Large Deviations and Wschebor's Theorems

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Abstract We revisit Wschebor's theorems on the a.s. convergence of small increments for processes with scaling and stationarity properties. We focus on occupation measures and proved that they satisfy large deviation principles.

Keywords Brownian motion · Stable processes · Scaling properties · Strong theorems · Large deviations

1 Introduction: Wschebor's Theorem and Beyond

In 1992, Mario Wschebor [24] proved the following remarkable property of the linear Brownian motion $(W(t), t \geq 0; W(0) = 0)$. Set

$$\mathcal{W}_1^\varepsilon = \varepsilon^{-1/2} W(\cdot + \varepsilon)$$

If λ is the Lebesgue measure on $[0, 1]$, then, almost surely, for every $x \in \mathbb{R}$ and every $t \in [0, 1]$:

$$\lim_{\varepsilon \to 0} \lambda\{s \leq t : \mathcal{W}_1^\varepsilon(s) \leq x\} = t\Phi(x)\,, \tag{1.1}$$

where Φ is the distribution function of the standard normal distribution $\mathcal{N}(0; 1)$.

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Let us give some notations. If $\Sigma = \mathbb{R}, \mathbb{R}^+ \times \mathbb{R}$ or $[0, 1] \times \mathbb{R}$, we denote by $\mathcal{M}^+(\Sigma)$ and $\mathcal{M}^r(\Sigma)$ the set of Borel measures on Σ positive and having total mass r, respectively.

If $\mathcal{Z}$ is a measurable function from $\mathbb{R}^+$ to $\mathbb{R}$, let $M_{\mathcal{Z}} \in \mathcal{M}^+(\mathbb{R}^+ \times \mathbb{R})$ be defined by

$$M_{\mathcal{Z}}(I \times A) = \lambda\{s \in I : \mathcal{Z}(s) \in A\}, \tag{1.2}$$

for every Borel subset $I \times A$ of $\mathbb{R}^+ \times \mathbb{R}$. The first marginal of $M_{\mathcal{Z}}$ is λ. The second marginal $\mu_{\mathcal{Z}}$ is the occupation measure

$$\mu_{\mathcal{Z}} = \int_0^1 \delta_{\mathcal{Z}(t)}\, dt\,,$$

defined either by its action on a Borel set A

$$\mu_{\mathcal{Z}}(A) = M_{\mathcal{Z}}([0, 1] \times A) = \lambda\{s \in [0, 1] : \mathcal{Z}(s) \in A\} \tag{1.3}$$

or, by its action on a test function $f \in \mathcal{C}_b(\mathbb{R})$

$$\int_{\mathbb{R}} f(x) d\mu_{\mathcal{Z}}(x) = \int_0^1 f(\mathcal{Z}(t))\, dt\,.$$

We will call $M_{\mathcal{Z}}$ the space-time occupation measure. In this framework,(1.1) may be considered as a law of large numbers (LLN):

$$M_{\mathcal{W}_1^\varepsilon} \Rightarrow \lambda \times \mathcal{N}(0; 1) \;\; (a.s.)$$

where $\Rightarrow$ stands for the weak convergence in $\mathcal{M}^+(\mathbb{R}^+ \times \mathbb{R})$.

It is then quite natural to ask for a possible Large Deviation Principle (LDP), i.e. an estimation of the form

$$\mathbb{P}(M_{W_1^\varepsilon} \simeq M) \approx \exp -I(M)/\varepsilon$$

for some nonnegative functional I called the rate function. (We refer to [13] for precise definition of LDP.)

Since the Brownian motion W is self-similar (Property P1) and has stationary increments (P2), it is possible to reduce the study of $\mu_{\mathcal{W}_1^\varepsilon}$ ($\varepsilon \to 0$) to the study of an occupation measure in large time ($T := \varepsilon^{-1} \to \infty$) for a process Y independent of ε. This new process is stationary and ergodic. Moreover, the independence of increments of W (P3) and its self-similarity induce a 1-dependence for Y, which allows to use the hypermixing property (see [8]) to get an LDP. This will be a consequence of our Theorem 2.3.

Actually, as the crucial properties (P1, P2, P3) are shared by α-stable Lévy processes, we are able to state the LDP in this last framework.

Besides, inspired by the extension of (1.1) in [24, 25], we consider mollified processes as follows.

Let BV be the set of bounded variation functions on $\mathbb{R}$ and also let $BV_c \subset BV$ be the subset of compactly supported functions. For $\psi \in BV_c$ let

$$\psi^\varepsilon(t) = \varepsilon^{-1}\psi\left(t\varepsilon^{-1}\right)$$

denote the rescaled version of ψ and for X a measurable function on $\mathbb{R}$, set $X_\psi^\varepsilon = X \star \psi^\varepsilon$, i.e.

$$X_\psi^\varepsilon(t) := \int \psi^\varepsilon(t-s)X(s)ds = \int \psi^\varepsilon(s)X(t-s)ds\,, \tag{1.4}$$

and

$$\dot{X}_\psi^\varepsilon(t) := \int X(t-s)d\psi^\varepsilon(s) = \varepsilon^{-1}\int X(t-\varepsilon s)d\psi(s)\,. \tag{1.5}$$

Taking for X an extension of W vanishing on $\mathbb{R}_-$, and denoting

$$\mathcal{W}_\psi^\varepsilon(s) := \sqrt{\varepsilon}\,\dot{W}_\psi^\varepsilon(s)\,, \tag{1.6}$$

the LLN reads

$$\lim_{\varepsilon\to 0} \lambda\{s \le t : \mathcal{W}_\psi^\varepsilon(s) \le x\} = t\Phi(x/||\psi||_2) \quad (a.s.)\,. \tag{1.7}$$

Notice that when $\psi = \psi_1 := 1_{[-1,0]}$, then $\mathcal{W}_\psi^\varepsilon = \mathcal{W}_1^\varepsilon$.

The fBM with Hurst index $H \neq 1/2$ shares also properties (P1, P2) but not (P3) with the above processes. Nevertheless, since it is Gaussian, with an explicit spectral density, we prove the LDP for (μ_ε) under specific conditions on the mollifier, thanks to a criterion of [7].

Let us give now the general framework needed in the sequel. Recall that a real-valued process $\{X(t), t \in \mathbb{R}\}$ is self-similar with index $H > 0$ if

$$(\forall a > 0)\ \ \{X(at), t \in \mathbb{R}\} \stackrel{(d)}{=} \{a^H X(t), t \in \mathbb{R}\}\,.$$

If X is a self-similar process with index H we set, if $\psi \in BV$

$$\mathcal{X}_\psi^\varepsilon = \varepsilon^{1-H}\dot{X}_\psi^\varepsilon\,, \tag{1.8}$$

where $\dot{X}_\psi^\varepsilon$ (see (1.5)) is assumed to be well defined. In particular

$$\mathcal{X}_\psi^1(t) = \int X(t-s)d\psi(s)\,. \tag{1.9}$$

The following lemma is the key for our study. Notice that we focus on the occupation measure. We let the easy proof to the reader.

Lemma 1.1 *Assume that X is self-similar with index H. For fixed ε and $\psi \in BV$, we have*

$$\left(\mathcal{X}_\psi^\varepsilon(t), t \in \mathbb{R}\right) \overset{(d)}{=} \left(\mathcal{X}_\psi^1(t\varepsilon^{-1}), t \in \mathbb{R}\right) \tag{1.10}$$

$$\mu_{\mathcal{X}_\psi^\varepsilon} \overset{(d)}{=} \varepsilon \int_0^{\varepsilon^{-1}} \delta_{\mathcal{X}_\psi^1(t)} dt\,. \tag{1.11}$$

From the above identity in law, it is clear that the asymptotic behavior of $\mu_{\mathcal{X}_\psi^\varepsilon}$ is connected to the long time asymptotics of the occupation measure of $\mathcal{X}_\psi^1$. We will focus on cases where the process $\mathcal{X}_\psi^1$ is stationary and ergodic, namely when the underlying process X is an α-stable Lévy process or a fractional Brownian motion. Both have stationary increments.

We give now a definition which will set the framework for the processes studied in the sequel. Recall that the τ-topology on $\mathcal{M}^1(\mathbb{R})$ is the topology induced by the space of bounded measurable functions on $\mathbb{R}$. It is stronger than the weak topology which is induced by $\mathcal{C}_b(\mathbb{R})$.

Definition 1.2 Let $\mathcal{F} \subset BV$. We say that a self-similar process X with index H has the $(LDP_w, \mathcal{F}, H)$ (resp. $(LDP_\tau, \mathcal{F}, H)$) property if the process $\mathcal{X}_\psi^1$ is well defined and if for every $\psi \in \mathcal{F}$, the family $(\mu_{\mathcal{X}_\psi^\varepsilon})$ satisfies the LDP in $\mathcal{M}^1(\mathbb{R})$ equipped with the weak topology (resp. the τ-topology), in the scale ε^{-1}, with good rate function

$$\Lambda_\psi^*(\mu) = \sup_{f\in\mathcal{C}_b(\mathbb{R})} \int f d\mu - \Lambda_\psi(f)\,, \tag{1.12}$$

(the Legendre dual of Λ_ψ) where for $f \in \mathcal{C}_b(\mathbb{R})$,

$$\Lambda_\psi(f) = \lim_{T\to\infty} T^{-1} \log \mathbb{E} \exp \int_0^T f(\mathcal{X}_\psi^1(t))dt\,, \tag{1.13}$$

in particular, the above limit exists.

Roughly speaking, this means that for ε small, the probability of seeing $\mu_{\mathcal{X}_\psi^\varepsilon}$ close to μ is of order $e^{-\Lambda_\psi^*(\mu)/\varepsilon}$. In this framework, here is the main result (the precise version is given in Sects. 2 and 3).

Theorem 1.3

1. *The* α*-stable Lévy process has the* $(LDP_\tau, BV_c, 1/\alpha)$ *property.*
2. *The fractional Brownian motion of index* $H \in [0, 1)$ *has the* $(LDP_w, \mathcal{G}_H, H)$ *property for some explicit* $\mathcal{G}_H$.

Before giving the outline of our paper, let us mention that there is a broad literature on the fluctuations around the LLN mentioned above. For example if g is a real even function g such that $\mathbb{E}[g^2(N)] < \infty$, then

$$\left(\varepsilon^{-1/2}\int_0^t \left(g(\mathcal{W}_\psi^\varepsilon(s)) - \mathbb{E}g(N/||\psi||_2)\right) ds, t \in [0, 1]\right) \Rightarrow (\sigma(g)W(t), t \in [0, 1]) \ , \tag{1.14}$$

where $\sigma(g)$ is an explicit positive constant [3]. In 2008, Marcus and Rosen in [19] have studied the convergence of the L^p norm (this is $g(x) = |x|^p$ in (1.14)) of the increments of stationary Gaussian processes and solved the problem in a somewhat definitive form. In another article [20] they said that their proofs were initially based on Wschebor's method, but afterwards they changed, looking for a more general and broadly used procedure.

Here is the outline. In Sect. 2 we prove the LDP for the occupation measure and the space-time occupation measure, covering in particular the Brownian motion. Section 3 is devoted to the fBm process, covering again the Brownian motion. In Sect. 4, we state a result for some "process level" empirical measure. At last, in Sect. 5 we study discrete versions of Wschebor's theorem using the Skorokhod embedding theorem.

Let us notice that except in a specific case in Sect. 3.3.2, we cannot give an explicit expression for the rate function. Moreover if one would be able to prove that the rate function is strictly convex and its minimum is reached at $\lambda \times \mathcal{N}(0; 1)$, this would give an alternate proof of Wschebor's results.

We let for a future work the study of increments for

- Gaussian random fields in $\mathbb{R}^d$
- multi-parameter indexed processes
- the Rosenblatt process.

2 The α-Stable Lévy Process

Let $\alpha \in (0, 2]$ fixed. The α-stable Lévy process $(S(t), t \geq 0; S(0) = 0)$ has independent and stationary increments and is $1/\alpha$-self-similar. If $\psi \in BV_c$, we set

$$\mathcal{S}_\psi^\varepsilon(t) := \varepsilon^{1-1/\alpha}\int S(t-s)d\psi_\varepsilon(s) \ ,$$

where we have extended S to zero on $\mathbb{R}_-$. As in (1.2) and (1.3), we may build the measures $M_{\mathcal{S}_\psi^\varepsilon}$ and $\mu_{\mathcal{S}_\psi^\varepsilon}$. In [1], Theorem 3.1, it is proved that a.s.

$$M_{\mathcal{S}_\psi^\varepsilon} \Rightarrow \lambda \times \Sigma_\alpha \ (a.s.)$$

where Σ_α is the law of $||\psi||_\alpha S(1)$.

2.1 LDP for $(\mu_{\mathcal{S}_\psi^\varepsilon})$

Proposition 2.1 *If $\mathcal{F} = BV_c$, then the α-stable Lévy process has the $(LDP_\tau, \mathcal{F}, 1/\alpha)$ property.*

Proof We apply Lemma 1.1 with $X = S$ and $H = 1/\alpha$.

Assume that the support of ψ is included in $[a, b]$. Since S has independent and stationary increments, a slight modification of the argument in [22] ex. 3.6.2 p.138 proves that the process $(\mathcal{S}_\psi^1(t), t \geq b)$ is stationary. Moreover the process $\mathcal{S}_\psi^1$ is $(b-a)$-dependent. This last property means that $\sigma(\mathcal{S}_\psi^1(u), u \in A)$ and $\sigma(\mathcal{S}_\psi^1(u), u \in B)$ are independent as soon as the distance between A and B is greater than $(b-a)$. Consequently, the process $(\mathcal{S}_\psi^1)$ is clearly hypermixing and so satisfies the LDP in the τ-topology (see [8] Theorem 2 p. 558) and the other conclusions hold. □

Remark 2.2 When $\alpha = 2$ we recover the Brownian case. In particular, when $\psi = \psi_1$

$$\mathcal{S}_\psi^1(u) = W(u+1) - W(u)\,. \tag{2.1}$$

This process is often called Slepian process; it is Gaussian, stationary and 1-dependent.

2.2 LDP for $(M_{\mathcal{S}_\psi^\varepsilon})$

We will now state a complete LDP, i.e. an LDP for $(M_{\mathcal{S}_\psi^\varepsilon})$.[1]

Following the notations of Dembo and Zajic in [12] we denote by $\mathcal{AC}_0$ the set of maps $\nu : [0, 1] \to \mathcal{M}^+(\mathbb{R})$ such that

- ν is absolutely continuous with respect to the variation norm,

[1] We could have presented the following Theorem 2.3 before Sect. 2.1 and then deduce an LDP as in Proposition 2.1 for $\mu_{\mathcal{X}_\psi^\varepsilon}$ by contraction. But this would have been in the weak topology, (and Proposition 2.1 is in the τ-topology), and we choose the present exposition for the sake of clarity.

- $\nu(0) = 0$ and $\nu(t) - \nu(s) \in \mathcal{M}^{t-s}(\mathbb{R})$ for all $t > s \geq 0$,
- for almost every $t \in [0, 1]$, $\nu(t)$ possesses a weak derivative.

(This last point means that $(\nu(t+\eta) - \nu(t))/\eta$ has a limit as $\eta \to 0$—denoted by $\dot{\nu}(t)$- in $\mathcal{M}^+(\mathbb{R})$ equipped with the topology of weak convergence).

Let F be the mapping

$$\begin{aligned} \mathcal{M}^+([0,1]\times\mathbb{R}) &\to D\left([0,1]; \mathcal{M}^+(\mathbb{R})\right) \\ M &\mapsto (t \mapsto F(M)(t) = M([0,t],\cdot)) \end{aligned} \tag{2.2}$$

or in other words $F(M)(t)$ is the positive measure on $\mathbb{R}$ defined by its action on $\varphi \in \mathcal{C}_b$:

$$\langle F(M)(t), \varphi\rangle = \langle M, 1_{[0,t]} \times \varphi\rangle \,.$$

Here $D([0, 1]; \cdot)$ is the set of càd-làg functions, equipped with the supremum norm topology. At last, let $\mathcal{E}$ be the image of $\mathcal{M}^1([0, 1] \times \mathbb{R})$ by F.

Theorem 2.3 *For $\psi \in BV_c$, the family $\left(M_{\mathcal{S}^\varepsilon_\psi}\right)$ satisfies the LDP in $\mathcal{M}^1([0,1]\times\mathbb{R})$ equipped with the weak topology, in the scale ε^{-1} with the good rate function*

$$\Lambda^*(M) = \begin{cases} \int_0^1 \Lambda^*_\psi(\dot{\gamma}(t))dt & \text{if } \gamma := F(M) \in \mathcal{AC}_0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof As in the above sections, it is actually a problem of large deviations in large time. For the sake of simplicity, set

$$Y = \mathcal{S}^1_\psi$$

and $T = \varepsilon^{-1}$. Using Lemma 1.1, the problem reduces to the study of the family $(M_{Y(\cdot T)})$. First, we study the corresponding distribution functions:

$$H_T(t) := F(M_{Y(\cdot T)})(t) = \int_0^t \delta_{Y(sT)}ds = T^{-1}\int_0^{tT} \delta_{Y(s)}ds \,. \tag{2.4}$$

In a first step we will prove that the family (H_T) satisfies the LDP, then in a second step we will transfer this property to $M_{Y(\cdot T)}$.

First Step We follow the method of Dembo-Zajic [12]. We begin with a reduction to their "discrete time" method by introducing

$$\eta_k = \int_{k-1}^k \delta_{Y(s)}ds \in \mathcal{M}^1(\mathbb{R}) \,,\ (k \geq 1) \text{ and } S_T(t) = \sum_1^{\lfloor tT\rfloor} \eta_k \,.$$

It holds that

$$H_T(t) - T^{-1}S_T = T^{-1}\int_{\lfloor tT \rfloor}^{tT} \delta_{Y(s)} ds \tag{2.5}$$

and this difference has a total variation norm less than T^{-1}, so that the families $(T^{-1}S_T)$ and (H_T) are exponentially equivalent (Def. 4.2.10 in [13]).

The sequence η_k is 1-dependent, hence hypermixing (see condition (S) in [12, p. 212]) which implies, by Th. 4 in the same paper that $(T^{-1}S_T)$ satisfies the LDP in $D([0, 1]; \mathcal{M}^+(\mathbb{R}))$ provided with the uniform norm topology, with the convex good rate function

$$I(\nu) = \int_0^1 \Lambda^\star_\psi(\dot\nu(t))dt \tag{2.6}$$

when $\nu \in \mathcal{AC}_0$ and ∞ otherwise.

We conclude, owing to Th. 4.2.13 in [13], that (H_T) satisfies the same LDP.

Second Step We now have to carry this LDP to $(M_{Y(\cdot T)})$ (see (2.4)). For every $T > 0$, $H_T \in \mathcal{E} \subset D([0, 1); \mathcal{M}^+(\mathbb{R}))$. We saw that the effective domain of I is included in $\mathcal{E}$. So, by Lemma 4.1.5 in Dembo-Zeitouni [13], (H_T) satisfies the same LDP in $\mathcal{E}$ equipped with the (uniform) induced topology.

Now, F is bijective from $\mathcal{M}^1([0, 1]\times\mathbb{R})$ to $\mathcal{E}$. Let us prove that F^{-1} is continuous from $\mathcal{E}$ (equipped with the uniform topology) to $\mathcal{M}^1([0, 1] \times \mathbb{R})$ equipped with the weak topology.

For $f : [0, 1] \to \mathbb{R}$, let

$$\|f\|_{BL} = \sup_x |f(x)| + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|} \tag{2.7}$$

$$d_{BL}(\mu, \nu) = \sup_{f:\|f\|_{BL}\leq 1} \left|\int f d\mu - \int f d\nu\right| \tag{2.8}$$

The space $\mathcal{M}^+(\mathbb{R})$ is a Polish space when equipped with the topology induced by d_{BL}, compatible with the weak topology.

It is known that $M_n \to M \in \mathcal{M}^1([0, 1] \times \mathbb{R})$ weakly as soon as

$$M_n(1_{[0,t]} \otimes f) \to M(1_{[0,t]} \otimes f) \tag{2.9}$$

for every $t \in [0, 1]$ and every f such that $\|f\|_{BL} < \infty$. But, for such t, f we have

$$\sup_t |M_n(1_{[0,t]} \otimes f) - M(1_{[0,t]} \otimes f)| \leq d_{BL}(F(M_n), F(M)) \tag{2.10}$$

which implies that F^{-1} is continuous from $\mathcal{E}$ to $\mathcal{M}^1([0, 1] \times \mathbb{R})$.

By the contraction principle (Th. 4.2.1 in [13]) we deduce that $M_{Y(\cdot T)}$ satisfies the LDP in $\mathcal{M}^1([0,1]\times\mathbb{R})$ with good rate function $J(M) = I(F(M))$, where I is given by (2.6). ◻

3 The Fractional Brownian Motion

3.1 General Statement

We now treat the case of self-similar Gaussian processes with stationary increments, i.e. fractional Brownian motion (fBm in short). The fBm with Hurst parameter $H \in [0,1)$ is the Gaussian process $(B_H(t), t \in \mathbb{R})$ with covariance

$$\mathbb{E}B_H(t)B_H(s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right).$$

It has a chaotic (or harmonizable) representation (see [22, Prop. 7.2.8])

$$B_H(t) = \frac{1}{C_H}\int_{\mathbb{R}}\left(e^{i\lambda t} - 1\right)|\lambda|^{-H-\frac{1}{2}}d\mathbf{W}(\lambda) \tag{3.1}$$

where $\mathbf{W}$ is a complex Brownian motion and

$$C_H^2 = \frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}.$$

This process has stationary increments and is self-similar of index H. When $H = 1/2$ we recover the Brownian motion, and it is the only case where the increments are independent.

All along this section, X will denote B_H.

When $\psi \in BV_c$, the LLN can be formulated as:

$$M_{\mathcal{X}_\psi^\varepsilon} \Rightarrow \lambda \times \mathcal{N}(0;\sigma_\psi^2) \quad (a.s.)\,, \tag{3.2}$$

where $\mathcal{N}(0;\sigma_\psi^2)$ is the centered normal distribution of variance

$$\sigma_\psi^2 = -\frac{1}{2}\iint |u-v|^{2H}d\psi(u)d\psi(v)\,,$$

(see [1]).

To get an LDP we first apply Lemma 1.1 with $X = B_H$. But now, for lack of independence of increments, we cannot use the method of Sect. 2. The process $\mathcal{X}_\psi^1$ is stationary and Gaussian. We will work with its spectral density and apply Theorem 2.1 in [7], which ensures the LDP as soon as the spectral density is in $\mathcal{C}_0(\mathbb{R})$, the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ that vanish at $\pm\infty$. Actually we can extend the set of admissible mollifiers.

From Fourier analysis we adopt the following notation: when $f, g \in L^1(\mathbb{R})$

$$\hat{f}(\theta) = \int e^{it\theta} f(t)dt \ , \ \check{g}(\gamma) = \frac{1}{2\pi} \int e^{-i\gamma x} g(x)dx \, .$$

Let, for $\psi \in L^2$

$$\ell_H^{\psi}(\lambda) = C_H^{-2} |\hat{\psi}(\lambda)|^2 |\lambda|^{1-2H} \, , \tag{3.3}$$

and

$$\tilde{\mathcal{G}}_H := \{\psi \in L^2(\mathbb{R}) : \ell_H^{\psi} \in L^1\} \, .$$

Notice that for $0 < H < 1/2$, $L^1 \cap L^2 \subset \tilde{\mathcal{G}}_H$. For $\psi \in \tilde{\mathcal{G}}_H$ we can define as in Pipiras and Taqqu [21]

$$\int \psi(t-s)dB_H(s)$$

as the limit of $\int \psi_n(t-s)dB_H(s) = \int B_H(t-s)d\psi_n(s)$ for ψ_n a sequence of simple functions (see Th. 3.1 therein). For these functions ψ_n we have

$$\int \psi_n(t-s)dB_H(s) = \mathrm{i}C_H^{-1} \int e^{\mathrm{i}t\lambda} \hat{\psi}_n(-\lambda)\lambda|\lambda|^{-H-\frac{1}{2}} d\mathbf{W}(\lambda)$$

Owing to the way of convergence of ψ_n we have, in the limit

$$\int \psi(t-s)dB_H(s) = \mathrm{i}C_H^{-1} \int e^{\mathrm{i}t\lambda} \hat{\psi}(-\lambda)\lambda|\lambda|^{-H-\frac{1}{2}} d\mathbf{W}(\lambda)$$

hence $\mathcal{X}_1^{\psi}$ is a Gaussian process and its spectral density is ℓ_H^{ψ}.

Applying the criterion on the continuity of the spectral density, we arrive at the following result on large deviations.

Theorem 3.1 *The process B_H has the $(LDP_w, \mathcal{G}_H, H)$ property, where*

$$\mathcal{G}_H = \{\psi \in L^2 : \ell_H^{\psi} \in L^1 \cap \mathcal{C}_0\} \subset \tilde{\mathcal{G}}_H \, .$$

3.2 Contraction

Since the mapping $\mu \mapsto \int |x|^p d\mu(x)$ is not continuous for the weak topology, we cannot obtain an LDP for the moments of $\mu_{\mathcal{X}^\varepsilon_\psi}$ by invoking the contraction principle (Th. 4.2.1 in [13])). Nevertheless, in the case of the fBm, the Gaussian stationary character of the process allows to conclude by a direct application of Corollary 2.1 in [7].

Proposition 3.2 *If either $H \le 1/2$ and $\psi \in \mathcal{G}$ or $H > 1/2$ and $\psi \in \mathcal{G} \cap \mathcal{G}_H$, then the family $\left(\int_0^1 |\mathcal{X}^\varepsilon_\psi(t)|^2 dt\right)$, where $X = B_H$, satisfies the LDP, in the scale ε^{-1} with good rate function*

$$I_\psi(x) = \sup_{-\infty < y < 1/(4\pi M)} \{xy - L(y)\},$$

where

$$L(y) = -\frac{1}{4\pi} \int \log(1 - 4\pi y \ell_H(s)) ds$$

ℓ_H is the spectral density given by (3.3) and

$$M = \sup_\lambda \ell_H(\lambda).$$

More generally, for $0 \le p \le 2$, the family $\left(\int_0^1 |\mathcal{X}^\varepsilon_\psi(t)|^p dt\right)$ satisfies the LDP at scale ε with a convex rate function.

3.3 Particular Cases

3.3.1 Remark: Two Basic Mollifiers

(1) As seen before, the function $\psi_1 = 1_{[-1,0]}$ is the most popular. It allows to study the first order increments $X(t+\varepsilon) - X(t)$. It belongs to $\mathcal{G}$ but since

$$|\hat{\psi}_1(\lambda)| = \frac{|\sin(\lambda/2)|}{|\lambda/2|},$$

it does not belong to $\mathcal{G}_H$ for $H > 1/2$.

For $H = 1/2$, we recover the Brownian motion and replace the notation $\mathcal{X}$ by $\mathcal{W}$. The process $\mathcal{W}^1_{\psi_1}$ is the Slepian process (2.1) with covariance

$$r(t) = (1 - |t|)^+ ,$$

and spectral density:

$$\check{r}(\lambda) = \frac{1}{2\pi}\left(\frac{\sin\frac{\lambda}{2}}{\frac{\lambda}{2}}\right)^2 .$$

As it is said above since $\check{r}$ is $\mathcal{C}_0$, the occupation measure satisfies a LDP in the weak topology in the scale ε^{-1}. This argument could have been used to prove the LDP, instead of the argument in Sect. 2 (but for the weak topology and not the τ-topology). Notice that although $\check{r}$ is differentiable, we cannot apply Theorem 5.18 in Chiyonobu and Kusuoka [9], since the condition (5.19) therein is violated in $x \in 2\pi\mathbb{Z}$.

(2) Another interesting function is

$$\psi_2 = \frac{1}{2}\left(1_{[-1,0]} - 1_{[0,1]}\right)$$

which yields

$$\dot{X}^{\varepsilon}_{\psi_2}(t) = \frac{X(t+\varepsilon) - 2X(t) + X(t-\varepsilon)}{2\varepsilon} . \tag{3.4}$$

Since

$$|\hat{\psi}_2(\lambda)| = \frac{\sin^2(\lambda/2)}{|\lambda/2|} ,$$

we see that $\psi_2 \in \mathcal{G} \cap \mathcal{G}_H$ for every $H \in (0,1)$ and then $(\mu_{\mathcal{X}^{\varepsilon}_{\psi_2}})$ satisfies the LDP.

In (3.4) we are faced with second order increments of the process X. These increments are linked with the behavior of the second derivative of X^{ε} when it exists. Let us consider ψ smooth enough so that X^{ε}_{ψ}, defined in (1.4), has a second derivative. For instance, let $\psi \in \mathcal{G}$ such that $\psi' \in \mathcal{G}$. Then the function X^{ε}_{ψ} is twice differentiable and

$$\ddot{X}^{\varepsilon}_{\psi}(t) = \varepsilon^{-2}\int X(t-\varepsilon s)d\psi'(s) = \varepsilon^{-1}\dot{X}^{\varepsilon}_{\psi'}(t) .$$

Now, $\psi' \in \mathcal{G}_H$ since

$$|\widehat{\psi}'(\lambda)||\lambda|^{\frac{1}{2}-H} = |\hat{\psi}(\lambda)||\lambda|^{\frac{3}{2}-H} \to 0$$

as $\lambda \to 0$.

Since $\mathcal{X}^\varepsilon_{\psi'} = \varepsilon^{2-H}\ddot{X}^\varepsilon_\psi$, we conclude that for every $H \in (0,1)$, the family $(\mu_{\varepsilon^{2-H}\ddot{X}^\varepsilon_\psi})$ satisfies the LDP in the scale ε^{-1} and good rate function $\Lambda^\star_{\psi'}$. The choice

$$\psi(t) = \frac{1}{2}(1-|t|)^+$$

allows to recover $\psi' = \psi_2$ and the second order increments.

3.3.2 Looking for an Explicit Rate Function

It is not easy to find examples of explicit rate functions for the occupation measures of the above stationary processes $\mathcal{X}^1_\psi$, since in general the limiting cumulant generating function Λ is not explicit. A particularly nice situation in the Gaussian case will occur if the process is also Markovian, i.e. if $\mathcal{X}^1_\psi$ is the Ornstein-Uhlenbeck (OU) process. Indeed, for the OU, the rate function for the LDP of the occupation measure is given by the Donsker-Varadhan theory [23, ex. 8.28]:

$$\Lambda^*(\mu) = \frac{1}{2}\int_{\mathbb{R}} |g'(x)|^2 d\Phi(x)$$

if $d\mu = g^2 d\Phi$. The goal is then to find a mollifier ψ such that $\mathcal{X}^1_\psi$ is distributed as OU.

For OU, the covariance and spectral density are, respectively

$$r(t) = e^{-|t|}\ ,\ \check{r}(\lambda) = \frac{1}{\pi(1+\lambda^2)}\,.$$

Let us assume that the underlying process is fBm. Remember that the process $\mathcal{X}^1_\psi$ is then stationary Gaussian with spectral density given in (3.3).

Owing to (3.3), the equation

$$\mathcal{X}^1_\psi \overset{(d)}{=} OU \tag{3.5}$$

may be turned into

$$\left|\hat{\psi}(\lambda)\right|^2 = C_H^2 \frac{|\lambda|^{2H-1}}{\pi(1+\lambda^2)}\,. \tag{3.6}$$

(1) For $H < 1/2$, this function is not continuous in 0, so it cannot be the Fourier transform of an integrable kernel.

(2) For $H = 1/2$, we present two answers.

(a) Let us choose

$$\hat{\psi}(\lambda) = \frac{\sqrt{2}}{1 - i\lambda}\,, \;\; \psi(x) = \sqrt{2} e^{-x} 1_{[0,\infty)}(x)\,,$$

and then, the formula (1.6) becomes

$$\mathcal{W}^1_\psi(t) = \sqrt{2}\int_{-\infty}^{t} e^{-(t-s)} dW_s .$$

This is the classical representation of the stationary OU process as a stochastic integral [22, p. 138].

(b) Let us choose ψ such that

$$\hat{\psi}(\lambda) = \frac{\sqrt{2}}{\sqrt{1+\lambda^2}}\,.$$

This function is in L^2 but not in L^1. We can recover it by the semi convergent integral:

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\lambda} \frac{\sqrt{2}}{\sqrt{1+\lambda^2}}\, d\lambda,$$

i.e.

$$\psi(x) = \frac{\sqrt{2}}{\pi}\int_0^{\infty} \frac{\cos(x\lambda)}{\sqrt{1+\lambda^2}}\, d\lambda = \frac{\sqrt{2}}{\pi} K_0(x),$$

where K_0 is the MacDonald (or modified Bessel) function (see [11, p. 369] or [14] formula 17 p. 9). This function can be expressed also as

$$K_0(x) = \sqrt{\pi} e^{-x} \Psi(1/2, 1; 2x)\,,$$

where Ψ is the confluent hypergeometric function (see [15, p. 265]), or (see [11, p. 369])

$$K_0(x) = \int_0^{\infty} e^{-x\cosh\theta} d\theta\,.$$

For these two kernels, (3.6) implies that $\psi \in \mathcal{G}_H$ (defined in Theorem 3.1).

Remark 3.3 It is clear that other solutions of (3.5) hence of (3.6) exist. For the general class of solutions corresponding to semimartingales see [2, Sec. 6].

(3) For $H > 1/2$ we have if ψ is even,

$$\hat{\psi}(\lambda) = C_H \frac{|\lambda|^{H-\frac{1}{2}}}{\sqrt{\pi(1+\lambda^2)}} . \tag{3.7}$$

This function is in L^2 but not in L^1 (again). The corresponding kernel is[2]

$$\psi(x) = \frac{C_H}{\pi} \int_0^\infty \cos(\lambda x) \frac{|\lambda|^{H-\frac{1}{2}}}{\sqrt{\pi(1+\lambda^2)}} d\lambda . \tag{3.8}$$

Again, (3.7) implies that $\psi \in \mathcal{G}_H$.

We have proved

Proposition 3.4 *When $X = B_H$ with $H \geq 1/2$ and ψ is given by (3.8), the family $(\mu_{\mathcal{X}_\psi^\varepsilon})$ satisfies the LDP, in the scale ε^{-1} with good rate function*

$$\Lambda^*(\mu) = \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 d\Phi(x)$$

if $d\mu = g^2 d\Phi$.

Remark 3.5 In this case, Λ^* has a unique minimum at $\mu = \mathcal{N}(0;1)$ which allows to recover Wschebor's result on a.s. convergence.

4 "Level Process" Study

In the study of strong convergence problems such as the almost sure CLT (see [17] and [18]), an interesting problem is the LDP of empirical measures at the level of processes. If we restrict us to the Brownian case to simplify, the corresponding problem could be the behavior of

$$\int_0^1 \delta_{\left\{\frac{W(s+\varepsilon)-W(s)}{\sqrt{\varepsilon}},\ s \geq t\right\}} dt .$$

Here we do not see clearly the interest of such a study for the Wschebor's theorem. It seems more natural to consider the family $(\xi_t^\varepsilon, t \geq 0)$ of shifted processes

$$\xi_t^\varepsilon : s \in [0,1] \mapsto \frac{W(t+\varepsilon s) - W(t)}{\sqrt{\varepsilon}} \in \mathcal{C}([0,1]) , \tag{4.1}$$

[2] We did not find this integral in the literature on special functions.

so that for every $t > 0$, ξ_t^ε is $\mathcal{C}([0,1])$-valued. The new occupation measure is now

$$\mathcal{L}_\varepsilon := \int_0^1 \delta_{\xi_t^\varepsilon}\, dt\,. \tag{4.2}$$

By the scaling invariance, for every $\varepsilon > 0$,

$$(\xi_{\varepsilon t}^\varepsilon, t \geq 0) \overset{(d)}{=} (\xi_t^1, t \geq 0)\,, \tag{4.3}$$

and then

$$\mathcal{L}_\varepsilon = \int_0^1 \delta_{\xi_t^\varepsilon}\, dt \overset{(d)}{=} \tilde{\mathcal{L}}_\varepsilon := \varepsilon \int_0^{\varepsilon^{-1}} \delta_{\xi_t^1} dt\,. \tag{4.4}$$

Since we have

$$\xi_t^1 = (W(t+s) - W(t), s \in [0,1]), \tag{4.5}$$

the process $(\xi_t^1, t \geq 0)$ will be called the *meta-Slepian* process in the sequel. For every t, the distribution of ξ_t^1 is the Wiener measure $\mathbb{W}$ on $\mathcal{C}([0,1])$.

The meta-Slepian process is clearly stationary and 1-dependent. Since it is ergodic, the Birkhoff theorem tells us that, almost surely when $\varepsilon \to 0$, $\tilde{\mathcal{L}}_\varepsilon$ converges weakly to $\mathbb{W}$. From the equality in distribution (4.4) we deduce that $(\mathcal{L}_\varepsilon)$ converges in distribution to the same limit. But this limit is deterministic, hence the convergence of $(\mathcal{L}_\varepsilon)$ holds in probability. We just proved:

Theorem 4.1 *When $\varepsilon \to 0$, the family of random probability measures $(\mathcal{L}_\varepsilon)$ on $\mathcal{C}([0,1])$ converges in probability weakly to the Wiener measure $\mathbb{W}$ on $\mathcal{C}([0,1])$.*

The problem of a.s. convergence raises some difficulties. We have obtained on the one hand a partial a.s. fidi convergence (which is no more than a multidimensional extension of Wschebor's theorem) and on the other hand an a.s. convergence when we plug $\mathcal{C}([0,1])$ into the Hilbert space $L^2([0,1])$, equipped with its norm.

To this last purpose, if μ is a measure on $\mathcal{C}([0,1])$, we will denote by μ^L its extension to $L^2([0,1)]$, i.e. that for every Borel set B of $L^2([0,1])$,

$$\mu^L(B) = \mu(B \cap \mathcal{C}([0,1]))\,.$$

Theorem 4.2

1. *When $\varepsilon \to 0$, for every integer d and every $s_1, \ldots, s_d \in [0,1]$, the family $(\mathcal{L}_\varepsilon \pi_{s_1,\ldots,s_d}^{-1})$ of random probability measures on $\mathbb{R}^d$ converges weakly to $\mathbb{W}\pi_{t_1,\ldots,t_d}^{-1}$ on $\mathcal{C}([0,1])$, where $\pi_{t_1,\ldots,t_d}$ is the projection: $f \in \mathcal{C}([0,1]) \mapsto (f(t_1), \ldots, f(t_d))$.*
2. *When $\varepsilon \to 0$, almost surely, the family of random probability measures $(\mathcal{L}_\varepsilon^L)$ on $L^2([0,1])$ converges weakly to the Wiener measure $\mathbb{W}^L$ on $L^2([0,1])$.*

Remark 4.3 We called 1. a *partial* fidi convergence, since we failed to prove a *full* almost sure fidi convergence, which would be: Almost surely, for every $d, t_1, \dots, t_d$ $(\mathcal{L}_\varepsilon \pi^{-1}_{s_1,\dots,s_d}) \Rightarrow \mathbb{W}\pi^{-1}_{t_1,\dots,t_d}$ on $\mathcal{C}([0,1])$. Nevertheless it is plausible that such a statement holds true.

To prove Theorem 4.2, we need the following lemma, which is straightforward in view of stationarity and 1-dependence.

Lemma 4.4 *If F is a bounded differentiable function with bounded derivative from $\mathcal{C}([0,1])$ (resp. $L^2([0,1])$) to $\mathbb{R}$. Then*

$$a.s. \lim_{\varepsilon\to 0} \int_0^1 F(\xi_t^\varepsilon)\, dt = \int_{\mathcal{C}([0,1])} F(\xi)\mathbb{W}(d\xi)\,. \tag{4.6}$$

Proof of Lemma 4.4 It is along the lines of the proof of Theorem 2.1 in [1]. We first claim a quadratic convergence as follows. By Fubini and stationarity

$$\mathbb{E}\left(\int_0^1 F(\xi_t^\varepsilon) dt\right) = \int_0^1 \mathbb{E}F(\xi_t^\varepsilon) dt = \int_{\mathcal{C}([0,1])} F(\xi)\mathbb{W}(d\xi)\,,$$

and by Fubini and 1-dependence,

$$\mathrm{Var}\left(\int_0^1 F(\xi_t^\varepsilon) dt\right) = \int\int_{|t-s|<2\varepsilon} \mathrm{Cov}\left(F(\xi_t^\varepsilon), F(\xi_s^\varepsilon)\right) dt ds \le 4\varepsilon ||F||_\infty^2\,. \tag{4.7}$$

The Borel-Cantelli lemma implies a.s. convergence of $\int_0^1 F(\xi_t^\varepsilon) dt$ along any sequence (ε_n) such that $\sum_n \varepsilon_n < \infty$.

To go on, take $\varepsilon_{n+1} < \varepsilon < \varepsilon_n$ and notice that

$$\left|\int_0^1 F(\xi_t^\varepsilon) - F(\xi_t^{\varepsilon_n}) dt\right| \le ||F'||_\infty \sup_{t,u\in[0,1]} \left|\xi_t^\varepsilon(u) - \xi_t^{\varepsilon_n}(u)\right|\,. \tag{4.8}$$

Now we use some properties of Brownian paths. On $[0, 2]$ the Brownian motion satisfies a.s. a Hölder condition with exponent $\beta < 1/2$, so that we can define the a.s. finite random variable

$$M := 2 \sup_{u,v\in[0,2]} \frac{|W(u) - W(v)|}{|v-u|^\beta}\,. \tag{4.9}$$

So,

$$\begin{aligned}\sup_{s\in[0,1]} |\xi_t^\varepsilon(s) - \xi_t^{\varepsilon_n}(s)| &\leq \frac{M}{2}\frac{(\varepsilon_n-\varepsilon)^\beta}{\varepsilon^{1/2}} + \frac{M}{2}(\varepsilon_n)^\beta\left(\varepsilon^{-1/2}-(\varepsilon_n)^{-1/2}\right)\\ &= \frac{M}{2}\frac{(\varepsilon_n)^\beta}{\varepsilon^{1/2}}\left[\left(1-\frac{\varepsilon}{\varepsilon_n}\right)^\beta + \left(1-\sqrt{\frac{\varepsilon}{\varepsilon_n}}\right)\right] \leq M\frac{\varepsilon_n^\beta-\varepsilon^\beta}{\varepsilon^{1/2}} \leq M\frac{\varepsilon_n^\beta-\varepsilon_{n+1}^\beta}{\varepsilon_{n+1}^{1/2}}\,.\end{aligned} \tag{4.10}$$

The choice of $\varepsilon_n = n^{-a}$ with $a > 1$ and $\beta \in \left(\frac{a}{2(a+1)}, \frac{1}{2}\right)$ ensures that the right hand side of (4.10), hence of (4.8) tends to 0 a.s., which ends the proof. □

Proof of Theorem 4.2

1. The (random) characteristic functional of the (random) probability measure $\mathcal{L}_\varepsilon\pi_{s_1,\dots,s_d}^{-1}$ is

$$(a_1,\dots,a_d)\mapsto \int F_{a_1,\dots,a_d}(\xi_t^\varepsilon)dt$$

where the function

$$F_{a_1,\dots,a_d}(\xi) := \exp \mathrm{i}\sum_1^d a_k\xi(s_k)$$

fulfills the conditions of Lemma 4.4. We have then, for every $(a_1,\dots,a_d)$, a.s.

$$\lim \int F_{a_1,\dots,a_d}(\xi_t^\varepsilon)dt = \int_{\mathcal{C}([0,1])} F_{a_1,\dots,a_d}(\xi)\mathbb{W}(d\xi) \tag{4.11}$$

Taking for A a countable dense subset of $\mathbb{R}^d$, we have that a.s. for every $a \in A$, (4.11) holds true This implies that, a.s. the family $\mathcal{L}_\varepsilon\pi_{s_1,\dots,s_d}^{-1}$ indexed by ε has $\mathbb{W}\pi_{s_1,\dots,s_d}^{-1}$ as its only limit point. It remains to prove tightness. Assume that $d = 1$ to simplify. A classical inequality [6, p. 359] gives:

$$\lambda\{t\in[0,1] \;:\; |\xi_t^\varepsilon(s)| > M\} \leq \frac{M}{2}\int_{-2/M}^{2/M}\left(1-\int F_a(\xi_t^\varepsilon)dt\right)\,da\,.$$

The integrand is bounded by 2 and converges for a.e. a. By Lebesgue's theorem, this yields to

$$\int_{-2/M}^{2/M}\left(1-\int F_a(\xi_t^\varepsilon)dt\right)\,da \to \int_{-2/M}^{2/M}\left(1-\int_{\mathcal{C}([0,1])} F_a(\xi)\mathbb{W}(\xi)\right)\,da\,. \tag{4.12}$$

The rest is routine.

2. We will use a method coming from [16, p. 46].[3] It consists in checking Billingsley's criterion on intersection of balls [6, p. 18] and approximating indicators by smooth functions. Let us give details for only one ball to shorten the proof.

   For $\delta \in (0, 1)$, define

$$\phi_\delta(t) = \mathbf{1}_{(0,1]}(t) + \mathbf{1}_{[1,(1+\delta)^2]}(t) \frac{1}{C} \int_0^{\frac{((1+\delta)^2 - t)}{(2\delta+\delta^2)}} e^{-\frac{1}{s(1-s)}} ds \ , \tag{4.13}$$

   where

$$C = \int_0^1 e^{-\frac{1}{s(1-s)}} ds \, .$$

   The function ϕ_δ has a bounded support and it is continuous and $||\phi_\delta||_\infty = 1$. Now we consider $\psi_\delta : L^2([0, 1] \to \mathbb{R}$ defined by

$$\psi_\delta(\xi) = \phi_\delta(||\xi||^2).$$

   This function is $\mathcal{C}^\infty$ and has all its derivatives bounded. For every $\xi_c \in L^2([0, 1])$, $r > 0$, $\delta \in (0, r)$ we have the nesting

$$\mathbf{1}_{B(\xi_c; r-\delta)}(\xi) \leq \psi_{\frac{\delta}{r-\delta}} \left(\frac{\xi - \xi_c}{r - \delta} \right) \leq \mathbf{1}_{B(\xi_c; r)}(\xi) \leq \psi_{\frac{\delta}{r}} \left(\frac{\xi - \xi_c}{r} \right) \leq \mathbf{1}_{B(\xi_c; r+\delta)}(\xi) \, . \tag{4.14}$$

   Take a sequence $\delta_n \to 0$.

   Let us remind that the measure $\mathcal{L}_\varepsilon^L$ is random. We did not write explicitly the item W for simplicity, although it is present in (4.1).

For every test function F as in Lemma 4.4, we have a null set N_F such that for $W \notin N_F$

$$\int_{L^2([0,1])} F(\xi) \mathcal{L}_\varepsilon^L(d\xi) \to \int_{\mathcal{C}([0,1])} F(\xi) \mathbb{W}(d\xi) \, . \tag{4.15}$$

Let $(g_k)_{k \geq 1}$ be a countable dense set in $L^2([0, 1])$, and for $q \in \mathbb{Q}$,

$$F^-_{n,k,q}(\xi) = \psi_{\delta_n/(q-\delta_n)} \left(\frac{\xi - g_k}{q - \delta_n} \right), \, F^+_{n,k,q}(\xi) = \psi_{\delta_n/q} \left(\frac{\xi - g_k}{q} \right)$$

[3] It is used there to prove that in Hilbert spaces, convergence in the Zolotarev metric implies weak convergence.

and

$$N = \bigcup_{n,k,q} \left(N_{F^-_{n,k,q}} \cup N_{F^+_{n,k,q}} \right) .$$

Take $W \notin N$. Assume that the ball $B(\xi_c; r)$ is given. For every $\gamma > 0$, one can find $k \geq 1$ and $q \in \mathbb{Q}^+$ such that

$$||\xi_c - g_k|| \leq \gamma \ , \ |r - q| \leq \gamma \, . \tag{4.16}$$

By (4.14) we have

$$\mathcal{L}^L_\varepsilon(B(\xi_c; r)) \leq \int \psi_{\delta_n/r} \left(\frac{\xi - \xi_c}{r} \right) \mathcal{L}^L_\varepsilon(d\xi) \, . \tag{4.17}$$

Besides, by (4.16) and by differentiability, there exists $C_n > 0$ such that

$$\psi_{\delta_n/r} \left(\frac{\xi - \xi_c}{r} \right) \leq F^+_{n,k,q}(\xi) + C_n \gamma \, . \tag{4.18}$$

Now, by (4.15),

$$\lim_\varepsilon \int_{L^2([0,1])} F^+_{n,k,q}(\xi) \mathcal{L}^L_\varepsilon(d\xi) = \int_{\mathcal{C}([0,1])} F^+_{n,k,q}(\xi) \mathbb{W}(d\xi) \, , \tag{4.19}$$

and by (4.14) again

$$\int_{\mathcal{C}([0,1])} F^+_{n,k,q}(\xi) \mathbb{W}(d\xi) \leq \mathbb{W}(B(g_k, q + \delta_n)) \, . \tag{4.20}$$

So far, we have obtained

$$\limsup_\varepsilon \mathcal{L}^L_\varepsilon(B(\xi_c; r)) \leq \mathbb{W}(B(g_k, q + \delta_n)) + C_n \gamma \, . \tag{4.21}$$

It remains, in the right hand side, to let $\gamma \to 0$ (hence $g_k \to \xi_c$ and $q \to r$), and then $n \to \infty$ to get

$$\limsup_\varepsilon \mathcal{L}^L_\varepsilon(B(\xi_c; r)) \leq \mathbb{W}(B(\xi_c, r)) \, . \tag{4.22}$$

With the same line of reasoning, using the other part of (4.14) we can obtain

$$\liminf_\varepsilon \mathcal{L}^L_\varepsilon(B(\xi_c; r)) \geq \mathbb{W}(B(\xi_c, r)) \, , \tag{4.23}$$

which ends the proof for one ball.

A similar proof can be made for functions approximating intersection of balls as in Theorem 2.2 of [16] and as a consequence the a.s. weak convergence follows. □

To end this section, we state the LDP for $(\mathcal{L}_\varepsilon)$ defined in (4.2). It is an extension of the scalar case (Proposition 2.1) and since the proof is similar, we omit it.

Proposition 4.5 *The family* $(\mathcal{L}_\varepsilon)$ *satisfies the LDP in* $\mathcal{M}_1(\mathcal{C}([0,1]))$ *equipped with the weak topology, in the scale* ε^{-1} *with good rate function*

$$\mathbf{\Lambda}^*(\mathcal{L}) = \sup_{F \in \mathcal{C}_b(\mathcal{C}([0,1]))} \int_{\mathcal{C}([0,1])} F(\xi)\mathcal{L}(d\xi) - \mathbf{\Lambda}(F), \tag{4.24}$$

(the Legendre dual of $\mathbf{\Lambda}$*) where for every* $F \in \mathcal{C}_b(\mathcal{C}([0,1]))$*,*

$$\mathbf{\Lambda}(F) = \lim_{T\to\infty} T^{-1} \log \mathbb{E} \int_0^T F(\xi_t^1)dt\,. \tag{4.25}$$

5 Discretization and Random Walks

For a possible discrete version of Wschebor's theorem and associated LDP, we can consider a continuous process S observed in a uniform mesh of $[0,1]$ and study the sequence $\{S\left(\frac{k+r}{n}\right) - S\left(\frac{k}{n}\right), k \le n-r\}$ where the lag r may depend on n. On that basis, there are two points of view. When r is fixed, there are already results of a.s. convergence of empirical measures of increments of fBm [4] and we explain which LDP holds. When r depends on n with $r_n \to \infty$ and $r_n/n \to 0$, we are actually changing t into k/n and ε into r_n/n in the above sections. It allows to obtain results on random walks.We state convergence (Theorem 5.1) and LDP (Theorem 5.2) under specific conditions.

All the LDPs mentioned take place in $\mathcal{M}^1(\mathbb{R})$ equipped with the weak convergence.

5.1 Fixed Lag

In [4], beyond the Wschbebor's theorem, there are results of a.s. convergence of empirical statistics built with the increments of fBm. The authors defined p. 39 the second order increments as

$$\Delta_n B_H(i) = \frac{n^H}{\sigma_{2H}} \left[B_H\left(\frac{i+2}{n}\right) - 2B_H\left(\frac{i+1}{n}\right) + B_H\left(\frac{i}{n}\right) \right].$$

and proved that as $n \to \infty$

$$\frac{1}{n-1}\sum_{0}^{n-2}\delta_{\Delta_n B_H(i)} \Rightarrow \mathcal{N}(0;1) \quad (a.s.)\,, \tag{5.1}$$

(Th. 3.1 p. 44 in [4]). Moreover, in a space-time extension, they proved that

$$\frac{1}{n-1}\sum_{0}^{n-2}\delta_{\frac{i}{n},\Delta_n B_H(i)} \Rightarrow \lambda \otimes \mathcal{N}(0;1) \quad (a.s.)\,, \tag{5.2}$$

(Th. 4.1 in [5]).

Let us restrict for the moment to the case $H = 1/2$. The empirical distribution in (5.1) has the same distribution as

$$\frac{1}{n-1}\sum_{0}^{n-2}\delta_{2^{-1/2}(X_{i+2}-X_{i+1})}$$

where the X_i are independent and $\mathcal{N}(0;1)$ distributed. We can deduce the LDP (in the scale n) from the LDP for the 2-empirical measure by contraction. If i is the mapping

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto (x_2 - x_1)/\sqrt{2} \end{aligned} \tag{5.3}$$

the rate function is

$$I(\nu) = \inf\{I_2(\mu);\, \mu \circ i^{-1} = \nu\}\,, \tag{5.4}$$

where I_2 is the rate function of the 2-empirical distribution (see [13, Th. 6.5.12]).

In the same vein, we could study the LDP for the empirical measure

$$\frac{1}{n-r}\sum_{0}^{n-r-1}\delta_{\frac{W(k+r)-W(k)}{\sqrt{r}}}$$

which looks like $\mathcal{W}_1^r$. When this lag r is fixed, the scale is n and the rate function is obtained also by contraction ($r = 1$ is just Sanov's theorem).

This point of view could be developed also for the fBm using stationarity instead of independence.

5.2 Unbounded Lag

Let (X_i) be a sequence of i.i.d. random variables and (S_i) the process of partial sums. Let (r_n) be a sequence of positive integers such that $\lim_n r_n = \infty$, and assume that

$$\varepsilon_n := \frac{r_n}{n} \searrow 0\,. \tag{5.5}$$

Set

$$V_k^n := \frac{S_{k+r_n} - S_k}{\sqrt{r_n}}\ ,\ m_n = \frac{1}{n}\sum_1^n \delta_{V_k^n}\,. \tag{5.6}$$

The next theorems state some extensions of Wschebor's theorem and give the associated LDPs. The a.s. convergence is obtained only in the Gaussian case under an additional condition. It seems difficult to find a general method.

Theorem 5.1

1. *If* $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1$, *then*

$$m_n \Rightarrow \mathcal{N}(0;1) \quad \textit{(in probability)}\,. \tag{5.7}$$

2. *If* $X_1 \sim \mathcal{N}(0;1)$ *and if* (ε_n) *is such that there exists* $\delta \in (0, 1/2)$ *and a subsequence* (n_k) *satisfying*

$$\sum_k \varepsilon_{n_k} < \infty \textit{ and } \varepsilon_{n_k} = \varepsilon_{n_{k+1}} + o(\varepsilon_{n_{k+1}}^{1+\delta})\,, \tag{5.8}$$

it holds that

$$m_n \Rightarrow \mathcal{N}(0;1) \quad (a.s.)\,. \tag{5.9}$$

Theorem 5.2

1. *Assume that* $X_1 \sim \mathcal{N}(0;1)$. *If* $\lim_n \varepsilon_n n^{1/2} = \infty$, *then* (m_n) *satisfies the LDP in the scale* ε_n^{-1} *with rate function given in (1.12) and (1.13) where* $\psi = \Psi_1$.
2. *Assume that* X_1 *has all its moments finite and satisfies* $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$ *and that*

$$0 < \liminf_n \varepsilon_n \log n \le \limsup_n \varepsilon_n \log n < \infty\,. \tag{5.10}$$

Then (m_n) *satisfies the LDP in the scale* ε_n^{-1} *with rate function given in (1.12) and (1.13) where* $\psi = \Psi_1$.

Remark 5.3 Two examples of (r_n) satisfying the assumptions of Theorem 5.1(2) are of interest, particularly in relation to the LDP of Theorem 5.2. The first one is $r_n = \lfloor n^\gamma \rfloor$ with $\gamma \in (0, 1)$ (hence $\varepsilon_n \sim n^{\gamma-1}$), for which we can choose $n_k = \lfloor k^{a(1-\gamma)} \rfloor$ with $a > 1$. The second one is $r_n = \lfloor n/\log n \rfloor$ (hence $\varepsilon_n \sim (\log n)^{-1}$), for which we can choose $n_k = \lfloor e^{k^2} \rfloor$.

Proof of Theorem 5.1 We use the method of the above Lemma 4.4 inspired by Azaïs and Wschebor [1]. For a bounded continuous test function f

$$\mathbb{E} \int f dm_n = \mathbb{E} f\left(\frac{S_{r_n}}{\sqrt{r_n}}\right) \to \int f d\Phi$$

thanks to the CLT. Moreover

$$\mathrm{Var}\left(\int f dm_n\right) = \frac{1}{n^2} \sum_{|j-k|\le r_n} \mathrm{Cov}\left(f\left(\frac{S_{j+r_n} - S_j}{\sqrt{r_n}}\right), f\left(\frac{S_{k+r_n} - S_k}{\sqrt{r_n}}\right)\right) \le \frac{8r_n}{n} ||f||_\infty^2 .$$

This gives the convergence in probability.

In the Gaussian case, it is possible to repeat the end of the proof of Lemma 4.4. Under our assumption, we see that for any $\beta \in (0, 1/2)$

$$\frac{\varepsilon_{n_k}^\beta - \varepsilon_{n_{k+1}}^\beta}{\varepsilon_{n_{k+1}}^{1/2}} = o\left(\varepsilon_{n_{k+1}}^{\delta+\beta-\frac{1}{2}}\right) ,$$

which implies that it is enough to choose $\beta \in \left(\frac{1}{2} - \delta, \frac{1}{2}\right)$. □

Proof of Theorem 5.2

(1) If $X_1 \sim \mathcal{N}(0; 1)$, then

$$(V_k^n, k = 1, \ldots, n) \overset{(d)}{=} \left((\varepsilon_n)^{-1/2}\left(W\left(\frac{k}{n} + \varepsilon_n\right) - W\left(\frac{k}{n}\right)\right), k = 1, \ldots, n\right)$$

and then it is natural to consider m_n as a Riemannian sum. We now have to compare m_n with

$$\mu_{\mathcal{W}_1^{\varepsilon_n}} = \int_0^1 \delta_{\varepsilon_n^{-1/2}(W(t+\varepsilon_n)-W(t))} dt .$$

It is known that $d_{BL}(\mu, \nu)$ given by (2.8) is a convex function of (μ, ν) so that:

$$d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) \leq \int_0^1 d_{BL}(\delta_{\varepsilon_n^{-1/2}(W(t+\varepsilon_n)-W(t))}, \delta_{V^n_{\lfloor nt \rfloor}})dt$$

$$\leq \varepsilon_n^{-1/2} \int_0^1 \left| W(t+\varepsilon_n) - W(t) - W\left(\frac{\lfloor nt \rfloor}{n} + \varepsilon_n\right) + W\left(\frac{\lfloor nt \rfloor}{n}\right) \right| dt$$

$$\leq 2(\varepsilon_n)^{-1/2} \sup_{|t-s|\leq 1/n} |W(t) - W(s)|$$

hence

$$\mathbb{P}(d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) > \delta) \leq \mathbb{P}\left(\sup_{|t-s|\leq 1/n} |W(t) - W(s)| > \frac{\delta(\varepsilon_n)^{1/2}}{2} \right) \leq 2 \exp - \frac{n\varepsilon_n \delta^2}{4}.$$

If $\lim_n \varepsilon_n n^{1/2} = \infty$ we conclude that

$$\lim_{n\to\infty} \varepsilon_n \log \mathbb{P}(d_{BL}(m_n, \mu_{\mathcal{W}_1^{\varepsilon_n}}) > \delta) = -\infty,$$

which means that (m_n) and $(\mu_{\mathcal{W}_1^{\varepsilon_n}})$ are exponentially equivalent in the scale ε_n^{-1} (Def. 4.2.10 in [13]).

Now, from our Proposition 2.1 or Theorem 3.1, $(\mu_{\mathcal{W}_1^{\varepsilon_n}})$ satisfies the LDP in the scale ε_n^{-1}. Consequently, from Th. 4.2.13 of [13], the family (m_n) satisfies the LDP at the same scale with the same rate function.

(2) Let us go to the case when X_1 is not normal. We use the Skorokhod representation, as in [17] or in [18] (see also [10] Th. 2.1.1 p.88).

When (X_i) is a sequence of independent (real) random variables such that $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$, there exists a probability space supporting a Brownian motion $(B(t); 0 \leq t < \infty)$ and an increasing sequence (τ_i) of stopping times such that

- $(\tau_{i+1} - \tau_i)$ are i.i.d., with $\mathbb{E}\tau_1 = 1$
- $(B(\tau_{i+1}) - B(\tau_i))$ are independent and distributed as X_1.

Moreover, if $\mathbb{E}X_1^{2q} < \infty$, then $\mathbb{E}\tau_1^q < \infty$.

We have

$$S_{j+r} - S_j \overset{(d)}{=} B(\tau_{j+r}) - B(\tau_j),$$

so that

$$m_n \overset{(d)}{=} \tilde{m}_n := \frac{1}{n} \sum_1^n \delta_{\tilde{V}_k^n} \text{ with } \tilde{V}_k^n = \frac{B(\tau_{k+r_n}) - B(\tau_k)}{\sqrt{r_n}}. \tag{5.11}$$

We will compare these quantities with

$$\pi_n = \frac{1}{n}\sum_1^n \delta_{U_k^n} \text{ with } U_k^n := \frac{B(k+r_n) - B(k)}{\sqrt{r_n}}\,, \tag{5.12}$$

which fall into the regime of the above part of the proof. We will prove that the sequences $(\tilde{m}_n)$ and (π_n) are exponentially equivalent.

Again by convexity of d_{BL}, we have

$$\begin{aligned} d_{BL}(\tilde{m}_n, \pi_n) &\le \sum_1^n \frac{1}{n} d_{BL}\left(\delta_{\tilde{V}_k^n}, \delta_{U_k^n}\right) \\ &\le \frac{1}{\sqrt{r_n}}\left(\sup_{k\le n} |B(\tau_{k+r_n}) - B(k+r_n)| + \sup_{k\le n} |B(\tau_k) - B(k)|\right) \end{aligned} \tag{5.13}$$

Our proof will be complete if we show that for all $\delta > 0$

$$\lim_n \frac{r_n}{n} \log \mathbb{P}\left(\max_{k\le n+r_n} |B(\tau_k) - B(k)| > \delta\sqrt{r_n}\right) = -\infty\,. \tag{5.14}$$

We will apply three times the following known result (cf. [18, Lemma 8] or [17, Lemma 2.9]).

If (ξ_i) are i.i.d. centered with $\mathbb{E}(\xi_1)^{2p}) < \infty$ for some $p \ge 1$, then there exists a universal constant $C > 0$ such that for all integers $n \ge 1$

$$\mathbb{E}(\xi_1 + \cdots + \xi_n)^{2p} \le C(2p)!\, \mathbb{E}(\xi_1^{2p}) n^p\,. \tag{5.15}$$

Actually, for $\alpha \in (0, 1)$ and $k \le r_n^\alpha$, by Markov's inequality and (5.15)

$$\mathbb{P}(|B(\tau_k)| > \delta\sqrt{r_n}) \le C(2p)!\, \delta^{-2p} r_n^{-p} \mathbb{E}((X_1')^{2p}) k^p \le C(2p)!\, \delta^{-2p} \mathbb{E}((X_1')^{2p}) r_n^{(\alpha-1)p}\,, \tag{5.16}$$

and for the same reasons

$$\mathbb{P}(B(k)| > \delta\sqrt{r_n}) \le C(2p)!\, \mathbb{E}(N^{2p})\delta^{2p} r_n^{(\alpha-1)p}\,. \tag{5.17}$$

Now, for $k \ge r_n^\alpha$, and $\beta > 1/2$

$$\mathbb{P}(|\tau_k - k| \ge k^\beta) \le C(2p)!\, \mathbb{E}((\tau_1 - 1)^{2p}) k^{p(1-2\beta)} \le C(2p)!\, \mathbb{E}((\tau_1 - 1)^{2p}) r_n^{\alpha p(1-2\beta)}\,.$$

Besides,

$$\mathbb{P}\left(|B(\tau_k)-B(k)|\geq 2\delta\sqrt{r_n}\,,\ |\tau_k-k|\leq k^{\beta}\right)\leq \mathbb{P}\left(\sup_{|t-k|\leq k^{\beta}}|B(t)-B(k)|>2\delta\sqrt{r_n}\right)$$

$$\leq 2\mathbb{P}\left(\sup_{t\in(0,k^{\beta})}|B(t)|>2\delta\sqrt{r_n}\right)\leq 4e^{-2\delta^2 r_n k^{-\beta}}\,,$$

which, for $k\leq n+r_n<2n$, yields

$$\mathbb{P}\left(|B_{\tau_k}-B_k|\geq 2\delta\sqrt{r_n}, |\tau_k-k|\leq k^{\beta}\right)\leq 4e^{-2^{1-\beta}\delta^2 r_n n^{-\beta}}\,. \tag{5.18}$$

Gathering (5.16), (5.17), and (5.18), we obtain, by the union bound,

$$\mathbb{P}\left(\max_{k\leq n+r_n}|B(\tau_k)-B(k)|>2\delta\sqrt{r_n}\right)\leq C_p\left(\delta^{2p}r_n^{1+(\alpha-1)p}+nr_n^{\alpha(1-2\beta)p}\right)$$

$$+8ne^{-2^{1-\beta}\delta^2 r_n n^{-\beta}}\,, \tag{5.19}$$

where the constant $C_p>0$ depends on p and on the distribution of X_1'.

Choosing $\beta>1/2$ and r_n such that

$$\liminf_n \frac{r_n}{n}\log r_n>0\,,\ \limsup_n \frac{r_n}{n}\log n<\infty\,,\ \liminf_n \frac{r_n^2}{n^{1+\beta}}>0\,, \tag{5.20}$$

we will ensure that for every $p>0$

$$\lim_n \frac{r_n}{n}\log\mathbb{P}\left(\max_{k\leq n+r_n}|B(\tau_k)-B(k)|>\delta\sqrt{r_n}\right)\leq -Cp \tag{5.21}$$

where C is a constant independent of p, which will prove (5.14).

Now, the set of sufficient conditions (5.20) is equivalent to the condition:

$$0<\liminf_n \frac{r_n}{n}\log n\leq\limsup_n \frac{r_n}{n}\log n<\infty\,,$$

which is exactly (5.10). □

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