On Generalized Convexity and Superquadracity

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Abstract In this paper we deal with generalized ψ-uniformly convex functions and with superquadratic functions and discuss some of their similarities and differences. Using the techniques discussed here, we obtain reversed and refined Minkowski type inequalities.

1 Introduction

Convex and convex type functions and their relations to mathematical inequalities play an important role in science, see, for instance, [3] about electrical engineering and [5] about statistical applications and their references.

In this paper we deal with generalized ψ-uniformly convex functions and with superquadratic functions and discuss some of their similarities and differences.

We start quoting the definition and properties of superquadratic functions from [1] which include the functions $f(x) = x^p$, $x \geq 0$, when $p \geq 2$, the functions $f(x) = -\left(1 + x^{\frac{1}{p}}\right)^p$ when $p > 0$ and $f(x) = 1 - \left(1 + x^{\frac{1}{p}}\right)^p$ when $p \geq \frac{1}{2}$. Also, we quote from [4] the definition of generalized ψ-uniformly convex functions.

In Sect. 2 we emphasize the importance of the general definition of superquadracity appearing in [1, Definition 2.1] compared with some of its special cases and with the generalized ψ-uniformly convex functions defined in [4].

In Sect. 3, by using the results discussed in Sect. 2 we refine and reverse the well known Minkowski inequality that says

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}, \tag{1}$$

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for $0 < p < 1,\ a_i, b_i \geq 0,\ i = 1, \ldots, n.$

Definition 1 ([1, Definition 2.1]) A function $f : [0, B) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0, B)$ there exists a constant $C_f(x) \in \mathbb{R}$ such that the inequality

$$f(y) \geq f(x) + C_f(x)(y - x) + f(|y - x|) \tag{2}$$

holds for all $y \in [0, B)$ (see [1, Definition 2.1], there $[0, \infty)$ instead $[0, B)$).

f is called subquadratic if $-f$ is superquadratic.

Theorem 1 ([1, Theorem 2.2]) *The inequality*

$$\int f(g(s))\, d\mu(s) \geq f\left(\int g d\mu\right) + f\left(\left|g(s) - \int g d\mu\right|\right)$$

holds for all probability measures and all non-negative, μ-integrable functions g if and only if f is superquadratic.

Corollary 1 ([1, 2]) *Suppose that f is superquadratic. Let $0 \leq x_i < B$, $i = 1, 2$ and let $0 \leq t \leq 1$. Then*

$$\begin{aligned} &tf(x_1) + (1 - t) f(x_2) - f(tx_1 + (1 - t) x_2) \\ &\geq tf((1 - t)|x_2 - x_1|) + (1 - t) f(t|x_2 - x_1|) \end{aligned} \tag{3}$$

holds.

More generally, suppose that f is superquadratic. Let $\xi_i \geq 0$, $i = 1, \ldots, m$, and let $\overline{\xi} = \sum_{i=1}^{m} p_i \xi_i$ where $p_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} p_i = 1$. Then

$$\sum_{i=1}^{m} p_i f(\xi_i) - f\left(\overline{\xi}\right) \geq \sum_{i=1}^{m} p_i f\left(\left|\xi_i - \overline{\xi}\right|\right) \tag{4}$$

holds.

If f is non-negative, it is also convex and Inequality (4) refines Jensen's inequality. In particular, the functions $f(x) = x^r$, $x \geq 0$, are superquadratic and convex when $r \geq 2$, and subquadratic and convex when $1 < r < 2$. Equality holds in inequalities (3) and (4) when $r = 2$.

Lemma 1 ([1, Lemma 2.1]) *Let f be superquadratic function with $C_f(x)$ as in Definition 1. Then:*

(i) $f(0) \leq 0$,
(ii) *if* $f(0) = f'(0) = 0$, *then* $C_f(x) = f'(x)$ *whenever* f *is differentiable at* $x > 0$,
(iii) *if* $f \geq 0$, *then* f *is convex and* $f(0) = f'(0) = 0$.

Lemma 2 ([1, Lemma 3.1]) *Suppose* $f : [0,\infty) \to \mathbb{R}$ *is continuously differentiable and* $f(0) \le 0$*. If* f' *is superadditive or* $\frac{f'(x)}{x}$ *is non-decreasing, then* f *is superquadratic and (according to its proof)* $C_f(x) = f'(x)$*, where* $C_f(x)$ *is as in Definition 1.*

Lemma 3 ([1, Lemma 4.1]) *A non-positive, non-increasing, and superadditive function is a superquadratic function and (according to its proof) satisfies* $C_f(x) = 0$*, where* $C_f(x)$ *is as in Definition 1.*

Example 1 ([1, Example 4.2]) Let

$$f_p(x) = -\left(1 + x^{\frac{1}{p}}\right)^p, \quad x \ge 0.$$

Then f_p is superquadratic for $p > 0$ with $C_{f_p}(x) = 0$ and $g = 1 + f_p$ is superquadratic for $p \ge \frac{1}{2}$ with $C_g(x) = g'(x) = f_p'(x)$.

Lemma 4 ([1, Section 3]) *Suppose that* f *is a differentiable function and* $f(0) = f'(0) = 0$*. If* f *is superquadratic, then* $\frac{f(x)}{x^2}$ *is non-decreasing.*

The definition of **generalized** ψ**-uniformly convex functions** as appears in [4] is the following:

Definition 2 ([4, Page 306]) Let $I = [a,b] \subset \mathbb{R}$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. A function $f : [a,b] \to \mathbb{R}$ is said to be *generalized* ψ*-uniformly convex* if:

$$tf(x) + (1-t) f(y) \ge f(tx + (1-t) y) + t(1-t)\psi(|x-y|)$$
$$\text{for } x, y \in I \text{ and } t \in [0,1]. \tag{5}$$

If in addition $\psi \ge 0$, then f is said to be ψ*-uniformly convex.*

Paper [4] deals with inequalities that extend the Levin-Stečkin's theorem. The main result in [4, Theorem 1] relates to the function ψ as appears in Definition 2, and depends on the fact that $\lim\limits_{t\to 0^+} \frac{\psi(t)}{t^2}$ is finite. We discuss this issue in Sect. 2.

In the unpublished [6] a companion inequality to Minkowski inequality is stated and proved:

Theorem 2 ([6, Th2.1]) *For* $0 < p < 1$*,* $a_i, b_i > 0$*,* $i = 1, \ldots, n$ *the inequality*

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}$$
$$\le \frac{\sum_{i=1}^n a_i b_i^{p-1}}{\left(\sum_{i=1}^n b_i^p\right)^{\frac{p-1}{p}}} + \frac{\sum_{i=1}^n b_i a_i^{p-1}}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{p-1}{p}}} \tag{6}$$

holds.

In Sect. 3 we refine Minkowski's inequality in four ways using generalized ψ-uniformly convexity, subquadracity and superquadracity properties of the functions discussed in Sect. 2. The proofs of Theorems 3 and 4 apply the technique employed in [6] to prove the right hand-side of (6) in Theorem 2, besides using superquadracity and subquadracity properties of the functions involved there.

2 Superquadracity and Generalized ψ-Uniformly Convexity

We start with emphasizing the importance of the definition of superquadracity as appears in [1] vis a vis its special cases. Definition 1 does not guarantee that $C_f(x) = f'(x)$. However, from Lemmas 1 and 2 we know that in the case that f is superquadratic and $f(0) = f'(0) = 0$, and in the case that the derivative of the superquadratic function is superadditive or $\frac{f'(x)}{x}$ is non-decreasing we get $C_f(x) = f'(x)$. On the other hand when the superquadratic function satisfies Lemma 3 we get that $C_f(x) = 0$.

Although the n-th derivative of $f_p(x) = -\left(1 + x^{\frac{1}{p}}\right)^p$, $x \geq 0$, $0 < p < 1$, as discussed in Example 1, is continuous on $[0, \infty)$, we get when inserting this function in Definition 1 that $C_{f_p}(x)$ satisfies $C_{f_p}(x) = 0 \neq f_p'(x) = -x^{\frac{1}{p}-1}\left(1 + x^{\frac{1}{p}}\right)^p$.

Therefore whenever

$$f(y) - f(x) \geq f'(x)(y - x) + f(|y - x|) \tag{7}$$

is used as the definition of superquadracity, it means that it deals not with the general case of superquadratic functions but it might, but not necessarily, deal with those superquadratic functions satisfying Lemma 1(ii) or Lemma 2. The following function f is an example of a superquadratic function that satisfies (7) but as proved in [1, Example 3.3] does not satisfy Lemma 2: This function is defined by $f(0) = 0$ and

$$f'(x) = \begin{cases} 0, & x \leq 1 \\ 1 + (x-2)^3, & x \geq 1. \end{cases}$$

For such superquadratic functions, Definition 1 translates into (7), but as explained above it does not hold for all superquadratic functions.

We point out now a difference between the superquadratic functions and the generalized ψ-uniformly convex functions:

According to the proof of Theorem 1 [1, Theorem 2.2] and Corollary 1 we get that inequalities (2) and (3) are equivalent. On the other hand, Inequality (5) that defines, according to [4], the generalized ψ-uniformly convex function f, when f is continuously differentiable, leads to the inequality

$$f(y) - f(x) \geq f'(x)(y - x) + \psi(|y - x|), \tag{8}$$

as proved in [4, Theorem 1], but Inequality (8) does not lead in general to Inequality (5) but to

$$\begin{aligned} & tf(x_1) + (1 - t) f(x_2) - f(tx_1 + (1 - t) x_2) \\ & \quad \geq t\psi((1 - t)|x_2 - x_1|) + (1 - t)\psi(t|x_2 - x_1|), \end{aligned} \tag{9}$$

for $0 \leq t \leq 1$.

More generally, it is easy to verify that, similarly to Inequality (4) for superquadratic functions, when f is a generalized ψ-uniformly convex function, then

$$\sum_{i=1}^{m} p_i f(\xi_i) - f(\overline{\xi}) \geq \sum_{i=1}^{m} p_i \psi\left(\left|\xi_i - \overline{\xi}\right|\right), \tag{10}$$

holds, where $\xi_i \geq 0$, $i = 1, \ldots, m$, $\overline{\xi} = \sum_{i=1}^{m} p_i \xi_i$, $p_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} p_i = 1$.

In addition, if ψ is non-negative, the function f is also convex and Inequality (10) refines Jensen's inequality.

Moreover, if instead of (5) in Definition 2 we have a set of functions f which satisfies

$$\begin{aligned} & tf(x_1) + (1 - t) f(x_2) - f(tx_1 + (1 - t) x_2) \\ & \quad \geq G(t)\psi(|x_1 - x_2|), \qquad t \in [0, 1], \end{aligned} \tag{11}$$

then (11) still leads to (8) when $\lim\limits_{t \to 0^+} \frac{G(t)}{t} = 1$.

However, for the special case where $\psi(x) = kx^2$, when k is constant, the inequalities (5) and (9) are the same.

Remark 1 By choosing $x = y$ in (5) or in (8) we get that ψ satisfies $\psi(0) \leq 0$.

From now on till the end of this section we deal with functions satisfying inequalities (7) and (8).

A similarity between convex superquadratic functions and ψ-uniformly convex functions is shown in Remark 2 below. The set of convex superquadratic functions f satisfies $f(0) = f'(0) = 0$. Also, the set f of ψ-uniformly convex functions satisfies $\psi(0) = \psi'(0) = 0$.

For the convenience of the reader a proof of Remark 2 is presented. This can easily be obtained by following the steps of the proof in [1] of Lemma 1(iii):

Remark 2 For a function $\psi : [0, b - a] \to \mathbb{R}$ and a continuously differentiable ψ-uniformly convex function f on $[a, b] \to \mathbb{R}$, we get that $\psi(0) = \psi'(0) = 0$.

Proof If $\psi \geq 0$, then $\psi(0) = 0$ because always as mentioned in Remark 1 $\psi(0) \leq 0$. Then by choosing in (8) first $y > x$ and then $y < x$ we get that

$$\limsup_{y \to x^-} \left(\frac{f(x) - f(y)}{x - y} + \frac{\psi(x - y)}{x - y} \right) \leq f'(x) \leq \limsup_{y \to x^+} \left(\frac{f(y) - f(x)}{y - x} + \frac{\psi(y - x)}{y - x} \right),$$

and hence

$$\limsup_{x \to 0^+} \frac{\psi(x)}{x} \leq 0.$$

Since ψ is non-negative, we have

$$0 \leq \limsup_{x \to 0^-} \frac{\psi(x)}{x} \leq \limsup_{x \to 0^+} \frac{\psi(x)}{x} \leq 0,$$

and therefore the one sided derivative at zero exists and $\psi'(0) = 0$.

We deal now with the behavior of $\frac{\psi(x)}{x^2}$ when f is generalized ψ-uniformly convex function, and with $\frac{f(x)}{x^2}$ when f is superquadratic.

Besides Lemma 4 we get the following lemma which is proved in [4, Proof of Theorem 1]:

Lemma 5 *If f is twice continuously differentiable generalized ψ-uniformly convex function, then $f''(x) \geq 2 \lim_{x \to 0^+} \frac{\psi(x)}{x^2}$.*

Corollary 2 *Let $I = [a, b]$ be an interval and $\psi : [0, b - a] \to \mathbb{R}$ be a twice differentiable function on $[0, b - a]$. Let $f : [a, b] \to \mathbb{R}$ be a continuously twice differentiable ψ-uniformly convex function, that is $\psi \geq 0$. Denote $\varphi(x) = \frac{\psi(x)}{x^2}$, $x > 0$. Then $\varphi(0) = \lim_{x = 0^+} \frac{\psi(x)}{x^2}$ is finite and non-negative.*

Indeed, Remark 2 says that $\psi(0) = \psi'(0) = 0$. Therefore,

$$\lim_{x = 0^+} \frac{\psi(x)}{x^2} = \varphi(0) = \lim_{x = 0^+} \frac{\psi'(x)}{2x} = \lim_{x = 0^+} \frac{\psi''(x)}{2} = \frac{\psi''(0)}{2}.$$

Remark 3 It is shown in Remark 1 that ψ satisfies $\psi(0) \leq 0$ and therefore when $\psi(0) < 0$ we get $\lim_{x \to 0^+} \frac{\psi(x)}{x^2} = -\infty$. Also, when ψ is differentiable on $[0, b - a]$ and $\psi(0) = 0$ but $\psi'(0) < 0$ then again

$$\lim_{x\to 0^+} \frac{\psi(x)}{x^2} = \lim_{x\to 0^+} \frac{\psi'(x)}{2x} = -\infty.$$

Example 2 shows that the conditions $\psi(0) = 0$, $\psi'(0) = 0$ do not guarantee that $\lim_{x\to 0^+} \frac{\psi(x)}{x^2}$ is finite:

Example 2 The superquadratic function $f(x) = x^2 \ln x$ for $x > 0$ and $f(0) = 0$, $f'(0) = 0$ is continuously differentiable but not twice continuously differentiable at $x = 0$. Therefore we deal now with an interval $[a, b]$, $a > 0$ for $f(x) = x^2 \ln x$ which is twice differentiable and $\psi(x) = x^2 \ln x$, $0 < x \leq b - a$. These f and ψ satisfy (8). In this case $\lim_{x\to 0^+} \frac{\psi(x)}{x^2} = -\infty$.

We show here an example where $\lim_{x\to 0^+} \frac{\psi(x)}{x^2}$ is finite, but the generalized ψ-uniformly convex function g is not necessarily convex.

Example 3 Let $g(x) = f(x) - (kx)^2$ where k is a constant and f is twice differentiable convex and superquadratic function satisfying $\lim_{x\to 0^+} \frac{f(x)}{x^2} = \varphi(0)$ and $\varphi(0) \geq 0$. In such cases $\frac{g(x)}{x^2} = \varphi(x) - k^2 \underset{x\to 0^+}{\to} \varphi(0) - k^2$ and because $\varphi(0)$ is finite and non-negative, and because equality holds in (3) for the function x^2, the function g is superquadratic satisfying (7) and therefore also (8) for $\psi = f$, but is not necessarily convex.

In addition to the monotonicity of $\frac{f(x)}{x^2}$ as proved in Lemma 4 for superquadratic functions satisfying $f(0) = f'(0) = 0$, it is easy to prove:

Remark 4 If Inequality (8) when $x \geq 0$ holds for $\psi \geq 0$ and $f(0) = 0$, then f is convex and $\left(\frac{f(x)}{x}\right)' \geq \frac{\psi(x)}{x^2} \geq 0$. In the special case that f is superquadratic and convex, we get that $\left(\frac{f(x)}{x}\right)' \geq \frac{f(x)}{x^2} \geq 0$.

Indeed, from (8) we get that

$$f(0) - f(x) \geq -xf'(x) + \psi(x)$$

holds.

From this, because $f(0) = 0$ we get that

$$\frac{xf'(x) - f(x)}{x^2} = \left(\frac{f(x)}{x}\right)' \geq \frac{\psi(x)}{x^2} \geq 0.$$

In the special case that f is superquadratic we get that

$$\left(\frac{f(x)}{x}\right)' \geq \frac{f(x)}{x^2} \geq 0.$$

We finish this section demonstrating a set of continuous differentiable functions satisfying Inequality (8). As explained above, (8) holds for continuous differentiable generalized ψ-uniformly convex functions.

Example 4 The functions $f_p = -\left(1 + x^{\frac{1}{p}}\right)^p$ where $\psi_t(x) = t - \left(1 + x^{\frac{1}{p}}\right)^p$, $p \geq \frac{1}{2}$, $0 \leq t \leq 1$, $x \geq 0$ are generalized ψ_t-uniformly convex functions and satisfy (8). In particular, when $t = 0$, the function f_p is superquadratic and when $t = 1$ the function $f^*(x) = 1 - \left(1 + x^{\frac{1}{p}}\right)^p$ where $\psi_1(x) = 1 - \left(1 + x^{\frac{1}{p}}\right)^p$ is also superquadratic.

Indeed, $f^*(x) = 1 - \left(1 + x^{\frac{1}{p}}\right)^p$, $p \geq \frac{1}{2}$ is superquadratic satisfying Inequality (7). Specifically as shown in Example 1 [1, Example 4.2] the inequality

$$1 - \left(1 + y^{\frac{1}{p}}\right)^p - \left(1 - \left(1 + x^{\frac{1}{p}}\right)^p\right) \geq -\left(1 + x^{-\frac{1}{p}}\right)^{p-1}(y - x) + \left(1 - \left(1 + |x - y|^{\frac{1}{p}}\right)^p\right)$$

holds, which is the same as Inequality (8)

$$-\left(1 + y^{\frac{1}{p}}\right)^p - \left(-\left(1 + x^{\frac{1}{p}}\right)^p\right) \geq -\left(1 + x^{-\frac{1}{p}}\right)^{p-1}(y - x) + \left(1 - \left(1 + |x - y|^{\frac{1}{p}}\right)^p\right),$$

for $f_p(x) = -\left(1 + x^{\frac{1}{p}}\right)^p$ and $\psi_1(x) = 1 - \left(1 + x^{\frac{1}{p}}\right)^p$.

Therefore, also

$$-\left(1 + y^{\frac{1}{p}}\right)^p - \left(-\left(1 + x^{\frac{1}{p}}\right)^p\right) \geq -\left(1 + x^{-\frac{1}{p}}\right)^{p-1}(y - x) + \left(t - \left(1 + |x - y|^{\frac{1}{p}}\right)^p\right)$$

holds when $t \leq 1$ and Inequality (8) is satisfied by $f_p(x) = -\left(1 + x^{\frac{1}{p}}\right)^p$ and $\psi_t(x) = t - \left(1 + x^{\frac{1}{p}}\right)^p$.

As shown in Example 1, when $t = 0$, the function $f_p(x) = -\left(1 + x^{\frac{1}{p}}\right)^p$ is also superquadratic but this time satisfying (7) with $C_f(x) = 0$, that is,

$$-\left(1+y^{\frac{1}{p}}\right)^{p}-\left(-\left(1+x^{\frac{1}{p}}\right)^{p}\right)\geq-\left(1+|x-y|^{\frac{1}{p}}\right)^{p}$$

holds.

3 Reversed and Refined Minkowski Inequality

In this section we use the properties discussed in Sect. 2 of superquadracity and of generalized ψ-uniformly convexity.

In Example 1 [1, Example 4.2] it is shown that $f_p(x)=\left(1+x^{\frac{1}{p}}\right)^{p}$ for $x\geq 0$, is subquadratic when $p>0$. Using this property and Corollary 1 together with the convexity of f_p when $p<1$ we get a refinement of Minkowski's inequality when $0<p<1$ (see also [1, Theorem 4.1]):

Lemma 6 *Let* $a_i, b_i \geq 0,\ i=1,\ldots,n$. *Then, when* $p>0$ *the inequality*

$$\begin{aligned}&\sum_{i=1}^{n}(a_i+b_i)^{p}\\&\leq\left(\left(\sum_{i=1}^{n}a_i^{p}\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^{p}\right)^{\frac{1}{p}}\right)^{p}+\sum_{i=1}^{n}a_i^{p}\left(1+\left|\frac{b_i^{p}}{a_i^{p}}-\frac{\sum_{j=1}^{n}b_j^{p}}{\sum_{j=1}^{n}a_j^{p}}\right|^{\frac{1}{p}}\right)^{p}\end{aligned}\tag{12}$$

holds, and when $0<p<1$ *the inequalities*

$$\begin{aligned}\left(\left(\sum_{i=1}^{n}a_i^{p}\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^{p}\right)^{\frac{1}{p}}\right)^{p}&\leq\sum_{i=1}^{n}(a_i+b_i)^{p}\\&\leq\left(\left(\sum_{i=1}^{n}a_i^{p}\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^{p}\right)^{\frac{1}{p}}\right)^{p}+\sum_{i=1}^{n}a_i^{p}\left(1+\left|\frac{b_i^{p}}{a_i^{p}}-\frac{\sum_{j=1}^{n}b_j^{p}}{\sum_{j=1}^{n}a_j^{p}}\right|^{\frac{1}{p}}\right)^{p}\end{aligned}\tag{13}$$

hold.

Proof From the subquadracity of $f_p=\left(1+x^{\frac{1}{p}}\right)^{p}$, $x\geq 0$, $p>0$, according to Lemma 3 and Example 1 we get that:

$$\sum_{i=1}^{n}x_i\left(1+\left(\frac{y_i}{x_i}\right)^{\frac{1}{p}}\right)^{p}=\sum_{i=1}^{n}\left(x_i^{\frac{1}{p}}+y_i^{\frac{1}{p}}\right)^{p}$$

$$\leq \left(\left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{p}}\right)^{p} + \sum_{i=1}^{n} x_i \left(1 + \left|\frac{y_i}{x_i} - \frac{\sum_{j=1}^{n} y_j}{\sum_{j=1}^{n} x_j}\right|^{\frac{1}{p}}\right)^{p} \tag{14}$$

is satisfied. Substituting $x_i^{\frac{1}{p}} = a_i$ and $y_i^{\frac{1}{p}} = b_i, i = 1, \ldots, n$, we get Inequality (12), and together with the convexity of f for $0 < p < 1$ we get from (14) that (13) holds.

The next lemma uses the generalized ψ-uniformly convex functions $g_p = -\left(1 + x^{\frac{1}{p}}\right)^{p}$ when $\psi(x) = t - \left(1 + x^{\frac{1}{p}}\right)^{p}$, $0 \leq t \leq 1$ for $p \geq \frac{1}{2}$ and the convexity of $f_p(x) = \left(1 + x^{\frac{1}{p}}\right)^{p}$ when $0 < p < 1$ as discussed in Example 4. Similar to Lemma 6 we get:

Lemma 7 *Let $a_i, b_i > 0$, $i = 1, \ldots, n$ and $0 \leq t \leq 1$ then when $p \geq \frac{1}{2}$ the inequality:*

$$\sum_{i=1}^{n} (a_i + b_i)^p \leq \left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right)^{p}$$
$$+ \sum_{i=1}^{n} a_i^p \left(1 + \left|\frac{b_i^p}{a_i^p} - \frac{\sum_{j=1}^{n} b_j^p}{\sum_{j=1}^{n} a_j^p}\right|^{\frac{1}{p}}\right)^{p} - t\sum_{i=1}^{n} a_i^p$$

holds, and when $\frac{1}{2} \leq p \leq 1$, the inequalities

$$\left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right)^{p} \leq \sum_{i=1}^{n} (a_i + b_i)^p$$
$$\leq \left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right)^{p} + \sum_{i=1}^{n} a_i^p \left(1 + \left|\frac{b_i^p}{a_i^p} - \frac{\sum_{j=1}^{n} b_j^p}{\sum_{j=1}^{n} a_j^p}\right|^{\frac{1}{p}}\right)^{p}$$
$$-t\sum_{i=1}^{n} a_i^p$$

hold.

We finish the paper by refining Inequality (6) in Theorem 2, and we get two new Minkowski type inequalities. In the proofs we use the technique employed in [6, Theorem 2.1] and the subquadracity of $f(x) = x^{\frac{1}{p}}$, $x \geq 0$, $\frac{1}{2} < p < 1$, the superquadracity of $f(x) = x^{\frac{1}{p}}$, $x \geq 0$, $0 < p \leq \frac{1}{2}$.

Theorem 3 *Let* $0 < p < \frac{1}{2}$, $a_i, b_i \geq 0$, $i = 1, \ldots, n$. *Then, the inequalities*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}$$
$$\leq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}$$
$$\leq \frac{\sum_{i=1}^{n} a_i b_i^{p-1}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}} + \frac{\sum_{i=1}^{n} b_i a_i^{p-1}}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{p-1}{p}}}$$
$$- \frac{\sum_{i=1}^{n} a_i^p \left|\frac{(a_i+b_i)^p}{a_i^p} - \frac{\sum_{j=1}^{n}(a_j+b_j)^p}{\sum_{j=1}^{n} a_j^p}\right|^{\frac{1}{p}}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}}} - \frac{\sum_{i=1}^{n} b_i^p \left|\frac{a_i^p}{b_i^p} - \frac{\sum_{j=1}^{n} a_j^p}{\sum_{j=1}^{n} b_j^p}\right|^{\frac{1}{p}}}{\left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}}} \tag{15}$$

hold. Equality holds in the right hand-side of inequality (15) when $p = \frac{1}{2}$.

Proof We use the superquadracity of $g(x) = x^{\frac{1}{p}}$, $x \geq 0$, $0 < p \leq \frac{1}{2}$ which by Corollary 1 leads to the inequality

$$\sum_{i=1}^{n} x_i \left(\frac{y_i}{x_i}\right)^{\frac{1}{p}} = \sum_{i=1}^{n} x_i^{1-\frac{1}{p}} y_i^{\frac{1}{p}}$$
$$\geq \left(\sum_{i=1}^{n} x_i\right)^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{p}} + \sum_{i=1}^{n} x_i \left|\frac{y_i}{x_i} - \frac{\sum_{j=1}^{n} y_j}{\sum_{j=1}^{n} x_j}\right|^{\frac{1}{p}}, \tag{16}$$

and we get from (16) that

$$\frac{\sum_{i=1}^{n} a_i b_i^{p-1}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}} = \frac{\sum_{i=1}^{n} \left(a_i^p\right)^{\frac{1}{p}} \left(b_i^p\right)^{1-\frac{1}{p}}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}}$$
$$\geq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \frac{\sum_{i=1}^{n} b_i^p}{\left(\sum_{j=1}^{n} b_j^p\right)^{\frac{p-1}{p}}} \left|\frac{a_i^p}{b_i^p} - \frac{\sum_{j=1}^{n} a_j^p}{\sum_{j=1}^{n} b_j^p}\right|^{\frac{1}{p}}. \tag{17}$$

By denoting $c_i = a_i + b_i$, $i = 1, \ldots, n$ we get also that

$$\begin{aligned}
\frac{\sum_{i=1}^{n} b_i a_i^{p-1}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} &= \frac{\sum_{i=1}^{n} \left(c_i - a_i\right) a_i^{p-1}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} \\
&= \frac{\sum_{i=1}^{n} c_i a_i^{p-1}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} - \frac{\sum_{i=1}^{n} a_i a_i^{p-1}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} \\
&= \frac{\sum_{i=1}^{n} c_i a_i^{p-1}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} - \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \\
&\geq \left(\sum_{i=1}^{n} c_i^p\right)^{\frac{1}{p}} + \frac{\sum_{i=1}^{n} a_i^p}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} \left| \frac{c_i^p}{a_i^p} - \frac{\sum_{j=1}^{n} c_j^p}{\sum_{j=1}^{n} a_j^p} \right|^{\frac{1}{p}} - \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}. \qquad (18)
\end{aligned}$$

Summing (17) with (18) and using $c_i = a_i + b_i$, $i = 1, \ldots, n$ we get that

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} a_i b_i^{p-1}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}} + \frac{\sum_{i=1}^{n} b_i a_i^{p-1}}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{p-1}{p}}} \\
&\quad \geq \left(\sum_{i=1}^{n} \left(a_i + b_i\right)^p\right)^{\frac{1}{p}} + \frac{\sum_{i=1}^{n} a_i^p}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{p-1}{p}}} \left| \frac{\left(a_i + b_i\right)^p}{a_i^p} - \frac{\sum_{j=1}^{n} \left(a_j + b_j\right)^p}{\sum_{j=1}^{n} a_j^p} \right|^{\frac{1}{p}} \\
&\qquad + \frac{\sum_{i=1}^{n} b_i^p}{\left(\sum_{j=1}^{n} b_j^p\right)^{\frac{p-1}{p}}} \left| \frac{a_i^p}{b_i^p} - \frac{\sum_{j=1}^{n} a_j^p}{\sum_{j=1}^{n} b_j^p} \right|^{\frac{1}{p}}. \qquad (19)
\end{aligned}$$

From (19) and from Minkowski inequality (1) for $0 < p < 1$ we get for $0 < p < \frac{1}{2}$ that (15) holds.

The proof is complete.

Theorem 4 *Let* $\frac{1}{2} \leq p \leq 1$, $a_i, b_i \geq 0$, $i = 1, \ldots, n$. *Then, the inequality*

$$\begin{aligned}
\max \Bigg(&\frac{\sum_{i=1}^{n} a_i b_i^{p-1}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}} + \frac{\sum_{i=1}^{n} b_i a_i^{p-1}}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{p-1}{p}}} \\
&- \frac{\sum_{i=1}^{n} a_i^p \left| \frac{(a_i+b_i)^p}{a_i^p} - \frac{\sum_{j=1}^{n} (a_j+b_j)^p}{\sum_{j=1}^{n} a_j^p} \right|^{\frac{1}{p}}}{\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}}} - \frac{\sum_{i=1}^{n} b_i^p \left| \frac{a_i^p}{b_i^p} - \frac{\sum_{j=1}^{n} a_j^p}{\sum_{j=1}^{n} b_j^p} \right|^{\frac{1}{p}}}{\left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}}},
\end{aligned}$$

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\Bigg) \\
&\leq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}} \\
&\leq \frac{\sum_{i=1}^{n} a_i b_i^{p-1}}{\left(\sum_{i=1}^{n} b_i^p\right)^{\frac{p-1}{p}}} + \frac{\sum_{i=1}^{n} b_i a_i^{p-1}}{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{p-1}{p}}} \qquad (20)
\end{aligned}
$$

holds.

Proof The proof of the inequalities in (20) is omitted because it is similar to the proof of Theorem 3 using here the subquadracity of $f(x) = x^{\frac{1}{p}}, x > 0, \frac{1}{2} < p < 1$.

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