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Maria Alberich-Carramiñana
Guillem Blanco
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Editors

Extended Abstracts GEOMVAP 2019

Geometry, Topology, Algebra, and Applications; Women in Geometry and Topology

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Preface

The Geometry of Varieties and Applications Group (GEOMVAP) is a group of researchers with interests in a wide range of fields, which include algebraic, differential and symplectic geometries, algebraic topology, commutative algebra and their applications. The group is composed of researchers rooted or formed at the Universitat Politècnica de Catalunya (UPC).

The main objective of GEOMVAP is to take a deep dive into the study of geometric structures and their applications. The geometric structures that are considered are algebraic varieties, symplectic and differentiable manifolds and the applications are mainly focused in the fields of Phylogenetics, Robotics, Mathematical Physics, Control Theory, Dynamical Systems and Celestial Mechanics. In order to achieve that end, a broad range of tools are used (geometric, algebraic, topological, arithmetic, differential and computational), and in many cases techniques from different fields are combined.

The members of the group work in interdisciplinary teams and transversal research topics. GEOMVAP promotes, in particular, Responsible Research and Innovation (RRI) within the framework of Horizon 2020. Among the RRI initiatives, it strives for gender equality, public engagement, science communication and the visibility of women in Science and Society.

The extended abstracts in this volume stem from the contributions in two events organized by GEOMVAP during the year 2019.

On January 23 and 24 of 2019, the GEOMVAP group organized an offside meeting on the *Parador de Cardona*, a large ninth century castle perched on a steep hill overlooking the town and salt mine of Cardona. The goal of that meeting was to share the different objectives, strategies and advances of the different research topics of GEOMVAP. The event consisted of 22 talks from different members of the group: Ph.D. students, postdoctoral members and professors.

A list of the contributed talks is included below.

List of Talks

- Maria Alberich Carramiñana, *Action of Cremona maps on planar polynomial differential systems.*
- Patricio Almirón Cuadros, *On the Tjurina number of plane curve singularities.*
- Josep Àlvarez Montaner, *D-modules over direct summands.*
- Miguel Angel Barja, *Clasificación de variedades irregulars y el Teorema Fundamental del Cálculo.*
- Guillem Blanco, *Bernstein-Sato polynomials of plane curves.*
- Roisin Braddell, *Group symmetries of cosymplectic and b-symplectic manifolds.*
- Joaquim Brugués, *La construcció de l'homologia de Floer*.
- Robert Cardona, *Estructures geomètriques en hidrodinàmica.*
- Franco Coltraro, *Mechanics of inextensible surfaces.*
- Josep Elgueta, *Representacions categòriques.*
- Marina Garrote López, *Semi-algebraic conditions for phylogenetic varieties.*
- Xavier Gràcia, *Hamilton-Jacobi theory and geometric mechanics.*
- Juan Margalef, *De la mecánica clásica a la mecánica cuántica.*
- Anastasiia Matveeva, *Group valued moment maps and equivariant cohomologies.*
- Eva Miranda, *From Celestial Mechanics to Fluid Dynamics: contact structures with singularities, part I*.
- Miguel C. Muñoz Lecanda, *Sobre distribucions no integrables.*
- Cédric Oms, *From Celestial Mechanics to Fluid Dynamics: contact structures with singularities, part II*.
- Alessandro Oneto, *Looking for equations of mixtures of phylogenetic models.*
- Arnau Planas, *A* b^m*-symplectic KAM theorem.*
- Xavier Rivas Guijarro, *Singular Lagrangian field theories and k-cosymplectic geometry.*
- Jordi Roca-Lacostena, *On the embedding problem for evolutionary Markov matrices.*
- Narciso Román Roy, *Multisimplectic formulation of Lagrangian models in gravitation (GR).*

The workshop *Women in Geometry and Topology* was an endeavor organized by the GEOMVAP research group and financed under the AGAUR project 2017SGR932. It took place at the *Centre de Recerca Matemètica*, Barcelona, from Semptember 25 to 27, 2019.

The workshop Women in Geometry and Topology featured nine plenary talks by top female mathematicians and several contributed talks and poster presentations by speakers of any gender identity. Two of the plenary lectures were addressed to the general public, not only for mathematicians but also for anyone with curiosity. A panel open to the public was also organized in order to discuss the situation of women in mathematics, the gender gap and strategies to break the glass-ceiling inside and outside academia.

Below there is a list of plenary, contributed talks and poster presentations.

List of Plenary Talks

- Başak Gürel (University of Central Florida), *From Hamiltonian systems with infinitely many periodic orbits to pseudo-rotations via symplectic topology.*
- Kathryn Hess (École Polytechnique Fédérale de Lausanne, EPFL SV BMI UPHESS), *What does topology have to do with neuroscience?*
- Ann Lemahieu (Laboratoire de Mathématiques J. A. Dieudonné), *On the monodromy conjecture for nondegenerate hypersurface singularities.*
- Marta Macho-Stadler (Universidad del País Vasco-Euskal Herriko Unibertsitatea), *Sesgos de género en la Academia: cuando las matemáticas no funcionan.*
- Catherine Meusburger (FAU Erlangen-Nürnberg), *Ideal tetrahedra and their duals.*
- Emmy Murphy (Northwestern University), *The Koras-Russel cubic and Weinstein flexibility.*
- Rita Pardini (Università di Pisa), *Deformations of semi-smooth varieties.*
- M. Eugenia Rosado Mar a (Universidad Politécnica de Madrid), *Second-order Lagrangians admitting a first-order Hamiltonian formalism.*
- Lidia Stoppino (Università degli Studi di Pavia), *Clifford-Severi inequalities for varieties of maximal Albanese dimension.*
- Ulrike Tillman (University of Oxford), *Geometric groups via homotopy theory.*
- Carme Torras (Institut de Robòtica i Informàtica Industrial (CSIC-UPC)), *Cloth manipulation in assistive robotics: Research challenges, ethics and fiction.*

List of Contributed Talks

- Daria Alekseeva (National Research University Higher School of Economics), *Presentations of symplectic mapping class group of rational 4-manifolds.*
- Patricio Almirón (Universidad Complutense de Madrid), *Milnor and Tjurina, a 4/3 relation.*
- Guillem Blanco Fernandez (Universitat de Politècnica de Catalunya), *Yano's conjecture.*
- Melanie Bondorevsky (Universidad de Buenos Aires & IMAS-CONICET), *Topological degree and periodic orbits of semi-dynamical systems.*
- Robert Cardona (Universitat de Politècnica de Catalunya), *A contact topology approach to Euler flows universality.*
- Joana Cirici (Universitat de Barcelona), *Hodge theory of almost Kähler manifolds.*
- Franco Coltraro (Institut de Robòtica i Informàtica Industrial, CSIC-UPC), *Collisions and friction for inextensible cloth simulatio.*
- Aina Ferrà Marcús (Universitat de Barcelona), *Localizations of models of theories with arities.*
- Marina Garrote-López (Universitat de Politècnica de Catalunya), *Distance to the stochastic part of phylogenetic varieties.*

- Jordi Gaset (Universitat de Politècnica de Catalunya), *A contact geometry approach to symmetries in systems with dissipation.*
- Debora Gil (Computer Vision Center and Computer Science Department at Universitat Autonoma de Barcelona), *Topological Radiomics (TOPiomics): Early Detection of Genetic Abnormalities in Cancer Treatment Evolution.*
- Matias V. Moya Giusti (Université Paris-Est), *Dimension formulas for the cohomology of arithmetic groups.*
- Cédric Oms (Universitat de Politècnica de Catalunya), *Do overtwisted contact manifolds admit infinitely many periodic Reeb orbits?*
- Sinem Onaran (Hacettepe University), *Legendrian knots in contact 3-manifolds.*
- Maryam Samavaki (University of Eastern Finland), *On several classes of Ricci tensor.*
- Julia Semikina (University of Bonn), *G-theory of group rings for finite groups.*
- Paola Supino (Roma Tre University), *On complete intersections containing a linear subspace.*
- M. Pilar Vélez (Universidad Antonio de Nebrija), *Automated proving and discovery in elementary Geometry by means of algebraic geometry.*

List of Poster Presentations

- Joaquim Brugués (Universitat de Politècnica de Catalunya), *Towards a Floer homology for singular symplectic manifolds.*
- Luciana Bonatto (University of Oxford), *Decoupling in Higher Dimensions.*
- Marta Mazzocco (Universitat de Birmingham), *Poisson Structures on Painlevé Monodromy Manifolds.*
- Pau Mir (Universitat de Politècnica de Catalunya), *Invariants in Semitoric Integrable Systems. Looking for a new interpretation.*
- Inasa Nakamura (Institute of Science and Engineering, Kanazawa University), *Branched covering surfaces in 4-space and simplifying numbers.*

We are very happy to attest that the atmosphere created by the participants of the workshop was very open and friendly, and we hope that it led to effective further collaborations.

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$\mathbb{Q}$-Hilbert Functions of Multiplier and Test Ideals

Josep Àlvarez Montaner and Luis Núñez-Betancourt

Abstract This is an extended abstract with some of the results that will appear in the forthcoming paper [3] in which we prove the rationality of the Poincaré series associated to multipliers and test ideals as long as we have discreteness and rationality of the corresponding jumping numbers and Skoda's theorem is available. In order to do so we extend the theory of Hilbert functions to the case of filtrations indexed over the rational numbers.

1 Introduction

Let A be a commutative Noetherian ring containing a field $\mathbb{K}$. Assume that A is either local or graded with maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Depending on the characteristic of the base field we may find two parallel sets of invariants associated to the pair $(A, \mathfrak{a}^c)$ where c is a real parameter. In characteristic zero we have the theory of *multiplier ideals* which play a prominent role in birational geometry and are defined using resolution of singularities (see [12] for more insight). In positive characteristic we may find the so-called *test ideals* which originated from the theory of tight closure [10, 11] and are defined using the Frobenius endomorphism. Despite its different origins, it is known that under some conditions on A, the reduction mod p of a multiplier ideal is the corresponding test ideal. Moreover, both theories share

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a lot of common properties which we summarize as saying that they form a filtration of $\mathfrak{m}$-primary ideals

$$\mathcal{J}: \quad A \supsetneq \mathcal{J}_{\alpha_1} \supsetneq \mathcal{J}_{\alpha_2} \supsetneq \cdots \supsetneq \mathcal{J}_{\alpha_i} \supsetneq \cdots$$

and the indices where there is an strict inequality is a discrete set of rational numbers. Following the ideas of [8] we define the *multiplicity* of $c \in \mathbb{R}_{>0}$ as $m(c) = \dim_{\mathbb{K}} (\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c)$, for $\varepsilon > 0$ small enough, and the *Poincaré series* of $\mathcal{J}$ as

$$P_{\mathcal{J}}(T) = \sum_{c \in \mathbb{R}_{>0}} \dim_{\mathbb{K}} (\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c) \ T^c.$$

The natural question is whether this is a rational function, in the sense that it belongs to the field of fractional functions $\mathbb{Q}(z)$ where the indeterminate z corresponds to a fractional power $T^{1/e}$ for a suitable $e \in \mathbb{N}_{>0}$. The only known results on this question were given in [9] and [1, 2] where the authors proved the rationality of the Poincaré series of multiplier ideals in rings of dimension two by giving an explicit formula for the multiplicities.

The approach that we are going to give is completely algebraic and will provide an unified proof of the rationality of the Poincaré series for both the multiplier and the test ideals in any dimension. To such purpose we will develop a theory of Hilbert functions indexed over $\mathbb{Q}$.

2 $\mathbb{Q}$-Good Filtrations

Let A be a commutative Noetherian ring. Assume that A is either local or graded with maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. The theory of *good* $\mathfrak{a}$-*filtrations* gives an approach to the study of Hilbert functions that covers most of the classical results in an unified way. We start recalling briefly this notion but we refer to the monograph [13] and the references therein for more insight.

Let M be a finitely generated A-module such that $\lambda(M/\mathfrak{a}M) < \infty$, where $\lambda(\cdot)$ denotes the length as A-module. A *good* $\mathfrak{a}$-*filtration* on M is a decreasing filtration

$$\mathcal{M}: \quad M = M_0 \supseteq M_1 \supseteq \cdots$$

by A-submodules of M such that $M_{j+1} = \mathfrak{a}M_j$ for $j >> 0$ large enough. Under these premises we may consider the *Hilbert* and the *Hilbert–Samuel function* of M with respect to the filtration $\mathcal{M}$ defined as

$$H_{\mathcal{M}}(j) := \lambda\left(M_j/M_{j+1}\right) \quad \text{and} \quad H^1_{\mathcal{M}}(j) := \lambda\left(M/M_j\right)$$

respectively. Moreover, we consider the *Hilbert* and the *Hilbert–Samuel series*

$$HS_{\mathcal{M}}(T) := \sum_{j\geq 0} \lambda\left(M_j/M_{j+1}\right) T^j \quad \text{and} \quad HS^1_{\mathcal{M}}(T) := \sum_{j\geq 0} \lambda\left(M/M_j\right) T^j.$$

Notice that we have $HS_{\mathcal{M}}(T) = (1-T)HS^1_{\mathcal{M}}(T)$. As a consequence of the Hilbert–Serre theorem we can express them as rational functions

$$HS_{\mathcal{M}}(T) = (1-T)HS^1_{\mathcal{M}}(T) = (1-T)\frac{h_{\mathcal{M}}(T)}{(1-T)^{d+1}},$$

where $h_{\mathcal{M}}(T) \in \mathbb{Z}[T]$ satisfies $h_{\mathcal{M}}(1) \neq 0$ and d is the Krull dimension of M. The polynomial $h_{\mathcal{M}}(T)$ is the *h-polynomial* of $\mathcal{M}$.

The aim of this section is to extend the notion of good $\mathfrak{a}$-filtrations by allowing filtrations indexed over $\mathbb{Q}$ and thus mimicking properties satisfied by filtrations given by multiplier and test ideals.

Definition 1 Let M be a finitely generated A-module such that $\lambda(M/\mathfrak{a}M) < \infty$. A $\mathbb{Q}$-good $\mathfrak{a}$-filtration is a decreasing filtration $\mathcal{M} := \{M_\alpha\}_{\alpha\geq 0}$ of submodules of $M_0 = M$, indexed by a discrete set of positive rational numbers such that $M_{\alpha+1} = \mathfrak{a}M_\alpha$ for all $\alpha > j$ with $j >> 0$ large enough.

Indeed, we may think of $\mathcal{M}$ as a filtration of submodules M_c indexed over the real numbers for which there exist an increasing sequence of rational numbers $0 < \alpha_1 < \alpha_2 < \cdots$ such that $M_{\alpha_i} = M_c \supsetneq M_{\alpha_{i+1}}$ for any $c \in \left[\alpha_i, \alpha_{i+1}\right)$. In particular we have a discrete filtration of submodules

$$\mathcal{M}: \quad M \supsetneq M_{\alpha_1} \supsetneq M_{\alpha_2} \supsetneq \cdots \supsetneq M_{\alpha_i} \supsetneq \cdots$$

and we say that the α_i are the *jumping numbers* of $\mathcal{M}$. A crucial observation is that, once we fix an index $c \in \mathbb{R}$, the filtration

$$\mathcal{M}_c: \quad M_c \supseteq M_{c+1} \supseteq M_{c+2} \supseteq \cdots$$

is a good $\mathfrak{a}$-filtration.

Definition 2 Let $\mathcal{M} := \{M_c\}_{c\geq 0}$ be a $\mathbb{Q}$-good $\mathfrak{a}$-filtration. We define the multiplicity of $c \in \mathbb{R}_{>0}$ as

$$m(c) := \lambda\left(M_{c-\varepsilon}/M_c\right)$$

for $\varepsilon > 0$ small enough. Clearly, c is a jumping number if and only if $m(c) > 0$.

Definition 3 Let $\mathcal{M} := \{M_c\}_{c\geq 0}$ be a $\mathbb{Q}$-good $\mathfrak{a}$-filtration. We define the Poincaré series of $\mathcal{M}$ as

$$P_{\mathcal{M}}(T) = \sum_{c\in\mathbb{R}_{>0}} m(c) \quad T^c.$$

The question that we want to address is whether the Poincaré series is rational in the sense that it belongs to the field of fractional functions $\mathbb{Q}(T^{1/e})$ where $e \in \mathbb{N}_{>0}$ is the least common multiple of the denominators of all the jumping numbers.

Proposition 4 *Let $\mathcal{M} := \{M_c\}_{c\geq 0}$ be a $\mathbb{Q}$-good $\mathfrak{a}$-filtration. Given $c \in \mathbb{R}_{>0}$ we have that*

$$\sum_{j\geq 0} m(c+j)T^j$$

is a rational function in $\mathbb{Q}(T^{1/e})$.

Theorem 5 *Let $\mathcal{M} := \{M_c\}_{c\geq 0}$ be a $\mathbb{Q}$-good $\mathfrak{a}$-filtration. Then, the Poincaré series $P_{\mathcal{M}}(T)$ is rational. Indeed we have*

$$P_{\mathcal{M}}(T) = \sum_{c\in(0,1]} \left(\frac{m(c)}{1-T} + \frac{h_{\mathcal{M}_c}(T) - h_{\mathcal{M}_{c-\varepsilon}}(T)}{(1-T)^{d+1}} \right) T^c$$

where $h_{\mathcal{M}_c}(T)$ is the h-polynomial associated to $\mathcal{M}_c$ and $d = \dim A$.

3 Poincaré Series of Multiplier and Test Ideals

In this Section we specialize the results obtained above to the case of multiplier and test ideals.

3.1 Multiplier Ideals

Let $(A, \mathfrak{m})$ be a normal local ring essentially of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathfrak{a} \subseteq A$ an $\mathfrak{m}$-primary ideal. Under these general assumptions we ensure the existence of canonical divisors K_X on $X = \operatorname{Spec} A$ which are not necessarily $\mathbb{Q}$-Cartier. Then we may find some effective boundary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index m large enough. Now, given a *log-resolution* $\pi : X' \to X$ of the triple $(X, \Delta, \mathfrak{a})$ we pick a canonical divisor $K_{X'}$ in X' such that $\pi^* K_{X'} = K_X$ and let F be an effective divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$.

The *multiplier ideal* associated to the triple $(X, \Delta, \mathfrak{a}^c)$ for some real number $c \in \mathbb{R}_{>0}$ is defined as

$$\mathcal{J}(X, \Delta, \mathfrak{a}^c) = \pi_* \mathcal{O}_{X'} \left(\left\lceil K_{X'} - \frac{1}{m}\pi^*(m(K_X + \Delta)) - cF \right\rceil \right).$$

This construction allowed de Fernex and Hacon [7] to define the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ associated to $\mathfrak{a}$ and c as the unique maximal element of the set of multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^c)$ where Δ varies among all the effective divisors such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The key point proved in [7] is the existence of such a divisor Δ that realizes the multiplier ideal as $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(X, \Delta, \mathfrak{a}^c)$.

From its construction we have that the multiplier ideals form a discrete filtration of $\mathfrak{m}$-primary ideals

$$A \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_2}) \supsetneq \cdots \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_i}) \supsetneq \cdots$$

and the α_i where we have a strict inclusion of ideals are the *jumping numbers* of the ideal $\mathfrak{a}$. Although Skoda's theorem still hold in this framework [7, Corollary 5.7], that is, for any $c > \dim A$

$$\mathcal{J}(X, \Delta, \mathfrak{a}^c) = \mathfrak{a} \cdot \mathcal{J}(X, \Delta, \mathfrak{a}^{c-1})$$

there are cases where the jumping numbers are not rational as shown by Urbinati in [16]. Moreover, the question about discreteness is still open despite some partial results in this direction. Is for this reason that, in order to have a $\mathbb{Q}$-good $\mathfrak{a}$-filtration $\mathcal{J} = \{\mathcal{J}(\mathfrak{a}^c)\}$ given by multiplier ideals we will have to restrict to the case that A is $\mathbb{Q}$-Gorenstein. In this situation, the canonical module K_X is $\mathbb{Q}$-Cartier so no boundary Δ is required in the definition of multiplier ideal.

Theorem 6 *Let $(A, \mathfrak{m})$ be a normal local $\mathbb{Q}$-Gorenstein ring essentially of finite type over a field $\mathbb{K}$ of characteristic zero and let $\mathfrak{a} \subseteq A$ be an $\mathfrak{m}$-primary ideal. Let $\mathcal{J} := \{\mathcal{J}(\mathfrak{a}^c)\}_{c \in \mathbb{R}_{>0}}$ be the filtration given by multiplier ideals. Then, the Poincaré series $P_{\mathcal{J}}(T)$ is rational.*

3.2 Test Ideals

Let A be a commutative Noetherian ring containing a field $\mathbb{K}$ of characteristic $p > 0$. The theory of test ideals has its origins in the work of Hochster and Huneke on tight closure [11]. In the case of A being a regular ring, Hara and Yoshida in [10] extended the notion of test ideals to pairs $(A, \mathfrak{a}^c)$ where $\mathfrak{a} \subseteq A$ is an ideal. Their construction has been generalized in subsequent works [4–6, 14] using the theory of *Cartier operators*. Roughly speaking, the test ideal $\tau(\mathfrak{a}^c)$ is the smallest nonzero ideal which is compatible with any Cartier operator $\phi \in \bigoplus_{e \geq 0} \mathrm{Hom}_A(F_*^e A, A) \cdot F_*^e \mathfrak{a}^{\lceil cp^e \rceil}$, where F_*^e is the Frobenius functor. In this situation we also have a filtration

$$A \supsetneq \tau(\mathfrak{a}^{\alpha_1}) \supsetneq \tau(\mathfrak{a}^{\alpha_2}) \supsetneq \cdots \supsetneq \tau(\mathfrak{a}^{\alpha_i}) \supsetneq \cdots$$

and the α_i where we have a strict inclusion of ideals are called the *F-jumping numbers* of the ideal $\mathfrak{a}$.

The big question in this setting is whether the F-jumping numbers are discrete and rational. There are some results in this direction but we are going to pay attention to the work of Schwede and Tucker [15] where they consider a normal $\mathbb{Q}$-Gorenstein local domain $(A, \mathfrak{m})$ essentially of finite type over a perfect field $\mathbb{K}$ of characteristic $p > 0$. Not only they proved the discreteness and rationality of F-jumping numbers

[15, Theorem 6.3] but also gave a version of Skoda's theorem that reads as $\tau(\mathfrak{a}^c) = \mathfrak{a} \cdot \tau(\mathfrak{a}^{c-1})$ for any $c > \dim A$. In the case that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal we have that any test ideal $\tau(\mathfrak{a}^c)$ is $\mathfrak{m}$-primary as well and thus the filtration $\tau = \{\tau(\mathfrak{a}^c)\}$ given by the test ideals is a $\mathbb{Q}$-good $\mathfrak{a}$-filtration.

Theorem 7 *Let $(A, \mathfrak{m})$ be a normal $\mathbb{Q}$-Gorenstein local domain essentially of finite type over a perfect field $\mathbb{K}$ of characteristic $p > 0$ and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Let $\tau := \{\tau(\mathfrak{a}^c)\}_{c \in \mathbb{R}_{>0}}$ be the filtration given by test ideals. Then, the Poincaré series $P_{\boldsymbol{\emptyset}}(T)$ is rational.*

References

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Up-to-Homotopy Algebras with Strict Units

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Abstract We prove the existence of Sullivan minimal models for operads with non trivial arity zero. So up-to-homotopy algebras with strict units are just operad algebras over these minimal models. As an application we give another proof of the formality of the *unitary* n-little disks operad over the rationals.

1 Introduction and Main Result

In the beginning, in Stasheff's seminal papers [17], A_∞-spaces (algebras) had points (units) in what was subsequently termed the zero arity of the unitary associative operad $\mathcal{A}ss_+$. They were still present in [1, 12], for instance, but after that, points or units generally disappeared, and for a while, people working with operads assumed as a starting point $P(0) = \emptyset$, in the topological setting, or $P(0) = 0$ in the algebraic one: see for instance [5]. This may have been caused because of the problems posed by those points (units); see [9], or [2] for two examples, or, more to the point, [14] (as well as [15]), where Markl constructs minimal models for operads of chain complexes over a field of zero characteristic, carefully excluding operads with non-trivial arity zero.

More recently, the situation changed, and people have turned their efforts to problems involving non-trivial arity zero. In the topological context, for instance, we have [16]; in the algebraic context, [3], or [10]; and dealing with both [4]. When introducing points (units) back in the theory of up-to-homotopy things, there are two main possibilities: either you consider *strict* ones, as in Stasheff's original papers [17], or in [2, 4, 12], or you consider *up-to-homotopy* ones, or other relaxed versions of them: [1, 10, 16], and many others. You can even do both: [11].

In this paper, we work in the algebraic and strict part of the subject. The contribution we add to the present panorama is to prove the existence of Sullivan minimal

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models P_∞ for operads P on cochain complexes over a characteristic zero field $\mathbf{k}$, with non-trivial arity zero in cohomology, $HP(0) = \mathbf{k}$. In doing so, we extend the work of Markl [14], (see also [15]) which proved the existence of such models for non-unitary operads, $P(0) = 0$. Our models include the one of [2] for the unitary associative operad $\mathcal{A}ss_+$. More precisely, our main result says:

Theorem 1 *Every cohomologically unitary, $HP(0) = \mathbf{k}$, cohomologically connected, $HP(1) = \mathbf{k}$, and with a unitary multiplication operad P, has a Sullivan minimal model $P_\infty \xrightarrow{\sim} P$.*

In the non-unitary case, the importance of such minimal models is well known. For instance, they provide a *strictification* of up-to-homotopy algebras, in that for an operad P (with mild hypotheses), up-to-homotopy P-algebras are the same as strict, regular P_∞-algebras. We show how up-to-homotopy associative algebras or A_∞-algebras with strict units are exactly $(\mathcal{A}ss_+)_\infty$-algebras. As a second application, we offer another proof of the formality of the *unitary* n-little disks operad $\mathcal{D}_{n+}$ over the rationals. This fills the gap in our paper [7] noticed by Willwacher in his speech at the 2018 Rio International Congress of Mathematicians [18].

2 Ingredients

Our result has been made possible thanks to two main ingredients: (1) the recently introduced Λ-modules and Λ-operads, of [4], and (2) the simplicial and Kan-like structures we found in an operad with unitary multiplication. Let us explain their role.

2.1 Restriction Operations

Sullivan minimal models are constructed by a cell-attaching inductive algorithm. In their original context of commutative dg algebras, the building blocks of this algorithm are called *Hirsch extensions* [6], or *KS-extensions* [8]. In the context of operads, they are called *principal extensions* [15]. Their definition in the non-unitary case is the following.

Definition 2 (*See* [15]) Let $n \geq 2$ be an integer. Let $P = \Gamma(M)$ be free as a graded operad, where M is a graded Σ-module, with $M(0) = M(1) = 0$. An *arity n principal extension* of P is the free graded operad

$$P \sqcup_d \Gamma(E) := \Gamma(M \oplus E) ,$$

where E is an arity-homogeneous Σ_n-module with zero differential and $d : E \longrightarrow ZP(n)^{+1}$ a map of Σ_n-modules of degree $+1$. The differential ∂ on $P \sqcup_d \Gamma(E)$ is built upon the differential of P, d, and the Leibniz rule.

Definition 3 Given an operad P, a *Sullivan minimal model* is a quasi-isomorphism $\rho : P_\infty \xrightarrow{\sim} P$, which is built inductively on the arity of the operad through consecutive principal extensions

$$P_2 = \Gamma(E(2)) \longrightarrow \dots \longrightarrow P_n = P_{n-1} \sqcup_{d_n} \Gamma(E(n)) \longrightarrow \dots \longrightarrow \varinjlim_n P_n = P_\infty$$
$$\rho_2 : P_2 \longrightarrow P, \quad \rho_n : P_n \longrightarrow P, \quad \rho : P_\infty \longrightarrow P$$

in such a way that, for all n, $\rho_n : P_n \longrightarrow P$ is a quasi-isomorphism up to arity n.

This works perfectly fine in Markl's non-unitary case. The success of the Sullivan algorithm relies on the fact that, when restricted to operads which are cohomologically non-unitary $HP(0) = 0$ and cohomologically connected $HP(1) = \mathbf{k}$, their minimal model is a free graded operad $P_\infty = \Gamma(M)$ over a Σ-module $M = \bigoplus_{n=2}^{\infty} E(n)$ which is trivial in arities 0 and 1, $M(0) = M(1) = 0$. As a consequence, $P_\infty(n) = P_n(n)$: generators added in arities $> n$ don't change what we have in lower ones. The problem in introducing units $1 \in \mathbf{k} = HP(0)$ of our cohomologically unitary operads as generators in the arity zero of their minimal model $P_\infty = \Gamma(M)$ would be that units give rise to *restriction operations* which lower the arity:

$$\delta_i = \_ \circ_i 1 : P(n) \longrightarrow P(n-1) , \quad \omega \mapsto \delta_i(\omega) = \omega \circ_i 1 \quad i = 1, \dots, n.$$

So, in the presence of units, new generators $\omega \in E(n)$ would produce *new* elements in lower arities $\delta_i(\omega) \in P_n(n-1)$; that is, in the previous steps of the induction process thus ruining it. Nevertheless, we can also assume that the generating module M also has trivial arities 0 and 1 in our unitary case. This possibility has been recently made feasible thanks to Fresse's Λ-modules and Λ-operads, [4]: to put it succinctly, we strip out of the operad all the structure carried by the elements of $P(0)$ and add it to the underlying category of Σ-modules, obtaining the category of Λ-modules. This way, we obtain a substitute for the general free operad functor with the bonus of getting our field $\mathbf{k}$ in arity zero, with no need of any generators in the risky arities zero and one.

But we must keep track of those units somewhere if we want to build minimal models for cohomologically *unitary* operads. This is how we do it: we add the restriction operations δ_i to the building blocks of our minimal model, the principal extensions, *without producing new generators in* P_∞, with a just slight modification of the principal extensions.

Definition 4 Let $n \geq 2$ be an integer. Let P be free as a unitary graded operad, $P = \Gamma(M)$, where M is a graded Σ-module, with $M(0) = M(1) = 0$. A *unitary arity n principal extension* of P is the free graded operad

$$P \sqcup_d^{\delta} \Gamma(E) := \Gamma(M \oplus E) ,$$

where E is an arity-homogeneous Σ_n-module with zero differential and:

(a) $d : E \longrightarrow ZP(n)^{+1}$ is a map of Σ_n-modules of degree $+1$; the differential ∂ on $P \sqcup_d \Gamma(E)$ is built upon the differential of P, d and the Leibniz rule.
(b) $\delta_i : E \longrightarrow M(n-1), i = 1, \dots, n$ are morphisms of $\mathbf{k}$-graded vector spaces, compatible with d and the differential of P, in the sense that, for all $i = 1, \dots, n$ we have commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{d} & ZP(n)^{+1} \\ \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \delta_i} \\ M(n-1) & \xrightarrow{\partial} & P(n-1)^{+1} \end{array} .$$

Which also have to be compatible with the Λ-structure of P, from arity $n-1$ downwards.

2.2 A Kan-Like Structure

However, once we put *unitary* principal extensions, with their extra restriction operations, in the Sullivan inductive algorithm, we have a new problem. To extend our "partial" quasi-isomorphism ρ_{n-1} to the next arity ρ_n, we now need to check that it is compatible with these new restriction operations. For this, we introduce *simplicial-like* and *Kan-like* structures in an operad with unitary multiplication. To the best of our knowledge, both structures are new.

We begin with the simplicial-like structure. Restriction operations δ_i give us the *face maps*. To obtain the *degeneracy maps*, we need a unitary multiplication on P; that is, a morphism of operads $\mathcal{M}ag_+ \longrightarrow P$. Here, $\mathcal{M}ag_+$ stands for the *unitary magmatic operad*. $\mathcal{M}ag_+$ is the operad of unitary magmas: algebras with a unit, a single operation, and just the unit relations. This morphism gives us elements $1 \in P(0)$ and $m_2 \in P(2)$. These elements behave as a unit and a product: $m_2 \circ_1 1 = \mathrm{id} = m_2 \circ_2 1$. This extra condition of a unitary multiplication has an easy and natural interpretation: we are only asking that the unit of our operad $1 \in P(0)$ not be an "idle" one: there needs to be an arity two operation m_2 for which 1 *actually* works as a unit. With this unitary multiplication in P, we define our degeneracy maps as

$$\sigma_i : P(n) \longrightarrow P(n+1) , \quad \sigma_i(\omega) = \omega \circ_i m_2 , \quad i = 1, \dots, n .$$

It is an easy exercise to check that these δ_i and σ_i fulfill the simplicial identities needed to prove Lemma 7 below, which says that every operad with unitary multiplication P is a *Kan-like* simplicial complex. Its proof follows the one of [13], theorem 17.1, verbatim. Notice that this proof does not use the simplicial identity $\sigma_i \sigma_j = \sigma_{j+1}\sigma_j$, $i \leq j$, which in our case would be false, since m_2 is not necessarily associative.

Definition 5 Let $\{\omega_i\}_{i=1,\dots,n}$ be a family of elements in $P(n-1)$. We say that they verify a *Kan-like condition* if $\delta_i \omega_j = \delta_{j-1}\omega_i$, for all $i < j$.

Example 6 Elements $\omega \in P(n)$, $n \geq 1$, produce families $\{\omega_i = \delta_i \omega\}_{i=1,\dots,n}$ in $P(n-1)$ that verify the *Kan-like condition*.

The reciprocal of this example is also true.

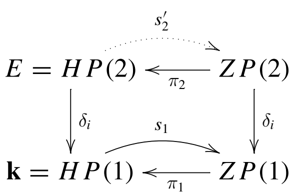
Lemma 7 *Let $\{\omega_i\}_{i=1,\dots,n}$ be a family of elements in $P(n-1)$ verifying the Kan-like condition. Then there exists an $\omega \in P(n)$ such that $\delta_i \omega = \omega_i$ for all $i = 1, \dots, n$.*

Moreover, we can prove that, if all the ω_i are cocycles, coboundaries, or belong to the kernel or the image of an operad morphism $\varphi : P \longrightarrow Q$, then ω can be chosen to be also a cocycle, a coboundary, or to belong to the kernel or the image of φ, respectively. Even more: if the $\omega_i = \omega_i(e)$ depend linearly on $e \in E(n)$, we can choose $\omega = \omega(e)$ to depend Σ_n-equivariantly on e. So much for the explanations. Let us now see all these constructions actually in action.

Sketch of the proof of theorem 1. To build $\rho_2 : P_2 \longrightarrow P$ in the non-unitary case, we take the generators in arity two to be $E = E(2) = HP(2)$. Then, we choose a Σ_2-equivariant section $s_2 : HP(2) \longrightarrow ZP(2) \subset P(2)$ of the projection $\pi_2 : ZP(2) \longrightarrow HP(2)$. And we get our first stage of the inductive algorithm as:

$$P_2 = \Gamma(E) \ , \qquad \partial_{2|E} = 0 \ , \qquad \text{and} \qquad \rho_2 : P_2 \longrightarrow P \ , \qquad \rho_{2|E} = s_2 \ .$$

In the unitary case, our section should make the following diagram to commute too:



Here, section s_1 is the unique $\mathbf{k}$-linear map sending $\mathrm{id} \in HP(1)$ to $\mathrm{id} \in ZP(1)$ and the restriction operations on E are the ones induced by $\delta_i : P(2) \longrightarrow P(1)$, $i = 1, 2$ on cohomology. This is not necessarily true for the section s_2 we have found in the non-unitary case. So, given $e \in E$, we study the differences $\omega_i(e) = \delta_i s_2(e) -$

$s_1\delta_i(e) \in P(1), i = 1, 2$. And we observe that they are coboundaries and verify our *Kan-like* condition in Definition 5. Therefore, because of Lemma 7, we get a coboundary $\partial\omega(e) \in P(2)$, such that $\delta_i\partial\omega(e) = \omega_i(e), i = 1, 2$. With this, we modify the section s_2 from the non-unitary case to a new one $s_2'(e) = s_2(e) - \partial\omega(e)$ which is compatible with the restriction operations δ_i. Finally, we average over Σ_2

$$\begin{aligned}\widetilde{s}_2(e) &= \frac{1}{2!}\sum_{\sigma\in\Sigma_2} \sigma \cdot s_2'(\sigma^{-1}\cdot e) \\ &= \frac{1}{2}\left(s_2'(e) + (2\ 1)\cdot s_2'((2\ 1)\cdot e)\right)\end{aligned}$$

and obtain a Σ_2-equivariant section, without losing anything we previously had for s_2 and s_2'. Therefore, we have our induced morphism of *unitary* operads $\rho_2 : P_2 \longrightarrow P$.
□

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Multisymplectic Lagrangian Models in Gravitation

Jordi Gaset and Narciso Román-Roy

Abstract We apply the multisymplectic formulation of classical field theories [12, 13, 15] to describe the Einstein–Hilbert and the Einstein–Palatini (or metric-affine) Lagrangian models of General Relativity.

1 Introduction

The geometrization of the theory of gravity, that is, *General Relativity* (GR), and in particular, the *multisymplectic framework*, allows us to understand several inherent characteristics of it. It is studied by different authors, such as [1–4, 8–11, 14, 16].

We present the main Lagrangian models for GR using the *multisymplectic framework*: first the *Einstein–Hilbert model* which is described by a 2nd-order singular Lagrangian (and so GR is formulated as a higher-order premultisymplectic field theory with constraints), and second the *Einstein–Palatini (metric-affine) model* described by a 1st-order singular Lagrangian (and so GR is formulated as a 1st-order premultisymplectic field theory with constraints).

2 Geometric Structures: Jet Bundles and Multivector Fields

First we introduce some fundamental geometrical tools which are used in the exposition.

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Let $\pi\colon E \longrightarrow M$ be a fiber bundle (with adapted coordinates (x^μ, y^i)). A *section* of π is a map $\phi\colon U \subset M \to E$ such that $\pi \circ \phi = Id_M$. The set of sections is denoted $\Gamma(\pi)$. Two sections $\phi_1, \phi_2 \in \Gamma(\pi)$ are *k-equivalent* at $x \in M$ if $\phi_1(x) = \phi_2(x)$ and their partial derivatives until order k at x are equal. This is an *equivalence relation* in $\Gamma_x(\pi)$ and each equivalence class is a *jet field* at x; denoted $j^k_x\phi$. The *kth-order jet bundle* of π is the set $J^k\pi := \{j^k_x\phi \,|\, x \in M,\ \phi \in \Gamma_x(\pi)\}$. Natural projections are:

$$\pi^k_r\colon J^k\pi \longrightarrow J^r\pi \ \ (r < k) \quad , \quad \pi^k\colon J^k\pi \longrightarrow E \quad , \quad \bar{\pi}^k\colon J^k\pi \longrightarrow M \ .$$

Definition 1 The *k***th-prolongation** of a section $\phi \in \Gamma(\pi)$ to $J^k\pi$ is the section $j^k\phi \in \Gamma(\bar{\pi}^k)$ defined as $j^k\phi(x) := j^k_x\phi\ ;\ x \in M$. A section $\psi \in \Gamma(\bar{\pi}^k)$ in $J^k\pi$ is **holonomic** if $\psi = j^k\phi$; that is, ψ is the kth prolongation of a section $\phi = \pi^k \circ \psi \in \Gamma(\pi)$.

If $\phi = (x, y^i(x))$, then $\psi = j^k\phi = \left(x, y^i(x), \dfrac{\partial y^i}{\partial x^\mu}(x), \dfrac{\partial^2 y^i}{\partial x^\mu \partial x^\nu}(x), \dots\right)$.

Definition 2 An m**-multivector field** in $J^k\pi$ is a skew-symmetric contravariant tensor of order m in $J^k\pi$. The set of m-multivector fields in $J^k\pi$ is denoted $\mathfrak{X}^m(J^k\pi)$. A multivector field $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ is said to be **locally decomposable** if, for every $p \in J^k\pi$, there is an open neighbourhood $U_p \subset J^k\pi$ and $X_1, \dots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \dots \wedge X_m$. Locally decomposable m-multivector fields $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ are locally associated with m-dimensional distributions $D \subset \mathrm{T}J^k\pi$. Then, $\mathbf{X}$ is **integrable** if its associated distribution is integrable. In particular, $\mathbf{X}$ is **holonomic** if it is integrable and its integral sections are holonomic sections of $\bar{\pi}^k$.

If $\Omega \in \Omega^r(J^k\pi)$ is a differential r-form in $J^k\pi$ and $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ is locally decomposable, the *contraction* between $\mathbf{X}$ and Ω is $i(\mathbf{X})\Omega\mid_U := i(X_1)\dots i(X_m)\Omega$.

3 Einstein–Hilbert Model (Without Sources)

The *configuration bundle* for the Einstein–Hilbert model is $\pi\colon E \to M$, where M is an oriented, connected 4-dimensional manifold representing space-time, with volume form $\omega \in \Omega^4(M)$, and E is the manifold of *Lorentzian metrics* on M. Thus $\dim E = 14$. Adapted fiber coordinates in E are $(x^\mu, g_{\alpha\beta})$, (with $0 \leq \alpha \leq \beta \leq 3$), where $g_{\alpha\beta}$ are the components of the metric, and such that $\omega = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \equiv \mathrm{d}^4x$. (We also use the notation $\mathrm{d}^3x_\mu \equiv i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^4x$).

The Lagrangian formalism is developed in $J^3\pi$, with the induced coordinates denoted as $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, g_{\alpha\beta,\mu\nu}, g_{\alpha\beta,\mu\nu\lambda})$, $(0 \leq \alpha \leq \beta \leq 3;\ 0 \leq \mu \leq \nu \leq \lambda \leq 3)$. The bundle $J^3\pi$ has some canonical structures; in particular, the *total derivatives* are

$$D_\tau = \frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + g_{\alpha\beta,\mu\nu\lambda\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \, .$$

The *Hilbert–Einstein Lagrangian function* (without energy-matter) is

$$L_{EH} = \sqrt{|det(g_{\alpha\beta})|}R \equiv \varrho R = \varrho g^{\alpha\beta}R_{\alpha\beta} \in \mathrm{C}^\infty(J^2\pi) \; ;$$

where $R_{\alpha\beta}$ are the *Ricci tensor* components and R is the *scalar curvature* (which, as the Levi–Civita connection is used, it contains 2nd-order derivatives of $g_{\mu\nu}$). The *Hilbert–Einstein Lagrangian density* is $\mathcal{L} = L\,\mathrm{d}^4x \in \Omega^4(J^3\pi)$, where $L = (\pi^3_2)^* L_{EH} \in C^\infty(J^3\pi)$. We denote

$$L^{\alpha\beta,\mu\nu} = \frac{1}{n(\mu\nu)}\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = \frac{n(\alpha\beta)}{2}\varrho(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu}) \, ,$$

$$L^{\alpha\beta,\mu} = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3}\frac{1}{n(\mu\nu)}D_\nu\left(\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}}\right) = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} D_\nu L^{\alpha\beta,\mu\nu} \, .$$

$$H = \sum_{\alpha\leq\beta;\mu\leq\nu} L^{\alpha\beta,\mu\nu}g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}g_{\alpha\beta,\mu} - L = \varrho\, g_{\alpha\beta,\mu}g_{kl,\nu}H^{\alpha\beta kl\mu\nu} \, ,$$

$$H^{\alpha\beta kl\mu\nu} = \frac{1}{4}g^{\alpha\beta}g^{kl}g^{\mu\nu} - \frac{1}{4}g^{\alpha k}g^{\beta l}g^{\mu\nu} + \frac{1}{2}g^{\alpha k}g^{l\mu}g^{\beta\nu} - \frac{1}{2}g^{\alpha\beta}g^{l\nu}g^{k\mu} \, ,$$

(where $n(\mu\nu) = 1$ if $\mu = \nu$, and $n(\mu\nu) = 2$ if $\mu \neq \nu$). Then, the *Poincaré–Cartan 5-form* associated with $\mathcal{L}$ is

$$\Omega_{\mathcal{L}} = \mathrm{d}H \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}L^{\alpha\beta,\mu}\mathrm{d}g_{\alpha\beta}\wedge\mathrm{d}^{m-1}x_\mu - \sum_{\alpha\leq\beta}\mathrm{d}L^{\alpha\beta,\mu\nu}\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^{m-1}x_\nu \in \Omega^5(J^3\pi) \, ,$$

and it is a premultisymplectic form because L is a singular Lagrangian.

The problem stated by the *Hamilton variational principle* for the system $(J^3\pi, \Omega_{\mathcal{L}})$ consists in finding holonomic sections $\psi_{\mathcal{L}} = j^3\phi \in \Gamma(\bar{\pi}^3)$ satisfying any of the following equivalent conditions:

(a) $\psi_{\mathcal{L}}$ is a solution to the equation $\psi_{\mathcal{L}}^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^3\pi)$.
(b) $\psi_{\mathcal{L}}$ is an integral section of a holonomic multivector field $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^4(J^3\pi)$ satisfying the equation $i(\mathbf{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$.

As $\Omega_{\mathcal{L}}$ is a premultisymplectic form, these field equations have no solution everywhere in $J^3\pi$. Applying the *premultisymplectic constraint algorithm* we obtain the following constraints (see [6]):

$$L^{\alpha\beta} := -\varrho\, n(\alpha\beta)\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = 0 \, . \tag{1}$$

$$D_\tau L^{\alpha\beta} = D_\tau(-\varrho\, n(\alpha\beta)\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)\right) = 0 \, . \tag{2}$$

They define the Lagrangian final constraint submanifold $S_f \hookrightarrow J^3\pi$ where solutions exist and, in particular,

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\tau=0}^{3} \sum_{\alpha\leq\beta} \sum_{\mu\leq\nu\leq\lambda} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + \right.$$
$$\left. g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + D_\tau D_\lambda (g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}))\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right)$$

is a holonomic multivector field solution to the equation in (b), tangent to S_f (here $\Gamma^\rho_{\mu\nu}$ are the *Christoffel symbols* of the *Levi–Civita connection* of g). Their integral sections are the solutions $\psi_{\mathcal{L}}(x) = (x^\mu,\ g_{\alpha\beta}(x),\ g_{\alpha\beta,\mu}(x),\ g_{\alpha\beta,\mu\nu}(x),\ g_{\alpha\beta,\mu\nu\lambda}(x))$ to the equation in (a), which gives

$$g_{\alpha\beta,\mu} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0\,, \tag{3}$$

$$g_{\alpha\beta,\mu\nu} - \frac{1}{n(\mu\nu)}\left(\frac{\partial g_{\alpha\beta,\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\beta,\nu}}{\partial x^\mu} \right) = 0\,, \tag{4}$$

$$\varrho\, n(\alpha\beta)(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R) = 0\,. \tag{5}$$

In this set of Eqs. (3) and (4) are (part of the) holonomy conditions; meanwhile (5) are the physical relevant equations, which are the constraints (1) evaluated on the image of sections, $L^{\alpha\beta}|_{\psi_{\mathcal{L}}} = 0$, and constitute the Euler–Lagrange equations of the theory; that is, the *Einstein equations*.

As a consequence of the singularity of $\mathcal{L}$, the form $\Theta_{\mathcal{L}}$ is π^3_1-projectable onto a form in $J^1\pi$ (but it is not the Poincaré–Cartan form of any 1st-order Lagrangian). Then, Einstein equations are 2nd-order PDE's, instead of $4th$-order as it correspond to a 2nd-order Lagrangian. So they are defined as a submanifold of $J^3\pi$ (and appear as constraints).

The constraints (2) are of geometrical nature and arise because we are using a manifold prepared for a theory of a 2nd-order Lagrangian that, really, is physically equivalent to a 1st-order Lagrangian. These constraints hold automatically when they are evaluated on the image of the sections $\psi_{\mathcal{L}}$ which are solutions to the Einstein equations. Furthermore, the Einstein–Hilbert model is a gauge theory (because L_{EH} is singular). Then, the constraints (1) and (2) fix partially the gauge. To remove the remaining gauge degrees of freedom leads to a submanifold of S_f diffeomorphic to $J^1\pi$.

4 Einstein–Palatini (Metric-Affine) Model (Without Sources)

The configuration bundle of the Einstein–Palatini (metric-affine) model is $\Pi\colon \mathcal{E} \to M$, where $\mathcal{E} = E \times_M C(LM)$, where E is the manifold of Lorentzian metrics on M and $C(LM)$ is the manifold of *linear connections* in $\mathrm{T}M$. Adapted fiber coordinates in E are $(x^\mu, g_{\alpha\beta}, \Gamma^\lambda_{\mu\nu})$, $(0 \leq \alpha \leq \beta \leq 3)$, and the induced coordinates in $J^1\Pi$ are $(x^\mu, g_{\alpha\beta}, \Gamma^\lambda_{\mu\nu}, g_{\alpha\beta,\mu}, \Gamma^\lambda_{\mu\nu,\rho})$. Thus $\dim E = 78$ and $\dim J^1\Pi = 374$.

The *Einstein–Palatini Lagrangian* (without energy-matter) is a singular 1st-order Lagrangian depending on the components of the metric g and of a connection Γ,

$$L_{EP} = \varrho\, g^{\alpha\beta} R_{\alpha\beta} = \varrho\, g^{\alpha\beta}(\Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha}) \in \mathrm{C}^\infty(J^1\Pi)\,.$$

The Lagrangian density is $\mathfrak{L} = L_{EP}\, \mathrm{d}^4x \in \Omega^4(J^1\Pi)$, and its *Poincaré–Cartan 5-form* is

$$\Omega_{\mathfrak{L}} = \mathrm{d}\left(\frac{\partial L_{\mathrm{EP}}}{\partial \Gamma^\alpha_{\beta\gamma,\mu}}\Gamma^\alpha_{\beta\gamma,\mu} - L_{\mathrm{EP}}\right) \wedge \mathrm{d}^4x - \mathrm{d}\frac{\partial L_{\mathrm{EP}}}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \wedge \mathrm{d}\Gamma^\alpha_{\beta\gamma} \wedge \mathrm{d}^3x_\mu \in \Omega^5(J^1\Pi)\,,$$

which is a premultisymplectic form since L_{EP} is also a singular Lagrangian.

The *Lagrangian problem* for the system $(J^1\Pi, \Omega_{\mathfrak{L}})$ consists in finding holonomic sections $\psi_{\mathfrak{L}} = j^1\phi \in \Gamma(\bar{\pi}^1)$ $(\phi \in \Gamma(\Pi))$ satisfying any of the following equivalent conditions:

(a) $\psi_{\mathfrak{L}}$ is a solution to the equation $\psi_{\mathfrak{L}}^*\, i(X)\Omega_{\mathfrak{L}} = 0$, for every $X \in \mathfrak{X}(J^1\Pi)$.
(b) $\psi_{\mathfrak{L}}$ is an integral section of a holonomic multivector field $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^4(J^1\Pi)$ satisfying the equation $i(\mathbf{X}_{\mathfrak{L}})\Omega_{\mathfrak{L}} = 0$.

Now, the premultisymplectic constraint algorithm leads to the constraints (see [7]):

$$0 = \frac{\partial H}{\partial g_{\mu\nu}} - \frac{\partial L^{\beta\gamma,\sigma}_\alpha}{\partial g_{\mu\nu}}\Gamma^\alpha_{\beta\gamma,\sigma}\,, \tag{6}$$

$$0 = g_{\rho\sigma,\mu} - g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} - g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} - \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu}\,, \tag{7}$$

$$0 = T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta}\,, \tag{8}$$

$$0 = T^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta,\nu}\,, \tag{9}$$

$$0 = g_{\rho\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\sigma} + g_{\sigma\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\rho} + g_{\rho\lambda}\Gamma^\lambda_{[\mu\sigma,\nu]} + g_{\sigma\lambda}\Gamma^\lambda_{[\mu\rho,\nu]} + \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda[\mu,\nu]}\,. \tag{10}$$

where $T^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}$. They define the submanifold $\mathcal{S}_f \hookrightarrow J^1\Pi$, where there are holonomic multivector fields solution to the equations in (b), tangent to $\mathcal{S}_f$.

A consequence of the singularity of $\mathfrak{L}$ is that $\Omega_{\mathfrak{L}}$ is Π^1-projectable onto a form in $\mathcal{E}$ and then, the Euler–Lagrange equations (Einstein's eqs.) are 1st-order PDE's, instead of 2nd-order. So they are defined as a submanifold of $J^1\pi$, and appear as constraints (6). On the other hand, the equalities (7) are related to the metricity condition for the Levi–Civita connection and they are called *pre-metricity constraints*. Furthermore, there are the *torsion constraints* that impose conditions on the torsion of the connection (8) and on their derivatives (9). Finally, the additional *integrability constraints* (10) appear as a consequence of demanding the integrability of the multivector fields which are solutions to the equations in (b).

The Einstein–Palatini model is a gauge theory (as $\mathfrak{L}$ is singular) with higher gauge freedom than in the Einstein–Hilbert model. The above constraints fix partially the gauge. To remove the remaining gauge degrees of freedom leads to a submanifold of $\mathcal{S}_f$ diffeomorphic to $J^1\pi$ in the Einstein–Hilbert model. The conditions of the connection to be *torsionless* and *metric* (which allows us to recover the Einstein–Hilbert model from the Einstein–Palatini model) are a consequence of the constraints and a partial fixing of this gauge freedom [5].

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Computing the Distance to the Stochastic Part of Phylogenetic Varieties

Marina Garrote-López

Abstract Phylogenetics and algebraic geometry are closely related since evolutionary models on phylogenetic trees describe algebraic varieties. However, only points in these varieties corresponding to stochastic parameters are of biological interest. We study whether restricting to these points could provide insight into the phylogenetic reconstruction problem.

1 Introduction

Phylogenetic studies the evolutionary relationships among a group of current species. These relationships are usually expressed in the form of a phylogenetic tree. For phylogenetic reconstruction and for theoretical analysis it is common to model evolution adopting a parametric statistical model. Then, the joint distribution at the leaves of the trees can be expressed as polynomials in terms of the model parameters, under some elementary assumptions in the models. The *phylogenetic invariants* are polynomial relationships among the entries of the joint distributions that vanish for any choice of the model parameters. They were introduced in 1987 by Cavender and Felsenstein in [5] and by Lake in [12]. Under this construction, algebraic geometry gets an important role and is used in both the study and computation of the ideals and the varieties defined by phylogenetic invariants. However, phylogenetic invariants not only describe points with biological sense and it is necessary to add semi-algebraic conditions in order to describe the regions of the varieties corresponding with biologically meaningful points.

This paper was originally motivated by the study of the semi-algebraic description of the model space that is done in [2]. We wish to investigate if the semi-algebraic conditions could make a positive contribution to phylogenetic inference. We focus on a case study close to the *long branch attraction* phenomenon. Our main goal is

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to understand the region of the varieties with biological meaning and to compute the Euclidean distance of a point into this subset. We describe this problem as an optimization problem and we propose a method using numerical algebraic geometry and computational algebra that goes through all critical points of the objective function and achieves the optimal solution.

2 Phylogenetic Varieties

We refer the reader to [1] for a good general overview into phylogenetic algebraic geometry. Here we introduce briefly basic concepts that will be needed later. Let T be a *quartet* tree topology, that is, an (unrooted) trivalent phylogenetic tree with its leaves labelled by $\{1, 2, 3, 4\}$. We choose an internal vertex as the root r. Suppose the evolutionary process on that tree follows a Markov process on 4 states $\{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{T}\}$ parametrized by a root distribution $\pi = \{\pi_{\mathtt{A}}, \pi_{\mathtt{C}}, \pi_{\mathtt{G}}, \pi_{\mathtt{T}}\}$ and Markov matrices M_e at each edge ($\sum_i \pi_i = 1$ and $\sum_j M_{ij} = 1\ \forall i$). We denote by $p_{s_1\cdots s_4}$ the probability of observing the state s_i at leaf i and by φ_T the parametrization map:

$$\begin{aligned}\psi_T : \mathbb{R}^{\ell} &\to \mathbb{R}^{4^4}\\ \{\pi, \{M_e\}_{e\in E(T)}\} &\mapsto P = (p_{s_1\cdots s_4})_{s_1\cdots s_4}\end{aligned}$$

which maps the ℓ parameters of the model to the vector of joint distribution of T. Define the *phylogenetic variety* associated to T as the smallest variety containing the image of φ_T, $\mathcal{V}_T = \overline{\mathrm{Im}\varphi_T}$. These varieties are characterized by the model but also by the topology of the tree. However only the points coming from stochastic parameters have biological sense. A vector is *stochastic* if all its entries are nonnegative and sum up to 1. A matrix is *stochastic* if all its rows are stochastic. Given a phylogenetic variety $\mathcal{V}_T$ we write $\mathcal{V}_T^+$ for the subset that contains the distributions arising from stochastic parameters and we call it the *stochastic phylogenetic region:* $\mathcal{V}_T^+ = \{P \in \mathcal{V}_T \mid P = \varphi_T(s) \text{ and } s \in S \subset [0, 1]^{\ell}\}$.

In this paper we restrict to the *Jukes–Cantor* (JC69) algebraic evolutionary model which assumes that on each edge, the conditional probabilities of substitution between different nucleotides are the same. The discrete Fourier transform introduced in [7] is a linear change of coordinates which diagonalizes group-based models, in particular the Jukes Cantor model. This simplifies the representation of the model: each transition matrix can be parametrized by a single parameter x_i (called *Fourier parameter*), which is the eigenvalue of M_i of multiplicity three different from 1 and the parametrization turns to be monomial. The coordinates of P after this transformation are called *Fourier coordinates*. From now on, for the JC69 model, we denote by φ_T the parameterization of the phylogenetic varieties from Fourier parameters x_i to Fourier coordinates:

$$\begin{aligned} \varphi_T : \mathbb{R}^5 &\longrightarrow \mathbb{R}^{4^4} \\ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5) &\mapsto P = \varphi_T(x_1, x_2, x_3, x_4, x_5). \end{aligned}$$

A JC69 matrix M_i is stochastic if and only if its Fourier parameter x_i lies in $I := [-1/3, 1]$.

3 Distance to the Stochastic Phylogenetic Regions

In this paper we deal with the *long branch attraction* (LBA) problem. This is a phenomenon which appears when fast evolving lineages are wrongly inferred as sister lineages even though they have been generated on a tree where they do not share a common node.

Consider a 4-leaf tree which has a JC69 non-stochastic matrix M at the interior edge with parameter m, a JC69 stochastic transition matrix K at edges going to leaves 1 and 3 with parameter k, and the identity matrix at the remaining edges. Write $P := \varphi_T\ (k, 1, k, 1, m)$ for the point in $\mathcal{V}_T$ corresponding to these parameters. Then we want to find the closest point(s) $\varphi_T(x)$ to P in the Euclidean distance. In other words, finding the closest point(s) on the stochastic phylogenetic region can be translated into the following optimization problem:

Problem 1 Minimize$_x$ $f(x) := \|P - \varphi_T(x)\|_2^2$ subject to $g_{1,i}(x) := x_i - 1 \leq 0$, and $g_{2,i}(x) := -x_i - 1/3 \leq 0$ for $i = 1, \ldots, 5$.

Theorem 1 If $(k, m) \in I \times (1, \omega]$, then $x^* = \big(\tilde{x}(k, m), 1, \tilde{x}(k, m), 1, 1\big)$ is a local minimum of the optimization Problem 1 where $\tilde{x}(k, m)$ is the minimum between 1 and the unique (real) critical point of $f(x, 1, x, 1, 1)$.

The value $\omega \approx 1.734$ is determined by the expression of $\tilde{x}(k, m)$, and characterizes the region $I \times (1, \omega]$ where $\tilde{x} : I \times (1, \omega] \to \mathbb{R}$ is a real continuous function. Theorem 1 can be proved using the *Karush-Kuhn-Tucker* conditions. We conjecture that this local minimum is actually the global minimum and we propose an algorithm to prove our conjecture for specific values of k and m.

Conjecture 2 Under the hypothesis of Theorem 1, $d(P, \mathcal{V}_T^+) = d\big(P, \varphi_T\big(x^*\big)\big)$.

Algorithm[1] To find the global minimum of this optimization problem we find all critical points in the interior and the boundary of $\Omega := I^5$ and then pick the one that minimizes $f(x)$. Similar approaches, where computational and numerical algebraic geometry are applied in mathematical biology can be found in [9, 11]. The algorithm falls naturally into two parts: first of all we find the critical points of the objective function f over all $\mathbb{C}^5$ and then we check the boundaries of Ω.

[1] The algorithm has been implemented with `Macaulay2` [8] and the code can be found in: https://github.com/marinagarrote/StochasticPhylogeneticVarieties.

It is know that the number of critical points of f whose image lies on the non-singular locus of $\mathcal{V}_T$ is finite and does not depend on the particular point P (Lemma 2.1 of [6]) and this value is called the *Euclidean distance degree* (EDd) of the variety. It is proved in [4] that the singular points of $\mathcal{V}_T$ are those that are the image of some null parameter. In other words, $\varphi_T(x_1, \dots, x_5)$ is singular if and only if $x_i = 0$ for some i. Points on the singular locus of $\mathcal{V}_T$ have not biological sense since they represent trees with infinite branch length. Hence, we can compute the number of critical points of our function f in the preimage of the smooth part of the variety as the degree of the saturation ideal $\mathcal{I} : (x_1 \cdots x_5)^\infty$, where $\mathcal{I}$ is generated by the partial derivatives of f.

On the other hand, the global minimum of f could be also on the boundary of Ω. Thus, we need to restrict f to all possible faces of Ω and find the critical points there. Write $S := (S_1, S_2)$ where $S_1, S_2 \subseteq \{1, \dots, 5\}$ are disjoint subsets. Set $\bar{x} := (x_{i_1}, \dots, x_{i_n})$ and set $F(\bar{x}) := f(x)$ where $x_i = 1$ if $i \in S_1$ and $x_j = -\frac{1}{3}$ if $j \in S_2$. Thus, we find the critical points of $F(\bar{x})$ for each pair S. Varieties described by the solutions of these multiple systems of equations are of dimension 0, and the solutions can be numerically approximated.

Algorithm 1: Projection into stochastic phylogenetic regions

Input: $f(x)$ for $k \in I$ and $m \geq 1$.
Output: Global minimum of Problem 1.
Compute $\mathcal{I} := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}\right)$;
$d := degree(\mathcal{I} : (x_1 \cdots x_5)^\infty)$;
while #{numerical solutions of $\nabla f(x) = 0 < d$} **do** Find the numerical solutions x^* of the system $\nabla f(x) = 0$;
foreach $S = (S_1, S_2)$ **do**
  Take $\bar{x} := (x_{i_1}, \dots, x_{i_n})$ where $i_1, \dots, i_n \notin S_1 \cup S_2$ and compute $F(\bar{x})$;
  Find the solutions $\bar{x}^*$ of $\nabla F(\bar{x}^*) = 0$ and complete the point $x^* = (x_1, \dots, x_5)$ with $x_i = 1$ if $i \in S_1$ and $x_i = \frac{-1}{3}$ $i \in S_2$;
Evaluate each $x^* \in \Omega$ into $f(x)$ and return the point x^* with minimum $f(x^*)$.

3.1 Computations and Conclusions

The computations were performed on a machine with 10 Dual Core Intel(R) Xeon(R) Silver 64 Processor 4114 (2.20 GHz, 13.75 M Cache) equipped with 256 GB RAM running Ubuntu 18.04.2. We have used `Macaulay2` version 1.3, `SageMath` [13] version 8.6 and `Magma` [3] version V2.24-8. The computations of the EDd of $\mathcal{V}_T$ where done with `Magma` and the output degree was 290. To find solutions to the different polynomial systems previously described we have used numerical algebraic geometry methods, in particularly homotopy continuation based methods. All computations have been done with the package `PHCpack.m2` [10, 14] which turned out to be the only numerical package capable to find the 290 points of $\mathcal{I} : (x_1 \cdots x_5)^\infty$.

The Conjecture 2 has been tested for 1000 data points with parameters $(k, m) \in (0, 1/4] \times (1, 3/2]$ randomly chosen in order to simulate points close to the LBA phenomenon. Every experiment has verified that the global minimum of the problem is the point $x^* = \big(\tilde{x}(k, m), 1, \tilde{x}(k, m), 1, 1\big)$ defined in Theorem 1 and which was proved to be a local minimum. Note that x_5 being 1 means that the matrix M at the interior edge is the identity matrix and therefore the point $\varphi_T(x^*)$ is in the intersection of the three phylogenetic varieties for the three 4-leaf tree topologies.

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Generating Embeddable Matrices Whose Principal Logarithm is Not a Markov Generator

Jordi Roca-Lacostena

Abstract Several results seem to point out that the embeddability of a Markov process may be determined by checking whether the principal logarithm of its transition matrix is a rate matrix. In this note, we provide a constructive method to produce a positive measure subspace of Markov matrices for which this is not true.

1 Introduction and Preliminaries

We introduce here the embedding problem of Markov matrices. A real square matrix M is a *Markov matrix* if its entries are non-negative and all its rows sum to 1. A real square matrix Q is a *rate matrix* if its off-diagonal entries are non-negative and its rows sum to 0. It is known that Q is a rate matrix if and only if $M(t) = exp(Qt)$ is a Markov matrix for all $t \geq 0$ [6]. In this case, we say that $M = exp(Q)$ is embeddable (since it can be embedded into a 1-dimensional semigroup) and we say that Q is a *Markov generator* for M. The *embedding problem* consists on deciding whether a given Markov matrix is embeddable or not [3]. Note that 1 is an eigenvalue of any Markov matrix because all its rows sum to 1. In the same way, 0 is an eigenvalue of any rate matrix.

We say that a matrix Q is a *logarithm* of a matrix M if $e^Q = M$. It follows from the exponential series of a matrix $e^Q = \sum_{n\geq 0} \frac{Q^n}{n!}$ that any matrix that diagonalizes Q does also diagonalize M, and the eigenvalues of M are the exponential of the eigenvalues of Q. Moreover, it is known that if $\det(M) \neq 0$ there is a unique matrix logarithm whose eigenvalues are the principal logarithm of the eigenvalues of M [4]. This is the so-called *principal logarithm of a matrix* M and will be denoted by $\text{Log}(M)$.

Despite the embedding problem is solved for 2×2 and 3×3 matrices, it remains open for bigger matrices. However, several results seem to indicate that $\text{Log}(M)$ is crucial to solve the embedding problem (check Theorems 5.1 and 5.2 in [5] for some

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of these results). In this note, we construct a set with positive measure containing embeddable matrices whose principal logarithm is not a rate matrix.

Our motivation comes from phylogenetics, where the embeddeding problem is equivalent to decide if the evolutionary process described by a Markov matrix can be explained by homogeneous continuous-time models or not. In this context, we work with 4×4 Markov matrices and the entries of the matrix represent the probability of mutation between the four different nucleotides. More precisely, we work with a DNA substitution model known as the *Strand Symmetric Model* [2], which is the simpler DNA substitution model that might be suitable for our purpose. DNA substitution models are given by different structures of the Markov matrix. In the case of the Strand Symmetric Model we will work with real matrices with the following symmetries :

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{pmatrix}.$$

We will refer to matrices with this structure as *SS matrices*. A straightforward computation shows that the product and sum of SS matrices is closed within the model. Hence, the exponential of a rate SS matrix is a Markov SS matrix. Moreover, it follows from Lemma 6.2 by [1] and Theorem 1.27 in [4] that any logarithm of a SS matrix with pairwise different eigenvalues is also a SS matrix. It is also immediate to check that SS matrices commute with each other.

2 SS Embeddable Matrices Whose Principal Logarithm is Not a Generator

We start by providing a parametrization of all SS matrices with rows summing to 0. Given $v = (v_1, \ldots, v_6) \in \mathbb{R}^6$ and $\theta \in \mathbb{R}$ we denote:

$$Q(\theta, v) := \begin{pmatrix} v_1 + v_2 - v_3 - \theta v_4 & -v_1 - v_2 + \theta v_5 & -v_1 - v_2 - \theta v_5 & v_1 + v_2 + v_3 + \theta v_4 \\ -v_1 + v_2 - \theta v_6 & v_1 - v_2 - v_3 + \theta v_4 & v_1 - v_2 + v_3 - \theta v_4 & -v_1 + v_2 + \theta v_6 \\ -v_1 + v_2 + \theta v_6 & v_1 - v_2 + v_3 - \theta v_4 & v_1 - v_2 - v_3 + \theta v_4 & -v_1 + v_2 - \theta v_6 \\ v_1 + v_2 + v_3 + \theta v_4 & -v_1 - v_2 - \theta v_5 & -v_1 - v_2 + \theta v_5 & v_1 + v_2 - v_3 - \theta v_4 \end{pmatrix}.$$

Note that $Q(\theta, v)$ is a SS matrix with rows summing to 0 and hence $exp(Q(\theta, v))$ is a matrix with rows summing to 1. However, the stochastic conditions of Markov and rate matrices requiring non-negative off-diagonal entries depend on the parameters θ and v. Further computations show that the spectrum of $Q(\theta, v)$ is

$$\sigma\big(Q(\theta, v)\big) = \left\{0, 4v_1, -2v_3 + 2\theta\sqrt{v_4^2 - v_6 v_5}, -2v_3 - 2\theta\sqrt{v_4^2 - v_6 v_5}\right\}. \quad (1)$$

Conversely, it can be seen that for any SS Markov matrix M with eigenvalues $1, x, z, \overline{z}$, its real logarithms with rows summing to 0 are all the $Q(\theta, v)$ with $\theta = Arg(z) + 2\pi k$ for some $k \in \mathbb{Z}$ and $v \in \mathbb{R}^6$ determined by the entries of the matrix. Moreover, it holds that $v \in \mathcal{V}$ where $\mathcal{V} \subseteq \mathbb{R}^6$ is the algebraic variety

$$\mathcal{V} = \{(v_1, \ldots, v_6) \in \mathbb{R}^6 \mid v_4^2 - v_5 v_6 = -1/4\}. \tag{2}$$

Given $\theta \in \mathbb{R}$ let us denote by $\mathcal{P}(\theta)$ the set of those $v \in \mathbb{R}^6$ such that $Q(\theta, v)$ is a rate matrix and by $\mathcal{P}(\theta)^c$ its complementary. Note that $\mathcal{P}(\theta)$ is an unbounded convex polyhedral cone because the entries of $Q(\theta, v)$ are linear expressions on the entries of v and hence if $Q(\theta, v)$ is a rate matrix so is $Q(\theta, \lambda\, v)$ for any $\lambda \geq 0$.

Next theorem uses the algebraic variety and the polyhedral cones introduced above to show that there are embeddable SS Markov matrices whose principal logarithm is not a rate matrix.

Theorem 1 *For any given $\theta_0 \in (-\pi, \pi)$ and $k \in \mathbb{Z}$, $k \neq 0$ let us denote $\theta_k = \theta_0 + 2\pi k$. Take $v \in \mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ and consider the matrices $L = Q(\theta_0, v)$ and $R = Q(\theta_k, v)$. Then, the following holds:*

(i) $M := exp(L) = exp(R)$ is a Markov matrix.
(ii) L is the principal logarithm of M but it is not a rate matrix.
(iii) R is a rate matrix. In particular, M is embeddable.

Proof Since SS matrices commute with each other, we have that R commutes with $L - R$ so that $exp(R)exp(L - R) = exp(R + (L - R))$. A straightforward computation shows that if $v \in \mathcal{V}$ then $exp(L - R) = Id$ and hence $exp(R) = exp(L)$. Since $v \in \mathcal{P}(\theta_k)$ we have that R is a rate matrix and hence $exp(R)$ is a Markov matrix, which concludes the proof of (i) and proves (iii). To prove statement (ii) we use the eigenvalues of L, which according to (1) are $0, -4|\theta_k|$ and $-2|\theta_k| \pm \theta_0 i$. As $M = exp(L)$, it follows from the uniqueness of the principal logarithm that L is its principal logarithm of. Moreover, it is not a rate matrix because $v \in \mathcal{P}(\theta_0)^C$. □

Next, we need to check that $\mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ is not empty. Using the same notation as in the previous theorem, consider the vector

$$v = (-|\theta_k|, -|\theta_k|/2, |\theta_k|, sign(k)/2, 1, 1/2). \tag{3}$$

It is immediate to check that $v \in \mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$. For instance, if we take $\theta_0 = \pi/2$ and $k = 1$ we get $v = (-5\pi/2, -5\pi/4, 5\pi/2, 1/2, 1, 1/2)$ and

$$L = \frac{\pi}{4}\begin{pmatrix} -26 & 17 & 13 & -4 \\ 4 & -14 & 4 & 6 \\ 6 & 4 & -14 & 4 \\ -4 & 13 & 17 & -26 \end{pmatrix} \qquad R = \frac{\pi}{4}\begin{pmatrix} -30 & 25 & 5 & 0 \\ 0 & -10 & 0 & 10 \\ 10 & 0 & -10 & 0 \\ 0 & 5 & 25 & -30 \end{pmatrix}.$$

To determine the dimension of $\mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ we will need the next lemma.

Lemma 2 *$\mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ has two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ is the set of solutions to the following inequalities:*

$$\begin{array}{ll} v_1 + v_2 + v_3 + \theta_0 v_4 < 0, & -v_1 - v_2 + \theta_k v_5 \geq 0, \\ v_1 + v_2 + v_3 + \theta_k v_4 \geq 0, & -v_1 - v_2 - \theta_k v_5 \geq 0, \\ v_1 - v_2 + v_3 - \theta_k v_4 \geq 0, & -v_1 + v_2 + \theta_k v_6 \geq 0, \\ & -v_1 + v_2 - \theta_k v_6 \geq 0. \end{array} \quad (4)$$

Moreover, $(v_1, v_2, v_3, v_4, v_5, v_6) \in \mathcal{C}_1$ *if and only if* $(v_1, -v_2, v_3, -v_4, v_6, v_5) \in \mathcal{C}_2$.

Proof Since the rows of $Q(\theta, v)$ sum to zero, $\mathcal{P}(\theta)$ is the convex polyhedral cone arising from the inequation system $Q(\theta, v)_{i,j} \geq 0$ for all pairs (i, j) with $i \neq j$. Moreover, due to the symmetries of SS matrices the sets of inequalities with $i \in \{1, 2\}$ and $i \in \{3, 4\}$ are the same.

As in Theorem 1 we take $L = Q(\theta_0, v)$ and $R = Q(\theta_k, v)$ and denote their entries by $l_{i,j}$ and $r_{i,j}$ respectively. According to the definition of $Q(\theta, v)$ we have that $r_{1,2} + r_{1,3} = l_{1,2} + l_{1,3} = 2(-v_1 - v_2)$, $r_{2,1} + r_{2,4} = l_{2,1} + l_{2,4} = 2(-v_1 + v_2)$ and $r_{1,4} + r_{2,3} = l_{1,4} + l_{2,3} = 2(v_1 + v_3)$. The off-diagonal entries of R are non-negative because it is a rate matrix and hence $(-v_1 - v_2), (-v_1 + v_2), (v_1 + v_3) \geq 0$. Since $|\theta_0| < |\theta_k|$ we have that $-v_1 - v_2 \pm \theta_k v_5 \geq 0$ implies $-v_1 - v_2 \pm \theta_0 v_5 \geq 0$ thus $l_{1,2}, l_{1,3} \geq 0$. Analogously, we can see that $l_{2,1}, l_{2,4} \geq 0$. Since L is not a rate matrix then $l_{1,4} < 0$ or $l_{2,3} < 0$ and we know that $l_{1,4} + l_{2,3} = 2(v_1 + v_3) \geq 0$ thus either $l_{1,4} \geq 0, l_{2,3} < 0$ or $l_{2,3} \geq 0, l_{1,4} < 0$ showing that $\mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ has two connected components. From the definition of $Q(\theta, v)$ one can immediately check that given $v = (v_1, v_2, v_3, v_4, v_5, v_6)$ such that the only negative off-diagonal entry of Q and L is $l_{1,4}$ then we get that for $(v_1, -v_2, v_3, -v_4, v_6, v_5)$ the only negative off-diagonal entry of Q and L is $l_{2,3}$. The linear inequalities system in (4) is the reduced system arising from the assumption that the only negative off-diagonal entry of Q and L is $l_{1,4}$. □

If we allow the first expression in (4) to vanish, a straightforward computation shows that for $k \neq 0$ the solution space is the convex hull of 10 different rays including the ones associated with the vectors

$$w_1 := (-|\theta_k|, 0, |\theta_k|, 0, 1, 1), \qquad w_2 := (-|\theta_k|, 0, |\theta_k|, 0, -1, 1),$$
$$w_3 := (-|\theta_k|, -|\theta_k|, |\theta_k|, sign(k), 2, 0).$$

Among these, w_3 satisfies that $Q(\theta_0, w_3)_{1,4} < 0$ and hence the interior of the convex hull of the rays associated with w_1, w_2 and w_3 is included in $\mathcal{C}_1$. In particular, for any $v \in \mathcal{V}$ such that $v = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$ with $\lambda_i > 0$ we have that $v \in \mathcal{V} \cap \mathcal{C}_1$. One example of this is the vector v in (3)

$$v = (-|\theta_k|, -|\theta_k|/2, |\theta_k|, sign(k)/2, 1, 1/2) = \frac{w_1}{4} + \frac{w_2}{4} + \frac{w_3}{2}.$$

Moreover, it follows from Lemma 2 that $\dim(\mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)) = \dim(\mathcal{C}_1) = 6$ and so $\dim(\mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)) \leq 5$. Indeed, any neighbourhood of v includes points in $\mathcal{V}$ that lie in the interior of $\mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ and hence we deduce that $\dim(\mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)) = 5$. Therefore, for any $k \in \mathbb{Z}$ $k \neq 0$, the set of embeddable SS matrices whose principal logarithm is not a rate matrices defined as

$$\mathcal{U}_k = \left\{ exp(Q(\theta_0, u)) : \ \theta_0 \in (-\pi, \pi), u \in \mathcal{V} \cap \mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k) \right\}$$

has dimension 6, and hence it has positive measure among SS Markov matrices. Not only that, but the matrices in $exp(Q(\theta_0, u)) \in \mathcal{U}_k$ arising from a point u in the interior of $\mathcal{P}(\theta_0)^c \cap \mathcal{P}(\theta_k)$ are actually embeddable SS matrix whose principal logarithm $Q(\theta_0, u)$ and its Markov generator $Q(\theta_k, u)$ do not have any null entry. For example, by taking $\theta_0 = \pi/2, k = -1, v$ as in (3) and $u = v + (-\pi/4, 0, \pi/2, 0, 0, 0)$ we get:

$$Q(\theta_0, u) = \frac{\pi}{4} \begin{pmatrix} -17 & 12 & 8 & -3 \\ 3 & -13 & 5 & 5 \\ 5 & 5 & -13 & 3 \\ -3 & 8 & 12 & -17 \end{pmatrix} \quad \text{and} \quad Q(\theta_{-1}, u) = \frac{\pi}{4} \begin{pmatrix} -21 & 4 & 16 & 1 \\ 7 & -9 & 1 & 1 \\ 1 & 1 & -9 & 7 \\ 1 & 16 & 4 & -21 \end{pmatrix}.$$

Since the entries of these matrices are not 0 and depend continuously on its eigenvalues and eigenvectors, we can apply a suitable perturbation on them so that the entries of the resulting matrices keep the same sign but no longer satisfy the symmetries of SS matrices. In this way, it is possible to construct a positive measure subset of 4×4 Markov matrices containing embeddable matrices whose principal logarithm is not a rate matrix.

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Hamilton-Jacobi Theory and Geometric Mechanics

Xavier Gràcia

Abstract From the viewpoint of geometric mechanics, any dynamical system should be described in a coordinate-independent way, by means of the tools of differential geometry. Thinking about the geometric structures involved may lead to new interesting questions. We focus on Hamilton–Jacobi equation, which is in some regards equivalent to Hamilton's equation. In particular, we will give an interpretation of Hamilton–Jacobi equation on a manifold in terms of a family of differential equations on a lower-dimensional manifold.

1 The Perspective of Geometric Mechanics

Analytical mechanics and differential geometry are closely related subjects. A costumary question in *geometric mechanics* is: "What is the geometric formulation of this?"

For example, a first-order, autonomous, ordinary, differential equation is described in elementary terms as a relation $x' = f(x)$ where $f: U \to \mathbf{R}^n$ is a vector function on an open set $U \subset \mathbf{R}^n$, and the solutions are paths $x: I \to U$ ($I \subset \mathbf{R}$ open interval). The change of dependent variable $y = \varphi(x)$ transforms the equation into $y' = g(y)$, where $g(y) = \mathrm{D}\varphi(\varphi^{-1}(y)) \cdot f(\varphi^{-1}(y))$. Whereas f *seems* to be a vector function, indeed it is the coordinate expression of a more complex object: a vector field.

The *geometric formulation* of a first-order, autonomous, ordinary differential equation on a manifold M is defined by a vector field X (that is, a section of the tangent bundle $\tau_M: \mathrm{T}M \to M$). A solution is an integral curve of X, that is, a path $\gamma: I \to M$ such that $\gamma' = X \circ \gamma$, where $\gamma': I \to \mathrm{T}M$ is the velocity (or canonical lift to $\mathrm{T}M$) of γ.

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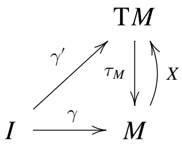


In general, there is not a unique way to give a geometric formulation of something. Often one can find several different ways that may correspond to different degrees of generality in the position of the problem, or in the properties of the solutions.

2 Hamiltonian Mechanics and Hamilton–Jacobi Theory

In this section we recall some notions and fix notations. A (time-independent) hamiltonian dynamical system is a triple (M, ω, H), where M is a smooth manifold (the phase space), $\omega \in \Omega^2(M)$ is a symplectic form (that is, closed and non-degenerate) and $H : M \to \mathbf{R}$ is a function (the hamiltonian). The symplectic form defines by left contraction a vector bundle isomorphism $\widehat{\omega} \colon \mathrm{T}M \to \mathrm{T}^*M$. With it we construct the hamiltonian vector field (or symplectic gradient) of H, $Z_H \in \mathfrak{X}(M)$, defined by the relation $\mathrm{i}_{Z_H}\omega = \mathrm{d}H$. Then Hamilton's equation reads $\xi' = Z_H \circ \xi$.

Since ω is closed, it is locally exact: $\omega = -\mathrm{d}\theta$, where $\theta \in \Omega^1(M)$. By Darboux's theorem, there exist canonical coordinates $(q^i; p_i)$ $(1 \le i \le n)$ with which $\theta = p_i\, \mathrm{d}q^i$ and $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$.

The basic example of symplectic manifold is the cotangent bundle of a manifold, $M = \mathrm{T}^*Q$, which is endowed with canonical differential forms $\theta_Q \in \Omega^1(M)$ and $\omega_Q = -\mathrm{d}\theta_Q \in \Omega^2(M)$. Natural coordinates $(q_i; p_i)$ of the cotangent bundle are canonical. Written in these coordinates, hamiltonian dynamics is just the Hamilton equations of classical textbooks on analytical mechanics.

In hamiltonian mechanics the Hamilton–Jacobi equation is associated with the theory of canonical transformations. Its main goal is to write the hamiltonian dynamics in a particularly simple way, so that one can integrate it. A particularly enlightening discussion can be found in the classical text by Arnol'd [2, pp. 258–261]. The time-independent Hamilton–Jacobi equation for a hamiltonian system is

$$H\left(q, \frac{\partial S}{\partial q}\right) = E\,,$$

where E is a constant, and we have the following result:

Theorem 1 (Jacobi) *Let $S_1(q, Q)$ be a solution of $H(q, \partial S_1(q, Q)/\partial q) = E$ depending on n parametres Q^j, and such that* $\det\big(\partial^2 S_1/\partial q^i\, \partial Q^j\big) \neq 0$.

Then Hamilton's equation is integrable through quadratures, and the functions $Q^j(q, p)$ determined by $\partial S_1/\partial q^i(q, Q) = p_i$ are first integrals of it.

The function $S_1(q, Q)$ is called a *complete solution* of the Hamilton–Jacobi equation.

On the other hand, we have [1, pp. 381–382]:

Theorem 2 (Hamilton–Jacobi) *Let $W(q)$ be a function. The following conditions are equivalent:*

- *W is a solution of the Hamilton–Jacobi equation.*
- *For every solution $q(t)$ of $\dot{q}^i = \frac{\partial H}{\partial p_i}\left(q, \frac{\partial W}{\partial q}\right)$, the path $\left(q^i(t), \frac{\partial W}{\partial q^i}(q(t))\right)$ is a solution of Hamilton's equation.*

The geometric formulation of the Hamilton–Jacobi equation is straightforward. Consider a function $W\colon Q \to \mathbf{R}$. Its differential is a map $\mathrm{d}W\colon Q \to \mathrm{T}^*Q$, and Hamilton–Jacobi equation reads

$$H \circ \mathrm{d}W \equiv (\mathrm{d}W)^*(H) = E. \tag{1}$$

Nevertheless, one can dig deeper on the meaning of the preceding theorems, and wonder about the relevance of the symplectic structure, or about a lagrangian counterpart of Hamilton–Jacobi theory. For instance, note that the preceding equation is locally equivalent to $\alpha^*(H) = E$, with $\alpha \in \Omega^1(Q)$ a closed 1-form; in other words, we can write the classical Hamilton–Jacobi equation as

$$\mathrm{d}\, \alpha^*(H) = 0\,. \tag{2}$$

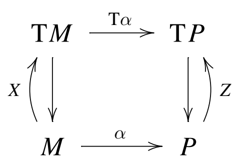
Notice also that α is closed iff $\alpha^*(\omega_Q) = 0$ iff $\alpha(Q) \subset \mathrm{T}^*Q$ is a lagrangian submanifold.

3 A General Framework for Hamilton–Jacobi Equation

In a long-term collaboration with Cariñena, Marmo, Martínez, Muñoz-Lecanda and Román-Roy [3–5] we have studied Hamilton–Jacobi equation aiming to better understand the geometric meaning of its ingredients and properties.

Consider manifolds M and P, with vector fields $X \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(P)$, and a map $\alpha\colon M \to P$. It is well-known that the following conditions are equivalent:

(1) γ integral curve of $X \implies \alpha \circ \gamma$ integral curve of Z.
(2) $\mathrm{T}\alpha \circ X = Z \circ \alpha$ (X is α-related with Z).



Suppose moreover that α is an injective immersion, thus inducing a diffeomorphism $\alpha_\circ \colon M \to \alpha(M)$ with an immersed submanifold. We have a third equivalent condition:

(3) Z is tangent to $\alpha(M)$, and X is given by $X = \alpha_\circ^*(Z|_{\alpha(M)})$.

In this case there is a bijection between the integral curves of X and the integral curves of Z that meet $\alpha(M)$.

Given a dynamical system (P, Z), we say that a triple (M, α, X) satisfying these conditions is a *slicing* for (P, Z). We omit X when it is determined by α, as for instance in condition (3).

A single solution of the slicing equation describes the integral curves of Z contained in $\alpha(M) \subset P$. To describe *all* of its integral curves we need a complete solution. A *complete slicing* of (P, Z) is given by

- a map $\overline{\alpha} \colon M \times N \to P$ and
- a vector field $\overline{X} \colon M \times N \to \mathrm{T}M$ along the projection $M \times N \to M$

such that:

- $\overline{\alpha}$ is surjective (or at least its image is an open dense subset), and
- for each $c \in N$, the map $\alpha_c \equiv \overline{\alpha}(\cdot, c) \colon M \to P$ and the vector field $X_c \equiv \overline{X}(\cdot, c) \colon M \to \mathrm{T}M$ constitute a slicing of Z.

On the other hand, a (generalised) *constant of motion* is a map $F \colon P \to N$ such that, for any integral curve $\zeta \colon I \to P$ of Z, $F \circ \zeta$ is constant.

The following theorem establishes a relation between both concepts, though a strong regularity assumption is required:

Theorem 3 *Let (P, Z) be a dynamical system, and $\overline{\alpha} \colon M \times N \to P$ a* diffeomorphism. *Then $\overline{\alpha}$ is a complete slicing for Z iff $F = \mathrm{pr}_2 \circ \overline{\alpha}^{-1} \colon P \to N$ is a constant of the motion for Z.*

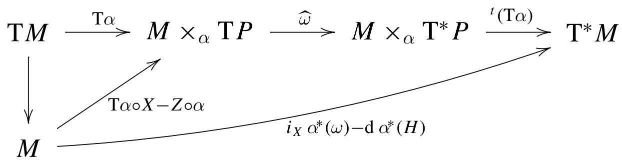
4 Slicing of Hamiltonian Systems

Now we consider a hamiltonian system (P, ω, H), with hamiltonian vector field $Z = Z_H$.

Proposition 4 *Let $\alpha \colon M \to P$ be a map, and X a vector field on M. If this is a solution of slicing equation ($\mathrm{T}\alpha \circ X - Z \circ \alpha = 0$), then*

$$i_X\, \alpha^*(\omega) - \mathrm{d}\, \alpha^*(H) = 0\,. \tag{3}$$

The proof of this statement is a consequence of this partly commutative diagram:



Remarkably, this proposition implies that a slicing α satisfies the Hamilton–Jacobi equation if we impose on it an additional condition,

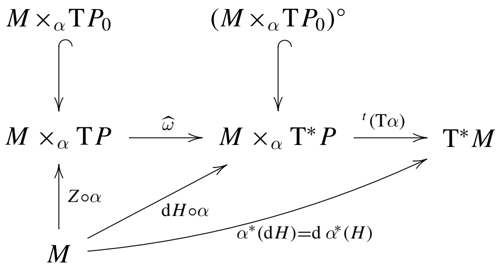
$$\alpha^*(\omega) = 0\,. \tag{4}$$

When α has constant rank this condition means that, locally, the image $\alpha(M) \subset P$ is an *isotropic submanifold*. Under which circumstances does the converse hold?

Theorem 5 *Let (P, ω, H) be a hamiltonian system. Suppose that $\alpha\colon M \to P$ is an embedding satisfying the isotropy condition $\alpha^*(\omega) = 0$ and that* $\dim P = 2 \dim M$. *Then α is a solution of the slicing equation iff it satisfies the classical Hamilton–Jacobi equation*

$$\mathrm{d}\,\alpha^*(H) = 0\,.$$

The following diagram includes some of the bundles and maps relevant to the proof:



Notice that the relation between dimensions means that $\alpha(M) \subset P$ is a *lagrangian submanifold*. So we say that α is a *lagrangian slicing*. In the same way, we speak about complete lagrangian slicings.

5 Slicings in Fibered Manifolds. Lagrangian and Hamiltonian Mechanics

From now on we consider a dynamical system (P, Z) where $\pi\colon P \to M$ is a fibre bundle, and consider the slicing equation only for the *sections* α of π. In this case we have an explicit formula that gives $X = \mathrm{T}\pi \circ Z \circ \alpha$ in terms of α. When the system is hamiltonian we have the following results, which improve Proposition 4 and Theorem 5:

Theorem 6 *With the preceding hypotheses, let* $\alpha \colon M \to P$ *be a section of* π. *Suppose that the fibres of* π *are isotropic. Then* α *is a slicing section iff*

$$i_X \, \alpha^*(\omega) - \mathrm{d}\, \alpha^*(H) = 0 \,.$$

Corollary 7 *For a hamiltonian system* (P, ω, H) *fibred over* M, *with isotropic fibres, suppose that* α *is a section with* $\alpha^*(\omega) = 0$. *Then* α *is a slicing section for* Z_H *iff*

$$\mathrm{d}\, \alpha^*(H) = 0 \,.$$

The preceding results apply directly to a hamiltonian system $(P = \mathrm{T}^*Q, \omega, H)$ on a cotangent bundle, with hamiltonian vector field $Z = Z_H$. In this case a section of $P \to Q$ is just a *differential 1-form* α on Q. Now the vector field X going with a slicing section α is $X = \mathcal{F}H \circ \alpha$, where $\mathcal{F}H \colon \mathrm{T}^*Q \to \mathrm{T}Q$ is the fibre derivative of H. Since the fibres of T^*Q are *isotropic* submanifolds, we are under the hypotheses of the previous theorem and corollary. In particular, the classical *Hamilton–Jacobi equation* is nothing but the slicing equation for a *closed* 1-form α (since this property is equivalent to $\alpha^*(\omega) = 0$).

The same applies to the lagrangian formulation of mechanics with a regular lagrangian function $L \colon \mathrm{T}Q \to \mathbf{R}$, where $P = \mathrm{T}Q$ is endowed with the symplectic structure $\omega_L = \mathcal{F}L^*(\omega_Q)$ and the dynamics Z is the Euler–Lagrange vector field (which is indeed the hamiltonian vector field of the energy function). Then the section α for the slicing equation is just a vector field on Q, and $X = \alpha$. When $X^*(\omega_L) = 0$ we recover a lagrangian counterpart of the classical Hamilton-Jacobi equation:

$$\mathrm{d}\, X^*(E_L) = 0 \,. \tag{5}$$

Of course, when the lagrangian is *hyperregular* the isomorphism $\mathcal{F}L \colon \mathrm{T}Q \to \mathrm{T}^*Q$ establishes an *equivalence* between the lagrangian and hamiltonian Hamilton–Jacobi theories.

This broad setting for Hamilton–Jacobi theory has been applied to other generalisations of hamiltonian mechanics, as for instance Poisson and Nambu mechanics, to systems with non-holonomic constraints, or to time-dependent systems, just to mention a few. Other authors have studied higher order systems, field theories, etc. —see for instance [6–8], not to speak of the deep connections with quantum mechanics [9]. So we believe that exploring generalisations of Hamilton–Jacobi theory is significantly clarifying for the purposes of geometric mechanics.

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Legendrian Knots in Contact 3-Manifolds

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Abstract The classification of Legendrian knots is one of the basic questions in 3-dimensional contact topology. In this note, the classification results for Legendrian knots are discussed. The first classification result for exceptional non-trivial knot types is studied and several open problems related to Legendrian knots are listed.

1 Definitions, Examples and Classification Results

A 2-plane field ξ on an orientable 3-manifold is a *contact structure* if there is a 1-form α such that locally $\xi = \texttt{ker}\alpha$ and $\alpha \wedge d\alpha \neq 0$. A 3-manifold M with a contact structure ξ is called a *contact* 3*-manifold* and it is denoted by (M, ξ). An embedded disk D in a contact 3-manifold (M, ξ) is called an *overtwisted disk* if D is tangent to the contact planes along its boundary. A contact structure ξ is *overtwisted* if ξ contains an overtwisted disk, otherwise ξ is called *tight*.

Example 1 On $\mathbb{R}^3$, the contact structure $\xi_{st} = \texttt{ker}(dz - y\,dx) = \langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \rangle$ is the standard contact structure and it is tight. See Fig. 1a. The standard tight contact structure ξ_{std} on S^3 is the contact structure where $(S^3 - \{p\}, \xi_{std}|_{S^3-p})$ is contactomorphic to $(\mathbb{R}^3, \xi_{st})$.

Example 2 On $\mathbb{R}^3$, the contact structure $\xi_{ot} = \texttt{ker}(\cos r\,dz + r\sin r\,d\theta)$ is overtwisted. The disk $D = \{(r, \theta, z)|z = 0, r \leq \pi\}$ is an overtwisted disk in $(\mathbb{R}^3, \xi_{ot})$. See Fig. 1b.

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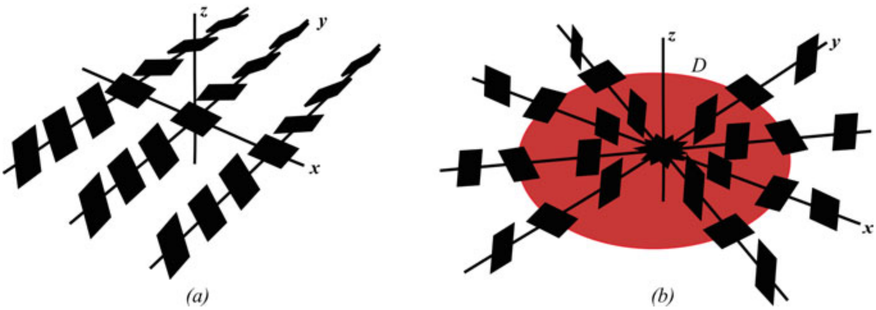
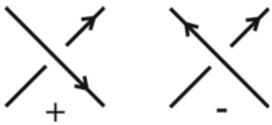


Fig. 1 **a** $(\mathbb{R}^3, \xi_{st})$ and **b** $(\mathbb{R}^3, \xi_{ot})$

Fig. 2 Legendrian unknot, right-handed trefoil, left-handed trefoil in $(\mathbb{R}^3, \xi_{st})$

Fig. 3 Positive and negative crossings



A knot L in (M, ξ) is called *Legendrian* if L is always tangent to ξ, that is $T_x L \in \xi_x$ for all $x \in L$. There are two types of Legendrian knots in overtwisted contact 3-manifolds: loose and exceptional/non-loose. A Legendrian knot is *loose* if its complement is overtwisted, otherwise it is called *exceptional/non-loose*.

Example 3 Front projections, projections to the xz-plane, of some Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ are given in Fig. 2.

The classical invariants of a null-homologous Legendrian knot L are the Thurston-Bennequin invariant and the rotation number. The *Thurston-Bennequin invariant* $\mathtt{tb}(L)$ measures the contact framing with respect to the framing given by a Seifert surface of L. The *rotation number* $\mathtt{rot}(L)$ of an oriented Legendrian knot L is the winding number of TL after trivializing ξ over a Seifert surface for L. For a Legendrian knot L in $(\mathbb{R}^3, \xi_{st})$, the Thurston-Bennequin invariant is computed by the formula $\mathtt{tb}(L) = writhe - \frac{1}{2}\#cusps$ where *writhe* is the signed count of crossings given in Fig. 3. The formulae $\mathtt{rot}(L) = \frac{1}{2}(c_d - c_u)$ computes the rotation number where c_d/c_u denotes the number of down/up cusps in the front projection of an oriented Legendrian knot L.

For a detailed introduction on contact 3-manifolds and knots in contact 3-manifolds, see [9].

Fig. 4 Legendrian unknots

Example 4 For the Legendrian unknots L_1, L_2 and L_3 in Fig. 4, the classical invariants are computed as $\mathtt{tb}(L_1) = -1$, $\mathtt{rot}(L_1) = 0$; $\mathtt{tb}(L_2) = -2$, $\mathtt{rot}(L_2) = 1$; and $\mathtt{tb}(L_3) = -2$, $\mathtt{rot}(L_3) = 1$.

Since two Legendrian unknots L_1 and L_2 in Example 4 have different classical invariants, they are different Legendrian unknots. Since the two Legendrian unknots L_2 and L_3 in Example 4 have the same classical invariants, the main question is: Are they the same or different Legendrian unknots?

Two Legendrian knots L_1 and L_2 of the same knot type in a contact 3-manifold (M, ξ) are called *coarsely equivalent* if there is a contactomorphism $\phi : (M, \xi) \to (M, \xi)$ such that $\phi(L_1) = L_2$. If further, the contactomorphism ϕ is contact isotopic to the identity, then the knots L_1 and L_2 are called *Legendrian isotopic*. The first classification result for Legendrian knots is due to Eliashberg and Fraser, [4].

Theorem 5 (Eliashberg and Fraser) (*a*) *Let L be a Legendrian unknot in the standard tight* S^3. *Then,* $\mathtt{tb}(L) = n < 0$ *and* $\mathtt{rot}(L)$ *lies in the range* $\{n+1, n+3, \ldots, -n-3, -n-1\}$. *Any pair* $(\mathtt{tb}, \mathtt{rot})$ *determines L up to Legendrian isotopy.*
(*b*) *Let L be an exceptional Legendrian unknot in an overtwisted* 3*-sphere* S^3. *Then,* $(\mathtt{tb}(L), \mathtt{rot}(L)) \in \{(n, \pm(n-1)) | n \in \mathbb{N}\}$. *These invariants determine L up to coarse equivalence.*

After Eliashberg and Fraser, the Legendrian classification problem is studied in various directions. Legendrian torus knots and the Legendrian figure eight knot in (S^3, ξ_{std}) are classified in [5] and knots in a cabled knot type are studied in [6]. Legendrian knots in other tight contact 3-manifolds are classified in [1, 2, 10, 11, 13, 15].

One of the first classification result for Legendrian knots in overtwisted contact 3-manifolds other than the overtwisted S^3 is due to Geiges and Onaran. In [10], the exceptional Legendrian rational unknots in overtwisted lens spaces are classified. The rational unknots in lens spaces are the spines of the Heegaard tori.

Theorem 6 (Geiges and Onaran) (*a*) *Let L be a rational unknot in a tight* $L(p, 1)$. *Then* $\mathtt{tb}_{\mathbb{Q}}(L) = n + \frac{1}{p}$ *with n negative integer and* $\mathtt{rot}_{\mathbb{Q}}(L) = r_0 + \frac{r_1}{p}$ *where* r_0 *lies in the range* $\{n+1, n+3, \ldots, -n-3, -n-1\}$, r_1 *lies in the range* $\{-p+2, -p+4, \ldots, p-4, p-2\}$. *Any pair* $(\mathtt{tb}, \mathtt{rot})$ *determines L up to coarse equivalence.*

(*b*) *Let* L *be an exceptional rational unknot in an overtwisted* $L(p,1)$. *Then* $\mathtt{tb}_{\mathbb{Q}}(L) = n + \frac{1}{p}$, $n \in \mathbb{N}_0$. *For* $n = 0$, *there is a single exceptional knot with* $\mathtt{rot}_{\mathbb{Q}}(L) = 0$. *For* $n = 1$, *there are exactly* $p + 1$ *exceptional knots with* $\mathtt{rot}_{\mathbb{Q}}(L)$ *in the range* $\{-1, -1 + \frac{2}{p}, -1 + \frac{4}{p}, \ldots, 1\}$; *and for* $n \geq 2$, *there are exactly* $2p$ *exceptional knots with* $\mathtt{rot}_{\mathbb{Q}}(L)$ *in the range* $\{\pm(n - 2 + \frac{2}{p}), \pm(n - 2 + \frac{4}{p}), \ldots, \pm(n - 2 + \frac{2p}{p}) = \pm n\}$. *These invariants determine* L *up to coarse equivalence.*

The first classification result for strongly exceptional non-trivial knot types is achieved by Geiges and Onaran. A Legendrian knot is called *strongly exceptional* if its complement has zero Giroux torsion. Strongly exceptional realisations of positive torus knots in Theorem 7 and negative torus knots in Theorem 8 below are completely classified and explicitly given in [11].

Theorem 7 (Geiges and Onaran) *Up to coarse equivalence,* $p \geq 2$ *and* $n \geq 1$ *there are exactly* $2p$ *strongly exceptional Legendrian realisations* L *of the* $(p, np + 1)$*-torus knot with* $\mathtt{tb}(L) = np^2 + p + 1$.

Theorem 8 (Geiges and Onaran) *Up to coarse equivalence, for* $p \geq 2$ *and* $n \geq 2$, *there are exactly* $2(p - 1)(n - 1)$ *strongly exceptional Legendrian realisations* L *of the* $(p, -(np - 1))$*-torus knot with* $\mathtt{tb}(L) = -np^2 + p + 1$.

The Legendrian classification problem is studied for links in [7, 8, 12]. The first complete Legendrian classification of a topological link type including Legendrian realisations in overtwisted contact structures is due to Geiges and Onaran. In [12], up to coarse equivalence, Legendrian Hopf links are completely classified in any contact 3-sphere S^3.

2 Questions and Open Problems

The classification problem for exceptional non-trivial knot types or link types in other overtwisted contact 3-manifolds may be studied. Along these lines:

Problem 9 Classify exceptional Legendrian torus knots in overtwisted lens spaces.

Problem 10 Classify Legendrian torus knots in small Seifert fibered 3-manifolds.

Problem 11 Classify exceptional rational Hopf links in lens spaces.

For the classification problem for topological link types other than the Hopf link, one may try to solve the following problems.

Problem 12 Classify Legendrian Whitehead links in the contact 3-sphere S^3.

Problem 13 Classify exceptional Legendrian realisations of the link $K \cup U$ where K is the trefoil knot and U is the meridional unknot of K in the contact 3-sphere S^3.

In all classification results mentioned in this note, the classical invariants suffice to distinguish Legendrian realisations. In general, this is not the case. The first example of two non-isotopic Legendrian knots in standard tight S^3 with the same classical invariants is given by Chekanov in [3] and Eliashberg. In [14], there are examples of exceptional non-isotopic knots having the same classical invariants that are constructed as connected sums. Thus it would be very interesting to know answers to the following questions.

Question 14 Are there any examples of exceptional, non-isotopic prime knots in overtwisted contact 3-manifolds?

Question 15 Are there any examples of exceptional, coarsely equivalent but non-isotopic knots in overtwisted contact 3-manifolds?

The analogue of Theorem 6 is expected to hold for general lens spaces $L(p, q)$ and the case for $L(5, 2)$ is illustrated in [10]. As a simpler first step towards classification results:

Problem 16 Construct the explicit diagrams for Legendrian realisations of rational unknots in arbitrary lens spaces $L(p, q)$.

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Topological Degree and Periodic Orbits of Semi-dynamical Systems

Pablo Amster and Melanie Bondorevsky

Abstract We study semi-dynamical systems associated to delay differential equations. Employing guiding type functions and topological degree, we show the existence of T-periodic closed orbits.

1 Introduction

With population models in mind [5], we consider the delayed differential system

$$x'(t) = f(t, x(t), x(t - \tau)) \tag{1}$$

where $f : [0, +\infty) \times [0, +\infty)^{2N} \to \mathbb{R}^N$ is continuous and $\tau > 0$ is the delay. An initial condition for (1) can be expressed in the following way

$$x_0 = \varphi, \tag{2}$$

where $\varphi : [-\tau, 0] \to [0, +\infty)^N$ is a continuous function and $x_t \in C([-\tau, 0], \mathbb{R}^N)$ is defined by $x_t(s) = x(t + s)$.

We are interested in the study of the dynamical behaviour of phenomena which, in this context, must remain non-negative for $t > 0$. If the solutions are defined over $[0, +\infty)$ and lie in $[0, +\infty)^N$ then the semi-flow associated to the system:

$$\Phi : [0, +\infty) \times C([-\tau, 0], [0, +\infty)^N) \to C([-\tau, 0], [0, +\infty)^N) \tag{3}$$

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given by

$$\Phi(t, \varphi) = x_t$$

induces a semi-dynamical system. An extra assumption is usually required for the analysis of *persistence* (see [2]):

(H1) If for some $j \in \{1, 2, ..., N\}$, $x_j = 0$ and $y \neq 0$, then $f_j(t, x, y) > 0$ for all $t > 0$.

This hypothesis assures that non-trivial solutions with non-negative initial data remain strictly positive.

In order to study the existence of T-periodic closed orbits, we might consider a *guiding-type function*

$$V : (0, +\infty)^N \to (0, +\infty),$$

i.e. there exist $t_0, r > 0$ such that

$$\langle \nabla V(x), f(t, x, y)\rangle > 0 \quad \text{for } t > t_0 \text{ and } V(x), V(y) < r$$

and such that

$$\lim_{|x|\to 0} V(x) = 0.$$

However, the previous condition is not fulfilled in many models which satisfy instead a weaker one, in terms of the vector field $f(\cdot, x, x)$:

(H2) There exist $t_0, \varepsilon{>}0$ such that $\langle \nabla V(x), f(t, x, x)\rangle{>}0$ for $t > t_0$, $V(x) < \varepsilon$.

We shall also take into account the following monotonicity condition:

(H3) $\langle \nabla V(x), f(t, x, y)\rangle \geq \langle \nabla V(x), f(t, x, x)\rangle$, whenever $V(x) \leq V(y)$.

The latter conditions guarantee the strong persistence of species in population dynamics. Namely,

$$\liminf_{t\to+\infty} ||\Phi(t, \varphi)||_\infty > 0 \qquad \forall \varphi \in C([-\tau, 0], (0, +\infty)^N).$$

Inspired by population models, in order to state our result we shall impose a condition that allows to find an upper bound for the flow. With this aim, we may choose a continuous function $a : [0, +\infty) \to (0, +\infty)$ and define:

$$\begin{aligned} F(t, x, y) &= \langle \nabla V(x), f(t, x, y)\rangle + a(t)V(x), \\ F^*(t, r) &= \sup_{V(x), V(y)\leq r} \frac{F(t,x,y)}{a(t)}. \end{aligned}$$

To ensure the existence of periodic orbits, we shall require:

(H4) There exist $R > 0$ such that $F^*(t, R) < R$ for $0 \leq t \leq T$.

We can now formulate our main result:

Theorem 1 *If f is T-periodic in the first coordinate and there exist positive constants such that $\varepsilon < R$ and* **(H2)***,* **(H3)** *and* **(H4)** *hold. Then there exists at least one T-periodic positive solution of (1)–(2) in $\Omega = \{x \in [0, +\infty)^N : V(x) \in (\varepsilon, R)\}$ provided that the Euler characteristic of Ω is non-zero.*

2 Periodic Orbits

When searching for periodic orbits of an ODE system, it is usual to employ a solution operator such as the Poincaré map and apply a standard procedure using the Brouwer degree to obtain fixed points. For the delayed case, since the space of initial conditions is infinite dimensional, the Brouwer degree cannot be applied: we shall use instead Leray-Schauder degree techniques. More precisely, inspired by [4], we shall work on the positive cone X of C_T, the Banach space of continuous T-periodic functions, for some $T > 0$, and define an appropriate fixed point operator $K : X \to C_T$. We shall see that if f is T-periodic in the first coordinate, then the fixed points of K determine T-periodic positive orbits of system (3).

Let us recall the Leray-Schauder degree is defined as follows [1]: Let $U \subseteq C_T$ be open and bounded, and let $K : \overline{U} \to C_T$ be compact with $Kx \neq x$ for $x \in \partial U$. Set $\varepsilon = \inf_{x\in\partial U} ||x - Kx||$. Then define

$$\deg_{L-S}(I - K, U, 0) = \deg_B((I - K_\varepsilon)|_{V_\varepsilon}, U \cap V_\varepsilon, 0),$$

where K_ε is an ε-approximation of K with $\mathrm{Im}(K_\varepsilon) \subseteq V_\varepsilon$ and $\dim(V_\varepsilon) < \infty$.

We will show that the Leray-Schauder degree of the operator $I - K$ is non-zero on an appropriate subset $U \subset X$ and therefore the set of fixed points of the compact operator K is non-empty.

The proof of our main theorem shall be based on the following crucial result (see e.g. [3]):

Theorem 2 (Hopf Theorem) *If ν is the outward normal on a compact, oriented manifold M, then the degree of ν equals the Euler characteristic of M.*

Proof (Proof of Theorem 1)

For convenience, a little of extra notation shall be introduced. For a function $x \in C_T$, let us write

$$\mathcal{I}x(t) := \int_0^t x(s)\,ds, \qquad \overline{x} := \frac{1}{T}\mathcal{I}x(T).$$

Moreover, denote by $\mathcal{N}$ the Nemitskii operator associated to the problem, namely

$$\mathcal{N}x(t) := f(t, x(t), x(t - \tau)).$$

Let us consider the open bounded sets $\Omega = \{x \in [0, +\infty)^N : V(x) \in (\varepsilon, R)\} \subseteq \mathbb{R}^N$, $U = \{x \in C_T : x(t) \in \Omega \text{ for all } t > 0\} \subseteq C_T$ and define the compact operator $K : C_T \to C_T$ by

$$Kx(t) := \overline{x} - t\overline{\mathcal{N}x} + \mathcal{I}\mathcal{N}x(t) - \overline{\mathcal{I}\mathcal{N}x}.$$

Via the Lyapunov-Schmidt reduction, if $x \in C_T$ is a fixed point of K then x is a solution of the equation.

Let $K_0 x := \overline{x} - \frac{T}{2}\overline{\mathcal{N}x}$ and consider for $s \in [0, 1]$, the homotopy $K_s := sK + (1-s)K_0$. We claim that K_s has no fixed points on ∂U. As mentioned, for $s > 0$ it is clear that $x \in \overline{U}$ is a fixed point of K_s if and only if $x'(t) = s\mathcal{N}x(t)$, that is:

$$x'(t) = sf(t, x(t), x(t-\tau)).$$

Observe that, if we identify $\mathbb{R}^N$ with the set of constant functions of C_T then $U \cap \mathbb{R}^N = \Omega$. Thus the image of K_0 is contained in $\mathbb{R}^N$, whence the Leray-Schauder degree of $I - K_0$ can be computed as the Brouwer degree of its restriction to Ω.

We apply another homotopy,

$$H(s, x) = sK_0(x) - (1-s)\nu(x), \qquad \text{for } (s, x) \in [0, 1] \times \overline{\Omega},$$

where ν is the outward normal, which does not have fixed points on $\partial\Omega$.

By the homotopy invariance of the degree and Hopf theorem, we conclude that

$$\deg_{LS}(I - K, U, 0) = \deg_B(I - K_0, \Omega, 0) = \deg_B(-\nu, \Omega, 0) = (-1)^N \chi(\Omega) \neq 0. \qquad \square$$

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Presentation of Symplectic Mapping Class Group of Rational 4-Manifolds

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Abstract We study the problem of description of the symplectic mapping class groups $\pi_0\big(Symp(X,\omega)\big)$ (SyMCG) of rational 4-manifolds $X = \mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$. We specify a certain class of symplectic forms ω on such X for which we give a finite presentation of the SyMCG with generators symplectic Dehn twists along Lagrangian spheres. This is a joint work with my scientific advisor Vsevolod Shevchishin.

1 Preliminaries and Previous Results

Denote by $Symp(X,\omega)$ the symplectomorphism group of a compact symplectic manifold (X,ω). The **symplectic mapping class group** (SyMCG) is the group $SMap(X,\omega) := \pi_0(Symp(X,\omega))$. Similarly we define the **smooth** MCG $Map(X) := \pi_0(Diff(X))$. The **pure symplectic MCG** (pure SyMCG) is kernel of the natural homomorphism $\pi_0(Symp(X,\omega)) \to \pi_0(Diff(X))$. Some authors called it the **symplectic Torelli group**.

A **rational** 4-**manifold** is either $S^2 \times S^2$, or $\mathbb{CP}^2$, or an l-fold blow-up $X_l = \mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$. It is known that two symplectic forms ω_1, ω_2 on a rational 4-manifold with equal cohomology classes (i.e., $[\omega_1] = [\omega_2]$) are isomorphic.

The first results on the topology of groups of symplectomorphisms were made by Gromov in his seminal paper [4]. He shows that the symplectomorphism groups of $\mathbb{CP}^2$ and $S^2 \times S^2$ equipped with the standard symplectic forms are homotopy equivalent to $\boldsymbol{PU}(2)$ and respectively $\mathbb{Z}_2 \ltimes \big(\boldsymbol{SO}(3) \times \boldsymbol{SO}(3)\big)$. The next important step was made by Abreu and McDuff [1, 8]. They describe rational homotopy type of the symplectomorphism group of $X = S^2 \times S^2$ and $X = S^2 \ltimes S^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with arbitrary symplectic form ω_λ, and how the rational homotopy type changes when the cohomology class $[\omega_\lambda]$ varies. The key conclusion is that under deformation of ω_λ when the cohomology class $[\omega_\lambda]$ varies in a certain way we obtain new elements in the higher homotopy groups $\pi_k(Symp(X,\omega_\lambda))$, see details in [1, 8].

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Let us compare their results with problems considered in this note. In the Abreu-McDuff case the manifolds are small (Betti number b_2 is 2), the groups $\pi_0(Symp(X, \omega_\lambda))$ are trivial and interesting phenomena occur in higher homotopy groups $\pi_k(Symp(X, \omega_\lambda))$. On the other hand, in our case the manifolds are large (Betti number $b_2 = l + 1$ is ≥ 6) and already the 0-group $\pi_0(Symp(X, \omega)) = SMap(X, \omega)$ could be large enough (e.g., infinite and having sufficiently sophisticated presentation). And as in the Abreu-McDuff case we study the dependence of the group on the symplectic form and try to describe this dependence.

By [5] in the case of smaller number of blow-ups $l \leq 4$ we still have nothing interesting: The pure SyMCG of (X_l, ω) is trivial for each symplectic form ω. But already for $l = 5$ the group is non-trivial: By Evans [3], for $X_5 = \mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$ and the monotone symplectic form (this means $[\omega] = c_1(X_5)$) the pure SyMCG is the MCG $Map(S^2, 5)$ of the sphere with 5 marked points. Its another description is the quotient of the pure braid group of the sphere $\mathsf{PBr}_5(S^2)$ by its center $\mathbb{Z}_2$.

I address the problem of finding a full description of the SyMCG (for example, in a form of a nice presentation) in the case when this group is rather big, in particular infinite. In this connection let us us notice that by [5] the group $SMap(X, \omega)$ is finite for every symplectic form ω in the case when X is the blow-up of $\mathbb{CP}^2$ at ≤ 4 points. It turns out that in some cases we have natural generators of SyMCG which are symplectic Dehn twists T_S along Lagrangian spheres $S \subset (X, \omega)$.

2 Main Results

In this abstract, I describe two particular types of classes of symplectic structures on rational 4-manifold, denoted by $\mathbb{D}_l$ and $\mathbb{E}_l$, for which can be found a presentation of SyMCG $SMap(X, \omega)$. They are characterised by the property that there exists a configuration of Lagrangian spheres in (X, ω) with the incidence graph that is a Dynkin graph of type $\mathbb{D}_l$ or resp. $\mathbb{E}_l$, and such that the symplectic Dehn twists along those spheres generate the group $SMap(X, \omega)$. The class $\mathbb{E}_l$ is the blow-up of $\mathbb{CP}^2$ in $l \leq 8$ points with the symplectic form ω of cohomology class $c_1(X)$. So, $\mathbb{E}_l$ is a symplectic analogue of del Pezzo surfaces (see e.g. [9]). The manifolds (X, ω) with $[\omega] = \lambda c_1(X)$ are also called **monotone symplectic manifolds**.

The notation $\mathbb{A}_l, \mathbb{D}_l, \mathbb{E}_l$ means the Dynkin diagram of the corresponding type and rank. Let us make here remarks about the class $\mathbb{E}_5$. As the Dynkin diagram it coincides with the diagram $\mathbb{D}_5$. As the result, the corresponding Weyl and (pure) braid groups are isomorphic: $W(\mathbb{E}_5) = W(\mathbb{D}_5)$ and $\mathsf{Br}(\mathbb{E}_5) = \mathsf{Br}(\mathbb{D}_5)$. However, *the classes of symplectic forms* $\mathbb{E}_5$ *and* $\mathbb{D}_5$ *are not the same.* Moreover, even the underlying rational manifolds are not diffeomorphic: $b_2(X) = 6$ in the case $\mathbb{E}_5$ and $b_2(X) = 7$ in the case $\mathbb{D}_5$.

If X_l is the l-fold blow-up of $\mathbb{CP}^2$ we denote by $[L]$ and $[E_i]$ ($i = 1, \ldots, l$) the homology classes of the line in $\mathbb{CP}^2$ and the exceptional curves. We identify the homology groups $H_2(X,\mathbb{R})$ and the cohomology groups $H^2(X,\mathbb{R})$ by means of the Poincaré duality.

A precise definition of the types $\mathbb{E}$ and $\mathbb{D}$ is as follows: A symplectic 4-manifold (X,ω) has type

- $\mathbb{E}_l$ with $l = 4,\dots,8$ if X is l-fold blow up of $\mathbb{CP}^2$ and $[\omega] = a\cdot c_1(X)$ for some $a > 0$;
- $\mathbb{D}_l$ with $l \geq 4$ if X is $(l+1)$-fold blow up of $\mathbb{CP}^2$ and $[\omega] = \lambda[L] - \sum_i \mu_i[E_i]$ such that $\mu_2 = \dots = \mu_l = \mu$, $\mu_1 = \lambda - 2\mu$, and

- $0 < \mu < \frac{\lambda}{3}$ if $l = 4,\dots,8$;
- $0 < \mu < \frac{2\lambda}{7}$ if $l = 9$;
- $0 < \mu < \frac{4\lambda}{l+2}$ if $l \geq 10$.

Theorem 1 ([2]) *Let (X,ω) be a rational symplectic 4-manifold of type $\mathbb{E}_l$ or $\mathbb{D}_l$.*
(1) *There exist Lagrangian spheres $S_1,\dots,S_l \subset X$ with the incidence diagram $\mathbb{E}_l$ or $\mathbb{D}_l$.*
(2) *The group $SMap(X,\omega)$ is generated by the Dehn twists T_{S_i} along the spheres S_i. In particular, $SMap(X,\omega)$ is the quotient of the braid group $\mathsf{Br}(\mathbb{E}_l)$ (resp. $\mathsf{Br}(\mathbb{D}_l)$).*
(3) *The image of the SyMCG $SMap(X,\omega)$ in the smooth MCG $Map(X)$ is the Weyl group $W(\mathbb{E}_l)$ (resp. $W(\mathbb{D}_l)$) and the standard generators are images of the twists T_{S_i}.*
(4) *Let (X,ω) be of the type $\mathbb{D}_l$ (or resp. $\mathbb{E}_5$). Then the pure SyMCG of (X,ω) is isomorphic to the MCG $Map(S^2,l)$ of the sphere with l marked points. This gives the extension*

$$1 \to Map(S^2,l) \to SMap(X,\omega) \to W(\mathbb{D}_l) \to 1.$$

Let us makes some comments about the meaning of the theorem. The incidence diagrams of the Lagrangian spheres are

$$\mathbb{E}_l:\quad S_1 - S_2 - \overset{\overset{\displaystyle S_0}{|}}{S_3} - \cdots - S_{l-1} \qquad\qquad \mathbb{D}_l:\quad S_2 - \overset{\overset{\displaystyle S_0}{|}}{S_3} - \cdots - S_l$$

and the homology classes $[S_i] = [E_i] - [E_{i+1}]$, $[S_0] = [L] - ([E_1] + [E_2] + [E_3])$.

Let us also notice that by Seidel [10] if two Lagrangian spheres S_i, S_j intersect transversally in a single point, then the corresponding Dehn twists satisfy the braid relation $T_{S_i}T_{S_j}T_{S_i} = T_{S_j}T_{S_i}T_{S_j}$.

The theorem gives almost complete description of the SyMCG $SMap(X,\omega)$ in the cases $\mathbb{D}_l$ and $\mathbb{E}_l$. What is missing is a system $\{R_1, R_2,\dots\}$ normally generating the kernel of the epimorphism $\mathsf{Br}(\mathbb{D}_l) \to SMap(X,\omega)$. Indeed, together with the commutativity and braid relations those elements R_i will form a defining system of relations between the generators T_{S_i} of the SyMCG, and this would be the desirable presentation. Let us describe those additional relations in the cases $\mathbb{D}_l$ and $\mathbb{E}_5$.

The first such relation is the equality $T^2_{S_0} = T^2_{S_2}$ in the cases $\mathbb{D}_l$ and $T^2_{S_0} = T^2_{S_4}$ in the cases $\mathbb{E}_5$. This relation defines an epimorphism $\mathsf{PBr}(\mathbb{D}_l) \twoheadrightarrow \mathsf{PBr}(\mathbb{A}_{l-1}) = \mathsf{PBr}_l$

in the sense that its kernel is normally generated in $\mathsf{Br}(\mathbb{D}_l)$ by the element $T_{S_0}^2 T_{S_2}^{-2}$. Here we recall that the (pure) braid group of $\mathbb{A}$-type is the usual (resp. pure) braid group.

The next relations arise in the description of the kernel of the epimorphism $\mathsf{PBr}_l \twoheadrightarrow \mathit{Map}(S^2, l)$. For this purpose we notice that the group PBr_l is naturally isomorphic to the MCG $\mathit{Map}(D, \partial D, l)$ of the disc with fixed boundary and l marked points. Then the epimorphism $\mathsf{PBr}_l \to \mathit{Map}(S^2, l)$ is induced by the embedding of the disc D in the sphere S^2. The kernel $\mathit{Ker}\big(\mathsf{PBr}_l \to \mathit{Map}(S^2, l)\big)$ is normally generated by the elements $(T_{S_2} \dots T_{S_l})^l$ and $(T_{S_2} \dots T_{S_{l-1}})^{l-1}$. The first element is well-known as the generator of center of Br_l. Let us notice that this element is the square of the **Garside element** Δ_l of the braid group Br_l, $(T_{S_2} \dots T_{S_l})^l = \Delta_l^2$. By this the other element $(T_{S_2} \dots T_{S_{l-1}})^{l-1}$ is the squared Garside element Δ_{l-1}^2 in the braid group Br_{l-1}. So finally we obtain the following presentation of the SyMCG in the case $\mathbb{D}_l$:

Theorem 2 ([2]) *In the case $\mathbb{D}_l$ the symplectic mapping class group admits a presentation*

$$\mathit{SMap}(\mathbb{D}_l) = \big\langle T_{S_0}; T_{S_2}, \dots, T_{S_l} \,\big|\, \mathbb{D}_l\text{-braid relations},\ T_{S_0}^2 = T_{S_2}^2,\ \Delta_l^2 = 1,\ \Delta_{l-1}^2 = 1 \big\rangle.$$

Notice that we can replace one of the relations $\Delta_l^2 = 1,\ \Delta_{l-1}^2 = 1$ by $\Delta_{l-1}^{-2}\Delta_l^2 = 1$. The latter is the **spherical relation** arising in the spherical braid group $\mathsf{Br}_l(S^2)$.

3 Sketch of the Proof

We use the approach of Abreu and McDuff based on Gromov's theory of pseudoholomorphic curves. For a 2-cohomology class $\eta \in H^2(X, \mathbb{R})$ we denote by $\Omega(X, \eta)$ the space of all symplectic forms ω on X such that $[\omega] = \eta$. Further, denote by $\mathcal{J}(X, \eta)$ the space of almost complex (a.cplx) structures J tamed by some $\omega \in \Omega(X, \eta)$. Finally, $\mathcal{J}^{int}(X, \eta)$ is the subspace of *integrable* a.cplx structures in $\mathcal{J}(X, \eta)$. Notice that by Moser's theorem ([12, Theorem 7.3]) the connected components of $\Omega(X, \eta)$ are orbits of the group of isotopies $\mathit{Diff}_0(X)$ (the connected component of Id_X in $\mathit{Diff}(X)$). Finally, let $\mathit{Diff}(X, \eta)$ be the stabiliser of the class η in $\mathit{Diff}(X)$. We assume that the class $\eta = [\omega]$ is of type $\mathbb{D}_l$ or $\mathbb{E}_l$.

The steps of the proof can be explained using the following **fundamental diagram.**

$$\begin{array}{ccccccccc}
\pi_1(\mathit{Diff}(X,\eta)) & \xrightarrow{\phi_1} & \pi_1(\Omega(X,\eta)) & \xrightarrow[\partial]{\phi_2} & \pi_0(\mathit{Symp}(X,\omega)) & \xrightarrow{\phi_3} & \pi_0(\mathit{Diff}(X,\eta)) & \xrightarrow{\phi_4} & \pi_0(\Omega(X,\eta)) \\
\uparrow{=} & & \phi_6\uparrow\cong & & \vdots\,\phi & & \uparrow{=} & & \\
\pi_1(\mathit{Diff}(X,\eta)) & \xrightarrow{\phi_5} & \pi_1(\mathcal{J}(X,\eta)) & & & & & & \\
\uparrow{=} & & \phi_8\uparrow\cong & & \vdots & & & & \\
\pi_1(\mathit{Diff}(X,\eta)) & \xrightarrow{\phi_7} & \pi_1(\mathcal{J}^{int}(X,\eta)) & \xrightarrow{\phi_9} & \boxed{\widehat{\pi}_1\big(\widehat{\mathcal{M}}(\eta)/W\big)} & \xrightarrow[\partial]{\phi_{10}} & \pi_0(\mathit{Diff}(X,\eta)) & &
\end{array}$$

Claim 1. The homomorphisms $\phi_6 : \pi_1(\mathcal{J}(X, \eta)) \to \pi_1(\Omega(X, \eta))$ and $\phi_8 : \pi_1(\mathcal{J}^{int}(X, \eta)) \to \pi_1(\mathcal{J}(X, \eta))$ are isomorphisms. (The first one is a classical result). $\square$

Claim 2. The action of $\mathit{Diff}(X, \eta)$ on $\Omega(X, \eta)$ is transitive and the stabiliser of a given ω is the symplectomorphism group $\mathit{Symp}(X, \omega)$. The first line of the diagram is a part of the long exact sequence of the fiber bundle $\mathit{Symp}(X, \omega) \to \mathit{Diff}(X, \eta) \to \Omega(X, \eta)$. □

We want to find a long exact sequence of homotopy groups also for the action of $\mathit{Diff}(X, \eta)$ on $\mathcal{J}^{int}(X, \eta)$. This is done in the following steps.

Claim 3. There exists a $\mathit{Diff}(X, \eta)$-invariant subspace $\mathcal{J}_0^{int}(X, \eta) \subset \mathcal{J}^{int}(X, \eta)$ such that $\pi_1(\mathcal{J}_0^{int}(X, \eta)) \cong \pi_1(\mathcal{J}^{int}(X, \eta))$ and such that $\mathcal{J}_0^{int}(X, \eta)$ admits a slice $\mathcal{X} \subset \mathcal{J}_0^{int}(X, \eta)$ which is finite-dimensional manifold. Moreover, the slice subgroup $\boldsymbol{G} \subset \mathit{Diff}(X, \eta)$ is a Lie group. The group $\pi_0(\boldsymbol{G})$ is the Weyl group $W = W(\mathbb{D}_l)$ in the $\mathbb{D}$-case and $W = W(\mathbb{E}_l)$ in the $\mathbb{E}$-case. □

Claim 4. The connected component $\boldsymbol{G}_0$ acts freely on the space $\mathcal{X}$. The quotient space $\widehat{\mathcal{M}}_l(\eta)$ is a complex manifold with an action of the Weyl group W. □

For the next step we need the definition of the **orbifold fundamental group**. We use approach of Looijenga [6]. Let $BW = K(W, 1)$ be the classifying space of the Weyl group W and $EW \to BW$ the universal covering. Define $\widehat{\pi}_1(\widehat{\mathcal{M}}_l(\eta)/W)$ as $\pi_1\big((\widehat{\mathcal{M}}_l(\eta) \times EW)\,/\,W\big)$.

Claim 5. The group $\widehat{\pi}_1(\widehat{\mathcal{M}}_l(\eta)/W)$ includes in the last line of the fundamental diagram, giving an exact sequence. Moreover, the homomorphism ϕ is an isomorphism. □

Claim 6. In the cases $\mathbb{D}_l$ and $\mathbb{E}_5$ the fundamental group $\pi_1(\widehat{\mathcal{M}}_l(\eta))$ is isomorphic to $\mathit{Map}(S^2, l)$. This gives a short exact sequence of groups

$$1 \longrightarrow \pi_1(\widehat{\mathcal{M}}_l(\eta)) = \mathit{Map}(S^2, l) \longrightarrow \widehat{\pi}_1(\widehat{\mathcal{M}}_l(\eta)/W) = \mathit{SMap}(X, \omega) \longrightarrow W(\mathbb{D}_l) \longrightarrow 1.$$

4 Some Remarks

1. The conditions on the class η in the definition of the type $\mathbb{D}_l$ are imposed to exclude **elliptic twists** which otherwise could give non-trivial elements in the SyMCG, see [11].
2. Our presentation of the SyMCG in the cases $\mathbb{D}_l$ and $\mathbb{E}_5$ resembles the one of the classical MCG $\mathit{Map}(S_g)$ of surfaces of genus g found by Matsumoto [7]. There he found a system of generators forming a Dynkin-like graph Γ, such that the first group of relations is a braid-like. The secondary relations are equalities between the Garside elements of certain subgraphs. For example, the so called lantern relation reads $\Delta^2(\mathbb{E}_6) = \Delta(\mathbb{E}_7)$.

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On Several Classes of Ricci Tensor

Maryam Samavaki and Jukka Tuomela

Abstract We survey some classes of Riemannian manifolds where the Ricci curvature satisfies some special property. In a local coordinate system the relevant condition is an overdetermined PDE where the unknowns are the components of the metric. Using the methods of the theory of overdetermined PDE we can produce a lot of explicit examples, which seem to be new, of such manifolds. We will also make some observations about relationships of various conditions to each other.

1 Introduction

We will consider below Ricci recurrent manifolds, pseudo Ricci symmetric manifolds and quasi Einstein manifolds. As far as we know these manifolds were introduced in papers [1–3]. There are of course a lot of articles devoted to these topics since then but there seems to be only few explicit examples known. In the present note we will describe large families of metrics which satisfy these conditions.

Let M be a smooth manifold with Riemannian metric g. The pointwise norm of a tensor T is denoted by $|T|$. The covariant derivative is denoted by ∇. The curvature tensor is denoted by R and the *Ricci tensor* is $\mathsf{Ri}_{jk} = R^i_{ijk}$ and the scalar curvature is $\mathsf{sc} = \mathsf{Ri}^k_k$.

In the following we will consider several classes of Riemannian manifolds. These classes are defined by requiring that the corresponding Ricci tensor satisfies some condition P. In this case we can also say that the manifold or the Riemannian metric is of the type P. Of course we will always assume that $\mathsf{Ri} \neq 0$.

The existence of various manifolds or Riemannian metrics on manifolds depend in general if certain overdetermined PDE have solutions. For a general overview of

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overdetermined PDE we refer to [6]. We will use below the command rifsimp which is implemented in Maple. The rif means *reduced involutive form*. The details of the algorithm are explained in [5]. In the examples there are typically several families of solutions. In the language of differential algebra one could say that typically the differential ideal corresponding to the relevant PDE system is not prime. Below we will only give one family of solutions in a given situation.

2 Geometry

Definition 1 Ricci tensor is

- *recurrent*, RR, if there is a nonzero one form β such that

$$\mathsf{Ri}_{ij;\ell} = \beta_\ell \mathsf{Ri}_{ij} \ . \tag{1}$$

- *pseudo Ricci symmetric*, PRS, if there is a nonzero one form α such that

$$\mathsf{Ri}_{ij;\ell} = 2\alpha_\ell \mathsf{Ri}_{ij} + \alpha_i \mathsf{Ri}_{\ell j} + \alpha_j \mathsf{Ri}_{i\ell} \tag{2}$$

- *quasi Einstein*, QE, if there are functions a and $b \neq 0$, and one form ω such that

$$\mathsf{Ri} = a\, g + b\, \frac{\omega \otimes \omega}{|\omega|^2} \tag{3}$$

Let us start with the RR case. A result in [4] implies that there is the following purely algebraic characterization of the RR condition:

$$\mathsf{Ri}_i^k \mathsf{Ri}_\ell^i = \tfrac{1}{2}\, \mathsf{sc}\, \mathsf{Ri}_\ell^k$$

But this leads easily to a complete description of the Ricci tensor.

Theorem 2 *Suppose that* Ri *is recurrent. Then it has a double eigenvalue* $\frac{\mathsf{sc}}{2}$ *and eigenvalue zero of multiplicity* $n-2$*. Moreover*

$$\beta = \nabla \ln(\mathsf{sc}) \quad , \quad \mathsf{sc}^2 = 2\,|\mathsf{Ri}|^2 \quad \text{and} \quad \mathsf{Ri}\,\beta = \tfrac{\mathsf{sc}}{2}\,\beta\ .$$

It turns out that the associated one form is almost the same in the PRS case.

Lemma 3 *Let the Ricci tensor be* PRS *with associated one form* α*. Then* $\mathsf{Ri}\,\alpha = 0$ *and*

$$\alpha = \tfrac{1}{4}\, \nabla \ln(|\mathsf{Ri}|^2)$$

If the scalar curvature is not zero then

$$\alpha = \frac{1}{2}\nabla \ln(\mathsf{sc}) \quad \text{and} \quad \mathsf{sc}^2 = c\,|\mathsf{Ri}|^2$$

for some constant c.

From this we get immediately that the Ricci tensor cannot be both RR and PRS at the same time.

Let us then analyze quasi Einstein structure. To this end it is convenient to formulate the condition (3) differently. Let us introduce the tensor $T = \mathsf{Ri}_{ij} - a\, g_{ij}$. Now if the symmetric tensor T is of matrix rank one then all 2×2 minors are zero and this gives us our system of $\binom{n}{2}$ PDE. In this way we see that the one form ω and the function b are actually quite irrelevant in the analysis of the existence of quasi Einstein structure.

Once the appropriate T is found b and ω can easily be computed. In fact $b = \mathsf{sc} - n\,a$ and ω can be solved from the linear system

$$T_{ij}\omega^j = \big(\mathsf{sc} - n\,a\big)\,\omega_i$$

Another way to characterize the QE case is that Ri has a simple eigenvalue $\mathsf{sc} - (n-1)a$ corresponding to the eigenvector ω, and all vectors orthogonal to ω are eigenvectors with eigenvalue a whose multiplicity is thus $n - 1$. But from this we get

Theorem 4 *If the Ricci tensor is recurrent and $n = 3$ then it is automatically quasi Einstein. If $n > 3$ the Ricci tensor cannot be both recurrent and quasi Einstein.*

Theorem 5 *Let us suppose that both* PRS *and* QE*.*
If $a \neq 0$ we have

$$\mathsf{Ri}_{ij} = \frac{\mathsf{sc}}{n-1}\Big(g_{ij} - \frac{\alpha_i\alpha_j}{|\alpha|^2}\Big)$$

If $a = 0$, then

$$g(\omega, \alpha) = 0 \quad \text{and} \quad \mathsf{Ri}_{ij} = \mathsf{sc}\,\frac{\omega_i\omega_j}{|\omega|^2}\ .$$

3 Examples

Let us consider a three dimensional manifold with the following metric:

$$g = f_1(x^1)(dx^1)^2 + f_2(x^1)h_2(x^2)(dx^2)^2 + f_3(x^1)h_3(x^2)q(x^3)(dx^3)^2 \qquad (4)$$

Example 1 Let us try to find RR metrics of the form (4). Let us define the tensor

$$P_{ij\ell} = \mathsf{sc}\,\mathsf{Ri}_{ij;\ell} - \mathsf{sc}_{;\ell}\mathsf{Ri}_{ij}$$

According to Theorem 2 the manifold is RR if $P = 0$. In this three dimensional case our PDE system $P = 0$ has a priori 18 independent equations but actually we have only 14 (not necessarily independent) nonzero equations. Computing with rifsimp reveals that the system splits into seven subsystems. The most general is as follows. We have three differential equations; first two are for f_j:

$$f_2'' = \frac{f_2'(f_1' f_2 + f_2' f_1)}{2 f_1 f_2} \quad \text{and} \quad f_3'' = \frac{f_2 f_3 f_1' f_3' + 2 f_1 f_2 (f_3')^2 - f_1 f_3 f_2' f_3'}{2 f_1 f_2 f_3}$$

Evidently now one can give f_1 arbitrarily and then solve the remaining functions. However, one can actually eliminate one of the functions by solving f_1 and f_3 in terms of f_2 which gives the family of solutions

$$f_1 = \frac{c_2 (f_2')^2}{f_2} \quad \text{and} \quad f_3 = c_1 f_2^m.$$

Note that m need not be an integer. Then we have the third differential equation which contain h_j and f_j. However, when we substitute the above formulas the functions f_j disappear and we are left with

$$h_3'' = \frac{(2m-1) c_2 h_2 (h_3')^2 + m c_2 h_3 h_2' h_3' - m^2 h_2^2 h_3^2}{2 m c_2 h_2 h_3}$$

Solving this for h_2 yields

$$h_2 = \frac{c_2 h_3^{1/m} (h_3')^2}{h_3^2 (c_2 c_3 - m^2 h_3^{1/m})} = \frac{c_2 m^2 (h')^2}{h (c_2 c_3 - m^2 h)}$$

where we have introduced a new function $h_3 = h^m$. Then writing f instead of f_2 we can write our final metric as

$$g = \frac{c_2 (f')^2}{f} (dx^1)^2 + \frac{m^2 c_2 f (h')^2}{h (c_2 c_3 - m^2 h)} (dx^2)^2 + c_1 f^m h^m q (dx^3)^2$$

Clearly one can choose constants and functions such that g is positive definite. For scalar curvature we get $\mathsf{sc} = (1-m) c_3 / (2 m f h)$ and thus $\beta = -\nabla \ln(fh)$. Note that $m \neq 1$ because otherwise also $\mathsf{Ri} = 0$.

Example 2 Let us consider again the metric (4), but now we try to find PRS metrics. Now our PDE system can be written as

$$S_{ij\ell} = 2\mathsf{sc}\,\mathsf{Ri}_{ij;\ell} - 2\mathsf{sc}_{;\ell}\mathsf{Ri}_{ij} - \mathsf{sc}_{;i}\mathsf{Ri}_{\ell j} - \mathsf{sc}_{;j}\mathsf{Ri}_{i\ell} = 0$$

Again we have 14 PDE, and computing with rifsimp we get three cases where $\alpha \neq 0$. In one case h_3 should be constant, and for f_j we obtain the equations

$$f_2'' = \frac{f_1 f_2 f_2' f_3' + f_1 f_3 (f_2')^2 + f_2 f_3 f_1' f_2'}{2 f_1 f_2 f_3} \quad \text{and} \quad f_3'' = \frac{f_1 f_2 (f_3')^2 - f_1 f_3 f_2' f_3' + f_2 f_3 f_1' f_3'}{2 f_1 f_2 f_3}$$

Solving this we obtain

$$f_1 = \frac{c_2 c_3 (f_3')^2 e^{c_1 f_3}}{f_3} \quad \text{and} \quad f_2 = c_2 e^{c_1 f_3} .$$

Then putting $f_3 = f$, $h_3 = c_4$ and $h_2 = h$ we can write our metric as

$$g = \frac{c_2 c_3 (f')^2 e^{c_1 f}}{f} (dx^1)^2 + c_2 h e^{c_1 f} (dx^2)^2 + c_4 f q (dx^3)^2$$

In this case $\mathsf{sc}^2 = |\mathsf{Ri}|^2$.

Example 3 Let us finally consider metric (4) and try to find solutions which satisfy the QE condition. Constructing the appropriate PDE system we obtain five nonzero equations. Since the function a appears algebraically and in some equations even linearly we can solve it and substitute back to the equations. rifsimp gives then us the following system

$$f_2''' = F_1(f_1, f_2, f_3), \quad f_3''' = F_2(f_1, f_2, f_3) \quad \text{and} \quad h_3'' = H(f_1, f_2, f_3, h_2, h_3)$$

The expressions for F_j and H are so big that we do not write them down explicitly. We can also solve it explicitly; denoting $f_2 = f$ the first two equations give

$$f_1 = \frac{c_1 c_3 (f')^2}{f_3 f} \quad \text{and} \quad f_3 = c_3 m^{-m} f^{1-m} \big(c_2 f - 1\big)^m$$

Substituting this into third equations yields

$$h_3'' = \frac{\big((3m-2) h_2 h_3' + 2(m-1) h_3 h_2'\big) h_3'}{4(m-1) h_2 h_3}.$$

Denoting $h_3 = h$ and solving for h_2 yields

$$h_2 = c_4 h^{(2-3m)/(2m-2)} (h')^2.$$

After this it is straightforward to compute a, b and ω which gives

$$\begin{aligned} a =& \frac{m}{8(1-m) c_4 f} h^{(2-m)/(2m-2)} - \frac{c_2^2 f^2 + (m-2) c_2 f + (m-1)^2}{2 c_1 m^m f^m} \big(c_2 f - 1\big)^{m-2} \\ b =& \frac{m}{8(m-1) c_4 f} h^{(2-m)/(2m-2)} + \frac{m(m-1)}{2 c_1 m^m f^m} \big(c_2 f - 1\big)^{m-2} \\ \omega =& f h' (c_2 f - 1) \partial_{x^1} + (2-2m) h f' \partial_{x^2} \end{aligned}$$

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Rank Conditions on Phylogenetic Networks

Marta Casanellas and Jesús Fernández-Sánchez

Abstract Less rigid than phylogenetic trees, phylogenetic networks allow the description of a wider range of evolutionary events. In this note, we explain how to extend the rank invariants from phylogenetic trees to phylogenetic networks evolving under the general Markov model and the equivariant models.

1 Introduction and Preliminaries

In order to model the evolution of a set of DNA sequences (each representing a species), one usually considers a phylogenetic tree (whose leaves are in correspondence with the living species and interior nodes correspond to ancestral species) and a Markov process governing the substitution of nucleotides on it. In phylogenetics, *invariants* is the name given to the polynomials that vanish on every distribution that arises as a Markov process on the phylogenetic tree. The main idea behind finding invariants is that they might help to distinguish phylogenetic trees and phylogenetic networks and they have been successfully used in phylogenetic reconstruction (see [4, 7]), in solving the identifiability of certain models [2] and in model selection [9].

Nevertheless, trees might be too restrictive to represent the evolutionary history as they cannot take into account processes such as hybridization or horizontal gene transfer. In order to incorporate them, one can use phylogenetic networks. Invariants for phylogenetic networks have been found for the JC69 substitution model [8] (for networks with a single reticulation vertex) and for the 2-state symmetric model on networks with four leaves [10, 11].

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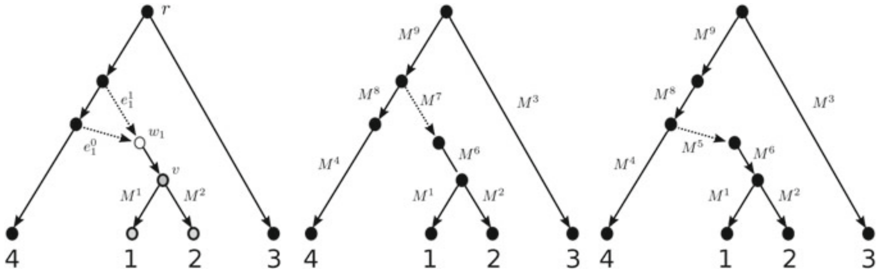

Fig. 1 On the left, a 4-leaf phylogenetic network $\mathcal{N}$ with one reticulation vertex w_1 painted white. The clade corresponding to leaves $A = \{1, 2\}$ has been coloured with gray. On the right, the two trees obtained when removing the edges e_1^0 and e_1^1 incident on w_1

We restrict to tree-child binary networks [13, §10]. That is, throughout the paper a *phylogenetic network* $\mathcal{N}$ is a rooted acyclic directed graph (with no edges in parallel) satisfying:

(1) the root r has out-degree two,
(2) every vertex with out-degree zero has in-degree one and is called a *leaf*,
(3) all other vertices have either in-degree one and out-degree two (which are called *tree vertices*) or in-degree two and out-degree one (which are called *reticulation vertices*)
(4) the child of a reticulation vertex is a tree vertex.

Following [8, 12], we introduce Markov processes on phylogenetic networks. We denote by $\mathcal{V}$ the set of vertices of the network and will assume that there is a discrete random variable assigned to each vertex taking values in $\Sigma := \{A, C, G, T\}$. We assign a distribution $\pi = (\pi_A, \pi_C, \pi_G, \pi_T)$ to the root r and to each edge e, a 4×4-transition matrix M^e. We write θ for the whole set of these parameters.

Let $\mathcal{N}$ be an n-leaf phylogenetic network and associate a 4×4 transition matrix from a nucleotide substitution model to each directed edge of $\mathcal{N}$. Suppose $\mathcal{N}$ has m reticulation vertices $\mathcal{R} = \{w_1, \ldots, w_m\}$. Each w_i has indegree two, and we denote by e_i^0 and e_i^1 the two edges directed into w_i. Figure 1 shows an example of a phylogenetic network with 4 leaves and only one reticulation vertex w_1 (painted white).

Each binary vector $\sigma \in \{0, 1\}^m$ encodes the possible choices for the reticulation edges, where a 0 or a 1 in the i-th coordinate indicates that the edge e_i^0 or e_i^1 was deleted, respectively. Any σ results in a n-leaf tree T_σ rooted at r with a collection of transition matrices corresponding to the particular edges in that tree. We call θ_σ the restriction of the parameters θ of the network to T_σ.

For $1 \leq i \leq m$, denote by δ_i the parameter corresponding to the probability that a particular site was inherited along edge e_i^1. We can then define a distribution on the set Σ^n (corresponding to characters at the leaves of the network) as follows

$$P_{\mathcal{N},\theta} = \sum_{\sigma\in\{0,1\}^m} \left(\prod_{i=1}^{m} \delta_i^{1-\sigma_i}(1-\delta_i)^{\sigma_i}\right) P_{T_\sigma,\theta_\sigma}.$$

Definition 1 Let $A|B$ be a bipartition of the set of leaves of $\mathcal{N}$. Given a distribution vector p on 4^n states, the flattening of p relative to the bipartition $A|B$ is the $4^{|A|} \times 4^{|B|}$ matrix $flatt_{A|B}(p)$ whose $(\mathbf{i}, \mathbf{j})$-entry is given by $p(\mathbf{k})$ where $\mathbf{k} = (\mathbf{i}, \mathbf{j})$ has entries matching those of $\mathbf{i}$ and $\mathbf{j}$ in the convenient order.

Let T be a tree and let $A|B$ be a bipartition of the leaves of T induced by removing an edge e of T. Let w be the vertex of e adjacent to A. If p is a distribution on T given by a distribution π at w and transition matrices at the edges of T oriented out from w, then $flatt_{A|B}(p)$ can be written as [1, 6]

$$flatt_{A|B}(p) = (M_A)^t D_\pi M_B, \tag{1}$$

where D_π is the 4×4 diagonal matrix with the entries of π at the diagonal, M_A is the $4 \times 4^{|A|}$ matrix whose entry $(x, \mathbf{i})$ is the probability in the subtree T_A of observing $\mathbf{i}$ at the leaves A given that the node w is at state x (and similarly for M_B). In the next sections we extend the well known edge invariants to phylogenetic networks. On a separate work we will study the consequences that this may have in distinguishing phylogenetic networks and phylogenetic trees.

2 Invariants for the General Markov Model

Assume that there is a clade T_A in $\mathcal{N}$ that does not contain any reticulation vertex (this is illustrated in the network of Fig. 1, where the clade T_A corresponds to leaves 1 and 2). Thus T_A is a subtree of $\mathcal{N}$ shared by all T_σ and the transition matrices at the edges of T_A are also shared by all T_σ. We call B the leaves in $\mathcal{N}$ that are not in A.

Theorem 2 *If $p = P_{\mathcal{N},\theta}$ is a distribution on a phylogenetic network $\mathcal{N}$ evolving under the GMM and T_A is a tree-clade in $\mathcal{N}$, then $flatt_{A|B}(p)$ has rank ≤ 4.*

Proof Let v be the root of T_A. To keep the proof simple we assume that v is different from r. By rerooting each T_σ at v, the edges of $\mathcal{N}$ that are not in T_A might change their orientation, but the corresponding transition matrices can also be changed so that the joint distribution does not change. If μ_σ is the new set of parameters for T_σ, which is composed of the distribution π^σ at the vertex v and the new transition matrices, then $P_{T_\sigma,\theta_\sigma} = P_{T_\sigma,\mu_\sigma}$. Note that after the rerooting process, the new transition matrices associated to the clade T_A are still the same for all T_σ (even if the distribution π^σ at v might be different for each T_σ). For each T_σ, we write M_A for the transition matrix from v to the leaves in A and write M_B^σ for the transition matrix from v to the leaves in B (as in Eq. (1)). Then, we have

$$flatt_{A|B}(p) = \sum_{\sigma}\left(\prod_{i=1}^{m}\delta_i^{1-\sigma_i}(1-\delta_i)^{\sigma_i}\right) flatt_{A|B}(P_{T_\sigma,\mu_\sigma})$$
$$= \sum_{\sigma}\left(\prod_{i=1}^{m}\delta_i^{1-\sigma_i}(1-\delta_i)^{\sigma_i}\right) M_A^t D_{\pi^\sigma} M_B^\sigma = M_A^t \sum_{\sigma}\left(\prod_{i=1}^{m}\delta_i^{1-\sigma_i}(1-\delta_i)^{\sigma_i}\right) D_{\pi^\sigma} M_B^\sigma,$$

where the second equality is obtained by using (1) for each T_σ. Therefore, $flatt_{A|B}(p)$ factorizes as a product of a $4^{|A|} \times 4$ and a $4 \times 4^{|B|}$ matrix, and hence has rank ≤ 4.

Corollary 3 *If $\mathcal{N}$ is a phylogenetic network with a tree-clade T_A as above and p is a distribution coming from a Markov process on $\mathcal{N}$, then the 5×5 minors of $flatt_{A|B}(p)$ are invariants for $\mathcal{N}$.*

Note that these invariants are shared by all the phylogenetic networks that have the same clade T_A. It is necessary to prove that the 5×5 minors above do not vanish for other networks before using them with the idea of distinguishing networks.

3 Invariants for Equivariant Models

The construction of the first section stands for the general Markov model (GMM), where no particular structure is assumed for the transition matrices or the root distribution. This construction can be adapted by taking the substitution model more restrictive and considering evolutionary submodels of the general Markov model. A large class of these submodels are the G-equivariant models, where the transition matrices satisfy some symmetries according to a permutation group $G < \mathcal{S}_4$. With precision, equivariant models only consider transition matrices that remain invariant after permuting rows and columns according to the permutations of some given permutation group (see [3, 5] for details). Among the G-equivariant models one finds the well known Jukes-Cantor model, Kimura 2 and 3 parameters and the strand symmetric model.

The result obtained in the previous section can be extended to G-equivariant models by using the tools introduced in [3]. We explain briefly the idea. Let $\mathcal{N}$ be a network with a tree-clade T_A. If p is a distribution on $\mathcal{N}$ arising from a G-equivariant model, then p actually lies in $(\mathbb{C}^{4^n})^G$, the set of points that remain invariant under the action of G. If we write N_i for the irreducible representations of G, the regular representation of G induces a decomposition of $W = \mathbb{C}^4$ into isotypic components: $W \cong \bigoplus_{i=1}^k N_i \otimes \mathbb{C}^{m_i}$, for some well-defined multiplicities $m_i \geq 0$, and similar decompositions for every tensor power $W^{\otimes l}$, $l \geq 1$ (Maschke's theorem). If $|\cdot|$ stands for cardinality, we can rewrite $flatt_{A|B}(p)$ in a convenient basis of $(\mathbb{C}^{4^n})^G \cong \mathrm{Hom}_G(W^{\otimes|A|}, W^{\otimes|B|})$ consistent with these decompositions, so that the resulting matrix becomes block diagonal:

$$\overline{flatt}_{A|B}(p) = (B_1, \ldots, B_k).$$

In this setting, we are able to prove the following result:

Theorem 4 *If p arises from the G-equivariant model on $\mathcal{N}$, then* $\operatorname{rank}(B_i) \le m_i$ *for each $i = 1, \ldots, k$.*

Corollary 5 *If $\mathcal{N}$ is a phylogenetic network with a tree-clade T_A as above and p is a distribution coming from a Markov process on $\mathcal{N}$, then the $(m_i + 1) \times (m_i + 1)$-minors of the block B_i of $flatt_{A|B}(p)$ are invariants for $\mathcal{N}$.*

The precise technical statement and the proof will be provided in a forthcoming paper. It will be interesting to check whether these invariants arising from rank conditions coincide with some of the invariants found in [8] for the Jukes-Cantor model.

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A Contact Geometry Approach to Symmetries in Systems with Dissipation

Jordi Gaset

Abstract Systems with dissipation can be described using contact geometry. We introduce the concepts of symmetries and dissipation laws for contact Hamiltonian systems and study the relation between them. This is an ongoing collaboration with Xavier Gràcia, Miguel C. Muñoz-Lecanda, Xavier Rivas and Narciso Román-Roy.

1 Introduction

In many mechanical systems without dissipation, we are interested in quantities (like energy or the different momenta) which are conserved along a solution. They are an effective tool to understand and integrate the system. From a physical point of view, if a system has dissipation, these quantities are not conserved. Since damped systems rarely have a standard Lagrangian or Hamiltonian formulation, this problem can not be studied with the usual tools.

There is a growing interest in describing the geometrical framework of dissipated or damped systems, specifically using contact geometry [1, 3–5]. All of them are described by ordinary differential equations to which some terms that account for the dissipation or damping have been added. In order to provide a variational formulation for these systems, contact geometry introduces a new variable or parameter, together with a new set of equations. It turns out that this variable is closely related to the action itself, and some authors consider these theories as described by an action-dependent Lagrangian. Contact Hamiltonian systems provide us the geometric framework we will use to analyse symmetries and (non)-conserved quantities.

First we will present the geometric structures of the contact formalism, and contact Hamiltonian systems. Then we will define several classes of symmetries for this kind of systems, which have different properties.

The analogous concept of conserved quantities are called dissipated quantities. In the contact formalism, the evolution of these quantities is determined by a dissi-

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pation law. We will also show how to construct conserved quantities from dissipated quantities.

It is possible to relate symmetries with dissipated quantities, in a result inspired in Noether's Theorem. We will briefly discuss how this relation works. Finally, we will apply this tools to the motion in a gravitational field with friction.

This framework can be extended to describe field theories with dissipation introducing the concept of k-contact structures [2].

2 Contact Manifolds and Contact Hamiltonian Systems

Definition 1 Let M be a $(2n+1)$-dimensional manifold. A **contact form** in M is a differential 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form in M. Then, (M, η) is said to be an **(exact) contact manifold**.

As a consequence of the condition that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form we have a decomposition of $\mathrm{T}M$, induced by η, in the form $\mathrm{T}M = \ker \mathrm{d}\eta \oplus \ker \eta$. Therefore, there exists a unique vector field $\mathcal{R} \in \mathfrak{X}(M)$, which is called **Reeb vector field**, such that

$$\begin{cases} i(\mathcal{R})\mathrm{d}\eta = 0, \\ i(\mathcal{R})\eta = 1. \end{cases} \tag{1}$$

This vector field generates the distribution $\ker \mathrm{d}\eta$, which is called the *Reeb distribution*. In a contact manifold one can prove the existence of Darboux-type coordinates:

Theorem 2 (Darboux theorem for contact manifolds) *Let (M, η) be a contact manifold. Then around each point $p \in M$ there exist a chart $(\mathcal{U}; q^i, p_i, s)$ with $1 \leq i \leq n$ such that*

$$\eta|_{\mathcal{U}} = \mathrm{d}s - p_i\, \mathrm{d}q^i \ .$$

*These are the so-called **Darboux** or **canonical coordinates** of the contact manifold (M, η).*

In Darboux coordinates, the Reeb vector field is $\mathcal{R}|_{\mathcal{U}} = \dfrac{\partial}{\partial s}$.

Theorem 3 *If (M, η) is a contact manifold and, for every $\mathcal{H} \in C^\infty(M)$, there exists a unique vector field $X_{\mathcal{H}} \in \mathfrak{X}(M)$ such that*

$$\begin{cases} i(X_{\mathcal{H}})\mathrm{d}\eta = \mathrm{d}\mathcal{H} - (\mathscr{L}_{\mathcal{R}}\mathcal{H})\eta \\ i(X_{\mathcal{H}})\eta = -\mathcal{H} \ . \end{cases} \tag{2}$$

The vector field $X_{\mathcal{H}}$ is the **contact Hamiltonian vector field** associated to $\mathcal{H}$ and the Eq. (2) are the **contact Hamiltonian equations** for this vector field. The triple $(M, \eta, \mathcal{H})$ is a **contact Hamiltonian system**.

As a consequence of the definition of $X_{\mathcal{H}}$ we have the following relation, which expresses the dissipation of the Hamiltonian:

$$\mathscr{L}_{X_{\mathcal{H}}}\mathcal{H} = -(\mathscr{L}_{\mathcal{R}}\mathcal{H})\mathcal{H}\,.$$

Indeed: $\mathscr{L}_{X_{\mathcal{H}}}\mathcal{H} = -\mathscr{L}_{X_{\mathcal{H}}} i(X_{\mathcal{H}})\eta = -i(X_{\mathcal{H}})\mathscr{L}_{X_{\mathcal{H}}}\eta = i(X_{\mathcal{H}})((\mathscr{L}_{\mathcal{R}}\mathcal{H})\eta) = -(\mathscr{L}_{\mathcal{R}}\mathcal{H})\mathcal{H}\,.$

Taking Darboux coordinates (q^i, p_i, s), the contact Hamiltonian vector field is

$$X_{\mathcal{H}} = \frac{\partial\mathcal{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial\mathcal{H}}{\partial q^i} + p_i\frac{\partial\mathcal{H}}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial\mathcal{H}}{\partial p_i} - \mathcal{H}\right)\frac{\partial}{\partial s}\,;$$

and its integral curves $\gamma(t) = (q^i(t), p_i(t), s(t))$ are solutions of

$$\begin{cases} \dot{q}^i = \frac{\partial\mathcal{H}}{\partial p_i}\,, \\ \dot{p}_i = -\left(\frac{\partial\mathcal{H}}{\partial q^i} + p_i\frac{\partial\mathcal{H}}{\partial s}\right)\,, \\ \dot{s} = p_i\frac{\partial\mathcal{H}}{\partial p_i} - \mathcal{H}\,. \end{cases} \tag{3}$$

3 Symmetries and Dissipation Laws for Contact Hamiltonian Systems

One can consider different concepts of symmetry in a dynamical system, which depend on which structure they preserve. We will define dynamical symmetries, which preserve the space of solutions, and contact symmetries, which preserve the geometric structure.

Let $(M, \eta, \mathcal{H})$ be a contact Hamiltonian system with Reeb vector field $\mathcal{R}$, and $X_{\mathcal{H}}$ the contact Hamiltonian vector field for this system; that is, the solution to the Hamilton equations (2).

Definition 4 Consider a diffeomorphism $\Phi\colon M \longrightarrow M$ and a vector field $Y \in \mathfrak{X}(M)$:

- Φ is a **dynamical symmetry** if $\Phi_* X_{\mathcal{H}} = X_{\mathcal{H}}$ (it maps solutions into solutions). Y is an **infinitesimal dynamical symmetry** if its local flows are dynamical symmetries; that is, $\mathscr{L}_Y X_{\mathcal{H}} = [Y, X_{\mathcal{H}}] = 0$.
- Φ is a **contact symmetry** if

$$\Phi^*\eta = \eta \quad , \quad \Phi_*\mathcal{H} = \mathcal{H}\,.$$

Y is an **infinitesimal contact symmetry** if its local flows are contact symmetries; that is,

$$\mathscr{L}_Y\eta = 0 \quad , \quad \mathscr{L}_Y\mathcal{H} = 0\,.$$

Every (infinitesimal) contact symmetry preserves the Reeb vector field; that is, $\Phi^*\mathcal{R} = \mathcal{R}$ (or $[Y, \mathcal{R}] = 0$). We have that an (infinitessimal) contact symmetry is an (infinitessimal) dynamical symmetry.

Associated with symmetries of contact Hamiltonian systems are the concepts of *dissipated* and *conserved quantities*:

Definition 5 A function $F \in C^\infty(M)$ is:

- A **conserved quantity** of a contact Hamiltonian system if $\mathscr{L}_{X_\mathcal{H}} F = 0$.
- A **dissipated quantity** of a contact Hamiltonian system if $\mathscr{L}_{X_\mathcal{H}} F = -(\mathscr{L}_\mathcal{R}\mathcal{H})\, F$.

For contact Hamiltonian systems, symmetries are associated with dissipated quantities as follows:

Theorem 6 (Dissipation theorem). *If Y is an infinitesimal dynamical symmetry, then $F = -i(Y)\eta$ is a dissipated quantity.*

In particular, the Hamiltonian vector field $X_\mathcal{H}$ is trivially a dynamical symmetry and its dissipated quantity is the energy, $F = -i(X_\mathcal{H})\eta = \mathcal{H}$; that is: $\mathscr{L}_{X_\mathcal{H}}\mathcal{H} = -(\mathscr{L}_\mathcal{R}\mathcal{H})\,\mathcal{H}$.

The Dissipation theorem is similar to the classical Noether's theorem. The converse of this result, that is, if every dissipated quantity is associated to an infinitesimal dynamical symmetry, is not true in general. Nevertheless, we can characterize them as follows: for any function F, we have an associated vector field: $F = -i(Y_F)\eta$, namely $Y_F = -F\mathcal{R}$. Then, the results follows using a theorem in [5]:

Theorem 7 *Let X be a vector field on M. Then $i(X)\eta$ is a dissipated quantity if, and only if, $i([X, X_\mathcal{H}])\eta = 0$.*

Every dissipated quantity changes with the same rate ($-\mathcal{R}(\mathcal{H})$), which suggests that the quotient of two dissipated quantities should be a conserved quantity. Indeed:

Proposition 8 *Given two functions $F_1,\ F_2 \in C^\infty(M)$:*

- *If F_1 and F_2 are dissipated quantities and $F_2 \neq 0$, then F_1/F_2 is a conserved quantity.*
- *If F_1 is a dissipated quantity and F_2 is a conserved quantity, then $F_1 G_2$ is a dissipated quantity.*

If $H \neq 0$, it is possible to assign a conserved quantity to an infinitesimal dynamical symmetry Y. Indeed, from Theorem 6 and Proposition 8, the function $-i(Y)\eta/H$ is a conserved quantity.

Finally, contact symmetries can be used to generate new dissipated quantities from a given dissipated quantity. In fact, as a corollary of Definitions 5 and 4 we obtain:

Proposition 9 *If $\Phi\colon M \to M$ is a contact symmetry and $F\colon M \to \mathbb{R}$ is a dissipated quantity, then so is $\Phi^* F$.*

4 Example: Motion in a Gravitational Field with Friction

Consider the motion of a particle in a vertical plane under the action of constant gravity, with friction proportional to velocity. The Hamiltonian system is $(M, \eta, \mathcal{H})$, where $M = \mathrm{T}^*\mathbb{R}^2 \times \mathbb{R}$. Considering coordinates (x, y, p_x, p_y, s) we have that:

$$\eta = \mathrm{d}s - p_x \mathrm{d}x - p_y \mathrm{d}y \,, \quad \mathcal{H} = \frac{1}{2}\frac{p_x^2 + p_y^2}{m} + mgy + \gamma s \,.$$

The contact Hamiltonian vector field is

$$X_{\mathcal{H}} = \Big(\frac{1}{2}\frac{p_x^2 + p_y^2}{m} - mgy - \gamma s\Big)\frac{\partial}{\partial s} + \frac{p_x}{m}\frac{\partial}{\partial x} + \frac{p_y}{m}\frac{\partial}{\partial y} - \gamma p_x \frac{\partial}{\partial p_x} - (mg + \gamma p_y)\frac{\partial}{\partial p_y} \,.$$

Which leads to the following equations for curves:

$$\ddot{x} + \gamma \dot{x} = 0 \quad , \quad \ddot{y} + \gamma \dot{y} + g = 0 \quad , \quad \dot{s} = \frac{1}{2}\frac{p_x^2 + p_y^2}{m} - mgy - \gamma s \,.$$

One can check that the Energy is dissipated: $\mathscr{L}_{X_{\mathcal{H}}}\mathcal{H} = -\gamma\mathcal{H}$. Moreover, we have that $\frac{\partial \mathcal{H}}{\partial x} = 0$, thus $\frac{\partial}{\partial x}$ is a contact symmetry. p_x is the associated dissipated quantity and $\mathscr{L}_{X_{\mathcal{H}}} p_x = -\gamma p_x$. Finally, we have the following conserved quantity: $\mathcal{H}/p_x = (mp^2/2 + mgy + \gamma s)/p_x$.

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Dimension Formulas for the Cohomology of Arithmetic Groups

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Abstract In this extended abstract we will describe a method to study the dimension of the cohomology of an arithmetic group. We will mainly use the Borel-Serre compactification, the theory of cuspidal and Eisenstein cohomology and the Euler characteristic.

1 Introduction

Let G be a semisimple algebraic group defined over $\mathbb{Q}$ and let $\Gamma \subset \mathrm{G}(\mathbb{Q})$ be an arithmetic subgroup. Given a representation $\mathcal{M}$ of G, we are interested in determining the dimension of $H^\bullet(\Gamma, \mathcal{M})$.

Let $K_\infty \subset \mathrm{G}(\mathbb{R})$ be a maximal compact subgroup. We denote by X the symmetric space $\mathrm{G}(\mathbb{R})/K_\infty$. The arithmetic group Γ acts on X and we denote by X_Γ the quotient space $\Gamma\backslash X$. The representation $\mathcal{M}$ of G determines in a natural way a sheaf $\widetilde{\mathcal{M}}$ on X_Γ.

One then has the following isomorphism in cohomology

$$H^\bullet(\Gamma, \mathcal{M}) \cong H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}),$$

where on the left hand side we are considering the group cohomology and on the right hand side we are considering the sheaf cohomology (see Sect. VII of [5] for a proof of this statement).

Arithmetic groups play a central role in Number Theory and their cohomology spaces encode very interesting arithmetic information. Already for the congruence arithmetic subgroups of SL_2, the cohomology is closely related to the space of modular forms. This is already one important motivation to study these objects.

From now on, we fix a system of simple roots Δ on G and we denote by $\mathcal{M}_\lambda$ the finite dimensional irreducible representation of G with highest weight λ. This repre-

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sentation will be defined over some number field F (i.e. over some finite extension of $\mathbb{Q}$).

In the next sections we will describe a method to study the dimension of $H^\bullet(\Gamma, \mathcal{M}_\lambda)$ that works mainly when λ is regular.

2 Borel-Serre Compactification

Let $\overline{X}_\Gamma$ denote the Borel-Serre compactification of X_Γ (see [4]). This object is a very important tool in the study of the cohomology of arithmetic groups. We will not give the precise definition of this compactification but we will list some of its most important properties. The structure of this compactification is strongly related with the $\mathbb{Q}$-structure of G. Let $\Delta_\mathbb{Q}$ be a system of simple $\mathbb{Q}$-roots on G. There is a natural bijection between the conjugacy classes in the set $\mathcal{P}_\mathbb{Q}(\mathrm{G})$ of $\mathbb{Q}$-parabolic subgroups and the subsets of $\Delta_\mathbb{Q}$. Let $rk_\mathbb{Q}\mathrm{G}$ denote the dimension of the maximal $\mathbb{Q}$-split torus in G. One defines the $\mathbb{Q}$-parabolic rank of $\mathrm{P} \in \mathcal{P}_\mathbb{Q}(\mathrm{G})$ whose conjugacy class is associated to $I \subset \Delta_\mathbb{Q}$ by $rk_\mathbb{Q}\mathrm{P} = rk_\mathbb{Q}\mathrm{G} - |I|$. The reader can see [3] for a detailed description of the theory of $\mathbb{Q}$-parabolic subgroups.

The Borel-Serre compactification verifies the following properties,

- Let $i : X_\Gamma \hookrightarrow \overline{X}_\Gamma$ be the inclusion, then i is an homotopy equivalence and one has an isomorphism in cohomology,

$$H^\bullet(\overline{X}_\Gamma, i_*(\widetilde{\mathcal{M}}_\lambda)) \cong H^\bullet(\overline{X}_\Gamma, \widetilde{\mathcal{M}}_\lambda),$$

  where i_* denotes the direct image functor defined by i. In what follows, we will denote $i_*(\widetilde{\mathcal{M}}_\lambda)$ simply by $\widetilde{\mathcal{M}}_\lambda$.
- Let $\partial\bar{X}_\Gamma = \bar{X}_\Gamma \setminus X_\Gamma$ be the boundary of the Borel-Serre compactification. One can write this boundary as a union

$$\partial\bar{X}_\Gamma = \bigcup_{\mathrm{P}\in\Gamma\backslash\mathcal{P}_\mathbb{Q}(\mathrm{G})} e'(\mathrm{P})$$

  indexed by the set of Γ-conjugacy classes of the set $\mathcal{P}_\mathbb{Q}(\mathrm{G})$ of $\mathbb{Q}$-parabolic subgroups of G. This expression of the boundary defines a spectral sequence abutting to the cohomology of the boundary

$$E_1^{p,q} = \bigoplus_{rk_\mathbb{Q}\mathrm{P}=p+1} H^q(e'(\mathrm{P}), \widetilde{\mathcal{M}}_\lambda) \Rightarrow H^{p+q}(\partial\bar{X}_\Gamma, \widetilde{\mathcal{M}}_\lambda).$$

Finally, one has a decomposition that reduces the study of the cohomology of the boundary $\partial\bar{X}_\Gamma$ to a study of the cohomology of certain arithmetic quotients of $\mathbb{Q}$-rank lower than $rk_\mathbb{Q}\mathrm{G}$, that are therefore easier to calculate. In fact, the cohomology of $e'(P)$ can be decomposed as a direct sum

$$H^k(e'(\mathrm{P}), \widetilde{\mathcal{M}}_\lambda) = \bigoplus_{w \in \mathcal{W}^{\mathrm{P}}(k)} H^p(X_\Gamma^{\mathrm{M_P}}, \widetilde{\mathcal{M}}_{w\cdot\lambda}),$$

where $X_\Gamma^{\mathrm{M_P}}$ denotes the locally symmetric space associated to the Levy quotient $\mathrm{M_P}$ of P, $\mathcal{W}^{\mathrm{P}}(k)$ is certain subset of the Weyl group $\mathcal{W}$ of G whose elements have length k and $\widetilde{\mathcal{M}}_{w\cdot\lambda}$ is the sheaf on $X_\Gamma^{\mathrm{M_P}}$ defined by the irreducible representation of $\mathrm{M_P}$ with highest weight $w \cdot \lambda$. See [11] for the details of this decomposition and [9] for the definitions of the sets $\mathcal{W}^{\mathrm{P}}(k)$ and the highest weights $w \cdot \lambda$.

3 Eisenstein and Cuspidal Cohomology

The representation $\mathcal{M}_\lambda$ is defined over a number field F and we denote $\mathcal{M}_{\lambda,\mathbb{C}} = \mathcal{M}_\lambda \otimes_F \mathbb{C}$. When extending scalars to $\mathbb{C}$, one has a decomposition of the cohomology of Γ into the direct sum of two subspaces, the cuspidal and the Eisenstein cohomology, defined analytically in terms of automorphic forms

$$H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^\bullet_{cusp}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \oplus H^\bullet_{Eis}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}). \quad (1)$$

On the other hand, the inclusion $\partial\overline{X}_\Gamma \hookrightarrow \overline{X}_\Gamma$ defines a morphism in cohomology, the restriction morphism

$$r : H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}_\lambda) \to H^\bullet(\partial X_\Gamma, \widetilde{\mathcal{M}}_\lambda).$$

One defines the inner cohomology by $H^\bullet_!(X_\Gamma, \widetilde{\mathcal{M}}_\lambda) = Ker(r)$. This space is defined over F and one has an isomorphism

$$H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet_!(X_\Gamma, \widetilde{\mathcal{M}}_\lambda) \oplus H^\bullet_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$$

where $H^\bullet_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ is the image of r.

In general $H^\bullet_{cusp}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \subset H^\bullet_!(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ and when the highest weight λ is regular, this inclusion is in fact an equality. This implies that in the regular case the restriction morphism r defines an isomorphism between $H^\bullet_{Eis}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ and $H^\bullet_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$.

Another important fact is that one has the natural nondegenerate pairings, usually referred to as Poincaré duality (see [7]),

$$H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}_\lambda) \times H_c^{d-\bullet}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda^*}) \to F,$$

and

$$H^\bullet(\partial X_\Gamma, \widetilde{\mathcal{M}}_\lambda) \times H^{d-1-\bullet}(\partial X_\Gamma, \widetilde{\mathcal{M}}_{\lambda^*}) \to F,$$

where $\mathcal{M}_{\lambda^*}$ denotes the dual representation of $\mathcal{M}_\lambda$ and d is the dimension of X.

These pairings respect some compatibility conditions involving the restriction morphism r, which implies the following

- The spaces $H^{\bullet}_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ are maximal isotropic subspaces of the boundary cohomology under the Poincaré duality. This means that $H^{\bullet}_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ is the orthogonal space of $H^{\bullet}_{inf}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda^*})$ under this pairing.
- Poincaré duality induces a duality between $H^{\bullet}_{!}(X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$ and $H^{\bullet}_{!}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda^*})$.

The description of the boundary cohomology and this property are very strong tools to study the Eisenstein cohomology. One could also use some important results in [12]. These methods to calculate the Eisenstein cohomology will work for most of the highest weights λ, including all the regular ones. In all the other cases one should use a study of the residual Eisenstein cohomology classes as in [6].

4 Euler Characteristic

The Euler characteristic of Γ with coefficients in the representation $\mathcal{M}_\lambda$ is defined by

$$\chi_h(\Gamma, \mathcal{M}_\lambda) = \sum_{k \geq 0} (-1)^k dim_F(H^q(\Gamma, \mathcal{M}_\lambda)).$$

One similarly defines the Eisenstein and the cuspidal Euler characteristic to be

$$\chi_{Eis}(\Gamma, \mathcal{M}_\lambda) = \sum_{k \geq 0} (-1)^k dim_{\mathbb{C}}(H^q_{Eis}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})),$$
$$\chi_{cusp}(\Gamma, \mathcal{M}_\lambda) = \sum_{k \geq 0} (-1)^k dim_{\mathbb{C}}(H^q_{cusp}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})),$$

and because of (1), one has

$$\chi_{cusp}(\Gamma, \mathcal{M}_\lambda) = \chi_h(\Gamma, \mathcal{M}_\lambda) - \chi_{Eis}(\Gamma, \mathcal{M}_\lambda).$$

There is a very useful formula, developed in [8], that can be used to calculate the Euler characteristic. In fact,

$$\chi_h(\Gamma, \mathcal{M}_\lambda) = \sum_{(T)} \chi(C_\Gamma(T)) Tr(T^{-1}, \mathcal{M}_\lambda),$$

where the sum runs over the Γ-conjugacy classes (T) of torsion elements in Γ, $C_\Gamma(T)$ is the centralizer in Γ of T and χ is the orbifold Euler characteristic.

The orbifold Euler characteristic $\chi(\Gamma')$ of an arithmetic group Γ' is determined by the following two properties:

- If Γ' is torsion free, then $\chi(\Gamma') = \chi_h(\Gamma')$, that is, the orbifold Euler characteristic and the homological Euler characteristic are the same in this case.
- If $\Gamma'' \subset \Gamma'$ are arithmetic subgroups such that $[\Gamma', \Gamma'']$ is finite then

$$\chi(\Gamma') = \frac{\chi(\Gamma'')}{[\Gamma', \Gamma'']}.$$

As a final remark, here is a list of two cases in which this theory is very useful in calculating the cohomology of the arithmetic group:

- If λ is not self dual, then $H_!^\bullet(\partial X_\Gamma, \widetilde{\mathcal{M}}_\lambda) = 0$. Therefore

$$H^\bullet(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = H^\bullet_{Eis}(\partial X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \cong H^\bullet_{inf}(\partial X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$$

  and one only has to calculate the dimension of $H^\bullet_{inf}(\partial X_\Gamma, \widetilde{\mathcal{M}}_\lambda)$.
- There are positive integers $a, b \in \mathbb{N}$ such that $H^k_{cusp}(\partial X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ could only be nonzero if $a \leq k \leq b$ (see for example [10]). Even more, in some cases the cuspidal cohomology is concentrated in one degree k. In those cases, one has

$$(-1)^k dim_{\mathbb{C}}(H^k_{cusp}(X_\Gamma, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})) = \chi_h(\Gamma, \mathcal{M}_\lambda) - \chi_{Eis}(\Gamma, \mathcal{M}_\lambda).$$

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Do Overtwisted Contact Manifolds Admit Infinitely Many Periodic Reeb Orbits?

Cédric Oms

Abstract In this note we discuss an approach to prove existence of infinitely many periodic Reeb orbits in overtwisted contact manifolds. The result is a combination of a plug-like construction and an adaptation of Hofer's J-holomorphic curve techniques in the case to b-contact manifold.

1 Introduction

The quest for existence results of periodic Reeb orbits associated to a contact form has a rich history. Weinstein conjectured in [19] that on a closed contact manifold, there is always at least one periodic Reeb orbit. The conjecture was motivated by the work of Rabinowitz [16], that proved the result for convex hypersurfaces in the standard symplectic Euclidean space. Although open in full generality, the question has seen steady but groundbreaking progress over the last three centuries, opening the door to influential techniques that are considered to be classical nowadays. One of the biggest steps in the mentioned development was carried out by Hofer [11], who established a narrow interplay between certain J-holomorphic curves and Reeb dynamics, applicable for a large class of contact manifolds, namely overtwisted ones. One century later, the conjecture was proved by [18] in full-generality in dimension 3.

A refinement of Weinstein conjecture is about the minimal number of periodic Reeb orbits. There are examples of compact contact manifolds exhibiting only finitely many periodic Reeb orbits, namely ellipsoids of irrational axis in the standard symplectic space. The only other examples known to the author are quotients thereof to lens spaces. The authors [1] proved the following dichotomy: in dimension 3, there are two or infinitely many periodic orbits. In the sequel [2], it is shown that there

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are always infinitely many, under the condition that the contact form satisfies some non-degeneracy condition and being torsion-free.

It is well-known that the two examples of contact manifold admitting finitely many periodic Reeb orbits are opposite of being overtwisted, namely *tight*.

Do compact overtwisted contact manifolds always admit infinitely many periodic Reeb orbits?

The novelty of this question is that there is no condition of non-degeneracy or being torsion-free involved. This short note outlines a strategy to attack this question which looks as follows. Succeeded by a brief introduction to the necessary definitions in contact geometry, we will generalize this notion to distributions which are everywhere non-integrable away from given hypersurface. The hypersurface however consists of an integrable submanifold. Those manifolds are called b-contact manifold, where b stands for *boundary*. We then prove that there exists a trap-like construction for those kind of manifolds and explain how this possibly can be generalized to a plug. This would prove that Weinstein conjecture as such is not true for this generalization of contact manifolds. We then outline a result of work in progress of [14] to adapt Hofer's machinery in this setting. A combination of those results would give a complete answer to the main question of this note.

2 Contact Geometry

In this section, we cover the basics of contact geometry. For more details see [6].

Definition 1 Let M be a manifold of dimension $2n+1$. A 1-form α is contact if $\alpha \wedge (d\alpha)^n \neq 0$. The hyperplane distribution given by $\xi = \ker \alpha$ is called a contact structure.

Note that the condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that ξ is at the opposite of being integrable (in the Frobenius sense). We include two examples.

- $\mathbb{R}^{2n+1}$ with the 1-form given by $\alpha_{\text{st}} = dz + \sum_{i=1}^n x_i dy_i$. This form satisfies the contact condition. This example is important: Darboux theorem states that all contact forms are locally diffeomorphic to this one and is therefore called standard contact form.
- Consider the unit sphere S^3 in the standard Euclidean space $(\mathbb{R}^4, \omega_{\text{st}})$. It can be shown that $\alpha = \omega_{\text{st}}(r\frac{\partial}{\partial r}, \cdot)$ is a contact form, where $r\frac{\partial}{\partial r}$ is the radial vector field.

Associated to a contact form α, there is a unique vector field R_α defined by the equations $\iota_{R_\alpha} d\alpha = 0$ and $\iota_{R_\alpha}\alpha = 1$, called the *Reeb vector field*. The Reeb vector field associated to the first example is just the linear vector field $\frac{\partial}{\partial z}$ so it does not admit any periodic orbit. In the second example the Reeb vector field defines the Hopf fibration, so all the orbits are closed.

Conjecture 2 (*Weinstein conjecture*) Let (M, α) be a compact contact manifold. Then the vector field R_α admits at least one periodic orbit.

As explained before, historically speaking an influential proof for this conjecture was carried out for overtwisted contact manifolds.

Definition 3 A 3-dimensional contact manifold $(M, \xi = \ker \alpha)$ is called overtwisted if there exists a an embedded disk F^2 such that the boundary of $T\partial F \subset \xi|_{\partial F}$ and $TF \cap \xi$ defines a 1-dimensional foliation except on a unique elliptic singular point $e \in \text{int}D$ with $T_e F = \xi_p$. The disk F is called overtwisted disk and we denote the overtwisted disk without the elliptic point by F^*.

Overtwisted contact structures play an important role in the classification of contact structures, see [4]. The two examples mentioned earlier do not admit an overtwisted disk and are therefore called *tight*. The only known example of compact contact manifold with finitely many periodic Reeb orbits are tight. An example is given by perturbing the contact form of the example on S^3.

Example 4 Consider the unit sphere S^3 in the symplectic manifold $(\mathbb{R}^4, \omega = dx_1 \wedge dy_1 + \epsilon dx_2 \wedge dy_2)$ where ϵ is irrational. As before, the contact form is given by contracting the symplectic form with the radial vector field. The only two Reeb orbits that are preserved under the change of the contact form are the ones given in the (x_i, y_i)-plane.

3 *b*-Contact Manifolds

In this section we give a generalization of contact structures, introduced in [13]. The generalization consists in a distribution that is contact away from a given hypersurface Z. However, Z is an integrable submanifold of the distribution. The case where $Z = \emptyset$ covers the contact case.

To formalize this definition, it is useful to work in the setting of vector fields that are tangent to Z and to extend differential calculus for those vector fields, as was done by Melrose [15] and later in [10].

Assume that the hypersurface Z in the manifold M is defined by the equation $f = 0$ for $f \in C^\infty(M)$. Locally, the vector fields that are tangent to Z are spanned by

$$\langle f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{m-1}}\rangle$$

where $m = \dim M$. By a theorem of Serre–Swan [17], there exists a vector bundle whose sections are given by those vector fields. We denote this vector bundle of rank m by ${}^bTM$ and call it b-tangent bundle. Here b stands for boundary. We denote the dual of the b-tangent bundle by ${}^bT^*M$, the b-cotangent bundle. Sections of wedge products of this bundle are called b-forms and denoted ${}^b\Omega^k(M)$. We put the structure of graded differential algebra on ${}^b\Omega(M)$ by expressing b-forms in terms of smooth forms and extending the de Rham differential using the following lemma.

Lemma 5 ([10]) *Let* $\omega \in {}^b\Omega^k(M)$. *Then there exists* $\alpha \in \Omega^{k-1}(M)$ *and* $\beta \in \Omega^k(M)$ *such that* $\omega = \frac{df}{f} \wedge \alpha + \beta$.

The differential on ${}^b\Omega(M)$ is then defined by $d\omega = \frac{df}{f} \wedge d\alpha + d\beta$. We are now able to formulate the definition of b-contact manifolds.

Definition 6 A manifold M of dimension $2n + 1$ with a marked hypersurface Z is b-contact if there exists $\alpha \in {}^b\Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n \neq 0$.

The differential in the definition is of course the one just defined for b-forms and the expression being non-zero needs to be understood as section of $\Lambda^{2n+1}({}^bT^*M)$. The Reeb vector field is defined analogously by the same equations, however it is important to note that the Reeb vector field associated to a b-contact form can vanish.

Example 7 Consider $\mathbb{R}^3$ with the b-form $\alpha = \frac{dz}{z} + xdy$. The Reeb vector field is given by $R_\alpha = z\frac{\partial}{\partial z}$ and is vanishing at Z.

This behaviour is in stark contrast to the non-vanishing of the Reeb vector field makes one wonder whether or not Weinstein conjecture may still hold for b-contact manifolds. In particular, relaxing the contact condition may permit the existence of plugs for the associated Reeb flow as we will discuss in the next section.

4 It Is a Trap

The theory of plugs has established many counterexample of compact manifolds equipped with a flow satisfying some geometric properties without admitting any periodic orbit, see for instance [9, 12]. The idea of a plug is to locally alter the flow and thereby break periodic orbits, without changing the global dynamics. Care needs to be taken to make this local modification satisfy geometry of the flow.

Definition 8 A *trap* is a smooth vector field on the manifold $D^{n-1} \times [0, 1]$ such that

(1) the flow of the vector field is given by $\frac{\partial}{\partial t}$ near the boundary of $\partial D \times [0, 1]$, where t is the coordinate on $[0, 1]$;
(2) there are no periodic orbits contained in $D \times [0, 1]$;
(3) the orbit entering at the origin of the disk $D \times \{0\}$ does not leave $D \times [0, 1]$ again.

If the vector field additionally satisfies *entrance-exit matching condition*, that is that the orbit entering at $(x, 0)$ leaves at $(x, 1)$ for all $x \in D \setminus \{0\}$, then the trap is called a *plug*.

Using the flow-box theorem for non-vanishing vector, traps and plugs can be locally inserted. It is clear that, due to the matching condition, plugs do not alter the global dynamics. Placing a plug at an isolated periodic orbit breaks the periodic orbit without creating any new ones. Traps, however, may alter the global dynamics and possibly create new periodic orbits.

It follows from the positive results of Weinstein conjecture that plugs can not exist for the Reeb vector field. Furthermore it was proved that traps do not exist in dimension 3 [5], but do exist in higher dimensions [7].

In the case of b-contact manifolds however, the construction of traps is true in any dimension.

Theorem 9 ([13]) *There exists b-contact traps in any dimension.*

The idea of the proof is to introduce a sphere in the standard contact form and to realise it as the critical hypersurface of a b-contact form. The b-contact form agrees with the standard contact form outside of a local neighbourhood of the sphere. This construction relies heavily on Giroux's convex hypersurface theory [8]. It can be checked that the altered Reeb vector field has an orbit that gets trapped.

Nevertheless, the matching condition is not satisfied and the global dynamics maybe altered. However the authors in [13] believe that the presence of singularities in the contact form and the Reeb vector field should make it possible to control this dynamics.

Conjecture 10 There exists b-contact plugs in any dimension.

In what follows, we will discuss the consequences of b-contact plugs. First of all, this construction would imply that Weinstein conjecture, stated as such, would not hold: indeed any example of contact manifold admitting finitely many periodic Reeb orbits could be changed to a b-contact manifold without any periodic Reeb orbits

We end this note with discussing the repercussion on the main question. □

5 Overtwisted Disk in b-Contact Manifolds

In this last section, we discuss the presence of overtwisted disk in b-contact manifolds and discuss the techniques introduced in [11] in this set-up. The result are contained in an upcoming paper [14]. First, we say that a b-contact manifold is overtwisted if there exists an overtwisted disk away from the hypersurface Z. Following [11], the elliptic singularity of the overtwisted disk gives rise to a family of pseudoholomorphic curves in the symplectization of the b-contact manifold. A careful analysis, similar to the original techniques, shows that this family either limits to either a finite energy plane away from the hypersurface, or either gives rise to a 1-parametric family of finite energy planes in the neighbourhood of the hyperplane, which is due to the $\mathbb{R}$-invariance in the direction of the hyperplane. Hence, compact overtwisted b-contact manifolds admit either a periodic Reeb orbits away from Z or a 1-parametric family of periodic Reeb orbits approaching Z.

This result is non-trivial because of non-compactness issues. However, the authors can deal with non-compactness using an approach resembling [3].

Surprisingly, this result in combination with the existence of plugs as conjectured would answer the main question of this short note. Indeed, assume that there is a compact overtwisted contact manifold with finitely many periodic Reeb orbits. The plug can be successively introduced away from the overtwisted disk to break every periodic Reeb orbit. This would result in a compact overtwisted b-contact manifold without any periodic Reeb orbits, which contradicts the main result of this section.

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Topological Radiomics (TOPiomics): Early Detection of Genetic Abnormalities in Cancer Treatment Evolution

Debora Gil, Oriol Ramos, and Raquel Perez

Abstract Abnormalities in radiomic measures correlate to genomic alterations prone to alter the outcome of personalized anti-cancer treatments. TOPiomics is a new method for the early detection of variations in tumor imaging phenotype from a topological structure in multi-view radiomic spaces.

1 Introduction

In the era of precision medicine, cancer therapies are tailored to the specific genetic makeup of a tumour. A main challenge during treatment is the early detection of variations in tumour phenotype that might alter the expected outcome. Radiomics [1] is an emerging area that converts medical imaging data into large amount of multiview measures (imaging phenotype) of the whole tumour correlated with genomics. Although abnormal radiomic features could be predictive early response biomarkers to cancer treatments, there are no methods specifically developed for detection of abnormalities (outliers). There are two main types of outliers in radiomic multi-view spaces [2]. Samples with inconsistent features with respect their class population (class outliers associated to a change in the mutation type) and samples with abnormal feature values not expected for any of the classes (attribute outlier associated to new unseen mutations).

Detection of abnormal radiomic features should model multi-view spaces with Small Sample Size (SSS) data prone to have a complex manifold structure. A main

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pitfall in current state of the art is the use of generic machine learning and statistical tools borrowed from other fields of application which fall short under the specific requirements of radiomics [3].

Existing methods for detection of outliers can be categorized into global approaches and local approaches. Global methods are population based and model the distribution in the feature space of a set of (annotated) samples. Global approaches are bad posed in the case of SSS unbalanced problems, which are common in many application areas like clinical decision support systems or personalized models. Local methods are based on a description of the structure of each sample's neighbors in the feature space. These description is used to compute measures of outlierness. A delicate requirement is the definition of sample's neighborhoods, which is mostly based on Euclidean distances. Such an approach can fail in the case of SSS problems in high dimensional spaces, which are prone to be arranged as a topological manifold.

The goal of TOPiomics is the early detection of variations in tumour imaging phenotype using a topological signature of abnormality obtained from the topological structure of SSS data in multi-view radiomic spaces.

2 Methods

TOPiomics is a local approach based on the communities (group of nodes with a given specific connectivity) of a graph encoding the structure of radiomics feature space. Features are given by quantities extracted from medical scans prone to correlate to treatment outcome, referred to as label. In the context of radiomics multimodal representations, there are two types of outliers: attribute outliers and class outliers. Attribute outliers are samples with abnormal feature values not expected for any of the classes, while class outliers are samples labelled differently across views.

Figure 1 sketches the main steps of TOPiomics. First, for each radiomic view (like the one shown in Fig. 1a), we encode the local structure of samples using the graph representing their mutual k-nearest neighbor (Fig. 1b). Second, we use methods for dynamical analysis of social networks to compute the graph communities (Fig. 1c) that define a set of neighborhoods. Isolated nodes not belonging to any community are attribute outliers, while class outliers should belong to communities with an heterogeneous distribution of labels. Finally, we define a local measure of abnormality from several probabilistic measures (Fig. 1d) of each sample heterogeneity computed in its set of neighborhoods.

The graph is given by the adjacency matrix of the mutual k-nearest neighbor of a set of samples. Let $D := \{(\mathbf{V}^i, \ell_{\mathbf{V}^i}) | \mathbf{V}^i = (v_1^i, \dots, v_n^i) \in \mathbb{R}^n, \ell_{\mathbf{V}^i} \in \{1, \dots, n_l\}\}_{i=1}^N$ be a set of N labelled points in an n-dimensional feature space endowed with a distance, namely d. For any positive integer, k, let $\text{kNN}(\mathbf{V}^i)$ denote the set of $\mathbf{V}^i$ k-nearest neighbors. Then, the graph connectivity is given by the following adjacency matrix:

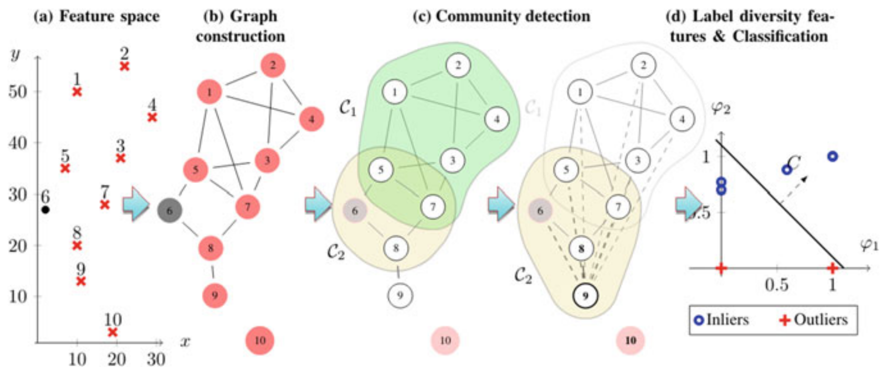

Fig. 1 TOPiomics workflow

$$a(\mathbf{V}_i, \mathbf{V}_j) = \begin{cases} \frac{1}{d(\mathbf{V}_i, \mathbf{V}_j)+1} & \text{if } \mathbf{V}_j \in \text{kNN}(\mathbf{V}_i) \text{ and } \mathbf{V}_i \in \text{kNN}(\mathbf{V}_j) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $d(\mathbf{V}_i, \mathbf{V}_j)$ the distance between $\mathbf{V}_j$ and $\mathbf{V}_i$.

In order to alleviate the impact of the parameters (the number of neighbors in this case) involved in the computation of (1), communities are computed using criteria for dynamic computation of communities [4] to extend an initial set of communities. The initial communities are given by Percolation clusters [5] which are defined as maximal unions of adjacent k-cliques (fully connected subgraphs of order k sharing (k-1)-nodes). Percolation communities are prone to exclude many points that are not actual attribute outliers [6]. An isolated node, $\mathbf{W}$, is added to an initial community, $\mathcal{C}$, if it fulfills that:

$$CS(\mathcal{C}, \mathbf{W}) \geq \delta IC(\mathcal{C}) \tag{2}$$

for $\delta \in [0, 1]$ a tolerance parameter, $IC(\mathcal{C})$ a measure of the community internal connectivity and $CS(\mathcal{C}, \mathbf{W})$ a measure of the connectivity between $\mathbf{W}$ and the community $\mathcal{C}$. Both measures are computed from a function of the degree of the community nodes as follows.

Let $G^{\mathcal{C}}$ be the subgraph induced by $\mathcal{C}$ and G^{σ} the subgraph induced by all nodes that belong to the set, namely σ, of initial communities. Then, for all $\mathbf{V} \in \mathcal{C}$ we can define the following function, $\rho_{\mathcal{C}}(\mathbf{V})$, measuring its belongingness to the community:

$$\rho_{\mathcal{C}}(\mathbf{V}) := \frac{\deg^{\mathcal{C}}(\mathbf{V})}{\deg^{\sigma}(\mathbf{V})} \tag{3}$$

being $\deg^{\mathcal{C}}(\mathbf{V})$ the degree of $\mathbf{V}$ in $G^{\mathcal{C}}$ and $\deg^{\sigma}(\mathbf{V})$ the degree of $\mathbf{V}$ in G^{σ}. The measure of $\mathcal{C}$ internal connectivity is defined from $\rho_{\mathcal{C}}(\mathbf{V})$ as:

$$IC(\mathcal{C}) := \sum_{\mathbf{V} \in \mathcal{C}} \rho_{\mathcal{C}}(\mathbf{V}) \tag{4}$$

Table 1 Assessment of Performance

DataSet	Method	Outlier configuration		
		2-8	5-5	8-2
Iris	HOAD	0.167 ± 0.057	0.309 ± 0.063	0.430 ± 0.055
	DMOD	0.909 ± 0.044	0.831 ± 0.038	0.799 ± 0.068
	TOPiomics	0.975 ± 0.024	0.971 ± 0.023	0.97 ± 0.021
Breast	HOAD	0.538 ±0.027	0.597 ± 0.038	0.643 ± 0.008
	DMOD	0.657 ± 0.017	0.720 ± 0.013	0.799 ± 0.016
	TOPiomics	0.838 ± 0.022	0.897 ± 0.020	0.91 ± 0.014
Ionosphere	HOAD	0.489 ± 0.079	0.477 ± 0.072	0.444 ± 0.065
	DMOD	0.818 ± 0.018	0.787 ± 0.039	0.784 ± 0.037
	TOPiomics	0.854 ± 0.019	0.827 ± 0.025	0.791 ± 0.036

The measure of the connectivity between $\mathbf{W}$ and $\mathcal{C}$ is defined from $\rho_{\mathcal{C}}(\mathbf{V})$ as:

$$CS(\mathcal{C}, \mathbf{W}) := \sum_{\mathbf{V} \in \mathcal{C}} \rho_{\mathcal{C}}(\mathbf{V}) \frac{1}{d(\mathbf{W}, \mathbf{V}) + 1} = \sum_{\mathbf{V} \in \mathcal{C}} \rho_{\mathcal{C}}(\mathbf{V}) a(\mathbf{W}, \mathbf{V}) \tag{5}$$

For the final measure of outlierness, we define a 2-dimensional feature space given by functions of node label entropy and probability in the communities it belongs to. Functions are normalized in $[0, 1]$ in such a way that inliers correspond to values around $(1, 1)$. A classifier provides our final score of outlier-ness.

3 Experiments

TOPiomics performance has been assessed in UCI[1] datasets altered to have different % of attribute and class outliers. We have followed the experimental settings described in [2]. In particular, we considered 3 combinations of percentages in attribute and class outliers ($\{(8\%, 2\%), (5\%, 5\%), (2\%, 8\%)\}$) and a multi-view setting. For each outlier configuration, we repeated the experiment 30 times for statistical analysis of results. TOPiomics has been compared to the state-of-art methods reported in [2] in terms of Area Under the ROC Curve (AUC).

Table 1 reports a statistical summary (average ± standard deviation) for the results obtained for TOPiomics, HOAD [7] and DMOD [2] in Iris (2-views), Breast (3-views) and Ionosphere (3-views) UCI datasets. Ranges indicate that TOPiomics is a better performance regardless database and outlier configuration.

[1] https://archive.ics.uci.edu/ml/datasets.php.

4 Conclusions

TOPiomics description is able to model the complex structure of radiomics SSS multi-view data. Its non-parametric local description endows TOPiomics with high robustness to detect abnormalities in SSS contexts, while its view-sensitive approach allows early detection of abnormal imaging phenotypes. Therefore, TOPiomics could be a unique specific technique to define robust imaging biomarkers for outcome in cancer treatment follow-up that will improve cancer patients care by optimizing treatment selection and sequence.

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Mixed Multiplier Ideals and the Topological Type of a Plane Curve

Ferran Dachs-Cadefau

Abstract Järviletho (see (Mem Am Math Soc 214(1009):viii+78 pp, (2011) [5])) and Tucker (see (Jumping numbers and multiplier ideals on algebraic surfaces, PhD thesis, University of Michigan (2010), [8])) studied in their respective Ph.D. Thesis the relation between the jumping numbers of a unibranch plane curve and its topological type. In this paper we study if we can infer the topological type of a general plane curve from its associated jumping walls.

1 Introduction

Järviletho presented in [5] a formula on how to infer the topological type of a unibranch plane curve based on its associated jumping numbers. Later, Tucker presented in [8] an example that this result no longer holds if we drop the condition of unibrach curve. The main goal of this paper is to understand the topological information of a general plane curves that can be deduced from its Jumping Walls, and what is the minimal information needed to determine it from its Jumping Walls.

2 Some Definitions

In this section we will present the definitions that we need for this short paper. Further insight can be found in [1, 2]. For this we will consider X to be a smooth complex surface and $\mathfrak{m} = \mathfrak{m}_{X,O}$ the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at a point O.

Given a tuple of curves $\mathfrak{a}=((f_1),\dots,(f_r)) \subseteq (\mathcal{O}_{X,O})^r$ with f_i irreducible, we will consider a common *log-resolution*, that is, a birational morphism $\pi : X' \to X$ such that

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- X' is smooth,
- the preimage of each of the (f_i) are locally principal, i.e., $(f_i) \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisor F_i for $i = 1, \ldots, r$, and
- $\sum_{i=1}^{r} F_i + E$ is a divisor with simple normal crossings, where $E = Exc(\pi)$ is the exceptional locus.

Since the point O is smooth in X, the exceptional locus E is a tree of smooth rational curves $E_1, \ldots, E_s$. Moreover, the intersection matrix $\left(E_i \cdot E_j\right)_{1 \le i,j \le s}$ is negative-definite. For any exceptional component E_j, we define the *excess* of (f_i) at E_j as $\rho_{i,j} = -F_i \cdot E_j$. We also recall the following notions:

- A component E_j of E is a *rupture* component if it intersects at least three more components of E (different from E_j).
- We say that E_j is *dicritical* if $\rho_{i,j} > 0$ for some i. Such components correspond to *Rees valuations* (see [6]).

Given a tuple of curves $\mathfrak{a} := ((f_1), \ldots, (f_r)) \subseteq (\mathcal{O}_{X,O})^r$ and a point $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r_{\ge 0}$, the corresponding *mixed multiplier ideal* is defined as[1]

$$\mathcal{J}(\mathfrak{a}) := \mathcal{J}\left((f_1)^{\lambda_1} \cdots (f_r)^{\lambda_r}\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_r F_r \rceil\right)$$

where the *relative canonical divisor* K_π is a divisor in X' defined as

$$K_\pi = K_{X'} - \pi^*(K_X) = \sum_{i=1}^{s} k_i E_i \in Div(X')$$

where $K_{X'}$ and K_X are the canonical divisors of X' and X respectively. K_π can be computed using the adjunction formula. As usual $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the operations of taking the *round-down* and *round-up* of a given $\mathbb{Q}$-divisor. The case $r = 1$ correspond to the usual *multiplier ideals*.

Associated to any point $\boldsymbol{c} \in \mathbb{R}^r_{\ge 0}$, we consider:

- The *region* of $\boldsymbol{c}$: $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c}) = \left\{\boldsymbol{c}' \in \mathbb{R}^r_{\ge 0} \,\middle|\, \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}'}\right) \supseteq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}}\right)\right\}$
- The *constancy region* of $\boldsymbol{c}$: $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{c}) = \left\{\boldsymbol{c}' \in \mathbb{R}^r_{\ge 0} \,\middle|\, \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}'}\right) = \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{c}}\right)\right\}$

The boundary of the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c})$ is what we call the *jumping wall* associated to $\boldsymbol{c}$. One usually refers to the jumping wall of the origin as the *log-canonical wall*. It follows from the definition of mixed multiplier ideals that the jumping walls must lie on *supporting hyperplanes* of the form

$$V_{j,\ell} : e_{1,j} z_1 + \cdots + e_{r,j} z_r = \ell + k_j, \tag{1}$$

for $j = 1, \ldots, s$ and a suitable $\ell \in \mathbb{Z}_{>0}$. Here we assume that the effective divisors F_i such that $(f_i) \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$, for $i = 1, \ldots, r$, are of the form $F_i =$

[1] By an abuse of notation, we will also denote $\mathcal{J}(\mathfrak{a})$ its stalk at O so we will omit the word "sheaf" if no confusion arises.

$\sum_{j=1}^{s} e_{i,j} E_j$. Notice that each supporting hyperplane $V_{j,\ell}$ is associated to a component E_j. Indeed, we may find other components associated to the same hyperplane, that is, we may find E_i and $\ell' \in \mathbb{Z}_{>0}$ such that $V_{j,\ell} = V_{i,\ell'}$.

It is proved in ([1], Theorem 3.3) that the region $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c})$ is (the interior of) a *rational convex polytope* defined by the inequalities

$$e_{1,j} z_1 + \cdots + e_{r,j} z_r < k_j + 1 + e_j^{\boldsymbol{c}}\,, \; for \; j = 1, \ldots, s$$

corresponding to either rupture or dicritical divisors E_j or a divisor intersecting any of the strict transforms and $D_{\boldsymbol{c}} = \sum e_j^{\boldsymbol{c}} E_j$ is the antinef closure of $\lfloor c_1 F_1 + \cdots + c_r F_r - K_\pi \rfloor$.

The intersection of the boundary of a connected component of a constancy region $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{c})$ with a supporting hyperplane of $\mathcal{R}_{\mathfrak{a}}(\boldsymbol{c})$ is what we call a $\mathcal{C}$-facet of $\mathcal{C}_{\mathfrak{a}}(\boldsymbol{c})$. Every facet of a jumping wall decomposes into several $\mathcal{C}$-facets associated to different mixed multiplier ideals.

The main result of [1] is an algorithm to compute all the constancy regions, and their corresponding mixed multiplier ideals, in any desired range of the positive orthant $\mathbb{R}^r_{\geq 0}$. In particular the set of jumping walls of $\mathfrak{a}$, that we will denote from now on as $\mathbf{JW}_{\mathfrak{a}}$, is precisely described. The points on the jumping walls, which we will denote with $\boldsymbol{\lambda}$ when we want to emphasize this fact, satisfy that $\mathcal{J}(\mathfrak{a}^{\boldsymbol{c}}) \supsetneq \mathcal{J}\left(\mathfrak{a}^{\boldsymbol{\lambda}}\right)$ for all $\boldsymbol{c} \in \{\boldsymbol{\lambda} - \mathbb{R}^r_{\geq 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ and $\varepsilon > 0$ small enough. In the sequel, we will refer to these points as the *jumping points* of the tuple of ideals $\mathfrak{a}$.

3 Topological Type

In this section we will briefly discuss the relation between the equisingularity class and the jumping numbers and jumping walls. We begin recalling the following result:

Theorem 1 (Järviletho [5]) *The jumping numbers of a unibranch plane curve C determine its equisingularity class.*

The proof of this result is constructive, namely, given a set of jumping numbers one can characterize the equisingularity class of the curve. However this result does no longer hold when we drop the condition of being unibranch. Namely, consider the following two curves given by Tucker in [8]:

- $C_1 = \{(y^5 - x^2)(y^3 - x^2)(y^3 - x^4)(y^2 - x^7) = 0\}$
- $C_2 = \{(y^5 - x^2)(x^3 - y^2)(x^3 - y^4)(y^2 - x^7) = 0\}$

Even though they have the same jumping numbers, they are not in the same equisingularity class. This may lead to the following question: "Do the jumping numbers of the germ of a plane curve determine the equisingularity classes of its branches?" (Tucker [8]). The answer to this question is no, as we could see from the following example:

- $C_1 = \{(y^4 - 2x^3y^2 + x^6 - 4x^{10}y - x^{17})(x^4 - 2x^2y^5 - 4xy^8 + y^{10} - y^{11}) = 0\}$
- $C_2 = \{(y^4 - 2x^3y^2 + x^6 - 4x^9y - x^{15})(x^4 - 2x^2y^5 + y^{10} - 4xy^9 - y^{13}) = 0\}$

The branches composing C_1 are not equisingular to any branch composing C_2, but both C_1 and C_2 have the same jumping numbers with the same associated multiplicities.

Our question is then the following:

Given a set of jumping walls, under which assumptions can one determine the equisingularity class of a plane curve?

It seems clear that a first assumption should be that every axis represents a unibranch curve, because, as already mentioned, in the opposite case it is not possible to determine the equisingularity class of each branch. In this case, using the results of Järviletho, it is clear that the only piece of information missing is the intersection number of the branches (see for example [3]).

Theorem 2 *The intersection multiplicity of two curves C_1 and C_2 is equal to the multiplicity of the branch C_1 in the exceptional divisor E_i such that $E_i \cdot \tilde{C}_2 \neq \emptyset$, where $\tilde{C}_2$ is the strict transform of C_2. Or equivalently, to the multiplicity of the branch C_2 in the exceptional divisor E_i such that $E_i \cdot \tilde{C}_1 \neq \emptyset$, where $\tilde{C}_1$ is the strict transform of C_1.*

Therefore, one can recover the topological type of a plane curve as follows: from the jumping numbers of each unibranch curve, i.e., the points of each axis, one can recover the topological type by using the results of Järviletho. Each of those jumping numbers are associated to a divisor. For a fixed unibranch curve C_j, consider the first jumping number associated to the divisor E_i satisfying $E_i \cdot \tilde{C}_j \neq \emptyset$, for this jumping number determine the hyperplane containing it. Using the previous Theorem, the intersection number with the other curves are the coefficients of the z_i's. Therefore, we have the following:

Theorem 3 *The jumping walls determine the equisingularity class of a plane curve.*

To show how it works, we consider the following example

Example 4 Assume that we have the jumping walls of Fig. 1. The cuts of the jumping walls with the axis, i.e., the jumping numbers of C_1 and C_2 are respectively:

- $\left\{\frac{5}{18}, \frac{25}{66}, \frac{14}{33}, \frac{31}{66}, \frac{17}{33}, \frac{37}{66}, \frac{20}{33}, \frac{11}{18}, \frac{43}{66}, \frac{23}{33}, \frac{47}{66}, \frac{49}{66}, \frac{25}{33}, \frac{26}{33}, \frac{53}{66}, \frac{5}{6}, \frac{28}{33}, \frac{29}{33}, \frac{59}{66}, \frac{61}{66}, \frac{31}{33}, \frac{17}{18}, \frac{32}{33}, \frac{65}{66}, \dots\right\}$
- $\left\{\frac{5}{24}, \frac{35}{124}, \frac{39}{124}, \frac{43}{124}, \frac{47}{124}, \frac{51}{124}, \frac{55}{124}, \frac{11}{24}, \frac{59}{124}, \frac{63}{124}, \frac{33}{62}, \frac{67}{124}, \frac{35}{62}, \frac{71}{124}, \frac{37}{62}, \frac{75}{124}, \frac{39}{62}, \frac{79}{124}, \frac{41}{62}, \frac{83}{124}, \frac{43}{62}, \dots\right\}$

From them, we get that the topological type of the first one has two Puiseux pairs that are $\{2, 3\}$, $\{3, 7\}$, and the second has $\{2, 3\}$, $\{4, 11\}$. Now we have to pick the last jumping number needed to determine the topological type of one of the curves, namely $\frac{25}{66}$ for the first one. It is contained in the hyperplane

$$66z_1 + 84z_2 = 25$$

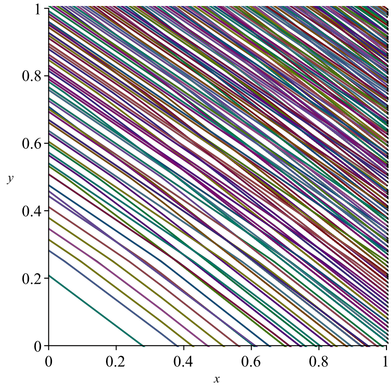

Fig. 1 Jumping Walls

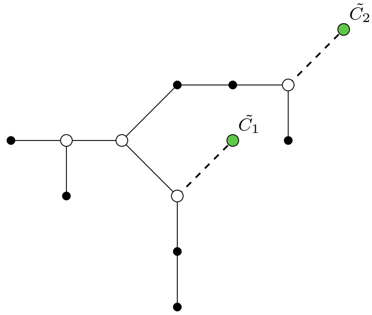

Fig. 2 Dual graph of the product of C_1 and C_2

therefore, the intersection multiplicity is 84. So, the product of C_1 and C_2 has the dual graph of Fig. 2, where the rupture divisors are represented by white dots, while the strict transforms are represented by green dots.

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Geometry of Non-holonomic Distributions

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Abstract We consider a non integrable regular distribution $\mathcal{D}$ in a Riemannian manifold (M, g). Using the Levi-Civita connection in M we extend the geometric notions of fundamental forms, curvature and geodesic curves from submanifolds of (M, g) to the distribution $\mathcal{D}$ and characterize the totally geodesic distributions in several ways.

1 Introduction

A regular distribution in a differentiable manifold M is a subbundle $\mathcal{D}$ of the tangent bundle $\mathrm{T}M$ of M. Integrability of distributions can be studied using Lie brackets of the set of sections $\Gamma(\mathcal{D})$. By Frobenius theorem it is known that the distribution $\mathcal{D}$ is integrable if and only if $\Gamma(\mathcal{D})$ is closed under the Lie bracket operation. The non-integrability implies the non existence of integral submanifolds whose dimension equals the rank of the subbundle, that is submanifolds $S \subset M$ such that its tangent space at every point $p \in S$ is $\mathcal{D}_p \subset \mathrm{T}_pM$. They are called maximal integral submanifolds of $\mathcal{D}$.

Non-integrable distributions appear in physics, were they are called non-holonomic, as constraints in the velocities in mechanical systems with some kind of contact between objects but without sliding, or in problems of control of systems were the controls can be modeled by families of vector fields, see [2, 13]. In mathematics, they appear when we have a geometric structure defined by a non closed differential form or in non null curvature situations.

If $\mathcal{D}$ is non integrable, we could think that there is no geometry as we do not have any space of points associated to the distribution. In fact this is not the real situation and it is possible to generalize the notions of fundamental forms, curvature, geodesics and others from a surface in $\mathbb{R}^3$ to non integrable distributions in a Riemannian manifold. Although we do not have points, we have tangent vectors at every point of

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the manifold, those belonging to $\mathcal{D}$, and hence vector fields, those in $\Gamma(\mathcal{D})$, and they are enough to develop the classical geometric structures associated to a submanifold.

The aim of this communication is to present some recent published results on the geometry of non integrable distributions in a Riemannian manifold. For other details see [12]. Previous approaches to this study can be seen in [5, 6, 15, 16], but in all of them the definition of the second fundamental form is different from the given here and contains only the symmetric part.

In the following section, we will generalize the first and second fundamental forms of a submanifold S of a Riemannian manifold (M, g) to a non integrable distribution; then we decompose the second one into symmetric and skew-symmetric parts and we prove that $\mathcal{D}$ is integrable if and only if the skew-symmetric component is identically zero. To study these properties we need to introduce a natural connection in $\mathcal{D}$ induced by the Levi-Civita connection in (M, g).

In Sect. 3 we study the curvature of curves in $\mathcal{D}$ and the characterization, in different ways, of the so called totally geodesic distributions. In particular we prove that a distribution $\mathcal{D}$ is totally geodesic if and only if the symmetric part of its second fundamental form is identically zero.

In the last section we comment briefly on other problems studied in [12] and possible future developments.

2 Notation, Natural Connection and Fundamental Forms

Let (M, g) be a smooth Riemannian manifold, $\dim M = m$, and ∇ the Levi-Civita connection associated to g. We denote by $\mathfrak{X}(M)$ the $\mathcal{C}^\infty(M)$-module of vector fields on M and by $\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$, the covariant derivative of Y with respect to X. All the manifolds and mappings will be regarded as being of $\mathcal{C}^\infty$ class.

Let $\mathcal{D} \subseteq TM$ be a fixed regular distribution on M with constant rank n. If necessary, we will assume that locally $\mathcal{D} = \text{span}\{X_1, \dots, X_n\}$, with $X_i \in \mathfrak{X}(M)$ linearly independent. If $p \in M$, then $\mathcal{D}_p \subset T_pM$ is the fibre of $\mathcal{D}$ at the point $p \in M$. We call $\mathcal{D}^\perp$ the **orthogonal distribution** of $\mathcal{D}$, $rank\, \mathcal{D}^\perp = m - n$. Its local generators will be denoted by $Z_1, \dots, Z_{m-n}$. Clearly, as $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, we have **natural projections** $\pi^{\mathcal{D}}$ and $\pi^{\mathcal{D}^\perp}$ from TM to $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively.

2.1 *The Metric* $g^{\mathcal{D}}$ *and the Connection* $\nabla^{\mathcal{D}}$

The **first fundamental form** of $\mathcal{D}$ is the restriction of the metric g to $\mathcal{D}$ and will be denoted by $g^{\mathcal{D}}$, a Riemannian metric on $\mathcal{D}$. Using the Levi-Civita connection ∇ in (M, g) and the projection $\pi^{\mathcal{D}}$, one can define a covariant derivative between sections of $\mathcal{D}$:

$$\nabla^{\mathcal{D}}_X Y = \pi^{\mathcal{D}}(\nabla_X Y), \quad X, Y \in \Gamma(\mathcal{D})$$

The connection $\nabla^{\mathcal{D}}$ is usually called the **intrinsic connection** of $\mathcal{D}$. It extends to all the tensor fields on the distribution and it is easy to prove that $\nabla^{\mathcal{D}}_X g^{\mathcal{D}} = 0$ for every $X \in \Gamma(\mathcal{D})$; that is $\nabla^{\mathcal{D}}$ is Riemannian with respect to $g^{\mathcal{D}}$, but **it is not torsionless** with respect to the ordinary Lie bracket unless the distribution $\mathcal{D}$ is involutive.

The initial idea of this connection goes back at least to [18, 19] where the description in geometric terms of the nonholonomic mechanical systems is studied. A modern view with some other applications can be found in [2, 4, 10].

2.2 The Second Fundamental Form (Gauss, ∼1830, Reinhart, 1977, 1983)

Consider the following $\mathcal{C}^\infty(M)$-bilinear application taking values in $\Gamma(\mathcal{D}^\perp)$:

$$B(X,Y) = \pi^{\mathcal{D}^\perp}(\nabla_X Y) = \nabla_X Y - \nabla^{\mathcal{D}}_X Y, \quad X, Y \in \Gamma(\mathcal{D}).$$

The mapping B is called the **second fundamental form** of $\mathcal{D}$. If $Z_1, \ldots, Z_{m-n}$ is a local orthonormal basis of $\Gamma(\mathcal{D}^\perp)$, then we obtain the well known Gauss formula for surfaces:

$$\nabla_X Y = \nabla^{\mathcal{D}}_X Y + \sum_{j=1}^{m-n} g(B(X,Y), Z_j) Z_j\,.$$

The second fundamental form was introduced by Gauss for a surface in $\mathbb{R}^3$ and generalized for a submanifold of a Riemannian or a pseudo-Riemannian manifold; see for example [4, 7, 8, 14, 17]. However, the oldest definition for a general distribution is given in [15].

Associated to B we have the map: $B_Z(X,Y) = g(B(X,Y), Z)$, for $Z \in \mathcal{D}^\perp$ and $X, Y \in \mathcal{D}$. Other interesting expressions for $B_Z(X,Y)$ are (L_Z is the Lie derivative with respect to Z):

$$\begin{aligned} B_Z(X,Y) &= g(B(X,Y),Z) = g(\nabla_X Y - \nabla^{\mathcal{D}}_X Y, Z) = g(\nabla_X Y, Z) \\ &= L_X(g(Y,Z)) - g(Y, \nabla_X Z) = -g(\nabla_X Z, Y) + g(\nabla_Z X, Y) - g(\nabla_Z X, Y) \\ &= g([Z,X],Y) - g(\nabla_Z X, Y) = g(L_Z X - \nabla_Z X, Y) \end{aligned}$$

The map B_Z is called the **second fundamental form along** Z.

We can decompose B_Z into its symmetric B^s_Z and skew-symmetric B^a_Z parts:

(1) $B^s_Z(X,Y) = (1/2)(B_Z(Y,X) + B_Z(Y,X)) = -(1/2)(L_Z\, g)(X,Y)$
(2) $B^a_Z(X,Y) = (1/2)(B_Z(X,Y) - B_Z(Y,X)) = -(1/2)(\mathrm{d}\,(\mathrm{i}_Z\, g))(X,Y)$
(3) $B_Z(Y,X) = B^s_Z(Y,X) + B^a_Z(Y,X)$

as can be easily proved using the different expressions above for $B_Z(X,Y)$.

The same decomposition can be made for B, being $B = B^s + B^a$, but there are not elegant expressions for these components of B as for B_Z. Observe that

$$B = 0, B^s = 0, B^a = 0 \Longleftrightarrow B_Z = 0, B^s_Z = 0, B^a_Z = 0, \ \forall Z \in \Gamma(\mathcal{D}^\perp),$$
respectively.

A direct consequence of the expression

$$(\mathrm{d}(\mathrm{i}_Z\, g))(X, Y) = -g(Z, [X, Y]), \ \forall Z \in \Gamma(\mathcal{D}^\perp), X, Y \in \Gamma(\mathcal{D}),$$

directly obtained from the definition of the exterior differential, is the following result:

Theorem 1 *The distribution $\mathcal{D}$ is involutive if and only if for every $Z \in \Gamma(\mathcal{D}^\perp)$ the tensor field B_Z is symmetric; that is, $\mathrm{d}(\mathrm{i}_Z\, g)$ is null **on** the sections of $\mathcal{D}$.*

In the classical theory of submanifolds of a Riemannian manifold, the second fundamental form is symmetric. In fact, as we have shown above, the non integrability of $\mathcal{D}$ is encoded in the skew-symmetric part of the second fundamental form.

3 Geodesics and Geodesic Invariance

Let $\gamma : I \subseteq \mathbb{R} \to M$ be a smooth curve parametrized by the arc. The **curvature** of γ in M is defined as $\mathbf{k}(\gamma) = ||\nabla_{\dot\gamma}\dot\gamma||$.

If $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in I$, we say that γ is a **curve of the distribution** $\mathcal{D}$, and we define the following functions of the parameter of the curve:

(1) The **geodesic curvature** of γ as $\mathbf{k}^{\mathcal{D}}(\gamma) = ||\nabla^{\mathcal{D}}_{\dot\gamma}\dot\gamma||$.
(2) The **normal curvature** of γ as $\mathbf{k}^{\mathcal{D}^\perp}(\gamma) = ||\pi^{\mathcal{D}^\perp}\nabla_{\dot\gamma}\dot\gamma|| = ||B(\dot\gamma, \dot\gamma)||$.

Using the second fundamental form and the Gauss formula, we have that

$$(\mathbf{k}(\gamma))^2 = (\mathbf{k}^{\mathcal{D}}(\gamma))^2 + (\mathbf{k}^{\mathcal{D}^\perp}(\gamma))^2 = (\mathbf{k}^{\mathcal{D}}(\gamma))^2 + ||B(\dot\gamma, \dot\gamma)||^2.$$

Observe that the normal curvature only depends on the symmetric part of the second fundamental form B, because we need to calculate only $B(\dot\gamma, \dot\gamma)$.

Let $\gamma : I \subseteq \mathbb{R} \mapsto M$ be a smooth curve. We say that

(1) The curve γ is **∇-geodesic** if $\nabla_{\dot\gamma}\dot\gamma = 0$.
(2) The curve γ is **$\nabla^{\mathcal{D}}$-geodesic** if $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$, for all $t \in I$, and $\nabla^{\mathcal{D}}_{\dot\gamma}\dot\gamma = 0$.

As usual, the geodesic curves are solutions to a second order ordinary differential equation whose solutions with initial condition points in TM for the ∇-geodesics or points in $\mathcal{D} \subset TM$ for the $\nabla^{\mathcal{D}}$-geodesics. The existence of solutions for such equations is a consequence of their regularity and the appropriate theorem for ordinary differential equations.

We are interested in the comparison between both classes of geodesics when they have initial condition in $\mathcal{D}$. The first relation is given by the following proposition whose proof is an easy consequence of the above formula for $(\mathbf{k}(\gamma))^2$.

Proposition 1 *Let γ be a smooth curve in the distribution, that is $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$, for all $t \in I$. Then it is a ∇-geodesic if and only if it is a $\nabla^{\mathcal{D}}$-geodesic and $B(\dot\gamma, \dot\gamma) = 0$.*

3.1 Totally Geodesic Distributions

We say that the distribution $\mathcal{D}$ is **totally geodesic** if every ∇-geodesic with initial condition in $\mathcal{D}$ is contained in $\mathcal{D}$.

For a submanifold of a Riemannian manifold this is a well known property: every geodesic with initial condition in the submanifold, lies locally in the submanifold, see [4, 8, 14, 17]. For integrable distributions it corresponds to the study of the leaves of the foliation defined by the distribution. In the non-integrable situation, apart from the geometric problems, the property is specially interesting in several other fields; for example in the study of controllability of dynamical control systems; see for instance [2] where they use the name **geodesically invariant** instead of totally geodesic. The characterization of this important property is given in the following

Theorem 2 *For a distribution $\mathcal{D}$, the following conditions are equivalent:*

(1) $\mathcal{D}$ is totally geodesic.
(2) Every $\nabla^{\mathcal{D}}$-geodesic in $\mathcal{D}$ is ∇-geodesic.
(3) The symmetric part of the second fundamental form is identically zero.
(4) If $X, Y \in \Gamma(\mathcal{D})$ then $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$.
(5) If $X \in \Gamma(\mathcal{D})$ then $\nabla_X X \in \Gamma(\mathcal{D})$.

A sketch of the proof of this important result is the following:

(1) From the expression $\nabla_{\dot{\gamma}}\dot{\gamma} = \nabla^{\mathcal{D}}_{\dot{\gamma}}\dot{\gamma} + B(\dot{\gamma}, \dot{\gamma})$ for a curve γ in M, it is easy to prove the equivalence between the first three items.
(2) The equivalence between (4) and (5) is direct.
(3) The expression $B^s_Z(X, Y) = (1/2)(B_Z(X, Y) + B_Z(Y, X)) = g(\nabla_X Y + \nabla_Y X, Z)$, for $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, proves the equivalence between (3) and (4).

The above expression $\nabla_X Y + \nabla_Y X$, for $X, Y \in \Gamma(\mathcal{D})$ or for vector fields in a manifold M with connection, is known as the **symmetric product**. It was introduced in [3] and has been deeply studied in [2, 9–11]. For other interpretation see [1].

Condition (3) in the above theorem is not equivalent to stating that the vector fields $Z \in \Gamma(\mathcal{D}^\perp)$ are Killing vector fields for the Riemannian metric g, because it refers only to the action on sections of $\mathcal{D}$, and not for every vector field in the manifold M.

4 Other Results and Commentaries

There are other interesting problems to state and solve and we will only indicate some of them. A more detailed development can be seen in [12] and references therein.

(1) To study the notions of curvature and curvature tensors and, as we have two connections, ∇ and $\nabla^{\mathcal{D}}$, on $\Gamma(\mathcal{D})$, the comparison of both notions and consequences.
(2) To understand the different formulations of analytical mechanics with constraints in terms of the above defined second fundamental form of the distribution of constraints. In particular vakonomic mechanics.
(3) To study the actions of the group of isometries of (M, g) on the distribution and look for the existence of invariants classifying distributions under this action.

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What If It Contains a Linear Subspace?

Paola Supino

Abstract Fano schemes of k-linear subspaces of projective hypersurfaces and complete intersections have been object of study since a long time; in particular, non-emptiness conditions are known. I will report how the non-emptiness of Fano schemes for projective hypersurfaces has been a tool in showing unirationality. Moreover, I will explain a numerical condition, depending on the dimension and the multidegree, for non-emptiness of Fano schemes of k-linear subspaces of complete intersections in a projective space.

1 Introduction

Detecting the existence of linear spaces sitting inside a subvariety of a projective space is more than a mathematical curiosity, since it can reveals informations on the shape of the variety. This is evident in the simple case of the two skew pencils of lines in a quadric surface in $\mathbb{P}^3$; or of the 27 lines in the cubic surface, which carries memory of its being a blown up plane in 6 general points. Thus, as the Grassmannian $\mathbb{G}(k, m)$ is a scheme parametrizing k-linear subspaces in $\mathbb{P}^m$, in general, given a closed subvariety $X \subset \mathbb{P}^m$ and an integer $k \geq 1$, it is natural to define the subscheme $F_k(X)$ of $\mathbb{G}(k, m)$ parametrizing those $[\Pi] \in \mathbb{G}(k, m)$ such that Π is contained in X. Such a $F_k(X)$ is called a *Fano scheme* of X. Fano schemes are fundamental in projective algebraic geometry and widely investigated. Just to mention some absolutely non exhaustive application in which Fano schemes and their geometry appear, we can cite, among others, rationality issues [13, 18, 19], Torelli type theorems for smooth cubic of high dimension (cf. [10, 11, 21]), enumerative formulas [7, 9, 15], covering gonality of hypersurfaces [2]. Here we report a classical application of non-emptiness of Fano schemes for complete intersections, and numerical formulas for such a non-emptiness.

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2 Unirational Hypersurfaces and Complete Intersections

Fix an algebraically closed field k of characteristic zero, a (quasi-projective) variety X is said to be rational if there is a birational map to some $\mathbb{P}^n$. This is equivalent to say that the function field $k(X)$ of X is equal to $k(\mathbb{P}^n) = k(x_1, \dots, x_n)$ for some n; in other words, one could give a parametric representation of X by means of rational functions. More weakly, one can say that a variety X is unirational if there is a dominant rational map from $\mathbb{P}^n$ to X for some n, equivalently, $k(X)$ can be embedded into some $k(\mathbb{P}^n) = k(x_1, \dots, x_n)$.

It is very easy to see that, on the complex field, projecting a quadric hypersurface in $\mathbb{P}^m$ from one of its points (a 0-linear subspace) gives a birational map with $\mathbb{P}^{m-1}$, proving that the quadric is a rational variety. Another simple example is given by any smooth cubic hypersurface $X \subset \mathbb{P}^{2k+1}$ containing two linear subspaces L_1 and L_2 of dimension k. This one is proved to be rational via the birational morphism $p : L_1 \times L_2 \to X$ such that $p(x_1; x_2) = y$, where the points $x_1,\ x_2,\ y$ are collinear. Note that, while any smooth cubic surface in $\mathbb{P}^3$ does contain two skew lines, a dimension count shows that the general cubic fourfold in $\mathbb{P}^5$ does not contains two planes.

In general, it turns out to be very difficult to determine whether a given variety is rational, or even unirational. Refined techniques have been developed along the times, for instance involving a closer study of the group of automorphisms of varieties, or Hodge theory, which associates complex tori to varieties, in particular intermediate Jacobians [8, 22], or studying cohomological torsion groups as obstruction to rationality [1].

Going back at pure projective geometry, one can describe some interesting constructions of rational and unirational varieties, which give ourselves another motivation to look for linear subspaces in projective varieties. In fact, in the forties of last century, Ugo Morin [17, 18] suggested to use k-linear subspaces to visualize the unirationality of a general hypersurface of suitable degree in $\mathbb{P}^m$.

The idea of Morin starts with an observation on a general cubic hypersurface $X \subset \mathbb{P}^m$ ($m \geq 3$): it contains a k-linear subspace Λ (where $k = 1$, or $m - 1 > k \geq 1$ if $m \geq 6$). Consider now the $(k + 1)$-planes containing Λ: they are parametrized by a projective space $B = \mathbb{P}^{m-k-1}$. Such a general $(k + 1)$-plane meets X in the union of Λ and a residual quadric hypersurface. Thus X is birational to a quadric bundle $\mathcal{Q} \to B = \mathbb{P}^{m-k-1}$, $\mathcal{Q}$ being the blow up of X along Λ.

The total space of a quadric bundle $\mathcal{Q} \to B$ over a rational base is not necessarily rational, unless one can choose rational parametrizations of the fibers consistently over (at least) a Zariski-open subset of the base, for instance, by finding a rational section of the bundle. This would furnish a point for any quadric fiber Q_b, $b \in B$, from where to project, so birationally identifying Q_b with a projective space. In the further alternative, one can try to make a base change $\bar{\mathcal{Q}} \to B$ for $\mathcal{Q} \to B$ and construct a (uni)rational space dominating both $\mathcal{Q}$ and $\bar{\mathcal{Q}}$. As a new base $\bar{\mathcal{Q}}$ one can take the total space of the family over B of the intersections $Q_b \cap \Lambda$, then is a natural map $\mathcal{Q} \to \bar{\mathcal{Q}}$. The fibration $\mathcal{Q} \times_B \bar{\mathcal{Q}} \to \bar{\mathcal{Q}}$ is now a quadric bundle endowed with a section, in a natural way. Thus, it remains to prove that the new base $\bar{\mathcal{Q}}$ is rational, in order to have $\mathcal{Q} \times_B \bar{\mathcal{Q}}$ rational, and hence $\mathcal{Q}$ unirational, via the natural projection $\mathcal{Q} \times_B \bar{\mathcal{Q}} \to \mathcal{Q}$. In fact, it is not difficult to prove that $\bar{\mathcal{Q}}$ is rational, since it fibers over Λ as a projective bundle. Observe that, with respect to the cubic X we started with, we are dealing with a problem where the degree is one less and the ambient space has a smaller dimension.

In some sense, a hypersurface X of any degree d of $\mathbb{P}^m$ is (uni)rational, as soon as it can be given a dominant map π from X to a projective space, whose fibers are (uni)rational, and the dominant maps from projective spaces to these fibers "glue" together. We can try to perform an induction on the degree, iterating the above idea, as follows. If X contains a linear subspace Λ of dimension k, the projection π_Λ from Λ is a dominant map onto $B = \mathbb{P}^{m-k-1}$, the Zariski closure of the fiber X_b over the point $b \in B$ is the residual of Λ in the intersection of X with the $(k+1)$-linear span of Λ and b, hence a hypersurface of degree $d-1$ in $\mathbb{P}^{k+1}$. Now, to perform the iteration, we have to ask whether, for the general b, X_b itself contains a linear subspace Λ'_b of a suitable dimension (smaller than k), the projection from which gives X_b as fibered over a projective space. We also ask that Λ'_b is contained in $\Lambda \cap X_b$, so that the existence of a section for a dominating bundle, after a suitable base change, is guaranteed. A detailed description of the procedure can be found in [13].

The result is a recursive formula involving the degree d, the dimension m and the dimension k which numerically ensures the existence of linear spaces in the sequel of general hypersurfaces along the iterations. In particular, Morin in [17] asserted in 1940 that a "general" hypersurface of degree d and dimension larger than a certain (recursively given) bound $m(d)$ is unirational. By necessity, the bound is greatly far from being optimal, even in low degrees: $m(d) = 10\,(1001)$ for $d = 3\,(4)$, while it was known to Morin that the best bound is 3 (7). The technique was extended to complete intersections by Predonzan [19] in 1949. Predonzan bounds have been improved by Ciliberto in [6], and then by Ramero in [20]. The story proceeds with [13], in which the genericity hypothesis drops, at the cost of worsening the bound: any smooth hypersurface $X \subset \mathbb{P}^m$ of degree d is unirational, if $m > m(d)$, an iterated exponential with respect to d, hence still very large. This has been recently improved in [4], where the bound is $m(d) = 2^{m!}$.

It is worth to point out that all the proofs require the non-emptiness of certain Fano schemes, not only of hypersurfaces, but also of complete intersections.

3 Numerical Conditions for K-Linear Spaces in Complete Intersections

Consider now a general complete intersection $X \subset \mathbb{P}^m$ of multi-degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \le s \le m-2$. Numerical assumptions on $\underline{d}$, k and $\dim X$ can be given in order to know if $F_k(X)$ is a non-empty, reduced scheme, and to know its dimension. Take

$$S^*_{\underline{d}} := \oplus_{i=1}^{s}\left(H^0\left(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)\right) \setminus \{0\}\right).$$

For any $u := (g_1, \dots, g_s) \in S^*_{\underline{d}}$, set

$$X_u := V(g_1, \dots, g_s) \subset \mathbb{P}^m$$

the closed subscheme defined by the vanishing of the s homogeneous polynomials $g_1, \dots, g_s$. If $u \in S^*_{\underline{d}}$ is a general point, X_u is a smooth, irreducible complete intersection of dimension $m - s \geq 2$ and multi-degree $\underline{d}$.

Consider $\mathbb{G} := \mathbb{G}(k, m)$, the Grassmannian of k-linear subspaces in $\mathbb{P}^m$, and the incidence correspondence

$$J := \left\{ ([\Pi], u) \in \mathbb{G} \times S^*_{\underline{d}} \middle| \Pi \subset X_u \right\} \subset \mathbb{G} \times S^*_{\underline{d}}$$

with the two natural projections

$$\begin{array}{ccc} J & \xrightarrow{\pi_2} & S^*_{\underline{d}} \\ {\scriptstyle \pi_1}\downarrow & & \\ \mathbb{G} & & \end{array}$$

The map $\pi_1 : J \to \mathbb{G}$ is surjective and, for any $[\Pi] \in \mathbb{G}$, one has $\pi_1^{-1}([\Pi]) = \oplus_{i=1}^s \left(H^0 \left(\mathcal{I}_{\Pi/\mathbb{P}^m}(d_i) \right) \setminus \{0\} \right)$, where $\mathcal{I}_{\Pi/\mathbb{P}^m}$ denotes the ideal sheaf of Π in $\mathbb{P}^m$. Thus J is irreducible and non-empty, its dimension can be computed, and compared with $\dim S^*_{\underline{d}}$. Let

$$t := \dim S^*_{\underline{d}} - \dim J.$$

Looking now at π_2, if $u \in S^*_{\underline{d}*}$ is in the image of π_2 its preimage $\pi_2^{-1}(u)$ is the Fano scheme $F_k(X_u)$. Define

$$W_{\underline{d},\, k} := \pi_2(J) = \left\{ u \in S^*_{\underline{d}} \middle| \; F_k(X_u) \neq \emptyset \right\} \subseteq S^*_{\underline{d}}.$$

One can study $W_{\underline{d},\, k}$ and look at non-emptiness, smoothness, irreducibility, dimension and enumerative properties of $F_k(X)$, when X is a general complete intersection of a given $u \in W_{\underline{d},\, k}$. Without loss of generality, one can suppose that $\Pi_{i=1}^s d_i > 2$, that is $\underline{d} \neq (1, \dots, 1)$, the projective space case, and $\underline{d} \neq (2, \dots, 1)$, the quadric case, this being fully understood (cf. e.g. [5, Prop. 2.1, Cor. 2.2, Thm. 4.1], [9, Thm. 2.1, Thm. 4.3] and [12, Ch. 6]). It has to be distinguished when $t \leq 0$, and when $t > 0$. In fact, only if $t \leq 0$, i.e. $\dim S^*_{\underline{d}} \leq \dim J$, there is room for the projection $J \xrightarrow{\pi_2} S^*_{\underline{d}}$ to be surjective, so that the k-Fano scheme on the general complete intersection of multidegree $\underline{d}$ has room to be non-empty. In fact, it is known the following (cf. [5, 9, 17, 19])

Theorem 1 *With the above notation, let* $t \leq 0$*, then*
(a) $W_{\underline{d},\,k} = S^*_{\underline{d}}$. *In particular* $W_{\underline{d},\,k}$ *is irreducible and rational. Moreover, for any* $u \in W_{\underline{d},\,k}$, $F_k(X_u) \neq \emptyset$.
(b) If $u \in W_{\underline{d},\,k}$ *is a general point,* $X_u \subset \mathbb{P}^m$ *is a smooth complete intersection of dimension* $m - s$*, and* $F_k(X_u)$ *is smooth, of (expected) dimension*

$$\dim\ F_k(X_u) = -t = (k+1)(m-k) - \sum_{i=1}^{s} \binom{d_i + k}{k} \geq 0,$$

when $\dim\ F_k(X_u) \geq 1$ *it is also irreducible.*

For the proofs, the different authors always use vector bundle on Grassmannians and zero-loci of their global sections. In [9, Thm. 4.3], the authors also compute via Schubert calculus $\deg\ F_k(X_u)$ under the Plücker embedding $F_k(X_u) \subset \mathbb{G}(k, m) \subset \mathbb{P}^N$, extending enumerative formulas of Libgober for Fano scheme of lines, when it is zero-dimensional (cf. [14]).

If $t > 0$, that is $\dim S^*_{\underline{d}} > \dim J$, the projection $J \xrightarrow{\pi_2} S^*_{\underline{d}}$ cannot be surjective, so the k-Fano scheme on the general complete intersection of multidegree $\underline{d}$ is empty, nevertheless, something can be said on the locus $W_{\underline{d},\,k}$ in which it is not.

Theorem 2 *With the above notation, let* $t > 0$*, then*
(a) $W_{\underline{d},\,k} \subset S^*_{\underline{d}}$ *is non-empty, irreducible and rational, with*

$$\operatorname{codim}_{S^*_{\underline{d}}}\ W_{\underline{d},\,k} = t = \sum_{i=1}^{s} \binom{d_i + k}{k} - (k+1)(m-k).$$

(b) For a general point $u \in W_{\underline{d},k}$*, the variety* $X_u \subset \mathbb{P}^m$ *is a complete intersection of dimension* $m - s$ *whose Fano scheme* $F_k(X_u)$ *is zero-dimensional of length one. Moreover,* X_u *has singular locus of dimension* $\max\{-1, 2k + s - m - 1\}$ *along its unique* k*-dimensional linear subspace (in particular* X_u *is smooth if and only if* $m - s \geq 2k$*).*

The proof of theorem 2 is contained in [9, Thm. 2.1 (a)], in [16, Cor. 1.2, Rem. 3.4]; and in [2]. The strategy used in [16] involves Flag Hilbert schemes, generalized normal sheaves and infinitesimal theory. In [2] extending [3, Prop. 2.3] to any $k \geq 1$, Miyazaki's results in [16, Cor. 1.2] is improved, via easier methods, avoiding his assumption $d_i \geq 2$, for any $1 \leq i \leq s$, and showing unicity of the k-linear subspace contained in X_u, and hence easily proving the rationality of $W_{\underline{d},\,k}$.

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Reeb Embeddings and Universality of Euler Flows

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Abstract We use a new geometrical approach to the universality of Euler flows. By proving flexibility results on embeddings for Reeb flows in contact topology, we deduce some new universal properties for Euler flows. As a byproduct, we deduce the Turing completeness of stationary Euler flows, answering an open question for steady solutions. The results contained in this article are an announcement and short version of [2], where the complete list of results and proofs can be found.

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1 Introduction

1.1 Hydrodynamics

The Euler equations model ideal fluids, and can be formulated on a Riemannian manifold (M, g) as follows,

$$\begin{cases} \partial_t u + \nabla_u u & = -\operatorname{grad}_g p \\ \operatorname{div}_g u & = 0 \end{cases},$$

where p is the pressure function and u the velocity field. In the articles [9, 10] Tao anticipated that Euler flows should be flexible enough to show any kind of dynamical behavior. Instead of using an analytical approach we will use a geometrical one to prove universality properties. In particular, in [10] the extendibility of flows to Euler solutions is studied, and existence of Turing complete Euler flows is left open. Turing completeness of Euler flows means that there is a solution to the Euler equations associated to a general Riemannian manifold encoding a universal Turing machine. We will address this question in the stationary case.

If we restrict ourselves to autonomous solutions to the Euler equations in odd dimensional manifolds, a rich geometric and topological study has been developed since the monograph [1]. Denoting $\alpha = \iota_u g$ the dual form to the velocity field, the stationary Euler equations can be written,

$$\begin{cases} \iota_u d\alpha & = -dB \\ d\iota_u \mu & = 0, \end{cases}$$

where $B = P + \iota_u \alpha$ is the Bernoulli function and μ is the Riemannian volume form. This formulation establishes a dichotomy. When the Bernoulli function is non-constant (and for instance analytical or C^2 and Morse-Bott) then by Arnold's structural theorem the flow lines have a very rigid structure and dynamics are similar to those exhibited in integrable systems. However, when B is constant the dynamics can be much more complicated. These solutions satisfy that the velocity field is parallel to the vorticity field ω defined by the equation

$$\iota_\omega \mu = (d\alpha)^n.$$

We call a vector field X a **Beltrami field** if it preserves μ and is everywhere parallel to its vorticity field ω. We call it rotational if the function $\lambda \in C^\infty(M)$ such that $\omega = \lambda X$ satisfies $\lambda \neq 0$.

1.2 Contact Geometry

At this point contact geometry appears as one of the interesting geometrical structures underlying the Euler equations. Let us recall some basic properties of contact structures.

Definition 1 Let M^{2n+1} be an odd dimensional manifold equipped with a hyperplane distribution ξ such that there is a 1-form $\alpha \in \Omega^1(M)$ with $\ker \alpha = \xi$ and $\alpha \wedge (d\alpha)^n \neq 0$ everywhere. Then (M^{2n+1}, ξ) is a (cooriented) contact manifold.

The contact structure ξ does not depend on the choice of the defining one form, called the contact form. There is a unique vector field R for a given α defined by the equations

$$\begin{cases} \iota_R\alpha & = 1 \\ \iota_R d\alpha & = 0 \end{cases}$$

called the Reeb vector field.

The relation of Reeb vector fields and the Euler equations is provided by the following theorem by Etnyre and Ghrist [4].

Theorem 2 *Any nonsingular rotational Beltrami field is a reparametrization of a Reeb vector field defined by some contact form. Any reparametrization of a Reeb vector field defined by a contact form is a nonsingular rotational Beltrami field for some metric and volume form.*

This theorem allows us to introduce the techniques used in contact geometry in the study of some steady Euler flows, for instance the flexibility shown through the h-principle techniques introduced by Gromov [5] (see [3] for a modern text and extension of these results).

2 Reeb Embeddings

In order to find new universality properties for Euler flows, we adapt the questions of Tao to this setting. We ask the following question: given an arbitrary non-vanishing vector field X on a closed manifold N, can we “embed” it in a Reeb flow of a bigger manifold for instance the standard contact sphere? More precisely, can we find a contact form α with Reeb vector field R in a bigger closed manifold M such that it has an invariant submanifold diffeomorphic to N with $R|_N = X$?

As we will see in a moment, X needs to satisfy at least one extra condition. Let us recall the following definition.

Definition 3 A vector field X in a manifold M is called **geodesible** if its orbits are geodesics for some metric in M.

We will assume from now on that X is geodesible of unit-length (i.e. it is of unit-length for the metric making its orbits geodesics). It is well known following works of Gluck [6] that an equivalent condition for X to be geodesible is that there exists a one form β such that

$$\iota_X\beta = 1, \text{ and } \iota_X d\beta = 0.$$

It becomes now clear that a Reeb vector field R is geodesible: the contact form α satisfies the necessary conditions. Furthermore if W is an invariant submanifold of R, then R is also geodesible in W since the restriction of the contact form to W satisfies also Gluck's conditions. This implies that vector fields that can potentially be embedded as a Reeb field need to be geodesible. However, we prove that this is the only condition: any geodesible vector field can be embedded as a Reeb vector field of the same manifold: the standard contact sphere. This is contained in the following theorem, which is a weaker version of the main Theorem 5 that will be discussed below.

Theorem 4 ([2]) *Let N be a closed manifold of dimension n and X a geodesible flow on it, then there is an embedding of N in (S^{4n-1}, ξ_{std}) and a form β defining ξ_{std} such that the Reeb field R of β satisfies $R|_N = X$.*

The proof of this theorem is less technical than the main theorem, let us give a sketch. The proofs with all the details can be found in [2].
Sketch of proof. We start with a geodesible vector field X on a closed manifold N. This means that for some form α we have $\iota_X\alpha > 0$ and $\iota_X d\alpha = 0$. This implies that X preserves $\eta = \ker\alpha$. In fact being geodesible is equivalent to saying that X preserves a transversal hyperplane field.

(1) We begin constructing an open contact manifold M containing N with a symplectic hyperplane distribution, that contains η when restricted to N. We consider the vector bundle $\pi : \eta^* \to N$ over N with the hyperplane distribution $\tilde{\eta} = \pi^*(\eta \oplus \eta^*)$, which is equipped with the canonical symplectic structure.
(2) We now obtain a contact structure that also contains η when restricted to N. To do this we perturb the symplectic form in an appropriate way and work chart by chart to find a contact structure $\bar{\alpha}$ satisfying $\ker \bar{\alpha}|_N \cap TN = \eta$
(3) We apply an h-principle result by Gromov on isocontact embeddings that tells us that M embeds isocontactly in (S^{4n-1}, ξ_{std}). This means that there is an embedding $e : N \hookrightarrow S^{4n-1}$ such that $e^*\xi_{std} = \ker\tilde{\alpha}$. We have that our vector field X in $e(N)$ preserves $\xi_{std} \cap TN = \eta$ by construction. By a characterization of Inaba [7], this condition is equivalent to the existence of a contact form defining ξ_{std} such that its Reeb vector field R satisfies $R|_N = X$.

□

This theorem is merely geometrical and allows the realization of any geodesible field on a manifold of dimension n as the Reeb vector field in the standard contact sphere S^{4n-1}. However, it is a particular instance of the following more general theorem.

Theorem 5 ([2]) *Let $e : (N, X) \hookrightarrow (M, \xi')$ be a smooth embedding of N into a contact manifold (M, ξ') where X is a geodesible vector field on N. Assume that* $\dim M \geq 3n + 2$. *Then e is isotopic to a Reeb embedding $\tilde{e}$, and $\tilde{e}$ can be taken C^0-close to e.*

This result is stronger, and interesting from the contact topology perspective. It relies on defining a formal counterpart of a Reeb embedding and proving that it satisfies a certain h-principle. Then Theorem 5 follows from proving that smooth embeddings in high enough codimension are formal Reeb embeddings satisfying the h-principle. We will use this last Theorem to deduce the universality properties of Euler flows, since the resulting dimension of the ambient manifold is smaller.

3 Applications

Realizing geodesible vector fields as Reeb vector will implies different universality properties for Euler flows, and we state some of them here. Because of the correspondence of Reeb vector fields and Beltrami fields, we know already that every geodesible vector field can be embedded in a stationary solution to the Euler equations. The following corollaries are obtained by finding the appropriate geodesible vector field and applying Theorem 5.

We define the suspension of a time-periodic vector field $X(t)$ on a manifold N (of dimension n) as the manifold $N \times S^1$ endowed with the vector field $Y = (X(p, \theta), \partial\theta)$. It is a geodesible field and hence

Corollary 6 ([2]) *Let $X(t)$ any periodic on time vector field on N. Then its suspension Y in $N \times S^1$ can be embedded as a Reeb flow on S^m with m the odd integer in* $\{3n + 5, 3n + 6\}$.

In particular this proves that any periodic in time vector field can be extended to an Euler solution in the sense of [10].

In a similar way, given an orientation preserving diffeomorphism, one can consider the manifold $\tilde{N} = N \times [0, 1]/\sim$ identifying $(x, 0)$ with $(\varphi(x), 1)$. Consider the horizontal flow on it

$$\phi_t(\theta, x) = (\theta + t, x).$$

The vector field obtained by this flow is geodesible and has as return map the given diffeomorphism.

Corollary 7 ([2]) *Let $\varphi : N \to N$ be an orientation preserving diffeomorphism. It can be realized as the return map on some cross section diffeomorphic to N in some Reeb flow on S^m with m the odd integer in* $\{3n + 5, 3n + 6\}$.

The Reeb flow can always be obtained on the standard contact sphere (S^{3n+5}, ξ_{std}) or (S^{3n+6}, ξ_{std}) depending on the parity of n (since the dimension of the sphere needs to be odd).

As a byproduct, we obtain the existence of Turing complete Euler flows. This is a consequence of the fact that there exists an orientation-preserving diffeomorphism of $\mathbb{T}^4$ encoding a universal Turing machine, see [8].

Corollary 8 ([2]) *There are Reeb flows (and hence Euler flows) which are Turing complete. Concretely, there is a Reeb flow on* (S^{17}, ξ_{std}) *encoding a universal Turing machine.*

In his papers, Tao speculates on using a Turing complete flow to construct a finite time blow up solution to the Euler or Navier-Stokes equations. We do not know how this solution evolves when taken as initial condition for the Navier-Stokes equation.

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The Continuous Rank Function for Varieties of Maximal Albanese Dimension and Its Applications

Lidia Stoppino

Abstract In this note, I review an aspect of some new techniques introduced recently in collaboration with Miguel Ángel Barja and Rita Pardini: the construction of the continuous rank function. I give a sketch of how to use this function to prove the Barja-Clifford-Pardini-Severi inequalities for varieties of maximal Albanese dimension and to obtain the classification of varieties satisfying the equalities.

1 Statement of the Results

We work over $\mathbb{C}$. Let X be a smooth projective n-dimensional variety and $a\colon X \to A$ a morphism to an abelian q-dimensional variety, such that the pullback homomorphism $a^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is injective; we call a morphism with such a property *strongly generating*. The main case to bear in mind is the one when $A = \operatorname{Alb}(X)$ is the Albanese variety and $a = \operatorname{alb}_X$ is its Albanese morphism: in this case alb_X^* is an isomorphism. We shall identify $\alpha \in \operatorname{Pic}^0(A)$ with $a^*\alpha \in \operatorname{Pic}^0(X)$.

Suppose moreover that X is of *maximal a-dimension*, i.e. that a is finite on its image. In particular this implies that $q(X) \geq q = \dim(A) \geq n$, where $q(X) = h^0(X, \Omega^1_X) = \dim(\operatorname{Pic}^0(X))$ is the irregularity of X.

Let L be any line bundle on X. Consider the following integer, which is called *the continuous rank of L* ([1, Def. 2.1]).

$$h^0_a(X, L) := \min\{h^0(X, L + \alpha),\ \ \alpha \in \operatorname{Pic}^0(A)\}. \tag{1}$$

Remark 1 By semicontinuity, $h^0_a(X, L)$ coincides with $h^0(X, L + \alpha)$ for α general in $\operatorname{Pic}^0(A)$, and by Generic Vanishing, if $L = K_X + D$ with D nef, then $h^0_a(X, L) = \chi(L)$, the Euler characteristic of L.

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We need also a restricted version of the rank function: for $M \subseteq X$ a smooth subvariety, there exists a non empty open subset of $\mathrm{Pic}^0(A)$ such that $h^0(X_{|M}, L+\alpha)$ is constant. We call this value the *restricted continuous rank* $h^0_a(X_{|M}, L)$. A first result that highlights the importance of this invariant is the following:

Proposition 2 (Barja, [1], Thm 3.6.) *If* $h^0_a(X, L) > 0$ *then* L *is big.*

Recall now that the *volume* $\mathrm{vol}_X(L)$ of L (see for instance [6]) is an invariant encoding positivity properties of the line bundle: for example $\mathrm{vol}_X(L) = L^n$ if L is nef, and $\mathrm{vol}_X(L) > 0$ if and only if L is big.

We start with the following general inequalities between the volume of L and its continuous rank.

Theorem 3 (Barja-Clifford-Pardini-Severi inequalities) *The following inequalities hold:*

(i) $\mathrm{vol}_X(L) \geq n! h^0_a(L)$;
(ii) If $K_X - L$ *is pseudoeffective, then* $\mathrm{vol}_X(L) \geq 2n! h^0_a(L)$.

For the case $n = 1$ the inequalities follow from Riemann-Roch and Clifford's Theorem ([3] Lem. 6.13). For the case $n = 2$ and $L = K_X$ inequality (ii) was stated by Severi in 1932, with a wrong proof, and eventually proven by Pardini in 2005 [9]. Barja in [1], proved the inequalities for any n and L nef. In [3] Barja, Pardini and myself proved the general version Theorem 3 for any line bundle L on X, in the form stated above. This is done via new techniques introduced in the same paper. Moreover, with our new methods it is possible to solve the problem of *classifying* the couples (X, L) that reach the BCPS equalities, obtaining the following general result [4].

Theorem 4 [[4], Thm 1.1, Thm 1.2] *Suppose* $h^0_a(X, L) > 0$.

(i) If $\lambda(L) = n!$, *then* $q = n$ *and* $\deg a = 1$ *(i.e. a is birational).*
(ii) If $K_X - L$ *is pseudoeffective and* $\lambda(L) = 2n!$, *then* $q = n$ *and* $\deg a = 2$.

This result was known for $n = 2$ and $L = K_X$ [2, 7] but a general classification was out of reach.

In this note I describe in particular one of the techniques of [3], i.e. the *continuous extension of the continuous rank*. I give an idea of the steps of the proof of Theorem 3 and of Theorem 4 that involve the rank function. Throughout this note, I make assumptions more restrictive than the ones of loc.cit., in order to simplify the exposition. Needless to say, I will hide some technicalities under the carpet.

2 Continuous Rank Function

2.1 Set Up: Pardini's Covering Trick

Let $\mu_d \colon A \to A$ be the multiplication by d on A. For any integer $d \geq 1$ consider the variety $X^{(d)}$ obtained by fibred product as follows:

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{\mu}_d} & X \\ {\scriptstyle a_d}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{\mu_d} & A \end{array} \tag{2}$$

In general, even if we start from $a = \mathrm{alb}_X$, the morphism a_d need not be $\mathrm{alb}_{X^{(d)}}$: what is still true is that a_d is strongly generating, as we see from the result below.

Lemma 5 ([3] Sect. 2.2 and [2] Lemma 2.3) *The variety $X^{(d)}$ is smooth and connected and the morphism $\widetilde{\mu}_d$ is étale with the same monodromy group of μ_d ($\cong (\mathbb{Z}/d)^{2q}$). We have the following chain of equalities:*

$$\ker((a_d \circ \mu_d)^*) = \ker((a \circ \widetilde{\mu}_d)^*) = \mathrm{Pic}^0(A)[d] = \ker(\widetilde{\mu}_d^*).$$

In particular, $\ker(a_d^*) = 0$.

Now, call $L^{(d)} := \widetilde{\mu}_d^*(L)$. Fix H a very ample divisor on A; let $M := a^*H$ and let M_d be a general smooth member of the linear system $|a_d^*H|$. By [8, Chap.II.8(iv)] we have $a_d^*H \equiv d^2H \mod \mathrm{Pic}^0(A)$, and hence

$$M^{(d)} = \widetilde{\mu}_d{}^*(a^*H) = a_d^*(\mu_d^*H) \equiv d^2M_d \mod \mathrm{Pic}^0(A). \tag{3}$$

Remark 6 Observe that the assumptions we have on X are verified by M_d for any $d \geq 1$. Precisely, the morphism $a_{d|M_d} \colon M_d \to A$ is strongly generating and M_d is of maximal $a_{d|M_d}$-dimension. Moreover, if we have the hypothesis of Theorem 3 (ii), i.e. that $K_X - L$ is pseudoeffective, then $K_{M_d} - L_{|M_d}$ is pseudoeffective.

2.2 Continuous Rank

A basic property of the continuous rank with respect to the construction above is the following (see [1, Prop. 2.8]):

$$\forall d \in \mathbb{N} \qquad h^0_{a_d}(X^{(d)}, L^{(d)}) = d^{2q} h^0_a(X, L). \tag{4}$$

This just follows from the fact that $\widetilde{\mu}_{d*}(\mathcal{O}_{X^{(d)}}) = \oplus_{\gamma \in \ker(\mu_d^*)} \gamma$ by Lemma 5.
Now we define an extension of the continuous rank for $\mathbb{R}$-divisors of the form

$$L_x := L + xM, \;\; x \in \mathbb{R}.$$

We start with the definition over the rationals.

Definition 7 Let $x \in \mathbb{Q}$, and let $d \in \mathbb{N}$ such that $d^2 x = e \in \mathbb{Z}$. We define

$$h^0_a(X, L_x) := \frac{1}{d^{2q}} h^0_{a_d}(X^{(d)}, L^{(d)} + eM_d). \tag{5}$$

Remark 8 Note that by (3) we have that M_d is an integer divisor on $X^{(d)}$ equivalent to $\frac{e}{d^2} M^{(d)}$ modulo $\mathrm{Pic}^0(A)$. For any $k \in \mathbb{N}$, by (3) and (4) we have:

$$h^0_{a_{dk}}(X^{(dk)}, L^{(dk)} + ek^2 M_{dk}) = h^0_{(a_d)_k}((X^{(d)})^{(k)}, (L^{(d)})^{(k)} + eM_d^{(k)}) = k^{2q} h^0_{a_d}(X^{(d)}, L^{(d)} + eM_d).$$

Using the above equality, it is immediate to see that given $d, d' \in \mathbb{N}$, $e, e' \in \mathbb{Z}$ such that $\frac{e}{d^2} = x = \frac{e'}{d'^2}$, the formula (5) with d and with d' agrees with the formula with dd', so the definition is independent of the chosen d.

By studying the properties of this function on $\mathbb{Q}$, we can in particular see that it has the midpoint property, and thus extend it:

Theorem 9 ([3], Theorem 4.2) *With the above assumptions, the function $h^0_a(X, L_x)$, extends to a continuous convex function $\phi \colon \mathbb{R} \to \mathbb{R}$. For any $x \in \mathbb{R}$ the left derivative has the following form:*

$$D^-\phi(x) = \lim_{d \to \infty} \frac{1}{d^{2q-2}} h^0_{a_d}(X^{(d)}_{|M_d}, (L_x)^{(d)}), \quad \forall x \in \mathbb{R}. \tag{6}$$

Remark 10 Let us here recall the formula for the derivative of the volume function for $\mathbb{R}$-divisors (see [6, Cor.C]). Fix $x_0 := \max\{x \mid \mathrm{vol}_X(L_x) = 0\}$. There is a continuous extension of the volume function for $\mathbb{Q}$-divisors, $\mathrm{vol}_X(L_x) = \psi(x) \colon \mathbb{R} \to \mathbb{R}$, which is differentiable for $x \neq x_0$ and

$$\psi'(x) = \begin{cases} 0 & x < x_0 \\ n\,\mathrm{vol}_{X|M}(L_x) & x > x_0 \end{cases} \tag{7}$$

where $\mathrm{vol}_{X|M}(L_x)$ is the *restricted volume*. So, similarly to what happens to the rank function, also the volume extends and the formula for the derivative involves a restricted function. We will soon use this formula.

3 Applications

The power of this new perspective is the following: if we study the BCPS inequalities as a particular case of inequalities between the rank function and the volume function, the proofs become strikingly simple, and we can apply induction via integration.

Now we state the main technical result (see [3, Sect. 2.4], [4, Sect. 2.5]).

Lemma 11 *There exists a* $\mathbb{Q}$*-divisor* P *on* X *such that for any* $x \in \mathbb{R}$ *and* d *high and divisible enough we have:*

$$\begin{aligned}
&\mathrm{vol}_{X|M}(L_x) \geq \mathrm{vol}_{X|M}(P_x) = P_x^{n-1}M = \frac{1}{d^{2q}}((P_x)^{(d)})^{n-1}M_d,\\
&\mathrm{vol}_{X^{(d)}|M_d}(P_x{}^{(d)}) = ((P_x)^{(d)})^{n-1}M_d,\\
&h^0_{a_d}(X^{(d)}{}_{|M_d}, (L_x)^{(d)}) = h^0_{a_d}(X^{(d)}{}_{|M_d}, (P_x)^{(d)})
\end{aligned}$$

The key result here is the so-called *continuous resolution of the base locus* introduced firstly in [1, Sect. 3].

3.1 BCPS Inequalities

Now we see how the induction step of the proof of the BCPS inequalities ends up in an application of the fundamental theorem of calculus. We prove here inequality (i) but the proof works exactly in the same way for (ii) (with the right first induction step). Consider as above the functions $\psi(x) := \mathrm{vol}_X(L_x)$ and $\phi(x) := h^0_a(X, L_x)$. Using Lemma 11 and formula (7) we have that

$$\psi'(x) = \frac{n}{d^{2q}}((P_x)^{(d)})^{n-1}M_d, \qquad D^-\phi(x) = \lim_{d\to\infty}\frac{1}{d^{2q-2}}h^0_{a_d}(X^{(d)}{}_{|M_d}, (P_x)^{(d)}).$$

Now, by Remark 6 M_d and $a_{d|M_d}$ satisfy the assumptions, and we can prove via the Lemma 11 that inequality (i) in dimension $n-1$ implies that

$$\psi'(x) \geq n!D^-\phi(x) \text{ for any } x \in \mathbb{R}^{\leq 0}.$$

We may thus apply the Fundamental Theorem of Calculus and compute

$$\mathrm{vol}_X(L) = \psi(0) = \int_{-\infty}^{0}\psi'(x)dx \geq n!\int_{-\infty}^{0}D^-\phi(x)dx = n!\,\phi(0) = n!\,h^0_a(X, L).$$

3.2 Classification of the Limit Cases

Both the BCPS inequalities are sharp: we have by Hirzebruch-Riemann-Roch theorem that equality in (i) holds for X an abelian variety and L any nef line bundle on it. As for (ii), consider an abelian variety A and a very ample line bundle N on it. Let $B \in |2N|$ a smooth divisor and let $a\colon X \to A$ be the double cover branched along B. let $L = a^*(N)$. We have

$$\mathrm{vol}_X(L) = 2\,\mathrm{vol}_A(H) = 2n!h^0_{id_A}(A, N) = 2n!h^0_a(X, L).$$

In Theorem 4 we see that essentially the cases above are the only ones reaching the equalities. Here we give an idea of a step of the proof of (ii). Consider the function

$$\nu(x) := \mathrm{vol}_X(L_x) - 2n!h^0_a(X, L_x), \ \ x \in \mathbb{R}.$$

One of the key points in the argument in [4] is to prove that $\nu(x) \equiv 0$ for $x \leq 0$. We have $\nu(0) = 0$ by assumption. From Theorem 3 (with some work) we can prove that $\nu(x) \geq 0$ for $x \leq 0$. Hence, it suffices to show that the left derivative $D^-\nu(x)$ is ≥ 0 for $x < 0$. Using Lemma 11 we have that for any real x smaller or equal than 0

$$D^-\nu(x) = \lim_{d\to\infty} \frac{n}{d^{2q-2}} \left(\mathrm{vol}_{M_d}(P_x{}^{(d)}_{|M_d}) - 2(n-1)!h^0_{a_d}(X^{(d)}{}_{|M_d}, (P_x)^{(d)})\right).$$

Now we prove that the right hand expression is greater or equal to 0 using the relative version of Theorem 3 again in dimension $n - 1$.

Remark 12 In Example 7.9 of [3] we proved that for any integer $m \geq 1$ there exist varieties X_m of maximal Albanese dimension such that $\mathrm{vol}(K_{X_m})/\chi(K_{X_m})$ is arbitrarily close to $2n!$ but with Albanese morphism of degree 2^m, hence far from being a double cover.

Remark 13 The continuous rank functions can be computed explicitly for abelian varieties, and in some cases for curves (see the Examples of [3]). There are examples where this function is not $\mathcal{C}^1$ ([3, Exercise 7.3]). The regularity properties of these functions, as well as the geometrical meaning of the points of discontinuity of their derivative, still have to be well understood. Some results in this direction can be found in [5].

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Developable Surfaces with Prescribed Boundary

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Abstract It is proved that a generic simple, closed, piecewise regular curve in space can be the boundary of only finitely many developable surfaces with nonvanishing mean curvature. The relevance of this result in the context of the dynamics of developable surfaces is discussed.

1 Introduction

The work presented here originated in the study by the authors of the motion and dynamics of pieces of cloth in the real world, with a view towards its robotic manipulation in a domestic, non-industrial, environment. Because in such an environment cloth is subject to low stresses, it makes sense to model garments as inextensible surfaces, and assume that their motion consists of isometries. The authors are currently developing such an isometric strain model, and its application to the control problem of cloth garments [2].

The original state of cloth in such a model is flat, so the set of possible states of a piece of cloth in our model is the set of developable surfaces isometric to a fixed one, which we may assume to be a domain R in the plane.

To study the dynamics of such a piece of cloth with the Lagrangian formalism, we need coordinates on the set of its states. This is usually done through a discretization scheme, but such schemes often introduce artifacts in the dynamics of cloth. Thus it would be interesting to have intrinsic, analytic coordinates on the space of states

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that are suitable for the formulation of a Lagrangian with a role analogous to that of the Helfrich Hamiltonian of membrane dynamics (see [3]). As explained in the concluding section, such coordinates should allow the formulation of discretization schemes where the resulting mechanics are more independent of how the garment is meshed, more frugal in computation time, and closer to reality.

Section 2 in this note discusses two candidates to the role of generalized coordinates in the space of states of a surface in our isometric strain model, and explains their common limitation from the viewpoint of their application.

Section 3 proposes an alternative approach: to track the motion of the surface by following its boundary. This is not straightforward because the boundary does not determine the position of the surface, but as we explain below our Main Theorem 5 is a step in this direction.

2 The Space of Developable Surfaces

Developable surface is a classical name for a smooth surface with Gaussian curvature 0. These are exactly the surfaces which are locally isometric to a domain in the Euclidean plane $\mathbb{R}^2$. Developability places a strong constrain on a surface (see [5]):

Theorem 1 (structure theorem for developable surfaces, classical) *A $\mathcal{C}^3$ developable surface S embedded in $\mathbb{R}^3$ has an open subset which is ruled, with unit normal vector constant along each line of the ruling but varying in a transverse direction. Every connected component of its complement is contained in a plane.*

This structure can be deduced from the Gauss map of the surface: Gaussian curvature 0 makes its rank 0 or 1, the latter rank being reached on an open subset of the surface. The normal vector is locally constant in the rank 0 subset.

The dichotomy in the rank of the Gauss map, and varied classical notations, motivate:

Definition 2 A developable surface is *torsal* if the Gauss map has rank 1 on a dense open subset.

Flat patches are connected subsets of a developable surface with nonempty interior where the Gauss map is constant, i.e. they are contained in a plane.

The subdivision of a developable surface into torsal and flat patches is given by the boundary of the vanishing locus of the mean curvature and is not necessarily simple.

With a view to our intended applications, fix a planar domain $R \subset \mathbb{R}^2$ which is compact, contractible and has a piecewise $\mathcal{C}^\infty$ boundary. Usually R will be a convex polygon. Define $\mathcal{S}$ to be the set of all $\mathcal{C}^3$ surfaces in $\mathbb{R}^3$ isometric to R. These surfaces are all developable, and $\mathcal{S}$ may be seen as the *space of states* of an inextensible (i.e. isometric for the inner distance) deformation of R in Euclidean space.

The space of states $\mathcal{S}$ can also be defined as the set of $\mathcal{C}^2$ maps from R to $\mathbb{R}^3$ which are isometries with the image. As such, it is endowed with the compact-open topology derived from the Euclidean one in R and $\mathbb{R}^3$. This topology furnishes valuable tips for the study of $\mathcal{S}$: the set of surfaces containing flat patches has an empty interior because there exist arbitrarily small deformations making the normal vector nonconstant on an open set. Torsal surfaces are *stably torsal* if the mean curvature function intersects transversely the zero function. These surfaces form an open subset $\mathcal{U} \subset \mathcal{S}$, and suffice for our practical study of $\mathcal{S}$.

We can try to develop coordinates for the stably torsal state space $\mathcal{U}$ based on the classical structure theorem. First, let us recall how to identify developable surfaces among the ruled ones:

Proposition 3 (classical, see [1]) *A ruled surface parametrized as $\phi(u, v) = \gamma(u) + v \cdot w(u)$, where γ is a regular parametrized curve and w a vector field over γ, is developable if and only if the 3 vectors $\gamma'(u), w(u), w'(u)$ are linearly dependent for all u.*

Given a regular $\mathcal{C}^2$ curve γ there is a way to obtain systematically such rulings over γ resulting in regular torsal surfaces:

Proposition 4 *Let n be a unit normal $\mathcal{C}^1$ vector field over a regular, $\mathcal{C}^2$ curve with $n' \neq 0$. Then $w = n \times n'$ defines a torsal surface in a neighbourhood of γ. Moreover, all regular, torsal rulings over γ are generated by such w, and only $n, -n$ define the same torsal surface.*

Proof n is normal to γ' and w by their definitions, and $w' = n \times n''$ so at every u the vectors γ', w, w' are normal to n. Also, note that $w \neq 0$ because $n' \neq 0$.

If $\tilde{n}$ is another unit normal vector field such that $\tilde{n} \times \tilde{n}' = \mu w$ for some function $\mu(u)$ then note that $\tilde{n}$ has to be normal to both w and γ', hence a multiple of n.

Finally, let us point out that if w is a nonvanishing tangent vector field over γ defining a torsal surface around it, then we can select a unit vector field n normal to γ', w, w'. The fact that n is normal to w and w' imply that n' is also normal to w, so $n \times n'$ is a multiple of w. □

To define coordinates in the space of stably torsal surfaces S isometric to a fixed bounded domain R, the pairs (γ, n) of Proposition 4 run into a practical difficulty: the condition that $n' \neq 0$ forces the Gauss map to have rank 1. If S is a stably torsal surface with mean curvature H of varying sign, we must subdivide it by the $H = 0$ curves and parametrize separately each component of the complement. To follow a motion of the surface, one has to track the boundary shifts, mergers and splits of these components.

Ushakov proposes in [7] an alternative, PDE based, coordinate scheme viewing developable surfaces as solutions of the trivial Monge-Ampère equation. But this requires a parametrization of the surface in the form $z = z(x, y)$. Such parametrizations exist only locally, so their use leads even more intensely to the problem of tracking boundaries, mergers and splits of their subdomains.

3 The Boundary of a Developable Surface

There is an alternative approach to study the dynamics of developable surfaces isometric to a fixed bounded planar domain R: follow the motion of the boundary ∂R in space, and derive from this the developable surface that fills it. This leads to:

Question. Given a piecewise smooth simple closed curve γ in $\mathbb{R}^3$, what are the developable surfaces with boundary γ?

The degeneracy nature of the trivial Monge-Ampère equation makes it fail to have a unique solution for this kind of boundary problem. Indeed, it is easy to find examples where there is more than one solution, as shown in Fig. 1.

Nevertheless, for problems such as the study of cloth dynamics it is not necessary for the boundary problem to have a unique solution. It suffices to know that it will always have a finite set of solutions, because this solution set is then discrete, with different solutions separated by a nontrivial jump in any tagging energy, local coordinate …In such case, once one has a developable ruling with a boundary γ_0 at time $t = 0$, the evolution γ_t of the boundary will determine the analytic continuation of the $t = 0$ developable ruling, and identify a unique ruling for every time t. Herein lies the interest of the authors in our

Theorem 5 (Main Theorem) *Let γ be a simple closed curve in $\mathbb{R}^3$ which is piecewise $\mathcal{C}^2$, has nonvanishing curvature, its torsion vanishes at finitely many points, and such that only for finitely many pairs $s \neq \tilde{s}$ does the tangent line to γ at s pass through $\gamma(\tilde{s})$. Then, there can be at most finitely many developable surfaces with boundary γ and nonzero mean curvature in its interior.*

Let us point out that the preconditions that we impose on γ are generic, i.e. satisfied by a dense open subset of the embeddings of S^1 in $\mathbb{R}^3$.

The starting idea to prove the theorem is another classical result, analogous to Proposition 3:

Lemma 6 *Let S be a torsal surface with boundary γ, and $l \subset S$ a segment with endpoints P, Q in γ. Then the common tangent plane to S along l is tangent to γ at both P, Q.*

The proof consists in pointing out that the normal vector to S stays constant over the segment l, and that γ is tangent to S.

Lemma 6 presents developable rulings as arcs of bitangent planes (i.e., tangent to γ at 2 points). Such planes are given by pairs $s \neq \tilde{s}$ whose tangent lines are coplanar:

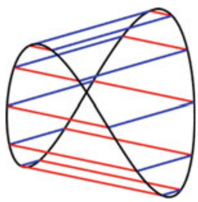

Fig. 1 A smooth simple closed curve (in black) which is the boundary of two developable surfaces (indicated in red and blue respectively)

Proposition 7 *Let $\gamma : [0, L] \subset \mathbb{R}^3$ be a simple, closed, arc-parametrized $\mathcal{C}^3$ curve. The function*

$$D : [0, L]^2 \longrightarrow \mathbb{R}$$
$$(s, \tilde{s}) \longmapsto \det\left(\gamma(s) - \gamma(\tilde{s}), \gamma'(s), \gamma'(\tilde{s})\right)$$

is a Morse function at a neighbourhood of its zeros $(s, \tilde{s})$ such that: $s \neq \tilde{s}$, γ has nonzero curvature and torsion at both $s, \tilde{s}$, and the tangent line to γ in each one does not pass through the other point of the curve.

Proof It is a straightforward computation. With coordinates $(s, \tilde{s})$ we have that

$$dD = \left(\det\left(\gamma(s) - \gamma(\tilde{s}), \gamma''(s), \gamma'(\tilde{s})\right), \det\left(\gamma(s) - \gamma(\tilde{s}), \gamma'(s), \gamma''(\tilde{s})\right)\right)$$

Let $(s, \tilde{s})$ be a zero of D with $s \neq \tilde{s}$, which is also a critical point of D. If any of the linear subspaces spanned by $\gamma(s) - \gamma(\tilde{s}), \gamma'(\tilde{s})$ and by $\gamma(s) - \gamma(\tilde{s}), \gamma'(s)$ has dimension less than 2, the tangent line to γ at one of the points $\gamma(s), \gamma(\tilde{s})$ contains the other.

When both linear subspaces have dimension 2, the conditions $D(s, \tilde{s}) = 0, dD(s, \tilde{s}) = (0, 0)$ show that γ has the same osculating plane to γ at the points $\gamma(s), \gamma(\tilde{s})$. Because of this, the second differential of D is

$$d^2 D = \begin{pmatrix} \kappa_s \tau_s \det\left(\gamma(s) - \gamma(\tilde{s}), B_s, \gamma'(\tilde{s})\right) & 0 \\ 0 & \kappa_{\tilde{s}} \tau_{\tilde{s}} \det\left(\gamma(s) - \gamma(\tilde{s}), \gamma'(s), B_{\tilde{s}}\right) \end{pmatrix}$$

Here κ, τ, B are respectively the curvature, torsion, binormal vector of the Frenet frame, at the point given by their subindex. The determinants in the diagonal of $d^2 D$ are nonzero because each consists of a binormal vector and a basis for the osculating plane at the same point of the curve. □

Proposition 7 has a version for piecewise $\mathcal{C}^3$ curves, saying just that D is Morse under the additional hypothesis that $s, \tilde{s}$ do not correspond to corner points, at which D has two different definitions. We are now ready for

Proof of Theorem 5.

A torsal developable surface S is foliated by segments which can only end at the boundary or at points of vanishing mean curvature. Having ruled out the latter, S is determined by an arc of bitangent planes $B(t)$, with $t \in [a, b]$, such that the curves $s(B(t)), \tilde{s}(B(t))$ formed by the two points of tangency of $B(t)$ cover γ.

Away from the finite set of horizontal and vertical lines in $[0, L]^2$ where one of the values $s, \tilde{s}$ corresponds to a corner point, point with vanishing torsion, or point whose tangent line intersects γ again, the pairs of values $s \neq \tilde{s}$ for which there exists at all a bitangent plane to γ through $\gamma(s), \gamma(\tilde{s})$ lie by Proposition 7 in the zero set of a Morse function D from an open subset of $[0, L]^2 \subset \mathbb{R}^2$ to $\mathbb{R}$. The function D is proper, therefore it is Morse over a suitably small range of values $(-\varepsilon, \varepsilon)$, which implies that D_0 is a finite union of smooth curves with transverse intersections in $[0, L]^2$.

The arc $B(t)$ is determined by its tangency points curve $(s(B(t)), \tilde{s}(B(t))) \subset [0, L]^2$, which must lie in the union of D_0 and finitely many vertical and horizontal lines, and cover γ, i.e. $\gamma = s(B) \cup \tilde{s}(B)$. There are only finitely many possibilities for that, once we specify a beginning point for the curves $s(B), \tilde{s}(B)$.

4 Future Continuation

The authors hope to carry out the program outlined in this note: to subdivide a developable surface S in patches according to the sign of its mean curvature, and follow its motion in a dynamical system by tracking the boundaries of the patches.

This approach is promising because it works with a 1-dimensional set of space coordinates which satisfy few restrictions, rather than with a 2-dimensional set of space coordinates that are heavily restricted because of the assumption of isometry.

There is a second interest which is more analytical: can we identify the developable surfaces which minimize a potential such as the gravitational potential? Fixing the boundary and the area of the surface leads to the Poisson equation which is not as straightforward to solve as its 1-dimensional analogue ([4]). What equation does one get if instead of fixing the area one fixes the Gaussian curvature to be zero? It is likely that global, i.e. parametrized by the fixed domain R, solutions will have in general singularities along points or curves.

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Configuration Space of a Textile Rectangle

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Abstract Given a rectangular piece of cloth on a planar surface, we aim to characterise its states based on the robot manipulations they would require. Considering the cloth as a set of n points in $\mathbb{R}^2$, we study its configuration space, $Conf_n(\mathbb{R}^2)$. We derive a stratification of $Conf_4(\mathbb{R}^2)$ using that of $\mathscr{Flag}$ (3), and we present some techniques that can be used to determine the adjacencies of $Conf_n(\mathbb{R}^2)$ and some group actions we can define on it.

1 Introduction

Given a rectangular cloth on a planar surface, we could consider it as a surface embedded in $\mathbb{R}^3$ with no self-intersection. Unfortunately considering the different states of such surface and studying their space bears difficulties, as we have to impose, on the already complex space of all possible surfaces with constant area and no self-intersections, constraints such as gravity force and cloth stiffness. In order to simplify, we consider instead the cloth as a set of points on the real plane. Since our aim is to distinguish states based on the types of robot manipulations they permit, we consider the configuration space of n ordered points in $\mathbb{R}^2$, namely $Conf_n(\mathbb{R}^2)$. This space belongs to the far more general family of configuration spaces of points on manifolds,

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$$Conf_k(X) = \left\{ \underline{p} = (p_1, \ldots, p_k) \in X^k \mid p_i \neq p_j, \text{ for } i \neq j \right\}.$$

Such spaces are interesting topological objects and their (co)homology type has been studied by several authors. In [2] some results regarding the homotopy type of $Conf_n(X)$ are obtained, assuming X is of dimension 2, while the real homotopy type of $Conf_n(X)$, when X is a smooth projective variety, was independently computed by Kriz [6] and Totaro [11]. Assuming $X = \mathbb{R}^n$, Cohen et al. computed the cohomology of $Conf_n(X)$, and in particular, they proved that $Conf_n(\mathbb{R}^n)$ is the classifying space of the n*-strand pure braid* group [3]. The action of $\mathbb{S}_n$ on $Conf_n(\mathbb{R}^n)$ is also studied in [3] and, in particular, the quotient of this action gives the *configuration space of n unordered points*, which is the classifying space of the n*-strand braid* group. Our interest lays mostly on the adjacency relations between the highest dimensional cells of $Conf_n(\mathbb{R}^2)$, when we regard it as a CW-complex. Such cells are "clusters" of similar point configurations and their adjacency information permits navigating between them. A *state* will then be a set of different cells, each one containing configurations of points, that permits similar types of robot manipulations.

In Sect. 2 we consider $n = 4$, for the 4 corner points of the rectangular cloth, and present a stratification of $Conf_4(\mathbb{R}^2)$ using that of $\mathscr{Flag}(3)$. We then move in Sect. 3 to the general case of $Conf_n(\mathbb{R}^2)$ and show some techniques to derive the adjacency structure of the space together with some group actions that are naturally defined on $Conf_n(\mathbb{R}^2)$.

2 Configuration Space of a Textile Rectangle Using 4 Points

In order to study the configuration space of the 4 corner points of the rectangular cloth we will make use of the flag manifold of $\mathbb{RP}^2$, $\mathscr{Flag}(3)$. If we consider the configuration $\underline{p} = (p_1, p_2, p_3, p_4)$ with $p_i \in \mathbb{R}^2$, we can embed them in $\mathbb{RP}^2$, mapping a point $p = (x, y)$ to $\widetilde{p} = [x : y : 1]$. The stratification of $\mathscr{Flag}(3)$ induces another on $Conf_4(\mathbb{R}^2)$, see Fig. 1.

If we consider $V = \{p_1, \overline{p_1 p_2}\}$ and $V^* = \{p_3, \overline{p_3 p_4}\}$, then the condition $v - l^*$ corresponds to the alignment of the three points $\{p_1, p_3, p_4\}$. Any alignment of three points p_i, p_j, p_k, with $i < j < k$ can be seen as a pure algebraic condition on the

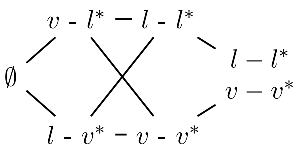

Fig. 1 We can stratify $\mathscr{Flag}(3)$ with respect to two flags, $V = \{v, l\}$ and $V^* = \{v^*, l^*\}$, using their incidence, indicated by—in the figure, see [5, 7]

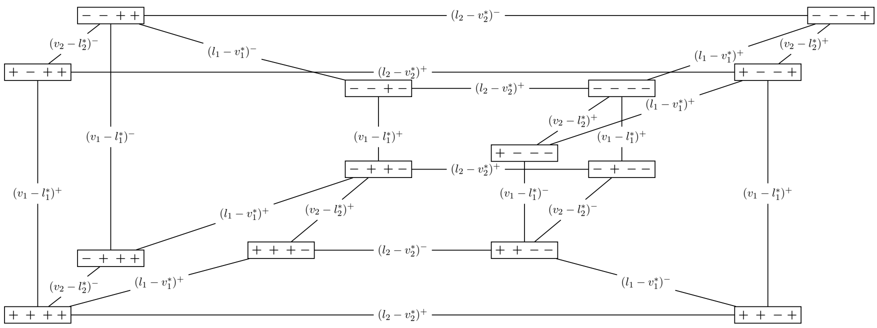

Fig. 2 We show the adjacency relations of $Conf_4(\mathbb{R}^2)$ using the stratification of the affine flag manifold [5] and as flags $V_1 = \{p_1, \overline{p_1, p_2}\}$, $V_1^* = \{p_3, \overline{p_3, p_4}\}$ and $V_2 = \{p_2, \overline{p_2, p_1}\}$, $V_2^* = \{p_4, \overline{p_4, p_3}\}$

points coordinates, given by the singularity of the determinant $d_{i,j,k} = |\widetilde{p_i}\ \widetilde{p_j}\ \widetilde{p_k}|$. The sign of $d_{i,j,k}$ depends on the clockwise or counter-clockwise position of the ordered triple (p_i, p_j, p_k). As the determinant is a continuous map onto $\mathbb{R}$, if two configurations $\underline{p}$ and $\underline{q}$ differ by one determinant sign, say $d_{i,j,k}$, then we know they belong to different cells. So any continuous path from $\underline{p}$ to $\underline{q}$ has to cross the singularity loci of $d_{i,j,k}$. We identify then a cell σ with the sequence of determinant signs of all triples of points belonging to any configuration $\underline{p}$ in σ. For us, the determinants signs are, in order, of $d_{1,2,3}$, $d_{1,2,4}$, $d_{1,3,4}$ and as last $d_{2,3,4}$. Moreover, an odd number of negative determinants tells us that one point lays inside the triangle spanned by the others. In such cases we call the configuration *internal*, otherwise *external*. One can prove easily that $d_{1,2,3} + d_{1,3,4} = d_{1,2,4} + d_{2,3,4}$, which means that not all sign sequences are admissible, as we can see in Fig. 2.

3 Configuration Space of an N-Points Textile Rectangle

Regarding $n > 4$, if we want to recover the stratification of $Conf_n(\mathbb{R}^2)$, similarly to Sect. 2, we would not consider more flags, as they will make less clear the description of the singularity loci. We have that singularities are given by the alignment of three points and again any cell can be identified by a sequence of $\binom{n}{3}$ determinant signs. In the general case, we do not know exactly which sign sequences are admissible, that is, how many cells are in $Conf_n(\mathbb{R}^2)$. Consider the arrangement of lines spanned by pairs of $n - 1$ points, we could deduce the cells of $Conf_n(\mathbb{R}^2)$ from the regions they divide $\mathbb{R}^2$ into. Line arrangements, both in the real and projective planes, have been studied extensively in various contexts [4] and references therein. Several authors have worked on how to bound the number of regions, triangles or polygons [8–10]. In [1], the authors consider the problem of characterising geometric graphs using the order type of their vertex set. Using the notion of *minimal representation*

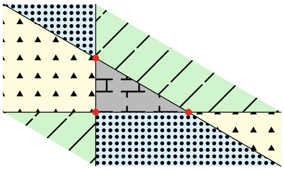

Fig. 3 Given any triple of points (not aligned), the lines they span divide the plane in 7 regions, that can be seen as three couples of *dual regions*, formed by external and internal configurations, which are coloured identically, and a self-dual internal region

of a graph, they identify which edges prevent the order type from changing via continuous deformations of the graph. Even if this approach is the closest to ours, to our knowledge in the literature there is not a detailed study of the adjacency relations of $Conf_n(\mathbb{R}^2)$. We present here two theorems that allow us to determine if and how we can move continuously a point (or more if needed) to change only one determinant sign. Due to lack of space and proof technicality, we give here only sketches of the proofs. The following theorem gives us a way to discern when an adjacency cannot exist.

Theorem 1 *Consider any configuration* $\underline{p} \in Conf_n(\mathbb{R}^2)$ *and a triple* $\{p_i, p_j, p_k\} \subset \underline{p}$*. If there exists either a point* $p_u \in \underline{p}$ *in the self-dual region of* $\{p_i, p_j, p_k\}$*, or two points* p_s, p_r *in two regions not dual w.r.t.* $\{p_i, p_j, p_k\}$*, then there does not exist a continuous movement of* $\underline{p}$ *that crosses only the singularity loci* $d_{i,j,k} = 0$.

Proof If $d_{i,j,k}$ is nullified via a continuous map, the 6 outer regions in Fig. 3 degenerate into 2 regions, corresponding to a pair of dual regions, depending on the map used, while the other ones degenerate to the line $\overline{p_i, p_j}$. In other words, if a point $p_u \in \underline{p}$ is inside the self-dual region then any continuous map that crosses the singularity loci $d_{i,j,k} = 0$ has to nullify at least one among $d_{i,j,u}$, $d_{i,k,u}$ and $d_{j,k,u}$. Similarly, if two points $p_s, p_r \in \underline{p}$ are in regions not dual w.r.t. $\{p_i, p_j, p_k\}$, then any continuous map crossing $d_{i,j,k}$ would also cross either $d_{i,j,s}$ or $d_{i,j,r}$. □

The following result tells us when instead it is possible to change sign.

Theorem 2 *Consider any configuration* $\underline{p} \in Conf_n(\mathbb{R}^2)$ *and a triple of points* $\{p_i, p_j, p_k\} \subset \underline{p}$*, such that they belong either to the same region or to two distinct and dual regions. If there exists a point* $p_u \notin \{p_i, p_j, p_k\}$*, such that for any pair* $p_s, p_r \notin \{p_i, p_j, p_k, p_u\}$ *in the same region, resp. dual regions, and for any pair* $p_a, p_b \in \{p_i, p_j, p_k\}$ *the configuration of* $\{p_a, p_b, p_s, p_r\}$ *is external, resp. internal, then the singularity loci can be crossed uniquely at* $d_{i,j,k} = 0$.

Proof If such point p_u exists, then the line $\overline{p_u\ p_v}$, with $v = i, j, k$, intersects any other line spanned by another two points outside the self-dual region. So we can move p_u along $\overline{p_u\ p_v}$ till $d_{i,j,k}$ changes sign, without crossing any other singularity. □

Note that Theorems 1 and 2 do not cover all cell adjacencies for $n > 6$. If $n \leq 6$ we can compute the exact number of cells. Such number is expected to rise quadratically [10], thus we want to group cells entailing similar robotic manipulations to form states. We consider also the action of the symmetric group $\mathbb{S}_n$. In terms of our stratification, such action induces an identification between cells whose determinant signs coincide after a permutation of the point labels, $\{1, \ldots, n\}$. For $n = 4, 5$ and 6, we obtain in total 2, 3 and 6 states, respectively, which are a lot fewer than we would hope for. In other words, such action induces an over-coarsened partition of the configuration space and we prefer to use instead the following refined partition. Let σ be a cell, i.e. a sign sequence, we define

$$\tau_1 \sim_\sigma \tau_2 \iff \exists g \in \mathbb{S}_n,\ g \cdot \tau_1 = \tau_2 \text{ and } d(\sigma, \tau_1) = d(\sigma, \tau_2),$$

where $d(\sigma, \tau_i)$ for $i = 1, 2$ is the number of different signs between cells σ and τ_i. Let Y_σ be the partition of the configuration space induced by the equivalence relation $\sim_\sigma$, which is a refinement of the one obtained via $\mathbb{S}_n$. That is, any equivalence class defined by $\sim_\sigma$ belongs to one and only one $\mathbb{S}_n$-equivalence class. The distance $d(\sigma, \cdot)$ is constant inside each class of Y_σ. We always have a unique state, $-\sigma$, which is $\sim_\sigma$-equivalent only to itself, and that realises the maximum distance from σ. When we consider G_σ, the Hesse diagram of Y_σ induced by the adjacency relation of $Conf_n(\mathbb{R}^2)$, we have that there exists an automorphism of G_σ, that maps σ to $-\sigma$.

In conclusion, given a configuration of n points, we are able to determine in which state τ is and how far it is from another (fixed) state σ. In addition, using G_σ, thanks to Theorems 1 and 2 and the stratification of $Conf_n(\mathbb{R}^2)$, we could be able to plan how to change state from one given state to either σ or $-\sigma$.

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The 4/3 Problem for Germs of Isolated Plane Curve Singularities

Patricio Almirón

Abstract In this survey we are going to overview the different approaches and solutions of a question posed by Dimca and Greuel about the quotient of the Milnor and Tjurina numbers.

1 Introduction

Analytic and topological invariants of germs of isolated plane curve singularities are central objects in Singularity Theory, see [8] and the references therein for an overview. One of the main objects of study is to find relations between them and to find topological constrains for analytical invariants. As one can see in [5, 8], two mainstream invariants are the Milnor number μ, and the Tjurina number τ. In fact, the Milnor number is a topological invariant and the Tjurina number an analytic invariant. If $C := \{f(x, y) = 0\}$ is a germ of isolated plane curve singularity, the easiest way to define these numbers is:

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(\partial f/\partial x, \partial f/\partial y)}, \quad \tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, \partial f/\partial x, \partial f/\partial y)}.$$

In 2017 on [5], Dimca and Greuel posed the following question:

Question 1 Is it true that $\mu/\tau < 4/3$ for any isolated plane curve singularity?

This guessed bound is inferred by Dimca and Greuel from some families of plane curves that asymptotically achieve this bound. From this point view, Question 1 can

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be divided into two questions: is it true? If it is true, can the $4/3$ bound be inferred from the geometry of the plane curve singularity?

In this survey, we will try to show the different approaches to this question and the main problems attached to them.

2 Deformation Theory

In this section, we are going to show that Milnor and Tjurina numbers are closely related to the theory of deformations. We refer to [8] for general deformation theory.

Definition 1 Let $(C, 0)$ be a germ of isolated plane curve singularity. A deformation of $(C, 0)$ is a germ of flat morphism $(\mathcal{Y}, 0) \to (S, 0)$ whose special fibre is isomorphic to $(C, 0)$. We call $(S, 0)$ the base space of the deformation. The deformation is called versal if any other deformation results from it by base change. It is called miniversal if it is versal and S has minimal possible dimension.

In [8], it is shown that an explicit way to construct versal and miniversal deformations of a plane curve is by using the Milnor algebra M_f, and the Tjurina algebra T_f.

$$M_f := \frac{\mathbb{C}\{x, y\}}{(\partial f/\partial x, \partial f/\partial y)}, \quad T_f := \frac{\mathbb{C}\{x, y\}}{(f, \partial f/\partial x, \partial f/\partial y)}.$$

Theorem 1 ((Tjurina) Corollary 1.17 [8]) *Let* $(C, 0)$ *be a germ of isolated plane curve singularity defined by* $f \in \mathcal{O}_{\mathbb{C}^2,0}$ *and* $g_1, \ldots, g_k \in \mathcal{O}_{\mathbb{C}^2,0}$ *be a* $\mathbb{C}$*–basis of* T_f *(resp. of* M_f*) If we sets,*

$$F(x, \mathbf{t}) := f(x) + \sum_{j=1}^{k} t_j g_j(x), \quad (\mathcal{X}, 0) := V(F) \subset (\mathbb{C}^2 \times \mathbb{C}^k, 0),$$

then $(C, 0) \hookrightarrow (\mathcal{X}, 0) \xrightarrow{\varphi} (\mathbb{C}^k, 0)$, *with* φ *the projection from the second component, is a miniversal (resp. versal) deformation of* $(C, 0)$.

Inside the base space of a miniversal deformation of a germ of plane curve singularity there is an interesting closed analytic subspace Δ^μ called the μ–constant stratum. This stratum can be defined as follows: take the miniversal deformation $\varphi : (\mathcal{Y}, 0) \to (S, 0)$ of a plane curve singularity C. Denote by μ the Milnor number of C and by $\mathcal{Y}_s := \varphi^{-1}(s)$ a fiber of the deformation, then

$$\Delta^\mu := \{s \in S \mid \mu(\mathcal{Y}_s) = \mu\}.$$

Then it can be proven that this stratum is smooth (see Theorem 2.61 in [8]) and its codimension can be computed from the embedded resolution of the plane curve by the following formula given by Wall in Sect. 8 of [11].

Theorem 2 (Theorem 8.1 in [11], (2.8.36) p. 373 in [8]) *If* $(C, 0)$ *is a germ of plane curve singularity,* e_p *is the sequence of multiplicities of the strict transform of the embedded resolution of* C *and* c *is the number of free points in the resolution then*

$$\operatorname{codim}(\Delta^{\mu}) = \sum_{p} \frac{e_p(e_p+1)}{2} - c - 1.$$

3 Solutions to Dimca and Greuel Question

Dimca and Greuel's question has been completely solved by the author in [3]. However, before this general solution, there has been several solutions from different points of view for some families of plane curve singularities. In this section, we will try to overview in a chronological order the different results until reaching the general solution of Dimca and Greuel's question.

The first result about Question 1 was given in 2018 by Blanco and the author in [2] for semi-quasi-homogeneous singularities. We recall that f is a semi-quasi-homogeneous singularity with weights $w = (n, m)$ such that $\gcd(n, m) \geq 1$ and $n, m \geq 2$ if $f = f_0 + g$ is a deformation of the initial term $f_0 = y^n - x^m$ such that $\deg_w(f_0) < \deg_w(g)$. For such singularities, Blanco and the author in [2] give a positive answer to this question. This answer is due to a formula for the minimal Tjurina number of the family of semi-quasi-homogeneous given by Briançon, Granger and Maisonobe in [4]. The idea here is to use the upper semicontinuity of the Tjurina number (Theorem 2.6 in [8]) to reduce the proof to show the inequality for $\mu/\tau_{\min}$. In fact, until the appearance of the general solution this was the only non–irreducible family of plane curve singularities for which Question 1 was solved.

In 2019, a series of three preprints [1, 7, 12] appeared in a short time. They give a positive answer for the case of irreducible germs of plane curve singularities. The three approaches are based on the explicit computation of the dimension of the generic component of the moduli space of irreducible plane curve singularities given by Genzmer in [6] in terms of the sequence of multiplicities of the strict transform of a resolution of the irreducible plane curve. It was shown by Zariski in [13] and Teissier in the appendix to the book of Zariski [9] that to compute the dimension of the generic component of the moduli space of irreducible plane curves is closely related to compute the minimal Tjurina number in the equisingularity class of a branch. The relation of the dimension of the generic component of the moduli and the minimal Tjurina number of irreducible plane curves is due to the properties of Teissier monomial curve (see [9]).

In April 2019, Alberich-Carramiñana, Blanco, Melle-Hernández and the author in [1] gave a positive answer to Question 1 through a formula for the minimal Tjurina number in an equisingularity class of irreducible plane curve singularities in terms of the sequence of multiplicities that can be obtained from Genzmer's formula in [6] together with Wall's formula (Theorem 2). A few days after, Genzmer and Hernandes in [7] provided an alternative proof of Dimca and Greuel's inequality. Even if the techniques used are quite different, both results are based on the explicit computations given by Genzmer in [6]. Finally, at the end of April, Wang in [12] gave another alternative proof for the irreducible case based also in Genzmer's result about the dimension of the generic component of the moduli space in [6]. However, Wang's approach is very interesting since he proves that $3\mu - 4\tau$ is a monotonic increasing invariant under blow–ups for irreducible plane curve singularities which provides a nice perspective in the possible applications of Dimca and Greuel's question.

After the previous discussion one realized that all the answers to Dimca and Greuel's question are based on the explicit computation of the minimal Tjurina number of certain families of singularities. However, they do not provide an answer to the second question that we formulated in the introduction: can the $4/3$ bound be inferred from the geometry of the plane curve singularity? One may think that Wang's approach gives the answer to this question for the irreducible case. However, one cannot prove that $3\mu - 4\tau$ is an increasing monotonic invariant under blow-ups if one does not have Genzmer's formula for the dimension of the generic component of the moduli space. Moreover, implicitly there is no reason to consider $a\mu - b\tau$ with $(a, b) \neq (3, 4)$. In this way, the reason about $4/3$ remained open after Wang's result even for irreducible plane curves.

Finally, based on the idea to provide a full answer to Dimca and Greuel's question, the author in [3] changed the point of view. This new approach is based on the theory of deformation for surface singularities. More concretely, in the geometry of normal two–dimension double point singularities. Normal two–dimension double point singularities have equation $\{z^2 + f(x, y) = 0\} \subset \mathbb{C}^3$ with $f(x, y) = 0$ defining a germ of plane curve singularity. Moreover, they have the same Milnor and Tjurina numbers than the associated plane curve singularity. From this point of view, one can use the upper bound for the difference $\mu - \tau$ given by Wahl in [10] in terms of the geometric genus of the double point singularity. After that, the good properties of the geometric genus of such a surface singularities allow the author to provide a full answer to Dimca and Greuel's question.

Theorem 3 ([3]) *For any germ of plane curve singularity*

$$\frac{\mu}{\tau} < \frac{4}{3}.$$

Moreover, in this setting the bound $4/3$ is inferred from the geometric genus of the surface singularity. Also, the author in [3] provides a general framework that allow to continue with the problem of finding bounds for the quotient of Milnor and Tjurina numbers in higher dimension.

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When Is a Complete Ideal in a Rational Surface Singularity a Multiplier Ideal?

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Abstract This is an extended abstract with some of the results that will appear in the forthcoming paper [1] in which we characterize when a given complete ideal in a two-dimensional local ring with a rational singularity can be realized as a multiplier ideal.

1 Introduction

Let X be a complex variety of dimension d which is $\mathbb{Q}$-Gorenstein and $\mathcal{O}_{X,O}$ its corresponding local ring at a point $O \in X$, with $\mathfrak{m} = \mathfrak{m}_{X,O}$ being the maximal ideal. Given an ideal $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$ and a parameter $\lambda \in \mathbb{R}$ we may consider its corresponding *multiplier ideal* $\mathcal{J}(\mathfrak{b}^{\lambda}) \subseteq \mathcal{O}_{X,O}$. It follows from its construction that multiplier ideals are complete so it is natural to wonder how special are multiplier ideals among all complete ideals.

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When X is smooth and $d = 2$, it was proved independently by Favre and Jonsson [2] and Lipman and Watanabe [7], that every complete ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ can be realized as a multiplier ideal; that is, we may find an ideal $\mathfrak{b}$ and a parameter λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda})$. This result is no longer true for $d \geq 3$ as it has been proved by Lazarsfeld and Lee in [4]. Indeed, they show some delicate properties regarding the vanishing of the syzygies of multiplier ideals which lead to the existence of complete ideals in higher dimension that cannot be realized as multiplier ideals.

Lazarsfeld, Lee and Smith [5] partially extended the results in [4] to the non-smooth case by giving some vanishing result on the first syzygy of multiplier ideals. This condition is still enough to cook up examples of complete ideals that cannot be realized as multiplier ideals when $d \geq 3$. They also quoted in [5, Question 3.12] the following question regarding the remaining case that is left open: *Is every complete ideal in a complex algebraic surface having a rational singularity a multiplier ideal?* A partial answer to this question was provided by Tucker in [8] by showing that this is indeed the case when X has a log-terminal singularity. In a forthcoming paper [1] we will give a characterization of complete ideals that can be realized as multiplier ideals by means of a new invariant that we introduce, the *limiting boundary* Δ^*_D, and we give examples where a complete ideal cannot be realized.

2 A Reformulation of the Problem via Antinef Closures

Let (Y, O) be a germ of complex surface with at worst a rational singularity. Let $\mathcal{O}_{Y,O}$ denote the local ring at O and let $\mathfrak{m} = \mathfrak{m}_{Y,O} \subseteq \mathcal{O}_{Y,O}$ be the maximal ideal. Let $\pi : X \to Y$ be a log-resolution of a $\mathfrak{m}$-primary complete ideal $\mathfrak{a} \subseteq \mathcal{O}_{Y,O}$. We say that $\mathfrak{a}$ is *realized as a multiplier ideal* in X if there exists another $\mathfrak{m}$-primary ideal $\mathfrak{b}$ such that π is also a log-resolution for $\mathfrak{b}$ and there is a rational number λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda})$. More precisely, let F and G be integral exceptional divisors such that $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-F)$ and $\mathfrak{b} \cdot \mathcal{O}_X = \mathcal{O}_X(-G)$. Let K_π be the relative canonical divisor which is a $\mathbb{Q}$-divisor with exceptional support. Then we want to find λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda}) := \pi_* \mathcal{O}_X(\lceil K_\pi - \lambda G \rceil)$, where $\lceil \cdot \rceil$ denotes the round up of any $\mathbb{Q}$-divisor and is nothing but rounding up its coefficients..

Lipman [6, Sect. 18] gave a correspondence between complete ideals and *antinef divisors* that will give us the right framework where we can address this question. Recall that an effective integral exceptional divisor $D \in \mathrm{EDiv}^{\geq 0}(X)$ is antinef if $D \cdot C_i < 0$ for all the irreducible components $C_1, \dots, C_r$ of the exceptional locus. Given any effective rational exceptional divisor $D \in \mathrm{EDiv}^{\geq 0}_{\mathbb{Q}}(X)$ we may either consider its:

· *Integral antinef closure :* $\widetilde{D} := \min \left\{ D' \in \mathrm{EDiv}^{\geq 0}(X) \mid D' \geq D, \quad D' \cdot C_i \leq 0 \quad \forall i \right\}$,

· *Rational antinef closure:* $\widetilde{D}^{\mathbb{Q}} = \min \left\{ D' \in \mathrm{EDiv}^{\geq 0}_{\mathbb{Q}}(X) \mid D' \geq D, \quad D' \cdot C_i \leq 0 \quad \forall i \right\}$.

The existence of the integral antinef closure can be found in [6, Sect. 18] and it can be computed using the *unloading procedure* described next: Set $D_0 = \lceil D \rceil$. For any

$k \geq 0$, whenever there is an exceptional component C_i such that $D_k \cdot C_i > 0$, define $D_{k+1} = D_k + C_i$. If there is no such C_i, then $\widetilde{D} = D_k$.

The existence of the $\mathbb{Q}$-antinef closure follows from the cone structure of the set of antinef divisors. To describe it we use the $\mathbb{Q}$-*unloading procedure*, which can be deduced from [3], and is described next: Set $D_0 = D$. For any $k \geq 0$, whenever there is an exceptional component C_i such that $D_k \cdot C_i > 0$, define $D_{k+1} = D_k + \sum x_i C_i$, where the x_i are the solutions of the system of equations $\sum \left(C_i \cdot C_j\right) x_i = -D \cdot C_j, \quad \forall i, j$. If there are no such C_i, then $\widetilde{D}^{\mathbb{Q}} = D_k$.

The main result of this section is a reformulation of our initial problem in terms of the following boundary $\mathbb{Q}$-divisors that measure the difference between a divisor and its $\mathbb{Q}$-antinef closure. Namely, given any rational exceptional divisor D, we define

$$\Delta_D = \widetilde{(D + K_\pi)}^{\mathbb{Q}} - (D + K_\pi) \geq 0.$$

Now, given a convenient log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$, we want to check whether there exists an antinef divisor G and a rational number λ such that

$$\lfloor \widetilde{\lambda G - K_\pi} \rfloor = F \tag{1}$$

Notice that the rational divisor λG is antinef as well and, denoting $D = \lfloor \lambda G - K_\pi \rfloor$, we have that $D + K_\pi \leq \lambda G$. Therefore, the $\mathbb{Q}$-antinef closure of $D + K_\pi$ satisfies

$$D + K_\pi \leq \widetilde{(D + K_\pi)}^{\mathbb{Q}} \leq \lambda G$$

and thus

$$D = \lfloor D + K_\pi - K_\pi \rfloor \leq \left\lfloor \widetilde{(D + K_\pi)}^{\mathbb{Q}} - K_\pi \right\rfloor \leq \lfloor \lambda G - K_\pi \rfloor = D.$$

Under these premises, Eq. 1 becomes

$$\left\lfloor \widetilde{\widetilde{(D + K_\pi)}^{\mathbb{Q}} - K_\pi} \right\rfloor = \lfloor \widetilde{D + \Delta_D} \rfloor = F. \tag{2}$$

Our approach to the problem is through the following

Proposition 1 *An* $\mathfrak{m}$*-primary complete ideal* $\mathfrak{a}$ *is realized as a multiplier ideal if and only if there is a log-resolution* $\pi : X \to Y$ *of* $\mathfrak{a}$ *with* $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ *and an integral exceptional divisor* D *such that* $D \geq \lfloor -K_\pi \rfloor$, $\widetilde{D} = F$, *and* $\lfloor \Delta_D \rfloor = 0$.

3 Working in a Fixed Log-Resolution

Let's start with a fixed log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$. It might well happen that we can not find an integral exceptional divisor D satisfying the conditions of Proposition 1. Indeed there are cases in which we may find such a divisor in a further log-resolution and cases where it will be impossible to find it, and thus giving examples of complete ideals that can not be realized as multiplier ideals (see Sect. 5). Even though working in a fixed log-resolution has a lot of shortcomings, the methods we present in this section will illustrate the main ideas behind our general method.

The starting point of our method comes from the unloading procedure. We can reach every $D \geq \lfloor -K_\pi \rfloor$ with $\widetilde{D} = F$ by starting with $D = F$ and then go backwards replacing D by $D - C$ for any exceptional component with $(D - C) \cdot C > 0$, and contained in the support of $D - \lfloor -K_\pi \rfloor$. If this is the case we say that going from D to $D - C$ is an *admissible subtraction*. Moreover, without getting into technical details, the multiplicities of Δ_{D-C} are smaller than the multiplicities of Δ_D when $(D + K_\pi + \Delta_D) \cdot C < 0$. We will say in this case that we have a *strict subtraction*. If a subtraction is admissible and strict we say that it is a *good subtraction*.

Our goal would be to find a chain of admissible subtractions $F > D_1 > \cdots > D_n = D$ such that $\lfloor \Delta_D \rfloor = 0$. In the case that every subtraction in the chain is also strict, hence good, we will say that $D < F$ is a *good subdivisor* and it is characterized as follows:

Proposition 2 *$D < F$ is a good subdivisor if and only if* $\mathrm{mult}_C(\Delta_D) < 1$ *for every subtracted component $C \subset$* $\mathrm{supp}(F - D)$.

It leads to the following characterisation:

Proposition 3 *An $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ is realized as a multiplier ideal if and only if there is a log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ and a good subdivisor $D < F$ such that* $\lfloor \Delta_D \rfloor = 0$.

This provides an efficient algorithm to decide whether a complete ideal can be realized as multiplier ideal in X. Obviously if $\lfloor \Delta_F \rfloor = 0$ then $\mathfrak{a}$ is a multiplier ideal. Otherwise, we can take F and consider recursively all the possible strict subtractions, until we either find some D with $\lfloor \Delta_D \rfloor = 0$ or we run out of divisors (in which case $\mathfrak{a}$ cannot be realized as multiplier ideal in X). We point out that we may find examples of surfaces with a log-terminal singularity and ideals that can not be realized in a given log-resolution. We already know, by Tucker's result [8], that they must be realized in a further log-resolution.

4 Comparing Log-Resolutions

In general, we have to study how the Δ_D behave in different log-resolutions, in order to obtain the best good chains possible. In order to get a minimal Δ_D we would consider only strict subtractions $D - C$ and, in the case that they are not admissible, it would require to blow-up $m = 1 - (D - C) \cdot C \geq 0$ smooth points of C to make them admissible, and thus good. This process can be quite involved but we can speed it up using what we call

Standard procedure with length N: Let $\pi : X \to Y$ be a log-resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ and consider (X, F) as our starting pair. Given a positive integer N we will produce a sequence $X_n^{(N)} \to \cdots \to X_1^{(N)} \to X \to Y$, hence a sequence of pairs $(X_n^{(N)}, D_n^{(N)})$ as follows:

- If some initial irreducible component $C_i \subset X$ is good-subtractible from $D_n^{(N)}$, then take $X_{n+1}^{(N)} = X_n^{(N)}$ and $D_{n+1}^{(N)}$ as the result of subtracting C_i and all subsequent possible good subtractions of non-initial components.
- If some initial irreducible component $C_i \subset X$ is strict-subtractible but not admissible, set $m_{n,i} = 1 - (D_n^{(N)} - C_i) \cdot C_i$. Then blow up C_i at $m_{n,i}$ smooth points, further blow-up each of the resulting $m_{n,i}$ exceptional components at a smooth point, and then blow-up each of the newest exceptional components, and so on until we have added $Nm_{n,i}$ exceptional components, forming $m_{n,i}$ *tails* of length N attached to the original exceptional divisor at C_i. Then subtract C_i and all subsequent possible good subtractions of non-initial components (including the newest ones).
- If no initial component is strict-subtratible, stop.

Remark 1 Each pair $(X_n^{(N)}, D_n^{(N)})$ is determined by data on the initial log-resolution $\pi : X \to Y$ if one also remembers how many tails have been created from each initial exceptional component. More precisely, each step can be codified by the pair $(\overline{D_n^{(N)}}, m_n)$, where $\overline{D_n^{(N)}}$ is the image of $D_n^{(N)}$ in X and $m_n = \left(m_{n,1}, \ldots, m_{n,r}\right) \in \mathbb{Z}^r_{\geq 0}$ is the vector such that at this step there are $m_{n,i}$ tails attached to the initial components $C_1, \ldots, C_r$.

At each step we may consider the corresponding $\Delta_{D_n^{(N)}}$ and its images $\overline{\Delta_{D_n^{(N)}}} \subset X$ decrease and have a limit Δ_n^* when $N \to \infty$ that can be computed as follows

Proposition 4 *Let* $\left(X^{(N)}, D^{(N)}\right)$ *be a pair computed using the standard procedure of length* N*, and for each* $i = 1, \ldots, r$ *let* m_i *be the number of tails attached to the initial exceptional component* $C_i \subset X$*. Then there exists* $\Delta_D^* = \lim_{N\to\infty} \overline{\Delta_{D^{(N)}}}$*, which can be computed as the smallest solution of the system of inequalities*

$$\left(\overline{D^{(N)}} + K_0 + \Delta_D^*\right) \cdot C_i < -m_i \quad i = 1, \ldots, r.$$

The fact that the limiting boundary Δ_D^* can be computed on the initial log-resolution by taking into account the tail-counting vector m motivates the following definitions.

Definition 1 A divisor $D \subset X$ is an *asymptotically good subdivisor* of F if for big enough $N \in \mathbb{N}$ there is a pair $\left(X^{(N)}, D^{(N)}\right)$ obtained by the standard procedure of length N such that the image of $D^{(N)}$ in X is D.

Let $D \leq F$ be an asymptotically good subdivisor and $C \subset X$ an (initial) exceptional component. We say that the subtraction $D > D - C$ is *asymptotically good* if for big enough $N \in \mathbb{N}$ there is a pair $\left(X^{(N)}, D^{(N)}\right)$ obtained by the standard procedure of length N such that the image of $D^{(N)}$ in X is D and $D^{(N)} > D^{(N)} - C$ is a good subtraction (where we identify $C \subset X_0$ with its strict transform in $X^{(N)}$).

Asymptotically good subtractions can be numerically characterized in the original log-resolution with the help of the tail-counting vector $m \in \mathbb{N}^r$.

Lemma 1 *Let (D, m) be a pair given by $D \subset X$ and $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$. The subtraction $D > D - C_i$ of the exceptional component C_i is asymptotically good with m_i tails constructed from each exceptional component C_i if and only if*

$$\left(D + K_\pi + \Delta_D^*\right) \cdot C_i < -m_i$$

It follows from the definition that a subdivisor $D \leq F \subset X$ is asymptotically good if it can be reached from F by a chain of asymptotically good subtractions

$$(F, 0) > (D_1, m_1) > \cdots > (D_n, m_n) = (D, m),$$

where the convention that we follow is that an *asymptotic subtraction* is $(D, m) > \left(D', m'\right)$ where $D' = D - C_i$ for some exceptional component C_i, $m_i \leq m'_i$ and $m_j = m'_j$ for all $j \neq i$. The main result of this work is

Theorem 1 *Let $\pi : X \to Y$ be a log-resolution of an $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$. The ideal $\mathfrak{a}$ is realized as a multiplier ideal in a further log-resolution if and only if there is an asymptotically good chain from $(F, 0)$ to a pair (D, m) such that $\lfloor \Delta_D^* \rfloor$=0.*

Example 1 Consider the rational singularity given by the intersection matrix

$$M = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

with relative canonical divisor $K_\pi = \left(-1, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, 0\right)$. In particular it is a log-canonical singularity. Consider the antinef divisor $F = (2, 1, 1, 1, 1, 4)$ and let's look for asymptotically good chains. We first compute $\Delta_F = \Delta_F^* = -K_\pi = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)$, with

$$\left(F + K_\pi + \Delta_F^*\right) \cdot M = F \cdot M = (0, 0, 0, 0, 0, -2) \leq (0, 0, 0, 0, 0, 0) = -m_0.$$

The only asymptotically strict subtraction is that of C_6, but since $F \cdot C_6 = -2 \leq -1 = C_6^2$, two tails need to be added. This means we have to take $D_1 = F -$

$C_6 = (2, 1, 1, 1, 1, 3)$ and $m_1 = (0, 0, 0, 0, 0, 2)$. Then we have $\Delta^*_{D_1} = K_\pi + C_6 = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$, with

$$\left(D_1 + K_\pi + \Delta^*_1\right) \cdot M = F \cdot M = (0, 0, 0, 0, 0, -2) = -m_1.$$

No further asymptotically strict subtraction is thus possible. Since both $\lfloor \Delta^*_F \rfloor$, $\lfloor \Delta^*_{D_1} \rfloor \neq 0$, the ideal defined by F is not a multiplier ideal.

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