# On the Relationship Between Geometric Objects and Figures in Euclidean Geometry

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**Abstract.** In this paper, we will make explicit the relationship existing between geometric objects and figures in planar Euclidean geometry. Geometric objects are defined in terms of idealizations of the corresponding figures of practical geometry. We name the relationship between them as a relation of idealization. It corresponds to a resemblance-like relationship between objects and figures. This relation is what enables figures to have a role in pure and applied geometry. That is, we can use a figure in pure geometry as a representation of geometric objects because of this relation. Moving beyond pure geometry, we will defend that there are two other 'layers' of representation at play in applied geometry.

**Keywords:** Figures · Diagrams · Geometry · Geometric objects

## 1 Introduction

The role of diagrams in geometry has been the subject of many philosophical inquires. Here, we endeavor to determine what kind of relationship exists between geometric objects and geometric figures in planar Euclidean geometry.

The rationale behind this work is the following. If there is a clear relation existing between geometric objects and geometric figures, then, this might condition or even determine what role geometric figures and diagrams (composite figures) can have when used in the context of pure or applied geometry.

The work unfolds as follows. First, in Sect. 2, we will address geometric figures. In Sect. 3, we will consider the treatment of geometric objects in Euclid's *Elements*. This will enable us to bring to light the relationship that geometric objects have with geometric figures. In Sect. 4, we will determine basic features that geometric figures or diagrams have due to this relationship when used in the context of pure or applied planar Euclidean geometry.

## 2 Geometric Figures in Practical Geometry

We can have geometric figures even without a clear indication of how they are conceptualized (see, e.g., [1, pp. 45–46]). A conceptualization proper of geometric figures arises in the context of a practical geometry where there are clear geometrical practices, and,


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importantly, the figures are named. This is already the case during the Old Babylonian period [2].

A good example is that of the rectangle. Each side is given a name that refers to agricultural field plots. They are named the 'long side' and the 'front'. The side called the 'front' is one of the small sides that is parallel to an irrigation channel [3, p. 34].

That Mesopotamian practical geometry arises in the context of field measurements has important implications regarding how the geometric figures were conceptualized. The rectangle, be it an actual field plot or a drawing (for example, a field plan), is conceptualized in terms of the boundary that establishes an inner space separated from the outside by it. The figure proper is what is inside the boundary [1, p. 64].

In Mesopotamia, the area of a field plot was calculated from the measurement of its boundary. Land surveyors could only rely on length measurements. For that purpose, they could use, e.g., ropes whose lengths were given in terms of a metrological length unit [4, pp. 296–297]. To calculate the area of quadrilateral field plots, surveyors applied the so-called surveyors' formula. This formula enables us to calculate what for us is the approximate value of a quadrilateral figure [5, pp. 106–107].

This boundary-oriented conceptualization of space lasted in Mesopotamian mathematics. In geometrical problems from the Old Babylonian period, one still finds "the assumption that the area of a quadrilateral is determined by the surveyors' formula" [5, p. 117]. We can say that the notion of area of geometric figures derives from the notion of practical geometry [5, pp. 115–117]. We see an example of this in the conception of circle in Mesopotamian mathematics. Like in the case of a rectangular figure, a circle is conceptualized in terms of its boundary: "a circle was the shape contained within an equidistant circumference" [2, p. 20]. The circle and its boundary were given the same name, something like "thing that curves" [2, p. 20]. Like in the case of quadrilateral figures the area of the circle is calculated, using a formula, from the length of its boundary (which can be measured) [2, p. 18].

Circle figures are well-attested in ancient Mesopotamian mathematical problems. The drawings can be very sketchy but also quite precise. In one example, there is a drawing of an equilateral triangle inscribed in a circle [6, pp. 207 and 488]. Not only the sides of the triangle are quite rectilinear, but also the circle is very precise since it was drawn using a compass [6, p. 207].

## 3 Relating Geometric Objects to Geometric Figures

In pure planar geometry as developed in Euclid's *Elements,* the geometric object called circle, like other geometric objects, is explicitly defined in the definitions.[1] At this point, it might be useful to make a silly question: Why do we name this geometric object with the same name as that of a figure?

To help to answer our silly question we will first consider another one: Why not take the definition of a circle as a geometric object as a definition of a circle figure? As it is, the definition of circle in the *Elements* seems to correspond to the practice of drawing a

[1] "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another; and the point is called the center of the circle" [7, pp. 153–4].

circle figure using a compass. The center of the circle is the needle point of the compass, and all points of the circumference drawn with the compass lead are at the same distance from this point as measured using, e.g., a ruler.[2] How can this be possible, using the same definition for a drawn figure and for a geometric object? For us, it comes down to semantics; in particular, the meaning of a few terms, which must be understood in the context of a particular geometrical practice.

A circle as a geometric object is instantiated in an idealized plane [9, p. 208] – an abstraction from a real physical plane [10, p. 19], like a dusted surface or a wax tablet [11, pp. 14–16]. As defined, the circle as a geometric object has all radii equal to one another. Here, ‘equal’ does not mean the same as ‘equal’ in practical geometry. In the latter case, the equality of different radii is a practical one; we simply neglect whatever small differences in lengths there are. In pure geometry, it is made an idealization of this practical approach and instead of conceiving of practically equal radii, these are conceived as exactly equal. We have what we might call an exactification of the equality of lengths. The relationship between the geometric circle and the circle figure is what we might call a relation of idealization: the abstract object is defined in terms of an idealization of the concrete figure.

This relation of idealization is made clearer by considering similar relations for lines and points [12]. As defined in the *Elements*, “a point is that which has no part” [7, p. 153]. This definition can be seen as arising from an idealization of the practice of practical geometry: “a point is characterized as a non-measurable entity, as it has no parts that can measure it” [13, p. 18]. In the same way, a “line is breadthless length” [7, p. 153]. Drawn lines have small breadths whose lengths are disregarded in practical terms. We idealize the concrete line as a geometric object that has an exact length and is breadthless. We can say that both the geometric line and the geometric point are in a relation of idealization with lines and points from practical geometry.

The relationship between geometric objects and figures is manifested very clearly in the definition of geometric line. We can see that the definition of geometric line is made by reference to an idealization at play. The geometric line is defined in relation to what is being implicitly idealized: a line from practical geometry. It only makes sense a definition in terms of breadthless length, in relation to something that has breadth. The definitions of geometric objects are dependent on the figures and result from an idealization of these. This enables us to say that between geometric objects and figures we have a relation of idealization.

[2] To the best of our knowledge, there is no extant text containing a practical definition of circle corresponding to that of geometric circle in the *Elements*. However, there are records of a conceptualization that approaches that in terms of radius [2, p. 20]. That this conceptualization can be ascribed to circle figures independently of having been adopted in the context of pure geometry is suggested, e.g., by a Greek third-century BC papyrus containing practical geometrical problems among others [8, pp. 70–2].

## 4 Basic Features of the Role of Diagrams in Pure and Applied Geometry

In the previous section, we have determined what we called the relation of idealization between geometric objects and figures. We expect that this relationship determines basic features regarding the role of figures or diagrams when used in the context of pure or applied geometry. To show this, we will consider two propositions, one from pure geometry and another from applied geometry. We will start with pure geometry.

In proposition 1 of book 1 of Euclid's *Elements* (proposition I.1), one constructs a geometric object – an equilateral triangle – by a particular procedure where one uses two circles that intersect each other. The text is accompanied by a lettered diagram (see Fig. 1), and with the letters one refers in the text to parts of the diagram.[3]

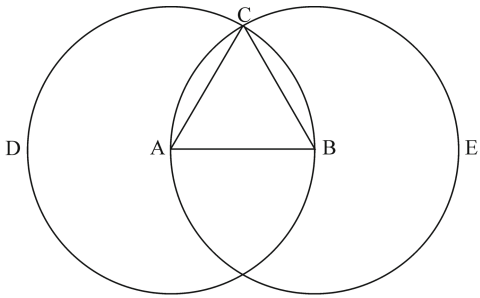


**Fig. 1.** The diagram in proposition I.1 of the *Elements*

The basic point we want to make here is to question how come that in the demonstration of a result in pure geometry we use a diagram which is a drawing consisting of several figures? The evident answer is that we take the diagram to represent the geometric objects. But what justifies using figures from practical geometry as a representation of geometric objects? More generally, we have to know what enables something to be a representation of something else. There are two features related to representation that are relevant here: intentionality and resemblance (see, e.g., [15–17]).

Regarding the intentionality in the adoption of a representation, what is relevant for us here is not so much that the intention of the author that adopts a particular representation is usually relevant in the interpretation of the representation, but that 'intentionality' underlies the possibility of choosing quite freely what we take to be the representation of something else. For example, we might decide that a hand-drawn line represents a segment drawn using a straightedge. That is, we intentionally take the sketchy line to represent the practical segment or segment figure.

The intentionality enables us to choose whatever we want as a symbol (representation) of something else. With an ad hoc representation, we would not go very far in the case under consideration. So, we rely on another concept related to representation. That of resemblance. We would go from just a symbol to a symbol that has iconic properties.

[3] We can expect that early versions of the *Elements* included at least unlettered diagrams [14].

That is, to a symbol that in some way resembles what it is symbolizing. But here we face a major problem. A geometric object is not something that we can see. It is instantiated in an idealized plane not in the space of our experience. There is no way in which we might say that a circle figure resembles a geometric circle. How do we overcome this difficulty?

A circle figure does not resemble a geometric circle; this simply has no meaning, unless we twist considerably the semantics of the word ‘resemblance’. However, we have another kind of relationship between geometric objects and figures. We can adopt the relation of idealization, e.g., between a geometric circle and a circle figure to take the second as a representation of the first. The relation of idealization works as a resemblance-like relation. While the circle figure does not resemble the geometric figure, we can nevertheless establish a simulacrum of a resemblance between them. The circle figure has as its center the needle point of the compass. To this concrete point corresponds the geometric point as the center of the geometric circle. To the drawn circumference corresponds a breadthless line. While the radii of the circle figure are equal only within a particular practice of practical geometry where we neglect small measurement differences, the radii of a geometric circle are equal exactly as if corresponding to an idealized measurement in which all lengths are exactly equal.

The relation of idealization is a sort of resemblance-like relation that enables us to take the circle figure to be a representation of the geometric circle. Since we have one-to-one resemblance-like relations between all relevant elements of the geometric circle and the circle figure (center, circumference, radii, diameter, etc.), the circle figure works as an avatar of the geometric object in the diagram. When in the text we refer to aspects of the diagram we can take these as referring to the corresponding aspects of geometric objects.

This is a very basic characteristic of the use of diagrams in pure geometry. We suggest that any account of the role of diagrams in pure geometry should be compatible with this feature and how it arises.

Let us now address the issue of the role of diagrams in applied geometry. We will consider Proposition 1 of Euclid’s *Optics*. What we want to determine here is in what way, if any, do we move beyond the representational role that a figure has in pure geometry. In that case, as we have just seen, we can establish a resemblance-like relationship between geometric object and figure.

When applying geometry like in the *Optics*, we take geometric objects to represent physical phenomena. The basic idea developed in Euclid’s *Optics* is that the eyes emit ‘visual fire’. It is the ‘visual fire’ that enables us to see the world around us. For example, the incidence of ‘visual fire’ in objects is what enables us to see them. ‘Visual fire’ is represented in the *Optics* by geometric segments [18, p. 8]. Proposition 1 of the *Optics* is as follows:

**Proposition 1:** No observed magnitude is seen simultaneously as a whole.

Call AD the observed magnitude, and B the eye from which the visual rays BA, BG, BK, BD fall. Since the visual rays diverge, they do not fall on the magnitude AD in a contiguous manner; so that there are intervals of this magnitude on which the visual rays do not fall. Consequently, the entire magnitude is not seen simultaneously. However, as the visual rays move rapidly, it is as if we saw [the entire magnitude] simultaneously. (Cited in [18, p. 10]) (Fig. 2).

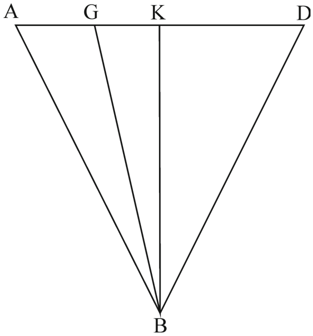


**Fig. 2.** The diagram in Proposition 1 of the *Optics*

Here, we are going to make some magic. We will use the above diagram to help us to interpret the text. We will take advantage of the representational roles of the diagram even if we have not clarified what these are. The geometric segment AD represents the physical object that we see. The geometric segments BA, BG, BK, BD represent the physical 'visual fire' emitted by an eye of the observer. The geometric point B represents an eye. Here, we are using the diagram to help us clarify the representational role of geometric figures. In fact, we are ascribing to the diagram these features. When we look, for example, at the drawn line BA, we take it to represent the 'visual fire'. The point is that since we have a resemblance-like relation established between the figure and its corresponding geometric object, and we take the geometric object to represent a physical entity, we can intentionally ascribe to the figure the representational role of its corresponding geometric object. We put another 'layer' of representation on top of the first one (see also [19]).

At this point, we can say that in applied geometry, the geometric figure has a double representational role. The geometric figure (or diagram) represents the geometric object, and this represents a physical entity. In this way, the geometric figure represents the physical entity, via the geometric object represented by the figure.[4]

There is in our view a third 'layer' of representation in the diagrams of Euclid's *Optics*. The geometric objects are given a representational character in the context of several assumptions. For example, it is assumed that "the straight lines drawn from the eye diverge to embrace the magnitudes seen" (cited in [18, p. 9]). How the visual rays 'diverge to embrace' is further specified in Proposition 1. There, the visual rays are taken to "move rapidly" [18, p. 10]. This corresponds to ascribing to the diagram a new 'layer'

[4] One might ask what justifies taking a geometric object to represent physical phenomena in the.first place. Again, it is due to the relation of idealization that we have between geometric objects and concrete objects. For instance, a geometric segment is in a relation of idealization not only with, e.g., a practically drawn segment but also, e.g., with a rod, a stretched rope, or with the 'visual fire' taken to be a sort of light beam [12].

of representation of the optical phenomena. We have the assumption that there is a sort of scanning of magnitudes by emitting successively the visual rays BA, BG, BK, and BD. The diagram as a whole is a static representation of a dynamic situation (see also [19]).

While this layer of representation relates, as the second one, to the geometric objects, it is only meaningful when taking into account the whole diagram. Like with the second layer of representation (where we can regard it as implemented directly on the figures), we can see this further 'layer' of representation as implemented directly on the diagram. An important difference with the second 'layer' is that it is not implemented so much on the figures that form the diagram but on the diagram as a whole.

For applied geometry, the situation is then as follows. The figures represent geometric objects due to the relation of idealization existing between them. Since we take the geometric objects to represent physical phenomena like, e.g., 'visual rays', we take the corresponding figures to represent the physical phenomena. This is a second 'layer' of representation that we ascribe to the figures. Besides this, the geometric objects on a whole have a dynamic relationship between them since they represent not only physical entities but also their dynamics. We must take into account that the visual rays 'move rapidly'. This corresponds to ascribing a third 'layer' of representation not to each figure individually but to the diagram as a whole since it is only at this 'level' that we can represent the dynamics. With this third 'layer' of representation, the diagram represents the dynamics of the physical phenomena.

These are basic aspects of the role of diagrams in applied geometry that follow from the relation of idealization that exists between geometric objects and figures and from taking geometric objects to represent physical phenomena. We suggest that any account of the role of diagrams in applied geometry should be compatible with this view.

## 5 Conclusions

In this work, we have tried to determine what kind of relationship there is between geometric figures of practical geometry and geometric objects of pure geometry. We have established that there is a relationship between geometric objects and figures. Geometric objects in the *Elements* are defined in terms of idealizations of the corresponding figures of practical geometry. We have named the relationship between them as a relation of idealization.

This relation existing between objects and figures is what in our view enables figures to have a role in pure and applied geometry. That is, we can use a figure or diagram as a representation of geometric objects or composite geometric objects because the relation of idealization corresponds to a resemblance-like relationship between objects and figures.

Moving beyond pure geometry we have defended that there are two other 'layers' of representation at play in applied geometry: 1) geometric figures can be ascribed as representing physical phenomena when we give this representational role to their corresponding geometric objects due to the relation of idealization existing between them; 2) The diagram as a whole can be taken to represent dynamical features of the physical phenomena also for the same reason.

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