The Došen Square Under Construction: A Tale of Four Modalities

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Abstract. In classical modal logic, necessity $\Box A$, possibility $\Diamond A$, impossibility $\Box\neg A$ and non-necessity $\Diamond\neg A$ form a Square of Oppositions (SO) whose corners are interdefinable using classical negation. The relationship between these modalities in intuitionistic modal logic is a more delicate matter since negation is weaker. Intuitionistic non-necessity $\boxminus$ and impossibility $\diamondminus$, first investigated by Došen, have received less attention and—together with their positive counterparts $\Box$ and $\Diamond$—form a square we call the *Došen Square*. Unfortunately, the core property of constructive logic, the Disjunction Property (DP), fails when the modalities are combined and, interpreted in birelational Kripke structures à la Došen, the Square partially collapses. We introduce the constructive logic CKD, whose four semantically independent modalities $\Box$, $\Diamond$, $\boxminus$, $\diamondminus$ prevent the Došen Square from collapsing under the effect of intuitionistic negation while preserving DP. The model theory of CKD involves a constructive Kripke frame interpretation of the modalities. A Hilbert deduction system and an equivalent cut-free sequent calculus are presented. Soundness, completeness and finite model property are proven, implying that CKD is decidable. The logics $\mathsf{HK}\boxminus$, $\mathsf{HK}\Box$, $\mathsf{HK}\Diamond$ and $\mathsf{HK}\diamondminus$ of Došen and other known theories of intuitionistic modalities are syntactic fragments or axiomatic extensions of CKD.

Being one world away from absurdity is very different from being in an absurd world. Being one step removed from disaster is often very different, and feels very different, from the disaster. (Routley 1983)

1 Introduction

The reader may recall the classical *square of opposition* (SO) [38] seen on the left side in Fig. 1, whose four corners express the distinction between contradictory and contrary oppositions, that were traditionally labelled with four letters A, E, I, O designating propositions, and connected by means of six edges. The SO has been applied to concepts in linguistics, mathematics and philosophy and can be generalised in a number of ways. From the vantage point of classical modal logic, the oppositions can be expressed in terms of the modal operators $\Diamond$ and $\Box$, which traditionally express *possibility* and *necessity*, and are interdefinable in terms of negation, i.e., $\Diamond A = \neg\Box\neg A$ and $\Box A = \neg\Diamond\neg A$. In

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constructive modal logic this is no longer the case, which results in four independent modal operators, complementing $\Diamond$ and $\Box$ with their opposing counterparts [10], namely *impossibility* $\diamondminus$ and *non-necessity* $\boxminus$. In this work we construct the *Došen square* (DS) seen on the right side in Fig. 1, by investigating the relationships between the modalities $\{\Diamond, \Box, \boxminus, \diamondminus\}$ in a constructive theory, in which they remain independent under (intuitionistic) negation ($\sim$) in the sense that they are not interdefinable anymore, unlike in classical logic. We will shortly discuss the interpretation of the DS.

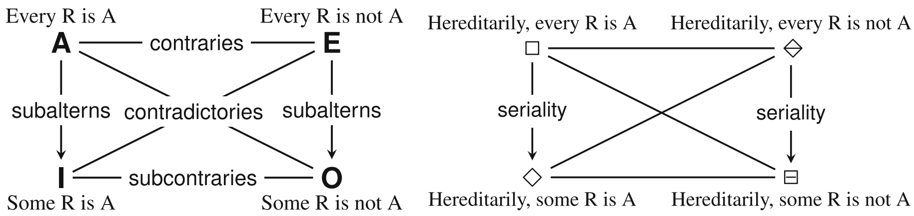

Fig. 1. The square of oppositions and the Došen Square.

1.1 State of the Art

In classical Kripke semantics, the modal operators $\Diamond$ and $\Box$ are interpreted w.r.t. frames $\mathfrak{F} = (S, R)$, consisting of a set of *states* S and a binary *accessibility* relation R on S. The satisfaction of formulas is defined relative to models $\mathfrak{M} = (\mathfrak{F}, V)$ extending a frame by a *valuation* $V : S \to \mathcal{P}(Var)$ that associates a set $V(s) \subseteq Var$ of propositional variables *satisfied at* a state s. Their interpretation is given by quantifying existentially and universally over states in the image of the relation R

$$\mathfrak{M}, s \models \Diamond A \Leftrightarrow \exists x. (sRx \;\&\; \mathfrak{M}, x \models A) \tag{1}$$

$$\mathfrak{M}, s \models \Box A \Leftrightarrow \forall x. (sRx \Rightarrow \mathfrak{M}, x \models A) \tag{2}$$

where $\mathfrak{M}, s \models A$ expresses that A *is satisfied at* state s *in* $\mathfrak{M}$. Standardly, in modal extensions of intuitionistic propositional logic (IPL), Kripke models are based on a birelational Kripke frame $\mathfrak{F} = (S, \sqsubseteq, R)$, where the accessibility relation R and the intuitionistic partial order $\sqsubseteq$ are relations on the same domain. Because the classical clauses (1) and (2) fail to ensure *intuitionistic heredity*:

$$s \sqsubseteq s' \text{ and } \mathfrak{M}, s \models A \text{ implies } \mathfrak{M}, s' \models A,$$

one common approach is to impose the frame conditions $(\sqsubseteq ; R) \subseteq (R ; \sqsupseteq)$ and $(\sqsubseteq ; R) \subseteq (R ; \sqsubseteq)$, where $R ; S =_{df} \{(x, z) \mid \exists y. x\, R\, y \text{ and } y\, S\, z\}$ denotes *sequential composition* of two binary relations R and S. In the Došen square we enforce heredity without any frame conditions by the following 'doubly quantified' (constructive) interpretation:

$$\mathfrak{M}, s \models \Diamond A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \exists x.\, (s' \; R \; x \;\&\; \mathfrak{M}, x \models A) \tag{3}$$

$$\mathfrak{M}, s \models \Box A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \forall x.\, (s' \; R \; x \Rightarrow \mathfrak{M}, x \models A). \tag{4}$$

We can pronounce $\Diamond A$ as "hereditarily, there is an R-accessible state at which A holds" and $\Box A$ as "hereditarily, for all R-accessible states A holds", hence the labelling of the Došen square in Fig. 1, in which such sentences have been still further abbreviated. The mainstream approach is to either adopt the 'singly quantified' approach (1) and (2) for both $\Box$ and $\Diamond$ [35,40,41] or to 'mix and match', adopting (1) for $\Diamond$ and (4) for $\Box$ [29,34]. The 'doubly quantified' approach for both modalities, first introduced by [39] and later used in the logic CK [3,20,25,33], is far less common, as it leads to non-normal modal logics invalidating the axiom $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$. Computationally, this makes sense (see [24,33]), and it has the consequence that $\sqsubseteq$ is not required to be antisymmetric as in standard intuitionistic Kripke frames. In CK, this gives rise to cyclic structures which are crucial in establishing the Finite Model Property (FMP) [25]. Furthermore, the nullary case $\sim \Diamond \bot$ is invalidated as well, because frames for CK include so-called *fallible* states which verify all formulas of the language. Fallible states may be accessible from other states via the modal accessibility relation in the clause for $\Diamond$ and so become 'visible' in the form of $\Diamond\bot$ statements and $\sqsubseteq$ is no longer reflexive. Constructive modal logics such as CK therefore allow for truth-value 'gluts' (i.e., they allow for the truth of formulas of the form $A \wedge {\sim}A$) as well as truth value 'gaps' (i.e., formulas of the form $A \vee {\sim}A$ fail to be verified at a state).

Consider now the *impossibility* and *non-necessity* operators [10] $\ominus$ and $\boxminus$ which occupy the right side of the squares in Fig. 1, where $\ominus$ (or $\boxminus$) is the negative counterpart of the positive modality $\Diamond$ (or $\Box$) and vice versa:[1]

$$\mathfrak{M}, s \models \ominus A \;\Leftrightarrow\; \forall x.\, (sRx \Rightarrow \mathfrak{M}, x \not\models A) \tag{5}$$

$$\mathfrak{M}, s \models \boxminus A \;\Leftrightarrow\; \exists x.\, (sRx \;\&\; \mathfrak{M}, x \not\models A). \tag{6}$$

Classically, these modalities can be expressed in terms of $\Diamond$ and $\Box$ as $\neg\Diamond A$ (or $\Box\neg A$) and $\neg\Box A$ (or $\Diamond\neg A$). Intuitionistically, this is no longer the case, because intuitionistic negation $\sim$ is weaker than classical negation $\neg$ as it fails Excluded Middle (EM).

To our knowledge, Došen was the first to pay extensive attention to the negative modalities in intuitionistic logic. For each $\otimes \in \{\Box, \Diamond, \boxminus, \ominus\}$, Došen produced

[1] Such negative modalities have been considered in the literature on FDE and Routley semantics as ways of capturing forms of negation [17–19,28,36] often called 'constructible' or 'strong' negation [26,37]. We do not suggest that the role of $\ominus$ and $\boxminus$ in the logic CKD is to capture forms of *negation*; rather, we are simply interested in how they behave in a constructive setting (i.e. in which the Disjunction Property holds) as *modal operators*.

a logic $HK\otimes$, combining $\otimes$ with IPL. In $HK\Box$, the classical truth conditions for $\Box$ in (2) are employed together with the frame condition $(\sqsubseteq; R) \subseteq (R; \sqsubseteq)$, whilst in $HK\Diamond$ the classical truth conditions for $\Diamond$ in (1) are employed together with $(\sqsupseteq; R) \subseteq (R; \sqsupseteq)$ [6]. In $HK \ominus\!\!\!\!\Diamond$, the truth conditions (5) are employed for $\ominus\!\!\!\!\Diamond$ and $(\sqsubseteq; R) \subseteq (R; \sqsupseteq)$ are imposed, and in $HK\boxminus$ the truth conditions (6) are employed for $\boxminus$ with the frame condition $(\sqsupseteq; R) \subseteq (R; \sqsubseteq)$ [9–11]. Each $HK\otimes$ for $\otimes \in \{\Box, \Diamond, \boxminus, \ominus\!\!\!\!\Diamond\}$ is a conservative extension of IPL which is sound and complete with respect to birelational frames, subject to the associated frame conditions. The work of Došen was very much out on a limb with respect to the mainstream in intuitionistic logic, which concentrated on the positive modalities almost entirely [42], and only in recent years have the negative modalities been given more attention in the literature on intuitionistic and constructive logic [15,16,28]. Curiously, Došen did not produce a logic which combines $\ominus\!\!\!\!\Diamond$, $\boxminus$, $\Diamond$ and $\Box$ with IPL on a single birelational frame $(S, \sqsubseteq, R)$ in which the modalities are interpreted with respect to the same R.

Some combinations of the modalities $\Box, \Diamond, \boxminus, \ominus\!\!\!\!\Diamond$ with each other and negation $\sim$ have been explored. For example, [6] consider a system $HK\,\Box\,\Diamond$, combining $\Diamond$ and $\Box$. They give two equivalent axiomatisations of $HK\,\Box\,\Diamond$, yet the theory does not have the DP, nor is it conservative over either $HK\Diamond$ and $HK\Box$ (see [6] for discussion). Drobyschevich [15] investigates the properties of the combined modality $\sim \ominus\!\!\!\!\Diamond A$ in an extension $\mathsf{N}^\star$ of IPL he calls $HKNR$ and he studies $\sim \ominus\!\!\!\!\Diamond\, A$ in $HK\boxminus$ in an extension he calls $HKN\boxminus$. $\mathsf{N}^\star$ is an extension of $HK\,\ominus\!\!\!\!\Diamond$ but without $\perp$, known as N [11]. In $\mathsf{N}^\star$, however, $\boxminus$ and $\ominus\!\!\!\!\Diamond$ collapse into a single modality, since R is a functional accessibility relation, called the 'Routley star' operation. Addition of $\Diamond$ to $HK\,\ominus\!\!\!\!\Diamond$ plus frame conditions imposed to ensure hereditariness, have the result that the modalities $\ominus\!\!\!\!\Diamond$ and $\Diamond$ become interdefinable as $\Diamond A \leftrightarrow \sim \ominus\!\!\!\!\Diamond A$ and $\ominus\!\!\!\!\Diamond A \leftrightarrow \sim \Diamond\, A$ via intuitionistic negation. But, from a constructive point of view, the directions of $\sim \ominus\!\!\!\!\Diamond A \rightarrow \Diamond A$ and $\sim \Diamond\, A \rightarrow \ominus\!\!\!\!\Diamond A$ are suspicious. If we can prove the absurdity of something being impossible (i.e., $\sim \ominus\!\!\!\!\Diamond\, A$), this doesn't mean we have a positive construction which will allow us to show that something is possible (i.e., $\Diamond A$). Likewise, if we can prove that a certain possibility is absurd (i.e., $\sim \Diamond\, A$), then we can't conclude that we have a proof that it is impossible. Similarly, addition of $\Box$ to $HK\boxminus$ plus frame conditions make $\Box$ and $\boxminus$ interdefinable ($\Box B \leftrightarrow \sim \boxminus\, B$ and $\boxminus B \leftrightarrow \sim \Box\, B$) and similar reservations regarding the constructive content of the implications $\sim \boxminus\, B \rightarrow \Box A$ and $\sim \Box\, B \rightarrow \boxminus B$ can be made. Adding $\Box$ and its associated heredity frame condition forces axiom $\boxminus B \vee \sim \Box\, B$ without $\boxminus B$ or $\sim \Box\, B$ being provable by itself. This breaks DP and thus constructiveness of non-necessity. This is a general side effect of the frame conditions: Each positive modality $\oplus$ induces the disjunction $\sim \ominus\, A \vee \ominus A$, where $\ominus$ is the corresponding negative modality, and each negative modality $\ominus$ induces the disjunction $\sim \oplus\, A \vee \oplus A$. Similar effects have been observed for $\mathsf{N}^\star$ [13], where the scheme $\ominus\!\!\!\!\Diamond A \vee \ominus\!\!\!\!\Diamond {\sim} A$ is valid and for $HK\,\Box\,\Diamond$, where $\Diamond A \vee \Box \neg A$ is an axiom, both in violation of the DP.

1.2 Contributions

The combination of the modalities $\Box$, $\Diamond$, $\diamondminus$ and $\boxminus$ so as to ensure a constructive logic is a delicate matter. Can the negative modalities $\diamondminus$ and $\boxminus$ live happily side-by-side with their 'positive' counterparts $\Diamond$ and $\Box$, within a constructive setting? According to consolidated tradition, a *constructive logic* means a logic in which the *Disjunction Property (DP)* holds: whenever $A \vee B$ is a theorem then either A is a theorem, or B is a theorem. Constructiveness thus construed is not a property of operators, but of logics. Our question is therefore whether we can combine the modalities whilst retaining the DP. In this paper we show that if we interpret $\diamondminus$ and $\boxminus$ constructively like $\Box$ and $\Diamond$ in (4) and (3),

$$\mathfrak{M}, s \ \models \diamondminus A \ \Leftrightarrow \forall s' \sqsupseteq s. \forall x. \, (s' \; R \; x \Rightarrow \mathfrak{M}, x \not\models \; A) \tag{7}$$

$$\mathfrak{M}, s \ \models \boxminus A \ \Leftrightarrow \forall s' \sqsupseteq s. \exists x. \, (s' \; R \; x \; \& \; \mathfrak{M}, x \not\models \; A) \tag{8}$$

then we can avoid the collapse of the modalities $\diamondminus$, $\boxminus$, $\Diamond$ and $\Box$, abandoning the frame conditions relating $\sqsubseteq$ and R:[2] The logic created by thus adding the negative modalities to CK [25,33], we dub CKD. CKD is both conservative over CK and constructive in the sense that it satisfies DP.

The Došen square is not supposed to be analogous to the SO; in fact, only certain features of the square of oppositions hold in CKD. The logic CKD will treat the relationships between the modalities in DS as follows. On the one hand, $\Diamond$ and $\diamondminus$ will be contradictories, i.e., $\sim(\Diamond A \wedge \diamondminus A)$ is valid. Similarly, necessity $\Box$ and unnecessity $\boxminus$ will be incompatible, i.e., $\sim(\Box A \wedge \boxminus A)$ is valid. Due to the absence of the Excluded Middle and fallibility, the modalities $\Diamond \sim A$ and $\boxminus A$ are independent in CKD, distinguishing the Došen square from the classical SO. In CKD $\Diamond \sim A \rightarrow \boxminus A$ follows from infallibility, expressed by $\diamondminus \bot$. Moreover, we have $\boxminus A \rightarrow \Diamond \sim A$ assuming $\Box(A \vee \sim A)$, which expresses the necessitation of the Excluded Middle. Similarly, $\Box \sim A$ and $\diamondminus A$ are independent. Again, the connection hinges on the absence of gluts and gaps: In CKD we have that infallibility $\diamondminus \bot$ entails $\Box \sim A \rightarrow \diamondminus A$ and similarly $\Box(A \vee \sim A)$ entails $\diamondminus \sim A \rightarrow \Box A$. Unless every state has an R-successor (seriality) – expressible by $\Diamond \top$ – the modality pairs $\Box$, $\Diamond$ and $\diamondminus$, $\boxminus$ are independent. However, like in the classical SO it holds that from seriality $\Diamond \top$ follows $\Box A \rightarrow \Diamond A$ and $\diamondminus A \rightarrow \boxminus A$.

In Sect. 2 the model theory of CKD is introduced and the DP is proven. In Sect. 3.1, an axiomatic Hilbert system, H_{CKD}, is provided for CKD, and its conservativity over CK and over N is sketched. In Sect. 3.2, a sequent calculus, G_{CKD}, for CKD is provided, proving its soundness and completeness with respect to C-frames , and its translation into H_{CKD} is obtained. As a corollary of completeness, it follows that the theory of CKD has the FMP, is cut-free and decidable. In Sect. 4 we end with Conclusions.

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[2] Our claim is that the doubly quantified truth conditions are a neat way out of the bind, not that they are necessary in order to provide a logic which combines $\Box$, $\Diamond$, $\diamondminus$ and $\boxminus$ interpreted with respect to the same relation.

2 The Došen Square CKD of Constructive Modalities

We begin by introducing the frames and models we will make use of.

Definition 1 (C-frame). *A* C-frame $\mathfrak{F} = (S, \leq, F, R)$ *consists of a set* $S \neq \varnothing$ *of* states, *a preordering* $\leq$ *(reflexive & transitive) on* S, *a subset* $F \subseteq S$ *of* fallible *states, s.t.* $s_1 \leq s_2$ *and* $s_1 \in F$ *implies* $s_2 \in F$ *and a binary relation* R *on* S. *On a C-frame we define the ordering* $\sqsubseteq =_{df} \{(s, s') \mid s \leq s' \,\&\, s' \notin F\}$ *and if* $F = \emptyset$ *then* $\mathfrak{F}$ *is called* infallible.

C-frames are non-standard in three ways. Firstly, we do not require any frame property to constrain the interaction of $\leq$ and R. In this way, we obtain a minimal logic to fuse the modalities $\Diamond$, $\Box$, $\diamondminus$ and $\boxminus$ on a single accessibility relation. Secondly, we only require $\leq$ to be a preorder rather than a partial ordering, i.e., omitting antisymmetry allows for the possibility of cyclic structures which are crucial in establishing the FMP. Thirdly, by adding the fallibility set $F \subseteq S$ we can declare frame states as 'internally exploded' and make states $s \in S$ such that $s \; R \; s' \in F$ border states "one world away from absurdity". This is instrumental to preserve constructiveness for certain extensions of CKD and amounts to working with an intuitionistic accessibility $\sqsubseteq$ that is not only not antisymmetric but also not reflexive.

Definition 2 (C-model). *A* C-model $\mathfrak{M} = (\mathfrak{F}, V)$ *consists of a C-frame* $\mathfrak{F} = (S, \leq, F, R)$ *together with a* valuation *function* $V : S \to \mathcal{P}(Var)$ *from* S *to the subset of propositional variables subject to* heredity *and* explosion *conditions: if* $s_1 \leq s_2$ *then (i)* $V(s_1) \subseteq V(s_2)$ *and (ii) if* $s \in F$ *then* $V(s) = Var$.

The language $\mathcal{L}_{\mathsf{CKD}}$ of CKD coincides with that of intuitionistic propositional logic (IPL) extended by the four modalities $\{\Box, \Diamond, \diamondminus, \boxminus\}$.

Definition 3 (Language $\mathcal{L}_{\mathsf{CKD}}$). *The language* $\mathcal{L}_{\mathsf{CKD}}$ *is based on a denumerable set of propositional variables* $Var = \{p, q, \ldots\}$. *The set of well-formed* CKD-*formulas over* Var *is inductively defined by the following grammar:*

$$A, B ::= p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A \mid \diamondminus A \mid \boxminus A$$

Note that $\sim A$ *abbreviates intuitionistic negation* $A \to \bot$, $A \leftrightarrow B$ *is expressed by* $(A \to B) \wedge (B \to A)$ *and implication* $\to$ *is right-associative.*

The interpretation of $\mathcal{L}_{\mathsf{CKD}}$ is by means of the following satisfaction relation:

Definition 4 (Satisfaction in C-models). *Let* $\mathfrak{M} = (S, \leq, F, R, V)$ *be a C-model. The notion of a formula* A *being* satisfied *in a C-model* $\mathfrak{M}$ *at a state* s *is* *defined inductively, for the modal operators as in* (3)*,* (4)*,* (7)*,* (8) *and for the other operators as in* IPL.[3]

$$\begin{aligned}
\mathfrak{M}, s &\models \top, \\
\mathfrak{M}, s &\models \bot && \textit{iff } s \in F, \\
\mathfrak{M}, s &\models p && \textit{iff } p \in V(s), \\
\mathfrak{M}, s &\models A \wedge B && \textit{iff } \mathfrak{M}, s \models A \textit{ and } \mathfrak{M}, s \models B, \\
\mathfrak{M}, s &\models A \vee B && \textit{iff } \mathfrak{M}, s \models A \textit{ or } \mathfrak{M}, s \models B, \\
\mathfrak{M}, s &\models A \to B && \textit{iff for all } s' \sqsupseteq s, \textit{ if } \mathfrak{M}, s' \models A \textit{ then } \mathfrak{M}, s' \models B.
\end{aligned}$$

The semantics of Definition 4 permits us to assume that each fallible state $f \in F$ is a dead end of the frame, i.e., there is no s with either $f\ R\ s$ or $f \sqsubseteq s$. Moreover, we may assume without loss of generality that every $f \in F$ is reachable by an R-step from a non-fallible state, i.e., there is $s \notin F$ with $s\ R\ f$. We call such frames $\bot$-*condensed*. In $\bot$-condensed frames we have $\mathfrak{M}, s \models \diamondminus\bot$ for all $s \in S$ iff $\mathfrak{M}$ is infallible, i.e., $F = \varnothing$.

Definition 5 (Validity). *A formula A is* valid in a C-model $\mathfrak{M}$*, written $\mathfrak{M} \models A$, if $\mathfrak{M}, s \models A$ for all $s \in S$. If $\mathfrak{M}$ is clear from the context, we will simply write $s \models A$. A formula A is* valid in a C-frame $\mathfrak{F}$*, written $\mathfrak{F} \models A$, if $\mathfrak{M} \models A$ for all models $\mathfrak{M} = (\mathfrak{F}, V)$ over $\mathfrak{F}$. We lift all the validity relations to sets of formulas Γ in the usual conjunctive way, for a state $\mathfrak{M}, s \models \Gamma$, a model $\mathfrak{M} \models \Gamma$ and frame $\mathfrak{F} \models \Gamma$.*

Lemma 1. *Satisfaction is hereditary and explosive, i.e., (i) $s \models A$ iff $\forall s' \sqsupseteq s.\, s' \models A$ and (ii) $s \in F$ implies $s \models A$.*

We define a semantic consequence relation axiomatising the semantic levels of the modal satisfaction relation at the frame, model and state level (global vs. local consequence) [21,31]. It allows us to map the semantic definition of a logical system to its syntactic axiomatisation in the form a Hilbert calculus, to be used in the discussion of the correspondences between Došen's logics and CKD (see Theorem 3).

Definition 6 (Semantic Entailment). *Let Ω (*frame hypotheses*), Φ (*model hypotheses*), Γ (*state hypotheses*) and Π (*state assertions*) be sets of formulas. We write $\Omega; \Phi; \Gamma \models \Pi$ iff for all C-frames $\mathfrak{F} = (S, \leq, F, R)$ with $\mathfrak{F} \models \Omega$ and all models $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{M} \models \Phi$ and all states $s \in S$ with $\mathfrak{M}, s \models \Gamma$, we have $\mathfrak{M}, s \models \Pi$.*

Let CKD be the set of all universally valid formulas, i.e., $\mathsf{CKD} = \{A \mid \emptyset; \emptyset; \emptyset \models A\}$. This set is a logical theory, i.e., closed under Modus Ponens and Substitution.

[3] As usual, we can take $\top =_{df} p \to p$ for a variable $p \in Var$. Interestingly, also absurdity $\bot$ is representable, viz. as the non-necessity of truth, i.e., $\bot =_{df} \boxminus\top$. First, $\mathfrak{M}, s \models \bot$ implies $\mathfrak{M}, s \models \boxminus\top$ since by definition there is no s' with $s \sqsubseteq s'$. Second, if $\mathfrak{M}, s \models \boxminus\top$ and $s \notin F$ we would have $s \sqsubseteq s$ and so by the truth condition for $\boxminus$ there must be s'' with $s\ R\ s''$ and $\mathfrak{M}, s'' \not\models \top$. This is impossible, hence $s \in F$ and so $\mathfrak{M}, s \models \bot$.

The theory CKD does not validate the axiom $\diamondminus A \vee \diamondminus {\sim} A$ of Drobyshevich nor any of the axiom schemes $\otimes A \vee {\sim} \otimes A$ for $\otimes \in \{\Diamond, \diamondminus, \Box, \boxminus\}$, as can be readily verified.

One of the hallmarks of constructive logics is the *disjunction property* (DP), stating that the proof of a disjunction $A \vee B$ requires positive evidence in the form of a proof of either A or B. The absence of frame conditions in CKD admits of a particularly simple model-theoretic argument for the Disjunction Property (Theorem 1) that proceeds completely analogously to IPL.

Theorem 1 (Disjunction Property). *The theory* CKD *has the Disjunction Property.*

A striking feature of CKD is that the Finite Model Property (Theorem 8) depends on permitting $\preceq$-cycles in C-frames. Consider the cyclic countermodel $\mathfrak{M}_c$ on the right in Fig. 2. The states s_0, s_1 each satisfy ${\sim}\boxminus A$, ${\sim}\boxminus B$ and $\boxminus(A \wedge B)$, being mutual $\sqsubseteq$-successors sharing the same theory. Yet, they cannot be condensed into a single state $s = \{s_0, s_1\}$, as s would have both s_0' and s_1' as immediate R-successors, and satisfy $s \models \boxminus A \wedge \boxminus B$ which is inconsistent with the properties of s_0 and s_1. Observe that $\mathfrak{M}_c$ does not satisfy Došen's HK$\boxminus$ frame condition [10] $(\sqsupseteq\, ; R) \subseteq (R\, ; \sqsubseteq)$ that generates the constructively disputable scheme ${\sim}\boxminus A \to \Box A$. Even more, $\mathfrak{M}_c$ provides a countermodel for the distribution axioms $\boxminus(A \wedge B) \to (\boxminus A \vee \boxminus B)$ and $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$. Their absence is characteristic of CKD as a non-normal modal logic, due to the 'doubly-quantified' truth conditions in the existential modalities $\boxminus$ (8) and $\Diamond$ (3).

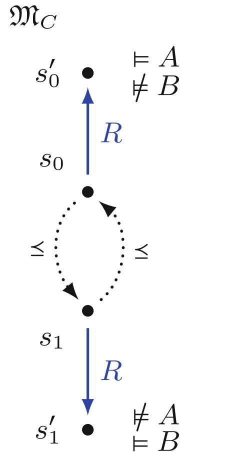

Fig. 2. Cyclic model.

Proposition 1. *The scheme* $({\sim}\boxminus A \wedge {\sim}\boxminus B) \to {\sim}\boxminus(A \wedge B)$ *is valid in* HK$\boxminus$ *[10] but not a theorem of* CKD*. Every* CKD *counter model for it is infinite or cyclic.*

3 Proof Systems for CKD

We develop the proof theory of CKD, in the form of the Hilbert calculus H_{CKD} and the Gentzen-style sequent calculus G_{CKD}. The calculus H_{CKD} captures semantic entailment $\Omega; \Phi; \Gamma \models \Pi$ where the set of state hypotheses $\Gamma = \varnothing$ is empty, which corresponds to the restriction [21] of rule *Nec* to apply to theorems only. In contrast, the sequent calculus G_{CKD} works entirely at the state level (i.e., $\Omega = \varnothing = \Phi$).

3.1 CKD Global Reasoning: The Hilbert Calculus H_{CKD}

Definition 7 (Hilbert Deduction and CKDAxioms). *Let Ω and Φ be sets of formulas. We write $\Omega; \Phi \vdash_H A$ if there is a sequence $A_0, A_1, \ldots A_{n-1}$ of formulas such that $A_{n-1} = A$ and each A_i $(i \in n)$ is either a* model hypothesis *from Φ,* *a substitution instance of some* frame hypothesis *or* axiom *in Ω, or arises by the rules of* Modus Ponens *(MP) or* Necessitation *(Nec) from formulas A_j $(j < i)$ appearing earlier. The set of axioms CKD_{ax} consist of those for* IPL *(see, e.g., [14]) and the modal axioms as depicted in the following. We write* $\mathsf{CKD}; \Phi \vdash_H A$ *for* $\mathsf{CKD}_{ax}; \Phi \vdash_H A$.

$$
\begin{array}{rlrl}
\Box K =_{df} & \Box(A \to B) \to \Box A \to \Box B & \diamondminus 2 =_{df} & \diamondminus A \to \diamondminus B \to \diamondminus(A \vee B) \\
\Diamond K =_{df} & \Box(A \to B) \to \Diamond A \to \Diamond B & \boxminus 2 =_{df} & \diamondminus A \to \boxminus B \to \boxminus(A \vee B) \\
\diamondminus K =_{df} & \Box(A \to B) \to \diamondminus B \to \diamondminus A & N5 =_{df} & \diamondminus(A \wedge B) \to \Diamond A \to \boxminus B \\
\boxminus K =_{df} & \Box(A \to B) \to \boxminus B \to \boxminus A & N6 =_{df} & \Box(A \vee B) \to \boxminus A \to \Diamond B \\
\Box 2 =_{df} & \diamondminus A \to \Box(A \vee B) \to \Box B & N7 =_{df} & \boxminus \top \to \bot \\
\Diamond 2 =_{df} & \diamondminus A \to \Diamond(A \vee B) \to \Diamond B & &
\end{array}
$$

Theorem 2 (Hilbert Soundness). *If* $\mathsf{CKD}; \Phi \vdash_H A$ *then* $\varnothing; \Phi; \varnothing \models A$.

The axioms $\Box K$, $\Diamond K$, $\diamondminus K$, $\boxminus K$ in combination with *Nec* ensure that the logic is extensional, i.e., satisfies the *Replacement Principle*: If $\mathsf{CKD}; \Phi \vdash_H A \leftrightarrow B$ then $\mathsf{CKD}; \Phi \vdash_H \phi[A] \leftrightarrow \phi[B]$ where $\phi[.]$ is an arbitrary formula context. In the axiomatisation by [10] replacement is achieved with the *R-Rules*

$$
\frac{\Omega; \Phi \vdash_H A \to B}{\Omega; \Phi \vdash_H \oplus A \to \oplus B}\, R\oplus \qquad \frac{\Omega; \Phi \vdash_H A \to B}{\Omega; \Phi \vdash_H \ominus B \to \ominus A}\, R\ominus
$$

for $\oplus \in \{\Diamond, \Box\}$ and $\ominus \in \{\diamondminus, \boxminus\}$. These are derivable from our axioms $\Box K$, $\Diamond K$, $\diamondminus K$, $\boxminus K$, Modus Ponens *MP* and Necessitation *Nec*.

The axioms $\otimes K$ (for $\otimes \in \{\Diamond, \Box, \diamondminus, \boxminus\}$) deal with the consequences of a necessary implication $\Box(A \to B)$ for statements made under modalities. Analogously, the axioms $\otimes 2$ express the consequences of an impossible property $\diamondminus A$ for modalised statements. The import of axiom $\Box 2$ is that if a disjunction $A \vee B$ is necessary and one of the disjuncts is impossible, then the other disjunct is necessary. The axiom $\Diamond 2$ says that if a disjunction $A \vee B$ is possible and one of the disjuncts is impossible, then the other disjunct is possible. The axiom $\diamondminus 2$ states that if two properties are impossible, then their disjunction is impossible, too. The axiom $\boxminus 2$ says that if one property is impossible and another is non-necessary, then its disjunction is non-necessary. $N5$ implies that if a conjunction $A \wedge B$ is impossible while one of the conjuncts is possible then the other conjunct is non-necessary. $N6$ is the statement that if a disjunction is necessary and one disjunct non-necessary then the other disjunct is possible. The final axiom $N7$ gives a representation of absurdity as non-necessity of truth.

Let us verify that possibility $\Diamond A$ and impossibility $\diamondminus A$ are contradictory, i.e., $\vdash_H \sim(\Diamond A \wedge \diamondminus A)$. Since $\vdash_H A \leftrightarrow (A \wedge \top)$ we obtain $\vdash_H (\Diamond A \wedge \diamondminus A) \leftrightarrow (\Diamond A \wedge \diamondminus(A \wedge \top))$ by the Replacement Principle. Then, instantiating $N5$ as $\vdash_H \diamondminus(A \wedge \top) \to \Diamond A \to \boxminus \top$, we can derive $\vdash_H (\Diamond A \wedge \diamondminus A) \to \boxminus \top$ by IPL. Finally chaining up in IPL with the implication $N7$ this implies $\vdash_H (\Diamond A \wedge \diamondminus A) \to \bot$.

As explained above, in the standard Kripke model theory, the presence of frame conditions force a collapse of the modalities and the loss of DP. In CKD

where we maintain their independence we can study existing theories as fragments and extensions. Došen's model theory of HK$\otimes$-frames [10] in the language $\mathcal{L}_{\otimes} = \{\bot, \wedge, \vee, \to, \otimes\}$ for fixed $\otimes \in \{\Diamond, \diamondminus, \Box, \boxminus\}$ generates the logic called HK$\otimes$. A HK$\otimes$-frame is an infallible C-frame satisfying the HK$\otimes$ frame condition (see Sect. 1). On such C-frames our truth conditions for $\otimes$ collapse to the ones of Došen for $\Diamond$, $\diamondminus$, $\Box$ and $\boxminus$. As a result, CKD is conservative over HK$\otimes$ in the language fragment $\mathcal{L}_{\otimes}$. However, the modalities $\otimes \in \{\diamondminus, \Diamond, \boxminus\}$ of CKD are weaker than the ones of HK$\otimes$. This is not surprising since we want to avoid the collapses arising from a naive fusion in the standard model theory. The properties of $\otimes$ in HK$\otimes$ can be regained in CKD by imposing frame conditions. Recall that N [11] is $HK\diamondminus$ in the language $\mathcal{L}_{\mathsf{N}} = \{\wedge, \vee, \to, \diamondminus\}$ without $\bot$. Now consider the axiom schemes:

$$(\boxminus 1) : \boxminus(A \wedge B) \to (\boxminus A \vee \boxminus B) \qquad (\Diamond 1) : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$$
$$(\diamondminus 2) : \diamondminus\bot \qquad (\Diamond 2) : \sim \Diamond \bot$$
$$(\Box\Diamond 1) : \Diamond A \vee \Box \sim A \qquad (\Box\Diamond 2) : \sim(\Diamond A \wedge \Box \sim A).$$

It can be shown that CKD in $\mathcal{L}_{\Box}$ corresponds to HK$\Box$ and in $\mathcal{L}_{\mathsf{N}}$ to N; HK$\diamondminus$ is CKD + $\diamondminus$2 restricted to $\mathcal{L}_{\diamondminus}$; CKD + $\boxminus$1 corresponds to HK$\boxminus$ in $\mathcal{L}_{\boxminus}$ and CKD + $\Diamond$1 + $\Diamond$2 generates the theory HK$\Diamond$ in $\mathcal{L}_{\Diamond}$. Finally, the extension CKD + $\diamondminus$2 + $\Box\Diamond$1 + $\Box\Diamond$2 coincides with the non-constructive theory HK$\Box\Diamond$ investigated by Božić & Došen [6] in $\mathcal{L}_{\Box\Diamond} =_{df} \{\bot, \wedge, \vee, \to, \Box, \Diamond\}$. In $\mathcal{L}_{\Box\Diamond}$ the logic CKD does not lose constructiveness like HK$\Box\Diamond$ does. In fact, CKD is conservative over CK [25] that combines the positive modalities $\Box$, $\Diamond$ by extending IPL with the axioms $\Box K$ and $\Diamond K$ and the *Nec* rule.

Theorem 3 (Conservativity). CKD *is a conservative extension of* N *and* CK *and* HK$\Box$. *The theories* HK$\otimes$ *for* $\otimes \in \{\Diamond, \diamondminus, \boxminus\}$ *and* HK $\Box\Diamond$ *are axiomatic extensions of* CKD*:*

For A in the language $\mathcal{L}_{\Box\Diamond}$: CK; $\varnothing \vdash_H A$ *iff* CKD; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\Box\Diamond}$: $HK\Box\Diamond$; $\varnothing \vdash_H A$ *iff* CKD, $\diamondminus$2, $\Box\Diamond$1, $\Box\Diamond$2; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\boxminus}$: $HK\boxminus$; $\varnothing \vdash_H A$ *iff* CKD, $\boxminus$1; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\Box}$: $HK\Box$; $\varnothing \vdash_H A$ *iff* CKD; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\diamondminus}$: $HK\diamondminus$; $\varnothing \vdash_H A$ *iff* CKD, $\diamondminus$2; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\mathsf{N}}$: N; $\varnothing \vdash_H A$ *iff* CKD; $\varnothing \vdash_H A$.
For A in the language $\mathcal{L}_{\Diamond}$: $HK\Diamond$; $\varnothing \vdash_H A$ *iff* CKD, $\Diamond$1, $\Diamond$2; $\varnothing \vdash_H A$.

3.2 Landing at Došen Square: The Sequent Calculus G_{CKD}

The proof theory of CK has previously been investigated in terms of a Natural Deduction system [3], multisequent calculi [22–24], nested sequents [2] and a tableaux-based calculus [33]. Our sequent calculus G_{CKD} is a refinement of the multisequent calculus of Dragalin [12] for IPL, similar to [22], that is enriched by additional scopes to cover *local* and *global* properties. This is required for the interpretation of the four modalities, and is consonant with Poggiolesi's remark that

> *[...] the failures of the search for a sequent calculus for modal logic gave rise to the idea that the standard Gentzen calculus could only account for classical and intuitionistic logics and should therefore be enriched.* [30][Sec. 2.3, p. 51]

In relation to the many variants explored in the literature (see [30]) G_{CKD} can be considered a *higher-arity* extension in the sense of Sato [32] and Blamey and Humberstone [5]. Notably, following Dragalin, we consider the *logical variant* of the Gentzen calculus (in the terminology of [30]) approach to sequents, where all structural rules are built into the axioms and logical rules. This is justified as we are dealing with a logical theory that has not been discussed before and thus are primarily interested in model-theoretic expressiveness, completeness, constructiveness and finite-model property.

A *sequent* in CKD is a structure $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ where the sets Γ and Π express direct truth and falsity at a state, as in a standard sequent. The sets Δ, Θ, Σ and Ψ are finite (possibly empty) sets of *signed* formulas each of which can be *strong* A^+ or *weak* A^-. With this structure, our sequents provide a formalisation of Došen square as visualised in Fig. 3. Note, that in Γ (Π) all formulas have no sign. Specifically, Δ and Θ contain positive existential and universal statements about modally reachable successors, while Σ and Ψ are negative existential and universal statements. Depending on the scope set, the sign $t \in \{+, -\}$ of a polarised formula A^t distinguishes *local* or hereditary *global* properties, where for a set X of signed formulas we write $X^t =_{df} \{A^t \mid A^t \in X\}$. For instance, $A^+ \in \Delta$ expresses the constraint that there *exists* an immediate R-successor satisfying A, while $A^- \in \Delta$ is the weaker statement that such a successor is reachable via $\sqsubseteq;R$, i.e., only after an initial intuitionistic step. Analogously, $A^- \in \Sigma$ says that A is false along immediate R-successors whereas $A^+ \in \Sigma$ is the stronger statement that A is false along all $\sqsubseteq;R$. This is captured by the following Definition 8.

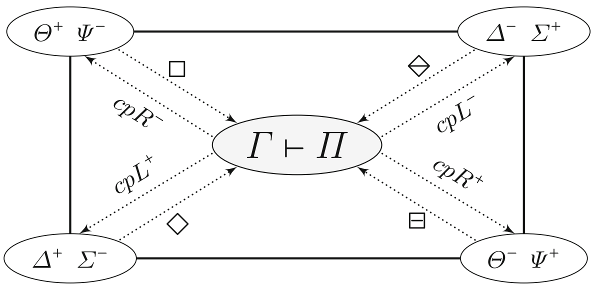

Fig. 3. The Došen square structure of G_{CKD} sequents.

Definition 8 (Refutability). *A sequent* $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ *is* refuted *in a state* s *of a C-model* $\mathfrak{M} = (S, \leq, F, R, V)$ *iff the following holds:*

- $\forall A \in \Gamma. \mathfrak{M}, s \models A.$
- $\forall B^- \in \Delta. \exists s'. s \sqsubseteq ; R\; s' \;\&\; \mathfrak{M}, s' \models B;$ $\forall B^+ \in \Delta. \exists s'. s \sqsubseteq s\; R\; s' \;\&\; \mathfrak{M}, s' \models B.$
- $\forall C^- \in \Theta, s'. s \sqsubseteq s\; R\; s' \Rightarrow \mathfrak{M}, s' \models C;$ $\forall C^+ \in \Theta, s'. s \sqsubseteq ; R\; s' \Rightarrow \mathfrak{M}, s' \models C.$
- $\forall D \in \Pi. \mathfrak{M}, s \not\models D.$
- $\forall E^- \in \Sigma, s'. s \sqsubseteq s\; R\; s' \Rightarrow \mathfrak{M}, s' \not\models E;$ $\forall E^+ \in \Sigma, s'. s \sqsubseteq ; R\; s' \Rightarrow \mathfrak{M}, s' \not\models E.$
- $\forall F^- \in \Psi. \exists s'. s \sqsubseteq ; R\; s' \;\&\; \mathfrak{M}, s' \not\models F;$ $\forall F^+ \in \Psi. \exists s'. s \sqsubseteq s\; R\; s' \;\&\; \mathfrak{M}, s \not\models F.$

A sequent is called refutable*, written* $\Gamma \star \Delta \star \Theta \not\models \Pi \star \Sigma \star \Psi$ *if there exists a C-model* $\mathfrak{M}$ *and a state* s *of* $\mathfrak{M}$ *in which it is refuted. A sequent is called* valid*, written* $\Gamma \star \Delta \star \Theta \models \Pi \star \Sigma \star \Psi$*, if it is not refutable.*

$$\frac{}{A, \Gamma \star \Delta \star \Theta \vdash A, \Pi \star \Sigma \star \Psi}\,Ax \qquad \frac{}{\bot, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\bot L^{\dagger} \qquad \frac{}{\Gamma \star \Delta \star \Theta \vdash \top, \Pi \star \Sigma \star \Psi}\,\top R$$

$$\frac{A, B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}{A \wedge B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\wedge L \qquad \frac{\Gamma \star \Delta \star \Theta \vdash A, \Pi \star \Sigma \star \Psi \quad \Gamma \star \Delta \star \Theta \vdash B, \Pi \star \Sigma \star \Psi}{\Gamma \star \Delta \star \Theta \vdash A \wedge B, \Pi \star \Sigma \star \Psi}\,\wedge R$$

$$\frac{A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi \quad B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}{A \vee B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\vee L \qquad \frac{\Gamma \star \Delta \star \Theta \vdash A, B, \Pi \star \Sigma \star \Psi}{\Gamma \star \Delta \star \Theta \vdash A \vee B, \Pi \star \Sigma \star \Psi}\,\vee R$$

$$\frac{\Gamma \star \Delta \star \Theta \vdash A, \Pi \star \Sigma \star \Psi \quad B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}{A \to B, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\to L \qquad \frac{A, \Gamma \star \varnothing \star \Theta^+ \vdash B \star \Sigma^+ \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash A \to B, \Pi \star \Sigma \star \Psi}\,\to R$$

$$\frac{B, \Theta^+ \star \varnothing \star \varnothing \vdash \Sigma^+ \star \varnothing \star \varnothing}{\Gamma \star B^-, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,cpL^- \qquad \frac{\Theta^+ \star \varnothing \star \varnothing \vdash F, \Sigma^+ \star \varnothing \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star F^-, \Psi}\,cpR^-$$

$$\frac{B, \Theta \star \varnothing \star \varnothing \vdash \Sigma \star \varnothing \star \varnothing}{\Gamma \star B^+, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,cpL^+ \qquad \frac{\Theta \star \varnothing \star \varnothing \vdash F, \Sigma \star \varnothing \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star F^+, \Psi}\,cpR^+$$

$$\frac{\Gamma \star A^+, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}{\Diamond A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\Diamond L^{\dagger} \qquad \frac{\Gamma \star \varnothing \star \Theta^+ \vdash \bot \star D^-, \Sigma^+ \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \Diamond D, \Pi \star \Sigma \star \Psi}\,\Diamond R$$

$$\frac{\Gamma \star \Delta \star A^+, \Theta \vdash \Pi \star \Sigma \star \Psi}{\Box A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\Box L \qquad \frac{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star D^-, \Psi}{\Gamma \star \Delta \star \Theta \vdash \Box D, \Pi \star \Sigma \star \Psi}\,\Box R$$

$$\frac{\Gamma \star \Delta \star \Theta \vdash \Pi \star A^+, \Sigma \star \Psi}{⟐ A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,⟐ L \qquad \frac{\Gamma \star D^-, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}{\Gamma \star \Delta \star \Theta \vdash ⟐ D, \Pi \star \Sigma \star \Psi}\,⟐ R$$

$$\frac{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star A^+, \Psi}{\boxminus A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\,\boxminus L^{\dagger} \qquad \frac{\Gamma \star \varnothing \star D^-, \Theta^+ \vdash \bot \star \Sigma^+ \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \boxminus D, \Pi \star \Sigma \star \Psi}\,\boxminus R$$

Fig. 4. G_{CKD} Sequent Rules. The sets Γ, Π are without sign. In the rules cpL^t and cpR^t all signs are dropped in the occurrences of the sets Θ, Θ^+ and Σ, Σ^+ in the premisses. Tagged rules (†) require its conclusion to be strict, i.e., $|\Delta \cup \Pi \cup \Psi| \geq 1$. We treat all scopes as sets with implicit duplication and permutation.

The sequent rules for CKD are seen in Fig. 4. In the top part, the rules Ax, $\bot L$, $\top R$, $\wedge L$, $\wedge R$, $\vee L$, $\vee R$, $\to L$ and $\to R$ are the left and right introduction rules for a (multisequent, logical [30]) Gentzen sequent calculus of IPL. These rules operate in the central $\Gamma \vdash \Pi$ scopes, leaving the corner scopes of the Došen square untouched. In the bottom part of Fig. 4 we list the left and right introduction rules $\Diamond L$, $\Diamond R$, $\Box L$, $\Box R$, $⟐ L$, $⟐ R$, $\boxminus L$ and $\boxminus R$ for the modalities.

These modal rules, applied in forward direction, take a signed formula from one of the corners Δ, Θ, Σ and Ψ of the Došen square (Fig. 3) and introduce an associated modal operator in the conclusion sequent, instead. From Ψ^- and Θ^+ we introduce the $\Box$ modalities in rules $\Box L$ and $\Box R$; From Ψ^+ and Θ^- we introduce $\boxminus$ via $\boxminus L$ and $\boxminus R$. No other rule depends on the presence of formulas in Ψ or Θ. From Δ^- and Σ^+ stem all occurrences of $\diamondminus$ through $\diamondminus R$ and $\diamondminus L$, while Δ^+ and Σ^- constitute a reservoir for $\Diamond$ introduced via $\Diamond L$ and $\Diamond R$. So far, G_{CKD} does not present surprises as a Gentzen-style calculus. The speciality of G_{CKD} lies in the four rules cpL^-, cpL^+, cpR^- and cpR^+ seen in the center of Fig. 4. The sign introduction rules cpL^t, cpR^t work in opposite direction to the modal introduction rules $\otimes L$, $\otimes R$. Together, they orchestrate the 'Grand Modal Dispatch' of the DS as suggested in Fig. 3.

Definition 9 (Derivability). *A derivation of a sequent $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ is either an axiom (rule Ax), an instance of $\bot L$ or $\top R$ or an application of a logical rule to derivations concluding its premises, that is built using the rules in Fig. 4. We say that a sequent is* underivable, *written $\Gamma \star \Delta \star \Theta \nvdash \Pi \star \Sigma \star \Psi$, if no derivation exists for it.*

G_{CKD} is conceived as a refutation system. Its purpose is to establish that a state specification (based on the six scopes) presented as a sequent is refutable. Refutability (Definition 8) and derivability (Definition 9) are linked in the sense that a sequent is underivable iff it is refutable, as established in the soundness and completeness proofs.

Theorem 4 (G_{CKD}Soundness). *If $\Gamma \star \Delta \star \Theta \not\models \Pi \star \Sigma \star \Psi$ then $\Gamma \star \Delta \star \Theta \nvdash \Pi \star \Sigma \star \Psi$.*

The proof of Theorem 4 is standard, by showing that for each sequent rule in Fig. 4 that if the conclusion is refutable then *at least one* of its premises is refutable as well.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \star \varnothing \star \varnothing \vdash A \star \varnothing \star \varnothing}\,Ax}{\varnothing \star A^+ \star \varnothing \vdash \bot \star A^+ \star \varnothing}\,cpL^+}{\Diamond A, \diamondminus A \star \varnothing \star \varnothing \vdash \bot \star \varnothing \star \varnothing}\,\diamondminus L\ \Diamond L}{\Diamond A \land \diamondminus A \star \varnothing \star \varnothing \vdash \bot \star \varnothing \star \varnothing}\,\land L}{\varnothing \star \varnothing \star \varnothing \vdash {\sim}(\Diamond A \land \diamondminus A) \star \varnothing \star \varnothing}\,{\sim}R
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{cannot be closed}}{A \star \varnothing \star \varnothing \vdash A \land B \star \varnothing \star \varnothing}\,?\,(2)}{\varnothing \star \varnothing \star A^- \vdash \bot \star \varnothing \star (A \land B)^+}\,cpR^+}{\boxminus(A \land B) \star \varnothing \star A^- \vdash \bot \star \varnothing \star \varnothing}\,\boxminus L}{\boxminus(A \land B) \star \varnothing \star \varnothing \vdash \boxminus A, \boxminus B \star \varnothing \star \varnothing}\,\boxminus R\,(1)}{\boxminus(A \land B) \star \varnothing \star \varnothing \vdash \boxminus A \lor \boxminus B \star \varnothing \star \varnothing}\,\lor R$$

Fig. 5. A successful G_{CKD} derivation (left) and a non-completable derivation (right).

As examples consider the G_{CKD} derivations in Fig. 5. The left derivation demonstrates the incompatibility of $\Diamond$ and $\diamondminus$ and the right indicates why a proof of the distribution $\boxminus(A \land B) \to (\boxminus A \lor \boxminus B)$ is doomed to fail. The application (1) of rule $\boxminus R$ on the right of Fig. 5, corresponding to an intuitionistic

$\leq$-step in backwards direction, must clear the Π-scope and drop the constraint $\boxminus B$. Because of this, the formula B is missing in situation (2) so that the sequent cannot be derived.

Theorem 5. *For each H_{CKD} derivation $\varnothing;\varnothing \vdash_H D$ there is a G_{CKD} derivation of the sequent $\varnothing \star \varnothing \star \varnothing \vdash D \star \varnothing \star \varnothing$ using the rules of Fig. 4 plus the cut rule: From $\Gamma\star\Delta\star\Theta \vdash D, \Pi\star\Sigma\star\Psi$ and $D, \Gamma\star\Delta\star\Theta \vdash \Pi\star\Sigma\star\Psi$ infer $\Gamma\star\Delta\star\Theta \vdash \Pi\star\Sigma\star\Psi$.*

A sequent $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ is called *strict* if $|\Delta \cup \Pi \cup \Psi| \geq 1$ and *polarised* if $|\Theta^- \cup \Sigma^-| \leq 1$. One can show that every derivable sequent is strict and that polarised sequents can be proven only using polarised sequents. For polarised and strict sequents the following 'hilbertification' provides a *translation* of G_{CKD} back into H_{CKD}.

Definition 10 (Hilbertification). *Let each sequent $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ be translated into the formula $\big(\hat{\Gamma} \wedge \hat{\Diamond}\Delta \wedge \hat{\diamondminus}\Sigma \wedge \hat{\Box}\Theta \wedge \hat{\boxminus}\Psi\big) \to \big(\check{\Pi} \vee \check{\diamondminus}\Delta \vee \check{\Diamond}\Sigma \vee \check{\boxminus}\Theta \vee \check{\Box}\Psi\big)$ where*

$$\begin{aligned}
\hat{\Gamma} &=_{df} \textstyle\bigwedge_{A\in\Gamma} A, & \check{\Pi} &=_{df} \textstyle\bigvee_{D\in\Pi} D,\\
\hat{\Diamond}\Delta &=_{df} \textstyle\bigwedge_{B^+\in\Delta} \Diamond B, & \check{\diamondminus}\Delta &=_{df} \textstyle\bigvee_{B^-\in\Delta} \diamondminus B,\\
\hat{\diamondminus}\Sigma &=_{df} \diamondminus \textstyle\bigvee_{E^+\in\Sigma} E, & \check{\Diamond}\Sigma &=_{df} \Diamond \textstyle\bigvee_{E^-\in\Sigma} E,\\
\hat{\Box}\Theta &=_{df} \textstyle\bigwedge_{D^+\in\Theta} \Box D, & \check{\boxminus}\Theta &=_{df} \check{\boxminus} \textstyle\bigwedge_{D^-\in\Theta} D,\\
\hat{\boxminus}\Psi &=_{df} \textstyle\bigwedge_{D^+\in\Psi} \boxminus F, & \check{\Box}\Psi &=_{df} \textstyle\bigvee_{D^-\in\Psi} \Box F,
\end{aligned}$$

and for empty sets we put $\hat{\Gamma} =_{df} \top$ if $\Gamma = \varnothing$, $\check{\Pi} =_{df} \bot$ if $\Pi = \varnothing$, and for $\otimes \in \{\Box, \Diamond, \diamondminus, \boxminus\}$ and X a set of signed formulas: $\hat{\otimes}X = \top$ if $X^+ = \varnothing$ and $\check{\otimes}X = \bot$ if $X^- = \varnothing$.

Theorem 6 *Let $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ be a polarised sequent, derivable using the rules of Fig. 4. Then, there exists a Hilbert derivation of*

$$\mathsf{CKD};\varnothing \vdash_H \big(\hat{\Gamma} \wedge \hat{\Diamond}\Delta \wedge \hat{\diamondminus}\Sigma \wedge \hat{\Box}\Theta \wedge \hat{\boxminus}\Psi\big) \to \big(\check{\Pi} \vee \check{\diamondminus}\Delta \vee \check{\Diamond}\Sigma \vee \check{\boxminus}\Theta \vee \check{\Box}\Psi\big).$$

Theorem 5 and 6 give us a back-and-forth translation of deductions in the Hilbert and Gentzen systems for CKD. However, this involves the *cut* rule, so neither calculus gives us a decision procedure. We address this by proving completeness of G_{CKD} and thus completeness of H_{CKD}, leading to our final completeness result that implies cut-elimination. First, let us introduce some technical definitions.

Definition 11 (Saturation). *A sequent $\Gamma\star\Delta\star\Theta \vdash \Pi\star\Sigma\star\Psi$ is called* saturated *if the following closure conditions hold:*

1. *If $M \wedge N \in \Gamma$ then both $M, N \in \Gamma$*
2. *If $M \vee N \in \Gamma$ then $M \in \Gamma$ or $N \in \Gamma$;*
3. *If $M \to N \in \Gamma$ then $M \in \Pi$ or $N \in \Gamma$*
4. *If $M \vee N \in \Pi$ then both $M, N \in \Pi$;*
5. *If $M \wedge N \in \Pi$ then $M \in \Pi$ or $N \in \Pi$*
6. *If $\diamondminus M \in \Gamma$ then $M^+ \in \Sigma$*
7. *If $\diamondminus M \in \Pi$ then $M^- \in \Delta$*
8. *If $\Box M \in \Gamma$ then $M^+ \in \Theta$*
9. *If $\Box M \in \Pi$ then $M^- \in \Psi$*
10. *If $\Pi = \varnothing$ and $\Delta = \varnothing$ then $\bot \in \Gamma$.*

In a saturated sequent the sets Γ and Π are coupled through the constraints (1)–(5). Closure conditions (6)–(9) are lower bounds on the presence of positive signs in Σ and Θ and on the negative signs in Δ and Ψ. If $\Gamma_1 \star \Delta_1 \star \Theta_1 \vdash \Pi \star \Sigma_1 \star \Psi_1$ is saturated then any sequent $\Gamma \star \Delta_2 \star \Theta_2 \vdash \Pi \star \Sigma_2 \star \Psi_2$ with $\Theta_1^+ \subseteq \Theta_2^+$, $\Sigma_1^+ \subseteq \Sigma_2^+$, $\Delta_1^- \subseteq \Delta_2^-$ and $\Psi_1^- \subseteq \Psi_2^-$ is saturated, too. In other words, we can add positive signs, or add and remove negative signs from Θ, Σ without losing saturation. Analogously, we can add negative signs or add and remove positive signs in Δ, Ψ and preserve saturation.

Definition 12. *A set SF of formulas is* subformula closed *if for every subformula A of a formula $M \in SF$ it holds that $A \in SF$. Let $SF^+ = SF \cup \{\bot\}$. We say that a sequent $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ is called a SF*-sequent *if $\Gamma \cup \Delta \cup \Theta \cup \Pi \cup \Sigma \cup \Psi \subseteq SF^+$. Moreover, a SF sequent is called* consistent *if it cannot be derived in the cut-free calculus. It is called SF*-complete *if for every $M \in SF^+$ we have $M \in \Gamma$ or $M \in \Pi$.*

For saturated, consistent and SF-complete sequents the essential information lies in Γ, in the positive signs $B^+ \in \Delta$, $F^+ \in \Psi$ and the negative signs $E^- \in \Sigma$, $C^- \in \Theta$. All of these express the existence and properties of *immediate* R-successors (see Definition 8).

Definition 13 (Canonical Interpretation). *Let SF be a subformula closed set. We define a basic* canonical *C-structure $\mathfrak{M}^c = (S^c, \leq^c, F^c, R^c, V^c)$ over SF as follows: The states $w \in S^c$ are the saturated and consistent SF sequents $w = \langle \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi \rangle$. Relating these canonical states, we define the intuitionistic accessibility relation $\leq^c$ and the compatibility relation R^c on S^c as follows:*

$$\langle \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi \rangle \;\leq^c\; \langle \Gamma' \star \Delta' \star \Theta' \vdash \Pi' \star \Sigma' \star \Psi' \rangle \\ \text{iff } \Gamma \subseteq \Gamma' \;\&\; \Theta^+ \subseteq \Theta' \quad \Sigma^+ \subseteq \Sigma' \tag{9}$$

$$\langle \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi \rangle \;R^c\; \langle \Gamma' \star \Delta' \star \Theta' \vdash \Pi' \star \Sigma' \star \Psi' \rangle \\ \text{iff } \Sigma \subseteq \Pi' \;\&\; \Theta \subseteq \Gamma'. \tag{10}$$

Let $w = \langle \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi \rangle \in S^c$ be an arbitrary state. The valuation of propositional variables p is given by stipulating $p \in V^c(w)$ iff $p \in \Gamma$ or $\bot \in \Gamma$. The state w is fallible $w \in F^c$ iff $\bot \in \Gamma$.

Lemma 1. *The canonical structure $\mathfrak{M}^c =_{df} (S^c, F^c, \leq^c, R^c, V^c)$ in Definition 13 is a C-model in the sense of Definition 2 such that for every sequent $w \in S^c$ the pair $(\mathfrak{M}^c, w)$ refutes w according to Definition 8.*

Theorem 7 (Gentzen Completeness). *Every underivable sequent is refutable, i.e., if* $\Gamma \star \Delta \star \Theta \nvdash \Pi \star \Sigma \star \Psi$ *then* $\Gamma \star \Delta \star \Theta \not\models \Pi \star \Sigma \star \Psi$.

The completeness proof proceeds in the standard fashion via canonical models (see Definition 13) constructed by saturation of unprovable end-sequents. Consistent saturation in all scopes Γ, Δ, Θ, Π, Σ and Ψ only involves subformulas (counting $\bot$ as a subformula) of the original sequent. The canonical model does not require maximal saturation or depends on the *cut* rule to achieve completeness of canonical states. Hence, the *cut* rule is admissible in CKD. Moreover, since all rules of CKD (not using *cut*) have the subformula property, it follows that CKD has the Finite Model Property. The Completeness Theorem 7 for our finite axiomatisation (Gentzen or Hilbert system) implies decidability. Therefore, we have the following theorem.

Theorem 8. *The theory* CKD *has the Finite Model Property, is cut-free and decidable.*

4 Conclusion

We have introduced a logic CKD, which combines the modalities $\Diamond, \Box, ⬖, ⊟$ with IPL. CKD is constructive since it has the Disjunction Property, and it is a conservative extension of the logics CK [25], N [11] and HK$\Box$ [6]. Technically, this is a clear contribution, since many extensions of N are not constructive, and combining the modalities $\Diamond, \Box, ⬖, ⊟$ with IPL can easily lead to loss of constructivity. But, we would add, this is also a contribution on another front: by combining the modalities $\Diamond, \Box, ⬖, ⊟$ with IPL we have constructed a logic in which all parts of the Došen square are included. Moreover, Došen's logics HK$\otimes$ for $\otimes \in \{\Diamond, ⬖, ⊟\}$ are axiomatic extensions of CKD.

The proof theory of CKD has been given in the form of a Hilbert calculus H_{CKD} and a sequent calculus G_{CKD}, and a constructive (bidirectional) translation between both proof systems is established. The soundness and completeness of H_{CKD} and G_{CKD} is proven, relative to a semantics based on C-frames and C-models. The structural complexity of G_{CKD} sequents arises from the aim to enforce the subformula property (analyticity) and to enable a Gentzen-style separation between left and right introduction rules for each operator (orthogonality). Finally, as a corollary of Gentzen completeness, it follows that the theory of CKD has the finite model property, is cut-free and decidable.

G_{CKD} is the first sequent calculus that combines all four modalities $\otimes \in \{\Box, \Diamond, ⊟, ⬖\}$ preserving the disjunction property of intuitionistic logic. It is instructive to look at special fragments: In the modal-free fragment IPL, i.e., without the rules $\otimes L$, $\otimes R$ for $\otimes \in \{\Box, \Diamond, ⊟, ⬖\}$, all scope sets except Γ and Π may be assumed empty. Hence, the dispatch rules cpL^t, cpR^t become obsolete and G_{CKD} reduces to the rules $\{Ax, \bot L, \top R, \wedge L, \wedge R, \vee L, \vee R, \rightarrow L, \rightarrow R\}$ corresponding to Dragalin's sequent calculus for IPL. In the $\Box$-fragment of G_{CKD} (i.e., IPL plus $\Box$), the modal rules $\Box L$, $\Box R$ generate only the positive signs Θ^+ and negative signs Ψ^- while $\Delta = \Sigma = \varnothing$. Hence, from the modal dispatch only cpR^- remains. The resulting sequents $\Gamma \star \varnothing \star \Theta^+ \vdash \Pi \star \varnothing \star \Psi^-$ correspond to an intuitionistic version of the 4-ary sequents $\Gamma \Rightarrow^{\Psi^-}_{\Theta^+} \Pi$ of Blamey and Humberstone's logic[4] K^4 [5], called H−ask by [30]. These K^4 sequents are translatable as formulas $(\bigwedge \Gamma \wedge \bigwedge \Box\Theta^+) \to (\bigvee \Pi \vee \bigvee \Box\Psi^-)$ (see [30] and also Definition 10). The constructive nature of CKD appears in the fact that the right introduction rules $\Diamond R$ and $\boxminus R$ are not obviously (locally) invertible, due to the restriction of the scopes in their premises. In classical logic, where $\sqsubseteq$ is the identity relation and there is no difference between positive and negative signs in the sequent's scope, the rule $\Diamond R$ could be replaced by the sound rule $\Gamma \star \Delta \star \Theta \vdash \Pi \star D, \Sigma \star \Psi \Rightarrow \Gamma \star \Delta \star \Theta \vdash \Diamond D, \Pi \star \Sigma \star \Psi$, which is invertible. Similarly, the rule $\boxminus R$ could be relaxed as the invertible rule $\Gamma \star \Delta \star D, \Theta \vdash \Pi \star \Sigma \star \Psi \Rightarrow \Gamma \star \Delta \star \Theta \vdash \boxminus D, \Pi \star \Sigma \star \Psi$. In such a classical collapse, G_{CKD} might be seen as a 6-ary multi-sequent calculus for the modalities $\otimes \in \{\Box, \Diamond, \boxminus, \diamondminus\}$ in the spirit of Blamey and Humberstone.

Two novel features of the semantics for CKD deserve to be highlighted for those unfamiliar with the literature on constructive logic: C-frames admit fallible states, and C-models adopt doubly-quantified truth conditions for modal operators, these latter explaining why $\Diamond$ does not distribute over disjunction, just like in CK [20,25,33]. We note that, fallible states appear to be relevant also in N. Došen [11] (see also [28,36]) proves completeness of N on HK$\diamondminus$-frames in the language $\mathcal{L}_{\mathsf{N}}$ which does not contain $\bot$. In the proof, however, canonical states with inconsistent theories must be permitted. As a result, the standard model theory via HK$\diamondminus$-frames is no longer adequate in the extended language $\mathcal{L}_{\mathsf{N}} \cup \{\bot\}$, since it would force the axiom $\diamondminus\bot$, which is not part of N. This problem does not re-occur in CKD since the definition of C-models permits fallible states to reject $\diamondminus\bot$. Hence, in CKD the fusion of N and *full* IPL can be studied.

There are various other logics in the vicinity of CKD which can be studied, too. For example, the theory of C-frames in which R is a transitive subrelation of $\leq$ that is reflexive on infallible states (if $s \notin F$ then $s\ R\ s$) generates *Propositional Lax Logic* PLL [20] also known as *Computational Logic* CL [4]. Both negative modalities $\diamondminus A$ and $\boxminus A$ collapse in this case, and become semantically equivalent to intuitionistic negation $\sim A$, whilst $\Box$ collapses since $\Box A \leftrightarrow A$. Only $\Diamond$ remains independent, yielding the (only) monadic modal operator $\bigcirc$ of Lax Logic, axiomatised by the single axiom $(A \to \bigcirc B) \leftrightarrow (\bigcirc A \to \bigcirc B)$, and the axiom $\sim \bigcirc \bot$ if additionally R is a subrelation of $\sqsubseteq$.

Other logics arise from CKD when the combined relation $\sqsubseteq;R$ is functional. C-frames in which $\sqsubseteq;R$ is functional collapse $\diamondminus A$ and $\boxminus A$ to a form of negation $\neg A$, known as *Routley negation* in the literature on FDE [17–19]. Routley negation is weaker than intuitionistic negation $\sim A$ in that it satisfies contraposition and DeMorgan laws while permitting gaps and gluts. In C-frames in which $\sqsubseteq;R$ is functional the theories $\mathsf{N}^\star$ and $\mathsf{N}^\star_i$ of Routley negation [27] can be developed.

[4] Blamey and Humberstone also use sets as scopes as we do, avoiding structural rules of duplication and permutation. However, [5] use an explicit weakening rule, which is built into the rules of G_{CKD}. Our dispatch rule cpR^- is named Switch in [5].

Specifically, if $\sqsubseteq;R$ is *weakly functional*[5] then we obtain the theory called N' [28] that extends IPL by axioms [27]

$$(N1) : \neg(A \wedge B) \to (\neg A \vee \neg B) \quad (N2) : (\neg A \wedge \neg B) \to \neg(A \vee B) \quad (N3) : \neg\top \to \bot$$

with derivation rules of Modus Ponens and Contraposition ("from $A \to B$ infer $\neg B \to \neg A$"). If we further assume that frames are infallible, the relation $\sqsubseteq;R$ becomes *functional*, and we arrive at Heyting-Ockham logic $\mathsf{N}^\star$ [7,27,28] (extended by quantifiers in [36]) that extends N' by the axiom $\neg\bot$. Note that CKD on functional frames also collapses the positive modalities $\Box A \leftrightarrow \Diamond A$ into a single modality $\Box$ that preserves the properties of $\Box$. This naturally generates an extension of $\mathsf{N}^\star$ with modality $\Box$ in a coherent theory that appears not to have been considered in the literature.

There are a number of open problems which could be considered in the future. The Correspondence Theory for CKD could be explored and a sequent calculus provided for extensions of CKD, such as $\mathsf{N}^\star$ and $\mathsf{N}_i^\star$ in language $\{\Box, \Diamond, \neg\}$ where $\neg$ collapses both $\diamondminus$ and $\boxminus$ into a single modality $\neg$. Following [36], the addition of quantifiers to CKD could be investigated. On the proof-theoretic front, means for termination control (such as invertibility of rules, duplication elimination, blocking conditions) of the sequent calculus G_{CKD} could be investigated, and the algorithmic complexity of the theory CKD determined. Since CKD is constructive, the question naturally arises of what lambda calculus is related to CKD via the Curry Howard isomorphism, and if there exists a natural deduction calculus for CKD. Recent work by [1] provides a novel semantics for proofs in CK, and could form the basis of constructing a semantics of proofs in CKD including negative modalities. Finally, it would be interesting to investigate if the neighbourhood semantics for CK and other non-normal extensions proposed by [8] could be used to interpret the negative modalities of CKD.

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[5] A frame is *weakly functional* if $\forall s \in S \setminus F.\, \exists s'.\, s \sqsubseteq;R\, s'$ and $\forall s, s_1', s_2'.\, s \sqsubseteq;R\, s_1' \,\&\, s \sqsubseteq;R\, s_2' \Rightarrow s_1' \cong s_2'$, where $s_1' \cong s_2'$ iff $s_1' \leq s_2'$ and $s_2' \leq s_1'$. The frame is *functional* if the existence condition holds in the stronger form $\forall s \in S.\, \exists s'.\, s \sqsubseteq;R\, s'$.

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