Historical and Didactical Roots of Visual and Dynamic Mathematical Models: The Case of "Rearrangement Method" for Calculation of the Area of a Circle

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1 Introduction

It has been widely acknowledged that the concept of areas of plane shapes and rigorous methods of their calculation are complex and difficult to learn and to teach.[1] Among these methods, the calculation of the area of a circle is particularly challenging as it requires dealing with infinitesimal methods and an irrational number (π). At the same time, it is considered crucial for middle school curriculum being embedded into a rich context of real-life applications along with introduction of ideas and concepts which are supposed to be used in later schooling.[2] Not surprisingly, the number of resources produced to help students deal with the complexity of explanations and justifications of the formula for the area of a circle has been growing, both in the traditional form of printed textbooks and teacher's manuals[3] and in novel, digital format. The latter category includes a number of online resources with useful information (explanations and exercises, learning scenarios and suggestions) for teachers, some of which are published by official curriculum bodies (e.g., NCTM,[4]

[1]See, for instance, Kordaki and Potari (1998), Rejeki and Putri (2018), among others.

[2]Cavanagh (2008), O'Dell et al. (2016).

[3]See, for instance, Van de Walle et al. (2020), p. 530. For a preliminary discussion of some of them, see Freiman and Volkov (2006).

[4]NCTM (National Council of Teachers of Mathematics), see nctm.org.

[5]LearnAlberta.ca is a website providing digital learning and teaching resources correlated with the educational programs from kindergarten to Grade 12 approved by the Government of Alberta Province, Canada.

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Learn Alberta,[5] and NZ Math[6]). With the raise of digital tools and environments, several applications (applets) were created allowing dynamic investigation of the formula of the area of a circle by the learners.[7]

In the present paper, we shall argue that at some point in the history the educators started adopting (sometimes with considerable modifications) methods coming from pre-modern (ancient and medieval) mathematical treatises available to them via translations and interpretations provided by historians of mathematics. The reason for this adoption usually was not explicitly stated by the educators and deserves a further study; for the time being we assume that the educators did it in hope that the "archaic" (and, therefore, presumably "simple") methods could stimulate interest and facilitate understanding of the learners.[8] It is also possible that this conviction of the educators may have been based on a tacit assumption concerning the growth of the child's understanding of geometrical concepts as mirroring the stages of their historical genesis.

In the context of teaching geometry, this vision led the educators to a transition from approaches found in Euclidean *Elements* (ca. 300 BC) and based on quasi-formal deduction system[9] to more practical, visual, intuitive, and dynamic ones.[10] However, while some of these methods indeed allowed a relatively simple visualization, the others involved concepts and ideas particularly difficult for young learners. The infinitesimal methods of calculation of the areas of a circle and its parts, as well as the methods of calculation of the volumes of a pyramid and a sphere, arguably belonged to the latter category. In the case of calculation of the area of a circle, the search for an appropriate didactical apparatus suitable for this case led some educators to the use of so-called "method of rearrangement of sectors" or, more precisely, a group of approaches based on the division of circle into sectors followed by rearrangements of different kinds and investigation of possible approximations based upon them.

Nowadays, this *rearrangement method* (we will use this term further in the chapter for the sake of convenience) seems to be the most often used by the authors of textbooks who suggest it for teaching calculation of the area of a circle; it can be

[6] See https://nzmaths.co.nz/, the website containing information about mathematics education in New Zealand.

[7] See, for instance, https://www.mathed.page/constructions/pi/index.html for an applet designed by Henri Picciotto. With the help of this applet a circle can be divided into $2n$ sectors, $n = 6, \ldots, 180$, and these sectors can be rearranged to form a "rectangle-like" figure.

[8] See, for instance, Francis (1976), Ernest (1998), Marshall and Rich (2000), Farmaki and Paschos (2007), Furinghetti (2007), Jankvist (2009), Mac an Bhaird (2011) (listed in chronological order). The earliest publications of this kind we are aware of were authored by D.E. Smith (1860–1944); yet, he was not the only one who claimed that history of mathematics can be used in classroom; see, for instance, Wiltshire (1930).

[9] For a recent study of the logical structure of Euclid's *Elements,* see Acerbi (2008).

[10] On this process see, for example, Montoito and Garnica (2015), Herbst et al. (2018).

43. JUSTIFYING THEOREM 11.9 You can follow the steps below to justify the formula for the area of a circle with radius *r*.

Divide a circle into 16 congruent sectors. Cut out the sectors.

Rearrange the 16 sectors to form a shape resembling a parallelogram.

a. Write expressions in terms of *r* for the approximate height and base of the parallelogram. Then write an expression for its area.

b. *Explain* how your answers to part (a) justify Theorem 11.9.

Fig. 1 A "justification" of the formula of the area of a circle. Larson et al. (2007), p. 755

found in almost all modern textbooks for middle school mathematics and in numerous resources available online.[11]

In particular, the *rearrangement method* explores the idea of dividing the circle into a number of equal sectors, then rearranging them in a way that makes it look like a familiar figure, namely, a polygon (e.g., parallelogram) for which the area can be easily found (see Fig. 1).

When facing this seemingly attractive, "visual," and "dynamic" justification of the formula, the learners are expected to follow the implied reasoning procedure and are supposed to realize that the area of the rearranged shape is "resembling a parallelogram," and, consequently, the area of the original shape (i.e., the circle) is equal to the product of the "height" and "base" of this "parallelogram." Further, the learners are guided toward expressing the area of a parallelogram using the radius, which is supposed to be the "approximate height" of the parallelogram, and the half of the circumference, identified as the "approximate base," and thus would obtain a "justification" (apparently, problematic[12]) of the area of the circle.

Why did the authors of this and other textbooks choose this particular approach? While seemingly appealing to students' intuition, it does contain hidden assumptions

[11] Freiman and Volkov (2006). The printed textbooks and teaching materials available online that feature this method are too numerous to be listed here; for the most recent ones see, for example, Johnson and Mowry (2016), pp. 574–575 and the website https://www.colorado.edu/csl/2017/03/23/slices-pi containing an animation of this kind. For an analysis of the attempts to use digital technology to estimate the value of π, see de Silva et al. (2021).

[12] There are two elements that make this kind of "justification" problematic: first, the left part of the picture in Fig. 1 shows that the sectors do not seem congruent while in the right part they are obviously congruent; second, the term "approximate" used in connection with the terms "base" and "height" of the "shape resembling a parallelogram" does not help the learner understand why the area of the circle is equal *exactly* to the product of the radius and circumference. Much of the work dealing with these subtleties is thus left to the teacher.

that are far from being obvious.[13] Stacey and Vincent (2009), for example, discuss this method among other methods that can be used for introducing the area of a circle (such as a "square grid" method or an "onion layers" model) to young learners; they stress that when using such methods the educators should be aware of "a difference in the mathematical quality of the arguments between the empirical *counting squares* method and the approaches that use deductive reasoning" (ibid., p. 9). The authors stress that these methods include "very complicated limiting processes involved in formally proving the formula," and the instructors need to take these complex elements into account when presenting students with intuitive models that might help shaping their conceptual understanding (ibid., p. 8).

In line with Tamborg (2017) who refers to classical work of Vérillon and Rabardel (1995), we look into the process of growth of conceptual understanding through a view of human cognition and knowledge building as artifact-mediated activities in which "artifacts (as opposed to natural objects) possess cultural and historical dimensions because they are constructed with a particular purpose and a particular way of fulfilling this purpose in mind" (op. cit., p. 3). These "cultural and historical dimensions" are not always made explicit, in our opinion, especially when the artifacts undergo transformations in the process of transition from traditional ("usual") "mathematical working spaces" (hereafter MWS) to modern ("new") MWS (e. g., in particular, those supported by digital technology).[14]

In our view, it appears promising to adopt a *genetic* paradigm that relates *developmental process* of mathematical work and the *manner* by which it was created and took shape. This paradigm dialectically unites mathematical, didactical, and historical aspects of an artifact which further could be transformed through, for example, an "instrumental genesis" (that is, an instrumentalization and instrumentation, as result of an interplay between a subject and an artifact, as a part of "didactical situation"; see, for example, Brousseau (1998) and Trouche (2005)). In this complex and transformational (eco)system of knowledge development, we consider the *rearrangement method* as an *artifact* whose history goes back to the Antiquity and whose genesis (or even "geneses") has/have not yet been made sufficiently explicit. While the ancient history of the *method* seems to be lost in time and space, as we will show in this chapter, it was later (approximately in the mid-second millennium CE) refined by mathematicians to help them solve some particular problems (related to the calculation of the area of circle), or to conduct investigations of other infinitesimal processes before being taken over by the nineteenth–early twentieth-century educators as "visual support" for learners. At that time, the *rearrangement method* was introduced propaedeutically, as a (novel) didactical approach in introductory geometry courses; in the case of the area of a circle, it was featured as an alternative to the relatively technical approaches of Euclid and Archimedes. Apparently, the

[13] This "justification" is apparently based on the unstated assumption that the length of any curved line getting closer to a given straight line tends to the length of this straight line, which is mathematically incorrect.

[14] On the concept of MWS see, for instance, Kuzniak and Richard (2014) and Richard et al. (2019).

educational reforms of the late 1990s–early 2000s (which coincided with the introduction of electronic calculators and other digital tools) placed it in the foreground of modern didactical innovations, yet without a clear explanation of the history of its genesis and possible pitfalls, as well as the challenges it might bear for teachers and students.

Therefore, we begin our chapter with a brief history of the origins of the *rearrangement method* and infinitesimal discourses that accompanied some of its versions. We will then investigate some didactical ideas brought by nineteenth–early twentieth-century educators into their textbooks. We will finally explore how these works shaped modern development of innovative spaces of reflection and practice in mathematics education. At the end of the chapter, we will outline epistemological and didactical issues that modern educators might need to be aware of when using the *method* and will also discuss how a further and deeper investigation of both historical and instrumental geneses could help dealing with these issues, both in theory and in a classroom practice.

2 Historical Roots of the *Rearrangement Method*

The origin of the *rearrangement method* is unknown. In his book on the history of π originally published in 1971,[15] Petr Beckmann (1924–1993) claimed that *rearrangement method* was used by some unidentified "ancient peoples":

> The ancient peoples had rules for calculating the area of a circle. […] They probably did it by a method of rearrangement. They calculated the area of a rectangle as length times width. To calculate the area of a parallelogram, they could construct a rectangle of equal area by rearrangement as in the figure below, and thus they found that the area of a parallelogram is given by base times height (1976, pp. 17–18),

and provided the diagrams shown in Fig. 2.

The word "probably" used by Beckmann in the above-sited quotation reveals that the author most probably *provided here his own reconstruction* that, he apparently believed, was not found in any particular extant historical documents produced by the unidentified "ancient peoples" (who, he claimed, devised this method "almost five millennia before the integral calculus was invented"). (Beckmann, 1976, p. 17).

[15] Even though the author himself admitted that he was "neither an historian nor a mathematician" (op. cit., p. 3), his book was praised by some of its reviewers (see, for instance, Brieske's 1977 review of its edition of 1976 and even more recent review of its edition of 1989 by Blank (2001), p. 155). However, mathematicians and historians of mathematics provided a detailed critique of this book accompanied with a list of numerous misprints and errors (some of them indeed quite surprising, such as, for example, the claim that $p! + 1$ which is not divisible by 2, 3,…, p, "is therefore a prime") that can be found in Gould 1974 who (quite justly, in our opinion) concluded that "[t]he book should be rewritten entirely and the manuscript should be examined by qualified mathematicians and historians before being committed to the printed page" (p. 327). Unfortunately, this work has never been done.

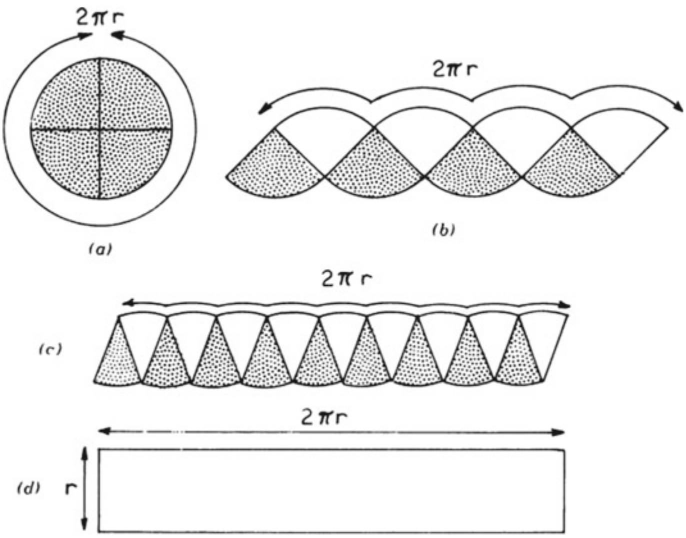

Fig. 2 Diagrams from p. 18 of Beckmann (1976) identical with the diagrams in the first edition of his book (1971). The caption reads: "Determination of the area of a circle by rearrangement. The areas of the figures (**b**), (**c**), (**d**) equal exactly double the area of circle (**a**)." The origin of the diagrams is not specified

It thus remains unknown from where Beckmann borrowed this diagram, apparently radically different from the one used by Archimedes of Syracuse (287–212 BC), to prove his statements concerning the area of a circle.[16] Some authors, when putting this method side by side with the one used by Archimedes, remind the reader that this method should be considered *less rigorous* and needs to be handled with care, for example, Bryan and Sangwin (2008, p. 145) write:

> We shall approximate the area of each sector by such a triangle. You would be well advised to retain a suspicious attitude towards the cavalier way we forge ahead with approximations such as these.

Similarly, (Casselman, 2012) states

> Roughly speaking, the reason the Theorem is true is that we can carve up both the circle and the triangle into very small regions that approximate each other closely in area.

A number of books, articles, and online resources featuring rearrangement diagrams (along with some descriptive texts) rarely contain the necessary references to the original works in which this method was introduced. Some of them mention David E. Smith's and Mikami Yoshio's 三上義夫 (1875–1950) *A History of Japanese Mathematics* (1914) where a similar diagram presumably dated of 1698 and credited to the authorship of Satō Moshun's 佐藤茂春 (whose name can also be spelled Satō Shigeharu; fl. late seventeenth century) can be found on p. 131, as

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[16] Archimedes (1544), pp. 44–46, 1615, pp. 128–133, 1676, pp. 81–97, 1880, pp. 257–271.

well as more recent books on the history of mathematics, in particular, the monographs authored by Carl B. Boyer (1906–1976) (Boyer, 1959) and Margaret Baron (1915–1996) (Baron, 1969). In her book, M. Baron refers (p. 110) to the approach used by J. Kepler (1571–1630) whereas C. Boyer, while also pointing at Kepler's work, mentions the works of Nicholas of Cusa (1401–1464) as the possible origin of Kepler's approach (Boyer, 1959, p. 108). In turn, Beckmann (1976, p. 19) conjectures that Leonardo da Vinci (1452–1519) may have used the same method; he, however, does not provide any references to Leonardo's works.

In his summary of Nicholas of Cusa's works on mathematics, Hofmann (1971) lists several treatises of Cusanus related to the squaring of circle.[17] The earliest statements of Nicholas concerning a circle and an inscribed polygon are found in his *De Docta Ignorantia* (On learned ignorance) completed in 1440.[18] Also, in Chap. 3 of Book 1 of the treatise titled *Quod praecisa veritas sit incomprehensibilis* (The precise truth is incomprehensible) Cusanus uses the relationship between these two figures as a metaphor of the process of approaching the absolute truth. It is interesting that Cusanus makes a mathematically correct statement: at any step the area of inscribed polygon is less than the area of the circle, and two of them cannot be equal to each other unless the polygon becomes identical with the circle (he, apparently, understands that this cannot happen at any finite step of the procedure). Later Cusanus returns to the topic in his other works including *De circuli quadratura, pars theologica* (On Squaring the Circle, Theological Part), completed in 1450, *Quadratura circuli* (Squaring the Circle), 1450; *De sinibus et chordis ou Dialogus de circuli quadratura* (On sines and chords, or Dialog on the Quadrature of the Circle), 1457, and *De caesarea circuli quadratura* (On Squaring the Caesarian Circle), 1457.[19]

Pierre Duhem (1861–1916) in his study of Leonardo (1909, vol. 2) devotes a short section titled "L'infiniment grand et l'infiniment petit dans les notes de Léonard de Vinci" (pp. 49–53) to Leonardo's concepts of infinitely small and infinitely large entities. Duhem quotes Leonardo's claim that "La Géométrie est infinie parce que toute quantité continue est divisible à l'infini dans l'un et l'autre sens" (p. 50);[20] he borrows this quote from Manuscript M of the Bibliothèque de l'Institut[21] and provides his own reconstruction of Leonardo's drawing that is supposed to illustrate

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[17] For more details, see Watanabe (2011), pp. xx–xxvi.

[18] Watanabe (2011), p. xxi.

[19] For details, see Watanabe (2011), pp. xxiv–xxv. On Cusanus' interest in mathematics and evaluation of his mathematical writings by his contemporaries, see the section "[Nicolaus von Cusa:] Die mathematischen Schriften" in Scharpff (1871), pp. 294–323; for more recent works, see Boyer (1959) (esp. see p. 91), Counet (2005) (esp. see pp. 286–289) and Vengeon 2006. Albertson (2014) on p. 172 cites a (negative) evaluation of Cusanus' geometry by Regiomontanus (1436–1476); see also idem, p. 255, n. 15 for relevant references. German translation of Cusanus' mathematical writings is available in Nikolaus von Kues (1980), while their French translation is found in Nicolas de Cues (2007).

[20] That is, "Geometry is infinite, because every continuous quantity is infinitely divisible from one direction to another" (da Vinci, 1964, p. 127). The original statement of Leonardo is found on the same page of da Vinci (1890).

[21] Da Vinci (1890), vol. 5, Ms M, fol. 18r.

this claim. Surprisingly, Duhem does not mention here Leonardo's fragment and diagrams directly related to the calculation of the area of a sector found in Manuscript E.[22] Leonardo gives his verbal formula for the area of a circle on the first page (verso) of this manuscript; Félix Ravaisson-Mollien (1813–1900) provides his French translation on the same page (1888, ibid), in our English translation it reads: "The circle is equal to a quadrilateral made of a half of the diameter of this circle, multiplied by a half of the circumference of the same circle."[23] Leonardo offers a description of his procedure accompanied by several pictures; its translation can be found in the mentioned works of C. Ravaisson-Mollien and V. Zubov.

In the context of infinitesimal (pre-calculus) debates of the sixteenth–seventeenth-century, Johannes Kepler's discussion of the circumference and area of a circle is found in his treatise *Nova Stereometria Doliorum Vinariorum* (New Stereometry of Wine Barrels) published in 1615; it makes important setback from Archimedes proof putting upfront practical utility against rigor. It opens with "Theorem 1" stating that the ratio Circumference: Diameter is close (but not equal!) to 22:7. Kepler provides his own proof using only inscribed and circumscribed regular hexagons and mentions that Archimedes found this ratio using polygons with 12, 24, and 48 sides. In his Theorem 2, Kepler states that the ratio of the area of a circle and the square built on its diameter is approximately equal to 11:14. It is surprising to find this outdated approximation in Kepler's work right after his mention of the much more precise approximation (3.141592653589793) attributed by him to Adriaan van Roomen (1561–1615), also known as Adrianus Romanus.[24] One can conjecture that for Kepler the main reason was not obtaining a good approximation of π; instead, he focused on the *method* which, as he stressed, *differed* from the method of Archimedes.[25]

3 The *Rearrangement Method* in Nineteenth–Early Twentieth-Century Western Textbooks

While logico-deductive approach to geometry based on Euclid's work was having significant influence on teaching geometry up to the modern times, alternative approaches were also making their way being grounded in its practical origins as "the art of measuring well" (Ramus, 1569, cited in Menghini, 2015). Hence, the activity of measuring, as argued by Marta Menghini (ibid.), was "preceded by a very interesting and original part of observation, a 'play' with figures" in which proofs were supported by geometric constructions and measurement activities by drawing. Later on, by the end of the eighteenth century, according to Menghini (2015, p. 568), one can observe in Western Europe an apparent shift from teaching practical geometry to teaching geometry "practically" (e.g., using problem-based approach). To

[22] Da Vinci (1888), vol. 3, Ms E, folios 24r-26v.

[23] For a Russian translation, see Zubov (1935), vol. 1, pp. 68–69, Zubov (1955), p. 73.

[24] Romanus (1593).

[25] For a Russian translation of these two theorems, see Kepler (1935), pp. 111–117.

support her claim, Menghini takes as example a textbook authored by Alexis Claude Clairaut (1713–1765):

> In 1741, again in France, Alexis Clairaut wrote his *Éléments* (sic.- V.F. & A.V.) *de Géométrie*. His first chapter is about the *measurement of fields*; nevertheless, Clairaut was *not* interested in teaching a practical geometry. With Clairaut we see a shift from measurement as a goal to measurement as *a means* to teach geometry via problems. This is seen by the fact that the part about measurement *doesn't contain numbers*; there is only a hint at the necessity for a comparison with a known measure (See Clairaut, 1743).
>
> The aim of Clairaut is to solve a problem 'constructing' the elements he wants to measure. The focus is on the *process* of constructing and in a *narrative* method.[26]

In their turn, Richard et al. (2019) argue that Clairaut, when building his proofs on inductive argument, "tries to make dynamic a figure by reasoning" in such a way that "the model reader will be able to visualize the animation, while in his demonstration, a classical figure like those in Euclid's *Elements* is proposed" (pp. 141–142).

The *rearrangement method* was not the only method used by mathematicians of the past that was taken over by mathematics educators of modern times. For instance, the so-called "onion" model was used by Rabbi Abraham bar Hiyya ha-Nasi (also known as Savasorda), a Jewish philosopher, mathematician, and astronomer active in the late eleventh and early twelfth centuries,[27] to justify the formula of the area of a circle (Figs. 3 and 4). This method was later used by Emma Castelnuovo (1913–2014) in her textbook on intuitive geometry (1966) as a part of a hands-on activity for elementary students as demonstration of the formula for the area of the circle (Fig. 5).

While the possible historical and didactical sources of the *rearrangement method* need to be investigated in more detail,[28] our analysis of geometry textbooks published in the nineteenth and early twentieth centuries shows that this model was already present in some of them thus becoming a *didactical artefact*; these archaic methods were introduced, we claim, because it was assumed that they may have helped articulate certain didactical ideas of modern "innovative" approaches to teaching geometry. The educators searching for alternatives to "Euclidean" methods of teaching geometry (that is, to the methods based on quasi-formal logic used in the *Elements*) in the second part of the nineteenth century often adopted an approach based on *Anschauung* (intuition) in preliminary courses on observational geometry and

[26] Menghini 2015, p. 568. Italics as in the original.-V.F.&A.V. On Clairaut and his approach, see also Schubring 2011, p. 81. In this paper, the author refers to Schubring 1987, 2003 (pp, 54–58) and Glaeser 1983. For a brief description of Clairaut's *Élémen(t)s de Géométrie* (1741) and its influence on European textbooks of the nineteenth and twentieth centuries, see Barbin and Menghini (2014), p. 481. Menghini (2015) provides a more detailed description of Clairaut's textbook on pages 568–571.

[27] The dates of birth and death of Abraham bar Hiyya are unknown. Some authors claim that he died in 1136, while other scholars consider this date not sufficiently justified; for more details, see Levey (1954), p. 50, n. 1. Levey himself suggests "fl. […] before 1136" (1981, p. 22).

[28] The results of our investigation will be published elsewhere.

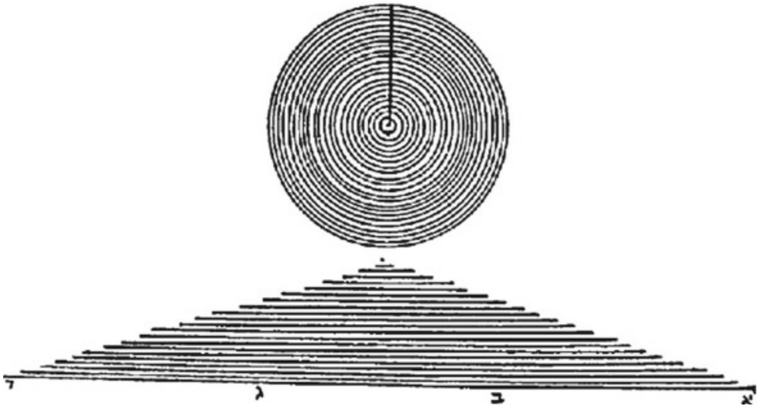


Fig. 3 The "onion model" used by Abraham bar Hiyya (Savasorda) (1116). The diagram reproduces Fig. 73 from Guttmann's (1903) edition of Savasorda's book

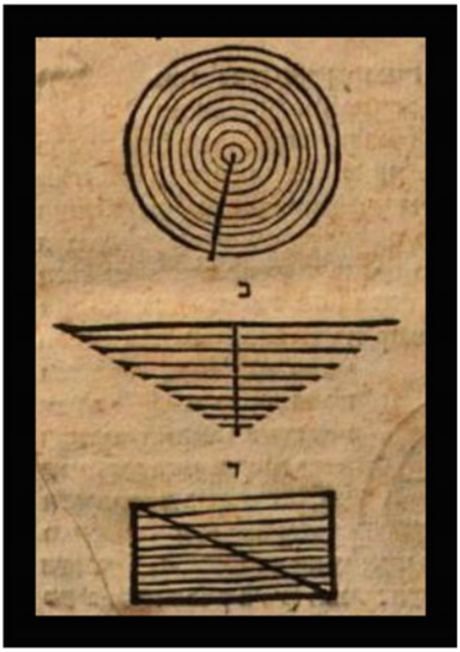


Fig. 4 The diagram from an earlier edition of Savasorda's treatise (Abraham, 1720). The authors thank Josep Maria Fortuny Aymemí for his suggestion to refer to Savasorda's work

drawing; this approach was based on the ideas of the pedagogical reformers J.H. Pestalozzi (1746–1827) and J.F. Herbart (1776–1841).[29]

Indeed, several geometry textbooks published during the time period from the early nineteenth century to the mid-twentieth century by the authors from several

[29] For more details, see Henrichi and Treutlein (1897); Castelnuovo (1948).

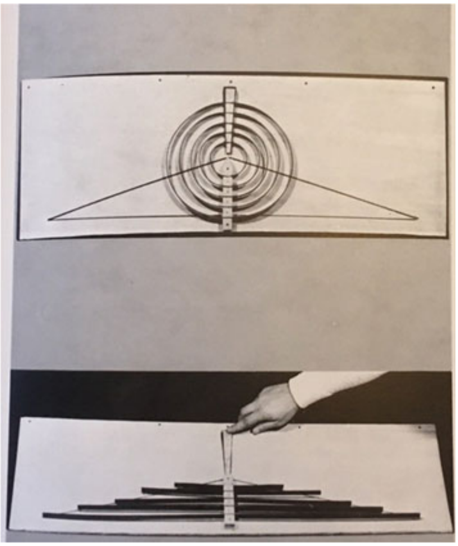


Fig. 5 A hands-on activity for elementary students learning the formula of the area of a circle (Castelnuovo, 1966, p. 93)

Western countries, namely, England, Austrian-Hungarian Empire, Germany, United States, and Russia, included various versions of the *rearrangement method*. In particular, its popularity might be attributed to the works of Dionysius Lardner (1793–1859), Peter Joseph Treutlein (1845–1912),[30] and Emma Castelnuovo,[31] among others.

We begin our analysis with Lardner's *The First Principles of Arithmetic and Geometry; Explained in a Series of Familiar Dialogues* (1835) to pursue with other textbooks that used the *method*, the latest of them was Zaitseva's teacher's manual (1952); none of these textbooks provides a reference to earlier sources (prior to the nineteenth century). The authors of these textbooks dealing with elementary geometry stressed various didactical aspects that were in the heart of innovative approaches at their time. In particular, the following elements were stated explicitly as the goals: the popularization of scientific (especially, mathematical) knowledge (clearly expressed by Lardner, 1835), development of the learners' intuition (Castelnuovo, 1948; Henrici & Treutlein, 1897), emphasizing "practical approach" (Stern & Topham, 1913),[32] and "hands-on experiments" (Willis, 1922). For instance, Treutlein's (1911) program contains three main principles which arguably influenced modern teaching: (1) to consider intuitive geometry as preliminary step to more formal, deductive instruction;

[30] On the reform of school mathematics education designed by P. Treutlein and the textbooks he co-authored with J.Henrici, see Becker (1994) and Weiss (2019).

[31] On Emma Castelnuovo's work, see Furinghetti and Menghini (2014); esp. see pp. 2–3, see also Castelnuovo (1977), pp. 42–44 on her interest in A.C. Clairaut's (1713–1765) didactical ideas.

[32] When dealing with the area of a circle (pp. 70–71), Stern and Topham describe the "matching sectors" method they borrowed from Earl (1894), p. 89–90 (see their footnote on p. 70). In turn, Earl does not provide any reference to his sources. It should be stressed that Earl's textbook was devoted to "lessons in *physical* measurement" (italics ours.- V.F. and A.V.).

(2) to conduct an implementation of modern ideas of projective and transformation geometry; and (3) to focus on spatial imagination and the fusion of spatial and plain geometry (Weiss, 2019, p. 112). The *method* (of rearrangement of the sectors) seems to align with this didactical vision.

The two books by Lardner that contain the method are *The First Principles of Arithmetic and Geometry; Explained in a Series of Familiar Dialogues* (1835) and *Lardner's Cabinet Cyclopedia* (1840). According to Peckham (1951), the audience of *Cyclopedia* were "those who seek that portion of information respecting technical and professional subjects which is generally expected from well-educated persons." The front page of the *First Principles…* ("Adapted for Preparatory Schools and Domestic Instruction; with Copious Examples and Illustrations") states that it was written as a mathematics textbook for the beginners. Lardner claimed that he considered geometry as part of public instruction having two objectives:

> First, it may be regarded as an exercise by which the faculty of thinking and reasoning may be strengthened and sharpened. It is peculiarly fitted for this purpose by the accuracy and clearness of which its investigations are susceptible, and the very high certitude which attends its conclusions. Secondly, it is the immediate and only instrument by which almost the whole range of physical investigation can be conducted; without it we could not advance a step beyond the surface of the earth in our knowledge of the universe; without it we could obtain no knowledge of the figure or dimensions of the earth itself, nor of the mutual mechanical operation, or influence of bodies upon it. (Lardner, 1840, pp. 2–3)

After having stated these two goals, Lardner complained that

> In the course of instruction followed by the great mass of students in our universities, geometry has been regarded almost exclusively as a system of intellectual gymnastics; while, on the other hand, owing to the very stinted portion of instruction attainable by those who are engaged in the useful arts, the science is with them almost degraded to a mass of rules, without reasons, and dicta, the truth of which is expected to be received on the authority of the writer, and of which the reader is not put in a condition to judge. (Lardner, 1840, p. 3)

Therefore, in his textbook, he uses a form of a (Socratic) dialogue between a student (Henry) and the teacher ("Mr. L.," with the capital "L" apparently, referring to Lardner himself). Let us consider one of such dialogues devoted to the area of a circle. It begins with the student's questioning:

> Henry (Student): […] How shall I compute the number of inches enclosed within the circle when I know the length of its diameter?[33]

Mr. L. presents then a picture of a circle divided into 16 equal sectors while mentioning that by further dividing the circle into many even smaller sectors the arches will be made "exceedingly" short so that their curvature would not perceptible. The student says that in this case, the sectors would look as "narrow triangles." The conversation arrives then at the point at which the teacher mentions that the space enclosed within the circle would be the same as within a rectangle formed by the circumference (length) and the radius (height). Further, the teacher brings even a more "evident" way of rearrangement cutting the two identical circles into sectors-triangles and then putting them one against the other (Fig. 6).[34]

[33] Lardner, 1835, p. 112.

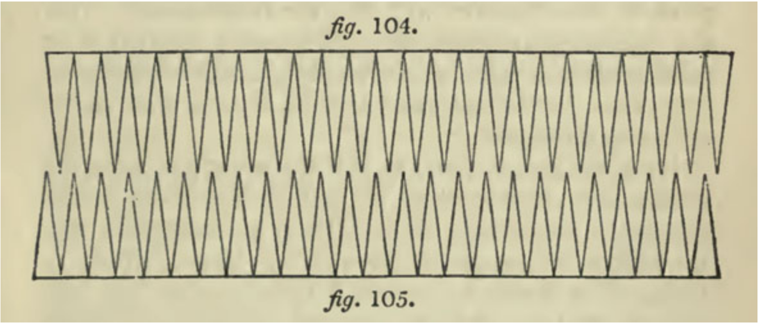


Fig. 6 Figures 104 and 105 from Lardner (1840), p. 101 originally published as Figs. 81–82 in Lardner (1835), p. 114. The author states that the formula for the area of a circle can be "made still more evident" with the help of this picture (1835, p. 113; 1840, pp. 100–101)

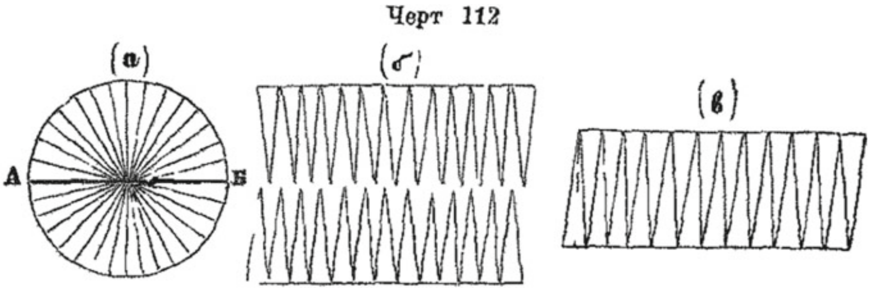


Fig. 7 Diagram 112 from Boryshkevich (1893), p. 68

A Russian textbook by Boryshkevich (?–1906) published in (1893), expressed a similar idea of using "saw-teeth," but this time with only one circle divided into 32 equal pieces; then from two half-circles, two "saws" with 16 teeth each were formed to be rearranged into a "parallelogram," see Fig. 7.[35]

The use of "rearrangement method" for the area of a circle was certainly a part of the larger didactical agenda focusing on "visualization" of geometrical statements. For instance, Max (Maximilian) Simon's (1844–1918) book (1889) provided some suggestions for teachers who intended to teach introductory geometry "intuitively." In the case of the area of the circle, he explains that one could imagine a circle divided into many triangles with a common vertex in the center of the circle and the bases on

[34] The regular polygon inscribed in the circle that Larnder used is rather unusual: it has 25 sides. It thus differs from the polygons of all the other authors whose works we inspected. It is possible that the division of the central angle into 25 equal parts was somehow related to the reform of measure units that took place in France after the French Revolution of 1789; according to the new system, the measure of the circumference equaled to 400°.

[35] The diagram reproduced in Fig. 7 is not clear: while the circle is divided into 32 sectors, the second (central) figure shows only $13 + 13 = 26$ triangles, while the right figure (composed of the upper and lower parts of the central diagram) contains only 24 triangles.

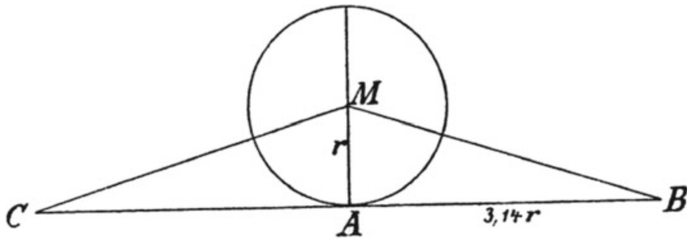


Fig. 8 Simon (1889), p. 19, Fig. 31

the circumference, so all of them, when combined, would form one big triangle with the base equal to the circumference and the height equal to the radius; see Fig. 8. So, its area should be equal to area of the circle.

Interestingly, a similar description of the method can be found in the abovementioned modern book by Bryan and Sangwin (2008), although in a slightly different manner: the authors subdivide the procedure into three "steps": (1) the circle is divided into 18 sectors; (2) the circle unfolds into a "teeth row" (making not very obvious the curvilinear shape of the "base" of each "tooth"); and then (3) suggesting a transformation of the "teeth row" into a set of triangles having equal bases and the same height (their vertices are joined together thus forming a big isosceles triangle, that is, a rectilinear figure). A warning accompanies this transition from a curvilinear to a rectilinear shape; it says that the sectors are arranged "along a straight line" whose length can be (when the number of sectors is large) "approximated very well by the circumference of the circle" (p. 145) and the height is "practically" equal to the radius of the circle, and, finally, that the "displacement of the apex of each triangle does not alter the area" (ibid.). The reader is thus supposed to be convinced that the circle was transformed into the large triangle, so the area of the circle equals to the area of the triangle.

Following the same line of thought, P. Treutlein's *Intuitive Geometry* (1911) introduces the process of calculation of the area of a circle by means of an even more sophisticated visualization passing through two consecutive transformations, one based on the division of the circle into six pieces and on the rearrangement ("gluing") of three of them on top of three others. Then a second circle, identical with the first one, would be divided into 12 pieces which would be similarly rearranged into two groups of six pieces completing each other. The student then is asked to make an observation concerning the differences between the surfaces of both shapes, and, moreover, is asked what would happen if the number of pieces grows larger. So, gradually the student would eventually come to the conclusion that the rearranged shape would become a parallelogram with the base equal to circumference, and the height, to the radius; see Fig. 9. All this serves to derive the formula for the area of the circle using the student's "imagination" (Treutlein, 1911; for a discussion, see Fujita et al., 2004).

In England, a book on practical mathematics by Stern and Topham (1913) focused on graphical and experimental processes in mathematics; the projected readership

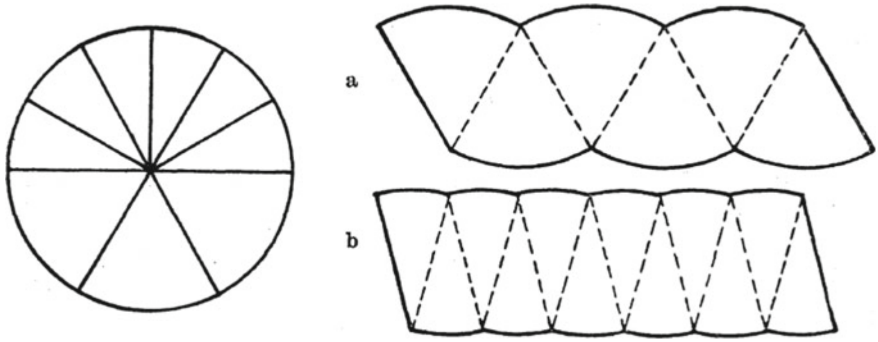


Fig. 9 Treutlein (1911), p. 186, Fig. 86[36]

was the militaries, but the book would also have been useful for more general audience. The authors called their method “approximate” (p. 70) while making reference to an earlier work by Earl (1894) who also suggested dividing a circle into a large number of equal sectors so that the obtained pieces, when put together, would form a figure whose area could be found approximately by treating it as a rectangle (Fig. 10).

As general introduction to his approach, Earl claimed that

> Public Schools should not be devoted simply to instilling into boys a certain amount of technical knowledge, but should rather train them to observe accurately, to reason rightly, and to front nature with an open and inquiring mind (Earl, 1894, p. v).

In the US, also in attempt to help teaching practical mathematics, C.I. Palmer (1871–1931) suggested in his *Practical mathematics for home study* (1919) a similar approach to approximation of the area of the circle using its division into 16 pieces. The illustration in his book reproduced below (Fig. 11) presents, however, a half of a circle divided into 8 pieces.

The ideas of laboratory methods of studying geometry, popular in the first half of the twentieth century, brought C.A. Willis (1869–?) to design an experiment-based lesson to help students “discovering” this formula using a model that suggested manipulations with sectors (1922, pp. 264–265); see Fig. 12. In his textbook, the formulas for areas of plane figures were supposed to be obtained as results of “experiments,” and the entire chapter 16 titled “Measurement of areas. Principles determined experimentally” was devoted to manipulations with paper models of geometrical figures.

The calculation of the area of a circle is preceded by a definition of π and a theorem stating that “The value of π can be calculated with as close an approximation as desired” (p. 249). His discussion of the matter reads as follows:

---

[36] Note that in this diagram the lower part of the circle shows its division into six equal sectors, while the upper part shows its further division into 12 sectors. Meanwhile, the results of the re-composition of the sectors shown on the right side are inversed: the upper figure shows the result of recombination of the six sectors, while the lower figure features 12 sectors.

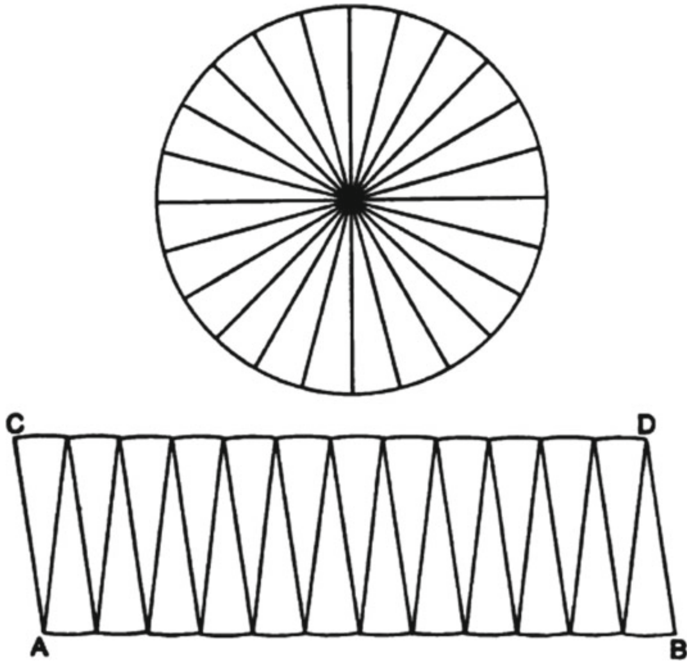


Fig. 10 Earl (1894), p. 89, Fig. 44

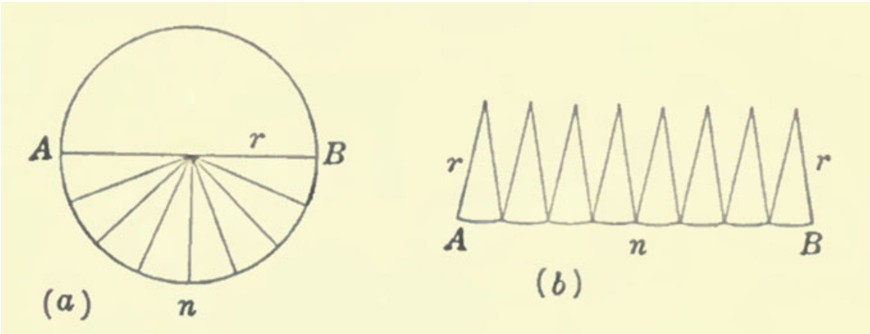


Fig. 11 Palmer (1919), p. 152, Fig. 90

356. Experiment V. The area of a circle.

Draw a circle about 4 inches in diameter and cut it out. Cut it on a diameter AB. Cut each semicircle into the same number of equal sectors; say into 8 sectors. Place the sectors as shown in Fig. 2. How may the resulting figure be made to approximate more and more closely to a rectangle as in Fig. 3? What are the base and altitude of this rectangle? What is therefore the area of a circle? (Willis, 1922, p. 264)

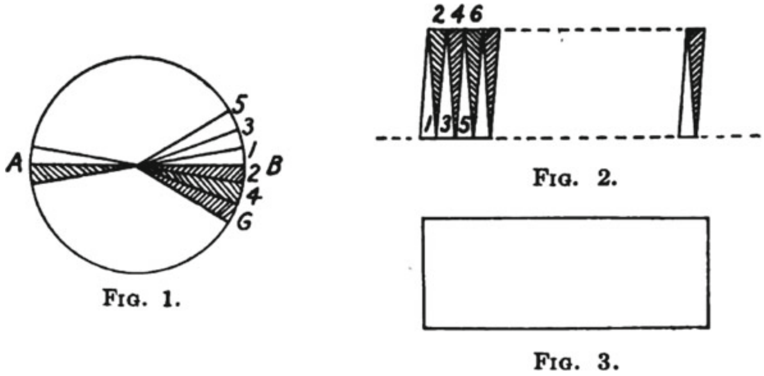


Fig. 12 Figures 1–3 from Willis (1922), p. 264[37]

The diagrams provided by Willis are quite remarkable. Firstly, they do not show the presumed division of the circle into 16 sectors, but into a larger number of sectors only 8 of which are shown. Our exploration of the diagram suggests that the diagrams shown in Fig. 12 represent a division of the circle into 32 sectors; this number is not specified in the text. Secondly, and this is even more interesting, the figure obtained as combination of sectors into which the circle is subdivided ("Fig. 2" of the diagram) is designed according to some particular strategy: the sectors from the lower half of the circle are coupled with the symmetric sectors from the upper half, for example, sector 1 is coupled with sector 2, sector 3 with sector 4, and so on. Apparently, the author of this diagram suggested that each sector from the upper half of part 1 is to be paired up with the symmetric sector from the lower half, and this operation should go from right to left to produce the couples placed in part 2 from left to right. Thirdly, the object shown in "Fig. 2" of the diagram has straight lines as its upper and lower sides, which is actually an approximation, since it is obtained as collection of sectors. So, using these diagrams would require some additional mathematical and didactical work from the teacher to avoid possible misconceptions.

Several Russian and Soviet authors used the method of rearrangement in the textbooks over the first half of the twentieth century often referring to Western sources (e.g., Treutlein's work which was translated into Russian). Some ideas of the above-mentioned experimental approach could be found, for instance, in Zaitseva (1952). On the one hand, the author used a diagram (Fig. 13) visually close to that used by Earl (1894) (however, the number of sectors, 16, was different). On the other hand, she suggested that students, after dividing a circle into 16 sectors, should have cut them out and superpose one half of them over the other half to form a figure resembling a parallelogram.

While visually convincing and eventually accessible to relatively young learners (the textbook was supposed to be used in grade 5 of Soviet school, that is, by students

[37] The lettering in Fig. 1 of the diagram contains a misprint: instead of number "6" the capital letter "G" is written to mark a sector, while Fig. 2 shows the same sector correctly marked with digit "6".

Fig. 13 Diagram from Zaitseva (1952), p. 70, Fig. 14

of 12 years of age), the complexity of the model and especially its infinitesimal part would have remained obscure for them, unless some extra explanations were provided by teacher.

As Earl (1894), Zaitseva suggests dividing the circle into 16 equal sectors then cutting them out and putting together as shown on the right picture to get "approximately a parallelogram"; its base will be half of circumference and its height will be equal to radius; so the area of the circle is half of circumference multiplied by radius.

In another part of the world, namely, in Quebec, Canada, the authors of a 1966 secondary school textbook on "new mathematics" (*Mathématiques Nouvelles*, Hamel et al., 1966) also were seemingly inspired by the "teeth model." On pages 203–204 of the textbook, we find first a regular hexagon divided into six triangles with the common vertex in the center of the circle. At the next step, this hexagon (which can be inscribed in a circle) is shown as decomposed into a chain of triangles; the area of one such triangle multiplied by 6 would give the area of the hexagon (Fig. 14). Then it is said that the same technique can be applied to a regular polygon with a very large number of sides, so it would "*practically coincide"* (highlighted by us. - V. F. & A.V.) with the circle. According to the authors, in the case of a circle, the base of each "triangle" (interestingly, the illustration shows figures looking rather like sectors) is "practically" a segment of a straight line and the height (of the triangles) is "practically" the radius (Fig. 15). This leads to the formula for the area of the circle.

Overall, in the 1950–60s, the bases of these "novel" approaches to initiation to geometry in the middle grades (starting from the age of 12) have been discussed,

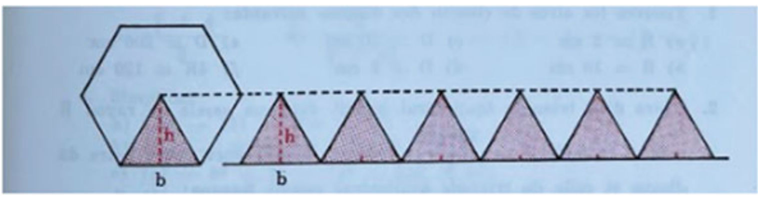


Fig. 14 Hamel et al. (1966), p. 203

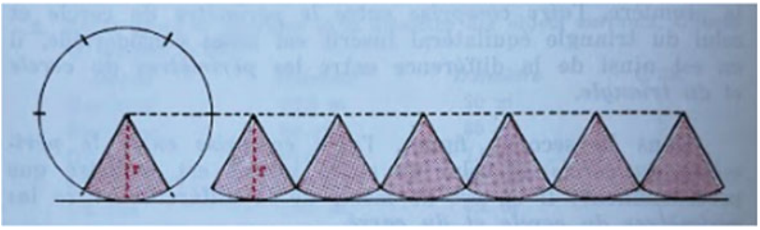


Fig. 15 Hamel et al. (1966), p. 204

according to Tessier and Beaugrand (1961) at the Congress of the International Commission for Mathematics Instruction (ICMI) in 1958 where the appropriate methods of teaching at entry level (first year of secondary school) where identified as intuitive, experimental, and empirical, based on observation and investigation, and providing an access to knowledge in a clear, direct, and immediate manner of perceiving the truth intuitively (without using deductive reasoning). The learners' access to rigorous modes of reasoning was therefore supposed to be based on experiment, and they were supposed to discover by themselves new properties of already familiar shapes as result of a process of active research (Tessier and Beaugrand (1961), see the section titled *Présentation*). We consider this essence of didactical genesis as representing an accumulation of the "innovative" elements of the didactical knowledge produced over the 150 years of reforms.

Not surprisingly, in their textbook, the authors of the "innovative approach" presented the familiar old rearrangement methods to introduce the area of the circle by suggesting to decompose the circle into "a big number of sectors, or figures that resemble to triangles" (the provided illustration shows 32 sectors) and then to operate with triangles in order to calculate the area of a sector (as base times height divided by 2 or, in other words, arc times radius divided by 2) which yields the area of the circle as the area of one sector multiplied by the total number of sectors. It remains however unclear why the formula of the area of the triangle can be applied to the area of the sector, and, moreover, why this *method* is believed to be helpful for transition to higher levels of rigor in later schooling?

This latter question can be considered fundamental in terms of the interplay between mathematical, historical, and didactical aspects of the development of the *method* as an artifact and its use to (presumably) support teaching and learning. In the next section, we will look at modern development to confirm that the issue appears to be well alive in twenty-first century's mathematical working spaces.

4 Looking into Modern Mathematical Working Spaces Through the Lens of Historical, Didactical, and Instrumental Genesis

From our genetic analysis of possible historical and didactical sources of *rearrangement method*, we can track a variety of ideas that reflect nineteenth–early twentieth-century innovations in teaching geometry by making it more visual, intuitive, and experimental, and therefore accessible to all learners at their early steps in mathematics. In particular, in Lardner's and Boryshkevich's textbooks, we find illustrations of a circle divided into equal sectors that are then rearranged into a new shape that "looks like" a parallelogram (or rectangle), eventually helping learner to *visualize* the process like it was done by the abovementioned authors centuries before.

An *intuitive approach* (Anschauung) which became the focus of innovative approaches at the end of the nineteenth–beginning of the twentieth centuries focused on the perception of a student who *imagines* "what happens if" a circle is divided into "many" sectors (considered as "triangles") that could be rearranged in what would look as a big "triangle" (Simon) or a "parallelogram" (Treutlein).

In the context of a *practical* (experimental) *approach*, the focus is put on *approximation* as the main tool used to grasp the concept of area of the circle by the learner (Earl; Stern and Topheim) who could also support her or his observation by physically *manipulating* with pieces of paper (like in Palmer, Willis, and Zaitseva); the latter approach was related to the idea of *mathematical laboratory* popular in the didactics of the first decades of the twentieth century.

All three approaches are apparently being used in the design of modern middle school geometry lessons which we will examine in the next section.

4.1 Rearrangement of the Circle in Modern Middle School Geometry Lessons: Possible Teaching Scenarios and Didactical Challenges

The approach based on the rearrangement of sectors apparently remains very attractive to the authors of geometry textbooks even nowadays, as we mentioned in the introductory section of our chapter (Larson et al., 2007; see Fig. 1).

In their quest for justification of the formula of the area of the circle, similar to the "approximate method" used in the textbooks of the late nineteenth and early twentieth centuries discussed above (e.g., Earl, 1894), the authors suggest an investigation based on the idea of dividing a circle into sectors, cutting them out and rearranging them into a new shape ("resembling a parallelogram"), and then ask the students to "write an expression" to calculate its area (using the radius of circle as the "approximate height of parallelogram.") Finally, students are asked to connect their "expression" to the area of the circle (eventually using the formula as "justification" of the theorem stating the mathematical expression for the area of a circle). However, the "task" does not provide any reference to the possibility of making the number of sectors grow indefinitely thus entirely omitting the infinitesimal part of the proof. Another difference with earlier textbooks is that the figure itself does not present a division of a circle into equal parts (sectors), for example, the circle in the left part of Fig. 1 looks like an orange sliced into pieces that do not look "congruent," while on the right side they, surprisingly, have the same size and shape. Again, in the case of Willis's (1922) scenario, some extra work from teacher would be needed to help students to grasp the subtlety of the process of "division" and "approximation."

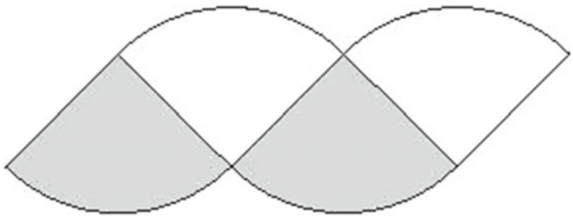
In a popular in the North America manual for (pre- and in-service) teachers which claims that *teaching* should be done *developmentally* (Van de Walle et al., 2020), it is suggested to challenge students to "figure out the area formula for circles on their own." Teachers are, however, advised to give a "hint" by showing the students how to "cut a circle apart into sectors and rearrange them to look like a parallelogram " (ibid., p. 530). According to the authors, the process of approximation (ibid., p. 530, Fig. 18.19) unfolds in three steps: (1) dividing a circle into 8 equal sectors and rearranging them a "near parallelogram"; (2) doing the same with 24 sectors ("even closer to a parallelogram"); (3) as the number of sectors grows, the figure "becomes closer and closer to a rectangle (a special parallelogram)". Here, the issue that draws our attention is that no mechanism is suggested to make students think about the reason why the fact of being "closer and closer to a rectangle" ensures the equality of areas (in other words, what justifies the transition from "being close" to "being equal").

Another manual for teachers' preparation (to make them *think*, as the authors state) by Brumbaugh et al. (2006) along with the method of "squares counting" suggests an activity of "cutting a circle" into "several pie-shaped wedges" (the authors suggest, as example, that wedges should have the central angle of 30°, that is, the circle is supposed to be cut into 12 wedges), then getting it "unrolled" and "interlaced" to have a shape which "approximates a parallelogram" (ibid., p. 187, Fig. 9.13). Again, the authors only mention that, presumably, an exploration of this fact ("approximating a parallelogram") would "lead to the area of the circle" (that is, the area as obtained by multiplying half of the circumference by the radius, idem.). The question of why one fact "leads" to the other seems to remain open (or even not formulated).

Finally, Small (2018, p. 167) provides an example of dividing a circle into multiple parts ("like pieces of a pie") to help students to see that the parts can be re-united to form a parallelogram. Her illustration (p. 167) shows first a circle divided into eight pieces and a rearranged shape ("parallelogram"), to finally come back to a circle (as a whole) whose area will be "equal" to Pi × radius × radius. Interestingly, the author does not mention *approximation* but puts straightly an equality sign for the area of the "parallelogram" (which is still curvilinear on the illustration) to claim that it is equal to the area of the circle.

A large number of similar approaches can be found in online resources available to teachers and learners nowadays. For instance, the method of calculation of the area of a circle based on the procedure of division of circle into sectors can be found on the *official* website of the National Council of Teachers of Mathematics suggesting a

Fig. 16 Picture from the NCTM website (accessed on November 01, 2020)



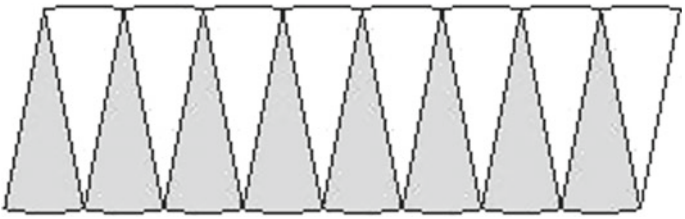


Fig. 17 Picture from the NCTM website (accessed on November 1, 2020)

number of said-to-be-high-quality resources.[38] The teaching scenario for "measuring circles" begins with students working in pairs or small groups; they would "cut the circle from the sheet and divide it into four wedges." The next step would be to ask the students to rearrange the wedges in the way shown in Fig. 16 (this configuration looks similar to Treutlein's, but contains only four pieces).

The authors suggest highlighting different parts of the circle (its radius and circumference) with a color. Then the authors argue that this shape would not be recognized by students as something familiar to them. Hence, the next step would be a further division of each wedge into two thinner wedges so that there would be eight wedges; the students have to continue the process to get 16 wedges and to finally produce a shape shown in Fig. 17 (similar to what we had in the above-cited examples, such as Larson et al. (2007), and in earlier sources).

This work, facilitated by the teacher's discussion eventually helping the students, would lead them to identify the shape resembling a parallelogram, which, when being "continually divided, […] will more closely resemble a rectangle."

Another step in the investigation (i.e., the further division of the wedges) would presumably lead the students to the conclusion that the length of the rectangle is equal to half the **circumference** of the **circle**, or πr. Additionally, students should recognize that the **height** of this rectangle is equal to the **radius** of the **circle**, r. And, similar to the previous examples, students would "try and generate a formula for **area** of this new rectangle formed by the pieces of the **circle."** Considering the **area** of this rectangle being equal to $\pi r \times r = \pi r^2$,[39] and knowing that "this rectangle is

[38] https://www.nctm.org/Classroom-Resources/ARCs/Measures-of-Circles/Circles-Lesson-2/.

[39] Interestingly enough, at the moment when we visited the official site of NCTM, it contained a misprint *at this crucial point*: instead of $\pi r \times r = \pi r^2$ it was typed "πr × r = πr2."

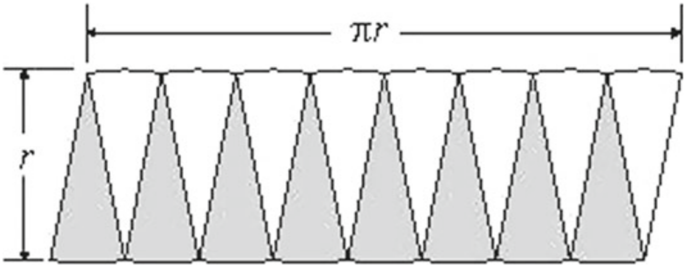


Fig. 18 Picture from the NCTM website (accessed on November 1, 2020)

equal in **area** to the original **circle**, this activity gives the **area** formula for a **circle**: $A = \pi r^2$"

The provided figure (Fig. 18) indicates dimensions of the shape to add more clarity to this conclusion (similar to Zaitseva's figure).

The final suggestion is to "have a class discussion with students explaining that total **area** is almost always an approximation."[40]

Again, as in previous examples, we can see the division of the circle into 16 pieces (yet, the authors encourage students into further exploration of situations with increasing numbers of pieces by saying that the process can be continued). The authors of this procedure also keep the idea of cutting out pieces of the circle and rearranging them into "parallelogram"; they also mention that the further division would transform the "parallelogram" into a "rectangle" without any specific suggestions of how to interpret and explain such transformation, again leaving it to the teacher's *didactical orchestration* (Drijvers, 2012). The website of NCTM does not mention any rationale of this method to explain why this activity is supposed to be relevant to teaching and learning mathematics in middle school.

4.2 Novel Approaches to Area Investigation Using Dynamic Digital Tools

The development of digital tools to support teaching and learning geometry in the late twentieth–early twenty-first centuries has brought novel didactical dimensions and tools that might be used by teachers. For instance, the Canadian website LearnAlberta provides an interactive animation which allows to increase the number of sectors (using a slider), so their rearrangement rapidly approaches the shape of a rectangle (see LearnAlberta.ca). Figures 19 and 20 show the division of the circle into 8 and 16 sectors.

[40] The exact meaning of this statement remains unknown. Does it refer to the area of a circle or to the area of *any* geometrical figure?

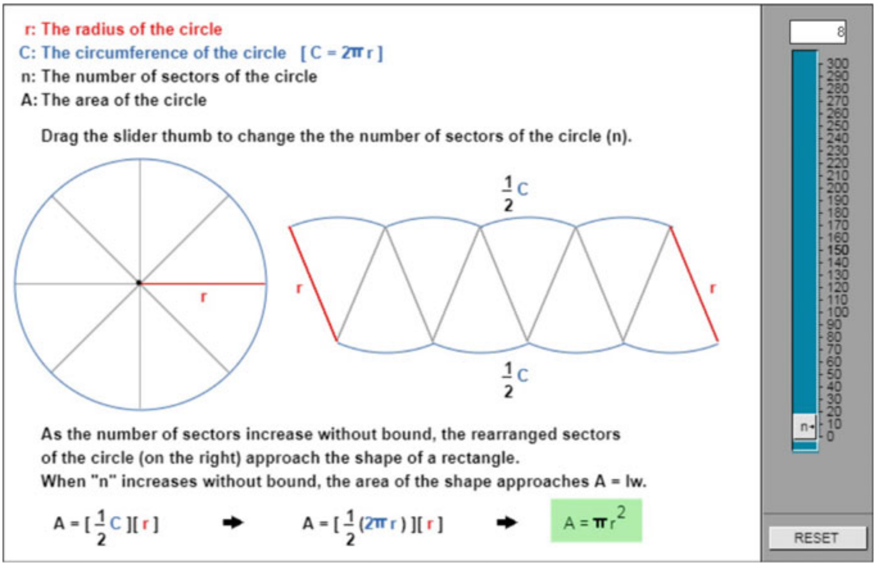


Fig. 19 LearnAlberta webpage, $n = 8$

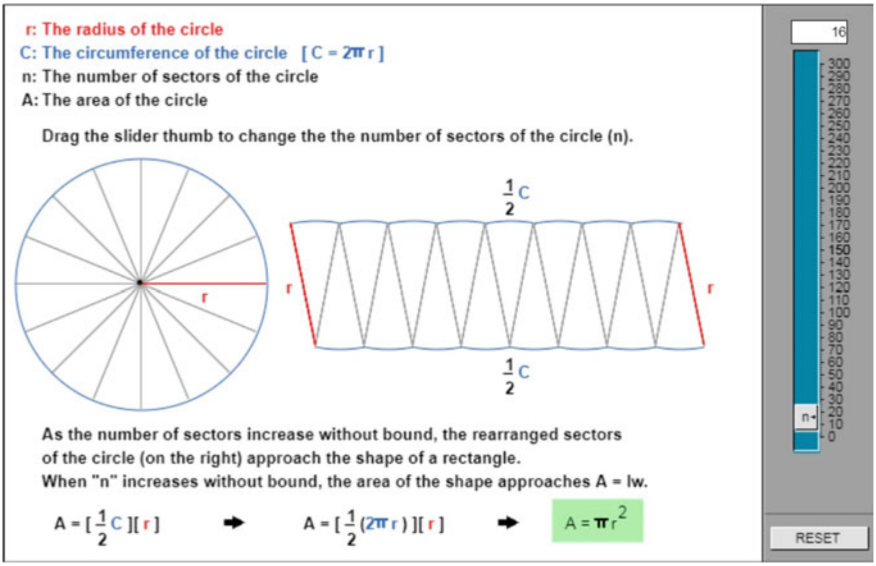


Fig. 20 LearnAlberta webpage, $n = 16$

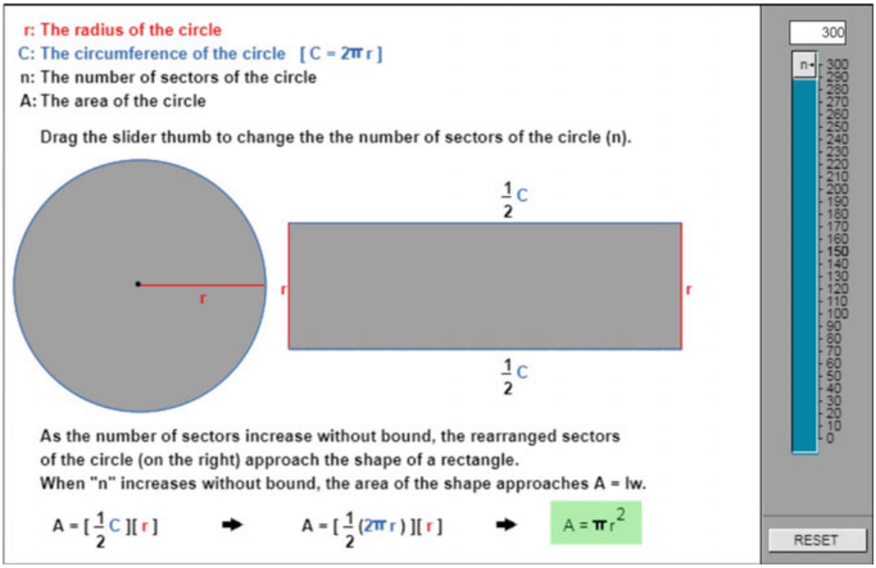


Fig. 21 LearnAlberta webpage, $n = 300$

The maximal possible number of sectors is 300, and when this number is reached, the figure composed of the sectors looks like a rectangle with the sides C/2 and r (Fig. 21).

The applet is accompanied by the following explanations:

> The area of the interior of a rectangle is the product of the length of the rectangle (l) and the width of the rectangle (w) [$A = lw$].
>
> When the number of sectors is large:
>
> – the width of the rectangle (w) is approximated by the radius of the circle (r);
>
> – the length of the rectangle (l) is approximated by half the circumference of the circle (πr).

This explanation is somewhat problematic. Firstly, at no step the figure obtained as rearrangement of sectors becomes "rectangle," so the figure referred to as "rectangle" is actually *not* a rectangle. Secondly, if l is used to note the length of a rectangle, say, drawn to approximate the "waved" figure, and w is used for its width, the relationships between these magnitudes and the half of circumference and radius are not very clear, for example, what can be l and w in Fig. 19? Thirdly, it is not very clear why the approximations suggested by the authors are valid. Again, as in above-cited textbooks, much of the work dealing with this complexity is left to the teacher.

Numerous interactive applets (like the one created using GeoGebra's affordances, see Fig. 22) allow for some more sophisticated exploration using several sliders to arrive at similar conjectures concerning the formula of area of the circle.

It can be argued that these and other similar recent computer-aided approaches to the introduction of the formula of area of the circle are largely based on the methods

Fig. 22 GeoGebra applet for exploring the formula of area of the circle, Authored by Ooi Soo Huat. (available at https://www.geogebra.org/m/AADN5Ruq, visited on October 24, 2021)

of teaching already found in the mathematics textbooks compiled in the twentieth century or even earlier. In other words, the creators of the mentioned software tools developed their programs to "animate" the procedures that had been found in earlier textbooks, and by doing so, to transform them, presumably, into more *dynamic and interactive* MWS, and, eventually, to make them more attractive and accessible to the learners. However, this transformation (from traditional "paper-and-pencil" methods to novel ones, based on digital tools) when carried out without a substantial historical analysis of the method, especially in regard to its epistemological complexity and didactical implications, could lead to limited conceptual understanding by students. One can question whether the application of software aiming at visualization of infinitesimal procedures can indeed enhance the ability of the learner to understand the rationale of the derivation of the formal expression for the area; this question will immediately lead to an even larger question concerning the application of computer-generated simulations in teaching mathematics (in particular, geometry).

4.3 Discussion and Conclusions

In our paper, we tried to track back the history of introduction of the "matching sectors method" (or "*rearrangement method*") recently used to justify the formula for the area of a circle in school curricula and resources (printed and available online). It appears plausible to conjecture that this method was originally used by professional mathematicians of Antiquity and Middle Ages in the West,[41] and only later it was borrowed by mathematics educators and placed into school textbooks and lessons as result of didactical transposition from "knowledge to be used" to "knowledge to be taught and learned" (Chevallard, 1985; Kang & Kilpatrick, 1992). In the case of the area of a circle, the earliest sources for "rearrangement method" that we were able

[41] The original version of this chapter included a discussion of the similar methods found in Chinese and Japanese mathematical treatises; it was removed as not directly related to the topic discussed in the present chapter. We plan to return to the collected materials in a future publication.

to identify in the Western textbooks were produced in the first half of the nineteenth century (e.g., Lardner, 1835, 1840). What were the particular reasons for the use of this archaic method for instruction in the nineteenth and especially in the twentieth century? Was it grounded in some pedagogical or didactical traditions or was it a result of innovative efforts to improve teaching and learning comparable to other newly introduced methods?

It remains equally unclear from where did the method of "matching sectors" come to the modern school textbooks. Why was it perceived as an especially "efficient" tool for teaching? Was it because Archimedes' "exhaustion method" involved sophisticated reasoning and thus was considered difficult for beginners, while the method of "matching sectors" used in textbooks may have seemed more didactically attractive since the procedure of "matching" the sectors looked easy to understand for even relatively young learners? This simplicity, along with visual and dynamic nature of rearrangements, apparently seemed to the modern educators to be the best way to construct lessons which are hands-on, practical, investigative, and providing students with initial intuition that could lead to more complex mathematical concepts.

In this chapter, we only briefly discussed possible historical roots of "rearrangement method" while mentioning some mathematical works authored by European scientists (Cusanus, da Vinci, Kepler), among others. We used mainly the primary sources cited in modern literature, and used secondary works, such as Baron's book on the history of calculus and Beckmann's *History of Pi*,[42] as well as in online resources (e.g., the *Cut-the-Knot* educational website created by Alexander Bogomolny).[43] A more detailed historico-didactical and genetic analysis of the process of calculation of the area of a circle mainly based on primary sources will be published elsewhere.

The problem of calculation of the area of a circle solved by mathematicians of the past was of a particular nature; in certain cases, they arguably dealt with it when focused on some specific problems as, for instance, the quadrature of the circle (Cusanus) or calculation of the volumes of the wine barrels (Kepler). Yet, when one turns to the process of transmission of scientific (in this particular case, mathematical) knowledge featured in the textbooks specifically designed for learning such as Clairaut's *Elements* of 1741 (1830)[44] or, later, the geometry textbooks of the nineteenth and twentieth centuries, then the task of transmission of theoretical and practical knowledge became essentially a didactical issue, and new considerations took place.

In the case of the area of the circle, the epistemic issue of dealing with infinity interacted with the need for visualization (in particular, using diagrams in textbooks), induction (based on intuitive approach to infinitely small and large entities), and approximation (dictated by practical considerations). In particular, we found two

[42] Although several authors cite Beckmann's work, especially the paragraphs concerning the procedures of calculation of the area of a circle, we find that certain parts of the book are not sufficiently elaborated and reliable.

[43] http://cut-the-knot.org/.

[44] On Clairaut's didactical ideas see, for example, Sander (1982) and Glaeser (1983).

different interpretations of the "making of a circumference a piece of straight line" (with the length equal to the circumference): one related to the category of methods based on the idea of "unfolding a circle" (with sectors becoming "teeth-like" row, as, for example, in the diagram constructed by Casselman, 2012 or in the one found in Beckmann, 1976 (Fig. 2)); the other is more like "cutting out the sectors of a circle" (each sector is to be moved and rearranged), when even the notion of "being equal" seems to be sacrificed to the goal of making a new arrangement of parts fitting into a rectangular shape (like in Lardner, 1835 and Willis, 1922).

We argue that the division of a circle into a number of equal sectors and their further rearrangement thus producing "teeth-like" diagrams in school textbooks was done mainly for the sake of visualization, even though minor differences between variants of this procedure can be identified. There is an aspect that needs deeper reflection as far as the "visual" part of the process is concerned: it is related to the number of sectors into which the circle is dissected. More specifically, there are two patterns that can be identified, the first one is based on a regular hexagon inscribed in the circle and then having its sides doubled (6–12–24…), the second one features powers of 2 (4–8–16…) and is apparently based on a square inscribed in the circle and then having its sides doubled. Both visual representations serve to convince the reader (learner) that the area of a given circle can indeed be calculated in the same way as the areas of all other circles, via an approximation by the polygons.

At some point, the circle itself is considered (or should one say "defined"?) as a polygon with an infinite number of sides. Then comes a "rearrangement part" arguably based on the assumption that the operations of division, decomposition, and re-composition of a shape do not change its area; several models that we investigated above reflect this dynamic process. Then comes even more complex (and complicated) issue which remains mathematically and didactically challenging: How to deal with the infinity? Some sources we analyzed provide a rather static explanation (providing a "sufficiently large" number of sectors) saying that the new shape "looks like" a triangle, or a rectangle, or a parallelogram. Others add the word "approximately" to this visualization. Finally, a number of authors bring dynamic aspects in the process, explicitly or implicitly, pointing at a possibility to increase the number of sectors and, consequentially, to make it intuitively clear that the area of the rearranged shape of the circle becomes (approximately) equal to that of a triangle, or a rectangle, or a parallelogram for which it is known how to calculate the area.

This "knowing" (most often coming from the previously proved formula for the circumference) allows for further manipulations with the formulas for the area of a triangle (or rectangle, or parallelogram) and, finally, for advancing conjectures concerning the formula of the area of the circle.

Considering geometry as a complex activity embracing various processes of interaction between practical knowledge (measuring) and its theoretical codification (e.g., the Euclidean deductive system used for proving), we are inclined to see in the use of this "rearrangement" method an attempt to introduce a twofold procedure explaining to the learners *how* to calculate the area of a circle and, at the same time, offering a plausible reasoning strategy explaining *why* this formula works. While considering it as "instrumental activity" (implying both signs and tools) from its very beginnings, Richard et al. (2019, p. 143) reflect on the complexity of distinguishing between mathematics as a science and mathematical thought in the context of a particular situation/task/activity. The authors further refer to Kuzniak and Richard's (2014) definition of mathematical work as a "progressively constructed process of bridging the epistemological and the cognitive aspects in accordance with three yet intertwined genetic developments as the *semiotic, instrumental and discursive* geneses" to introduce their model of MWS allowing to "report on mathematical activity, potential or real, during problem solving or mathematical tasks" (Richard et al., 2019, p. 144).

From the historico-didactical perspective, it appears to be difficult, or even impossible, to make direct connections between the mathematicians' work prior to eighteenth century and didactical innovations of the later period; however, some of the issues that they were dealing with certainly merit attentive look of modern educators. For instance, the idea of da Vinci of separating "the angles of the sectors from each other in such a way that the space between the vertices of these angles become equal to flattened bases of the sectors" (Ravaisson-Mollien 1888, Ms E, folios 24r-26v) might be resonating with that of transforming rearranged sectors into configurations where sectors (approximately) become triangles, and, when put together, form rectilinear shapes (triangles, rectangles, or parallelograms). The idea of Cusanus that the area of the circle may be found by the same means as that employed for any other polygon, that is, by dividing it up into a number (in this case, an infinite number) of triangles (Boyer), is also fruitful in terms of the treatment of "teeth-like" representations in later (modern) sources.[45] Kepler's work, besides following Cusanus' method of "indivisibles," provides an insight into the procedure that points at the approximation as a way of approaching the circle by polygons with a large (even infinite) number of sides.

In our study, we found that the modern authors (e.g., Boyer, Baron, Beckmann) seem to add details which were *not* found in the cited works. Indeed, neither Cusanus, nor Kepler explicitly discussed the "rearrangement" of sectors. Moreover, despite a seeming similarity of the way in which the rearrangement method was described in the nineteenth–early twentieth-century sources that we analyzed in this chapter, we noticed some substantial *differences* in representations that need to be reflected upon from the "teaching and learning" perspective. While the idea of dividing the circle into sectors is present in all models, there are differences in terms of the number of pieces, as well as in their pictorial representations. Some authors provided illustration of the whole circle divided into equal sectors (e.g., Boryshkevich, 1893; Earl, 1894). Others presented a division of a half of a circle (Palmer, 1919) or showed a part of the sectors (Willis, 1922). One model (Henrici & Treutlein, 1897) shows division of one half of a circle into three sectors and the other half into six sectors. When representing the result of a rearrangement, some authors show the final result of one transformation (Earl, 1894; Hall & Stevens, 1921; Palmer, 1919), or even leave the "matching" incomplete (like in Lardner, 1835, 1840), where two rows of "teeth" are getting close to each other but not yet stuck together (similar to the

[45] For discussions of Cusanus' quadrature, see Uebinger 1985, 1896, 1897, Wertz 2001, Nicolle 1996; 2001; 2020.

middle configuration in Boryshkevich's drawing). Other authors might show two consecutive transformations (like Treutlein's model with three and six pieces, or Willis's showing a part of sectors put together and a plain rectangle) or even three (similar to Boryshkevich). This variety of the representations of the one and the same model might have had an impact on its use (in terms of instrumentation) which also needs a deeper investigation.

Finally, the development of recent didactical models, especially employing the dynamic aspects of new digital technology (e.g., dynamic geometry) adds more complexity to the existing MWS while pointing at the fertility of genetic investigation of their historical and didactical antecedents.

References

A. Primary materials

Abraham Bar Hiyya. (Savasorda, 1116). *Hibbur ha-mêsihá wé-ha-tidbóret (Tractat de geometria i mesurament).* Barcelona.

Abraham B., H. (1720). 地球の形 (*Séfer Surat ha-aretz*), 1720, p. 15. Offenbach. Biblioteca Virtual del Patrimonio Bibliográfico. https://bvpb.mcu.es/es/catalogo_imagenes/grupo.do?path=154236.

[Archimedes]. (1544). *Archimedis Syracvsani Philosophi ac Geometriae excellentissimi Opera* [Greek edition]. Basileae: Ioannes Heruagius.

[Archimedes]. (1615). *Archimēdus Panta Sōzomena Novis Demonstrationibvs Commentariisqve Illvstrata.* Paris: Morellus.

[Archimedes]. (1676). *Archimedis Siracusani. Arenarius et dimensio circuli.* Oxford: Sheldonian Theatre.

[Archimedes]. (1880). *Archimedis Opera Omnia cum Commentariis Eutocii.* Lipsiae: B.G. Teubner.

Boryshkevich, M. (1893). [Борышкевич, М[ихаил Федотович]] Kurs elementarnoi geometrii s prakticheskimi zadachami dlya gorodskih uchilishch [Курс элементарной геометрии с практическими задачами для городских училищ (A course of elementary geometry accompanied by practical problems for educational institutions located in towns). Kiev: Kryzhanosvkii and Avdyushenko (Типография И.Крыжановского и В.Авдюшенко), (second edition).

Castelnuovo, E. (1948). *Geometria intuitiva, per le scuole medie inferiori* (Intuitive geometry for lower secondary schools). Roma: Carabba [reprint 1949].

Castelnuovo, E. (1966). *The way of Mathematics: Geometry*. Florence: The New Italy.

Clairaut [Alexis Claude]. (1741). Élémens de géométrie. Paris: Lambert & Durand.

Clairaut [Alexis Claude]. (1830). Élémens de géométrie. Nouvelle édition, revue et corrigée. Paris: Bachelier.

Clairaut, A.-C. (1743). *Théorie de la figure de la terre, tirée des principes de l'hydrostatique.* Paris: David fils.

Da Vinci, L. (1888). *Les manuscrits de Leonard de Vinci.* Tome 3 [Manuscrits C, E, & K de la Bibliothèque de l'Institut]. Ed., transl., Charles Ravaisson-Mollien. Paris: Maison Quantin.

Da Vinci, L. (1890). *Les manuscrits de Leonard de Vinci.* Tome 5 [Manuscrits G, L & M de la Bibliothèque de l'Institut]. Ed., transl., Charles Ravaisson-Mollien. Paris: Maison Quantin.

Da Vinci, L. (1964). *Leonardo da Vinci on Painting. A Lost Book. Reassembled from* Codex Vaticanus Urbinas 1270 *and from the* Codex Leicester *by Carlo Pedretti.* Berkeley and Los Angeles: University of California Press.

Earl, A. G. (1894). *Practical lessons in physical measurement*. London and New York: Macmillan and Co.

Guttmann, M. (1903). Chibbur ha-Meschicha we-ha-Tishboret. Berlin: Mekis.e Nirdamim. https://www.hebrewbooks.org/38608.

Hamel, O., Richard, G. W., Hébert, M., Labrie, J.-M. (1966). *Mathématiques nouvelles: Cours secondaire -2*. Laprairie, Qc.: Les éditions F.I.C. La Mennais.

Hall, H. S., & Stevens, F. H. (1921). *School Geometry*, *Parts I, II, and III*. London: Macmillan.

Henrici, J., & Treutlein, P. (1987). *Lehrbuch der Elementar-Geometrie: Erster Teil. Gleichheit der Planimetrischen Grössen. Kongruente Abbildung in der Enene. Pesnum der Tertia.* Leipzig: Teubner.

Kepler, J. (1615). *Nova Stereometria Dolorium Vinariorum.* [New Solid Geometry of Wine Barrels] […] *Accessit Stereometriae Archimedeae Svpplementvm* [[to which] A Supplement to the Archimedean Solid Geometry Has Been Added]. Lincii [=Linz]: J. Plancus.

Kepler, J. [= Johannes]. (1935). *Новая Стереометрия Винных Бочек* [New Stereometry of Wine Barrels, annotated Russian translation]. Translated by G.N. Sveshnikov, with an introduction by M.Ya. Vygodskiï. Moscow and Leningrad: GTTI.

Lardner, D. (1835). *The First Principles of Arithmetic and Geometry; Explained in a Series of Familiar Dialogues, Adapted for Preparatory Schools and Domestic Instruction; with Copious Examples and Illustrations.* [Part 2:] *Conversations on Geometry*. London: Longman and Taylor.

Lardner, D. (1840). *A Treatise on Geometry and its Application to the Arts*. Longman etc.

Larson, R., Boswell, L., Kanold, T. D., & Stiff, L. (2007). *Geometry*. Evanston (IL): McDougal Littell.

Nicolas de Cues (2007). *Les écrits mathématiques.* Introduction, translation and notes by Jean-Marie Nicolle. Paris: Honoré Champion.

Nikolaus von Kues (Nicolaus Cusanus) (1980). *Die mathematischen Schriften.* Josepha Hofmann (transl.). *Philosophische Bibliothek* 231. Hamburg: Felix Meiner Verlag.

Palmer, C. I. (1919). *Practical mathematics for home study, being the essentials of arithmetic, geometry, algebra and trigonometry*. New York: McGraw-Hill Book Company.

Ravaisson-Mollien, C. (ed., transl.). (1888). *Les manuscrits de Leonard de Vinci.* Tome 3 [Manuscrits C, E, & K de la Bibliothèque de l'Institut]. Paris: Maison Quantin.

Romanus, A. [= Adriaan van Roomen]. (1593). *Ideae mathematicae pars prima, sive Methodus polygonorum, qua laterum, perimetrorum & arearum cujiuscunque polygoni investigandorum ratio exactissima & certissima; unà cum circuli quadratura continentur.* Antwerpen: Ioannem Keerbergium.

Simon, Max (Maximilian). (1889). *Der erste Unterricht in der Raumlehre: ein methodischer Leitfaden für die unteren Klassen höherer Lehranstalten, sowie für die Volksschule in heuristischer Darstellung*. Berlin: Springer.

Small, M. (2018). *Grandes idées pour l'enseignement des mathématiques 9 à 14 ans.* Chenelière.

Stern, H. A., & Topham, W. H. (1913). *Practical Mathematics.* London: G. Bell and Sons (4th edition).

Tessier, G. J.-M., & Beaugrand, R. (1961). *Initiation à la géométrie: Géométrie intuitive*. Montréal: Centre de psychologie et de pédagogie.

Treutlein, P. J. (1911). *Der Geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichtes an unseren höheren Schulen.* Leipzig und Berlin: B.G. Teubner.

Van de Walle, J. A., Karp, K. S., Bay-Williams, J. M. (2020). *Elementary and middle school mathematics: Teaching developmentally*. 10th Edition. Harlow, etc.: Pearson.

Willis, C. A. (1922). *Plane Geometry: Experiment, Classification, Discovery, Application.* Philadelphia: Blakiston's Son & Company.

Zaitseva, N. Y. (1952). *Plany urokov po arifmetike v V classe. Is opyta raboty.* [Plans of arithmetic classes in the 5th grade. [Lessons] drawn from working experience.] Moscow: Uchpedgiz.

Zubov, V. P. (1935). *Леонардо да Винчи. Избранные естественнонаучные произведения.* Leonardo Da Vinci. [Selected works on natural sciences.] Translated and commented by V.P.

Zubov. Moscow: Academy of Sciences, 1935 [reproduced fac-simile by A.Lebedev' Studio Publishers (Moscow, 2010)].
Zubov, V. P. (1955). *Леонардо да Винчи. Избранные естественнонаучные произведения.* Leonardo Da Vinci. [Selected works on natural sciences.] Translated and commented by V.P. Zubov. 2 vols. Moscow: Academy of Sciences.

B. Secondary works

Acerbi, F. (2008). Disjunction and conjunction in Euclid's *Elements*. *Histoire Épistémologie Langage*, 30(1), 21–47.
Albertson, D. (2014). *Mathematical theologies: Nicholas of Cusa and the legacy of Thierry of Chartres*. Oxford University Press.
Barbin, E., & Menghini, M. (2014). History of teaching geometry. In A. Karp & G. Schubring (Eds.), *Handbook on history of mathematics education* (pp. 473–492). New York: Springer.
Baron, M. E. (1969). *The origins of the infinitesimal calculus.* Oxford etc.: Pergamon Press.
Becker, G. (1994). Das Unterrichtswerk "Lehrbuch der Elementargeometrie" von J. Henrici und P. Treutlein: Entstehungsbedingungen, Konzeption, Wirkung. In J. Schönbeck, H. Struve, & K. Volkert (Eds.), *Der Wandel im Lehren und Lernen von Mathematik und Naturwissenschaften.* Band 1, pp. 89–112. Weinheim: Deutscher Studienverlag.
Beckmann, P. (1976). *A history of π (pi).* New York: St. Martin's Press, 1976 [originally published in 1971 by Golem Press, a publishing house established by the author himself].
Blank, B. (2001). *A history of Pi* by Petr Beckmann; *The joy of Pi* by David Blatner; *The nothing that is* by Robert Kaplan; *The story of a number* by Eli Maor; *An imaginary tale* by Paul Nahin; *Zero: The biography of a dangerous idea* by Charles Seife. *The College Mathematics Journal*, 32(2), 155–160.
Boyer, C. B. (1959). *The history of the calculus and its conceptual development (The Concepts of the Calculus).* New York: Dover.
Brousseau, G. (1998). *Théorie des situations didactiques.* Grenoble: La Pensée Sauvage.
Brumbaugh, D. K., Ortiz, E., & Gresham, G. (2006). *Teaching Middle School Mathematics.* Mahwah, NJ: Lawrence Erlbaum.
Bryan, J., & Sangwin, C. (2008). *How round is your circle? Where engineering and mathematics meet*. Princeton University Press.
Casselman, B. (2012). Archimedes on the circumference and area of a circle. Feature column from the AMS. Posted on Internet in February 2012. http://www.ams.org/publicoutreach/feature-column/fc-2012-02.
Castelnuovo, E. (1977). L'enseignement des mathématiques. *Educational Studies in Mathematics, 8*(1), 41–50.
Cavanagh, M. (2008). Area measurement in year 7. *Reflections, 33*(1), 55–58.
Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage.
Counet, J.-M. (2005). Mathematics and the divine in Nicholas of Cusa. In T. Koetsier & L. Bergmans (Eds.), *Mathematics and the divine: A historical study* (pp. 273–290). Elsevier.
da Silva, R. S. R., Barbosa, L. M., Borba, M. C., & Andrejew Ferreira, A. L. (2021). The use of digital technology to estimate a value of pi: Teachers' solutions on squaring the circle in a graduate course in Brazil. *ZDM–Mathematics Education*, 23, 1–15. https://doi.org/10.1007/s11858-021-01246
Duhem, P. (1909). *Études sur Léonard de Vinci: Ceux qu'il a lus et ceux qui l'ont lu.* A. Hermann et fils.

Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 265–281). Springer.

Ernest, P. (1998). The history of mathematics in the classroom. *Mathematics in School*, 27(4), 25–31. [Reprinted in *Mathematics in School*, 40 (2011), No. 4, 14–20.]

Farmaki, V., & Paschos, T. (2007). Employing genetic "moments" in the history of mathematics in classroom activities. *Educational Studies in Mathematics, 66*(1), 83–106.

Francis, R. L. (1976). History of mathematics in the training program for elementary teachers. *The Arithmetic Teacher*, 23(4), 248–250.

Freiman, V., & Volkov, A. (2006). Infinitesimal procedures in modern and medieval mathematics textbooks. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the twenty eighth annual meeting of the north American chapter of the international group for the psychology of mathematics education* (Vol. 2, pp. 519–520). Mérida, Mexico: Universidad Pedagógica Nacional. http://www.pmena.org/pmenaproceedings/PMENA%2028%202006%20Proceedings.pdf

Fujita, T., Jones, K., & Yamamoto, S. (2004). Geometrical intuition and the learning and teaching of geometry. Paper presented at the 10th International Congress on Mathematical Education (ICME-10). Copenhagen, Denmark, pp. 4–11 July 2004.

Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics, 66*(2), 131–143.

Furinghetti, F., & Menghini, M. (2014). The role of concrete materials in Emma Castelnuovo's view of mathematics teaching. *Educational Studies in Mathematics, 87*(1), 1–6.

Glaeser, G. (1983). A propos de la pédagogie de Clairaut: Vers une nouvelle orientation dans l'histoire de l'éducation. *Recherches En Didactique Des Mathématiques, 4*(3), 332–344.

Gould, H.W. (1974). A History of π by Petr Beckmann [review article]. *Mathematics of Computation, 28*(125), 325–327.

Herbst, P., Cheah, U. H., Richard, P. R., & Jones, K. (2018). *International perspectives on the teaching and learning of geometry in secondary schools.* (ICME-13 Monographs). Cham: Springer.

Hofmann, J. E. (1971). Cusa, Nicholas. In C. Coulston Gillispie (Ed.), *Dictionary of scientific biography* (Vol. 3, pp. 512–516). New York: Charles Scribner's Sons.

Jankvist, U. T. (2009). A categorization of the "Whys" and "Hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Johnson, D. B., & Mowry, T. A. (2016). *Mathematics: A practical odyssey*. Cengage Learning.

Kang, W., & Kilpatrick, J. (1992). Didactic transposition in mathematics textbooks. *For the Learning of Mathematics, 12*(1), 2–7.

Kordaki, M., & Potari, D. (1998). Children's approaches to area measurement through different contexts. *The Journal of Mathematical Behavior, 17*(3), 303–316.

Kuzniak, A., Richard, P. R. (2014). Espaces de travail mathématique. Point de vues et perspectives. *Revista latinoamericana de investigación en matemática educativa*, 17(4(I)), 5–40.

Levey, M. (1954). Abraham Savasorda and his algorism: A study in early European logistic. *Osiris, 11*, 50–64.

Levey, M. (1981). Abraham bar Hiyya. In C. Coulston Gillespie (Ed.), *Dictionary of scientific biography* (Vol. 1, pp. 22–23), New York: Charles Scribner's Sons.

Mac an Bhaird, C. (2011). The provision of mathematics support and the role of the history of mathematics. *Hermathena*, No. 191 (2011), Philosophy and Mathematics II, pp. 53–67.

Marshall, G. L., & Rich, B. S. (2000). The role of history in a mathematics class. *The Mathematics Teacher, 93*(8), 704–706.

Menghini, M. (2015). *From practical geometry to the laboratory method: The search for an alternative to Euclid in the history of teaching geometry*. In: S. J. Cho (Ed.). *Selected Regular Lectures from the 12th International Congress on Mathematical Education*, (pp. 561–587) Cham: Springer.

Montoito, R., & Garnica, A. V. M. (2015). Lewis Carroll, education and the teaching of geometry in Victorian England. *História da Educação*, *19*(45) (January/April), 2015. (Online publication available at https://www.scielo.br/scielo.php?pid=S2236-34592015000100009&script=sci_arttext&tlng=en; retrieved on October 28, 2020).

Nicolle, J.-M. (1996). Les transsomptions mathématiques du Cardinal Nicolas de Cues. *Actes de l'Université d'été 95: Épistémologie et Histoire des Mathématiques (Besançon)* (pp. 359–372). I.R.E.M. de Franche-Comté.

Nicolle, J.-M. (2001). *Mathématiques et métaphysique dans l'œuvre de Nicolas de Cues*. Villeneuve d'Ascq: Presses Universitaires du Septentrion.

Nicolle, J.-M. (2020). *Le laboratoire mathématique de Nicolas de Cues*. Paris: Beauchesne.

O'Dell, J. R., Rupnow, T., Cullen, C. J., Barrett, J. E., Clements, D. H., & Sarama, J. (2016). Developing an Understanding of Children's Justifications for the Circle Area Formula, paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (38th, Tucson, AZ, Nov 3–6, 2016), 235–242.

Peckham. (1951). Dr. Lardner's "cabinet cyclopaedia." *The Papers of the Bibliographical Society of America*, *45*(1), 37–58. https://www.jstor.org/stable/24298686

Rejeki, S., & Putri RII. (2018). Models to support students' understanding of measuring area of circles. *Journal of Physics (Conf. Series)*, no. 948. https://iopscience.iop.org/article/https://doi.org/10.1088/1742-6596/948/1/012058/pdf.

Richard, P. R., Venant, F., & Gagnon, M. (2019). Issues and Challenges in Instrumental Proof. In P. R. Richard, F. Venant, & M. Gagnon (Eds.), *Proof technology in mathematics research and teaching* (pp. 139–172). Springer.

Sander, H. J. (1982). Die Lehrbücher «Eléments de Géométrie» und «Eléments d'Algèbre» von Alexis-Claude Clairaut. Dissertation zur Erlangung des Grades eines Doktors der Erziehungswissenschaften an der Universitàt Dortmund

Scharpff, F. A. (Ed.). (1871). *Der Cardinal und Bischof Nicolaus von Cusa als Reformator in Kirche, Reich und Philosophie des 15. Jahrhunderts*. Tubingen: Laupp.

Smith, D. E., & Mikami, Y. (1914). *A History of Japanese Mathematics*. Chicago: Open Court

Stacey, K., & Vincent, J. (2009). Finding the area of a circle. Didactic explanations in school mathematics. *The Australian Mathematics Teacher*, *65*(3), 6–9.

Tamborg, A. L. (2017). Origins, Transformations, and Key Foci in Instrumental Genesis. In *Proceedings of Eighth Nordic Conference on Mathematics Education (Norma'17). Nordic Society for Research in Mathematics Education* (Stockholm, Sweden), 1–9.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In: Dominique, G., Kenneth, R., Luc, T. (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). Boston etc.: Springer Science+Business Media.

Uebinger, J. (1895). Die mathematischen Schriften des Nik. Cusanus. *Philosophisches Jahrbuch.* Band 8 (1895), S. 301–317, 403–422. Band 9 (1896), S. 54–66, 391–410. Band 10 (1897), S. 144–159.

Vengeon, F. (2006). Mathématiques, création et humanisme chez Nicolas de Cues. *Revue D'histoire Des Sciences, Tome, 59*(2), 219–244.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

Watanabe, M. (2011). *Nicholas of Cusa—A companion to his life and his times*. Edited by Gerald Christianson and Thomas M. Izbicki. Farnham and Burlington: Ashgate, 2011.

Weiss, Y. (2019). Introducing history of mathematics education through its actors: Peter Treutlein's intuitive geometry. In: Weigand, H.-G., McCallum, W., Menghini, M., Neubrand, M., Schubring, G. (Eds.), *The legacy of Felix Klein.* (*ICME-13 monographs*) (pp. 107–116). Cham: Springer.

Wertz, W. F., Jr. (2001). Nicolaus of Cusa's 'on the quadrature of the circle'. *Fidelio, 10*(2) (Summer 2001), 30–40.

Wiltshire, B. (1930). History of Mathematics in the Classroom. *The Mathematics Teacher, 23*(8), 504–508.