A Decision Making Tool for Mathematics Curricula Formal Verification

Eugenio Roanes-Lozano and Angélica Martínez-Zarzuelo

1 Introduction

An appropriate organization of the educational contents taught and learned in today's classrooms is fundamental. Educational laws usually establish a possible organization of educational contents, making it very clear that it should be considered a brief general guide. In Martínez-Zarzuelo et al. (2017) we take advantage of the degree of freedom that these laws grant and we propose a grouping and organization of contents considering a criterion based on the meaningful learning theory (Ausubel, 1963; Ausubel & Barberán, 2002; Ausubel et al., 1976; Moreira, 2000).

The authors have been working on mathematics curricula organization for a long time. The basic idea of these investigations is to consider as a starting point a set of mathematical educational contents and to establish two binary relations among the contents: the relations "to be a prerequisite" and "to be an immediate prerequisite". More precisely, if Content_1 and Content_2 are two educational contents, we have used the following definitions (Martínez Zarzuelo, 2015; Martínez-Zarzuelo et al., 2016):

- Content_1 is a prerequisite of Content_2, denoted Content_1 $\blacktriangleright$ Content_2, if and only if understanding Content_1 is required to understand Content_2.
- Content_1 is an immediate prerequisite of Content_2, denoted Content_1 $\triangleright$ Content_2, if and only if

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(i) understanding Content_1 is required to understand Content_2 (that is, Content_1 ► Content_2),
and
(ii) there is no Content_3 such that Content_1 ► Content_3 ► Content_2.

(from the formal point of view, it is smarter to define $\triangleright$ from ►).

Therefore, the key idea is to address the different types of curricula as partially ordered sets (or directed graphs). This way it is possible to use a computer to perform calculations about the dependence among contents.

2 A Theoretical Proposal

In Roanes-Lozano et al. (2020b), a theoretical approach to the verification of a certain "official curriculum development" (in the sense that it matches a given "preprocessed official curriculum" set as reference) is developed.

Let us try to summarize that article. Three curricula are distinguished in it:

- There is an "official curriculum" (O), a very general small set of contents imposed by the education authorities.
- We suppose that a team of experts has detailed O and there exists a "preprocessed official curriculum" (C), that details O, and introduces the "immediate prerequisite" relation ($\triangleright$) among contents. The transitive closure of $\triangleright$, that could be denoted "prerequisite" relation, is represented by a ►. (C is not normally available, but must be prepared for our purposes from O.) Observe that, when facing the real situation, that is, from a constructive point of view, the "prerequisite" relation, ►, is obtained as the transitive closure of the "immediate prerequisite" relation, $\triangleright$, defined by the team of experts that have prepared the "preprocessed official curriculum", C.
- Someone else proposes an "official curriculum development" (for instance, a textbook or a project-based learning proposal), with another "immediate prerequisite" relation (G,$\succ$).

The following comprehensive verification process is proposed:

- Step 1a: contents soundness: all the contents in G appear in C.
- Step 1b: contents completeness: all the contents in O can be found in G.
- Step 2a: relation soundness: all "immediate prerequisite" relation ordered pairs of G appear in the transitive closure of the "immediate prerequisite" relation proposed by C (that is, $\succ \subseteq$ ►).
- Step 2b: relation completeness: all "immediate prerequisite" relation ordered pairs of G appear as "immediate prerequisite" in C ($\succ \subseteq \triangleright$).
- Step 3: absence of cycles in G.

Let us clarify steps 2a and 2b. Step 2b is clearly a stronger condition than 2a, but their meaning in this context is different.

In Step 2a it has to be checked that all ordered pairs in $(G, \succ)$ are either explicitly included by the experts in $(C, \rhd)$ or can be inferred from the ordered pairs considered in $(C, \rhd)$. As $\blacktriangleright$ is the transitive closure of $\rhd$, this is equivalent to checking whether $\succ \subseteq \blacktriangleright$ or not. If it does not hold it can be an error of the experts that developed $(C, \rhd)$ or something wrong in $(G, \succ)$.

Meanwhile, if Step 2a has been passed, Step 2b checks if any intermediate content in C is bypassed in G (as G is an detailed extension of the brief O, this does not mean that G is wrong, but the absence of this ordered pair deserves a careful analysis).

3 Design and Implementation of the Theoretical Proposal

We consider that this theoretical proposal could be useful and have various applications in the educational context. If an "official curriculum" is provided, for instance, by the educational authorities, an "official curriculum development" could be proposed by authors of educational resources such as textbooks. Thus, these textbooks that are adjusted to the "official curriculum" could be evaluated with our proposal in a simple computational way once the mathematical contents and the "prerequisite" relation among them have been set. This would avoid manual checking and, above all, would provide a guarantee of the completeness and soundness of the educational resources based on what is approved by the academic authorities. Other examples of current educational interest are "official curriculum developments" corresponding to project-based learning proposals. With our idea, it could be automatically checked in a simple way if, for example, a project-based learning proposal complies with the educational contents planned for a certain educational period and if it is complete and sound.

We shall exemplify the approaches hereinafter with a real case taken from Martínez Zarzuelo (2015). For this, we will consider 112 different algebraic educational contents of the Spanish Compulsory Secondary Education (corresponding to Grades 7–10 in the K-12 system) and 261 ordered pairs of the "immediate prerequisite" relation. We focus on mathematical contents because the hierarchical structure of the mathematical discipline allows its concepts to be organized coherently according to a prerequisite relation, but the same ideas can also be applied to educational contents from other disciplines.

3.1 First Approach (Rule Based Expert System)

The inspiration to this work is the process used for Rule Based Expert Systems (RBES) knowledge extraction and verification. There are several different computational methods for this goal. We have considered one based on moving to an algebraic model of Boolean logic (Alonso & Briales, 1995; Chazarain et al., 1991; Hsiang, 1985; Kapur & Narendran, 1985; Roanes-Lozano et al., 1998):

$$A = (Z/2Z)[x_1, \ldots, x_n]/ < x_1^2 - x_1, \ldots, x_n^2 - x_n >$$

where $x_1,\ldots,x_n$ are polynomial variables, image of the propositional variables in the isomorphism between the Boolean algebra of logic and the polynomial Boolean algebra (depending on the operations considered, either a Boolean ring isomorphism or a Boolean algebra isomorphism can be considered).

The main result states that (Roanes-Lozano et al., 1998, 2010):

- The logical proposition Y is a tautological consequence of $\{Y_1,\ldots,Y_n\}$ if and only if $y + 1 \in < y_1 + 1,\ldots,y_n + 1 >$
- $\{Y_1,\ldots,Y_n\}$ is consistent if and only if the ideal $< y_1 + 1,\ldots,y_n + 1 >$ is not the whole ring (that is, $< y_1 + 1,\ldots,y_n + 1 > \neq < 1 >$)

(where $< y_1 + 1,\ldots,y_n + 1 >$ denotes the polynomial ideal generated by $y_1 + 1,\ldots,y_n + 1$ and the lowercase polynomial variables denote the image of the corresponding uppercase propositional variables in the isomorphism mentioned above). The ideal membership and the non-degeneracy of the ideal can be computed using "normal forms" and "Gröbner bases", respectively.

Note that if x is the polynomial translation of proposition X, $x + 1$ is the polynomial translation of the negation of Y. The reason for including the negations in the results above is that propositions are normally stated as "true", what corresponds in the algebraic model to stating that the value of their algebraic translation is 1, meanwhile what is convenient in algebra is to decide whether an expression vanishes, that is, it is equal to 0, or not.

If we denote $I = < x_1^2 - x_1,\ldots,x_n^2 - x_n >$, a RBES where the facts in a certain set are stated as true is modeled by:

$$A/(J + K) = (Z/2Z)[x_1, \ldots, x_n]/(I + J + K)$$

where J is the ideal generated by the negation of the rules and K is the ideal generated by the negation of the facts stated as true (see Roanes-Lozano et al. (2010) for details). A recent related paper is Alonso-Jiménez et al. (2018).

We could use the *Maple*[1] implementation used in the recent Roanes-Lozano et al. (2020a) for dealing with knowledge extraction and formal verification of RBES whose underlying logic is Boolean logic:

[1] *Maple* is a trademark of *Waterloo Maple Inc.*, Waterloo, ON, Canada.

```
> with(Groebner):
> with(Ore_algebra):
> SV:=x||(1..112);
> fu:=var->var^2-var:
> iI:=map(fu,[SV]);
> A:=poly_algebra(SV,characteristic=2):
> Orde:=MonomialOrder(A,'plex'(SV)):
> fu:=var->var^2-var:
> iI:=map(fu,[SV]):
> NEG:=(m::algebraic)->NormalForm(1+`m`,iI,Orde):
> `&AND`:=(m::algebraic,n::algebraic)->
>         NormalForm(expand(m*n),iI,Orde):
> `&OR`:=(m::algebraic,n::algebraic)->
>         NormalForm(expand(m+n+m*n),iI,Orde):
> `&IMP`:=(m::algebraic,n::algebraic)->
>         NormalForm(expand(1+m+m*n),iI,Orde):
> `&XOR`:=(m::algebraic,n::algebraic)->
>         (m &OR n) &AND NEG(m &AND n):
```

where the *Maple* functions `NEG` (prefix) and `&AND`, `&OR`, `&IMP` and `&XOR` (infix) are the algebraic translation of the logical connectives “negation”, “conjunction”, “disjunction”, “implies” and “exclusive disjunction”. Note that `iI` is the ideal of the squares of variables minus variables introduced in the algebraic model of RBES to force idempotency (denoted I above).

If we identify the “immediate prerequisite” relation with the logical implication, we can introduce the former in the following form (not all 261 rules are listed and an ellipsis is used for the sake of brevity):

```
> R1:=incognita &IMP exp_algebraica:
> R2:=incognita &IMP parte_literal_exp_algebraica:
  ...
> R261:=sistema_ecuaciones_grado_1_equivalente &IMP
       metodo_Gauss:
```

(the contents are in Spanish because they come from the study of the Spanish case Martínez Zarzuelo (2015), but most of them are very similar). They are stored in file `Edges_Algebra_GB.txt`.

We can use nicknames to shorten the names of the contents:

```
> incognita:=x1:
> exp_algebraica:=x2:
  ...
> ecuacion_explicita_recta:=x112:
```

(the nicknames are stored in file `Vertices_Nicknames.txt`).

So, after reading these two files from the *Maple* session:

```
> read(`c:/.../Vertices_Nicknames.txt`);
> read(`c:/.../Edges_Algebra_GB.txt`);
```

it is possible to define the ideal of rules, `iJ`:

```
> iJ:=[NEG(R1),NEG(R2),NEG(R3),...,NEG(R261)]:
```

We can now check if, for instance, content “exp_algebraica” (x_2) follows from content “suma_monomios” (x_6). `iK` denotes the ideal of what is stated as true:

```
> iK:=[NEG(x6)]:
> B:=Basis([op(iJ),op(iI),op(iK)],Orde):
> NormalForm(NEG(x2),B,Orde);
                              x2 + 1
```

The answer is not 0, so it does not follow. Once the Gröbner basis `B` is computed (what takes about 30 s on a standard laptop), each question takes very little (hundredths of a second).

It is now very easy, for instance, to exhaustively check all what follows from “suma_monomios” (x_6):

```
> W:=[]:
> for i from 1 to 112 do
           if NormalForm(NEG(x||i),B,Orde)=0
                then W:=[op(W),x||i]
           fi;
   od;
> W;
```

(the answer consists of 60 variables and is computed in slightly more than 3 s).

Reciprocally, “suma_monomios” (x_6) does follow from “exp_algebraica” (x_2):

```
> iK:=[NEG(x2)]:
> B:=Basis([op(iJ),op(iI),op(iK)], Orde):
> NormalForm(NEG(x6),B,Orde);
                                 0
```

(and, as done above for x_6, it can be easily checked that 109 contents follow from x_2).

Nevertheless, the RBES approach, although is the inspiration for the present article and does work, does not take advantage of its potential, as it is designed to deal with RBES complex rules (that are logic propositions involving negations, disjunctions and conjunctions, that do not arise in this particular case, where all rules are of the form $Yi \rightarrow Yj$). Let us try another approach.

3.2 Second Approach (Graph Theory)

The idea of this second approach is based on using graph theory to model the educational contents and ordered pairs of the binary relation using a graph structure. More precisely, using a directed graph (also called digraph) structure.

Maple offers an efficient package for dealing with graphs named *GraphTheory*. We can approach the same questions of the previous subsection from this other approach. Now we have to begin by loading the package and the data of the graph:

```
> restart;
> with(GraphTheory):
> read(`c:/.../Vertices_Nicknames.txt`);
> read(`c:/.../Edges_Algebra_Networks.txt`);
```

As in the previous subsection, the first file introduces the nicknames of the vertices and the second one introduces the directed edges of the digraph, now as a set of ordered pairs:

```
> LC:={[incognita,exp_algebraica],
       [incognita,parte_literal_exp_algebraica],
       [exp_algebraica,termino_exp_algebraica],
       ...
       [sistema_ecuaciones_grado_1_equivalente,
        metodo_Gauss]}:
```

Let us suppose that this is the "preprocessed official curriculum" (*C*) that is set as reference. It is straightforward to define the corresponding graph in *Maple* and to plot it:

```
> C:=Digraph([x||(1..112)],LC);
    C := Graph 2: a directed unweighted graph with
              112 vertices and 261 arc(s)
> DrawGraph(C);
```

The output of this last line of code can be found in Fig. 1 (there are too many vertices to display their names). Surprisingly, two contents are clearly isolated.

We can easily look for them constructing the set $\{x_1, x_2,\ldots,x_{112}\}$ and using the set difference operator `minus` and the command `indets` (that returns the variables in an expression):

```
> {x||(1..112)} minus indets(LC);
                          {x69, x70}
```

Variables x_{69} and x_{70} are "funcion_valor_absoluto" (absolute value function) and "función trigonométrica" (trigonometric function), respectively. The experts that developed the "preprocessed official curriculum" should be contacted to confirm that their isolation is correct.

There is a very convenient command in the *GraphTheory* package, `IsReachable`, that checks whether a vertex is reachable from another one or

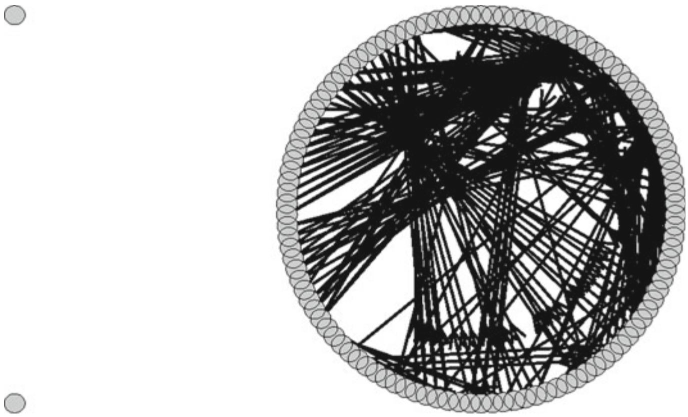

Fig. 1 Plot of the digraph "preprocessed official curriculum"

not. For instance, it can be used in this case to check whether x_2 is reachable from x_6 and vice versa:

```
> IsReachable(C,x6,x2);
                              false
> IsReachable(C,x2,x6);
                              true
```

These results are obtained in 0 s. This time is also obtained for computing which of the 112 contents follow from x_2:

```
> W:=[]:
> for i from 1 to 112 do
     if IsReachable(C,x2,x||i)
           then W:=[op(W),x||i]
     fi;
   od;
> W;
```

and `W` has, as computed in Sect. 3.2, 109 contents.

Therefore it is clear that this approach is much better in this case.

3.3 Case Study

Let us suppose that the list of vertices and the list of oriented edges (`LC`) used in Sect. 3.2 are those corresponding to the "preprocessed official curriculum" (the reference). The list of vertices is supposed to be known by the authors of educational resources, but the list of oriented edges (`LC`) is not.

Let us suppose that a textbook has been analyzed and the corresponding "official curriculum development" has been obtained and written in the same format as LC, and is denoted LG. And let us imagine that the author has reached almost the same graph as the set of experts but has forgotten to include the content "division_fraccion_algebraica" (x_{41}), and the corresponding ordered pairs. Let us proceed to see what happens when trying to verify this "official curriculum development".

Let us proceed as in Sect. 3.2, by also loading the file containing the set VO of vertices in the "official curriculum" (O) of Spanish Secondary Compulsory Education (ESO) and the file containing the set LG of edges corresponding to the "official curriculum development" proposal:

```
> restart;
> with(GraphTheory):
> read(`c:/.../Vertices_Nicknames.txt`);
> read(`c:/.../Edges_Algebra_Networks.txt`);
> read(`c:/.../Vertices_Algebra_RealDecretoESO.txt`);
> read(`c:/.../Edges_Algebra_Networks_Case.txt`);
```

We have to define two graphs, corresponding to the "preprocessed official curriculum" (C) and the "official curriculum development" (G):

```
> C:=Digraph([x||(1..112)],LC);
> G:=Digraph([x||(1..112)],LG);
```

`> DrawGraph(G);` and we can easily plot the latter (Fig. 2):

We can now carry out the process proposed:

- STEP 1a: Check whether all the contents in the proposed "official curriculum development" (G) appear in the "preprocessed official curriculum" (C). The set difference of the vertices of G and C should be the empty set:

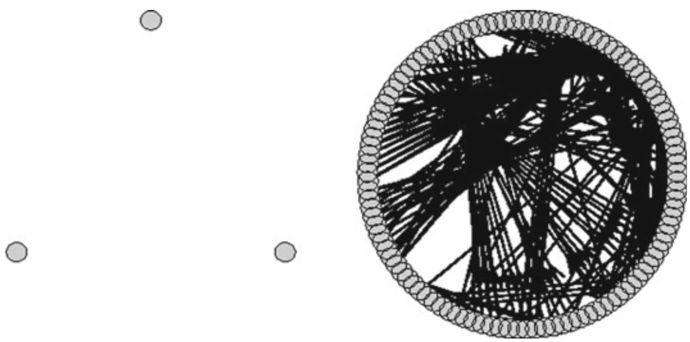

Fig. 2 Plot of the digraph "official curriculum development" proposal

```
> indets(LG) minus indets(LC);
                        {}
```

- STEP 1b: Check whether all the contents in the "official curriculum" (*O*) appear in the proposed "official curriculum development" (the set difference should be the empty set):

```
> VO minus indets(LG);
                       {x41}
```

In this case, the "official curriculum development" does not comply with the "official curriculum", as there is a content missing.

- STEP 2a: Do all the ordered pairs of the "immediate prerequisite" relation considered in the proposed "official curriculum development" (*G*) appear in the transitive closure of the "immediate prerequisite" relation of the "preprocessed official curriculum" (*C*)? Computing the transitive closure of the whole relation is computationally expensive, but to check whether certain directed edges are in the transitive closure of the relation is not (in fact `IsReachable` really checks that). The answer to the next lines of code (that stores in set `H` the edges that are not in the transitive closure of *C*) should be the empty set):

```
> H:={}:
> for i in LG do
       if not IsReachable(C,op(i))
             then H:={op(H),i}
       fi;
   od;
> H;
                        {}
```

- STEP 2b: Do all the ordered pairs of the "immediate prerequisite" relation considered in the proposed "official curriculum development" (*G*) appear as ordered pairs of the "immediate prerequisite" relation of the "preprocessed official curriculum" (*C*)? (the set difference should be the empty set):

```
> LG minus LC;
                        {}
```

- STEP 3: Are there cycles in the graph? There is a command in the *GraphTheory* that checks whether a graph is acyclic or not (so we expect a *true* answer):

```
> IsAcyclic(G);

                       true
```

We could even go further:

- STEP 1b (further): We could also check if all the contents in the "preprocessed official curriculum" (*C*) appear in the "official curriculum development" (*G*):

```
> indets(LC) minus indets(LG);
                       {x41}
```

In this case there are no content that the experts have detailed (added) to *O* (in *C*) that the author has not considered (in *G*). The author could be informed of this fact (if it arose), and the author could perhaps like to include it (although it would not be mandatory, as it complies with "official curriculum" (*O*)).

- STEP 2b (further): We could also check if all the ordered pairs in the "preprocessed official curriculum" (*G*) appear in the "official curriculum development" (*G*). In this case, at least all directed edges including x_{41} must arise:

```
> LC minus LG;
 {[x21, x41], [x24, x41], [x25, x41], [x41, x98]}
```

4 Conclusions

In RBES verification, what is checked is the logical coherence. For instance, in a medical RBES, a patient cannot be advised that he/she should take a drug that has a certain active ingredient, and, at the same time, that he/she should not take a drug that has that same active ingredient. What cannot be detected are errors or errata within the medical knowledge implemented. For instance, it cannot be detected in the verification process a (very serious) error such as "if the patient is dehydrated then do not give him any liquid". That is, we start from a supposedly correct knowledge and verify its logical coherence.

Similarly, in this case we suppose that a brief "official curriculum" (*O*) exists and that a team of experts has extended and detailed *O* in a "preprocessed official curriculum" (*C*), that will act as reference to check ulterior "official curriculum developments" (*G*). *C* is supposed to be correct. What can be checked is the correctness of *G* compared to *C*.

Developing the "preprocessed official curriculum" (*C*) from an "official curriculum" (*O*) is neither a short nor a trivial task. As said above, we had available the "preprocessed official curriculum" corresponding to the mathematics subjects of the Spanish Compulsory Secondary Education (Grades 7–10 in the K-12 system) developed by the second author for her Ph.D. Thesis (Martínez Zarzuelo, 2015). It is mainly based on different available "official curriculum developments" (textbooks) corresponding to that "official curriculum" and previous ones and required a long hard work.

The theoretical aspects of this proposal were discussed in Roanes-Lozano et al. (2020b). In this new article it has been proved that it is possible to implement the proposal. Obviously other implementations are possible (here, one closer to the inspiration of this work and other, much more efficient, have been detailed).

As in any computational field, an application is not design, developed and implemented if it is planned to be used just once. A proposal like this makes sense, for instance, in a country where textbooks (and perhaps project-based learning proposals) have to be checked and authorized. We believe that in such scenario our proposal is very convenient. It proposes a step forward, analogous to the one taken from manual RBES verification by a panel of experts, to computational verification of RBES.

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