Approximate Solution of Fredholm Integral and Integro-Differential Equations with Non-Separable Kernels

E. Providas

Abstract This chapter deals with the approximate solution of Fredholm integral equations and a type of integro-differential equations having non-separable kernels, as they appear in many applications. The procedure proposed consists of firstly approximating the non-separable kernel by a finite partial sum of a power series and then constructing the solution of the degenerate equation explicitly by a direct matrix method. The method, which is easily programmable in a computer algebra system, is explained and tested by solving several examples from the literature.

1 Introduction

Integral and integro-differential equations appear in many applications in sciences and engineering. Integral equations have been studied extensively and there is today accumulated knowledge which one can find in good treatises, see, for example, [6, 10, 16]. Integro-differential equations are a less researched topic and usually they occupy a separate chapter in integral equations text books [13, 15]. Integral and Integro-differential equations are usually solved by numerical methods, see, for example, the monograph [1]. Direct solution methods have also been used, as it can be seen in the above-mentioned references, in the cases where the kernels are degenerate. Recently, the author with his co-authors developed a direct matrix method for solving exactly integro-differential equations with separable kernels [7–9, 12]. However, in many engineering applications, such as nonlocal or gradient elasticity [4, 5, 11, 14] and hydrodynamics [2], integral and integro-differential equations emerge with non-separable kernels. The aim of this article is to propose a procedure by which the non-separable kernel is approximated by a degenerate one and then solving the integral or integro-differential equation explicitly by the direct matrix method above.

E. Providas (✉)
Department of Environmental Sciences, University of Thessaly, Larissa, Greece
e-mail: providas@uth.gr

N. J. Daras, Th. M. Rassias (eds.), *Approximation and Computation in Science and Engineering*, Springer Optimization and Its Applications 180,
https://doi.org/10.1007/978-3-030-84122-5_38

In Sect. 2, we present a direct matrix method for obtaining in closed form the unique solution of the Fredholm integral equation

$$\mathcal{I}u(x) = u(x) - \int_a^b K(x,s)u(s)ds = f(x), \quad x \in [a,b], \tag{1}$$

where $\mathcal{I} : C[a,b] \to C[a,b]$ is a linear operator, $K(x,s)$ is a given kernel function which is assumed to be continuous on the closed square $Q(a,b) = \{(x,s) : a \leq x \leq b,\ a \leq s \leq b\}$ and separable, $f(x) \in C[a,b]$ is an input free function, and $u(x)$ is the unknown function describing the response of the system modeled by (1). Also, we propose a technique for establishing uniqueness and constructing in closed form the solution of the Fredholm integro-differential equation

$$\begin{aligned} &Bu(x) = \widehat{A}u(x) - \int_a^b K(x,s)\widehat{A}u(s)ds = f(x), \quad x \in [a,b], \\ &D(B) = D(\widehat{A}), \end{aligned} \tag{2}$$

where $\widehat{A} : C[a,b] \to C[a,b]$ is a bijective linear differential operator incorporating initial or boundary conditions, and $B : C[a,b] \to C[a,b]$ is a linear operator with $D(B) = D(\widehat{A})$. As an example of equations of this kind, we refer to the case of modeling the Euler-Bernoulli beams using Eringen's integral formulation [14].

In Sect. 3, we find approximate solutions to Fredholm integral equations with non-separable kernel functions $K(x,s)$. The approach we follow consists of representing $K(x,s)$ as a power series at a point and replacing $K(x,s)$ in the integral equation by the partial sum $K_n(x,s)$ of the power series. The resulting degenerate integral equation is then solved by the direct matrix method.

The same procedure is employed in Sect. 4 to acquire an approximate solution of Fredholm integro-differential equations with non-separable kernels.

Finally, some conclusions regarding the efficiency of the method proposed are quoted in Sect. 5.

2 Direct Matrix Methods

Let the integral equation (1) and assume that the kernel $K(x,s)$ is a separable function which has the specific form

$$K(x,s) = \sum_{k=1}^{n} g_k(x)h_k(s), \quad x, s \in [a,b], \tag{3}$$

where $g_k(x),\ h_k(s) \in C[a,b]$. Also, it is assumed without loss of generality that the sets of the functions $\{g_k(x)\}$ and $\{h_k(x)\}$ are linearly independent; otherwise, the number of functions should be lessened. Then the integral equation (1) becomes

$$\mathcal{I}u(x) = u(x) - \sum_{k=1}^{n} g_k(x) \int_a^b h_k(s)u(s)ds = f(x), \quad x \in [a,b]. \tag{4}$$

We introduce the vector of functions

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_n \end{pmatrix}, \quad g_k = g_k(x) \in C[a,b], \quad k = 1,2,\dots,n, \tag{5}$$

and the vector of linear bounded functionals

$$\Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \\ \vdots \\ \Phi_n(u) \end{pmatrix}, \quad \Phi_k(u) = \int_a^b h_k(s)u(s)ds, \quad k = 1,2,\dots,n, \tag{6}$$

and write Eq. (4) as

$$\mathcal{I}u = u - \mathbf{g}\Phi(u) = f, \tag{7}$$

where $f = f(x),\ u = u(x) \in C[a,b]$.

For the solution of (7), we state and prove the next theorem where use is made of the notations

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1) & \Phi_1(g_2) & \cdots & \Phi_1(g_n) \\ \Phi_2(g_1) & \Phi_2(g_2) & \cdots & \Phi_2(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_n(g_1) & \Phi_n(g_2) & \cdots & \Phi_n(g_n) \end{bmatrix}, \quad \Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_n(f) \end{pmatrix}, \tag{8}$$

$\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{0}$ the zero column vector. We note that

$$\Phi(\mathbf{gN}) = \Phi(\mathbf{g})\mathbf{N}, \tag{9}$$

where $\mathbf{N}$ is an $n \times m,\ m \in \mathbb{N}$, constant matrix. Finally, it is recalled that a linear operator $P : C[a,b] \to C[a,b]$ is said to be *correct* if P is bijective and its inverse P^{-1} is bounded on $C[a,b]$.

Theorem 1 *In $C[a,b]$, let the vectors $\mathbf{g}$ and Φ be defined as in (5) and (6), respectively, and $\mathcal{I} : C[a,b] \to C[a,b]$ be the linear operator*

$$\mathcal{I}u = u - \mathbf{g}\Phi(u). \tag{10}$$

Then the operator $\mathcal{I}$ is bijective on $C[a,b]$ if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{11}$$

and the unique solution of the integral equation $\mathcal{I}u = f$, for any $f \in C[a,b]$, is given by the formula

$$u = \mathcal{I}^{-1} f = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{12}$$

The operator $\mathcal{I}$ is correct.

Proof

(i) Let $\det \mathbf{W} \neq 0$ and $u \in \ker \mathcal{I}$. Then,

$$\mathcal{I}u = u - \mathbf{g}\Phi(u) = 0, \tag{13}$$

and by acting by the vector Φ on both sides of (13), we get

$$\Phi\left(u - \mathbf{g}\Phi(u)\right) = [\mathbf{I}_n - \Phi(\mathbf{g})]\,\Phi(u) = \mathbf{W}\Phi(u) = \mathbf{0}, \tag{14}$$

which implies that $\Phi(u) = \mathbf{0}$. Substitution into (13) yields $\mathcal{I}u = u = 0$, which means that the $\ker \mathcal{I} = \{0\}$ and hence the operator $\mathcal{I}$ is injective. Conversely, we prove that if $\mathcal{I}$ is an injective operator then $\det \mathbf{W} \neq 0$, or equivalently, if $\det \mathbf{W} = 0$, then $\mathcal{I}$ is not injective. Let $\det \mathbf{W} = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_n)$ such that $\mathbf{W}\mathbf{c} = \mathbf{0}$. Let the element $u_0 = \mathbf{g}\mathbf{c}$ and note that $u_0 \neq 0$; otherwise, $u_0 = \mathbf{g}\mathbf{c} = 0$ implies $\mathbf{W}\mathbf{c} = [\mathbf{I}_n - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{c} - \Phi(\mathbf{g}\mathbf{c}) = \mathbf{c} = \mathbf{0}$. From Eq. (13), we get

$$\mathcal{I}u_0 = \mathbf{g}\mathbf{c} - \mathbf{g}\Phi(\mathbf{g})\mathbf{c} = \mathbf{g}[\mathbf{I}_n - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{g}\mathbf{W}\mathbf{c} = \mathbf{g}\mathbf{0} = 0, \tag{15}$$

which means that $\ker \mathcal{I} \neq 0$ and so $\mathcal{I}$ is not injective.

By applying now the vector Φ on $\mathcal{I}u = f$, we have

$$[\mathbf{I}_n - \Phi(\mathbf{g})]\,\Phi(u) = \mathbf{W}\Phi(u) = \Phi(f). \tag{16}$$

Since $\det \mathbf{W} \neq 0$ it follows that $\Phi(u) = \mathbf{W}^{-1}\Phi(f)$ and hence

$$\mathcal{I}u = u - \mathbf{g}\mathbf{W}^{-1}\Phi(f) = f, \tag{17}$$

from where formula (12) is obtained. Moreover, since the input function $f \in C[a,b]$ is arbitrary, we have $R(\mathcal{I}) = C[a,b]$ which means that $\mathcal{I}$ is bijective.

Lastly, in (12) the functionals Φ_k are bounded on $C[a,b]$ and hence the operator $\mathcal{I}^{-1}$ is bounded. Thus, if the operator $\mathcal{I}$ is bijective then it is correct. □

Let now the mth order linear differential operator $A : C[a,b] \to C[a,b]$:

$$Au = a_m(x)\frac{d^m u}{dx^m} + a_{m-1}(x)\frac{d^{m-1}u}{dx^{m-1}} + \cdots + a_1(x)\frac{du}{dx} + a_0(x), \tag{18}$$

where the coefficients $a_i(x) \in C[a,b]$, $i = 0,\ldots,m$, $a_m(x) \neq 0$, and $\widehat{A} : C[a,b] \to C[a,b]$ be a restriction of A on $D(\widehat{A})$ by specifying initial or boundary conditions. We assume that $\widehat{A}$ is a bijective operator and that the inverse $\widehat{A}^{-1}$ is known. Further, let $K(x,s)$ be degenerate as in (3), and the vectors $\mathbf{g}$ and Φ be as in (5) and (6), respectively. Then the Fredholm integro-differential equation (2) can be put in the form

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u) = f, \quad D(B) = D(\widehat{A}). \tag{19}$$

The existence and uniqueness criteria and the solution of the integro-differential equation (19) are provided by the following theorem.

Theorem 2 *Let the restriction $\widehat{A} : C[a,b] \to C[a,b]$ be a bijective linear operator and $\widehat{A}^{-1}$ its inverse, the vectors $\mathbf{g}$ and Φ as in (5) and (6), respectively, and $B : C[a,b] \to C[a,b]$ the linear operator*

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u), \quad D(B) = D(\widehat{A}). \tag{20}$$

Then the following statements are true:

(i) The operator B is bijective on $C[a,b]$ if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{21}$$

and the unique solution to problem $Bu = f$, for any $f \in C[a,b]$, is given by the formula

$$u = B^{-1}f = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{22}$$

(ii) If in addition the inverse operator $\widehat{A}^{-1}$ is bounded on $C[a,b]$, then the operator B correct.

Proof

(i) Set $\widehat{A}u = y$, $y \in C[a,b]$, and express $Bu = f$ as

$$y - \mathbf{g}\Phi(y) = f. \tag{23}$$

This is an integral equation of the type (7). From Theorem 1 follows that Eq. (23) has a unique solution if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{24}$$

and that its unique solution is given by

$$y = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{25}$$

Acting by the operator $\widehat{A}^{-1}$ on both sides of (25), we get

$$\widehat{A}^{-1}y = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{26}$$

and hence

$$u = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f), \tag{27}$$

which is the solution formula (22). Furthermore, since $f \in C[a, b]$ is arbitrary, we have $R(B) = C[a, b]$ which means that B is bijective.

(ii) Suppose that (21) is true and that the operator $\widehat{A}^{-1}$ is bounded on $C[a, b]$. Then by (i) the operator B is bijective and the unique solution to $Bu = f$ is given by (22). Additionally, in (22) the operator $\widehat{A}^{-1}$ and the functionals $\Phi_1, \ldots, \Phi_n$ are bounded on $C[a, b]$ and hence the operator B^{-1} is bounded too. Therefore the operator B is correct.

□

3 Approximate Solution of Integral Equations with Non-Separable Kernels

Let the integral equation (1) and suppose the kernel function $K(x, s)$ is non-separable, but it can be represented as a power series in s at a point s_0 such that

$$K(x, s) = \sum_{k=0}^{\infty} p_k(x)(s - s_0)^k, \tag{28}$$

where the functions $p_k(x)$ are continuous functions. We truncate this series and take the partial sum of the first $n + 1$ terms, namely

$$K_n(x, s) = \sum_{k=1}^{n+1} p_{k-1}(x)(s - s_0)^{k-1}. \tag{29}$$

We replace the kernel $K(x, s)$ in (1) by (29) to obtain the degenerate Fredholm integral equation

$$\mathcal{I}_n\tilde{u}(x) = \tilde{u}(x) - \sum_{k=1}^{n+1} p_{k-1}(x)\int_a^b (s - s_0)^{k-1}\tilde{u}(s)ds = f(x), \quad x \in [a, b], \tag{30}$$

where $\mathcal{I}_n : C[a, b] \to C[a, b]$ is a linear operator. Further, we define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_{n+1} \end{pmatrix} = \begin{pmatrix} p_0(x) & p_1(x) & \dots & p_n(x) \end{pmatrix}, \tag{31}$$

and

$$\Phi(\tilde{u}) = \begin{pmatrix} \Phi_1(\tilde{u}) \\ \Phi_2(\tilde{u}) \\ \vdots \\ \Phi_{n+1}(\tilde{u}) \end{pmatrix}, \quad \Phi_k(\tilde{u}) = \int_a^b (s - s_0)^{k-1} \tilde{u}(s) ds, \quad k = 1, 2, \dots, n+1, \tag{32}$$

and write Eq. (30) in the compact form

$$\mathcal{I}_n \tilde{u} = \tilde{u} - \mathbf{g}\Phi(\tilde{u}) = f. \tag{33}$$

The solution $\tilde{u} = \mathcal{I}_n^{-1} f$ of (33) can be obtained by applying Theorem 1. This solution is an approximate solution to (1) having a non-separable kernel $K(x, s)$ which was expressed as in (29).

An estimation of the error $|u - \tilde{u}|$ can be found by using standard analysis techniques [6, 16]. A similar procedure would have resulted if we had used a power series in x or a double power series.

Example 1 Let us derive an approximate solution of the Fredholm integral equation of the second kind

$$u(x) - \int_0^{1/2} e^{-x^2 s^2} u(s) ds = f(x), \quad 0 \le x \le \frac{1}{2}, \tag{34}$$

for any $f(x) \in C[0, \frac{1}{2}]$. The kernel is non-separable and therefore we take its Taylor series expansion in the variable s (or in x) about the point 0, viz.

$$\begin{aligned} K(x, s) = e^{-x^2 s^2} &= 1 - x^2 s^2 + \frac{1}{2} x^4 s^4 - \frac{1}{6} x^6 s^6 \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} s^{2k}. \end{aligned}$$

By taking the partial sum

$$K_n(x, s) = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!} x^{2(k-1)} s^{2(k-1)},$$

and placing it in (34), we get the companion equation

$$\tilde{u}(x) - \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!} x^{2(k-1)} \int_0^{1/2} s^{2(k-1)} \tilde{u}(s) ds = f(x), \quad 0 \le x \le \frac{1}{2}. \tag{35}$$

We define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \dots & g_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & -x^2 & \dots & \frac{(-1)^n}{n!} x^{2n} \end{pmatrix},$$

and

$$\Phi(\tilde{u}(s)) = \begin{pmatrix} \Phi_1(\tilde{u}(s)) \\ \Phi_2(\tilde{u}(s)) \\ \vdots \\ \Phi_{n+1}(\tilde{u}(s)) \end{pmatrix} = \begin{pmatrix} \int_0^{1/2} \tilde{u}(s) ds \\ \int_0^{1/2} s^2 \tilde{u}(s) ds \\ \vdots \\ \int_0^{1/2} s^{2n} \tilde{u}(s) ds \end{pmatrix},$$

and write (35) as

$$\mathcal{I}_n \tilde{u}(x) = \tilde{u}(x) - \mathbf{g}(x) \Phi(\tilde{u}(s)) = f(x). \tag{36}$$

Then, we construct the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_1(1) & \Phi_1(-s^2) & \cdots & \Phi_1\left(\frac{(-1)^n}{n!} s^{2n}\right) \\ \Phi_2(1) & \Phi_2(-s^2) & \cdots & \Phi_2\left(\frac{(-1)^n}{n!} s^{2n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(1) & \Phi_{n+1}(-s^2) & \cdots & \Phi_{n+1}\left(\frac{(-1)^n}{n!} s^{2n}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{1/2} ds & -\int_0^{1/2} s^2 ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{2n} ds \\ \int_0^{1/2} s^2 ds & -\int_0^{1/2} s^4 ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{2(n+1)} ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^{1/2} s^{2n} ds & -\int_0^{1/2} s^{2(n+1)} ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{4n} ds \end{bmatrix},$$

and thus the matrix

$$\mathbf{W} = \mathbf{I_{n+1}} - \Phi(\mathbf{g}).$$

If $\det \mathbf{W} \neq 0$, then Eq. (36) has exactly one solution. To obtain the solution, we put up the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_{n+1}(f) \end{pmatrix} = \begin{pmatrix} \int_0^{1/2} f(s)ds \\ \int_0^{1/2} s^2 f(s)ds \\ \vdots \\ \int_0^{1/2} s^{2n} f(s)ds \end{pmatrix},$$

and by Theorem 1 compute

$$\tilde{u} = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f).$$

Let $f(x) = 1$ [10]. Then for $n = 2$, $n = 4$, and $n = 6$, we have

$$K_2(x,s) = 1 - x^2s^2 + \frac{1}{2}x^4s^4,$$
$$K_4(x,s) = 1 - x^2s^2 + \frac{1}{2}x^4s^4 - \frac{1}{6}x^6s^6 + \frac{1}{24}x^8s^8,$$
$$K_6(x,s) = 1 - x^2s^2 + \frac{1}{2}x^4s^4 - \frac{1}{6}x^6s^6 + \frac{1}{24}x^8s^8 - \frac{1}{120}x^{10}s^{10} + \frac{1}{720}x^{12}s^{12},$$

and the approximate solutions

$$\tilde{u}_2 = 1.993199 - 0.082541x^2 + 0.006183x^4,$$
$$\tilde{u}_4 = 1.993198 - 0.082541x^2 + 0.006183x^4 - 0.000368x^6 + 0.000018x^8,$$
$$\tilde{u}_6 = 1.993198 - 0.082541x^2 + 0.006183x^4 - 0.000368x^6 + 0.000018x^8$$
$$-7.309486 \times 10^{-7}x^{10} + 2.576526 \times 10^{-8}x^{12},$$

respectively, where all coefficients have been rounded up to six decimal digits. The results are in very good agreement with those obtained in [10] where the same problem has been solved for $n = 2$.

Example 2 Consider the inhomogeneous Fredholm integral equation

$$u(x) - \frac{1}{2}\int_{-1}^{1} \sin\left(\frac{\pi s x}{2}\right) u(s)ds = f(x), \quad -1 \leq x \leq 1, \tag{37}$$

where $f(x) \in C[-1, 1]$. The kernel is non-separable, but it can be represented in Taylor series in x (or in s) about the point 0, namely

$$K(x,s) = \sin\left(\frac{\pi s x}{2}\right) = \frac{\pi s x}{2} - \frac{\pi^3 s^3 x^3}{48} + \frac{\pi^5 s^5 x^5}{3840} \cdots$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1} s^{2k+1} x^{2k+1}}{2^{2k+1}(2k+1)!}.$$

After replacing $K(x,s)$ in (37) with the partial sum

$$K_n(x,s) = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\pi^{2k-1} s^{2k-1} x^{2k-1}}{2^{2k-1}(2k-1)!},$$

we get the auxiliary equation

$$\tilde{u}(x) - \frac{1}{2} \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\pi^{2k-1} x^{2k-1}}{2^{2k-1}(2k-1)!} \int_{-1}^{1} s^{2k-1} \tilde{u}(s) ds = f(x), \quad -1 \leq x \leq 1. \tag{38}$$

We set up the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \ldots & g_{n+1}(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\pi x}{2} & -\frac{\pi^3 x^3}{48} & \ldots & (-1)^n \frac{\pi^{2n+1} x^{2n+1}}{2^{2n+1}(2n+1)!} \end{pmatrix},$$

and

$$\Phi(\tilde{u}(s)) = \begin{pmatrix} \Phi_1(\tilde{u}(s)) \\ \Phi_2(\tilde{u}(s)) \\ \vdots \\ \Phi_{n+1}(\tilde{u}(s)) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} s \tilde{u}(s) ds \\ \int_{-1}^{1} s^3 \tilde{u}(s) ds \\ \vdots \\ \int_{-1}^{1} s^{2n+1} \tilde{u}(s) ds \end{pmatrix},$$

and write (38) as

$$\mathcal{I}_n \tilde{u}(x) = \tilde{u}(x) - \mathbf{g}(x) \Phi(\tilde{u}(s)) = f(x). \tag{39}$$

Then, we form the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$

and compute the matrix

$$\mathbf{W} = \mathbf{I}_{n+1} - \Phi(\mathbf{g}).$$

If $\det \mathbf{W} \neq 0$, then Eq. (39) admits exactly one solution. By setting up the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_{n+1}(f) \end{pmatrix},$$

we can determine the solution from Theorem 1, which is

$$\tilde{u} = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f).$$

Let $f(x) = x^3$ [16], which is continuous in $[-1, 1]$. Then, for $n = 2$, $n = 4$, and $n = 6$, we get

$$K_2(x,s) = \frac{\pi s x}{2} - \frac{\pi^3 s^3 x^3}{48} + \frac{\pi^5 s^5 x^5}{3840},$$
$$K_4(x,s) = K_2(x,s) - \frac{\pi^7 s^7 x^7}{645120} + \frac{\pi^9 s^9 x^9}{185794560},$$
$$K_6(x,s) = K_4(x,s) - \frac{\pi^{11} s^{11} x^{11}}{81749606400} + \frac{\pi^{13} s^{13} x^{13}}{51011754393600},$$

and the approximate solutions

$$\tilde{u}_2(x) = 0.565621x + 0.847692x^3 + 0.014047x^5,$$
$$\tilde{u}_4(x) = 0.565421x + 0.847751x^3 + 0.014042x^5 - 0.000660x^7 + 0.000019x^9,$$
$$\tilde{u}_6(x) = 0.565421x + 0.847751x^3 + 0.014042x^5 - 0.000660x^7 + 0.000019x^9$$
$$-3.627733 \times 10^{-7} x^{11} + 5.024528 \times 10^{-9} x^{13},$$

respectively, where the coefficients have been rounded up to six decimal places. The same problem is solved in [16] for $n = 2$ using other techniques such as the resolvent kernel, Simpson's rule, or Gaussian quadrature. The results obtained here for $n = 2$ are identical with those reported in [16] with six decimal digits.

4 Approximate Solution of Integro-Differential Equations with Non-Separable Kernels

Let the integro-differential equation (2) with a kernel function $K(x,s)$ which is non-separable, but it can be expanded in a power series in x at a point x_0

$$K(x,s) = \sum_{k=0}^{\infty} h_k(s)(x - x_0)^k, \qquad (40)$$

where the functions $h_k(s)$ are continuous functions. We consider the partial sum of the first $n+1$ terms

$$K_n(x,s) = \sum_{k=1}^{n+1} h_{k-1}(s)(x-x_0)^{k-1}, \tag{41}$$

and place it in (2) instead of $K(x,s)$. As a result, we obtain the degenerate Fredholm integro-differential equation

$$B_n\tilde{u}(x) = \widehat{A}\tilde{u}(x) - \sum_{k=1}^{n+1}(x-x_0)^{k-1}\int_a^b h_{k-1}(s)\widehat{A}\tilde{u}(s)ds = f(x), \quad x \in [a,b], \tag{42}$$

where $B_n : C[a,b] \to C[a,b]$ is a linear operator with $D(B_n) = D(\widehat{A})$. Define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & x-x_0 & \dots & (x-x_0)^n \end{pmatrix}, \tag{43}$$

and

$$\Phi(\widehat{A}\tilde{u}) = \begin{pmatrix} \Phi_1(\widehat{A}\tilde{u}) \\ \Phi_2(\widehat{A}\tilde{u}) \\ \vdots \\ \Phi_{n+1}(\widehat{A}\tilde{u}) \end{pmatrix}, \ \Phi_k(\widehat{A}\tilde{u}) = \int_a^b h_{k-1}(s)\widehat{A}\tilde{u}(s)ds, \ k = 1,2,\dots,n+1, \tag{44}$$

and formulate Eq. (42) as

$$B_n\tilde{u} = \widehat{A}\tilde{u} - \mathbf{g}\Phi(\widehat{A}\tilde{u}) = f. \tag{45}$$

By using Theorem 2, we can compute the solution $\tilde{u} = B_n^{-1}f$ of (45), which is an approximate solution of Eq. (2) having the non-separable kernel $K(x,s)$ approximated by (41).

As before, an evaluation of the error $|u - \tilde{u}|$ can be found by using standard analysis techniques [6, 16]. A similar procedure results if one uses a power series in s or a double power series.

Example 3 Consider the Fredholm integro-differential equation

$$u'(x) - \int_0^1 e^{xs}u'(s)ds = f(x), \quad 0 \le x \le 1, \quad u(0) = 1, \tag{46}$$

for an input function $f(x) \in C[0,1]$. By means of $v(x) = u(x) - 1$, we can transform this equation to the following one with a homogeneous condition

$$v'(x) - \int_0^1 e^{xs} v'(s)ds = f(x), \quad 0 \leq x \leq 1, \quad v(0) = 0. \tag{47}$$

The kernel is non-separable, but it can be represented as Taylor series in x (or in s) about 0 as

$$K(x,s) = e^{xs} = 1 + sx + \frac{1}{2}s^2x^2 + \frac{1}{6}s^3x^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{s^k x^k}{k!}.$$

Let the partial sum

$$K_n(x,s) = \sum_{k=1}^{n+1} \frac{s^{k-1}x^{k-1}}{(k-1)!},$$

which when is placed in (47) instead of $K(x,s)$ yields the auxiliary equation

$$\tilde{v}'(x) - \sum_{k=1}^{n+1} x^{k-1} \int_0^1 \frac{s^{k-1}}{(k-1)!} \tilde{v}'(s)ds = f(x), \quad 0 \leq x \leq 1. \tag{48}$$

Take the operator $\widehat{A} : C[0,1] \to C[0,1]$ to be

$$\widehat{A}\tilde{v}(x) = \tilde{v}'(x), \quad D(\widehat{A}) = \{\tilde{v}(x) \in C^1[0,1] : \ \tilde{v}(0) = 0\},$$

which is bijective and its inverse is

$$\widehat{A}^{-1} f(x) = \int_0^x f(s)ds, \quad f(x) \in C[0,1].$$

Set up the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \ldots & g_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & \ldots & x^n \end{pmatrix},$$

and

$$\Phi(\widehat{A}\tilde{v}(s)) = \begin{pmatrix} \Phi_1 & (\widehat{A}\tilde{v}(s)) \\ \Phi_2 & (\widehat{A}\tilde{v}(s)) \\ & \vdots \\ \Phi_{n+1} & (\widehat{A}\tilde{v}(s)) \end{pmatrix} = \begin{pmatrix} \int_0^1 & \widehat{A}\tilde{v}(s)ds \\ \int_0^1 s & \widehat{A}\tilde{v}(s)ds \\ & \vdots \\ \int_0^1 \frac{s^n}{n!} & \widehat{A}\tilde{v}(s)ds \end{pmatrix},$$

and write (48) as

$$\widehat{A}\tilde{v}(x) - \mathbf{g}(x)\Phi(\widehat{A}\tilde{v}(s)) = f(x). \tag{49}$$

Form the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_1(1) & \Phi_1(s) & \cdots & \Phi_1(s^n) \\ \Phi_2(1) & \Phi_2(s) & \cdots & \Phi_2(s^n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(1) & \Phi_{n+1}(s) & \cdots & \Phi_{n+1}(s^n) \end{bmatrix}$$

and then the matrix

$$\mathbf{W} = \mathbf{I_{n+1}} - \Phi(\mathbf{g}).$$

If $\det \mathbf{W} \neq 0$, then Eq. (49) has exactly one solution. To obtain the solution, we construct the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1 & (f) \\ \Phi_2 & (f) \\ & \vdots \\ \Phi_{n+1} & (f) \end{pmatrix} = \begin{pmatrix} \int_0^1 & f(s)ds \\ \int_0^1 & sf(s)ds \\ & \vdots \\ \int_0^1 & \frac{s^n}{n!} f(s)ds \end{pmatrix},$$

and by Theorem 2 compute

$$\tilde{v} = \widehat{A}^{-1} f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f) \quad \text{and then} \quad \tilde{u} = \tilde{v} + 1.$$

Let

$$f(x) = e^x + \frac{1 - e^{x+1}}{x+1}, \quad 0 \le x \le 1,$$

as in a comparable problem in [3]. Then Eq. (46) admits the exact solution $u(x) = e^x$. We take Taylor series expansions for both $K(x, s)$ and $f(x)$ in x around 0. For $n = 2$, we have

$$K_2(x,s) = 1 + sx + \frac{1}{2}s^2x^2, \quad f_2(x) = 2 - e - \frac{(e-3)x^2}{2},$$

and analogous expressions for $n = 4$ and $n = 8$. The corresponding solutions are as follows

$$\tilde{u}_2 = 1.0 + 1.184093x + 0.542764x^2 + 0.175518x^3,$$
$$\tilde{u}_4 = 1.0 + 1.005793x + 0.501306x^2 + 0.166926x^3 + 0.041710x^4 + 0.008340x^5,$$
$$\tilde{u}_8 = 1.0 + 1.000001x + 0.500000x^2 + 0.166667x^3 + 0.0416667x^4$$
$$+0.008333x^5 + 0.001389x^6 + 0.000198x^7 + 0.000025x^8 + 0.000003x^9,$$

respectively, where all coefficients have been rounded up to six decimal digits. The results are of high accuracy and agree with the exact solution $u(x) = e^x$.

5 Conclusions

An efficient matrix procedure for solving Fredholm integral and integro-differential equations has been presented. The procedure involves the approximation of the non-separable kernel by a degenerate one, such as the partial sum of a power series, and the application of a direct matrix method to obtain the solution. We have programmed the method into Maxima computer algebra system and solved several example problems. In all cases the results obtained are of very high accuracy. The novelty and the main advantage of the method is the management of the computations involved and that it can be repeated many times with easiness and a large number of terms of the series.

References

1. K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind* (Cambridge University, Cambridge, 1997). https://doi.org/10.1017/CBO9780511626340
2. W. Benz, Smooth particle hydrodynamics: A review, in *The Numerical Modelling of Nonlinear Stellar Pulsations*, ed. by J.R. Buchler. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol. 302 (Springer, Dordrecht, 1990). https://doi.org/10.1007/978-94-009-0519-1_16
3. A.H. Bhrawy, E. Tohidi, F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals. Appl. Math. Comput. **219**, 482–497 (2012). https://doi.org/10.1016/j.amc.2012.06.020
4. A.C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. **54**, 4703–4710 (1983). https://doi.org/10.1063/1.332803
5. C.Chr. Koutsoumaris, K.G. Eptaimeros, G.J. Tsamasphyros, A different approach to Eringen's nonlocal integral stress model with applications for beams. Int. J. Solids Struct. **112**, 222–238 (2017). https://doi.org/10.1016/j.ijsolstr.2016.09.007

6. R. Kress, *Linear Integral Equations* (Springer, New York, 2014). https://doi.org/10.1007/978-1-4614-9593-2
7. I.N. Parasidis, E. Providas, Resolvent operators for some classes of integro-differential equations, in *Mathematical Analysis, Approximation Theory and Their Applications*, ed. by T. Rassias, V. Gupta. Springer Optimization and Its Applications, vol 111 (Springer, Cham, 2016), pp 535–558. https://doi.org/10.1007/978-3-319-31281-1
8. I.N. Parasidis, E. Providas, Extension operator method for the exact solution of integro-differential equations, in *Contributions in Mathematics and Engineering*, ed. by P. Pardalos, T. Rassias (Springer, Cham, 2016), pp. 473–496. https://doi.org/10.1007/978-3-319-31317-7
9. I.N. Parasidis, E. Providas, On the exact solution of nonlinear integro-differential equations, in *Applications of Nonlinear Analysis*, ed. by T. Rassias. Springer Optimization and Its Applications, vol 134 (Springer, Cham, 2018), pp. 591–609. https://doi.org/10.1007/978-3-319-89815-5
10. A.D. Polyanin, A.V. Manzirov, *Handbook of Integral Equations* (CRC Press LLC, New York, 1998)
11. E. Providas, M.A. Kattis, Finite element method in plane Cosserat elasticity. Comput. Struct. **80**, 2059–2069 (2002). https://doi.org/10.1016/S0045-7949(02)00262-6
12. E. Providas, I.N. Parasidis, On the solution of some higher-order integro-differential equations of special form. Vestnik SamU. Estestvenno-Nauchnaya Ser. (Vestnik of Samara University. Natural Science Series), **26**(1), 14–22 (2020). (to appear)
13. M. Rahman, *Integral Equations and Their Applications* (WIT Press, Southampton, 2007)
14. Y. Wang, X. Zhu, H. Dai, Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. AIP Adv. **6**, 085114-1–085114-22 (2016). https://doi.org/10.1063/1.4961695
15. A.M. Wazwaz, *Linear and Nonlinear Integral Equations* (Springer, Heidelberg, 2011). https://doi.org/10.1007/978-3-642-21449-3
16. S.M. Zemyan, *The Classical Theory of Integral Equations* (Birkhäuser, Basel, 2012). https://doi.org/10.1007/978-0-8176-8349-8