x_3) = x_{2:3} = \max(\min(x_1, x_2), \min(x_1, x_3), \min(x_2, x_3)).$$

Note that this system cannot be plotted in a plane graph similar to that showed in Fig. 1.1 (we need to repeat the components). An alternative representation as a network will be showed in Sect. 1.3.

Other authors prefer to consider the *k-out-of-n:F systems* (here F means "failed") that fail when at least k of their n components fail. Its structure function is $\phi_{k:n}$ as defined in (1.1) for $k = 1, \ldots, n$. The minimal cut sets of the k-out-of-n:F system are all the sets with exactly k components. So it has $\binom{n}{k}$ minimal cut sets. In this case, the k-out-of-n system considered in the preceding definition can also be called k-out-of-n:G system (here G means "good"). Of course, the dual system of the k-out-of-n:G system is the k-out-of-n:F system and vice versa. Moreover, the k-out-of-n:F system coincides with the $(n - k + 1)$-out-of-n system for $k = 1, \ldots, n$. So we do not need to use the concept of k-out-of-n:F system. However, this notation is needed in the concepts of linear and circular systems defined as follows.

Definition 1.10 For $k = 1, \ldots, n$, the **k-out-of-n:G linear system** is the system that works when at least k consecutive components work, that is, its structure function $\phi_{k:n:G|l}(x_1, \ldots, x_n) = 1$ iff there exists $i \in \{0, \ldots, n-k\}$ such that $x_{i+1} = \cdots = x_{i+k} = 1$. The **$k$-out-of-$n$:$F$ linear system** is the system that fails when at least k consecutive components fail, that is, its structure function $\phi_{k:n:F|l}(x_1, \ldots, x_n) = 0$ iff there exists $i \in \{0, \ldots, n-k\}$ such that $x_{i+1} = \cdots = x_{i+k} = 0$.

The circular systems $\phi_{k:n:G|c}$ and $\phi_{k:n:F|c}$ are defined in a similar way but placing the components in a circle (that is, in this case the first and the last components are also consecutive).

These systems have several applications in practice. For example, the k-out-of-n:F linear systems are used to represent transportation systems as oil or gas pipeline systems and k-out-of-n:F circular systems can represent particle accelerators.

In this case, some k-out-of-n:F linear systems cannot be represented as k-out-of-n:G linear systems. For example, the 2-out-of-3:F linear system is

$$\phi_{2:3:F|l}(x_1, x_2, x_3) = \max(x_2, \min(x_1, x_3)).$$

Its minimal path sets are $P_1 = \{2\}$ and $P_2 = \{1, 3\}$ and its minimal cut sets are $C_1 = \{1, 2\}$ and $C_2 = \{2, 3\}$. So it cannot be represented as a k-out-of-3:G linear system. It is the dual system of the 2-out-of-3:G linear system given by

$$\phi_{2:3:G|l}(x_1, x_2, x_3) = \min(x_2, \max(x_1, x_3)).$$

We conclude this section by computing all the coherent and semi-coherent systems with orders 1-3. Of course, if $n = 1$, then we just have a component and a coherent system $\phi_{1:1}(x_1) = x_1$. If $n = 2$, then we have two coherent systems, the series system $\phi_{1:2}(x_1, x_2) = \min(x_1, x_2)$ and the parallel system $\phi_{2:2}(x_1, x_2) = \max(x_1, x_2)$ of order 2, and the two semi-coherent systems formed with each component. If $n = 3$, then we obtain all the semi-coherent systems given in Table 1.1. Only the nine systems in lines 1, 5, 6, 7, 11, 12, 13, 14 and 18 are coherent systems of order 3. The others are just semi-coherent systems or coherent system of order 1 or 2. The horizontal lines determine the systems that are equivalent under permutations (we have 5 coherent systems and 3 that are just semi-coherent). Note that the system in line $18 - i + 1$ is the dual system of that in line i for $i = 1, \ldots, 7$. The dual systems of the systems in lines 8, 9, 10, 11 are themselves.

1.2 Main Properties

The coherent systems (as Boolean functions) can be written by using different (equivalent) representations. Let us see some of them. The first one is called the *pivotal decomposition* in Barlow and Proschan (1975), p. 5, and can be stated as follows. We shall use the following notation. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $i \in \{1, \ldots, n\}$, then

$$\mathbf{1}_i(\mathbf{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

Table 1.1 Semi-coherent systems of order 3

N	$\phi_N(x_1, x_2, x_3)$	Minimal path sets	Minimal cut sets
1	$x_{1:3} = \min(x_1, x_2, x_3)$	{1, 2, 3}	{1}, {2}, {3}
2	$\min(x_1, x_2)$	{1, 2}	{1}, {2}
3	$\min(x_1, x_3)$	{1, 3}	{1}, {3}
4	$\min(x_2, x_3)$	{2, 3}	{2}, {3}
5	$\min(x_1, \max(x_2, x_3))$	{1, 2}, {1, 3}	{1}, {2, 3}
6	$\min(x_2, \max(x_1, x_3))$	{1, 2}, {2, 3}	{2}, {1, 3}
7	$\min(x_3, \max(x_1, x_2))$	{1, 3}, {2, 3}	{3}, {1, 2}
8	x_3	{3}	{3}
9	x_2	{2}	{2}
10	x_1	{1}	{1}
11	$x_{2:3}$	{1, 2}, {1, 3}, {2, 3}	{1, 2}, {1, 3}, {2, 3}
12	$\max(x_3, \min(x_1, x_2))$	{3}, {1, 2}	{1, 3}, {2, 3}
13	$\max(x_2, \min(x_1, x_3))$	{2}, {1, 3}	{1, 2}, {2, 3}
14	$\max(x_1, \min(x_2, x_3))$	{1}, {2, 3}	{1, 2}, {1, 3}
15	$\max(x_2, x_3)$	{2}, {3}	{2, 3}
16	$\max(x_1, x_3)$	{1}, {3}	{1, 3}
17	$\max(x_1, x_2)$	{1}, {2}	{1, 2}
18	$x_{3:3} = \max(x_1, x_2, x_3)$	{1}, {2}, {3}	{1, 2, 3}

and

$$\mathbf{0}_i(\mathbf{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

Theorem 1.1 (Pivotal decomposition) *Let ϕ be a system of order n, then*

$$\phi(\mathbf{x}) = x_i \phi(\mathbf{1}_i(\mathbf{x})) + (1 - x_i)\phi(\mathbf{0}_i(\mathbf{x})) \tag{1.2}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$ and all $i = 1, \ldots, n$. Moreover,

$$\phi(\mathbf{x}) = \sum_{\mathbf{y} \in \{0,1\}^n} \left(\phi(\mathbf{y}) \prod_{j=1}^{n} x_j^{y_j} (1 - x_j)^{1-y_j} \right) \tag{1.3}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$.

Proof Clearly, (1.2) holds in the two possible cases, $x_i = 1$ and $x_i = 0$. Expression (1.3) is obtained by repeated applications of (1.2). For example, we can start with x_1 obtaining

$$\phi(\mathbf{x}) = x_1 \phi(1, x_2, \ldots, x_n) + (1 - x_1)\phi(0, x_2, \ldots, x_n).$$

Then we apply (1.2) to $\phi(1, x_2, \ldots, x_n)$ and $\phi(0, x_2, \ldots, x_n)$ for $i = 2$ and so on. □

Expression (1.3) proves that ϕ can be written as a multinomial of degree n. This representation will be used in the next chapter to compute the reliability of systems with independent components.

For example, the pivotal decomposition for the system

$$\phi_{14}(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3))$$

(see Table 1.1) is

$$\begin{aligned}\phi_{14}(x_1, x_2, x_3) &= x_1(1-x_2)(1-x_3) + (1-x_1)x_2x_3 + x_1x_2(1-x_3)\\ &\quad + x_1(1-x_2)x_3 + x_1x_2x_3\\ &= x_1 + x_2x_3 - x_1x_2x_3.\end{aligned}$$

The second representation is based on minimal path or minimal cut sets (defined in the preceding section). It is stated in the following theorem. It will be used in the next chapter to compute the system lifetime and the system reliability.

Theorem 1.2 (Minimal path/cut sets' representations) *Let ϕ be a coherent (or semi-coherent) system of order n and let $P_1, \ldots, P_r$ and $C_1, \ldots, C_s$ be its minimal path and minimal cut sets, respectively. Then*

$$\phi(\mathbf{x}) = \max_{i=1,\ldots,r} \min_{j \in P_i} x_j \tag{1.4}$$

and

$$\phi(\mathbf{x}) = \min_{i=1,\ldots,s} \max_{j \in C_i} x_j \tag{1.5}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$.

Proof The first expression (1.4) holds since a coherent system works iff at least one of the series systems obtained from its minimal path sets works. Analogously, (1.5) holds since a coherent system fails iff at least one of the parallel systems obtained from its minimal cut sets fails. □

Remark 1.1 The preceding theorem can also be stated by using path or cut sets. However, the expressions obtained in this way are more complicated than that stated above (so we will not use them).

The preceding theorem shows that any coherent system can be decomposed as series systems connected in parallel or as parallel systems connected in series (with some possible common components). Here we can use the notation introduced in the preceding section for series and parallel systems and write (1.4) and (1.5) as

$$\phi(\mathbf{x}) = \max_{i=1,\ldots,r} \phi_{P_i}(\mathbf{x})$$

and

$$\phi(\mathbf{x}) = \min_{i=1,\ldots,s} \phi^{C_i}(\mathbf{x}),$$

respectively. They can also be written by using products and coproducts as

$$\phi(\mathbf{x}) = \coprod_{i=1}^{r} \prod_{j \in P_i} x_j \tag{1.6}$$

and

$$\phi(\mathbf{x}) = \prod_{i=1}^{s} \coprod_{j \in C_i} x_j. \tag{1.7}$$

Note that we obtain again multinomials of degre n and that these representations are more "efficient" than the pivotal decomposition. For example, for the system ϕ_{14} considered above, we obtain

$$\phi_{14}(x_1, x_2, x_3) = x_1 \coprod x_2 x_3 = 1 - (1 - x_1)(1 - x_2 x_3) = x_1 + x_2 x_3 - x_1 x_2 x_3.$$

In this chapter, representations (1.4)–(1.7) for the Boolean function ϕ are equivalent. However, in the next chapter, they will be used to extend ϕ to real numbers and then they will provide different expressions (that will be used to different purposes). For example, the series system of order 2 can be written as $\phi_{2:2}(x_1, x_2) = \min(x_1, x_2)$ or as the multinomial $\psi_{2:2}(x_1, x_2) = x_1 x_2$. If $x_1, x_2 \in \{0, 1\}$, then $\phi_{2:2}(x_1, x_2) = \psi_{2:2}(x_1, x_2)$. However, they are different as real functions. For example, $\phi_{2:2}(1/2, 1/2) = 1/2 \neq 1/4 = \psi_{2:2}(1/2, 1/2)$.

The minimal path and minimal cut set representations can also be used to determine all the coherent systems of order n. They show that a system is completely determined by its minimal path sets (or by its minimal cut sets). So a system can also be seen as a finite sequence of subsets of $[n] := \{1, \ldots, n\}$ with the properties given in the following proposition.

Proposition 1.4 *The non-empty sets $P_1, \ldots, P_r \subseteq [n]$ are the minimal path (or cut) sets of a coherent system iff the two following properties hold:*

(i) P_i *is not contained in* P_j *for all* $i \neq j$;
(ii) $P_1 \cup \cdots \cup P_r = [n]$.

Proof Clearly, (i) holds when $P_1, \ldots, P_r$ are the minimal path (or cut) sets of a semi-coherent system (by definition). Moreover, if $i \notin P_1 \cup \cdots \cup P_r$ then, from (1.4), the ith component is irrelevant for the system. Therefore, (ii) holds for the minimal path sets of any coherent system. From (1.5), (ii) also holds for the minimal cut sets of a coherent system.

Conversely, if the sets $P_1, \ldots, P_r$ satisfy (i) and (ii), then we can consider the system (Boolean function) ϕ defined by (1.4). Clearly, ϕ is increasing. Moreover, we can prove that any component is relevant due to (ii). Thus, if $i \in [n] = P_1 \cup \cdots \cup P_r$, then, from (ii), there exists a $j \in [r]$ such that $i \in P_j$. Now we consider the point $\mathbf{x} = (x_1, \ldots, x_n)$ defined as $x_k = 1$ if $k \in P_j$ and $x_k = 0$ if $k \notin P_j$. Hence,

$$\phi(\mathbf{0}_i(\mathbf{x})) = 0 < 1 = \phi(\mathbf{1}_i(\mathbf{x}))$$

since $\phi_{P_j}(\mathbf{x}) = x_i$ and $\phi_{P_\ell}(\mathbf{x}) = 0$ for all $\ell \neq j$ (from (i)). Therefore ϕ is a coherent structure. Moreover, it is easy to see that $P_1, \ldots, P_r$ are its minimal path sets. The proof for the minimal cut sets is similar. □

Note that the characteristic properties of minimal path sets and minimal cut sets of coherent systems coincide. This is an expectable property since the minimal path sets of a system are the minimal cut sets of its dual system (and vice versa). However, in the first case we use (1.4) to determine the system while in the second we use (1.5). Moreover, as we have seen in the proof, the minimal path (or cut) sets of semi-coherent systems are just characterized by property (i).

The systems can also be represented by using their paths (or cut) sets. However, as mentioned above, these representations are always more complicated. So we do not include these properties here. Both structures (path/cut sets and minimal path/cut sets) can be used in Set Theory (see Ramamurthy 1990).

The preceding proposition can be used jointly with the following algorithm, extracted from Navarro and Rubio (2010), to determine all the coherent systems of order n. They are determined by their minimal path sets. We use a recursive method on the number k of minimal path sets. We use the notation $|A|$ for the cardinality of the set A. The coherent system ϕ is represented here by the sequence $\phi = (P_1, \ldots, P_k)$ of its minimal path sets with $|P_1| \leq \cdots \leq |P_k|$. Some systems can be written in different ways (we avoid repetitions).

Algorithm 1.2.1:

Step 0: Generate the set $\mathcal{S}$ with all the non-empty subsets of $[n]$ (there are $m = |\mathcal{S}| = 2^n - 1$ subsets).

Step 1: Generate the unique coherent system with $k = 1$ minimal path set (the series system with $P_1 = [n]$). Let $\mathcal{S}_1 = \{([n])\}$.

Step 2: Generate all the coherent systems with $k = 2$ minimal path sets by studying (using Proposition 1.4) all the couples of sets from $\mathcal{S}$ (there are $\binom{m}{2} = m(m-1)/2$ different couples). Their sequences (P_1, P_2) of minimal path sets are included in the set $\mathcal{S}_2$ with $|P_1| \leq |P_2|$ (avoiding repetitions).

Step k (for k = 3, 4, ...): For any sequence $(P_1, P_2, \ldots, P_{k-1}) \in \mathcal{S}_{k-1}$, generate all the different coherent systems obtained by replacing P_{k-1} with a couple of subsets $A, B \in \mathcal{S}$ such that $|P_{k-2}| \leq |A| \leq |B|$. Their sequences of minimal path sets are included in $\mathcal{S}_k$ with $|P_1| \leq |P_2| \leq \cdots \leq |P_k|$ (avoiding repetitions).

Final step: Stop when $\mathcal{S}_k = \emptyset$.

Theorem 1.3 *The preceding algorithm generates all the coherent systems of order n.*

Proof Clearly, from the preceding algorithm, $\mathcal{S}_1$ and $\mathcal{S}_2$ contain all the coherent systems with $k = 1$ and $k = 2$ minimal path sets.

Let us see that the set S_3 obtained in step 3 contains all the coherent systems with $k = 3$ minimal path sets. Let P_1, P_2, P_3 be the minimal path sets of a coherent system of size n and let us assume that $|P_1| \leq |P_2| \leq |P_3|$. Then we consider two cases:

Case I: If $P_1 \cup P_2 = [n]$, then, from Proposition 1.4, P_1, P_2 are the minimal path sets of a coherent system of size n, that is, $(P_1, P_2) \in \mathcal{S}_2$ or $(P_2, P_1) \in \mathcal{S}_2$. Hence (P_1, P_2, P_3) is generated in step 3 when in (P_1, P_2) we delete P_2 and we add the couple (P_2, P_3) or when in (P_2, P_1) we delete P_1 and we add the couple (P_1, P_3).

Case II: If $P_1 \cup P_2 \neq [n]$, we define $A = [n] - (P_1 \cup P_2)$ and $Q = P_2 \cup A$. Clearly, $P_1 \cup Q = [n]$ and $|P_1| \leq |P_2| < |Q|$. Hence (P_1, Q) are the minimal path sets of a coherent system of size n, with $|P_1| < |Q|$, that is, $(P_1, Q) \in \mathcal{S}_2$ (in that order). Hence (P_1, P_2, P_3) is generated in step 3 when in (P_1, Q) we delete Q and we add the couple (P_2, P_3).

By induction, let us assume that $\mathcal{S}_{k-1}$ contains all the coherent systems with $k-1$ minimal path sets. We want to prove that the same happen for $\mathcal{S}_k$ by using a procedure similar to that used in step $k = 3$. Let ϕ be a coherent system with minimal path sets $P_1, \ldots, P_k$ satisfying $|P_1| \leq \cdots \leq |P_k|$. As above we consider two cases:

Case I: If $P_1 \cup \cdots \cup P_{k-1} = [n]$, then, from Proposition 1.4, $P_1, \ldots, P_{k-1}$ are the minimal path sets of a coherent system of size n, that is, $(P_1, \ldots, P_{k-1}) \in \mathcal{S}_{k-1}$ (in this way or in a permuted version). Hence $(P_1, \ldots, P_k)$ is generated in step 3 when in $(P_1, \ldots, P_{k-1})$ we delete P_{k-1} (or the last set P_j) and we add the couple (P_{k-1}, P_k) (we add the couple (P_j, P_k)).

Case II: If $P_1 \cup \cdots \cup P_{k-1} \neq [n]$, we define $A = [n] - (P_1 \cup \cdots \cup P_{k-1})$ and $Q = P_{k-1} \cup A$. Clearly, $P_1 \cup \cdots \cup P_{k-2} \cup Q = [n]$ and $|P_1| \leq \cdots \leq |P_{k-2}| < |Q|$. Hence $P_1, \ldots, P_{k-2}, Q$ are the minimal path sets of a coherent system of order n, that is, $(P_1, \ldots, P_{k-2}, Q) \in \mathcal{S}_{k-1}$ (in this way or in a permuted version). Moreover, in all these permuted versions, Q is the last set in the sequence since $|P_i| \leq |P_{k-1}| < |Q|$ for $i = 1, \ldots, k-2$. Hence $(P_1, \ldots, P_k)$ is generated in step k when in $(P_1, \ldots, P_{k-2}, Q)$ (or in any of its permuted versions) we delete Q and we add the couple (P_{k-1}, P_k). □

The preceding theorem can be used to obtain all the coherent systems of order n (we can use a computer to do so). Let us see an example.

Example 1.1 As we have mentioned in the preceding section, there are 9 coherent system of order 3 (see Table 1.1) that are reduced to just 5 coherent system classes of equivalent systems under permutations. They can be obtained by using the preceding algorithm as follows.

Step 0: If $n = 3$, then $\mathcal{S} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ (with cardinality $2^3 - 1 = 7$) is the set with all the possible minimal path sets.

Step 1: The unique system with $k = 1$ is the series system

$$\phi_1 = (\{1, 2, 3\}) = \min(x_1, x_2, x_3), \ \mathcal{S}_1 = \{\phi_1\}.$$

Step 2: For $k = 2$, we consider the $\binom{7}{2} = 21$ couples of sets from $\mathcal{S}$, obtaining six coherent systems:

$$\mathcal{S}_2 = \{\phi_{14}, \phi_{13}, \phi_{12}, \phi_5, \phi_6, \phi_7\},$$

where we use the notation of Table 1.1, that is, $\phi_{14} = (\{1\}, \{2, 3\})$, $\phi_{13} = (\{2\}, \{1, 3\})$, $\phi_{12} = (\{3\}, \{1, 2\})$, $\phi_5 = (\{1, 2\}, \{1, 3\})$, $\phi_6 = (\{1, 2\}, \{2, 3\})$, $\phi_7 =$

$(\{1, 3\}, \{2, 3\})\}$. For example, the first one is obtained as follows. First we consider all the couples that contain the first set $P_1 = \{1\}$. The first option is $(P_1, P_2 = \{2\})$. It does not determine a proper coherent system since $P_1 \cup P_2 \neq \{1, 2, 3\}$. The same happen with $(P_1, P_2 = \{3\})$. The next options are $(P_1, P_2 = \{1, 2\})$ and $(P_1, P_2 = \{1, 3\})$. They do not determine coherent systems since $P_1 \subset P_2$. The next one is $(P_1, P_2 = \{2, 3\})$ that leads us to system ϕ_{14}.

Step 3: For $k = 3$, we consider the systems in $\mathcal{S}_2$. With the first one $\phi_{14} = (\{1\}, \{2, 3\})$, we delete $\{2, 3\}$ and when we add the couple $(\{2\}, \{3\})$, we obtain the parallel system $\phi_{18} = (\{1\}, \{2\}, \{3\})$. Analogously, with the fourth $\phi_5 = (\{1, 2\}, \{1, 3\})$, we delete $\{1, 3\}$, and when we add the pair $\{1, 3\}, \{2, 3\}$, we obtain the 2-out-of-3 system

$$\phi_{2:3} = \phi_{11} = (\{1, 2\}, \{1, 3\}, \{2, 3\}).$$

In the other options we do not obtain new coherent systems.

Step 4: For $k = 4$, we consider the systems in $\mathcal{S}_3 = \{\phi_{18}, \phi_{11}\}$. With the first one $\phi_{18} = (\{1\}, \{2\}, \{3\})$, we delete $\{3\}$ but we cannot obtain coherent systems by adding $A, B \in \mathcal{S}$ with $1 \leq |A| \leq |B|$. The same happen with the second one $\phi_{11} = (\{1, 2\}, \{1, 3\}, \{2, 3\})$ when we delete $\{2, 3\}$ and we add $A, B \in \mathcal{S}$ with $2 \leq |A| \leq |B|$. Therefore $\mathcal{S}_4 = \emptyset$ and so we stop here. ◀

Shaked and Suárez–Llorens (2003) proved that there are 20 classes of order 4. Navarro and Rubio (2010) used the preceding theorem to compute the 180 and 16145 classes of coherent systems of order 5 and 6. The systems of order 5 can be seen in that paper and those with 6 components in:

https://webs.um.es/jorgenav/miwiki/doku.php?id=coherent_systems.

The last representation is based on the Möbius transform of ϕ. First, we note that a system ϕ can be seen as a set function

$$\phi : 2^{[n]} \to \{0, 1\},$$

where $2^{[n]}$ represents the set (or class) of all the subsets of $[n]$ and for $J \subseteq [n]$ we have

$$\phi(J) := \phi(\mathbf{1}_J)$$

and $\mathbf{1}_J := (x_1, \ldots, x_n)$ with $x_i = 1$ if $i \in J$ and $x_i = 0$ if $i \notin J$. Note that the condition "ϕ is increasing" can be written now as

$$I \subseteq J \Rightarrow \phi(I) \leq \phi(J)$$

(i.e., ϕ is increasing as a set function). Analogously, the conditions $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$, can be written now as

$$\phi(\emptyset) = 0 \text{ and } \phi([n]) = 1.$$

Hence, a semi-coherent system ϕ can be seen as a normalized (or regular) *fuzzy measure* (see Fantozzi and Spizzichino 2015; Grabisch 2016). In this sense (see, e.g., Grabisch 2016), the **Möbius transform** $\widehat{\phi}$ of ϕ is defined as

$$\widehat{\phi}(I) := \sum_{J \subseteq I} (-1)^{|I|-|J|} \phi(J). \tag{1.8}$$

It satisfies the following property: if $\phi(I) = 0$, then $\widehat{\phi}(I) = 0$ (since $\phi(J) = 0$ for all $J \subseteq I$). Moreover the inverse relation

$$\phi(J) = \sum_{I \subseteq J} \widehat{\phi}(I) \tag{1.9}$$

holds. Thus we obtain the following representation.

Theorem 1.4 (Möbius representation) *The structure function of a coherent system ϕ can be written as*

$$\phi(x_1, \ldots, x_n) = \sum_{I \subseteq [n]} \widehat{\phi}(I) \prod_{i \in I} x_i \tag{1.10}$$

for all $x_1, \ldots, x_n \in \{0, 1\}$, where $\widehat{\phi}$ is Möbius transform of ϕ defined by (1.8).

The proof is immediate from (1.9) taking into account that if $(x_1, \ldots, x_n) = \mathbf{1}_J$, then $I \subseteq J$ iff $\prod_{i \in I} x_i = 1$. The main advantage of this representation is that it gives us directly the coefficients of the multinomial representation (in the other representations, we have to do some calculations). Let us see an example.

Example 1.2 Let us consider again the coherent system

$$\phi_{14}(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3)).$$

Its Möbius transform is given by

$$\widehat{\phi}_{14}(\{1\}) = \sum_{J \subseteq \{1\}} (-1)^{1-|J|} \phi(J) = (-1)^0 \phi(\{1\}) = 1,$$

$$\widehat{\phi}_{14}(\{2,3\}) = \sum_{J \subseteq \{2,3\}} (-1)^{2-|J|} \phi(J) = (-1)^0 \phi(\{2,3\}) = 1,$$

$$\begin{aligned}\widehat{\phi}_{14}(\{1,2,3\}) &= \sum_{J \subseteq \{1,2,3\}} (-1)^{3-|J|} \phi(J) \\ &= (-1)^{3-1} \phi(\{1\}) + (-1)^{3-2} \phi(\{2,3\}) + (-1)^{3-2} \phi(\{1,2\}) \\ &\quad + (-1)^{3-2} \phi(\{1,3\}) + (-1)^{3-3} \phi(\{1,2,3\}) \\ &= -1\end{aligned}$$

and $\widehat{\phi}_{14}(I) = 0$ for the other subsets I. Therefore, from (1.10), we obtain

$$\phi_{14}(x_1, x_2, x_3) = x_1 + x_2 x_3 - x_1 x_2 x_3$$

as in the preceding examples. ◀

For more properties on systems' structures we refer the readers to Barlow and Proschan (1975), Marichal et al. (2011) and Ramamurthy (1990).

1.3 Related Concepts

Coherent system structures are similar to other concepts considered in different mathematical and engineering subjects. Let us see some of them.

1.3.1 Simple Games

Let N be a finite set and let 2^N be its power set (with all the subsets of N). Here the elements of N are called **players** and the elements of 2^N are called **coalitions** (see Ramamurthy 1990, p. 37). Then a **simple game** (or a **voting system**) on N is defined as follows.

Definition 1.11 A **simple game** on N is $\lambda : 2^N \to \{0, 1\}$ such that:

(i) $\lambda(\emptyset) = 0$;
(ii) $\lambda(N) = 1$;
(iii) $\lambda(A) \leq \lambda(B)$ for all $A \subseteq B$.

A coalition A is a **winning (losing) coalition** if $\lambda(A) = 1$ (0). It is a **blocking coalition** if $\lambda(A^c) = 0$ where $A^c = N - A$. A winning (blocking) coalition is minimal if it does not contain other winning (blocking) coalitions. To simplify, we can assume $N = [n]$. A player $i \in N$ is called a **dictator** if $\{i\}$ is winning and it is called a **veto-player** if $\{i\}$ is blocking. A player $i \in N$ is called a **dummy** if $\lambda(\{i\} \cap A) = \lambda(A)$ for all A.

Clearly, simple games are equivalent to semi-coherent systems, replacing players with components, winning coalitions with path sets, blocking coalitions with cut sets and dummy players with irrelevant components.

The axioms (properties) that must satisfy a simple game are the following (see Ramamurthy 1990, p. 37).

A1. Every coalition is either winning or losing.
A2. The empty set is losing.
A3. The all player set N is winning.
A4. No losing coalition contains a winning coalition.

Sometimes, the following axioms are also added:

A5. If A is winning, then A^c is losing (proper games).
A6. If A is losing, then A^c is winning (strong games).

The simple games can be classified (see Ramamurthy 1990, p. 42–43) as follows.

1. **Proper games.** Every winning coalition is also a blocking coalition. In this case N cannot be divided in two disjoint winning coalitions. This prevent to get different decisions from disjoint coalitions.
2. **Strong games.** Every blocking coalition is also a winning coalition. In this case N cannot be divided in two disjoint blocking coalitions. This prevent to get a blocking situation from disjoint coalitions.
3. **Decisive games.** They are both proper and strong games.
4. **Symmetric games.** There exists an integer number k such that A is winning iff $|A| \geq k$. These games are equivalent to k-out-of-n systems.
5. **Weighted majority games.** There exist a non-negative vector of weights $(w_1, \ldots, w_n)$ and a real number r such that A is winning iff $\sum_{i \in A} w_i \geq r$. In particular it is also **homogeneous** if all the minimal winning coalitions have the same weights.

1.3.2 Networks

The networks are everywhere today. There are several problems related with networks. Here we just consider connectivity problems. From a mathematical point of view, they can be defined as follows. The main results of this section have been obtained from Gertsbakh and Shpungin (2010, 2020). These references can also be used to get more results.

Definition 1.12 A **network** is $\mathbf{N} = (V, E)$ where V is the vertex (or node) set and E is the edge (or link) set.

Here we just consider networks with a finite set V with $|V| = m$ and a finite set E with $|E| = n$. Usually, the set E is written as $E = \{e_i = \{u_i, v_i\} : u_i, v_i \in V, i = 1, \ldots, n\}$ (undirected networks) or as $E = \{e_i = (u_i, v_i) : u_i, v_i \in V, i = 1, \ldots, n\}$ (directed networks). We assume that the vertices do not fail but that the edges can fail. As in the case of systems, we just consider two possible states for the edges (up and down). A network is **connected** (all connectivity criterion) if all the nodes are connected by a chain of edges. Sometimes, we might fix a set of **terminal** vertices $T \subseteq V$ and just consider connectivity problems between these terminal vertices. All the concepts studied for systems can be translated to these connectivity problems by defining the structure (or state) function of the network

$$\phi : \{0, 1\}^n \to \{0, 1\},$$

where $\phi(x_1, \ldots, x_n) = 1$ (resp. 0) if the network satisfies (does not satisfy) the connectivity conditions when just the edges with $x_i = 1$ work.

For example, the network with $V = \{1, 2, 3\}$ and $E = \{e_1 = \{1, 2\}, e_2 = \{1, 3\}, e_3 = \{2, 3\}\}$ might represent three islands connected with three bridges (or three cities connected by regular lines of airplanes), see Fig. 1.3, left. Then the structure function for the all connectivity criterion is $\phi_{2:3}$, that is, we need at least two working edges. In this case, the minimal path sets are $P_1 = \{e_1, e_2\}$, $P_2 = \{e_1, e_3\}$, and $P_3 = \{e_2, e_3\}$. Remember that this coherent system cannot be plotted as a plane

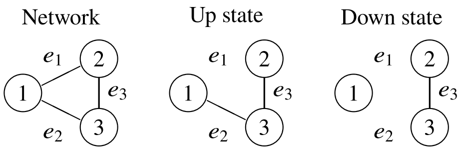

Fig. 1.3 Network of three islands connected with three bridges

system (without repeating components). However, if we just consider the terminal vertices $T = \{1, 3\}$, then the structure function for the connectivity of these two terminal vertices is $\phi_{13}(x_1, x_2, x_3) = \max(x_2, \min(x_1, x_3))$. In this case, the minimal path sets are $P_1 = \{e_1, e_3\}$ and $P_2 = \{e_2\}$. For other criterion see Gertsbakh and Shpungin (2010, 2020).

1.3.3 Mixed Systems

The concept of *mixed system* was introduced by Boland and Samaniego (2004). They can be used to represent systems that should fulfill different requirements in different periods of time. They can be defined as follows.

Definition 1.13 We say that ϕ is a **mixed system** of order n if it is equal to ϕ_j with probability $p_j \geq 0$ for $j = 1, \ldots, m$, where $\phi_1, \ldots, \phi_m$ are systems of order n and $p_1 + \cdots + p_m = 1$. We say that a mixed system ϕ is **semi-coherent** if $\phi_1, \ldots, \phi_m$ are semi-coherent systems. We say that a mixed system ϕ is **coherent** if $\phi_1, \ldots, \phi_m$ are semi-coherent systems and every component is relevant in at least a system with a positive probability.

Any (deterministic) system ϕ_1 can be seen as a mixed system ϕ that takes the value $\phi = \phi_1$ with probability 1. However, the reverse is not true. A mixed system ϕ can written as a map $\phi : \{0, 1\}^n \to \{0, 1\}$ but note that here $\phi(x_1, \ldots, x_n)$ represents a discrete random variable that takes the value $\phi_j(x_1, \ldots, x_n) \in \{0, 1\}$ with probability p_j, for $j = 1, \ldots, m$. If ϕ is semi-coherent, then $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$ (since the same properties hold for any j). However, we cannot assure that ϕ is increasing (due to the randomness). For example, we can consider the coherent mixed system ϕ defined as

$$\phi(x_1, x_2, x_3) = \phi_{1:3}(x_1, x_2, x_3) = \min(x_1, x_2, x_3), \text{ with probability } 1/2$$

and

$$\phi(x_1, x_2, x_3) = \phi_{3:3}(x_1, x_2, x_3) = \max(x_1, x_2, x_3), \text{ with probability } 1/2.$$

This mixed system might represent a system (situation) in which we need the three components half the time (by the day, say) and just one of them in the other half time (by night). Note that $\phi(0, 0, 0) = 0 \leq \phi(1, 1, 1) = 1$. However, we cannot assure that $\phi(1, 0, 0) = 0 \leq \phi(1, 1, 0) = 1$ since the following event might happen

$\phi(1, 0, 0) = 1 > \phi(1, 1, 0) = 0$ (with probability 1/4). Instead we have the following property. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, we say that $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$ for all i.

Proposition 1.5 *If ϕ is a semi-coherent mixed system and $\mathbf{x} \leq \mathbf{y}$, then*

$$E(\phi(\mathbf{x})) \leq E(\phi(\mathbf{y})).$$

Proof From the definition we have

$$E(\phi(\mathbf{x})) = \sum_{j=1}^{m} p_j \phi_j(\mathbf{x}) \leq \sum_{j=1}^{m} p_j \phi_j(\mathbf{y}) = E(\phi(\mathbf{y})),$$

where the inequality holds since $\phi_1, \ldots \phi_m$ are semi-coherent systems. □

1.4 Multi-state Systems with Binary Components

In this section we assume that, for a fixed $m \in \mathbb{N}$, the set of possible states of a system is

$$\mathcal{S} := \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\right\},$$

where, as above, 1 represents the perfect functioning state and 0 the state of failure. In the middle, we have $m - 1$ intermediate states. The evolution in time of the performance of the system can then be seen as a stochastic process starting from 1 (perfect functioning) and eventually going to 0 (failure) as $t \to \infty$.

This representation is clearly equivalent to the classical representation using the levels $\{0, 1, \ldots, m\}$ for a given integer number m. We could of course consider systems with more general levels $\ell_0 = 0 < \ell_1 < \cdots < \ell_m$ by using the set

$$\mathcal{S}^* := \left\{h_0 = 0, h_1 = \frac{\ell_1}{\ell_m}, h_2 = \frac{\ell_2}{\ell_m}, \ldots, h_{m-1} = \frac{\ell_{m-1}}{\ell_m}, h_m = 1\right\}.$$

This general case can be studied in a similar way.

Thus we define the structure of a multi-state system with binary components as follows.

Definition 1.14 A **multi-state system** with binary components is a function

$$\varphi : \{0, 1\}^n \to \mathcal{S}.$$

It is semi-coherent if φ is increasing, $\varphi(0, \ldots, 0) = 0$ and $\varphi(1, \ldots, 1) = 1$. It is coherent if all the components are relevant (i.e. φ is strictly increasing in all the variables in at least a point).

Then we notice that φ has the properties of a normalized (or regular) **fuzzy measure**. As for binary systems, φ can be considered as a set function defined over the family $2^{[n]}$ of all the subsets of $[n]$ where for $J \subseteq [n]$,

$$\varphi(J) := \varphi(\mathbf{1}_J)$$

and $\mathbf{1}_J := (x_1, \ldots, x_n)$ with $x_j = 1$ for $j \in J$ and $x_j = 0$ for $i \notin J$. In this sense (see, e.g., Grabisch 2016), the Möbius transform $\widehat{\varphi}$ of φ is

$$\widehat{\varphi}(I) := \sum_{J \subseteq I} (-1)^{|I|-|J|} \varphi(J)$$

and it is such that the inverse relation

$$\varphi(J) = \sum_{I \subseteq J} \widehat{\varphi}(I) \tag{1.11}$$

holds. It is also useful for our purposes below to rewrite the previous equation (1.11) in a slightly different form. For $\mathbf{x} \in \{0, 1\}^n$ and $I \subseteq [n]$ such that $\mathbf{x} = \mathbf{1}_I$, we can write

$$\varphi(x_1, \ldots, x_n) = \sum_{J \subseteq I} \widehat{\varphi}(J) = \sum_{J \subseteq [n]} \widehat{\varphi}(J) \prod_{j \in J} x_j. \tag{1.12}$$

This expression is similar to the one obtained for binary systems, see (1.10).

1.4.1 Binary Systems Associated to a Multi-state System

Given a multi-state structure φ, we can consider (see Block and Savits 1982; Marichal et al. 2017) the associated binary systems with the following structures

$$\varphi_i(x_1, \ldots, x_n) = \begin{cases} 1, \text{ if } \varphi(x_1, \ldots, x_n) \geq \frac{i}{m} \\ 0, \text{ if } \varphi(x_1, \ldots, x_n) < \frac{i}{m} \end{cases} \tag{1.13}$$

for $i = 1, \ldots, m$. If φ is semi-coherent, the binary structures $\varphi_1, \ldots, \varphi_m$ are semi-coherent binary systems and satisfy $\varphi_1 \geq \cdots \geq \varphi_m$. Moreover, we have

$$\varphi(x_1, \ldots, x_n) = \frac{1}{m} \sum_{i=1}^{m} \varphi_i(x_1, \ldots, x_n). \tag{1.14}$$

Thus any multi-level system can be associated to a mixed system (see the definition in the preceding subsection) as follows. Note that if ϕ is a mixed system, then $E(\phi)$ is a semi-coherent multi-level system (see the preceding subsection).

Definition 1.15 The mixed system ϕ associated to a multi-level system with structure function φ is the one that is equal to the binary system φ_i with probability $1/m$, for $i = 1, \ldots, m$.

Note that

$$E(\phi(x_1, \ldots, x_n)) = \frac{1}{m} \sum_{i=1}^{m} \varphi_i(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_n).$$

1.4.2 Multi-state System Associated to a Binary System

Conversely, if ψ is a semi-coherent binary system, then we can define an associated multi-state system. First, we consider the semi-coherent series systems associated to the minimal path sets $P_1, \ldots, P_r$ of ψ defined as

$$\psi_{P_j}(x_1, \ldots, x_n) = \min_{i \in P_j} x_i$$

for $j = 1, \ldots, r$. Then we can use these systems to define the multi-state system with binary components associated to ψ as follows.

Definition 1.16 Let ψ be a semi-coherent system. Then the multi-state system $\widetilde{\psi} : \{0, 1\}^n \to [0, 1]$ associated to ψ is defined by

$$\tilde{\psi}(x_1, \ldots, x_n) = \frac{1}{r} \sum_{j=1}^{r} \psi_{P_j}(x_1, \ldots, x_n). \tag{1.15}$$

Note that the set of possible states of system $\tilde{\psi}$ is

$$\mathcal{S} := \left\{0, \frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-1}{r}, 1\right\}.$$

As above, $\tilde{\psi}$ can be seen as a normalized (or regular) fuzzy measure. The meaning of $\tilde{\psi}$ is clear, it represents the proportion of working minimal path sets in the system (note that the multi-state system $\tilde{\psi}$ could also be defined over the set $\{0, \ldots, r\}$ as the number of working minimal path sets). In particular, $\tilde{\psi} = 1$ means that all the minimal path sets are working and $\tilde{\psi} = 0$ that the system has failed (all the minimal path sets have failed). Therefore, $\tilde{\psi}$ is a risk measure for the system that can be used to describe the *system failure process* from the initial state $\tilde{\psi} = 1$ to the final failure state $\tilde{\psi} = 0$ with intermediate states $(r-1)/r, \ldots, 1/r$. This process could also be used to determine replacement or repair policies in the system.

As in the preceding subsection we can define the associated semi-coherent systems $\psi_j : \{0, 1\} \to \{0, 1\}$ for $j = 1, \ldots, r$, defined by

$$\psi_j(x_1, \ldots, x_n) = 1 \Leftrightarrow \tilde{\psi}(x_1, \ldots, x_n) \geq j/r.$$

Then

$$\tilde{\psi}(x_1, \ldots, x_n) = \frac{1}{r} \sum_{j=1}^{r} \psi_j(x_1, \ldots, x_n)$$

and we can define (as in the preceding section) the mixed system associated to $\tilde{\psi}$.

Problems

1. Prove that if ϕ is a coherent system and $A \subseteq [n]$, then either A is a path set or $A^c := [n] - A$ is a cut set.
2. Compute the structure function of a plain with four engines, two in each wing, that can fly whenever at least an engine is working in each wing.
3. Compute all the coherent systems of order 4 and the number of equivalence classes.
4. Compute all the semi-coherent systems of order 4.
5. Compute the structure functions of all the k-out-of-5 systems.
6. Compute the structure function of a k-out-of-4 linear or circular system. Could some of them be written in a simplified way?
7. Obtain the pivotal decomposition of a coherent system of order 3.
8. Obtain the minimal path set representation of a system of order 4.
9. Obtain the minimal cut set representation of a system of order 4.
10. Obtain the Möbius transform representation of a system of order 4.
11. Prove (1.9).
12. Obtain all the coherent systems of order 5 with 3 minimal path sets.
13. Given a coherent system of order 4, obtain an equivalent network.
14. Given a network, obtain an equivalent coherent system.
15. In a parliament, the parties A, B, C and D have 50, 26, 22 and 11 deputies, respectively. If in a majority decision voting system, they can just vote 'yes' or 'no', obtain the associated simple game (system).
16. Obtain the binary systems associated to a multi-state system.
17. Obtain the multi-state system associated to a binary coherent system.