Tractarian Notations

Francesco Bellucci(✉)

University of Bologna, Via Azzo Gardino 23, 40122 Bologna, Italy
francesco.bellucci4@unibo.it

Abstract. In the *Tractatus* Wittgenstein presents two different notations for logic: the truth-tabular notation introduced at TLP 4.442, and the so-called N operator notation at TLP 5.502 (plus a third notation, the so called *ab*-notation, at TLP 6.1203). Gregory Landini (2007) has argued that both the truth-tabular notation and the N operator notation fulfill the Wittgensteinian ideal of having a language in which all and only logical equivalents have exactly one and the same expression. In this paper, I show that Landini's argument is mistaken, for it overlooks the crucial Tractarian distinction between truth-operation and truth-function.

Keywords: Ludwig Wittgenstein · Gregory Landini · Truth-tables · Notation · Logical equivalence

1 Tractarian Extensionality

At TLP 5.25–5.251 Wittgenstein distinguishes between a truth-operation and a truth-function. In the context of the sentential calculus, a proposition is a truth-function of elementary propositions (TLP 5) and is the result of truth-operations on elementary propositions (TLP 5.3). A truth-operation is the way in which a truth-function (a proposition) results from another truth-function; so '~ *p*' results from the application of the truth-operation '~' to the elementary proposition '*p*'. Truth-operations are iterative; thus '~~ *p*' is the result of two successive applications of the truth-operation '~' to the elementary proposition '*p*'. One single truth-operation, the joint denial or Sheffer stroke, is capable of expressing all possible results of truth-operations on elementary propositions, i.e. the Sheffer stroke is a sole sufficient operator for the sentential calculus. A generalization of the Sheffer stroke to *n* elementary propositions is the so-called N operator, which Wittgenstein presents at TLP 5.5: 'every truth-function is a result of the successive application of the operation (– – – – – T) (ξ, …..) to elementary propositions'. The N operator generalizes the Sheffer stroke, which is a binary operation, to the simultaneous negation (represented by the left-hand parentheses) of *n* elementary propositions (represented by the right-hand parentheses). At TLP 5.502 Wittgenstein proposes to write 'N $\overline{\xi}$', where '$\overline{\xi}$' is a 'variable whose values are the terms of the expression in brackets, and the line over the variable indicates that it stands for all its values in the bracket'.

It is a central claim of the *Tractatus* that an operation does not characterize the *sense* of a proposition (TLP 5.25). The sense of a proposition is its agreement and disagreement with the possibilities of truth and falsity (or truth-possibilities) of elementary propositions (TLP 4.2); a proposition is the expression of agreement and disagreement with

A. Basu et al. (Eds.): Diagrams 2021, LNAI 12909, pp. 183–187, 2021.
https://doi.org/10.1007/978-3-030-86062-2_15

the truth-possibilities of elementary propositions (TLP 4.4); the expression of agreement and disagreement with the truth-possibilities of elementary propositions is the expression of its truth-conditions (TLP 4.431); therefore, the sense of a proposition is its truth-conditions, and the proposition expresses (i.e. shows, TLP 4.022) it, and expresses nothing else. An operation does not characterize the sense of a proposition because one and the same sense (agreement and disagreement with the truth-possibilities of elementary propositions) may be obtained by application of distinct truth-operations on elementary propositions. For example, '~ (p & ~ q)' and '$p \supset q$' agree and disagree with the truth-possibilities of the elementary propositions 'p' and 'q' in precisely the same cases, i.e. they have the same truth-conditions, and thus the same sense. If '~' really characterized the sense of '~ (p & ~ q)', it should equally characterize the sense of '$p \supset q$', because by the definition of sense '~ (p & ~ q)' and '$p \supset q$' have the same sense. But no operation can be said to characterize the sense of a proposition in which it does not occur. Thus, '~' does not characterize the sense of '$p \supset q$', and therefore it neither characterizes the sense of '~ (p & ~ q)', because the sense of these two propositions is the same (TLP 5.43).

2 Propositional Signs

Since the sense of a proposition is given by its truth-conditions, and since the expression of the truth-conditions of a proposition is the expression of its agreement and disagreement with the truth-possibilities of elementary propositions, the sign that expresses agreement and disagreement with the truth-possibilities of elementary propositions is a propositional sign, i.e. it expresses the truth-conditions of that proposition, or its sense. Thus a truth-table like that in Fig. 1, since it expresses, in the right-hand column, the proposition's agreement and disagreement with the truth-possibilities of the elementary propositions 'p' and 'q' given in the left-hand column, is a propositional sign (TLP 4.442). The propositional sign in Fig. 1 does not represent the truth-operation on elementary propositions from which the truth-function expressed in the right-hand column has been obtained. It only represents the *result* of some such truth-operation on elementary propositions. Truth-operations on elementary propositions which have one and the same truth-function as result are not distinguished in a truth-table. The reason is that since a truth-operation does not characterize the sense of a proposition, but only the result of truth-operations does, and this is the sense of the proposition, a propositional sign that should express the truth-operation by which a truth-function is obtained from elementary propositions would express something *foreign to the sense of the proposition.*

Landini (2007) takes Wittgenstein's goal to have been a notation in which all and only logical equivalent propositions have the same representation: "He hoped to demonstrate that a deductive calculus for logic can be supplanted by a representational system in which all and only logical equivalents have exactly one and the same expression. The representation of quantifier-free sentences in terms of their truth-conditions (or, alternatively, Venn's representation) offers just such a notation. As Wittgenstein sees matters, systems that employ different logical particles '&,' '∨,' ' ⊃,' '~,' etc., hide their formal ('internal') nature. Wittgenstein attempted to exploit the truth-table representation of propositions as evidence for his view that a proper representation would reveal that

‘p	q	’
T	T	T
F	T	T
T	F	
F	F	T.

Fig. 1. A truth-table.

tautologies and contradictions are scaffolding.” (Landini 2007, 124). In a notation for the sentential calculus in which the truth-table is the propositional sign, all and only logically equivalent propositions will be represented by the same propositional sign. In this notation, ‘$P \supset Q$’, ‘$\sim (P \,\&\, \sim Q)$’, ‘$(P \supset Q) \,\&\, (Q \vee \sim Q)$’, ‘$\sim\sim (P \supset Q)$’, etc. will have the same representation, namely the truth-table in Fig. 1.

Landini suggests, correctly in my opinion, that Wittgenstein hoped to extend this result to quantification theory with identity, and that he hoped to do it by means of the N operator. In fact, the truth-tabular method constitutes a decision procedure for sentential logic. Landini notes that Wittgenstain’s project was doomed to failure in light of Church’s later result that quantificational logic is undecidable. Here I want to focus on the sentential fragment, in order to understand whether Landini’s idea is correct that the N operator notation is able to do what the truth-tabular notation does, i.e. to represent all and only propositional logical equivalents by the same propositional sign.

3 Tabular and Operational Notations

If the truth-tabular notation presented by Wittgenstein at TLP 4.442 is admitted as a legitimate notation for the sentential fragment of logic, we are at once provided with a means of distinguishing two different kinds of notations. I label them *tabular* and *operational*. Tabular notations are those in which only the *result* of a truth-operation on elementary propositions is represented. Operational notations are those in which truth-operations themselves are represented. Whenever an operation is represented, there is a possibility of representing the result of that operation by means of other operations, while when only the result of an operation is represented, this possibility is excluded.

The distinction between tabular and operational notations shows that, while it is correct that the truth-tabular notation that Wittgenstein presents at TLP 4.442 conforms to the notational ideal of having all and only logically equivalent propositions represented by the same sign, it is not the case that the N operator notation that Wittgenstein introduces at *TLP* 5.502 conforms to that ideal. In a nutshell, I want to argue that the N operator notation does not conform to the notational ideal because that notation is operational, not tabular, and in operational notations there is always the possibility of representing the result of one operation by means of some other operation, and therefore of representing logically equivalent propositions by *syntactically distinct* propositional signs.

Here is the explanation. In the N operator notation every truth-function is a result of successive application of the N operator to elementary propositions (TLP 5.5). The application of the N operator to one propositional argument '*P*', represented as 'N(*P*)', results in its negation, '~ *P*'; applied to two propositional arguments '*P*' and '*Q*', represented as 'N(*P, Q*)', it results in their joint denial, '~ *P* & ~ *Q*'; applied to an arbitrary number of propositional arguments, it results in the joint denial of all the arguments (TLP 5.502). As I mentioned, the N operator is more powerful than the Sheffer stroke because whereas the Sheffer stroke is a binary connective, the N operator can be applied to any number of propositional arguments as the joint denial of all of them.

Now, Landini takes Wittgenstein to have thought that the N operator notation has a truth-tabular nature akin to the truth-tabular representation of propositional logic which Wittgenstein had presented at TLP 4.442, because in truth-tabular notation all and only logically equivalents are represented by the same sign. This cannot be quite right, however. In the N operator notation, 'NNN(*P*)' and 'N(*P*)' are logically equivalent but syntactically distinct formulas. The same is true, for example, of 'NN(*P,Q*)' and 'NN(*Q,P*)', which are logically equivalent but syntactically distinct. In order to make justice to Wittgenstein's claim concerning the N operator notation, Landini proposes – attributing the origins of these proposals to Wittgenstein himself – five equational 'rules of operation' (Landini 2007, 129–130) by means of which the equivalence of the representation of logically equivalent propositions is meant to be achieved:

(L1) $\mathrm{N}(\xi_1, \ldots \xi_n) = \mathrm{N}(\xi_i, \ldots, \xi_j)$, $1 \leq i \leq n$, and $1 \leq j \leq n$.
(L2) $\mathrm{N}(\ldots\xi, \ldots, \xi\ldots) = \mathrm{N}(\ldots\xi, \ldots)$.
(L3) $\mathrm{N}(\ldots \mathrm{NN}(\xi_1, \ldots, \xi_n)\ldots) = \mathrm{N}(\ldots\xi_1, \ldots, \xi_{n,}\ldots)$.
(L4) $\mathrm{N}(\ldots\mathrm{N}(\ldots\xi, \ldots, \mathrm{N}\xi,\ldots)\ldots) = \mathrm{N}(\ldots)$.
(L5) $\mathrm{NN}(\gamma, \mathrm{N}(\xi_1, \ldots, \xi_n)) = \mathrm{N}(\mathrm{N}(\gamma, \mathrm{N}\xi_1), \ldots, \mathrm{N}(\gamma, \mathrm{N}\xi_n))$.

Clause (L1) corresponds to a generalization of the commutation rule '$\xi\gamma = \gamma\xi$'. Clause (L2) is a rule of elimination of equivalents. Clause (L3) corresponds to the rule of insertion and omission of double negation in whatever context it occurs ('$\xi = \sim \sim \xi$'). Clause (L4) allows us to delete or insert any argument of the form '$\mathrm{N}(\ldots\xi, \ldots, \mathrm{N}\xi,\ldots)$', which is a tautology, in whatever context it occurs. Clause (L5) is a distribution rule. Landini takes Wittgenstein to have thought that (L1–5) make the N operator notation tabular: 'by application of (1)–(5) we can see how Wittgenstein thought that the N-operator recovers the features of truth-table representations' (2007, 130). Yet, Landini argues, (L1–5) are not to be taken as rules of logical equivalence in the proper sense: '[t]hese rules assert the sameness of certain *practices* of operation. They are not, therefore, identity statements' (*ibid.*).

It is far from clear in what sense (L1–5) are not rules of logical equivalence. A notation in which some rule concerning double negation is applicable is a notation in which in contexts like '$\xi = \sim\sim \xi$', syntactically distinct sentences appear on both sides of the '='. The '=' in fact states that syntactically distinct sentences are logically equivalent. On both sides of Landini's '=' in (L3), two syntactically distinct sentences in the N operator notation must appear. He says: 'N(NN(*p*, *q*, *r*) is to be regarded, *in some sense*, as the same as N(*p*, *q*, *r*)' (2007, 129). Now, the sense in which the former sentence is to be regarded as the same as the latter is that they are *logically* equivalent, not that they are *syntactically* equivalent. For were they syntactically equivalent, (L3) could not be applied, because (L3) states the logical equivalence of syntactically distinct sentences. One cannot apply it to sentences which are not syntactically distinct. By the same token, a notation in which a rule of commutation is applicable is a notation in which on the sides of '=' in '$\xi\gamma = \gamma\xi$' syntactically distinct sentences appear which the rule declares to be logically equivalent. And therefore on both sides of '=' in (L1) two distinct sentences in the N operator notation must appear.

Whenever a rule of logical equivalence of syntactically distinct sentences applies, it cannot be true that logical equivalent propositions have the same representation. It is not sufficient to do what Landini thinks Wittgenstein should do, namely to declare that these clauses are not rules of logical equivalence but rules of syntactical equivalence. Such a declaration is merely nominal and *cannot transform a rule of logical equivalence into a rule of syntactical equivalence*. Labeling them 'rules of practices of operation' rather than 'rules of logical equivalence' does not change the fact that they must succeed in capturing exactly what syntactically distinct sentences are logically equivalent. Landini is correct in saying that Wittgenstein's tabular notation fulfills the ideal of having all and only logical equivalents represented by the same sign. But he is wrong that the same is true of the N operator notation. In our terms, though they are equivalently expressive of the same fragment of classical logic, the truth-tabular notation of TLP 4.442 is tabular, whereas the N operator notation of TLP 5.502 is operational. Landini's (L1–5), being rules of logical equivalence, cannot turn an operational notation into a tabular one.

References

Landini, G.: Wittgenstein's Apprenticeship with Russell. Cambridge University Press, Cambridge (2007)

Wittgenstein, L.: Tractatus Logico-Philosophicus. Routledge & Kegan Paul, London (1922). Translated by C. K. Ogden