Linear and Geometrically Nonlinear Frequencies and Mode Shapes of Point Supported Rectangular Plates at the Free Corner Whose Opposite Edges Are Simply Supported

Ahmed Babahammou(✉) and Rhali Benamar

EMI, Mohamed V University in Rabat, BP 765, Agdal, Rabat, Morocco

Abstract. The linear and geometrically nonlinear flexural vibration of simply supported simply supported free rectangular plates punctually supported at the free corner is investigated. First, the frequency parameters and mode shapes are calculated with the efficient Rayleigh-Ritz method (RRM). The RRM is used here to study the geometrically nonlinear vibrations occurring at large amplitudes of the plates examined. The test plate functions used are the products of beam functions with appropriate end conditions, i.e. simply supported-free beam functions, in each direction and the point support is modeled by a factious translational spring with a stiffness tending to infinity. The solutions obtained for various plate aspect ratios compare well with available solutions based on different approaches. The nonlinear vibrations have been then examined using spectral analysis and Hamilton's principle to determine the backbone curves of SSFF plates with various aspect ratios via the so-called the second formulation in order to determine the fundamental nonlinear frequency parameter and its mode shape.

Keywords: Linear vibration · Non-linear vibrations · Rayleigh-Ritz method · Benamar's method · Backbone curves

1 Introduction

Rectangular plates are commonly used as structural components in many engineering fields. The knowledge of their natural frequencies and mode shapes is necessary to determine their response under the working loads and to estimate properly the induced strains and stresses in order to make an optimal design. In spite of the very large number of research work devoted to this topic, due to the variety of edge conditions, there are still numerous situations which are not yet covered by the literature, particularly in the non-linear regime. Free vibration of rectangular plates punctually supported at a corner has been studied by numerous researches using various laborious methods, such as [1, 2], but yet the studies were restricted to linear vibration. The present work investigated the vibration of plates simply supported at two adjacent edges and free at the two other

T. Bouraoui et al. (Eds.): CoTuMe 2021, LNME, pp. 355–361, 2022.
https://doi.org/10.1007/978-3-030-86446-0_47

edges with a point support at the free corner, denoted as (SSFFRPSC). The Rayleigh-Ritz method (RRM) has been used to study the plate linear vibrations for various values of the aspect ratio. Benamar's method (BM) [3, 4] has then been to investigate the nonlinear vibration for large vibration amplitudes, leading to the plotted backbone curves.

2 General Formulation

The studied SSFFRPSC is plotted in Fig. 1. The transverse displacement function W of the current point P(x,y) is given by the following expression:

$$W(x, y, t) = \sum_{k=1}^{N} a_k w_k(x, y)\sin(\omega t) \tag{1}$$

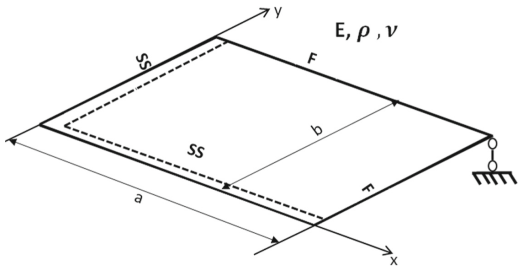

Fig. 1. Plates simply supported at the edges $x = y = 0$ and free at the edges $x = a$ and $y = b$ with a point support at the corner $x = a,\ y = b$ (SSFFRPSC)

a_k represents the contribution coefficient of test plate function w_k. This test functions w_k used, for $k = 1, 2\ldots n^2$, are obtained as products of n linear simply supported-free beam mode X_i in the x- and y- directions:

$$w_k(x, y) = w_{ij}(x, y) = X_i(x)X_j(y), k = n(i-1) + j \tag{2}$$

With:

$$X_i(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) + C_3 \sinh(\lambda x) + C_4 \cosh(\lambda x) \tag{3}$$

n being the number of trial beam functions used. The simply supported-free beam frequency parameters $\lambda = \frac{L}{\pi}\sqrt[4]{\frac{\rho\omega_b^2}{EI}}$, listed in Table 1, are analytical solution of the differential equation which governs the beam vibration [5].

The nonlinear vibrations have been described in many papers by Benamar and al [6–8] by a set of nonlinear algebraic equation involving the rigidity and the mass matrices $[K]$ and $[M]$ and a fourth order tensor $[B(\{A\})]$ expressing the structure nonlinearity. A similar study is applied in the present work leads to:

$$[K]\{A\} - \Omega^2[M] \cdot \{A\} + \frac{2}{3}[B(\{A\})]\{A\} = 0 \tag{4}$$

Table 1. Lowest ten frequency parameters $\lambda = \frac{L}{\pi}\sqrt[4]{\frac{\rho\omega_b^2}{EI}}$ of a simply supported-free beam.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.499	1.2499	2.25	3.25	4.25	5.25	6.25	7.25	8.25	9.25

$\{A\}$ being the vector of the trial function contribution coefficients. The expressions for $[M]$ and $[B]$ are identical to those given in [6] but the parameters are calculated here using the functions defined above which satisfy the present edge conditions. On the other hand, the strain energy due to the point support, modeled by a translational spring of rigidity K_{Sp}, is given by:

$$V_{Spr} = \frac{K_{Sp}}{2}W^2(a,b) = \frac{1}{2}a_i a_j K_{ij}^{Sp}\sin(\omega t) = \frac{1}{2}a_i a_j K_{Sp} w_i(a,b)\cdot w_j(a,b) \tag{5}$$

K_{ij}^{Sp} is the rigidity term associated to the elastic energy stored in the spring to be included in the tensor $[K]$. For very large values of K_{Sp}, $W(a,b)$ tends to zero.

3 Numerical Results

3.1 Numerical Results for Linear Vibration

The free linear vibration of the point supported plate is found by eliminating the nonlinear tensor B({A}). The Eq. (4) becomes:

$$[K]\{A\} - \Omega^2[M]\cdot\{A\} = 0 \tag{6}$$

The results obtained from solution of Eq. (6) are summarized giving The lowest seven frequency parameters of SSFFRPSC are listed in Table 2 for several values of $\alpha = \frac{a}{b}$. These results are compared with those given by Li in [2]. The differences remain less than 0.6%. The lowest four mode shapes are shown in Fig. 2 for $\alpha = 0.6$. The cross-sections at the plate diagonal $\frac{x}{a} = \frac{y}{b}$ are given in Fig. 3, showing a zero displacement at the simply supported corner.

3.2 Numerical Results for Nonlinear Vibration

In order to investigate the nonlinear vibrations of the studied plate the fourth order tensor [B({A})] is not neglected in Eq. (4). Benamar's method is applied to find the amplitude dependent nonlinear fundamental frequency parameters for plates having different values of the plate aspect ratio $\alpha = \frac{a}{b}$. Equation (4), allows to calculate the frequency parameters: both sides of the equation are pre-multiplied by $\{A\}^T$ which becomes [6]:

$$\Omega_{NL}^2 = \frac{a_i a_j m_{ij} + 1.5 a_i a_j a_k a_l b_{ijkl}}{a_i a_j k_{ij}}. \tag{7}$$

Table 2. Lowest seven frequency parameters $\Omega_i = \omega_i \sqrt{\frac{\rho H}{D}} a^2$ of a SSFFRPSC for several values of $\alpha = a/b$

α		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
1	Present	9.6387	17.3790	30.6260	43.8479	51.2420	64.3751	73.0867
	Li [2]	9.6079	17.3160	30.5960	43.6520	51.0350	64.3440	72.9660
0.5	Present	3.9943	10.2852	15.0008	19.8891	25.4359	30.7094	39.3289
	Li [2]	3.9872	10.2440	14.9390	19.8740	25.3470	30.6630	39.1590
1/3	Present	2.3999	5.9854	10.6275	14.3218	17.3415	20.2735	22.9328
	Li [2]	2.3976	5.9694	10.5780	14.2560	17.3300	20.2070	22.9000
0.25	Present	1.7204	4.0177	7.1548	10.8415	13.9386	16.4236	18.0810
	Li [2]	1.7193	4.0100	7.1301	10.7800	13.8660	16.4130	18.0200
0.2	Present	1.3453	3.0058	5.1828	7.9251	10.9966	13.6731	16.0045
	Li [2]	1.3447	3.0016	5.1687	7.8893	10.9180	13.5900	15.9900

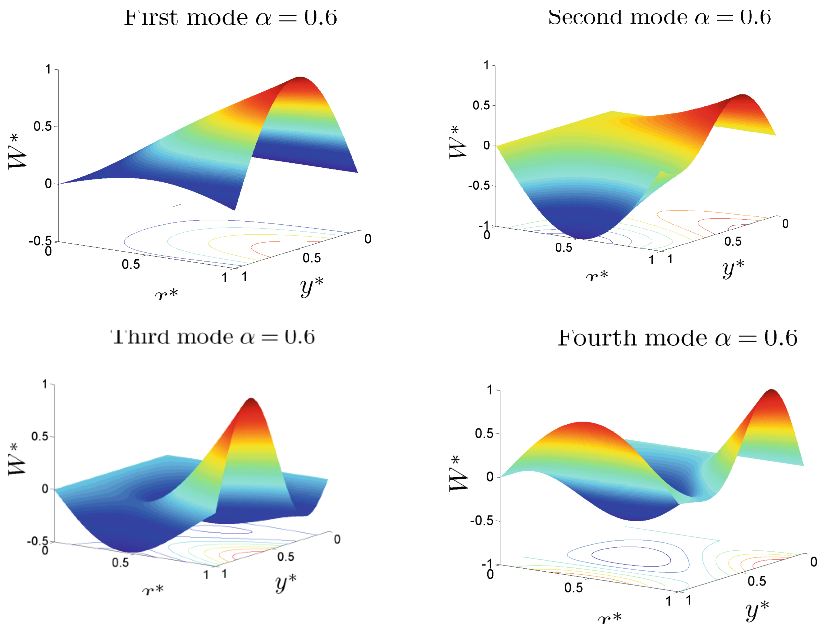

Fig. 2. Lowest four modes of SSFFRPSC for $\alpha = 0.6$

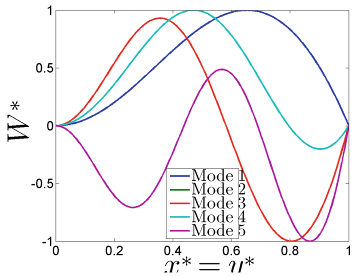

Fig. 3. Cross-sections of the four modes at the plate diagonal ($x/a = y/b$). $\alpha = 0.6$.

The so-called second formulation (SF) was used in this work to solve Eq. (4) and find the a_i's, i = 2 to N, for various values of the first component a_{11}, and consequently the amplitude dependent nonlinear mode shapes. The bases of this approximate method, developed by El Kadiri–Benamar, are detailed in Ref [6] for the nonlinear free vibration problem formulated in the MFB (see Appendix B in Ref [6]), leads to $(m \times n) - 1$ equations, the i^{th} equation is:

$$-\Omega^2 \ \bar{a}_s \ \bar{m}_{si} + \bar{a}_s \ \bar{k}_{si} + 1.5 \ \bar{a}_s \ \bar{a}_u \ \bar{a}_v \ \bar{b}_{suvi} = 0, i = 2, 3, 4, \ldots n \times m \qquad (8)$$

Examples of numerical results obtained by the SF are listed in Table 3 corresponding to a SSFFRPC for$\alpha = 0.6$, and in Table 4 for $\alpha = 0.3, 0.6, 0.9$ and for several values of a_{11} showing the nonlinear amplitude dependence of the contribution coefficient. Figure 4 plots the backbone curves of SSFFR by the second formulation for several values of the aspect ratio indicating a nonlinear behavior of the hardening type. The aspect ratio influence on the fundamental mode shape is shown in Fig. 5. It appears that the hardening effect is more accentuated with increasing the aspect ratioα. Figure 6 gives the corresponding normalized cross sections of the amplitude dependent nonlinear first mode, for increasing values of the first componenta_{11}, at the plate diagonal $\frac{y}{b} = \frac{x}{a}$ and at the plate middle line $y = \frac{b}{2}$, showing a nonlinear increase of curvature at the simply supporter corner with increasing the vibration amplitude.

Table 3. $\Omega_{NL}/.\Omega$, $W_{max}$and a_{ij} as functions of a_{11} for $\alpha = 0.6$ ($\Omega = 5.10929$)

W_{Max}	Ω_{NL}/Ω	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}
0.1404	1.0090	0.05	3.46E−02	−1.95E−03	3.68E−04	4.31E−03	−6.32E−04
0.2775	1.0354	0.10	6.82E−02	−3.96E−03	6.69E−04	9.44E−03	−1.21E−03
0.5334	1.1316	0.20	1.30E−01	−8.43E−03	9.18E−04	2.42E−02	−2.11E−03
0.7637	1.2691	0.30	1.84E−01	−1.38E−02	7.00E−04	4.51E−02	−2.67E−03
0.9753	1.4324	0.40	2.30E−01	−2.01E−02	1.82E−04	7.07E−02	−3.07E−03

a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}	a_{44}
3.80e−4	−1.53e−4	−0.90e−4	2.13e−4	−0.73e−5	4.07e−5	−3.10e−6	−0.84e−5	2.63e−5	−0.57e−5
8.50e−4	−2.94e-4	−4.62e−4	4.26e−4	−1.70e−4	8.61e−5	1.32e−5	−1.77e−4	5.96e−5	−3.40e−5
2.33e−3	−5.16e-4	−1.43e−3	8.41e−4	−4.37e−4	2.14e−4	1.44e−4	−3.50e−4	1.66e−4	−8.95e−5
4.72e−3	−6.66e−4	−2.95e−3	1.22e−3	−8.5e−4	4.32e−4	3.89e−4	−5.11e−4	3.33e−4	−1.89e−4
8.3e−3	−7.83e−4	−4.77e−3	1.53e−3	−1.22e−3	7.86e−4	6.75e−4	−6.48e−4	5.55e−4	−3.53e−4

Table 4. Ω_{NL}/Ω and W_{max} as functions of a_{11} for several values of α

a_{11}	$\alpha = 0.3$ ($\Omega = 2.1218$)		$\alpha = 0.6$ ($\Omega = 5.10929$)		$\alpha = 0.9$ ($\Omega = 8.6366$)	
	W_{max}	Ω_{NL}/Ω	W_{max}	Ω_{NL}/Ω	W_{max}	Ω_{NL}/Ω
0.05	0.1549	1.0238	0.1404	1.0090	0.1008	1.0052
0.10	0.3066	1.0916	0.2775	1.0354	0.2004	1.0206
0.20	0.5910	1.3238	0.5334	1.1316	0.3923	1.0797
0.30	0.8435	1.6280	0.7637	1.2691	0.5748	1.1708
0.40	1.0673	1.9618	0.9753	1.4324	0.7513	1.2866

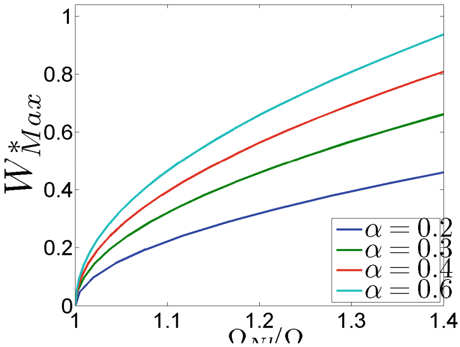

Fig. 4. Backbone curves of SSFFRPSC by the second formulation with various aspect ratios

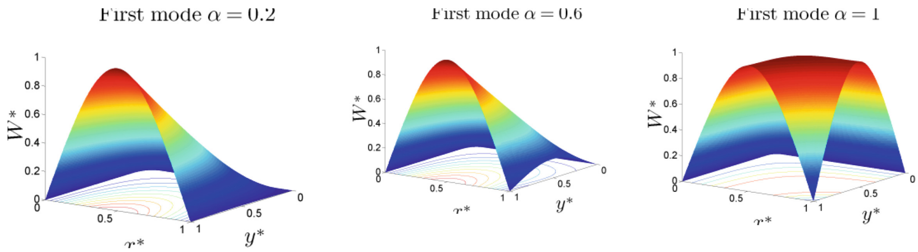

Fig. 5. Fundamental mode shape of SSFFRPSC with various aspect ratios $\alpha = 0.2, 0.6, 1$.

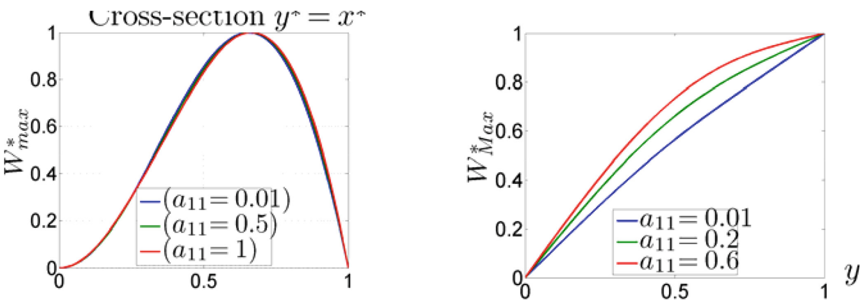

Fig. 6: Normalized cross sections of the amplitude dependent nonlinear first mode shape of SSFFRPSC for increasing values of a_{11}; (a) at the diagonal $\frac{y}{b} = \frac{x}{a}$,(b) at middle plate $y = \frac{b}{2}$

4 Conclusion

Linear frequencies and mode shapes of point supported rectangular plates at the free corner whose opposite edges are simply supported has been studied using the Rayleigh-Ritz method with appropriate plate functions. The efficiency of the Rayleigh-Ritz method used in such plate is established by the accuracy of found results which are compared with the available bibliography. The results are given for several values of the aspect ratio. The mode shapes not only were given up to the fourth mode, but their normalized cross sections are plotted. The Benamar's method was applied to study the nonlinear vibrations. The nonlinear frequency parameter Ω_{NL} has been calculated by the second formulation developed by Benamar and al for various values of the maximum displacement. The Backbone has been plotted for many values of the plate aspect ratio $\alpha = a/b$. The first nonlinear mode shape which depends on the amplitude has been determined and plotted for several values of first component a_{11} of the vector of the trial function contribution coefficients $\{A\}$.

References

1. Li, R., Wang, B., Li, P.: Hamiltonian system-based benchmark bending solutions of rectangular thin plates with a corner point-supported. Int. J. Mech. Sci. **85**, 212–218 (2014)
2. Li, R., Wang, B., Li, G., Jiahui, D., An, X.: Analytic free vibration solutions of rectangular thin plates point-supported at a corner. Int. J. Mech. Sci. **96**, 199–205 (2015)
3. Lewandowski, R.: Free vibration of structures with cubic non-linearity-remarks on amplitude equation and Rayleigh quotient. Comput. Methods Appl. Mech. Eng. **192**(13–14), 1681–1709 (2003)
4. Eddanguir, A., Beidouri, Z., Benamar, R.: Nonlinear transverse steady-state periodic forced vibration of 2-DOF discrete systems with cubic nonlinearities. Eur. J. Comput. Mech./Revue Européenne de Mécanique Numérique **20**(1–4), 143–166 (2011)
5. Igor, A.K., Olga, L.: Formulas structural dynamics (2001)
6. El Kadiri, M., Benamar, R.: Improvement of the semi-analytical method, for determining the geometrically non-linear response of thin straight structures: Part II first and second non-linear mode shapes of fully clamped rectangular plates. J. Sound Vibr. **257**(1), 19–62 (2002)
7. Benamar, R., Bennouna, M., White, R.G.: The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, part II: fully clamped rectangular isotropic plates. J. Sound Vibr. **164**(2), 295–316 (1993)
8. Babahammou, A., Benamar, R.: Geometrically non-linear free vibrations of simply supported rectangular plates connected to two distributions of rotational springs at two opposite edges. In: Advances in Materials, Mechanics, and Manufacturing, pp. 166–174. Springer (2020)