Creative Use of Dynamic Mathematical Environment in Mathematics Teacher Training

Roman Hašek

1 Introduction

The chapter does not deal with the concept of artificial intelligence (AI) as it is usually understood, i.e. as a comprehensive agent, as it is dealt with, for example, in Russell et al. (2010) or du Sautoy (2019). It relates to AI indirectly, through the particular properties of currently available free-of-charge dynamic geometry software (DGS), namely GeoGebra (2020) and OK Geometry (2020), that undoubtedly move the capabilities of this software into the realm of AI. With full awareness of the relevant difference in the conceptual behaviour of this software and of the fact that the practices we present are not the only solution, the possibility of a purely automatic solution using GeoGebra is becoming more and more real, Botana et al. (2020), we specifically discuss the use of the software's ability to independently assess an individual geometric sketch, applying the specific method of automated observation of a dynamic construction and the principles of the automatic proving and deriving of geometric theorems. Based on this assessment, the software provides a user with specific feedback on her or his approach to the solution of a particular task. Such features definitely link this software to the world of AI. We believe in the usefulness of the connection of the worlds of DGS and AI, and we are convinced that the wide fulfilment of their potential in educational practice is very real. An educational environment with AI in the background and equipped with the features of reading dynamic constructions and automated reasoning could be the right means to balance the teacher's supervisory role in the classroom with the individualisation of learning in terms of the goals formulated by Bloom in his well-known treatise on mastery learning, Bloom (1968, 1971), Levin (2017).

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From the above perspective, using examples from geometry based on historical problems, we will show specific educational procedures utilising the environment of the mentioned software providing users with feedback that leads them in their independent creative work, the result of which can be both a dynamic geometric model of the respective phenomenon and a real object, e.g. the physical model printed on a 3D printer. All are based on practical findings from the preparation of prospective mathematics teachers of lower and upper secondary schools. The use of the software to interpret historical geometrical subject matter from the perspective of up to date mathematics, to create a dynamic model of the respective phenomenon and also to serve as a basis to create its physical model has proven to be a functioning component of mathematics teacher education, Dennis (2000), Clark (2012), Furinghetti (2007), Hašek et al. (2017).

2 Historical Context

The story underlying the chapter is based on two contributions submitted by Josef Rudolf Vaňaus (1839–1910), a Czech grammar school mathematics teacher and one of the leading personalities of both the professional and social life of his time, to the *Journal for the Cultivation of Mathematics and Physics*; a paper *Trisektorie* (*Trisectrix* in English) on the use of an oblique strophoid to trisect an angle, Vaňaus (1881), published in 1881, and an assignment of a geometry task for the journal's problem corner, Vaňaus (1902), the solution of which was based on the trisection of an angle, Ostermann and Wanner (2012), published in 1902.

Josef Rudolf Vaňaus was born in 1839. In 1862, he graduated from the Faculty of Arts of Charles University in Prague, and then, for more than thirty years, he worked as a grammar school teacher. He died in 1910, after fourteen years of retirement. All his life J. R. Vaňaus was very active in promoting mathematics and its teaching. Starting as a young university student, in 1862, he became one of four founders of the Union of Czech Mathematicians and Physicists and he continued doing research and publishing papers on findings in mathematics and its teaching in relevant Czech journals. He paid significant attention to supporting students with mathematical talent at the secondary school level through assigning them problems in the problem corner of the Czech *Journal for the Cultivation of Mathematics and Physics* (with an original Czech title *Časopis pro pěstování mathematiky a fysiky*), Folta and Šišma (2003).

A beneficial and a creative way of using GeoGebra and OK Geometry to solve the problem assigned by Vaňaus and to model and analyse his original method of trisecting an angle with contemporary students of mathematics teaching is treated in this chapter.

3 Problem 36

The following problem authored by J. R. Vaňaus was, as Problem No. 36, set in the problem corner of the third issue of the *Journal for the Cultivation of Mathematics and Physics* Vaňaus (1902), published in Czech. The target group of the problem assignment was students of an upper secondary school, ages 15–18.

> Given a line segment AB. Circular arcs, both with the radius $|AB|$, are drawn around points A and B, passing through points B and A, respectively, and intersecting at point C. The task is to set points M and N at arcs AC and BC, respectively, so that the line segment MN is parallel to AB and the angle $\angle MAN$ is equal to a given acute angle; see Fig. 1.

Solution: First, we add a few more elements to our sketch, see Fig. 2, an angle $\beta = \angle BAN$, the knowledge of which would immediately lead to point N, and segment MB, the diagonal of trapezoid $ABNM$ and at the same time the leg of the isosceles triangle $\triangle MAB$, which has two sides $|AB|$ and $|MB|$ of equal length due to the fact that both given arcs have the same radii.

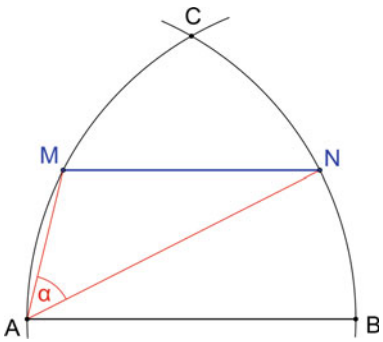


Fig. 1 Problem 36: Determine the line segment MN; $MN \parallel AB$, for a given angle α

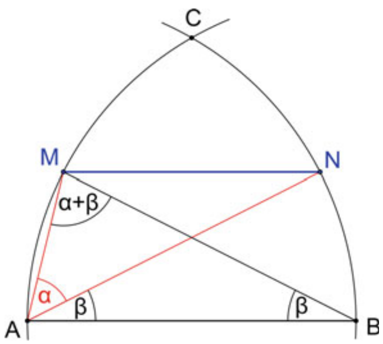


Fig. 2 Problem 36: Elements β and MB added for solution, $\triangle MAB$ is an isosceles triangle

Following Fig. 2, we express β in terms of α as follows. Since it applies to the interior angles of $\triangle MAB$ that $2\alpha + 3\beta = 180^\circ$, angle β can be written as

$$\beta = 60^\circ - 2\frac{\alpha}{3}. \quad (1)$$

Therefore, obtaining angle β, which leads to solving the problem, is subject to the trisection of an angle, namely the given angle α, a task that is not solvable by using just a straightedge and compass.

Trisecting an angle together with squaring the circle and doubling the cube are three classical problems of Greek mathematics that were proved to be impossible constructions using just a straightedge and compass. The impossibility of this so-called Euclidean construction of a trisection of an angle was proved by French mathematician Pierre Laurent Wantzel in 1873. For more information, see Impossible constructions (2020), Ostermann and Wanner (2012).

Three solutions to this problem, leading to the trisection of an angle, all in a similar manner to the one above, sent by students of upper secondary schools, were published in the last issue of the journal volume Vaňaus (1902). All three authors were aware that the solution is not constructable using only a straightedge and compass. One of them offered to complete the solution analytically, converting it into the problem of the intersection of conic sections, namely the circle and hyperbola, as recorded in Vaňaus (1902). Obviously, the upper secondary school students at that time were familiar with the non-Euclidean techniques of the trisection of the angle, among others using curves called 'trisectrix', namely, for example, the trisectrix of Maclaurin, Trisectrix of Maclaurin (2020), see Fig. 3, named after Scottish mathematician Colin Maclaurin

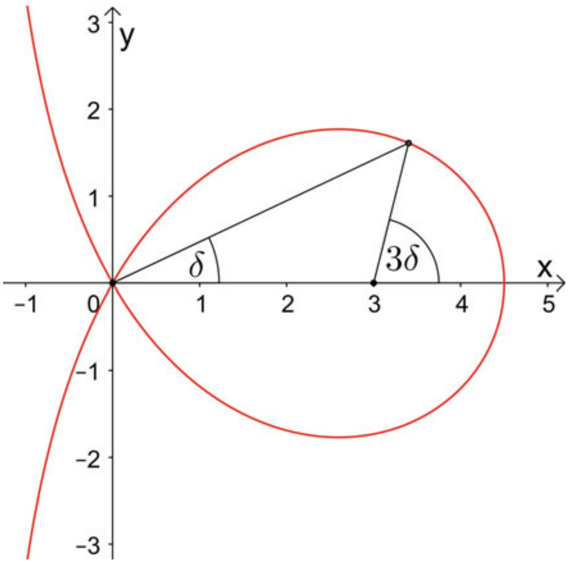


Fig. 3 Trisectrix of Maclaurin given by the Cartesian equation $2x(x^2 + y^2) = a(3x^2 - y^2)$, for $a = 3$. Processed in GeoGebra

(1698–1746). A list of curves that can be used as a trisectrix, i.e. as an additional tool with compass and ruler to trisect an arbitrary angle, can be found at Wikipedia, Trisectrix (2020). An angle can also be trisected using other non-Euclidean methods; see Angle trisection (2020) and Ostermann and Wanner (2012).

4 Problem 36 from the Contemporary Perspective

The possible use of the current free software in the field of Problem 36 will be presented in this section through the two software mentioned in the introduction, GeoGebra and OK Geometry. Our ambition is simply to share our specific findings and experience with the reader. We do not aim to provide a comparative evaluation of the software used and do not claim that this is the only possible way to solve the discussed problems with the currently available software.

4.1 GeoGebra

We assigned Problem No. 36, as a problem for volunteers, to students of the first year of the study of mathematics teaching at lower secondary school. Like the authors of the solutions published in the journal, in 1902, most of the current solvers arrived at a solution corresponding to (1). Yet there was a difference. Contemporary students are not as familiar with the non-Euclidean ways of trisecting an angle as their peers from the early twentieth century. On the contrary, they are well acquainted with the available mathematical software, which was clearly reflected in their approach to the solution of the problem. Once they found relation (1), they used GeoGebra to construct a solution based on the numerically calculated trisection of a given angle. GeoGebra, thus, served primarily as an environment for creating a dynamic model of a numerically calculated solution to a given problem, one such model being shown in Fig. 4.

Another approach to the use of GeoGebra to solving the problem, which we could identify among the students' solutions, went to the essence of a dynamic geometry system. Its author employed the dynamic features of GeoGebra to try to find a solution by manipulating the construction; see Fig. 5. She created a movable transversal MN between given arcs, $M \in \overset{\frown}{CA}$, $N \in \overset{\frown}{BC}$, visible from A at a given angle α. Moving M the midpoint S of MN draws a curve, the locus of point S. The intersection of this curve with the axis of symmetry of the line segment AB determines the position of MN we are looking for to solve the problem. This dynamic investigation of the nature of the locus curve gives rise to the question of which curve it is. Can we determine its equation? Yes, using GeoGebra CAS, or any other suitable computer algebra system, it is possible, without any special knowledge of differential geometry of curves.

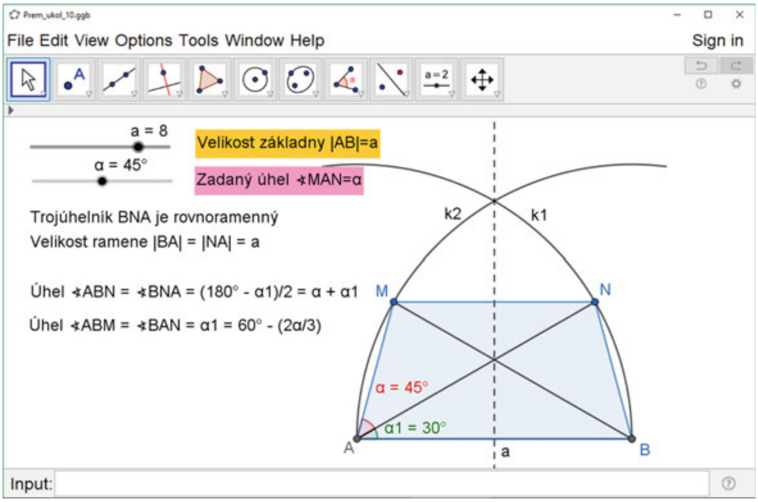
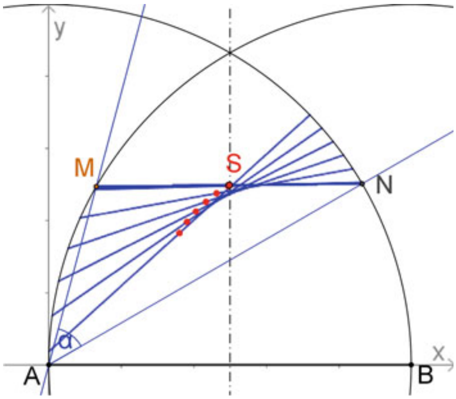


Fig. 4 Problem 36: Student's solution, a dynamic model created in GeoGebra

Fig. 5 Problem 36: Dynamic investigation of the solution. Processed in GeoGebra



First, we place the given construction into the coordinate system appropriately so that $A[0, 0]$, $B[r, 0]$, $M[m_1, m_2]$, $N[n_1, n_2]$, $k = \tan\beta$ and $a = \tan\alpha$, see Fig. 6, and express its configuration by symbolic equations $e_1, e_2, \ldots, e_6$ as follows: From the right triangle with hypotenuse AM and an internal angle $\alpha + \beta$, we get the first equation $e_1 : \tan(\alpha + \beta) = \dfrac{m_2}{m_1}$, which can be written as

$$e_1 : (a + k)m_1 - (1 - ak)m_2 = 0, \tag{2}$$

where $k = \tan\beta$ and $a = \tan\alpha$. Analogously, from a right triangle with the hypotenuse AN and an internal angle β, we get the second equation

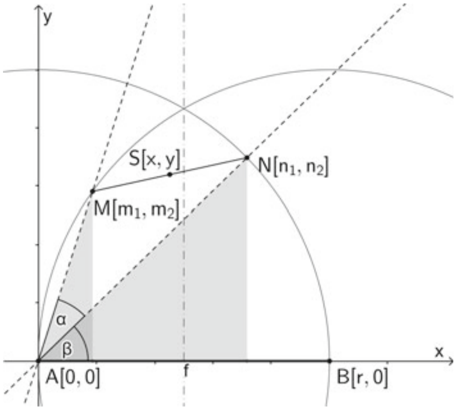


Fig. 6 Problem 36: Placement in the coordinate system

$$e_2 : kn_1 - n_2 = 0. \tag{3}$$

The other two equations e_3 and e_4 are based on the fact that $S[x, y]$ is the midpoint of the line segment MN:

$$e_3 : m_1 + n_1 - 2x = 0, \tag{4}$$

$$e_4 : m_2 + n_2 - 2y = 0. \tag{5}$$

The last two equations reflect the fact that M and N are the points at circles $k(B; r)$ and $l(A; r)$, respectively, where $r = |AB|$:

$$e_5 : m_1^2 - 2m_1 r + m_2^2 = 0, \tag{6}$$

$$e_6 : n_1^2 + n_2^2 - r^2 = 0. \tag{7}$$

The notation of these equations and their further processing in the *CAS* environment are shown in Fig. 7. For equations, see lines 1 to 6. To get the general algebraic polynomial representation of the locus curve, we use the `Eliminate` command, based on the method of the Groebner bases, Kovács (2017), Hašek (2019). Part of its result can be seen on line 7 of the *CAS* view. The complete resulting algebraic equation of the locus curve is as follows:

$$\begin{aligned}
&16x^6a^2 - 32x^5ra^2 + 8x^4r^2a^2 + 16x^3r^3a^2 - 7x^2r^4a^2 - 2xr^5a^2 + r^6a^2 \\
&\quad + 48x^4y^2a^2 - 64x^3ry^2a^2 + 16xr^3y^2a^2 - 3r^4y^2a^2 + 48x^2y^4a^2 \\
&- 32xry^4a^2 - 8r^2y^4a^2 + 16y^6a^2 + 8x^2r^3ya - 2r^5ya + 8r^3y^3a \\
&\quad + 16x^6 - 32x^5r + 8x^4r^2 + 8x^3r^3 - 3x^2r^4 + 48x^4y^2 - 64x^3ry^2 \\
&\qquad + 8xr^3y^2 + r^4y^2 + 48x^2y^4 - 32xry^4 - 8r^2y^4 + 16y^6 = 0.
\end{aligned} \tag{8}$$

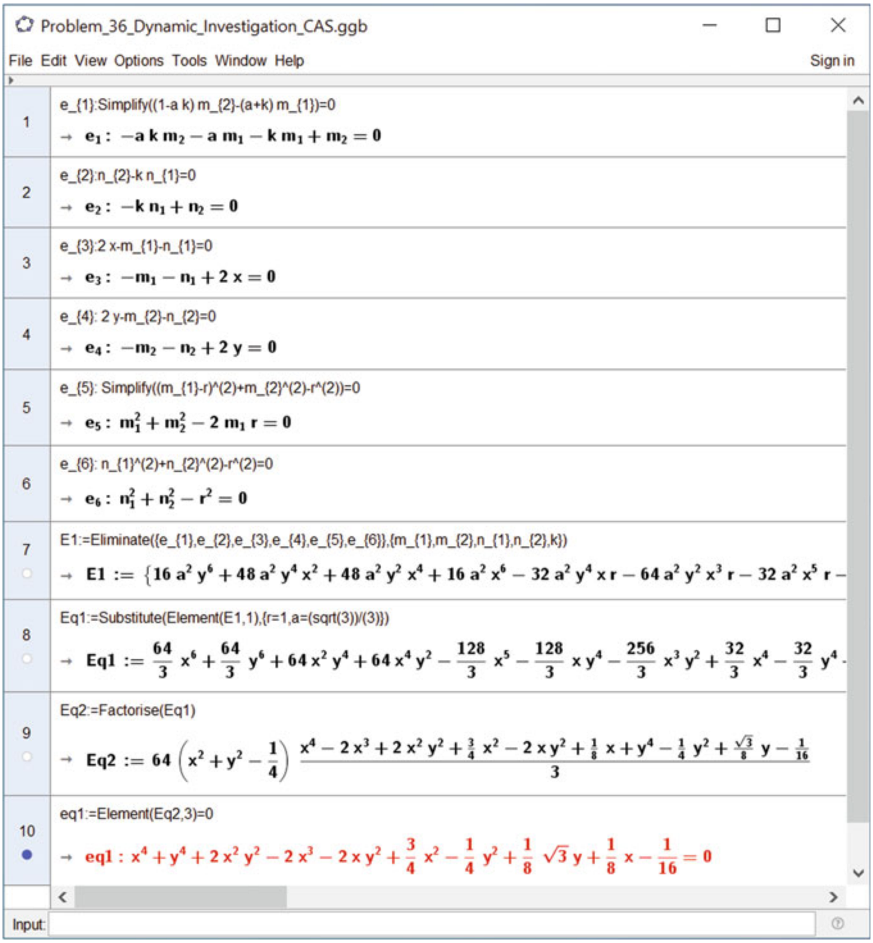


Fig. 7 Equation of the locus curve of S derived in GeoGebra CAS

Equation (8), thus, completes the phase of the symbolic solution of the problem. Now, in order to plot a particular curve given by (8) in the *Graphics View* of GeoGebra, we have to substitute some specific values of its parameters into this equation, i.e. the radius r of arcs and the angle α, precisely, the tangent of the angle α as the value of a; see line 8 in Fig. 7.

Let us use the values $r = 1$ and $a = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$. Factorising the resulting polynomial using the `Factorise` command, we get two curves, a circle drawn by S when M coincides with A and the curve that interests us, which appears to be the limacon, Limacon (2020), plotted in the Graphics view of GeoGebra; see Fig. 8.

The solutions mentioned so far have always been based on an idea. But what if we have no idea? Can any software assist us in such a way that it gives us an impetus to

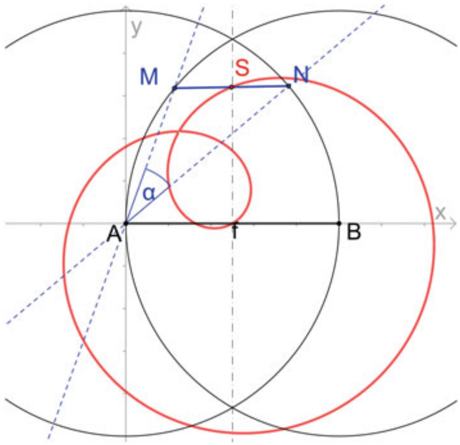


Fig. 8 Problem 36: Limacon, the locus of midpoints of MN for $\alpha = \frac{\pi}{6}$. Processed in GeoGebra

start a solution? And, moreover, can such software reveal to us a hitherto unknown solution? In the next section, we will show how OK Geometry can serve us in this way.

4.2 OK Geometry

OK Geometry is a dynamic geometry software that has a unique ability to analyse a dynamic geometric construction, either created directly in it or imported from another DGS, and to provide a list of properties of this construction, not proved, but determined with a high degree of probability. OK Geometry was conceived by Zlatan Magajna and is available free of charge from OK Geometry (2020), where an interested user can find all the necessary information on its use and functionality.

By drawing just a sketch of the assignment of Problem 36 and letting OK Geometry analyse it, a user receives a number of properties that with high probability pay for this geometric construction. Focussing only on those of them that relate to the stated task, she or he almost certainly obtains a base for developing some ideas on the problem's solution, some of which may be not entirely obvious even to an experienced solver. A small portion of the result of such an analysis is shown in Fig. 9. The software indicates that the sizes of angles $\angle CAM$ and $\angle NAM$, where $|\angle NAM|$ is the given acute angle, see α in Fig. 1, are in the ratio 1 : 3. If we manage to prove this hypothesis, we can design a new way of constructing the segment MN according to Problem 36 assignment, of course, again based on the trisection of a given angle. Having the line AC determined with the fixed points A and C, we simply find the point M, so that $|\angle CAM| = \frac{1}{3}\alpha$. So, is the relationship between angles $\angle CAM$ and $\angle NAM$, stated by OK Geometry, true? If it is not obvious from Fig. 9, we can 'ask' the software for more information on the relations of involved angles. Among

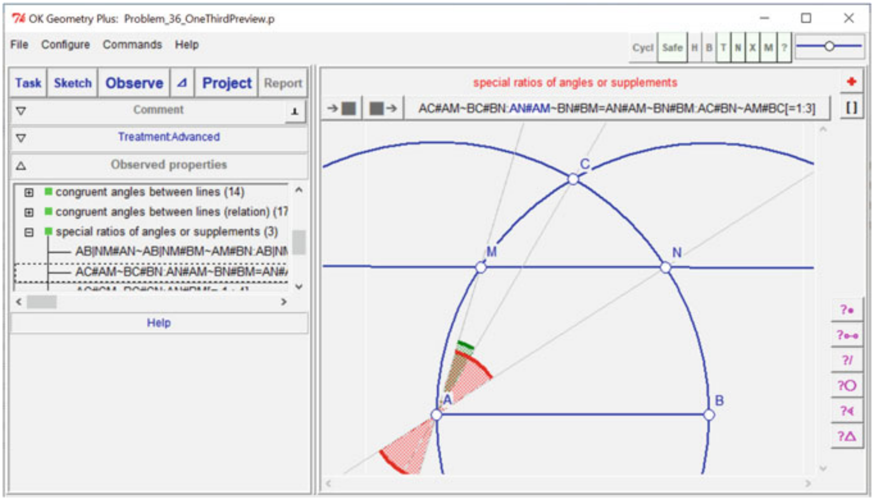


Fig. 9 Problem 36: Analysis using OK Geometry; $|\angle NAM| = 3|\angle CAM|$

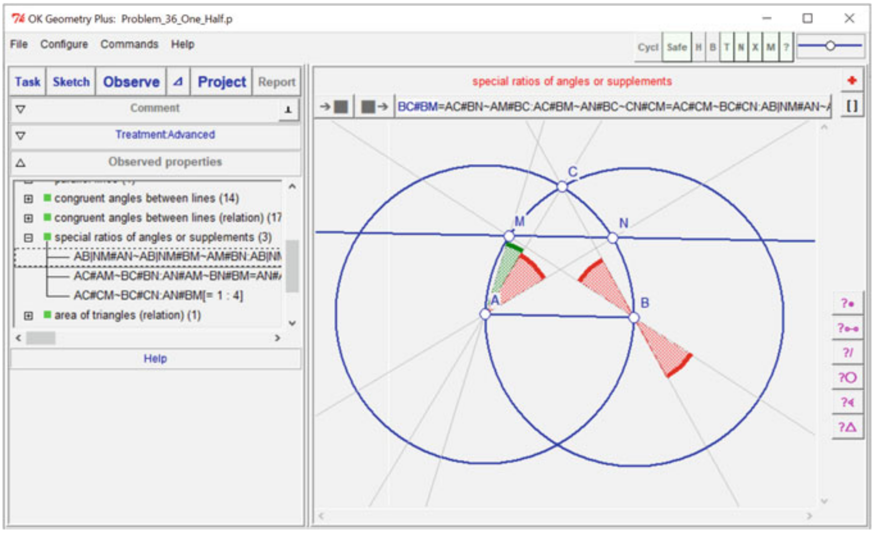


Fig. 10 Problem 36: Analysis using OK Geometry; $|\angle NAC| = 2|\angle CAM|$

them, we then find the key to the proof, 1 : 2 ratio of measures of angles $\angle CAM$ and $\angle NAC$, where $\angle NAC \cup \angle CAM = \angle NAM$; see Fig. 10. Both relationships between angles are, therefore, equivalent, with the latter being a clear consequence of the relationship between the inscribed angle $\angle CAM$ and the central angle $\angle CBM$, where $|\angle CBM| = 2|\angle CAM|$. Consequently, due to $|\angle NAC| = |\angle CBM|$, the relation $|\angle NAC| = 2|\angle CAM|$ pays. Therefore, the former, $|\angle NAM| = 3|\angle CAM|$, also holds. We have shown how OK Geometry can assist us, both in the recognition of some geometric properties and in indicating the prospective way of proving them.

5 Vaňaus' Trisectrix

In Sect. 4, the part dealing with the 'dynamic geometry' approach to solving Problem No. 36, we used the dynamic geometry and computer algebra features of GeoGebra to create a dynamic model of the respective geometric construction and to derive an equation of the corresponding locus curve.

Here, in the section devoted to the first of two Vaňaus' publications covered in this text, a paper *Trisektorie* from 1881, Vaňaus (1881), we will apply this approach again and complete it with the creation of a dynamic geometric model of Vaňaus' trisector, a mechanical linkage implementing his method of trisection.

Let's move back to 1902 to complete the story of solving Problem No. 36. In his comment to the solutions of the three students, published in Vaňaus (1902), Vaňaus recommended his 1881 paper in which he introduced a method of doing a trisection using the cubic curve shown in Fig. 11. This cubic curve, currently known as the *oblique strophoid*, Gibson (1998), Lockwood (2007), Strophoid (2020), is presented by Vaňaus as follows.

> The locus of points M for B moving along the line l, a secant to the circle c, so that $|MD| = |DB|$, where D is the intersection of the line OB with c.

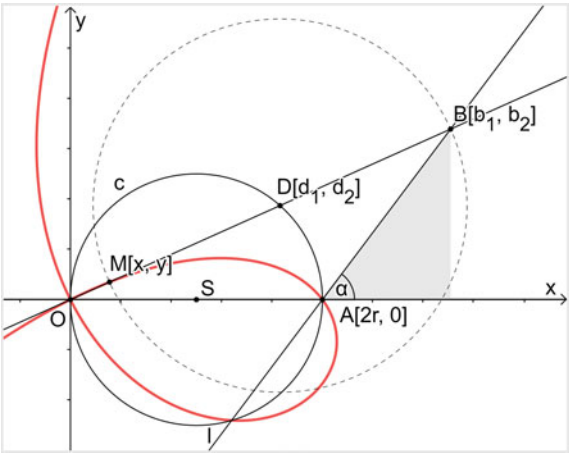


Fig. 11 Vaňaus' trisectrix, the *oblique strophoid*, for $r = 1$ and $a = 3$. Processed in GeoGebra

He derives the Cartesian equation of this curve

$$a(y^2(2r + x) - x^2(2r - x)) = y(y^2 + x^2 - 4rx), \quad (9)$$

where r is the radius of the circle c and a is the slope of the line l (i.e. a is the tangent of the angle of incline α of the line l), and describes a simple way of using it to trisect an angle, namely the angle u in Fig. 15. To learn which position of the curve with respect to the Cartesian coordinate system corresponds to (9), see Fig. 11.

Analysing this curve as a locus curve with students, GeoGebra allows us to apply different approaches to the derivation of its algebraic equation. On the one hand, we let the software do it automatically, from the perspective of a user in the hidden 'black box' mode, and simply ask it to derive the equation based on the geometric construction created in the 'Graphics' view. To do so, we apply the `LocusEquation` command, which utilises the algorithms of the automated theorem proving to compute an equation of the locus of a given point; see Fig. 12. On the other hand, we can do it manually, using the 'CAS' of GeoGebra, with its functions and tools, as the environment to control the process. In Fig. 13, the use of the `Eliminate` command to derive the locus equation in a manner analogous to the derivation of the limacon equation in Sect. 4 is shown.

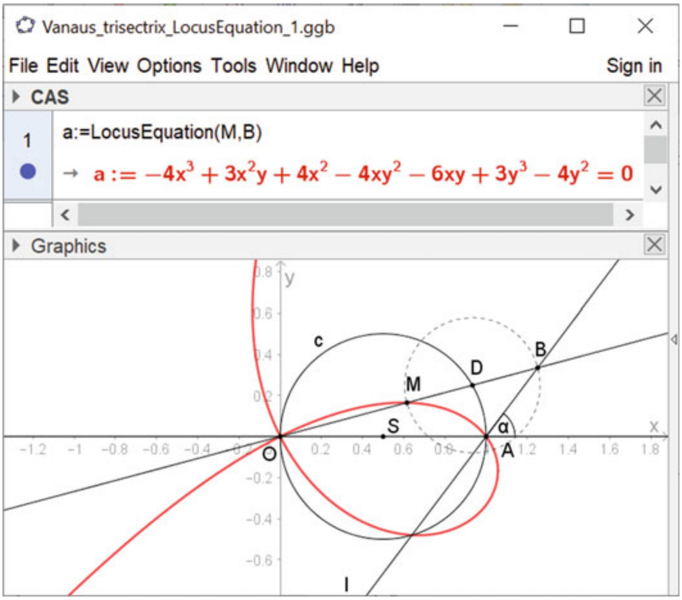


Fig. 12 Derivation of the algebraic equation of Vaňaus' trisectrix using the `LocusEquation` command of GeoGebra

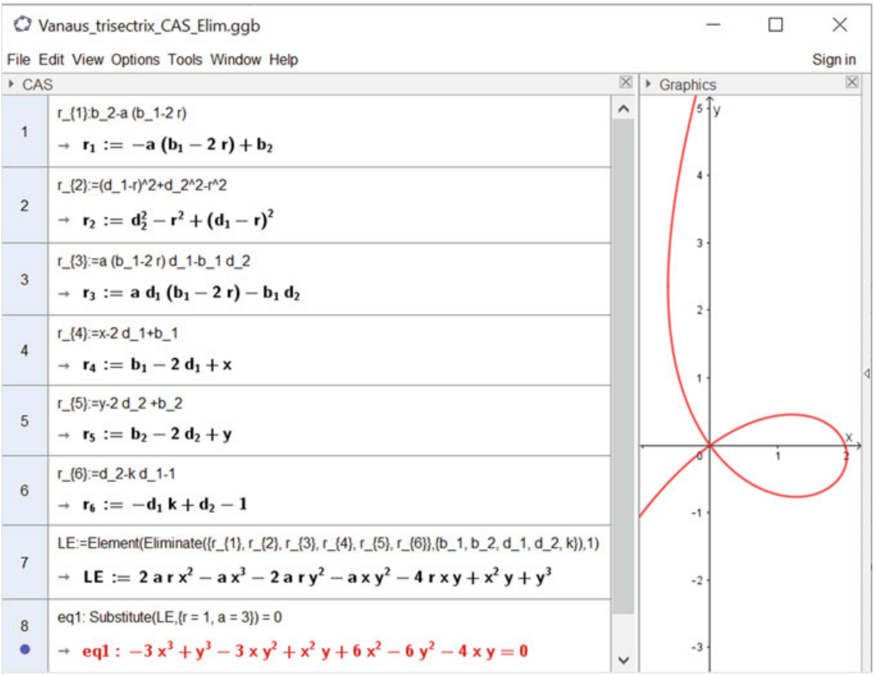


Fig. 13 Derivation of the algebraic equation of Vaňaus' trisectrix using the `Eliminate` command of GeoGebra

To get acquainted with the meaning of the equations $r_1, r_2, \ldots r_6$ that we used to symbolic derivation of the locus curve equation, see Fig. 11. They are derived as follows: From the right triangle with hypotenuse AB and an internal angle α, we get the first equation $r_1 : \tan(\alpha) = \dfrac{b_2}{b_1 - 2r}$, which can be written as

$$r_1 : (b_1 - 2r)a - b_2 = 0, \tag{10}$$

where $a = \tan\alpha$. The fact that the point $D[d_1, d_2]$ lies on a circle c with centre $S[r, 0]$ and radius r led to the second equation

$$r_2 : (d_1 - r)^2 + d_2^2 - r^2 = 0. \tag{11}$$

The third equation reflects the condition of collinearity of points O, D and B, i.e. $\frac{d_2}{d_1} = \frac{b_2}{b_1}$, where the expression $a(b_1 - 2r)$ is substituted for b_2:

$$r_3 : ad_1(b_1 - 2r) - b_1 d_2 = 0. \tag{12}$$

The other two equations r_4 and r_5 are based on the fact that $D[d_1, d_2]$ is the centre of the line segment MB, with the coordinates $M[x, y]$, $B[b_1, b_2]$ of its endpoints:

$$r_4: \ b_1 - 2d_1 + x = 0, \tag{13}$$
$$r_5: \ b_2 - 2d_2 + y = 0. \tag{14}$$

The last equation represents a non-degenerate condition preventing the case $D[0, 0]$, which would lead to the curve's asymptote instead of the curve itself:

$$r_6: \ -d_1 k + d_2 - 1 = 0, \tag{15}$$

where k is the real parameter.

As already mentioned, Vaňaus identified this curve as a trisectrix, i.e. the curve that can be used, together with compass and ruler, to trisect an arbitrary angle. Specifically, he uses the property of equidistance among the points of the given circle, its secant and the curve, respectively. For a detailed illustration, see Fig. 14. Let us remember that B is the mover in Vaňaus' definition of trisectrix; moving B the point M draws the curve. Then, decisive for the trisection is a configuration of B and consequently M, where M lies on a circle centred at O passing through A. Only in this position, it holds that $|\angle HOM| = \frac{1}{3}|\angle HOA|$. To prove it, we will deal with Fig. 15, where the trisectrix is rotated, in comparison with the position used so far, so that the ray OH of the angle $\angle HOA$, the trisection of which is the subject of our interest, is horizontal. Let us focus on the triangle $\triangle OBA$ and its exterior angle $\angle MBA$, the measure of which is $|\angle MBA| = \alpha + 2x$. Due to the exterior angle theorem, which states that an exterior angle of a triangle is equal to the sum of the opposite interior angles,

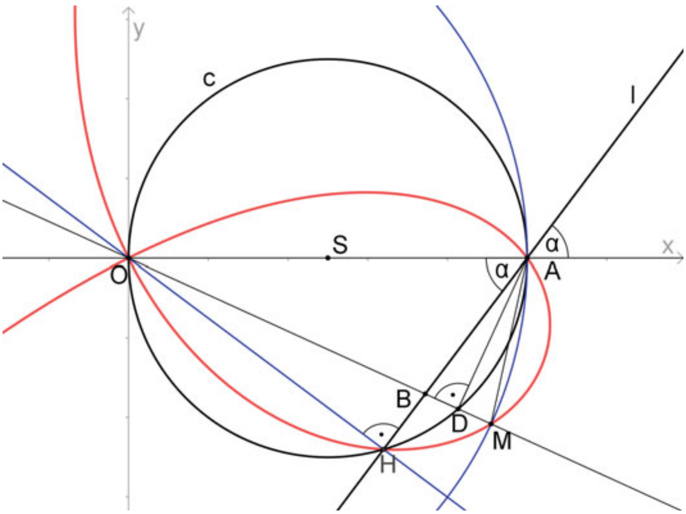


Fig. 14 Vaňaus' trisectrix; determining properties for trisection. Processed in GeoGebra

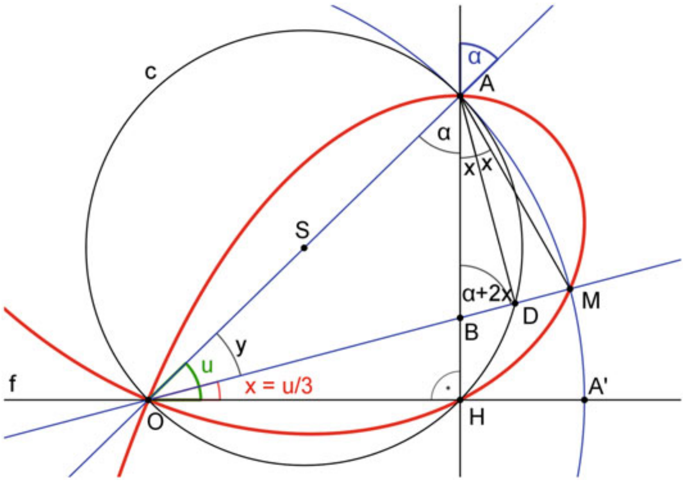
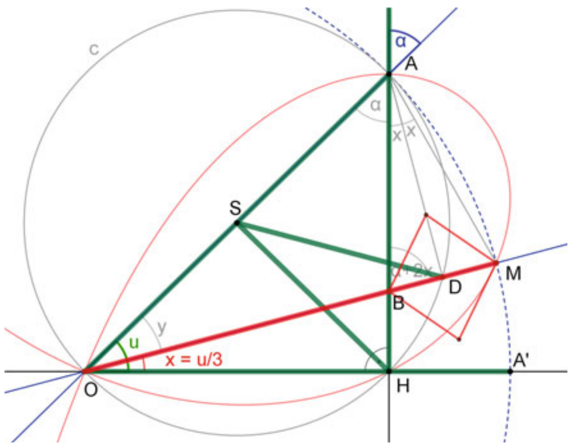


Fig. 15 Using the 'Vaňaus curve' to trisect an angle u

Fig. 16 Proposal of a possible design of Vaňaus' mechanism



we know that $\alpha + 2x = \alpha + y$. After simplification, we get $y = 2x$. Consequently, because $u = x + y$, it follows that $x = \frac{1}{3}u$.

In conclusion to his paper, Vaňaus (1881), Vaňaus mentions that he designed a quite simple mechanism to implement his method of trisection which he had made with a satisfactory result by a skilled mechanic. However, no such mechanism was found in his estate, so nobody knows what it looked like. A possible design of this mechanism is indicated in Fig. 16 using Vaňaus' trisection method in the background. Its dynamic model created in GeoGebra is available at Hašek (2020). The author is not entirely satisfied with this proposal as he believes that there could be a simpler mechanical implementation of Vaňaus' method. Of course, there are a number of other mechanisms for angle trisection; see, e.g. Soluzione (2021).

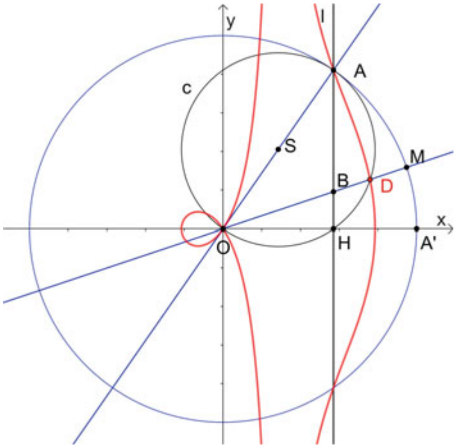


Fig. 17 Conchoid of Nicomedes; the locus of points D, the midpoints of MB, when M moves along the circle centred at O and passing through A

The use of DGS brought us another interesting revelation in this story of a method of trisection. When searching for the way of the mechanical trisection of an angle using Vaňaus' strophoid, a question appeared; what is the locus of points D, the midpoints of the segment BM, when M moves along the circle centred at O and passing through A; see Fig. 12. The resulting curve, shown in Fig. 17, is the conchoid of Nicomedes, Lawrence (2014), the Cartesian equation of which is $(x^2 + y^2)(x - a)^2 = b^2x^2$, where $a = \frac{1}{2}|OH|$ and $b = \frac{1}{2}|OA|$. Thus, applying the GeoGebra Tools, we have found a close relationship of Vaňaus' trisectrix to this conchoid, a curve which is well known for its use in trisecting an angle Lockwood (2007).

6 Conclusion

Through a real story from the history of the study and teaching of mathematics concerning the trisection of an angle and related problems, we have shown the properties of contemporary dynamic geometric software, such as the ability to immediately respond to user's demands, the ability to provide individual feedback tailored to the user's needs, equipment with the environment supporting creative approach to solve the given problem and to find new ways of doing it, the possibility of sharing ideas and approaches, among others. This all predetermines this software for its use in contemporary mathematics teaching. Its potential to be implemented into an educational environment controlled by artificial intelligence is obvious and undoubtedly calls for detailed research.

Theoretical bases of AI implementation in mathematics education are stated in Balacheff (1993). We are convinced that now all the necessary components of this implementation, whether of a technical, software or didactic nature, are sufficiently mature to assemble them all together for the application of AI to assist a teacher in supporting and streamlining the school education of pupils according to their individual needs and to focus on their skills, knowledge and demands.

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