



Is Optimal Velocity Constant During Running?

Ryszard Maroński^(✉)

Institute of Aeronautics and Applied Mechanics, Warsaw University of Technology,
ul. Nowowiejska 24, 00-665 Warsaw, Poland
maron@meil.pw.edu.pl

Abstract. The minimum-time competitive run is reconsidered in the paper. The problem is formulated and solved in optimal control (calculus of variations). A non-classical method of Miele is applied – the method of extremization of line integrals by Green’s theorem. This method gives necessary and sufficient conditions of optimality. Two models of the energy conversions in competitor’s body are considered: the model of Keller and the model of Bonnans and Aftalion. It is shown that the optimal race may be broken into three phases: acceleration, cruise (along so-called singular arc), and final slowing down. The fundamental finding is that the optimal cruising velocity is constant for Keller’s model but it decreases for Bonnans and Aftalion’s model.

Keywords: Minimum-time running · Singular arc

1 Introduction

The paper is devoted to investigation of optimal strategies during competitive running applying optimal control (calculus of variations). A pioneering mathematical work is that of Keller [1, 2]. He applied calculus of variations. From his investigations it follows that the distance to be covered may be broken into three sections:

- 1) initial acceleration where competitor uses his maximal abilities,
- 2) cruise with constant velocity in the middle part of the distance along so-called singular arc,
- 3) negative kick at the end, where the runner’s velocity decreases.

For distances not longer than 291 m, where the energy reserves are not depleted, only the first section appears. On the other hand, observations of pacing strategies show that for 400 m and 800 m running events the races are always won by people who run the first half of the race faster than the second half. It is not true for shorter races, or for longer, where the second half of the race is faster. The constant-speed pacing strategy is observed for races of a mile or longer [3].

The questions may arise: 1) is the pacing strategy optimal during real races? or 2) are the models of running used in optimal control suitable for the process under consideration?

The models base on two ordinary differential equations. The first one derives from Newton's second law of motion. This equation is generally accepted. Different models of resistive force are used: a linear function of velocity or proportional to the square velocity. The second differential equation describes the energy conversions in runner's body and this is the area where different models may be employed. A progress has been made in this area the last years. Bonnans and Aftalion [4] propose more realistic model that relies on the assumption of variable oxygen uptake comparing with Keller's model of constant oxygen uptake. Aftalion [5] found the optimal solution for such a model along so-called singular arc where the optimal cruising velocity decreases with the time. The applied reasoning is very sophisticated, however. In this paper, the same problem for Bonnans and Aftalion's model is reconsidered using the method of Miele [6] – the method of extremization of line integrals by Green's theorem. The method gives a solution in the energy-velocity plane, it gives necessary and sufficient conditions of optimality including singular arcs of linear type, and it gives a clear graphical interpretation. It is easy to consider the inequality constraints imposed on the state variables and that may generate problems in the classical approach. The difficulty with application of the method is such that not always it is possible to perform the problem in the form required by the theory – as a line integral of two variables.

2 Mathematical Models

Many authors have considered the problem of constant velocity in sports events. Keller [1, 2] using calculus of variations proves that the runner's velocity is constant during distance running in the middle part of the race (cruise), omitting the starting phase and the finish. Maroński [7] shows that the optimal velocity is constant for cycling even if the slope angle of the trace varies with the distance. Such finding is not true for the race with variable wind heading. In the paper of Maroński and Samoraj [8] it is shown that for Keller's model of running, but for variable slope of the trace, the optimal velocity is constant during the cruise. As it is mentioned above, such results do not agree with observations of real events [3]. Bonnans and Aftalion [4] propose an improved model of metabolism where the oxygen uptake is not constant as in Keller's model, but it depends on the energy reserves staying at competitor's disposal. The current paper bases on this model.

Consider Keller's model of running. The racer is modelled as a particle of the mass m – his mass centre. The competitor moves on a linear horizontal track. The vertical displacements of his mass centre resulting from the cyclic nature of the stride pattern and the displacement at the start are omitted.

The first equation relies on Newton's second law (Fig. 1)

$$\frac{dv}{dt} = f(t) - \frac{v}{\tau}, \quad (1)$$

where all quantities refer to the unit mass of the runner, $v(t)$ is the instantaneous runner's velocity, $f(t)$ is the variable propulsive force (a control variable in optimal control), and v/τ is the resistive force linearly depending on the velocity v (τ is a constant coefficient).

The second equation describes the energy conversions in the competitor's body

$$\frac{de}{dt} = \sigma - f v , \tag{2}$$

where: e is the available energy per unit mass, σ is a constant energy flow rate (energy equivalent per unit mass of $V\dot{O}_2$), $f v$ is the actual mechanical power per unit mass used by the athlete to overcome the inertia and the resistance of motion.

Both ordinary differential Eqs. (1) and (2) should be supplemented by the initial conditions at the start to the race:

$$v(0) = 0 , \quad e(0) = e_0 . \tag{3}$$

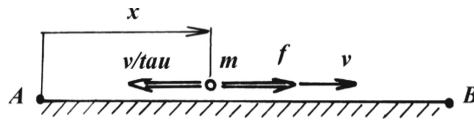


Fig. 1. The particle model of the competitor of the mass m . The forces exerted on his body per unit mass are: f – propulsive force (control variable), v/τ – resistive force. Furthermore: v – velocity of the runner, τ – constant coefficient, x is the actual covered distance, symbols A and B stand for initial and final points.

The hydraulic analogy of the energy flow model is given in Fig. 2. There are two containers representing different types of human energy. The upper vessel on the left contains a fluid representing an aerobic energy. The aerobic energy container is of infinite capacity. The fluid flows through the pipe to the lower container with a constant flow rate σ (energy equivalent per unit mass of $V\dot{O}_2$). The energy flow rate does not depend on the fluid levels in both containers. The lower vessel contains the fluid representing the anaerobic energy e_0 at the beginning. Symbol $e(t)$ denotes the energy per unit mass at a given instant of time. Bonnans and Aftalion [4] identify the energy $e(t)$ as an anaerobic energy. It is equal to anaerobic energy at the beginning of the process. It is a mixture of both energies (aerobic and anaerobic) later; therefore it is regarded as the available energy in this paper. The difference ($e_0 - e(t)$) may be regarded as the accumulated oxygen deficit (AOD). There is another pipe at the bottom of the lower container. The energy flow rate ($f v$) may be regulated by a tap and adjusted to the actual conditions. The algorithm of regulation is not known at that moment and it should be found. Further details of the hydraulic analogy of the energy flow one can find in Bonnans and Aftalion [4].

From the point of view of optimal control Eqs. (1) and (2) are the state equations, v and e are the state variables, f is the control variable. Two inequality constraints should be satisfied. The first one:

$$0 \leq f(t) \leq f_{max} , \tag{4}$$

where f_{max} is the maximal propulsive force per unit mass depending on the abilities of the runner, and the second one:

$$e(t) \geq 0 , \tag{5}$$

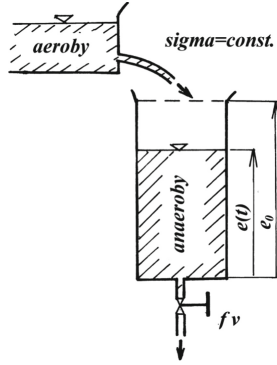


Fig. 2. Keller’s model of the power balance. Symbol $e(t)$ - energy per unit mass, e_0 – initial energy, σ energy flow rate from the aerobic container to the anaerobic one, here $\sigma = \text{const.}$, $f v$ – power per unit mass, f – propulsive force per unit mass (control variable), v – velocity.

which means that the reserves of energy per unit mass cannot be negative during the race – the lower container may be filled with or empty.

The distance D to be covered is given, therefore:

$$D = \int_0^T v dt , \tag{6}$$

where: T is the time of the event.

The problem may be formulated in the following manner. The runner should vary his speed $v(t)$ during the race over a given distance D to minimize the time of the event T

$$T = \int_0^T dt . \tag{7}$$

The state Eqs. (1) and (2), the initial conditions (3), the inequality constraints (4), (5) and the isoperimetric constraint (6) should be satisfied. Aftalion considers an equivalent formulation of the problem: maximization of the distance D for the given time T of the event [5].

Now, consider another model of the energy flow in competitor’s body [5] that is given in Fig. 3.

Here the fluid representing the energy per unit mass $e(t)$ is contained in communicating vessels. The left vessel contains the aerobic energy of infinite capacity. The right vessel contains the anaerobic energy at the beginning e_0 . A pipe of finite diameter connects both vessels; therefore the fluid flow rate $\sigma(e)$ is limited and depending on the fluid level in the right container. This pipe is placed at the bottom of the right container. At the beginning of the process the fluid level is the same in both containers (the competitor is rested) – the fluid does not flow through the bottom pipe, $\sigma = 0$. If the right vessel is empty the fluid flows with maximal intensity $\bar{\sigma}$ from left container to the right one.

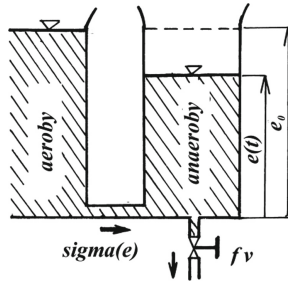


Fig. 3. Aftalion’s model of the power balance. Symbol $e(t)$ - energy per unit mass, e_0 – initial energy, $\sigma(e)$ energy flow rate from the aerobic container to the anaerobic one, here σ depends on the fluid levels in both containers, $f v$ – power per unit mass, f – propulsive force per unit mass (control variable), v – velocity.

The model assumes that $\sigma(e)$ is a linear function of the energy per unit mass e and it is proportional to the accumulated oxygen deficit $AOD = e_0 - e$:

$$\sigma(e) = \bar{\sigma} \left(1 - \frac{e}{e_0} \right), \tag{8}$$

where $\bar{\sigma}$ is the maximal value of σ . Now, Eq. (2) describing the energy conversions takes the form:

$$\frac{de}{dt} = \sigma(e) - f v. \tag{9}$$

Here, the constant σ from Keller’s model is replaced by the function $\sigma(e)$ given by Eq. (8). Further details of the model and the problem formulation are the same.

3 Method

3.1 Method of Miele

The method of Miele [6] was developed in the study of trajectories of high-speed aircraft and missiles, which could not be handled by conventional methods of performance analysis at the beginning of sixties. Here, a particular class of variational problems is considered, where the functional form to be extremized and the possible isoperimetric constraint are linear in the derivative of the unknown function $y(x)$.

Consider a functional linear in the derivative y' :

$$J = \int_A^B [\phi(x, y) + y' \psi(x, y)] dx = \int_A^B [\phi(x, y) dx + \psi(x, y) dy] \tag{10}$$

that is minimized. Symbols ϕ and ψ denote known functions of two arguments (x, y) . Symbols A and B stand for initial and final points. Assume that the process under consideration is represented by a curve joining points A and B in the (x, y) -plane (Fig. 4).

We assume that all solutions are within or along the border of the admissible domain represented by a region limited by a closed curve $\varepsilon(x,y) = 0$, and the points A and B also belong to this curve. Now, we can determine the difference in values of the integral (10) for two arbitrarily taken curves AQB and APB .

$$\Delta J = J_{AQB} - J_{APB} = \int_{AQB} (\phi dx + \psi dy) - \int_{APB} (\phi dx + \psi dy) = \oint_{AQBPA} (\phi dx + \psi dy) \tag{11}$$

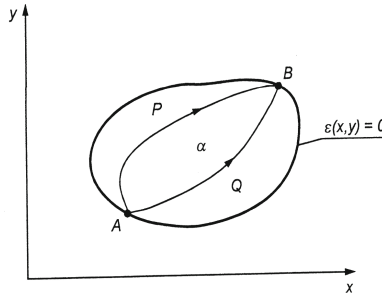


Fig. 4. The admissible domain is an area within the closed curve $\varepsilon(x,y) = 0$. Curves AQB and APB represent different admissible strategies.

Employing Green’s theorem we can transform the cyclic integral (11) into a surface integral:

$$\Delta J = \iint_{\alpha} \omega(x, y) dx dy \tag{12}$$

where α represents a region limited by these two curves and ω is so-called fundamental function. It takes the form:

$$\omega(x, y) = \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \tag{13}$$

Three cases are possible:

- a) Fundamental function ω is identically equal to zero. That means $\Delta J = 0$, and the functional is independent of the curve in the (x,y) -plane – the process is irrespective of the strategy.
- b) Fundamental function ω has the same sign, for example $\omega > 0$. For such the case $\Delta J > 0$ or $J_{AQB} > J_{APB}$. It means that every curve to the left gives smaller value of the functional. In the limit the minimizing curve belongs to the border of the admissible domain.
- c) Fundamental function ω changes its sign. It means that there is a curve along which $\omega = 0$ that divides admissible domain into two subregions, where $\omega > 0$ and $\omega < 0$.

0 respectively. The optimal path contains subarcs along the border of admissible domain and along the curve $\omega = 0$. This last subarc refers to the singular arc in calculus of variations (Fig. 5).

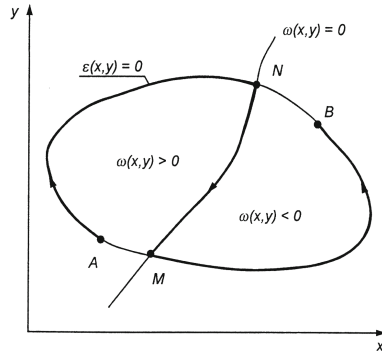


Fig. 5. The fundamental function ω changes its sign. The optimal solution contains two subarcs along the border of admissible domain AN, MB and the subarc within the admissible domain NM, where $\omega(x,y) = 0$. The functional is minimized along the curve ANMB.

A modification of the previous problem occurs when the integral to be extremized (10) must satisfy also the linear isoperimetric constraint:

$$J = \int_A^B [\phi_1(x, y) + y' \psi_1(x, y)] dx = C, \tag{14}$$

where: ϕ_1 and ψ_1 denote known functions and C is a given constant. Now, the function ω must be replaced by the augmented function ω^* of the form:

$$\omega^*(x, y, \lambda) = \frac{\partial \psi^*}{\partial x} - \frac{\partial \phi^*}{\partial y}, \tag{15}$$

where

$$\phi^* = \phi + \lambda \phi_1, \quad \psi^* = \psi + \lambda \psi_1. \tag{16}$$

The equation $\omega = 0$ of the simple problem must be replaced by the equation for augmented function $\omega^* = 0$. The equation for the isoperimetric problem represents a family of curves – one curve for each value of Lagrange’s multiplier λ . The particular value of λ is to be determined from the given isoperimetric constraint (14).

3.2 Solution

Keller [1, 2] applies the classical methods of calculus of variations for the problem solution. Aftalion [5] uses the classical approach of optimal control applying Pontryagin’s

maximum principle with Hamiltonian linear in the control function – here the propulsive force $f(t)$. Aftalion's reasoning is very sophisticated. Maroński [10], for Keller's model of running, uses non-classical method of Miele. This method gives necessary and sufficient conditions of optimality including singular arcs of linear type. It gives clear graphical interpretation. This method is not popular because the problem under investigation should be reduced to the form required by the method of Miele. For Aftalion's model of running it is possible, however. The detailed description of the method is out of the scope of this paper. The details one can find in Maroński [9], pp. 24–29.

The minimum-time running problem for Aftalion's model [5] may be reduced to a line integral as it follows. Equations (1) and (2) may be expressed using x , the actual covered distance, instead of the time t , using the definition of the velocity:

$$v = \frac{dx}{dt}. \quad (17)$$

Now, the state equations take the forms:

$$\frac{dv}{dx} = \frac{f(x)}{v} - \frac{1}{\tau}, \quad (18)$$

$$\frac{de}{dx} = \frac{\sigma(e)}{v} - f(x). \quad (19)$$

The isoperimetric constraint – the distance to be covered – is of the form:

$$D = \int_A^B dx, \quad (20)$$

and the time of the run to be minimized:

$$T = \int_A^B \frac{1}{v} dx, \quad (21)$$

where: A, B are the initial and final points.

Now, we can find the propulsive force $f(x)$ from Eq. (19) and put it into Eq. (18), then

$$\frac{dv}{dx} = \frac{\sigma(e)}{v^2} - \frac{1}{v} \frac{de}{dx} - \frac{1}{\tau}. \quad (22)$$

The differentials appearing in the integrals Eqs. (20) and (21) are as follows:

$$dx = \frac{\tau v}{a(e, v)} de + \frac{v^2 \tau}{a(e, v)} dv, \quad (23)$$

$$\frac{dx}{v} = \frac{\tau}{a(e, v)} de + \frac{v \tau}{a(e, v)} dv, \quad (24)$$

where:

$$a(e, v) = \sigma(e) \tau - v^2 \neq 0. \quad (25)$$

The isoperimetric constraint (20) and the time to be minimized (21) may be performed as line integrals

$$D = \int_A^B dx = \int_A^B \phi_1(e, v) de + \psi_1(e, v) dv, \quad (26)$$

where

$$\phi_1 = \frac{v \tau}{a(e, v)}, \quad \psi_1 = \frac{v^2 \tau}{a(e, v)}, \quad (27)$$

and

$$T = \int_A^B \frac{dx}{v} = \int_A^B \phi(e, v) de + \psi(e, v) dv, \quad (28)$$

where

$$\phi = \frac{\tau}{a(e, v)}, \quad \psi = \frac{v \tau}{a(e, v)}. \quad (29)$$

According to the method of Miele [6], the augmented functions ϕ^* and ψ^* take the forms:

$$\phi^* = \phi + \lambda \phi_1 = \frac{\tau}{a(e, v)} (1 + \lambda v), \quad (30)$$

$$\psi^* = \psi + \lambda \psi_1 = \frac{v \tau}{a(e, v)} (1 + \lambda v), \quad (31)$$

where: λ is constant Lagrange's multiplier.

The augmented fundamental function is of the form

$$\omega^*(e, v, \lambda) = \frac{\partial \psi^*}{\partial e} - \frac{\partial \phi^*}{\partial v} = \frac{1}{a^2(e, v)} \left[-\tau v (1 + \lambda v) \left(\tau \frac{d\sigma}{de} + 2 \right) - \lambda \tau (\sigma(e) \tau - v^2) \right]. \quad (32)$$

Expression (32), after equating to zero, yields the optimal solution along so-called singular arc

$$\tau \frac{d\sigma}{de} (1 + \lambda v) + \frac{\lambda \sigma(e) \tau}{v} + \lambda v + 2 = 0. \quad (33)$$

It is an algebraic equation depending on the functions of the covered distance x : $e(x)$, $v(x)$, and constants. If there is a subarc, where

$$\sigma(e) = \bar{\sigma} = \text{const.}, \quad (34)$$

then $d\sigma/de = 0$ and the equation of the singular arc is an algebraic equation of a simpler form

$$\lambda v^2 + 2v + \lambda \bar{\sigma} \tau = 0. \quad (35)$$

The parameters λ , $\bar{\sigma}$, τ are constant therefore from Eq. (35) it follows that along such arc $v = v_{\text{cruise}} = \text{const.}$, and this result is significantly consistent with the results of Keller [1,2] and Maroński [10].

From Miele's method it follows, that the admissible domain is bordered by the curves obtained from integration of Eqs. (1) and (2) for $f(t) = f_{\text{max}}$ on the right, the equality $e(t) = 0$ on the left, and the equality (25), $a(e, v) = 0$, from the bottom (cf. Figs. 6 and 7). The distance of the race may be broken into three phases:

1. The early phase of the race (acceleration), where the competitor moves using his maximal propulsive force, $f(t) = f_{\text{max}}$. The velocity rapidly increases. For the distances short enough only such a phase occurs (sprints) – the energy reserves are not depleted. In the (e, v) -plane this phase represents the arc along the border of admissible domain joining the initial point A and the singular arc.
2. The middle phase of the race (cruise). Here the competitor moves employing his partial propulsive force, $0 < f(t) < f_{\text{max}}$. His velocity moderately varies with the time or even it is constant. In the (e, v) -plane this phase corresponds to the arc within the admissible domain. The arc extends from the border where $f(t) = f_{\text{max}}$, to the border where $e(t) = 0$. This section corresponds to the singular arc in calculus of variations or optimal control.
3. The finish with decreasing velocity (negative kick). The border of the admissible domain is reached. Here the energy from the anaerobic container is depleted, $e(t) = 0$. The energy is supplemented by the flow from aerobic container, but the intensity of the flow is relatively low. The propulsive force $f(t)$ follows from Eqs. (2) or (9), $f = \sigma(e)/v$. The energy flow rate $\sigma(e)$ is too small to sustain the cruising velocity. The final point B may be reached along this curve.

Originally, the method of Miele was worked out for the given final point B (given final velocity v_B). This difficulty may be overcome in the manner described in the book of Maroński [9], where the problem with unspecified final velocity is considered as a sequence of the problems with fixed final velocity and minimizing the time of the event with respect to v_B .

4 Results

An example of the minimum-time run for constant σ over the distance $D = 400$ m using the method of Miele is given in Maroński [10]. Originally this problem solved Keller [1, 2] using the classical methods of calculus of variations. The data for computations were adjusted to the results at that time and they are: $e_0 = 2409.25 \text{ m}^2/\text{s}^2$, $f_{\text{max}} = 12.2 \text{ m/s}^2$, $\tau = 0.892 \text{ s}$, $\sigma = 41.61 \text{ m}^2/\text{s}^3$. The track is inclined to the horizontal and the inclination angle is 2° . The admissible domain is given in Fig. 6 (undashed area). The optimal path consists of the subarcs: the acceleration with $f(t) = f_{\text{max}}$ on the right, the cruise with

constant velocity in the middle, and the negative kick at the end. The details one can find in Maroński [9].

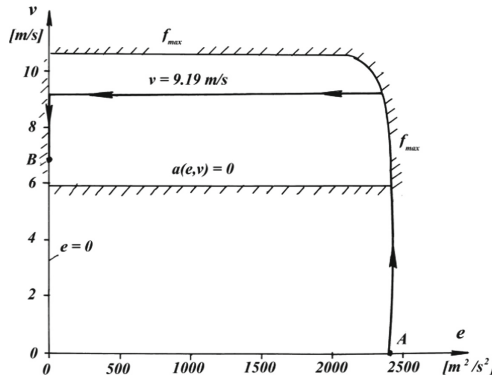


Fig. 6. The admissible domain (undashed area) for Keller’s model of metabolism ($\sigma = const.$), the 400 m run and the slope angle of the track 2^0 . The velocity is constant along the singular arc (in the middle).

The second example employs Aftalion’s model of metabolism [5], where the energy flow rate from the aerobic container to the anaerobic one is not constant, but it depends on the energy according to Eq. (8). The data for computations are as follows (cf [5, 11]): $e_0 = 2409 \text{ m}^2/\text{s}^2$, $f_{max} = 9 \text{ m/s}^2$, $\tau = 0.89 \text{ s}$, $\bar{\sigma} = 22 \text{ m}^2/\text{s}^3$. The track is horizontal. Here also the maximum propulsive force $f(t) = f_{max}$ is developed at the early phase of the race (acceleration). The optimal velocity slightly decreases along the singular arc then. The finish is with decreasing velocity along the subarc, where $e(t) = 0$. The difference, comparing with Keller’s model, is along the singular arc, where the cruising velocity is not constant. There is the family of singular arcs for different values of Lagrange’s multiplier in Fig. 7. The singular arc for $\lambda = -0.19 \text{ s/m}$ is on the outside of the admissible domain – it is not active. The singular arc for $\lambda = -0.23 \text{ s/m}$ does not reach the left

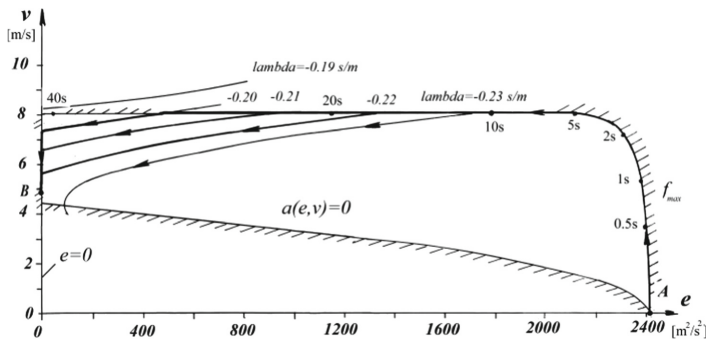


Fig. 7. The admissible domain (undashed area) for Aftalion’s model of metabolism ($\sigma = \sigma(e)$), and the run over different distances represented by different values of constant Lagrange’s multiplier λ . Here the velocity slightly decreases along the singular arc (in the middle).

border of the admissible domain, $e(t) = 0$. It cuts the bottom border of the admissible domain, where $a(e, v) = 0$. The method of Miele disappoints for such the case.

5 Discussion

The minimum-time competitive run is reconsidered in the paper. Such a problem may be formulated and solved in optimal control (calculus of variations). The question formulated at the beginning is this: What is the runner's optimal velocity profile along the distance minimizing the time of the event? Another question may be put: Is such a section of the race where the optimal velocity is constant? Such a problem solved Keller [1, 2]. He employs relatively simple model of motion basing on Newton's second law and an equation of the power balance. In his model the energy flow rate from the aerobic container to the anaerobic one is assumed to be constant. The result of Keller's considerations is that the race may be broken into three phases:

- acceleration,
- cruise with constant competitor's velocity,
- and final slowing down.

Such the result does not fit well the results observed during real events. Aftalion [5] reconsidered the problem improving the model of the power balance in the competitor's body. Her model assumes that the energy flow rate between the aerobic container and the anaerobic one is not further constant, but it depends on the energy level in the anaerobic vessel (cf. Eq. (8)). For the problem solution she applies classical Pontryagin's maximum principle with the Hamiltonian linear in the control function – here the propulsive force $f(t)$. Such an approach is very sophisticated and it gives only necessary conditions of optimality. In the present paper the non-classical method of Miele is applied – the method of extremization of line integrals by Green's theorem. This method gives necessary and sufficient conditions of optimality and it gives a simple graphical interpretation. The difficulty is that not all problems may be performed in the form required by this method – as a line integral of two variables. It was the reason that the minimum-time problem for running over variable slope trace was solved using a numerical method - direct Chebyshev's pseudospectral method [8]. For Keller's and Aftalion's models of running such transformation is possible, however. In this paper it is proved that the optimal path consists of three sections: acceleration, cruise and final slowing down. The optimal cruising velocity is not constant, as in Keller's model – it slightly decreases. The equation of the singular arc is an algebraic Eq. (33) and not an ordinary differential equation (cf. [5], Eq. (17)). The qualitative result is the same – the optimal velocity decreases along the singular arc (during the cruise). The answer for the question put in the title of the paper is: the optimal cruising velocity may be constant or decreasing. It depends on the model of the power balance used, and not on the distance to be covered. According to Reardon [3], it seems that the model used by Aftalion [5] is better for middle distance running comparing with Keller's model.

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