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Approximation and Computation in Science and Engineering

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Preface

Approximation and Computation in Science and Engineering presents a wide spectrum of research and research-survey papers in several subjects of pure and applied mathematics with various applications. Emphasis is given to the study of topics of approximation theory, functional equations and inequalities, fixed point theory, numerical analysis, theory of wavelets, convex analysis, topology, operator theory, differential operators, fractional integral operators, integro-differential equations, ternary algebras, super and hyper relators, variational analysis, discrete mathematics, cryptography, and a variety of applications in interdisciplinary topics. Several of these domains have a strong connection with both theories and problems of linear and nonlinear optimization. Therefore, it is hoped that this collective effort will be particularly useful to researchers who are focusing on applications of theories and methods of the above-mentioned subjects for optimization. Overall, the works published within this book will be of particular value for both theoretical and applicable interdisciplinary research.

We would like to express our sincere thanks to the contributing authors of the book chapters. We would also like to warmly thank the staff at Springer for their valuable assistance throughout the preparation of this book.

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Rearrangements, L-Superadditivity and Jensen-Type Inequalities

Shoshana Abramovich

Abstract We deal here with the minimum and the maximum of

$$\sum_{i=1}^{n} F\left(a_{2i-1}, a_{2i}\right), (\mathbf{a}) \in \mathbb{R}^{2n}$$

and of

$$\sum_{i=1}^{n} F\left(a_{i}, a_{i+1}\right), \quad a_{n+1} = a_{1}, (\mathbf{a}) \in \mathbb{R}^{n}$$

obtained by using rearrangement techniques. The results depend on the arrangement of $(\mathbf{a})$ and are used in proving Jensen-type inequalities.

1 Introduction

We deal here with the minimum and the maximum of $\sum_{i=1}^{n} F\left(a_{2i-1}, a_{2i}\right)$, $(\mathbf{a}) \in \mathbb{R}^{2n}$ and $\sum_{i=1}^{n} F\left(a_{i}, a_{i+1}\right)$, where $a_{n+1} = a_{1}$, $(\mathbf{a}) \in \mathbb{R}^{n}$ when $F(x, y)$, $(x, y) \in \mathbb{R}^{2}$ is L-superadditive function. These extrema are obtained by using rearrangement techniques. The results are used in proving Jensen-type inequalities.

In [5], the authors prove the following theorem:

Theorem A ([5, Lemma2]) *Let $y_i \in \mathbb{R}_{+}$, $i = 1, \ldots, 2n$. Then, for $t > 0$, the sum $\sum_{i=1}^{n} \ln\left(y_{2i-1} y_{2i} + t\right)$ gets its maximum value when $(\mathbf{y}) = (y_1, \ldots, y_{2n})$ is arranged in a decreasing order.*

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A generalization of this result is as follows:

Theorem B ([1, Theorem 2]) *Let* $(\mathbf{a}) = (a_1, \dots, a_m)$ *and* $(\mathbf{y}) = (y_1, \dots, y_{2n})$ *be sets of non-negative numbers, where* $(\mathbf{y})$ *is given except its arrangements. Then,* $\sum_{i=1}^{n} \ln\left[(y_{2i-1}y_{2i})^m + a_1 (y_{2i-1}y_{2i})^{m-1} + \dots + a_m\right]$ *attains its maximum when* $(\mathbf{y})$ *is arranged in a decreasing order.*

Definition 1 ([8, Page 150]) A real-valued function F defined on $\mathbb{R}^2$ is said to be L-superadditive if it satisfies the condition that

$$F(x+\alpha, y) - F(x, y) \quad \text{is increasing in } y \text{ for all } x \text{ and all } \alpha > 0.$$

Corollary 1 ([7]) *Let* F *have second partial derivatives. Then,* $\frac{\partial}{\partial y} F(x, y)$ *is increasing in* x*, which is equivalent to* $\frac{\partial^2}{\partial x \partial y} F(x, y) \geq 0$*, if and only if* F *is* L*-superadditive function.*

For clarity, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ are $[0, \infty)$ and $(0, \infty)$, respectively.

In the sequel, we show that the following arrangements of a given set of real numbers have a role in getting our extrema.

Definition 2 For any $(\mathbf{x}) = (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$x_{[1]} \geq \dots \geq x_{[n]}$$

denote the terms of $(\mathbf{x})$ in decreasing order, and let

$$\left(\mathbf{x}_{\downarrow}\right) = \left(x_{[1]} \dots x_{[n]}\right)$$

denote the decreasing rearrangement of $(\mathbf{x})$.

Similarly, let

$$x_{(1)} \leq \dots \leq x_{(n)}$$

denote the terms of $(\mathbf{x})$ in increasing order, and let

$$\left(\mathbf{x}_{\uparrow}\right) = \left(x_{(1)} \dots x_{(n)}\right)$$

denote the increasing rearrangement of $(\mathbf{x})$.

Definition 3 ([4, 6]) An ordered set $(\mathbf{x}) = (x_1, \dots, x_n)$ of n real numbers is arranged in **symmetrical decreasing order** if

$$x_1 \leq x_n \leq x_2 \leq \dots \leq x_{[(n+2/2)]} \tag{1}$$

or if

$$x_n \leq x_1 \leq x_{n-1} \leq \ldots \leq x_{[(n+1/2)]}. \tag{2}$$

Definition 4 ([4, 6]) A **circular rearrangement** of an ordered set $(\mathbf{x})$ is a cyclic rearrangement of $(\mathbf{x})$ or a cyclic rearrangement followed by inversion. For example, the circular rearrangements of the ordered set $(1, 2, 3, 4)$ are the sets

$$(1, 2, 3, 4), (2, 3, 4, 1), (3, 4, 1, 2), (4, 1, 2, 3),$$
$$(4, 3, 2, 1), (1, 4, 3, 2), (2, 1, 4, 3), (3, 2, 1, 4).$$

Definition 5 ([4, 6]) A set $(\mathbf{x})$ is arranged in **circular symmetrical order** if one of its circular rearrangements is symmetrically decreasing.

Definition 6 ([4]) An ordered set $(\mathbf{x}) = (x_1, \ldots, x_n)$ of n real numbers is arranged in an **alternating order** if

$$x_1 \leq x_{n-1} \leq x_3 \leq x_{n-3} \leq \ldots \leq x_{\left[\frac{n+1}{2}\right]} \leq \ldots \leq x_4 \leq x_{n-2} \leq x_2 \leq x_n, \tag{3}$$

or if

$$x_n \leq x_2 \leq x_{n-2} \leq x_4 \leq \ldots \leq x_{\left[\frac{n+1}{2}\right]} \leq \ldots \leq x_{n-3} \leq x_3 \leq x_{n-1} \leq x_1. \tag{4}$$

Definition 7 ([4]) A set $(\mathbf{x})$ is arranged in a **circular alternating order** if one of its circular rearrangements is arranged in an alternating order.

We denote in the sequel $\left(\mathbf{x}_{\uparrow}\right) = \left(x_{(1)} \ldots x_{(n)}\right)$, $\left(\mathbf{x}_{\downarrow}\right) = \left(x_{[1]} \ldots x_{[n]}\right)$, $(\widetilde{\mathbf{x}}) = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)$ and $(\widehat{\mathbf{x}}) = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)$ to be the increasing order, the decreasing order, the circular alternating order and the circular symmetrical order of a given set $(\mathbf{x}) \in \mathbb{R}^n$, respectively.

2 L-Superadditivity Applications to Rearrangements

We extend Theorems A and B as follows:

Theorem 1 *Let $F(u, v)$ be a real function symmetric in u and v defined on $\alpha < u, v < \beta$, $-\infty \leq \alpha < \beta \leq \infty$, and assume that*

$$G(u, v, w) = F(u, w) - F(u, v), \quad \alpha < u, v, w < \beta, \tag{5}$$

is increasing in u for $w \geq v$, that is, F is L-superadditive function. Then, for any set $(\mathbf{x}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$, $\alpha < x_i < \beta$, $i = 1, \ldots, 2n$, given except its *arrangement, the inequalities*

$$\begin{aligned}\sum_{i=1}^{n} F\left(x_{(i)}, x_{[i]}\right) &= \sum_{i=1}^{n} F\left(x_{(i)}, x_{(2n-i+1)}\right) \\ &\le \sum_{i=1}^{n} F\left(x_{2i-1}, x_{2i}\right) \\ &\le \sum_{i=1}^{n} F\left(x_{(2i-1)}, x_{(2i)}\right) = \sum_{i=1}^{n} F\left(x_{[2i-1]}, x_{[2i]}\right)\end{aligned} \tag{6}$$

hold when $\left(\mathbf{x}_{\uparrow}\right) = \left(x_{(1)}, x_{(2)}, \ldots, x_{(2n-1)}, x_{(2n)}\right)$ *is the increasing rearrangement of* $(\mathbf{x}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$*, and* $\left(\mathbf{x}_{\downarrow}\right) = \left(x_{[1]} \ldots x_{[2n]}\right)$ *is the decreasing rearrangement of* $(\mathbf{x})$*.*

Proof We first prove that the minimum of $\sum_{i=1}^{n} F\left(x_{2i-1}, x_{2i}\right)$ is obtained when $x_{2i-1} = x_{(i)},\ i = 1, \ldots, n$ and $x_{2i} = x_{(2n-i+1)},\ i = 1, \ldots, n$.

We may assume that $x_1 = x_{(1)}$. If $x_2 \neq x_{(2n)}$, and $x_{(2n)}$ appears as x_{2j-1} or x_{2j}, we may assume that $x_{2j} = x_{(2n)}$ because of the symmetry of $F(u, v)$. We now rearrange $(\mathbf{x}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ and get $\left(\mathbf{x}''\right) = \left(x_1'', x_2'', \ldots, x_{2n-1}'', x_{2n}''\right) = \left(x_{(1)}, x_{(2n)}, \ldots, x_{2j-1}, x_2, x_{2j+1}, x_{2j+2}, \ldots, x_{2n}\right)$. We exchange here only the locations of x_2 with $x_{2j} = x_{(2n)}$ and compute the difference

$$\begin{aligned}&\sum_{i=1}^{n} F\left(x_{2i-1}, x_{2i}\right) - \sum_{i=1}^{n} F\left(x_{2i-1}'', x_{2i}''\right) \\ &= F\left(x_{(1)}, x_2\right) + F\left(x_{2j-1}, x_{(2n)}\right) - \left[F\left(x_{(1)}, x_{(2n)}\right) + F\left(x_{2j-1}, x_2\right)\right] \\ &= F\left(x_{2j-1}, x_{(2n)}\right) - F\left(x_{2j-1}, x_2\right) - \left[F\left(x_{(1)}, x_{(2n)}\right) - F\left(x_{(1)}, x_2\right)\right].\end{aligned} \tag{7}$$

As $x_{2j-1} \ge x_{(1)}$ and $x_{(2n)} \ge x_2$, we get by using the conditions of the theorem on $G(u, v, w)$ that

$$F\left(x_{2j-1}, x_{(2n)}\right) - F\left(x_{2j-1}, x_2\right) \ge \left[F\left(x_{(1)}, x_{(2n)}\right) - F\left(x_{(1)}, x_2\right)\right]. \tag{8}$$

From (7) and (8), we get that

$$\sum_{i=1}^{n} F\left(x_{2i-1}, x_{2i}\right) \ge \sum_{i=1}^{n} F\left(x_{2i-1}'', x_{2i}''\right).$$

As a result, we get that the two terms $x_{(1)}$ and $x_{(2n)}$ appear as $F\left(x_{(1)}, x_{(2n)}\right)$ in $\sum_{i=1}^{n} F\left(x_{2i-1}'', x_{2i}''\right)$ without increasing the sum $\sum_{i=1}^{n} F\left(x_{2i-1}, x_{2i}\right)$. We continue now with the other $2n-2$ terms in $\left(\mathbf{x}''\right)$ and by the same procedure bring $x_{(2)}$ and

$x_{(2n-1)}$ next to each other in F. After at most n steps, we get the left-hand side of inequalities of (6).

To get the maximum of $\sum_{i=1}^{n} F(x_{2i-1}, x_{2i})$, we use the same procedure as in the proof of its minimum.

Without loss of generality, we may assume that the minimal term in $(\mathbf{x})$ is $x_1 = x_{(1)}$. If $x_2 \neq x_{(2)}$, but $x_{2j} = x_{(2)}$, we create a new rearrangement $\left(\mathbf{x}^{'}\right) = \left(x_1^{'}, x_2^{'}, \ldots, x_{2n-1}^{'}, x_{2n}^{'}\right)$ in which $x_2^{'} = x_{(2)} = x_{2j}$, and $x_{2j}^{'} = x_2$, and all other terms are as in $(\mathbf{x})$.

We compute now the difference

$$\begin{aligned} &\sum_{i=1}^{n} F\left(x_{2i-1}^{'}, x_{2i}^{'}\right) - \sum_{i=1}^{n} F(x_{2i-1}, x_{2i}) \\ &= F\left(x_{(1)}, x_{(2)}\right) + F\left(x_{2j-1}, x_2\right) - \left[F\left(x_{(1)}, x_2\right) + F\left(x_{2j-1}, x_{(2)}\right)\right] \\ &= F\left(x_{2j-1}, x_2\right) - F\left(x_{2j-1}, x_{(2)}\right) - \left[F\left(x_{(1)}, x_2\right) - F\left(x_{(1)}, x_{(2)}\right)\right]. \end{aligned} \tag{9}$$

As it is given in (5) that $G(u, v, w)$ is increasing in u when $w \geq v$ and as $x_{2j-1} \geq x_{(1)}$ and $x_2 \geq x_{(2)}$, we get that

$$F\left(x_{2j-1}, x_2\right) - F\left(x_{2j-1}, x_{(2)}\right) - \left[F\left(x_{(1)}, x_2\right) - F\left(x_{(1)}, x_{(2)}\right)\right] \geq 0. \tag{10}$$

From (9) and (10), we obtain that

$$\sum_{i=1}^{n} F\left(x_{2i-1}^{'}, x_{2i}^{'}\right) - \sum_{i=1}^{n} F(x_{2i-1}, x_{2i}) \geq 0,$$

which means that the two smallest terms $x_{(1)}$ and $x_{(2)}$ appear as $F\left(x_{(1)}, x_{(2)}\right)$ in $\sum_{i=1}^{n} F\left(x_{2i-1}^{'}, x_{2i}^{'}\right)$ without decreasing $\sum_{i=1}^{n} F(x_{2i-1}, x_{2i})$.

Repeating now the procedure by bringing the two minimal terms $x_{(3)}$ and $x_{(4)}$ of the remaining $2n-2$ x_ith to be in the same F. After at most n similar steps, we get the right-hand side of inequalities of (6).

This completes the proof of the theorem.

It is easy to see that the following functions satisfy Theorem 1.

Example 1 Let $F(u, v) = \frac{1}{uv+1}$, then according to Theorem 1, we get by simple computations that

$$\sum_{i=1}^{n} \frac{1}{x_{(i)}x_{(2n+1-i)} + 1} \leq \sum_{i=1}^{n} \frac{1}{x_{2i-1}x_{2i} + 1} \leq \sum_{i=1}^{n} \frac{1}{x_{(2i-1)}x_{(2i)} + 1}$$

when $(\mathbf{x}) \geq (\mathbf{1})$.

Example 2 Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be increasing and convex function. Then, $F(u, v) = f(uv)$ satisfies Theorem 1.

Remark 1 Theorems A and B are special cases of the right-hand side of (6). Also, in [10], the author shows that for L-superadditive function F the right-hand side of (6) can be derived from [7].

Lately, Theorems 2 and 3 were proved in [4]. However, it was proved earlier in [1] that the maximum of $\sum_{i=1}^{n} F(x_i, x_{i+1}), \ x_{n+1} = x_1$, is attained when $(\mathbf{x})$ is arranged in a circular symmetrical order (which generalizes a theorem in [6], there on $f(|x-y|)$ for concave decreasing f). The proof uses the induction procedure. It relies on the fact that if the n numbers $(x_1, x_2, \ldots, x_{n-1}, x_n)$ satisfy (1), then the $n-1$ numbers $(x_2, \ldots, x_{n-1}, x_n)$ satisfy (2).

The arrangement of $\sum_{i=1}^{n} F(x_i, x_{i+1}), \ x_{n+1} = x_1$, for which its minimum value is attained is stated in Theorem 2 and proved in [4, Theorem 1]. A brief outline of this proof appears also in [2]. The proof uses the induction procedure. The validity of the minimum value is assumed for the set of $n-2$ numbers, and it is shown that this implies its validity for the set of n numbers. This relies on the fact that if the n numbers $(x_1, x_2, \ldots, x_{n-1}, x_n)$ satisfy (3), then the $n-2$ numbers $(x_2, \ldots, x_{n-1})$ satisfy (4).

Theorem 2 ([4, Theorem 1]) *Let $F(u, v)$ be differentiable and symmetric real function in u and v defined on $\alpha \le u, v, w \le \beta$, and assume that*

$$\frac{\partial F(v, u)}{\partial v} \le \frac{\partial F(v, w)}{\partial v}$$

for $u \le \min(w, v)$.

Then, for any set $(\mathbf{x}) = (x_1, x_2, \ldots, x_n)$, $\alpha \le x_i \le \beta$, $i = 1, \ldots, n$, given except its arrangements, the sum

$$\sum_{i=1}^{n} F(x_i, x_{i+1}), \qquad x_{n+1} = x_1,$$

is maximal if $(\mathbf{x})$ is arranged in a circular symmetrical order and minimal if $(\mathbf{x})$ is arranged in the circular alternating order as defined above.

Theorem 3 ([4, Theorem 2]) *Let $F = F(u, v)$ be a real function defined on $\alpha \le u, v \le \beta$, which is symmetric in u and v. Then, $\sum_{i=1}^{n} F(x_i, x_{i+1})$, where $x_{n+1} = x_1$, is maximal if $(\mathbf{x})$ is arranged in a circular symmetrical order and minimal if $(\mathbf{x})$ is arranged in the circular alternating order in each of the following cases:*

Case (a) $F(x, y) = f(x + y)$, *where f is convex on $\mathbb{R}_+$,*
Case (b) $F(x, y) = f(|x - y|)$, *where f is concave and decreasing on $\mathbb{R}_+$.*
Case (c) $F(x, y) = f\left(\frac{x+y}{2}\right) + f\left(\left|\frac{x-y}{2}\right|\right)$, *where f' is convex and differentiable on $\mathbb{R}_+$ and $f'(0) = 0$,*

Case (d) $F(x, y) = f\left(\frac{x+y}{2}\right) + C \times \left(\frac{x-y}{2}\right)^2$, *where the constant* C *satisfies* $C \leq \varphi'\left(\frac{x_m+x_j}{2}\right)$, *with* $0 \leq x_m \leq x_j \leq x_i$, $i \neq m, j$, $i = 1, \ldots, n$, *and* $f(x) = x\varphi(x)$, *and* φ *is twice differentiable and convex function on* $0 \leq x \leq b$.

Case (e) $F(x, y) = f\left(\frac{x+y}{2}\right) + \varphi'\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)^2$, *where* φ *and* φ' *are twice differentiable and convex on* $\mathbb{R}_+$ *and* $f(x) = x\varphi(x)$.

Also, it has been proved lately in [2] that $F(x, y) = x\varphi\left(\frac{y}{x}\right) + y\varphi\left(\frac{x}{y}\right)$ when φ is concave on $\mathbb{R}_+$ and $\lim_{x\to 0_+}\left(x\varphi'(x) - \varphi(x)\right) = 0$ and in particular when $F(x, y) = x^s y^t + x^t y^s$, $x, y, t, s \in \mathbb{R}_+$ satisfy Theorem 2.

Theorem 4 ([2, Theorem 2.7]) *If* φ *is a concave differentiable function on* $\mathbb{R}_+$ *and* $\lim_{x\to 0_+}\left(x\varphi'(x) - \varphi(x)\right) = 0$. *Then, the inequalities*

$$\sum_{i=1}^{n}\left(\widetilde{x}_i\varphi\left(\frac{\widetilde{x}_{i+1}}{\widetilde{x}_i}\right) + \widetilde{x}_{i+1}\varphi\left(\frac{\widetilde{x}_i}{\widetilde{x}_{i+1}}\right)\right) \tag{11}$$

$$\leq \sum_{i=1}^{n}\left(x_i\varphi\left(\frac{x_{i+1}}{x_i}\right) + x_{i+1}\varphi\left(\frac{x_i}{x_{i+1}}\right)\right)$$

$$\leq \sum_{i=1}^{n}\left(\widehat{x}_i\varphi\left(\frac{\widehat{x}_{i+1}}{\widehat{x}_i}\right) + \widehat{x}_{i+1}\varphi\left(\frac{\widehat{x}_i}{\widehat{x}_{i+1}}\right)\right)$$

hold where $\widetilde{x}_{n+1} = \widetilde{x}_1$, $x_{n+1} = x_1$ *and* $\widehat{x}_{n+1} = \widehat{x}_1$.

If φ *is a convex differentiable function on* $\mathbb{R}_+$ *and* $\lim_{x\to 0_+}\left(x\varphi'(x) - \varphi(x)\right) = 0$, *then the reverse of inequalities of (11) holds.*

Corollary 2 ([2, Theorem 2.8]) *Let* $F(x, y) = x^s y^t + x^t y^s$.

(a) If $x, y, s, t \in \mathbb{R}_+$, *then for* $(\mathbf{x}) \in \mathbb{R}^n_+$, *the inequalities*

$$\sum_{i=1}^{n}\left(\widetilde{x}_i^t\widetilde{x}_{i+1}^s + \widetilde{x}_{i+1}^t\widetilde{x}_i^s\right) \tag{12}$$

$$\leq \sum_{i=1}^{n}\left(x_i^s x_{i+1}^t + x_{i+1}^t x_i^s\right)$$

$$\leq \sum_{i=1}^{n}\left(\widehat{x}_i^t\widehat{x}_{i+1}^s + \widehat{x}_{i+1}^t\widehat{x}_i^s\right),$$

hold, where $\widetilde{x}_{n+1} = \widetilde{x}_1$, $x_{n+1} = x_1$, $\widehat{x}_{n+1} = \widehat{x}_1$, *and* $(\widetilde{\mathbf{x}})$ *is the circular alternating order of* $(\mathbf{x})$ *and* $(\widehat{\mathbf{x}})$ *is the circular symmetrical order of* $(\mathbf{x})$. *In particular, (12) holds when* $t + s = 1$, $0 \leq t \leq 1$.

(b) If $s \le 0$, $t \ge 0$, the reverse of (12) holds for $(\mathbf{x}) \in \mathbb{R}^n_{++}$. *In particular, the reverse of (12) holds when $t + s = 1$, $t \ge 1$.*

Remark 2 It is easy to verify that in addition to Examples 1 and 2, the functions $F(x, y)$, which appear in Theorem 3 cases (a), (d) and (e), Theorem 4 and Corollary 2, are twice differentiable L-superadditive functions satisfying $\frac{\partial^2 F(x,y)}{\partial x \partial y} \ge 0$, and therefore also Theorem 1 holds.

3 Jensen-Type Inequalities and Rearrangements

Using the results of Sect. 2, we get in this section Jensen-type inequalities. The methods employed here can be obtained similarly to those derived in [4, Section 3] and [2] using Theorem 3 [4, Theorem 2].

We demonstrate and prove two refinements of Jensen-type inequalities and quote one from [4]. For more inequalities related to Theorem 2, see [4, Section 3] and [2, Theorem 2.11].

We first refine the Jensen inequality for convex function $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, which reads

$$\sum_{i=1}^{n} a_i f(x_i) \ge f\left(\sum_{i=1}^{n} a_i x_i\right)$$

for all $a_i \ge 0$, $x_i \in I$, $i = 1, \dots, n$, $\sum_{i=1}^{n} a_i = 1$. The refinement is obtained by using Theorem 1 for the L-superadditive function $F(x, y) = f(x + y)$ when f is a convex function.

Theorem 5 *Let f be a convex function on an interval, and let $(x_1, \dots, x_{2n})$ be a given $2n$ real numbers. Then,*

$$\begin{aligned}
\sum_{i=1}^{2n} \frac{f(x_i)}{2n} &\ge \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) \qquad (13)\\
&\ge \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_{2i-1} + x_{2i}}{2}\right) \ge \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_{(i)} + x_{(2n+1-i)}}{2}\right)\\
&\ge f\left(\frac{\sum_{i=1}^{2n} x_i}{2n}\right).
\end{aligned}$$

Proof From the identity $\sum_{i=1}^{2n} \frac{f(x_i)}{2n} = \frac{1}{n}\sum_{i=1}^{n} \frac{f(x_{(2i-1)})+f(x_{(2i)})}{2}$, as a result of Jensen inequality, we get the first inequality in (13), and because of the L-superadditivity of $F(x, y) = f(x + y)$, we get the second and third inequalities of Equation (13). The last inequality follows again from the Jensen inequality for convex functions.

We quote now some definitions and lemmas that we need for the theorems presented below.

The following is a definition of the Jensen-type inequality for strongly convex functions with modulus C.

Definition 8 ([9]) Let $n \in \mathbb{N}$. *The function* $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, is called strongly convex with modulus C if for all $x_i \in I$, and all $a_i \geq 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} a_i = 1$, the Jensen-type inequality

$$\sum_{i=1}^{n} a_i f(x_i) - f(\overline{x}) \geq C \sum_{i=1}^{n} a_i (x_i - \overline{x})^2,$$

where $C \geq 0$ and $\overline{x} = \sum_{i=1}^{n} a_i x_i$, holds.

Definition 9 ([3]) A real-valued function ψ_1 defined on an interval $[a, b)$ with $0 \leq a < b \leq \infty$ is called 1-quasiconvex if it can be represented as the product of a convex function φ and the function $p(x) = x$.

Corollary 3 *A* 1*-quasiconvex function* ψ_1*, as defined in Definition 9, satisfies the inequalities*

$$\sum_{i=1}^{n} a_i \psi_1(x_i) \geq \psi_1(\overline{x}) + \varphi'(\overline{x}) \sum_{i=1}^{n} a_i (x_i - \overline{x})^2 \tag{14}$$

$$\geq \psi_1(\overline{x}) + C \sum_{i=1}^{n} a_i (x_i - \overline{x})^2,$$

where $C \leq \min \varphi'(x_i)$, $a_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} a_i = 1$ *and* $\overline{x} = \sum_{i=1}^{n} a_i x_i$.

If, in addition, φ *is increasing, then* ψ_1 *is also a strongly convex function.*

Theorems 6, 7 and 8 show the use of rearrangements for refinements of Jensen-type inequality for 1-quasiconvex functions by using Corollary 1 and Theorem 1 for the function

$$F(x, y) = f\left(\frac{x+y}{2}\right) + \varphi'\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)^2,$$

when $f(x) = x\varphi(x)$ and where φ and φ' are convex. In this case, $F(x, y)$ is L-superadditive and therefore satisfies Theorem 1 as well as Theorem 2.

Theorem 6 *Let* x_i, $i = 1, \ldots, 2n$, $n \in \mathbb{N}$, *be a sequence of real non-negative numbers, and let* φ *and* φ' *be convex on* $\mathbb{R}_+$ *and* $f(x) = x\varphi(x)$, $x \in \mathbb{R}_+$. *Then,* *denoting* $\overline{x} = \frac{1}{2n}\sum_{j=1}^{2n} x_j$, *it yields that*

$$\sum_{i=1}^{2n} f(x_i) - (2n-1) f(\overline{x}) - (2n-1)\varphi'(\overline{x}) \sum_{i=1}^{2n} \frac{1}{2n}(x_i - \overline{x})^2 \tag{15}$$
$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) + \varphi'\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right)\left(\frac{x_{(2i)} - x_{(2i-1)}}{2}\right)^2\right)$$
$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{2i-1} + x_{2i}}{2}\right) + \varphi'\left(\frac{x_{2i-1} + x_{2i}}{2}\right)\left(\frac{x_{2i} - x_{2i-1}}{2}\right)^2\right)$$
$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(i)} + x_{(2n+1-i)}}{2}\right) + \varphi'\left(\frac{x_{(i)} + x_{(2n+1-i)}}{2}\right)\left(\frac{x_{(2n+1-i)} - x_{(i)}}{2}\right)^2\right).$$

If also

$$\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) + \varphi'\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right)\left(\frac{x_{(2i)} - x_{(2i-1)}}{2}\right)^2\right) \tag{16}$$
$$\geq f(\overline{x}) + \frac{1}{2n}\varphi'(\overline{x})\sum_{i=1}^{2n}(x_i - \overline{x})^2,$$

then

$$\sum_{i=1}^{2n} f(x_i) - 2nf(\overline{x}) - \varphi'(\overline{x})\sum_{i=1}^{2n}(x_i - \overline{x})^2 \tag{17}$$
$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) + \varphi'\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right)\left(\frac{x_{(2i)} - x_{(2i-1)}}{2}\right)^2\right)$$
$$- f(\overline{x}) - \varphi'(\overline{x})\sum_{i=1}^{2n}\frac{1}{2n}(x_i - \overline{x})^2 \geq 0$$

refines Inequality (14).

Proof

$$
\begin{aligned}
\sum_{i=1}^{2n} f\left(x_i\right) &= \sum_{i=1}^{2n} f\left(x_{(i)}\right) \\
&= \frac{1}{2n} \sum_{i=1}^{2n} f\left(x_{(i)}\right) + \frac{2n-1}{2n} \sum_{i=1}^{2n} f\left(x_i\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{f\left(x_{(2i-1)}\right) + f\left(x_{(2i)}\right)}{2} + \frac{2n-1}{2n} \sum_{i=1}^{2n} f\left(x_i\right) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) + \varphi^{'}\left(\frac{x_{(2i-1)} + x_{(2i)}}{2}\right) \right. \\
&\quad \left. \times \left(\frac{x_{(2i)} - x_{(2i-1)}}{2}\right)^2 \right) \\
&\quad + \frac{2n-1}{2n} \sum_{i=1}^{2n} f\left(x_i\right) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{x_{2i-1} + x_{2i}}{2}\right) + \varphi^{'}\left(\frac{x_{2i-1} + x_{2i}}{2}\right) \left(\frac{x_{2i} - x_{2i-1}}{2}\right)^2 \right) \\
&\quad + \frac{2n-1}{2n} \sum_{i=1}^{2n} f\left(x_i\right) \\
&\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{x_{(i)} + x_{(2n+1-i)}}{2}\right) + \varphi^{'}\left(\frac{x_{(i)} + x_{(2n+1-i)}}{2}\right) \right. \\
&\quad \left. \times \left(\frac{x_{(2n+1-i)} - x_{(i)}}{2}\right)^2 \right) \\
&\quad + \frac{2n-1}{2n} \sum_{i=1}^{2n} f\left(x_i\right).
\end{aligned} \tag{18}
$$

Indeed, Inequality (18) follows from Inequality (14) for $a_i = \frac{1}{2}$, $i = 1, 2$.

Using again Inequality (14), this time for $a_i = \frac{1}{2n}$, $i = 1, \ldots, 2n$, we get that the inequalities

$$\begin{aligned}
&\sum_{i=1}^{2n} f(x_i) \\
&\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)+\varphi'\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)\left(\frac{x_{(2i)}-x_{(2i-1)}}{2}\right)^2\right) \\
&\quad+\frac{2n-1}{2n}\sum_{i=1}^{2n} f(x_i) \\
&\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)+\varphi'\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)\left(\frac{x_{(2i)}-x_{(2i-1)}}{2}\right)^2\right) \\
&\quad+(2n-1)\sum_{i=1}^{2n}\left(f(\overline{x})+\varphi'(\overline{x})\frac{1}{2n}(x_i-\overline{x})^2\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{2i-1}+x_{2i}}{2}\right)+\varphi'\left(\frac{x_{2i-1}+x_{2i}}{2}\right)\left(\frac{x_{2i}-x_{2i-1}}{2}\right)^2\right) \\
&\quad+(2n-1)\sum_{i=1}^{2n}\left(f(\overline{x})+\varphi'(\overline{x})\frac{1}{2n}(x_i-\overline{x})^2\right) \\
&\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{(i)}+x_{(2n+1-i)}}{2}\right)+\varphi'\left(\frac{x_{(i)}+x_{(2n+1-i)}}{2}\right)\left(\frac{x_{(2n+1-i)}-x_{(i)}}{2}\right)^2\right) \\
&\quad+(2n-1)\sum_{i=1}^{2n}\left(f(\overline{x})+\varphi'(\overline{x})\frac{1}{2n}(x_i-\overline{x})^2\right)
\end{aligned}$$

hold, from which Inequality (15) follows.

The first inequality in (15) can be rewritten as Inequality (17). Therefore, when (16) holds, the inequality (17) refines (14) for $a_i = \frac{1}{2n}$, $i = 1, \ldots, 2n$. This completes the proof of the theorem.

Example 3 shows a case where (16) holds, and therefore in this case, (17) is a refinement of (14):

Example 3 $F(x, y) = f\left(\frac{x+y}{2}\right) + \varphi'\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)^2$ satisfies Theorem 3 Case (e) for $f(x) = x^4 = x\varphi(x)$, where $\varphi(x) = x^3$. It is easy to check that for $x_i = i$,

$i = 1, 2, 3, 4,$

$$\frac{1}{2}\sum_{i=1}^{2}\left(f\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)+\varphi^{'}\left(\frac{x_{(2i-1)}+x_{(2i)}}{2}\right)\left(\frac{x_{(2i)}-x_{(2i-1)}}{2}\right)^{2}\right)$$

$$\geq f\left(\overline{x}\right)+\frac{1}{4}\varphi^{'}\left(\overline{x}\right)\sum_{i=1}^{4}\left(x_{i}-\overline{x}\right)^{2},$$

and therefore (17) is a refinement of (14) in this case.

Similarly to Theorem 6, but this time using the L-superadditivity of $F(x,y) = f\left(\frac{x+y}{2}\right)+\varphi^{'}\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right)^{2}$ as appears in Theorem 3 case (e), we get the following theorem:

Theorem 7 ([4, Theorem 5]) *Let x_i, $i = 1,\ldots,n$, $n \in \mathbb{N}$, be a sequence of real non-negative numbers, and let φ and $\varphi^{'}$, be convex on $\mathbb{R}_+$ where $f(x) = x\varphi(x)$, $x \in \mathbb{R}_+$ and $\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$. Then,*

$$\sum_{i=1}^{n} f\left(x_{i}\right)-(n-1)\, f\left(\overline{x}\right)-(n-1)\,\varphi^{'}\left(\overline{x}\right)\sum_{i=1}^{n}\frac{1}{n}\left(x_{i}-\overline{x}\right)^{2}$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_{i}+\widehat{x}_{i+1}}{2}\right)+\varphi^{'}\left(\frac{\widehat{x}_{i}+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_{i}-\widehat{x}_{i+1}}{2}\right)^{2}\right)$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{x_{i}+x_{i+1}}{2}\right)+\varphi^{'}\left(\frac{x_{i}+x_{i+1}}{2}\right)\left(\frac{x_{i}-x_{i+1}}{2}\right)^{2}\right)$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widetilde{x}_{i}+\widetilde{x}_{i+1}}{2}\right)+\varphi^{'}\left(\frac{\widetilde{x}_{i}+\widetilde{x}_{i+1}}{2}\right)\left(\frac{\widetilde{x}_{i}-\widetilde{x}_{i+1}}{2}\right)^{2}\right),$$

where $\widetilde{x}_{n+1} = \widetilde{x}_1$, $x_{n+1} = x_1$, $\widehat{x}_{n+1} = \widehat{x}_1$.

If also

$$\frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_{i}+\widehat{x}_{i+1}}{2}\right)+\varphi^{'}\left(\frac{\widehat{x}_{i}+\widehat{x}_{i+1}}{2}\right)\left(\frac{\widehat{x}_{i}-\widehat{x}_{i+1}}{2}\right)^{2}\right)$$

$$\geq f\left(\overline{x}\right)+\frac{1}{n}\varphi^{'}\left(\overline{x}\right)\sum_{i=1}^{n}\left(x_{i}-\overline{x}\right)^{2},$$

then

$$\sum_{i=1}^{n} f(x_i) - nf(\overline{x}) - \varphi^{'}(\overline{x}) \sum_{i=1}^{n} (x_i - \overline{x})^2$$
$$\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2} \right) + \varphi^{'}\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2} \right) \left(\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2} \right)^2 \right)$$
$$-f(\overline{x}) - \varphi^{'}(\overline{x}) \sum_{i=1}^{n} \frac{1}{n} (x_i - \overline{x})^2 \geq 0.$$

Theorem 8 (Using [4, Theorem 2, Case (d)]) *Let x_i, $i = 1, \ldots, n$, $n \in \mathbb{N}$, be a sequence of real non-negative numbers, and let φ be convex on $x \geq 0$ and $f(x) = x\varphi(x)$. Let $C \leq \min \varphi^{'}(x_i)$, $i = 1, \ldots, n$. Then, the 1-quasiconvex function f (which is strongly convex when $C \geq 0$) satisfies*

$$\sum_{i=1}^{n} f(x_i) - (n-1) f\left(\frac{\sum_{j=1}^{n} x_j}{n} \right)$$
$$- \frac{n-1}{n} \varphi^{'}\left(\frac{\sum_{j=1}^{n} x_j}{n} \right) \sum_{i=1}^{n} \left(x_i - \frac{\sum_{j=1}^{n} x_j}{n} \right)^2$$
$$\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2} \right) + C \times \left(\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2} \right)^2 \right)$$
$$\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{x_i + x_{i+1}}{2} \right) + C \times \left(\frac{x_i - x_{i+1}}{2} \right)^2 \right)$$
$$\geq \frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{\widetilde{x}_i + \widetilde{x}_{i+1}}{2} \right) + C \times \left(\frac{\widetilde{x}_i - \widetilde{x}_{i+1}}{2} \right)^2 \right),$$

where $\widetilde{x}_{n+1} = \widetilde{x}_1$, $x_{n+1} = x_1$, $\widehat{x}_{n+1} = \widehat{x}_1$.

If, in addition,

$$\frac{1}{n} \sum_{i=1}^{n} \left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2} \right) + C \times \left(\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2} \right)^2 \right)$$
$$- \left(f\left(\frac{\sum_{j=1}^{n} x_j}{n} \right) + C \sum_{i=1}^{n} \frac{1}{n} \left(x_i - \frac{\sum_{j=1}^{n} x_j}{n} \right)^2 \right) \geq 0,$$

then

$$\sum_{i=1}^{n} f(x_i) - nf\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) - C \times \sum_{i=1}^{n}\left(x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right)^2$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i + \widehat{x}_{i+1}}{2}\right) + C \times \left(\frac{\widehat{x}_i - \widehat{x}_{i+1}}{2}\right)^2\right)$$

$$-\left(f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right) + C\sum_{i=1}^{n}\frac{1}{n}\left(x_i - \frac{\sum_{j=1}^{n} x_j}{n}\right)^2\right).$$

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Approximate Generalized Jensen Mappings in 2-Banach Spaces

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Abstract Our aim is to investigate the generalized Hyers-Ulam-Rassias stability for the following general Jensen functional equation:

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x),$$

where $n \in \mathbb{N}_2$, $b_k = \exp(\frac{2i\pi k}{n})$ for $0 \leq k \leq n-1$, in 2-Banach spaces by using a new version of Brzdęk's fixed point theorem. In addition, we prove some hyperstability results for the considered equation and the general inhomogeneous Jensen equation

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + G(x, y).$$

1 Introduction and Preliminaries

In the middle of the 1960s, S. Gähler [10, 11] introduced the basic concepts of linear 2-normed spaces. In the following definitions and lemmas, we present some facts concerning the linear 2-normed spaces.

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Definition 1 Let X be a real linear space with $dimX > 1$ and $\|.,.\| : X \times X \longrightarrow [0,\infty)$ be a function satisfying the following properties:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\lambda x, y\| = |\lambda|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called *a 2-norm* on X, and the pair $(X, \|.,.\|)$ is called a *linear 2-normed space*. Sometimes, the condition (4) is called the *triangle inequality*.

Example 1 For $x = (x_1, x_2),\ y = (y_1, y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_{\mathbb{R}^2}$ is defined by

$$\|x, y\|_{\mathbb{R}^2} = |x_1y_2 - x_2y_1| .$$

Lemma 1 *Let* $(X, \|.,.\|)$ *be a 2-normed space. If* $x \in X$ *and* $\|x, y\| = 0$*, for all* $y \in X$*, then* $x = 0$.

Definition 2 A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$, and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 3 A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space*.

Now, we state the following results as lemma (see [15] for the details).

Lemma 2 *Let* X *be a 2-normed space. Then,*

1. $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,
2. *if* $\|x, z\| = 0$ *for all* $z \in X$*, then* $x = 0$,
3. *for a convergent sequence* x_n *in* X,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

The stability problem of functional equations originated from a question of S. M. Ulam [18] concerning the stability of group homomorphisms. D. H. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

Throughout this chapter, we will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ with $\mathbb{R}_+ := [0\ ,\ \infty)$, and the set of complex numbers by $\mathbb{C}$. By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

Let $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B.

Definition 4 Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{X^n}$, and $\mathscr{F}_1$, $\mathscr{F}_2$ be operators mapping from a nonempty set $\mathscr{D} \subset Y^{X^n}$ into Y^{X^n}. We say that the operator equation

$$\mathscr{F}_1\varphi(x_1, \ldots, x_n) = \mathscr{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathscr{D}$ which satisfies

$$d\left(\mathscr{F}_1\varphi_0(x_1, \ldots, x_n), \mathscr{F}_2\varphi_0(x_1, \ldots, x_n)\right) \le \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X)$$

fulfills Eq. (1).

J. Brzdęk et al. [5] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 1 ([5]) *Let X be a nonempty set, (Y, d) be a complete metric space, and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty} \delta_n = 0.$$

Suppose that $\mathscr{T}\colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathscr{T}\xi(x), \mathscr{T}\mu(x)\big) \le \Lambda\big(\Delta(\xi, \mu)\big)(x), \quad \xi, \mu \in Y^X,\ x \in X, \tag{2}$$

where $\Delta : Y^X \times Y^X \to \mathbb{R}_+^X$ *is a mapping which is defined by*

$$\Delta(\xi, \mu)(x) := d\big(\xi(x), \mu(x)\big) \quad \xi, \mu \in Y^X,\ x \in X. \tag{3}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *such that*

$$d\Big((\mathscr{T}\varphi)(x), \varphi(x)\Big) \le \varepsilon(x) \tag{4}$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n \varepsilon\big)(x) < \infty \tag{5}$$

for all $x \in X$, *then the limit*

$$\lim_{n\to\infty} \big((\mathscr{T}^n\varphi)\big)(x) \tag{6}$$

exists for each $x \in X$. *Moreover, the function* $\psi \in Y^X$ *defined by*

$$\psi(x) := \lim_{n\to\infty} \big((\mathscr{T}^n\varphi)\big)(x) \tag{7}$$

is a fixed point of $\mathscr{T}$ *with*

$$d\big(\varphi(x), \psi(x)\big) \le \varepsilon^*(x) \tag{8}$$

for all $x \in X$.

In 2013, Brzdęk [4] gave the fixed point result by applying Theorem 1 as follows:

Theorem 2 *Let* X *be a nonempty set,* (Y, d) *a complete metric space, and* $f_1, \ldots, f_s\colon X \to X$ *and* $L_1, \ldots, L_s\colon X \to \mathbb{R}_+$ *be given mappings. Let* $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ *be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)), \tag{9}$$

for $\delta \in \mathbb{R}_+^X$ *and* $x \in X$. *If* $\mathscr{T}\colon Y^X \to Y^X$ *is an operator satisfying the inequality*

$$d\big(\mathscr{T}\xi(x), \mathscr{T}\mu(x)\big) \le \sum_{i=1}^{s} L_i(x)d\big(\xi(f_i(x)), \mu(f_i(x))\big), \quad \xi, \mu \in Y^X, x \in X,$$

and a function $\varepsilon\colon X \to \mathbb{R}_+$ *and a mapping* $\varphi\colon X \to Y$ *satisfy*

$$d\left(\mathscr{T}\varphi(x), \varphi(x)\right) \le \varepsilon(x), \quad (x \in X),$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k \varepsilon(x) < \infty, \quad (x \in X),$$

then for every $x \in X$*, the limit*

$$\psi(x) := \lim_{n\to\infty} \mathscr{T}^n \varphi(x)$$

exists, and the function $\psi \in Y^X$ *so defined is a unique fixed point of* $\mathscr{T}$ *with*

$$d\left(\varphi(x), \psi(x)\right) \le \varepsilon^*(x), \qquad (x \in X).$$

In the following theorem, we extend the fixed point theorem (Theorem 2) in 2-Banach spaces.

Theorem 3 *Let* X *be a nonempty set,* $\left(Y, \|\cdot,\cdot\|\right)$ *be a 2-Banach space,* $g : X \to Y$ *be a mapping such that the set* $g(X) \subseteq Y$ *contains two linearly independent vectors, and* $f_1, \ldots, f_r : X \to X$ *and* $L_1, \ldots, L_r : X \to \mathbb{R}_+$ *be given mappings. Suppose that* $\mathscr{T} : Y^X \to Y^X$ *and* $\Lambda : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ *are two operators satisfying the conditions*

$$\left\| \mathscr{T}\xi(x) - \mathscr{T}\mu(x), g(z) \right\| \le \sum_{i=1}^{r} L_i(x) \left\| \xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g(z) \right\| \tag{10}$$

for all $\xi, \mu \in Y^X$*,* $x, z \in X$ *and*

$$\Lambda\delta(x,z) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x), z\big), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \tag{11}$$

If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *such that*

$$\left\| \mathscr{T}\varphi(x) - \varphi(x), g(z) \right\| \le \varepsilon(x,z) \tag{12}$$

and

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x,z) < \infty \tag{13}$$

for all $x, z \in X$*, then the limit*

$$\lim_{n\to\infty}\big((\mathscr{T}^n\varphi)\big)(x) \tag{14}$$

exists for each $x \in X$*. Moreover, the function* $\psi : X \to Y$ *defined by*

$$\psi(x) := \lim_{n\to\infty}\big((\mathscr{T}^n\varphi)\big)(x) \tag{15}$$

is a fixed point of $\mathscr{T}$ *with*

$$\big\|\varphi(x) - \psi(x), g(z)\big\| \le \varepsilon^*(x, z) \tag{16}$$

for all $x, z \in X$*.*

There are two other versions of Theorem 3 in 2-Banach space given in [1] and [6]. In addition, Brzdęk et al. [7] gave important related results in generalized metric spaces. In this chapter, we discuss the generalized Hyers-Ulam-Rassias stability problem for the following generalization of Jensen functional equation:

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x), \tag{17}$$

where $n \in \mathbb{N}_2$ and $b_k = \exp(\frac{2i\pi k}{n})$ for $0 \le k \le n-1$, in 2-Banach spaces by using Theorem 3 as a basic tool. The general solution and stability of this equation and its generalizations were studied by numerous researchers; see, for example, [3, 8, 9, 14] and [17].

2 Main Results

Let X be a complex normed space. We will denote by $Aut(X)$ the family of all automorphisms of X. Moreover, for each $u \in X^X$, we write $ux := u(x)$ for $x \in X$, and we define u' by $u'x := x - ux$ for $x \in X$.
The following theorem is the main result concerning the stability of the functional equation (17).

Theorem 4 *Let* X *be a* $\mathbb{C}$*-normed space,* Y *be a 2-Banach space,* $\varepsilon : \Big(X\setminus\{0\}\Big)^3 \to \mathbb{R}_+$*, and*

$$l(X) := \Big\{u \in Aut(X) : u', (u' + b_k u) \in Aut(X),$$

$$\alpha_u := n\lambda(u') + \sum_{k=1}^{n-1}\lambda(u' + b_k u) < 1\Big\} \neq \varnothing \tag{18}$$

where

$$\lambda(u) := \inf\Big\{t \in \mathbb{R}_+ : \varepsilon(ux, uy, z) \le t\varepsilon(x, y, z), \quad \forall x, y, z \in X \setminus \{0\}\Big\}$$

for all $u \in Aut(X)$. Assume that $f : X \longrightarrow Y$ satisfies the inequality

$$\left\| f(x+y) - nf(x) + \sum_{k=1}^{n-1} f(x + b_k y)\ ,\ g(z) \right\| \le \varepsilon(x, y, z) \tag{19}$$

for all $x, y, z \in X \setminus \{0\}$ such that $x + b_k y \neq 0$ for $0 < k < n-1$, where $g : X \to Y$ is a mapping such that the set $g(X) \subseteq Y$ contains two linearly independent vectors. Then, for each nonempty subset $\mathcal{U} \subset l(X)$ such that

$$u \circ v = v \circ u, \quad \forall u, v \in \mathcal{U}, \tag{20}$$

there exists a unique function $J : X \longrightarrow Y$ which satisfies Eq. (17) and

$$\|f(x) - J(x)\ ,\ g(z)\| \le \tilde{\varepsilon}(x, z) \quad x, z \in X \setminus \{0\}, \tag{21}$$

where

$$\tilde{\varepsilon}(x, z) := inf\left\{\frac{\varepsilon(u'x, ux, z)}{1 - \alpha_u} : u \in \mathcal{U}\right\} \quad x, z \in X \setminus \{0\}.$$

Proof Write $X_0 := X \setminus \{0\}$, and let us fix $u \in \mathcal{U}$. Replacing x with $u'x$ and y with ux in (19), we get

$$\left\| f(x) - nf(u'x) + \sum_{k=1}^{n-1} f\big((u' + b_k u)x\big)\ ,\ g(z) \right\| \le \varepsilon(u'x, ux, z) := \varepsilon_u(x, z) \tag{22}$$

for all $x, z \in X_0$. We define the operators $\mathcal{T}_u : Y^{X_0} \to Y^{X_0}$ and $\Lambda_u : \mathbb{R}_+^{X_0 \times X_0} \to \mathbb{R}_+^{X_0 \times X_0}$ by

$$\mathcal{T}_u \xi(x) := n\xi(u'x) - \sum_{k=1}^{n-1} \xi\big((u' + b_k u)x\big), \tag{23}$$

$$\Lambda_u \delta(x, z) := n\delta\big(u'x, z\big) + \sum_{k=1}^{n-1} \delta\Big((u' + b_k u)x, z\Big)$$

for all $x, z \in X_0$, $\xi \in Y^{X_0}$ and $\delta \in \mathbb{R}_+^{X_0 \times X_0}$. Then the inequality (22) becomes

$$\left\| f(x) - \mathscr{T}_u f(x)\ ,\ g(z) \right\| \le \varepsilon_u(x, z)$$

for all $x, z \in X_0$ and $u \in \mathscr{U}$.

The operator Λ_u has the form given by (11) with $s = n$ and $f_n(x) = u'x$, $f_i(x) = (u' + b_i u)x$, $L_n(x) = n$, $L_i(x) = 1$, $i \in \{1, 2, \dots, n-1\}$ for all $x \in X_0$ and $u \in \mathscr{U}$.
Further,

$$\begin{aligned} \left\| \mathscr{T}_u \xi(x) - \mathscr{T}_u \mu(x)\ ,\ g(z) \right\| &= \Big\| n\xi(u'x) - \sum_{k=1}^{n-1} \xi\big((u' + b_k u)x\big) - n\mu(u'x) \\ &\quad + \sum_{k=1}^{n-1} \mu\big((u' + b_k u)x\big)\ ,\ g(z) \Big\| \\ &\le n \left\| \xi(u'x) - \mu(u'x)\ ,\ g(z) \right\| \\ &\quad + \sum_{k=1}^{n-1} \left\| \xi\big((u' + b_k u)x\big) - \mu\big((u' + b_k u)x\big)\ ,\ g(z) \right\| \end{aligned}$$

for all $x, z \in X_0$, $u \in \mathscr{U}$, and $\xi, \mu \in Y^{X_0}$.

Note that, in view of the definition of $\lambda(u)$,

$$\varepsilon(ux, uy, z) \le \lambda(u)\varepsilon(x, y, z), \qquad x, y, z \in X_0.$$

By mathematical induction on $s \in \mathbb{N}$, it is easy to show that

$$\Lambda_u^s \varepsilon_u(x, z) \le \alpha_u^s \varepsilon(u'x, ux, z),$$

for all $x, z \in X_0$ and all $u \in \mathscr{U}$, where

$$\alpha_u = n\lambda(u') + \sum_{k=1}^{n-1} \lambda(u' + b_k u).$$

Hence,

$$\varepsilon^*(x, z) := \sum_{r=0}^{\infty} \Lambda_u^r \varepsilon_u(x, z) \le \varepsilon(u'x, ux, z) \sum_{r=0}^{\infty} \alpha_u^r = \frac{\varepsilon(u'x, ux, z)}{1 - \alpha_u} < \infty \quad (24)$$

for all $x, z \in X_0$. According to Theorem 3, there exists a unique solution $J_u : X \to Y$ of the equation

$$J_u(x) = nJ_u(u'x) - \sum_{k=1}^{n-1} J_u\big((u' + b_k u)x\big) \tag{25}$$

for all $x \in X_0$ and all $u \in \mathscr{U}$, which is a fixed point of $\mathscr{T}_u$ such that

$$\Big\| J_u(x) - f(x)\ ,\ g(z) \Big\| \le \frac{\varepsilon(u'x, ux, z)}{1 - \alpha_u} \tag{26}$$

for all $x \in X_0$ and all $u \in \mathscr{U}$. Moreover,

$$J_u(x) = \lim_{r\to\infty} \mathscr{T}_u^r f(x)$$

for all $x \in X_0$ and all $u \in \mathscr{U}$.

To prove that J_u satisfies the functional equation (17) on X_0, we just prove the following inequality:

$$\Big\| \mathscr{T}_u^r f(x+y) - n\mathscr{T}_u^r f(x) + \sum_{k=1}^{n-1} \mathscr{T}_u^r f(x + b_k y)\ ,\ g(z) \Big\| \le \alpha_u^r \varepsilon(x, y, z) \tag{27}$$

for all $r \in \mathbb{N}$, all $u \in \mathscr{U}$ and all $x, y, z \in X_0$ such that $x + b_k y \ne 0$.
Indeed, if $r = 0$, then (27) is simply (19). So, take $r \in \mathbb{N}_1$, and suppose that (27) holds for r and $x, y, z \in X_0$. Then, by using (23) and the triangle inequality, we have

$$\begin{aligned}
&\Big\| \mathscr{T}_u^{r+1} f(x+y) - n\mathscr{T}_u^{r+1} f(x) + \sum_{k=1}^{n-1} \mathscr{T}_u^{r+1} f(x + b_k y)\ ,\ g(z) \Big\| \\
&= \Bigg\| n\mathscr{T}_u^r f\Big(u'(x+y)\Big) + n\mathscr{T}_u^r f\Big(b_k u(x+y)\Big) - \sum_{k=1}^{n-1} \mathscr{T}_u^r f\Big((u' + b_k u)(x+y)\Big) \\
&\quad -n^2 \mathscr{T}_u^r f(u'x) + n\sum_{k=1}^{n-1} \mathscr{T}_u^r f\Big((u' + b_k u)x\Big) \\
&\quad + \sum_{k=1}^{n-1} \Bigg\{ n\mathscr{T}_u f\Big(u'(x + b_k y)\Big) - \sum_{k=1}^{n-1} \mathscr{T}_u f\Big((u' + b_k u)(x + b_k y)\Big) \Bigg\}\ ,\ g(z) \Bigg\| \\
&\le n \Bigg\| \mathscr{T}_u^r f\Big(u'(x+y)\Big) - n\mathscr{T}_u^r f(u'x) + \sum_{k=1}^{n-1} \mathscr{T}_u^r f\Big(u'(x + b_k y)\Big)\ ,\ g(z) \Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=1}^{n-1}\Big\|\mathcal{T}_u^r f\Big(u'+b_k u)(x+y)\Big)-n\mathcal{T}_u^r f\Big((u'+b_k u)x\Big)\\
&+\sum_{k=1}^{n-1}\mathcal{T}_u^r f\Big((u'+b_k u)(x+b_k y)\Big)\ ,\ g(z)\Big\|\\
\leq\ &\alpha_u^r\left(n\varepsilon(u'x,u'y,z)+\sum_{k=1}^{n-1}\varepsilon\Big((u'+b_k u)x,(u'+b_k u)y,z\Big)\right)\\
\leq\ &\alpha_u^r\left(n\lambda(u')+\sum_{k=1}^{n-1}\lambda(u'+b_k u)\right)\varepsilon(x,y,z)\\
=\ &\alpha_u^{r+1}\varepsilon(x,y,z).
\end{aligned}
$$

By induction, we have shown that (27) holds for all $r\in\mathbb{N}$, all $u\in\mathcal{U}$, and all $x,y,z\in X_0$ such that $x+b_k y\neq 0$. Letting $r\to\infty$ in (27), we get

$$
\sum_{k=0}^{n-1}J_u(x+b_k y)=nJ_u(x)
$$

for all $x,y\in X_0$ such that $x+b_k y\neq 0$ for $0\leq k\leq n-1$. Thus, we have proved that for every $u\in\mathcal{U}$, there exists a function $J_u : X_0\to Y$ which is a solution of the functional equation (17) on X_0 and satisfies

$$
\Big\|f(x)-J_u(x)\ ,\ g(z)\Big\|\leq\frac{\varepsilon(u'x,ux,z)}{1-\alpha_u}
$$

for all $x,z\in X_0$. Next, we prove that each solution $J : X\to Y$ of (17) satisfying the inequality

$$
\|f(x)-J(x)\ ,\ g(z)\|\leq L\ \varepsilon(v'x,vx,z),\qquad x,z\in X_0 \tag{28}
$$

with some $L>0$ and $v\in\mathcal{U}$ is equal to J_w for each $w\in\mathcal{U}$. So, fix $v,w\in\mathcal{U}$, $L>0$ and $J : X\to Y$ a solution of (17) satisfying (28). Note that, by (26) and (28), there is $L_0>0$ such that

$$
\begin{aligned}
\|J(x)-J_w(x)\ ,\ g(z)\|&\leq\|J(x)-f(x)\ ,\ g(z)\|+\|f(x)-J_w(x)\ ,\ g(z)\|\\
&\leq L_0\ \big(\varepsilon(v'x,vx,z)+\varepsilon(w'x,wx,z)\big)\cdot\sum_{r=0}^{\infty}\alpha_w^r
\end{aligned}
\tag{29}
$$

for all $x,z\in X_0$. In other side, J and J_w are solutions of (25) because they satisfy (17).

We show that, for each $j \in \mathbb{N}$,

$$\|J(x) - J_w(x)\ ,\ g(z)\| \\ \leq L_0\ \big(\varepsilon(v'x, vx, z) + \varepsilon(w'x, wx, z)\big) \cdot \sum_{r=j}^{\infty} \alpha_w^r, \quad (x, z \in X_0). \tag{30}$$

The case $j = 0$ is exactly (29). So fix $\gamma \in \mathbb{N}_0$, and assume that (30) holds for $j = \gamma$. Then, in view of definition of $\lambda(u)$,

$$\begin{aligned}
\big\|J(x) - J_w(x)\ ,\ g(z)\big\| &= \Big\|nJ(w'x) - \sum_{k=1}^{n-1} J\big((w' + b_k w)x\big) \\
&\quad -nJ_w(w'x) + \sum_{k=1}^{n-1} J_w\big((w' + b_k w)x\big)\ ,\ g(z)\Big\| \\
&\leq n\big\|J(w'x) - J_w(w'x)\ ,\ g(z)\big\| \\
&\quad + \sum_{k=1}^{n-1} \big\|J\big((w' + b_k w)x\big) - J_w\big((w' + b_k w)x\big)\ ,\ g(z)\big\| \\
&\leq n\ L_0\ \big(\varepsilon(v'w'x, vw'x, z) + \varepsilon(w'w'x, ww'x, z)\big) \cdot \sum_{r=\gamma}^{\infty} \alpha_w^r \\
&\quad + L_0\ \sum_{k=1}^{n-1} \Big(\varepsilon\big(v'(w' + b_k w)x, v(w' + b_k w)x, z\big) \\
&\quad + \varepsilon\big(w'(w' + b_k w)x, w(w' + b_k w)x, z\big)\Big) \cdot \sum_{r=\gamma}^{\infty} \alpha_w^r \\
&\leq L_0\ \big(\varepsilon(v'x, vx, z) + \varepsilon(w'x, wx, z)\big)\Big(n\lambda(w') + \sum_{k=1}^{n-1} \lambda(w' + b_k w)\Big) \cdot \sum_{r=\gamma}^{\infty} \alpha_w^r \\
&= L_0\Big(\varepsilon(v'x, vx, z) + \varepsilon(w'x, wx, z)\Big) \cdot \sum_{r=\gamma+1}^{\infty} \alpha_w^r.
\end{aligned}$$

Hence, we have shown (30). Now, letting $j \to \infty$ in (30), we get

$$J(x) = J_w(x) \quad \forall x \in X_0. \tag{31}$$

By similar method, we also prove that $J_u = J_w$ for each $u \in \mathcal{U}$, which yields

$$\|f(x) - J_w(x)\ ,\ g(z)\| \le \frac{\varepsilon(u'x, ux, z)}{1 - \alpha_u} \quad x, z \in X_0,\ u \in \mathcal{U}.$$

This implies (21) with $J := J_w$ and the uniqueness of J is given by (31).

In the following theorem, we prove the hyperstability of Eq. (17) in 2-Banach spaces.

Theorem 5 *Let X, Y, and ε be as in Theorem 4. Suppose that there exists a nonempty set $\mathcal{U} \in l(X)$ such that $u \circ v = v \circ u \quad \forall u, v \in \mathcal{U}$ and*

$$\begin{cases} inf_{u\in\mathcal{U}}\ \varepsilon(u'x, ux, z) = 0 \quad \forall x, z \in X_0, \forall u \in \mathcal{U} \\ sup_{u\in\mathcal{U}}\alpha_u < 1. \end{cases} \tag{32}$$

Then every $f : X \to Y$ satisfying (19) is a solution of (17).

Proof Suppose that $f : X \to Y$ satisfies (19). Then, by Theorem 4, there exists a mapping $J : X \to Y$ which satisfies (17) and

$$\|f(x) - J(x)\ ,\ g(z)\| \le \tilde{\varepsilon}(x, z) \quad \forall x, z \in X_0.$$

In view of (32), $\tilde{\varepsilon}(x, z) = 0\ \ \forall x, z \in X_0$. This means that $f(x) = J(x)\ \ \forall x \in X_0$, whence

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x),$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$ for $0 < k < n - 1$ which implies that f satisfies the functional equation (17) on X_0.

3 Applications

In this section, we discuss some hyperstability results for Eq. (17) and the inhomogeneous functional equation

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + G(x, y). \tag{33}$$

Namely, from Theorems 4 and 5, we can obtain the following corollaries as natural results.

Corollary 1 *Let X, Y, and ε be as in Theorem 4 and $G : X^2 \to Y$. Suppose that*

$$\left\| G(x, y)\ ,\ g(z) \right\| \leq \varepsilon(x, y, z), \quad x, y, z \in X_0, \tag{34}$$

where $g : X \to Y$ is a mapping such that the set $g(X) \subseteq Y$ contains two linearly independent vectors, $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X_0$, and there exists a nonempty $\mathscr{U} \subset l(X)$ such that (20) and (32) hold. Then the inhomogeneous equation

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + G(x, y) \tag{35}$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$ for $0 < k < n-1$ has no solutions in the class of functions $f : X \to Y$.

Proof Suppose that $f : X \to Y$ is a solution to (35). Then

$$\begin{aligned} \left\| \sum_{k=0}^{n-1} f(x + b_k y) - nf(x)\ ,\ g(z) \right\| &= \left\| nf(x) + G(x, y) - nf(x)\ ,\ g(z) \right\| \\ &= \left\| G(x, y)\ ,\ g(z) \right\| \\ &\leq \varepsilon(x, y, z), \end{aligned}$$

for all $x, y, z \in X_0$ such that $x + b_k y \neq 0$ for $0 < k < n-1$. Consequently, by Theorem 5, f is solution of (17). Therefore,

$$G(x_0, y_0) = \sum_{k=0}^{n-1} f(x_0 + b_k y_0) - nf(x_0) = 0,$$

which is contradiction.

Corollary 2 *Let X, Y, and ε be as in Theorem 4. Assume that $G : X^2 \to Y$ and $f : X \to Y$ satisfy the inequality*

$$\left\| f(x + y) - nf(x) + \sum_{k=1}^{n-1} f(x + b_k y) - G(x, y)\ ,\ g(z) \right\| \leq \varepsilon(x, y, z) \tag{36}$$

for all $x, y, z \in X \setminus \{0\}$ such that $x + b_k y \neq 0$ for $0 < k < n-1$, where $g : X \to Y$ is a mapping such that the set $g(X) \subseteq Y$ contains two linearly independent vectors and there exists a nonempty $\mathscr{U} \subset l(X)$ such that (20) and (32) hold. If the functional equation (35) has a solution $f_0 : X \to Y$, then f is a solution to (35).

Proof In view of (36), we obtain that $J := f - f_0$ satisfies (19). According to Theorem 5, we conclude that J is a solution to Eq. (17). Therefore,

$$\begin{aligned}\sum_{k=0}^{n-1} f(x+b_k y) - nf(x) - G(x,y) &= \sum_{k=0}^{n-1} J(x+b_k y) + \sum_{k=0}^{n-1} f_0(x+b_k y) \\ &\quad -nJ(x) - nf_0(x) - G(x,y) \\ &= 0,\end{aligned}$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$ for $0 < k < n-1$ which means that f is a solution to (35).

Corollary 3 *Let X and Y be a $\mathbb{C}$-normed space and a 2-Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p < 0$, $q < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| \sum_{k=0}^{n-1} f(x+b_k y) - nf(x)\ ,\ g(z) \right\| \leq \theta\Big(\|x\|^p + \|y\|^q \Big)\|z\|^r \tag{37}$$

for all $x, y, z \in X_0$ such that $x + b_k y \neq 0$, then f satisfies the functional equation (17) on X_0.

Proof The proof follows from Theorem 5 by taking

$$\varepsilon(x,y,z) = \theta\Big(\|x\|^p + \|y\|^q \Big)\|z\|^r, \quad x, y, z \in X_0,$$

with some real numbers $\theta, r \geq 0$, $p < 0$, and $q < 0$. For each $m \in \mathbb{N}$, define $u_m : X_0 \to X_0$ by $u_m x := -mx$ and $u'_m : X_0 \to X_0$ by $u'_m x := (1+m)x$. Then

$$\begin{aligned}\varepsilon(u_m x, u_\ell y, z) &= \varepsilon(-mx, -\ell y, z) \\ &= \theta\Big(\|-mx\|^p + \|-\ell y\|^q \Big)\|z\|^r \\ &= \theta\Big(m^p\, \|x\|^p + \ell^q\, \|y\|^q \Big)\|z\|^r \\ &\leq \Big(m^p + \ell^q\Big)\,\theta\Big(\|x\|^p + \|y\|^q \Big)\|z\|^r \\ &= \Big(m^p + \ell^q\Big)\,\varepsilon(x,y,z)\end{aligned}$$

for all $x, y, z \in X_0$ such that $x + b_k y \neq 0$ with $\ell, m \in \mathbb{N}$. Hence,

$$\begin{aligned}\lim_{m\to\infty} \varepsilon(u'_m x, u_m y, z) &\leq \lim_{m\to\infty} \Big((1+m)^p + m^q\Big)\varepsilon(x,y,z) \\ &= 0\end{aligned}$$

for all $x, y, z \in X_0$ such that $x+b_k y \neq 0$. Then (32) is valid with $\lambda(u_m) = m^p+m^q$ for $m \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $m \geq n_0$ and

$$n\Big((1+m)^p + (1+m)^q\Big) + \sum_{k=1}^{n-1}\Big(\big|1+m+b_k m\big|^p + \big|1+m+b_k m\big|^q\Big) < 1.$$

So, it easily seen that (18) is fulfilled with

$$\mathscr{U} := \left\{u_m \in Aut\ X : m \in \mathbb{N}_{n_0}\right\}.$$

Therefore, by Theorem 5, every $f : X \to Y$ satisfying (37) is a solution of the functional equation (17) on X_0.

Corollary 4 *Let X and Y be a* $\mathbb{C}$*-normed space and a 2-Banach space, respectively. Assume that* $p, q \in \mathbb{R}$, $p+q < 0$, *and* $\theta, r \geq 0$. *If* $f : X \to Y$ *satisfies*

$$\left\|\sum_{k=0}^{n-1} f(x+b_k y) - nf(x)\ ,\ g(z)\right\| \leq \theta\ \|x\|^p\ \|y\|^q\ \|z\|^r$$

for all $x, y, z \in X_0$ *such that* $x+b_k y \neq 0$ *for* $0 < k < n-1$, *then* f *satisfies the functional equation (17) on* X_0.

Proof It is easily seen that the function ε given by

$$\varepsilon(x, y, z) = \theta\ \|x\|^p\ \|y\|^q\ \|z\|^r \qquad x, y, z \in X\backslash\{0\}$$

satisfies (32) and

$$\begin{aligned}\varepsilon(mx, \ell y, z) &= \theta\ \|mx\|^p\ \|\ell y\|^q\ \|z\|^r\\ &= \theta m^p \ell^q\ \|x\|^p\ \|y\|^q\ \|z\|^r\\ &= m^p \ell^q \varepsilon(x, y, z)\end{aligned}$$

for all $x, y, z \in X_0$ such that $x+b_k y \neq 0$ with $\ell, m \in \mathbb{N}$ and $\ell m \neq 0$.

The remainder of the proof is similar to the proof of Corollary 3.

By an analogous conclusion, the function ε given by

$$\varepsilon(x, y, z) = \theta\Big(\ \|x\|^p + \|y\|^q + \|x\|^p\ \|y\|^q\ \Big)\ \|z\|^r \qquad x, y, z \in X_0,$$

satisfies (32) and

$$\begin{aligned}\varepsilon(mx, \ell y, z) &= \theta\Big(\|mx\|^p + \|\ell y\|^q + \|mx\|^p \, \|\ell y\|^q \Big) \, \|z\|^r \\ &= \theta\Big(m^p \, \|x\|^p + \ell^q \, \|y\|^p + m^p \ell^q \, \|x\|^p \, \|y\|^q \Big) \, \|z\|^r \\ &\le \Big(m^p + \ell^q + m^p \ell^q\Big)\varepsilon(x, y, z)\end{aligned}$$

for all $x, y, z \in X_0$ such that $x + b_k y \neq 0$ with $\ell, m \in \mathbb{N}$ and $\ell m \neq 0$. So, we have the following corollary.

Corollary 5 *Let X and Y be a $\mathbb{C}$-normed space and a 2-Banach space, respectively. Assume that $p < 0$, $q < 0$, $p + q < 0$, and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| \sum_{k=0}^{n-1} f(x + b_k y) - nf(x) \ , \ g(z) \right\| \leq \theta \left(\|x\|^p + \|y\|^q + \|x\|^p \, \|y\|^q \right) \|z\|^r$$

for all $x, y, z \in X_0$ such that $x + b_k y \neq 0$ for $0 < k < n - 1$, then f satisfies the functional equation (17) on X_0.

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The Asymptotic Expansion for a Class of Convergent Sequences Defined by Integrals

Dorin Andrica and Dan Ştefan Marinescu

Abstract We obtain the complete asymptotic expansion of the sequence defined by $\int_0^1 f(x)g(x^n)dx$, where the functions f and g satisfy various conditions. The main result is applied in Sect. 4 to find the complete asymptotic expansion of some classical sequences.

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1 Introduction

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Following Poincaré (see, e.g., [8, 10, 25]), a series $\sum_{j=1}^{\infty} \frac{b_j}{n^j}$, convergent or divergent, is called a complete asymptotic expansion of the sequence $(a_n)_{n\geq 1}$ if for each integer $k \geq 0$, we have

$$a_n = \sum_{j=1}^{k} \frac{b_j}{n^j} + o(\frac{1}{n^k}) \text{ as } n \to \infty,$$

where we have used the Landau "little-o" symbol. In this case, we also write

$$a_n \sim \sum_{j=1}^{\infty} \frac{b_j}{n^j} \text{ as } n \to \infty,$$

and the numbers $b_0, b_1, \ldots,$ are called the coefficients of the expansion.

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Previously, several authors have studied the behavior of some sequences defined by Riemann integrals obtaining the limit and the first two convergence orders. Let us review few such results.

One of the most popular result in this direction is so-called Riemann-Lebesgue lemma which asserts that if $g : [0, \infty) \to \mathbb{R}$ is a continuous and T-periodic function, then for any continuous function $f : [a, b] \to \mathbb{R}$, where $0 \leq a < b$, the following relation holds:

$$\lim_{n\to\infty} \int_a^b f(x)g(nx)dx = \frac{1}{T} \int_0^T g(x)dx \int_a^b f(x)dx.$$

For the proof, we refer to [11] (in special case $a = 0, b = T$) and [23]. In the paper [2], we have proved that a similar relation holds for all continuous and bounded functions $g : [0, \infty) \to \mathbb{R}$ having finite Cesaro mean.

In the paper [2] (see also the book [21]), the authors proved that given a continuous and T-periodic function $g : [0, \infty) \to \mathbb{R}$ having continuous derivative on the interval $[0, T]$, then for every function $f \in C^1[0, T]$, the following relation holds:

$$\lim_{n\to\infty} \int_0^T f(x)g(nx)dx = \frac{1}{T}(f(T) - f(0)) \left(G(T) - \int_0^T G(x)dx \right),$$

where $G(x) = \int_0^T g(t)dt$.

The same authors have obtained in [3] the following result. If $f : [0, \infty) \to \mathbb{R}$ is continuous such that the limit $\lim_{x\to\infty} xf(x)$ exists and is finite, then

$$\lim_{n\to\infty} \int_1^a f(x^n)dx = \int_1^\infty \frac{f(x)}{x}dx,$$

for every $a > 1$.

Another result given by the same authors [4] is the following. Consider the continuous function $f : [0, 1] \to \mathbb{R}$ such that $\lim_{x\searrow 0} \frac{f(x)}{x}$ exists and is finite. Then for any continuous function $f : [0, 1] \to \mathbb{R}$, the following relation holds:

$$\lim_{n\to\infty} n \int_0^1 g(x)f(x^n)dx = g(1) \int_0^1 \frac{f(x)}{x}dx,$$

In the papers [9, 16] and [17], the following result is proved with small variations. If $f : [0, 1] \to \mathbb{R}$ is continuous and $g : [0, 1] \to \mathbb{R}$ is continuously differentiable, then

$$\lim_{n\to\infty} n \int_0^1 x^n f(x^n)g(x)dx = g(1) \int_0^1 f(x)dx.$$

The second order of convergence of $\int_0^1 x^n f(x^n)g(x)dx$ is obtained in [20] in the special case $g = 1$. More precisely, the following result is proved. If $f : [0, 1] \to \mathbb{R}$ is continuous, then

$$\int_0^1 x^n f(x^n)dx = \frac{1}{n}\int_0^1 f(x)dx + \frac{1}{n^2}\int_0^1 f(x)\ln x dx + o\left(\frac{1}{n^3}\right).$$

The complete asymptotic expansion of $\int_0^1 h(x^n)dx$ is obtained in the paper [18]. If $h : [0, 1] \to \mathbb{R}$ is continuous on the interval $[0, 1]$ and differentiable at 0, then

$$\int_0^1 h(x^n)dx \sim h(0) + \sum_{j=0}^{\infty} \frac{1}{n^{j+1} j!}\int_0^1 (\ln t)^j \frac{h(t) - h(0)}{t} dt \quad \text{as} \quad n \to \infty. \tag{1}$$

Clearly, when $h(x) = xf(x)$, $x \in [0, 1]$, where the function f is continuous on $[0, 1]$ and differentiable at 0, from this formula, one obtains the complete asymptotic expansion of $\int_0^1 x^n f(x^n)dx$ as

$$\int_0^1 x^n f(x^n)dx \sim \sum_{j=0}^{\infty} \frac{1}{n^{j+1} j!}\int_0^1 (\ln t)^j h(t)dt \quad \text{as} \quad n \to \infty.$$

The main purpose of this paper is to extend formula (1). In Sect. 3, we obtain the complete asymptotic expansion of the sequence defined by $\int_0^1 f(x)g(x^n)dx$, where the functions f and g satisfy various conditions. We apply the main result to find in Sect. 4 the complete asymptotic expansion of some classical sequences.

2 Preliminaries

We need the following helpful results. The first is so-called the bounded convergence theorem formulated in 1885 by C. Arzelá for Riemann integrable functions [5] and, independently, in 1897 by W. F. Osgood for continuous functions [19].

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be a function and $(f_n)_{n\geq 1}$, $f_n : [a, b] \to \mathbb{R}$ be sequence of functions such that the following conditions are satisfied :*

(i) For every $n \geq 1$, f_n is Riemann integrable on the interval $[a, b]$;
(ii) For every $x \in [a, b]$, we have $\lim_{n\to\infty} f_n(x) = f(x)$;
(iii) There is a constant M such that $|f_n(x)| \leq M$, for all $x \in [a, b]$ and every $n \geq 1$;
(iv) The function f is Riemann integrable.

Then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

For the standard proof and some historical issues of this result, we refer to the book of S. Lang [13]. For a simple elementary proof, we refer to the paper [14] and for some interesting comments and extensions to [24].

Lemma 1 *Let* $g : [0, 1] \to \mathbb{R}$ *be a Riemann integrable function on the interval* $[0, 1]$. *Then, for every positive integer* $n \geq 1$*, the function* $u_n : [0, 1] \to \mathbb{R}$*, defined by* $u_n(x) = g(x^n)$*, is Riemann integrable on the interval* $[0, 1]$.

Proof Because the function g is Riemann integrable, it is bounded; hence, the function u_n is bounded. We will show that the set $D(u_n)$ of points where the function u_n is discontinuous has zero Lebesgue measure. It is easy to see that g is continuous at x_0 if and only if u_n is continuous at $\sqrt[n]{x_0}$, that is, $D(u_n) = \{\sqrt[n]{x} : x \in D(g)\}$.

Now, we can assume that $D(u_n) \subset (0, 1]$. We can write

$$D(u_n) = D(u_n) \cap (0, 1] = D(u_n) \cap \cup_{k=1}^{\infty}\left[\frac{1}{k}, 1\right]$$

$$= \cup_{k=1}^{\infty}(D(u_n) \cap \left[\frac{1}{k}, 1\right]) = \cup_{k=1}^{\infty}(D_k(u_n),$$

where $D_k(u_n) = D(u_n) \cap [\frac{1}{k}, 1]$. It suffices to prove that every set $D_k(u_n)$ is of zero Lebesgue measure. In this respect, observe that the set $D_{k^n}(g) = D(g) \cap [\frac{1}{k^n}, 1]$ has zero Lebesgue measure, as a subset of $D(g)$. Therefore, for every $\epsilon > 0$, there is a sequences of open intervals $\left(I_m^k\right)_{m\geq 1}$, $I_m^k = \left(a_m^k, b_m^k\right) \subset [\frac{1}{k^n}, 1]$ such that $D_{k^n}(g) \subset \cup_{m=1}^{\infty} I_m^k$ and

$$\sum_{m=1}^{\infty} l(I_m^k) = \sum_{m=1}^{\infty}(b_m^k - a_m^k) < \frac{\epsilon}{nk^{n-1}}.$$

Denoting $J_m^k = (\sqrt[n]{a_m^k}, \sqrt[n]{b_m^k})$, we obtain $D_k(u_n) = \sqrt[n]{D_{k^n}}(g) \subset \cup_{m=1}^{\infty} J_m^k$. Using the inequalities $a_m^k, b_m^k \geq k^{-n}$, it follows

$$\sum_{m=1}^{\infty} l(J_m^k) = \sum_{m=1}^{\infty}\left(\sqrt[n]{b_m^k} - \sqrt[n]{a_m^k}\right) = \sum_{m=1}^{\infty}\frac{b_m^k - a_m^k}{\sqrt[n]{(b_m^k)^{n-1}} + \cdots + \sqrt[n]{(a_m^k)^{n-1}}}$$

$$< nk^{n-1}\sum_{m=1}^{\infty}\left(b_m^k - a_m^k\right) < nk^{n-1}\frac{\epsilon}{nk^{n-1}} = \epsilon;$$

hence, $D_k(u_n)$ has zero measure, and the desired result is obtained from the Lebesgue integrability criterion. □

The following general result, extending Lemma 1, was proved by elementary methods in the paper [7]. It was only stated in [15, Proposition 2].

Theorem 2 *Let $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ be two functions satisfying the following conditions :*

(i) $f([a, b]) \subset [c, d]$;
(ii) The function f is continuous;
(iii) The function f is differentiable on the open interval (a, b) and its derivative is continuous;
(iv) $f'(x) \neq 0$, *for all* $x \in (a, b)$;
(v) The function g is Riemann integrable on $[c, d]$.

Then, the composition function $g \circ f$ is Riemann integrable on the interval $[a, b]$.

3 The Main Results

In what follows, assume that the functions $f, g : [0, 1] \to \mathbb{R}$ are Riemann integrable on the interval [0, 1]. From Lemma 1, the sequence given by

$$\int_0^1 f(x)g(x^n)dx, n \geq 1 \tag{2}$$

is well-defined because a product of two Riemann integrable functions is Riemann integrable. The next result is in the spirit of Lemma 1 in [3].

Proposition 1 *If the function g is continuous at* 0*, then*

$$\lim_{n\to\infty} \int_0^1 f(x)g(x^n)dx = g(0) \int_0^1 f(x)dx.$$

Proof Let $(f_n)_{n\geq 1}$ be the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = f(x)g(x^n)$, $n \geq 1$. Then, we have

$$\lim_{n\to\infty} f_n(x) = \begin{cases} f(x)g(0) & \text{if} \quad x \in [0, 1) \\ f(1)g(1) & \text{if} \quad x = 1. \end{cases} \tag{3}$$

Clearly, the hypotheses in Theorem 1 are verified; hence, the relation

$$\lim_{n\to\infty} \int_0^1 f(x)g(x^n)dx = g(0) \int_0^1 f(x)dx$$

holds. □

For a function $h : [0, 1] \to \mathbb{R}$, which is differentiable at 0, consider the function h^* defined by

$$h^*(t) = \begin{cases} h'(0) & \text{if} \quad t = 0 \\ \frac{h(t)-h(0)}{t} & \text{if} \quad t \in (0, 1]. \end{cases} \tag{4}$$

The following result gives, under supplementary assumptions, the first asymptotic order for the limit in Proposition 1.

Proposition 2 *If the functions* f, g *are continuous on the interval* $[0, 1]$*, and the function* g *is differentiable at* 0*, then*

$$\lim_{n\to\infty} n[\int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx] = f(1)\int_0^1 g^*(x)dx.$$

Proof Obviously, the function g^* is continuous; therefore, the integral in the right-hand side is well-defined. On the other hand, we have

$$n\left[\int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx\right] = n\left[\int_0^1 f(x)(g(x^n) - g(0))dx\right]$$
$$= n\int_0^1 f(x)x^n g^*(x^n)dx = \int_0^1 \sqrt[n]{x} f(\sqrt[n]{x})g^*(x)dx.$$

Because

$$\lim_{n\to\infty} \sqrt[n]{x} f(\sqrt[n]{x})g^*(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ f(1)g^*(x) & \text{if} \quad x \in (0, 1], \end{cases} \tag{5}$$

it follows that the hypotheses in Theorem 1 are fulfilled; hence, the desired conclusion holds. □

In what follows, we are interested to determine, under additional assumptions, all asymptotic convergence orders of the sequence defined by (1). For a positive integer k, consider $f \in C^{k+1}[0, 1]$ and g a continuous function on the interval $[0, 1]$ which is differentiable at 0. For $i = 1, \ldots, k + 2$, define the functions $f_i : [0, 1] \to \mathbb{R}$ by $f_1 = f$ and

$$f_{i+1}(x) = [xf_i(x)]', x \in [0, 1].$$

Also, we introduce the functions $g_0, g_1, \ldots, g_{k+1} : [0, 1] \to \mathbb{R}$ by $g_0 = g$ and $g_{i+1}(x) = \int_0^x g_i^*(t)dt$, for all $i \geq 0$ and for every $x \in [0, 1]$.

Consider the infinite matrix $A = (a_i^j)$, having real entrances defined by $a_i^1 = 1$ for all $i = 1, 2, \ldots$, $a_1^j = 0$ for all $j = 2, 3, \ldots,$, and $a_{i+1}^j = ja_i^j + a_i^{j-1}$ for all $i = 1, 2, \ldots,$ and $j = 2, 3, \ldots$. Clearly, the matrix A is inferior triangular, and its entrances verify a nonlinear recursive relation of order 1. We will prove the following formula for a_i^j:

$$a_i^j = \frac{(-1)^j}{j!} s_{j,i},$$

where

$$s_{j,i} = \sum_{u=0}^{j} (-1)^u \binom{j}{u} u^i.$$

In order to prove the above relation, we need the following auxiliary result proved in [1]

Lemma 2 *For every i, j, the following relation holds:*

$$s_{j,i+1} = j(s_{j,i} - s_{j-1,i}).$$

Proof Let us observe that

$$(e^x - 1)^j = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} e^{kx} = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \left(\sum_{u=0}^{\infty} \frac{1}{u!} k^u x^u \right)$$
$$= \sum_{u=0}^{\infty} \frac{(-1)^j}{u!} s_{j,u} x^u.$$

Considering the derivative, we have

$$j(e^x - 1)^{j-1} e^x = \sum_{u=1}^{\infty} \frac{(-1)^j}{u!} u s_{j,u} x^{u-1}.$$

That is,

$$j \left[(e^x - 1)^j + (e^x - 1)^{j-1} \right] = \sum_{u=1}^{\infty} \frac{(-1)^j}{u!} u s_{j,u} x^{u-1};$$

hence,

$$j \left(\sum_{u=0}^{\infty} \frac{(-1)^j}{u!} s_{j,u} x^u + \sum_{u=0}^{\infty} \frac{(-1)^{j-1}}{u!} s_{j-1,u} x^u \right) = \sum_{u=1}^{\infty} \frac{(-1)^j}{u!} u s_{j,u} x^{u-1}.$$

Identifying the coefficient of x^i in both sides of the above equality, we get the desired relation. □

To prove the relation $a_i^j = \frac{(-1)^j}{j!} s_{j,i}$, let us define the infinite matrix $B = (b_i^j)$, where $b_i^j = \frac{(-1)^j}{j!} s_{j,i}$. Clearly, we have $b_i^1 = 1 = a_i^1$ for all $i = 1, 2, \ldots$ and $b_1^j = 0 = a_1^j$ for all $j = 2, 3, \ldots$. Now, it suffice to show that the entries of the matrix (b_i^j) satisfy the same recursive relation. Indeed, according to the relation in Lemma 2, we have

$$b_{i+1}^j = \frac{(-1)^j}{j!} s_{j,i+1} = \frac{(-1)^j}{j!} j(s_{j,i} - s_{j-1,i}) = jb_i^j + \frac{(-1)^{j-1}}{(j-1)!} s_{j-1,i} = jb_i^j + b_i^{j-1},$$

and we are done.

According to the well-known formulas due by Euler, we have (see also [6])

$$a_i^j = \frac{(-1)^j}{j!} s_{j,i} = \begin{cases} 0 \ \text{ if } \ 0 \le i < j \\ 1 \ \text{ if } \ i = j \\ \frac{j(j+1)}{2} \ \text{ if } \ i = j+1 \\ \frac{j(j+1)(j+2)(3j+1)}{24} \ \text{ if } \ i = j+2 \\ \frac{j^2(j+1)^2(j+2)(j+3)}{48} \ \text{ if } \ i = j+3 \\ \frac{j(j+1)(j+2)(j+3)(j+4)(15j^3+30j^2+5j+1)}{1152} \ \text{ if } \ i = j+4. \end{cases}$$

In order to prove the above formulas, we may consider the expansion

$$(e^x - 1)^j = \left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)^j = x^j + \frac{j}{2}x^{j+1} + \frac{j(3j+1)}{24}x^{j+2} + \frac{j^2(j+1)}{48}x^{j+3} + \frac{j(15j^3 + 30j^2 + 5j + 1)}{1152}x^{j+4} + \cdots,$$

and identify the coefficients in the two series

$$(e^x - 1)^j = \sum_{i=0}^{\infty} \frac{(-1)^j}{i!} s_{j,i} x^i = \sum_{i=j}^{\infty} \frac{(-1)^j}{i!} s_{j,i} x^i = \sum_{i=j}^{\infty} \frac{(-1)^j}{j!(j+1)\cdots i} s_{j,i} x^i = \sum_{i=j}^{\infty} \frac{a_i^j}{(j+1)\cdots i} x^i.$$

That is, we have the general formula

$$a_i^j = (j+1)\cdots i \cdot \alpha_i^j,$$

where α_i^j is the coefficient of x^i in the expansion $(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)^j$, and obviously we have $\alpha_i^j = 0$ if $i < j$.

In what follows, we denote by A_{k+2} the block square matrix of dimension $(k + 2) \times (k + 2)$ obtained from the infinite matrix A in the upper left position. For instance, for $k = 0$, we have

$$A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For $k = 1$, we obtain

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

For $k = 2$ and $k = 3$, we obtain

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix},$$

respectively

$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}.$$

Generally, we have the block formula

$$A_{k+2} = \left(\begin{array}{c|c} A_{k+1} & 0_{k+1,1} \\ \hline a_{k+2}^1 \;\cdots & a_{k+2}^{k+2} \end{array}\right).$$

Lemma 3 *With the above notations, for all* $i = 1, \ldots, k+2$, *and for every* $x \in [0, 1]$, *the following result holds:*

$$f_i(x) = \sum_{j=1}^{k+2} a_i^j x^{j-1} f^{(j-1)}(x).$$

Proof We proceed by induction on i. If $i = 1$, then we have $f_1(x) = f(x) = \sum_{j=1}^{k+2} a_i^j x^{j-1} f^{(j-1)}(x) = f(x)$, since $a_1^1 = 1$ and $a_1^j = 0$ for all $j = 2, \ldots, k+2$. Suppose that the relation holds for i, where $i \in \{1, \ldots, k\}$. Taking into account the definition, for every $x \in [0, 1]$, we can write

$$f_{i+1}(x) = [xf_i(x)]' = (\sum_{j=1}^{k+2} a_i^j x^j f^{(j-1)}(x))'$$

$$= \sum_{j=1}^{k+2} (ja_i^j x^{j-1} f^{(j-1)}(x) + a_i^j x^j f^{(j)}(x)) = \sum_{j=1}^{k+2} a_{i+1}^j x^{j-1} f^{(j-1)}(x),$$

because of the recursive relations satisfied by the entrances of the matrix A defined before the statement. If we have $i = k+1$, then the passing to $i+1$ is doing in the same way. □

Theorem 3 *Under the above conditions, the following relation holds:*

$$\int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{n^i} f_i(1)g_i(1)$$

$$+ \frac{(-1)^{k+1}}{n^{k+1}} \int_0^1 g_{k+1}(x^n) f_{k+2}(x)dx.$$

Proof By successively integration by parts, and using the relation $g_1(0) = 0$, we obtain

$$\int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx = \int_0^1 [g(x^n) - g(0)]f(x)dx$$

$$= \int_0^1 g_0^*(x^n)x^{n-1}xf(x)dx = \frac{1}{n}\int_0^1 (g_1(x^n))'(xf(x))dx = \frac{1}{n}(g_1(x^n))(xf(x))|_0^1$$

$$- \int_0^1 g_1(x^n) f_2(x)dx = \frac{1}{n} g_1(1) f_1(1) - \int_0^1 g_1(x^n) f_2(x)dx$$

$$= \frac{1}{n} g_1(1) f_1(1) - \frac{1}{n}[\int_0^1 g_1^*(x^n)x^{n-1}xf_2(x)dx] = \frac{1}{n} g_1(1) f_1(1)$$

$$- \frac{1}{n}[\frac{1}{n} g_2(1) f_2(1) - \frac{1}{n}\int_0^1 g_2(x^n) f_3(x)dx] = \frac{1}{n} g_1(1) f_1(1)$$

$$- \frac{1}{n^2} g_2(1) f_2(1) + \frac{1}{n^2}\int_0^1 g_2(x^n) f_3(x)dx].$$

Continuing this process and using the relation $g_i(0) = 0$, we obtain the desired formula. □

Remark 1 Using Lemma 2, the conclusion of Theorem 3 becomes

$$\int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx = \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{n^i}(\sum_{j=1}^{k+1} a_i^j f^{(j-1)}(1))g_i(1) + \frac{(-1)^{k+1}}{n^{k+1}}\int_0^1 g_{k+1}(x^n)f_{k+2}(x)dx.$$

The following application is given in Theorem 2 in the paper [4].

Corollary 1 *Let $f : [0,1] \to \mathbb{R}$ be a function of class $C^1[0,1]$, and let $h : [0,1] \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{n\to\infty} n[\int_0^1 f(x)h(x^n)dx - f(1)\int_0^1 h(x)dx] = -(f(1)+f'(1))\int_0^1 \frac{H(x)}{x}dx,$$

where $H(x) = \int_0^x h(t)dt$.

Proof Let us consider $k = 0$ in Theorem 2 and the function $g : [0,1] \to \mathbb{R}$ defined by $g(x) = xh(x)$. Then, we have $f_1(x) = f(x)$ and $f_2(x) = f(x) + xf'(x)$, $g_0(x) = g(x) = xh(x)$, $g_1(x) = \int_0^x (t)dt$. Applying the result in Theorem 2 and using the relation $g(0) = 0$, we obtain

$$\int_9^1 x^n f(x)h(x^n)dx = \int_0^1 f(x)g(x^n)dx - g(0)\int_0^1 f(x)dx = \frac{1}{n}f_1(1)g_1(1)$$
$$-\frac{1}{n}\int_0^1 g_1(x^n)f_2(x)dx = \frac{1}{n}f_1(1)\int_0^1 h(t)dt$$
$$-\frac{1}{n}\int_0^1 (f(x)+xf'(x))g_1(x^n)dx;$$

hence,

$$n\left[\int_0^1 f(x)h(x^n)dx - f(1)\int_0^1 h(x)dx\right] = n\int_0^1 (f(x)+xf'(x))g_1(x^n)dx.$$

But, from Proposition 1, we have

$$\lim_{n\to\infty}\int_0^1 (f(x)+xf'(x))g_1(x^n)dx = g_1(0)\int_0^1 (f(x)+xf'(x))dx = 0;$$

therefore, using Proposition 2, it follows

$$\lim_{n\to\infty}\int_0^1 (f(x)+xf'(x))g_1(x^n)dx = (f(1)+f'(1))\int_0^1 g_1^*(x)dx$$
$$= (f(1)+f'(1))\int_0^1 \frac{H(x)}{x}dx,$$

and we are done. □

Corollary 2 *In the hypotheses of Theorem 3, there exists a convergent sequence* $(a_n)_{n\geq 1}$ *of real numbers such that*

$$\int_0^1 f(x)g(x^n)dx = g(0)\int_0^1 f(x)dx + \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{n^i}f_i(1)g_i(1) + \frac{a_n}{n^{k+2}}.$$

Proof From Proposition 2, we have

$$\lim_{n\to\infty} n\int_0^1 g_{k+1}(x^n)f_{k+2}(x)dx = f_{k+2}(1)\int_0^1 g_{k+1}^*(x)dx.$$

Let us consider $a_n = (-1)^{k+1}n\int_0^1 g_{k+1}(x^n)f_{k+2}(x)dx$, and the desired property follows via Theorem 3. □

Remark 2 The result in Corollary 1 shows that

$$\int_0^1 f(x)g(x^n)dx = g(0)\int_0^1 f(x)dx + \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{n^i}f_i(1)g_i(1) + o\left(\frac{1}{n^{k+1}}\right).$$

Corollary 3 *If the function* f *is of class* $C^\infty[0,1]$*, then*

$$\int_0^1 f(x)g(x^n)dx = g(0)\int_0^1 f(x)dx + \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{n^i}f_i(1)g_i(1).$$

Proof Just apply Corollary 1. □

Remark 3

(1) If in Corollary 1, we consider $f(x) = 1$, $x \in [0,1]$, then we get the asymptotic formula

$$\int_0^1 g(x^n)dx = g(0) + \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{n^i}g_i(1),$$

which appears in a different form in the paper [18] (see also formula (1)).

(2) If in Corollary 1, we consider $g(x) = x, x \in [0, 1]$, then we get $g_i(x) = x$ for $i = 1, 2, \ldots$, hence $g_i(1) = 1, i = 1, 2, \ldots$, and we obtain the following asymptotic formula:

$$\int_0^1 x^n f(x)dx = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} f_i(1)}{n^i}.$$

(3) If in Corollary 1, we consider $f(x) = 1, g(x) = xh(x), x \in [0, 1]$, where h is continuous on $[0, 1]$, then we obtain the main result in [18].

4 The Complete Asymptotic Expansion of Some Classical Sequences

4.1 The Sequence $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{(n-1)}}{n}$

Let us consider the functions $f(x) = \frac{1}{x+1}$, $g(x) = x, x \in [0, 1]$. We have

$$\int_0^1 \frac{x^n}{x+1} dx = (-1)^{n-1}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{(n-1)}}{n} - \ln 2\right).$$

Using the result in Corollary 2, we obtain the formula

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} = \ln 2 + \sum_{i=1}^{\infty} \frac{(-1)^{n+i}}{n^i} f_i(1) g_i(1) = \ln 2 + \sum_{i=1}^{\infty} \frac{(-1)^{n+i}}{n^i} f_i(1).$$

In order to get an explicit formula, we need to determine the sequence $f_1(1), f_2(1), \ldots$. After simple computation, we get $f_1(1) = \frac{1}{2}$, $f_2(1) = \frac{1}{4}$, $f_3(1) = 0$, $f_4(1) = -\frac{1}{8}, \ldots$, and we can continue this process. It follows

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{(n-1)}}{n} = \ln 2 + (-1)^{n+1} \frac{1}{2n} + (-1)^{n+2} \frac{1}{4n^2} - (-1)^{n+4} \frac{1}{8n^4} + o\left(\frac{1}{n^4}\right) = \ln 2 + (-1)^n \left(-\frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{8n^4}\right) + o\left(\frac{1}{n^4}\right).$$

4.2 Other Expansion for $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{(n-1)}}{n}$

If $g(x) = x$ and f belongs to $C^{k+1}[0, 1]$, then there exists a convergent sequence $(a_n)_{n\geq 1}$ such that

$$\int_0^1 f(x)x^n dx = \frac{D_1}{n} + \frac{D_2}{n^2} + \cdots + \frac{D_k}{n^k} + \frac{a_n}{n^{k+1}}, n \geq 1,$$

where

$$\begin{aligned} D_i &= (-1)^{i-1}\left[a_i^1 f^{(0)}(1) + a_i^2 f^{(1)}(1) + \cdots + a_i^i f^{(i-1)}(1)\right] \\ &= (-1)^{i-1}\sum_{k=1}^{i}\frac{(-1)^k}{k!}(\sum_{j=1}^{k}\binom{k}{j}k^i)f^{(k-1)}(1). \end{aligned}$$

If $f(x) = \frac{1}{x+1}$, then we obtain the result in [22]. In this case, $f^{(p)}(1) = \frac{(-1)^p p!}{2^{p+1}}$, and it follows

$$\begin{aligned} D_i &= (-1)^{i-1}\sum_{k=1}^{i} a_i^k f^{(k-1)}(1) \\ &= (-1)^{i-1}\sum_{k=1}^{i}\frac{(-1)^{k-1}(k-1)!}{2^k}\cdot\frac{(-1)^k}{k!}\left(\sum_{j=1}^{k}(-1)^j\binom{k}{j}j^i\right) \\ &= (-1)^{i-1}\sum_{k=1}^{i}\frac{1}{2^k}\sum_{j=1}^{k}(-1)^j\binom{k-1}{j-1}j^{i-1} = C_i, \end{aligned}$$

using the notation in [12].

Therefore,

$$\int_0^1 \frac{x^n}{x+1}dx = \frac{D_1}{n} + \frac{D_2}{n^2} + \cdots + \frac{D_k}{n^k} + \frac{a_n}{n^{k+1}}, n \geq 1.$$

Because

$$\int_0^1 \frac{x^n}{x+1}dx = (-1)^n\left(\ln 2 - \sum_{k=1}^{n}\frac{(-1)^{(k-1)}}{k}\right),$$

it follows

$$\ln 2 - \sum_{k=1}^{n} \frac{(-1)^{(k-1)}}{k} = (-1)^n \left(\frac{D_1}{n} + \frac{D_2}{n^2} + \cdots + \frac{D_k}{n^k} + \frac{a_n}{n^{k+1}} \right),$$

and then

$$| \ln 2 - \sum_{k=1}^{n} \frac{(-1)^{(k-1)}}{k} | = \frac{D_1}{n} + \frac{D_2}{n^2} + \cdots + \frac{D_k}{n^k} + \frac{a_n}{n^{k+1}}.$$

Remark 4

(1) As it is mentioned in [12], Ulrich Abel has noted that it is possible to generalize the result to obtain the asymptotic expansion

$$\left| \ln(1-x) + \sum_{k=1}^{n} \frac{x^k}{k} \right| \sim |x|^k \sum_{k=1}^{\infty} \frac{C_k(x)}{n^k}, \quad -1 \le x < 1,$$

where the coefficients $C_k(x)$ are given by

$$C_k(x) = \sum_{i=1}^{k} \frac{1}{(1-x)^i} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} j^{k-1}.$$

(2) In Tauraso (Personal communication, 2014) is given a different proof obtained by using the Euler-Maclaurin summation formula

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n = \gamma + \frac{1}{2n} - \sum_{k=1}^{m} \frac{B_k}{k} \cdot \frac{1}{n^k} + o\left(\frac{1}{n^m}\right),$$

where γ is the well-known Euler-Mascheroni constant and B_k are the Bernoulli numbers.

4.3 *The Sequence* $b_n = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{4n+1}$

Let us consider the functions $f(x) = \frac{1}{x^2+1}$, $g(x) = x$, $x \in [0, 1]$. In this case, we have

$$\int_0^1 \frac{x^{4n+2}}{x^2+1} dx = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{4n+1} - \frac{\pi}{4}.$$

Using the result in Corollary 2 and the fact that $g_i(1) = 1, i = 1, 2, \ldots$, we obtain the formula

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{4n+1} = \frac{\pi}{4} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(4n+1)^i} f_i(1).$$

After simple computation, we get $f_1(1) = \frac{1}{2}$, $f_2(1) = 0$, $f_3(1) = -\frac{1}{2}, \ldots$, and we can continue this process. It follows

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{4n+1} = \frac{\pi}{4} + \frac{1}{2(4n+1)} - \frac{1}{2(4n+1)^3} + o\left(\frac{1}{(4n+1)^3}\right).$$

4.4 The Sequence $c_n = 1 - \frac{1}{1!} + \frac{1}{2} - + \cdots + \frac{(-1)^n}{n!}$

If we consider the functions $f(x) = e^x, g(x) = x, x \in [0, 1]$, then we get the integral $I_n = \int_0^n e^x x^n dx, n = 0, 1, \ldots$. A simple integration by parts gives the recursive formula $I_n + nI_{n-1} = e$, where $I_0 = e - 1$. It follows

$$I_n = (-1)^n n! e \left(1 - \frac{1}{1!} + \frac{1}{2} - \cdots + \frac{(-1)^n}{n!}\right) + (-1)^{n+1} n!.$$

Using the result in Corollary 2 and the relations $g_i(1) = 1, i = 1, 2, \ldots$, we obtain the formula

$$(-1)^n n! e \left(1 - \frac{1}{1!} + \frac{1}{2} - \cdots + \frac{(-1)^n}{n!}\right) + (-1)^{n+1} n! = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{n^i} f_i(1).$$

That is,

$$1 - \frac{1}{1!} + \frac{1}{2} - \cdots + \frac{(-1)^n}{n!} = \frac{1}{e} + \frac{1}{e} \sum_{i=1}^{\infty} \frac{(-1)^{n+i-1}}{n^i n!} f_i(1).$$

In this case, we have $f_1(1) = e$, $f_2(1) = 2e$, $f_3(1) = 5e$, $f_4(1) = 15e, \ldots$. It follows

$$1 - \frac{1}{1!} + \frac{1}{2} - \cdots + \frac{(-1)^n}{n!} = \frac{1}{e} + (-1)^n \left(\frac{1}{nn!} - \frac{2}{n^2 n!} + \frac{5}{n^3 n!} - \frac{15}{n^4 n!}\right) + o\left(\frac{1}{n^4 n!}\right),$$

and the expansion can be improved.

4.5 The Sequence $e_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$

If we consider the functions $f(x) = e^{-x}$, $g(x) = x$, $x \in [0, 1]$, then we get the integral $J_n = \int_0^1 e^x x^n dx$, $n = 0, 1, \ldots$. A simple integration by parts gives the recursive formula $J_n - nJ_{n-1} = -\frac{1}{e}$, where $J_0 = 1 - \frac{1}{e}$. It follows

$$J_n = -\frac{n!}{e}\left(1 + \frac{1}{1!} + \frac{1}{2} + \cdots + \frac{1}{n!}\right) + n!;$$

hence, from Corollary 2 and the relations $g_i(1) = 1$, $i = 1, 2, \ldots$, we obtain the formula

$$1 + \frac{1}{1!} + \frac{1}{2} + \cdots + \frac{1}{n!} = e - e\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{n^i n!} f_i(1).$$

For this choice of functions, we have $f_1(1) = \frac{1}{e}$, $f_2(1) = 0$, $f_3(1) = -\frac{1}{e}$, $f_4(1) = -\frac{1}{e}, \ldots$. It follows

$$1 + \frac{1}{1!} + \frac{1}{2} + \cdots + \frac{1}{n!} = e - \frac{1}{nn!} + \frac{1}{n^3 n!} + \frac{1}{n^4 n!} + o\left(\frac{1}{n^4 n!}\right),$$

and the expansion can be improved.

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Weak Pseudoprimality Associated with the Generalized Lucas Sequences

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Abstract Pseudoprimes are composite integers which share properties of the prime numbers, and they have applications in many areas, as, for example, in public-key cryptography. Numerous types of pseudoprimes are known to exist, many of them defined by linear recurrent sequences. In this material, we present some novel classes of pseudoprimes related to the generalized Lucas sequences. First, we present some arithmetic properties of the generalized Lucas and Pell–Lucas sequences and review some classical pseudoprimality notions defined for Fibonacci, Lucas, Pell, and Pell–Lucas sequences and their generalizations. Then we define new notions of pseudoprimality which do not involve the use of the Jacobi symbol and include many classical pseudoprimes. For these, we present associated integer sequences recently added to the Online Encyclopedia of Integer Sequences, identify some key properties, and propose a few conjectures.

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1 Introduction

Let a and b be integers. The *generalized Lucas* $(U_n(a,b))_{n\geq 0}$ and their companion, the *generalized Pell–Lucas* sequences $(V_n(a,b))_{n\geq 0}$ (often denoted by $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ for simplicity), are defined by

$$U_{n+2} = aU_{n+1} - bU_n, \quad U_0 = 0,\ U_1 = 1, \quad n = 0, 1, \ldots, \tag{1}$$

$$V_{n+2} = aV_{n+1} - bV_n, \quad V_0 = 2,\ V_1 = a, \quad n = 0, 1, \ldots. \tag{2}$$

The general term of these sequences can be written explicitly in terms of the Binet-type formulae below

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{D}}\left(\alpha^n - \beta^n\right), \quad n = 0, 1, \ldots, \tag{3}$$

$$V_n = \alpha^n + \beta^n, \quad n = 0, 1, \ldots, \tag{4}$$

where $D = a^2 - 4b \neq 0$ and $\alpha = \frac{a+\sqrt{D}}{2}$, $\beta = \frac{a-\sqrt{D}}{2}$ are the roots of the quadratic equation $z^2 - az + b = 0$. By Viéte's relations, one obtains that $\alpha + \beta = a$, $\alpha\beta = b$ and $\alpha - \beta = \sqrt{D}$.

Notice that the formula (3) could also be expressed using bivariate cyclotomic polynomials in α and β [15, p. 99], as

$$U_n = \prod_{d|n, d\geq 2} \Phi_d(\alpha, \beta),$$

where

$$\Phi_d(\alpha, \beta) = \prod_{j=1, \gcd(j,n)=1}^{n} (\alpha - \zeta^j \beta)$$

and ζ is a primitive n-th root of unity. As $\Phi_d(\alpha, \beta)$ is an integer for $d \geq 2$, this formula is useful in the study arithmetic properties of the integers U_n. Similarly for V_n, if ω is an n-th root of -1, we have

$$V_n = \prod_{d|n} \Phi_d(\alpha, \omega\beta),$$

The utility of this formula is limited, as $\Phi_d(\alpha, \omega\beta)$ is not an integer.

The Binet-type formulae extend to negative indices, and we have

$$U_{-n} = \frac{1}{\sqrt{D}}\left(\alpha^{-n} - \beta^{-n}\right) = -\frac{1}{b^n}U_n, \quad V_{-n} = \alpha^{-n} + \beta^{-n} = \frac{1}{b^n}V_n, \quad n \geq 0.$$

Note that U_n and V_n are integers for all $n \in \mathbb{Z}$, if and only if $|b| = 1$, and for this reason, we will focus on this case. These sequences are called *balanced* and have interesting divisibility properties [13].

For $b = -1$, if k is a positive real number, then we obtain the k-Fibonacci and k-Lucas numbers for $F_{k,n} = U_n(k, -1)$ and $L_{k,n} = V_n(k, -1)$, where $D = k^2 + 4$. Clearly, for $k = 1$, we get the Fibonacci and Lucas numbers $F_n = U_n(1, -1)$ and $L_n = V_n(1, -1)$ with $D = 5$, and for $k = 2$, we recover the Pell and Pell–Lucas numbers $P_n = U_n(2, -1)$ and $Q_n = V_n(2, -1)$, where $D = 8$.

When $b = 1$, the integers $U_n(a, 1)$ have interesting combinatorial interpretations, while $V_n(a, 1)$ are linked to the number of solutions for certain Diophantine equations [3], as well as to important classes of polynomials, including the Chebyshev polynomials of the first and second kinds [2, Chapter 2.2].

Some density results concerning these sequences are obtained in [7], together with results allowing the identification of generalized Lucas and Pell–Lucas numbers. The theorems below concerning arithmetic properties of the generalized Lucas and Pell–Lucas sequences have been proved in [3].

Theorem 1 (Theorem 3.1, [3]) *Let p be an odd prime, k a non-negative integer, and r an arbitrary integer. If $b = \pm 1$ and a is an integer such that $D = a^2 - 4b > 0$ is not perfect square, then the sequences U_n and V_n defined by (1) and (2) satisfy the following relations:*

$$(1) \quad 2U_{kp+r} \equiv \left(\frac{D}{p}\right) U_k V_r + V_k U_r \pmod{p};$$

$$(2) \quad 2V_{kp+r} \equiv D \left(\frac{D}{p}\right) U_k U_r + V_k V_r \pmod{p},$$

where $\left(\frac{D}{p}\right)$ is the Legendre symbol [1].

Theorem 2 (Theorem 3.5, [3]) *Let p be an odd prime, and let $k > 0$ and a be integers so that $D = a^2 + 4 > 0$ is not a perfect square. If $U_n = U_n(a, -1)$, $V_n = V_n(a, -1)$, then*

1. $U_{kp-\left(\frac{D}{p}\right)} \equiv U_{k-1} \pmod{p}$;
2. $V_{kp-\left(\frac{D}{p}\right)} \equiv \left(\frac{D}{p}\right) V_{k-1} \pmod{p}$.

Theorem 3 (Theorem 3.7, [3]) *Let p be an odd prime, and let $k > 0$ and a be integers so that $D = a^2 - 4 > 0$ is not a perfect square. If $U_n = U_n(a, 1)$, $V_n = V_n(a, 1)$, then the following relations hold:*

1. $U_{kp-\left(\frac{D}{p}\right)} \equiv \left(\frac{D}{p}\right) U_{k-1} \pmod{p}$;
2. $V_{kp-\left(\frac{D}{p}\right)} \equiv V_{k-1} \pmod{p}$.

Applying Theorem 1 for $k = 1$ and $r = 0$, we obtain

$$U_p \equiv \left(\frac{D}{p}\right) \pmod{p}; \tag{5}$$

$$V_p \equiv a \pmod{p}. \tag{6}$$

Also, since $U_0 = 0$ and $V_0 = 2$, and by using $k = 1$ in Theorems 2 and 3, one has

$$U_{p-\left(\frac{D}{p}\right)} \equiv 0 \pmod{p}; \tag{7}$$

$$V_{p-\left(\frac{D}{p}\right)} \equiv 2\left(\frac{D}{p}\right)^{\frac{1-b}{2}}. \tag{8}$$

These relations were known even by E. Lucas (see, e.g., [28]).

In this paper, we define pseudoprimality notions related to the generalized Lucas sequences. These are related to the Fibonacci pseudoprimes of level k defined by Andrica et al. [6], which were further generalized and analyzed in the recent papers by Andrica and Bagdasar [4, 5].

In Sect. 2, we present some classical pseudoprimality notions. In Sect. 3, we investigate the generalized Bruckman–Lucas pseudoprimes (previously known for Lucas sequences). In Sect. 4, we define the weak generalized Lucas pseudoprimes which do not require the Jacobi symbol. Combining these two notions, we define the weak generalized Lucas–Bruckner pseudoprimes explored in Sect. 5. Throughout this material, we review numerous recent entries to the *Online Encyclopedia of Integer Sequences* (OEIS) [26] and formulate some conjectures.

As indicated in recent studies, there could be many possible links between these pseudoprimality notions and public key cryptography [19], computational number theory [20], and IT security [27].

2 Some Pseudoprimality Properties

Pseudoprimes are composite numbers which under certain conditions behave as the prime numbers, which have applications in the factorization of large integers, primality testing, and public-key cryptography. Numerous classes of pseudoprimes are defined by recurrent sequences.

2.1 Classical Pseudoprimes Involving Generalized Lucas Sequences

Many pseudoprimality notions involving generalized Lucas sequences $(U_n(a,b))_{n\geq 0}$ and $(V_n(a,b))_{n\geq 0}$ defined by (1) and (2) are based on the relations (5), (6), (7), and (8). For historical details and pseudoprimality tests for generalized Lucas sequences, one may check the papers [8, 9] or the classical books by Ribenboim [21, 22].

Definition 1 A composite integer n is called a *Lucas pseudoprime* of parameters a and b if $\gcd(n,b)=1$ and n divides $U_{n-\left(\frac{D}{n}\right)}$, where $\left(\frac{D}{n}\right)$ is the Jacobi symbol.

More divisibility results involving sequences U_n and V_n are found in [8, Section 2].

Proposition 1 *If n is an odd composite number such that $\gcd(n, 2abD) = 1$, then any two of the following statements imply the other two.*

1. $U_n \equiv \left(\frac{D}{n}\right) \pmod{n}$;
2. $V_n \equiv V_1 = a \pmod{n}$;
3. $U_{n-\left(\frac{D}{n}\right)} \equiv U_0 = 0 \pmod{n}$;
4. $V_{n-\left(\frac{D}{n}\right)} \equiv 2b^{\frac{1-\left(\frac{D}{n}\right)}{2}} \pmod{n}$ *(valid whenever* $\gcd(n,D)=1$*).*

Grantham [16] unified many notions of pseudoprimality under the name of Frobenius pseudoprimes. Rotkiewicz [24] presents several types of Lucas and Frobenius pseudoprimes, together with detailed historical information.

2.2 Fibonacci and Bruckman–Lucas Pseudoprimes

For a prime $p \geq 3$, from (5) and (7) applied for $a=1$ and $b=-1$, we obtain

$$F_p \equiv \left(\frac{p}{5}\right) \pmod{p}; \tag{9}$$

$$F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p}. \tag{10}$$

By the law of quadratic reciprocity, we have

$$\left(\frac{p}{5}\right) = \left(\frac{5}{p}\right).$$

A composite number n is called a *Fibonacci pseudoprime* if $n \mid F_{n-\left(\frac{n}{5}\right)}$. The even such pseudoprimes are indexed as A141137 in OEIS [26] (where each sequence is indexed by a six-digit A-code), while the odd Fibonacci pseudoprimes indexed as A081264 start with the terms

$$323, 377, 1891, 3827, 4181, 5777, 6601, 6721, 8149, 10877, 11663, 13201, 13981,$$
$$15251, 17119, 17711, 18407, 19043, 23407, 25877, 27323, 30889, 34561, 34943,$$
$$35207, 39203, 40501, 50183, 51841, 51983, 52701, 53663, 60377, \ldots.$$

By the relations (6) and (8) applied for $a = 1$ and $b = -1$, we obtain

$$L_p \equiv 1 \pmod{p}; \tag{11}$$

$$L_{p-\left(\frac{p}{5}\right)} \equiv 2\left(\frac{p}{5}\right) \pmod{p}. \tag{12}$$

A composite integer n satisfying $n \mid L_n - 1$ is called a *Bruckman–Lucas pseudoprime*. The sequence is indexed in the OEIS as A005845 and begins with

$$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251, 24465, 29281, 34561,$$
$$35785, 51841, 54705, 64079, 64681, 67861, 68251, 75077, 80189, 90061, 96049, \ldots.$$

In 1964, Lehmer [18] proved that Fibonacci and Bruckman–Lucas pseudoprimes are infinite, while Bruckman [11] showed that Bruckman–Lucas pseudoprimes are odd.

Other classes of pseudoprimes are defined by combining multiple properties. For example, a composite integer n is a *Fibonacci–Bruckner–Lucas pseudoprime* if it satisfies $n \mid F_{n-\left(\frac{p}{5}\right)}$ and $n \mid L_n - 1$. These numbers give the sequence A212424

$$4181, 5777, 6721, 10877, 13201, 15251, 34561, 51841, 64079, 64681, 64681, 67861,$$
$$68251, 75077, 90061, 96049, 97921, 100127, 113573, 118441, 146611, 161027, \ldots.$$

Bruckman [10] showed that there are infinitely many Fibonacci–Bruckner–Lucas pseudoprimes. These numbers correspond to the Frobenius pseudoprimes for the quadratic equation $x^2 - x - 1$ [14, 24]. By Bruckman's result [11], it follows that all the Fibonacci–Bruckner–Lucas pseudoprimes are odd.

2.3 Pell and Pell–Lucas Pseudoprimality

Similar notions of primality and pseudoprimality have been defined for Pell and Pell–Lucas numbers. In the proof, we will use Euler's identity which states that $\left(\frac{8}{p}\right) = \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$ [1, Theorem 9.1.2]. For a prime number p, the following relations are obtained from (5), (7), (6), and (8) applied for $a = 2$ and $b = -1$ (see

Corollary 4.2 and Proposition 4.4 [3])

$$P_p \equiv \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} \pmod{p};$$
$$P_{p-\left(\frac{2}{p}\right)} \equiv 0 \pmod{p};$$
$$Q_p \equiv 2 \pmod{p};$$
$$Q_{p-\left(\frac{p}{2}\right)} \equiv 2\left(\frac{2}{p}\right) = 2(-1)^{\frac{p^2-1}{8}} \pmod{p}.$$

An odd composite integer n is called a *Pell pseudoprime* if n divides $P_{n-(-1)^{\frac{n^2-1}{8}}}$. The Pell pseudoprimes indexed as A099011 in OEIS start with the terms

$$169, 385, 741, 961, 1121, 2001, 3827, 4879, 5719, 6215, 6265, 6441, 6479, 6601,$$
$$7055, 7801, 8119, 9799, 10945, 11395, 13067, 13079, 13601, 15841, 18241, 19097,$$
$$20833, 20951, 24727, 27839, 27971, 29183, 29953, 31417, 31535, 34561, 35459, \ldots.$$

In 1986, Kiss, Phong, and Lieuwen [17] showed that this sequence is infinite.

A composite integer n which satisfies the relation $n \mid Q_n - 2$ is called a *Pell–Lucas pseudoprime* (see [2, Chapter 3.2] and [3]). The odd such pseudoprimes are indexed as A330276 in OEIS and start with the terms

$$169, 385, 961, 1105, 1121, 3827, 4901, 6265, 6441, 6601, 7107, 7801, 8119, 10945,$$
$$11285, 13067, 15841, 18241, 19097, 20833, 24727, 27971, 29953, 31417, 34561,$$
$$35459, 37345, 37505, 38081, 39059, 42127, 45451, 45961, 47321, 49105, \ldots.$$

It has been conjectured that this sequence is infinite (see [2] and [3]).

The even Pell–Lucas pseudoprimes starting with

$$4, 8, 16, 24, 32, 48, 64, 72, 96, 120, 128, 144, 168, 192, 216, 240, 256, 264, 272, 288,$$
$$336, 360, 384, 432, 480, 504, 512, 528, 544, 576, 600, 648, 672, 720, 768, 792, 816,$$
$$840, 864, 960, 1008, 1024, 1056, 1080, 1088, 1152, 1176, 1200, 1296, \ldots,$$

were indexed as A335668. The sequence of odd and even Pell–Lucas pseudoprimes recover A270345, which is defined in OEIS as the sequence of composite integers n for which $n \mid P_0 + P_1 + \cdots + P_{n-1}$. The two different interpretations of this sequence have suggested the following result.

Theorem 4 *If n is composite, then $n \mid Q_n - 2$ if and only if $n \mid (P_0 + \cdots + P_{n-1})$.*

Proof If m, k are non-negative integers, then by using the Binet-like formula for Pell numbers and the sum of geometric series, we immediately get [12]

$$\sum_{i=0}^{m} P_{k+i} = \frac{1}{4}\left(Q_{k+m+1} - Q_k\right).$$

When $k = 0$ and $m = n - 1$, we obtain that $Q_n - 2 = 4(P_0 + \cdots + P_{n-1})$. By this identity, the conclusion holds when n is odd. Whenever $n \mid P_0 + \cdots + P_{n-1}$, we also have $n \mid Q_n - 2$.

We now prove that whenever n is even and $n \mid Q_n - 2$, then $n \mid P_0 + \cdots + P_{n-1}$. Let $n = 2^m M$, where $m \geq 1$ and M is odd. Since $Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$, one can find the integers a_n and b_n with the properties

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2}, \quad (1-\sqrt{2})^n = a_n - b_n\sqrt{2}.$$

We deduce that $Q_n = 2a_n$, and moreover, we have the relation

$$a_n - 1 = \binom{n}{2}\left(\sqrt{2}\right)^2 + \binom{n}{4}\left(\sqrt{2}\right)^4 + \cdots + \binom{n}{2l}\left(\sqrt{2}\right)^{2l} + \cdots + \left(\sqrt{2}\right)^n.$$

It is sufficient to prove that once $n = 2^m M \mid Q_n - 2$, we have $2^{m+1}M \mid a_n - 1$. Clearly, this is trivial for the odd factor M; hence, one requires to prove that the divisibility relation $2^{m+1} \mid a_n - 1$ holds.

First, for $l \geq m+1$, it easily follows that $2^{m+1} \mid \binom{n}{2l}\left(\sqrt{2}\right)^{2l}$.

We now have to prove that

$$2^{m+1} \mid \binom{n}{2}\left(\sqrt{2}\right)^2 + \binom{n}{4}\left(\sqrt{2}\right)^4 + \cdots + \binom{n}{2m}\left(\sqrt{2}\right)^{2m}. \tag{13}$$

For $k = 1, \ldots, m$, we have $2k \leq 2m \leq 2^m$; hence, $\gcd(2^m, n-l) = \gcd(2^m, l)$ for all $l = 1, \ldots, 2k-1$. For $k = 1, \ldots, m$, we get $k = 2^{r_k} R_k$, with $r_k \geq 0$ and $R_k \geq 1$ odd. By the above property, we deduce that the power of 2 in the binomial coefficient

$$\binom{n}{2k} = \frac{n(n-1)\cdots(n-2k+1)}{1\cdot 2\cdots(2k-1)\cdot 2k} = \frac{(n-1)(n-2)\cdots(n-2k+1)}{1\cdot 2\cdots(2k-1)}\cdot\frac{n}{2k}$$

can be computed as $m - (r_k + 1)$. For each $k = 1, \ldots, m$, we may write

$$\binom{n}{2k}\left(\sqrt{2}\right)^{2k} = 2^{m_k} M_k = 2^{m+k-(r_k+1)} M_k, \tag{14}$$

where M_k is odd.

We now show that $m_1 = m_2 = m$ and that $m_k \geq m + 1$ for $k \geq 3$. For $k = 1$, we have $r_1 = 0$ and $R_1 = 1$, which gives $m_1 = m$. Also, for $k = 2$, one obtains $r_2 = 1$ and $R_2 = 1$, which yields $m_2 = m + 2 - (1 + 1) = m$. Therefore,

$$\binom{n}{2}\left(\sqrt{2}\right)^2 + \binom{n}{4}\left(\sqrt{2}\right)^4 = 2^m \left(M_1 + M_2\right).$$

Whenever $k \geq 3$, one has $r_k \geq 2$ and $R_k \geq 3$; hence, $2^{r_k} R_k \geq r_k + 2$; therefore,

$$m_k = m + 2^{r_k} R_k - (r_k + 1) \geq m + 1.$$

We deduce that 2^{m+1} divides all the terms in (14) for $k = 3, \ldots, m$.

Since M_1 and M_2 are both odd, we conclude that (13) holds. □

An odd composite integer n is called a *Pell–Pell–Lucas pseudoprime* if it satisfies

$$n \mid P_{n-(-1)^{\frac{n^2-1}{8}}} \text{ and } n \mid Q_n - 2.$$

The list of such pseudoprimes is indexed as A327652 and starts with

169, 385, 961, 1121, 3827, 6265, 6441, 6601, 7801, 8119, 10945, 13067, 15841, 18241,

19097, 20833, 24727, 27971, 29953, 31417, 34561, 35459, 37345, 38081, 39059, ...

It has been conjectured that this sequence is infinite [3].

3 Generalized Bruckman–Lucas Pseudoprimes

Proposition 1 motivates the following notion.

Definition 2 A composite integer n is said to be a *generalized Bruckman–Lucas pseudoprime* of parameters a and b if $n \mid V_n(a, b) - a$.

For $b = \pm 1$ and $(a, b) \neq (1, 1)$, there are infinitely many odd composite integers n with $V_n \equiv a \pmod{n}$ [23], so there are infinitely many generalized Bruckner–Lucas pseudoprimes.

3.1 Results for $b = -1$

We obtained known integer sequences for $b = -1$, $a = 1$ (Bruckman–Lucas pseudoprimes) and for $b = -1$, $a = 2$, while the others were new. We first present sequences of odd terms.

- $a = 1$. We obtain the Bruckman–Lucas pseudoprimes indexed A005845.
- $a = 2$. Here we get the odd Pell–Lucas pseudoprimes A330276.
- $a = 3$. The sequence $V_n(3, -1)$ gives the 3-Lucas sequences A006497. The odd numbers n for which $n \mid V_n(3, -1) - 3$ are indexed as A335669. The first few are

$$33, 65, 119, 273, 377, 385, 533, 561, 649, 1105, 1189, 1441, 2065, 2289, 2465,$$
$$2849, 4187, 4641, 6545, 6721, 11921, 12871, 13281, 14041, 15457, 16109, \dots.$$

- $a = 4$. The sequence $V_n(4, -1)$ is A014448 for which $V_n(4, -1) = L(3n)$. The odd composite integers n for which $n \mid V_n(4, -1) - 4$ starting with the terms

$$9, 85, 161, 341, 705, 897, 901, 1105, 1281, 1853, 2465, 2737, 3745, 4181, 4209,$$
$$4577, 5473, 5611, 5777, 6119, 6721, 9701, 9729, 10877, 11041, 12209, 12349, \dots,$$

were indexed as A335670.
- $a = 5$. The sequence $V_n(5, -1)$ is A087130, related to the fifth metallic mean. The odd composite numbers n for which $n \mid V_n(5, -1) - 5$ start with

$$9, 27, 65, 121, 145, 377, 385, 533, 1035, 1189, 1305, 1885, 2233, 2465, 4081,$$
$$5089, 5993, 6409, 6721, 7107, 10877, 11281, 11285, 13281, 13369, 13741, \dots,$$

were added to OEIS as A335671.
- $a = 6$. The sequence $V_n(6, -1)$ recovers A085447. The odd composite numbers n for which $n \mid V_n(6, -1) - 6$ are indexed as A338078 and start with

$$57, 185, 385, 481, 629, 721, 779, 1121, 1441, 1729, 2419, 2737, 5665, 6721,$$
$$7471, 8401, 9361, 10465, 10561, 11285, 11521, 11859, 12257, 13585, 14705, \dots.$$

- $a = 7$. The sequence $V_n(7, -1)$ is A086902. The odd composite numbers n for which $n \mid V_n(7, -1) - 7$ have been indexed as A338079. The sequence starts with

$$25, 51, 91, 161, 265, 325, 425, 561, 791, 1105, 1113, 1325, 1633, 1921, 1961, 2001,$$
$$2465, 2599, 2651, 2737, 3445, 4081, 4505, 4929, 7345, 7685, 8449, 9361, \dots.$$

Now we present some sequences of even pseudoprimes obtained for $b = -1$.

- $a = 1$. A classical result by Bruckman [11] shows that this set is empty.
- $a = 2$. As seen in Section 2.3, in this case, we obtain A335668 in OEIS.
- $a = 3$. The even numbers $n \leq 25000$ for which $n \mid V_n - 3$ are $n = 4, 116, 938$.
- $a = 4$. The even numbers $n \leq 10000$ for which $n \mid V_n - 4$ are $n = 14, 1442$.
- $a = 5$. The even numbers $n \leq 15000$ for which $n \mid V_n - 5$ is $n = 22$.

- $a = 6$. The even numbers $n \leq 15000$ for which $n \mid V_n - 6$ are $n = 4, 682, 1436, 8618$.
- $a = 7$. The first even numbers for which $n \mid V_n - 7$ are

$$4, 8, 22, 88, 472, 5588, 10408, 20648, 34568, 123076, 1783976, 3677228, 4609418,$$
$$4857688, 6027208, 9906578, 16508152, 19995308, 20226572, 32039062, \ldots,$$

  indexed in OEIS as A338310.

3.2 Results for $b = 1$

- $a = 3$. The sequence $V_n(3, 1) = L_{2n}$ is the bisection of Lucas numbers A005248. The odd composite numbers n for which $n \mid V_n(3, 1) - 3$ start with

$$15, 105, 195, 231, 323, 377, 435, 665, 705, 1443, 1551, 1891, 2465, 2737, 2849,$$
$$3289, 3689, 3745, 3827, 4181, 4465, 4879, 5655, 5777, 6479, 6601, 6721, \ldots.$$

  We have added this sequence to OEIS A335672.
- $a = 4$. The sequence $V_n(4, 1)$ is A003500, starting with

$$2, 4, 14, 52, 194, 724, 2702, 10084, 37634, 140452, 524174, 1956244, 7300802,$$
$$27246964, 101687054, 379501252, 1416317954, 5285770564, 19726764302, \ldots.$$

  The composite integers n for which $n \mid V_n(4, 1) - 4$ are indexed as A335673.

$$10, 209, 230, 231, 399, 430, 455, 530, 901, 903, 923, 989, 1295, 1729, 1855, 2015,$$
$$2211, 2345, 2639, 2701, 2795, 2911, 3007, 3201, 3439, 3535, 3801, 4823, \ldots.$$

  The odd such numbers are indexed as A330206, where they are called Chebyshev pseudoprimes to base 2. The even terms are not yet indexed in OEIS.
- $a = 5$. The sequence $V_n(5, 1)$ is indexed as A003501 and starts with the terms

$$2, 5, 23, 110, 527, 2525, 12098, 57965, 277727, 1330670, 6375623, 30547445, \ldots.$$

  The odd composite numbers n for which $n \mid V_n(5, 1) - 5$ recover the sequence A335674 and start with the terms

$$15, 21, 35, 105, 161, 195, 255, 345, 385, 399, 465, 527, 551, 609, 741, 897, 1105,$$
$$1295, 1311, 1807, 1919, 2001, 2015, 2071, 2085, 2121, 2415, 2737, 2915, \ldots.$$

- $a = 6$. The sequence $V_n(6, 1)$ is A003499, starting with

$$2, 6, 34, 198, 1154, 6726, 39202, 228486, 1331714, 7761798, 45239074, \ldots.$$

The first few odd numbers n for which $n \mid V_n(6, 1) - 6$ recover A337233 and are

$$35, 119, 169, 385, 741, 779, 899, 935, 961, 1105, 1121, 1189, 1443, 1479, 2001,$$
$$2419, 2555, 2915, 3059, 3107, 3383, 3605, 3689, 3741, 3781, 3827, 4199, 4795, \ldots.$$

- $a = 7$. The sequence $V_n(7, 1)$ is indexed as A056854 and starts with the terms

$$2, 7, 47, 322, 2207, 15127, 103682, 710647, 4870847, 33385282, 228826127,$$
$$1568397607, 10749957122, 73681302247, 505019158607, 3461452808002, \ldots.$$

This represents the quadrisection of Lucas numbers, i.e., $V_n = L_{4n}$. The odd numbers n for which $n \mid V_n(7, 1) - 7$ define A338082 added to OEIS and start with

$$9, 15, 21, 35, 45, 63, 99, 105, 195, 231, 315, 323, 329, 369, 377, 423, 435, 451, 595,$$
$$665, 705, 805, 861, 903, 1081, 1189, 1443, 1551, 1819, 1833, 1869, 1891, 1935, \ldots.$$

Now we present some sequences of even pseudoprimes obtained for $b = 1$.

- $a = 3$. The even numbers satisfying $n \mid V_n - 3$ were indexed as A337777

$$4, 44, 836, 1364, 2204, 7676, 7964, 9164, 11476, 12524, 23804, 31124, 32642,$$
$$39556, 73124, 80476, 99644, 110564, 128876, 156484, 192676, 199924, \ldots.$$

- $a = 4$. The even numbers $3 \le n \le 36000$ for which $n \mid V_n - 4$ are

$$10, 230, 430, 530, 9890, 35626.$$

- $a = 5$. The even numbers $3 \le n \le 30000$ for which $n \mid V_n - 5$ are

$$6554, 11026, 26506.$$

- $a = 6$. The even numbers for which $n \mid V_n - 6$ give A338311 and start with

$$4, 14, 28, 164, 434, 574, 1106, 5084, 5572, 7874, 8386, 13454, 13694, 19964,$$
$$21988, 33166, 39934, 40132, 95122, 103886, 113918, 148994, 157604, 215326, \ldots$$

The combination of even and odd sequences is not currently indexed.

4, 14, 28, 35, 119, 164, 169, 385, 434, 574, 741, 779, 899, 935, 961, 1105, 1106, 1121, 1189, 1443, 1479, 2001, 2419, 2555, 2915, 3059, 3107, 3383, 3605, ...

- $a = 7$. The even numbers satisfying $n \mid V_n - 7$ recover A338312 starting with

4, 8, 10, 20, 40, 44, 104, 136, 152, 170, 190, 232, 260, 286, 442, 580, 740, 836, 890, 1364, 1378, 1990, 2204, 2260, 2584, 2626, 2684, 2834, 3016, 3160, 3230, 3926, 4220, 4636, 5662, 6290, 7208, 7384, 7540, 7676, 7964, 8294, 8420, 9164, 9316, 9320, 10070, 11476, 12524, 14824, 15224, 17324, 19720, . . .

Remark 1 Direct calculations can show that the following identities hold true:

$$V_n(1, -1) = L_n, \quad V_n(3, 1) = L_{2n}, \quad V_n(7, 1) = L_{4n}.$$

4 Weak Generalized Lucas Pseudoprimes

By Proposition 1, whenever p is prime, one has

$$U_p \equiv \left(\frac{D}{p}\right) \pmod{p}.$$

Clearly, $U_p^2 \equiv 1 \pmod{p}$, and we can define some weak pseudoprimality notions for generalized Lucas and Pell–Lucas sequences $U_n(a, b)$ and $V_n(a, b)$.

Definition 3 A composite integer n for which $n \mid U_n^2 - 1$ is called a *weak generalized Lucas pseudoprime* of parameters a and b.

Theorem 5 *If $U_n = U_n(a, b)$ with $b = \pm 1$. If a is even, then the weak generalized Lucas pseudoprimes $n \geq 3$ are all odd.*

Proof By the recurrence relation (1), we have

$$U_{n+2} + bU_n = aU_{n+1}, \quad n \geq 0. \tag{15}$$

Since a is even and $b = \pm 1$, we deduce that U_n and U_{n+2} have the same parity. As $U_0 = 0$ and $U_1 = 1$, it follows that U_{2m} is even, and U_{2m+1} is odd, for all $m \geq 0$. Therefore, the divisibility relation $n \mid U_n^2 - 1$ may only hold when n is odd. □

In particular, for $(a, b) = (1, -1)$ and $(a, b) = (2, -1)$, one obtains the following new notions of pseudoprimality.

Definition 4 A composite integer n satisfying the property $n \mid F_n^2 - 1$ is called *weak Fibonacci pseudoprime*.

The first few odd composite numbers in this list are

$$231, 323, 377, 1443, 1551, 1891, 2737, 2849, 3289, 3689, 3827, 4181, 4879, 5777,$$
$$6479, 6601, 6721, 7743, 8149, 9879, 10877, 11663, 13201, 13981, 15251, 15301, \ldots.$$

The first even numbers with this property are

$$4, 8, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 88, 94, 106, 118, 122, 134, 142, 146, 158,$$
$$166, 178, 194, 202, 206, 214, 218, 226, 254, 262, 274, 278, 298, 302, 314, 326, 334, 346, \ldots.$$

These sequences are indexed in OEIS as A337231 and A337232, respectively.

Conjecture 1 Results for $n \leq 20000$ suggest that these numbers are square-free.

Definition 5 A composite integer n satisfying the property $n \mid P_n^2 - 1$ is called a *weak Pell pseudoprime*.

By Theorem 5, all weak Pell pseudoprimes are odd, and the first few terms are

$$35, 119, 169, 385, 741, 779, 899, 935, 961, 1105, 1121, 1189, 1443, 1479, 2001, 2419,$$
$$2555, 2915, 3059, 3107, 3383, 3605, 3689, 3741, 3781, 3827, 4199, 4795, 4879, \ldots.$$

This sequence contains the squares 169 and 961 and is indexed A337233 in OEIS. Notice that this sequence also has some other interpretations.

4.1 Results for $b = -1$

We now present some sequences of odd numbers.

- $a = 3$. In this case, $(U_n)_{n\geq 0}$ is indexed as A006190, called *bronze Fibonacci sequence*, used for enumerating classes of fatty acids [25] and starts with

  $$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, 141481, \ldots.$$

  The sequence of odd integers n for which $n \mid U_n^2 - 1$ starts with

  $$9, 33, 55, 63, 99, 119, 153, 231, 385, 399, 561, 649, 935, 981, 1023, 1071, 1179,$$
  $$1189, 1199, 1441, 1595, 1763, 1881, 1953, 2001, 2065, 2255, 2289, 2465, 2703,$$
  $$2751, 2849, 2871, 3519, 3599, 3655, 3927, 4059, 4081, 4187, 5015, 5151, \ldots.$$

  This sequence was added to OEIS as A337234.

- $a = 4$. The sequence $U_n(4, -1)$ is A001076. The sequence of odd composite integers n for which $n \mid U_n^2 - 1$ was added to OEIS as A337236 and starts with

$$9, 63, 99, 119, 161, 207, 209, 231, 279, 323, 341, 377, 391, 549, 589, 671, 759, 779, 799, 897, 901, 1007, 1159, 1281, 1443, 1449, 1551, 1853, 1891, 2001, 2047, 2071, 2379, 2407, 2501, 2737, 2743, 2849, 2871, 2961, 3069, 3289, 3689, 3827, \ldots.$$

- $a = 5$. The sequence $(U_n)_{n\geq 0}$ satisfies $U_n =$ A052918$(n-1)$, for $n \geq 1$. The sequence of odd integers n for which $n \mid U_n^2 - 1$ recovers A337237 and starts with

$$9, 15, 25, 27, 35, 45, 65, 75, 91, 121, 135, 143, 175, 225, 275, 325, 385, 455, 533, 595, 615, 675, 935, 1035, 1107, 1325, 1359, 1431, 1495, 1547, 1573, 1935, \ldots.$$

- $a = 6$. The sequence $U_n(6, -1)$ is A005668. The sequence of odd integers n for which n divides $U_n^2 - 1$ recovers A338080 and starts with

$$9, 57, 63, 143, 171, 247, 323, 399, 407, 481, 629, 703, 721, 779, 899, 927, 1121, 1239, 1407, 1441, 1463, 1703, 1729, 2419, 2529, 2639, 2737, 3289, 3367, 3689, 4081, 4847, 4879, 4921, 5291, 5339, 5871, 6061, 6479, 6489, 6601, 6721, \ldots.$$

- $a = 7$. The sequence $(U_n)_{n\geq 0}$ is linked to A054413 and starts with

$$0, 1, 7, 50, 357, 2549, 18200, 129949, 927843, 6624850, 47301793, \ldots.$$

  The sequence of odd integers n for which $n \mid U_n^2 - 1$ recovers A338081 given by

$$21, 25, 35, 49, 51, 65, 85, 91, 119, 147, 161, 175, 221, 231, 245, 325, 357, 377, 391, 399, 425, 455, 539, 559, 561, 575, 595, 629, 637, 759, 791, 833, 1001, 1105, 1127, 1225, 1247, 1295, 1309, 1495, 1547, 1633, 1763, 1775, 1921, 2001, 2015, \ldots.$$

We now present sequences of even pseudoprimes for $b = -1$. By Theorem 5, when a is even, all weak generalized Lucas pseudoprimes are odd.

- $a = 3$. The composite even integers n for which $n \mid U_n^2 - 1$ start with

$$4, 8, 16, 68, 1208, 1424, 3056, 3824, 3928, 20912, 52174, 63716, 88708, 123148, 161872, 582224, 887566, 17083292, 18900412, 34648888, 39991684, 44884912, \ldots.$$

  This sequence was added A337235.

- $a = 5$. The composite even numbers for which $n \mid U_n^2 - 1$ recover sequence A338313 in OEIS, which starts with the terms

$$4, 8, 16, 32, 68, 248, 268, 544, 1328, 4216, 4768, 9112, 9376, 12664, 20128,$$
$$22112, 24536, 25544, 30488, 43262, 61574, 125792, 148004, 304792, 398248, \dots.$$

- $a = 7$. The composite even numbers $n \le 15000$ for which $n \mid U_n^2 - 1$ are

$$4, 8, 356, 716, 12626.$$

4.2 Results for $b = 1$

Notice that for $b = 1$, $a = 1$, and $D = -3$, the sequences $(U_n)_{n\ge 0}$ and $(V_n)_{n\ge 0}$ have the period 6, as follows:

$$0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, \dots$$
$$2, 1, -1, -2, -1, 1, 2, 1, -1, -2, -1, 1, 2, 1, -1, -2, -1, 1, 2, 1, -1, \dots.$$

Also, for $b = 1$, $a = 2$, and $D = 0$, one obtains $U_n = n$ and $V_n = 2$, for $n = 0, 1, 2, \dots$.

- $a = 3$. The sequence $(U_n)_{n\ge 0}$ starts with the terms

$$0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368, 121393, 317811, \dots,$$

representing the bisection of Fibonacci numbers, i.e., $U_n = F_{2n}$ A001906. The odd composite integers n for which $n \mid U_n^2 - 1$ generate A338007 and start with

$$9, 21, 63, 99, 231, 323, 329, 369, 377, 423, 451, 861, 903, 1081, 1189, 1443, 1551,$$
$$1819, 1833, 1869, 1891, 2033, 2211, 2737, 2849, 2871, 2961, 3059, 3289, \dots.$$

- $a = 4$. The sequence $(U_n)_{n\ge 0}$ starts with the terms

$$0, 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719, 2107560, \dots,$$

indexed as A001353. The sequence of odd composite integers n which satisfy the relation $n \mid U_n^2 - 1$ recovers A338008, starting with

$$35, 65, 91, 209, 455, 533, 595, 629, 679, 901, 923, 989, 1001, 1241, 1295, 1495,$$
$$1547, 1729, 1769, 1855, 1961, 1991, 2015, 2345, 2431, 2509, 2555, 2639, \dots.$$

- $a = 5$. The sequence $(U_n)_{n\geq 0}$ starts with the terms

  $$0, 1, 5, 24, 115, 551, 2640, 12649, 60605, 290376, 1391275, 6665999, \ldots,$$

  and is indexed as A004254. The sequence of positive integers n for which the number n divides $U_n^2 - 1$ gives A338009 and starts with

  $$25, 55, 115, 209, 253, 275, 319, 391, 425, 527, 551, 575, 713, 715, 775, 779, 935,$$
  $$1105, 1111, 1265, 1705, 1807, 1919, 2015, 2035, 2071, 2575, 2627, 2893, 2915, \ldots.$$

- $a = 6$. The sequence $(U_n)_{n\geq 0}$ starts with the terms

  $$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, 46611179, \ldots,$$

  indexed as A001109. The sequence of odd composite integers n for which we have $n \mid U_n^2 - 1$ gives A338010 and starts with the terms

  $$9, 35, 51, 55, 77, 85, 119, 153, 169, 171, 187, 209, 261, 319, 369, 385, 451, 531,$$
  $$551, 595, 649, 715, 741, 779, 899, 935, 961, 969, 989, 1105, 1121, 1189, 1241, \ldots.$$

- $a = 7$. The sequence $(U_n)_{n\geq 0}$ starts with the terms

  $$0, 1, 7, 48, 329, 2255, 15456, 105937, 726103, 4976784, 34111385, \ldots,$$

  and recovers A004187. The sequence of positive integers n for which n divides $U_n^2 - 1$ gives A338011 and starts with

  $$49, 161, 323, 329, 377, 451, 539, 989, 1081, 1127, 1189, 1771, 1819, 1891, 2009,$$
  $$2033, 2047, 2303, 2737, 2849, 3059, 3289, 3619, 3653, 3689, 3827, 4181, \ldots.$$

We now present some sequences of even pseudoprimes for $b = 1$. By Theorem 5, when a is even, all weak generalized Lucas pseudoprimes are odd.

- $a = 3$. The composite even integers n for which $n \mid U_n^2 - 1$ give A337782

  $$4, 8, 44, 104, 136, 152, 232, 286, 442, 836, 1364, 1378, 2204, 2584, 2626, 2684,$$
  $$2834, 3016, 3926, 4636, 5662, 7208, 7384, 7676, 7964, 8294, 9164, 9316, \ldots.$$

- $a = 5$. The composite even numbers n for which $n \mid U_n^2 - 1$ give A338314

  $$4, 8, 76, 104, 116, 296, 872, 1112, 1378, 2204, 2774, 2834, 3016, 4472, 5174, 5624,$$
  $$6364, 6554, 8854, 9164, 9976, 10564, 11026, 11324, 11476, 12644, 14356, \ldots.$$

- $a = 7$. The composite even numbers n for which $n \mid U_n^2 - 1$ give A337783

$$4, 8, 16, 44, 104, 136, 152, 164, 176, 232, 286, 442, 496, 656, 836, 856, 976, 1072,$$
$$1364, 1378, 1394, 1804, 1826, 2204, 2248, 2584, 2626, 2684, 2834, 3016, 3268,$$
$$3536, 3926, 4264, 4346, 4636, 5084, 5104, 5146, 5662, 7208, 7216, 7384, \ldots.$$

5 Weak Generalized Lucas–Bruckner Pseudoprimes

Since $V_p \equiv a \pmod{p}$, we may also introduce the following notion, which combines the notions discussed in Sects. 3 and 4.

Definition 6 A composite integer n for is called a *weak generalized Lucas–Bruckner pseudoprime* of parameters a and b if it satisfies the relations $n \mid U_n^2 - 1$ and $n \mid V_n - a$.

Clearly, this definition does not involve the Jacobi symbol. Rotkiewicz [23] proved that when $b = \pm 1$ and $(a, b) \neq (1, 1)$, there are infinitely many odd composite numbers n satisfying simultaneously the relations 1, 2, and 3 from Proposition 1.

If n is an odd composite number satisfying the three relations $\gcd(n, 2abD) = 1$, $U_n \equiv \left(\frac{D}{n}\right) \pmod{n}$, and $V_n \equiv V_1 = a \pmod{n}$, then $U_n^2 \equiv 1 \pmod{n}$; hence, n is a weak generalized Lucas pseudoprime. We obtain the following result.

Proposition 2 *There are infinitely many weak generalized Lucas–Bruckner pseudoprimes.*

For $(a, b) = (1, -1)$ and $(a, b) = (2, -1)$, one obtains the following notions.

Definition 7 A composite integer n is called a *weak Fibonacci–Lucas–Bruckner pseudoprime* if it satisfies the relations $n \mid F_n^2 - 1$ and $n \mid L_n - 1$.

Since the Bruckman–Lucas numbers satisfying $n \mid L_n - 1$ start with the values

$$705, 2465, 2737, 3745, 4181, 5777, 6721, 10877, 13201, 15251, \ldots,$$

the first few weak Fibonacci–Lucas–Bruckner pseudoprimes are

$$2737, 4181, 5777, 6721, 10877, 13201, 15251, 29281, 34561, 51841, 64079, 64681,$$
$$67861, 68251, 75077, 80189, 90061, 96049, 97921, 100127, 105281, 113573, \ldots.$$

This sequence is indexed to OEIS as A337625.

Definition 8 The composite integers n which satisfy the properties $n \mid P_n^2 - 1$ and $n \mid Q_n - 2$ are called *weak Pell–Lucas–Bruckner pseudoprimes*.

This sequence only has odd terms, and it is indexed as A330276 (see Sect. 2.3). The numbers in this sequence are also called *NSW pseudoprimes*.

Conjecture 2 If the integer n satisfies $n \mid Q_n - 2$, then we also have $n \mid P_n^2 - 1$.

In other words, a composite integer n is a Pell–Lucas pseudoprime if and only if it is a weak Pell–Lucas–Bruckner pseudoprime.

5.1 Results for $b = -1$

We begin with some sequences of odd numbers, recently indexed to OEIS.

- $a = 3$. The first odd composite numbers satisfying $n \mid U_n^2 - 1$ and $n \mid V_n - 3$ are

  $$33, 119, 385, 561, 649, 1189, 1441, 2065, 2289, 2465, 2849, 4187, 6545, 12871,$$
  $$13281, 14041, 16109, 18241, 22049, 23479, 24769, 25345, 28421, 31631, \ldots.$$

  This sequence corresponds to A337626.
- $a = 4$. The first odd composite numbers such that $n \mid U_n^2 - 1$ and $n \mid V_n - 4$ are

  $$9, 161, 341, 897, 901, 1281, 1853, 2737, 4181, 4209, 4577, 5473, 5611, 5777, 6119,$$
  $$6721, 9701, 9729, 10877, 11041, 12209, 12349, 13201, 13481, 14981, 15251, \ldots.$$

  This sequence corresponds to A337627.
- $a = 5$. The first odd composite numbers for which $n \mid U_n^2 - 1$ and $n \mid V_n - 5$ are

  $$9, 27, 65, 121, 385, 533, 1035, 4081, 5089, 5993, 6721, 7107, 10877, 11285,$$
  $$13281, 13741, 14705, 16721, 18901, 19601, 19951, 20705, 24769, 25345, \ldots.$$

  This sequence was added to OEIS as A337628.
- $a = 6$. The first odd composite numbers satisfying $n \mid U_n^2 - 1$ and $n \mid V_n - 6$ are

  $$57, 481, 629, 721, 779, 1121, 1441, 1729, 2419, 2737, 6721, 7471, 8401, 9361,$$
  $$10561, 11521, 11859, 12257, 15281, 16321, 16583, 18849, 24721, 25441, \ldots.$$

  This sequence is indexed to OEIS as A337629.
- $a = 7$. The first odd composite numbers such that $n \mid U_n^2 - 1$ and $n \mid V_n - 7$ are

  $$25, 51, 91, 161, 325, 425, 561, 791, 1105, 1633, 1921, 2001, 2465, 2599, 2651,$$
  $$2737, 7345, 8449, 9361, 10325, 10465, 10825, 11285, 12025, 12291, 13021, \ldots.$$

  This sequence recovers the OEIS sequence A337630.

We now present sequences of even pseudoprimes with $b = -1$. From Theorem 5, when a is even, all weak generalized Lucas pseudoprimes are odd.

- $a = 3$. For n smaller that 20000, the only even composite integer which satisfies the relations $n \mid U_n^2 - 1$ and $n \mid V_n - 3$ is $n = 4$.
- $a = 5$. There are no even numbers $3 \leq n \leq 15000$ which satisfy the relations $n \mid U_n^2 - 1$ and $n \mid V_n - 5$.
- $a = 7$. The even numbers $3 \leq n \leq 15000$ for which $n \mid U_n^2 - 1$ and $n \mid V_n - 7$ are only $n = 4$ and $n = 8$.

5.2 *Results for* $b = 1$

We present examples for $a = 3, 4, 5, 6, 7$, starting with sequences having odd terms. All these sequences are new and have been recently added to OEIS.

- $a = 3$. The odd composite numbers with $n \mid U_n^2 - 1$ and $n \mid V_n - 3$ are A337231

$$231, 323, 377, 1443, 1551, 1891, 2737, 2849, 3289, 3689, 3827, 4181, 4879, 5777,$$
$$6479, 6601, 6721, 7743, 8149, 9879, 10877, 11663, 13201, 13981, 15251, \ldots.$$

- $a = 4$. The odd composite numbers with $n \mid U_n^2 - 1$ and $n \mid V_n - 4$ recover A337778

$$209, 455, 901, 923, 989, 1295, 1729, 1855, 2015, 2345, 2639, 2701, 2795, 2911,$$
$$3007, 3439, 3535, 4823, 5291, 5719, 6061, 6767, 6989, 7421, 8569, 9503, \ldots.$$

- $a = 5$. The odd composite numbers with $n \mid U_n^2 - 1$ and $n \mid V_n - 5$ are A337779

$$527, 551, 1105, 1807, 1919, 2015, 2071, 2915, 3289, 4031, 4033, 4355, 5291, 5777,$$
$$5983, 6049, 6061, 6479, 6785, 7645, 8695, 9361, 9889, 11285, 11663, 11951, \ldots.$$

- $a = 6$. The odd composite numbers with $n \mid U_n^2 - 1$ and $n \mid V_n - 6$ give A337233

$$35, 119, 169, 385, 741, 779, 899, 935, 961, 1105, 1121, 1189, 1443, 1479, 2001,$$
$$2419, 2555, 2915, 3059, 3107, 3383, 3605, 3689, 3741, 3781, 3827, 4199, \ldots.$$

- $a = 7$. The odd composite numbers such that $n \mid U_n^2 - 1$ and $n \mid V_n - 7$ are A337781

$$323, 329, 377, 451, 1081, 1189, 1819, 1891, 2033, 2737, 2849, 3059, 3289, 3653,$$
$$3689, 3827, 4181, 4879, 5671, 5777, 6479, 6601, 6721, 8149, 8533, 8557, \ldots.$$

We now present some sequences of pseudoprimes of even numbers for $b = 1$. By Theorem 5, when a is even, all weak generalized Lucas pseudoprimes are odd.

- $a = 3$. The even integers for which $n \mid U_n^2 - 1$ and $n \mid V_n - 3$ recover A337777

  4, 44, 836, 1364, 2204, 7676, 7964, 9164, 11476, 12524, 23804, 31124, 32642, 39556, 73124, 80476, 99644, 110564, 128876, 156484, 192676, 199924, ...

- $a = 5$. The even numbers $3 \le n \le 15000$ for which $n \mid U_n^2 - 1$ and $n \mid V_n - 5$ are

  6554, 11026, 26506.

- $a = 7$. The even numbers $3 \le n \le 20000$ satisfying $n \mid U_n^2 - 1$ and $n \mid V_n - 7$ recover the sequence indexed as A337782, which starts with the terms

  4, 8, 44, 104, 136, 152, 232, 286, 442, 836, 1364, 1378, 2204, 2584, 2626, 2684, 2834, 3016, 3926, 4636, 5662, 7208, 7384, 7676, 7964, 8294, 9164, 9316, ...

6 Conclusions and Future Work

Here we summarize the newly indexed sequences related to the weak pseudoprimality notions presented in this paper, for which we also propose some conjectures.

The following abbreviations will be used in the tables.

- gBL-psp: generalized Bruckner–Lucas pseudoprimes, i.e., n with $V_n \equiv a \pmod{n}$;
- w-gL-psp: weak generalized Lucas pseudoprimes, i.e., n with $U_n^2 \equiv 1 \pmod{n}$;
- w-gBL-psp: weak generalized Bruckman–Lucas pseudoprimes, i.e., integers n satisfying the properties $V_n \equiv a \pmod{n}$ and $U_n^2 \equiv 1 \pmod{n}$.

Table 1 shows OEIS [26] indices for $(U_n(a, b))_{n\ge0}$, $(V_n(a, b))_{n\ge0}$ obtained for $b = -1$ and $a = 1, \ldots, 7$, and the corresponding gBL-psp, w-gL-psp and w-gBL-psp sequences of odd pseudoprimes.

Table 2 presents OEIS [26] indices for $(U_n(a, b))_{n\ge0}$, $(V_n(a, b))_{n\ge0}$ obtained for $b = 1$ and $a = 3, \ldots, 7$, and the corresponding gBL-psp, w-gL-psp and w-gBL-psp pseudoprimes.

Conjecture 3 The odd integers n satisfying the relation $n \mid U_n^2(1, -1) - 1$ recover the odd integers for which one has $n \mid U_n^2(3, 1) - 1$ and $n \mid V_n(3, 1) - 3$, indexed A337231. Since $U_n(3, 1) = F_{2n}$ and $V_n(3, 1) = L_{2n}$, it means that $n \mid F_n^2 - 1$ if and only if $n \mid F_{2n}^2 - 1$ and $n \mid L_{2n} - 3$.

Table 1 OEIS indices of sequences $(U_n(a,b))_{n\geq 0}$, $(V_n(a,b))_{n\geq 0}$ and pseudoprime sequences obtained for $b=-1$ and $a=1,\dots,7$

(a,b)	D	$U_n(a,b)$	$V_n(a,b)$	gBL-psp	w-gL-psp	w-gBL-psp
$(1,-1)$	5	A000045	A000032	A005845	A337231	A337625
$(2,-1)$	8	A000129	A002203	A330276	A337233	A330276
$(3,-1)$	13	A006190	A006497	A335669	A337234	A337626
$(4,-1)$	20	A001076	A014448	A335670	A337236	A337627
$(5,-1)$	29	A052918	A087130	A335671	A337237	A337628
$(6,-1)$	40	A005668	A085447	A338078	A338080	A337629
$(7,-1)$	53	A054413	A086902	A338079	A338081	A337630

Table 2 OEIS indices for the sequences $(U_n(a,b))_{n\geq 0}$, $(V_n(a,b))_{n\geq 0}$ and for the pseudoprime sequences obtained for $b=1$ and $a=3,\dots,7$

(a,b)	D	$U_n(a,b)$	$V_n(a,b)$	gBL-psp	w-gL-psp	w-gBL-psp
$(3,1)$	5	A001906	A005248	A335672	A338007	A338007
$(4,1)$	12	A001353	A003500	A335673	A338008	A337778
$(5,1)$	21	A004254	A003501	A335674	A338009	A337779
$(6,1)$	32	A001109	A003499	A337233	A338010	A337233
$(7,1)$	45	A004187	A056854	A338082	A338011	A337781

Conjecture 4 The set of odd integers n satisfying $n \mid P_n^2 - 1$ where P_n is the nth Pell number is the set of odd integers for which $n \mid U_n^2 - 1$ and $n \mid V_n - 6$ when $a = 6$ and $b = 1$. The sequence is indexed A337233. In this case, U_n is A001109 satisfying $U_n = \frac{P_{2n}}{2}$, while V_n is A003499, given by $V_n = 2T_n(3)$, where T_n are the Chebyshev polynomials of first kind. Moreover, it seems that for $a = 6$ and $b = 1$, if $n \mid V_n - 6$, then $n \mid U_n^2 - 1$.

Remark 2 1° The sequence A335673 contains both even and odd terms.

We also formulate two conjectures for some sequences of even integers.

Conjecture 5 The set of even numbers n satisfying $n \mid U_n^2(3,1) - 1$ coincides with the set of even numbers for which $n \mid U_n^2(7,1) - 1$ and $n \mid V_n(7,1) - 7$. Since $U_n(7,1) = F_{4n}$ and $V_n(7,1) = L_{4n}$, this means that $n \mid F_{2n}^2 - 1$ if and only if $F_{4n}^2 - 1$ and $n \mid L_{4n} - 3$. This represents A337782.

Conjecture 6 The set of even numbers n satisfying $n \mid U_n^2 - 1$ and $n \mid V_n - 3$ coincides with the set of even integers for which $n \mid V_n - 3$. Since $U_n(3,1) = F_{2n}$, while $V_n(3,1) = L_{2n}$, this entails that $n \mid F_{2n}^2 - 1$ and $n \mid L_{2n} - 3$ if and only if $n \mid L_{2n} - 3$. Alternatively, one could prove that whenever is even, whenever $n \mid L_{2n} - 3$, we also have $n \mid F_{2n}^2 - 1$. This sequence represents A337777.

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Finite Shift-Invariant Subspaces of Periodic Functions: Characterization, Approximation, and Applications

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Abstract We discuss approximations of square integrable periodic functions by their projections in finite shift-invariant subspaces and highlight the role of principal shift invariance. We also show how we may produce a variety of sampling representations based on finite frame theory and we discuss some applications.

1 Introduction

Let $L_2 := L_2(\mathbb{T})$ be the space of all measurable square integrable periodic functions on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with usual inner product $\langle \cdot, \cdot \rangle_{L_2}$ and norm $\| \cdot \|_2$, and let $\ell_2(I_N)$ (or simply ℓ_2 if the index set is clear from the context) be the space of all complex valued sequences over the index set $I_N = \{0, \ldots, N-1\}$ with standard inner product $\langle \cdot, \cdot \rangle_{\ell_2}$ and norm $\| \cdot \|_{\ell_2}$. Clearly, every element of ℓ_2 can be considered as an N-periodic sequence over $\mathbb{Z}$.

For any $N \in \mathbb{N}$, we say that a subspace $V_N \subset L_2$ is $\frac{1}{N}$-*shift-invariant*, if for any $f \in V_N$, we have that $\tau_{n/N} f := f(\cdot - \frac{n}{N}) \in V_N$, for all $n \in I_N$. The simplest case,

$$V_N(\phi) = \operatorname{span}\big\{\tau_{n/N}\phi : \ n = 0, \ldots, N-1\big\}, \tag{1}$$

which is generated from shifts of a single function $\phi \in L_2$, is called an $\frac{1}{N}$-*principal shift-invariant subspace* of L_2. In other words,

$$V_N(\phi) = \Big\{ f = \sum_{n=0}^{N-1} c_f(n)\phi(\cdot - n/N) : \ c_f = (c_f(n)) \in \ell_2 \Big\}.$$

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Shift invariance is important because it exploits nicely properties of Fourier analysis. Many spaces encountered in approximation theory are generated by shifts of one or more generators, e.g., in wavelet theory. Associated with any shift-invariant space V_N and any function $f \in L_2$ is the unique *best term approximation error*

$$E(f, V_N) = \|f - P_N f\|_2,$$

where

$$P_N : L_2 \to V_N$$

is the orthogonal projection from L_2 onto V_N. We mention here that the Fourier coefficients of a function $f \in L_1$ are defined by

$$\widehat{f}(n) = \int_{\mathbb{T}} f(\gamma) e^{-2\pi i n \gamma} d\gamma, \quad n \in \mathbb{Z}.$$

In addition, we define by $\widehat{c} = (\widehat{c}_j)_{j \in I_N}$ the discrete Fourier transform of a complex valued sequence $c \in \ell_2$:

$$\widehat{c}_j = \sum_{k=0}^{N-1} c_k e^{-2\pi i jk/N}, \ j \in I_N.$$

The inverse discrete Fourier transform of $\widehat{c}$ is computed from the formula

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} \widehat{c}_j e^{2\pi i jk/N}, \ k \in I_N.$$

Below, we review results from [6] in the periodic setting, involving a characterization for elements in V_N (and $V_N(\phi)$), and we see how approximation properties of V_N are reduced to the study of approximation properties of a suitable principal shift-invariant subspace of V_N. For $f, g \in L_2$, we define a sequence $\left(\left[\widehat{f}, \widehat{g}\right]_N\right)_{n \in I_N}$ by

$$\left[\widehat{f}, \widehat{g}\right]_N(n) = \sum_{l \in \mathbb{Z}} \widehat{f}(n + lN) \overline{\widehat{g}(n + lN)}.$$

Then, it is easy to show that

$$f \perp V_N \Longleftrightarrow \langle f, g(\cdot - k/N) \rangle_{L_2} = 0 \Longleftrightarrow \left[\widehat{f}, \widehat{g}\right]_N(n) = 0, \ \forall g \in V_N \text{ and for any } n \in I_N.$$

Therefore, for the case of a principal shift-invariant space $V_N(\phi)$, we immediately obtain

$$f \perp V_N(\phi) \Longleftrightarrow \big[\widehat{f}, \widehat{\phi}\big]_N(n) = 0, \ \forall n \in I_N.$$

Let $\Omega_\phi = \big\{n \in I_N : \big[\widehat{\phi}, \widehat{\phi}\big]_N(n) \neq 0\big\}$, and let $\big(\widehat{c}_f(n)\big)_{n \in I_N}$ be such that

$$\widehat{c}_f(n) = \frac{\big[\widehat{f}, \widehat{\phi}\big]_N(n)}{\big[\widehat{\phi}, \widehat{\phi}\big]_N(n)}, \quad n \in \Omega_\phi. \tag{2}$$

Then we may prove the following modification of [6, Theorem 2.9, page 793]:

Proposition 1 *For any $f \in L_2(\mathbb{T})$, we have*

$$f \in V_N(\phi) \Longleftrightarrow \widehat{f}(n) = \widehat{c_f}(n)\widehat{\phi}(n), \ \forall n \in \mathbb{Z}, \tag{3}$$

where $\widehat{c_f} = (\widehat{c_f}(n))$ is a N-periodic sequence in $\mathbb{Z}$, whose elements $\widehat{c_f}(n)$ are defined in (2) for $n \in \Omega_\phi$, whereas $\widehat{c_f}(n)$ can be arbitrarily defined for $n \notin \Omega_\phi$.

Proof

$$f \in V_N(\phi) \Longleftrightarrow f = \sum_{k=0}^{N-1} c_f(k)\phi\big(\cdot - \frac{k}{N}\big) \Longleftrightarrow \widehat{f}(n) = \widehat{c_f}(n)\widehat{\phi}(n), \ \forall n \in \mathbb{Z}.$$

Hence,

$$\widehat{f}(n + lN) = \widehat{c_f}(n)\widehat{\phi}(n + lN), \ \forall n \in I_N \text{ and for all } l \in \mathbb{Z}, \tag{4}$$

so

$$\sum_{l \in \mathbb{Z}} \widehat{f}(n + lN)\overline{\widehat{\phi}(n + lN)} = \widehat{c_f}(n) \sum_{l \in \mathbb{Z}} |\widehat{\phi}(n + lN)|^2, \ \forall n \in \Omega_\phi.$$

Therefore, $\widehat{c}_f(n) = \frac{\big[\widehat{f}, \widehat{\phi}\big]_N(n)}{\big[\widehat{\phi}, \widehat{\phi}\big]_N(n)}, \ \forall n \in \Omega_\phi$. If $n \notin \Omega_\phi$, then by (4) we obtain $\widehat{f}(n + lN) = 0 \ \forall l$ and so, $\widehat{c_f}(n)$ can be arbitrarily defined.

Remark 1 As we showed in the proof of Proposition (1), the definition of $\widehat{c_f}$ may be not unique. However, whenever the set $\{\phi(\cdot - n/N) : \ n \in I_N\}$ is a basis for its span $V_N(\phi)$, then $\sum_{l \in \mathbb{Z}} |\widehat{\phi}(n + lN)|^2 \neq 0 \ \ \forall n \in I_N$; hence, $\Omega_\phi = I_N$, and the representation of $\widehat{c_f}$ is unique.

Remark 2 We notice here that the orthogonal complement $V_N^\perp$ of V_N is an $\frac{1}{N}$ shift-invariant space as well. If $f \in V_N^\perp$, then $\tau_{k/N} f \in V_N^\perp$ as well; otherwise, we would have contradiction.

Now we can prove the following:

Proposition 2 *Let P_N be the orthogonal projector from L_2 onto an $\frac{1}{N}$-shift-invariant space V_N as above. Define by $P_N(W)$ to be the image of the restriction of P_N on a subspace W of L_2. Let $U_N(f)$ be a principal $\frac{1}{N}$-shift-invariant subspace of L_2, associated with some function $f \in L_2$. Then:*

$$P_N(U_N(f)) = V_N(P_N f)$$

and so, for any element $g \in U_N(f)$, we have

$$\widehat{P_N g}(n) = \widehat{c_g}(n)\widehat{P_N f}(n),\ \forall n \in \mathbb{Z},$$

where $\widehat{c_g}$ is some N-periodic sequence.

Proof For any $g \in U_N(f)$, we have

$$P_N(g) = \sum_{k=0}^{N-1} c_g(k) P_N f(\cdot - k/N) \in V_N(P_N f);$$

hence, $P_N(U_N(f)) \subseteq V_N(P_N f)$. The reverse inclusion is proved in a similar manner. By taking the discrete Fourier transform and use (3), we obtain the result.

Proposition 3 ([6, Theorem 3.3, pg 797]) *Let P_N^g be the orthogonal projection from L_2 onto a principal $\frac{1}{N}$-shift-invariant subspace $U_N(g)$ of L_2, and let $P_N : L_2 \to V_N$ be the above orthogonal projection. Then, for any $f \in L_2$, we have*

$$\|f - P_N f\|_2 \leq \|f - P_N P_N^g f\|_2 \leq \|f - P_N f\|_2 + \|f - P_N^g f\|_2. \tag{5}$$

Proof According to the previous proposition, the image of $P_N P_N^g$ is exactly $V_N(P_N g)$ which is a subspace of V_N. Hence, the first inequality is immediately obtained. Now, for the second inequality, we have

$$\|f - (P_N P_N^g) f\|_2 \leq \|f - P_N f\|_2 + \|P_N\| \|f - P_N^g f\|_2 \leq \|f - P_N f\|_2 + \|f - P_N^g f\|_2.$$

Assume now that $g_0 \in L_2$ is a function whose Fourier transform $\widehat{g_0} = (\widehat{g_0}(n))_{n\in\mathbb{Z}}$ is non-vanishing only for $n \in \{-(N-1)/2, \ldots, (N-1)/2\}$, if n is odd, or only for $n \in \{-N/2+1, \ldots, N/2\}$, if n is even. Then, by using (2) and by using Parseval's identity, we obtain the following estimate for the best term approximation error:

$$\|f - P_N^{g_0} f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \Big(1 - \frac{|\widehat{g_0}(n)|^2}{\big[\widehat{g_0}, \widehat{g_0}\big]_N(n)}\Big)^2 = \sum_{|n|>N/2} |\widehat{f}(n)|^2. \tag{6}$$

Now the following theorem is a direct consequence of Proposition 3:

Theorem 1 *Let g_0 be a square integrable function on $\mathbb{T}$ as in (6), and let P_N, $P_N^{g_0}$ be the above orthogonal projections. Then:*

$$\lim_{N\to\infty} \|f - P_N f\|_2 = 0 \Longleftrightarrow \lim_{N\to\infty} \|f - P_N P_N^{g_0} f\|_2 = 0.$$

In other words, the space V_N is dense in $L_2(\mathbb{T})$ if and only if its principal $\frac{1}{N}$-shift-invariant subspace $V_N(P_N g_0)$ is dense in $L_2(\mathbb{T})$.

Proof
$\Rightarrow$: We use (6) together with the second inequality of (5) to obtain the result.
$\Leftarrow$: It is a direct application of the first inequality in (5).

The above theorem highlights the role of principal shift invariance for approximating square integrable periodic functions. Therefore, for the rest of this work, we are concerned with alternate representations for functions in principal shift spaces. From now on, we consider

$$N = pq$$

for some pair (p, q) of natural numbers. We aim to provide a wide variety of representations for functions $f \in V_N(\phi)$ in the following form:

$$f = \sum_{j=0}^{r-1}\sum_{k=0}^{p-1} \langle f, \psi_j(\cdot - k/p)\rangle_{L_2}\, S_j(\cdot - k/p), \tag{7}$$

with respect to a finite set $\{S_j\}_{j=0}^{r-1} \subset V_N(\phi)$ of *generators*, whose corresponding set of dual generators is $\Psi = \{\psi_j\}_{j=0}^{r-1} \subset V_N(\phi)$. Equation (7) can be considered as an *average sampling formula* [1, 13]. Reconstruction schemes similar to (7) were first studied by Papoulis on spaces of band-limited functions [11], and then the results were extended to infinite shift-invariant subspaces of $L_2(\mathbb{R})$ by using the theory of Riesz bases and z-transform techniques [5, 7–9, 14, 15]. Since in a wide variety of applications we consider time-limited signals and we use a finite number of samples for reconstruction (i.e., we work with spaces of type (1)), the existence of (7) is useful because it not only serves as a reconstruction equation for time-limited signals in $V_N(\phi)$ but at the same time it gives us the opportunity to design and implement representations with desirable properties; see Sect. 4 for details. Our main tool toward (7) is finite frame theory [2, 4, 12]. Indeed, the spanning set $\mathbf{T}_\phi^N = \{\tau_{k/N}\phi : k = 0, \dots, N-1\}$ is a *finite frame* for $V_N(\phi)$. The space $V_N(\phi)$ is isomorphic with a subspace W of ℓ_2, and so the existence of a stable decomposition (7) is related with the ability of expressing the set of sampled values $L_\Psi(f) = \Big\{\langle f, \psi_j(\cdot - k/p)\rangle_{L_2} : 0 \le j < r,\ 0 \le k < p\Big\}$ as frame coefficients of some sequence $c_f \in W$ with respect to an appropriate frame for W. Then our problem is transformed into constructing an appropriate frame for W, and the simplicity of this construction dictates the simplicity of (7). In Sect. 2, we obtain a canonical frame decomposition of the isomorphic image of $V_N(\phi)$ into ℓ_2; see Propositions 4 and 5 for details. Based on this result, in Sect. 3, we prove (7); see Theorem 2. Finally, in Sect. 4, we present some applications of (7). More specifically, we construct sampling generators with desirable properties and a communication packet network with erasures [3, 10].

2 Shift-Invariant Frames for Subspaces of $\ell_2(I_N)$

Let $N = pq$ be a natural number as above, $E \subseteq I_N = \{0, \ldots, N-1\}$ and $H = (h_j)_{j=0}^{r-1} \subset \ell_2(I_N)$. In this section, we establish necessary and sufficient conditions such that the shift-invariant set

$$\mathbf{T}_H^q = \{\tau_{kq} h_j = h_j(\cdot - kq) : \; j = 0, \ldots, r-1, \; k = 0, \ldots, p-1\}$$

is a frame for the subspace

$$W_E = \left\{c \in \ell_2(I_N) : \; \widehat{c}_j = 0 \text{ for all } j \in E\right\}. \tag{8}$$

Let

$$\mathscr{S} : \ell_2(I_N) \to \ell_2^r(I_p) : \; c \mapsto \mathscr{S}c = \mathbf{x} = (x_0, \ldots, x_{r-1}) : \; x_j = \left(\langle c, \tau_{kq} h_j \rangle_{\ell_2(I_N)}\right)_{k=0}^{p-1}$$

be the operator associated with the above set $\mathbf{T}_H^q$, where

$$\ell_2^r(I_p) = \left\{\mathbf{c} = (c_0, \ldots, c_{r-1}) : \; c_j \in \ell_2(I_p)\right\}$$

is a Hilbert space with inner product $\langle \mathbf{c}, \mathbf{d} \rangle_{\ell_2^r(I_p)} = \sum_{j=0}^{r-1} \langle c_j, d_j \rangle_{\ell_2(I_p)}$. Then the adjoint operator of $\mathscr{S}$ is defined by

$$\mathscr{S}^* : \ell_2^r(I_p) \to \ell_2(I_N) : \; \mathbf{x} \mapsto \mathscr{S}^*\mathbf{x} = \sum_{j=0}^{r-1} \sum_{k=0}^{p-1} x_{j,k} \tau_{kq} h_j .$$

For the above selection of E, we consider the sets

$$E_n = \left\{m \in I_q = \{0, \ldots, q-1\} : n + mp \notin E\right\}, \quad n = 0, \ldots, p-1 \tag{9}$$

and we define the $|E_n| \times |E_n|$ *Gram* matrices

$$G_{n,H} = J_{n,H}^* J_{n,H}, \tag{10}$$

where

$$J_{n,H} = \left\{ \left(J_{n,H}\right)_{j,m} = \widehat{h}_j(n+pm) : h_j \in H,\ m \in E_n \right\} \tag{11}$$

and $J^*_{n,H}$ is the Hermitian transpose of $J_{n,H}$. Then we have

Proposition 4 *Under the above definitions and assumptions, the set $\mathbf{T}^q_H$ is a frame for W_E if and only if $rank(J_{n,H}) = |E_n|$ for every $n = 0, \ldots, p-1$, provided that the set of generators H consists of at least $r \geq \max\{|E_n|\}_{n=0}^{p-1}$ elements. The optimal frame bounds are $A = \frac{Nd^-}{p}$ and $B = \frac{Nd^+}{p}$, respectively, where d^- and d^+ are the smallest and biggest positive real numbers over the set of eigenvalues of all Gramian matrices $G_{n,H}$.*

Proof Let $h_j \in H \subset W_E$. Taking into account (9), for any $c \in W_E$, we have

$$\begin{aligned}
\|\mathscr{S}c\|^2_{\ell_2^r(I_p)} &= \frac{1}{p}\|\widehat{\mathscr{S}c}\|^2_{\ell_2^r(I_p)} = \frac{1}{p}\sum_{j=0}^{r-1}\sum_{n=0}^{p-1}\Big|\sum_{m=0}^{q-1}\widehat{c}_{n+mp}\overline{\widehat{h}_j(n+mp)}\Big|^2 \\
&= \frac{1}{p}\sum_{j=0}^{r-1}\sum_{n=0}^{p-1}\Big(\sum_{m,m'\in E_n}\widehat{c}_{n+mp}\overline{\widehat{h}_j(n+mp)}\widehat{h}_j(n+m'p)\overline{\widehat{c}_{n+m'p}}\Big) \\
&= \frac{1}{p}\sum_{n=0}^{p-1}\widehat{\mathbf{c}}_n J^*_{n,H} J_{n,H}\widehat{\mathbf{c}}^*_n = \frac{1}{p}\sum_{n=0}^{p-1}\widehat{\mathbf{c}}_n G_{n,H}\widehat{\mathbf{c}}^*_n,
\end{aligned} \tag{12}$$

where $\widehat{\mathbf{c}}_n = \left\{\widehat{c}_{n+pm} : m \in E_n\right\}$ are $1 \times |E_n|$ row vectors and $G_{n,H}$ are $|E_n| \times |E_n|$ Gramian matrices as in (10).

Suppose that $rank(J_{n,H}) = |E_n|$ for every $n = 0, \ldots, p-1$. Then necessarily $r \geq \max\{|E_n|\}_{n=0}^{p-1}$. Let d^- and d^+ be the minimum and the maximum values over the set of all eigenvalues of all positive definite matrices $G_{n,H}$. Then the right-hand side of the last equality of (12) is bounded by

$$\frac{d^-}{p}\sum_{n=0}^{p-1}\|\widehat{\mathbf{c}}_n\|^2_{\ell_2(E_n)} \leq \frac{1}{p}\sum_{n=0}^{p-1}\widehat{\mathbf{c}}_n G_{n,H}\widehat{\mathbf{c}}^*_n \leq \frac{d^+}{p}\sum_{n=0}^{p-1}\|\widehat{\mathbf{c}}_n\|^2_{\ell_2(E_n)}$$

as a result of Rayleigh-Ritz theorem. Since $c \in W_E$, we have $\|\widehat{\mathbf{c}}_n\|^2_{\ell_2(E_n)} = \|\widehat{\mathbf{c}}_n\|^2_{\ell_2(I_q)}$, and so by using this observation, the above double inequality becomes

$$\frac{d^-N}{p}\|c\|^2_{\ell_2(I_N)} = \frac{d^-}{p}\|\widehat{c}\|^2_{\ell_2(I_N)} \leq \|\mathscr{S}c\|^2_{\ell_2^r(I_p)} \leq \frac{d^+}{p}\|\widehat{c}\|^2_{\ell_2(I_N)} = \frac{d^+N}{p}\|c\|^2_{\ell_2(I_N)}.$$

Therefore, the set $\mathbf{T}^q_H$ is a frame for W_E, and the frame bounds are optimal.

Suppose now that the set $\mathbf{T}_H^q$ is a frame for W_E, but $rank(J_{n_0,H}) < |E_{n_0}|$ for some index $0 \leq n_0 \leq p-1$. If $\mathcal{N}_{G_{n_0,H}}$ is the kernel of $G_{n_0,H}$ considered as an operator from $\ell_2(E_{n_0})$ to $\ell_2(E_{n_0})$, then there exist an element $\left\{a_k : k \in E_{n_0}\right\} \in \mathcal{N}_{G_{n_0,H}}$ and consequently an element

$$c \in W_E : \widehat{c}_m = \begin{cases} a_k, & m = n_0 + pk \ (k \in E_{n_0}) \\ 0, & \text{elsewhere} \end{cases}$$

such that $\|\mathscr{S}c\|^2_{\ell_2^r(I_p)} = 0$. Therefore, the set $\mathbf{T}_H^q$ is not a frame for W_E, contradiction.

If the set $\mathbf{T}_H^q$ is a frame for W_E, then it is well known that its corresponding *frame operator*

$$(\mathscr{S}^*\mathscr{S}) : W_E \to W_E : \mathscr{S}^*\mathscr{S}c = \sum_{j=0}^{r-1}\sum_{k=0}^{p-1} \langle c, \tau_{kq}h_j\rangle_{\ell_2(I_N)}\tau_{kq}h_j$$

is invertible on W_E. Let

$$\mathbf{T}_{H^\dagger}^q = \{\tau_{kq}h_j^\dagger : h_j^\dagger \in H^\dagger, k = 0, \ldots, p-1\}$$

is the *canonical dual frame* of $\mathbf{T}_H^q$ produced from a set of dual generators

$$H^\dagger = \{h_j^\dagger = (\mathscr{S}^*\mathscr{S})^{-1}h_j : h_j \in H\}.$$

Proposition 5 *Let* $\left(\mathbf{T}_H^q, \mathbf{T}_{H^\dagger}^q\right)$ *be a pair of canonical dual frames for* W_E. *Then, every dual generator* $h_j^\dagger \in H^\dagger$ *can be computed from its discrete Fourier transform by the formula*

$$\widehat{h_j^\dagger}(l) = \begin{cases} 0, & l \in E \\ \left(J^\dagger_{Mod(l,p),H}\right)_{j,[\frac{l}{p}]}, & l \notin E \end{cases}, \quad l = 0, \ldots, N-1, \tag{13}$$

where

$$J^\dagger_{n,H} = J_{n,H}(J^*_{n,H}J_{n,H})^{-1}, \ (n = 0, \ldots, p-1).$$

Note that the symbol $[x]$ *is used to denote the floor of a real number* x.

Proof Let $h_j^\dagger = (\mathscr{S}^*\mathscr{S})^{-1}h_j$, i.e., $h_j = (\mathscr{S}^*\mathscr{S})h_j^\dagger$. Then, using the above definition of $\mathscr{S}^*\mathscr{S}$ and working in the Fourier domain, we have

$$\widehat{h}_j(l) = (\widehat{\mathscr{S}^*\mathscr{S}h_j^\dagger})_l = \sum_{j'=0}^{r-1}\Big(\sum_{k=0}^{p-1}\langle h_j^\dagger, \tau_{kq}h_{j'}\rangle_{\ell_2(I_N)}e^{-2\pi ikl/p}\Big)\widehat{h}_{j'}(l).$$

By substituting $l = n + pm : n \in I_p,\ m \in E_n$ (see (9)) and using Parseval equality, we obtain

$$\begin{aligned}\widehat{h}_j(n+pm) &= \sum_{j'=0}^{r-1}\Big(\sum_{m'\in E_n}\widehat{h}_j^\dagger(n+pm')\overline{\widehat{h}_{j'}(n+pm')}\Big)\widehat{h}_{j'}(n+pm)\\ &= \sum_{m'\in E_n}\widehat{h}_j^\dagger(n+pm')\Big(\sum_{j'=0}^{r-1}\overline{\widehat{h}_{j'}(n+pm')}\widehat{h}_{j'}(n+pm)\Big).\end{aligned}$$

Then, from (11), the above equality can be written in matrix form by

$$J_{n,H} = J_{n,H}^\dagger(J_{n,H}^*J_{n,H})$$

where $J_{n,H}^\dagger = \Big\{\big(J_{n,H}^\dagger\big)_{j,m} = \widehat{h}_j^\dagger(n+pm),\ h_j^\dagger \in H^\dagger,\ m \in E_n\Big\}$. The rest follow easily.

3 Stable Reconstruction on V_ϕ

Let $V_N(\phi)$ be a principal shift-invariant space as in (1) and

$$K_\phi = \Big\{n \in \{0,\dots,N-1\} : \sum_{l\in\mathbb{Z}}|\widehat{\phi}(n+lN)|^2 = 0\Big\}.$$

In this section, we obtain simple stable reconstruction formulas (7) based on the fact that the space $V_N(\phi)$ is isomorphic with the subspace W_{K_ϕ} (see (8) for the particular selection $E = K_\phi$). This fact allows us to use the frame expansions of the previous section as the main tool for our construction. By definition, the space $V_N(\phi)$ can be considered as the range of the operator

$$U : W_{K_\phi} \to V_N(\phi) : f = \sum_{n=0}^{N-1} c_n\phi\Big(\cdot - \frac{n}{N}\Big),$$

i.e., U is invertible. Based on this observation, we can prove the following:

Proposition 6 *Let U be as above. If $H = \{h_j\} \subset W_{K_\phi}$ is a finite set and if $G = \{g_j = Uh_j : \ h_j \in H\}$, then the set $\mathbf{T}_G^{q/N} = \{\tau_{kq/N} g_j : g_j \in G, k = 0, \dots, p-1\}$ is a frame for V_ϕ, if and only if the set $\mathbf{T}_H^q$ is a frame for W_{K_ϕ}.*

Proof Let $\mathbf{T}_H^q$ be a frame for W_{K_ϕ}. Since the operator U^*U which is identified with its matrix representation $U^*U := \left\{\langle \tau_{m/N}\phi, \tau_{n/N}\phi\rangle_{L_2}\right\}_{m,n=0}^{N-1}$ is invertible, then the set $\left\{(U^*U)\tau_{kq}h_j : h_j \in H, k = 0, \dots, p-1\right\}$ is a frame for W_{K_ϕ} as a result of [4, Corollary 5.3.2], and so there exist positive constants A, B such that

$$A\|c\|_{\ell_2(I_N)}^2 \leq \sum_{j=0}^{r-1}\sum_{k=0}^{p-1} \left|\langle c, U^*U\tau_{kq}h_j\rangle_{\ell_2}\right|^2 \leq B\|c\|_{\ell_2(I_N)}^2$$

for all $c \in W_{K_\phi}$. For $f = Uc$, we have $\lambda_{\min}\|c\|_{\ell_2(I_N)} \leq \|f\|_{L_2} \leq \lambda_{\max}\|c\|_{\ell_2(I_N)}$, where $\lambda_{\min}$ ($\lambda_{\max}$) are the smallest (biggest) positive eigenvalues of the above Gramian matrix U^*U. Therefore,

$$A\lambda_{\max}^{-2}\|f\|_{L_2}^2 \leq \sum_{j=0}^{r-1}\sum_{k=0}^{p-1} \left|\left\langle f, \tau_{kq/N}(Uh_j)\right\rangle_{L_2}\right|^2 \leq B\lambda_{\min}^{-2}\|f\|_{L_2}^2 \text{ for all } f \in V_N(\phi)$$

and by defining $g_j = Uh_j$, the proof is complete. The inverse claim can be proven by using similar arguments. We omit the proof.

Theorem 2 *Let $N = pq$, $(p, q \in \mathbb{N})$ and $V_N(\phi)$ be a principal $\frac{1}{N}$-shift-invariant subspace of L_2 as in (1), which is associated with a set K_ϕ as above and with its corresponding space W_{K_ϕ} as in (8). Consider the operator $U : W_{K_\phi} \to V_N(\phi)$ as above. Then for any finite set $\Psi = \{\psi_j\}_{j=0}^{r-1} \in V_N\phi$, if the set*

$$\mathbf{T}_\Psi^q = \{\tau_{nq}(U^*\psi_j) : \ j = 0, \dots, r-1, \ n = 0, \dots, p-1\}$$

is a frame for W_{K_ϕ}, then every function $f \in V_N(\phi)$ can be stably reconstructed from the set of measurements $\left\{\langle f, \tau_{kq/N}\psi_j\rangle_{L_2} : 0 \leq j < r, \ 0 \leq k < p\right\}$ by the formula

$$f = \sum_{j=0}^{r-1}\sum_{k=0}^{p-1}\langle f, \tau_{kq/N}\psi_j\rangle_{L_2}\tau_{kq/N}S_j,$$

where

$$S_j = \sum_{n=0}^{N-1} \psi_j^\dagger(n)\tau_{n/N}\phi. \tag{14}$$

Here, the elements $\psi_j^\dagger$ *are generators of the canonical dual frame* $\mathbf{T}_{\Psi^\dagger}^q$ *of* $\mathbf{T}_\Psi^q$*, and they are computed from their discrete Fourier transform as in* (13)*.*

Proof Assume that the sets $\mathbf{T}_\Psi^q$ and $\mathbf{T}_{\Psi^\dagger}^q$ are dual frames for the space W_{K_ϕ} and U be as above. Then, for any $f \in V_N(\phi)$, there exists a unique $c \in W_{K_\phi}$ such that $f = Uc$ and so

$$f = \sum_{n=0}^{N-1} c_n \tau_{n/N}\phi = \sum_{j=0}^{r-1}\sum_{m=0}^{p-1} \langle c, \tau_{mq} U^* \psi_j \rangle_{\ell_2} \Big(\sum_{n=0}^{N-1} \tau_{mq} \psi_j^\dagger(n) \tau_{n/N}\phi \Big)$$
$$= \sum_{j=0}^{r-1}\sum_{m=0}^{p-1} \langle c, \tau_{mq/N} U^*(\psi_j) \rangle_{\ell_2} \tau_{mq/N} S_j = \sum_{j=0}^{r-1}\sum_{m=0}^{p-1} \langle f, \tau_{mq/N} \psi_j \rangle_{L_2} \tau_{mq/N} S_j$$

Therefore, our reconstruction formula is obtained, and the sampling functions S_j are computed from (14). Since $S_j = U(H_{\Psi^\dagger})$, the set $\{\tau_{q/N} S_j : j = 0, \dots, r-1,\ k = 0, \dots, p-1\}$ is a frame for V_ϕ as a direct result of Proposition 6, and the dual generators $\psi_j^\dagger$ are computed from (13) for the particular selection $H = \Psi$.

Given the space W_{K_ϕ}, we may define a wide variety of trigonometric polynomials $\Psi = \{\psi_j\}_{j=0}^{r-1}$ by the formula

$$\psi_j(\gamma) = \sum_{n \in K_\phi^c} \frac{\widehat{h}_j(n)\overline{\widehat{\phi}(n + Nl_n)}}{|\widehat{\phi}(n + Nl_n)|^2} e^{2\pi i (n+Nl_n)\gamma}, \tag{15}$$

where:

- **(i)** $H = (h_j)_{j=0}^{r-1}$ is a certain set of generators in W_{K_ϕ} so that the set T_H^q is a frame for W_{K_ϕ} and
- **(ii)** $l_n \in L_n = \{l \in \mathbb{Z} : \widehat{\phi}(n + Nl) \neq 0\}$. Here, the set K_ϕ^c is the complement of K_ϕ in I_N.

Then we have

Corollary 1 *Let* Ψ *be a set of trigonometric polynomials as in* (15)*. Then every function* $f \in V_N(\phi)$ *admits a stable reconstruction formula as in Theorem* 2*, whose sampling function has the form*

$$S_j = \sum_{n=0}^{N-1} h_j^\dagger(n)\phi(\cdot - n/N),$$

and the elements $h_j^\dagger$ *are defined in* (13)*.*

Proof By direct calculation using Proposition 1, we deduce that $U^*\psi_j = h_j$. The rest follow from Theorem 2.

4 Applications

In this section, we present some applications to demonstrate our theory.

Application 1 Let $V_N(\phi)$ be as in (1). Then every $f \in V_N(\phi)$ can be stably reconstructed by the formula

$$f = \sum_{n=0}^{N-1} \langle f, \tau_{n/N}\widetilde{\phi}\rangle_{L_2} \tau_{n/N}\phi,$$

where $\widetilde{\phi} = \sum_{n=0}^{N-1} a_\phi(n)\tau_{n/N}\phi$ is the canonical dual generator of ϕ and the sequence of coefficients $a_\phi = (a_\phi(n))$ is computed from its discrete Fourier transform by

$$\widehat{a}_\phi(n) = \begin{cases} \frac{N}{\sum_{l\in\mathbb{Z}} |\widehat{\phi}(n+lN)|^2}, & n \notin K_\phi \\ 0, & n \in K_\phi \end{cases}, \quad n = 0, \ldots, N-1.$$

If the above sum (in the denominator) is infinite, then the calculation of $\widetilde{\phi}$ via the sequence of coefficients $(a_\phi(n))_{n=0}^{N-1}$ may be not be precise due to truncation and round-off errors. Therefore, we may use Theorem 2 for simplifying the reconstruction process, provided that N is not prime. Indeed, if N is composite, we consider a divisor q of N, and we define a set of generators $H = \{h_j\}_{j=0}^{r-1}$ of length $r = q$ by the formula

$$\widehat{h}_j(n) = \begin{cases} \frac{e^{2\pi i j n/N}}{\sqrt{q}}, & n \notin K_\phi \\ 0, & n \in K_\phi \end{cases}, \quad n = 0, \ldots, N-1.$$

Then the matrices $J_{n,H}$ in (11) have orthonormal columns and so the set $\mathbf{T}_H^q$ is a Parseval frame for the space W_{K_ϕ}. For this selection of H, let Ψ be a set of trigonometric polynomials as in (15), and for simplicity, let us assume that all elements $l_n = 0$. Then by using Corollary 1, we obtain

$$S_j = \frac{1}{\sqrt{q}} \sum_{n\notin K_\phi} e^{2\pi i j n/N}\tau_{n/N}\phi, \; j = 0, \ldots, q-1$$

and the related reconstruction formula of Theorem 2 implies that we have to evaluate the sequence $(\langle f, \tau_{kq/N}\psi_j\rangle_{L_2})_{k,j}$ which can be easier calculated (than $(\langle f, \tau_{n/N}\widetilde{\phi}\rangle_{L_2})$), because (ψ_j) is a sequence of trigonometric polynomials.

Application 2 Let $\{\tau_{n/N}\phi : n \in I_N\}$ be a basis for the space $V_N(\phi)$, and assume that we want to transmit a signal $f \in V_N(\phi)$ by using convenient r-filtered versions of f of the form $L_j f(t) = \langle f, \psi_j(\cdot - t)\rangle_{L_2}$ sampled at time instants $t = kq/N$, $(k = 0, \dots, p-1)$, where q is a pre-determined divisor of N. This process can be considered as a communication packet network transporting packets of data (in this case, sequences of length p) from a source to a recipient. By allowing dependencies between transmitted packets, we are led to use suitable frame expansions for encoding data, so that reconstruction is obtained even if some packet is lost (provided that we know its position). It is well known that harmonic frames are robust to erasures [3, 10], and below, we exploit this fact in order to construct a frame $\mathbf{T}_H^q$ for $\ell_2(I_N)$ which allows us to reconstruct a signal $f \in V_N(\phi)$ if any packet of data produced from a specific linear filter L_{j_0} is lost during transmission. To do that, we need to define convenient impulse responses ψ_j, $j = 0, \dots, r-1$, provided that $r > q$. We shall use the trigonometric polynomials (15) with the following selection of generators h_j:

$$\widehat{h}_j(n) = \frac{1}{\sqrt{r}} e^{2\pi i j [\frac{n}{p}]/r}, \; n = 0, \dots, N-1 \text{ and } N = pq.$$

Then we may obtain an overcomplete frame for ℓ_2. If, for example, we have $r = q+1$ generators h_j as above, then if any packet associated with transmission of any (specific) filter $L_{j_0} f$ is lost, then the signal f can be reconstructed from Theorem 2 by putting $\psi_{j_0} = 0$, because the set of generators $\{h_0, \dots, h_{j_0-1}, 0, h_{j_0+1}, \dots, h_q\}$ continues to span ℓ_2. Similarly, if we use $r > q+1$ generators, then reconstruction is obtained even if any $r-q$ packets are lost.

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Generalized Intensity-Dependent Multiphoton Jaynes–Cummings Model

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Abstract In this chapter, we study the Jaynes–Cummings model under multiphoton excitation and in the general case of intensity-dependent coupling strength given by an arbitrary function f. The Jaynes–Cummings theoretical model is of great interest to atomic physics, quantum optics, solid-state physics, and quantum information theory with several applications in coherent control and quantum information processing. As the initial state of the radiation mode, we consider a squeezed state, which is the most general Gaussian pure state. The time evolution of the mean photon number and the dispersions of the two quadrature components of the electromagnetic field are calculated for an arbitrary function f. The mean value of the inversion operator of the atom is also calculated for some simple forms of the function f.

1 Introduction

The Jaynes–Cummings model [1–3] is a theoretical model that describes the system of a two-level atom interacting with a single mode of the quantum electromagnetic field. The model is considered to be of great importance in quantum optics because it

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is the simplest solvable model that describes the interaction of radiation with matter. The model allows for a fully quantum mechanical treatment of atoms interacting with an electromagnetic field, thus revealing a number of novel features, in contrast to the semi-classical approximation, in which only the atom is treated quantum mechanically and the electromagnetic field is assumed to behave according to the classical electromagnetic theory.

The mathematical formulation of the model is based on the Hamiltonian formalism of the full system, which after the rotating wave approximation [3] it can be expressed in terms of the inversion, raising, and lowering operators of the atom, denoted by $\sigma_3, \sigma_+, \sigma_-$ and annihilation and creation operators a, a^+ of the radiation mode. The full system's Hamiltonian consists of the atomic excitation Hamiltonian, the free field Hamiltonian, and the Jaynes–Cummings interaction Hamiltonian:

$$H = \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega a^+ a + \hbar\lambda(\sigma_+ a + \sigma_- a^+) \tag{1}$$

Here ω_0 is the transition frequency of the atom, and ω is the single mode angular frequency. The parameter λ is the coupling constant for the radiation–atom interaction. The operators $\sigma_3, \sigma_+, \sigma_-$ are 2×2 Pauli matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

The σ and a obey the following algebra:

$$[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm,\ \ [\sigma_+, \sigma_-] = \sigma_3,\ \ \left[a, a^+\right] = 1 \tag{3}$$

$$\sigma_+\sigma_- = \frac{1}{2}(1 + \sigma_3),\ \sigma_-\sigma_+ = \frac{1}{2}(1 - \sigma_3),\ \sigma_3^2 = 1 \tag{4}$$

In a series of articles [4–7], Sukumar, Buck, and Singh considered two generalized Jaynes–Cummings models with the following interaction Hamiltonians:

$$H_{\text{int}} = \hbar\lambda(\sigma_+ a\sqrt{a^+ a} + \sigma_-\sqrt{a^+ a}\,a^+) \tag{5}$$

$$H_{\text{int}} = \hbar\lambda(\sigma_+ a^m + \sigma_- a^{+m}) \tag{6}$$

We note that in the first model (5), the coupling strength depends on the number operator $n = a^+ a$ (or otherwise on the radiation intensity), whereas in the second model (6), the transmission of the atom from one level to the other is accompanied by the absorption or emission of m photons. The model described by Eq. (6) has been studied by Nayak and Mohanty [8] with $m = 2$ in order to obtain the steady-state photon statistics in a two-photon laser in which the decay of the lasing levels was taken into account. In addition, Haroche et al. [9] have observed the two-photon laser emission in Rydberg atoms of Rb, and Eq. (6) has been also widely applied to study the dynamics of the field and atomic variables in Rydberg atoms [10, 11].

Bartzis [12] has already studied the intensity-dependent two photon Jaynes–Cummings model with interaction Hamiltonian

$$H_{\text{int}} = \hbar\lambda(\sigma_+ a^2\sqrt{a^+a} + \sigma_-\sqrt{a^+a}\ \mathrm{a}^{+2}) \tag{7}$$

and the Jaynes–Cummings model with atomic motion [13].

N. Nayak and V. Bartzis [14, 15] have also used the three-level and the two-level Rydberg atom interacting with two nondegenerate modes, thus showing the differences in the dynamics. In another work, Bartzis, Patargias, and Jannussis have presented results in the case of one or two cavity modes interacting with both a three-level atom and Kerr-like medium [16, 17].

The atomic spin squeezing of N two-level and three-level atoms has been observed by Nayak et al. [18–20]. In recent years, more generalized Jaynes–Cummings models have been proposed [21–29], and the intensity dependence has also been considered in the work of Saha et al. [23]. In the case of multilevel atomic systems and multiphoton processes, the theoretical description is easier using the semi-classical approximation, for example, in potassium atoms in order to study two-photon excitation, multiphoton emissions, and other nonlinear processes [30–33].

In this work, we continue the generalization of the Jaynes–Cummings model by considering the interaction Hamiltonian that has the form

$$H_{\text{int}} = \hbar\lambda(\sigma_+ a^m f(a^+a) + \sigma_- f(a^+a)\ \mathrm{a}^{+m}) \tag{8}$$

This Hamiltonian describes a multiphoton process, since the transmission of the atom from one level to the other is accompanied by absorption or emission of m photons. In addition, the coupling strength in Eq. (8) is intensity dependent with the dependency described by an arbitrary function $f(a^+a)$. In the standard Jaynes–Cummings model, the coupling strength is considered to have a constant value. However, it is reasonable to assume that the coupling strength depends on the intensity since radiation intensity is observed to depend on time. As initial state of the radiation mode, we consider a squeezed state [34–40], the most general Gaussian pure state, which is defined as

$$|\alpha, z\rangle = S(z)D(\alpha)|0\rangle \tag{9}$$

where $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$ is the Weyl displacement operator and

$$S(z) = \exp\left[\frac{1}{2}(\mathrm{za}^2 - z^* a^{+2})\right], \mathrm{z} = \mathrm{re}^{-i\theta} \tag{10}$$

represents the squeeze operator.

In the n-representation, the squeeze state takes the form [35]

$$|\alpha, z\rangle = \sum_n C_n |n\rangle,$$

$$C_n = \frac{1}{\sqrt{n!\mu}} \left(\frac{\nu}{2\mu}\right)^{n/2} H_n \left[\alpha(2\mu\nu)^{-1/2}\right] \exp\left[-\frac{1}{2}|\alpha|^2 + \frac{\nu^*}{2\mu}\alpha^2\right] \tag{11}$$

where $\mu = \cosh r$, $v = e^{i\theta} \sinh r$

The mean photon number for a squeezed state has the form

$$\bar{n} = |\hat{\alpha}|^2 + |\nu|^2, \quad where \quad \hat{\alpha} = \mu^*\alpha - \nu\alpha^* \tag{12}$$

The two quadrature components are defined as

$$X_1 = \frac{1}{2}(a + a^+) \tag{13}$$

$$X_2 = \frac{1}{2i}(a - a^+) \tag{14}$$

Consequently, the electric field of the radiation mode has the form

$$E(t) = X_1 \cos\omega t + X_2 \sin\omega t \tag{15}$$

The dispersions of X_1 and X_2 for a squeezed state with $\theta = 0$ are

$$\langle(\Delta X_1)^2\rangle = \frac{1}{4}e^{-2r} \tag{16}$$

$$\langle(\Delta X_2)^2\rangle = \frac{1}{4}e^{2r} \tag{17}$$

The squeezing phenomenon is observed in Eqs. (16) and (17), since the quantum noise is lower in one quadrature component than that of the coherent state ($\langle(\Delta X_i)^2\rangle = 1/4, i = 1,2$) and higher in the other.

2 Time Evolution of the Atom Inversion Operator

In order to compute the time evolution of the system, we use Eq. (8) for the interaction, and the Hamiltonian of our model takes the form

$$H = \hbar\omega\left(a^+a + \frac{m}{2}\sigma_3\right) + \frac{\hbar\Delta}{2}\sigma_3 + \hbar\lambda(\sigma_+ a^m f(a^+a) + \sigma_- f(a^+a)a^{+m}) \tag{18}$$

where $\Delta = \omega_0 - m\omega$.

We can define the operators

$$C = a^{+}a + \frac{m}{2}\sigma_3 \tag{19}$$

$$B = \hbar\lambda(\sigma_{+}a^{m} f(a^{+}a) - \sigma_{-} f(a^{+}a)a^{+m}) \tag{20}$$

$$D = \hbar\lambda(\sigma_{+}a^{m} f(a^{+}a) + \sigma_{-} f(a^{+}a)a^{+m}) \tag{21}$$

so finally the Hamiltonian (18) takes the form

$$H = \hbar\omega C + \frac{\hbar\Delta}{2}\sigma_3 + D \tag{22}$$

It is easy to prove that $[C, H] = [C, B] = 0$, $[\sigma_3, B] = 2D$, $[\sigma_3, D] = 2B$

For the calculation of the time evolution of the operator σ_3 (represents the atom population inversion), we will work in the Heisenberg picture. The Heisenberg equations of motion for the operators σ_3 and B are

$$i\hbar\dot{\sigma}_3 = [\sigma_3, H] = [\sigma_3, D] = 2B \tag{23}$$

$$i\hbar\dot{B} = [B, H] = -\hbar\Delta D + [B, D] \tag{24}$$

The commutator of B and D is calculated to be

$$[B, D] = \hbar^2\lambda^2 \left\{\sigma_3 \left[a^{m} f(a^{+}a), f(a^{+}a)a^{+m}\right]_{+} + \left[a^{m} f(a^{+}a), f(a^{+}a)a^{+m}\right]\right\} \tag{25}$$

where the symbol $[,]_{+}$ represents the anticommutator.

So solving the above system of Eqs. (23) and (24), we obtain the following differential equation for σ_3:

$$\begin{aligned}\ddot{\sigma}_3 =& \frac{2\Delta}{\hbar}(H - \hbar\omega C - \frac{\hbar\Delta}{2}\sigma_3) \\ &- 2\lambda^2 \left\{(\sigma_3 + 1)a^{m} f^2(a^{+}a)a^{+m} + (\sigma_3 - 1) f(a^{+}a)a^{+m}a^{m} f(a^{+}a)\right\}\end{aligned} \tag{26}$$

The differential equation (26) cannot be solved in general for any arbitrary function $f(a^{+}a)$ so in the following discussion, we present the solution considering the two simple cases of $f(a^{+}a)$=1 and $f(a^{+}a)$=$\sqrt{a^{+}a}$. For these cases, Eq. (26) takes the form

$$\ddot{\sigma}_3 + \omega'^2\sigma_3 = \frac{2\Delta}{\hbar}(H - \hbar\omega C) \tag{27}$$

where $\omega'^2 = \left\{\begin{matrix} 4\lambda^2\kappa(\kappa-1)\cdots(\kappa+1-m) + \Delta^2 \text{ for } f(a^{+}a) = 1 \\ 4\lambda^2\kappa^2(\kappa-1)\cdots(\kappa+1-m) + \Delta^2 \text{ for } f(a^{+}a) = \sqrt{a^{+}a} \end{matrix}\right\}$ and $\kappa = C + \frac{1}{2}m$.

The operator C is a constant of motion, so the solution of the above equation is

$$\sigma_3(t) = \sigma_3(0)\cos\omega' t + \frac{2B(0)}{i\hbar\omega'}\sin\omega' t + \frac{2\Delta}{\hbar\omega'^2}(H - \hbar\omega C)(1 - \cos\omega' t) \tag{28}$$

We suppose that the atom is initially at the excited state and the field in a squeezed state; thus, the solution takes the form

$$\langle\sigma_3(t)\rangle = \sum_n \left\{ \frac{\Delta^2}{\omega_n'^2} + \left(1 - \frac{\Delta^2}{\omega_n'^2}\right)\cos\omega_n' t \right\} |C_n|^2 \tag{29}$$

where $\omega_n'^2 = \begin{Bmatrix} 4\lambda^2\kappa_n(\kappa_n - 1)\cdots(\kappa_n + 1 - m) + \Delta^2 \text{ for } f(a^+a) = 1 \\ 4\lambda^2{\kappa_n}^2(\kappa_n - 1)\cdots(\kappa_n + 1 - m) + \Delta^2 \text{ for } f(a^+a) = \sqrt{a^+a} \end{Bmatrix}$
and $\kappa_n = n + m$.

3 Field Statistics of the Generalized Intensity-Dependent Multiphoton Jaynes–Cummings Model

The Hamiltonian of the system is given by Eq. (18). We define the operators

$$C = a^+a + \frac{m}{2}\sigma_3 \tag{30}$$

$$N = \frac{\Delta}{2}\sigma_3 + \lambda(\sigma_+ a^m f(a^+a) + \sigma_- f(a^+a)a^{+m}) \tag{31}$$

We easily can prove that

$$[C, N] = [H, N] = [H, C] = 0 \tag{32}$$

Consequently, the time evolution operator can be written in the form

$$U(t, 0) = e^{(-i/\hbar)\mathrm{Ht}} = e^{-i\omega\mathrm{Ct}}e^{-\mathrm{iNt}} \equiv U_1(t, 0)U_2(t, 0) \tag{33}$$

In the two-dimensional atomic subspace, the matrix representation of the operators U_1 and U_2 has the form

$$U_1(t, 0) = e^{-i\omega\alpha^+\alpha t}\sum_{n=0}^{\infty}\frac{(-\frac{\mathrm{im}\omega t}{2})^n}{n!}\sigma_3^{(n)} \tag{34}$$

or

$$U_1(t, 0) = e^{-i\omega\alpha^+\alpha t}\begin{pmatrix} e^{\frac{-\mathrm{im}\omega t}{2}} & 0 \\ 0 & e^{\frac{\mathrm{im}\omega t}{2}} \end{pmatrix} \tag{35}$$

Similarly, we can prove that operator

$$U_2(t,0) = e^{-\mathrm{iNt}} = \sum_{n=0}^{\infty} \frac{(-\mathrm{it})^n}{n!} N^{(n)} \tag{36}$$

has the following form:

$$U_2(t,0) = \begin{pmatrix} K & L \\ M & Q \end{pmatrix} \tag{37}$$

where *K, L, M,* and *Q* are calculated as

$$\begin{aligned} K =& \cos\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^m f^2(a^+a)a^{+m}}\right\} \\ &- i\frac{\Delta}{2}\frac{\sin\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^m f^2(a^+a)a^{+m}}\right\}}{\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^m f^2(a^+a)a^{+m}}} \end{aligned} \tag{38}$$

$$L = -i\lambda \frac{\sin\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^m f^2(a^+a)a^{+m}}\right\}}{\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^m f^2(a^+a)a^{+m}}} a^m f(a^+a) \tag{39}$$

$$M = -i\lambda \frac{\sin\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^{+m} f^2(a^+a+m)a^m}\right\}}{\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^{+m} f^2(a^+a+m)a^m}} f(a^+a)a^{+m} \tag{40}$$

$$\begin{aligned} Q =& \cos\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^{+m} f^2(a^+a+m)a^m}\right\} \\ &+ i\frac{\Delta}{2}\frac{\sin\left\{t\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^{+m} f^2(a^+a+m)a^m}\right\}}{\sqrt{\frac{\Delta^2}{4} + \lambda^2 a^{+m} f^2(a^+a+m)a^m}} \end{aligned} \tag{41}$$

In addition, from Eq. (33), the operator $U(t,0)$ is written as

$$U(t,0) = e^{-i\omega a^+ a} \begin{pmatrix} \Psi & Z \\ Y & W \end{pmatrix} \tag{42}$$

where $\Psi = e^{\frac{-\mathrm{im}\omega t}{2}} K \quad Z = e^{\frac{-\mathrm{im}\omega t}{2}} L \quad Y = e^{\frac{\mathrm{im}\omega t}{2}} M \quad W = e^{\frac{\mathrm{im}\omega t}{2}} Q$

We can easily show that

$$UU^{+} = U^{+}U = 1$$

Assuming that the atom is initially the excited state, we have the density operator of the field as

$$\begin{aligned}
\rho_f(t) &= \mathrm{Tr}_{\mathrm{atom}}\left\{U(t,0)\begin{pmatrix}\rho_f(0) & 0\\ 0 & 0\end{pmatrix}U^{+}(t,0)\right\}\\
&= e^{-i\omega a^{+}at}\Bigg[\Bigg(\cos\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}\right]\\
&\quad -i\frac{\Delta}{2}\frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}\right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}}\Bigg)\rho_f(0)\\
&\quad \Bigg(\cos\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}\right]\\
&\quad +i\frac{\Delta}{2}\frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}\right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a+m)!}{(a^{+}a)!}f^2(a^{+}a+m)}}\Bigg)\\
&\quad +\lambda^2\frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a)!}{(a^{+}a-m)!}f^2(a^{+}a)}\right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a)!}{(a^{+}a-m)!}f^2(a^{+}a)}}f(a^{+}a)a^{+m}\rho_f(0)a^{m}f(a^{+}a)\\
&\quad \frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a)!}{(a^{+}a-m)!}f^2(a^{+}a)}\right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2\frac{(a^{+}a)!}{(a^{+}a-m)!}f^2(a^{+}a)}}\Bigg]e^{i\omega\alpha^{+}\alpha t}
\end{aligned} \tag{43}$$

We consider as initial state of the system a squeezed state (9–11). So we can calculate the matrix elements of $\rho_f(t)$ in the $|n\rangle$-basis

$$
\begin{aligned}
\langle n|\rho_f(t)|n'\rangle = C_n C_{n'}^* e^{-i\omega(n-n')t} & \left(\cos\left[t\sqrt{\frac{\Delta^2}{4} + \lambda^2 \frac{(n+m)!}{(n)!} f^2(n+m)} \right] \right. \\
& \left. - i\frac{\Delta}{2} \frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n+m)!}{(n)!} f^2(n+m)} \right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n+m)!}{(n)!} f^2(n+m)}} \right) \times \\
& \times \left(\cos\left[t\sqrt{\frac{\Delta^2}{4} + \lambda^2 \frac{(n'+m)!}{(n')!} f^2(n'+m)} \right] \right. \\
& \left. + i\frac{\Delta}{2} \frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n'+m)!}{(n')!} f^2(n'+m)} \right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n'+m)!}{(n')!} f^2(n'+m)}} \right) + \\
& + \lambda^2 C_{n-m} C_{n'-m}^* e^{-i\omega(n-n')t} f(n) f(n') \\
& \times \sqrt{\frac{n!n'!}{(n-m)!(n'-m)!}} \frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n)!}{(n-m)!} f^2(n)} \right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n)!}{(n-m)!} f^2(n')}} \\
& \times \frac{\sin\left[t\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n')!}{(n'-m)!} f^2(n')} \right]}{\sqrt{\frac{\Delta^2}{4}+\lambda^2 \frac{(n')!}{(n'-m)!} f^2(n')}}
\end{aligned}
\tag{44}
$$

Finally, the time evolution of the mean photon number is calculated as

$$
\overline{n} = \mathrm{Tr}_{\text{field}}\left[a^+ a \rho_f(t)\right] = \sum_n n\langle n|\rho_f(t)|n\rangle \tag{45}
$$

We next consider the time of the dispersions of the quadrature operators

$$
X_1 = \frac{1}{2}(a + a^+) \tag{46}
$$

$$
X_2 = \frac{1}{2i}(a - a^+) \tag{47}
$$

which are finally calculated to have the form

$$
\begin{aligned}
\langle(\Delta X_1)^2\rangle = \tfrac{1}{4} \Bigg\{ 1 + & \\
\sum_n \left[2n\langle n|\rho_f(t)|n\rangle + \sqrt{(n+1)(n+2)}(\langle n+2|\rho_f(t)|n\rangle + \langle n|\rho_f(t)|n+2\rangle) \right] & \\
- (\sum_n [\sqrt{n+1}(\langle n+1|\rho_f(t)|n\rangle + \langle n|\rho_f(t)|n+1\rangle)])^2 \Bigg\} &
\end{aligned}
\tag{48}
$$

$$\langle(\Delta X_2)^2\rangle = \frac{1}{4}\Bigg\{1 + \sum_n \left[2n\langle n|\rho_f(t)|n\rangle - \sqrt{(n+1)(n+2)}(\langle n+2|\rho_f(t)|n\rangle + \langle n|\rho_f(t)|n+2\rangle)\right] + (\sum_n[\sqrt{n+1}(\langle n+1|\rho_f(t)|n\rangle - \langle n|\rho_f(t)|n+1\rangle)])^2\Bigg\} \quad (49)$$

4 Conclusions

The mathematical formalism for the generalized intensity-dependent multiphoton Jaynes–Cummings model is presented for an arbitrary mathematical function f describing the dependency on the intensity. The time evolution of the mean value of the atom inversion operator is calculated for two simple cases of the function f. The mean photon number and the dispersions of the two quadrature components are also calculated for an arbitrary function f in the case of a squeezed state as initial state of the electromagnetic field.

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Functional Inequalities for Multi-additive-Quadratic-Cubic Mappings

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Abstract In this chapter, a new version of multi-quadratic mappings are characterized. By this characterization, every multi-additive-quadratic-cubic mapping which is defined as system of functional equations can be unified as a single equation. In addition, by applying two fixed point theorems, the generalized Hyers-Ulam stability of multi-additive-quadratic-cubic mappings in normed and non-Archimedean normed spaces are studied. A few corollaries corresponding to some known stability and hyperstability outcomes for multi-additive, multi-quadratic, multi-cubic, and multi-additive-quadratic-cubic mappings (functional equations) are presented.

1 Introduction

Throughout this chapter, $\mathbb{N}$ and $\mathbb{Q}$ are the set of all positive integers and rationals, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. Moreover, for the set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n\text{-times}}$ by X^n. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $t = (t_1, \cdots, t_n) \in \{-1, 1\}^n$, and $x = (x_1, \cdots, x_n) \in V^n$, we write $lx := (lx_1, \cdots, lx_n)$ and $tx := (t_1x_1, \cdots, t_nx_n)$, where lx stands, as usual, for the lth power of an element x of the commutative group V.

Let $n \in \mathbb{N}$ and $n \geq 2$. Let us recall that a function $f : V^n \longrightarrow W$ is called multi-additive if it is additive (satisfies Cauchy's functional equation $A(x + y) = A(x) + A(y)$) in each variable. Some basic facts on such mappings can be found, for instance, in [24], where their application to the representation of polynomial

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functions is also presented (see also [25]). In [14], Ciepliński proved that a mapping f is multi-additive if and only if the following relation holds:

$$f(x_1 + x_2) = \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_n n}). \tag{1}$$

Moreover, $f : V^n \longrightarrow W$ is said to be *multi-quadratic* if it is quadratic in each variable, namely, it satisfies the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \tag{2}$$

in each variable. It is shown in [34] that the system of functional equations defining a multi-quadratic mappings can be unified as a single equation. Indeed, Zhao et al. proved that a mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if it satisfies the equation

$$\sum_{q \in \{-1,1\}^n} f(x_1 + qx_2) = 2^n \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{3}$$

where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. For the Jensen type and some generalized of multi-quadratic mappings which are recently studied, we refer to [7, 9] and [29].

A mapping $f : V^n \longrightarrow W$ is also called a *multi-cubic* if it is cubic in each variable, i.e., satisfies the equation

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x) \tag{4}$$

in each variable [8]. It is shown in [8] that every multi-cubic mapping can be unified as an equation. For other forms of multi-cubic mappings, we refer to [17] and [26].

Roughly speaking, nowadays, we say that an equation is *stable* in some class of functions if any function from that class, satisfying the equation approximately (in some sense), is near (in some way) to an exact solution of the equation. In 1940, Ulam [31] asked the question concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [21] for the linear functional equation of Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by the second author [28] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [19] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. Next, many of mathematicians were attracted and motivated to investigate the stability problems of functional equations in various spaces. For instance, the stability of multi-additive, multi-quadratic, multi-cubic, multi-quartic, multi-Jensen-cubic, and multi-additive-quadratic can be found in [2, 6, 8, 13, 14, 27, 34].

It is worth mentioning that the fixed point theorems have been considered and applied as some tools for the investigation of stability and hyperstability of various mappings and functional equations; for example, see [3, 4, 10, 15, 16, 18, 22, 30].

The organization of the paper is as follows. In Sect. 2, motivated by the quadratic functional equation

$$\mathfrak{Q}(2x+y)+\mathfrak{Q}(2x-y)=\mathfrak{Q}(x+y)+\mathfrak{Q}(x-y)+6\mathfrak{Q}(x), \tag{5}$$

we introduce a multi-quadratic mapping and present a characterization of such mappings. Section 3 begins with definition of the multi-additive-quadratic-cubic mappings which are additive in each of some k variables, are quadratic in each of some p variables, and are cubic (in sense of satisfies Eq. (4)) in each of the other variables. Furthermore, we reduce the system of n equations defining the multi-additive-quadratic-cubic mappings to obtain a single functional equation. In Sect. 4, we prove the generalized Hyers-Ulam stability for such mappings by using the fixed point method. Finally, we indicate some direct consequences of stability and hyperstability of multi-quadratic and multi-additive-quadratic-cubic mappings in Banach spaces setting. Section 5 is devoted to study of the generalized Hyers-Ulam stability for multi-additive-quadratic-cubic mappings by applying the fixed point method in non-Archimedean normed spaces which is introduced in [11]; for more applications of this approach for the stability of multi-Cauchy-Jensen and multi-additive-quadratic mappings, see [5]. In addition, for the stability of multi-Jensen and multi-additive mappings in non-Archimedean spaces, we refer to [32] and [33], respectively.

2 Characterization of Multi-quadratic Mappings

In this section, we characterize the multi-quadratic mappings. Here, we indicate an elementary result as follows. Since the proof is routine, we include it without proof.

Proposition 1 *Let V and W be vector spaces over $\mathbb{Q}$. Then, a mapping $Q: V \longrightarrow W$ satisfies functional equation (2) if and only if it satisfies Eq. (5).*

From now on, let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$, and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. In what follows, we shall denote x_i^n by x_i unless otherwise stated explicitly. Let $x_1, x_2 \in V^n$ and $T \in \mathbb{N}_0$ with $0 \leq T \leq n$. Put

$$\mathbb{A}^n = \left\{\mathfrak{A}_n = (A_1, A_2, \ldots, A_n) \mid A_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\right\},$$

where $j \in \{1, \ldots, n\}$. To achieve our aim in this section, set

$$\mathbb{A}_T^n := \left\{\mathfrak{A}_n = (A_1, A_2, \ldots, A_n) \in \mathbb{A}^n \mid \operatorname{Card}\{A_j : A_j = x_{1j}\} = T\right\}.$$

Definition 1 A mapping $f : V^n \longrightarrow W$ is said to be *n-multi-quadratic* or briefly *multi-quadratic* if f satisfies (5) in each variable.

For multi-quadratic mappings, we use the following notations:

$$f\left(\mathbb{A}_T^n\right) := \sum_{\mathfrak{A}_n \in \mathbb{A}_T^n} f(\mathfrak{A}_n), \tag{6}$$

$$f\left(\mathbb{A}_T^n, z\right) := \sum_{\mathfrak{A}_n \in \mathbb{A}_T^n} f(\mathfrak{A}_n, z) \qquad (z \in V).$$

We wish to show that if a mapping $f : V^n \longrightarrow W$ satisfies the equation

$$\sum_{s \in \{-1,1\}^n} f(2x_1 + sx_2) = \sum_{l=0}^{n} 6^l f\left(\mathbb{A}_l^n\right), \tag{7}$$

where $f\left(\mathbb{A}_l^n\right)$ is defined in (6), then it is multi-quadratic and vice versa.

We say a mapping $f : V^n \longrightarrow W$ satisfying *the r-power condition* in the jth component if

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $(z_1, \ldots, z_n) \in V^n$. In particular, 2-power and 3-power conditions are also called the quadratic condition and the cubic condition, respectively. In the sequel, $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$.

Theorem 1 *For a mapping $f : V^n \longrightarrow W$, the following assertions are equivalent:*

(i) *f is multi-quadratic;*
(ii) *f satisfies Eq. (7) and the quadratic condition in each variable.*

Proof (i)⇒(ii) It is easily verified that f satisfies the quadratic condition in all variables. We now prove that f satisfies Eq. (7) by induction on n. For $n = 1$, it is trivial that Eq. (5) holds for f. If (7) is true for some positive integer $n > 1$, then

$$\begin{aligned}\sum_{s \in \{-1,1\}^{n+1}} f(2x_1^{n+1} + sx_2^{n+1}) = &\sum_{s \in \{-1,1\}^n} f(2x_1^n + sx_2^n, x_{1n+1} + x_{2n+1}) \\ &+ \sum_{s \in \{-1,1\}^n} f(2x_1^n + sx_2^n, x_{1n+1} - x_{2n+1}) \\ &+ 6 \sum_{s \in \{-1,1\}^n} f(2x_1^n + sx_2^n, x_{1n+1})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n} \sum_{s \in \{-1,1\}} 6^l f\left(\mathbb{A}_l^n, x_{1n+1} + s x_{2n+1}\right) \\
&\quad + 6 \sum_{l=0}^{n} 6^l f\left(\mathbb{A}_l^n, x_{1n+1}\right) \\
&= \sum_{l=0}^{n+1} 6^l f\left(\mathbb{A}_l^{n+1}\right).
\end{aligned}
$$

This means that (7) holds for $n+1$.

(ii)⇒(i) Fix $j \in \{1, \ldots, n\}$. Putting $x_{2k} = 0$ for each $k \in \{1, \ldots, n\} \backslash \{j\}$ in the left side of (7) and using the assumption, we get

$$
\begin{aligned}
&2^{n-1} \times 2^{2(n-1)} \Big[f\left(x_{11}, \ldots, x_{1j-1}, 2x_{1j} + x_{2j}, x_{1j+1}, \ldots, x_{1n}\right) \\
&+ f\left(x_{11}, \ldots, x_{1j-1}, 2x_{1j} - x_{2j}, x_{1j+1}, \ldots, x_{1n}\right) \Big] \\
&= 2^{n-1} \Big[f\left(2x_{11}, \ldots, 2x_{1j-1}, 2x_{1j} + x_{2j}, 2x_{1j+1}, \ldots, 2x_{1n}\right) \\
&+ f\left(2x_{11}, \ldots, 2x_{1j-1}, 2x_{1j} - x_{2j}, 2x_{1j+1}, \ldots, 2x_{1n}\right) \Big]. \qquad (8)
\end{aligned}
$$

Set

$$
\begin{aligned}
f^*(x_{1j}, x_{2j}) := {} & f\left(x_{11}, \ldots, x_{1j-1}, x_{1j} + x_{2j}, x_{1j+1}, \ldots, x_{1n}\right) \\
& + f\left(x_{11}, \ldots, x_{1j-1}, x_{1j} - x_{2j}, x_{1j+1}, \ldots, x_{1n}\right).
\end{aligned}
$$

By the above substitutions in (7), it concludes from (8) that

$$
\begin{aligned}
&2^{n-1} \times 2^{2(n-1)} [f\left(x_{11}, \ldots, x_{1j-1}, 2x_{1j} + x_{2j}, x_{1j+1}, \ldots, x_{1n}\right) \\
&+ f\left(x_{11}, \ldots, x_{1j-1}, 2x_{1j} - x_{2j}, x_{1j+1}, \ldots, x_{1n}\right)] \\
&= \sum_{l=0}^{n-1} \left[\binom{n-1}{l} 2^{n-l-1} \times 6^l \right] f^*(x_{1j}, x_{2j}) \\
&+ \sum_{l=1}^{n} \left[\binom{n-1}{l-1} 2^{n-l} \times 6^l \right] f(x_{11}, \ldots, x_{1n}) \\
&= (6+2)^{n-1} f^*(x_{1j}, x_{2j}) + 6 \left[\sum_{l=0}^{n-1} \binom{n-1}{l} 2^{n-l-1} \times 6^l \right] f(x_{11}, \ldots, x_{1n}) \\
&= 8^{n-1} f^*(x_{1j}, x_{2j}) + 6 \times 8^{n-1} f(x_{11}, \ldots, x_{1n}) \\
&= 2^{n-1} \times 2^{2(n-1)} [f^*(x_{1j}, x_{2j}) + 6 f(x_{11}, \ldots, x_{1n})]. \qquad (9)
\end{aligned}
$$

Now, relation (9) shows that

$$\begin{aligned}&f\left(x_{11},\ldots,x_{1j-1},2x_{1j}+x_{2j},x_{1j+1},\ldots,x_{1n}\right)\\&+f\left(x_{11},\ldots,x_{1j-1},2x_{1j}-x_{2j},x_{1j+1},\ldots,x_{1n}\right)\\&=f^{*}(x_{1j},x_{2j})+6f\left(x_{11},\ldots,x_{1n}\right).\end{aligned}$$

This means that f is quadratic (satisfying (5)) in the jth variable. Since j is arbitrary, we obtain the desired result.

3 Characterization of Multi-additive-Quadratic-Cubic Mappings

Let V and W be linear spaces, $n \in \mathbb{N}$, and $k, p \in \{0, \cdots, n\}$. A mapping $f : V^n \longrightarrow W$ is called *k-additive*, *p-quadratic*, and $n-k-p$*-cubic* (briefly, *multi-additive-quadratic-cubic*) if f is additive in each of some k variables, is quadratic in each of some p variables (see Eq. (5)), and is cubic in each of the other variables (see Eq. (4)). In this note, we suppose for simplicity that f is additive in each of the first k variables, is cubic in each of the last $n-k-p$ variables, and is quadratic in each of the middle p variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$, $p = n$, and $k, p = 0$, the above definition leads to the so-called multi additive, multi-quadratic, and multi-cubic mappings, respectively.

In this section, we identify $x = (x_1, \ldots, x_n) \in V^n$ with

$$(x^k, x^p, x^{n-k}) \in V^k \times V^p \times V^{n-k-p},$$

where $x^k := (x_1, \ldots, x_k)$, $x^p := (x_{k+1}, \ldots, x_{k+p})$ and $x^{n-k-p} := (x_{k+p+1}, \ldots, x_n)$, and we adopt the convention that $(x^n, x^0, x^0) := (x^0, x^n, x^0) := (x^0, x^0, x^n) := x^n$. Put $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$, $x_i^p = (x_{i,k+1} \ldots, x_{i,k+p}) \in V^p$ and $x_i^{n-k-p} = (x_{i,k+p+1} \ldots, x_{in}) \in V^{n-k-p}$ where $i \in \{1, 2\}$. We remember the denotation x_i instead of x_i^n for the rest of the paper. In addition, we put

$$\mathbb{A}^{k+p} = \left\{\mathfrak{A}_{k+p} = (A_{k+1}, \ldots, A_{k+p})|\ A_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\right\},$$

where $j \in \{k+1, \ldots, k+p\}$. Consider

$$\mathbb{A}_T^{k+p} := \left\{\mathfrak{A}_{k+p} = (A_{k+1}, \ldots, A_{k+p}) \in \mathbb{A}^{k+p}|\ \mathrm{Card}\{A_j :\ A_j = x_{1j}\} = T\right\}.$$

Moreover, we put $\mathbb{B}^{n-k-p} = \{\mathfrak{B}_n = (B_{k+p+1}, \ldots, B_n) |\ B_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, where $j \in \{k+p+1, \ldots, n\}$. Besides

$$\mathbb{B}_T^{n-k-p} := \left\{\mathfrak{B}_n = (B_{k+p+1}, \ldots, B_n) \in \mathbb{B}^{n-k-p} |\ \mathrm{Card}\{B_j : B_j = x_{1j}\} = T\right\}.$$

For the multi-additive-quadratic-cubic mappings, we use some notations as follows:

$$f\left(x_i^k, \mathbb{A}_T^{k+p}, x_i^{n-k-p}\right) := \sum_{\mathfrak{A}_{k+p} \in \mathbb{A}_T^{k+p}} f\left(x_i^k, \mathfrak{A}_{k+p}, x_i^{n-k-p}\right),$$

$$f\left(x_i^k, x_i^p, \mathbb{B}_T^{n-k-p}\right) := \sum_{\mathfrak{B}_n \in \mathbb{B}_T^{n-k-p}} f\left(x_i^k, x_i^p, \mathfrak{B}_n\right),$$

$$f\left(x_i^k, \mathbb{A}_T^{k+p}, \mathbb{B}_T^{n-k-p}\right) := \sum_{\mathfrak{A}_{k+p} \in \mathbb{A}_T^{k+p}} \sum_{\mathfrak{B}_n \in \mathbb{B}_T^{n-k-p}} f\left(x_i^k, \mathfrak{A}_{k+p}, \mathfrak{B}_n\right) \qquad (i \in \{1, 2\}).$$

Here, we reduce the system of n equations defining the multi-additive-quadratic-cubic mapping to obtain a single functional equation.

Proposition 2 *Let $n \in \mathbb{N}$ and $k, p \in \{0, \ldots, n\}$. If a mapping $f : V^n \longrightarrow W$ is multi-additive-quadratic-cubic, then f satisfies the equation*

$$\sum_{s \in \{-1,1\}^p} \sum_{t \in \{-1,1\}^{n-k-p}} f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)$$
$$= \sum_{l=0}^{p} \sum_{m=0}^{n-k-p} \sum_{i \in \{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right) \tag{10}$$

for all $x_i^{n-k-p} = (x_{i,k+p+1} \ldots, x_{in}) \in V^{n-k-p}$, $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^p = (x_{i,k+1} \ldots, x_{i,k+p}) \in V^p$.

Proof Since for $k, p \in \{0, n\}$ our assertion follows from [14, Theorem 2], [8, Proposition 2.2] and Theorem 1, we can assume that $k, p \in \{1, \ldots, n-1\}$. For any $x^p \in V^p, x^{n-k-p} \in V^{n-k-p}$, define the mapping $\mathscr{T}_{x^p, x^{n-k-p}} : V^k \longrightarrow W$ by $\mathscr{T}_{x^p, x^{n-k-p}}(x^k) := f\left(x^k, x^p, x^{n-k-p}\right)$ for $x^k \in V^k$. By assumption, $\mathscr{T}_{x^p, x^{n-k-p}}$ is k-additive, and hence Theorem 2 from [14] implies that

$$\mathscr{T}_{x^p, x^{n-k-p}}\left(x_1^k + x_2^k\right) = \sum_{j_1, j_2, \cdots, j_k \in \{1,2\}} \mathscr{T}_{x^p, x^{n-k-p}}\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}\right)$$

for all $x_1^k, x_2^k \in V^k$. It now follows from the above equality that

$$f\left(x_1^k + x_2^k, x^p, x^{n-k-p}\right) = \sum_{j_1, j_2, \dots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \dots, x_{j_k k}, x^p, x^{n-k-p}\right) \tag{11}$$

for all $x_1^k, x_2^k \in V^k$, $x^p \in V^p$, $x^{n-k-p} \in V^{n-k-p}$. Similar to the above, for any $x^k \in V^k$, $x^{n-k-p} \in V^{n-k-p}$, consider the mapping $\mathscr{G}_{x^k, x^{n-k-p}} : V^p \longrightarrow W$ defined through $\mathscr{G}_{x^k, x^{n-k-p}}(x^p) := f\left(x^k, x^p, x^{n-k-p}\right)$ where $x^p \in V^p$ which is in fact p-quadratic. By Theorem 1, we get

$$\sum_{s \in \{-1,1\}^p} \mathscr{G}_{x^k, x^{n-k-p}}(2x_1^p + sx_2^p) = \sum_{l=0}^{p} 6^l \mathscr{G}_{x^k, x^{n-k-p}}\left(\mathbb{A}_l^{k+p}\right) \tag{12}$$

for all $x_1^p, x_2^p \in V^p$. Thus, relation (12) implies that

$$\sum_{s \in \{-1,1\}^p} f\left(x^k, 2x_1^p + sx_2^p, x^{n-k-p}\right) = \sum_{l=0}^{p} 6^l f\left(x^k, \mathbb{A}_l^{k+p}, x^{n-k-p}\right) \tag{13}$$

for all $x_1^p, x_2^p \in V^p$ and $x^k \in V^k$, $x^{n-k-p} \in V^{n-k-p}$. Next, for any $x^k \in V^k$ and $x^p \in V^p$, define the mapping $\mathscr{H}_{x^k, x^p} : V^{n-k-p} \longrightarrow W$ via $\mathscr{H}_{x^k, x^p}\left(x^{n-k-p}\right) := f\left(x^k, x^p, x^{n-k-p}\right)$, for any $x^{n-k-p} \in V^{n-k-p}$ which is $n-k-p$-cubic. Hence, we conclude from Proposition 2.2 of [8] that

$$\sum_{t \in \{-1,1\}^{n-k-p}} H_{x^k, x^p}\left(2x_1^{n-k-p} + tx_2^{n-k-p}\right) = \sum_{m=0}^{n-k-p} 2^{n-k-p-m} \times 12^m \mathscr{H}_{x^k, x^p}\left(\mathbb{B}_m^{n-k-p}\right) \tag{14}$$

for all $x_1^{n-k-p}, x_2^{n-k-p} \in V^{n-k-p}$. By the definition of $\mathscr{H}_{x^k, x^p}$, relation (14) is equivalent to

$$\sum_{t \in \{-1,1\}^{n-k-p}} f\left(x^k, x^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right) = \sum_{m=0}^{n-k-p} 2^{n-k-p-m} \times 12^m f\left(x^k, x^p, \mathbb{B}_m^{n-k-p}\right) \tag{15}$$

for all $x_1^{n-k-p}, x_2^{n-k-p} \in V^{n-k-p}$ and $x^k \in V^k$, $x^p \in V^p$. Plugging equalities (11) and (13) into (15), we obtain

$$\sum_{s\in\{-1,1\}^p} \sum_{t\in\{-1,1\}^{n-k-p}} f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)$$
$$= \sum_{s\in\{-1,1\}^p} \sum_{m=0}^{n-k-p} 2^{n-k-p-m} \times 12^m f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, \mathbb{B}_m^{n-k-p}\right)$$
$$= \sum_{l=0}^{p} 6^l \sum_{m=0}^{n-k-p} 2^{n-k-p-m} \times 12^m f\left(x_1^k + x_2^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right)$$
$$= \sum_{l=0}^{p} \sum_{m=0}^{n-k-p} \sum_{i\in\{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right)$$

for all $x_i^{n-k-p} = (x_{i,k+p+1} \dots, x_{in}) \in V^{n-k-p}$, $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$, and $x_i^p = (x_{i,k+1} \dots, x_{i,k+p}) \in V^p$, which proves that f satisfies Eq. (10).

It is easily verified that the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(z_1, z_2, \dots, z_n) = \prod_{j=1}^{k} \prod_{i=1}^{p} \prod_{r=1}^{n-k-p} z_j z_i^2 z_r^3$ is multi-additive-quadratic-cubic and so it satisfies (10) and so this equation is called multi-additive-quadratic-cubic functional equation.

Remark 1 [26] We mention that if a mapping f satisfies the cubic condition in each variable, it does not imply that f is multi-cubic. Let $(\mathscr{A}, \|\cdot\|)$ be a Banach algebra. Fix the vector a_0 in $\mathscr{A}$ (not necessarily unit). Define the mapping $h : \mathscr{A}^n \longrightarrow \mathscr{A}$ by $h(a_1, \dots, a_n) = \prod_{j=1}^{n} \|a_j\|^3 a_0$ for $(a_1, \dots, a_n) \in \mathscr{A}^n$. It is easy to check that the mapping h satisfies the cubic condition in all variables but h is not multi-cubic even for $n = 1$, that is, h does not satisfy in Eq. (4). Similarly, we have for the quadratic condition.

In the next result, we show that if a mapping f satisfies Eq. (10), then it is multi-additive-quadratic-cubic under some mild conditions.

Proposition 3 *If a mapping $f : V^n \longrightarrow W$ satisfies Eq.* (10) *and the cubic condition in the last $n-k-p$ and the quadratic condition in the middle p variables, then it is multi-additive-quadratic-cubic.*

Proof Putting $x_2^p = (0, \dots, 0)$ and $x_2^{n-k-p} = (0, \dots, 0)$ in (10) and applying the hypothesis, we see the left side of (10) will be as follows:

$$2^p \times 2^{2p} \times 2^{n-k-p} \times 2^{3(n-k-p)} f(x_1^k + x_2^k, x_1^p, x_1^{n-k-p})$$
$$= 2^{4(n-k)-p} f(x_1^k + x_2^k, x_1^p, x_1^{n-k-p}) \tag{16}$$

for all $x_1^k, x_2^k \in V^k$ and $x_1^p \in V^p, x_1^{n-k-p} \in V^{n-k-p}$. On the other hand, the right side of (10) under assumptions above is as

$$\Gamma(n) \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^p, x_1^{n-k-p}\right),$$

where

$$\Gamma(n) = \sum_{l=0}^{p} \binom{p}{l} 6^l \times 2^{p-l} \sum_{m=0}^{n-k-p} \binom{n-k-p}{m} 2^{n-k-p-m} \times 12^m \times 2^{n-k-p-m}. \tag{17}$$

By an easy computation, we have $\Gamma(n) = (6+2)^p (12+4)^{n-k-p} = 2^{4(n-k)-p}$, and so the right side of (10) is

$$2^{4(n-k)-p} \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^p, x_1^{n-k-p}\right) \tag{18}$$

for all $x_1^k, x_2^k \in V^k$ and $x_1^p \in V^p, x_1^{n-k-p} \in V^{n-k-p}$. Comparing relations (16) and (18), we find

$$f(x_1^k + x_2^k, x_1^p, x_1^{n-k-p}) = \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x_1^p, x_1^{n-k-p}\right) \tag{19}$$

for all $x_1^k, x_2^k \in V^k$ and $x_1^p \in V^p, x_1^{n-k-p} \in V^{n-k-p}$. In light of [14, Theorem 2], we see that f is additive in each of the k first variables. Similar to the previous, by putting $x_1^k = x_2^k, x_2^{n-k-p} = (0, \ldots, 0)$ in (10), using the assumptions and this fact that $f\left(2x_1^k, x_1^p, x_1^{n-k-p}\right) = 2^k f\left(x_1^k, x_1^p, x_1^{n-k-p}\right)$, we obtain the left side of (10) as follows:

$$\begin{aligned} &2^k \times 2^{n-k-p} \times 2^{3(n-k-p)} \sum_{s \in \{-1,1\}^k} f\left(x_1^k, 2x_1^p + sx_2^p, x_1^{n-k-p}\right) \\ &= 2^{4(n-p)-3k} \sum_{s \in \{-1,1\}^k} f\left(x_1^k, 2x_1^p + sx_2^p, x_1^{n-k-p}\right) \end{aligned} \tag{20}$$

for all $x_1^k \in V^k, x_1^p, x_2^p \in V^p$ and $x_1^{n-k-p} \in V^{n-k-p}$. On the other hand, the right side of (10) will be

$$2^k \sum_{m=0}^{n-k-p} \binom{n-k-p}{m} 2^{n-k-p-m} \times 12^m \times 2^{n-k-p-m} \sum_{l=0}^{p} 6^l f\left(x_1^k, \mathbb{A}_l^{k+p}, x_1^{n-k-p}\right)$$

$$= 2^k \times 2^{4(n-k-p)} \sum_{l=0}^{p} 6^l f\left(x_1^k, \mathbb{A}_l^{k+p}, x_1^{n-k-p}\right)$$

$$= 2^{4(n-p)-3k} \sum_{l=0}^{p} 6^l f\left(x_1^k, \mathbb{A}_l^{k+p}, x_1^{n-k-p}\right) \tag{21}$$

for all $x_1^k, x_1^p, x_2^p \in V^p$ and $x_1^{n-k-p} \in V^{n-k-p}$. It follows from (20) and (21) that

$$\sum_{s\in\{-1,1\}^p} f\left(x_1^k, 2x_1^p + sx_2^p, x_1^{n-k-p}\right) = \sum_{l=0}^{p} 6^l f\left(x_1^k, \mathbb{A}_l^{k+p}, x_1^{n-k-p}\right)$$

for all $x_1^k, x_2^k \in V^k$ and $x_1^{n-k-p} \in V^{n-k-p}$. Now, Theorem 1 implies that f is quadratic in each of the p middle variables. In a similar way, by putting $x_1^k = x_2^k$, $x_2^p = (0, \ldots, 0)$ in (10), one can show that

$$\sum_{t\in\{-1,1\}^{n-k-p}} f\left(x_1^k, x_1^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)$$
$$= \sum_{m=0}^{n-k-p} 2^{n-k-p-m} \times 12^m f\left(x_1^k, x_1^p, \mathbb{B}_m^{n-k-p}\right)$$

for all $x_1^k \in V^k$, $x_1^p \in V^p$ and $x_1^{n-k-p}, x_2^{n-k-p} \in V^{n-k-p}$, and thus Proposition 2.2 of [8] now completes the proof.

4 Stability of the Multi-additive-Quadratic-Cubic Mappings

In this section, we prove the generalized Hyers-Ulam stability of multi-additive-quadratic-cubic functional equation (10) by a fixed point result (Theorem 2) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. To reach our purpose in this section, we present the next theorem which is a fundamental result in fixed point theory [10, Theorem 1].

Theorem 2 *Let the following hypotheses hold.*

(A1) *Y is a Banach space, $\mathscr{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \ldots, g_j : \mathscr{S} \longrightarrow \mathscr{S}$, and $L_1, \ldots, L_j : \mathscr{S} \longrightarrow \mathbb{R}_+$,*

(A2) *$\mathscr{T}: Y^{\mathscr{S}} \longrightarrow Y^{\mathscr{S}}$ is an operator satisfying the inequality*

$$\|\mathscr{T}\lambda(x) - \mathscr{T}\mu(x)\| \leq \sum_{i=1}^{j} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^{\mathscr{S}}, x \in \mathscr{S},$$

(A3) *$\Lambda : \mathbb{R}_+^{\mathscr{S}} \longrightarrow \mathbb{R}_+^{\mathscr{S}}$ is an operator defined through*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^{\mathscr{S}}, x \in \mathscr{S}.$$

Let also a function $\theta : \mathscr{S} \longrightarrow \mathbb{R}_+$ and a mapping $\phi : \mathscr{S} \longrightarrow Y$ fulfill the following two conditions:

$$\|\mathscr{T}\phi(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathscr{S}).$$

Then, there exists a unique fixed point ψ of $\mathscr{T}$ such that

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x) \qquad (x \in \mathscr{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathscr{T}^l\phi(x)$ for all $x \in \mathscr{S}$.

Here and subsequently, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathscr{D}_{aqc} f : V^n \times V^n \longrightarrow W$ by

$$\mathscr{D}_{aqc} f(x_1, x_2) := \sum_{s\in\{-1,1\}^p} \sum_{t\in\{-1,1\}^{n-k-p}} f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)$$
$$- \sum_{l=0}^{p} \sum_{m=0}^{n-k-p} \sum_{i\in\{1,2\}} 6^l 2^{n-k-p-m} 12^m f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right)$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$, $x_i^p = (x_{i,k+1} \ldots, x_{i,k+p}) \in V^p$, and $x_i^{n-k-p} = (x_{i,k+p+1} \ldots, x_{in}) \in V^{n-k-p}$ where $i \in \{1, 2\}$.

We now have the following stability result for the multi-additive-quadratic-cubic functional equations.

Theorem 3 *Let $\beta \in \{-1, 1\}$, V be a linear space, and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a function satisfying*

$$\lim_{r\to\infty}\left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \phi(2^{\beta r}x_1, 2^{\beta r}x_2) = 0 \tag{22}$$

for all $x_1, x_2 \in V^n$ and

$$\Phi(x) = \frac{1}{2^{n-k}\times 2^{\frac{\beta+1}{2}(3n-2k-p)}} \sum_{r=0}^{\infty}\left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \phi\left(2^{\beta r+\frac{\beta-1}{2}}x, \left(2^{\beta r+\frac{\beta-1}{2}}x_1^k, 0, 0\right)\right) < \infty \tag{23}$$

for all $x = x_1 = (x_1^k, x_1^p, x_1^{n-k-p}) \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping satisfying the inequality

$$\|\mathscr{D}_{aqc} f(x_1, x_2)\| \leqslant \phi(x_1, x_2) \tag{24}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x) - \mathscr{F}(x)\| \le \Phi(x) \tag{25}$$

for all $x \in V^n$.

Proof Putting $x_1^k = x_2^k$ and $x_2^p = x_2^{n-k-p} = 0$ in (24), we have

$$\left\|2^{n-k} f(2x) - 2^k\Gamma(n) f(x)\right\| \le \phi(x, (x_1^k, 0, 0)) \tag{26}$$

for all $x = x_1 = (x_1^k, x_1^p, x_1^{n-k-p}) \in V^n$, where $\Gamma(n)$ is defined in (17). Since $\Gamma(n) = 2^{4(n-k)-p}$, (26) is as follows:

$$\left\|f(2x) - 2^{3n-2k-p} f(x)\right\| \le \frac{1}{2^{n-k}}\phi(x, (x_1^k, 0, 0)) \tag{27}$$

for all $x \in V^n$. Now, relation (27) can be modified as

$$\|f(x) - \mathscr{T} f(x)\| \le \xi(x) \tag{28}$$

for all $x \in V^n$, where

$$\xi(x) := \frac{1}{2^{n-k}\times 2^{\frac{\beta+1}{2}(3n-2k-p)}}\phi\left(2^{\frac{\beta-1}{2}}x, \left(2^{\frac{\beta-1}{2}}x_1^k, 0, 0\right)\right)$$

and $\mathscr{T}\xi(x) := \frac{1}{2^{(3n-2k-p)\beta}}\xi(2^\beta x)$ in which $\xi \in W^{V^n}$. For each $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$, define $\Lambda\eta(x) := \frac{1}{2^{(3n-2k-p)\beta}}\eta(2^\beta x)$. We now see that Λ has the form described in (A3) with $\mathscr{S} = V^n$, $g_1(x) = 2^\beta x$, and $L_1(x) = \frac{1}{2^{(3n-2k-p)\beta}}$ for all $x \in V^n$. Furthermore, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\begin{aligned}\|\mathscr{T}\lambda(x) - \mathscr{T}\mu(x)\| &= \left\|\frac{1}{2^{(3n-2k-p)\beta}}\left[\lambda(2^\beta x) - \mu(2^\beta x)\right]\right\| \\ &\leq L_1(x)\,\|\lambda(g_1(x)) - \mu(g_1(x))\|.\end{aligned}$$

The relation above shows that the hypothesis (A2) of Theorem 2 holds. By induction on r, one can check that

$$\begin{aligned}\Lambda^r\xi(x) &:= \left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \xi(2^{\beta r}x) \\ &= \frac{1}{2^{n-k}\times 2^{\frac{\beta+1}{2}(3n-2k-p)}}\left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \phi\left(2^{\beta r+\frac{\beta-1}{2}}x, \left(2^{\beta r+\frac{\beta-1}{2}}x_1^k, 0, 0\right)\right)\end{aligned} \tag{29}$$

for all $r \in \mathbb{N}_0$ and $x \in V^n$. It follows from relations (23) and (29) that all assumptions of Theorem 2 are satisfied. Hence, there exists a solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\mathscr{F}(x) = \lim_{l\to\infty}(\mathscr{T} f)(x) = \frac{1}{2^{(3n-2k-p)\beta}}\mathscr{F}(2^\beta x) \qquad (x \in V^n),$$

and (25) holds. We shall to show that

$$\|\mathscr{D}_{aqc}(\mathscr{T} f)(x_1, x_2)\| \leq \left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \phi(2^{\beta r}x_1, 2^{\beta r}x_2) \tag{30}$$

for all $x_1, x_2 \in V^n$ and $r \in \mathbb{N}_0$. We argue by induction on r. The inequality (30) holds for $r = 0$ by (24). Assume that (30) is valid for a $r \in \mathbb{N}_0$. Then

$$\begin{aligned}&\left\|\mathscr{D}_{aqc}(\mathscr{T}^{+1} f)(x_1, x_2)\right\| \\ &= \Bigg\| \sum_{s\in\{-1,1\}^p}\sum_{t\in\{-1,1\}^{n-k-p}}(\mathscr{T}^{+1} f)\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right) \\ &\quad - \sum_{l=0}^{p}\sum_{m=0}^{n-k-p}\sum_{i\in\{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m(\mathscr{T}^{+1} f)\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right)\Bigg\| \\ &= \frac{1}{2^{(3n-2k-p)\beta}}\Bigg\| \sum_{s\in\{-1,1\}^p}\sum_{t\in\{-1,1\}^{n-k-p}}\end{aligned}$$

$$
\begin{aligned}
&(\mathscr{T} f)\left(2^{\beta}\left(x_1^k + x_2^k\right), 2^{\beta}\left(2x_1^p + sx_2^p\right), 2^{\beta}\left(2x_1^{n-k-p} + tx_2^{n-k-p}\right)\right)\\
&-\sum_{l=0}^{p}\sum_{m=0}^{n-k-p}\sum_{i\in\{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m (\mathscr{T} f)\left(2^{\beta}x_i^k, 2^{\beta}\mathbb{A}_l^{k+p}, 2^{\beta}\mathbb{B}_m^{n-k-p}\right)\Big\|\\
&= \frac{1}{2^{(3n-2k-p)\beta}}\left\|\mathscr{D}_{aqc}(\mathscr{T} f)(2^{\beta}x_1, 2^{\beta}x_2)\right\|\\
&\le \left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^{r+1}\phi\left(2^{\beta(r+1)}x_1, 2^{\beta(r+1)}x_2\right) \qquad (31)
\end{aligned}
$$

for all $x_1, x_2 \in V^n$. Letting $r \to \infty$ in (30) and applying (22), we find $\mathscr{D}_{aqc}\mathscr{F}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathscr{F}$ satisfies (10). Finally, assume that $\mathscr{F}' : V^n \longrightarrow W$ is another solution of equation (10) and inequality (25), and fix $x \in V^n$, $j \in \mathbb{N}$. Then

$$
\begin{aligned}
&\|\mathscr{F}(x) - \mathscr{F}'(x)\|\\
&= \left\|\frac{1}{2^{(3n-2k-p)\beta j}}\mathscr{F}\left(2^{\beta j}x\right) - \frac{1}{2^{(3n-2k-p)\beta j}}\mathscr{F}'\left(2^{\beta j}x\right)\right\|\\
&\le \frac{1}{2^{(3n-2k-p)\beta j}}\left(\left\|\mathscr{F}\left(2^{\beta j}x\right) - f\left(2^{\beta j}x\right)\right\| + \left\|\mathscr{F}'\left(2^{\beta j}x\right) - f\left(2^{\beta j}x\right)\right\|\right)\\
&\le \frac{2}{2^{(3n-2k-p)\beta j}}\Phi\left(2^{\beta j}x\right)\\
&\le \frac{2}{2^{n-k} \times 2^{\left(\frac{\beta+1}{2}+\beta j\right)(3n-2k-p)}}\sum_{r=j}^{\infty}\left(\frac{1}{2^{(3n-2k-p)\beta}}\right)^r \phi\left(2^{\beta r+\frac{\beta-1}{2}}x, \left(2^{\beta r+\frac{\beta-1}{2}}x_1^k, 0, 0\right)\right).
\end{aligned}
$$

Consequently, letting $j \to \infty$ and using the fact that series (23) is convergent for all $x \in V^n$, we obtain $\mathscr{F}(x) = \mathscr{F}'(x)$ for all $x \in V^n$, which finishes the proof.

The upcoming corollary is a direct consequence of Theorem 3 concerning the stability of multi-additive-quadratic-cubic mappings when the norm of $\mathscr{D}_{aqc} f(x_1, x_2)$ is bounded by the sum of powers of norms.

Corollary 1 *Let $\theta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 3n - 2k - p$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$
\|\mathscr{D}_{aqc} f(x_1, x_2)\| \le \theta \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha}
$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x) - \mathscr{F}(x)\| \leq \frac{\theta}{2^{n-k}|2^{3n-2k-p} - 2^{\alpha}|} \left(2\sum_{j=1}^{k} \|x_{1j}\|^{\alpha} + \sum_{j=k+1}^{n} \|x_{1j}\|^{\alpha} \right)$$

for all $x \in V^n$.

Theorem 3 with $k = n$ gives a result on the generalized Hyers-Ulam stability of an equation characterizing multi-additive mappings which is studied in [14, Theorem 3]. Moreover, by putting $k = p = 0$ in Corollary 1, we obtain [8, Corollary 3.5] on the stability of multi-cubic mappings. Furthermore, by considering $p = n$ in Corollary 1, we obtain the below result on the stability of multi-quadratic functional equations.

Corollary 2 *Let* $\theta > 0$ *and* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 2n$. *Let also* V *be a normed space and* W *be a Banach space. If* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\left\| \sum_{s\in\{-1,1\}^n} f(2x_1 + sx_2) - \sum_{l=0}^{n} 6^l f\left(\mathbb{A}_l^n\right) \right\| \leq \theta \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha}$$

for all $x_1, x_2 \in V^n$, *then there exists a unique solution* $\mathscr{Q} : V^n \longrightarrow W$ *of* (7) *such that*

$$\|f(x) - \mathscr{Q}(x)\| \leq \frac{\theta}{2^n|2^{2n} - 2^{\alpha}|} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}$$

for all $x \in V^n$.

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathscr{F}_1, \mathscr{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathscr{F}_1\varphi(a_1, \ldots, a_n) = \mathscr{F}_2\varphi(a_1, \ldots, a_n) \tag{32}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathscr{F}_1\varphi_0(a_1, \ldots, a_n), \mathscr{F}_2\varphi_0(a_1, \ldots, a_n)) \leq \psi(a_1, \ldots, a_n), \qquad a_1, \ldots, a_n \in A,$$

fulfills (32); this definition is introduced in [12]. In other words, a functional equation $\mathscr{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathscr{F}$ approximately is a true solution of $\mathscr{F}$.

Under some conditions, the multi-additive-quadratic-cubic mappings are hyperstable. It is shown in the next corollary.

Corollary 3 *Let* V *be a normed space and* W *be a Banach space. Suppose that* $\theta_{ij} > 0$ *for* $i \in \{1, 2\}$ *and* $j \in \{1, \ldots, n\}$ *fulfill* $\sum_{i=1}^{2}\sum_{j=1}^{n} \theta_{ij} \neq 3n - 2k - p$. *If* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathscr{D}_{aqc}f(x_1, x_2)\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\theta_{ij}}$$

for all $x_1, x_2 \in V^n$, then f satisfies (10). Moreover, if f satisfies the cubic condition in the last $n-k-p$ and the quadratic condition in the middle p variables, then it is multi-additive-quadratic-cubic.

Putting $p = n$ in Corollary 3, we see a multi-quadratic mapping can be hyperstable as follows.

Corollary 4 *Let V be a normed space and W be a Banach space. Suppose that $\theta_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n}\theta_{ij} \neq 2n$. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\|\sum_{s\in\{-1,1\}^n} f(2x_1 + sx_2) - \sum_{l=0}^{n} 6^l f\left(\mathbb{A}_l^n\right)\right\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\theta_{ij}}$$

for all $x_1, x_2 \in V^n$, then f satisfies (10). In particular, if f satisfies the quadratic condition in all variables, then it is multi-quadratic.

5 Stability Results for (10) in Non-Archimedean Normed Spaces

We firstly express some basic facts concerning non-Archimedean spaces and some preliminary results. Let us recall that a metric d on a nonempty set X is said to be non-Archimedean (or an ultrametric) provided $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for $x, y, z \in X$. By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$, and $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let $\mathscr{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathscr{X} \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|, \quad (x \in \mathscr{X}, a \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in \mathscr{X}).$$

Then, $(\mathscr{X}, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathscr{X}$. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent. If $(\mathscr{X}, \|\cdot\|)$ is a non-Archimedean normed space, then it is easily verified that the function $d_{\mathscr{X}} : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{R}_+$, given by $d_{\mathscr{X}}(x, y) := \|x - y\|$, is a non-Archimedean metric on $\mathscr{X}$ that is invariant (i.e., $d_{\mathscr{X}}(x+z, y+z) = d_{\mathscr{X}}(x, y)$ for $x, y, z \in X$). Hence, non-Archimedean normed spaces are also special cases of metric spaces with invariant metrics.

The most important examples of non-Archimedean normed spaces are the p-adic numbers, which have gained the interest of physicists because of their connections with some problems coming from quantum physics, p-adic strings, and superstrings [23]. Indeed, Hensel [20] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1 + \cdots + 1}^{n-\text{times}} = 0$.

In this section, we prove the generalized Hyers-Ulam stability for multi-additive-quadratic-cubic in non-Archimedean normed spaces. The proof is based on a fixed point result that can be derived from [11, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,

(H2) $\mathscr{T}: Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathscr{T}\lambda(x) - \mathscr{T}\mu(x)\| \leq \max_{i\in\{1,\ldots,j\}} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|,$$

for all $\lambda, \mu \in Y^E$, $x \in E$,

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [11, Theorem 1]. This result plays a key tool to obtain our goal in this section.

Theorem 4 *Let hypotheses* (H1)-(H3) *hold and the function* $\theta : E \longrightarrow \mathbb{R}_+$ *and the mapping* $\varphi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathscr{T}\varphi(x) - \varphi(x)\| \leq \theta(x), \quad \lim_{l\to\infty} \Lambda^l\theta(x) = 0 \qquad (x \in E).$$

Then, for every $x \in E$, *the limit* $\lim_{l\to\infty} \mathscr{T}^l\varphi(x) =: \psi(x)$ *exists and the mapping* $\psi \in Y^E$, *defined in this way, is a fixed point of* $\mathscr{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq sup_{l\in\mathbb{N}_0} \Lambda^l\theta(x) \qquad (x \in E).$$

We now are ready to indicate the upcoming result which is the main result in this section.

Theorem 5 *Let* $\beta \in \{-1, 1\}$ *be fixed,* V *be a linear space, and* W *be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2. *Suppose that* $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying the equality*

$$\lim_{r\to\infty} \left(\frac{1}{|2|^{(3n-2k-p)\beta}}\right)^r \varphi(2^{\beta r}x_1, 2^{\beta r}x_2) = 0 \tag{33}$$

for all $x_1, x_2 \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathscr{D}_{aqc} f(x_1, x_2)\| \leq \varphi(x_1, x_2) \tag{34}$$

for all $x_1, x_2 \in V^n$. *Then, there exists a unique solution* $\mathscr{F} : V^n \longrightarrow W$ *of* (10) *such that*

$$\|f(x) - \mathscr{F}(x)\| \leq sup_{r\in\mathbb{N}_0} \frac{1}{|2|^{n-k} \times |2|^{\frac{\beta+1}{2}(3n-2k-p)}} \left(\frac{1}{|2|^{(3n-2k-p)\beta}}\right)^r \varphi\left(2^{\beta r+\frac{\beta-1}{2}}x, \left(2^{\beta r+\frac{\beta-1}{2}}x_1^k, 0, 0\right)\right) \tag{35}$$

for all $x \in V^n$.

Proof Putting $x_1^k = x_2^k$ and $x_2^p = x_2^{n-k-p} = 0$ in (34), we get

$$\left\|2^{n-k} f(2x) - 2^k \Gamma(n) f(x)\right\| \leq \varphi(x, (x_1^k, 0, 0)) \tag{36}$$

for all $x = x_1 = (x_1^k, x_1^p, x_1^{n-k-p}) \in V^n$, where $\Gamma(n)$ is defined in (17). Hence, (36) can be rewritten as

$$\left\| f(2x) - 2^{3n-2k-p} f(x) \right\| \leq \frac{1}{|2|^{n-k}} \varphi(x, (x_1^k, 0, 0)) \tag{37}$$

for all $x \in V^n$. Set

$$\theta(x) := \frac{1}{|2|^{n-k} \times |2|^{\frac{\beta+1}{2}(3n-2k-p)}} \varphi\left(2^{\frac{\beta-1}{2}} x, \left(2^{\frac{\beta-1}{2}} x_1^k, 0, 0\right)\right)$$

and $\mathscr{T}\theta(x) := \frac{1}{|2|^{(3n-2k-p)\beta}} \theta(2^\beta x)$ in which $\xi \in W^{V^n}$ for all $\theta \in W^{V^n}$ and $x \in V^n$. Thus, relation (37) shows that

$$\| f(x) - \mathscr{T} f(x) \| \leq \theta(x) \tag{38}$$

for all $x \in V^n$. Define $\Lambda\eta(x) := \frac{1}{|2|^{(3n-2k-p)\beta}} \eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$. It is easy to see that Λ has the form described in (H3) with $E = V^n$, $g_1(x) := 2^\beta x$ for all $x \in V^n$ and $L_1(x) = \frac{1}{|2|^{(3n-2k-p)\beta}}$. Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we obtain

$$\begin{aligned} \| \mathscr{T}\lambda(x) - \mathscr{T}\mu(x) \| &= \left\| \frac{1}{2^{(3n-2k-p)\beta}} \left[\lambda(2^\beta x) - \mu(2^\beta x) \right] \right\| \\ &\leq L_1(x) \left\| \lambda(g_1(x)) - \mu(g_1(x)) \right\| . \end{aligned}$$

The last relation shows that the hypothesis (H2) is valid. By induction on r, one can check that for any $r \in \mathbb{N}$ and $x \in V^n$ that

$$\begin{aligned} \Lambda^r \theta(x) &:= \left(\frac{1}{|2|^{(3n-2k-p)\beta}} \right)^r \theta(2^{\beta r} x) \\ &= \frac{1}{|2|^{n-k} \times |2|^{\frac{\beta+1}{2}(3n-2k-p)}} \left(\frac{1}{|2|^{(3n-2k-p)\beta}} \right)^r \varphi\left(2^{\beta r + \frac{\beta-1}{2}} x, \left(2^{\beta r + \frac{\beta-1}{2}} x_1^k, 0, 0\right)\right) \end{aligned} \tag{39}$$

for all $x \in V^n$. The relations (38) and (39) necessitate that all assumptions of Theorem 4 are satisfied. Hence, there exists a unique mapping $\mathscr{F} : V^n \longrightarrow W$ such that $\mathscr{F}(x) = \lim_{r\to\infty}(\mathscr{T} f)(x)$ for all $x \in V^n$, and also (35) holds. We also can verify by induction on r that

$$\| \mathscr{D}_{aqc}(\mathscr{T} f)(x_1, x_2) \| \leq \left(\frac{1}{|2|^{(3n-2k-p)\beta}} \right)^r \varphi(2^{\beta r} x_1, 2^{\beta r} x_2) \tag{40}$$

for all $x_1, x_2 \in V^n$. Letting $r \to \infty$ in (40) and applying (33), we arrive at $\mathscr{D}_{aqc}\mathscr{F}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping satisfies Eq. (10) and the proof is now completed.

In the sequel, we assume that $|2| < 1$. The following corollaries are some direct applications of Theorem 5 concerning the stability of (10).

Corollary 5 *Let $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathscr{D}_{aqc} f(x_1, x_2)\| \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x) - \mathscr{F}(x)\| \leq \frac{1}{|2|^{n-k}}\delta$$

for all $x \in V^n$.

Proof Letting $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 5, we have

$$\lim_{r\to\infty} \left(\frac{1}{|2|^{(3n-2k-p)\beta}}\right)^r \delta = 0.$$

Therefore, one can obtain the desired result.

In the next result, we study the stability of multi-additive, multi-quadratic, multi-cubic, and multi-additive-quadratic-cubic functional equations when the norm of corresponding difference operators is bounded by the sum of powers of norms.

Corollary 6 *Let V be a non-Archimedean normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. If $\alpha \in \mathbb{R}$ fulfills $\alpha \neq 3n - 2k - p$ and $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathscr{D}_{aqc} f(x_1, x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^\alpha$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (10) such that

$$\|f(x)-\mathscr{F}(x)\| \leq \begin{cases} \frac{1}{|2|^{4n-3k-p}}\left(2\sum_{j=1}^{k}\|x_{1j}\|^{\alpha}+\sum_{j=k+1}^{n}\|x_{1j}\|^{\alpha}\right) & \alpha>3n-2k-p \\ \frac{1}{|2|^{n-k-\alpha}}\left(2\sum_{j=1}^{k}\|x_{1j}\|^{\alpha}+\sum_{j=k+1}^{n}\|x_{1j}\|^{\alpha}\right) & \alpha<3n-2k-p \end{cases} \quad (41)$$

for all $x=x_1\in V^n$. In particular,

(i) *if $\alpha\neq n$ and f satisfying the inequality*

$$\left\|f(x_1+x_2)-\sum_{j_1,\ldots,j_n\in\{1,2\}}f\left(x_{j_11},\ldots,x_{j_nn}\right)\right\|\leq\sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$$

for all $x_1,x_2\in V^n$, then there exists a unique multi-additive mapping $\mathscr{A}:V^n\longrightarrow W$ such that

$$\|f(x)-\mathscr{A}(x)\| \leq \begin{cases} \frac{2}{|2|^{n}}\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} & \alpha>n \\ \frac{2}{|2|^{-\alpha}}\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} & \alpha<n; \end{cases}$$

(ii) *if $\alpha\neq 2n$ and f satisfying the inequality*

$$\left\|\sum_{s\in\{-1,1\}^n}f(2x_1+sx_2)-\sum_{l=0}^{n}6^l f\left(\mathbb{A}_l^n\right)\right\|\leq\sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$$

for all $x_1,x_2\in V^n$, then there exists a unique solution $\mathscr{Q}:V^n\longrightarrow W$ of (7) *such that*

$$\|f(x)-\mathscr{Q}(x)\| \leq \begin{cases} \frac{1}{|2|^{3n}}\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} & \alpha>2n \\ \frac{1}{|2|^{n-\alpha}}\sum_{j=1}^{k}\|x_{1j}\|^{\alpha} & \alpha<2n; \end{cases}$$

(iii) *if $\alpha\neq 3n$ and f satisfying the inequality*

$$\left\|\sum_{t\in\{-1,1\}^n}f(2x_1+tx_2)-\sum_{m=0}^{n}2^{n-m}\times 12^m f\left(\mathbb{B}_m^n\right)\right\|\leq\sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha}$$

for all $x_1,x_2\in V^n$, then there exists a unique solution $\mathscr{C}:V^n\longrightarrow W$ of (2.3) *from* [8] *such that*

$$\|f(x) - \mathscr{C}(x)\| \quad \leq \begin{cases} \frac{1}{|2|^{4n}} \sum_{j=1}^{k} \|x_{1j}\|^{\alpha} & \alpha > 3n \\ \\ \frac{1}{|2|^{n-\alpha}} \sum_{j=1}^{k} \|x_{1j}\|^{\alpha} & \alpha < 3n. \end{cases}$$

Proof Putting $\varphi(x_1, x_2) = \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^p$, we have $\varphi(2^r x_1, 2^r x_2) = |2|^{r\alpha}\varphi(x_1, x_2)$. The first and second inequalities of (41) follow from Theorem 5 in the cases $\beta = 1$ and $\beta = -1$, respectively. Other inequalities are taken from (41).

Under some conditions, the multi-additive-quadratic-cubic mappings can be hyperstable as follows.

Corollary 7 *Suppose that $\alpha_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \cdots, n\}$ fulfill $\sum_{i=1}^{2} \sum_{j=1}^{n} \alpha_{ij} \neq 3n - 2k - p$. Let V be a normed space and W be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathscr{D}_{aqc} f(x_1, x_2)\| \leq \prod_{k=i}^{2} \prod_{j=1}^{n} \|x_{ij}\|^{\alpha_{ij}}$$

for all $x_1, x_2 \in V^n$, then it satisfies (10).

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Generalizations of Truncated M-Fractional Derivative Associated with (p, k)-Mittag-Leffler Function with Classical Properties

Mehar Chand and Praveen Agarwal

Abstract In the present chapter, we have generalized the truncated M-fractional derivative. This new differential operator denoted by ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}$, where the parameter σ associated with the order of the derivative is such that $0 < \sigma < 1$ and M is the notation to designate that the function to be derived involves the truncated (p, k)-Mittag-Leffler function. The operator ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}$ satisfies the properties of the integer-order calculus. We also present the respective fractional integral from which emerges, as a natural consequence, the result, which can be interpreted as an inverse property. Finally, we obtain the analytical solution of the M-fractional heat equation, linear fractional differential equation, and present a graphical analysis.

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1 Introduction and Preliminaries

Gehlot in [1], presented the following two parameter Pochhammer symbol defined as:

Definition 1 Let $w \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \ p, k \in \mathbb{R}^+ - \{0\};\ \ n \in \mathbb{N};$ *and* $\Re(w) > 0$; then (p-k) Pochhammer symbol is defined as

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$$
\begin{aligned}
{}_p(w)_{n,k} &= \left(\frac{wp}{k}\right)\left(\frac{wp}{k}+p\right)\left(\frac{wp}{k}+2p\right)\cdots\left(\frac{wp}{k}+(n-1)p\right) \\
&= \frac{{}_p\Gamma_k(w+nk)}{{}_p\Gamma_k(w)}.
\end{aligned} \tag{1.1}
$$

Gehlot in [1], introduced the two parameter Gamma function defined as:

Definition 2 Let $w \in \mathbb{C} \setminus k\mathbb{Z}^-$; $p, k \in \mathbb{R}^+ - 0$; $n \in \mathbb{N}$; *and* $\Re(w) > 0$; then (p-k) Gamma function is defined as

$$
{}_p\Gamma_k(w) = \int_0^\infty e^{-\frac{t^k}{p}} t^{w-1} dt. \tag{1.2}
$$

Recently in [2], Gehlot introduced the (p-k) Mittag-Leffler function defined as:

Definition 3 Let $p, k \in \mathbb{R}^+ - \{0\}$; $\alpha, \beta, \gamma \in \mathbb{C} \setminus k\mathbb{Z}^-$; *and* $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$; then (p-k) Mittag-Leffler function is defined as

$$
{}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!}. \tag{1.3}
$$

where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol defined in Eq. (1.1).

1.1 Special Cases

1. For $q = 1$, Eq. (1.3), gives the following form of (p, k)-Mittag-Leffler function as follows:

$$
{}_pE^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!}. \tag{1.4}
$$

2. For $p = k$, Eq. (1.3), yields the following form of (p, k)-Mittag-Leffler function as follows [3]:

$$
{}_kE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}}{{}_k{}_k\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = GE^{\gamma,q}_{k,\alpha,\beta}(z). \tag{1.5}
$$

3. For $p = k$ and $q = 1$, Eq. (1.3), gives the following form of (p, k)-Mittag-Leffler function as follows [4]:

$$
{}_kE^{\gamma,1}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = E^{\gamma}_{k,\alpha,\beta}(z). \tag{1.6}
$$

4. For $p = k$ and $k = 1$, Eq. (1.3), yields the following form of Mittag-Leffler function as follows [5]:

$$ {}_1E^{\gamma,q}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!} = E^{\gamma,q}_{\alpha,\beta}(z). \tag{1.7} $$

5. For $p = k, q = 1$, and $k = 1$, Eq. (1.3), gives the following form of Mittag-Leffler function as follows [6]:

$$ {}_1E^{\gamma,1}_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!} = E^{\gamma}_{\alpha,\beta}(z). \tag{1.8} $$

6. For $p = k, q = 1, k = 1$, and $\gamma = 1$, Eq. (1.3), gives the following form of Mittag-Leffler function as follows [4]:

$$ {}_1E^{1,1}_{1,\alpha,\beta}(z) = \sum_{0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z). \tag{1.9} $$

7. For $p = k, q = 1, k = 1, \gamma = 1$, and $\beta = 1$, Eq. (1.3), reduces to the following form of Mittag-Leffler function as follows [7]:

$$ {}_1E^{1,1}_{1,\alpha,1}(z) = \sum_{0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{1.10} $$

Following lemmas are required for our present study as follows:

Lemma 1 *For the (p-k) Pochhammer symbol and the k-Pochhammer symbol and the classical Pochhammer symbol, it has*

$$ {}_p(w)_{n,k} = \left(\frac{p}{k}\right)^n (w)_{n,k} = p^n \left(\frac{w}{k}\right)_n. \tag{1.11} $$

Lemma 2 *For the (p-k) Gamma function, the k-Gamma function, and the classical Gamma function, it has [1]*

$$ {}_p\Gamma_k(w) = \left(\frac{p}{k}\right)^{\frac{w}{k}} {}_p\Gamma_k(w) = \frac{p^{\frac{w}{k}}}{k} \Gamma\left(\frac{w}{k}\right). \tag{1.12} $$

2 Truncated M-Fractional Derivative Type

In this section, we define a truncated M-fractional derivative type and obtain several results that have a great similarity with the results found in the classical calculus.

From the definition, we present a theorem showing that this truncated M-fractional derivative type is linear and obeys the product rule and the composition of two α-differentiable functions, the quotient rule, and the chain rule. It is also shown that the derivative of a constant is zero, as well as versions for Rolle's theorem, the mean value theorem, and an extension of the mean value theorem. Further, the continuity of this truncated M-fractional derivative type is shown as in integer-order calculus. Also, we introduce the concept of M-fractional integral of a function f. From the definition, we shown the Inverse theorem.
We define the truncated (p, k)-Mittag-Leffler function by

$$ {}_{i,p}\mathbb{E}_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{r=0}^{i} \frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)} \frac{z^r}{r!}, \tag{2.1} $$

with $z, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

From Eq. (2.1), we define a truncated M-fractional derivative type that unifies other four fractional derivatives that refer to classical properties of the integer-order calculus.

In this work, if a truncated M-fractional derivative type of order σ as defined in (2.2), of a function f exists, we say that the function f is σ-differentiable.

Thus, let us begin with the following definition, which is a generalization of the usual definition of integer-order derivative.

Definition 4 Let $f : [0, \infty) \to \mathbb{R}$. For $0 < \sigma < 1$, a truncated M-fractional derivative type of f of order σ denoted by ${}_{i,p}\mathscr{D}_{M,k,\alpha,\beta}^{\sigma,\gamma,q}$ is

$$ {}_{i,p}\mathscr{D}_{M,k,\alpha,\beta}^{\sigma,\gamma,q} f(t) = \lim_{\varepsilon \to 0} \frac{f(t\ {}_{i,p}\mathbb{E}_{k,\alpha,\beta}^{\gamma,q}(\varepsilon t^{-\sigma})) - f(t)}{\varepsilon}, \tag{2.2} $$

$\forall\, t > 0$ and ${}_{i,p}\mathbb{E}_{k,\alpha,\beta}^{\gamma,q}(.)$ is a truncated (p, k)-Mittag-Leffler function, as defined in Eq. (2.1).

Note that, if function f is σ-differentiable in some open interval $(0, a), a > 0$, and $\lim_{t\to 0^+}\left({}_{i,p}\mathscr{D}_{M,k,\alpha,\beta}^{\sigma,\gamma,q} f(t)\right)$ exist, then

$$ {}_{i,p}\mathscr{D}_{M,k,\alpha,\beta}^{\sigma,\gamma,q} f(0) = \lim_{t\to 0^+}\left({}_{i,p}\mathscr{D}_{M,k,\alpha,\beta}^{\sigma,\gamma,q} f(t)\right). \tag{2.3} $$

Theorem 1 *If a function f:*$[0, \infty) \to \mathbb{R}$ *is* σ*- differentiable for* $t_0 > 0$ *with* $0 < \sigma < 1; \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0; \Re(\beta) > 0;$ *and* $\Re(\gamma) > 0$, *then f is continuous at* t_0.

Proof Let us consider the identity

$$ f(t_0\ {}_{i,p}\mathbb{E}_{k,\alpha,\beta}^{\gamma,q}(\varepsilon t_0^{-\sigma})) - f(t_0) = \left(\frac{f(t_0\ {}_{i,p}\mathbb{E}_{k,\alpha,\beta}^{\gamma,q}(\varepsilon t_0^{-\sigma})) - f(t_0)}{\varepsilon}\right)\varepsilon \tag{2.4} $$

Applying the $\lim \varepsilon \to 0$ on both sides of Eq. (2.4), we get

$$\lim_{\varepsilon\to 0} f(t_0\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t_0^{-\sigma})) - f(t_0) = \lim_{\varepsilon\to 0}\left(\frac{f(t_0\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t_0^{-\sigma})) - f(t_0)}{\varepsilon}\right)\lim_{\varepsilon\to 0}\varepsilon$$
$$= {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t_0)\lim_{\varepsilon\to 0}\varepsilon$$
$$= 0. \tag{2.5}$$

Then, f is continuous at t_0.
Using the definition of truncated (p, k)-Mittag-Leffler function, we have

$$f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) = f\left(t\sum_{r=0}^{i}\frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^r}{r!}\right). \tag{2.6}$$

Applying the $\lim\limits_{\varepsilon\to 0}$ on both sides of (2.5), and since function f is a continuous function, we have

$$\lim_{\varepsilon\to 0} f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) = \lim_{\varepsilon\to 0} f\left(t\sum_{r=0}^{i}\frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^r}{r!}\right)$$
$$= f\left(t\lim_{\varepsilon\to 0}\sum_{r=0}^{i}\frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^r}{r!}\right). \tag{2.7}$$

Further, we have

$${}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma}) = \sum_{r=0}^{i}\frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^r}{r!}$$
$$= 1 + \frac{{}_p(\gamma)_{q,k}}{{}_p\Gamma_k(\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})}{1!} + \ldots + \frac{{}_p(\gamma)_{iq,k}}{{}_p\Gamma_k(i\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^i}{i!} \tag{2.8}$$

Applying the $\lim_{\varepsilon\to 0}$ it $\varepsilon \to 0$ on both sides of (2.8), we have

$$\lim_{\varepsilon\to 0}\sum_{r=0}^{i}\frac{{}_p(\gamma)_{rq,k}}{{}_p\Gamma_k(r\alpha+\beta)}\frac{(\varepsilon t^{-\sigma})^r}{r!} = 1 \tag{2.9}$$

In this way, we conclude from Eq. (2.5), that

$$\lim_{\varepsilon\to 0} f(t_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) = f(t) \tag{2.10}$$

Here, we present the theorem that encompasses the main classical properties of integer-order calculus. For the chain rule, it is verified through an example, as we will see next. We will do here, only the demonstration of the product and chain rule, for other items, follow the same steps of Theorem 2 found in the paper by Sousa and Oliveira [8].

Theorem 2 *Let* $0 < \sigma \leq 1; \alpha, \beta, \gamma \in \mathbb{C}$*; such that* $\Re(\alpha) > 0; \Re(\beta) > 0; \Re(\gamma) > 0$*; and* f, g *be* σ*-differentiable at a point* $t > 0$*. Then*

(1) (Linearity) ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(af+bg)(t) = a\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}f(t) + b\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}g(t)$.

(2) (Product rule) ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(f.g)(t) = f(t)\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}g(t) + g(t)\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)$.

Proof Using Definition 4, we have

$$
\begin{aligned}
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(f.g)(t) &= \lim_{\varepsilon\to 0}\frac{f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})).g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) - f(t).g(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\left(\frac{\begin{matrix} f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})).g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) + f(t)g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) \\ -f(t)g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) - f(t).g(t)\end{matrix}}{\varepsilon}\right)
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\left(\frac{f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) - f(t)}{\varepsilon}\right)\lim_{\varepsilon\to 0} g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) \\
&+ \lim_{\varepsilon\to 0}\left(\frac{g(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) - g(t)}{\varepsilon}\right)\lim_{\varepsilon\to 0} f(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma})) \\
&= g(t)_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}f(t) + f(t)_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}g(t).
\end{aligned}
$$

(3) ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left(\frac{f}{g}\right)(t) = \dfrac{g(t)\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}f(t) - f(t)\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}g(t)}{[g(t)]^2}$.

(4) ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(c) = 0$, where $f(t) = c$ is a constant.

(5) (Chain rule) If f is differentiable, then ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(f)(t) = \dfrac{{}_p(\gamma)_{q,k}\ t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)}\dfrac{df(t)}{dt}$

Proof From Eq. (2.8), we have

$$
t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{-\sigma}) = t + \frac{{}_p(\gamma)_{q,k}(\varepsilon t^{1-\sigma})}{{}_p\Gamma_k(\alpha+\beta)} + O(\varepsilon^2). \tag{2.12}
$$

and introducing the following change

$$h = \varepsilon t^{1-\sigma}\left(\frac{{}_p(\gamma)_{q,k}}{{}_p\Gamma_k(\alpha+\beta)} + O(\varepsilon)\right) \Rightarrow \varepsilon = \frac{h\,{}_p\Gamma_k(\alpha+\beta)}{t^{1-\sigma}\left({}_p(\gamma)_{q,k} + {}_p\Gamma_k(\alpha+\beta)O(\varepsilon)\right)}. \tag{2.13}$$

Employing the value of ε from above Eq. (2.13), the definition given in Eq. (2.2) reduces to the following form:

$$\begin{aligned} {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t) &= \lim_{\varepsilon\to 0} \frac{f(t+h)-f(t)}{h} \frac{t^{1-\sigma}\left({}_p(\gamma)_{q,k} + {}_p\Gamma_k(\alpha+\beta)O(\varepsilon)\right)}{{}_p\Gamma_k(\alpha+\beta)} \\ &= \frac{{}_p(\gamma)_{q,k}\, t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} \lim_{\varepsilon\to 0} \frac{f(t+h)-f(t)}{h} \\ &= \frac{{}_p(\gamma)_{q,k}\, t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} \frac{df(t)}{dt}, \end{aligned} \tag{2.14}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, and $t > 0$.

(6) ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(fog)(t) = f^{'}(g(t))\ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} g(t)$, for f is differentiable at $g(t)$.

Now, it is necessary to know if, in addition to the previous Theorem 2, that contains important properties similar to integer-order calculus, this truncated M-fractional derivative type Eq. (2.2), also has important theorems related to the classical calculus. We shall now see that Rolle's theorem and the mean value theorem and its extension coming from the integer-order calculus can be extended to σ-differentiable functions, i.e., that admit truncated M-fractional derivative as introduced in Eq. (2.2).

Theorem 3 *Let* $0 < \sigma < 1$; $\alpha, \beta, \gamma \in \mathbb{C}$*; such that* $\Re(\alpha) > 0$; $\Re(\beta) > 0$ *such that* $\Re(\gamma) > 0$*; and* f, g *be* σ*-differentiable at a point* $t > 0$*. Then*

$$\begin{aligned} &{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(1) = 0, \\ &{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(\exp(at)) = \frac{{}_p(\gamma)_{q,k}\, t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} a\exp(at), \\ &{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(\sin(at)) = \frac{{}_p(\gamma)_{q,k}\, t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} a\cos(at), \\ &{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(\cos(at)) = -\frac{{}_p(\gamma)_{q,k}\, t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} a\sin(at), \\ &{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left(\frac{t^\sigma}{\sigma}\right) = \frac{{}_p(\gamma)_{q,k}}{{}_p\Gamma_k(\alpha+\beta)}. \end{aligned} \tag{2.15}$$

Theorem 4 *Let* $0 < \sigma < 1$; $\alpha, \beta, \gamma \in \mathbb{C}$*; such that* $\Re(\alpha) > 0$; $\Re(\beta) > 0$; $\Re(\gamma) > 0$*; and* f, g *be* σ*-differentiable at a point* $t > 0$. *Then*

$$
\begin{aligned}
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left(\exp\left(\frac{t^\sigma}{\sigma}\right)\right) &= \frac{{}_p(\gamma)_{q,k}\,t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)}\exp\left(\frac{t^\sigma}{\sigma}\right),\\
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left(\sin\left(\frac{t^\sigma}{\sigma}\right)\right) &= \frac{{}_p(\gamma)_{q,k}\,t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)}\cos\left(\frac{t^\sigma}{\sigma}\right),\\
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left(\cos\left(\frac{t^\sigma}{\sigma}\right)\right) &= -\frac{{}_p(\gamma)_{q,k}\,t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)}\sin\left(\frac{t^\sigma}{\sigma}\right).
\end{aligned}
\tag{2.16}
$$

Theorem 5 *(Rolle's Theorem for Fractional* σ*-Differentiable Functions) Let* $a > 0$ *and* $f : [a; b] \to \mathbb{R}$ *be a function with the properties:*

1. f *is continuous on [a,b].*
2. f *is* σ*-differentiable on (a, b) for some* $\sigma \in (0, 1)$.
3. $f(a) = f(b)$.

Then, $\exists\, c \in (a, b)$*, such that* ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(c) = 0$*, with* $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0,$ *and* $\Re(\gamma) > 0$.

Proof Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there exist $c \in (a, b)$, at which the function has a local extreme. Then,

$$
\begin{aligned}
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(c) &= \lim_{\varepsilon\to 0^-}\frac{f(c\;{}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon c^{-\sigma})) - f(c)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0^+}\frac{f(c\;{}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon c^{-\sigma})) - f(c)}{\varepsilon}.
\end{aligned}
\tag{2.17}
$$

But, the two limits have opposite signs. Hence, ${}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(c) = 0$.

The proof of Theorems 6 and 7, will be omitted, but follow the same reasoning of the respective theorems demonstrated in Sousa and Oliveira [8].

Theorem 6 *(Mean Value Theorem for Fractional* σ*-Differentiable Functions) Let* $a > 0$ *and* f *:*$[a; b] \to \mathbb{R}$ *be a function with the properties:*

1. *f is continuous on [a, b].*
2. *f is* σ*-differentiable on (a, b) for some* $\sigma \in (0, 1)$.

Then, $\exists\, c \in (a, b)$*, such that*

$$
{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(c) = \frac{f(b) - f(a)}{\frac{b^\sigma}{\sigma} - \frac{a^\sigma}{\sigma}}
\tag{2.18}
$$

with $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0,$ *and* $\Re(\gamma) > 0$.

Theorem 7 *(Extension Mean Value Theorem for Fractional σ-Differentiable Functions) Let $a > 0$ and $f, g : [a, b] \to \mathbb{R}$ function that satisfy:*

1. *f, g is continuous on $[a, b]$.*
2. *f, g is σ-differentiable on (a, b) for some $\sigma \in (0, 1)$.*

Then, $\exists\, c \in (a, b)$, such that

$$\frac{{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(c)}{{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \tag{2.19}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\Re(\gamma) > 0$.

Definition 5 Let $\sigma \in (n, n + 1]$, for some $n \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{C}$; such that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and f is n times differentiable for $t > 0$. Then the σ-fractional derivative of f is defined by

$${}_{i,p}\mathscr{D}^{\sigma,\gamma;n}_{M,k,\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t\ {}_{i,p}\mathbb{E}^{\gamma,q}_{k,\alpha,\beta}(\varepsilon t^{n-\alpha})) - f^{(n)}(t)}{\varepsilon}, \tag{2.20}$$

since the $\lim\limits_{\varepsilon \to 0}$ on it exist.

From Definition 2, and the chain rule, that is, from item 5 of Theorem 2, by induction on n, we can prove that

$${}_{i,p}\mathscr{D}^{\sigma,\gamma;n}_{M,k,\alpha,\beta} f(t) = \frac{{}_p(\gamma)_{q,k} t^{n+1-\sigma}}{{}_p\Gamma_k(\alpha + \beta)} f^{(n+1)}(t),$$

$\sigma \in (n, n + 1]$ and f is $(n + 1)$-differentiable for $t > 0$.

Now, we know that this truncated M-fractional derivative type Eq. (2.2), has a corresponding M-fractional integral. Then, we will present the definition and a theorem that corresponds to the inverse property. For other results involving integrals, one can consult [8, 9].

3 Generalized M-Integral

Definition 6 Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined in (a, t] and $0 < \sigma < 1$. Then, the M-fractional integral of order σ of function f is defined by [8]

$${}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t) = \frac{{}_p\Gamma_k(\alpha + \beta)}{(\gamma)_{i,k}} \int_\alpha^t \frac{f(x)}{x^{1-\sigma}} dx, \tag{3.1}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\Re(\gamma) > 0$.

Theorem 8 (Inverse) *Let $a \geq 0$, $0 < \sigma < 1$, and let f be a continuous function such that ${}_i\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)$ exist. Then*

$$ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left({}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) = f(t) \tag{3.2} $$

with $t \geq \alpha$ and $\alpha, \beta, \gamma \in \mathbb{C}$; such that $\Re(\alpha) > 0, \Re(\beta) > 0$, and $\Re(\gamma) > 0$.

Proof In fact, using the chain rule as seen in Theorem 2, we have

$$ \begin{aligned} {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left({}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) &= \frac{{}_p(\gamma)_{q,k} t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} \frac{d}{dt}\left({}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) \\ &= \frac{{}_p(\gamma)_{q,k} t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} \frac{d}{dt}\left(\frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}} \int_a^t \frac{f(x)}{x^{1-\sigma}} dx\right) \\ &= \frac{{}_p(\gamma)_{q,k} t^{1-\sigma}}{{}_p\Gamma_k(\alpha+\beta)} \left(\frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k} t^{1-\sigma}} f(t)\right) \\ &= f(t). \end{aligned} \tag{3.3} $$

With the condition $f(a) = 0$, by Theorem 8, that is, 3.3, we have ${}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left[{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right] = f(t)$

Theorem 9 (Fundamental Theorem of Calculus) *Let $f : (a,b) \to \mathbb{R}$ be an α-differentiable function and $0 < \alpha \leq 1$. Then, for all $t > 0$, we have*

$$ {}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left({}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) = f(t) - f(a), \tag{3.4} $$

with $\beta > 0$.

Proof In fact, since function f is differentiable, using the chain rule of Theorem 2, and the fundamental theorem of calculus for the integer-order derivative, we have

$$ \begin{aligned} {}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left({}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) &= \frac{{}_p\Gamma_k(\alpha+\beta)}{(\gamma)_{i,k}} \int_a^t \frac{{}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)}{x^{1-\sigma}} dx \\ &= \frac{{}_p\Gamma_k(\alpha+\beta)}{(\gamma)_{i,k}} \int_a^t \frac{(\gamma)_{i,k} x^{1-\sigma}}{\Gamma(\beta+1) x^{1-\sigma}} \frac{df(t)}{dt} dx \\ &= \int_a^t \frac{df(t)}{dt} dx = f(t) - f(a), \end{aligned} \tag{3.5} $$

If the condition $f(a) = 0$ holds, then by Theorem 9, we have ${}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}\left({}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} f(t)\right) = f(t)$

4 Relation with Other Fractional Derivative Types

In this section, we will discuss the relationship between the fractional conformable derivative proposed by Khalil et al. [10], the alternative fractional derivative and the generalized alternative fractional derivative proposed by Katugampola [11], and the M-fractional derivative proposed by Sousa and Oliveira [8], with our truncated M-fractional derivative type.

Khalil et al. [10], proposed a definition of a fractional derivative, called conformable fractional derivative that refers to the classical properties of integer order calculus, given by

$$f^{(\sigma)}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\sigma}) - f(t)}{\varepsilon}, \tag{4.1}$$

with $\sigma \in (0, 1)$ and $t > 0$.

In 2014, Katugampola [11], proposed another definition of a fractional derivative, called an alternative fractional derivative which also refers to the classical properties of integer-order calculus, given by

$$\mathcal{D}^{\sigma} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\sigma}}\right) - f(t)}{\varepsilon}, \tag{4.2}$$

with $\sigma \in (0, 1)$ and $t > 0$.

In the same paper, Katugampola [11], by means of a truncated exponential function, that is, ${}_r e^x$, proposed another generalized fractional derivative, given by

$$\mathcal{D}_r^{(\sigma)} f(t) = \lim_{\varepsilon \to 0} \frac{f\left({}_r e^{\varepsilon t^{-\sigma}} t\right) - f(t)}{\varepsilon}, \tag{4.3}$$

with $\sigma \in (0, 1)$ and $t > 0$.

Recently, Sousa and Oliveira [8], introduced the M-fractional derivative $\mathscr{D}_M^{\alpha,\beta}$ where the parameter $\beta > 0$ and M is the notation to designate that the function to be derived involves the Mittag-Leffler function of one parameter, given by

$$\mathscr{D}_M^{\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f(tE_{\beta}(\varepsilon t^{\sigma}) - f(t)}{\varepsilon}, \tag{4.4}$$

$\sigma \in (0, 1)$ and $t > 0$.

It is clear that our definition of truncated M-fractional derivative type (2.2), is more general than the fractional derivative Eqs. (4.1), (4.2), (4.3), and (4.4).

5 Application

5.1 Solution Heat Equation

In this section, we obtain the solution of the heat equation using a truncated M-fractional derivative type with $0 < \alpha < 1$ and present some graphs about the behavior of the solution. Consider the heat equation in one dimension given by

$$\frac{\partial u(x,t)}{\partial t} = \omega \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < L, t > 0, \tag{5.1}$$

where ω is a positive constant. Using a M-fractional derivative type, we propose an M-fractional heat equation given by

$$_p\left(\frac{\partial}{\partial t}\right)^{\sigma,\gamma,q}_{M,k,\alpha,\beta} u(x,t) = \omega \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < L, t > 0, \tag{5.2}$$

where $0 < \alpha < 1$ and with the initial condition and boundary conditions given by

$$u(0,t) = 0, t \geq 0,$$

$$u(L,T) = 0, t \geq 0,$$

$$u(x,0) = f(x), 0 \leq x \leq L$$

We start considering the so-called M-fractional linear differential equation with constant coefficients

$$_p\left(\frac{\partial}{\partial t}\right)^{\sigma,\gamma,q}_{M,k,\alpha,\beta} v(x,t) \pm \mu^2 v(x,t) = 0, \tag{5.3}$$

where μ^2 is a positive constant.

Using the item 5 in Theorem 2, (5.2), can be written as follows:

$$\frac{_p(\gamma)_{q,k}\, t^{1-\sigma}}{_p\Gamma_k(\alpha+\beta)} \frac{dv(x,t)}{dt} \pm \mu^2 v(x,t) = 0,$$

whose solution is given by

$$v(t) = c \exp\left(\pm \frac{_p\Gamma_k(\alpha+\beta)}{_p(\gamma)_{q,k}} \frac{\mu^2 t^\sigma}{\sigma}\right), \tag{5.4}$$

with $0 < \alpha < 1$ and $\beta > 0$.

Now, we will use separation of variables method to obtain the solution of the M-fractional heat equation. Then, considering $u(x, t) = P(x)Q(t)$ and replacing in (5.2), we get

$$_p\left(\frac{\partial}{\partial t}\right)^{\sigma,\gamma,q}_{M,k,\alpha,\beta} Q(t)P(x) = \omega\left(\frac{d}{dx}\right)^2 P(x)Q(t),$$

which implies

$$\frac{1}{\omega Q(t)}\ _p\left(\frac{\partial}{\partial t}\right)^{\sigma,\gamma,q}_{M,k,\alpha,\beta} Q(t) = \frac{1}{P(x)}\left(\frac{d}{dx}\right)^2 P(x) = \alpha. \tag{5.5}$$

From (5.5), we obtain a system of differential equations, given by

$$_p\left(\frac{\partial}{\partial t}\right)^{\sigma,\gamma,q}_{M,k,\alpha,\beta} Q(t) - \omega\alpha Q(x) = 0. \tag{5.6}$$

$$\left(\frac{d}{dx}\right)^2 P(x) - \alpha P(x) = 0. \tag{5.7}$$

First, let's find the solution of (5.7). For this, we must study three cases, that is, $\xi = 0$ $\xi = -\mu^2$ and $\xi = -\mu^2$.

Case 1 $\xi = 0$. Substituting $\xi = 0$ into (5.7), we have

$$\left(\frac{d}{dx}\right)^2 P(x) - \xi P(x) = 0 \tag{5.8}$$

whose solution is given by $P(x) = c_1 x + c_2$, with c_1 and c_2 arbitrary constant. Using the initial conditions given by Eq. (5.2), we obtain that $c_1 = c_2 = 0$. Like this, $P(x) = 0$, which implies $u(x, t) = 0$ trivial solution.

Case 2 $\xi = -\mu^2$ Substituting $\xi = -\mu^2$ into (5.7), we get

$$\left(\frac{d}{dx}\right)^2 P(x) + \xi^2 P(x) = 0 \tag{5.9}$$

whose solution is given by $P(x) = c_2 \sin(\mu x) + c_1 \cos(\mu x)$, with c_1 and c_2 arbitrary constant. Using the initial conditions Eq. (5.2), we obtain $c_1 = 0$ and $0 = c_2 \sin(\mu x)$ which implies that $\mu = \dfrac{n\pi}{L}$ with n=1,2,...L ; with n = 1, 2, ... Then, we obtain

$$P_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right) \text{ and } \mu = \frac{n\pi}{L} \tag{5.10}$$

Case 3 $\xi = \mu^2$ Substituting $\xi = \mu^2$ into (5.7), we get

$$\left(\frac{d}{dx}\right)^2 P(x) - \xi^2 P(x) = 0 \tag{5.11}$$

whose solution is given by $P(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} = A\cosh(\mu x) + B\sinh(\mu x)$, with c_1, c_2, A, B arbitrary constant. Using the boundary conditions (5.3), we have $A = 0$ and $0 = B\sinh(\mu x)$. As $\lambda = -\mu^2 < 0$ and $\lambda L \neq 0$, then $\sinh(\mu x) \neq 0$. Like this, we get B = 0 and then $P_n(x) = 0$, which implies u(x, t) = 0, trivial solution.

Therefore, the solution of (5.7) is given by

$$P_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right) \text{ and } \mu = \frac{n\pi}{L} \tag{5.12}$$

Using (5.4) and (5.5) in Eq. (5.6), we have

$$Q_n(t) = b_n \exp\left(-\frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}}\left(\frac{n\pi}{L}\right)^2 \frac{\omega}{\sigma} t^{\sigma}\right) \tag{5.13}$$

where b_n are constant coefficients.

So, using (5.9), and (5.11), the partial solutions of (5.2), is given by

$${}_p u^{\gamma,q}_{M,k,\alpha,\beta}(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}}\left(\frac{n\pi}{L}\right)^2 \frac{\omega}{\sigma} t^{\sigma}\right). \tag{5.14}$$

Using Eq. (5.3), we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

which provides c_n through

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

So, we conclude that the solution of M-fractional heat equation (5.2), satisfying the conditions (5.3), is given by

$$\begin{aligned} {}_p u^{\gamma,q}_{M,k,\alpha,\beta}(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx\right] \sin\left(\frac{n\pi x}{L}\right) \\ \times \exp\left(-\frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}}\left(\frac{n\pi}{L}\right)^2 \frac{\omega}{\sigma} t^{\sigma}\right). \end{aligned} \tag{5.15}$$

Taking the Limit $\alpha, \beta, \gamma, k \to 1$, in the last equation and using (2.2), we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \frac{\omega}{\alpha} t^{\alpha}\right)\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right), \tag{5.16}$$

which is exactly the solution of the fractional heat equation proposed by Cenesiz et al.

Further, if we choose $\sigma = 1$, the above Eq. (5.16), produces the solution of heat equation of integral order.

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \omega t\right)\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right). \tag{5.17}$$

5.1.1 Graphical and Numerical Results

Graphical presentations of the solution in Eqs. (5.14) and (5.17) are established in Figs. 1 and 2 respectively. We choose here the values of function and parameters as $f(x) = log(x)$; $\sigma = 0.5$ and $f(x) = exp(-x)$; $\sigma = 1.5$ with $y = 1.5$, $q = 3$, $k = 0.2$, $\alpha = 0.2$, $\beta = 0.3$, $\omega = 0.5$ for Eq. (5.14) and $f(x) = (1 + x + 2x2)\sigma = 1.5$, $y = 2.5$, $q = 3$, $k = 0.2$, $\alpha = 0.2$, $\beta = 0.3$, $\omega = 0.5$ for the Eq. (5.17). Also we present here numerical results of our finding in Eq. (5.17) for different values of parameters as shown in Table 1.

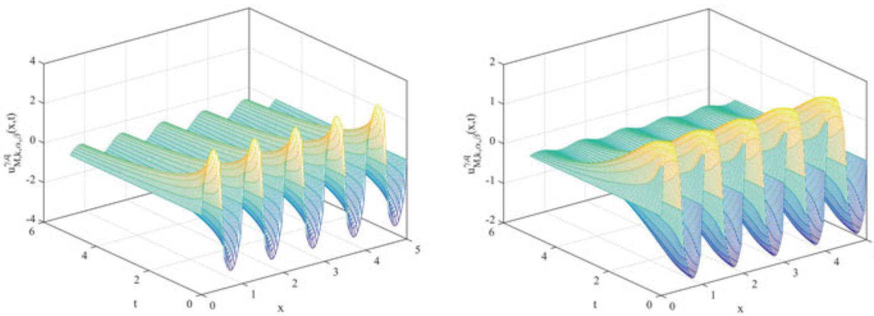

Fig. 1 Graph for $\gamma = 1.5, q = 3, k = 0.2, \alpha = 0.2, \beta = 0.3, \omega = 0.5$. (**a**) Plot for $f(x) = log(x)$; $\sigma = 0.5$. (**b**) Plot for $f(x) = exp(-x)$; $\sigma = 1.5$

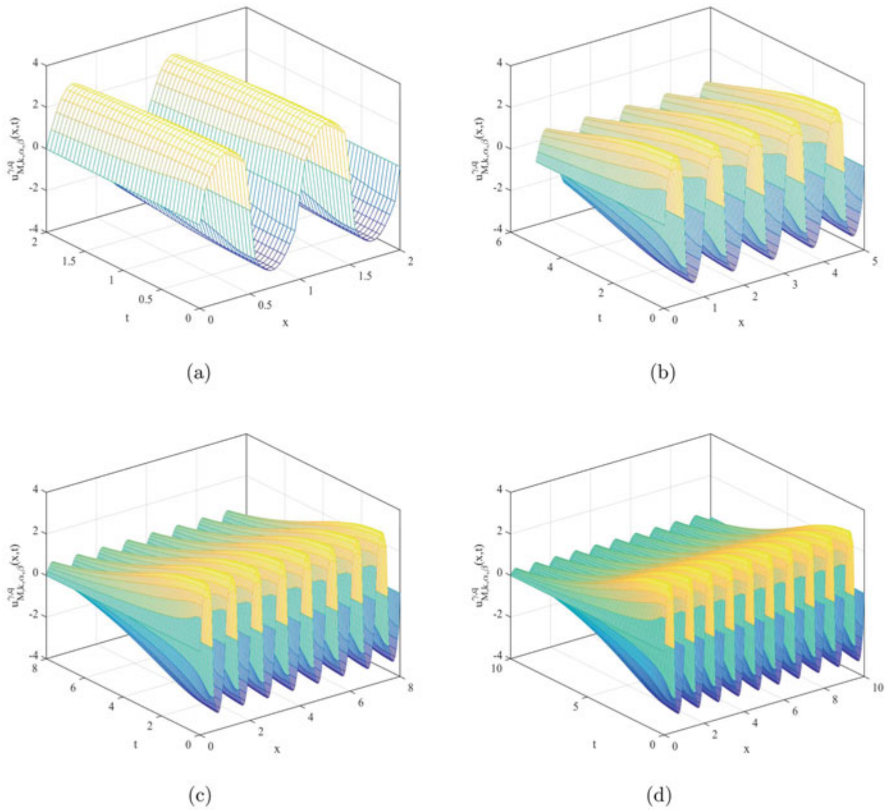

Fig. 2 Graph for $f(x) = (1 + x + 2x^2)$ and $\sigma = 1.5, \gamma = 2.5, q = 3, k = 0.2, \alpha = 0.2, \beta = 0.3, \omega = 0.5$. (**a**) $x, t = 0 : .05 : 2$. (**b**) $x, t = 0 : .05 : 5$. (**c**) $x, t = 0 : .05 : 8$. (**d**) $x, t = 0 : .05 : 10$

5.2 Solution First-Order Differential Equation

The general first-order differential equation based on generalized M-derivative is represented as

$$ {}_{i,p}\mathscr{D}^{\sigma,\gamma,q}_{M,k,\alpha,\beta} u(t) + P(t)u(t) = Q(t), \tag{5.18} $$

where $P(t)$ and $Q(t)$ are $\sigma-$differentiable function and $u(t)$ is unknown.

Using the chain rule from Theorem 2, the above Eq. (5.18), reduces to

$$ \frac{d}{dt}u(t) + \frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}\, t^{1-\sigma}} P(t)u(t) = \frac{{}_p\Gamma_k(\alpha+\beta)}{{}_p(\gamma)_{q,k}\, t^{1-\sigma}} Q(t), \tag{5.19} $$

Table 1 $f(x) = (1 + x + 2x^2)$ and $\gamma = 2.5, q = 3, k = 0.2, \alpha = 0.2, \beta = 0.3, \omega = 0.5; L = 10$

x	t	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$	$\sigma = 0.7$	$\sigma = 0.9$
0.0	0.0	0.000	0.000	0.000	0.000	0.000
0.5	0.5	15.081	15.063	15.052	15.048	15.045
1.0	1.0	29.804	29.775	29.769	29.766	29.765
1.5	1.5	44.854	44.848	44.847	44.847	44.846
2.0	2.0	59.971	59.988	59.991	59.992	59.993
2.5	2.5	74.885	74.921	74.928	74.930	74.931
3.0	3.0	89.351	89.403	89.413	89.416	89.416
3.5	3.5	103.161	103.226	103.237	103.240	103.240
4.0	4.0	116.148	116.222	116.235	116.237	116.236
4.5	4.5	128.189	128.269	128.282	128.283	128.281
5.0	5.0	139.211	139.292	139.304	139.304	139.300
5.5	5.5	149.187	149.265	149.276	149.276	149.270
6.0	6.0	158.143	158.215	158.224	158.223	158.216
6.5	6.5	166.154	166.216	166.224	166.222	166.215
7.0	7.0	173.343	173.392	173.398	173.396	173.389
7.5	7.5	179.878	179.912	179.915	179.913	179.908
8.0	8.0	185.967	185.983	185.984	185.983	185.980
8.5	8.5	191.855	191.850	191.850	191.850	191.851
9.0	9.0	197.757	197.788	197.787	197.789	197.794
9.5	9.5	189.399	201.524	202.030	201.007	198.389
10.0	10.0	0.000	0.000	0.000	0.000	0.000

The solution of the above Eq. (5.19), is given by

$$u(t) = \exp\left(-\frac{{}_p\Gamma_k(\alpha+\beta)}{p(\gamma)_{q,k}} \int \frac{P(t)}{t^{1-\sigma}} dt\right) \times \left\{ \frac{{}_p\Gamma_k(\alpha+\beta)}{p(\gamma)_{q,k}} \int \frac{Q(t)}{t^{1-\sigma}} \exp\left(\frac{{}_p\Gamma_k(\alpha+\beta)}{p(\gamma)_{q,k}} \int \frac{P(t)}{t^{1-\sigma}} dt\right) dt + C \right\}. \tag{5.20}$$

Further, using the definition of integral operator, we can obtain from the above the equation

$$u(t) = \exp\left(-{}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(P(t))\right) \times \left\{ {}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(Q(t)) \exp\left({}_{a,i}\mathscr{I}^{\sigma,\gamma,q}_{M,k,\alpha,\beta}(P(t))\right) + C \right\}. \tag{5.21}$$

6 Conclusion

We conclude the present study by remarking that Mittag-Leffler functions play a very vital role in determining the solution of fractional differential and integral equations which are associated with an extensive variety of problems in diverse areas of sciences and engineerings [15–24]. All the finding in this paper are general in nature. Various results can be easily obtained by employing the particular values to the parameters involving in our findings.

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On Hyers-Ulam-Rassias Stability of a Volterra-Hammerstein Functional Integral Equation

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Abstract The aim of this paper is to study the Hyers-Ulam-Rassias stability for a Volterra-Hammerstein functional integral equation in three variables via Picard operators.

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1 Introduction

Ulam stability is an important concept in the theory of functional equations. The origin of Ulam stability theory was an open problem formulated by Ulam, in 1940, concerning the stability of homomorphism [46]. The first partial answer to Ulam's question came within a year, when Hyers [8] proved a stability result, for the additive Cauchy equation in Banach spaces. The first result on Hyers-Ulam stability of differential equations was given by Obloza [36]. Alsina and Ger investigated the stability of the differential equation $y' = y$ [2]. The result of Alsina and Ger was extended by many authors (cf. [6, 13, 16–18, 38–40, 42, 44, 45]) to the stability of the first-order linear differential equation and linear differential equations of higher order. For a broader study of Hyers-Ulam stability for functional

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equations, the reader is also referred to the following books and papers: [1, 5, 9–12, 19, 24, 25, 33, 34, 37, 46].

The first result proved on the Hyers-Ulam stability of partial differential equations is due to A. Prastaro and Th.M. Rassias [41]. Some results regarding Ulam-Hyers stability of partial differential equations were given by S.-M. Jung [10]; S.-M. Jung and K.-S. Lee [15]; N. Lungu and S.A. Ciplea [26]; N. Lungu and D. Popa [29–31]; N. Lungu and C. Craciun [27]; N. Lungu and D. Marian [28]; D Marian, S.A. Ciplea, and N. Lungu [32]; and I.A. Rus and N. Lungu [43]. In [4], Brzdek, Popa, Rasa, and Xu presented a unified and systematic approach to the field. Some recent results regarding stability analysis and their applications were established by H. Khan, A. Khan, T. Abdeljawad, and A. Alkhazzan [20]; A. Khan, J.F. Gómez-Aguilar, T.S. Khan, and H. Khan [21]; H. Khan, T. Abdeljawad, M. Aslam, R.A. Khan, and A. Khan [22]; and H. Khan, J.F. Gómez-Aguilar, A. Khan, and T.S. Khan [23]. Results regarding fixed point theory and the Ulam stability can be found in [3].

In this paper, we consider the following Volterra-Hammerstein functional integral equation in three variables:

$$\begin{aligned} u(x,y,z) &= g(x,y,z,h(u)(x,y,z)) \\ &+ \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,f_1(u)(r,s,t))\,drdsdt \\ &\quad + \int_0^\infty\int_0^\infty\int_0^\infty F(x,y,z,r,s,t,f_2(u)(r,s,t))\,drdsdt \end{aligned} \tag{1}$$

via Picard operators.

The present paper is motivated by a recent paper [35] of L.T.P. Ngoc, T.M. Thuyet, and N.T. Long in which is studied the existence of asymptotically stable solution for a Volterra-Hammerstein integral equation in three variables. Equation (1) is a generalization of equation (1.1) from [35].

2 Existence and Uniqueness

In what follows, we consider some conditions relative to Eq. (1).

Let $(E, |\cdot|)$ be a Banach space and

$$\Delta = \left\{(x,y,z,r,s,t) \in \mathbb{R}_+^6 : r \le x, s \le y, t \le z\right\}.$$

Let $\tau > 0$ and the set

$$X_\tau := \left\{u \in C\left(\mathbb{R}_+^3, E\right) \mid \exists M(u) > 0 : |u(x,y,z)|\, e^{-\tau(x+y+z)} \le M(u), \forall (x,y,z) \in \mathbb{R}_+\right\}.$$

On the set X_τ, we consider Bielecki's norm

$$\|u\|_\tau := \sup_{x,y,z\in R_+} \left(|u(x,y,z)|\, e^{-\tau(x+y+z)} \right).$$

It is clear that $(X_\tau, \|\cdot\|_\tau)$ is a Banach space. In what follows, we assume, relative to (1), the conditions

(C1) $g \in C\left(\mathbb{R}_+^3 \times E, E\right), K \in C(\Delta \times E, E), F \in C(\Delta \times E, E), h \in C(X_\tau, X_\tau), f_1 \in C(X_\tau, X_\tau), f_2 \in C(X_\tau, X_\tau);$

(C2) there exists $l_h > 0$ such that

$$|h(u)(x,y,z) - h(v)(x,y,z)| \le l_h \|u - v\|_\tau\, e^{\tau(x+y+z)}, \forall x,y,z \in \mathbb{R}_+, \forall u, v \in X_\tau;$$

(C3) there exists $l_g > 0$ such that

$$|g(x,y,z,e_1) - g(x,y,z,e_2)| \le l_g |e_1 - e_2|, \forall x,y,z \in \mathbb{R}_+, \forall e_1, e_2 \in E;$$

(C4) there exists $l_K \in C(\Delta, \mathbb{R}_+)$ such that

$$|K(x,y,z,r,s,t,e_3) - K(x,y,z,r,s,t,e_4)| \le l_K(x,y,z,r,s,t)\,|e_3 - e_4|,$$

$\forall (x,y,z,r,s,t) \in \Delta, \forall e_3, e_4 \in E;$

(C5) there exists $l_F \in C(\Delta, \mathbb{R}_+)$ such that

$$|F(x,y,z,r,s,t,e_5) - F(x,y,z,r,s,t,e_6)| \le l_F(x,y,z,r,s,t)\,|e_5 - e_6|,$$

$\forall (x,y,z,r,s,t) \in \Delta, \forall e_5, e_6 \in E;$

(C6) there exist $l_{f_1} > 0$ and $l_{f_2} > 0$ such that

$$|f_1(u)(r,s,t) - f_1(v)(r,s,t)| \le l_{f_1} |u(r,s,t) - v(r,s,t)|, \forall (r,s,t) \in \mathbb{R}_+^3, \forall u, v \in X_\tau,$$

$$|f_2(u)(r,s,t) - f_2(v)(r,s,t)| \le l_{f_2} |u(r,s,t) - v(r,s,t)|, \forall (r,s,t) \in \mathbb{R}_+^3, \forall u, v \in X_\tau;$$

(C7) there exist $l_1 > 0$ and $l_2 > 0$ such that

$$\int_0^x\int_0^y\int_0^z l_{f_1} l_K(x,y,z,r,s,t)\, e^{\tau(r+s+t)} drdsdt \le l_1 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty l_{f_2} l_F(x,y,z,r,s,t)\, e^{\tau(r+s+t)} drdsdt \le l_2 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta;$$

(C8) $l_g l_h + l_1 + l_2 < 1;$

(C9)

$$|g(x,y,z,h(u)(x,y,z))| + \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,f_1(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty |F(x,y,z,r,s,t,f_2(0)(r,s,t))|\,drdsdt \le \alpha\exp(\tau(x+y+z)),$$

$\forall(x,y,z,r,s,t)\in\Delta$;

(C10) there exists $m>0$ such that

$$\int_0^\infty\int_0^\infty\int_0^\infty \left[l_{f_1}l_K(x,y,z,r,s,t) + l_{f_2}l_F(x,y,z,r,s,t)\right]drdsdt \le m, \forall(x,y,z,r,s,t)\in\Delta.$$

Theorem 2.1 *Under the conditions* $(C1)-(C9)$, *Eq.* (1) *has in the set* X_τ *a unique solution* u^*.

Proof We consider the operator

$$A: X_\tau \to X_\tau, A(u)(x,y,z) := \text{ second part of (1)}.$$

First, we prove that $A(u)$ maps X_τ in X_τ. For $u\in X_\tau$, we have

$$|A(u)(x,y,z)| \le |g(x,y,z,h(u)(x,y,z))| + \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,f_1(u)(r,s,t))|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty |F(x,y,z,r,s,t,f_2(u)(r,s,t))|\,drdsdt \le |g(x,y,z,h(u)(x,y,z))|$$
$$+ \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,f_1(u)(r,s,t)) - K(x,y,z,r,s,t,f_1(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty |F(x,y,z,r,s,t,f_2(u)(r,s,t)) - F(x,y,z,r,s,t,f_2(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,f_1(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty |F(x,y,z,r,s,t,f_2(0)(r,s,t))|\,drdsdt.$$

We obtain

$$|A(u)(x,y,z)| \le |g(x,y,z,h(u)(x,y,z))| + \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,f_1(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty |F(x,y,z,r,s,t,f_2(0)(r,s,t))|\,drdsdt$$
$$+ \int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)\,|f_1(u)(r,s,t) - f_1(0)(r,s,t)|\,drdsdt$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)\,|f_2(u)(r,s,t) - f_2(0)(r,s,t)|\,drdsdt$$

$$\leq \alpha \exp\left(\tau\left(x+y+z\right)\right)+\int_0^x\int_0^y\int_0^z l_K\left(x,y,z,r,s,t\right)l_{f_1}\left|u\left(r,s,t\right)\right|e^{\tau(r+s+t)}\cdot e^{-\tau(r+s+t)}drdsdt$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty l_F\left(x,y,z,r,s,t\right)l_{f_2}\left|u\left(r,s,t\right)\right|e^{\tau(r+s+t)}\cdot e^{-\tau(r+s+t)}drdsdt$$

$$\leq \alpha \exp\left(\tau\left(x+y+z\right)\right)+\|u\|_\tau\int_0^x\int_0^y\int_0^z l_K\left(x,y,z,r,s,t\right)l_{f_1}e^{\tau(r+s+t)}drdsdt$$

$$+\|u\|_\tau\int_0^\infty\int_0^\infty\int_0^\infty l_F\left(x,y,z,r,s,t\right)l_{f_2}e^{\tau(r+s+t)}drdsdt.$$

Thus, we obtain

$$\left|A\left(u\right)\left(x,y,z\right)\right|\leq\left[\alpha+\|u\|_\tau\left(l_1+l_2\right)\right]\exp\left(\tau\left(x+y+z\right)\right),$$

and hence $A(u)\in X_\tau$.

The operator A is a contraction in X_τ with respect to $\|\cdot\|_\tau$. Indeed, for $u,v\in X_\tau$, we have

$$\left|A\left(u\right)\left(x,y,z\right)-A\left(v\right)\left(x,y,z\right)\right|\leq\left|g\left(x,y,z,h\left(u\right)\left(x,y,z\right)\right)-g\left(x,y,z,h\left(v\right)\left(x,y,z\right)\right)\right|$$

$$+\int_0^x\int_0^y\int_0^z\left|K\left(x,y,z,r,s,t,f_1\left(u\right)\left(r,s,t\right)\right)-K\left(x,y,z,r,s,t,f_1\left(v\right)\left(r,s,t\right)\right)\right|drdsdt$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\left|F\left(x,y,z,r,s,t,f_2\left(u\right)\left(r,s,t\right)\right)-F\left(x,y,z,r,s,t,f_2\left(v\right)\left(r,s,t\right)\right)\right|drdsdt$$

$$\leq l_g\left|h\left(u\right)\left(x,y,z\right)-h\left(v\right)\left(x,y,z\right)\right|$$

$$+\int_0^x\int_0^y\int_0^z l_K\left(x,y,z,r,s,t\right)\left|f_1\left(u\right)\left(r,s,t\right)-f_1\left(v\right)\left(r,s,t\right)\right|drdsdt$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty l_F\left(x,y,z,r,s,t\right)\left|f_2\left(u\right)\left(r,s,t\right)-f_2\left(v\right)\left(r,s,t\right)\right|drdsdt$$

$$\leq l_gl_h\|u-v\|_\tau e^{\tau(x+y+z)}+\int_0^x\int_0^y\int_0^z l_K\left(x,y,z,r,s,t\right)l_{f_1}\|u-v\|_\tau e^{\tau(r+s+t)}drdsdt$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty l_F\left(x,y,z,r,s,t\right)l_{f_2}\|u-v\|_\tau e^{\tau(r+s+t)}drdsdt$$

$$\leq l_gl_h\|u-v\|_\tau e^{\tau(x+y+z)}+l_1\|u-v\|_\tau e^{\tau(x+y+z)}+l_2\|u-v\|_\tau e^{\tau(x+y+z)}.$$

Then we get

$$\|A\left(u\right)-A\left(v\right)\|_\tau\leq\left(l_gl_h+l_1+l_2\right)\|u-v\|_\tau$$

for all $u,v\in X_\tau$. From $(C8)$, we have that A is a contraction. Hence, A is a $c-$Picard operator, with

$$c=\frac{1}{1-l_gl_h-l_1-l_2}.$$

Hence, Eq. (1) has a unique solution in the set X_τ.

3 Hyers-Ulam-Rassias stability

In what follows, we consider the equation

$$
\begin{aligned}
u(x,y,z) &= g(x,y,z,h(u)(x,y,z)) \\
&+ \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,\ f_1(u)(r,s,t))\,drdsdt \\
&\quad + \int_0^\infty\int_0^\infty\int_0^\infty F(x,y,z,r,s,t,\ f_2(u)(r,s,t))\,drdsdt
\end{aligned} \tag{2}
$$

and the inequality

$$
\begin{aligned}
&|u(x,y,z) - g(x,y,z,h(u)(x,y,z)) \\
&- \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,\ f_1(u)(r,s,t))\,drdsdt - \\
&- \left.\int_0^\infty\int_0^\infty\int_0^\infty F(x,y,z,r,s,t,\ f_2(u)(r,s,t))\,drdsdt\right| \le \varphi(x,y,z),
\end{aligned} \tag{3}
$$

where $(E, |\cdot|)$ is a Banach space and $\varphi \in C\left([0,a)^3, \mathbb{R}_+\right)$ is increasing, $g \in C\left([0,a)^3 \times E, E\right)$, $K \in C\left([0,a)^6 \times E, E\right)$, $F \in C\left([0,a)^6 \times E, E\right)$, $h \in C(X_\tau, X_\tau)$, $f_1 \in C(X_\tau, X_\tau)$, $f_2 \in C(X_\tau, X_\tau)$.

Theorem 3.1 *Under the conditions* $(C1) - (C10)$ *and*

(i) *there exists* $N > 0$ *such that*

$$
\begin{aligned}
|h(u)(x,y,z) - h(v)(x,y,z)| &\le N\,|u(x,y,z) \\
&-v(x,y,z)|,\ \forall x,y,z \in [0,a),\ \forall u,v \in X_\tau;
\end{aligned}
$$

(ii) $l_g N < 1$,

if u is a solution of (3) *and* u^* *is the unique solution of* (2)*, we have*

$$
\left|u(x,y,z) - u^*(x,y,z)\right| \le C_{KFghf_1f_2}\varphi(x,y,z)
$$

where

$$
C_{KFghf_1f_2}\varphi(x,y,z) = \frac{1}{1-l_gN}\exp\left(\frac{m}{1-l_gN}\right),
$$

i.e., Eq. (2) *is Hyers-Ulam-Rassias stable.*

Proof We have

$$\left|u(x,y,z)-u^*(x,y,z)\right|$$
$$\leq \left|u(x,y,z)-g(x,y,z,h(u)(x,y,z))-\int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,f_1(u)(r,s,t))\,drdsdt\right.$$
$$\left.-\int_0^\infty\int_0^\infty\int_0^\infty F(x,y,z,r,s,t,f_2(u)(r,s,t))\,drdsdt\right|$$
$$+\left|g(x,y,z,h(u)(x,y,z))-g\left(x,y,z,h\left(u^*\right)(x,y,z)\right)\right|$$
$$+\int_0^x\int_0^y\int_0^z \left|K(x,y,z,r,s,t,f_1(u)(r,s,t))-K\left(x,y,z,r,s,t,f_1\left(u^*\right)(r,s,t)\right)\right|drdsdt$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty \left|F(x,y,z,r,s,t,f_2(u)(r,s,t))-F\left(x,y,z,r,s,t,f_2\left(u^*\right)(r,s,t)\right)\right|drdsdt$$
$$\leq \varphi(x,y,z)+l_g\left|h(u)(x,y,z)-h\left(u^*\right)(x,y,z)\right|$$
$$+\int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)\left|f_1(u)(r,s,t)-f_1\left(u^*\right)(r,s,t)\right|drdsdt$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)\left|f_2(u)(r,s,t)-f_2\left(u^*\right)(r,s,t)\right|drdsdt.$$

From conditions (i), (ii), we have

$$\left|u(x,y,z)-u^*(x,y,z)\right|\leq \varphi(x,y,z)+l_gN\left|u(x,y,z)-u^*(x,y,z)\right|$$
$$+\int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)\,l_{f_1}\left|u(r,s,t)-u^*(r,s,t)\right|drdsdt$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)\,l_{f_2}\left|u(r,s,t)-u^*(r,s,t)\right|drdsdt$$
$$\leq \varphi(x,y,z)+l_gN\left|u(x,y,z)-u^*(x,y,z)\right|$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty \left(l_K(x,y,z,r,s,t)\,l_{f_1}+l_F(x,y,z,r,s,t)\,l_{f_2}\right)\left|u(r,s,t)-u^*(r,s,t)\right|drdsdt.$$

Then

$$\left(1-l_gN\right)\left|u(x,y,z)-u^*(x,y,z)\right|$$
$$\leq \varphi(x,y,z)$$
$$+\int_0^\infty\int_0^\infty\int_0^\infty \left(l_K(x,y,z,r,s,t)\,l_{f_1}+l_F(x,y,z,r,s,t)\,l_{f_2}\right)\left|u(r,s,t)-u^*(r,s,t)\right|drdsdt$$

and we have

$$\left|u(x,y,z)-u^*(x,y,z)\right|\leq \frac{\varphi(x,y,z)}{1-l_gN}$$
$$+\frac{1}{1-l_gN}\int_0^\infty\int_0^\infty\int_0^\infty \left(l_K(x,y,z,r,s,t)\,l_{f_1}+l_F(x,y,z,r,s,t)\,l_{f_2}\right)$$
$$\left|u(r,s,t)-u^*(r,s,t)\right|drdsdt.$$

From Wendorf lemma [7], for an unbounded domain, it follows that

$$\left|u(x,y,z)-u^{*}(x,y,z)\right| \leq \frac{\varphi(x,y,z)}{1-l_gN}\exp\left[\frac{1}{1-l_gN}\int_0^\infty\int_0^\infty\int_0^\infty \left(l_K(x,y,z,r,s,t)\,l_{f_1}+l_F(x,y,z,r,s,t)\,l_{f_2}\right)drdsdt\right]$$

and we have

$$\left|u(x,y,z)-u^{*}(x,y,z)\right| \leq \frac{1}{1-l_gN}\exp\left[\frac{m}{1-l_gN}\right]\cdot\varphi(x,y,z)$$

and

$$\left|u(x,y,z)-u^{*}(x,y,z)\right| \leq C_{KFghf_1f_2}\cdot\varphi(x,y,z)$$

where

$$C_{KFghf_1f_2}\varphi(x,y,z)=\frac{1}{1-l_gN}\exp\left(\frac{m}{1-l_gN}\right),$$

and Eq. (2) is Hyers-Ulam-Rassias stable. □

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Analysis of Electroencephalography (EEG) Signals Based on the Haar Wavelet Transformation

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Abstract EEG recordings give extremely noisy signals that do not allow classical methods to clearly display such as the existence of power laws or even more so the critical state that is a signature of the normal operation of biological tissues (Contoyiannis et al., Phys Rev Lett 93:098101, 2004; Contoyiannis et al., Nat Hazards Earth Syst Sci 13:125–139, 2013; Kosmidis et al., Eur J Neurosci, 2018. https://doi.org/10.1111/ejn.14117). We have recently introduced a method, based on Haar wavelet transformation (Contoyiannis et al. Phys. Rev. E 101:052104, 2020), that completely ignores noise and thus can reveal the information of the power law in EEGs. It calculates the exponent of the power law and thus gives us the ability to determine whether the brain is in critical state in terms of physics, i.e., in a state of normal biological function. Pathological conditions, such as epilepsy, are quantified through this method so we can observe their evolution.

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1 Introduction

We have shown in recent years that biological tissues such as the heart [1, 2] and neurons [3], when operating normally, then obey the dynamics of critical intermittency. This means that the appropriately defined waiting times, laminar length L, in terminology of intermittency [4], have power law distribution of the form:

$$P(L) = p_1 L^{-p_l} \tag{1}$$

where $p_l \in [1, 2]$[6].

These dynamics belong to the category of critical dynamics of the systems we found in Nature, in the economy, in society. The meaning of the power law [1] in biological systems is that due to its scale-free character , the biological systems have all time scales to respond to all stimuli, which is one of the main features of normal operation.

Therefore, it is an important diagnostic tool that can be used to determine whether the tissue demonstrates behaviors such as Eq. [1]. For this reason, we have proposed the Method of Critical Fluctuations (MCF) which reveals not only the critical state but also how far or how close we are to it [5–8]. A series of biological signals are the records of EEG coupled by the wide variety of noise sources external as well internal.

Four general strategies [9] are employed to deal with the issue of noise in EEG recording and analysis: elimination of noise sources, averaging, rejection of noisy data, and noise removal. Specifically, in elimination of noise sources, the easiest sources of noise to deal with are external, environmental sources of noise, such as AC power lines, lighting a large array of electronic equipment (from computers, displays, and TVs to wireless routers, notebooks, and mobile phones). The averaging method [9] based on the random character of noise is a simple and powerful way of dealing with noise, but it has a number of limitations and caveats [9]. The most straightforward procedure for rejection of noisy data is by visual inspection [9]. Other features of the data can be used to identify and reject specific segments of the recording. EEGLAB analysis package [10] provides a number of such options. In the category noise removal, the easiest way to remove noise from the raw data is by filtering. Such method has been developed in [11]. Another source of noise in EEG recording is physiological noise that can be caused by various noise generators. Such a case is ocular signal caused by eyeball movement (electrooculogram, EOG) [12–15].

A number of methods for estimating specific sources of EEG signal developed and tested in the recent years fall under the umbrella of blind source separation (BSS). The key assumption of BBS is that the observed signal can be interpreted as a mixture of original source signals [16]. A series of the improving of devices has been accomplished like [17]. Various linear or non-linear noise mathematical methods of removal have been developed at the level of EEG signal analysis. One such linear method that refers to wavelet-based analysis is the [18]. Our approach presented in this paper is based on the coefficients of the analysis on the basis of Haar wavelet, but it has a peculiarity. It does not show any sensitivity to noise. In other words, it ignores noise, no matter how strong it may be, and it reveals the power law or, more generally, the critical state if this exists. The scope of this work is to extract information from the operation of brain tissue from EEG regardless of how noisy they are.

2 Data Collection

The data used, in the present work, have been introduced by Andrzejak et al. [19] and are analyzed by statistical methods like Tsallis entropy and Shannon block entropy, in [20]. Two sets, denoted “A” and “E,” respectively, each containing 100 single-channel EEG segments of 23.6 s. duration, were used for this study. Set “A” contains EEGs of healthy people. Set “E” contains volunteers seizure activities. The segments fulfill the criterion of weak stationarity [19]. After 12-bit analog-to-digital conversion, the data is written continuously onto the disk of a data acquisition computer system at a sampling rate of 173.61 Hz. Band-pass filter settings were 0.53–40 Hz (12 dB/oct.).

3 The MCF Analysis

We will present the procedure of analysis for file A. The procedure for analyzing the E file, for which we will present only the results, is similar. The MCF produces the distribution of waiting times, and its application is explained in Fig. 1. As we show in [6], the critical dynamics are described by a non-linear intermittent map. So we can see in Fig. 1 the time series as a portrait of this intermittent map. Details of the MCF can be found in [6–8].

Figure 1 shows the two lines, which extend along the entire length of the 400,000-point time series and define the so-called laminar region. E_0 is the fixed point value which is usually the lowest values level; therefore, a stationary behavior is necessary, which is something that happens in EEG. The green line determines the end of the laminar region. The value of E_l is a free parameter. Hence, the green line sweeps across y-axis. Due to the symmetric form around the zero of the distribution of EEG values, we start the sweep of green line after the zero. For a width of EEG values between −200 and 200, the research for the end of laminar region could be restricted in a zone from 20 up to 50 by a step of 10 units. The laminar lengths are defined as the waiting times in the zone, $E_o < E < E_l$. The sweep is finished when we locate the largest array in the laminar length distribution table (this will be explained later). The largest lengths are favored by the distribution of power law

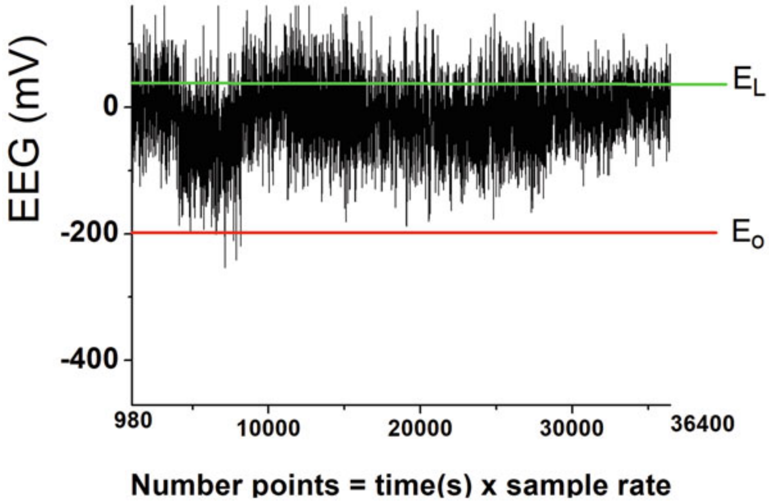

Fig. 1 A segment (980–36,400 points) of the EEG 400,000 recording points by healthy people

(1) which is also required in our research. The distribution of the laminar lengths is fitted by the function:

$$P(L) = p_1 L^{-p_2} e^{-p_3 L} \tag{2}$$

We focus in the exponents p_2 and p_3. If p_3 is zero, then the exponent p_2 is equal to p_l in Eq. 1. The intermittency is characterized as critical intermittency when $p_3 \approx 0$, $p_2 \in [1, 2]$. It is clear that when p_3 is not close to zero, the exponential factor in (2) is significant and cuts large lengths L. So the more important this factor becomes, the further we move away from the dynamics of critical intermittency. In the case of EEGs, the very strong noise destroys the distribution of laminar lengths, and so it is impossible with classical methods, like the fitting by function like (2), to extract the information of the power law, if this exist. Figure 2a shows the distribution of laminar lengths for the EEG signal. This distribution shows that it is far from being considered a power law. But this distribution has come from healthy people and normal operation requires it to be scale-free. Obviously, what we are seeing is the destruction by very strong noise. How will this noise be removed so that the dynamics are revealed? Or, on the other hand, maybe some methodologies could ignore (and not remove) the noise so that the information of power law can be shown. The answer to these questions is given by a method which comes from the development on a wavelet basis and that reveals the power law, if it exists, no matter how strong is the noise.

4 The Haar Wavelet Analysis: The Steps

The wavelet base is a linear base suitable for phenomena that exhibit self-similar properties such as critical phenomena. The wavelet undergoes two transformations, the change of scale j and their displacement k. Thus, the coefficients of the analysis are $d_{j,k}$. When $j = k = 0$, we have the coarse graining description of the analysis. In the framework of this description, the coefficients of the analysis might ignore the noise of the analyzed signal [21]. We use this behavior to develop an algorithm that applies to each distinct numerical or real signal $f(i)$, $i = 1,\ldots \Delta_{max}$, with the Δ_{max} the maximum length of the signal. This algorithm answers the question of whether a signal is a power law and how close or far it is from the power law and calculates the corresponding exponent p_2. In other words, it creates a fitting function without carrying the pathogenicity of the fitting function due to noise, especially at the high values of the laminar lengths. In order to avoid the tail of distributions for the fitting function, where the strong noise makes the results precarious, we usually keep the small scales. But this is not correct because it removes the information that we can obtain from the great scales. The new method uses all scales. The base we use to develop the algorithm is the Haar wavelet base, which has as a mother function, the following function which defined by the theta functions for spaces [0, Δ]:

$$\psi_H = \Theta\left(\frac{\Delta}{2} - x\right)\Theta(x - 0) - \Theta\left(x - \frac{\Delta}{2}\right)\Theta(\Delta - x) \tag{3}$$

We define the quantities [4]:

$$\lambda = \frac{\frac{d_{00}}{d_{10}}}{\frac{d_{10}}{d_{20}}} = \frac{d_{00}d_{20}}{d_{10}^2} = \frac{\left(\sum_{i=0}^{\frac{\Delta}{2}} f(i) - \sum_{\frac{\Delta}{2}}^{\Delta} f(i)\right)\cdot\left(\sum_{i=0}^{\frac{\Delta}{8}} f(i) - \sum_{\frac{\Delta}{8}}^{\frac{\Delta}{4}} f(i)\right)}{\left(\sum_{i=0}^{\frac{\Delta}{4}} f(i) - \sum_{\frac{\Delta}{4}}^{\frac{\Delta}{2}} f(i)\right)^2} \tag{4}$$

and

$$R = \frac{d_{00}}{d_{10}} = \frac{1}{\sqrt{2}}\Big(\sum_{i=1}^{\frac{\Delta}{2}} f(i) - \sum_{\frac{\Delta}{2}}^{\Delta} f(i)\Big)/\Big(\sum_{i=1}^{\frac{\Delta}{4}} f(i) - \sum_{\frac{\Delta}{4}}^{\frac{\Delta}{2}} f(i)\Big) \tag{5}$$

The proposed method for revealing the criticality and finding the exponent of the power law of distribution of laminar lengths has the following steps:

1. We apply the algorithm (4) to calculate λ as a function of Δ up to Δ_{max}. As we can see from (4), the minimum Δ that can give information is $\Delta = 8$. Because some zeros are also included in these distributions between the digits, we do not cut the length if they are less than 8 consecutive, but we cut it when more than

8 consecutive ones appear (including 8). We make the plot λ vs Δ, and because we are interested in the convergence of λ [21], the last 10 points are enough to deduce conclusion.

2. Obviously, the closer the D_λ is to the value 0, the closer to the power law is the distribution. We quantify the previous step by calculating the distance of λ from the value $\lambda = 1$, which is the perfect power law, by calculating the quantity:

$$D_\lambda = \frac{1}{10}\sum_{i=1}^{10}(1-\lambda_i)^2 \quad (6)$$

3. We produce the plot R vs Δ. From the convergence region of the diagram (≤ 10), an average value for the quantity R is obtained.
4. We consider $f(i) = ci^{-p}$, $i = 1, 2, 3 \ldots \Delta_{max}$ as a test function in (5) (we have to mention here that c vanished in relation (5) and by solving numerically Eq. (5) we calculate the exponent p for R closer to the average value which we found in step 3).

5 The Results

As shown in Fig. 2, the distribution of the waiting times for the EEG of the healthy set is very close to a power law, something that is impossible to obtain from the distribution of Fig. 2a. This information is extracted using the new wavelet method. The exponent p2 of the power law in EQ. 2, has been calculated in the value range of the critical state, which means that this is the normal operation of the brain.
Following the procedure we presented, we analyze the data of the set E that refers to patients with seizures. We present in Fig. 3 the results.

As shown in Fig. 3, the convergence of λ has been removed from the 1. Thus, in the case of patients, no power law is expected for the distribution of waiting times. So R does not make sense for the calculation of a power law exponent.

One question that arises is whether the above results depend on the time scale we are analyzing. To answer this, we divided the set A into two consecutive subsets of 200,000 points and repeated the calculations. Then we divided it into four subsets of 100,000 points. The results for the amount $\mathbf{D}_\lambda$ are shown in Fig. 4 with the green color.

As we can see in Fig. 4, the change in scale, in the Healthy case, does not significantly affect the quantitative results that refer to the criticality and the power law. This is perfectly consistent with the scale-free character of the power law, but also with the self-similarity of the critical state. On the contrary, in the case of patients, we see that the change in the scale is associated with the reinforcement of the exponential damping in relation 2, which means a removal from the power law and the scale-free property.

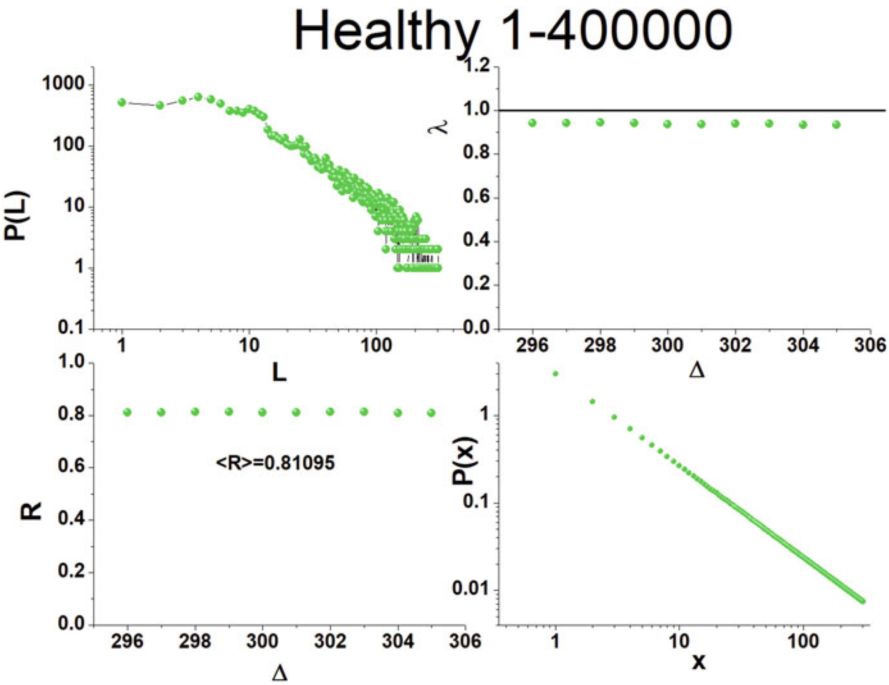

Fig. 2 (**a**) The distribution of laminar lengths for the healthy EEG with maximum length $L_{max} = \Delta_{max} = 305$. (**b**) The diagram λ vs Δ of the last 10 points ($296 < \Delta < 305$) showing the convergence of the values λ very close to the unit. The laminar region is $[-200, 40]$. The value $D_\lambda = 3.75 \cdot 10^{-3}$ is very close to zero. (**c**) The diagram R vs Δ for the same area as (**b**) showing the convergence of the values of R. The diagram shows the mean value of R for this period. (**d**) The test function for the distribution is the power law with exponent $p = 1.05 \in [1, 2]$ corresponding to diagram 2a. The parameter $p_1 = 3$ is determined by the condition of normalization of the distribution of laminar lengths $\int_1^{305} p_1 x^{-1.05} dx = 1$

6 Conclusions

Let us now come to our conclusions. An important source of information on whether or not a biological tissue in a normal state is the appropriate waiting time for biological signals. Based on the MCF application on EEG signal, we can find the distribution of these times . These distributions in the EEG, regardless of whether they came from healthy people or patients, are far from being power law, if they are analyzed by classical methods. This is the result of strong noise on EEG due to many sources. The new method we have recently introduced, based on special processing of wavelet analysis, ignores the noise and thus extracts from the laminar lengths distribution and information about the existence of power laws. This, we can say, is a strong indication of the normal operation of the brain tissue. The method we present in this work not only understands the healthy state but also

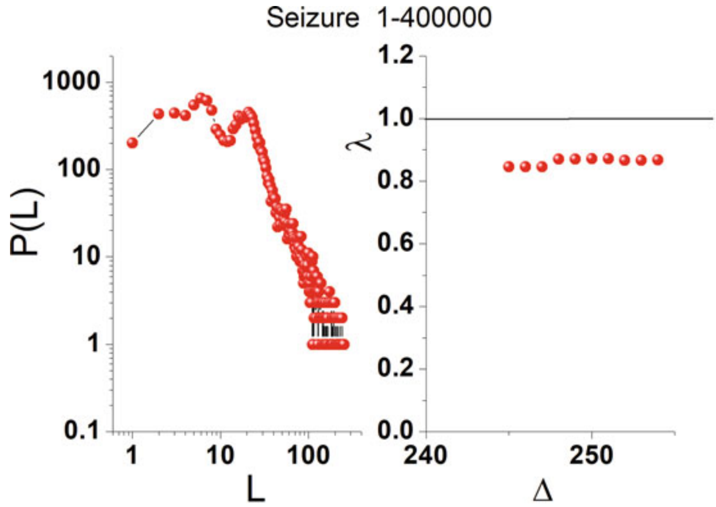

Fig. 3 (**a**) The distribution of laminar lengths for the EEG of patients with maximum length $L_{max} = \Delta_{max} = 254$ for laminar region $[-1500, 200]$. (**b**) The diagram λ vs Δ of the last 10 points ($235 < \Delta < 254$) showing the convergence of the values λ. It is clear that the unit is removed in relation to the healthy ones. The value $D_\lambda = 2 \cdot 10^{-2}$ quantifies this removal

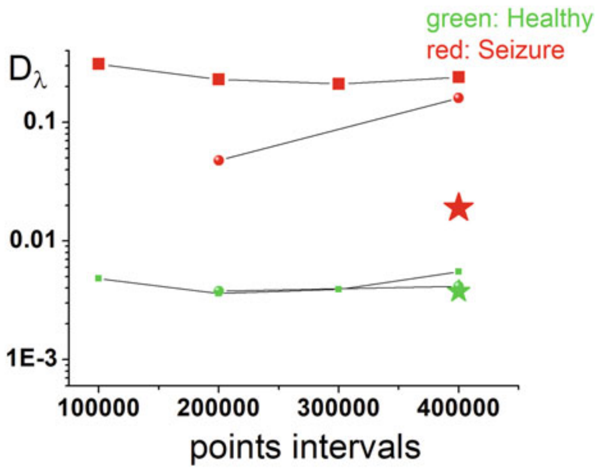

Fig. 4 Healthy with the green color. Star, set 1–400,000; circle, two sets from 200,000 points; square, four sets from 100,000 points. The D_λ values are almost equal between them. Patient with the red color. The symbolisms are the same as in the previous case. Now the D_λ values depend a lot on the time scale

identifies unhealthy states, such as epilepsy. Investigating the most specialized brain diseases or predispositions of them, we have to remark that the methodology we have presented in this work is a very important tool.

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Perov-Type Contractions

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Abstract Fixed point theory is rapidly growing in various directions, so the goal of this chapter is to collect and underline recent results on Perov-type contractions and talk about various generalizations of this result. Perov contraction is defined on generalized metric space firstly introduced by Russian mathematician A.I. Perov in the 1960s. The main difference and strength of this result is in changed view on contractive constant since, in Perov results, that role is played by a matrix with positive entries. The question is what do we gain in this case? And also can we talk about scientific novelty of this concrete results and all other generalizations published in the last 10 years? We will try to answer at least partially on these questions and gather most important results regarding Perov contractions.

1 Introduction

Famous Banach fixed point theorem [13] was generalized in numerous ways by changing a setting, contractive condition, or both. Among large quantity of fixed point theorems, it is important to demand some applications of these results and

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independence of previously presented theorems. Russian mathematician A.I. Perov [61] in 1964 published a paper, in Russian, dealing with a Cauchy problem for a system of ordinary differential equations. In this paper, he presented a concept of generalized metric space (in a sense of Perov) and gave a proof of a new type of fixed point theorems. From that point of view, we can say that Perov theorem was created as a tool in the area of differential equations and therefore fulfilled the application goal. It was used once again in Perov's paper in 1966, and then were no significant results on this topic till the 2000s. In the meantime, Polish mathematician S. Czerwik [31] in 1976 published a similar result as a generalization of Edelstein's fixed point theorem. In 1992, M. Zima [86], who also works in the area of differential equations, published a paper, quoting different work of Czerwik, which gave fixed point result on Banach space that could be related to Perov fixed point theorem. G. Petruşel [65] in 2005 did some research on Perov contractions for multivalued operators that was followed by results for Perov multivalued operators by A. Petruşel and A.D. Filip in 2010 ([35]). This led to several published papers on this topic [6, 34, 39, 40, 78]. N. Jurja [50] proved version of Perov theorem for partially ordered generalized metric space. In 2014, M. Cvetković and V. Rakočević published a generalization of Perov fixed point theorem on cone metric spaces, and this result obtained many extensions such as quasi-contraction, Fisher contraction, θ-contraction, F-contraction, coupled fixed point problem, common fixed point problem, etc. [2, 3, 22–30, 38, 41, 45, 63, 69, 73]. Many papers were published in the 2010s citing Perov work, adjusting and generalizing that idea for multivalued operators, spaces endowed with a graph, ω-distance, etc., but will not be the main topic of this chapter.

We will focus on three different frameworks: metric space, generalized metric space, and cone metric space. Thus, we present some basic definitions and properties. As one of the examples, we will present a system of operatorial equations that transforms into coupled fixed point problem.

Definition 1 Let X be nonempty set and $d : X \times X \mapsto \mathbb{R}$ mapping such that

(d_1) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(d_2) $d(x, y) = d(y, x)$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$.

Mapping d is a metric on X and (X, d) is called a metric space.

If $f : X \mapsto X$ is a mapping, then $x \in X$ is a fixed point of f if $f(x) = x$. Set of all fixed point of mapping f is denoted with $Fix(f)$. S. Banach in [13] published in 1922 gave a proof of famous fixed point result regarding existence of a unique fixed point for a class of contractive mappings.

Definition 2 The mapping f on a metric space X is named contraction (contractive mapping) if there exists some constant $q \in (0, 1)$ such that

$$d(f(x), f(y)) \leq qd(x, y), \; x, y \in X.$$

The constant q is known as the contractive constant.

Clearly, every contraction is a non-expansive mapping. For any self-mapping, we define a sequence (x_n), $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, for arbitrary $x_0 \in X$. It is called a sequence of successive approximations or iterative sequence.

Theorem 1 ([13]) *Let (X, d) be a nonempty complete metric space with a contraction mapping $f : X \mapsto X$. Then f admits a unique fixed point in X, and for any $x_0 \in X$, the iterative sequence (x_n) converges to the fixed point of f.*

Russian mathematician A.I. Perov [61] defined generalized metric space where metric has values in $\mathbb{R}^n$. Then, this concept of metric space allowed him to define a new class of mappings, known as Perov contractions, which satisfy contractive condition similar to Banach's, but with a matrix A with non-negative entries instead of a constant q.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 3 ([61]) A mapping $d : X \times X \mapsto \mathbb{R}^m$ is called a *vector-valued metric* on X if the following statements are satisfied for all $x, y, z \in X$.

(d_1) $d(x, y) \geq 0_n$ and $d(x, y) = 0_m \Leftrightarrow x = y$, where $0_m = (0, \ldots, 0) \in \mathbb{R}^m$;
(d_2) $d(x, y) = d(y, x)$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$.

If $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, then notation $x \leq y$ means $x_i \leq y_i$, $i = \overline{1, m}$.

Denote by $\mathscr{M}_{n,n}$ the set of all $n \times n$ matrices and by $\mathscr{M}_{n,n}(\mathbb{R}_+)$ the set of all $n \times n$ matrices with non-negative entries. We write O_n for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix, and further on, we identify row and column vector in $\mathbb{R}^n$.

A matrix $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if $A^n \to O_m$, as $n \to \infty$, or, equivalently, if the matrix norm is less than 1.

Theorem 2 ([61, 62]) *Let (X, d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A(d(x, y)), \quad x, y \in X. \tag{1}$$

Then:

(i) f has a unique fixed point $x^* \in X$;
(ii) the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to x^* for any $x_0 \in X$;
(iii) $d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;

(*iv*) *if* $g : X \mapsto X$ *satisfies the condition* $d(f(x), g(x)) \preceq c$ *for all* $x \in X$ *and some* $c \in \mathbb{R}^n$, *then by considering the sequence* $y_n = g^n(x_0), n \in \mathbb{N}$, *then*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

This result has main application in solving differential and integral equations ([61, 62, 68, 79]).

Even though the first fixed point theorems in cone metric spaces were obtained by Schröder [80, 81] in 1956, cone metric spaces are in the focus of the research in metric fixed point theory in the last decades (see, e.g., [1, 5, 7, 14, 37, 47, 49, 52–56, 71, 74], for more details). Serbian mathematician Đ. Kurepa [58] presented the idea of pseudometrics and cone metric in 1934, but most authors in fixed point theory cite Huang and Zhang's paper [43] from 2007 as a pioneer paper in cone metric fixed point theory.

Definition 4 Let E be a real Banach space with a zero vector θ. A subset P of E is called a cone if:

(*i*) P is closed, nonempty, and $P \neq \{\theta\}$;
(*ii*) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(*iii*) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in (\text{int})\, P$ where $\text{int}(P)$ is the interior of P.
The cone P in a real Banach space E is called normal if

$$\inf\{\|x + y\| \mid x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\, \|y\|\,. \tag{2}$$

The least positive number satisfying (2) is called the normal constant of P. It has been shown that we can consider only case $K = 1$ for normal cone metric spaces. The cone P is called solid if $\text{int}\,(P) \neq \emptyset$.

Introducing a concept of cone in a real Banach space allows us to present a different type of pseudometric related to defined partial ordering induced by observed cone.

Definition 5 Let X be a nonempty set, and let P be a cone on a real Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(d_1) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces. Note that the generalized metric space is a normal cone metric space.

Example 1 Defined partial ordering on $\mathbb{R}^n$ as in the definition of generalized metric in the sense of Perov determines a normal cone

$$P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = \overline{1, n}\}$$

on $\mathbb{R}^n$, with the normal constant $K = 1$. Evidently, $A(P) \subseteq P$ if and only if $A \in \mathscr{M}_{n,n}(\mathbb{R}_+)$. It appears possible to adjust and probably broadly modify Perov's idea on a concept of cone metric space. Preferably, we will get some existence results. Nevertheless, forcing the transfer of contractive condition on cone metric space would be possible for some operator A instead of a matrix.

We present some well-known examples of cone metric spaces.

Example 2 Let $X = \mathbb{R}$, $E = \mathbb{R}^n$, and

$$P = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = \overline{1, n}\right\}.$$

It is easy to see that $d : X \times X \mapsto E$ defined by

$$d(x, y) = (|x - y|, k_1|x - y|, \ldots, k_{n-1}|x - y|),\ x, y \in X$$

is a cone metric on X, where $k_i \geq 0$ for $i = \overline{1, n - 1}$.

Example 3 For $X = E = C[0, 1]$ where E is equipped with the supremum norm, a function $d : X \times X \mapsto E$ defined with $d(f, g)(x) = |f(x) - g(x)|$, $x \in [0, 1]$, $f, g \in X$, is a cone metric on $C[0, 1]$.

Example 4 ([33]) Let $E = C^{(1)}[0, 1]$ with a norm $\|x\| = \|x\|_\infty + \left\|x'\right\|_\infty$, for any $x \in E$, and $P = \{x \in E \mid\ x(t) \geq 0,\ t \in [0, 1]\}$. Consider, for example,

$$x_n(t) = \frac{1 - \sin nt}{n + 2} \quad \text{and} \quad y_n(t) = \frac{1 + \sin nt}{n + 2},\ n \in \mathbb{N}.$$

Deducing $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$ as $n \to \infty$, so it is a non-normal cone.

Presumably, convergent and Cauchy sequences are naturally defined. Suppose that E is a Banach space; P is a solid cone in E, whenever it is not normal; and $\preceq$ is the partial order on E with respect to P.

Definition 6 The sequence $(x_n) \subseteq X$ is convergent in X if there exists some $x \in X$ such that

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n \geq n_0 \implies d(x_n, x) \ll c.$$

We say that a sequence $(x_n) \subseteq X$ converges to $x \in X$ and denote that with $\lim_{n\to\infty} x_n = x$ or $x_n \to x, n \to \infty$. Point x is called a limit of the sequence (x_n).

Definition 7 The sequence $(x_n) \subseteq X$ is a Cauchy sequence if

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n, m \geq n_0 \implies d(x_n, x_m) \ll c.$$

Every convergent sequence is a Cauchy (fundamental) sequence, but reverse do not hold. If any Cauchy sequence in a cone metric space (X, d) is convergent, then X is a complete cone metric space.

As proved in [43], if P is a normal cone, not related to if it is solid, a sequence $(x_n) \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, $n \to \infty$. Similarly, $(x_n) \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, $n, m \to \infty$. Also, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $d(x_n, y_n) \to d(x, y)$, $n \to \infty$. Let us emphasize that these equivalences do not hold if P is a non-normal cone.

The following properties of normal cone metric spaces are often used:

(1) If E is a real Banach space with a cone P and if $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.
(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$, we have $a_n \ll c$.

It follows that the sequence (x_n) converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$ and (x_n) is a Cauchy sequence if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$. In the situation with a non-normal cone, we have Lemmas 1 and 4 from [43] just partially. Also, in this case, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable. As shown in [73], if the cone is Archimedean, meaning $x \preceq ny$, $n \in \mathbb{N}$ implies $x \preceq \theta$, then any decreasing sequence is convergent if and only if it is Cauchy and has an infimum.

A mapping $f : X \mapsto X$ is a continuous mapping on X if for any $x \in X$ and a sequence $(x_n) \subseteq X$ such that $\lim_{n\to\infty} x_n = x$, it follows $\lim_{n\to\infty} f(x_n) = f(x)$. For the purpose of proof that will be presented in the sequent, we will also recall the following lemma [26].

Lemma 1 *Let (X, d) be a cone metric space. Suppose that x_n is a sequence in X and that b_n is a sequence in E. If $\theta \preceq d(x_n, x_m) \preceq b_n$ for $m > n$ and $b_n \to \theta, n \to \infty$, then (x_n) is a Cauchy sequence.*

The question which raises when we work on generalization of Perov theorem for cone metric spaces is could we and under which conditions replace matrix with an operator in a role of contractive constant. And that was the main idea of previously mentioned Perov-type results on cone metric spaces. Therefore, we will recall some basic properties of operators on a Banach space and try to examine

sufficient condition which operator should satisfy to suit our needs in generalized Perov theorem for cone metric spaces.
Observe that with $\mathscr{B}(E)$ is denoted the set of all bounded linear operators on a Banach space E and with $\mathscr{L}(E)$ the set of all linear operators on E. As usual, $r(A)$ is a spectral radius of an operator $A \in \mathscr{B}(E)$,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}.$$

If $r(A) < 1$, then the series $\sum_{n=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathscr{B}(E)$, and

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

Also, $r((I - A)^{-1}) \leq \frac{1}{1-r(A)}$.
If $A, B \in \mathscr{B}(E)$ and $AB = BA$, then $r(AB) \leq r(A)r(B)$.
Furthermore, if $\|A\| < 1$, then $I - A$ is invertible, and

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Results in this area can be divided in a several ways, but mostly we will focus on generalizations of Perov's result on generalized metric spaces, cone metric spaces, partially ordered metric spaces, and several applications, leaving out-of-focus results for multivalued operators, common fixed point problem, and Perov-type fixed point results on many other types of spaces.

2 Fixed Point Theorems of Perov Type on Generalized Metric Space

As mentioned, Perov presented fixed point result on generalized metric space in 1962 in the paper *On Cauchy problem for a system of ordinary differential equations* (in Russian). But he used different techniques and did not state his result in the way that was later used and as we stated in Theorem 2.

Theorem 3 *Let* (X, d) *be a complete generalized metric space,* $f : X \mapsto X$ *a mapping satisfying*

$$d(f(x), f(y)) \leq A(d(x, y)), \quad x, y \in X,$$

for some non-negative a-matrix A. Then a mapping f has a single fixed point x^* in X that can be obtained by the method of successive approximations (x_n), where $x_n = f^n(x)$, $n \in \mathbb{N}$. Following estimation holds:

$$d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Proof It is easy to see that the inequality

$$d(x_m, x_{m+k}) \leq A^m(I_n - A)^{-1}(d(x_1, x_2)),\ k \in \mathbb{N}$$

is satisfied for any $m \in \mathbb{N}$. Thus, the sequence (x_n) is fundamental and therefore convergent in X. Let $x^* = \lim_{n\to\infty} x_n$. Then

$$\begin{aligned} d(f(x^*), x^*) &\leq d(f(x^*), x_{m+1}) + d(x_{m+1}, x^*) \\ &\leq A(d(x^*, x_m)) + d(x_{m+1}, x^*), \end{aligned}$$

so $f(x^*) = x^*$.
Assume that, in addition to x^*, there exists some $x \in X$, another fixed point of f. In that case,

$$d(x^*, x) = d(f(x^*), f(x)) \leq A(d(x^*, x)),$$

yields to contradiction and, as a consequence, uniqueness of a fixed point of mapping f.

Matrix $A \in \mathscr{M}_{m,m}$ with non-negative entries is a-matrix if determinants of all principle minors of matrix $I_m - A$ are positive.
Fixed point theorems presented in that paper by Perov were used in solving many differential equations and systems of differential equations, par example for Cauchy problem [61].

Precup [70] has shown that the main advantage of Perov theorem over Banach fixed point theorem and the most valuable impact of this result is application to different fixed point problems with a much better estimation and faster convergence of the iterative sequence. Some advantages of a vector-valued norm over the usual scalar norms were pointed out. Throughout presented examples in [70], one can show that, in general, the condition that A is a matrix convergent to zero is weaker than the contraction conditions for operators given in terms of the scalar norms on X of the following type:

$$\begin{aligned} &\|x\|_M := \|x_1\| + \|x_2\|, \\ &\|x\|_C := \max\{\|x_1\|, \|x_2\|\} \text{ or} \\ &\|x\|_E := \left(\|x_1\|^2 + \|x_2\|^2\right)^{1/2}. \end{aligned}$$

Also, in [66], the extension of Perov theorem is applied on solving Hammerstein integral equation on $\mathbb{R}^n$

$$u(x) = \int_{\Omega} k(x, y) f(y, u(y)) \mathrm{dy}, \ x \in \Omega,$$

in the case that kernel k has matrix-values, i.e.,

$$k : \Omega^2 \mapsto \mathscr{M}_{n,n}.$$

The usual Hammerstein integral equation on $\mathbb{R}^n$ is a special case for $k = \lambda I_n$. As one of the results, appropriate iterative method is presented along with abstract results. This is just one chosen example among many showing impact of Perov theorem and generalizations on area of differential equations. Hence, there are more than enough reasons to apply Perov fixed point theorem before Banach's in expectance of better estimations and weaker requirements.
Chronologically, after Perov presented his results in the 1960s, it took almost 50 years for the mathematical community to take further mass interest in this problem, and that happened in the setting of cone metric space.

Jurja ([50]) gives a Perov-type fixed point theorem in generalized ordered metric spaces. In this setting, the map is assumed to be monotone and to satisfies a Lipschitz-type condition with a matrix A. This condition is supposed to hold only on elements that are comparable with respect to the partial order. It is also presumed that f is continuous. It has been shown that under such conditions, a Perov-type fixed point theorem still holds, meaning that a mapping has a fixed point on generalized metric space.

Theorem 4 ([50]) *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a generalized complete metric space $\left(d(x, y) \in \mathbb{R}_+^m\right)$. If the map $f : X \to X$ is continuous, monotone (i.e., increasing or decreasing) such that*

(1) f satisfies a Lipschitz-type condition with a matrix $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$

$$d(f(x), f(y)) \leq Ad(x, y), \ x \geq y; \tag{3}$$

(2) $A^n \to O_m, n \to \infty$;
(3) $\exists x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$,

then,

(i) $Fix(f) = \{x^*\}$;
(ii) the sequence of successive approximations $x_n = f^n(x)$ is convergent and $\lim_{n\to\infty} f^n(x) = x^*$, for any $x \in X$.

Proof Due to statement of the theorem, there exists some $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $f(x_0) \leq x_0$ and then, based on monotonicity of f, $f^n(x_0) \leq f^{n+1}(x_0)$ or

$f^{n+1}(x_0) \leq f^n(x_0)$, for any $n \in \mathbb{N}$. Therefore, the inequality (3) holds:

$$d(f^n(x_0), f^{n+1}(x_0)) \leq A(d(f^{n-1}(x_0), f^n(x_0))),\ n \in \mathbb{N}.$$

We will prove, by the principle of mathematical induction, that

$$d(f^n(x_0), f^{n+1}(x_0)) \leq A^n(d(x_0, f(x_0))),\ n \in \mathbb{N}. \tag{4}$$

Since the base case obviously holds, let us assume that (4) holds for $n-1$. Then

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leq A(d(f^{n-1}(x_0), f^n(x_0))) \\ &\leq A(A^{n-1}(d(f(x_0), x_0))) \\ &= A^n(d(x_0, f(x_0))). \end{aligned}$$

Hence, (4) holds for any $n \in \mathbb{N}$. Furthermore, we will prove that the sequence $(f^n(x_0))$ is a Cauchy sequence and so convergent in X. Let $n, m \in \mathbb{N}$ and $n \leq m$.

$$\begin{aligned} d(f^n(x_0), f^m(x_0)) &\leq \sum_{i=n}^{m-1} d(f^i(x), f^{i+1}(x)) \\ &\leq \sum_{i=n}^{m-1} A^i(d(x_0, f(x_0))) \\ &\leq \sum_{i=n}^{\infty} A^i(d(x_0, f(x_0))) \\ &= A^n(I-A)^{-1}(d(x_0, f(x_0))). \end{aligned}$$

As $A^n \to O_m$, there exists $x^* \in X$, a limit of $(f^n(x_0))$. Since the mapping f is continuous, x^* is a fixed point of f. If we assume that $x \in Fix(f)$ and then let y be an upper bound and z a lower bound of $\{x^*, x\}$, then

$$d(f^n(a), x^*) = d(f^n(a), f^n(x^*)) \leq A^n(d(a, x^*)),\ n \in \mathbb{N},\ a \in \{y, z\}.$$

Taking into the account that $A^n \to O_m$, we have $\lim_{n\to\infty} f^n(y) = \lim_{n\to\infty} f^n(z) = x^*$ and $f^n(z) \leq x = f^n(x) \leq f^n(y)$, $n \in \mathbb{N}$, so $x = x^*$. Hence, f has a unique fixed point in X.

Remark 1 Condition (3) is weaker than the condition (1) in Perov original fixed point theorem, where it is required that (1) is satisfied for all $x, y \in X$.

Perov-Maia theorem can also be stated for generalized ordered metric spaces ([59, 62]). We will not present proof of this and all other theorems that use the same proof techniques as previously stated results.

Theorem 5 ([50]) *(Perov-Maia) Let X be a nonempty set, partially ordered, such that every pair $x, y \in X$ has a lower and an upper bound. Let d and ρ be two metrics on X and $f : X \to X$ a mapping. We suppose that*

(1) $d(x, y) \leq \rho(x, y), \quad x \geq y$;
(2) (X, d) *is a generalized ordered complete metric space;*
(3) $f : (X, d) \to (X, d)$ *is a continuous mapping;*
(4) f *is a monotone mapping;*
(5) there exists a matrix $A \in \mathscr{M}_{m\times m}(\mathbb{R}_+)$ *convergent to zero, such that*

$$\rho(f(x), f(y)) \leq A\rho(x, y), \quad x \geq y;$$

(6) $\exists x_0 \in X$ *such that* $x_0 \leq f(x_0)$ *or* $x_0 \geq f(x_0)$.

Then, $Fix(f) = \{x^*\}$.

Filip and Petruşel [35] considered application of local fixed point theorem for a class of generalized single-valued contractions stated and proved in [34]. Their main goal was to solve semilinear inclusion systems that model evolution of macrosystems under uncertainty or lack of precision, from control theory, biology, economics, artificial intelligence, etc.
Since the next result has local character, we will define term of closed ball in generalized metric space as usual. Let (X, d) be a generalized metric space, $x_0 \in X$, and $r = (r_1, r_2, \ldots, r_m) > 0_m$; then

$$\widetilde{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}.$$

Also, $\mathbb{R}^m_+$ denotes a set of all positive m-tuples, i.e.,

$$\mathbb{R}^m_+ = \{r = (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m \mid r_i > 0, \ i = \overline{1, m}\}.$$

Theorem 6 ([34]) *Let (X, d) be a complete generalized metric space, $x_0 \in X$, $r = (r_i)_{i=1}^m \in \mathbb{R}^m_+$, and $f : \widetilde{B}(x_0, r) \to X$ having the property that there exist $A, B \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ such that*

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(y, f(x)) \tag{5}$$

for all $x, y \in \widetilde{B}(x_0, r)$. We suppose that

(1) A is a matrix that converges to zero;
(2) if $u \in \mathbb{R}^m_+$ is such that $(I - A)^{-1}u \leq (I - A)^{-1}r$, then $u \leq r$;
(3) $(I - A)^{-1}d(x_0, f(x_0)) \leq r$.

Then Fix $(f) \neq \emptyset$.
In addition, if the matrix $A + B$ *converges to zero, then* $\text{Fix}(f) = \{x^*\}$.

Remark 2 By similitude to [18], a mapping $f : Y \subseteq X \to X$ satisfying the condition

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(y, f(x)),\ x, y \in Y, \tag{6}$$

for some matrices $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ where A is a matrix that converges toward zero, could be called an almost contraction of Perov type.

We have also a global version of Theorem 6 expressed by the following result.

Corollary 1 *Let* (X, d) *be a complete generalized metric space. Let* $f : X \to X$ *be a mapping having the property that there exist* $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that*

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(y, f(x)), \quad \forall x, y \in X.$$

If A is a matrix that converges toward zero, then

(i) $\text{Fix}(f) \neq \emptyset$;
(ii) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *given by* $x_n = f^n(x_0)$ *converges toward a fixed point of* f, *for all* $x_0 \in X$;
(iii) *one has the estimation*

$$d\left(x_n, x^*\right) \leq A^n(I - A)^{-1}d(x_0, x_1)$$

where $x^* \in \text{Fix}(f)$.
In addition, if the matrix $A + B$ *converges to zero, then* $\text{Fix}(f) = \{x^*\}$.

Remark 3 Any matrix $A = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, where $a, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$, satisfies the assumptions (1)-(2) in Theorem 6.

Theorem 7 *Let* $(X, |\cdot|)$ *be a Banach space, and let* $T_1, T_2 : X \times X \to X$ *be two operators. Suppose that there exist* $a_{ij}, b_{ij} \in \mathbb{R}_+, i, j \in \{1, 2\}$ *such that, for each* $x = (x_1, x_2), y = (y_1, y_2) \in \text{X} \times \text{X}$, *one has*

(1) $|T_1(x_1, x_2) - T_1(y_1, y_2)| \leq a_{11}|x_1 - y_1| + a_{12}|x_2 - y_2| + b_{11}|x_1 - T_1(y_1, y_2)| + b_{12}|x_2 - T_2(y_1, y_2)|$
(2) $|T_2(x_1, x_2) - T_2(y_1, y_2)| \leq a_{21}|x_1 - y_1| + a_{22}|x_2 - y_2| + b_{21}|x_1 - T_1(y_1, y_2)| + b_{22}|x_2 - T_2(y_1, y_2)|$.

In addition, assume that the matrix $A = \begin{pmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{pmatrix}$ *converges to* O_2. *Then, the system*

$$x_1 = T_1(x_1, x_2), \quad x_2 = T_2(x_1, x_2)$$

has at least one solution $x^* = (x_1^*, x_2^*) \in X \times X$. *Moreover, if, in addition, the matrix* $A + B$ *converges to zero, then the above solution is unique.*

In correlation to Theorem 7, we will discuss results from [64] where authors considered a system of operatorial equations

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases}$$

where $T_1, T_2 : X \times X \to X$ are two given operators and X is a nonempty set.
By definition, a solution $(x, y) \in X \times X$ of the above system is called a coupled fixed point for the operators T_1 and T_2. Notice that if $S : X \times X \to X$ is an operator and we define

$$T_1(x, y) := S(x, y) \quad \text{and} \quad T_2(x, y) := S(y, x),$$

then we get the classical concept of a coupled fixed point for the operator S introduced by Opoitsev and then studied in some papers by Lakshmikantham and al. Coupled fixed point for operatorial inclusion is defined in a similar way, namely, by using the symbol $\in$ instead of $=$. The concept of a coupled fixed point for a multivalued operator S is accordingly defined [83].

We present another result in the case of a generalized metric space but endowed with two metrics.

Theorem 8 *Let* X *be a nonempty set, and let* d, ρ *be two generalized metrics on* X. *Let* $f : \mathrm{X} \to \mathrm{X}$ *be an operator. We assume that*

(i) *there exists* $\mathrm{C} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that* $d(f(x), f(y)) \leq C\rho(x, y)$;
(ii) (X, d) *is a complete generalized metric space;*
(iii) $f : (X, d) \to (X, d)$ *is continuous;*
(iv) *there exists* $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that for all* $x, y \in X$, *one has* $\rho(f(x), f(y)) \leq A\rho(x, y) + B\rho(y, f(x))$.

If the matrix A *converges toward zero, then* $\mathrm{Fix}(f) \neq \emptyset$.
In addition, if the matrix $A + B$ *converges to zero, then fixed of point of* f *is unique.*

A. Petruşel, G. Petruşel, and C. Urs presented an extension of Perov's theorem along with some theorems on multivalued operators that extend results from [15].

Theorem 9 *Let* (X, d) *be a generalized complete metric space, and let* $f : X \to X$ *be an almost contraction with matrices* A, B, *and* C, *i.e., the matrix* $A + C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *converges to zero,* $B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, *and*

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(y, f(x)) + Cd(x, f(x)), \quad \textit{for all } x, y \in X.$$

Then, the following conclusions hold.

(i) *f has at least one fixed point in X, and, for any $x_0 \in X$, the sequence $(f^n(x_0))$ of successive approximations starting from x_0 converges to $x^*(x_0) \in \mathrm{Fix}(f)$ as $n \to \infty$;*

(ii) *For arbitrary $x_0 \in X$, we have*

$$d\left(x_n, x^*(x_0)\right) \le A^n(I - A)^{-1} d\left(x_0, f(x_0)\right),\ n \in \mathbb{N},$$

and

$$d\left(x_0, x^*(x_0)\right) \le (I - A)^{-1} d\left(x_0, f(x_0)\right);$$

(iii) *If, additionally, the matrix $A + B$ converges to zero, then f has a unique fixed point in X.*

Remark 4 Theorem 9 above extends Theorem 6 where the case of almost contractions with matric $C = O_m$ is treated.

Two important abstract concepts are given now.

Definition 8 ([77, 78]) If (X, d) is a generalized metric space, then a mapping $f : X \to X$ is called a weakly Picard operator if and only if the sequence $(f^n(x))$ of successive approximations of f converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.
If f is weakly Picard operator, then we define the operator $f^\infty : X \to X$ by

$$f^\infty(x) = \lim_{n\to\infty} f^n(x),\ x \in X.$$

Notice that, in this case, $f^\infty(X) = Fix(f)$. Moreover, f^∞ is a set retraction of X to $Fix(f)$.
If f is weakly Picard operator and $\mathrm{Fix}(f) = \{x^*\}$, then by definition f is a Picard operator. In this case, f^∞ is the constant operator, i.e., $f^\infty(x) = x^*$ for all $x \in X$.

Definition 9 ([78]) Let (X, d) be a generalized metric space, and let $f : X \to X$ be an operator. Then, f is said to be a ψ -weakly Picard operator if and only if f is a weakly Picard operator and $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is an increasing operator, continuous in O_m with $\psi(O_m) = O_m$ such that

$$d\left(x, f^\infty(x)\right) \le \psi(d(x, f(x)), \quad \text{for all } x \in X.$$

Moreover, a ψ -weakly Picard operator $f : X \to X$ with a unique fixed point is said to be a ψ -Picard operator. In particular, if $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is given by $\psi(t) = M \cdot t$ (with $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)\big)$, then we say that f is M-weakly Picard operator (respectively, a M-Picard operator).

Stability of functional equations [44, 72, 82] is another interesting topic in correlation with fixed point theory and could be implemented and deduced from some of previous theorems. We can prove the following abstract result (see also [78]) concerning the Ulam-Hyers stability of the fixed point equation (7).

Definition 10 Let (X, d) be a generalized metric space, and let $f : X \to X$ be an operator. Then, the fixed point equation

$$x = f(x) \tag{7}$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$, continuous in O_m with $\psi(O_m) = O_m$ such that for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ with $\varepsilon_i > 0$ for $i \in \{1, \ldots, m\}$ and any ε-solution $y^* \in X$ of (7), i.e.,

$$d\left(y^*, f\left(y^*\right)\right) \le \varepsilon,$$

there exists a solution x^* of (7) such that

$$d\left(x^*, y^*\right) \le \psi(\varepsilon).$$

In particular, if $\psi(t) = C \cdot t$, $t \in \mathbb{R}^m_+$ (where $C \in \mathscr{M}_{\mathrm{m,m}}\left(\mathbb{R}_+\right)\big)$, then the fixed point equation (7) is called Ulam-Hyers stable.

Direct consequence of Perov theorem is Ulam-Hyers stability of the equation $f(x) = x$ where f is a Perov contraction.

Theorem 10 *Let (X, d) be a generalized metric space, and let $f : X \to X$ be an operator with the property that there exists a matrix $A \in \mathscr{M}_{m,m}(\mathbb{R})$ such that A converges to zero and*

$$d(f(x), f(y)) \le Ad(x, y), \text{ for all } x, y \in X.$$

Then the fixed point equation

$$x = f(x), x \in X$$

is Ulam-Hyers stable.

Proof From Perov's theorem, the equation $f(x) = x$, where f is a Perov contraction, has a unique fixed point $x^* \in X$. If we assume that, for arbitrary $\varepsilon \in \mathbb{R}^m_+$, $y \in X$ is ε-solution of this equation, i.e., $d(y, f(y)) \le \varepsilon$, then let us estimate $d(x^*, y)$. First, the inequality

$$\begin{aligned} d(y, f^n(y)) &\le \sum_{i=0}^{n-1} d(f^i(y), f^{i+1}(y)) \\ &\le \sum_{i=0}^{n-1} A^i \left(d(y, f(y))\right), \end{aligned}$$

indicating

$$\begin{aligned} d(x^*, y) &\le d(x^*, f^n(y)) + \sum_{i=0}^{n-1} d(f^i(y), f^{i+1}(y)) \\ &\le d(x^*, f^n(y)) + \sum_{i=0}^{n-1} A^i\left(d(y, f(y))\right). \end{aligned}$$

In accordance to Perov's theorem, the iterative sequence $(f^n(y))$ converges to the fixed point x^*; thus, we may choose $n \in \mathbb{N}$ such that $d(x^*, f^n(y)) \le A(\varepsilon)$. Therefore,

$$\begin{aligned} d(x^*, y) &\le A(\varepsilon) + \sum_{i=0}^{n-1} A^i\left(d(y, f(y))\right) \\ &\le A(\varepsilon) + \sum_{i=0}^{\infty} A^i\left(\varepsilon\right)) \\ &= (A + (I - A)^{-1})(\varepsilon). \end{aligned}$$

Denote $\psi = A + (I - A)^{-1}$; then it is increasing and continuous at zero, and $\psi(O_m) = O_m$. Consequently, $f(x) = x$ is a generalized Ulam-Hyers-stable equation.

We can also talk about stability of fixed point equation $f(x) = x$ where f is a weakly Picard operator.

Theorem 11 *Let (X, d) be a generalized metric space, and let $f : X \to X$ be a ψ-weakly Picard operator. Then, the fixed point equation (7) is generalized Ulam-Hyers stable.*

In [83] are proved several results on existence, uniqueness, and Ulam-Hyers stability results for the coupled fixed point of a pair of contractive-type operators on complete generalized metric spaces observing Perov-type almost contractions and similar results.

Definition 11 Let (X, d) be a metric space, and let $T_1, T_2 : X \times X \to X$ be two operators. Then the operatorial equation system

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases} \tag{8}$$

is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each solution-pair $(u^*, v^*) \in X \times X$ of the inequations

$$d\left(u^*, T_1\left(u^*, v^*\right)\right) \le \varepsilon_1$$
$$d\left(v^*, T_2\left(u^*, v^*\right)\right) \le \varepsilon_2$$

there exists a solution $(x^*, y^*) \in X \times X$ of (8) such that

$$d\left(u^*, x^*\right) \le c_1\varepsilon_1 + c_2\varepsilon_2$$

$$d\left(v^*, y^*\right) \le c_3\varepsilon_1 + c_4\varepsilon_2$$

We recall the following existence, uniqueness, data dependence, and Ulam-Hyers stability theorem for the coupled fixed point of a pair of single-valued operators that more extensively deals with Ulam-Hyers stability of coupled fixed point problem for Perov contraction.

Theorem 12 ([84]) *Let* (X, d) *be a complete metric space, and let operators* $T_1, T_2 : X \times X \to X$ *be such that*

$$\begin{aligned} d\left(T_1(x, y), T_1(u, v)\right) &\le k_1 d(x, u) + k_2 d(y, v) \\ d\left(T_2(x, y), T_2(u, v)\right) &\le k_3 d(x, u) + k_4 d(y, v) \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$. *We suppose that* $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ *converges to zero. Then*

(i) *there exists a unique element* $(x^*, y^*) \in X \times X$ *such that*

$$\begin{cases} x^* = T_1\left(x^*, y^*\right) \\ y^* = T_2\left(x^*, y^*\right) \end{cases}$$

(ii) *the sequence* $\left(T_1^n(x, y), T_2^n(x, y)\right)$ *converges to* (x^*, y^*) *as* $n \to \infty$, *where*

$$\begin{aligned} T_1^{n+1}(x, y) &= T_1^n\left(T_1(x, y), T_2(x, y)\right) \\ T_2^{n+1}(x, y) &= T_2^n\left(T_1(x, y), T_2(x, y)\right) \end{aligned}$$

for all $n \in \mathbb{N}^*$

(iii) *we have the following estimation:*

$$\begin{pmatrix} d\left(T_1^n\left(x_0, y_0\right), x^*\right) \\ d\left(T_2^n\left(x_0, y_0\right), y^*\right) \end{pmatrix} \le A^n (I - A)^{-1} \begin{pmatrix} d\left(x_0, T_1\left(x_0, y_0\right)\right) \\ d\left(y_0, T_2\left(x_0, y_0\right)\right) \end{pmatrix}$$

(iv) *let* $F_1, F_2 : X \times X \to X$ *be two operators such that there exist* $\eta_1, \eta_2 > 0$ *with*

$$\begin{aligned} d\left(T_1(x, y), F_1(x, y)\right) &\le \eta_1 \\ d\left(T_2(x, y), F_2(x, y)\right) &\le \eta_2 \end{aligned}$$

for all $(x, y) \in X \times X$. *If* $(a^*, b^*) \in X \times X$ *is such that*

$$\begin{cases} a^* = F_1\left(a^*, b^*\right) \\ b^* = F_2\left(a^*, b^*\right) \end{cases}$$

then

$$\begin{pmatrix} d\left(a^*, x^*\right) \\ d\left(b^*, y^*\right) \end{pmatrix} \leq (I - A)^{-1}\eta$$

where $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$

(v) *let* $F_1, F_2 : X \times X \to X$ *be two operators such that there exist* $\eta_1, \eta_2 > 0$ *with*

$$\begin{aligned} d\left(T_1(x, y), F_1(x, y)\right) &\leq \eta_1 \\ d\left(T_2(x, y), F_2(x, y)\right) &\leq \eta_2 \end{aligned}$$

for all $(x, y) \in X \times X$, *and considering the sequence* $\left(F_1^n(x, y), F_2^n(x, y)\right)$ *where*

$$\begin{aligned} F_1^{n+1}(x, y) &= F_1^n\left(F_1(x, y), F_2(x, y)\right) \\ F_2^{n+1}(x, y) &= F_2^n\left(F_1(x, y), F_2(x, y)\right) \end{aligned}$$

for all $n \in \mathbb{N}^*$ *and* $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, *then*

$$\begin{pmatrix} d\left(F_1^n\left(x_0, y_0\right), x^*\right) \\ d\left(F_2^n\left(x_0, y_0\right), y^*\right) \end{pmatrix} \leq (I - A)^{-1}\eta + A^n(I - A)^{-1}\begin{pmatrix} d\left(x_0, T_1\left(x_0, y_0\right)\right) \\ d\left(y_0, T_2\left(x_0, y_0\right)\right) \end{pmatrix}$$

(vi) *the operatorial equation system*

$$\begin{aligned} x &= T_1(x, y) \\ y &= T_2(x, y) \end{aligned}$$

is Ulam-Hyers stable.

We will return to Ulam-Hyers stability issue once again when talking about generalized Ulam-Hyers stability on cone metric spaces.

By considering the recent technique of Jleli and Samet [48], Altun et al. [8] present a new generalization of the Perov fixed point theorem. It is known in the literature as a θ-contraction, and several results are published on this topic for metric and b-metric spaces.

Let $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ be a function, where $\mathbb{R}^m_{+j}$ is the set of $m \times 1$ real matrices with every element being greater than j.
For the sake of completeness, we will consider the following conditions:

(Θ1) Θ is nondecreasing in each variable, i.e., for all $\alpha = (\alpha_i)_{i=1}^m$ and for $\beta = (\beta_i)_{i=1}^m \in \mathbb{R}^m_{+0}$ such that $\alpha \leq \beta$, then $\Theta(\alpha) \leq \Theta(\beta)$.

(Θ2) For each sequence $(\alpha_n) = \left(\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(m)}\right)$ of $\mathbb{R}^m_{+0}$

$$\lim_{n\to\infty} \alpha_n^{(i)} = 0^+ \text{if and only if } \lim_{n\to\infty} \beta_n^{(i)} = 1$$

for each $i \in \{1, 2, \ldots, m\}$, where

$$\Theta\left(\left(\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(m)}\right)\right) = \left(\beta_n^{(1)}, \beta_n^{(2)}, \ldots, \beta_n^{(m)}\right).$$

(Θ3) There exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{\alpha_i \to 0^+} \frac{\beta_i - 1}{\alpha_i^r} = l$ for each $i \in \{1, 2, \ldots, m\}$, where

$$\Theta\left((\alpha_1, \alpha_2, \ldots, \alpha_m)\right) = (\beta_1, \beta_2, \ldots, \beta_m).$$

We denote by Ξ^m the set of all functions Θ satisfying (Θ1) − (Θ3).
Let us define notation $\Lambda^{[k]} := \left(\Lambda_i^{k_i}\right)_{i=1}^m$ for $\Lambda = (\Lambda_i)_{i=1}^m \in \mathbb{R}^m_+$ and $k = (k_i)_{i=1}^m \in \mathbb{R}^m_+$. By considering the class Ξ^m, we introduce the concept of Perov-type Θ-contraction as follows:

Definition 12 Let (X, d) be a generalized metric space and $T : X \to X$ a mapping. If there exist $\Theta \in \Xi^m$ and $k = (k_i)_{i=1}^m \in \mathbb{R}^m_+$with $k_i < 1$ for all $i \in \{1, 2, \ldots, m\}$ such that

$$\Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^{[k]}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0_m$, then a mapping T is called a Perov-type Θ-contraction.

For this class of mappings, we can prove existence and uniqueness theorem.

Theorem 13 ([9]) *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a Perov-type Θ-contraction; then T has a unique fixed point.*

Proof For some $x_0 \in X$, define sequence a $x_n = T^n(x_n)$, $n \in \mathbb{N}$, and assume that $x_n \neq x_{n+1}$, $n \in \mathbb{N}_0$; otherwise, $x_n \in Fix(T)$.
For the convenience, introduce notations $d(x_n, x_{n+1}) = (a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(m)})$ and $\Theta(d(x_n, x_{n+1})) = (b_n^{(1)}, b_n^{(2)}, \ldots, b_n^{(m)})$, $n \in \mathbb{N}$; then

$$\begin{aligned}(b_n^{(1)}, b_n^{(2)}, \ldots, b_n^{(m)}) &= \Theta(d(T(x_{n-1}), T(x_n))) \\ &\leq [\Theta(d(x_{n-1}, x_n))]^k \\ &= \left(b_{n-1}^{(1)}, b_{n-1}^{(2)}, \ldots, b_{n-1}^{(m)}\right)^{[k]},\end{aligned}$$

that yields to the conclusion

$$b_n^{(i)} \leq (b_{n-1}^{(i)})^{k_i} \leq \ldots \leq (b_0^{(i)})^{{k_i}^n}, \ i = \overline{1, m}, \ n \in \mathbb{N}. \tag{9}$$

This means $\lim_{n\to\infty} b_n^{(i)} = 1$ and $\lim_{n\to\infty} a_n^{(i)} = 0^+$, $i = \overline{1, m}$.
Taking into the account $(\Theta 2)$, there exists some $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{b_n^{(i)} - 1}{(a_n^{(i)})^r} = l.$$

If $l < \infty$, then there exists some $n_0 \in \mathbb{N}$ that inequality

$$\left| \frac{b_n^{(i)} - 1}{(a_n^{(i)})^r} - l \right| \leq \frac{l}{2}$$

for any $n \geq n_0$.
Furthermore, $\frac{b_n^{(i)}-1}{(a_n^{(i)})^r} \geq \frac{l}{2}$ and, if $L = \frac{l}{2}$

$$L(a_n^{(i)})^r \leq b_n^{(i)} - 1. \tag{10}$$

If $l = \infty$, for L from the previous case, there exists $n_0 \in \mathbb{N}$

$$\left| \frac{b_n^{(i)} - 1}{(a_n^{(i)})^r} - 2L \right| \leq L, \ n \geq n_0,$$

and (10) holds. Considering (9) and multiplying Eq. (10) with n,

$$Ln(a_n^{(i)})^r \leq n((b_n^{(i)})^{k_i^n} - 1),$$

we get $\lim_{n\to\infty} n(a_n^{(i)})^r = 0$. For $i \in \{1, \ldots, n\}$, there exists $n(i) \in \mathbb{N}$ such that $n(a_n^{(i)})^r \leq 1$ for all $n \geq n(i)$. If $n_0 = \max\{n(i) \mid i = \overline{1, m}\}$, then

$$a_n^{(i)} \leq \frac{1}{n^{1/r}}, \ i = \overline{1, m}, \ n \geq n_0.$$

For $p, q \geq n_0$, the following inequalities hold:

$$d(x_p, x_q) \leq \sum_{j=p}^{q-1} d(x_j, x_{j+1})$$

$$= \left(\sum_{j=p}^{q-1} a_j^{(1)}, \ldots, \sum_{j=p}^{q-1} a_j^{(m)} \right)$$

$$\leq \left(\sum_{j=p}^{\infty} a_j^{(1)}, \ldots, \sum_{j=p}^{\infty} a_j^{(m)} \right)$$

$$\leq \left(\sum_{j=p}^{\infty} \frac{1}{j^{1/r}}, \ldots, \sum_{j=p}^{\infty} \frac{1}{j^{1/r}} \right).$$

Convergence of the series $\sum_{j=1}^{\infty} \frac{1}{j^{1/r}}$ gives us a conclusion that (x_n) is a Cauchy sequence, thus convergent, and observe the limit of (x_n), $x^* \in X$. Function Θ is nondecreasing in each variable and

$$\Theta(d(Tx_n, Tx^*)) \leq [\Theta(d(x_n, x^*))]^k \leq \Theta(d(x_n, x^*)).$$

Accordingly, x^* is a fixed point of T, and if $Ty = y$, then $\Theta(d(y, x^*)) = \Theta(d(Ty, Tx^*)) \leq [\Theta(d(y, x^*))]^k \leq \Theta(d(y, x^*))$, so fixed point of T is unique.

Remark 5 If $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ is defined with

$$\Theta\left((\alpha_1, \alpha_2, \ldots, \alpha_m)\right) = \left(e^{\sqrt{\alpha_1}}, e^{\sqrt{\alpha_2}}, \ldots, e^{\sqrt{\alpha_m}}\right), \ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m_{+0}$$

in Theorem 13, we obtain Theorem 2 with

$$A = \begin{pmatrix} k_1^2 & 0 & \ldots & 0 \\ 0 & k_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & k_m^2 \end{pmatrix}_{m \times m}.$$

Here, since $\max\left\{k_i \mid i = \overline{1, m}\right\} < 1$, the matrix A is convergent to zero.

It is natural to apply Theorem 13 on solving semilinear operator system. It will also be done for some other types of generalized Perov contractions, so it is left to the reader to compare these results.
Let $(E, \|\cdot\|)$ be a Banach space and $T_1, T_2 : E^2 \to E$ be two nonlinear operators. In this section, we will give an existence result for a semilinear operator system of the form (8).

Again, we discuss about coupled fixed point problem. Since initial or boundary value problems for nonlinear differential systems can be written in the operator form of Eq. (8), such systems appear very often in different fields of mathematics and science overall. We can see that various fixed point theorems such as Schauder, Leray-Schauder, Krasnoselskii, and Perov fixed point theorems were applied in proving the existence of solutions of such systems in [67]. Let $X = E^2$, and define $d : X \times X \to \mathbb{R}^2$, for $u = (x_1, y_1), v = (x_2, y_2) \in X \times X$ by $d(u, v) = (\|x_1 - x_2\|, \|y_1 - y_2\|)$. Then it can be seen that (X, d) is a complete generalized metric space. If we define a mapping $S : X \to X$ by $Su = (T_1 u, T_2 u)$, then Eq. (8) can be written as a fixed point problem

$$Su = u \tag{11}$$

in the space X. Therefore, we will use Theorem 13 to investigate the sufficient conditions that guarantee the existence of a solution of the fixed point problem (11).

Theorem 14 *Let $(E, \|\cdot\|)$ be a Banach space and $A, B : E^2 \to E$ be two nonlinear operators. Assume that there exist a function $\Theta \in \Xi^2$ and a constant $\gamma \in (0, 1)$ such that*

$$\Theta(\|Au - Av\|, \|Bu - Bv\|) \leq [\Theta(\|x_1 - x_2\|, \|y_1 - y_2\|)]^{[k]} \tag{12}$$

where $k = (\gamma, \gamma)$, for all $u = (x_1, y_1)$ and $v = (x_2, y_2) \in E^2$ with $Au \neq Av$. Then Eq. (8) has a unique solution in E^2.

The proof goes similarly as the proof of Theorem 13 since this could be observed as a direct consequence.

Remark 6 Note that, if there exists a constant $\gamma < 1$ such that

$$\max \left\{ \begin{array}{l} \frac{\|A(x_1,y_1)-A(x_2,y_2)\|}{\|x_1-x_2\|}, \\ \frac{\|B(x_1,y_1)-B(x_2,y_2)\|}{\|y_1-y_2\|} e^{\{\|B(x_1,y_1)-B(x_2,y_2)\|-\|y_1-y_2\|\}} \end{array} \right\} \leq \gamma$$

for all $u = (x_1, y_1)$ and $v = (x_2, y_2) \in E^2$ with $x_1 \neq x_2$ and $y_1 \neq y_2$, then we get Eq. (12) with the function

$$\Theta(\alpha_1, \alpha_2) = \left(e^{\sqrt{\alpha_1}}, e^{\sqrt{\alpha_2 e^{\alpha_2}}} \right)$$

and $k = (\sqrt{\gamma}, \sqrt{\gamma})$.

Altun and Olgun ([9]) followed the idea of Wardowski [85] and obtained a class of F-Perov contractions.
Let $F : \mathbb{R}_+^m \to \mathbb{R}^m$ be a function. Consider the following conditions:

(F1) F is strictly increasing in each variable, i.e., for all $\alpha = (\alpha_i)_{i=1}^m, \beta = (\beta_i)_{i=1}^m \in \mathbb{R}_+^m$, such that $\alpha < \beta$, and then , $F(\alpha) < F(\beta)$

(F2) For each sequence $\{\alpha_n\} = \left(\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(m)}\right)$ of $\mathbb{R}_+^m$, assume that $\lim_{n\to\infty} \alpha_n^{(i)} = 0$ if and only if $\lim_{n\to\infty} \beta_n^{(i)} = -\infty$, $i \in \{1, 2, \ldots m\}$, where

$$F\left(\left(\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(m)}\right)\right) = \left(\beta_n^{(1)}, \beta_n^{(2)}, \ldots, \beta_n^{(m)}\right), \ n \in \mathbb{N}.$$

(F3) There exists $k \in (0,1)$, such that $\lim_{\alpha_i \to 0^+} \alpha_i^k \beta_i = 0$ for each $i \in \{1, 2, \dots m\}$, where

$$F\left((\alpha_1, \alpha_2, \dots, \alpha_m)\right) = \left(\beta_1, \beta_2, \cdots, \beta_m\right).$$

We denote by $\mathscr{F}^m$ the set of all functions F satisfying (F1) − (F3).

Taking into the account a class $\mathscr{F}^m$, Perov-type F-contraction is defined as follows:

Definition 13 Let (X, d) be a generalized metric space and $T : X \to X$ be a map. If there exist $F \in \mathscr{F}^m$ and $\tau = (\tau_i)_{i=1}^m \in \mathbb{R}_+^m$, such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$ with $d(Tx, Ty) > 0_m$, then a mapping T is called a Perov-type F-contraction.

We will state a fixed point theorem for F-contractions, and later on for Ψ-contraction, but not the proof since the proof idea and techniques are very similar to Θ-contraction case.

Theorem 15 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a Perov-type F-contraction. Then, T has a unique fixed point.*

Theorem 15 will be used to find sufficient conditions that guarantee the existence of a solution of the fixed point problem (8).

Theorem 16 *Let $(E, \|\cdot\|_E)$ be a Banach space and $A, B : E^2 \to E$ be two nonlinear operators. Assume that there exist positive numbers $\tau_i \in \mathbb{R}_+$ for $i = 1, 2$, such that*

$$\frac{\|A(x_1, y_1) - A(x_2, y_2)\|_E}{\|x_1 - x_2\|_E} e^{\|x_1 - x_2\|_E - \|A(x_1,y_1) - A(x_2,y_2)\|_E} \leq e^{-\tau_1}$$

and

$$\|B(x_1, y_1) - B(x_2, y_2)\|_E \leq e^{-\tau_2} \|y_1 - y_2\|_E$$

for all $u = (x_1, y_1)$ and $v = (x_2, y_2) \in E^2$ with $x_1 \neq x_2$. Then, the system (15) has a unique solution in E^2.

Altun and Qasim [10], similar to the case of Θ-contraction and F-contraction of Perov type, and by following [48], gave a new generalization of Perov fixed point theorem combining Perov result with a ψ-contractions. This result is applied on solving complex partial differential equation

$$\partial_{\bar{z}} w = F(z, w, \partial_z w).$$

Let $\Psi : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ be a function. For the sake of completeness, we will consider the following conditions:

(Ψ_1) Ψ is nondecreasing in each variable, that is, for all $u = (u_i)\,, v = (v_i) \in \mathbb{R}^m_{+0}$ such that $u \leqslant v$, then $\Psi(u) \leqslant \Psi(v)$.

(Ψ_2) For each sequence $(u_v) = \left(u_v^{(1)}, u_v^{(2)}, \ldots, u_v^{(m)}\right)$ of $\mathbb{R}^m_{+0}$

$$\lim_{v\to+\infty} u_v^{(i)} = 0^+ \text{ if and only if } \lim_{v\to+\infty} v_v^{(i)} = 1$$

for each $i \in \{1, 2, \ldots m\}$, where

$$\Psi\left(\left(u_v^{(1)}, u_v^{(2)}, \ldots, u_v^{(m)}\right)\right) = \left(v_v^{(1)}, v_v^{(2)}, \ldots, v_v^{(m)}\right).$$

(Ψ_3) There exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that $\lim_{u_i\to 0^+} \frac{v_i - 1}{u_i^r} = l$ for each $i \in \{1, 2, \ldots m\}$, where

$$\Psi\left((u_1, u_2, \ldots, u_m)\right) = (v_1, v_2, \ldots, v_m)\,.$$

Denote by Ψ^m the set of all functions Ψ satisfying $(\Psi_1) - (\Psi_3)$.

Definition 14 Let X be a nonempty set, $\Lambda : X \times X \to \mathscr{M}_{m,m}(\mathbb{R})$ and $F : X \to X$ mappings. The function F is called Λ -admissible if for all $x, y \in X$

$$\Lambda(x, y) \gg I \text{ implies } \Lambda(Fx, Fy) \gg I$$

where I is the $m \times m$ identity matrix.

Definition 15 Let (X, d) be a generalized metric space, and let $\Lambda : X \times X \to \mathscr{M}_{m,m}(\mathbb{R})$ be a function. If for each sequence $(t_n) \subseteq X$ such that $\lim_{n\to\infty} t_n = t$ and $\Lambda\left(t_n, t_{n+1}\right) \gg I$ for all $n \in \mathbb{N}$ implies that there exists $n_0 \in \mathbb{N}$ such that $\Lambda\left(t_n, t\right) \gg I$ for all $n \geq n_0$, then the space (X, d) is called Λ-regular.

Let (X, d) be a generalized metric space, $F : X \mapsto X$ be a mapping, and $\Lambda : X \times X \to \mathscr{M}^m_{m,m}(\mathbb{R})$ be a function. We will include the following set in the contractive condition:

$$\Omega(\Lambda, F) = \{(t, s) \subseteq X \times X : \Lambda(t, s) \gg I \text{ and } d(Ft, Fs) > 0_m\}.$$

Perov-type Ψ-contraction, under some strict conditions, has a fixed point.

Theorem 17 ([8]) *Let (X, d) be a complete generalized metric space, $F : X \to X$ be a mapping, $\Psi \in \Psi^m$, and $\Lambda : X \times X \to \mathscr{M}_{m,m}(\mathbb{R})$ be a function. Suppose that there exists $\gamma = (\gamma_i) \in \mathbb{R}^m_+$ with $\gamma_i < 1$ for all $i \in \{1, 2, \ldots, m\}$ such that*

$$\Psi(d(Ft, Fs)) \leq [\Psi(d(t, s))]^{[\gamma]}$$

for all $(t, s) \in \Omega(\Lambda, F)$*. Then* F *has a fixed point in* X *assuming that the following conditions hold:*

(1) There exists $t_0 \in X$ *such that* $\Lambda(t_0, Ft_0) \gg I$*;*
(2) F *is* Λ *-admissible;*
(3) F *is continuous or* (X, d) *is* Λ *-regular.*

Corollary 2 ([8]) Let(X, d) *be a complete generalized metric space,* $F : X \to X$ *be a mapping, and* $\Psi \in \Psi^m$*. Suppose that there exists* $\gamma = (\gamma_i) \in \mathbb{R}^m_+$ *with* $\gamma_i < 1$ *for all* $i \in \{1, 2, \ldots, m\}$ *such that*

$$\Psi(d(Ft, Fs)) \leqslant [\Psi(d(t, s))]^{[\gamma]}$$

for all $(t, s) \in X \times X$ *such that* $d(Ft, Fs) > 0_m$*. Then* F *has a fixed point in* X*.*

Corollary 3 *Let* (X, d) *be a complete generalized metric space,* $F : X \to X$ *be a mapping, and* $\Lambda : X \times X \to \mathscr{M}_{m,m}(\mathbb{R})$ *be a function. Suppose that*

$$d(Ft, Fs) \preccurlyeq Ad(t, s)$$

hold for all $(t, s) \in \Omega(\Lambda, F)$*. Then* F *has a fixed point in* X *provided that the following conditions hold:*

(1) There exists $t_0 \in X$ *such that* $\Lambda(t_0, Ft_0) \gg I$*;*
(2) F *is* Λ *-admissible;*
(3) F *is continuous, or* (X, d) *is* Λ*-regular.*

3 Perov Fixed Point Theorem on Cone Metric Spaces

In the setting of a cone metric space (X, d), distance is a vector in a Banach space E, and therefore contractive constant q from the well-known Banach contraction can be replaced with some operator $A : E \mapsto E$. Accordingly, for some $f : X \mapsto X$, the inequality

$$d(f(x), f(y)) \preceq A(d(x, y)),\ x, y \in X,$$

defines a new kind of contractions which we will name Perov-type contractions. It remains to determine necessary conditions for the operator A that will guarantee existence of a fixed point of a Perov-type contractions. Uniqueness of the fixed point will be also discussed. Taking into the account previously stated Perov theorem, we may suppose that A should be positive operator on cone metric space and A^n should tend to zero operator when $n \to \infty$.

Discussing linear operators on a Banach space E, it is important to emphasize that the class of positive and the class of increasing operators coincide. Remark that, without linearity, only inclusion remains.

Lemma 2 ([26]) *Let E be Banach space, $P \subseteq E$ cone in E, and $A : E \mapsto E$ a linear operator. The following conditions are equivalent:*

(i) A is increasing, i.e., $x \preceq y$ implies $A(x) \preceq A(y)$.

(ii) A is positive, i.e., $A(P) \subseteq P$.

Proof If A is monotonically increasing and $p \in P$, then, by definitions, it follows $p \succeq \theta$ and $A(p) \succeq A(\theta) = \theta$. Thus, $A(p) \in P$, and $A(P) \subseteq P$.
To prove the other implication, let us assume that $A(P) \subseteq P$ and $x, y \in E$ are such that $x \preceq y$. Now $y - x \in P$, and so $A(y - x) \in P$. Thus, $A(x) \preceq A(y)$.

The results of the following theorem apply to the cone metric spaces in the case when cone is not necessary normal, and Banach space should not be finite dimensional. This extends the results of Perov for matrices [61, 62] and as a corollary generalizes Theorem 1 of Zima [86].

Theorem 18 ([26]) *Let (X, d) be a complete solid cone metric space, $d : X \times X \mapsto E$, and $f : X \mapsto X$, $A \in \mathscr{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \preceq Ad(x, y), \quad x, y \in X. \tag{13}$$

Then:

(i) f has a unique fixed point $x^* \in X$;
(ii) For any $x_0 \in X$, the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to x^* and

$$d(x_n, x^*) \preceq A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N};$$

(iii) Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \preceq c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0)$, $n \in \mathbb{N}$,

$$d(y_n, x^*) \preceq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Proof (i) For $n, m \in \mathbb{N}$, $m > n$, the inequality

$$\theta \preceq d(x_n, x_m) \preceq \sum_{i=n}^{m-1} A^i(d(x_0, x_1)) \preceq \sum_{i=n}^{\infty} A^i(d(x_0, x_1)),$$

along with $r(A) < 1$, implies

$$\|\sum_{i=n}^{\infty} A^i(d(x_0, x_1))\| \leq \sum_{i=n}^{\infty} \|A^i\| \|(d(x_0, x_1))\| \to 0,\ n \to \infty.$$

Thus, $a_n = \sum_{i=n}^{\infty} A^i(d(x_0, x_1)) \to \theta$, $n \to \infty$, and, by Lemma 1, (x_n) is a Cauchy sequence. Since X is a complete cone metric space, there exists the limit $x^* \in X$ of sequence (x_n).

Let us prove that $f(x^*) = x^*$. Set $p = d(x^*, f(x^*))$, and suppose that $c \gg \theta$ and $\varepsilon \gg \theta$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, x_n) \ll c \text{ and } d(x^*, x_n) \ll \varepsilon \text{ for all } n \geq n_0.$$

Therefore, $p = d(x^*, f(x^*)) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, f(x^*)) \preceq d(x^*, x_{n+1}) + A(d(x^*, x_n)) \preceq c + A(\varepsilon)$ for $n \geq n_0$. Thus, $p \preceq c + A(\varepsilon)$ for each $c \gg 0$, and so $p \preceq A(\varepsilon)$. For $\varepsilon = \varepsilon/n$, $n \in \mathbb{N}$, we get

$$\theta \preceq p \preceq A\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n},\ n \in \mathbb{N}.$$

Due to $\frac{A(\varepsilon)}{n} \to \theta$, $n \to \infty$, it follows $p = \theta$. Consequently, $x^* = f(x^*)$.

If $f(y) = y$, for some $y \in X$, then $d(x^*, y) \preceq A(d(x^*, y))$. Accordingly, $d(x^*, y) \preceq A^n(d(x^*, y))$ for each $n \in \mathbb{N}$ and $r(A) < 1$ imply

$$\|A^n(d(x^*, y)\| \leq \|A^n\| \|(d(x^*, y)\| \to 0,\ n \to \infty,$$

so, $d(x^*, y) = \theta$ and $x^* = y$.

(ii) Depending on (i), for any $n \in \mathbb{N}$, we have

$$d(x_n, x^*) \preceq A(d(x_{n-1}, x^*)) \preceq \cdots \preceq A^n(d(x_0, x^*)).$$

On the other hand,

$$\begin{aligned} d(x_0, x^*) &\preceq d(x_0, x_n) + d(x_n, x^*) \\ &\preceq \sum_{i=0}^{n-1} d(x_i, x_{i+1}) + A^n(d(x_0, x_1)) + A^n(d(x_1, x^*)) \\ &\preceq \sum_{i=0}^{i=n} A^i(d(x_0, x_1)) + A^n(d(x_1, x^*)). \end{aligned}$$

Since $A^n(d(x_1, x^*)) \to \theta$, $n \to \infty$, we get

$$d(x_0, x^*) \leq \sum_{i=0}^{\infty} A^i(d(x_0, x_1)) = (I - A)^{-1}(d(x_0, x_1)),$$

and $d(x_n, x^*) \preceq A^n(I - A)^{-1}(d(x_0, x_1))$.

(iii) For any $n \in \mathbb{N}$, $d(y_n, x^*) \preceq d(y_n, x_n) + d(x_n, x^*)$, along with (ii), implies

$$d(y_n, x^*) \preceq d(y_n, x_n) + A^n(I - A)^{-1}(d(x_0, x_1)).$$

Hence,

$$\begin{aligned} d(y_n, x_n) &\preceq d(y_n, f(y_{n-1})) + d(f(y_{n-1}), x_n) \\ &\preceq c + A(d(y_{n-1}, x_{n-1})) \\ &\preceq c + A\Big(d(y_{n-1}, f(y_{n-2})) + d(f(y_{n-2}), x_{n-1})\Big) \\ &\preceq c + A(c) + A^2(d(y_{n-2}, x_{n-2})) \\ &\preceq \ldots \preceq \sum_{i=0}^{n-1} A^i(c) \\ &\preceq (I - A)^{-1}(c). \end{aligned}$$

Remark 19 *Let us remark that the initial assumption $A \in \mathscr{M}_{n,n}(\mathbb{R}_+)$, in Perov theorem, is unnecessary. This will be illustrated by the following example.*

Example 5 Let

$$A = \begin{bmatrix} ccc\frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$X = \left\{ \begin{bmatrix} cx_1 \\ 1 \\ x_3 \end{bmatrix} \mid x \in \mathbb{R} \right\} \text{ and } f : X \mapsto X,\ f\left(\begin{bmatrix} cx_1 \\ 1 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} c\frac{x_1+1}{2} \\ 1 \\ \frac{x_3+2}{3} \end{bmatrix}.$$

Set $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ for $x = \begin{bmatrix} cx_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x_i \in \mathbb{R}$, $i = 1, 2, 3$.

For arbitrary $x \in X$,

$$\begin{aligned} \|Ax\| &= \max\left\{ |\frac{1}{2}x_1 - \frac{1}{4}x_2|, |\frac{1}{4}x_1 - \frac{1}{2}x_2|, |\frac{1}{2}x_3| \right\} \\ &\le \max\left\{ \frac{1}{2}\|x\| + \frac{1}{4}\|x\|, \frac{1}{4}\|x\| + \frac{1}{2}\|x\|, \frac{1}{2}\|x\| \right\} = \frac{3}{4}\|x\|. \end{aligned}$$

Therefore, $\|A\| \le \frac{3}{4}$. If $x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\|x\| = 1$, then $\|Ax\| = \frac{3}{4}$. Thus, $\|A\| = \frac{3}{4}$.

Evidently, $r(A) \leq \|A\| = 3/4$ and $d(f(x), f(y)) \preceq A(d(x, y))$, $x, y \in X$. Despite $A(P) \nsubseteq P$, $(1, 1, 1)$ is a unique fixed point of f in X.

Based on the previous comments, we obtain the next result, where we do not suppose that $A(P) \subseteq P$.

Theorem 20 ([26]) *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with normal constant K, $A \in \mathscr{B}(E)$, and $K\|A\| < 1$. If the condition (13) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $x^* \in X$, and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$ converges to x^* for any $x_0 \in X$.*

Proof Let $x_0 \in X$ be arbitrary, $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Inequality

$$d(x_n, x_{n+1}) \preceq A(d(x_{n-1}, x_n)),\ n \in \mathbb{N}$$

implies

$$\begin{aligned}\|d(x_n, x_{n+1})\| &\leq K\|A(d(x_{n-1}, x_n))\| \leq K\|A\|\|d(x_{n-1}, x_n)\| \\ &\leq K^2\|A\|^2\|d(x_{n-2}, x_{n-1})\| \leq \ldots \leq K^n\|A\|^n\|d(x_0, x_1)\|.\end{aligned}$$

If $n, m \in \mathbb{N}, n < m$, then

$$\|d(x_n, x_m)\| \leq \sum_{i=n}^{m-1} \|d(x_i, x_{i+1})\| \leq \sum_{i=n}^{m-1} K^i\|A\|^i\|d(x_0, x_1)\|.$$

Clearly, $K\|A\| < 1$ implies that the series $\sum_{i=0}^{\infty} K^i\|A\|^i$ is convergent, and it means that $\|d(x_n, x_m)\| \to 0$, as $n, m \to \infty$. Consequently, (x_n) is a Cauchy sequence, and there is $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Let us prove that $f(x^*) = x^*$. From $d(f(x^*), x_{n+1}) \preceq A(d(x^*, x_n))$, we get

$$\|d(f(x^*), x_{n+1})\| \leq K\|A(d(x^*, x_n))\| \leq K\|A\|\|d(x^*, x_n)\|.$$

Thus, $\lim_{n\to\infty} x_n = f(x^*)$, and $f(x^*) = x^*$.
It remains to show that x^* is a unique fixed point of f.
If $f(y) = y$, for some $y \in X$ then $d(x^*, y) = d(f(x^*), f(y)) \preceq A(d(x^*, y))$ it follows $d(x^*, y) \leq KAd(x^*, y)$. Now, $K\|A\| < 1$ implies $x^* = y$.

If we take into consideration that we could ask for normal constant K to be equal to exactly 1, then Theorem 20 could be understood as an equivalent of Banach fixed point theorem [23].

Following the work of Berinde [16, 17], the existence of the fixed point for the class of Perov-type weak contraction is investigated.

Theorem 21 ([26]) *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathscr{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, $B \in \mathscr{L}(E)$ with $B(P) \subseteq P$, such that*

$$d(f(x), f(y)) \preceq A(d(x, y)) + B(d(x, f(y))),\ x, y \in X.$$

Then

(i) $f : X \mapsto X$ has a fixed point in X, and for any $x_0 \in X$, the sequence of successive approximations $(f^n(x_0))$ converges to a fixed point of f.

(ii) If additionally,

$$B \in \mathscr{B}(E) \text{ and } r(A + B) < 1,$$

or, for some $n_0 \in \mathbb{N}$,

$$d(f(x), f(y)) \preceq Ad(x, y) + B(d(x, f^{n_0}(x))),\ x, y \in X,$$

then f has a unique fixed point.

The following two theorems generalize Theorem 1 of [11] and, consequently, Theorem 2 of [60]. Recall results of Jurja and Filip and Petrusel, Theorems 4 and 6 are just a special case of Theorem 21.

Theorem 22 *Let (X, d) be a cone metric space, $P \subseteq E$ a cone, and $T : X \mapsto X$. If there exists a point $z \in X$ such that $\overline{O(z)}$ is complete, $A \in \mathscr{B}(E)$ a positive operator with $r(A) < 1$, and*

$$d(Tx, Ty) \preceq A(d(x, y)),\ \text{holdsforany}\ x, y = T(x) \in O(z), \tag{14}$$

then $(T^n z)$ converges to some $x^* \in \overline{O(z)}$ and

$$d(T^n z, x^*) \preceq A^n(I - A)^{-1}(d(z, Tz)),\ n \in \mathbb{N}.$$

If (14) holds for any $x, y \in \overline{O(z)}$, then x^* is a fixed point of T.

Theorem 22 as a corollary has Theorem 6.

Corollary 4 *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \preceq A(d(Tx, x) + d(Ty, y)),\ x, y \in X,$$

for some positive operator $A \in \mathscr{B}(E)$ with $r(A) < \frac{1}{2}$. Then T has a unique fixed point $x^* \in X$ and $(T^n x)$ converges to x^* for any $x \in X$.

Corollary 5 *Let (X, d) be a complete cone metric space and $T : X \mapsto X$ a mapping satisfying*

$$d(Tx, Ty) \preceq A(d(x, T^m z) + d(y, T^m z)),$$

for some $m \in \mathbb{N}$*,* $A \in \mathscr{B}(E)$ *positive operator,* $r(A) < 1$ *and for all* $x, y, z \in X$*. Then the iterative sequence* $(T^n x)$ *converges to a unique fixed point of* T *for any* $x \in X$*.*

There are many extensions of well-known Banach contractive condition, and most of them could be altered to suit Perov contraction. It is also important to mention that many of them are equivalent or imply each other, and that is why we define Perov-type quasi-contraction as one of the widest classes of contractive mappings.
Serbian mathematician Lj. Ćiric studied new kind of mappings such that, for some $q \in (0, 1)$ and any $x, y \in X$,

$$d(f(x), f(y)) \leq q \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\right\},$$

known as quasi-contraction or Ćirić quasi-contraction. In [32], he proved the following statement:

Theorem 23 *If* (X, d) *is a complete metric space and* $f : X \mapsto X$ *a quasi-contraction, then it possesses a unique fixed point, and the iterative sequence converges to the fixed point of* f*.*

Example that convinces us that the class of quasi-contraction is strict superset of contractions is presented in [32].

Ilić and Rakočević [46] defined a quasi-contractive mapping on a normal cone metric space and proved existence and uniqueness of a fixed point. Kadelburg, Radenović, and Rakočević [51], without the normality requirement, proved related results, but only in the case when contractive constant $q \in (0, 1/2)$. Later, Haghi, Rezapour, and Shahzad [75] and also Gajić and Rakočević [37] gave proof of the same result without the additional normality assumption and for $q \in (0, 1)$ by applying two different proof techniques. For more informations about quasi-contraction on cone metric spaces, see [38, 42, 57], etc.

Definition 16 Let (X, d) be a cone metric space. A mapping $f : X \mapsto X$ such that for some bounded linear operator $A \in \mathscr{B}(E)$, $r(A) < 1$ and for each $x, y \in X$, there exists

$$u \in C(f, x, y) \equiv \left\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\right\},$$

such that

$$d(f(x), f(y)) \preceq A(u), \tag{15}$$

is called a quasi-contraction of Perov type.

If $f : X \mapsto X$, and $n \in \mathbb{N}$, set

$$O(x;n) = \Big\{x, f(x), f^2(x), \ldots, f^n(x)\Big\},$$

and

$$O(x;\infty) = \Big\{x, f(x), f^2(x), \ldots\Big\}.$$

Denote by $\delta(O(x,n)) = \max\{\|d(a,b)\| : a, b \in O(x,n)\}$, $n \in \mathbb{N}, x \in X$ and $\delta(O(x,\infty)) = \sup_{n\in\mathbb{N}} \delta(O(x,n))$.

Theorem 24 *Let (X,d) be a complete solid cone metric space. If a mapping $f : X \mapsto X$ is a quasi-contraction and $A(P) \subseteq P$, then f has a unique fixed point, and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point of f.*

Proof The following two inequalities hold for arbitrary $x \in X$:

(i) $d(f^n(x), f(x)) \preceq (I-A)^{-1}A(d(f(x),x))$ $n \in \mathbb{N}$,
(ii) $d(f^n(x), x) \preceq (I-A)^{-1}(d(f(x),x))$ $n \in \mathbb{N}$.

Evidently, (i) is true for $n = 1$. Suppose that it is true for each $m \leq n$.
Since

$$d(f^{n+1}(x), f(x)) \preceq A(u),$$

where

$$u \in \Big\{d(f^n(x),x), d(f^n(x), f(x)), d(x, f(x)),$$
$$d(x, f^{n+1}(x)), d(f^n(x), f^{n+1}(x))\Big\},$$

we have to consider the following five different cases.

(1) If $u = d(f^n(x),x)$, then

$$\begin{aligned} d(f^{n+1}(x), f(x)) &\preceq A(d(f^n(x),x)) \\ &\preceq A(d(f^n(x), f(x))) + A(d(f(x),x)) \\ &\preceq A(I-A)^{-1}A(d(f(x),x)) + A(d(f(x),x)) \\ &= (I-A)^{-1}A(d(f(x),x)). \end{aligned}$$

(2) If $u = d(f^n(x), f(x))$, then $A(P) \subseteq P$ implies

$$d(f^{n+1}(x), f(x)) \preceq A(d(f^n(x), f(x)))$$

$$
\begin{aligned}
&\preceq A(I-A)^{-1}A(d(f(x),x))\\
&=[(A-I)+I](I-A)^{-1}A(d(f(x),x))\\
&=-A(d(f(x),x))+(I-A)^{-1}A(d(f(x),x))\\
&\preceq (I-A)^{-1}A(d(f(x),x)).
\end{aligned}
$$

(3) Clearly, for $u=d(f(x),x)$, the inequality (i) holds.
(4) Suppose that $u=d(x,f^{n+1}(x))$; then

$$d(x,f^{n+1}(x))\leq d(x,f(x))+d(f(x),f^{n+1}(x))$$

and, since $A(P)\subseteq P$,

$$d(f^{n+1}(x),f(x))\preceq A(d(x,f(x)))+A(d(f(x),f^{n+1}(x))).$$

Therefore,

$$d(f^{n+1}(x),f(x))\preceq (I-A)^{-1}A(d(x,f(x))).$$

(5) If $u=d(f^{n}(x),f^{n+1}(x))$, then

$$d(f^{n+1}(x),f(x))\preceq A(d(f^{n}(x),f^{n+1}(x)),$$

and since f is a quasi-contraction, there exist some $i,j\in\{0,1,\dots,n\}$ such that

$$d(f^{n}(x),f^{n+1}(x))\preceq A^{n-1+i}(d(f(x),f^{j}(x))).$$

However,

$$
\begin{aligned}
d(f^{n+1}(x),f(x))&\preceq A^{n+i}(d(f(x),f^{j}(x)))\\
&\preceq A^{n+i}(I-A)^{-1}A(d(f(x),x))\\
&=(I-A)^{-1}A(d(f(x),x))-\sum_{j=1}^{n+i}A^{j}(d(f(x),x))\\
&\preceq (I-A)^{-1}A(d(f(x),x)),
\end{aligned}
$$

unless $j=n+1$.
If $j=n+1$, then $d(f^{n+1}(x),f(x))\preceq A^{n+i}(d(f(x),f^{n+1}(x)))$ and more $d(f^{n+1}(x),f(x))=\theta$.
Indeed, since $I-A^{n+i}$ is an invertible operator and $A^{n+i}(P)\subseteq P$, we have

$$d(f^{n+1}(x),f(x))\preceq (I-A^{n+i})^{-1}(\theta)=\theta,$$

along with $d(f^{n+1}(x), f(x)) = \theta$.

By using the method of the mathematical induction, we have proved that the inequality (i) holds for each $n \in \mathbb{N}$.

The inequality (ii) proceeds from (i):

$$\begin{aligned} d(f^n(x), x) &\preceq d(f^n(x), f(x)) + d(f(x), x) \\ &\preceq (I - A)^{-1} A(d(f(x), x)) + d(f(x), x) \\ &= (I - A)^{-1}(d(f(x), x)),\ n \in \mathbb{N}. \end{aligned}$$

Let us prove that $(f^n(x))$ is a Cauchy sequence in X; thus, it is convergent. Suppose that $n, m \in \mathbb{N}, m > n$. Mapping f is a quasi-contraction of Perov type, so there exist $i, j \in \mathbb{N}$ such that , $1 \leq i \leq n,\ 1 \leq j \leq m$,

$$d(f^n(x), f^m(x)) \preceq A^{n-1}(d(f^i(x), f^j(x))).$$

By (i), this implies

$$d(f^n(x), f^m(x)) \preceq 2A^n(I - A)^{-1}(d(f(x), x)).$$

Nevertheless, $2A^n(I - A)^{-1}(d(f(x), x)) \to \theta,\ n \to \infty$; by Lemma 1, $(f^n(x))$ is a Cauchy sequence in X. Hence, there exists $x^* \in X$ such that $\lim_n f^n(x) = x^*$. Let us prove that x^* is a fixed point of f.
Suppose that $c \gg \theta$ and $\varepsilon \gg \theta$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(x^*, f^n(x)) \ll c,\ d(f^n(x), f^m(x)) \ll \varepsilon \\ \text{and } d(x^*, f^n(x)) \ll \varepsilon \text{ for all } n, m \geq n_0. \end{gathered} \tag{16}$$

Also, for any $n > n_0$,

$$\begin{aligned} d(x^*, f(x^*)) &\preceq d(x^*, f^n(x)) + d(f^n(x), f(x^*)) \\ &\preceq c + d(f^n(x), f(x^*)). \end{aligned} \tag{17}$$

Yet, because f is a quasi-contraction, we have

$$d(f^n(x), f(x^*)) \preceq A(u), \tag{18}$$

for some

$$u \in \Big\{ d(f^{n-1}(x), x^*), d(f^{n-1}(x), f^n(x)), d(f^{n-1}(x), f(x^*)), \\ d(x^*, f(x^*)), d(x^*, f^n(x)) \Big\}.$$

If

$$u \in \Big\{ d(f^{n-1}(x), x^*), d(f^{n-1}(x), f^n(x)), d(x^*, f^n(x)) \Big\},$$

for infinitely many $n > n_0$, then (16), (17), and (18) imply

$$d(x^*, f(x^*)) \preceq c + A(\varepsilon). \tag{19}$$

Because the inequality (19) is true for each $c \gg \theta$, we get

$$d(x^*, f(x^*)) \preceq A(\varepsilon). \tag{20}$$

If $u = d(f^{n-1}(x), f(x^*))$, then

$$d(f^{n-1}(x), f(x^*)) \preceq d(f^{n-1}(x), x^*) + d(x^*, f(x^*)),$$

and $A(P) \subseteq P$ imply

$$A(u) \preceq A(d(f^{n-1}(x), x^*)) + A(d(x^*, f(x^*))).$$

Taking into the account (16), (17), and (18), we have

$$d(x^*, f(x^*)) \preceq c + A(\varepsilon) + A(d(x^*, f(x^*))),$$

and, how $c \gg \theta$ is arbitrary,

$$(I - A)(d(x^*, f(x^*))) \preceq A(\varepsilon). \tag{21}$$

Since $(I - A)^{-1}$ is increasing, (21) implies

$$d(x^*, f(x^*)) \preceq (I - A)^{-1} A(\varepsilon). \tag{22}$$

Finally, in the case $u = d(x^*, f(x^*))$, (17) and (18) imply

$$d(x^*, f(x^*)) \preceq c + A(d(x^*, f(x^*))),$$

that is,

$$(I - A)(d(x^*, f(x^*))) \preceq c. \tag{23}$$

Again, because $(I - A)^{-1}$ is increasing, (23) implies

$$d(x^*, f(x^*)) \preceq (I - A)^{-1}(c). \tag{24}$$

Now, by (20), (22), and (24), for $\varepsilon = \varepsilon/n$ and $c = c/n$, $n \in \mathbb{N}$, it follows, respectively,

$$\theta \preceq d(x^*, f(x^*)) \preceq A\left(\frac{\varepsilon}{n}\right) = \frac{A(\varepsilon)}{n} \to \theta,\ n \to \infty,$$

$$\theta \preceq d(x^*, f(x^*)) \preceq (I - A)^{-1} A\left(\frac{\varepsilon}{n}\right) = \frac{(I - A)^{-1} A(\varepsilon)}{n} \to \theta,\ n \to \infty,$$

and

$$\theta \preceq d(x^*, f(x^*)) \preceq (I - A)^{-1} A\left(\frac{c}{n}\right) = \frac{(I - A)^{-1} A(c)}{n} \to \theta,\ n \to \infty, .$$

Hence, $d(x^*, f(x^*) = \theta$, i.e., $f(x^*) = x^*$.
If y is a fixed point of f, then

$$d(x^*, y) = d(f(x^*), f(y)) \preceq A(d(x^*, y)),$$

that is,

$$(I - A)(d(x^*, y)) \preceq \theta \implies d(x^*, y) \preceq (I - A)^{-1}(\theta) = \theta,$$

so $x^* = y$.

Theorem 24 could be combined with P property presented in [5]. It is said that the mapping f has the property P if $F(f) = F(f^n)$ for each $n \in \mathbb{N}$ (if it has no periodic points). From the proof of the previous theorem, we obtain as a corollary the extension of the known results Theorem 3.2 of [51] and Corollary 3.4 of [37].

Corollary 6 *Let (X, d) be a complete solid cone metric space. Let $f : X \mapsto X$ be a quasi-contraction of Perov type with $A(P) \subseteq P$ and $\|A\| < \frac{1}{2}$. Then f has the property* P.

Observe that the first part of Theorem 18 follows directly from Theorem 24.

Example 6 Let $X = [0, 3] \cup [4, 5]$ and $E = C^{(1)}[0, 1]$ with a non-normal cone P as in Example 4. Let us define cone metric $d : X \times X \mapsto E$ by

$$d(x, y) = |x - y| \cdot \exp,\ x, y \in X.$$

If $f : X \mapsto X$ is defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in [0, 3], \\ 3, & \text{if} \quad x \in [4, 5], \end{cases}$$

then for each $x \in [4, 5]$, we have $d(x, f(x)) \le 2 \cdot \exp$, $d(f(x), f^2(x)) = 3 \cdot \exp$. Thus, $d(f(x), f^2(x)) > d(x, f(x))$, and f does not obey the condition (13). Let us show that f fulfills the condition (15).

It is enough to consider $x \in [0, 3]$ and $y \in [4, 5]$. Now $d(f(x), f(y)) = 3 \exp$ and $d(y, f(x)) \ge 4 \cdot \exp$. Hence,

$$d(f(x), f(y)) = \frac{3}{4} \cdot 4 \cdot \exp \le \frac{3}{4} \max\{x - f(y)|, |y - f(x)|\} \cdot \exp .$$

Thus, a mapping $f : X \mapsto X$ satisfies the condition (15), where operator $A : E \mapsto E$ is a bounded linear operator defined with $A(f) = (3/4)f$, $f \in E$. Clearly, $\|A\| = 3/4$, and all the assumptions from Theorem 24 are satisfied. Accordingly, f has a unique fixed point $x = 0 \in X$.

The question that raises looking at the presented theorems on cone metric spaces is do only x and $f(x)$ have influence on existence of a fixed point. Obvious answer follows from the sequence of successive approximations that converges to a fixed point in any initial point $x \in X$. Having that in mind, it is important to somehow include more values from the orbit of f in the contractive condition. Thus, we introduce the concept of (p, q)-quasi-contraction of Perov type known also as Fisher contraction ([36]).

Fixed point in both Banach and Ćirić theorem, and many similar, is a limit of the iterative sequence. Having that in mind along with Ćiric's result, it induces a different condition:

$$d(f^p(x), f^q y) \le q \max \Big\{ d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y) \mid 0 \le r, r' \le p \quad \text{and} \quad 0 \le s, s' \le q \Big\}.$$

for some $p, q \in \mathbb{N}$ and any $x, y \in X$, determining a (p, q)-quasi-contraction. Fisher ([36]) proved that continuous (p, q)-quasi-contraction on a complete metric space possesses a unique fixed point. If $p = 1$ or $q = 1$, continuity is not necessary. Ćirić quasi-contraction is a special case for $p = q = 1$.
As in the case of generalized quasi-contraction, bounded linear operator in (p, q)-contractive condition will replace a number.

Definition 17 Let (X, d) be a cone metric space. A mapping $f : X \mapsto X$ such that for some $A \in \mathscr{B}(E)$, $r(A) < 1$ and for some fixed positive integers p and q and every $x, y \in X$, there exists

$$u \in \mathrm{F}_f^{p,q}(x, y) \equiv \Big\{ d(f^r(x), f^s(y)), d(f^r(x), f^{r'}(x)), d(f^s(y), f^{s'}(y)) \mid 0 \le r, r' \le p \text{ and } 0 \le s, s' \le q \Big\},$$

such that $d(f^p(x), f^q(y)) \preceq A(u)$, is called a (p, q)-quasi-contraction (Fisher's quasi-contraction, F quasi-contraction) of Perov type.

This theorem extends the results of Perov for matrices and also, as a corollary, generalizes Theorem 1 of Zima ([86]) along with Theorem 24.

Theorem 25 *Let (X, d) be a complete cone metric space and P a solid cone. Suppose that the mapping $f : X \mapsto X$ is a $(p, q)-$quasi-contraction of Perov type, $A(P) \subseteq P$, and f continuous. Then f has a unique fixed point in X, and for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point.*

Proof Without loss of generality, assume that $p \geq q$. If $x \in X$ be arbitrary and $\omega(x) = \sum_{0 \leq i < p} d(f^i(x), f^p(x))$, we prove that

$$d(f^n(x), f^p(x)) \preceq (I - A)^{-1} A(\omega(x)),\ n \geq p. \tag{25}$$

Obviously, (25) is true for $n = p$. Suppose that it holds for $m \leq n_0 - 1$, and observe $m = n_0 \geq p + 1$.
Because f is (p, q)- quasi-contraction, there exist some $i, j \in \mathbb{N}$, in a way that

$$d(f^{n_0}(x), f^p(x)) \preceq A(d(f^i(x), f^j x)). \tag{26}$$

(1) If $i, j \leq p$, then

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^i(x), f^p(x)) + d(f^p(x), f^j(x))) \\ &\preceq A(\omega(x)) \\ &\preceq (I - A)^{-1} A(\omega(x)). \end{aligned}$$

Remark that we have used that $i \neq j$ in this inference, but if $i = j$, (25) is fulfilled.

(2) If $p < i < n_0$, $j \leq p$, then (25) and (26) imply

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^i(x), f^p x)) + A(d(f^p(x), f^j(x))) \\ &\preceq A(I - A)^{-1} A(\omega(x)) + A(\omega(x)) \\ &= (I - A)^{-1} A(\omega(x)). \end{aligned}$$

(3) For $p < i < n_0$, $p < j < n_0$, we have

$$d(f^{n_0} x, f^p(x)) \preceq A^k(d(f^{i_0}(x), f^{j_0}(x))),$$

where $i_0 < p$ or $j_0 < p$ and $1 < k$.
Assume that at least $i_0 < p$.

$$d(f^{n_0}(x), f^p(x)) \preceq A^k(d(f^{i_0}(x), f^p x)) + A^k(d(f^p(x), f^{j_0}(x)))$$

$$\preceq A^k(\omega(x)) + A^k(I - A)^{-1}A(\omega(x))$$
$$\preceq (I - A)^{-1}A(\omega(x)),$$

since $j_0 \leq j < n_0$, so the inequality (25) holds in this case.

(4) In the case $i = n_0$, $j \leq p$, the triangle inequality, $A(P) \subseteq P$ and (26) imply

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A(d(f^{n_0}(x), f^p(x))) + A(d(f^p(x), f^j x)) \\ &\preceq A(d(f^{n_0}(x), f^p x)) + A(\omega(x)). \end{aligned}$$

Looking at previous case (3), (25) easily follows.

(5) Finally, consider $i = n_0$ and $p < j \leq n_0$.
If $j = n_0$, it follows $d(f^{n_0}(x), f^p(x)) \preceq A(\theta)$ and $d(f^{n_0}(x), f^p(x)) = \theta$.
Otherwise,

$$d(f^{n_0}(x), f^p(x)) \preceq A(d(f^j(x), f^{n_0}(x))) \tag{27}$$

and there exist $i_0 \leq j_0 \leq n_0$, $i_0 < p$, and some $k_0 > 1$ such that

$$d(f^j(x), f^{n_0}(x)) \preceq A^{k_0}(d(f^{i_0}(x), f^{j_0}(x))).$$

If $j_0 \leq p$, then (25) follows by the last inequality and (27). Notice that if $p < j_0 < n_0$, then

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A^{1+k_0}(d(f^{i_0}(x), f^{j_0}(x))) \\ &\preceq A^{1+k_0}(d(f^{i_0}(x), f^p(x))) + A^{1+k_0}(d(f^p x, f^{j_0}(x))) \\ &\preceq A^{1+k_0}(\omega(x)) + A^{1+k_0}(I - A)^{-1}A(\omega(x)) \\ &= A^{1+k_0}(I - A)^{-1}(I - A + A)(\omega(x)) \\ &\preceq (I - A)^{-1}A(\omega(x)). \end{aligned} \tag{28}$$

But if $j_0 = n_0$, then

$$d(f^{n_0}(x), f^p(x)) \preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0}(d(f^p(x), f^{n_0}(x))). \tag{29}$$

For some $k_1 \geq 1$ and $i_1 \leq j_1 \leq n_0$, $i_1 < p$, $d(f^p(x), f^{n_0}(x)) \preceq A^{k_1}(d(f^{i_1}(x), f^{j_1}(x)))$, so by (29) we get

$$d(f^{n_0}(x), f^p(x)) \preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0+k_1}(d(f^{i_1}(x), f^{j_1}(x))).$$

Obviously, if $j_1 < n_0$, as in (28), we have (25). Otherwise,

$$d(f^{n_0}(x), f^p(x)) \preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + A^{1+k_0+k_1}(d(f^{i_1}(x), f^p x)) \\ +A^{1+k_0+2k_1}(d(f^{i_1}(x), f^{n_0}(x))).$$

Thus, for arbitrary $n \in \mathbb{N}$,

$$\begin{aligned} d(f^{n_0}(x), f^p(x)) &\preceq A^{1+k_0}(d(f^{i_0}(x), f^p x)) + \sum_{m=1}^{n-1} A^{1+k_0+mk_1}(d(f^{i_1}(x), f^p x)) \\ &\quad +A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ &\preceq \sum_{m=0}^{n-1} A^{1+k_0+mk_1} A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ &\preceq (I-A)^{-1} A^{1+k_0}(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \\ &\preceq (I-A)^{-1} A(\omega(x)) + A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))). \end{aligned}$$

However, $A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \to \theta, n \to \infty$. For each $c \gg \theta$, there exists $n_c \in \mathbb{N}$ such that $A^{1+k_0+nk_1}(d(f^{i_1}(x), f^{n_0}(x))) \ll c$ for $n > n_c$, so

$$d(f^{n_0}(x), f^p(x)) \preceq (I-A)^{-1} A(\omega(x)) + c,\ c \gg \theta,$$

and $d(f^{n_0}(x), f^p(x)) \preceq (I-A)^{-1} A(\omega(x))$.
Consequently, (25) is true for any $n \in \mathbb{N}$. The inequality

$$\begin{aligned} d(f^n x, f^j(x)) &\preceq d(f^n x, f^p(x)) + d(f^p(x), f^j(x)) \\ &\preceq (I-A)^{-1} A(\omega(x)) + \omega(x) \\ &= (I-A)^{-1}(\omega(x)). \end{aligned}$$

proceeds from(25). Mapping f is a (p, q)-quasi-contraction; thus, for any $n > m \geq p, m = kp + r,\ 0 \leq r < p, k \geq 1$,

$$d(f^n(x), f^m(x)) \preceq A^k(d(f^i(x), f^j(x))) \preceq A^k(I-A)^{-1}(\omega(x)),$$

where $0 \leq i \leq j \leq n$ and $i \leq p$.

Observe that $A^k(I-A)^{-1}(\omega(x)) \to \theta, k \to \infty$ $(m \to \infty)$, so $(f^n(x))$ is a Cauchy sequence in X and $z = \lim_{n\to\infty} f^n(x) \in X$ is a fixed point of f since f is a continuous it follows that $f(z) = z$. The uniqueness of z follows from the definition of a (p, q)-quasi-contraction.

As in the case of Perov theorem and coupled fixed problem on a generalized metric spaces, we can also discuss generalized Ulam-Hyers stability of the fixed point problem on cone metric spaces.

In order to discuss application of Perov-type result, we will present generalization of Ulam-Hyers stability for a class of cone metric spaces.

Definition 18 ([76]) Let (X, d) and (Y, ρ) be two cone metric spaces and $f, g : X \mapsto Y$ mappings. The coincidence equation $f(x) = g(x)$ is generalized Ulam-Hyers stable if there exists a linear increasing operator $\psi : E \mapsto E$ such that for any $\varepsilon \gg \theta$ and each x fulfilling the inequality $\rho(f(x), g(x)) \preceq \varepsilon$, there exists some solution z of the coincidence equation such that $d(x, z) \preceq \psi(\varepsilon)$.

Particular case of the coincidence equation is fixed point problem for $X = Y$, choosing $g(x) = x, x \in X$.

Definition 19 Let (X, d) be a solid cone metric space, $f : X \mapsto X$, and $\psi : P \to P$ a nondecreasing function such that $\psi(\theta) = \theta$. The equation $f(x) = x$ is generalized Ulam-Hyers stable with respect to ψ if for any $\varepsilon \gg \theta$ and y such that $d(f(y), y) \preceq \varepsilon$, there exists some solution z of this equation such that

$$d(z, y) \preceq \psi(\varepsilon).$$

Theorem 26 *If (X, d) is a solid cone metric space and a mapping $f : X \mapsto X$ satisfies condition (13) for some increasing operator $A \in \mathscr{B}(E)$ with spectral radius less than 1, then the equation $f(x) = x$ is Ulam-Hyers stable.*

Proof Due to Theorem 18, f has a unique fixed point $x^* \in X$. Accordingly,

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), f(x^*)) \\ &\leq d(x, f(x)) + A(d(x, x^*)) \\ &\leq 2I - A)^{-1}(\varepsilon). \end{aligned}$$

Taking $\psi = (I - A)^{-1}$, since it is nondecreasing linear function, the equation is Ulam-Hyers stable.

Concerning recent results in this area [12, 20, 21], there are obvious tendencies to incorporate fixed point results and, in that way, obtain Ulam-Hyers stability of functional, operator, differential, or integral equations of higher order or with several variables. Many of those results are obtainable from Perov-type theorems included in this thesis and without any extensive proof or complicated proof approach despite what was presented in [20].

Theorem 27 *Let S be a nonempty set, (X, d) be a complete metric space, $k \in \mathbb{N}$, $f_i : S \mapsto S$, $L_i : S \to \mathbb{R}_+$, $i = \overline{1, k}$, and $\Lambda : {\mathbb{R}_+}^S \mapsto {\mathbb{R}_+}^S$ given by*

$$(\Lambda(\delta))(t) = \sum_{i=1}^{k} L_i(t)\delta(f_i(t)), \quad t \in S.$$

If operator $\mathcal{T}\colon X^S \mapsto X^S$ *satisfying the inequality*

$$\Delta(\mathcal{T}(u), \mathcal{T}(v))(t) \le \Lambda(\Delta(u, v))(t), \quad u, v \in X^S,\ t \in S,$$

and functions $g \in X^S$ *and* $\varepsilon \in \mathbb{R}^S$ *such that*

$$\Delta(\mathcal{T}(g), g)(t) \le \varepsilon(t), \quad t \in \mathbb{R}^+,$$

and

$$\sum_{n=1}^{\infty} \Lambda^n(\varepsilon(t))\sigma(t) < \infty, \quad t \in S,$$

then for every $t \in S$*, the limit* $\lim_{n\to\infty} (\mathcal{T}^n(g))(t) = f(t)$ *exists, and the function* $f \in X^S$ *defined in this way is a unique fixed point of* $\mathcal{T}$ *with*

$$\Delta(g, f)(t) \le \sigma(t), \quad t \in S$$

Proof Observe that $\Delta : \left(X^S\right)^2 \mapsto r_+^S$ defined with

$$\Delta(u, v)(t) = d(u(t), v(t)), \quad u, v \in X^S,\ t \in S,$$

is just an example of a cone metric and operator Λ has properties of operator A of Theorem 29 so this result can be obtained as a direct consequence of Theorem 27.

We will recall one more result on cone metric spaces, but the proof will be omitted due to similarity of applied techniques. Abbas, Rakočević, and Iqbal [4] defined cyclic operators of Perov type in the setting of cone metric spaces and gave proof of existence of a fixed point for this class of operators.

Definition 20 Let (X, d) be a cone metric space, n be a positive integer, and $\{X_i \mid i = \overline{1, n}\}$ be a family of nonempty closed subsets of X. Let $f : \cup_{i=1}^n X_i \to \cup_{i=1}^n X_i$ be a mapping. We say that family $\left\{X_i \mid i = \overline{1, n}\right\}$ is a cyclic representation of X with respect to f if $X = \cup_{i=1}^n X_i$ and

$$f(X_1) \subset X_{2,\ldots,} f(X_{n-1}) \subset X_n, f(X_n) \subset X_1.$$

Theorem 28 *Let* (X, d) *be a complete cone metric space over a solid cone* C, p *be a positive integer,* $\left\{X_i \mid i = \overline{1, n}\right\}$ *be a family of nonempty closed subsets of* X*, and* $f : X \to X$*. Assume that* $\left\{X_i \mid i = \overline{1, n}\right\}$ *is a cyclic representation of* X *with* *respect to* f. *Suppose that there exists* $A \in \mathscr{B}(E)$ *with* $r(A) < 1$ *such that for each* $x \in X_i$ *and* $y \in X_{i+1}$, $i = \overline{1, n-1}$, *we have*

$$d(fx, fy) \preceq A(d(x, y))$$

where $X_{\mathrm{n+1}} := X_1$. *Then* f *has a unique fired point* $x^* \in \cap_{i=1}^n X_i$ *and* $f^n x \to x^*$ *for each* $x \in \cup_{i=1}^n X_j$.

4 Nonlinear Operatorial Contractions

Recall that Perov fixed point theorem on cone metric space requires for contractive operator A to satisfy (13), to be bounded, linear and with spectral radius less than 1. The goal is to weaken these requirements and still obtain existence and uniqueness of a fixed point of mapping f. The focus will be on omitting a linearity condition.

Theorem 29 *Let* (X, d) *be complete cone metric space with a solid cone* P *and* $f : X \mapsto X$ *a continuous mapping. If there exists an increasing operator* $A : E \mapsto E$ *such that* $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, *and, for any* $x, y \in X$,

$$d(f(x), f(y)) \preceq A(d(x, y)), \tag{30}$$

then a mapping f *has a unique fixed point in* X.

Proof Let $x_0 \in X$ and $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Due to (30),

$$\theta \preceq d(x_n, x_{n+1}) \preceq A^n(d(x_0, x_{n_0})),\ \ n \in \mathbb{N},$$

we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = \theta$.
For some $c \gg \theta$, let $n_0 \in \mathbb{N}$ be such that $A^n(c) \preceq \frac{c}{8}$, $n \geq n_0$, and $n_1 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) \prec \frac{c}{8n_0}$ for $n \geq n_1 n_0$. Observe a sequence $y_k = f^{kn_0}(x)$, $k \in \mathbb{N}$. Then

$$d(y_k, y_{k+1}) \preceq A^{kn_0}(d(x_0, x_1)),\ \ k \in \mathbb{N},$$

and $\lim_{n\to\infty} d(y_k, y_{k+1}) = \theta$. Furthermore, there exists some $k_0 \in \mathbb{N}$ such that $d(y_k, y_{k+1}) \prec \frac{c}{8}$ holds also for any index greater than k_0 and choose such $k \geq n_1$. Denote with S a closed ball $K[y_k, \frac{c}{4}] = \{x \in X \mid d(y_k, x) \preceq \frac{c}{4}\}$. It means that $f^{n_0}(S) \subseteq S$ since, for any $x \in S$,

$$\begin{aligned} d(y_k, f^{n_0}(x)) &\preceq d(y_k, y_{k+1}) + d(y_{k+1}, f^{n_0}(x)) \\ &\preceq \frac{c}{8} + A^{n_0}(d(y_k, x)) \\ &\prec \frac{c}{4}. \end{aligned}$$

Hence, $y_n \in S$ for any $n \geq k$.
If $m \geq kn_0$, let $m = qn_0 + r$ for some $q \geq k$ and $0 \leq r < n_0$; then the inequalities

$$\begin{aligned} d(y_k, x_m) &\preceq d(y_k, y_q) + d(y_q, x_m) \\ &\preceq d(y_k, y_q) + \sum_{i=qn_0}^{m-1} d(x_i, x_{i+1}) \\ &\preceq \frac{c}{4} + \sum_{i=qn_0}^{m-1} \frac{c}{8n_0} \\ &\preceq \frac{3c}{8}, \end{aligned}$$

lead to

$$d(x_n, x_m) \preceq d(x_n, y_k) + d(y_k, x_m) \preceq \frac{3c}{4} \prec c, \;\; n, m \geq kn_0.$$

Thus, (x_n) is a Cauchy sequence in X and therefore convergent in X with a limit $x^* \in X$. Since the mapping f is continuous,

$$x^* = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(x^*).$$

If $f(u) = u$, then

$$d(x^*, u) = d(f^n(x^*), f^n(u)) \preceq A^n(x^*, u),$$

along with $\lim_{n\to\infty} A^n(x^*, u) = \theta$ gives $u = x^*$. Uniqueness of fixed point implies that $\lim_{n\to\infty} f^n(x)) = x^*$ for any $x \in X$ since the first part of the proof induces that $(f^n(x))$ converges to the point with fixed point property.

Remark 30 *Comparing this theorem with Theorem 24, notice that A is not assumed to be linear. Continuity condition of f is implicitly requested in Perov theorem and Theorem 24. Condition $r(A) < 1$ means $\lim_{n\to\infty} \|A^n\| = 0$ and $\lim_{n\to\infty} \|A^n(e)\| = 0$, for any $e \in E$. Hence, this result generalizes Theorem 24 and Perov theorem.*

Instead of requesting that $\lim_{n\to\infty} A^n(e) = \theta$, for any $e \in E$, it is enough to assume that for all $e \in P$.

Theorem 31 *Let (X, d) be complete cone metric space with a solid cone P and $f : X \mapsto X$ a continuous mapping. If there exists an increasing operator $A : E \mapsto E$ such that $\lim_{n\to\infty} A^n(e) = \theta$, $e \in P$, and (30) holds for any $x, y \in X$, then a mapping f has a unique fixed point in X.*

It is also possible to ask for a restriction $A \restriction_P : P \mapsto P$ to be an increasing operator instead of $A : E \mapsto E$.

In the case that $A \in \mathscr{B}(E)$, it is equivalent for A to be increasing or positive. Boundedness does not have impact on this conclusion, only linearity. If A is a nonlinear operator, but increasing and satisfies (30), then for $x = y$, $\theta \preceq A(\theta)$ and for $x \in P$, $\theta \preceq A(\theta) \preceq A(x) \in P$, so A is a positive operator. On the other hand, positivity of A does not imply that A is increasing.

Example 7 Let E be a Banach space with a solid cone P and $c \in \text{int}(P)$. Define an operator $A : E \mapsto E$ with

$$A(x) = \begin{cases} \frac{c}{2}, & x = \theta \\ \theta, & x \in E \setminus \{\theta\} \end{cases}.$$

Operator A is positive, but it is not increasing. Also, $A^n(x) = \theta$, $x \in E$, for any $n \geq 2$.

Comparing requirements $r(A) < 1$ and $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, for a bounded linear operator A, it is obvious that $r(A) < 1$ implies other condition, but reverse do not hold. Moreover, the condition $\lim_{n\to\infty} A^n(e) = \theta, e \in E$, (or $e \in P$) of Theorem 31 is less strict that corresponding condition of Theorem 24.

5 Conclusion

Presented results are just a few selected ones on this topic and only concentrated on single-valued operator on generalized metric spaces and cone metric spaces. Since this topic is in the focus of the fixed point theory research in the last decade, there have been published very interesting articles with different techniques and angle of approach. It is also worth mentioning that some results do have questionable novelty, but, in several cases, have some interesting applicability. In the future, we would expect to differentiate these kinds of results, to find new areas of application, and to talk about unification. There are several open questions like correlation between nonlinear operatorial contraction and Perov theorem on cone metric spaces, explaining better estimations obtained in this way and deepening influence of Perov type results in the area of differential and difference equations. Since this problem was not considered from the numerical point of view, it would be also meaningful to consider some iterative methods based on new types of contractions. Nevertheless, this is still a very interesting topic for research with a possible wide impact to different areas of mathematics and science overall.

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On a Logarithmic Equation by Primes

S. I. Dimitrov

Abstract Let $[\,\cdot\,]$ be the floor function. In this paper, we show that every sufficiently large positive integer N can be represented in the form

$$N = [p_1 \log p_1] + [p_2 \log p_2] + [p_3 \log p_3],$$

where $p_1,\ p_2,\ and p_3$ are prime numbers. We also establish an asymptotic formula for the number of such representations, when $p_1,\ p_2,\ and p_3$ do not exceed given sufficiently large positive number.

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1 Introduction and Main Result

A remarkable moment in analytic number theory is 1937, when Vinogradov [10] proved the ternary Goldbach problem. He showed that every sufficiently large odd integer N can be represented in the form

$$N = p_1 + p_2 + p_3,$$

where $p_1,\ p_2,\ and p_3$ are prime numbers.

The consequences of Vinogradov's [11] ingenious method for estimating exponential sums over primes continue to this day in analytic number theory.

A detailed proof of Vinogradov's theorem, beginning with a historical perspective along with an overview of essential lemmas and theorems, can be found in the monograph of Rassias [7].

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In 1995, Laporta and Tolev [6] investigated an analogue of the Goldbach-Vinogradov theorem. They considered the diophantine equation

$$N = [p_1^c] + [p_2^c] + [p_3^c]\,,$$

where $p_1,\ p_2,\ and p_3$ are primes. For $1 < c < 17/16$, they showed that for the sum

$$R(N) = \sum_{N=[p_1^c]+[p_2^c]+[p_3^c]} \log p_1 \log p_2 \log p_3$$

the asymptotic formula

$$R(N) = \frac{\Gamma^3(1+1/c)}{\Gamma(3/c)} N^{3/c-1} + \mathcal{O}\Big(N^{3/c-1} \exp\big(-(\log N)^{1/3-\varepsilon}\big)\Big) \tag{1}$$

holds.

Subsequently, the result of Laporta and Tolev was sharpened by Kumchev and Nedeva [5] to

$$1 < c < \frac{12}{11},$$

by Zhai and Cao [12] to

$$1 < c < \frac{258}{235},$$

and by Cai [2] to

$$1 < c < \frac{137}{119}.$$

Overcoming all difficulties, Zhang and Li [15] improved the result of Cai to

$$1 < c < \frac{3113}{2703}$$

and this is the best result up to now.

On the other hand, recently, the author [3] showed that when N is a sufficiently large positive number and $\varepsilon > 0$ is a small constant, then the logarithmic inequality

$$\big|p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N\big| < \varepsilon$$

has a solution in prime numbers $p_1,\ p_2,\ and p_3$.

Motivated by these results, in this paper, we introduce new diophantine equation with prime numbers.

Consider the logarithmic equation

$$N = [p_1 \log p_1] + [p_2 \log p_2] + [p_3 \log p_3], \tag{2}$$

where N is a sufficiently large positive integer. Having the arguments of the aforementioned marvelous mathematicians and [3], we expect that (2) has a solution in primes $p_1,\ p_2,\ and p_3$. Define the sum

$$\Gamma = \sum_{N=[p_1 \log p_1]+[p_2 \log p_2]+[p_3 \log p_3]} \log p_1 \log p_2 \log p_3\,. \tag{3}$$

We make the first attempt and prove the following theorem.

Theorem 1 *Let N is a sufficiently large positive integer. Let X is a solution of the equality*

$$X \log X = N.$$

Then the asymptotic formula

$$\Gamma = \frac{X^2}{1+\log X} + \mathcal{O}\Big(X^2 \exp\big(-(\log X)^{1/3-\varepsilon}\big)\Big) \tag{4}$$

holds.

As usual, the corresponding binary problem is out of reach of the current state of analytic number theory. In other words, we have the following challenge.

Conjecture 1 Let N is a sufficiently large positive integer. Then the logarithmic equation

$$N = [p_1 \log p_1] + [p_2 \log p_2]$$

is solvable in prime numbers $p_1 and\ p_2$.

Needless to say, we believe that in the near future, we will see the solution of this binary logarithmic hypothesis.

2 Notations

The letter p with or without subscript will always denote prime number. We denote by $\Lambda(n)$ von Mangoldt's function. Moreover, $e(y) = e^{2\pi \imath y}$. As usual, $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. We recall that $t = [t] + \{t\}$ and $\|t\| = \min(\{t\}, 1 - \{t\})$. By ε, we denote an arbitrary small positive constant, not the same in all appearances. Let N be a sufficiently large positive integer. Let X is a solution of the equality

$$X \log X = N. \tag{5}$$

Let y be an implicit function of t defined by

$$y \log y = t. \tag{6}$$

The first derivative of y is

$$y' = \frac{1}{1 + \log y}. \tag{7}$$

Denote

$$\tau = X^{-\frac{23}{25}}\,; \tag{8}$$

$$S(\alpha) = \sum_{p \le X} e\big(\alpha[p \log p]\big) \log p\,; \tag{9}$$

$$\Theta(\alpha) = \sum_{m \le N} \frac{1}{1 + \log y(m)}\, e(m\alpha)\,; \tag{10}$$

$$\Gamma_1 = \int\limits_{-\tau}^{\tau} S^3(\alpha) e(-N\alpha)\, d\alpha\,; \tag{11}$$

$$\Gamma_2 = \int\limits_{\tau}^{1-\tau} S^3(\alpha) e(-N\alpha)\, d\alpha\,; \tag{12}$$

$$\Psi_k = \int\limits_{-1/2}^{1/2} \Theta^k(\alpha) e(-N\alpha)\, d\alpha, \quad k = 1, 2, 3, \dots\,; \tag{13}$$

$$\widetilde{\Psi} = \int\limits_{-\tau}^{\tau} \Theta^3(\alpha) e(-N\alpha)\, d\alpha\,. \tag{14}$$

3 Lemmas

Lemma 1 *Let $f(x)$ be a real differentiable function in the interval $[a, b]$. If $f'(x)$ is a monotonous and satisfies $|f'(x)| \le \theta < 1$, then we have*

$$\sum_{a<n\le b} e(f(n)) = \int\limits_a^b e(f(x))\,dx + \mathcal{O}(1)\,.$$

Proof See [8, Lemma 4.8]. □

Lemma 2 *Let $x, y \in \mathbb{R}$ and $H \ge 3$. Then the formula*

$$e(-x\{y\}) = \sum_{|h|\le H} c_h(x)e(hy) + \mathcal{O}\left(\min\left(1, \frac{1}{H\|y\|}\right)\right)$$

holds. Here

$$c_h(x) = \frac{1-e(-x)}{2\pi i(h+x)}\,.$$

Proof See [1, Lemma 12]. □

Lemma 3 (Van der Corput) *Let $f(x)$ be a real-valued function with continuous second derivative in $[a, b]$ such that*

$$|f''(x)| \asymp \lambda, \quad (\lambda > 0) \quad \textit{for } x \in [a, b].$$

Then

$$\left|\sum_{a<n\le b} e(f(n))\right| \ll (b-a)\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

Proof See [4, Ch. 1, Th. 5]. □

Lemma 4 *For any real number t and $H \ge 1$, there holds*

$$\min\left(1, \frac{1}{H\|t\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(ht)\,,$$

where

$$a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2},\right).$$

Proof *See [13, Lemma 2].* □

4 Proof of the Theorem

From (3), (9), (11), and (12), we have

$$\Gamma = \int_0^1 S^3(\alpha)e(-N\alpha)\,d\alpha = \Gamma_1 + \Gamma_2. \tag{15}$$

Estimation of Γ_1
We write

$$\Gamma_1 = (\Gamma_1 - \widetilde{\Psi}) + (\widetilde{\Psi} - \Psi_3) + \Psi_3. \tag{16}$$

Bearing in mind (10) and (13), we obtain

$$\Psi_1 = \int_{-1/2}^{1/2} \Theta(\alpha)e(-N\alpha)\,d\alpha = \frac{1}{1+\log y(N)}.$$

Suppose that

$$\Psi_k = \frac{1}{1+\log y(N)} X^{k-1} + \mathcal{O}\big(X^{k-2}\big) \quad \text{for} \quad k \geq 2. \tag{17}$$

Then

$$\begin{aligned}
\Psi_{k+1} &= \sum_{m\leq N} \frac{1}{1+\log y(m)} \Bigg(\sum_{m_1\leq N-m} \cdots \sum_{\substack{m_k\leq N-m \\ m_1+\cdots+m_k=N-m}} \frac{1}{1+\log y(m_1)} \cdots \frac{1}{1+\log y(m_k)} \Bigg) \\
&= \sum_{m\leq N} \frac{1}{1+\log y(m)} \left(\frac{1}{1+\log y(N-m)} X^{k-1} + \mathcal{O}\big(X^{k-2}\big) \right) \\
&= \sum_{m\leq N} \frac{1}{1+\log y(m)} \cdot \frac{1}{1+\log y(N-m)} X^{k-1} + \mathcal{O}\big(X^{k-1}\big) \\
&= \frac{1}{1+\log y(N)} X^{k} + \mathcal{O}\big(X^{k-1}\big).
\end{aligned}$$

Consequently, the supposition (17) is true.
From (5) and (6), it follows that

$$y(N) = X. \tag{18}$$

Bearing in mind (17) and (18), we conclude that

$$\Psi_k = \frac{X^{k-1}}{1+\log X} + \mathcal{O}\big(X^{k-2}\big) \quad \text{for} \quad k \geq 2. \tag{19}$$

Now the asymptotic formula (19) gives us

$$\Psi_3 = \frac{X^2}{1+\log X} + \mathcal{O}\big(X\big). \tag{20}$$

From (11) and (14), we get

$$\begin{aligned} |\Gamma_1 - \widetilde{\Psi}| &\ll \int\limits_{-\tau}^{\tau} \big|S^3(\alpha) - \Theta^3(\alpha)\big| \, d\alpha \\ &\ll \max_{|\alpha|\leq\tau} \big|S(\alpha) - \Theta(\alpha)\big| \Bigg(\int\limits_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha + \int\limits_{-1/2}^{1/2} |\Theta(\alpha)|^2 \, d\alpha \Bigg). \end{aligned} \tag{21}$$

Arguing as in [3, Lemma 8], we find

$$\int\limits_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha \ll X \log X. \tag{22}$$

Square out and integrate, we obtain

$$\int\limits_{-1/2}^{1/2} |\Theta(\alpha)|^2 \, d\alpha \ll \frac{N}{\log^2 N} \ll X. \tag{23}$$

Now we shall estimate from above $|S(\alpha) - \Theta(\alpha)|$ for $|\alpha| \leq \tau$.
Our argument is a modification of Zhang's and Li's [14] argument.
From (8) and (9), we get

$$\begin{aligned} S(\alpha) &= \sum_{p\leq X} e(\alpha p \log p) \log p + \mathcal{O}\big(\tau X\big) \\ &= \sum_{n\leq X} \Lambda(n) e(\alpha n \log n) + \mathcal{O}\big(X^{1/2}\big) + \mathcal{O}\big(\tau X\big) \\ &= \sum_{n\leq X} \Lambda(n) e(\alpha n \log n) + \mathcal{O}\big(X^{1/2}\big). \end{aligned} \tag{24}$$

From $|\alpha| \leq \tau$, $y \geq 2$ and Lemma 1, we have that

$$\sum_{1<m\leq y} e(m\alpha) = \int_1^y e(\alpha t)\,dt + \mathcal{O}(1). \tag{25}$$

Using (6), (7), (8), (10), (25), and partial summation, we find

$$\begin{aligned}
\sum_{n\leq X} \Lambda(n)e(\alpha n\log n) &= \int_1^X e(\alpha y\log y)\,d\Big(\sum_{n\leq y}\Lambda(n)\Big)\\
&= \int_1^X e(\alpha y\log y)\,dy + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big)\\
&= \int_1^N e(\alpha t)\frac{1}{1+\log y(t)}\,dt + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big)\\
&= \int_1^N \frac{1}{1+\log y(t)}\,d\Big(\int_1^t e(\alpha u)\,du\Big) + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big)\\
&= \int_1^N \frac{1}{1+\log y(t)}\,d\Big(\sum_{1<m\leq t} e(m\alpha) + \mathcal{O}(1)\Big)\\
&\quad + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big)\\
&= \sum_{m\leq N}\frac{1}{1+\log y(m)}e(m\alpha) + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big)\\
&= \Theta(\alpha) + \mathcal{O}\Big(X\exp\big(-(\log X)^{1/3}\big)\Big).
\end{aligned} \tag{26}$$

From (24) and (26), it follows that

$$\max_{|\alpha|\leq\tau} |S(\alpha)-\Theta(\alpha)| \ll X\exp\big(-(\log X)^{1/3}\big). \tag{27}$$

Taking into account (21), (22), (23), and (27), we conclude

$$\Gamma_1 - \widetilde{\Psi} \ll X^2\exp\big(-(\log X)^{1/3-\varepsilon}\big). \tag{28}$$

Using (8), (13), and (14) and working as in [9, Lemma 2.8], we deduce

$$\big|\Psi_3 - \widetilde{\Psi}\big| \ll \int_{\tau\leq|\alpha|\leq 1/2} |\Theta(\alpha)|^3\,d\alpha \ll \int_\tau^{1/2}\frac{d\alpha}{\alpha^3} \ll X^{\frac{46}{25}}. \tag{29}$$

Summarizing (16), (20), (28), and (29), we obtain

$$\Gamma_1 = \frac{X^2}{1+\log X} + \mathcal{O}\Big(X^2 \exp\big(-(\log X)^{1/3-\varepsilon}\big)\Big). \tag{30}$$

Estimation of Γ_2
From (12), we get

$$\Gamma_2 \ll \max_{\tau\le\alpha\le1-\tau} |S(\alpha)| \int_0^1 |S(\alpha)|^2\,d\alpha \ll X(\log X) \max_{\tau\le\alpha\le1-\tau} |S(\alpha)|. \tag{31}$$

By (9) and Lemma 2 with $x=\alpha$, $y=n\log n$, and

$$H = X^{\frac{1}{25}} \tag{32}$$

it follows

$$\begin{aligned} S(\alpha) &= \sum_{n\le X} \Lambda(n)e(\alpha n\log n)e(-\alpha\{n\log n\}) + \mathcal{O}(X^{1/2}) \\ &= \sum_{|h|\le H} c_h(\alpha) \sum_{n\le X} \Lambda(n)e\big((h+\alpha)n\log n\big) \\ &+ \mathcal{O}\left((\log X)\sum_{n\le X}\min\left(1, \frac{1}{H\|n\log n\|}\right)\right). \end{aligned}$$

Therefore,

$$\max_{\tau\le\alpha\le1-\tau} |S(\alpha)| \ll (S_1+S_2)\log X, \tag{33}$$

where

$$S_1 = \max_{\tau\le\alpha\le H+1} \Big|\sum_{n\le X} \Lambda(n)e(\alpha n\log n)\Big|, \tag{34}$$

$$S_2 = \sum_{n\le X}\min\left(1, \frac{1}{H\|n\log n\|}\right). \tag{35}$$

Bearing in mind (8), (32), and (34), according to [3, Lemma 9], we conclude

$$S_1 \ll X^{24/25}\log^3 X. \tag{36}$$

By (32), (35), Lemmas 3 and 4, and $Y \le X/2$, we obtain

$$
\begin{aligned}
S_2 &\ll (\log X) \sum_{Y<n\le 2Y} \min\left(1, \frac{1}{H\|n\log n\|}\right) \\
&\le (\log X) \sum_{h=-\infty}^{+\infty} |a_h| \left| \sum_{Y<n\le 2Y} e(hn\log n) \right| \\
&\ll (\log X)\left(\frac{Y\log 2H}{H} + \frac{Y^{1/2}\log 2H}{H}\sum_{h\le H} h^{1/2} + Y^{1/2}H\sum_{h>H} h^{-3/2}\right) \\
&\ll XH^{-1}\log^2 X \\
&\ll X^{24/25}\log^2 X. \qquad (37)
\end{aligned}
$$

From (31), (33), (36), and (37), we find

$$\Gamma_2 \ll X^{49/25}\log^5 X. \qquad (38)$$

The End of the Proof
Bearing in mind (15), (30), and (38), we establish the asymptotic formula (4). The theorem is proved.

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Hermite-Hadamard Trapezoid and Mid-Point Divergences

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Abstract In this paper, we introduce the concepts of Hermite-Hadamard trapezoid and mid-point divergences that are closely related to the Jensen divergence considered by Burbea and Rao in 1982. The joint convexity of these divergences and several inequalities involving these measures are established. Various examples concerning the Csiszár, Lin-Wong, and HH f-divergence measures are also given.

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1 Introduction

For a function f defined on an interval I of the real line $\mathbb{R}$, by following the paper by Burbea and Rao [1], we consider the $\mathcal{J}$*-divergence* between the vectors $x, y \in I^n$ given by

$$\mathcal{J}_{n,f}(x, y) := \sum_{i=1}^{n} \left(\frac{1}{2} [f(x_i) + f(y_i)] - f\left(\frac{x_i + y_i}{2} \right) \right).$$

As important examples of such divergences, we can consider [1],

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$$\mathcal{J}_{n,\alpha}(x,y) := \begin{cases} (\alpha-1)^{-1}\sum_{i=1}^{n}\left[\frac{1}{2}\left(x_i^{\alpha}+y_i^{\alpha}\right)-\left(\frac{x_i+y_i}{2}\right)^{\alpha}\right], & \alpha\neq 1 \\ \frac{1}{2}\sum_{i=1}^{n}\left[x_i\ln(x_i)+y_i\ln(y_i)-(x_i+y_i)\ln\left(\frac{x_i+y_i}{2}\right)\right], & \alpha=1. \end{cases}$$

If f is convex on I, then $\mathcal{J}_{n,f}(x,y)\geq 0$ for all $(x,y)\in I^n\times I^n$.

The following result concerning the joint convexity of $\mathcal{J}_{n,f}$ also holds:

Theorem 1 (Burbea-Rao, 1982 [1]) *Let f be a C^2 function on an interval I. Then $\mathcal{J}_{n,f}$ is convex (concave) on $I^n\times I^n$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

We define the *Hermite-Hadamard trapezoid* and *mid-point divergences*

$$\mathcal{T}_{n,f}(x,y) := \sum_{i=1}^{n}\left(\frac{1}{2}\left[f(x_i)+f(y_i)\right]-\int_0^1 f((1-t)x_i+ty_i)\,dt\right) \tag{1.1}$$

and

$$\mathcal{M}_{n,f}(x,y) := \sum_{i=1}^{n}\left(\int_0^1 f((1-t)x_i+ty_i)\,dt - f\left(\frac{x_i+y_i}{2}\right)\right) \tag{1.2}$$

for all $(x,y)\in I^n\times I^n$.

We observe that

$$\mathcal{J}_{n,f}(x,y)=\mathcal{T}_{n,f}(x,y)+\mathcal{M}_{n,f}(x,y) \tag{1.3}$$

for all $(x,y)\in I^n\times I^n$.

If f is convex on I, then by *Hermite-Hadamard inequalities*

$$\frac{f(a)+f(b)}{2}\geq\int_0^1 f((1-t)a+tb)\,dt\geq f\left(\frac{a+b}{2}\right)$$

for all $a,b\in I$, we have the following fundamental facts:

$$\mathcal{T}_{n,f}(x,y)\geq 0 \text{ and } \mathcal{M}_{n,f}(x,y)\geq 0 \tag{1.4}$$

for all $(x,y)\in I^n\times I^n$.

Using *Bullen's inequality* (see, for instance, [6, p. 2]),

$$0\leq\int_0^1 f((1-t)a+tb)\,dt-f\left(\frac{a+b}{2}\right)$$

$$\leq \frac{f(a)+f(b)}{2} - \int_0^1 f((1-t)a+tb)\,dt$$

we also have

$$0 \leq \mathcal{M}_{n,f}(x,y) \leq \mathcal{T}_{n,f}(x,y). \tag{1.5}$$

Let us recall the following special means:

(a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a,b>0,$$

(b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a,b \geq 0,$$

(c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \quad a,b>0,$$

(d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases}; \quad a,b>0$$

(e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases}; \quad a,b>0$$

(f) The *p-logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if } b \neq a, \quad p \in \mathbb{R}\setminus\{-1,0\} \\ a & \text{if } b = a \end{cases}; \quad a,b>0.$$

If we put $L_0(a,b) := I(a,b)$ and $L_{-1}(a,b) := L(a,b)$, then it is well known that the function $\mathbb{R} \ni p \mapsto L_p(a,b)$ is monotonic increasing on $\mathbb{R}$.

We observe that for $p \in \mathbb{R}\setminus\{-1,0\}$, we have

$$\int_0^1 [(1-t)a+tb]^p \, dt = L_p^p(a,b), \quad \int_0^1 [(1-t)a+tb]^{-1} \, dt = L^{-1}(a,b)$$

and

$$\int_0^1 \ln[(1-t)a+tb] \, dt = \ln I(a,b).$$

Using these notations, we can define the following divergences for $(x,y) \in I^n \times I^n$ where I is an interval of positive numbers

$$\mathcal{T}_{n,p}(x,y) := \sum_{i=1}^n \left[A\left(x_i^p, y_i^p\right) - L_p^p(x_i,y_i)\right]$$

and

$$\mathcal{M}_{n,p}(x,y) := \sum_{i=1}^n \left[L_p^p(x_i,y_i) - A^p(x_i,y_i)\right]$$

for all $p \in \mathbb{R}\setminus\{-1,0\}$,

$$\mathcal{T}_{n,-1}(x,y) := \sum_{i=1}^n \left[H^{-1}(x_i,y_i) - L^{-1}(x_i,y_i)\right]$$

and

$$\mathcal{M}_{n,-1}(x,y) := \sum_{i=1}^n \left[L^{-1}(x_i,y_i) - A^{-1}(x_i,y_i)\right]$$

for $p = -1$, and

$$\mathcal{T}_{n,0}(x,y) := \ln\left[\prod_{i=1}^n \left(\frac{G(x_i,y_i)}{I(x_i,y_i)}\right)\right]$$

and

$$\mathcal{M}_{n,0}(x,y) := \ln\left[\prod_{i=1}^n \left(\frac{I(x_i,y_i)}{A(x_i,y_i)}\right)\right]$$

for $p = 0$.

Since the function $f(t) = t^p$, $t > 0$ is convex for $p \in (-\infty, 0) \cup (1, \infty)$, then we have

$$\mathcal{T}_{n,p}(x, y), \ \mathcal{M}_{n,p}(x, y) \geq 0 \tag{1.6}$$

for all $(x, y) \in I^n \times I^n$.

For $p \in (0, 1)$ the function $f(t) = t^p$, $t > 0$ and for $p = 0$, the function $f(t) = \ln t$ are concave, then we have for $p \in [0, 1)$ that

$$\mathcal{T}_{n,p}(x, y), \ \mathcal{M}_{n,p}(x, y) \leq 0 \tag{1.7}$$

for all $(x, y) \in I^n \times I^n$.

Finally, for $p = 1$, we have both $\mathcal{T}_{n,p}(x, y) = \mathcal{M}_{n,p}(x, y) = 0$ for all $(x, y) \in I^n \times I^n$.

In this paper, we establish the joint convexity of the *Hermite-Hadamard trapezoid* and *mid-point divergences* $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ and also provide several inequalities involving these measures. Several examples concerning the Csiszár, Lin-Wong, and HH f-divergence measures are also given.

2 General Results

We start with the following convexity result that is a consequence of Burbea-Rao theorem above:

Theorem 2 *Let f be a C^2 function on an interval I. Then $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ are convex (concave) on $I^n \times I^n$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

Proof If $\mathcal{T}_{n,f}$ and $\mathcal{M}_{n,f}$ are convex on $I^n \times I^n$, then the sum $\mathcal{T}_{n,f} + \mathcal{M}_{n,f} = \mathcal{J}_{n,f}$ is convex on $I^n \times I^n$, which, by Burbea-Rao theorem, implies that f is convex and $\frac{1}{f''}$ is concave on I.

Now, if f is convex and $\frac{1}{f''}$ is concave on I, then by the same theorem, we have that the function $\mathcal{J}_f : I \times I \to \mathbb{R}$

$$\mathcal{J}_f(x, y) := \frac{1}{2}[f(x) + f(y)] - f\left(\frac{x + y}{2}\right)$$

is convex.

Let $x, y, u, v \in I$. We define

$$\varphi(t) := \mathcal{J}_f((1 - t)(x, y) + t(u, v)) = \mathcal{J}_f(((1 - t)x + tu, (1 - t)y + tv))$$

$$= \frac{1}{2}\left[f\left((1-t)x+tu\right)+f\left((1-t)y+tv\right)\right]$$
$$- f\left(\frac{(1-t)x+tu+(1-t)y+tv}{2}\right)$$
$$= \frac{1}{2}\left[f\left((1-t)x+tu\right)+f\left((1-t)y+tv\right)\right]$$
$$- f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)$$

for $t \in [0,1]$.

Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta=1$. By the convexity of $\mathcal{J}_f$, we have

$$\begin{aligned}
&\varphi\left(\alpha t_1+\beta t_2\right)\\
&=\mathcal{J}_f\left(\left(1-\alpha t_1-\beta t_2\right)(x,y)+\left(\alpha t_1+\beta t_2\right)(u,v)\right)\\
&=\mathcal{J}_f\left(\left(\alpha+\beta-\alpha t_1-\beta t_2\right)(x,y)+\left(\alpha t_1+\beta t_2\right)(u,v)\right)\\
&=\mathcal{J}_f\left(\alpha\left(1-t_1\right)(x,y)+\beta\left(1-t_2\right)(x,y)+\alpha t_1(u,v)+\beta t_2(u,v)\right)\\
&=\mathcal{J}_f\left(\alpha\left[\left(1-t_1\right)(x,y)+t_1(u,v)\right]+\beta\left[\left(1-t_2\right)(x,y)+t_2(u,v)\right]\right)\\
&\leq\alpha\mathcal{J}_f\left(\left(1-t_1\right)(x,y)+t_1(u,v)\right)+\beta\mathcal{J}_f\left(\left(1-t_2\right)(x,y)+t_2(u,v)\right)\\
&=\alpha\varphi\left(t_1\right)+\beta\varphi\left(t_2\right),
\end{aligned}$$

which proves that φ is convex on $[0,1]$ for all $x, y, u, v \in I$.

Applying the Hermite-Hadamard inequality for φ, we get

$$\frac{1}{2}\left[\varphi(0)+\varphi(1)\right] \geq \int_0^1 \varphi(t)\,dt \tag{2.1}$$

and since

$$\varphi(0)=\frac{1}{2}\left[f(x)+f(y)\right]-f\left(\frac{x+y}{2}\right),$$

$$\varphi(1)=\frac{1}{2}\left[f(u)+f(v)\right]-f\left(\frac{u+v}{2}\right)$$

and

$$\int_0^1 \varphi(t)\,dt=\frac{1}{2}\left[\int_0^1 f\left((1-t)x+tu\right)dt+\int_0^1 f\left((1-t)y+tv\right)dt\right]$$

$$-\int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt,$$

hence, by (2.1), we get

$$\frac{1}{2}\left\{\frac{1}{2}[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)+\frac{1}{2}[f(u)+f(v)]-f\left(\frac{u+v}{2}\right)\right\}$$
$$\geq\frac{1}{2}\left[\int_0^1 f((1-t)x+tu)\,dt+\int_0^1 f((1-t)y+tv)\,dt\right]$$
$$-\int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt.$$

Re-arranging this inequality, we get

$$\frac{1}{2}\left[\frac{f(x)+f(u)}{2}-\int_0^1 f((1-t)x+tu)\,dt\right]$$
$$+\frac{1}{2}\left[\frac{f(y)+f(v)}{2}-\int_0^1 f((1-t)y+tv)\,dt\right]$$
$$\geq\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{u+v}{2}\right)-\int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt\right]$$

which is equivalent to

$$\frac{1}{2}\left[\mathcal{T}_f(x,u)+\mathcal{T}_f(y,v)\right]\geq\mathcal{T}_f\left(\frac{x+y}{2},\frac{u+v}{2}\right)$$
$$=\mathcal{T}_f\left(\frac{1}{2}(x,u)+\frac{1}{2}(y,v)\right),$$

for all $(x,u),(y,v)\in I\times I$, which shows that $\mathcal{T}_f$ is Jensen's convex on $I\times I$. Since $\mathcal{T}_f$ is continuous on $I\times I$, hence $\mathcal{T}_f$ is convex in the usual sense on $I\times I$. Further, by summing over i from 1 to n, we deduce that $\mathcal{T}_{n,f}$ is convex on $I^n\times I^n$.

Now, if we use the second Hermite-Hadamard inequality for φ on $[0,1]$, we have

$$\int_0^1\varphi(t)\,dt\geq\varphi\left(\frac{1}{2}\right). \tag{2.2}$$

Since

$$\varphi\left(\frac{1}{2}\right)=\frac{1}{2}\left[f\left(\frac{x+u}{2}\right)+f\left(\frac{y+v}{2}\right)\right]-f\left(\frac{1}{2}\frac{x+y}{2}+\frac{1}{2}\frac{u+v}{2}\right)$$

hence, by (2.2), we have

$$\frac{1}{2}\left[\int_0^1 f\left((1-t)x+tu\right)dt+\int_0^1 f\left((1-t)y+tv\right)dt\right]$$
$$-\int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt$$
$$\geq \frac{1}{2}\left[f\left(\frac{x+u}{2}\right)+f\left(\frac{y+v}{2}\right)\right]-f\left(\frac{1}{2}\left(\frac{x+y}{2}+\frac{u+v}{2}\right)\right),$$

which is equivalent to

$$\frac{1}{2}\left[\int_0^1 f\left((1-t)x+tu\right)dt-f\left(\frac{x+u}{2}\right)\right]$$
$$+\frac{1}{2}\left[\int_0^1 f\left((1-t)y+tv\right)dt-f\left(\frac{y+v}{2}\right)\right]$$
$$\geq \int_0^1 f\left((1-t)\frac{x+y}{2}+t\frac{u+v}{2}\right)dt-f\left(\frac{1}{2}\left(\frac{x+y}{2}+\frac{u+v}{2}\right)\right)$$

that can be written as

$$\frac{1}{2}\left[\mathcal{M}_f(x,u)+\mathcal{M}_f(y,v)\right]\geq \mathcal{M}_f\left(\frac{x+y}{2},\frac{u+v}{2}\right)$$
$$=\mathcal{M}_f\left(\frac{1}{2}(x,u)+\frac{1}{2}(y,v)\right)$$

for all $(x,u),(y,v)\in I\times I$, which shows that $\mathcal{M}_f$ is Jensen's convex on $I\times I$. Since $\mathcal{M}_f$ is continuous on $I\times I$, hence $\mathcal{M}_f$ is convex in the usual sense on $I\times I$. Further, by summing over i from 1 to n, we deduce that $\mathcal{M}_{n,f}$ is convex on $I^n\times I^n$.

The following reverses of the Hermite-Hadamard inequality hold:

Lemma 1 (Dragomir, 2002 [4] and [5]) *Let* $h:[a,b]\to\mathbb{R}$ *be a convex function on* $[a,b]$. *Then*

$$0\leq \frac{1}{8}\left[h_+\left(\frac{a+b}{2}\right)-h_-\left(\frac{a+b}{2}\right)\right](b-a) \tag{2.3}$$
$$\leq \frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)\,dx$$
$$\leq \frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a)$$

and

$$0 \le \frac{1}{8}\left[h_{+}\left(\frac{a+b}{2}\right)-h_{-}\left(\frac{a+b}{2}\right)\right](b-a) \tag{2.4}$$
$$\le \frac{1}{b-a}\int_a^b h(x)\,dx - h\left(\frac{a+b}{2}\right)$$
$$\le \frac{1}{8}\left[h_{-}(b)-h_{+}(a)\right](b-a).$$

The constant $\frac{1}{8}$ *is best possible in all inequalities from (2.3) and (2.4).*

We also have:

Theorem 3 *Let* f *be a* C^1 *convex function on an interval* I. *If* $\mathring{I}$ *is the interior of* I, *then for all* $(x, y) \in \mathring{I}^n \times \mathring{I}^n$, *we have*

$$0 \le \mathcal{M}_{n,f}(x, y) \le \mathcal{T}_{n,f}(x, y) \le \frac{1}{8}\mathcal{C}_{n,f'}(x, y) \tag{2.5}$$

where

$$\mathcal{C}_{n,f'}(x, y) := \sum_{i=1}^{n}\left[f'(x_i)-f'(y_i)\right](x_i-y_i). \tag{2.6}$$

Proof Since for $b \neq a$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 f((1-t)a+tb)\,dt,$$

then from (2.3), we get

$$\frac{f(x_i)+f(y_i)}{2} - \int_0^1 f((1-t)x_i+ty_i)\,dt \le \frac{1}{8}\left[f'(x_i)-f'(y_i)\right](x_i-y_i)$$

for all $i \in \{1, \ldots, n\}$, and this inequality also holds if $x_i = y_i$.

By summing these inequalities over $i \in \{1, \ldots, n\}$, we get the last inequality in (2.5).

Remark 1 If

$$\gamma = \inf_{t\in\mathring{I}} f'(t) \text{ and } \Gamma = \sup_{t\in\mathring{I}} f'(t)$$

are finite, then

$$\mathcal{C}_{n,f'}(x, y) \le (\Gamma-\gamma)\sum_{i=1}^{n}|x_i-y_i|$$

and by (2.5), we get the simpler upper bound

$$0 \le \mathcal{M}_{n,f}(x,y) \le \mathcal{T}_{n,f}(x,y) \le \frac{1}{8}(\Gamma - \gamma)\sum_{i=1}^{n}|x_i - y_i|.$$

Moreover, if $x_i, y_i \in [a,b] \subset \mathring{I}$ for all $i \in \{1,\ldots,n\}$ and since f' is increasing on $\mathring{I}$, then we have the inequalities

$$0 \le \mathcal{M}_{n,f}(x,y) \le \mathcal{T}_{n,f}(x,y) \le \frac{1}{8}\left[f'(b) - f'(a)\right]\sum_{i=1}^{n}|x_i - y_i|. \tag{2.7}$$

Since $\mathcal{J}_{n,f}(x,y) = \mathcal{T}_{n,f}(x,y) + \mathcal{M}_{n,f}(x,y)$, hence

$$0 \le \mathcal{J}_{n,f}(x,y) \le \frac{1}{4}\left[f'(b) - f'(a)\right]\sum_{i=1}^{n}|x_i - y_i|.$$

Corollary 1 *With the assumptions of Theorem 3 and if the derivative f' is Lipschitzian with the constant $K > 0$, namely,*

$$\left|f'(t) - f'(s)\right| \le K|t-s| \text{ for all } t,\ s \in \mathring{I},$$

then we have the inequality

$$0 \le \mathcal{M}_{n,f}(x,y) \le \mathcal{T}_{n,f}(x,y) \le \frac{1}{8}Kd_2^2(x,y), \tag{2.8}$$

where $d_2(x,y)$ is the Euclidean distance between x and y, namely,

$$d_2(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}.$$

Also, we have

$$0 \le \mathcal{J}_{n,f}(x,y) \le \frac{1}{4}Kd_2^2(x,y).$$

3 Related Results

We have the following Jensen's type inequality:

Theorem 4 *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I,, then for all $(x_i, y_i) \in I \times I$, $i \in \{1,\ldots,n\}$ and $p_i \ge 0$, $i \in \{1,\ldots,n\}$*

with $\sum_{i=1}^{n} p_i = 1,$, *we have*

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(x_i) - f'\left(\frac{x_i+y_i}{2}\right)\right](x_i - u) \\
&+ \frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'\left(\frac{x_i+y_i}{2}\right)\right](y_i - v) \\
&\ge \frac{1}{2}\sum_{i=1}^{n} p_i \left[f(x_i) + f(y_i)\right] - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&- \frac{1}{2}\left[f(u) + f(v)\right] + f\left(\frac{u+v}{2}\right) \\
&\ge \frac{1}{2}\left[f'(u) - f'\left(\frac{u+v}{2}\right)\right]\left(\sum_{i=1}^{n} p_i x_i - u\right) \\
&+ \frac{1}{2}\left[f'(v) - f'\left(\frac{u+v}{2}\right)\right]\left(\sum_{i=1}^{n} p_i y_i - v\right)
\end{aligned}
\tag{3.1}
$$

for all $(u, v) \in I \times I$.

In particular,

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(x_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]\left(x_i - \sum_{j=1}^{n} p_j x_j\right) \\
&+ \frac{1}{2}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'\left(\frac{x_i+y_i}{2}\right)\right]\left(y_i - \sum_{j=1}^{n} p_j y_j\right) \\
&\ge \frac{1}{2}\sum_{i=1}^{n} p_i \left[f(x_i) + f(y_i)\right] - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\
&- \frac{1}{2}\left[f\left(\sum_{i=1}^{n} p_i x_i\right) + f\left(\sum_{i=1}^{n} p_i y_i\right)\right] + f\left(\frac{\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i y_i}{2}\right) \\
&\ge 0.
\end{aligned}
\tag{3.2}
$$

Proof It is well known that if the function of two independent variables $F : D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex on the convex domain D and has partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ on D then for all $(x, y), (u, v) \in D$ we have the gradient inequalities

$$\begin{aligned}&\frac{\partial F(x,y)}{\partial x}(x-u)+\frac{\partial F(x,y)}{\partial y}(y-v)\\&\geq F(x,y)-F(u,v)\\&\geq\frac{\partial F(u,v)}{\partial x}(x-u)+\frac{\partial F(u,v)}{\partial y}(y-v).\end{aligned}\tag{3.3}$$

Now, if we take $F : I \times I \to \mathbb{R}$ given by

$$F(x,y)=\frac{1}{2}[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)$$

and observe that

$$\frac{\partial F(x,y)}{\partial x}=\frac{1}{2}\left[f'(x)-f'\left(\frac{x+y}{2}\right)\right]$$

and

$$\frac{\partial F(x,y)}{\partial y}=\frac{1}{2}\left[f'(y)-f'\left(\frac{x+y}{2}\right)\right]$$

and since F is convex on $I \times I$, then by (3.3), we get

$$\begin{aligned}&\frac{1}{2}\left[f'(x)-f'\left(\frac{x+y}{2}\right)\right](x-u)+\frac{1}{2}\left[f'(y)-f'\left(\frac{x+y}{2}\right)\right](y-v)\\&\geq\frac{1}{2}[f(x)+f(y)]-f\left(\frac{x+y}{2}\right)-\frac{1}{2}[f(u)+f(v)]+f\left(\frac{u+v}{2}\right)\\&\geq\frac{1}{2}\left[f'(u)-f'\left(\frac{u+v}{2}\right)\right](x-u)+\frac{1}{2}\left[f'(v)-f'\left(\frac{u+v}{2}\right)\right](y-v).\end{aligned}\tag{3.4}$$

Moreover, if $(x_i, y_i) \in I \times I$, $i \in \{1, \ldots, n\}$, then by (3.4), we get

$$\begin{aligned}&\frac{1}{2}\left[f'(x_i)-f'\left(\frac{x_i+y_i}{2}\right)\right](x_i-u)+\frac{1}{2}\left[f'(y_i)-f'\left(\frac{x_i+y_i}{2}\right)\right](y_i-v)\\&\geq\frac{1}{2}[f(x_i)+f(y_i)]-f\left(\frac{x_i+y_i}{2}\right)-\frac{1}{2}[f(u)+f(v)]+f\left(\frac{u+v}{2}\right)\\&\geq\frac{1}{2}\left[f'(u)-f'\left(\frac{u+v}{2}\right)\right](x_i-u)+\frac{1}{2}\left[f'(v)-f'\left(\frac{u+v}{2}\right)\right](y_i-v)\end{aligned}\tag{3.5}$$

for all $i \in \{1, \ldots, n\}$ and $(u, v) \in I \times I$.

Let $p_i \geq 0$ for all $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$. If we multiply (3.5) by $p_i \geq 0$ and sum over i from 1 to n, then we get the desired result (3.1).

Corollary 2 *With the assumptions of Theorem 4, we have*

$$\frac{1}{2}\sum_{i=1}^{n}\left[f'(x_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) \tag{3.6}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left[f'(y_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]\left(y_i - \frac{1}{n}\sum_{j=1}^{n} y_j\right)$$

$$\geq \mathcal{J}_{n,f}(x, y)$$

$$-\frac{1}{2}n\left[f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)\right] + nf\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i + y_i}{2}\right)\right)$$

$$\geq 0.$$

Similar results hold for the *Hermite-Hadamard trapezoid* and *mid-point divergences.*

Theorem 5 *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I, then for all $(x_i, y_i) \in I \times I$, $i \in \{1, \ldots, n\}$ and $p_i \geq 0$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$, we have*

$$\sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t)\left[f'(x_i) - f'((1-t)x_i + ty_i)\right] dt \tag{3.7}$$

$$+\sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t\left[f'(y_i) - f'((1-t)x_i + ty_i)\right] dt$$

$$\geq \frac{\sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} p_i f(y_i)}{2} - \sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i + ty_i)\, dt$$

$$-\frac{f(u) + f(v)}{2} + \int_0^1 f((1-t)u + tv)\, dt$$

$$\geq \sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t)\left[f'(u) - f'((1-t)u + tv)\right] dt$$

$$+\sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t\left[f'(v) - f'((1-t)u + tv)\right] dt$$

for all $(u, v) \in I \times I$.

In particular,

$$\sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j \right) \int_0^1 (1-t) \left[f'(x_i) - f'((1-t) x_i + t y_i) \right] dt \tag{3.8}$$
$$+ \sum_{i=1}^{n} p_i \left(y_i - \sum_{j=1}^{n} p_j y_j \right) \int_0^1 t \left[f'(y_i) - f'((1-t) x_i + t y_i) \right] dt$$
$$\geq \frac{\sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} p_i f(y_i)}{2} - \sum_{i=1}^{n} p_i \int_0^1 f((1-t) x_i + t y_i) dt$$
$$- \frac{f\left(\sum_{j=1}^{n} p_j x_j\right) + f\left(\sum_{j=1}^{n} p_j y_j\right)}{2} + \int_0^1 f\left(\sum_{j=1}^{n} p_j \left[(1-t) x_j + t y_j \right] \right) dt$$
$$\geq 0.$$

Proof Let $(x, y), (u, v) \in I \times I$. If we take $F : I \times I \to \mathbb{R}$ given by

$$F(x, y) = \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t) x + t y) dt$$

then

$$\frac{\partial F(x, y)}{\partial x} = \frac{1}{2} f'(x) - \int_0^1 (1-t) f'((1-t) x + t y) dt$$
$$= \int_0^1 (1-t) \left[f'(x) - f'((1-t) x + t y) \right] dt$$

and

$$\frac{\partial F(x, y)}{\partial y} = \frac{1}{2} f'(y) - \int_0^1 t f'((1-t) x + t y) dt$$
$$= \int_0^1 t \left[f'(y) - f'((1-t) x + t y) \right] dt$$

and since F is convex on $I \times I$, then by (3.3), we get

$$\int_0^1 (1-t) \left[f'(x) - f'((1-t) x + t y) \right] dt (x - u) \tag{3.9}$$
$$+ \int_0^1 t \left[f'(y) - f'((1-t) x + t y) \right] dt (y - v)$$

$$\geq \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt$$
$$- \frac{f(u)+f(v)}{2} + \int_0^1 f((1-t)u+tv)\,dt$$
$$\geq \int_0^1 (1-t)\left[f'(u) - f'((1-t)u+tv)\right]dt\,(x-u)$$
$$+ \int_0^1 t\left[f'(v) - f'((1-t)u+tv)\right]dt\,(y-v).$$

Therefore, if $(x_i, y_i) \in I \times I$, $i \in \{1, \ldots, n\}$, then by (3.9), we get

$$(x_i - u)\int_0^1 (1-t)\left[f'(x_i) - f'((1-t)x_i+ty_i)\right]dt \tag{3.10}$$
$$+ (y_i - v)\int_0^1 t\left[f'(y_i) - f'((1-t)x_i+ty_i)\right]dt$$
$$\geq \frac{f(x_i)+f(y_i)}{2} - \int_0^1 f((1-t)x_i+ty_i)\,dt$$
$$- \frac{f(u)+f(v)}{2} + \int_0^1 f((1-t)u+tv)\,dt$$
$$\geq (x_i - u)\int_0^1 (1-t)\left[f'(u) - f'((1-t)u+tv)\right]dt$$
$$+ (y_i - v)\int_0^1 t\left[f'(v) - f'((1-t)u+tv)\right]dt$$

for all $i \in \{1, \ldots, n\}$ and $(u, v) \in I \times I$.

Let $p_i \geq 0$ for all $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^n p_i = 1$. If we multiply (3.10) by $p_i \geq 0$ and sum over i from 1 to n, then we get the desired result (3.7).

Corollary 3 *With the assumptions of Theorem 4, we have*

$$\sum_{i=1}^n \left(x_i - \frac{1}{n}\sum_{j=1}^n x_j\right)\int_0^1 (1-t)\left[f'(x_i) - f'((1-t)x_i+ty_i)\right]dt \tag{3.11}$$
$$+ \sum_{i=1}^n \left(y_i - \frac{1}{n}\sum_{j=1}^n y_j\right)\int_0^1 t\left[f'(y_i) - f'((1-t)x_i+ty_i)\right]dt$$
$$\geq \mathcal{T}_{n,f}(x, y)$$

$$- n \frac{f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)}{2} + n\int_0^1 f\left(\frac{1}{n}\sum_{i=1}^{n}[(1-t)x_i + ty_i]\right)dt$$

$$\geq 0.$$

We also have:

Theorem 6 *Let f be a C^2 function on an interval I. If f is convex and $\frac{1}{f''}$ is concave on I, then for all $(x_i, y_i) \in I \times I$, $i \in \{1, \ldots, n\}$ and $p_i \geq 0$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$, we have*

$$\sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t)\left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]dt \tag{3.12}$$

$$+ \sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t\left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]dt$$

$$\geq \sum_{i=1}^{n} p_i \int_0^1 f((1-t)x_i + ty_i)\,dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i + y_i}{2}\right)$$

$$- \int_0^1 f((1-t)u + tv)\,dt + f\left(\frac{u+v}{2}\right)$$

$$\geq \sum_{i=1}^{n} p_i (x_i - u) \int_0^1 (1-t)\left[f'((1-t)u + tv) - f'\left(\frac{u+v}{2}\right)\right]dt$$

$$+ \sum_{i=1}^{n} p_i (y_i - v) \int_0^1 t\left[f'((1-t)u + tv) - f'\left(\frac{u+v}{2}\right)\right]dt$$

for all $(u, v) \in I \times I$.

In particular,

$$\sum_{i=1}^{n} p_i \left(x_i - \sum_{j=1}^{n} p_j x_j\right) \int_0^1 (1-t)\left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]dt \tag{3.13}$$

$$+ \sum_{i=1}^{n} p_i \left(y_i - \sum_{j=1}^{n} p_j y_j\right) \int_0^1 t\left[f'((1-t)x_i + ty_i) - f'\left(\frac{x_i + y_i}{2}\right)\right]dt$$

$$\geq \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\,x_i + ty_i\right) dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i + y_i}{2}\right)$$
$$- \int_0^1 f\left(\sum_{j=1}^{n} p_i \left[(1-t)\,x_i + ty_i\right]\right) dt + f\left(\sum_{j=1}^{n} p_i \left(\frac{x_i + y_i}{2}\right)\right)$$
$$\geq 0.$$

Proof Let $(x, y), (u, v) \in I \times I$. If we take $F : I \times I \to \mathbb{R}$ given by

$$F(x, y) = \int_0^1 f\left((1-t)\,x + ty\right) dt - f\left(\frac{x+y}{2}\right)$$

then

$$\frac{\partial F(x, y)}{\partial x} = \int_0^1 (1-t)\, f'\left((1-t)\,x + ty\right) dt - \frac{1}{2} f'\left(\frac{x+y}{2}\right)$$
$$= \int_0^1 (1-t)\left[f'\left((1-t)\,x + ty\right) - f'\left(\frac{x+y}{2}\right)\right] dt$$

and

$$\frac{\partial F(x, y)}{\partial y} = \int_0^1 t f'\left((1-t)\,x + ty\right) dt - \frac{1}{2} f'\left(\frac{x+y}{2}\right)$$
$$= \int_0^1 t\left[f'\left((1-t)\,x + ty\right) - f'\left(\frac{x+y}{2}\right)\right] dt.$$

The rest of the proof follows in the similar way to the one from above, and we omit the details.

Corollary 4 *With the assumptions of Theorem 4, we have*

$$\sum_{i=1}^{n} \left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) \int_0^1 (1-t)\left[f'\left((1-t)\,x_i + ty_i\right) - f'\left(\frac{x_i + y_i}{2}\right)\right] dt \tag{3.14}$$
$$+ \sum_{i=1}^{n} \left(y_i - \frac{1}{n}\sum_{j=1}^{n} y_j\right) \int_0^1 t\left[f'\left((1-t)\,x_i + ty_i\right) - f'\left(\frac{x_i + y_i}{2}\right)\right] dt$$
$$\geq \mathcal{M}_{n,f}(x, y)$$

$$- n \int_0^1 f \left(\frac{1}{n} \sum_{j=1}^{n} [(1-t) x_i + t y_i] \right) dt + n f \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_i + y_i}{2} \right) \right)$$
$$\geq 0.$$

4 Some Results for f-Divergences

Consider the probability distributions p and q. Assume that $f : (0, \infty) \to \mathbb{R}$ is convex, and define the *Csiszár's f-divergence measure* [2] and [3]

$$\mathcal{C}_{n,f}(p,q) := \sum_{i=1}^{n} p_i f \left(\frac{q_i}{p_i} \right)$$

and the *Lin-Wong f-divergence measure* [9]

$$\mathcal{LW}_{n,f}(p,q) := \sum_{i=1}^{n} p_i f \left(\frac{q_i + p_i}{2 p_i} \right).$$

If $f : (0, \infty) \to \mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0, \infty)$, then we get from (3.2) for $x_i = \frac{q_i}{p_i}$ and $y_i = 1$, $i \in \{1, \dots, n\}$ that

$$\frac{1}{2} \sum_{i=1}^{n} \left[f' \left(\frac{q_i}{p_i} \right) - f' \left(\frac{q_i + p_i}{2 p_i} \right) \right] (q_i - p_i)$$
$$\geq \frac{1}{2} \sum_{i=1}^{n} p_i \left[f \left(\frac{q_i}{p_i} \right) + f(1) \right] - \sum_{i=1}^{n} p_i f \left(\frac{q_i + p_i}{2 p_i} \right) \geq 0,$$

namely,

$$0 \leq \frac{1}{2} \left[\mathcal{C}_{n,f}(p,q) + f(1) \right] - \mathcal{LW}_{n,f}(p,q)$$
$$\leq \frac{1}{2} \sum_{i=1}^{n} \left[f' \left(\frac{q_i}{p_i} \right) - f' \left(\frac{q_i + p_i}{2 p_i} \right) \right] (q_i - p_i) \qquad (4.1)$$

for any probability distributions p and q.

If there exists $0 < r < 1 < R < \infty$ and $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, \dots, n\}$ and for some $K > 0$, we have

$$\left|f'(s) - f'(t)\right| \leq K|s - t| \tag{4.2}$$

for any $s, t \in [r, R]$, then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\left[f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i + p_i}{2p_i}\right)\right](q_i - p_i) \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\left|f'\left(\frac{q_i}{p_i}\right) - f'\left(\frac{q_i + p_i}{2p_i}\right)\right||q_i - p_i| \\
&\leq \frac{1}{2}K\sum_{i=1}^{n}\left|\frac{q_i}{p_i} - \frac{q_i + p_i}{2p_i}\right||q_i - p_i| = \frac{1}{4}K\sum_{i=1}^{n}\frac{(q_i - p_i)^2}{p_i} \\
&= \frac{1}{4}K\sum_{i=1}^{n}\frac{q_i^2 - 2p_i q_i + p_i^2}{p_i} = \frac{1}{4}K\left(\sum_{i=1}^{n}\frac{q_i^2}{p_i} - 1\right) = D_{\chi^2}(p, q)
\end{aligned}$$

where $D_{\chi^2}(p, q)$ is the well-known χ^2-*divergence.*

By utilizing the inequality (4.1), we get

$$0 \leq \frac{1}{2}\left[\mathcal{C}_{n,f}(p, q) + f(1)\right] - \mathcal{LW}_{n,f}(p, q) \leq \frac{1}{4}K D_{\chi^2}(p, q). \tag{4.3}$$

Since f is a C^2 convex function on $[r, R]$, then we can take $K = \max_{t\in[r,R]}\left|f''(t)\right|$ in the inequality (4.3).

In the same paper [9], the authors introduced the *Hermite-Hadamard (HH) f-divergence* by

$$\mathcal{D}^f_{n,HH}(p, q) := \sum_{i=1}^{n} p_i \frac{\int_1^{\frac{q_i}{p_i}} f(t)\,dt}{\frac{q_i}{p_i} - 1} = \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)\frac{q_i}{p_i} + t\right)dt.$$

If $f : (0, \infty) \to \mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0, \infty)$, then we get from (3.8) for $x_i = \frac{q_i}{p_i}$ and $y_i = 1$, $i \in \{1, \ldots, n\}$ that, namely,

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\mathcal{C}_{n,f}(p, q) + f(1)\right] - \mathcal{D}^f_{n,HH}(p, q) \\
&\leq \sum_{i=1}^{n}(q_i - p_i)\int_0^1 (1-t)\left[f'\left(\frac{q_i}{p_i}\right) - f'\left((1-t)\frac{q_i}{p_i} + t\right)\right]dt
\end{aligned} \tag{4.4}$$

for any probability distributions p and q.

If there exists $0 < r < 1 < R < \infty$ and $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, \ldots, n\}$ and for some $K > 0$, we have the condition (4.2), then

$$\sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left(\frac{q_i}{p_i}\right)-f'\left((1-t)\frac{q_i}{p_i}+t\right)\right]dt$$
$$\le\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|f'\left(\frac{q_i}{p_i}\right)-f'\left((1-t)\frac{q_i}{p_i}+t\right)\right|dt$$
$$\le K\sum_{i=1}^{n}|q_i-p_i|\int_0^1(1-t)\left|\frac{q_i}{p_i}-(1-t)\frac{q_i}{p_i}-t\right|dt$$
$$=K\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}\int_0^1(1-t)\,tdt=\frac{1}{6}K\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}=\frac{1}{6}KD_{\chi^2}(p,q).$$

Therefore, if $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,\ldots,n\}$ and $K=\max_{t\in[r,R]}\left|f''(t)\right|$, then

$$0\le\frac{1}{2}\left[\mathcal{C}_{n,f}(p,q)+f(1)\right]-\mathcal{D}^{f}_{n,HH}(p,q)\le\frac{1}{6}KD_{\chi^2}(p,q). \tag{4.5}$$

If $f:(0,\infty)\to\mathbb{R}$ is a C^2 convex function and such that $\frac{1}{f''}$ is concave on $(0,\infty)$, then we get from (3.13) for $x_i=\frac{q_i}{p_i}$ and $y_i=1$, $i\in\{1,\ldots,n\}$ that

$$\sum_{i=1}^{n}p_i\left(\frac{q_i}{p_i}-1\right)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{\frac{q_i}{p_i}+1}{2}\right)\right]dt$$
$$\ge\sum_{i=1}^{n}p_i\int_0^1 f\left((1-t)\frac{q_i}{p_i}+t\right)dt-\sum_{i=1}^{n}p_i f\left(\frac{\frac{q_i}{p_i}+1}{2}\right)\ge 0,$$

namely,

$$0\le\mathcal{D}^{f}_{n,HH}(p,q)-\mathcal{LW}_{n,f}(p,q) \tag{4.6}$$
$$\le\sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{q_i+p_i}{2p_i}\right)\right]dt$$

for any probability distributions p and q.

If there exists $0<r<1<R<\infty$ and $\frac{q_i}{p_i}\in[r,R]$ for any $i\in\{1,\ldots,n\}$ and for some $K>0$, we have the condition (4.2), then

$$\sum_{i=1}^{n}(q_i-p_i)\int_0^1(1-t)\left[f'\left((1-t)\frac{q_i}{p_i}+t\right)-f'\left(\frac{q_i+p_i}{2p_i}\right)\right]dt$$

$$\le \sum_{i=1}^{n} |q_i - p_i| \int_0^1 (1-t) \left| f'\left((1-t)\frac{q_i}{p_i} + t\right) - f'\left(\frac{q_i + p_i}{2p_i}\right)\right| dt$$

$$\le K \sum_{i=1}^{n} |q_i - p_i| \int_0^1 (1-t) \left|(1-t)\frac{q_i}{p_i} + t - \frac{q_i + p_i}{2p_i}\right| dt$$

$$= K \sum_{i=1}^{n} \frac{(q_i - p_i)^2}{p_i} \int_0^1 (1-t) \left|t - \frac{1}{2}\right| dt = \frac{1}{8} K D_{\chi^2}(p, q).$$

Therefore, if $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, \dots, n\}$ and $K = \max_{t \in [r,R]} \left|f''(t)\right|$, then

$$0 \le \mathcal{D}^f_{n,HH}(p, q) - \mathcal{LW}_{n,f}(p, q) \le \frac{1}{8} K D_{\chi^2}(p, q). \tag{4.7}$$

5 Some Examples

Consider the power function $f_\alpha : [0, \infty) \to \mathbb{R}$, $f_\alpha(t) = (\alpha - 1)^{-1} t^\alpha$ with $\alpha \in (1, 2]$. This function is convex on $[0, \infty)$ and $\frac{1}{f''_\alpha}$ is concave on $(0, \infty)$ and therefore

$$\mathcal{J}_{n,\alpha}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} \left[A\left(x_i^\alpha, y_i^\alpha\right) - A^\alpha(x_i, y_i)\right] \tag{5.1}$$

is jointly convex on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, where $\mathbb{R}_+ := [0, \infty)$.

The *Hermite-Hadamard trapezoid* and *mid-point divergences* associated to f_α are

$$\mathcal{T}_{n,\alpha}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} \left[A\left(x_i^\alpha, y_i^\alpha\right) - L_\alpha^\alpha(x_i, y_i)\right] \tag{5.2}$$

and

$$\mathcal{M}_{n,f}(x, y) := (\alpha - 1)^{-1} \sum_{i=1}^{n} \left[L_\alpha^\alpha(x_i, y_i) - A^\alpha(x_i, y_i)\right] \tag{5.3}$$

for $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

According to Theorem 2, these divergences are jointly convex on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. From Theorem 3, we have the inequalities

$$0 \le \mathcal{M}_{n,\alpha}(x, y) \le \mathcal{T}_{n,\alpha}(x, y) \le \frac{1}{8}\alpha^2(\alpha-1)^{-1}\sum_{i=1}^{n} L_{\alpha-1}^{\alpha-1}(x_i, y_i)(x_i - y_i)^2 \tag{5.4}$$

for $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$.

If $[a, b] \subset \mathbb{R}_+^n$ and $(x, y) \in [a, b]^n \times [a, b]^n$, then by (2.7), we have

$$0 \le \mathcal{M}_{n,f}(x, y) \le \mathcal{T}_{n,f}(x, y) \le \frac{1}{8}(\alpha-1)^{-1}\alpha\left(b^{\alpha-1} - a^{\alpha-1}\right)\sum_{i=1}^{n}|x_i - y_i|. \tag{5.5}$$

We have for $[a, b] \subset \mathbb{R}_{++}^n := (0, \infty)$ that

$$K := \max_{t\in[a,b]} f_\alpha''(t) = (\alpha-1)^{-1}\alpha(\alpha-1)\max_{t\in[a,b]} t^{\alpha-2} = \frac{\alpha}{a^{2-\alpha}}$$

and by the inequality (2.8), we have

$$0 \le \mathcal{M}_{n,f}(x, y) \le \mathcal{T}_{n,f}(x, y) \le \frac{\alpha}{8a^{2-\alpha}} d_2^2(x, y). \tag{5.6}$$

For f_α, we have

$$\mathcal{C}_{n,f_\alpha}(p, q) := (\alpha-1)^{-1}\sum_{i=1}^{n} p_i^{1-\alpha} q_i^\alpha,$$

$$\mathcal{LW}_{n,f_\alpha}(p, q) := (\alpha-1)^{-1}\sum_{i=1}^{n} p_i^{1-\alpha}\left(\frac{q_i + p_i}{2}\right)^\alpha$$

and

$$\mathcal{D}_{n,HH}^{f_\alpha}(p, q) := (\alpha-1)^{-1}\sum_{i=1}^{n} p_i \frac{\int_1^{\frac{q_i}{p_i}} f_\alpha(t)\,dt}{\frac{q_i}{p_i} - 1} = (\alpha-1)^{-1}\sum_{i=1}^{n} p_i L_\alpha\left(\frac{q_i}{p_i}, 1\right).$$

Let $0 < r < 1 < R < \infty$. If $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, \dots, n\}$, then from the inequality (4.3), we get

$$0 \le \frac{1}{2}\left[\mathcal{C}_{n,f_\alpha}(p, q) + (\alpha-1)^{-1}\right] - \mathcal{LW}_{n,f_\alpha}(p, q) \le \frac{1}{4}\frac{\alpha}{r^{2-\alpha}} D_{\chi^2}(p, q), \tag{5.7}$$

and from (4.5), we have

$$0 \le \frac{1}{2}\left[\mathcal{C}_{n,f_\alpha}(p, q) + (\alpha-1)^{-1}\right] - \mathcal{D}_{n,HH}^{f_\alpha}(p, q) \le \frac{1}{6}\frac{\alpha}{r^{2-\alpha}} D_{\chi^2}(p, q) \tag{5.8}$$

while from (4.7), we obtain

$$0 \le \mathcal{D}_{n,HH}^{f_\alpha}(p,q) - \mathcal{LW}_{n,f_\alpha}(p,q) \le \frac{1}{8}\frac{\alpha}{r^{2-\alpha}} D_{\chi^2}(p,q). \tag{5.9}$$

Consider now the function $f_1 : (0,\infty) \to \mathbb{R}$, $f_1(t) = t\ln t$. The function f_1 is convex on $(0,\infty)$ and $\frac{1}{f_1''}$ is concave on $(0,\infty)$. Then the function

$$\mathcal{J}_{n,1}(x,y) := \frac{1}{2}\sum_{i=1}^{n}\left[x_i \ln(x_i) + y_i \ln(y_i) - (x_i + y_i)\ln\left(\frac{x_i + y_i}{2}\right)\right]$$

is jointly convex on $\mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$.

Observe that

$$\begin{aligned}
\frac{1}{b-a}\int_a^b t\ln t\,dt &= \frac{1}{2}\frac{1}{b-a}\int_a^b \ln t\,d\left(t^2\right) \\
&= \frac{1}{2}\frac{1}{b-a}\left[t^2\ln t\Big|_a^b - \int_a^b t\,dt\right] \\
&= \frac{1}{2}\frac{1}{b-a}\left[b^2\ln b - a^2\ln a - \frac{b^2-a^2}{2}\right] \\
&= \frac{1}{2}\frac{1}{b-a}\left[\frac{b^2\ln b^2 - a^2\ln a^2}{2} - \frac{b^2-a^2}{2}\right] \\
&= \frac{1}{2}\frac{1}{b-a}\frac{b^2-a^2}{2}\left[\frac{b^2\ln b^2 - a^2\ln a^2}{b^2-a^2} - 1\right] \\
&= \frac{1}{4}(b+a)\ln I\left(a^2,b^2\right),
\end{aligned}$$

where I is the *identric mean*.

Therefore,

$$\begin{aligned}
\mathcal{T}_{n,1}(x,y) &:= \sum_{i=1}^{n}\left(\frac{1}{2}\left[f(x_i) + f(y_i)\right] - \int_0^1 f((1-t)x_i + ty_i)\,dt\right) \qquad (5.10)\\
&= \frac{1}{2}\sum_{i=1}^{n}\left[\frac{x_i\ln\left(x_i^2\right) + y_i\ln\left(y_i^2\right)}{2} - A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right)\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\left[A\left(x_i\ln\left(x_i^2\right), y_i\ln\left(y_i^2\right)\right) - A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right)\right]
\end{aligned}$$

and

$$\mathcal{M}_{n,1}(x,y) := \sum_{i=1}^{n}\left(A(x_i,y_i)\ln I\left(x_i^2,y_i^2\right) - A(x_i,y_i)\ln A(x_i,y_i)\right)$$
$$= \sum_{i=1}^{n} A(x_i,y_i)\left[\ln I\left(x_i^2,y_i^2\right) - \ln A(x_i,y_i)\right]$$

for $(x,y) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$.

According to Theorem 2, these divergences are jointly convex on $\mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$. From Theorem 3, we have the inequalities

$$0 \le \mathcal{M}_{n,1}(x,y) \le \mathcal{T}_{n,1}(x,y) \le \frac{1}{8}\sum_{i=1}^{n}\frac{(x_i - y_i)^2}{L(x_i,y_i)}. \tag{5.11}$$

From the inequality (2.7), we have

$$0 \le \mathcal{M}_{n,1}(x,y) \le \mathcal{T}_{n,1}(x,y) \le \frac{1}{8}(\ln b - \ln a)\sum_{i=1}^{n}|x_i - y_i| \tag{5.12}$$

for $(x,y) \in [a,b]^n \times [a,b]^n$, where $[a,b] \subset (0,\infty)$.

We also have from (2.8) that

$$0 \le \mathcal{M}_{n,1}(x,y) \le \mathcal{T}_{n,1}(x,y) \le \frac{1}{8}\frac{b-a}{ba}d_2^2(x,y) \tag{5.13}$$

for $(x,y) \in [a,b]^n \times [a,b]^n$, where $[a,b] \subset (0,\infty)$.

Consider the divergences

$$\mathcal{C}_{n,f_1}(p,q) := \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right),$$

Kullback-Leibler divergence [7],

$$\mathcal{LW}_{n,f_1}(p,q) := \sum_{i=1}^{n}\frac{q_i + p_i}{2}\ln\left(\frac{q_i + p_i}{2p_i}\right),$$

Lin-Wong divergence measure [8],

and

$$\mathcal{D}_{n,HH}^{f_1}(p,q) := \sum_{i=1}^{n} p_i \frac{\int_1^{\frac{q_i}{p_i}} t\ln t\,dt}{\frac{q_i}{p_i} - 1} = \frac{1}{2}\sum_{i=1}^{n} A(q_i,p_i)\ln I\left(\left(\frac{q_i}{p_i}\right)^2, 1\right).$$

Let $0 < r < 1 < R < \infty$. If $\frac{q_i}{p_i} \in [r, R]$ for any $i \in \{1, \ldots, n\}$, then from the inequality (4.3), we get

$$0 \leq \frac{1}{2}\mathcal{C}_{n,f_1}(p,q) - \mathcal{LW}_{n,f_1}(p,q) \leq \frac{1}{4r} D_{\chi^2}(p,q), \tag{5.14}$$

and from (4.5), we have

$$0 \leq \frac{1}{2}\mathcal{C}_{n,f_1}(p,q) - \mathcal{D}^{f_1}_{n,HH}(p,q) \leq \frac{1}{6r} D_{\chi^2}(p,q), \tag{5.15}$$

while from (4.7), we obtain

$$0 \leq \mathcal{D}^{f_1}_{n,HH}(p,q) - \mathcal{LW}_{n,f_1}(p,q) \leq \frac{1}{8r} D_{\chi^2}(p,q). \tag{5.16}$$

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Hermite-Hadamard-Type Integral Inequalities for Perspective Function

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Abstract Let $f:(0,\infty)\to\mathbb{R}$ be a convex function on $(0,\infty)$. The associated two variables *perspective function* $P_f:(0,\infty)\times(0,\infty)\to\mathbb{R}$ is defined by

$$P_f(x,y):=xf\left(\frac{y}{x}\right).$$

In this paper, we establish some basic and double integral inequalities for the perspective function P_f defined above. Some double integral inequalities in the case of rectangles, squares, and circular sectors are also given.

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1 Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)\le\int_a^b f(x)dx\le(b-a)\frac{f(a)+f(b)}{2},\quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

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It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality. For a monograph devoted to this inequality, see [6]. Related results can be also found in [4].

In 1990, [3] the author established the following refinement of Hermite-Hadamard inequality for double and triple integrals for the convex function $f : [a, b] \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \le \frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left((1-t)\,x+ty\right)dtdxdy \le \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.2}$$

More recently, [5] we obtained a different double integral inequality of Hermite-Hadamard type for the convex function $f : [a, b] \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{(d-c)^2}\int_c^d\int_c^d f\left(\frac{\alpha a+\beta b}{\alpha+\beta}\right)d\beta d\alpha \le \frac{f(a)+f(b)}{2} \tag{1.3}$$

where $0 < c < d$.

Let $f : (0, \infty) \to \mathbb{R}$ be a convex function on $(0, \infty)$. The associated two variables *perspective function* $P_f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is defined by

$$P_f(x, y) := xf\left(\frac{y}{x}\right). \tag{1.4}$$

In this paper, we establish some basic and double integral inequalities for the perspective function P_f defined above. Some integral inequalities in the case of rectangles, squares, and circular sectors are also given.

2 General Results

We start with the following fundamental fact.

Lemma 1 *Let* $f : (0, \infty) \to \mathbb{R}$ *be a convex function on* $(0, \infty)$. *Then the perspective function* $P_f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ *defined by (1.4) is convex on* $(0, \infty) \times (0, \infty)$.

Proof Let $(x_1, y_1), (x_2, y_2) \in (0, \infty) \times (0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$; then

$$\begin{aligned} P_f(\alpha(x_1, y_1) + \beta(x_2, y_2)) &= P_f(\alpha x_1 + \beta x_2, \alpha y_1 + y_2\beta) \\ &= (\alpha x_1 + \beta x_2) f\left(\frac{\alpha y_1 + y_2\beta}{\alpha x_1 + \beta x}\right) \\ &= (\alpha x_1 + \beta x_2) f\left(\frac{\alpha x_1 \frac{y_1}{x_1} + \beta x_2 \frac{y_2}{x_2}}{\alpha x_1 + \beta x_2}\right) \\ &\leq (\alpha x_1 + \beta x_2)\left[\frac{\alpha x_1}{\alpha x_1 + \beta x_2} f\left(\frac{y_1}{x_1}\right) + \frac{\beta x_2}{\alpha x_1 + \beta x_2} f\left(\frac{y_2}{x_2}\right)\right] \\ &= \alpha x_1 f\left(\frac{y_1}{x_1}\right) + \beta x_2 f\left(\frac{y_2}{x_2}\right) = \alpha P_f(x_1, y_1) + \beta P_f(x_2, y_2), \end{aligned}$$

which proves the joint convexity. □

We have the following basic inequality for two values of the perspective function:

Theorem 1 *Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable convex function on $(0, \infty)$. Then for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$, we have the double inequality*

$$\begin{aligned} f\left(\frac{y}{x}\right)(x - u) + f'\left(\frac{y}{x}\right)\left(\frac{yu - xv}{x}\right) &\geq P_f(x, y) - P_f(u, v) \\ &\geq f\left(\frac{v}{u}\right)(x - u) + f'\left(\frac{v}{u}\right)\left(\frac{yu - vx}{u}\right). \end{aligned} \tag{2.1}$$

The inequality (2.1) is equivalent to the following two inequalities:

$$P_f(x, y) \geq xf\left(\frac{v}{u}\right) + f'\left(\frac{v}{u}\right)\left(\frac{yu - vx}{u}\right) \tag{2.2}$$

and

$$P_f(u, v) \geq uf\left(\frac{y}{x}\right) - f'\left(\frac{y}{x}\right)\left(\frac{yu - xv}{x}\right) \tag{2.3}$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

The inequality (2.1) is also equivalent to the double inequality

$$f'\left(\frac{y}{x}\right)\left(y - \frac{xv}{u}\right) + xf\left(\frac{v}{u}\right) \geq P_f(x, y) \geq xf\left(\frac{v}{u}\right) + f'\left(\frac{v}{u}\right)\left(y - \frac{xv}{u}\right) \tag{2.4}$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

Proof Observe that the following partial derivatives exist and for all $(x, y) \in (0, \infty) \times (0, \infty)$

$$\begin{aligned}\frac{\partial P_f(x, y)}{\partial x} &= \frac{d}{dx}\left(xf\left(\frac{y}{x}\right)\right) = f\left(\frac{y}{x}\right) + x\frac{d}{dx}\left(f\left(\frac{y}{x}\right)\right)\\ &= f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\frac{d}{dx}\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial P_f(x, y)}{\partial y} &= \frac{d}{dy}\left(xf\left(\frac{y}{x}\right)\right) = x\frac{d}{dy}\left(f\left(\frac{y}{x}\right)\right)\\ &= xf'\left(\frac{y}{x}\right)\frac{d}{dy}\left(\frac{y}{x}\right) = f'\left(\frac{y}{x}\right).\end{aligned}$$

Also, for all $(u, v) \in (0, \infty) \times (0, \infty)$, we have

$$\frac{\partial P_f(u, v)}{\partial x} = f\left(\frac{v}{u}\right) - \frac{v}{u}f'\left(\frac{v}{u}\right)$$

and

$$\frac{\partial P_f(u, v)}{\partial y} = f'\left(\frac{v}{u}\right).$$

Since P_f is a convex function on $(0, \infty) \times (0, \infty)$, then for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$, we have the gradient inequality

$$\begin{aligned}&\frac{\partial P_f(x, y)}{\partial x}(x-u) + \frac{\partial P_f(x, y)}{\partial y}(y-v)\\ &\geq P_f(x, y) - P_f(u, v)\\ &\geq \frac{\partial P_f(u, v)}{\partial x}(x-u) + \frac{\partial P_f(u, v)}{\partial y}(y-v),\end{aligned}$$

namely, by the calculations above,

$$\begin{aligned}&\left[f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u) + f'\left(\frac{y}{x}\right)(y-v)\\ &\geq P_f(x, y) - P_f(u, v)\\ &\geq \left[f\left(\frac{v}{u}\right) - \frac{v}{u}f'\left(\frac{v}{u}\right)\right](x-u) + f'\left(\frac{v}{u}\right)(y-v).\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}
&\left[f\left(\frac{v}{u}\right)-\frac{v}{u}f'\left(\frac{v}{u}\right)\right](x-u)+f'\left(\frac{v}{u}\right)(y-v)\\
&=f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)(y-v)-\frac{v}{u}f'\left(\frac{v}{u}\right)(x-u)\\
&=f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)\left[y-v-\frac{v}{u}(x-u)\right]\\
&=f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left[f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u)+f'\left(\frac{y}{x}\right)(y-v)\\
&=f\left(\frac{y}{x}\right)(x-u)-\frac{y}{x}f'\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)(y-v)\\
&=f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)(y-v)-\frac{y}{x}f'\left(\frac{y}{x}\right)(x-u)\\
&=f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right),
\end{aligned}$$

hence, by (2.5), we get (2.1).

Now, observe that

$$\begin{aligned}
&f\left(\frac{v}{u}\right)(x-u)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)\\
&=f\left(\frac{v}{u}\right)x-f\left(\frac{v}{u}\right)u+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)\\
&=f\left(\frac{v}{u}\right)x-P_f(u,v)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)
\end{aligned}$$

and by the second inequality in (2.1), we get

$$P_f(x,y)-P_f(u,v)\geq f\left(\frac{v}{u}\right)x-P_f(u,v)+f'\left(\frac{v}{u}\right)\left(\frac{yu-vx}{u}\right)$$

namely, (2.2).

Also,

$$\begin{aligned} & f\left(\frac{y}{x}\right)(x-u)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right) \\ & = xf\left(\frac{y}{x}\right)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right) \\ & = P_f(x,y)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right) \end{aligned}$$

and by the first inequality in (2.1), we have

$$P_f(x,y)-uf\left(\frac{y}{x}\right)+f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)\geq P_f(x,y)-P_f(u,v),$$

namely, (2.3).

The inequality (2.3) can also be written as

$$f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{x}\right)+uf\left(\frac{v}{u}\right)\geq uf\left(\frac{y}{x}\right).$$

By multiplying this inequality by x and dividing with u, we get

$$f'\left(\frac{y}{x}\right)\left(\frac{yu-xv}{u}\right)+xf\left(\frac{v}{u}\right)\geq P_f(x,y),$$

which proves the last part of the theorem. □

Corollary 1 *With the assumptions of Theorem 1, we have*

$$\begin{aligned} & \left[f\left(\frac{y}{x}\right)-f'\left(\frac{y}{x}\right)\left(\frac{y+x}{x}\right)\right](x-y)\geq P_f(x,y)-P_f(y,x) \\ & \qquad \geq \left[f\left(\frac{x}{y}\right)-f'\left(\frac{x}{y}\right)\left(\frac{y+x}{y}\right)\right](x-y), \end{aligned} \tag{2.6}$$

$$f'\left(\frac{y}{x}\right)(y-v)\geq P_f(x,y)-P_f(x,v)\geq f'\left(\frac{v}{x}\right)(y-v) \tag{2.7}$$

and

$$\begin{aligned} & \left[f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right](x-u)\geq P_f(x,y)-P_f(u,y) \\ & \qquad \geq \left[f\left(\frac{y}{u}\right)-\frac{y}{u}f'\left(\frac{y}{u}\right)\right](x-u). \end{aligned} \tag{2.8}$$

If f is normalized, namely, $f(1) = 0$, then

$$f\left(\frac{y}{x}\right)(x-u) + \frac{u}{x} f'\left(\frac{y}{x}\right)(y-x) \geq P_f(x, y) \geq f'(1)(y-x) \tag{2.9}$$

and

$$f'\left(\frac{y}{x}\right)(y-x) \geq P_f(x, y) \geq f'(1)(y-x). \tag{2.10}$$

Remark 1 From the inequality (2.4), we have for $u = y$ and $v = x$ that

$$f'\left(\frac{y}{x}\right)\left(\frac{y^2-x^2}{y}\right) + xf\left(\frac{x}{y}\right) \geq P_f(x, y) \geq xf\left(\frac{x}{y}\right) + f'\left(\frac{x}{y}\right)\left(\frac{y^2-x^2}{y}\right). \tag{2.11}$$

By taking $u = x$ in (2.4), we get

$$f'\left(\frac{y}{x}\right)(y-v) + xf\left(\frac{v}{x}\right) \geq P_f(x, y) \geq xf\left(\frac{v}{x}\right) + f'\left(\frac{v}{x}\right)(y-v). \tag{2.12}$$

Also, for $v = y$ in (2.4), we get

$$f'\left(\frac{y}{x}\right)\left(\frac{u-x}{u}\right)y + xf\left(\frac{y}{u}\right) \geq P_f(x, y) \geq xf\left(\frac{y}{u}\right) + f'\left(\frac{y}{u}\right)\left(\frac{u-x}{u}\right)y. \tag{2.13}$$

Consider the convex function $f(t) = -\ln t$, $t > 0$. Then by the inequality (2.4), we get

$$\frac{x^2 v}{yu} - x + x\ln\left(\frac{u}{v}\right) \geq x\ln\left(\frac{x}{y}\right) \geq x\ln\left(\frac{u}{v}\right) + x - \frac{yu}{v} \tag{2.14}$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

If we divide by $x > 0$, then we get

$$\frac{xv}{yu} - 1 + \ln\left(\frac{u}{v}\right) \geq \ln\left(\frac{x}{y}\right) \geq \ln\left(\frac{u}{v}\right) + 1 - \frac{yu}{xv} \tag{2.15}$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

Also, consider the convex function $f(t) = t\ln t$, $t > 0$. Then by the inequality (2.4), we have

$$\left(\ln\left(\frac{y}{x}\right) + 1\right)\left(y - \frac{xv}{u}\right) + \frac{xv}{u}\ln\left(\frac{v}{u}\right) \geq y\ln\left(\frac{y}{x}\right)$$
$$\geq \frac{xv}{u}\ln\left(\frac{v}{u}\right) + \left(\ln\left(\frac{v}{u}\right) + 1\right)\left(y - \frac{xv}{u}\right)$$

namely, by division with $y > 0$,

$$\left(\ln\left(\frac{y}{x}\right)+1\right)\left(1-\frac{xv}{yu}\right)+\frac{xv}{yu}\ln\left(\frac{v}{u}\right) \tag{2.16}$$
$$\geq \ln\left(\frac{y}{x}\right)$$
$$\geq \frac{xv}{yu}\ln\left(\frac{v}{u}\right)+\left(\ln\left(\frac{v}{u}\right)+1\right)\left(1-\frac{xv}{yu}\right)$$

for all $(x, y), (u, v) \in (0, \infty) \times (0, \infty)$.

3 Double Integral Inequalities

Consider G a closed and bounded subset of $(0, \infty) \times (0, \infty)$. Define

$$A_G := \int\int_G dxdy$$

the *area* of G and $(\overline{x_G}, \overline{y_G})$ the *center of mass* for G, where

$$\overline{x_G} := \frac{1}{A_G}\int\int_G xdxdy,\ \overline{y_G} := \frac{1}{A_G}\int\int_G ydxdy.$$

Observe that if $f : (0, \infty) \to \mathbb{R}$ is convex and G a closed and bounded subset of $(0, \infty) \times (0, \infty)$, then the double integral

$$\int\int_G P_f(x, y)\,dxdy = \int\int_G xf\left(\frac{y}{x}\right)dxdy$$

exists.

We have the following main result:

Theorem 2 *If* $f : (0, \infty) \to \mathbb{R}$ *is differentiable convex on* $(0, \infty)$ *and* G *a closed and bounded subset of* $(0, \infty) \times (0, \infty)$*, then*

$$\frac{1}{A_G}\left[\int\int_G f'\left(\frac{y}{x}\right)ydxdy-\frac{v}{u}\int\int_G f'\left(\frac{y}{x}\right)xdxdy\right]+\overline{x_G}f\left(\frac{v}{u}\right) \tag{3.1}$$
$$\geq \frac{1}{A_G}\int\int_G P_f(x, y)\,dxdy \geq \overline{x_G}f\left(\frac{v}{u}\right)+\left(\overline{y_G}-\overline{x_G}\frac{v}{u}\right)f'\left(\frac{v}{u}\right)$$

for all $(u, v) \in G$.

Proof By taking the integral in the inequality (2.4) over (x, y) on G, we get

$$\int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{xv}{u}\right)dxdy + \int\int_G xf\left(\frac{v}{u}\right)dxdy \tag{3.2}$$
$$\geq \int\int_G P_f(x, y)\,dxdy$$
$$\geq \int\int_G xf\left(\frac{v}{u}\right)dxdy + \int\int_G f'\left(\frac{v}{u}\right)\left(y - \frac{xv}{u}\right).$$

Observe that

$$\int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{xv}{u}\right)dxdy$$
$$= \int\int_G f'\left(\frac{y}{x}\right)ydxdy - \frac{v}{u}\int\int_G f'\left(\frac{y}{x}\right)xdxdy,$$

$$\int\int_G xf\left(\frac{v}{u}\right)dxdy = f\left(\frac{v}{u}\right)\int\int_G xdxdy = \overline{x_G}A_G f\left(\frac{v}{u}\right)$$

and

$$\int\int_G f'\left(\frac{v}{u}\right)\left(y - \frac{xv}{u}\right)dxdy = f'\left(\frac{v}{u}\right)\int\int_G\left(y - \frac{xv}{u}\right)dxdy$$
$$= A_G\left(\overline{y_G} - \overline{x_G}\frac{v}{u}\right)f'\left(\frac{v}{u}\right).$$

By replacing these values in (3.2) and dividing by the area A_G, we obtain the desired result (3.1). □

Corollary 2 *With the assumptions of Theorem 2, we have*

$$0 \leq \frac{1}{A_G}\int\int_G P_f(x, y)\,dxdy - \overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)$$
$$\leq \frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy. \tag{3.3}$$

The proof follows by taking $\frac{v}{u} = \frac{\overline{y_G}}{\overline{x_G}}$ in (3.1).

We define for $f : (0, \infty) \to \mathbb{R}$ a differentiable function on $(0, \infty)$ the quantity

$$\ell_G(f') := \frac{\int\int_G f'\left(\frac{y}{x}\right)ydxdy}{\int\int_G f'\left(\frac{y}{x}\right)xdxdy},$$

provided that the denominator is nonzero.

Corollary 3 *With the assumptions of Theorem 2 and if* $\ell_G\left(f'\right)>0$, *then*

$$0\le\overline{x_G}f\left(\ell_G\left(f'\right)\right)-\frac{1}{A_G}\int\int_G P_f\left(x,y\right)dxdy\ge\left(\overline{x_G}\ell_G\left(f'\right)-\overline{y_G}\right)f'\left(\ell_G\left(f'\right)\right). \tag{3.4}$$

The proof follows by taking $\frac{v}{u}=\ell_G\left(f'\right)$ in (3.1).

We observe that the condition f is strictly increasing on $(0,\infty)$ implies that $\ell_G\left(f'\right)>0$.

In 2002, Cerone and Dragomir [2] obtained the following refinement of Grüss inequality for the general Lebesgue integral:

Lemma 2 *Let* $w, f, g:\Omega\to\mathbb{R}$ *be* μ*-measurable functions on* Ω *and* $w\ge 0$ μ*-almost everywhere on* Ω. *If there exist the constants* δ,Δ *such that*

$$-\infty<\delta\le g\le\Delta<\infty,$$

μ*-almost everywhere on* Ω, *then*

$$\left|\frac{\int_\Omega w\left(x\right)f\left(x\right)g\left(x\right)d\mu\left(x\right)}{\int_\Omega w\left(x\right)d\mu\left(x\right)}-\frac{\int_\Omega w\left(x\right)g\left(x\right)d\mu\left(x\right)}{\int_\Omega w\left(x\right)d\mu\left(x\right)}\frac{\int_\Omega w\left(x\right)f\left(x\right)d\mu\left(x\right)}{\int_\Omega w\left(x\right)d\mu\left(x\right)}\right|$$
$$\le\frac{1}{2}\frac{\Delta-\delta}{\int_\Omega w\left(x\right)d\mu\left(x\right)}\int_\Omega\left|g\left(y\right)-\frac{1}{\int_\Omega w\left(x\right)d\mu\left(x\right)}\int_\Omega w\left(x\right)g\left(x\right)d\mu\left(x\right)\right|d\mu\left(y\right). \tag{3.5}$$

The constant $\frac{1}{2}$ *is best possible.*

We have:

Theorem 3 *Let* $f:(0,\infty)\to\mathbb{R}$ *be differentiable convex on* $(0,\infty)$ *and* G *a closed and bounded subset of* $(0,\infty)\times(0,\infty)$. *Assume that there exist the constants* γ,Γ *such that*

$$-\infty<\gamma\le f'\left(\frac{y}{x}\right)\le\Gamma<\infty \tag{3.6}$$

for almost every $(x,y)\in G$; *then*

$$0\le\frac{1}{A_G}\int\int_G P_f\left(x,y\right)dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)$$
$$\le\frac{1}{2}\left(\Gamma-\gamma\right)\frac{1}{A_G}\int\int_G\left|y-\frac{\overline{y_G}}{\overline{x_G}}x\right|dxdy\le\frac{1}{2}\left(\Gamma-\gamma\right)I_G^{1/2}, \tag{3.7}$$

where

$$I_G := \frac{1}{A_G}\int\int_G y^2 dxdy - 2\frac{\overline{y_G}}{\overline{x_G}}\frac{1}{A_G}\int\int_G xydxdy + \left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2 \frac{1}{A_G}\int\int_G x^2 dxdy.$$

Proof Observe that

$$\frac{1}{A_G}\int\int_G \left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right) dxdy = \frac{1}{A_G}\left(\int\int_G ydxdy - \frac{\overline{y_G}}{\overline{x_G}}\int\int_G xdxdy\right)$$
$$= \overline{y_G} - \frac{\overline{y_G}}{\overline{x_G}}\overline{x_G} = 0.$$

Then by the inequality (3.5) for functions defined on G, we get

$$\left|\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right) dxdy\right| = \left|\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right) dxdy\right.$$
$$\left. - \frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right) dxdy \frac{1}{A_G}\int\int_G \left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right) dxdy\right|$$
$$\leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{A_G}\int\int_G \left|y - \frac{\overline{y_G}}{\overline{x_G}}x - \frac{1}{A_G}\int\int_G \left(u - \frac{\overline{y_G}}{\overline{x_G}}v\right) dudv\right| dxdy$$
$$\leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{A_G}\int\int_G \left|y - \frac{\overline{y_G}}{\overline{x_G}}x\right| dxdy.$$

By utilizing (3.3), we get

$$0 \leq \frac{1}{A_G}\int\int_G P_f(x, y)\, dxdy - \overline{x_G} f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)$$
$$\leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{A_G}\int\int_G \left|y - \frac{\overline{y_G}}{\overline{x_G}}x\right| dxdy,$$

which proves the second inequality in (3.7).

Using Cauchy-Schwarz inequality for the double integral, we have

$$\frac{1}{A_G}\int\int_G \left|y - \frac{\overline{y_G}}{\overline{x_G}}x\right| dxdy \leq \left(\frac{1}{A_G}\int\int_G \left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right)^2 dxdy\right)^{1/2}. \quad (3.8)$$

Since

$$\int\int_G \left(y - \frac{\overline{y_G}}{\overline{x_G}}x\right)^2 dxdy = \int\int_G \left(y^2 - 2\frac{\overline{y_G}}{\overline{x_G}}xy + \left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2 x^2\right) dxdy$$
$$= \int\int_G y^2 dxdy - 2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G xydxdy + \left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2 \int\int_G x^2 dxdy,$$

hence by (3.8), we get the last part of (3.7). □

Corollary 4 *With the assumptions of Theorem 3 and if there exists* $0 < m < M < \infty$ *such that*

$$\frac{y}{x} \in [m, M] \text{ for all } (x, y) \in G, \tag{3.9}$$

then

$$\begin{aligned} 0 &\leq \frac{1}{A_G} \int \int_G P_f(x, y)\, dxdy - \overline{x_G} f\left(\frac{\overline{y_G}}{\overline{x_G}}\right) \\ &\leq \frac{1}{2}\left[f'(M) - f'(m)\right] \frac{1}{A_G} \int \int_G \left| y - \frac{\overline{y_G}}{\overline{x_G}} x \right| dxdy \leq \frac{1}{2}\left[f'(M) - f'(m)\right] I_G^{1/2}. \end{aligned} \tag{3.10}$$

Proof Since f' is increasing, then by (3.9), we have $f'(m) \leq f'\left(\frac{y}{x}\right) \leq f'(M)$, and by (3.7), we get the desired result. □

We have:

Theorem 4 *Let* $f : (0, \infty) \to \mathbb{R}$ *be differentiable convex on* $(0, \infty)$ *and G a closed and bounded subset of* $(0, \infty) \times (0, \infty)$. *Assume that there exist the constants* γ, Γ *such that*

$$\left| f'\left(\frac{y}{x}\right) - f'\left(\frac{u}{v}\right) \right| \leq \Lambda \left| \frac{y}{x} - \frac{u}{v} \right| < \infty \tag{3.11}$$

for almost every $(x, y) \in G$; *then*

$$0 \leq \frac{1}{A_G} \int \int_G P_f(x, y)\, dxdy - \overline{x_G} f\left(\frac{\overline{y_G}}{\overline{x_G}}\right) \leq \Lambda J_G, \tag{3.12}$$

where

$$J_G := \frac{1}{A_G} \int \int_G \frac{y^2}{x} dxdy - \frac{(\overline{y_G})^2}{\overline{x_G}}.$$

Proof Observe that

$$\begin{aligned} &\frac{1}{A_G} \int \int_G f'\left(\frac{y}{x}\right) \left(y - \frac{\overline{y_G}}{\overline{x_G}} x\right) dxdy \\ &\qquad = \frac{1}{A_G} \int \int_G \left[f'\left(\frac{y}{x}\right) - f'\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\right] \left(y - \frac{\overline{y_G}}{\overline{x_G}} x\right) dxdy. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{A_G}\int\int_G f'\left(\frac{y}{x}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)dxdy \\
&\quad\le \frac{1}{A_G}\int\int_G \left|\left[f'\left(\frac{y}{x}\right)-f'\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\right]\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)\right|dxdy \\
&\quad\le \frac{1}{A_G}\Lambda\int\int_G \left|\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\left(y-\frac{\overline{y_G}}{\overline{x_G}}x\right)\right|dxdy \\
&\quad= \frac{1}{A_G}\Lambda\int\int_G \left|\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)\right|xdxdy \\
&\quad= \frac{1}{A_G}\Lambda\int\int_G \left(\frac{y}{x}-\frac{\overline{y_G}}{\overline{x_G}}\right)^2 xdxdy \\
&\quad= \frac{1}{A_G}\Lambda\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy. \qquad (3.13)
\end{aligned}$$

Since

$$\begin{aligned}
&\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy \\
&= \int\int_G \frac{y^2}{x}dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G \frac{y}{x}xdxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\int\int_G xdxdy \\
&= \int\int_G \frac{y^2}{x}dxdy-2\frac{\overline{y_G}}{\overline{x_G}}\int\int_G ydxdy+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\int\int_G xdxdy \\
&= \int\int_G \frac{y^2}{x}dxdy-2A_G\frac{\overline{y_G}}{\overline{x_G}}\overline{y_G}+A_G\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\overline{x_G} = \int\int_G \frac{y^2}{x}dxdy-A_G\frac{(\overline{y_G})^2}{\overline{x_G}},
\end{aligned}$$

hence

$$\frac{1}{A_G}\int\int_G \left[\frac{y^2}{x^2}-2\frac{\overline{y_G}}{\overline{x_G}}\frac{y}{x}+\left(\frac{\overline{y_G}}{\overline{x_G}}\right)^2\right]xdxdy = \frac{1}{A_G}\int\int_G \frac{y^2}{x}dxdy-\frac{(\overline{y_G})^2}{\overline{x_G}}$$

and by (3.13), we get (3.12). □

Corollary 5 *If $f:(0,\infty)\to\mathbb{R}$ is twice differentiable convex on $(0,\infty)$ and if there exists $0<m<M<\infty$ such that the condition (3.9) holds, then we have*

$$0\le \frac{1}{A_G}\int\int_G P_f(x,y)\,dxdy-\overline{x_G}f\left(\frac{\overline{y_G}}{\overline{x_G}}\right)\le \left\|f''\right\|_{[m,M],\infty}J_G, \qquad (3.14)$$

where

$$\left\| f'' \right\|_{[m,M],\infty} := \sup_{t\in[m,M]} \left| f'(t) \right| < \infty.$$

4 Examples for Functions Defined on Rectangles

If $G = [a, b] \times [c, d]$ is a rectangle from $(0, \infty) \times (0, \infty)$, then

$$\int_a^b \int_c^d P_f(x, y)\, dxdy = \int_a^b x \left(\int_c^d f\left(\frac{y}{x}\right) dy \right) dx = \int_a^b x^2 \left(\int_{\frac{c}{x}}^{\frac{d}{x}} f(u)\, du \right) dx,$$

and

$$A_G = (b-a)(d-c), \ \overline{x_G} = \frac{a+b}{2} \text{ and } \overline{y_G} = \frac{c+d}{2}.$$

If F is an antiderivative for f, namely, $F'(x) = f(x)$, then integrating by parts we have the following identity that can be used in applications to calculate $\int_a^b \int_c^d P_f(x, y)\, dxdy$

$$\begin{aligned}
\int_a^b x^2 \left(\int_{\frac{c}{x}}^{\frac{d}{x}} f(u)\, du \right) dx &= \int_a^b x^2 \left(F\left(\frac{d}{x}\right) - F\left(\frac{c}{x}\right) \right) dx \\
&= \frac{1}{3} \int_a^b \left(F\left(\frac{d}{x}\right) - F\left(\frac{c}{x}\right) \right) d\left(x^3\right) \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad - \frac{1}{3} \int_a^b x^3 \left(-F'\left(\frac{d}{x}\right) \left(\frac{d}{x^2}\right) + F'\left(\frac{c}{x}\right) \frac{c}{x^2} \right) dx \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad - \frac{1}{3} \int_a^b x \left(-df\left(\frac{d}{x}\right) + cf\left(\frac{c}{x}\right) \right) dx \\
&= \frac{1}{3} \left[\left(F\left(\frac{d}{b}\right) - F\left(\frac{c}{b}\right) \right) b^3 - \left(F\left(\frac{d}{a}\right) - F\left(\frac{c}{a}\right) \right) a^3 \right] \\
&\quad + \frac{1}{3} d \int_a^b xf\left(\frac{d}{x}\right) dx - \frac{1}{3} c \int_a^b xf\left(\frac{c}{x}\right) dx.
\end{aligned}$$

We also have

$$I_{[a,b]\times[c,d]} = \frac{(b-a)\left(d^3-c^3\right)}{3(b-a)(d-c)} - 2\frac{c+d}{a+b}\frac{\left(b^2-a^2\right)\left(d^2-c^2\right)}{4(b-a)(d-c)} + \left(\frac{c+d}{a+b}\right)^2 \frac{(d-c)\left(b^3-a^3\right)}{3(b-a)(d-c)}$$

$$= \frac{\left(d^2+dc+c^2\right)}{3} - \frac{c+d}{a+b}\frac{(b+a)(d+c)}{2} + \left(\frac{c+d}{a+b}\right)^2 \frac{\left(b^2+ba+a^2\right)}{3}$$

$$= \frac{1}{6(a+b)^2} \times \left[2\left(d^2+dc+c^2\right)(a+b)^2 - 3(b+a)^2(d+c)^2 + 2(d+c)^2\left(b^2+ba+a^2\right)\right]$$

$$= \frac{1}{6(a+b)^2} \times \left[2\left((d+c)^2-dc\right)(a+b)^2 - 3(b+a)^2(d+c)^2 + 2(d+c)^2\left((b+a)^2-ba\right)\right]$$

$$= \frac{1}{6(a+b)^2}\left[(d+c)^2(a+b)^2 - 2dc(a+b)^2 - 2ba(d+c)^2\right].$$

On the other hand,

$$J_{[a,b]\times[c,d]} := \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{y^2}{x}dxdy - \frac{(c+d)^2}{2(a+b)} = \frac{(\ln b-\ln a)\left(d^2+dc+c^2\right)}{3(b-a)} - \frac{(c+d)^2}{2(a+b)}.$$

If $(x, y) \in [a, b] \times [c, d] \subset (0, \infty) \times (0, \infty)$, then

$$m = \frac{c}{b} \le \frac{y}{x} \le \frac{d}{a} = M$$

From the inequality (3.10), we have for a differentiable convex function $f : (0, \infty) \to \mathbb{R}$

$$0 \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d P_f(x, y)\,dxdy - \frac{a+b}{2} f\left(\frac{c+d}{a+b}\right) \le \frac{1}{2}\left[f'\left(\frac{d}{a}\right) - f'\left(\frac{c}{b}\right)\right]\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left|y - \frac{c+d}{a+b}x\right| dxdy$$

$$\leq \frac{1}{2\sqrt{6}(a+b)}\left[f'\left(\frac{d}{a}\right)-f'\left(\frac{c}{b}\right)\right]$$
$$\times\left[(d+c)^2(a+b)^2-2dc(a+b)^2-2ba(d+c)^2\right]^{1/2}. \quad (4.1)$$

If $f:(0,\infty)\to\mathbb{R}$ is twice differentiable convex function, then by (3.14)

$$0\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d P_f(x,y)\,dxdy-\frac{a+b}{2}f\left(\frac{c+d}{a+b}\right)$$
$$\leq \|f''\|_{\left[\frac{c}{b},\frac{d}{a}\right],\infty}\left[\frac{(\ln b-\ln a)\left(d^2+dc+c^2\right)}{3(b-a)}-\frac{(c+d)^2}{2(a+b)}\right]. \quad (4.2)$$

The case of squares $[a,b]\times[a,b]$ provides simpler forms as follows:

$$0\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b P_f(x,y)\,dxdy-\frac{a+b}{2}f(1)$$
$$\leq \frac{1}{2}\left[f'\left(\frac{b}{a}\right)-f'\left(\frac{a}{b}\right)\right]\frac{1}{(b-a)^2}\int_a^b\int_a^b|y-x|\,dxdy$$
$$=\frac{1}{6}\left[f'\left(\frac{b}{a}\right)-f'\left(\frac{a}{b}\right)\right](b-a) \quad (4.3)$$

for a differentiable convex function $f:(0,\infty)\to\mathbb{R}$ and $[a,b]\subset(0,\infty)$, and

$$0\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b P_f(x,y)\,dxdy-\frac{a+b}{2}f(1)$$
$$\leq \|f''\|_{\left[\frac{a}{b},\frac{b}{a}\right],\infty}\left[\frac{(\ln b-\ln a)\left(a^2+ab+b^2\right)}{3(b-a)}-\frac{a+b}{2}\right] \quad (4.4)$$

if $f:(0,\infty)\to\mathbb{R}$ is twice differentiable convex function and $[a,b]\subset(0,\infty)$.

5 Examples for Functions Defined on Circular Sectors

We consider the first quarter of the circle

$$Q(R):=\left\{(x,y)\mid x=r\cos\theta,\ \ y=r\sin\theta \text{ with } r\in[0,R],\ \theta\in\left[0,\frac{\pi}{2}\right]\right\}.$$

Using the polar coordinates change of variable, we have

$$\int\int_{Q(R)} P_f(x,y)\,dxdy = \int\int_{Q(R)} xf\left(\frac{y}{x}\right)dxdy$$
$$= \int_0^R \int_0^{\frac{\pi}{2}} r^2 \cos\theta f(\tan(\theta))\,drd\theta = \frac{R^3}{3}\int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta$$

where $f:(0,\infty)\to\mathbb{R}$ is convex and the integral $\int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta$ is finite.

We have

$$A_{Q(R)} = \int\int_{Q(R)} dxdy = \int_0^R \int_0^{\frac{\pi}{2}} rdrd\theta = \frac{\pi R^2}{4}$$

$$\overline{x_{Q(R)}} := \frac{1}{A_{Q(R)}}\int\int_{Q(R)} xdxdy = \frac{1}{\frac{\pi R^2}{4}}\int_0^R\int_0^{\frac{\pi}{2}} r^2\cos\theta drd\theta = \frac{4}{3\pi}R$$

and

$$\overline{y_{Q(R)}} := \frac{1}{A_{Q(R)}}\int\int_{Q(R)} ydxdy = \frac{1}{\frac{\pi R^2}{4}}\int_0^R\int_0^{\frac{\pi}{2}} r^2\sin\theta drd\theta = \frac{4}{3\pi}R.$$

From the inequality (3.3), we have

$$0 \le \int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta - f(1) \le \int_0^{\frac{\pi}{2}} f'(\tan(\theta))(\sin\theta - \cos\theta)\,d\theta, \tag{5.1}$$

for $f:(0,\infty)\to\mathbb{R}$ convex and provided that the involved integrals exist.

Consider

$$\ell_{Q(R)}\left(f'\right) := \frac{\int_0^{\frac{\pi}{2}} f'(\tan(\theta))\cos\theta d\theta}{\int_0^{\frac{\pi}{2}} f'(\tan(\theta))\sin\theta d\theta}, \tag{5.2}$$

provided the involved integrals exists, and assume that $\ell_{Q(R)}\left(f'\right) > 0$; then by (3.4), we get

$$0 \le f\left(\ell_{Q(R)}\left(f'\right)\right) - \int_0^{\frac{\pi}{2}} \cos\theta f(\tan(\theta))\,d\theta \le \left(\ell_{Q(R)}\left(f'\right) - 1\right) f'\left(\ell_{Q(R)}\left(f'\right)\right), \tag{5.3}$$

for $f:(0,\infty)\to\mathbb{R}$ convex and provided that the involved integrals exist.

We can also consider the circular sector

$$Q(R,\theta_1,\theta_2) := \{(x,y) \mid x = r\cos\theta,\ y = r\sin\theta \text{ with } r \in [0,R],\ \theta \in [\theta_1,\theta_2]\},$$

where $[\theta_1,\theta_2] \subset \left[0,\frac{\pi}{2}\right]$.

Then

$$\int\int_{Q(R,\theta_1,\theta_2)} P_f(x,y)\,dxdy = \frac{R^3}{3}\int_{,\theta_1}^{\theta_2} \cos\theta f(\tan(\theta))\,d\theta$$

$$A_{Q(R,\theta_1,\theta_2)} = \frac{R^2}{2}(\theta_2-\theta_1),$$

$$\overline{x_{Q(R,\theta_1,\theta_2)}} = \frac{2R}{3}\frac{\sin\theta_2-\sin\theta_1}{\theta_2-\theta_1}$$

and

$$\overline{y_{Q(R,\theta_1,\theta_2)}} = \frac{2R}{3}\frac{\cos\theta_1-\cos\theta_2}{\theta_2-\theta_1}.$$

We also have

$$J_{Q(R,\theta_1,\theta_2)} := \frac{2R}{3}\frac{1}{\theta_2-\theta_1}\left[\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta - \frac{(\cos\theta_2-\cos\theta_1)^2}{\sin\theta_2-\sin\theta_1}\right].$$

Since

$$\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta = \ln\left(\frac{\tan\left(\frac{\theta_2}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2}+\frac{\pi}{4}\right)}\right) - (\sin\theta_2-\sin\theta_1),$$

hence

$$\int_{\theta_1}^{\theta_2}\frac{\sin^2\theta}{\cos\theta}d\theta - \frac{(\cos\theta_2-\cos\theta_1)^2}{\sin\theta_2-\sin\theta_1}$$
$$= \ln\left(\frac{\tan\left(\frac{\theta_2}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2}+\frac{\pi}{4}\right)}\right) - \frac{(\sin\theta_2-\sin\theta_1)^2+(\cos\theta_2-\cos\theta_1)^2}{\sin\theta_2-\sin\theta_1}.$$

Moreover,

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan(\theta) \in [\tan(\theta_1),\tan(\theta_2)]$$

and by (3.14), we get

$$0 \le \frac{1}{\theta_2 - \theta_1} \int_{,\theta_1}^{\theta_2} \cos\theta f\left(\tan(\theta)\right) d\theta - \frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1} f\left(\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_2 - \sin\theta_1}\right)$$
$$\le \left\| f'' \right\|_{[\tan(\theta_1),\tan(\theta_2)],\infty}$$
$$\times \frac{1}{\theta_2 - \theta_1}\left[\ln\left(\frac{\tan\left(\frac{\theta_2}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{\theta_1}{2} + \frac{\pi}{4}\right)}\right) - \frac{(\sin\theta_2 - \sin\theta_1)^2 + (\cos\theta_2 - \cos\theta_1)^2}{\sin\theta_2 - \sin\theta_1}\right] \tag{5.4}$$

provided $f : (0, \infty) \to \mathbb{R}$ is twice differentiable convex on $(0, \infty)$ and $[\theta_1, \theta_2] \subset \left[0, \frac{\pi}{2}\right]$.

By utilizing the above general results, the interested reader may obtain other inequalities for the integral of perspective on the circular sectors. The details are not presented here.

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On the Maximum Value of a Multi-variable Function

Soheila Emamyari and Mehdi Hassani

Abstract In this paper, we find the maximum value of a multi-variable function, related by an optimization problem. Our method of maximizing this function is geometric, without applying the partial derivatives tests and the concept of Hessian matrix.

1 Introduction

Because of applications of canals to convey water for irrigation, industrial, and domestic uses around the world, some of scientists study canals in various points of view, including design of minimum water loss canal sections. Canals with trapezoidal section consisting of cement bricks are the most common and practical kind of canals, and a big number of abovementioned studies are about their cross sections. In this paper, we consider the problem of maximizing cross section of the canal consisting of $2n + 1$ bricks, symmetrically, as in Fig. 1. We assume that the length of similar bricks is 1.

The case $n = 1$ is related to the canal consisting of three bricks and maximizing the function $f(x) = (1 + \cos x)\sin x$, where x denote the angle between slope of canal and horizon. A simple calculus argument shows that

$$\max_{x\in[0,\frac{\pi}{2}]} f(x) = f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{4}.$$

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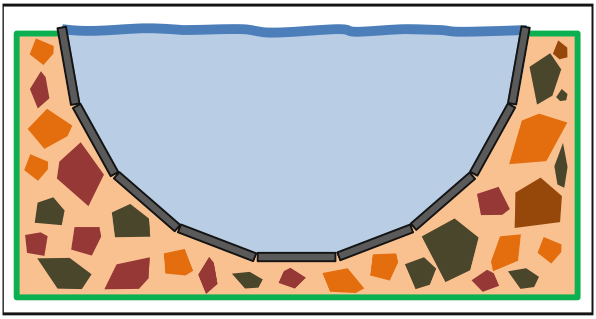

Fig. 1 Cross section of a canal consisting of $2n + 1$ similar bricks

For $n \geq 2$, maximizing cross section of the symmetric canal consisting of $2n + 1$ bricks is related to maximizing the multi-variable function $F : \mathbf{R}^n \to \mathbf{R}$ defined by

$$F(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} (\sin x_k)\,(1 + \cos x_k) + 2\sum_{k=2}^{n} \left((\sin x_k) \sum_{j=1}^{k-1} \cos x_j \right). \quad (1)$$

In Sect. 2, we study F for the case $n = 2$ by using partial derivative test. We observe that applying the partial derivative tests and the concept of Hessian matrix for arbitrary n is not easy. Instead, we suggest a geometric argument, based on the fact (see [1], page 129) asserting that "Of all convex n-gons of a given perimeter, the one which maximizes area is the regular n-gon." This fact, which is a variant of isoperimetric theorem formulated for convex n-gons, allows us to prove the following.

Theorem 1 *Let $n \geq 2$. For $x_1, x_2, \ldots, x_n \in [0, \frac{\pi}{2}]$, the maximum value of the multi-variable function F defined by (1) occurs at*

$$x_k = \frac{k\pi}{2n+1} \qquad (1 \leq k \leq n).$$

Moreover, as $n \to \infty$, we have

$$\max_{x_1, x_2, \ldots, x_n \in [0, \frac{\pi}{2}]} F = \frac{2n+1}{4} \cot \frac{\pi}{2(2n+1)} = \frac{2}{\pi} n^2 + \frac{2}{\pi} n + \left(\frac{1}{2\pi} - \frac{\pi}{24} \right) + O(n^{-2}).$$

2 Theorem 1 for the Case $n = 2$

We consider a symmetric shape as in Fig. 2. We assume that chords are similar with length 1, and we denote its area by A.

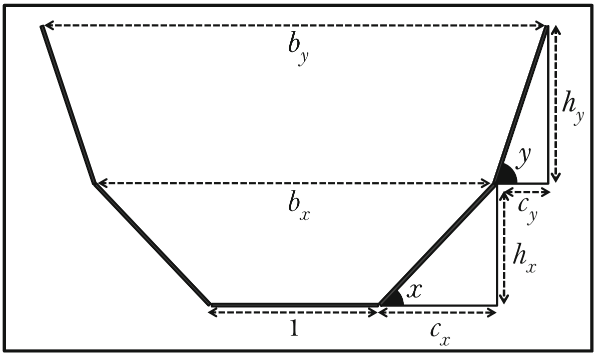

Fig. 2 A symmetric shape with area $A = F(x, y)$

Let A_1 and A_2 be areas of lower and upper isosceles trapezoids, respectively. We have

$$A_1 = \frac{h_x}{2}(1 + b_x),$$

and

$$A_2 = \frac{h_y}{2}(b_x + b_y).$$

Since $h_x = \sin x$ and $b_x = 1 + 2c_x = 1 + 2\cos x$, we obtain

$$A_1 = (\sin x)(1 + \cos x).$$

Similarly, by considering $h_y = \sin y$ and $b_y = b_x + 2c_y = 1 + 2\cos x + 2\cos y$, we get

$$A_2 = (\sin y)(1 + 2\cos x + \cos y).$$

Since $A = A_1 + A_2$, we have

$$A = (\sin x)(1 + \cos x) + (\sin y)(1 + \cos y) + 2\cos x \sin y = F(x, y).$$

The maximum value of the function $F(x, y)$ appears in some point inside the region $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. To find this point, we should solve the system of equations given by

$$\nabla F(x, y) = \mathbf{0}.$$

Figure 3 shows surface of the function $z = F(x, y)$ on the domain $[0, \pi] \times [0, \pi]$ and its level curves. This figure shows that most likely the maximum value of $F(x, y)$

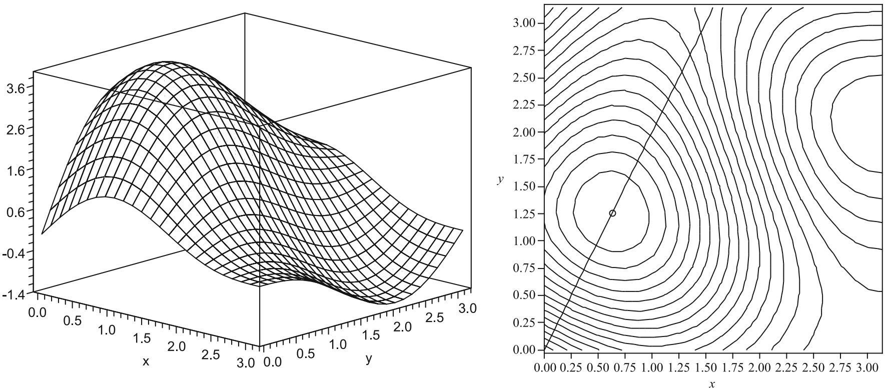

Fig. 3 Left: surface of the function $z = F(x, y)$ on $[0, \pi] \times [0, \pi]$. Right: level curves of $F(x, y)$, the line $y = 2x$, and the point $(\frac{\pi}{5}, \frac{2\pi}{5})$, where the maximum value of $F(x, y)$ occurs

occurs at a point (x, y) satisfying $y = 2x$. We put $y = 2x$ in $\frac{\partial}{\partial x} F(x, y) = 0$, and we apply the change of the variable $t = \cos x$, from which we obtain the cubic equation $4t^3 + 2t^2 - 3t - 1 = 0$, with solutions -1, $\frac{1 \pm \sqrt{5}}{4}$. Since $x \in (0, \frac{\pi}{2})$, acceptable value is $\cos x = t = \frac{1+\sqrt{5}}{4}$. Thus, $x = \frac{\pi}{5}$ and $y = 2x = \frac{2\pi}{5}$. Moreover, we observe that

$$\nabla F\left(\frac{\pi}{5}, \frac{2\pi}{5}\right) = \mathbf{0}.$$

Thus, the maximum value of $F(x, y)$ is

$$\max_{x,y \in [0, \frac{\pi}{2}]} F(x, y) = F\left(\frac{\pi}{5}, \frac{2\pi}{5}\right) = \frac{5}{4}\sqrt{5 + 2\sqrt{5}}.$$

3 Proof of Theorem 1

We consider a symmetric shape as in the left graph of Fig. 4. We assume that chords are similar with length 1, and we denote its area by A. We show that the area of cross cutting is $F(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are the slope angles. Thus, the proof of Theorem 1 is based on finding $x_1, x_2, \ldots, x_n$ such that we achieve the maximum area of a shape consisting of n isosceles trapezoids with areas $A_1, A_2, \ldots, A_n$.

We set $b_0 = 1$. Also, we let b_k be the top side of isosceles trapezoid A_k and h_k its height. For $1 \leq k \leq n$, we have $A_k = \frac{h_k}{2}(b_{k-1} + b_k)$. By considering Fig. 4, the right graph, we have

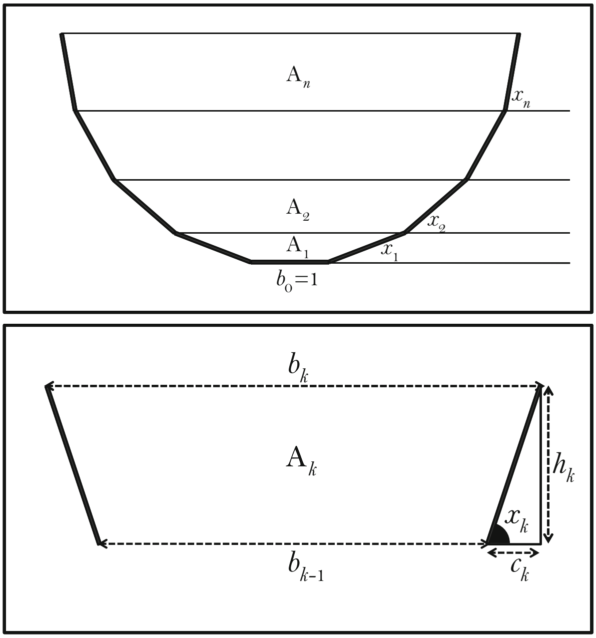

Fig. 4 Up: a symmetric shape with area $A = F(x_1, \ldots, x_n)$. Down: kth isosceles trapezoid and its related parameters

$$b_k = b_{k-1} + 2c_k = b_{k-1} + 2\cos x_k.$$

Thus, for $1 \le k \le n$, we obtain

$$A_k = (\sin x_k)(b_{k-1} + \cos x_k).$$

For $1 \le j \le k$, we have $b_j - b_{j-1} = 2\cos x_j$. Summing over $j = 1, \ldots, k$, we imply that $b_k = 1 + 2\sum_{j=1}^{k} \cos x_j$ for $k \ge 1$. Hence,

$$A_k = (\sin x_k)\Big(1 + \cos x_k + 2\sum_{j=1}^{k-1} \cos x_j\Big),$$

for $2 \le k \le n$. Also, we note that $A_1 = (\sin x_1)(1 + \cos x_1)$. Thus, for $n \ge 2$, we obtain

$$A = \sum_{k=1}^{n} A_k = F(x_1, x_2, \ldots, x_n),$$

where F is defined by (1).

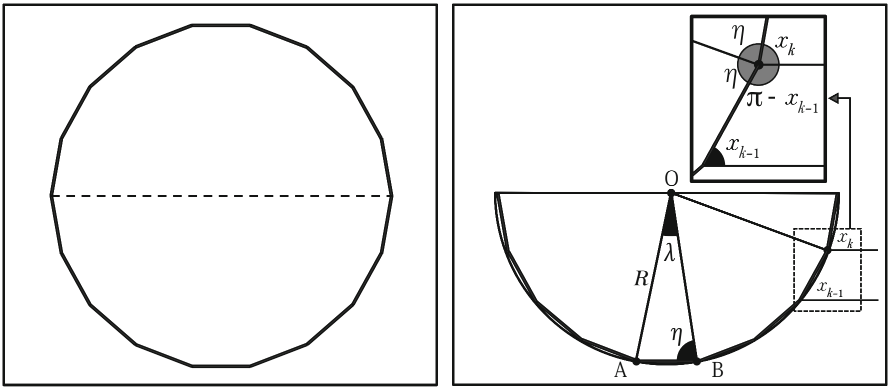

Fig. 5 A regular convex $2(2n+1)$-gon

Now, we consider Fig. 5. The isoperimetric theorem implies that twice of the maximum of A is related by a regular convex $2(2n+1)$-gon. Based on this fact, we compute slope angles $x_1, x_2, \ldots, x_n$ for which maximum value of the function F occurs. Considering the right-hand side graph of Fig. 5, we let

$$\lambda = \frac{\pi}{2n+1},$$

and

$$\eta = \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right).$$

Also, if we let $x_0 = 0$, then for $1 \le k \le n$, we imply that

$$x_k + (\pi - x_{k-1}) + 2\eta = 2\pi,$$

and this gives

$$x_k = k\lambda = \frac{k\pi}{2n+1}, \qquad (1 \le k \le n).$$

Therefore, we obtain

$$\max_{x_1,x_2,\ldots,x_n \in [0,\frac{\pi}{2}]} F(x_1, x_2, \ldots, x_n) = F\left(\frac{\pi}{2n+1}, \frac{2\pi}{2n+1}, \ldots, \frac{n\pi}{2n+1}\right) := G(n),$$

say. We have

$$G(n) = \sum_{k=1}^{n} \left(\sin \frac{k\pi}{2n+1}\right)\left(1 + \cos \frac{k\pi}{2n+1}\right) + 2\sum_{k=2}^{n}\left(\left(\sin \frac{k\pi}{2n+1}\right)\sum_{j=1}^{k-1} \cos \frac{j\pi}{2n+1}\right).$$

On the other hand, by considering the right-hand side of Fig. 5, we have $R = \frac{1}{2} \csc \frac{\lambda}{2}$. Since $G(n) = \frac{1}{2}$ area of regular $2(2n+1)$-gon, or $G(n) = \frac{2(2n+1)}{2}$ area of triangle OAB, we obtain

$$G(n) = \frac{2(2n+1)}{2}\left(\frac{1}{4} \cot \frac{\lambda}{2}\right).$$

Hence,

$$G(n) = \frac{2n+1}{4} \cot \frac{\pi}{2(2n+1)}.$$

It remains to obtain asymptotic expansion for $G(n)$. As $n \to \infty$, we have

$$\cot \frac{\pi}{2(2n+1)} = \frac{1}{\tan \frac{\pi}{2(2n+1)}} = \frac{1}{\frac{\pi}{2(2n+1)}\left(1 + \frac{1}{3}\left(\frac{\pi}{2(2n+1)}\right)^2 + O(n^{-4})\right)}.$$

Thus,

$$\cot \frac{\pi}{2(2n+1)} = \frac{2(2n+1)}{\pi}\left(1 - \frac{1}{3}\left(\frac{\pi}{2(2n+1)}\right)^2 + O(n^{-4})\right).$$

Hence,

$$G(n) = \frac{2}{\pi}n^2 + \frac{2}{\pi}n + \left(\frac{1}{2\pi} - \frac{\pi}{24}\right) + O(n^{-2}).$$

This completes the proof of Theorem 1.

Remark 1 We observe that the above geometrical argument allows us to compute the summations $\sum_{k=1}^{n} \cos \frac{k\pi}{2n+1}$ and $\sum_{k=1}^{n} \sin \frac{k\pi}{2n+1}$. We consider the right-hand side graph of Fig. 5 to write

$$\sum_{k=1}^{n} c_k = R - \frac{1}{2} = \frac{1}{2}\left(\csc \frac{\pi}{2(2n+1)} - 1\right).$$

This implies

$$\sum_{k=1}^{n} \cos \frac{k\pi}{2n+1} = \frac{1}{2}\left(\csc \frac{\pi}{2(2n+1)} - 1\right).$$

Also, we have $\sum_{k=1}^{n} h_k = h$, where h is the distance of O from AB; hence, we have $h = \frac{1}{2}\cot\frac{\pi}{2(2n+1)}$. Thus, we obtain

$$\sum_{k=1}^{n} \sin\frac{k\pi}{2n+1} = \frac{1}{2}\cot\frac{\pi}{2(2n+1)}.$$

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Image Reconstruction for Positron Emission Tomography Based on Chebyshev Polynomials

George Fragoyiannis, Athena Papargiri, Vassilis Kalantonis, Michael Doschoris, and Panayiotis Vafeas

Abstract The study of the functional characteristics of the brain plays a crucial role in modern medical imaging. An important and effective nuclear medicine technique is positron emission tomography (PET), whose utility is based upon the noninvasive measure of the in vivo distribution of imaging agents, which are labeled with positron-emitting radionuclides. The main mathematical problem of PET involves the inverse Radon transform, leading to the development of several methods toward this direction. Herein, we present an improved formulation based on Chebyshev polynomials, according to which a novel numerical algorithm is employed in order to interpolate exact simulated values of the Randon transform via an analytical Shepp–Logan phantom representation. This approach appears to be efficient in calculating the Hilbert transform and its derivative, being incorporated within the final analytical formulae. The numerical tests are validated by comparing the presented methodology to the well-known spline reconstruction technique.

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1 Introduction

Modern brain medical imaging is directly associated with positron emission tomography (PET), a state-of-the-art technique that allows the study of a wide range of physiological and pathological processes *in vivo*, and it is frequently combined with computed tomography (CT) and magnetic resonance imaging (MRI) to provide additional anatomical and metabolic information. In fact, PET is used to measure brain metabolism via gamma-ray emissions from radioactively labeled and metabolically active chemical agents that have been injected into the bloodstream, whereas the measured emission data are numerically processed in order to construct multidimensional images of the distribution of the radiopharmaceutical in the brain [17, 20, 27].

PET is mathematically based on the Radon transform, representing the output of a tomographic scan, and its inverse, being responsible for reconstructing the original data [7, 19]. However, the numerical implementation of the explicitly defined inverse Radon transform is in general complicated, since it depends on the occurrence of the Hilbert transform [9, 12], which is necessary for the aforementioned data processing. The difficulties in the computation of the Hilbert transform are closely related to the singularities involved, wherein special methods must be adopted to overcome the indeterminacies. Actually, the main goal of this process is to achieve a satisfactory image reconstruction, which is an essential feature in tomographic medical imaging, allowing tomographic images to be acquired from a set of two-dimensional projection data. Doing so, critical information about the functional characteristics of the brain is recovered, providing the adequate insight that is vital in the wide area of clinical medicine, including neurology, oncology, and cardiology.

In this study, we propose an efficient deterministic method that is designed to reconstruct images from real Radon transform data, taking much advantage from the Chebyshev polynomials [21, 25]. In particular, our first task is to provide the appropriate tools in order to construct the Radon transform and obtain easy-to-handle analytical formulae. Then, we introduce an effective, fast, and accurate algorithm to numerically implement the integral relationships and process the given data via Chebyshev interpolation. This technique is based on the numerical evaluation of the Hilbert transform associated with the Radon transform, which is crucial for the manipulation of the implicated relations. Further analysis includes the comparison of the presented algorithm and its efficiency against a variation of the spline reconstruction technique [7, 16], where the outcome gives rise to the fact that the proposed methodology is a solid and credible tool in the direction of brain image reconstruction. The technical part of this chapter incorporates many basic mathematical and computational tools, whose references are available in [4, 6, 7, 10, 15, 22, 23], providing adequate information to the interested reader.

The rest of this chapter is organized as follows. In Sect. 2, we present a brief overview of the relative mathematical background, wherein a more detailed analysis can be found in [4]. Section 3 renders the spline reconstruction technique, wherein Sect. 4 is devoted to the Chebyshev reconstruction method. Finally, in Sect. 5, we provide a numerical implementation of our semi-analytical methodology, while we end up with a summary of this project in Sect. 6.

2 Mathematical Formulation

The Radon transform of the continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ is the function $\mathrm{R}\{g\}$, defined by the line integral of g along each line L on the $2D$ space of all straight lines in the x_1x_2 plane with coordinates (ρ and θ) and parameter $\tau \in (-\infty, +\infty)$. The coordinate $\rho \in (-\infty, +\infty)$ represents the distance of L from the origin, while $\theta \in [0, 2\pi)$ is the angle of this line with respect to the x_1-axis, as shown in Fig. 1. Hence, the Cartesian coordinates of a point (x_1, x_2) on each of these lines are

$$\begin{aligned} x_1 &= \tau \cos\theta - \rho \sin\theta \\ x_2 &= \tau \sin\theta + \rho \cos\theta, \end{aligned} \tag{1}$$

which implies that

$$\begin{aligned} \rho &= x_2 \cos\theta - x_1 \sin\theta \\ \tau &= x_2 \sin\theta + x_1 \cos\theta, \end{aligned} \tag{2}$$

and the Radon transform is written as [24]

$$\begin{aligned} \mathrm{R}\{g(\rho, \theta)\} &= \int_L g(x_1, x_2)\, d\tau \\ &= \int_{-\infty}^{\infty} g(\tau \cos\theta - \rho \sin\theta, \tau \sin\theta + \rho \cos\theta)\, d\tau. \end{aligned} \tag{3}$$

The inverse formula of (3) is given by Fokas et al. [7] and Radon [24]

$$\begin{aligned} g(x_1, x_2) &= \frac{1}{4i\pi^2} \int_0^{2\pi} e^{i\theta} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \\ &\quad \times \left(\text{p.v.} \int_{-\infty}^{+\infty} \frac{\mathrm{R}\left\{g\left(\rho', \theta\right)\right\}}{\rho' - (x_2 \cos\theta - x_1 \sin\theta)} d\rho' \right) d\theta \\ &= -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{\partial}{\partial \rho} \left(\text{p.v.} \int_{-\infty}^{+\infty} \frac{\mathrm{R}\left\{g\left(\rho', \theta\right)\right\}}{\rho' - \rho} d\rho' \right) d\theta \\ &= -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{\partial}{\partial \rho} \mathrm{H}\{\mathrm{R}\{g(\rho, \theta)\}\}\, d\theta, \end{aligned} \tag{4}$$

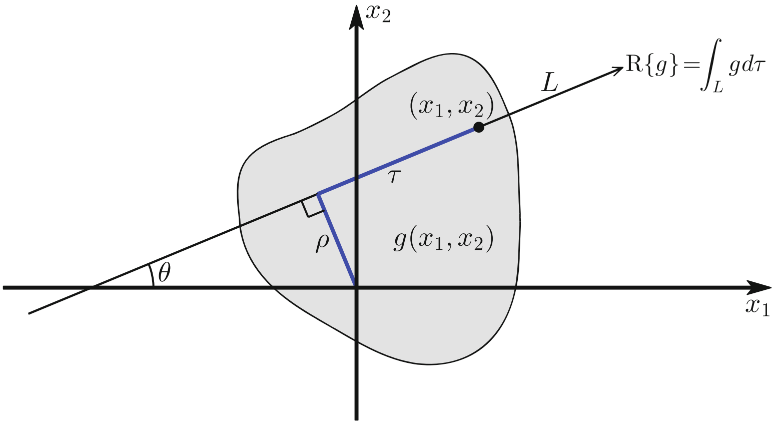

Fig. 1 The coordinates (ρ and θ) of an arbitrary line L. The line integral of function $g(x_1, x_2)$ along this line is the Radon transform $\mathrm{R}\{g(\rho, \theta)\}$

where p.v. is the Cauchy principal value [18] and H stands for the Hilbert transform, while the derivative $\partial_\rho \mathrm{H}$ is known as the Hadamard transform [9].

For simplicity, in what follows, we denote the function $\mathrm{R}\{g(\rho, \theta)\}$ by $f(\rho)$, as the case may be, where it is implied that f is also a function of θ. Furthermore, we assume that g vanishes outside the circle $x_1^2 + x_2^2 = \rho^2 + \tau^2 = 1$, and therefore we can consider that $\rho \in [-1, 1]$. Hence, the Hilbert transform of the function $f : \mathbb{R} \to \mathbb{R}$ is given by the generalized integral

$$\mathrm{H}\{f(c)\} = \text{p.v.} \int_{-1}^{1} \frac{f(x)}{x - c} dx, \tag{5}$$

in which we take the Cauchy principal value for the singular point $x = c$, yielding

$$\mathrm{H}\{f(c)\} = \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{c-\varepsilon} \frac{f(x)}{x - c} dx + \int_{c+\varepsilon}^{1} \frac{f(x)}{x - c} dx \right]. \tag{6}$$

Relationship (6) can be written as [9–11]

$$\begin{aligned} \mathrm{H}\{f(c)\} &= \int_{-1}^{1} \frac{f(x) - f(c)}{x - c} dx + f(c) \int_{-1}^{1} \frac{1}{x - c} dx \\ &= \int_{-1}^{1} \frac{f(x) - f(c)}{x - c} dx + f(c) \ln\left(\frac{1 - c}{1 + c}\right), \end{aligned} \tag{7}$$

wherein the singularity at the point $x = c$ has been eliminated.

As far as PET is concerned, the main problem is the reconstruction of the original image function $g(x_1, x_2)$, given a number of measured values of its projections $f(\rho,\theta)$ for specific values of ρ and θ. The difficulty in the numerical implementation of this inversion is the accurate evaluation of the Hilbert transform and its derivative.

In order to evaluate the image reconstruction methods via the inverse Radon transform relation (4), we utilize noiseless simulated data that correspond to the standard Shepp–Logan phantom (see Fig. 2). The corresponding sinogram (as usually reported) can be constructed via a closed-form expression in terms of the coordinates ρ and θ. Indeed, the Shepp–Logan phantom is constituted by the superposition of ellipses with distinct centers, orientations, and semi-axes [15]. Inside these ellipses, which lie on the domain $[-1, 1] \times [-1, 1]$, the intensity takes specific constant values, which are shown in Table 1. Assuming a single ellipse with center at (x_0, y_0) and semi-axes A and B, whose large semi-axis A is forming angle ϕ with respect to the x_1-axis, we can show that the Radon transform (along the line with coordinates ρ and θ) of the relative intensity function $g(x_1, x_2) = \lambda$ is given by

$$\mathrm{R}\{g(\rho,\theta)\} = \lambda \frac{2AB\sqrt{A^2\sin^2(\theta-\phi) + B^2\cos^2(\theta-\phi) - (\rho + x_0\sin\theta - y_0\cos\theta)^2}}{A^2\sin^2(\theta-\phi) + B^2\cos^2(\theta-\phi)}. \tag{8}$$

Hence, substituting A, B, x_0, y_0, and ϕ in the above relation with the corresponding values for each ellipse (see Table 1) and then summing the corresponding functions $\mathrm{R}\left\{g_j(\rho,\theta)\right\}$, we obtain the closed-form expression of the Radon transform of the

Fig. 2 The exact Shepp–Logan phantom. It is used to produce the simulated PET scan data via the corresponding analytic expression of $\mathrm{R}\{g(\rho,\theta)\}$

Table 1 Parameters of the ellipses that constitute the Shepp–Logan phantom

Center (x_0, y_0)	Semi-major axis A	Semi-minor axis B	Angle ϕ	Intensity λ
(0,0)	0.92	0.69	90°	1
(0,−0.0184)	0.874	0.6624	90°	−0.8
(0.22,0)	0.31	0.11	72°	−0.1
(−0.22,0)	0.41	0.16	108°	−0.1
(0,0.35)	0.25	0.21	90°	0.2
(0,0.1)	0.046	0.046	0°	0.2
(0,−0.1)	0.046	0.046	0°	0.2
(−0.08,−0.605)	0.046	0.023	0°	0.2
(0.06,−0.605)	0.046	0.023	90°	0.2
(0,−0.605)	0.023	0.023	0°	0.2

Shepp–Logan phantom, which will be later utilized in the numerical implementation.

3 Spline Reconstruction

We assume that for a given projection angle θ, the function f is approximated by piecewise cubic polynomials in the interval $[-1, 1]$, divided into $N-1$ equally spaced subintervals of length $h = x_{i+1} - x_i$, i.e.,

$$f(x) \approx f_i(x) = A_i(x - x_i)^3 + B_i(x - x_i)^2 + C_i(x - x_i) + D_i, \quad x \in [x_i, x_{i+1}]. \tag{9}$$

The coefficients A_i, B_i, C_i, and D_i are functions of the coordinate θ, while they depend upon the type of spline that is utilized [3, 14]. Here, we consider the cubic Hermite spline and the B-spline.

In the case of the cubic Hermite spline, the approximation of $f(x)$ in the subinterval $[x_i, x_{i+1}]$ is given by

$$\begin{aligned} f(x) \approx& \left(2t^3 - 3t^2 + 1\right) f(x_i) + \left(t^3 - 2t^2 + t\right) f'(x_i)\, h \\ &+ \left(-2t^3 + 3t^2\right) f(x_{i+1}) + \left(t^3 - t^2\right) f'(x_{i+1})\, h, \end{aligned} \tag{10}$$

where $t = (x - x_i)/h$, while the terms $f'(x_i)$ are unknown. Expression (10) is equivalent to the relation (9), if we assume that

$$A_i = \frac{2f(x_i) - 2f(x_{i+1}) + hf'(x_i) + hf'(x_{i+1})}{h^3}, \tag{11}$$

$$B_i = \frac{3f(x_{i+1}) - 3f(x_i) - 2hf'(x_i) - hf'(x_{i+1})}{h^2} \tag{12}$$

and

$$C_i = f'(x_i), \quad D_i = f(x_i). \tag{13}$$

For the derivatives $f'(x_i)$, we consider the fourth-order centered difference approximation

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O\left(h^4\right), \tag{14}$$

where at the end points we consider that

$$f'(x_1) = \frac{-f(x_3) + 4f(x_2) - 3f(x_1)}{2h}, \quad f'(x_2) = \frac{f(x_3) - f(x_1)}{2h} \tag{15}$$

and

$$f'(x_N) = \frac{3f(x_N) - 4f(x_{N-1}) + f(x_{N-2})}{2h}, \quad f'(x_{N-1}) = \frac{f(x_N) - f(x_{N-2})}{2h}. \tag{16}$$

In the case of the uniform B-spline, the terms A_i, B_i, C_i, and D_i within (9) take the form

$$A_i = \frac{3f(x_i) - f(x_{i-1}) - 3f(x_{i+1}) + f(x_{i+2})}{6h^3}, \tag{17}$$

$$B_i = \frac{3f(x_{i-1}) - 6f(x_i) + 3f(x_{i+1})}{6h^2}, \tag{18}$$

$$C_i = \frac{3f(x_{i+1}) - 3f(x_{i-1})}{6h}, \tag{19}$$

and

$$D_i = \frac{f(x_{i-1}) + 4f(x_i) + f(x_{i+1})}{6}, \tag{20}$$

respectively, while at the end points we consider that $f(x_0) = f(x_1)$ and $f(x_{N+1}) = f(x_N)$.

Next, from the relation (7), we obtain [7]

$$\mathrm{H}\{f(c)\} = \sum_{i=1}^{N-1} I_i + f(c)\ln\left(\frac{1-c}{1+c}\right), \tag{21}$$

where

$$I_i = \int_{x_i}^{x_i+h} \frac{f_i(x) - f(c)}{x-c} dx, \tag{22}$$

and therefore, if $c \in \left[x_j, x_{j+1}\right]$, we get

$$\begin{aligned} I_j &= \int_{x_j}^{x_j+h} \frac{f_j(x) - f(c)}{x-c} dx \\ &= \int_{x_j}^{x_j+h} \frac{f_j(x) - f_j(c)}{x-c} dx, \end{aligned} \tag{23}$$

which, due to (9), becomes

$$\begin{aligned} I_j &\approx \int_{x_j}^{x_j+h} \frac{A_j\left[\left(x-x_j\right)^3 - \left(c-x_j\right)^3\right] + B_j\left[\left(x-x_j\right)^2 - \left(c-x_j\right)^2\right] + C_j\,(x-c)}{x-c} dx \\ &= A_j h\left(c-x_j\right)^2 + \left(B_j h + \frac{1}{2}A_j h^2\right)\left(c-x_j\right) + \frac{1}{3}A_j h^3 + \frac{1}{2}B_j h^2 + C_j h, \end{aligned} \tag{24}$$

while for $i \neq j$ and according to (9), it holds that

$$\begin{aligned} I_i &\approx \int_{x_i}^{x_i+h} \frac{A_i(x-x_i)^3 + B_i(x-x_i)^2 + C_i\,(x-x_i) + D_i - f(c)}{x-c} dx \\ &= A_i h(c-x_i)^2 + \left(B_i h + \frac{1}{2}A_i h^2\right)(c-x_i) + \frac{1}{3}A_i h^3 + \frac{1}{2}B_i h^2 + C_i h \\ &\quad + \left[A_i(c-x_i)^3 + B_i(c-x_i)^2 + C_i\,(c-x_i) + D_i - f(c)\right]\ln\left(\frac{x_i+h-c}{x_i-c}\right). \end{aligned} \tag{25}$$

Hence, we obtain

$$\mathrm{H}\{f(c)\} \approx \sum_{i=1, i\neq j}^{N-1} I_i + I_j + f(c)\ln\left(\frac{1-c}{1+c}\right), \tag{26}$$

where

$$f(c) = A_j\left(c-x_j\right)^3 + B_j\left(x-x_j\right)^2 + C_j\left(x-x_j\right) + D_j, \tag{27}$$

and finally

$$\frac{d}{dc}\mathrm{H}\{f(c)\} \approx \sum_{i=1}^{N-1} J_i, \tag{28}$$

where

$$J_i = 2A_i h\,(c - x_i) + \left(B_i h + \frac{1}{2}A_i h^2\right) + \left[3A_i(c - x_i)^2 + 2B_i\,(c - x_i) + C_i\right]\ln\left|\frac{x_i + h - c}{x_i - c}\right| + \left[A_i(c - x_i)^3 + B_i(c - x_i)^2 + C_i\,(c - x_i) + D_i\right]\frac{h}{(c - x_i)\,(c - x_i - h)}. \tag{29}$$

4 The Proposed Chebyshev Reconstruction Method

In this method, the function $f\,(x)$ (for a specific projection angle θ) is approximated via a Chebyshev interpolating polynomial $P(x)$ [5, 21], given a set of samples $f\,(x_n)$ at the points x_n, that is,

$$f\,(x) \approx P\,(x) = \sum_{k=1}^{N} w_k a_k T_{k-1}\,(x), \quad x \in [-1, 1] \tag{30}$$

with

$$w_k = \begin{cases} \sqrt{\frac{1}{N}} & k = 1 \\ \sqrt{\frac{2}{N}} & k \geq 2, \end{cases} \tag{31}$$

where T_k are the Chebyshev polynomials of the first kind,

$$a_k = w_k \sum_{n=1}^{N} f\,(x_n)\cos\left(\frac{(k-1)\,(2n-1)}{2N}\pi\right), \quad k = 1, 2, \ldots, N, \tag{32}$$

are the coefficients, which depend on θ and

$$x_n = \cos\left(\frac{2n-1}{2N}\pi\right), \quad n = 1, 2, \ldots, N, \tag{33}$$

are the Chebyshev nodes (roots of T_N). The function f is assumed to be fairly smooth, while the sampling is performed at the non-equally spaced interpolation points x_n, in order to minimize the effect of Runge's phenomenon [26].

If we substitute the polynomial (30) into relation (7), we readily obtain

$$\begin{aligned} \mathrm{H}\{P(c)\} &= \int_{-1}^{1} \frac{P(x) - P(c)}{x - c} dx + P(c) \ln\left(\frac{1-c}{1+c}\right) \\ &= \sqrt{\frac{2}{N}} \sum_{k=2}^{N} a_k \int_{-1}^{1} \frac{T_{k-1}(x) - T_{k-1}(c)}{x - c} dx + P(c) \ln\left(\frac{1-c}{1+c}\right), \end{aligned} \tag{34}$$

and then we utilize the identity [11, 25]

$$T_{k+1}(x) - T_{k+1}(c) = 2(x - c) \sum_{n=0}^{k}{}' U_{k-n}(x) T_n(c), \quad k \geq 0, \tag{35}$$

where U_k are the Chebyshev polynomials of the second kind, while the prime in the sum denotes that the first term is multiplied with $1/2$. The relation (35) is written as

$$\frac{T_{k-1}(x) - T_{k-1}(c)}{x - c} = 2 \sum_{n=0}^{k-2}{}' U_{k-n-2}(x) T_n(c), \quad k \geq 2, \tag{36}$$

and therefore, given that

$$\int_{-1}^{1} U_k(x)\, dx = \begin{cases} \frac{2}{k+1} & k \text{ even} \\ 0 & k \text{ odd}, \end{cases} \tag{37}$$

we obtain

$$\begin{aligned} \int_{-1}^{1} \frac{T_{k-1}(x) - T_{k-1}(c)}{x - c} dx &= 2 \sum_{n=0}^{k-2}{}' \int_{-1}^{1} U_{k-n-2}(x)\, dx T_n(c) \\ &= 2 \sum_{n=0}^{k-2}{}' \frac{1 - (-1)^{(k-n-1)}}{k - n - 1} T_n(c). \end{aligned} \tag{38}$$

Substituting the relation (38) into (34), we have

$$\mathrm{H}\{P(c)\} = \sqrt{\frac{2}{N}} \sum_{k=2}^{N} a_k \left(2 \sum_{n=0}^{k-2}{}' \frac{1 - (-1)^{(k-n-1)}}{k - n - 1} T_n(c) \right) + P(c) \ln\left(\frac{1-c}{1+c}\right), \tag{39}$$

where we can change the order of summation to obtain

$$\mathrm{H}\left\{P\left(c\right)\right\} = 2\sqrt{\frac{2}{N}}\sum_{n=0}^{N-2}{}'\left(\sum_{k=n+2}^{N}\frac{1-(-1)^{(k-n-1)}}{k-n-1}a_k\right)T_n\left(c\right) + P\left(c\right)\ln\left(\frac{1-c}{1+c}\right) \tag{40}$$

or

$$\mathrm{H}\left\{P\left(c\right)\right\} = \sum_{n=1}^{N-1}{}'A_n T_{n-1}\left(c\right) + P\left(c\right)\ln\left(\frac{1-c}{1+c}\right), \tag{41}$$

where

$$A_n = 2\sqrt{\frac{2}{N}}\sum_{k=n+1}^{N}\frac{1-(-1)^{(k-n)}}{k-n}a_k. \tag{42}$$

In the sequel, the derivative of (41) with respect to c leads to

$$\frac{d}{dc}\mathrm{H}\left\{P\left(c\right)\right\} = \sum_{n=2}^{N-1}A_n\left(n-1\right)U_{n-2}\left(c\right) + P'\left(c\right)\ln\left(\frac{1-c}{1+c}\right) - P\left(c\right)\frac{2}{1-c^2}, \tag{43}$$

with

$$P'\left(c\right) = \sqrt{\frac{2}{N}}\sum_{n=2}^{N}a_n\left(n-1\right)U_{n-2}\left(c\right), \tag{44}$$

and finally, in view of the recurrence relation [25],

$$2T_n\left(x\right) = \frac{1}{n+1}\frac{d}{dx}T_{n+1}\left(x\right) - \frac{1}{n-1}\frac{d}{dx}T_{n-1}\left(x\right), \quad n = 2, 3, \ldots, \tag{45}$$

the derivative of the Hilbert transform is approximated by

$$\frac{d}{dc}\mathrm{H}\left\{f\left(c\right)\right\} \approx \sum_{n=1}^{N-2}{}'A'_n T_{n-1}\left(c\right) + P'\left(c\right)\ln\left(\frac{1-c}{1+c}\right) - P\left(c\right)\frac{2}{1-c^2}, \tag{46}$$

with

$$P'\left(c\right) = \sqrt{\frac{2}{N}}\sum_{n=1}^{N-1}{}'a'_n T_{n-1}\left(c\right), \quad P\left(c\right) = \sum_{k=1}^{N}w_k a_k T_{k-1}\left(c\right) \tag{47}$$

and [22]

$$a'_N = 0, \;\; a'_{N-1} = 2\,(N-1)\,a_N, \quad a'_k = a'_{k+2} + 2ka_{k+1}, \\ k = N-2, N-3, \ldots, 1, \tag{48}$$

$$A'_{N-1} = 0, \;\; A'_{N-2} = 2\,(N-2)\,A_{N-1}, \quad A'_k = A'_{k+2} + 2kA_{k+1}, \\ k = N-3, N-4, \ldots, 1. \tag{49}$$

5 Numerical Implementation and Results

In order to implement the above methods, first we construct the exact simulated projection data for the Shepp–Logan phantom for various discrete values of ρ and θ via the analytic formula (8). For the reconstruction of this phantom, we consider a grid of 250×250 points. For each point on the grid (x_1, x_2) with $x_1^2 + x_2^2 < 1$, we evaluate $M = 150$ distinct values of $\theta_k \in [0, 2\pi)$ for every $k = 1, 2, \ldots, M$ and the corresponding coordinates $\rho \equiv c_k = x_2 \cos\theta_k - x_1 \sin\theta_k$.

Then, for each one of the values θ_k, we utilize the precomputed discrete values of the Radon transform $R\{g(\rho_\ell, \theta_k)\}$ at the N distinct values $\rho_\ell \in [-1, 1]$, $\ell = 1, 2, \ldots, N$, in order to calculate the coefficients (32) of the Chebyshev interpolating polynomial (30) or the coefficients of the piecewise polynomials (9) for the Chebyshev or spline method, respectively. In order to obtain similar resolution for the reconstructed image for the Chebyshev method, we consider $N = 200$ Chebyshev–Gauss nodes ρ_ℓ (since the Chebyshev points are denser at the end points), while for the spline method we consider $N = 150$ points equidistantly distributed in the same interval. It is worth considering that, despite the fewer points in the spline method (for both the cubic Hermite and the B-spline cases), the execution time for an equivalent implementation is longer, probably due to the greater number of operations required.

Next, the computed Chebyshev or spline coefficients are utilized for the evaluation of the approximated derivative at $\rho = c_k$, of the Hilbert transform of the simulated Radon transform, via the relation (46) or (28), respectively. Finally, all the M values of the derivatives, which correspond to each one of the M values of θ, are used for the calculation of the image function $g(x_1, x_2)$ at the considered point (x_1, x_2) via the relation (4) and with the aid of the trapezoidal rule. The sought reconstruction of the Shepp–Logan phantom via the spline technique and the Chebyshev method is illustrated in Figs. 3 and 4, respectively.

Note that, in order to suppress the Gibbs phenomenon [8] in the Chebyshev method, due to the step discontinuities of the Shepp–Logan phantom (the oscillations are propagated away from the discontinuities), we apply appropriate spectral filtering with no additional computational cost by multiplying the coefficients of the Chebyshev interpolating polynomial a_k (see (30)) with the coefficients

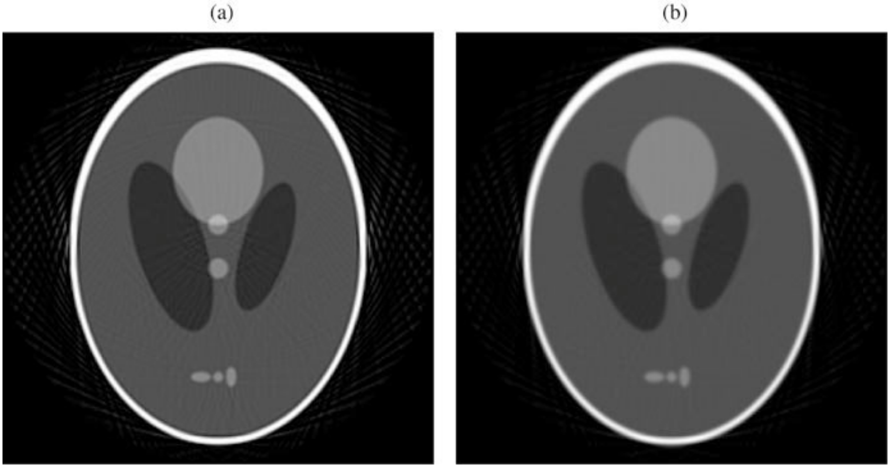

Fig. 3 Reconstruction of the Shepp–Logan phantom via the spline reconstruction technique utilizing (**a**) Hermite cubic splines and (**b**) B-splines

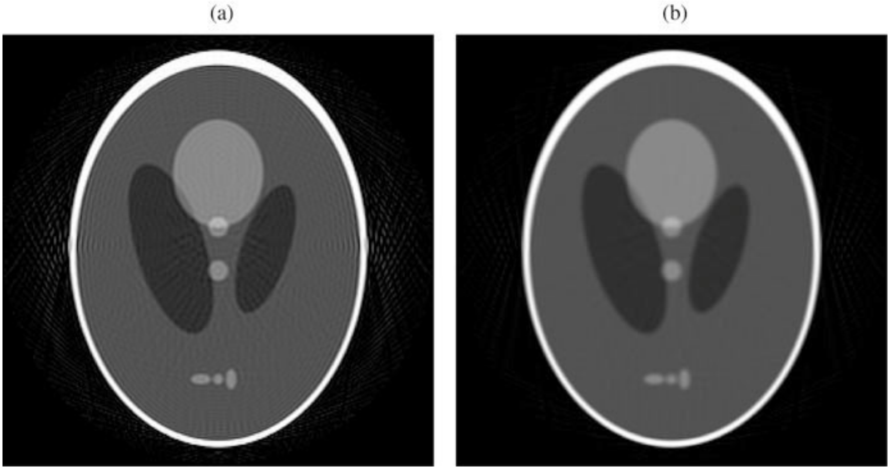

Fig. 4 Reconstruction of the Shepp–Logan phantom via the Chebyshev reconstruction method (**a**) without filtering and (**b**) with Lanczos spectral filtering

$\sigma_k = \sin(\pi\eta)/\pi\eta$ with $\eta = (k-1)/(N-1)$ (the Lanczos spectral filter [8]). Figure 4b illustrates the reconstructed image incorporating the Lanczos filter, where it is clear that the occurred oscillations almost vanish, providing a clean image, similar to that of the B-spline method, as shown in Fig. 3b (the B-spline does not pass through the control points, and hence it approximates without oscillations at the discontinuities [13]).

Finally, we mention that the Chebyshev coefficients (32) can be computed via the discrete cosine transform (DCT) [1], while the Chebyshev summation (46) can be computed recursively via the Clenshaw algorithm [2, 22].

6 Conclusion

PET is an important modern medical imaging technique, in which the reconstruction of the tomographic image can be achieved via the inverse Radon transform formula. The numerical implementation of the inverse Radon transform is in general complicated, since it depends on the derivative of the Hilbert transform of the projection data, which involves singularities. These singularities can be effectively subtracted, if we approximate the input data via appropriate piecewise or global polynomials.

Hence, in this work, we propose an efficient PET reconstruction algorithm based on Chebyshev interpolation, and we compare it to an older similar technique based on cubic splines. For the evaluation of the proposed algorithm, we utilize noiseless simulated data that were constructed analytically from the standard Shepp–Logan phantom, while we compare the results with the corresponding results of the spline method. The numerical experiments validate the efficiency of the proposed method, as this requires relatively few operations for the computation of the coefficients and the final sum of Chebyshev polynomials, while it has the additional advantage of possible hardware acceleration via the DCT algorithm. Furthermore, it is easy to implement and can facilitate the application of various filters for image denoising.

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Approximation by Mixed Operators of Max-Product–Choquet Type

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Abstract The main aim of this chapter is to introduce several mixed operators between Choquet integral operators and max-product operators and to study their approximation, shape preserving, and localization properties. Section 2 contains some preliminaries on the Choquet integral. In Sect. 3, we obtain quantitative estimates in uniform and pointwise approximation for the following mixed type operators: max-product Bernstein–Kantorovich–Choquet operator, max-product Szász–Mirakjan–Kantorovich–Choquet operators, nontruncated and truncated cases, and max-product Baskakov–Kantorovich–Choquet operators, nontruncated and truncated cases. We show that for large classes of functions, the max-product Bernstein–Kantorovich–Choquet operators approximate better than their classical correspondents, and we construct new max-product Szász–Mirakjan–Kantorovich–Choquet and max-product Baskakov–Kantorovich–Choquet operators, which approximate uniformly f in each compact subinterval of $[0, +\infty)$ with the order $\omega_1(f; \sqrt{\lambda_n})$, where $\lambda_n \searrow 0$ arbitrary fast. Also, shape preserving and localization results for max-product Bernstein–Kantorovich–Choquet operators are obtained. Section 4 contains quantitative approximation results for discrete max-product Picard–Kantorovich–Choquet, discrete max-product Gauss–Weierstrass–Kantorovich–Choquet operators, and discrete max-product Poisson–Cauchy–Kantorovich–Choquet operators. Section 5 deals with the approximation properties of the max-product Kantorovich–Choquet operators based on (φ, ψ)-kernels. It is worth to mention that with respect to their max-product correspondents, while they keep their good properties, the mixed max-product Choquet operators present, in addition, the advantage of a great flexibility by the many possible choices for the families of set functions used in their definitions. The

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results obtained present potential applications in sampling theory, neural networks, and learning theory.

1 Introduction

In the recent years, two new directions of research in approximation of functions were developed: (I) approximation by the so-called max-product operators and (II) approximation by Choquet integral operators. Both directions of research generate nonlinear approximation operators, which can produce better approximation estimates than their linear (classical) counterparts.

(I) The max-product approximation operators can naturally be obtained by using the possibility theory in an analogous manner with Feller's scheme in probability theory used for generating linear and positive approximation operators (see [6, 25]), by replacing the probability (σ-additive), with a maxitive set function and the classical integral with the possibilistic integral (see, e.g., [3], Chapter 10, Section 10.2). Formally, the max-product operators are attached to the classical linear and positive operators, by replacing in their expression the sum by the maximum (supremum). Their construction can be well illustrated in the discrete case as follows. Let $f : I \to [0, +\infty)$, $p_{n,k}(x)$ be various kinds of function basis on the interval I satisfying $\sum_{k\in I_n} p_{n,k}(x) = 1$, I_n finite or infinite families of indices, $\{x_{n,k}; k \in I_n\}$ representing a division of I and consider the notation $\bigvee_{k\in A} a_k = \sup_{k\in A} a_k$. Starting from the general form of discrete linear positive operators

$$D_n(f)(x) = \sum_{k\in I_n} p_{n,k}(x) f(x_{n,k}), x \in I, n \in \mathbb{N},$$

and their Kantorovich-type variants

$$DK_n(f)(x) = \sum_{k\in I_n} p_{n,k}(x) \cdot \frac{1}{x_{n,k+1} - x_{n,k}} \cdot \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt, x \in I, n \in \mathbb{N}, \tag{1}$$

one replaces the sum by maximum (supremum), obtaining thus the max-product operators

$$D_n^{(M)}(f)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot f(x_{n,k})}{\bigvee_{k\in I_n} p_{n,k}(x)}, x \in I, n \in \mathbb{N} \tag{2}$$

and their Kantorovich-type variants

$$DK_n^{(M)}(f)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot (1/(x_{n,k+1} - x_{n,k})) \cdot \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt}{\bigvee_{k\in I_n} p_{n,k}(x)},$$
$$x \in I, n \in \mathbb{N}. \tag{3}$$

Note that if, for example, $p_{n,k}(x)$, $n \in \mathbb{N}$, $k = 0, \ldots, n$, is a polynomial basis, then the operators $D_n^{(M)}(f)(x)$ and $DK_n^{(M)}(f)(x)$ become piecewise rational functions.

In a long series of papers, collected in the recent research monograph [3], were introduced and studied approximation properties (including upper estimates, saturation, localization, inverse results, shape preservation, and global smoothness preservation) of the max-product operators of the form in (2), attached to Bernstein-type operators, like the Bernstein polynomials, Favard–Szász–Mirakjan operators (truncated and nontruncated cases), Baskakov operators (truncated and nontruncated cases), Meyer-König and Zeller operators, Bleimann–Butzer–Hahn operators, to interpolation polynomials of Lagrange and Hermite–Feéjer on various special knots and to sampling operators based on various kernels, like those of Whittaker type based on sinc-type kernels and those based on Fejér-type kernels.

After the appearance of this research monograph [3], the study of the max-product operators of the form (2) has been continued by other authors in many papers, like, for example, [7, 14, 16, 39, 39–45, 47]–[49].

It is worth mentioning that the max-product operators give for large classes of functions better estimates in approximation than their linear counterparts. For example, for concave functions, the order of approximation by the max-product Bernstein operators (which are piecewise rational functions) is $\mathcal{O}\left(\omega_1\left(f; \frac{1}{n}\right)\right)$, or more general, for strictly positive continuous functions, the order of approximation is $\mathcal{O}\left(\left\{n\left[\omega_1(\left(f; \frac{1}{n}\right)\right]^2 + \omega_1\left(f; \frac{1}{n}\right)\right\}\right)$. Also, it is worth noting that the max-product Bernstein operators present essentially better localization results than their linear counterparts. In this sense, if $f = g$ on $[a, b]$ with $0 < a < b < 1$ and $0 < a < c < d < b < 1$ are arbitrary, then there exists $\tilde{n}$ such that while $B_n^{(M)}(f) - B_n^{(M)}(g) = 0$ on $[c, d]$ for all $n \geq \tilde{n}$, see, e.g., Theorem 2.4.1, p. 76 in [3]), for the classical Bernstein polynomials, we have only the much weaker result $B_n(f) - B_n(g) = o(1/n)$ on $[c, d]$, for all $n \geq \tilde{n}$ (see, e.g., [22], p. 308, relation (3.3)).

For another relevant example, in the case of interpolation operators, the degree of approximation given by the max-product operators is of Jackson type $\omega_1(f; 1/n)$ and they keep the same points of interpolation, while for the classical corresponding interpolation operators of Hermite–Fejér and Lagrange on special nodes is $\omega_1(f; \ln(n)/n)$ or even the divergence phenomenon holds, see, e.g., [3], Chapter 7.

For other papers on approximation by max-product operators of the form (2), which are not mentioned in the monograph [3], including applications to signal theory, neural networks, and learning rates of algorithms, see, e.g., [7–18, 20, 23, 40, 42–45, 47].

Also, notice that the study of the max-product Kantorovich operators of the form in (3) is missing from the research monograph [3] and has been completed only in the very recent papers [7–10, 12–15, 18–20].

(II) The second direction of research deals with approximation by the so-called Choquet integral operators, consisting in the replacement of the classical linear integral in the expressions of the integral operators, with the nonlinear Choquet integral. The use of this integral in approximation theory was motivated by its potential, due to its many applications to statistical mechanics and potential theory, to the study of cooperative games, to decision-making under risk and uncertainty, in finance, economics, portofolio problems, and insurance.

Thus, in the recent papers [30, 34, 36, 37] for the multivariate Bernstein–Durrmeyer polynomial operators defined in terms of the nonlinear Choquet integral with respect to a family of strictly positive, monotone, and submodular set function, $\mu_{n,k,x} \in \Gamma_{n,x}$ on the standard simplex $S^d \subset \mathbb{R}^d$, by

$$\mathscr{D}_{n,\Gamma_{n,x}}(f)(x) = \sum_{|\alpha|=n} \frac{(C)\int_{S^d} f(t)B_\alpha(t)d\mu_{n,k,x}(t)}{(C)\int_{S^d} B_\alpha(t)d\mu_{n,k,x}(t)} \cdot B_\alpha(x),\ x \in S^d,\ n \in \mathbb{N},$$

qualitative and quantitative approximation results were obtained. Here, $B_\alpha(x)$ denote the multivariate fundamental Bernstein polynomials.

Qualitative and better quantitative approximation results for other Choquet integral operators obtained by using a Feller kind scheme (and including discrete Bernstein–Choquet polynomials and Picard–Choquet operators) were obtained in [26]. Approximation results in approximation by continuous (i.e., non-discrete) convolution integral operators of Picard–Choquet type, Poisson–Cauchy–Choquet type, Gauss–Weierstrass–Choquet type, Landau–Choquet type, and Angheluţă–Choquet type were obtained in [28, 31–33].

Also, approximation results for various Kantorovich–Choquet-type operators with respect to a family of strictly positive, monotone, and submodular set functions $\mu_{n,k,x} \in \Gamma_{n,x}$, obtained from the expression in (1) by replacing the usual integral with the Choquet integral, i.e., for Choquet-type operators of the form

$$DK_{n,\Gamma_{n,x}}(f)(x) = \sum_{k \in I_n} p_{n,k}(x) \cdot \frac{(C)\int_{x_{n,k}}^{x_{n,k+1}} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([x_{n,k}, x_{n,k+1}])}, \tag{4}$$

$x \in I$, $n \in \mathbb{N}$, were obtained in the papers [27, 29, 38].

The main aim of this chapter is to introduce and study quantitative approximation results, localization, and shape preserving results for various mixed operators between Choquet integral operators and max-product operators, of the general form obtained by replacing the usual integral in the expression of (3) by the Choquet integral, i.e., for operators of the general form

$$DK^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k\in I_n} p_{n,k}(x) \cdot (C)\int_{x_{n,k}}^{x_{n,k+1}} f(t)d\mu_{n,k,x}(t)/\mu_{n,k,x}([x_{n,k}, x_{n,k+1}])}{\bigvee_{k\in I_n} p_{n,k}(x)}, \tag{5}$$

$x \in I, n \in \mathbb{N}$.

The plan of the chapter goes as follows. Section 2 contains some preliminaries on the Choquet integral. In Section 3, we obtain quantitative estimates in uniform and pointwise approximation for the following mixed type operators: max-product Bernstein–Kantorovich–Choquet operator, max-product Szász–Mirakjan–Kantorovich–Choquet operators, nontruncated and truncated cases, and max-product Baskakov–Kantorovich–Choquet operators, nontruncated and truncated cases. We show that for large classes of functions, the max-product Bernstein–Kantorovich–Choquet operators approximate better than their classical correspondents, and we construct new max-product Szász–Mirakjan–Kantorovich–Choquet and max-product Baskakov–Kantorovich–Choquet operators, which approximate uniformly f in each compact subinterval of $[0, +\infty)$ with the order $\omega_1(f; \sqrt{\lambda_n})$, where $\lambda_n \searrow 0$ arbitrary fast. Also, shape preserving and localization results for max-product Bernstein–Kantorovich–Choquet operators are obtained. Section 4 contains quantitative approximation results for discrete max-product Picard–Kantorovich–Choquet, discrete max-product Gauss–Weierstrass–Kantorovich–Choquet operators and discrete max-product Poisson–Cauchy–Kantorovich–Choquet operators. Section 5 deals with the approximation properties of the max-product Kantorovich–Choquet operators based on (φ, ψ)-kernels. It is worth mentioning that with respect to their max-product correspondents, while they keep their good properties mentioned above, the mixed max-product–Choquet operators present, in addition, the advantage of a great flexibility by the many possible choices for the families of set functions used in their definitions.

2 Preliminaries on Choquet Integral

In this section, we present some concepts and results on the Choquet integral which will be used in the next sections.

Definition 1 Let Ω be a nonempty set and $\mathscr{C}$ be a σ-algebra of subsets in Ω.

(i) (see, e.g., [50], p. 63) Let $\mu : \mathscr{C} \to [0, +\infty]$. If $\mu(\emptyset) = 0$ and $A, B \in \mathscr{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$, then μ is called a monotone set function (or capacity). Also, if

$$\mu(A \bigcup B) + \mu(A \bigcap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathscr{C},$$

then μ is called submodular. Finally, if $\mu(\Omega) = 1$, then μ is called normalized.

(ii) (see [5], or [50], p. 233) Let μ be a normalized monotone set function on $\mathscr{C}$.

If $f : \Omega \to \mathbb{R}$ is $\mathscr{C}$-measurable, i.e., for any Borel subset $B \subset \mathbb{R}$, we have $f^{-1}(B) \in \mathscr{C}$, then for any $A \in \mathscr{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f)\bigcap A)d\beta + \int_{-\infty}^0 [\mu(F_\beta(f)\bigcap A) - \mu(A)]d\beta,$$

where $F_\beta(f) = \{\omega \in \Omega;\ f(\omega) \geq \beta\}$. If $(C)\int_A f d\mu \in \mathbb{R}$, then f is called the Choquet integrable on A. Notice that if $f \geq 0$ on A, then in the above formula, we get $\int_{-\infty}^0 = 0$.

If μ is the Lebesgue measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the Lebesgue integral.

In what follows, we list some known properties of the Choquet integral.

Remark 1 If $\mu : \mathscr{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold:

(i) For all $a \geq 0$, we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (if $f \geq 0$, then see, e.g., [50], Theorem 11.2, (5), p. 228, and if f is of arbitrary sign, then see, e.g., [21], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [50], pp. 232-233, or [21], p. 65)

$$(C)\int_A (f + c)d\mu = (C)\int_A f d\mu + c \cdot \mu(A).$$

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded, we have (see, e.g., [21], p. 75, Theorem 6.3)

$$(C)\int_A (f + g)d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu,$$

that is, the Choquet integral is sublinear.

(iii) If $f \leq g$ on A, then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [50], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f and g are of arbitrary sign).

(iv) Let $f \geq 0$. By the definition of the Choquet integral, it is immediate that if $A \subset B$, then

$$(C)\int_A f d\mu \leq (C)\int_B f d\mu,$$

and if, in addition, μ is finitely subadditive, then

$$(C)\int_{A\bigcup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu.$$

(v) By the definition of the Choquet integral, it is immediate that

$$(C)\int_A 1 \cdot d\mu(t) = \mu(A).$$

(vi) The formula $\mu(A) = \gamma(m(A))$, where $\gamma : [0, 1] \to [0, 1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and m is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $m(\emptyset) = 0$, $m(\Omega) = 1$ and m is countably additive), gives simple examples of monotone and submodular set functions (see, e.g., [21], pp. 16-17, Example 2.1). Such set functions μ are also called distortion of countably normalized additive measures (or distorted measures). For example, we can take $\gamma(t) = t^\alpha$, $0 < \alpha < 1$, or $\gamma(t) = \frac{2t}{1+t}$, or $\gamma(t) = \frac{\sin(t)}{\sin(1)}$, to choose only a few.

3 Approximation by Max-Product Kantorovich–Choquet-Type Operators

Denoting by $\mathscr{B}_I$ the sigma algebra of all Borel measurable subsets in $\mathscr{P}(I)$, everywhere in this section, $(\Gamma_{n,x})_{n\in\mathbb{N},x\in I}$, $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^n$, will be a collection of families of monotone, submodular, and strictly positive set functions $\mu_{n,k,x}$ on $\mathscr{B}_I$, with I a compact interval, or $I = [0, +\infty)$ or $I = \mathbb{R}$, depending on the operator studied. Note here that a set function on $\mathscr{B}_I$ is called strictly positive, if for any open subset $A \subset \mathbb{R}$ with $A \cap I \neq \emptyset$, we have $\mu(A \cap I) > 0$.

Starting from the classical forms of the linear and positive operators of Bernstein–Kantorovich (see, e.g., [46]), Szász–Mirakjan–Kantorovich (see, e.g., [2, 4]) and Baskakov–Kantorovich, in the recent paper [27], were introduced and studied the Bernstein-Kantorovich–Choquet, Szász–Mirakjan–Kantorovich–Choquet, and Baskakov–Kantorovich–Choquet operators.

On the other hand, starting from the classical linear operators, in another recent paper [12], were introduced and studied the max-product operators of Bernstein–Kantorovich kind, Szász–Mirakjan–Kantorovich nontruncated and truncated kinds, Baskakov–Kantorovich nontruncated and truncated kinds, Meyer-König and Zeller kind, Hermite–Fejér kind, discrete Picard kind, discrete Poisson–Cauchy kind, and Gauss–Weierstrass kind.

In this chapter, we introduce and study the corresponding mixed type operators between those of max-product types and those of Choquet types, as follows.

Definition 2 Firstly, we define the following types of max-product–Choquet operators, with respect to $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^n$:

the max-product Bernstein–Kantorovich–Choquet operators

$$K^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t) d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k}},$$

the max-product Szász–Mirakjan–Kantorovich–Choquet operators, nontruncated case

$$S^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/n,(k+1)/n])}}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}},$$

the max-product Szász–Mirakjan–Kantorovich–Choquet operators, truncated case

$$TS^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/n,(k+1)/n])}}{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!}},$$

the max-product Baskakov–Kantorovich–Choquet operators, nontruncated case

$$V^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/n,(k+1)/n])}}{\bigvee_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}},$$

and the max-product Baskakov–Kantorovich–Choquet operators, truncated case

$$TV^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/n,(k+1)/n])}}{\bigvee_{k=0}^{n} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}}.$$

In order to be well defined these operators, it is good enough if, for example, we suppose that $f : I \to \mathbb{R}_+$ is a $\mathscr{B}_I$-measurable function, bounded on I, where $I = [0, 1]$ for $K^{(M)}_{n,\Gamma_{n,x}}(f)(x)$, $TS^{(M)}_{n,\Gamma_{n,x}}(f)(x)$, $TV^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ and $I = [0, +\infty)$ for $S^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ and $V^{(M)}_{n,\Gamma_{n,x}}(f)(x)$.

Remark 2 It is clear that if $\mu_{n,k,x} = m$, for all n, k and x, where m is the Lebesgue measure, then the above operators become the max-product Kantorovich-type operators studied in [12].

Also, if $\mu_{n,k,x} = \delta_{k/n}$ (the Dirac measures), since $k/n \in (k/(n+1), (k+1)/(n+1))$, it is immediate that $K^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ become the max-product Bernstein operators, $S^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ become the nontruncated max-product Szász–Mirakjan operators, $TS^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ become the truncated max-product Szász–Mirakjan operators, $V^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ become the nontruncated max-product Baskakov operators, and finally, $TV^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ become the truncated max-product Baskakov operators, studied in detail in a series of papers, all collected by the book [3], Chapters 2, 3, and 4. This fact shows the great flexibility of the formulas of these operators. More exactly, we can generate very many kinds of max-product–Choquet approximation operators, by choosing for some $\mu_{n,k,x}$ the Lebesgue measure, for some others $\mu_{n,k,x}$, the Dirac measures, and for the others $\mu_{n,k,x}$, some Choquet measures.

In what follows, denoting for simplicity by $L_n^{(M)}(f)$ any from the operators $K^{(M)}_{n,\Gamma_{n,x}}(f)$, $S^{(M)}_{n,\Gamma_{n,x}}(f)$, $TS^{(M)}_{n,\Gamma_{n,x}}(f)(x)$, $V^{(M)}_{n,\Gamma_{n,x}}(f)$, and $TV^{(M)}_{n,\Gamma_{n,x}}(f)$, we can state the first result of this section.

Theorem 1 *Let $I = [0,1]$ for $K^{(M)}_{n,\Gamma_{n,x}}(f)(x)$, $TS^{(M)}_{n,\Gamma_{n,x}}(f)(x)$, and $TV^{(M)}_{n,\Gamma_{n,x}}(f)$ and $I = [0,+\infty)$ for $S^{(M)}_{n,\Gamma_{n,x}}(f)(x)$ and $V^{(M)}_{n,\Gamma_{n,x}}(f)(x)$. Define by $C_+^b(I)$ the space of all bounded, continuous, and positive-valued functions defined on I. For all $f \in C_+^b(I)$, $x \in I$ and $n \in \mathbb{N}$, we have*

$$|L_n^{(M)}(f)(x) - f(x)| \le 2\omega_1(f; L_n^{(M)}(\varphi_x)(x))_I, \, x \in I, n \in \mathbb{N}, \tag{6}$$

where $\varphi_x(t) = |t-x|$ and $\omega_1(f;\delta)_I = \sup\{|f(x) - f(y)|; x, y \in I, |x-y| \le \delta\}$.

Proof For $x \in I$, $n, k \in \mathbb{N}$, let us consider $T_{n,k,x} : C_+^b(I) \to \mathbb{R}_+$ defined by

$$T_{n,k,x}(f) = (C)\int_{I_{k,n}} f(t)d\mu_{n,k,x}(t)/\mu_{n,k,x}(I_{n,k}), \, f \in C_+^b(I),$$

where $I_{k,n} = [k/(n+1), (k+1)/(n+1)]$ for $K_{n,\Gamma_{n,x}}(f)(x)$ and $I_{k,n} = [k/n, (k+1)/n]$ for $S_{n,\Gamma_{n,x}}(f)(x)$ and $V_{n,\Gamma_{n,x}}(f)(x)$.

Firstly, let us denote $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. Since from the properties of the Choquet integral in Remark 1, it is easy to see that $T_{n,k,x}$ is positively homogeneous, sublinear, and monotonically increasing, multiplying it by $p_{n,k}(x)$, passing to supremum after k and finally dividing by $\bigvee_{k=0}^n p_{n,k}(x)$, we immediately get that $L_n^{(M)}(f)$ shares the same properties, that is, $L_n^{(M)}(\lambda f) = \lambda L_n^{(M)}(f)$, $L_n^{(M)}(f+g) \le L_n^{(M)}(f) + L_n^{(M)}(g)$, $f \le g$ on I implies $L_n^{(M)}(f) \le L_n^{(M)}(g)$ on I, for all $\lambda \ge 0$, $f, g \in C_+^b(I)$, $n \in \mathbb{N}$, $x \in I$. Then, by Lemma 3.1 in [34] and by its proof, we immediately have that

$$|L_n^{(M)}(f)(x) - L_n^{(M)}(g)(x)| \le L_n^{(M)}(|f-g|)(x). \tag{7}$$

Denoting $e_0(t) = 1$ for all $t \in I$, since obviously $L_n(e_0)(x) = 1$ for all $x \in I$ and taking into account the property in Remark 1, (i) and (7), for any fixed x, we obtain

$$\begin{aligned} &|L_n^{(M)}(f)(x) - f(x)| \\ &= |L_n^{(M)}(f(t))(x) - L_n^{(M)}(f(x))(x)| \le L_n^{(M)}(|f(t) - f(x)|)(x). \end{aligned} \tag{8}$$

But taking into account the properties of the modulus of continuity, for all $t, x \in I$ and $\delta > 0$, we get

$$|f(t) - f(x)| \leq \omega_1(f; \|t - x\|)_I \leq \left[\frac{1}{\delta}\|t - x\| + 1\right]\omega_1(f; \delta)_I. \quad (9)$$

Now, from (8) and applying $L_n^{(M)}$ to (9), by the properties of $L_n^{(M)}$ mentioned after the inequality (7), we immediately get

$$|L_n^{(M)}(f)(x) - f(x)| \leq \left[\frac{1}{\delta}L_n^{(M)}(\varphi_x)(x) + 1\right]\omega_1(f; \delta)_I.$$

Choosing here $\delta = L_n(\varphi_x)(x)$, we obtain the desired estimate. □

Remark 3 The positivity of function f in Theorem 1 is necessary because of the positive homogeneity of the Choquet integral used in its proof. However, if f is of arbitrary sign on I, then the statement of Theorem 1 can be restated for the slightly modified operator defined by

$$L_n^{(M*)}(f)(x) = L_n(f - d)(x) + d,$$

where $d \in \mathbb{R}$ is a lower bound for f, that is, $f(x) \geq d$, for all $x \in I$.

Indeed, this is immediate from the fact that $\omega_1(f - m; \delta)_I = \omega_1(f; \delta)_I$ and from the equality

$$L_n^{(M*)}(f)(x) - f(x) = L_n(f - d)(x) - (f(x) - d).$$

Remark 4 It is worth noting that due to the nonlinearity of the Choquet integral in Remark 2.2, (ii), unlike the classical cases, the $L_n^{(M)}$ operators in Theorem 1 are nonlinear. In the particular case when $\Gamma_{n,x}$ reduces to one element (i.e., $\mu_{n,\alpha,x} = \mu$ for all n, x and α), we will denote $K_{n,\Gamma_{n,x}}^{(M)}(f) := K_{n,\mu}^{(M)}(f)$, $S_{n,\Gamma_{n,x}}^{(M)}(f) := S_{n,\mu}^{(M)}(f)$, $TS_{n,\Gamma_{n,x}}^{(M)}(f) := TS_{n,\mu}^{(M)}(f)$, $V_{n,\Gamma_{n,x}}^{(M)}(f) := V_{n,\mu}^{(M)}(f)$, and $TV_{n,\Gamma_{n,x}}^{(M)}(f) := V_{n,\mu}^{(M)}(f)$.

In what follows, the estimate (6) allows to obtain concrete quantitative results for some particular choices of $\Gamma_{n,x}$.

Corollary 1 *Suppose that $\mu_{n,k,x} = \mu := \sqrt{m}$, for all n, k, and x, where m is the Lebesgue measure. We have*

(i) $|K_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{6}{\sqrt{n+1}} + \frac{1}{n}\right)_{[0,1]}$, *for all* $n \in \mathbb{N}$, $x \in [0, 1]$, $f \in C_+([0, 1])$ *;*

(ii) $|S_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{4\sqrt{x}}{\sqrt{n}} + \frac{1}{n}\right)_{[0,+\infty)}$, *for all* $n \in \mathbb{N}$, $x \in [0, +\infty)$, $f \in UC_+^b([0, +\infty))$, *where*

$$UC_+^b([0, +\infty))$$

$$= \{f : [0, +\infty) \to \mathbb{R}_+;\ f \text{ is uniformly continuous and bounded on } [0, +\infty)\};$$

(iii) $|TS_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{3}{\sqrt{n}} + \frac{1}{n}\right)_{[0,1]}$, *for all* $n \in \mathbb{N}$, $x \in [0, 1]$, $f \in C_+([0, 1])$ *;*

(iv) $|V_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; 6\frac{\sqrt{x(1+x)}}{\sqrt{n}} + \frac{1}{n}\right)_{[0,+\infty)}$, *for all* $n \in \mathbb{N}$, $x \in [0, +\infty)$, $f \in UC_+^b([0, +\infty))$*;*

(v) $|TV_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{12}{\sqrt{n+1}} + \frac{1}{n}\right)_{[0,1]}$, *for all* $n \in \mathbb{N}$, $x \in [0, 1]$, $f \in C_+([0, 1])$.

Proof According to Remark 1, (vi), $\mu = \sqrt{m}$ is a monotone and submodular set function. Also, it is clear that μ is strictly positive.

(i) Denote $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. In order to estimate $K_{n,\mu}^{(M)}(\varphi_x)(x)$, let us denote

$$C_{n,k}(x) = \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} |t - x| d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1), (k+1)/(n+1)])}.$$

According to the proof of Corollary 3.6 in [27], we have

$$C_{n,k}(x) \leq \left|\frac{k}{n} - x\right| + \frac{1}{n},$$

which immediately implies

$$K_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=0}^{n} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right| + \frac{1}{n}\right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} \leq \frac{\bigvee_{k=0}^{n} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} + \frac{1}{n} \leq \frac{6}{\sqrt{n+1}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 2.1.5, p. 31 in [3])

$$\frac{\bigvee_{k=0}^{n} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} \leq \frac{6}{\sqrt{n+1}}.$$

Now, using the estimate (6) in Theorem 1 too, we obtain the desired conclusion.

(ii) Denote $s_{n,k}(x) = \frac{(nx)^k}{k!}$. Using the estimate for $C_{n,k}(x)$ from the above point (i), we get

$$S_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=0}^{+\infty} s_{n,k}(x)\left[\left|\frac{k}{n} - x\right| + \frac{1}{n}\right]}{\bigvee_{k=0}^{+\infty} s_{n,k}(x)} \leq \frac{\bigvee_{k=0}^{+\infty} s_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=0}^{+\infty} s_{n,k}(x)} + \frac{1}{n} \leq \frac{4\sqrt{x}}{\sqrt{n}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 3.1.4, p. 163 in [3])

$$\frac{\bigvee_{k=0}^{+\infty} s_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{+\infty} s_{n,k}(x)} \le \frac{4\sqrt{x}}{\sqrt{n}}.$$

Now, using (6) too, we obtain the desired estimate.

(iii) Denote $s_{n,k}(x) = \frac{(nx)^k}{k!}$. Using the estimate for $C_{n,k}(x)$ from the above point (i), we get

$$TS_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\le \frac{\bigvee_{k=0}^{n} s_{n,k}(x)\left[\left|\frac{k}{n}-x\right|+\frac{1}{n}\right]}{\bigvee_{k=0}^{n} s_{n,k}(x)} \le \frac{\bigvee_{k=0}^{n} s_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{n} s_{n,k}(x)} + \frac{1}{n} \le \frac{3}{\sqrt{n}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 3.2.5, p. 179 in [3])

$$\frac{\bigvee_{k=0}^{n} s_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{n} s_{n,k}(x)} \le \frac{3}{\sqrt{n}}.$$

Now, using (6) too, we obtain the desired estimate.

(iv) Denote $v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$. As at the above points (i) and (ii), we easily obtain

$$V_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\le \frac{\bigvee_{k=0}^{\infty} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|+\frac{1}{n}\right]}{\bigvee_{k=0}^{\infty} v_{n,k}(x)} \le \frac{\bigvee_{k=0}^{\infty} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{\infty} v_{n,k}(x)} + \frac{1}{n}$$

$$\le 6\frac{\sqrt{x(1+x)}}{\sqrt{n}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 4.1.6, p. 198 in [3])

$$\frac{\bigvee_{k=0}^{\infty} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{\infty} v_{n,k}(x)} \le 6\frac{\sqrt{x(1+x)}}{\sqrt{n}}.$$

Finally, by using (6) too, we obtain the desired estimate.

(v) Denote $v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$. As at the above points (i) and (ii), we easily obtain

$$TV^{(M)}_{n,\mu}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=0}^{n} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|+\frac{1}{n}\right]}{\bigvee_{k=0}^{n} v_{n,k}(x)} \leq \frac{\bigvee_{k=0}^{n} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{n} v_{n,k}(x)} + \frac{1}{n}$$
$$\leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 4.2.6, pp. 217-218 in [3])

$$\frac{\bigvee_{k=0}^{n} v_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=0}^{n} v_{n,k}(x)} \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{n+1}} \leq \frac{12}{\sqrt{n+1}}.$$

Finally, by using (6) too, we obtain the desired estimate. □

Remark 5 Comparing the orders of approximation for the operators in Corollary 1, one sees that they are of the same order as for their only max-product correspondents (see [12]), as for their only Choquet correspondents (see [27]) and as for their classical correspondents (see, e.g., [1], p. 296, p. 300, and pp. 339-340).

However, in what follows, we can give, for example, two concrete examples of max-product Bernstein–Kantorovich–Choquet operators $K^{(M)}_{n,\Gamma_{n,x}}$, which, due to their great flexibility of the form given by Definition 2, approximate better large classes of functions than the classical ones.

Example 1 Let us take $\mu_{n,k,x} = \delta_{k/n}$—the Dirac measures, $k = 0, \ldots, n-1$, and $\mu_{n,n,x} = \sqrt{m}$, with m the Lebesgue measure. Since $k/n \in [k/(n+1), (k+1)/(n+1)]$, we get

$$K^{(M)}_{n,\Gamma_{n,x}}(f)(x)$$

$$= \max\left\{\frac{\bigvee_{k=0}^{n-1} p_{n,k}(x)\cdot\frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\delta_{k/n}(t)}{\delta_{k/n}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=0}^{n} p_{n,k}(x)},\right.$$

$$\left.\frac{p_{n,n}(x)\sqrt{n+1}\cdot(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\right\}.$$

Suppose now that f is nonnegative and nonincreasing on [0, 1]. It follows

$$(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t) \geq f(1)\cdot\sqrt{1-n/(n+1)} = f(1)\cdot\frac{1}{\sqrt{n+1}},$$

which immediately implies

$$K_{n,\Gamma_{n,x}}(f)(x) \geq \frac{\bigvee_{k=0}^{n} p_{n,k}(x)f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(x)} = B_n^{(M)}(f)(x),$$

where $B_n^{(M)}(f)(x)$ denotes the max-product Bernstein operator intensively studied in, for example, [3], Chapter 2.

On the other hand, by

$$\frac{p_{n,n}(x)\sqrt{n+1}\cdot(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)} = \frac{p_{n,n}(x)f(1)}{\bigvee_{k=0}^{n} p_{n,k}(x)}$$

$$+\frac{p_{n,n}(x)\sqrt{n+1}\cdot(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)} - \frac{p_{n,n}(x)f(1)}{\bigvee_{k=0}^{n} p_{n,k}(x)}$$

and by the simple double inequality

$$\max\{A, B\} \leq \max\{A, B+C\} \leq \max\{A, B\} + C, \text{ for all } A, B, C \geq 0, \tag{10}$$

choosing

$$A = \frac{\bigvee_{k=0}^{n-1} p_{n,k}(x)\cdot\frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\delta_{k/n}(t)}{\delta_{k/n}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=0}^{n} p_{n,k}(x)}$$

and

$$B = \frac{p_{n,n}(x)\sqrt{n+1}\cdot(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)} = \frac{p_{n,n}(x)f(1)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

it is immediate that

$$B_n^{(M)}(f)(x) \leq K_{n,\Gamma_{n,x}}^{(M)}(f)(x) \leq B_n^{(M)}(f)(x) + C,$$

where

$$0 \leq C = \frac{p_{n,n}(x)\sqrt{n+1}\cdot(C)\int_{n/(n+1)}^{1} f(t)d\sqrt{m}(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)} - \frac{p_{n,n}(x)f(1)}{\bigvee_{k=0}^{n} p_{n,k}(x)}$$

$$
\begin{aligned}
&= \frac{p_{n,n}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)} \left(\sqrt{n+1} \cdot (C) \int_{n/(n+1)}^{1} f(t) d\sqrt{m}(t) - f(1) \right) \\
&\le \sqrt{n+1} \cdot (C) \int_{n/(n+1)}^{1} f(t) d\sqrt{m}(t) - f(1) \\
&\le \sqrt{n+1} f(n/(n+1)) \cdot (C) \int_{n/(n+1)}^{1} 1 \cdot d\sqrt{m}(t) - f(1) \\
&= f(n/(n+1)) - f(1) \\
&\le \omega_1(f; 1/(n+1))_{[0,1]}.
\end{aligned}
$$

Therefore, if, in addition, we suppose that f is strictly positive too (with $m_f > 0$ its minimum), by Coroianu and Gal [12] (see also Theorem 2.2.18, p. 63 in [3]), we obtain

$$
\begin{aligned}
&|K^{(M)}_{n,\Gamma_{n,x}}(f)(x) - f(x)| \\
&\quad \le |K_{n,\Gamma_{n,x}}(f)(x) - B_n^{(M)}(f)(x)| + |B_n^{(M)}(f)(x) - f(x)| \\
&\quad \le \omega_1(f; 1/(n+1))_{[0,1]} + \left(\frac{n \cdot \omega_1(f; 1/n)_{[0,1]}}{m_f} + 4 \right) \omega_1(f; 1/n)_{[0,1]}.
\end{aligned}
$$

For example, if f is strictly positive, nonincreasing, and Lipschitz function on [0, 1], by the above inequality, it follows that the order of approximation by the max-product Bernstein–Kantorovich–Choquet operator $K^{(M)}_{n,\Gamma_{n,x}}(f)$ is $\mathcal{O}\left(\frac{1}{n}\right)$.

It is worth mentioning that this order of approximation is not achieved in the case of classical Bernstein–Kantorovich operators.

Example 2 Let us take $\mu_{n,k,x} = m$—the Lebesgue measures, $k = 0, \ldots, n-1$ and $\mu_{n,n,x} = \delta_1$—the Dirac measure. We get

$$
\begin{aligned}
&K^{(M)}_{n,\Gamma_{n,x}}(f)(x) \\
&\quad = \max\left\{ \frac{\bigvee_{k=0}^{n-1} p_{n,k}(x) \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dm(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \frac{p_{n,n}(x) f(1)}{\bigvee_{k=0}^{n} p_{n,k}(x)} \right\} \\
&\quad = \max\left\{ \frac{\bigvee_{k=0}^{n-1} p_{n,k}(x) \cdot (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dm(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \right. \\
&\qquad \frac{p_{n,n}(x)(n+1) \int_{n/(n+1)}^{1} f(t) dm(t)}{\bigvee_{k=0}^{n} p_{n,k}(x)}
\end{aligned}
$$

$$+\frac{p_{n,n}(x)f(1)-p_{n,n}(x)(n+1)\int_{n/(n+1)}^{1}f(t)dm(t)}{\bigvee_{k=0}^{n}p_{n,k}(x)}\Bigg\}.$$

Suppose now that f is nonnegative and nondecreasing on [0, 1]. It immediately follows

$$C:=\frac{p_{n,n}(x)\left[f(1)-(n+1)\int_{n/(n+1)}^{1}f(t)dm(t)\right]}{\bigvee_{k=0}^{n}p_{n,k}(x)}\geq 0,$$

which from the above formula for $K_{n,\Gamma_{n,x}}^{(M)}(f)(x)$ immediately implies

$$K_{n,\Gamma_{n,x}}(f)(x)\geq\frac{\bigvee_{k=0}^{n}p_{n,k}(x)(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)}f(t)dm(t)}{\bigvee_{k=0}^{n}p_{n,k}(x)}=K_{n}^{(M)}(f)(x),$$

where $K_{n}^{(M)}(f)(x)$ denotes the usual max-product Bernstein–Kantorovich operator intensively studied in [12].

On the other hand, by choosing in the inequality (10)

$$A=\frac{\bigvee_{k=0}^{n-1}p_{n,k}(x)\cdot(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)}f(t)dm(t)}{\bigvee_{k=0}^{n}p_{n,k}(x)}$$

and

$$B=\frac{p_{n,n}(x)(n+1)\cdot\int_{n/(n+1)}^{1}f(t)dm(t)}{\bigvee_{k=0}^{n}p_{n,k}(x)},$$

it is immediate that

$$K_{n}^{(M)}(f)(x)\leq K_{n,\Gamma_{n,x}}^{(M)}(f)(x)\leq K_{n}^{(M)}(f)(x)+C,$$

where from the above formula for C, we get

$$0\leq C\leq f(1)-(n+1)\cdot\int_{n/(n+1)}^{1}f(t)dm(t)\leq\omega_{1}(f;1/(n+1))_{[0,1]}.$$

Therefore, if, in addition, we suppose that f is strictly positive too (with $m_f>0$ its minimum), by Theorem 2.4, (iii) in [12], we obtain

$$|K_{n,\Gamma_{n,x}}^{(M)}(f)(x)-f(x)|$$
$$\leq|K_{n,\Gamma_{n,x}}(f)(x)-K_{n}^{(M)}(f)(x)|+|K_{n}^{(M)}(f)(x)-f(x)|$$

$$\leq \omega_1(f; 1/(n+1))_{[0,1]} + 2\omega_1(f; 1/n)_{[0,1]} \left(\frac{n \cdot \omega_1(f; 1/n)_{[0,1]}}{m_f} + 5 \right).$$

For example, if f is strictly positive, nondecreasing, and Lipschitz function on [0, 1], by the above inequality, it follows that the order of approximation by the max-product Bernstein–Kantorovich–Choquet operator $K^{(M)}_{n,\Gamma_{n,x}}(f)$ is $\mathscr{O}\left(\frac{1}{n}\right)$.

Again, we observe that this order of approximation is not achieved in the case of classical Bernstein–Kantorovich operators.

Remark 6 To the above operators $S^{(M)}_{n,\mu}$ and $V^{(M)}_{n,\mu}$, we can apply the idea in the papers [24, 35]. Let λ_n be with $\lambda_n \searrow 0$. For simplicity, we will define the slightly different operators

$$S^{(M)}_{n,\Gamma_{n,x}}(f; \lambda_n)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{x^k}{\lambda_n^k k!} \cdot \frac{(C)\int_{k\lambda_n}^{(k+1)\lambda_n} f(t) d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k\lambda_n,(k+1)\lambda_n])}}{\bigvee_{k=0}^{\infty} \frac{x^k}{\lambda_n^k k!}},$$

$$V^{(M)}_{n,\Gamma_{n,x}}(f; \lambda_n)(x)$$

$$= \frac{\bigvee_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{\lambda_n}\left(1 + \frac{1}{\lambda_n}\right) \cdot \ldots \cdot \left(k-1+\frac{1}{\lambda_n}\right) \frac{x^k}{(1+x)^k} \cdot \frac{(C)\int_{k\lambda_n}^{(k+1)\lambda_n} f(t) d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k\lambda_n,(k+1)\lambda_n])}}{\bigvee_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{\lambda_n}\left(1 + \frac{1}{\lambda_n}\right) \cdot \ldots \cdot \left(k-1+\frac{1}{\lambda_n}\right) \frac{x^k}{(1+x)^k}},$$

respectively, where by convention $\frac{1}{\lambda_n}\left(1 + \frac{1}{\lambda_n}\right) \cdot \ldots \cdot \left(k-1+\frac{1}{\lambda_n}\right) = 1$ for $k = 0$.

In this case, the estimates in Corollary 1 (ii) and (iii) easily become as follows.

Corollary 2 *Let* $\mu = \sqrt{m}$. *For all* $n \in \mathbb{N}$, $x \in [0, +\infty)$, $f \in UC^b_+([0, +\infty))$, *we have*

$$|S^{(M)}_{n,\mu}(f; \lambda_n)(x) - f(x)| \leq 2\omega_1\left(f; 4\sqrt{x} \cdot \sqrt{\lambda_n} + \lambda_n\right)_{[0,+\infty)}$$

and

$$|V^{(M)}_{n,\mu}(f; \lambda_n)(x) - f(x)| \leq 2\omega_1\left(f; 6\sqrt{x(1+x)} \cdot \sqrt{\lambda_n} + \lambda_n\right)_{[0,+\infty)},$$

where the sequence $(\lambda_n)_{n\in\mathbb{N}}$ *with* $\lambda_n \searrow 0$ *can be chosen of an arbitrary order.*

Proof Firstly, we consider the case of $S^{(M)}_{n,\mu}(f; \lambda_n)(x)$. Thus, denoting

$$C_{\lambda_n,k}(x) = \frac{(C)\int_{k\lambda_n}^{(k+1)\lambda_n} |t - x| d\mu(t)}{\sqrt{\lambda_n}},$$

by the proof of Corollary 3.8 in [27], we get

$$C_{\lambda_n,k}(x) \le |x - k\lambda_n| + \lambda_n.$$

Now, denoting by $S_n^{(M)}(f;\lambda_n)$ the operator obtained from the usual max-product Szász–Mirakjan operator, by replacing in its formula of definition $1/n$ by λ_n and taking into account the estimate in, e.g., the book [3], p. 165, finally we easily arrive at $S_{n,\mu}^{(M)}(\varphi_x;\lambda_n)(x) \le 4\sqrt{x}\cdot\sqrt{\lambda_n}$.

Applying the estimate (6) in Theorem 1, we arrive at the desired conclusion.

The proof for $V_{n,\mu}^{(M)}(f;\lambda_n)(x)$ is similar with that in Corollary 1. We omit the details. □

Remark 7 In other words, Corollary 2 shows that the order of uniform approximation by $S_{n,\mu}^{(M)}(f;\lambda_n)(x)$ and $V_{n,\mu}^{(M)}(f;\lambda_n)(x)$ in each compact subinterval of $[0,+\infty)$ can be chosen as fast we want, that is, $\mathcal{O}(\omega_1(f;\sqrt{\lambda_n})_{[0,+\infty)})$, with $\lambda_n \searrow 0$ arbitrary fast.

At the end of this section, we present the shape preserving properties, direct results, and localization results of the max-product Bernstein–Kantorovich–Choquet operators given by Definition 2.

They can be deduced from the corresponding results of the usual max-product Bernstein operator

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\cdot f(k/n)}{\bigvee_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}},$$

based on the remark that the operator $K_{n,\mu_n}^{(M)}$ can be obtained from the operator $B_n^{(M)}$, as follows. Suppose that f is arbitrary in $C_+([0,1])$. Let us consider

$$f_n(x) = (C)\int_{nx/(n+1)}^{(nx+1)/(n+1)} f(t)d\mu_n(t)/\mu_n([nx/(n+1),(nx+1)/(n+1)]), \tag{11}$$

where μ_n is a strictly positive set function.

It is readily seen that $B_n^{(M)}(f_n)(x) = K_{n,\mu_n}^{(M)}(f)(x)$, for all $x \in [0,1]$. We also notice that $f_n \in C_+([0,1])$. What is more, if f is strictly positive, then so is f_n.

The following two shape preserving results hold.

Theorem 2 *Let μ_n, $n \in \mathbb{N}$, be strictly positive set functions and $f \in C_+([0,1])$.*

(i) If f is nondecreasing (nonincreasing) on $[0,1]$, then for all $n \in \mathbb{N}$, $K_{n,\mu_n}^{(M)}(f)$ is nondecreasing (nonincreasing, respectively) on $[0,1]$.

(ii) If f is quasi-convex on $[0,1]$, then for all $n \in \mathbb{N}$, $K_{n,\mu_n}^{(M)}(f)$ is quasi-convex on $[0,1]$. Here, quasi-convexity on $[0,1]$ means that $f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$, for all $x, y, \lambda \in [0,1]$.

Proof

(i) If $f \in C_+([0,1])$, then the integral mean value theorem holds for the Choquet integral too. Indeed, there exist $m, M \geq 0$ such that $m \leq f(x) \leq M$ for all $x \in [0,1]$. Then, by the properties of the Choquet integral in Remark 1, (iii), (v), it easily follows

$$m \leq (C)\int_{k/(n+1)}^{(k+1)/(n+1)} f d\mu_n(t)/\mu_n([k/(n+1),(k+1)/(n+1)]) \leq M,$$

which by the continuity of f implies the existence of

$$\xi_{n,k} \in [k/(n+1),(k+1)/(n+1)]$$

such that

$$(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f d\mu_n(t)/\mu_{n,k}([k/(n+1),(k+1)/(n+1)]) = f(\xi_{n,k}).$$

By using this formula, we can write $K_{n,\mu_n}^{(M)}(f)$ under the form

$$K_{n,\mu_n}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \cdot f(\xi_{n,k})}{\bigvee_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}},$$

where $\xi_{n,k} \in [k/(n+1),(k+1)/(n+1)]$, for all $k = 0, \ldots, n$.

Then, by analogy with the proofs for the Bernstein max-product operators (see, e.g., [3], pp. 39-41), the proofs for the Bernstein–Kantorovich–Choquet max-product operators will be based on the properties of the functions

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(\xi_{n,k}).$$

Now, analyzing the proofs of Lemma 2.1.13, Corollary 2.1.14, Theorem 2.1.15, and Corollary 2.1.16 in [3], pp. 39-41, it is easy to see that they work identically for the above $f_{k,n,j}$ too, and we immediately obtain the required conclusions.

(ii) Since as in the case of the max-product Bernstein operators in Corollary 2.1.18, p. 41 in [3], this point is based on the properties from the above point (i), we easily get the required conclusion for this point too.

□

In what follows, we will prove that $K_{n,\mu_n}^{(M)}$ preserves quasi-concavity too. Recall that a continuous function $f : [a,b] \to \mathbb{R}$ is quasi-concave, if and only if there exists $c \in [a,b]$ such that f is nondecreasing on $[a,c]$ and nonincreasing on $[c,b]$. This property holds in the case of the operator $B_n^{(M)}$ (see, e.g., Corollary 2.2.23, p.

68 in [3]). However, it is difficult to adapt the proof to our case. Instead, we can prove this property by finding a direct correspondence between the operators $B_n^{(M)}$ and $K_{n,\mu_n}^{(M)}$.

We are now in a position to prove that $K_{n,\mu_N}^{(M)}$ preserves quasi-concavity too.

Theorem 3 *Let μ_n, $n \in \mathbb{N}$, be strictly positive set functions and $f \in C_+([0, 1])$. If f is quasi-concave on* $[0, 1]$, *then for all $n \in \mathbb{N}$, $K_{n,\mu_n}^{(M)}(f)$ is quasi-concave on* $[0, 1]$.

Proof For some arbitrary $n \geq 1$, let us consider the function f_n given by (11). Moreover, let $c \in [0, 1]$ such that f is nondecreasing on $[0, c]$ and nonincreasing on $[c, 1]$. Then, let $j(c) \in \{0, \ldots, n\}$ such that

$$\frac{j(c)}{n+1} \leq c \leq \frac{j(c)+1}{n+1}.$$

Next, we consider the function g_n, which interpolates f_n at all the knots $\frac{k}{n}$, $k = 0, 1, \ldots, n$, and which is continuous on $[0, 1]$ and affine on any interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0, 1, \ldots, n-1$. It means that g_n is the continuous polygonal line, which interpolates f_n at all the knots $\frac{k}{n}$, $k = 0, 1, \ldots, n$. This easily implies that $B_n^{(M)}(f_n)(x) = B_n^{(M)}(g_n)(x)$, $x \in [0, 1]$, and hence, $K_{n,\mu_n}^{(M)}(f)(x) = B_n^{(M)}(g_n)(x)$, $x \in [0, 1]$. Let us now choose arbitrary $0 \leq k_1 < k_2 \leq j(c) - 1$. We have

$$g_n\left(\frac{k_1}{n}\right) = (C)\int_{k_1/(n+1)}^{(k_1+1)/(n+1)} f(t)d\mu_n(t)/\mu_n([k_1/(n+1), (k_1+1)/(n+1)])$$

and

$$g_n\left(\frac{k_2}{n}\right) = (C))\int_{k_2/(n+1)}^{(k_2+1)/(n+1)} f(t)d\mu_n(t)/\mu_n([k_2/(n+1), (k_2+1)/(n+1)]).$$

As $\frac{k_1+1}{n+1} \leq \frac{k_2}{n+1}$ and f is nondecreasing on $[0, \frac{k_2+1}{n+1}]$, we easily obtain (after applying the mean value theorem) that $g_n\left(\frac{k_1}{n}\right) \leq g_n\left(\frac{k_2}{n}\right)$. The construction of g_n easily implies that g_n is nondecreasing on $\left[0, \frac{j(c)-1}{n}\right]$. By similar reasoning, we get that g_n is nonincreasing on $\left[\frac{j(c)+1}{n}, 1\right]$. Now, suppose that $f\left(\frac{j(c)}{n+1}\right) \geq f\left(\frac{j(c)+1}{n+1}\right)$. The quasi-concavity of f implies that $f(x) \geq f\left(\frac{j(c)+1}{n+1}\right)$ for any $x \in \left[\frac{j(c)}{n+1}, \frac{j(c)+1}{n+1}\right]$. Since there exists $x_0 \in \left[\frac{j(c)}{n+1}, \frac{j(c)+1}{n+1}\right]$ such that

$$(C)\int_{j(c)/(n+1)}^{(j(c)+1)/(n+1)} f(t)d\mu_n(t)/\mu_n([f(c)/(n+1), (f(c)+1)/(n+1)])$$

$$= f(x_0) = g_n\left(\frac{j(c)}{n}\right),$$

and since $f\left(\frac{j(c)+1}{n+1}\right) \geq g_n\left(\frac{j(c)+1}{n}\right)$ (this is true indeed as f is nondecreasing on $\left[\frac{j(c)+1}{n+1}, 1\right]$), we get that $g_n\left(\frac{j(c)}{n}\right) \geq g_n\left(\frac{j(c)+1}{n}\right)$. Therefore, g_n is nonincreasing on $\left[\frac{j(c)}{n}, \frac{j(c)+1}{n}\right]$. This implies that g_n is nondecreasing on $\left[0, \frac{j(c)-1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n}, 1\right]$. But f is affine on $\left[\frac{j(c)-1}{n}, \frac{j(c)}{n}\right]$, which means that it is monotone on this interval. Clearly, this implies that g_n is either nondecreasing on $\left[0, \frac{j(c)-1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)-1}{n}, 1\right]$ or it is nondecreasing on $\left[0, \frac{j(c)}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n}, 1\right]$. It means that g_n is quasi-concave on [0, 1]. By similar reasonings, we get to the same conclusion if $f\left(\frac{j(c)}{n+1}\right) \leq f\left(\frac{j(c)+1}{n+1}\right)$. The only difference is that now g_n is either nondecreasing on $\left[0, \frac{j(c)}{n}\right]$ and nonincreasing on $\left[\frac{j(c)}{n}, 1\right]$ or it is nondecreasing on $\left[0, \frac{j(c)+1}{n}\right]$ and nonincreasing on $\left[\frac{j(c)+1}{n}, 1\right]$. Thus, we just proved that g_n is quasi-concave on [0, 1]. By Theorem 2.2.22, p. 67 in the book [3], it follows that $B_n^{(M)}(g_n)$ is quasi-concave on [0, 1]. As $B_n^{(M)}(g_n) = K_{n,\Gamma_n}^{(M)}(f)$, it follows that $K_{n,\Gamma_n}^{(M)}(f)$ is quasi-concave on [0, 1]. □

Remark 8 As an important side remark, let us note that in Theorem 2.2.22, p. 67 in the book [3]), it is proved that if f is quasi-concave and c is a maximum point of f, then there exists a maximum point of $B_n^{(M)}(f)$ such that $|c - c\prime| \leq \frac{1}{n+1}$. By the construction of g_n, it follows that one maximum point of g_n is between the values $\frac{j(c)-1}{n}$, $\frac{j(c)}{n}$, or $\frac{j(c)+1}{n}$. If we denote this value with c_n, then one can easily check that $|c_n - c| \leq \frac{2}{n}$. Now, applying the previous property, let $c\prime$ be a maximum point of $B_n^{(M)}(g_n) = K_{n,\Gamma_n}^{(M)}(f)$, such that $|c\prime - c_n| \leq \frac{1}{n+1}$. This easily implies that $|c\prime - c| \leq \frac{3}{n}$. So, we obtained a quite similar result for the operator $K_{n,\Gamma_n}^{(M)}$ in comparison with the operator $B_n^{(M)}$.

Let us return to the functions f_n given in (11), and let us find now an upper bound for the approximation of f by f_n in terms of the uniform norm. For some $x \in [0, 1]$, using the mean value theorem, there exists $\xi_x \in \left[\frac{nx}{n+1}, \frac{nx+1}{n+1}\right]$ such that $f_n(x) = f(\xi_x)$. We also easily notice that $|\xi_x - x| \leq \frac{1}{n+1}$. It means that

$$|f(x) - f_n(x)| \leq \omega_1(f; 1/(n+1)), x \in \mathbb{R}, n \in \mathbb{N}. \tag{12}$$

In what follows, we deal with the localization properties of the max-product Bernstein–Kantorovich–Choquet operator.

We firstly prove a very strong localization property of the operator $K_{n,\mu_n}^{(M)}$.

Theorem 4 *Suppose that* μ_n, $n \in \mathbb{N}$ *are strictly positive set functions. Let* $f, g : [0, 1] \to [0, \infty)$ *be both bounded on* [0, 1] *with strictly positive lower bounds, and suppose that there exist* $a, b \in [0, 1]$, $0 < a < b < 1$ *such that* $f(x) = g(x)$ *for all* $x \in [a, b]$. *Then, for all* $c, d \in [a, b]$ *satisfying* $a < c < d < b$, *there exists* $\widetilde{n} \in \mathbb{N}$

depending only on f, g, a, b, c, and d such that $K_{n,\mu_n}^{(M)}(f)(x) = K_{n,\mu_n}^{(M)}(g)(x)$ for all $x \in [c, d]$ and $n \in \mathbb{N}$ with $n \geq \widetilde{n}$.

Proof Let us choose arbitrary $x \in [c, d]$, and for each $n \in \mathbb{N}$, let $j_x \in \{0, 1, \ldots, n\}$ be such that $x \in [j_x/(n+1), (j_x+1)/(n+1)]$. Then, by the last relation on page 33 in the book [3], we have

$$K_{n,\mu_n}^{(M)}(f)(x) = B_n^{(M)}(f_n)(x) = \bigvee_{k=0}^{n} (f_n)_{k,n,j_x}(x), \tag{13}$$

where for $k \in \{0, 1, \ldots, n\}$, we have

$$(f_n)_{k,n,j_x} = \frac{\binom{n}{k}}{\binom{n}{j_x}} \left(\frac{x}{1-x}\right)^{k-j_x} f_n\left(\frac{k}{n}\right), \tag{14}$$

and each f_n is given by (11). Let us denote with m_f, M_f and m_{f_n}, M_{f_n}, respectively, the minimums and maximum values of the functions f and f_n. By the mean value theorem for the Choquet integral, one can easily notice that for any $x \in [0, 1]$, there exists $\xi_{n,x} \in [0, 1]$ such that $f_n(x) = f(\xi_{n,x})$. It means that $0 < m_f \leq m_{f_n} \leq M_{f_n} \leq M_f$. In what follows, the proof is very similar to the proof of Theorem 2.4.1 in [3]. However, as often, we will use f_n instead of f, especially, since the constants obtained in the proof of Theorem 2.4.1 in [3] depend on f, in our setting these constants would depend on f_n; hence, they would depend on n, if we would apply directly the results in [3]. Therefore, there are some differences in the two proofs as our intention is to obtain constants that do not depend on f_n.

We need the set $I_{n,x} = \{k \in \{0, 1, \ldots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$, where $a_n = \left[\sqrt[3]{n^2}\right]$ (here, $[a]$ denotes the integer part of a). Now, suppose that $k \notin I_{n,x}$, and let us discuss first the case when $k < j_x - a_n$. If we look over the proof of Theorem 2.4.1 in [3], we observe that this proof is split into cases (i) and (ii). Case (i) corresponds to the case when $k < j_x - a_n$. Furthermore, this case is divided into two subcases (i$_a$) and (i$_b$). In subcase (i$_a$), the inequality $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{f(j_x/n)}{f(k/n)}$ is obtained, which then gives $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f}$. Applying this reasoning but considering f_n instead of f, we get $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq \left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{f_n(j_x/n)}{f_n(k/n)}$. But since $m_f \leq m_{f_n} \leq M_{f_n} \leq M_f$, we get $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq \left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f}$. We get the same conclusion for all the cases and subcases, that is, any lower bound for $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)}$ is also a lower bound for $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)}$, for any k outside of $I_{n,x}$. Since in the proof of Theorem 2.4.1 in [3], on page 79, it was proved that there exists $N_0 \in \mathbb{N}$, which may depend only on f, a, b, c, and d, such that for any $n \geq N_0$, $k \in \{0, 1, \ldots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$, we have $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq 1$, it follows that $\frac{(f_n)_{j_x,n,j_x}(x)}{(f_n)_{k,n,j_x}(x)} \geq 1$, for any $n \geq N_0$, $k \in \{0, 1, \ldots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$. Combining this fact with relations (13)–(14), we get that

$$K_{n,\mu_n}^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} (f_n)_{k,n,j_x}(x), x \in [c,d],\ n \geq N_0.$$

Using a similar reasoning as in the proof of Theorem 2.4.1 in [3], in what follows, we will prove that N_0 can be replaced if necessary with a larger value $\widetilde{N}_1$ such that $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$ for any $k \in I_{n,x}$. Let us choose arbitrary $x \in [c,d]$ and $n \in \mathbb{N}$ so that $n \geq N_0$. If there exists $k \in I_{n,x}$ such that $k/(n+1) \notin [c,d]$, then we distinguish two cases. Either $\frac{k}{n+1} < c$ or $\frac{k}{n+1} > d$. In the first case, we observe that

$$0 < c - \frac{k}{n+1} \leq x - \frac{k}{n+1} \leq \frac{j_x+1}{n+1} - \frac{k}{n+1} \leq \frac{j_x+1}{n+1} - \frac{k}{n+1} \leq \frac{a_n+1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{a_n+1}{n+1} = 0$, it results that for sufficiently large n, we necessarily have $\frac{a_n+1}{n+1} < c - a$, which clearly implies that $\frac{k}{n+1} \in [a,c]$. In the same manner, when $\frac{k}{n+1} > d$, for sufficiently large n, we necessarily have $\frac{k}{n+1} \in [d,b]$. By similar reasoning, it results that for sufficiently large n, we necessarily have $\frac{k}{n+1} \in [a,b]$. Summarizing, there exists a constant $\widetilde{N}_1 \in \mathbb{N}$ independent of any $x \in [c,d]$ such that

$$K_{n,\mu_n}^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} (f_n)_{k,n,j_x}(x), x \in [c,d],\ n \geq \widetilde{N}_1,$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_1$ and $k \in I_{n,x}$, we have $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$. Also, it is easy to check that $\widetilde{N}_1$ depends only on a, b, c, d, and f.

Now, for $k \in \{0, 1, \ldots, n\}$, taking

$$(g_n)_{k,n,j_x} = \frac{\binom{n}{k}}{\binom{n}{j_x}} \left(\frac{x}{1-x}\right)^{k-j_x} g_n\left(\frac{k}{n}\right),$$

and applying the same reasoning, there exists $\widetilde{N}_2 \in \mathbb{N}$, which may depend only on a, b, c, d, and g, such that

$$K_{n,\mu_n}^{(M)}(g)(x) = \bigvee_{k \in I_{n,x}} (g_n)_{k,n,j_x}(x), x \in [c,d],\ n \geq \widetilde{N}_2,$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_2$ and $k \in I_{n,x}$, we have $[\frac{k}{n+1}, \frac{k+1}{n+1}] \subseteq [a,b]$. Since $f(x) = g(x)$, $x \in [a,b]$, we get that for any $n \geq \widetilde{n} = \max\{\widetilde{N}_1, \widetilde{N}_2\}$, $k \in I_{n.x}$ and $x \in [c,d]$, it holds that $(f_n)_{k,n,j_x}(x) = (g_n)_{k,n,j_x}(x)$. Thus, for any

$n \geq \widetilde{n}$ and $x \in [c, d]$, we have $K_{n,\mu_n}^{(M)}(f)(x) = K_{n,\mu_n}^{(M)}(g)(x)$. The proof is complete now. □

Previously, by Theorem 2, we proved that $K_{n,\mu_n}^{(M)}$ preserves monotonicity and more generally quasi-convexity. By the localization result in Theorem 4 and then applying a very similar reasoning to the one used in the proof of Theorem 2, we obtain local versions for these shape preserving properties. For this reason, we omit the proofs of the following corollaries (see also the corresponding local shape preserving properties proved for the operator $B_n^{(M)}$ in Corollary 2.4.4 and Corollary 2.4.5, pp. 81-82 in the book [3]).

Corollary 3 *Suppose that* μ_n, $n \in \mathbb{N}$ *are strictly positive set functions. Let* $f : [0, 1] \to [0, \infty)$ *be bounded on* $[0, 1]$ *with strictly positive lower bound, and suppose that there exist* $a, b \in [0, 1]$, $0 < a < b < 1$, *such that* f *is nondecreasing (nonincreasing) on* $[a, b]$. *Then, for any* $c, d \in [a, b]$ *with* $a < c < d < b$, *there exists* $\widetilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d, *and* f, *such that* $K_{n,\mu_n}^{(M)}(f)$ *is nondecreasing (nonincreasing) on* $[c, d]$ *for all* $n \in \mathbb{N}$ *with* $n \geq \widetilde{n}$.

Corollary 4 *Suppose that* μ_n, $n \in \mathbb{N}$ *are strictly positive set functions. Let* $f : [0, 1] \to [0, \infty)$ *be a continuous and strictly positive function, and suppose that there exist* $a, b \in [0, 1]$, $0 < a < b < 1$, *such that* f *is quasi-convex on* $[a, b]$. *Then, for any* $c, d \in [a, b]$ *with* $a < c < d < b$, *there exists* $\widetilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d, *and* f *such that* $K_{n,\mu_n}^{(M)}(f)$ *is quasi-convex on* $[c, d]$ *for all* $n \in \mathbb{N}$ *with* $n \geq \widetilde{n}$.

Corollary 5 *Suppose that* μ_n, $n \in \mathbb{N}$ *are strictly positive set functions. Let* $f : [0, 1] \to [0, \infty)$ *be a continuous and strictly positive function, and suppose that there exist* $a, b \in [0, 1]$, $0 < a < b < 1$, *such that* f *is quasi-concave on* $[a, b]$. *Then, for any* $c, d \in [a, b]$ *with* $a < c < d < b$, *there exists* $\widetilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d *and* f, *such that* $K_{n,\mu_n}^{(M)}(f)$ *is quasi-concave on* $[c, d]$ *for all* $n \in \mathbb{N}$ *with* $n \geq \widetilde{n}$.

Remark 9 As in the cases of Bernstein-type max-product operators studied in the research monograph [3], for the max-product Bernstein–Kantorovich–Choquet-type operators, we can find natural interpretation as possibilistic operators, which can be deduced from the Feller scheme written in terms of the possibilistic integral. These approaches also offer new proofs for the uniform convergence, based on a Chebyshev-type inequality in the theory of possibility.

4 Approximation by Max-Product Discrete Singular Integrals of Choquet Type

In this section, we consider discrete variants of the convolution integrals of Picard, Gauss–Weierstrass, and Poisson–Cauchy type.

Firstly, let us consider $p_{n,k}(x) = e^{-n|x-k/n|}$, $I = (-\infty, +\infty)$. The max-product Picard–Kantorovich–Choquet operators, $\mathscr{P}K_n^{(M)}$, are defined by

$$\mathscr{P}K_{n,\Gamma_{n,x}}^{(M)}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} e^{-n|x-k/n|} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=-\infty}^{+\infty} e^{-n|x-k/n)|}}, \tag{15}$$

where $\mu_{n,k,x}, k = 1, \ldots, n, x \in \mathbb{R}$, is a collection of families of monotone, submodular, and strictly positive set functions.

We can state the following result.

Theorem 5 *Suppose that $\mu_{n,k,x} = \mu := \sqrt{m}$, for all n, k and x, where m is the Lebesgue measure. If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathscr{P}K_{n,\mu}^{(M)}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n)_{\mathbb{R}}, x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof As in the proofs of the results in the previous section, let us consider the quantity

$$C_{n,k}(x) = \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} |t-x|d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])}.$$

According to the proof of Corollary 3.6 in [27], we have

$$C_{n,k}(x) \leq \left|\frac{k}{n} - x\right| + \frac{1}{n},$$

which immediately implies

$$\mathscr{P}K_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right| + \frac{1}{n}\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} + \frac{1}{n} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 10.3.1, p. 424 in [3])

$$\frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{1}{n}.$$

Using the estimate (6) in Theorem 1 too, we obtain the desired conclusion. □

Now, let us choose $p_{n,k}(x) = e^{-n(x-k/n)^2}$, $I = (-\infty, +\infty)$. In this case, the max-product Gauss–Weierstrass–Kantorovich–Choquet operators are defined by

$$\mathscr{W}K^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=-\infty}^{+\infty} e^{-n(x-k/n))^2} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=-\infty}^{+\infty} e^{-n(x-k/n)^2}}. \tag{16}$$

We have the following result.

Theorem 6 *Suppose that $\mu_{n,k,x} = \mu := \sqrt{m}$, for all n, k and x, where m is the Lebesgue measure. If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathscr{W}K^{(M)}_{n,\mu}(f)(x) - f(x)| \leq 2\omega_1(f; 1/\sqrt{n} + 1/n), x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof As in the proof of Theorem 5, we get

$$\mathscr{W}K^{(M)}_{n,\mu}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right| + \frac{1}{n}\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} + \frac{1}{n} \leq \frac{1}{\sqrt{n}} + \frac{1}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 10.3.3, p. 426 in [3])

$$\frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n} - x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{1}{\sqrt{n}}.$$

Using the estimate (6) in Theorem 1 too, we obtain the desired conclusion. □

Finally, let us choose $p_{n,k}(x) = \frac{1}{n^2(x-k/n)^2+1}$, $I = (-\infty, +\infty)$. In this case, the max-product Poisson–Cauchy–Kantorovich–Choquet operators are defined by

$$\mathscr{C}K^{(M)}_{n,\Gamma_{n,x}}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{1}{n^2(x-k/n)^2+1} \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([k/(n+1),(k+1)/(n+1)])}}{\bigvee_{k=0}^{\infty} \frac{1}{n^2(x-k/n)^2+1}}. \tag{17}$$

Concerning these operators, the following result holds.

Theorem 7 *Suppose that $\mu_{n,k,x} = \mu := \sqrt{m}$, for all n, k and x, where m is the Lebesgue measure. If $f : \mathbb{R} \to [0, +\infty)$ is bounded and uniformly continuous on $\mathbb{R}$, then we have*

$$|\mathscr{C}K^{(M)}_{n,\mu}(f)(x) - f(x)| \leq 4\omega_1(f; 1/n), x \in \mathbb{R}, n \in \mathbb{N}.$$

Proof As in the proof of Theorem 6, we get

$$\mathscr{C}K_{n,\mu}^{(M)}(\varphi_x)(x)$$

$$\leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n}-x\right|+\frac{1}{n}\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)}+\frac{1}{n} \leq \frac{1}{n}+\frac{1}{n}=\frac{2}{n},$$

where we have used the estimate (see, e.g., the proof of Theorem 10.3.5, p. 427 in [3])

$$\frac{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)\left[\left|\frac{k}{n}-x\right|\right]}{\bigvee_{k=-\infty}^{+\infty} p_{n,k}(x)} \leq \frac{1}{n}.$$

Using the estimate (6) in Theorem 1 too, we obtain the desired estimate. □

5 Max-Product Kantorovich–Choquet Operators Based on (ϕ, ψ)-Kernels

In the paper [11], the truncated max-product sampling operators based on sinc-Fejér kernels were generalized to truncated max-product Kantorovich operators based on generalized type kernels depending on two functions φ and ψ satisfying a set of suitable conditions and convergence results were obtained. In the same paper, as particular cases previous results in sampling and neural network approximation are recaptured and new results for many concrete examples are obtained.

In this section, we introduce the more general max-product Kantorovich–Choquet with respect to a family of strictly positive and submodular set functions $\Gamma_{n,x} = \{\mu_{n,k,x}, k = 0, \ldots, n, x \in [0, b]\}$ and based on a generalized (φ, ψ)-kernel, by the formula

$$K_{n,\Gamma_{n,x}}^{(M)}(f;\varphi,\psi)(x) = \frac{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)} \cdot \left[\frac{(C)\int_{kb/(n+1)}^{(k+1)b/(n+1)} f(v)d\mu_{n,k,x}(v)}{\mu_{n,k,x}([kb/(n+1),(k+1)b/(n+1)])}\right]}{\bigvee_{k=0}^{n} \frac{\varphi(nx-kb)}{\psi(nx-kb)}}, \quad (18)$$

where $b > 0$, $f : [0, b] \to \mathbb{R}_+$ is a bounded, the Choquet integrable function on $[0, b]$ with respect to each $\mu_{n,k,x}$ and φ and ψ satisfy some properties specific to max-product operators and required to prove convergence results, as follows:

Definition 3 We say that (φ, ψ) forms a generalized kernel if satisfy some (not necessary all, depending on the type of convergence intended for study) of the following properties:

(i) $\varphi, \psi : \mathbb{R} \to \mathbb{R}_+$ are continuous on $\mathbb{R}$, $\varphi(x) \neq 0$ for all $x \in (0, b/2]$ and $\psi(x) \neq 0$ for all $x \neq 0$, $\frac{\varphi(x)}{\psi(x)}$ is an even function on $\mathbb{R}$, and $\lim_{x\to 0} \frac{\varphi(x)}{\psi(x)} = \alpha \in (0, 1]$.
(ii) There exists a constant $C \in \mathbb{R}$ such that $\varphi(x) \leq C \cdot \psi(x)$, for all $x \in \mathbb{R}$.
(iii) There exist the positive constants $M > 0$ and $\beta > 0$, such that $\frac{\varphi(x)}{\psi(x)} \leq \frac{M}{x^\beta}$, for all $x \in (0, \infty)$.
(iv) For any $n \in \mathbb{N}$, $j \in \{0, \dots, n\}$ and $x \in \left[\frac{jb}{n}, \frac{(j+1)b}{n}\right]$,

$$\bigvee_{k=0}^{n} \frac{\varphi(nx - kb)}{\psi(nx - kb)} = \max\left\{\frac{\varphi(nx - jb)}{\psi(nx - jb)}, \frac{\varphi(nx - (j+1)b)}{\psi(nx - (j+1)b)}\right\}.$$

(v) $\int_{-\infty}^{+\infty} \frac{\varphi(y)}{\psi(y)} dy = c$, where $c > 0$ is a positive real constant.

Remark 10 The use of the two functions in the generalized kernels offers a large flexibility in finding many concrete examples.

Remark 11 Let us note that if properties (i) and (iii) hold simultaneously, then (ii) holds too. Indeed, firstly if (i) holds, clearly that we may extend the continuity of $\frac{\varphi(x)}{\psi(x)}$ in the origin too, that is, we take $\frac{\varphi(0)}{\psi(0)} = \lim_{x\to 0} \frac{\varphi(x)}{\psi(x)}$. This means that $\frac{\varphi(x)}{\psi(x)}$ is continuous on the whole $\mathbb{R}$. Secondly, from (iii), it is readily seen that there exists a constant $a > 0$ such that $\frac{M}{x^\beta} \leq 1$, for all $x \in [a, \infty)$. It means that $\varphi(x) \leq \psi(x)$, for all $x \in [a, \infty)$. This fact combined with the continuity of $\frac{\varphi(x)}{\psi(x)}$ on $[-a, a]$ easily implies that (ii) holds.

Remark 12 Another important remark is that if (i) and (iii), $\beta > 1$ case, hold simultaneously, then (v) holds too. Indeed, since $\frac{\varphi(x)}{\psi(x)}$ is an even function on $\mathbb{R}$, it suffices to prove that $\int_0^{+\infty} \frac{\varphi(y)}{\psi(y)} dy$ is finite. From the continuity of $\frac{\varphi(x)}{\psi(x)}$, this later integral is finite if and only if $\int_1^{+\infty} \frac{\varphi(y)}{\psi(y)} dy$ is finite. Now, since $\frac{\varphi(x)}{\psi(x)} \leq \frac{M}{x^\beta}$, for all $x \in [0, \infty)$, and since we easily note that $\int_1^{+\infty} \frac{M}{x^\beta} dx$ is finite, we conclude that $\int_1^{+\infty} \frac{\varphi(y)}{\psi(y)} dy$ is finite. Thus, $\int_{-\infty}^{+\infty} \frac{\varphi(y)}{\psi(y)} dy$ is finite, which means that (v) holds.

Remark 13 If in the pair (φ, ψ), we consider that ψ is a strictly positive constant function, then in order that (φ, ψ) be a generalized kernel satisfying all the properties (i)–(v) in Definition 3, it is good enough if $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a continuous even function, satisfying $\varphi(x) > 0$, for all $x \in (0, b/2)$, $\varphi(0) \neq 0$ (this implies (i)), $\varphi(x)$ is bounded on $\mathbb{R}$ (this implies (ii)), $\varphi(x) = \mathcal{O}\left(\frac{1}{x^\beta}\right)$, $x \in [0, +\infty)$, $\beta > 0$ (this implies (iii)), $\varphi(x)$ is nonincreasing on $[0, +\infty)$ (this implies (iv)), and $\int_0^{+\infty} \varphi(x) dx < +\infty$ (this implies (v)). Note that this particular type of choice for the generalized kernel (φ, ψ) may cover some sampling approximation operators (see Application 3 below) and neural network operators (see Application 6 below).

In what follows, we prove a quantitative estimate for a particular variant of the max-product Kantorovich–Choquet operators defined by formula (18), which involves the modulus of continuity, as follows.

Theorem 8 *Suppose that $f : [0, b] \to \mathbb{R}_+$ is continuous on $[0, b]$ and that properties (i), (iii), and (iv) are fulfilled by φ and ψ. Also, take $\mu_{n,k,x} = \mu := \sqrt{m}$, for all n, k, and x, where m is the Lebesgue measure. Then, for any $n \in \mathbb{N}$, we have*

$$\left\| K_{n,\mu}^{(M)}(f; \varphi, \psi) - f \right\| \leq 2\omega_1 \left(f; \frac{M \cdot (2b)^{1-\beta}}{c_1 \cdot n^{\beta}} + \frac{b}{n} \right)_{[0,b]}.$$

Here, c_1 denotes a constant which follows from relations (6) and (7) in [11], and $\| \cdot \|$ denotes the uniform norm.

Proof Denote $p_{n,k}(x) = \frac{\varphi(nx-kb)}{\psi(nx-kb)}$. In order to estimate $K_{n,\mu}^{(M)}(\varphi_x; \varphi, \psi)(x)$, let us denote

$$C_{n,k}(x) = \frac{(C) \int_{kb/(n+1)}^{(k+1)b/(n+1)} |t - x| d\mu(t)}{\mu([k/(n+1), (k+1)/(n+1)])}$$
$$= \frac{\sqrt{n+1}}{\sqrt{b}} \cdot (C) \int_{kb/(n+1)}^{(k+1)b/(n+1)} |t - x| d\mu(t).$$

We have three possibilities: (a) $x \in [kb/(n+1), (k+1)b/(n+1)]$, (b) $0 \leq x < kb/(n+1)$, and (c) $(k+1)b/(n+1) < x$.

Case (a) Since $|t - x| \leq (k+1)b/(n+1) - kb/(n+1) = b/(n+1)$, for all $t, x \in I_{n,k} = [kb/(n+1), (k+1)b/(n+1)]$, we get

$$C_{n,k}(x) \leq \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \frac{b}{n+1} \cdot (C) \int_{kb/(n+1)}^{(k+1)b/(n+1)} 1 \cdot d\mu$$
$$= \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \frac{b}{n+1} \cdot \frac{\sqrt{b}}{\sqrt{n+1}} = \frac{b}{n+1} < \frac{b}{n}.$$

Case (b) We have $|t - x| = t - x$, and denoting $E(n, k, x, \beta) := \mu(\{t \in I_{n,k}; t \geq x + \beta\})$, we get

$$C_{n,k}(x) = \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \int_0^{\infty} E(n, k, x, \beta) d\beta = \frac{\sqrt{n+1}}{\sqrt{b}}$$
$$\cdot \int_0^{(k+1)b/(n+1)-x} E(n, k, x, \beta) d\beta$$
$$= \frac{\sqrt{n+1}}{\sqrt{b}} \Big(\int_0^{kb/(n+1)-x} E(n, k, x, \beta) d\beta$$
$$+ \int_{kb/(n+1)-x}^{(k+1)b/(n+1)-x} E(n, k, x, \beta) d\beta \Big)$$

$$\leq \left(\frac{kb}{n+1} - x\right) + \frac{\sqrt{n+1}}{\sqrt{b}}$$

$$\cdot \int_{kb/(n+1)-x}^{(k+1)b/(n+1)-x} \sqrt{(k+1)b/(n+1) - x - \beta}\, d\beta$$

$$= \left(\frac{kb}{n+1} - x\right) + \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \int_0^{b/(n+1)} \sqrt{\eta}\, d\eta = \left(\frac{kb}{n+1} - x\right) + \frac{2b}{3(n+1)}$$

$$= \frac{n}{n+1}\left(\frac{kb}{n} - x\right) + \frac{2b - 3x}{3(n+1)} \leq \frac{n}{n+1}\left|\frac{kb}{n} - x\right| + \frac{2b}{3(n+1)}$$

$$\leq \left|\frac{kb}{n} - x\right| + \frac{2b}{3(n+1)}$$

$$< \left|\frac{kb}{n} - x\right| + \frac{2b}{3n}.$$

Case (c) Since $|t - x| = x - t$, denoting $E(n, k, x, \beta) := \mu(\{t \in I_{n,k}; t \leq x - \beta\})$, and reasoning as in the case (b), we obtain

$$C_{n,k}(x) = \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \int_0^{\infty} E(n, k, x, \beta)\, d\beta = \frac{\sqrt{n+1}}{\sqrt{b}}$$

$$\cdot \int_0^{x-kb/(n+1)} E(n, k, x, \beta)\, d\beta$$

$$= \frac{\sqrt{n+1}}{\sqrt{b}} \left(\int_0^{x-(k+1)b/(n+1)} E(n, k, x, \beta)\, d\beta\right.$$

$$\left. + \int_{x-(k+1)b/(n+1)}^{x-kb/(n+1)} E(n, k, x, \beta)\, d\beta\right)$$

$$\leq \left(x - \frac{(k+1)b}{n+1}\right) + \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \int_{x-(k+1)b/(n+1)}^{x-kb/(n+1)} \sqrt{x - kb/(n+1) - \beta}\, d\beta$$

$$= \left(x - \frac{(k+1)b}{n+1}\right) + \frac{\sqrt{n+1}}{\sqrt{b}} \cdot \int_0^{b/(n+1)} \sqrt{\eta}\, d\eta$$

$$= \left(x - \frac{(k+1)b}{n+1}\right) + \frac{2b}{3(n+1)} = \frac{n}{n+1}\left(x - \frac{kb}{n}\right) + \frac{x - b/3}{(n+1)}$$

$$\leq \frac{n}{n+1}\left|x - \frac{kb}{n}\right| + \frac{2b}{3(n+1)} \leq \left|x - \frac{kb}{n}\right| + \frac{b}{n+1} < \left|x - \frac{kb}{n}\right| + \frac{b}{n}.$$

Collecting the estimates in the three cases (a), (b), and (c), it follows that for all $x \in [0, 1]$, we get

$$C_{n,k}(x) \le \left| \frac{kb}{n} - x \right| + \frac{b}{n},$$

which immediately implies

$$K_{n,\mu}^{(M)}(\varphi_x; \varphi, \psi)(x)$$

$$\le \frac{\bigvee_{k=0}^{n} p_{n,k}(x) \left[\left| \frac{kb}{n} - x \right| + \frac{b}{n} \right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} \le \frac{\bigvee_{k=0}^{n} p_{n,k}(x) \left[\left| \frac{kb}{n} - x \right| \right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} + \frac{b}{n} \le \frac{M 2^{1-\beta}}{c_1 n^{\beta}} + \frac{b}{n},$$

where we have used the estimate (see the estimate (10) and the reasonings in the proof of Theorem 3.3 in [11])

$$\frac{\bigvee_{k=0}^{n} p_{n,k}(x) \left[\left| \frac{kb}{n} - x \right| \right]}{\bigvee_{k=0}^{n} p_{n,k}(x)} \le \frac{M \cdot (2b)^{1-\beta}}{c_1 \cdot n^{\beta}},$$

for all $x \in [0, b]$.

Now, using the estimate (6) in Theorem 1 too, we obtain the desired conclusion. □

Remark 14 The estimate in the statement of Theorem 8 remains valid for functions of arbitrary sign, lower bounded. Indeed, if $m \in \mathbb{R}$ is such that $f(x) \ge m$ for all $x \in [0, b]$, then it is easy to see that defining the new max-product operator $\overline{K}_n^{(M)}(f; \varphi, \psi)(x) = K_n^{(M)}(f - m; \varphi, \psi)(x) + m$, for $|f(x) - \overline{K}_n^{(M)}(f; \varphi, \psi)(x)|$, we get the same estimate as in the statement of Theorem 8.

In the next lines, we present some concrete examples of (φ, ψ)-kernels satisfying the conditions in Definition 3.

Application 1 Let us choose $\varphi(x) = \sin^{2r}(x)$, $\psi(x) = x^{2r}$, with $r \in \mathbb{N}$. In this case, $\frac{\varphi(x)}{\psi(x)}$ represents in fact the so-called generalized Jackson kernel. Now, in Definition 3 by taking $b = \pi$, condition (i) is evidently satisfied with $\alpha = 1$, condition (ii) is evidently satisfied with $C = 1$, $c_1 = \left(\frac{2}{\pi}\right)^{2r}$), condition (iii) holds with $M = 1$ and $\beta = 2r$, and condition (v) is satisfied with $c = \frac{\pi}{(2r-1)!} \cdot e_r$, where e_r is the so-called Eulerian number given by

$$e_r = \sum_{j=0}^{r} (-1)^j \binom{2r}{j} (r - j)^{2r-1}.$$

Due to the fact that $\sin^{2r}(nx - k\pi) = \sin^{2r}(nx)$, the equality in condition (iv) in Definition 3, one reduces to

$$\bigvee_{k=0}^{n} \frac{1}{(nx-kb)^{2r}} = \max\left\{\frac{1}{(nx-jb)^{2r}}, \frac{1}{(nx-(j+1)b)^{2r}}\right\}, \tag{19}$$

for all $x \in \left[\frac{jb}{n}, \frac{(j+1)b}{n}\right]$, which follows by simple calculation; see Application 5.1 in [11].

Concluding, Theorem 8 is valid for the max-product Kantorovich–Choquet sampling operators based on this kernel (φ, ψ) and given by (18).

Application 2 Let us choose $\varphi(x) = \sin(x/2)\sin(3x/2)$, $\psi(x) = 9x^2/4$. We note that $\frac{\varphi(x)}{\psi(x)}$ represents in fact the so-called de la Vallée-Poussin kernel used in approximation by sampling operators. Similar reasonings with those in Application 1 easily lead to the fact that in this case too, conditions (i)–(v) in Definition 3 hold and that the max-product Kantorovich–Choquet sampling operators in (18) based on this (φ, ψ)-kernel satisfy Theorem 8.

Application 3 Let us choose as $\varphi(x)$ the B-spline of order 3 given by

$$\varphi(x) = \frac{3}{4} - x^2, \text{ if } |x| \le \frac{1}{2}, \; \varphi(x) = \frac{1}{2}(\frac{3}{2} - |x|)^2, \text{ if } \frac{1}{2} < |x| \le \frac{3}{2},$$
$$\varphi(x) = 0, \text{ if } |x| > \frac{3}{2}.$$

Choosing, for example, $\psi(x) = 1$, for all $x \in \mathbb{R}$, it is easy to see that (φ, ψ) verifies all the conditions in Definition 3, as follows: condition (i) with $b = \frac{1}{2}$, condition (ii) with a sufficiently large constant $C > 0$, condition (iii) with $\beta = 2$ and $M > 0$ sufficiently large, and evidently condition (iv), and condition (v); see Application 5.3 in [11].

In conclusion, Theorem 8 holds for the max-product Kantorovich–Choquet operator in (18) based on this kernel (φ, ψ).

In fact, if we choose for $\varphi(x)$ any B-spline of an arbitrary order n and $\psi(x) = 1$, $x \in \mathbb{R}$, then (φ, ψ) verifies, as in the previous lines, all the conditions in Definition 3; see again Application 5.3 in [11].

This means that Theorem 8 holds for the max-product Kantorovich–Choquet operators in (18) based on this (φ, ψ)-kernel.

Application 4 Let us consider $\varphi(x) = 2\arctan\left(\frac{1}{x^2}\right)$, $x \ne 0$, $\varphi(0) = \pi$, and $\psi(x) = \pi$, $x \in \mathbb{R}$, where $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\lim_{y\to\infty} \arctan(y) = \frac{\pi}{2}$. We check the conditions in Definition 3. Indeed, it is clear that condition (i) is satisfied for $b = 1$ and with $\alpha = 1$, while since $0 \le \arctan(y) \le \frac{\pi}{2}$ for all $y \ge 0$, condition (ii) follows with $C = 1$. Note here that since $\arctan(y) \ge \frac{\pi}{4}$, for all $y \in [1, +\infty)$, putting $y = \frac{1}{x^2}$, we immediately obtain that in Definition 3 $b = 1$ and $c_1 = \frac{1}{4}$. By $2\arctan(1/x^2) \le \frac{2}{x^2}$, for all $x > 0$, we obtain that condition (iii) holds too with $\beta = 2$ and $M = 2\pi$.

Then, since $\arctan(y) \le y$ for all $y \in [0 + \infty)$, we get

$$\int_{-\infty}^{+\infty} \frac{\varphi(x)}{\psi(x)} dx = \frac{2}{\pi} \int_0^{+\infty} \arctan(1/x^2) dx$$
$$= \frac{2}{\pi} \int_0^1 \arctan(1/x^2) dx + \frac{2}{\pi} \int_1^{+\infty} \arctan(1/x^2) dx$$
$$\leq 1 + \frac{2}{\pi} \int_1^{+\infty} \arctan(1/x^2) dx$$
$$\leq 1 + \frac{2}{\pi} \int_1^{+\infty} \frac{1}{x^2} dx = 1 + \frac{2}{\pi} < +\infty,$$

which shows that condition (v) holds.

Now, since for $x \in [j/n, (j+1)/n]$, we evidently have (see also the similar relation (19))

$$\frac{1}{(x-k/n)^2} \leq \frac{1}{(x-j/n)^2} \text{ and } \frac{1}{(x-k/n)^2} \leq \frac{1}{(x-(j+1)/n)^2}, \text{ for } 0 \leq k \leq n; \tag{20}$$

applying here the increasing function arctan, we immediately obtain (iv).

In conclusion, for this choice of the (φ, ψ)-kernel, Theorem 8 remains valid for the max-product operators given by (18).

Application 5 Let us choose $\varphi(x) = |x|$ and $\psi(x) = e^{|x|} - 1$. We will check the conditions in Definition 3. Firstly, it is easy to see that condition (i) is satisfied with, e.g., $b = \ln(2)$, since by using l'Hospital's rule, we have $\lim_{x\to 0} \frac{\varphi(x)}{\psi(x)} = 1$. Then, by $|x| \leq e^{|x|} - 1$ for all $x \in \mathbb{R}$, it follows that condition (ii) holds with $C = 1$ and $c_1 = \frac{1}{2}$ and $b = \ln(2)$). Condition (iii) obviously holds for $M = 2$ and $\beta = 1$.

Then, condition (v) is also satisfied, since

$$\int_{-\infty}^{+\infty} \frac{|x|}{e^{|x|} - 1} dx = 2 \int_0^{+\infty} \frac{x}{e^x - 1} dx$$
$$= 2 \int_0^1 \frac{x}{e^x - 1} dx + 2 \int_1^{+\infty} \frac{x}{e^x - 1} dx = c > 0, \ c \text{ finite },$$

since $\int_1^{+\infty} \frac{x}{e^x-1} dx \leq \int_1^{+\infty} x \cdot e^{-x/2} dx < +\infty$.

It remains to check condition (iv). Firstly, by similar reasonings to those used for the proofs of relations (19) and (20), for all $x \in [jb/n, (j+1)b/n]$, we get

$$|nx - kb| \geq |nx - jb| \text{ and } |nx - kb| \geq |nx - (j+1)b|, \text{ for all } k = 0, \ldots, n.$$

Now, denote $F(u) = \frac{u}{e^u - 1}$, $u \geq 0$. If we prove that F is nonincreasing on $[0, +\infty)$, then we immediately get that condition (iv) in Definition 3 is satisfied.

In this sense, by $F'(u) = \frac{e^u - 1 - ue^u}{(e^u-1)^2} = \frac{G(u)}{(e^u-1)^2}$, with $G(u) = e^u - 1 - ue^u$, since $G(0) = 0$ and $G'(u) = -ue^u \le 0$, we immediately obtain $G(u) \le 0$, for all $u \ge 0$ and consequently $F'(u) \le 0$, for all $u \ge 0$. See Application 5.4 in [11].

In conclusion, in the case of this (φ, ψ)-kernel too, Theorem 8 remains valid for the max-product operators given by (18).

Application 6 Starting with a so-called sigmoidal function σ (as, for example, the hyperbolic sigmoidal function or the sigmoidal logistic function or the ramp function), one can define the "centered bell-shaped function" $\Phi_\sigma(x) = \frac{\sigma(x+1)-\sigma(x-1)}{2}$ and the corresponding max-product Kantorovich–Choquet neural network operator on [0, 1]. Then, taking into account that $\Phi_\sigma(x)$ is a continuous, positive, even function on $\mathbb{R}$, nonincreasing for $x \ge 0$ and $\Phi_\sigma(x) = \mathcal{O}(|x|^{-\beta})$, with $\beta > 1$, we may consider the (Φ_σ, ψ)-kernel with ψ a positive constant function. Then, it follows that (Φ_σ, ψ) satisfies all the properties in Definition 3; see Application 5.6 in [11].

Therefore, Theorem 8 remains valid for this max-product neural network operator of Choquet type.

Application 7 It is worth mentioning that if (φ_1, ψ_1) and (φ_2, ψ_2) are two kernels satisfying the conditions (i), (ii), (iii), and (v) in Definition 3, then the new kernel $(\varphi_1 \cdot \varphi_2, \psi_1 \cdot \psi_2)$ also satisfies these conditions.

The only problem is that the condition (iv) is not, in general, satisfied by the $(\varphi_1 \cdot \varphi_2, \psi_1 \cdot \psi_2)$-kernel.

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On the Approximation of Extinction Time for the Discrete-Time Birth–Death Circuit Chains in Random Environments

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Abstract We investigate suitable expressions for the mean time to extinction of the corresponding "adjoint" circuit chains describing uniquely the discrete-time birth–death model in random environments.

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1 Introduction

It is known that the *discrete probability theory* deals with events that occur in measurable sample spaces, which can be modeled through the discrete probability distributions (cf. [17, 22]), as well as that the graph theory is considered as a part of the combinatorics (cf. [32]). In particular, graphs are one of the most important subjects studied in discrete mathematics because it is one of the most common models of physical and artificial structures since they can model several types of relationships and dynamic processes in physical, biological, and social systems ([cf. 33]), such as the *classical discrete-time birth–death model*, which is an important special case of homogeneous, irreducible Markov chain (cf. [30, 35]) introduced by W. Feller in 1939 [3, 4], where the possible state changes can only occur between neighboring states at discrete time points $0, 1, 2, \ldots, n, n \in N$. This means that the state transitions are of only two types: *"births,"* which increase the state variable by one, and *"deaths,"İ* which decrease the state variable by one; that is, if the current state at discrete time instant n is $X_n = i$, then the state at the next time instant $(n+1)$ can only be $X_{n+1} = i + 1$ or $(i - 1)$. It is assumed that the birth and death events are independent of each other. Loosely speaking, this is a process which combines

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the property of a random walk with reflection at zero [6, 7, 9–11] and discrete-time nature of the transition times. (Here, the states are represented as non-negative integer-valued without loss of generality (cf. [28, 36]).)

The model's name has been arisen by a common application, the use of such models to represent the current size of a population where the transitions are literal births and deaths. In particular, biology and epidemiology birth–death chains are useful for studying processes of biological populations such as the model growth of biological populations since the variety of dynamic behavior exhibited by many species of bacteria, insects, and animals has stimulated a great interest in the development of such mathematical models. In particular, in many ecological problems such as animal populations, epidemics, and competition between species, their patterns of growth are influenced by population size (cf. [2, 27, 31]). Furthermore, they can also be used to model the states of chemical systems such as the radioactive transformations, where the radioactive atoms are unstable and disintegrate stochastically. Then, the new atoms that are also unstable and could emit radioactive particles will decay with specified rates from one state to the adjacent state through a process which can be modeled by a birth–death chain. Besides, the queuing model is an important application of the birth–death chains in a wide range of areas such as computer networks and telecommunications since it can be used to optimize the size of the storage space, to determine the trade-off between throughput and inventory as well as to exhibit the propagation of blockage (cf. [29, 34]). We will consider the discrete-time birth–death model with state space $S = N$ and transition probabilities given by $p_{ij} = 0$, if $j \neq i-1, i+1$, $p_{i,i-1} = q_i$, $p_{i,i} = 0$, $p_{i,i+1} = p_i$, $p_i + q_i = 1$, $i \geq 1$, with $p_0 = 1$, allowed to depend on the current population size, which describes the size of a "population" at discrete instant time periods [19].

Usually, the discrete-time birth–death chains are studied from the Markov chain point of view, where the random mechanism of spatial motion is determined by the given transition probabilities (probabilities of jumps) at each state in a *non-random (fixed) environment* (cf. [1, 5, 15, 21]). Although they provide a simple conventional model to describe various transport processes in many cases, the medium where the system evolves is highly irregular due to many irregularities (defects, fluctuations, etc.) known as random environments which lead to the choice of the local characteristics of the motion at random according to certain probability distribution. Such models are referred to as *birth–death chains in random environments*. The definition of these chains involves two special ingredients: the *environment* (randomly chosen but still fixed throughout the time evolution) and the *birth–death chain* (whose transition probabilities are determined by the environment) (cf.[18]).

It is also known that *extinction* means the termination of a kind of organism or of a group of kinds usually species. The moment of extinction is generally to be the death of the last individual of the species although the capacity to recover may have been lost before this point. Since a species' potential range may be very large, the determination of this moment is difficult and it is usually done retrospectively ([cf. 16]). Knowing that the discrete-time birth–death chains are special cases of the birth–death processes, they incorporate the possibility of reductions in population of arbitrary size. A central question in the theory of birth–death processes is the probability of ultimate extinction where no individuals exist after some finite number of births–deaths. To this direction, the determination of a measure which possesses logical properties is very fundamental. It is known as the *expected time to extinction* also called *"average extinction time"* or *"mean survival time"*. The expected time to extinction of a population is intimately related to the probability of its occurrence in such a way so as to ensure the validity of certain common inference patterns found optimization, adaptation, and similar types of evolutionary reasoning (cf. [25, 26]).

In parallel, in recent years, a systematic research has been developed (Kalpazidou [20], MacQueen [23], Qian Minping and Qian Min [24], Zemanian [37], and others) in order to investigate representations of the finite-dimensional distributions of Markov processes (with discrete or continuous parameter) having an invariant measure, as decompositions in terms of the *circuit* (or *cycle*) *passage functions*

$$J_c(i,j) = \begin{cases} 1, & \text{if } i, j \text{are consecutive states of} c, \\ 0, & \text{otherwise,} \end{cases}$$

for any directed sequence $c = (i_1, i_2, \ldots, i_v, i_1)$ (or $\hat{c} = (i_1, i_2, \ldots, i_v)$) of states, called a *circuit* (or a *cycle*), $v > 1$, of the corresponding Markov process. This research has stimulated a motivation toward the representation of Markov processes through directed circuits (or cycles) and weights in terms of circuit (or cycle) passage functions in fixed or random environments as well as the study of specific problems associated with Markov processes in a different way. The representations are called *circuit* (or *cycle*) *representations*, while the corresponding discrete parameter Markov chains generated by directed weighted circuits are called *circuit chains*.

More specifically, let S be a denumerable set. The directed sequence $c = (i_1, i_2, \ldots, i_v, i_1)$ modulo the cyclic permutations, where $i_1, i_2, \ldots, i_v \in S$, $v > 1$, completely defines a *directed circuit* in S. The ordered sequence $\hat{c} = (i_1, i_2, \ldots, i_v)$ associated with the given directed circuit c is called a *directed cycle* in S. A directed circuit may be considered as$c = (c(m), c(m+1), \ldots, c(m+v-1), c(m+v))$, if there exists an m $\in Z$, such that $i_1 = c(m+0), i_2 = c(m+1), \ldots, i_v = c(m+v-1)$, $i_1 = c(m+v)$, that is, a periodic function from Z to S. The corresponding directed cycle is defined by the ordered sequence $\hat{c} = (c(m), c(m+1), \ldots, c(m+v-1))$. The values $c(k)$ are the *points* of c, while the directed pairs $(c(k), c(k+1))$, $k \in Z$, are the directed edges of c. The smallest integer $p \equiv p(c) \geq 1$ satisfying the equation $c(m+p) = c(m)$, for all m$\in Z$, is the *period* of c. A directed circuit c such that $p(c) = 1$ is called a *loop*. (In the present work, we shall use directed circuits with distinct point elements.) Let a directed circuit c with period $p(c) > 1$. Then, we may define by

$$J_c^{(n)}(i,j) = \begin{cases} 1, & \text{if there exists an } m \in Z \text{ such that } i = c(m), j = c(m+n), \\ 0, & \text{otherwise,} \end{cases}$$

the *n-step passage function* associated with the directed circuit c, for any i, j$\in S$, n$\in N^* = \{1, 2, \ldots\}$.

We may also define by

$$J_c(i, j) = \begin{cases} 1, & \text{if there exists an m } \in Z \text{ such that} i = c(m), \\ 0, & \text{otherwise,} \end{cases}$$

the *passage function* associated with the directed circuit c, for any i$\in S$. The above definitions are due to MacQueen [23] and Kalpazidou [20].

Given a denumerable set S and an infinite denumerable class C of overlapping directed circuits (or directed cycles) with distinct points (except for the terminals) in S such that all the points of S can be reached from one another following paths of circuit edges, that is, for each two distinct points i and j of S, there exists a finite sequence $c_1, c_2, \ldots, c_k, k \in N^*$, of circuits (or cycles) of C such that i lies on c_1 and j lies on c_k and any pair of consecutive circuits (c_n, c_{n+1}) have at least one point in common. We may assume also that the class C contains, among its elements, circuits (or cycles) with period greater than or equal to 2. With each directed circuit (or directed cycle) $c \in C$, let us associate a strictly *positive weight* w_c which must be independent of the choice of the representative of c, that is, it must satisfy the consistency condition $w_{cot_k} = w_c, k \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, where t_k is the translation of length k.

For a given class C of overlapping directed circuits (or cycles) and for a given sequence $(w_c)_{c\in\mathbb{C}}$ of weights, we may define by

$$p_{ij} = \frac{\sum_{c\in C} w_c \cdot J_c^{(1)}(i, j)}{\sum_{c\in C} w_c \cdot J_c(i)} \tag{1.1}$$

the elements of a Markov transition matrix on S, if and only if $\sum_{c\in C} w_c \cdot J_c(i) < \infty$, for any $i \in S$. This means that a given Markov transition matrix $P = (p_{ij}), i, j \in S$, can be represented by directed circuits (or cycles) and weights if and only if there exist a class of overlapping directed circuits (or cycles) C and a sequence of positive weights $(w_c)_{c\in C}$ such that the formula (1.1) holds. In this case, the representation of the distribution of Markov process (with discrete or continuous parameter) having an invariant measure as decomposition in terms of the *circuit* (or *cycle*) passage functions is called *circuit* (or *cycle*) *representation*, while the corresponding discrete parameter Markov chain generated by directed circuits (or cycles) is called *circuit* (or *cycle*) chain with Markov transition matrix P given by (1.1) and unique stationary distribution p (a solution of $p.\ P = p$) defined by

$$p(i) = \sum_{c\in C} w_c \cdot J_c(i), i \in S.$$

The following classes of Markov chains may be represented uniquely by directed circuits (or cycles) and weights: (1) *the recurrent Markov chains* [24] and (2) *the reversible Markov chains*.

By considering the importance of the study of extinction times for different classes of birth–death processes in general and by using the context of circuit representation theory of Markov processes, the present work arises as an attempt to give approximations of the exact expressions of the mean time to extinction, that is, the mean first passage time to the state $n = 0$ starting at $n = k$, $k \in N^*$, of the corresponding "adjoint" discrete-time Markov chains (circuit chains) describing uniquely the discrete-time birth–death model by directed weighted circuits in random environments giving a new perspective in the study of specific problems associated with birth–death chains [8, 12, 14].

The work is organized as follows. In Sect. 2, the abovementioned discrete-time birth–death model is considered, and the unique representations by directed circuits and weights of the corresponding Markov chains (circuit chains) are investigated in fixed random environments. These representations will give us the possibility to find approximations of the exact expressions of the mean time to extinction through the unique representations by directed circuit and weights of the corresponding "adjoint" Markov chains describing uniquely the discrete-time birth–death model especially in random environments, as it is given in Sect. 3.

Throughout the chapter, we shall need the following notations: $N = \{0, 1, 2, \ldots\}$, $Z_+^* = \{1, 2, 3, \ldots\}$, $Z_-^* = \{\ldots, -2, -1\}$.

2 Circuit and Weight Representations of Discrete-Time Birth–Death Chains

2.1 Fixed Environments

Let us consider the Markov chain $(X_n)_{n\in N}$ on N (X_n expresses the current size of a "population" at discrete instant time n, $n \in N$), which describes a discrete-time birth–death chain in a fixed environment. Since the state transitions are of only two types, that is, $k \to (k+1)$ and $k \to (k-1)$, the elements of the corresponding Markov transition matrix (transition probabilities) are defined by

$$P(X_{n+1} = k + 1/X_n = k) = p_k,$$
$$P(X_{n+1} = k - 1/X_n = k) = q_k,$$

such that $p_k + q_k = 1$, $0 < p_k \leq 1$, for every $k \in N^*$, with $p_0 = 1$, as it is shown in Fig. 1.

Assume that $(p_k)_{k\in N}$ is an arbitrary fixed sequence with $0 < p_k \leq 1$, for every $k \in N^*$, with $p_0 = 1$. If we consider the directed circuits $c_k = (k, k+1, k)$, $k \in N$, and the collection of weights $(w_{c_k})_{k\in N}$, we may obtain the corresponding transition

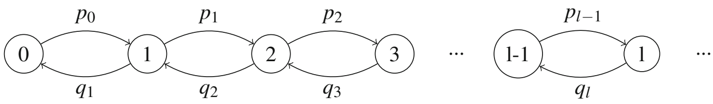

Fig. 1 The Markov chain $(X_n)_{n\in N}$ (fixed environments)

probabilities

$$p_k = \frac{w_{c_k}}{w_{c_{k-1}} + w_{c_k}},$$

$$q_k = 1 - p_k = \frac{w_{c_{k-1}}}{w_{c_{k-1}} + w_{c_k}},$$

with $p_0 = 1$, for every $k \in N^*$. Here, the class C(k) contains the directed circuits $c_k = (k, k+1, k)$, $c_{k-1} = (k-1, k, k-1)$. Equivalently, the transition matrix P=(p_{ij}) with

$$p_{ij} = \frac{\sum_{k=0}^{\infty} w_{c_k} \cdot J_{c_k}^{(1)}(i, j)}{\sum_{k=0}^{\infty} w_{c_k} \cdot J_{c_k}(i)}, \qquad \text{for i} \neq j, \tag{2.1}$$

$$p_{ii} = 0$$

where $J_{c_k}^{(1)}(i, j) = 1$, if i and j are consecutive points of the circuit c_k, and $J_{c_k}(i) = 1$, if i is a point of the circuit c_k,

expresses the representation of the Markov chain $(X_n)_{n\in N}$ by directed circuits and weights.

Similarly, let us consider the *"adjoint"* Markov chain $(X'_n)_{n\in N}$ on N whose elements of the corresponding Markov transition matrix are defined by

$$P(X'_{n+1} = k+1/X'_n = k) = q'_k,$$
$$P(X'_{n+1} = k-1/X'_n = k) = p'_k,$$

such that $p'_k + q'_k = 1, 0 < q'_k \leq 1$, for every $k \in N^*$, with $q'_0 = 1$, as it is shown in Fig. 2.

Assume that $(q'_k)_{k\in N}$ is an arbitrary fixed sequence with $0 < q'_k \leq 1$, for every $k \in N^*$, with $q'_0 = 1$. If we consider the directed circuits $c'_k = (k+1, k, k+1)$, $k \in N$ and the collection of weights $(w_{c'_k})_{k\in N}$, then we may have that

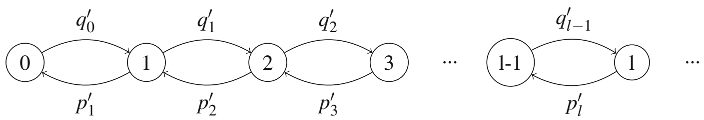

Fig. 2 The "adjoint" Markov chain $(X'_n)_{n\in N}$ (fixed environments)

$$q'_k = \frac{w_{c'_k}}{w_{c'_{k-1}} + w_{c'_k}},$$

$$p'_k = 1 - q'_k = \frac{w_{c'_{k-1}}}{w_{c'_{k-1}} + w_{c'_k}},$$

with $q'_0 = 1$, such that $p'_k + q'_k = 1$, $0 < q'_k \leq 1$, for every $k \in N^*$. Here, the class $C'(k)$ contains the directed circuits $c'_k = (k+1, k, k+1)$, $c'_{k-1} = (k, k-1, k)$. As a consequence, the transition matrix $P' = (p'_{ij})$ with elements equivalent to that given by the abovementioned formulas (2.1) expresses also the representation of the *"adjoint"* Markov chain $(X'_n)_{n\in N}$ by directed circuits and weights.
So, we have the following:

Proposition 1 *The "adjoint" Markov chains $(X_n)_{n\in N}$ and $(X'_n)_{n\in N}$ have unique representations by directed circuits and weights.*

For the proof of the above proposition, see Ganatsiou [13].

2.2 Random Environments

Let us now consider a discrete-time birth–death chain on N with transitions $k \to (k-1)$ and $k \to (k+1)$, whose transition probabilities $(p_k)_{k\in N}$ constitute a stationary ergodic sequence. A realization of this sequence is called a *random environment* for this chain. In order to investigate the unique circuit and weight representation of this chain in random environments, for almost every environment, let us consider a probability space (Ω, F, μ), a measure preserving ergodic automorphism of this space $\theta : \Omega \to \Omega$ and a measurable function $p : \Omega \to (0, 1)$ such that every $\omega \in \Omega$ generates the random environment $p_k \equiv p(\theta^\kappa \omega)$, $k \in N$. Since θ is measure preserving and ergodic, the sequence $(p_k)_{k\in N}$ is a stationary ergodic sequence of random variables.

Let also $S = (N)^N$ be the infinite product space with coordinates $(X_n)_{n\in N}$. Then, we may define a family $(P^\omega)_{\omega\in\Omega}$ of probability measures on S such that, for every $\omega \in \Omega$, the sequence $(X_n)_{n\in N}$ is a Markov chain on N whose elements of the corresponding Markov transition matrix are defined by

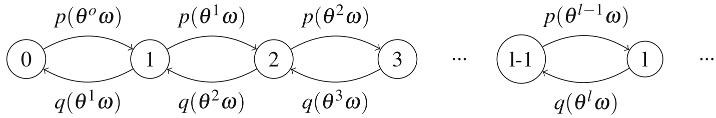

Fig. 3 The Markov chain $(X_n)_{n\in N}$ (random environments)

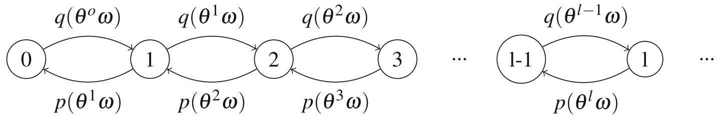

Fig. 4 The "adjoint" Markov chain $(X'_n)_{n\in N}$ (random environments)

$$P^\omega(X_0 = 0) = 1,$$
$$P^\omega(X_{n+1} = k+1/X_n = k) = p(\theta^k\omega),$$
$$P^\omega(X_{n+1} = k-1/X_n = k) = 1 - p(\theta^k\omega) \equiv q(\theta^k\omega), k \in N^*,$$

as it is shown in Fig. 3.

Let us now introduce the *"adjoint"* discrete-time birth–death chain in random environment $(X'_n)_{n\in N}$. For every $\omega \in \Omega$ and for the family $(P^\omega)_{\omega\in\Omega}$ of probability measures on S, the sequence $(X'_n)_{n\in N}$ is a Markov chain on N whose elements of the corresponding Markov transition matrix are defined by

$$P^\omega(X'_0 = 0) = 1,$$
$$P^\omega(X'_{n+1} = k+1/X'_n = k) = q(\theta^k\omega),$$
$$P^\omega(X'_{n+1} = k-1/X'_n = k) = 1 - q(\theta^k\omega) \equiv p(\theta^k\omega), \qquad k \in N^*,$$

as it is shown in Fig. 4.
We have the following (Ganatsiou [14]):

Proposition 2 *For μ-almost every environment $\omega \in \Omega$, the chains $(X_n)_{n\in N}$ and $(X'_n)_{n\in N}$ have unique circuit and weight representations.*

Proof Following an analogous way of that given in subsection 2.1, let us consider the set of directed circuits $c_k = (k, k+1, k)$, for every $k \in N$, since only the transitions from $k \to (k+1)$ and $k \to (k-1)$ are possible. There are two circuits through each point $k \in N$: c_{k-1} and c_k.

In order to define the weights of the circuits c_k, $k \in N$, we may symbolize by $w_k(\omega)$ the weight of the circuit c_k, for every $k \in N$. For the definition of weights, let us consider the sequence $(b_k(\omega))_{k\in N^*}$ defined by

$$b_k(\omega) = \frac{w_k(\omega)}{w_{k-1}(\omega)}, k \in N^*.$$

Consequently, we may have

$$b_k(\omega) = \frac{p(\theta^k\omega)}{1 - p(\theta^k\omega)} = \frac{p(\theta^k\omega)}{q(\theta^k\omega)} \equiv \frac{p}{q}(\theta^k\omega), \qquad k \in N^*. \tag{2.2}$$

Given the stationary ergodic sequence $(p_k)_{k\in N}$, for which every $\omega \in \Omega$ generates the random environment $p_k \equiv p(\theta^\kappa\omega)$, $k \in N$, we have that the preceding equation (2.2) gives a unique definition of the sequence $(b_k(\omega))_{k\in N^*}$, for μ-almost every $\omega \in \Omega$, by ergodicity of θ. Hence, the sequence of weights $(w_k(\omega))_{k\in N^*}$ is defined uniquely by

$$w_k(\omega) = w_0(\omega)b_1(\omega)b_2(\omega)\dots b_k(\omega), \qquad k \in N^*$$

(the unicity of the weight sequence $(w_k(\omega))_{k\in N^*}$ is understood up to the constant factor $w_0(\omega)$).

Similarly, let us consider the set of directed circuits $c_k = (k+1, k, k+1)$, for every $k \in N$, since only the transitions from $k \to (k+1)$ and $k \to (k-1)$ are possible. There are two circuits, through each point $k \in N^*$: c'_{k-1} and c'_k. The problem we have also to manage here is the definition of the weights of the circuits. To this direction, we may denote by $w'_k(\omega)$ the weight of the circuit c'_k, for every $k \in N$. By using an analogous way of that given before for the chain $(X_n)_{n\in N}$, let us consider the sequence $(l_k(\omega))_{k\in N^*}$ defined by

$$\ell_k(\omega) = \frac{w'_{k-1}(\omega)}{w'_k(\omega)}, \qquad k \in N^*,$$

such that

$$l_k(\omega) = \frac{1 - q(\theta^k\omega)}{q(\theta^k\omega)} = \frac{p(\theta^k\omega)}{q(\theta^k\omega)} \equiv \frac{p}{q}(\theta^k\omega), \qquad \text{for every } k \in N^*.$$

Then, the sequence of weights $(w'_k(\omega))_{k\in N^*}$ is defined uniquely by

$$w'_k(\omega) = \frac{w'_0(\omega)}{l_1(\omega)l_2(\omega)\dots l_k(\omega)}, \qquad k \in N^*$$

(the unicity of the weight sequence $(w'_k(\omega))_{k\in N^*}$ is understood up to the constant factor $w'_0(\omega)$).

3 The Mean Time to Extinction of the Discrete-Time Birth–Death Circuit Chains in Random Environments

3.1 For the Circuit Chain $(X_n)_{n \in N}$

Let us consider that the state 0 is a recurrent absorbing state, that is, $p(\theta^0\omega) = 0$, $\omega \in \Omega$. This means that the population cannot recover once it has been extinct (Fig. 3). Let $t_k(\omega)$, $\omega \in \Omega$, be the expected time before the population hits zero, conditioned on an initial population of size k, $k \in N$. We have that

$$t_0(\omega) = 0,\ p(\theta^0\omega) = 0,\ p(\theta^k\omega) + q(\theta^k\omega) = 1, k \in N^*, \omega \in \Omega.$$

Then, we may obtain that

$$t_k(\omega) = p(\theta^k\omega)[1 + t_{k+1}(\omega)] + q(\theta^k\omega)[1 + t_{k-1}(\omega)]$$

or

$$t_{k+1}(\omega) = t_k(\omega) + \frac{q(\theta^k\omega)}{p(\theta^k\omega)}\left[t_k(\omega) - t_{k-1}(\omega) - \frac{1}{q(\theta^k\omega)}\right], k \in N^*, \omega \in \Omega. \tag{3.1}$$

Iterating the above Eq. (3.1), for every $k \in N^*$, $\omega \in \Omega$, and since $t_0(\omega) = 0$, we may have that

$$t_m(\omega) = t_1(\omega) + \sum_{k=1}^{m-1} \frac{q(\theta^1\omega)\ldots q(\theta^k\omega)}{p(\theta^1\omega)\ldots p(\theta^k\omega)}\Bigg[t_1(\omega) - \frac{1}{q(\theta^1\omega)} - \sum_{i=2}^{k} \frac{p(\theta^1\omega)\ldots p(\theta^{i-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^i\omega)}\Bigg], m \geq 2, \omega \in \Omega. \tag{3.2}$$

In order to determine the exact value of $t_1(\omega)$, $\omega \in \Omega$, we modify the circuit chain $(X_n)_{n \in N}$ such that $p(\theta^0\omega) = 1$. Since

$$t_1(\omega) = E(T_0(\omega)) - 1,$$

where $T_0(\omega)$ is the first return time, for every $\omega \in \Omega$, with

$$E(T_0(\omega)) = \frac{1}{\pi_0(\omega)},$$

it remains to determine the exact value of $\pi_0(\omega)$. To this direction, let $\pi_k(\omega)$, $\omega \in \Omega$, $k \in N$, be the stationary distribution of the modified circuit chain $(X_n)_{n \in N}$ satisfying the following relation:

$$\pi_k(\omega) = p(\theta^{k-1}\omega)\pi_{k-1}(\omega) + q(\theta^{k+1}\omega)\pi_{k+1}(\omega), \tag{3.3}$$

with

$$\pi_0(\omega) = q(\theta^1\omega)\pi_1(\omega).$$

By rearranging Eq. (3.3), we may obtain that

$$\pi_{k+1}(\omega) = \frac{p(\theta^k\omega)}{q(\theta^{k+1}\omega)}\pi_k(\omega), k \in N^*, \omega \in \Omega,$$

or equivalently,

$$\pi_k(\omega) = \frac{p(\theta^{k-1}\omega)}{q(\theta^k\omega)}\pi_{k-1}(\omega) = \ldots = \frac{p(\theta^1\omega)\ldots p(\theta^{k-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^k\omega)}\pi_0(\omega), k \in N^*, \omega \in \Omega.$$

Since

$$\sum_{k=0}^{+\infty}\pi_k(\omega) = 1,$$

we obtain that

$$\pi_0(\omega) = \left[1 + \sum_{k=1}^{+\infty}\frac{p(\theta^1\omega)\ldots p(\theta^{k-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^k\omega)}\right]^{-1} \text{ if and only if } \sum_{k=1}^{+\infty}\frac{p(\theta^1\omega)\ldots p(\theta^{k-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^k\omega)} < \infty, \omega \in \Omega.$$

Hence, we have that

$$t_1(\omega) = E(T_0(\omega)) - 1 = \frac{1}{\pi_0(\omega)} - 1 = \frac{1}{q(\theta^1\omega)} + \sum_{k=2}^{+\infty}\frac{p(\theta^1\omega)\ldots p(\theta^{k-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^k\omega)}, \omega \in \Omega.$$

So, finally Eq. (3.2) turns into

$$t_m(\omega) = t_1(\omega) + \sum_{k=1}^{m-1}\left[\frac{q(\theta^1\omega)\ldots q(\theta^k\omega)}{p(\theta^1\omega)\ldots p(\theta^k\omega)}\sum_{i=k+1}^{+\infty}\frac{p(\theta^1\omega)\ldots p(\theta^{i-1}\omega)}{q(\theta^1\omega)\ldots q(\theta^i\omega)}\right], \omega \in \Omega. \tag{3.4}$$

Since

$$b_1(\omega)b_2(\omega)\ldots b_k(\omega) = \frac{p(\theta^1\omega)p(\theta^2\omega)\ldots p(\theta^k\omega)}{q(\theta^1\omega)q(\theta^2\omega)\ldots q(\theta^k\omega)} = \frac{w_k(\omega)}{w_0(\omega)}, k \in N^*, \omega \in \Omega,$$

from relations (3.2) and (3.4), we may obtain that

$$t_m(\omega) = t_1(\omega) + \sum_{k=1}^{m-1} \frac{w_0(\omega)}{w_k(\omega)}[t_1(\omega) - \frac{1}{q(\theta^1\omega)} - \sum_{i=2}^{k} \frac{1}{p(\theta^i\omega)} w_i(\omega)/w_0(\omega)], m \geq 2, \omega \in \Omega, \tag{3.5}$$

or equivalently,

$$t_m(\omega) = t_1(\omega) + \sum_{k=1}^{m-1} \left[\frac{1}{w_k(\omega)} \cdot \sum_{i=k+1}^{+\infty} \frac{1}{p(\theta^i\omega)} w_i(\omega) \right], m \geq 2, \omega \in \Omega. \tag{3.6}$$

Equations (3.5) and (3.6) are suitable expressions of the mean time to extinction $t_m(\omega)$, for every $m \geq 2$, of the discrete-time birth–death chain $(X_n)_{n\in N}$ through its unique representation by the sequences of the directed circuits $(c_k)_{k\in N}$ and weights $(w_{c_k})_{k\in N}$, for every random environment $\omega \in \Omega$.

3.2 For the Circuit Chain $(X'_n)_n$

Following an analogous way of that given in Sect. 3.1 for the circuit chain $(X_n)_{n\in N}$, let us consider that the state 0 is a recurrent absorbing state, that is, $q(\theta^0\omega) = 0$. This means that the population will extinct at some point (Fig. 4). Let also $t'_k(\omega)$, $\omega \in \Omega$, be the expected time before the population hits zero, conditioned on an initial population of size k, $k \in N$. We have that

$$t'_0(\omega) = 0, q(\theta^0\omega) = 0, p(\theta^k\omega) + q(\theta^k\omega) = 1, k \in N^*, \omega \in \Omega.$$

Then, we may take

$$t'_k(\omega) = q(\theta^k\omega)\big[1 + t'_{k+1}(\omega)\big] + p(\theta^k(\omega)\big[1 + t'_{k-1}(\omega)\big]$$

or

$$t'_{k+1}(\omega) = t'_k(\omega) + \frac{p(\theta^k\omega)}{q(\theta^k\omega)}[t'_k(\omega) - t'_{k-1}(\omega) - \frac{1}{p(\theta^k\omega)}], k \in N^*, \omega \in \Omega. \tag{3.7}$$

Iterating Eq. (3.7), for every $k \in N^*$, $\omega \in \Omega$, and since $t'_0(\omega) = 0$, we may have that

$$t'_m(\omega) = t'_1(\omega) + \sum_{k=1}^{m-1} \frac{p(\theta^1\omega)\dots p(\theta^k\omega)}{q(\theta^1\omega)\dots q(\theta^k\omega)} \Bigg[t'_1(\omega) - \frac{1}{p(\theta^1\omega)}$$

$$- \sum_{i=2}^{k} \frac{q(\theta^1\omega)q(\theta^2\omega)\dots q(\theta^{i-1}\omega)}{p(\theta^1\omega)p(\theta^2\omega)\dots p(\theta^i\omega)}\Bigg], m \geq 2, \omega \in \Omega. \tag{3.8}$$

In order to determine the exact value of $t'_1(\omega)$, $\omega \in \Omega$, we modify the circuit chain $(X'_n)_{n\in N}$ such that $q(\theta^0\omega) = 1$. Since

$$t'_1(\omega) = E(T'_0(\omega)) - 1,$$

where $T'_0(\omega)$ is the first return time, for every $\omega \in \Omega$, with

$$E(T'_0(\omega)) = \frac{1}{\pi'_0(\omega)},$$

it remains to determine the exact value of $\pi'_0(\omega)$. To this direction, let $\pi'_k(\omega)$, $\omega \in \Omega$, $k \in N$, be the stationary distribution of the modified circuit chain $(X'_n)_{n\in N}$ satisfying the following relation:

$$\pi'_k(\omega) = q(\theta^{k-1}\omega)\pi'_{k-1}(\omega) + p(\theta^{k+1}\omega)\pi'_{k+1}(\omega), k \in N, \omega \in \Omega$$

with $\pi'_0(\omega) = p(\theta^1\omega)\pi'_1(\omega)$.
Equivalently, we have that

$$\pi'_{k+1}(\omega) = \frac{q(\theta^k\omega)}{p(\theta^{k+1}\omega)}\pi'_k(\omega), k \in N^*, \omega \in \Omega$$

or

$$\pi'_k(\omega) = \frac{q(\theta^{k-1}\omega)}{p(\theta^k\omega)}\pi'_{k-1}(\omega) = \dots = \frac{q(\theta^1\omega)\dots q(\theta^{k-1}\omega)}{p(\theta^1\omega)\dots p(\theta^k\omega)}\pi'_0(\omega), k \in N^*, \omega \in \Omega.$$

Since

$$\sum_{k=0}^{+\infty} \pi'_k(\omega) = 1,$$

we obtain that

$$\pi'_0(\omega) = \left(1 + \sum_{k=1}^{+\infty} \frac{q(\theta^1\omega)\dots q(\theta^{k-1}\omega)}{p(\theta^1\omega)\dots p(\theta^k\omega)}\right)^{-1} \text{ if and only if } \sum_{k=1}^{+\infty} \frac{q(\theta^1\omega)\dots q(\theta^{k-1}\omega)}{p(\theta^1\omega)\dots p(\theta^k\omega)} < +\infty, \omega \in \Omega.$$

Hence, we have that

$$t_1'(\omega) = E(T_0'(\omega))-1 = \frac{1}{\pi_0'(\omega)}-1 = \frac{1}{p(\theta^1\omega)}+\sum_{k=2}^{+\infty}\frac{q(\theta^1\omega)\dots q(\theta^{k-1}\omega)}{p(\theta^1\omega)\dots p(\theta^k\omega)}, \omega \in \Omega.$$

Finally, by substituting $t_1'(\omega)$ in relation (3.8), we may obtain that

$$t_m'(\omega) = t_1'(\omega) + \sum_{k=1}^{m-1}\left[\frac{p(\theta^1\omega)\dots p(\theta^k\omega)}{q(\theta^1\omega)\dots q(\theta^k\omega)}\sum_{i=k+1}^{+\infty}\frac{q(\theta^1\omega)\dots q(\theta^{i-1}\omega)}{p(\theta^1\omega)\dots p(\theta^i\omega)}\right], m \geq 2, \omega \in \Omega. \tag{3.9}$$

Furthermore, since

$$l_1(\omega)l_2(\omega)\dots l_k(\omega) = \frac{p(\theta^1\omega)p(\theta^2\omega)\dots p(\theta^k\omega)}{q(\theta^1\omega)q(\theta^2\omega)\dots q(\theta^k\omega)} = \frac{w_0'(\omega)}{w_k'(\omega)}, k \in N^*, \omega \in \Omega,$$

from relations (3.8) and (3.9), we may obtain that

$$t_m'(\omega) = t_1'(\omega) + \sum_{k=1}^{m-1}\frac{w_0'(\omega)}{w_k'(\omega)}\left[t_1'(\omega) - \frac{1}{p(\theta^1\omega)} - \sum_{i=2}^{k}\frac{1}{q(\theta^i\omega)}w_i'(\omega)/w_0'(\omega)\right], m \geq 2, \omega \in \Omega, \tag{3.10}$$

or equivalently,

$$t_m'(\omega) = t_1'(\omega) + \sum_{k=1}^{m-1}\left[\frac{1}{w_k'(\omega)}\cdot\sum_{i=k+1}^{+\infty}\frac{1}{q(\theta^i\omega)}w_i'(\omega)\right], m \geq 2, \omega \in \Omega. \tag{3.11}$$

Equations (3.10) and (3.11) are suitable expressions of the expected extinction time $t_m'(\omega)$, for every $m \geq 2$, of the discrete-time birth–death chain $(X_n')_{n\in N}$ through its unique representation by the sequences of the directed circuits $(c_k')_{k\in N}$ and weight $(w_{c_k'})_{k\in N}$, for every random environment $\omega \in \Omega$.

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Some Hyperstability Results in Non-Archimedean 2-Banach Space for a σ-Jensen Functional Equation

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Abstract By combining the two versions of Brzdęk's fixed point theorem in non-Archimedean Banach spaces Brzdęk and Ciepliński (Nonlinear Analy 74:6861–6867, 2011) and that in 2-Banach spaces Brzdęk and Ciepliński (Acta Math Sci 38(2):377–390, 2018), we will investigate the hyperstability of the following σ-Jensen functional equation:

$$f(x+y) + f(x+\sigma(y)) = 2f(x),$$

where $f : X \to Y$ such that X is a normed space, Y is a non-Archimedean 2-Banach space, and σ is a homomorphism of X. In addition, we prove some interesting corollaries corresponding to some inhomogeneous outcomes and particular cases of our main results in C^*-algebras.

Mathematics Subject Classification Primary: 39B82; Secondary: 39B62, 47H14, 47J20, 47H10

1 Introduction

The stability of functional equations has been a very popular subject of investigation for the last nearly 80 years. There are published hundreds of papers and many books on very active domain of research (Cf. J.Aczel and J.Dhombre [3], S. Czerwik [21], D.H. Hyers [29], S.-M. Jung [32], Pl. Kannappan [36], Th. M. Rassias [48]). Its main motivation was given by the problem of S. M. Ulam [53] concerning the stability of group homomorphisms that he posed in his lecture delivered in the University of Wisconsin in 1940.

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Ulam's Problem 1940

Given a group G, a metric group H with metric $d(.,.)$, and a positive number ε, does there exists a $\delta > 0$ such that if $f : G \to H$ satisfies:

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then a homomorphism $\Phi : G \to H$ exists with $d(f(x), \Phi(x)) < \varepsilon$ for $x \in G$?

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

In 1941, D. H. Hyers [27] gave a first partial answer to Ulam's question and introduced the stability result as follows:

Theorem 1.1 *[27] Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Afterwards several mathematicians extended this result. The following theorem is one of the most classical result concerning the Hyers–Ulam stability of Cauchy functional equation.

Theorem 1.2 *Let E_1 and E_2 be two normed spaces, $c \geq 0$ and $p \neq 1$ be fixed real numbers. Let $f : E_1 \to E_2$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in E_1 \setminus \{0\}.$$

Then the following statements are valid:

(1) If $p \geq 0$ and E_2 is complete, then there exists a unique additive function $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|}, \quad x \in E_1 \setminus \{0\}.$$

(2) If $p < 0$, then f is additive.

In the case $p = 0$, we get the first answer of Ulam's problem [53] which has been given by Hyers [27]. Moreover, the case $0 < p < 1$ was the contribution of T. Aoki [9]. Also, Z. Gajda [24] proved this result for $p > 1$. Th. M. Rassias [45] proved this result for the case $p < 0$. An example in [24] showed that this result is not true when $p = 1$ thus answering a question that had been posed by Th. M. Rassias. A few years later J.M. Rassias in [46, 47] following the spirit of the approach of Th. M. Rassias [45] studied the case when the sum of two p-norms is replaced by the product of a p-norm with a q-norm.

In 1994, P. Găvruţa [25] following Th. M. Rassias' approach replaced $c\left(\|x\|^p + \|y\|^p\right)$ and $c\left(\|x\|^p\|y\|^q\right)$ by $\varphi(x, y)$ where φ is a map from $E_1 \times E_1$ into $\mathbb{R}^+$. Some extensive account of further results as well as applications and numerous references can be found for example in [1–3, 12, 20, 21, 28, 29, 31–36, 38, 39, 41, 44, 48, 49, 51, 52].

Note that, in Theorem 3.1, for the case $p < 0$, we already have f is additive. This result is known as the hyperstability result, (see [14]). However, the term of hyperstability was introduced for the first time probably in [40], and it was developed with fixed point theorem of Brzdęk in [15] and thereafter, the hyperstability of a several functional equation has been studied by many authors. For more information about the hyperstability, see for example [14–16]. In 2011, Brzdęk et al. [18, 19] proved the existence of fixed point theorem for nonlinear operator. Also, they used this result to study the stability of functional equations in non-Archimedean metric spaces and obtained the fixed point result in arbitrary metric spaces. On the other hand, in 2018, Brzdęk [17] proved the fixed point theorem in 2-Banach spaces and studied the Ulam stability of Cauchy functional equations.

Let X be a vector space, Y be a non-Archimedean 2-Banach space, and let σ be a homomorphism of X such that $\sigma \circ \sigma(x) = x$ for all $x \in X$. A function $f : X \to Y$ is called a σ-Jensen if it satisfies the following functional equation:

$$f(x + y) + f(x + \sigma(y)) = 2f(x). \tag{1.1}$$

P. Sinopoulos [50] proved that the general solution $f : S \to G$ of the Eq. (1.1) is of the form $f(x) = A(x) + \alpha$ where $(S, +)$ is a commutative semi-group, G is a 2-cancellative abelian group, A is an additive function, and $\alpha \in G$ is an arbitrary constant such that $A \circ \sigma = -A$. The functional equation (1.1) is a generalization of the Cauchy–Jensen functional equation. In fact, the Eq. (1.1) becomes a Cauchy–Jensen functional equation by taking $\sigma = -Id$. The Hyers–Ulam stability of the Cauchy–Jensen functional equation has been studied by various authors (cf.[10, 42]).

The purpose of this paper is to exploit the results of Brzdęk's fixed point theorem in 2-Banach spaces [17] and in non-Archimedean Banach spaces [18] to establish the hyperstability of the functional equation (1.1) in non-Archimedean 2-

Banach spaces. Some interesting consequences of our main results concerning the hyperstability in the C^*-algebras will be presented in the last section of this paper.

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{m_0}$ the set of all integers greater than or equal to m_0 ($m_0 \in \mathbb{N}$), $\mathbb{R}_+ = [0, \infty)$, X is a normed space, Y is a non-Archimedean 2-Banach space, σ is a homomorphism of X, and we use X_0 for the set $X \setminus \{0\}$.

We need to recall some basic definitions and properties concerning non-Archimedean 2-normed spaces. For more details (cf. [37]).

2 Background

Note that the theory of the 2-normed spaces was first developed by Gähler [22] in the mid-1960s, while that of 2-Banach spaces was studied later by Gähler [23].

In this section, we need to recall some basic definitions and properties concerning non-Archimedean 2-Banach spaces.

Definition 2.1 By a *non-Archimedean* field, we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$,
(2) $|rs| = |r||s|$,
(3) $|r + s| \leq \max\{|r|, |s|\}$.

The pair $(\mathbb{K}, |.|)$ is called a *valued field.*

Remark 2.2 In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}$.

Example 2.3 In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial valuation*. The most important examples of non-Archimedean field are p-adic numbers which have gained the interest of physicists for their research in some problems deriving from quantum physics, p-adic strings, and superstrings.

Let p be a fixed prime number and x a non-rational number, there exists a unique integer $v_p(x) \in \mathbb{Z}$ such that $x = p^{v_p(x)}\ \frac{a}{b}$ where a and b are integers co-prime to p. The function defined in $\mathbb{Q}$ by $|x|_p = p^{v_p(x)}$ is called a p-adic, an Ultrametric, or simply a non-Archimedean absolute value on $\mathbb{Q}$. The completion, denoted by $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the metric defined by the p-adic absolute is called p-adic numbers.

Definition 2.4 Let X be a vector space (with $dimX > 1$) over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||.,.|| : X^2 \to \mathbb{R}_+$ is called a *non-Archimedean 2-norm (valuation)* if it satisfies the following conditions:

(1) $\|x, y\| = 0$ if and only if x and y are linearly independent, $x, y \in X$,
(2) $\|x, y\| = \|y, x\| \quad x, y \in X$,
(3) $\|rx, y\| = |r|\ \|x, y\| \quad (r \in \mathbb{K}, x, \in X)$,
(4) $\|x, y + z\| \leq \max\left\{\|x, y\|, \|x, z\|\right\} \quad x, y, z \in X$.

Then $(X, \|\cdot, \cdot\|)$ is called *a non-Archimedean 2-normed space* or *an Ultrametric 2-normed space*.

Example 2.5 Let p be a fixed prime number. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we define the non-Archimedean 2-norm in ${\mathbb{Q}_p}^2$ by $\|x, y\|_p = |x_1 y_2 - x_2 y_1|_p$.

Definition 2.6 Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space X.

(1) A sequence$\{x_n\}_{n=1}^{\infty}$ is *a Cauchy sequence* if there are linearly independent $y, z \in X$ such that

$$\lim_{n\to\infty} \|x_{n+1} - x_n, y\| = 0 = \lim_{n\to\infty} \|x_{n+1} - x_n, z\|$$

(2) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ (called limit of this sequence and denoted by $\lim_{n\to\infty} x_n$) such that

$$\lim_{n\to\infty} \|x_n - x, y\| \quad y \in X$$

(3) If every Cauchy sequence in X converges, then the non-Archimedean 2-normed space X is called a *non-Archimedean 2-Banach space* or an *Ultrametric 2-Banach space*.

Lemma 2.7 ([43])

(1) Let X be a non-Archimedean 2-Banach space over a non-Archimedean field $\mathbb{K}$ and $x, y, z \in X$ such that y and z are linearly independent and $\|x, y\| = 0 = \|x, z\|$. Then $x = 0$.
(2) If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of X, then:

$$\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\| \quad y \in X.$$

3 A Fixed Point Theorem

In 2011 and in 2018, J. Brzdęk and K. Ciepliński [17, 18] proved a fixed point theorem in non-Archimedean Banach space and in 2-Banach spaces, respectively.

By combining those two results, we get Theorem 3.1. Before we present this theorem, we need the following hypotheses:

(**H1**) X is a non-empty set, $(Y, \|.,.\|)$ is a non-Archimedean 2-Banach space over a non-Archimedean field, Y_0 is a subset of Y containing two linearly independent vectors, $f_1, \ldots, f_k : X \longrightarrow X$, $g_1, \ldots, g_k : Y_0 \longrightarrow Y_0$, and $L_1, \ldots, L_k : X \times Y_0 \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality:

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\right\| \leq \max_{1\leq i\leq k} \left\{L_i(x, y)\left\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g_i(y)\right\|\right\},$$

$$\xi, \mu \in Y^X, x \in X, y \in Y_0.$$

(**H3**) $\Lambda : \mathbb{R}_+^{X\times Y_0} \longrightarrow \mathbb{R}_+^{X\times Y_0}$ is a non-decreasing linear operator defined by

$$\Lambda\delta(x, y) := \max_{1\leq i\leq k} \left\{L_i(x, y)\delta\big(f_i(x), g_i(y)\big)\right\}, \quad \delta \in \mathbb{R}_+^{X\times Y_0}, \quad x \in X, y \in Y_0.$$

Theorem 3.1 *Let hypotheses (***H1***)–(***H3***) are valid and let $\varepsilon : X \times Y_0 \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ be functions fulfilling the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x), y\| \leq \varepsilon(x, y), \qquad x \in X, y \in Y_0, \tag{3.1}$$

$$\lim_{n\to\infty} \Lambda^n\varepsilon(x, y) = 0, \qquad x \in X, y \in Y_0. \tag{3.2}$$

Then, for every $x \in X$, the limit

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$$

exists and defines a fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x), y\| \leq \sup_{n\in\mathbb{N}_0} \Lambda^n\varepsilon(x, y) = \gamma(x, y), \qquad x \in X, y \in Y_0. \tag{3.3}$$

Moreover, if

$$(\Lambda\gamma)(x, y) \leq \sup_{n\in\mathbb{N}_0} \Lambda^{n+1}\varepsilon(x, y), \qquad x \in X, x \in Y_0, \tag{3.4}$$

then ψ is a unique fixed point of $\mathcal{T}$ satisfying (3.3).

4 Hyperstability of σ-Jensen Functional Equation in Non-archimedean 2-Banach Space

Using the fixed point Theorem 3.1 as a basic tool, we investigate the hyperstability of the σ-Jensen functional equation (1.1) in a non-Archimedean 2-Banach space. In the remaining part of this paper, we assume that X' is a non-empty subset of X.

Theorem 4.1 *Let $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \Big\{ n \in \mathbb{N} : \alpha_n = \max\{\lambda_1(n+1)\lambda_2(n+1)\ ,\ \lambda_1^\sigma(n)\lambda_2^\sigma(n)\} < 1 \Big\},$$

where

$$\lambda_i(n) = \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leq t h_i(x, z)\} \text{ and}$$

$$\lambda_i^\sigma(n) = \inf\{t \in \mathbb{R}_+ : h_i((n+1)x - n\sigma(x), z) \leq t h_i(x, z), \}$$

for all $x \in X'$,$z \in Y$ and $n \in \mathbb{N}$, where $i = 1, 2$ such that

$$\lim_{n\to\infty} \lambda_1(n+1)\lambda_2(-n) = 0.$$

Suppose that $f : X' \to Y$ satisfies the inequality

$$\| f(x+y) + f(x+\sigma(y)) - 2f(x), z\| \leq h_1(x, z)h_2(y, z), \tag{4.1}$$

for all $x, y \in X'$ and $z \in Y$. Then f is a σ-Jensen on X'.

Proof Replacing x by $(m+1)x$ and y by $-mx$ in (4.1) where $m \in \mathcal{U}$, we get

$$\begin{aligned} &\|2f((m+1)x) - f((m+1)x - m\sigma(x)) - f(x), z\| \\ &\leq h_1((m+1)x, z)h_2(-mx, z) \quad x \in X', z \in Y. \end{aligned} \tag{4.2}$$

We consider

$$\mathcal{T}_m\xi(x) := 2\xi((m+1)x) - \xi((m+1)x - m\sigma(x)), \quad \xi \in Y^{X'},\ x \in X',\ z \in Y,$$

and

$$\varepsilon_m(x, z) := h_1((m+1)x, z)h_2(-mx, z), \quad x \in X',\ z \in Y.$$

Therefore, the inequality (4.1) becomes

$$\|\mathcal{T}_m f(x) - f(x), z\| \leq \varepsilon_m(x, z) \ \ x \in X', z \in Y, m \in \mathcal{U}.$$

Now, for each $\xi, \mu \in Y^{X'}$, $x \in X'$, $z \in Y$ and $m \in \mathcal{U}$, we have

$$\begin{aligned}
&\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\| \\
&= \|2\xi((m+1)x) - \xi((m+1)x - m\sigma(x)) - 2\mu((m+1)x) \\
&\quad + \mu((m+1)x - m\sigma(x)), z\| \\
&\le \max\left\{|2|\|\xi((m+1)x) - \mu((m+1)x), z\|,\right. \\
&\quad \left.\|\xi((m+1)x - m\sigma(x)) - \mu((m+1)x - m\sigma(x)), z\|\right\} \\
&\le \max\left\{\|\xi((m+1)x) - \mu((m+1)x), z\|,\right. \\
&\quad \left.\|\xi((m+1)x - m\sigma(x)) - \mu((m+1)x - m\sigma(x)), z\|\right\}.
\end{aligned}$$

Next, we define

$$\begin{aligned}
&\Lambda_m\delta(x, z) \\
&\quad := \max\left\{\delta((m+1)x, z), \delta((m+1)x - m\sigma(x), z)\right\}, \ \delta \in \mathbb{R}_+^{X'\times Y}, \ x \in X', z \in Y.
\end{aligned}$$

Therefore, for each $m \in \mathcal{U}$, the operator $\Lambda := \Lambda_m$ has the form described in (**H3**) with $k = 2$, $f_1(x) = (m+1)x$, $f_2(x) = (m+1)x - m\sigma(x)$, $L_1(x, z) = L_2(x, z) = 1$, $g_i = Id_Y$, $i = 1, 2$ for all $x \in X'$ and $z \in Y$. It is easy to see that

$$\varepsilon_m(x, z) \le \lambda_1(m+1)\lambda_2(-m)h_1(x, z)h_2(x, z), \quad x \in X', z \in Y. \tag{4.3}$$

By induction, we will show that for each $n \in \mathbb{N}_0$

$$\Lambda_m^n\varepsilon_m(x, z) \le \lambda_1(m+1)\lambda_2(-m)\alpha_m^n h_1(x, z)h_2(x, z), \quad x \in X', z \in Y. \tag{4.4}$$

Indeed, for $n = 0$ it is evident that (4.4) is exactly (4.3). Next, we fix $k \in \mathbb{N}$ and assume that (4.4) holds for $n = k$. Then, using the non-decreasing of Λ_m, we have

$$\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x, z) &= \Lambda_m(\Lambda_m^k\varepsilon_m(x, z)) \\
&= \max\{\Lambda_m^k\varepsilon_m((m+1)x, z), \Lambda_m^k\varepsilon_m((m+1)x - m\sigma(x), z)\} \\
&\le \lambda_1(m+1)\lambda_2(-m)\alpha_m^k \max\{h_1((m+1)x, z)h_2((m+1)x, z), \\
&\quad h_1((m+1)x - m\sigma(x), z)h_2((m+1)x - m\sigma(x), z)\} \\
&\le \lambda_1(m+1)\lambda_2(-m)\alpha_m^k h_1(x, z)h_2(x, z) \max\{\lambda_1(m+1)\lambda_2(m+1), \\
&\quad \lambda_1^\sigma(m)\lambda_2^\sigma(m)\} \\
&= \lambda_1(m+1)\lambda_2(-m)\alpha_m^{k+1} h_1(x, z)h_2(x, z),
\end{aligned}$$

for all $x \in X'$ and $z \in Y$. Letting $n \to \infty$ in (4.4), we get

$$\lim_{n\to\infty} \Lambda_m^n \varepsilon_m(x,z) = 0$$

for all $x \in X'$,$z \in Y$ and all $m \in \mathcal{U}$. Then, by applying the Theorem 3.1, there exists, for each $m \in \mathcal{U}$, a fixed point J_m of $\mathcal{T}_m$ such that

$$\|f(x) - J_m(x), z\| \le \sup_{n\in\mathbb{N}_0} \Lambda_m^n \varepsilon_m(x,z), \tag{4.5}$$

for all $x \in X'$ and all $z \in Y$ and

$$\lim_{n\to\infty} \mathcal{T}_m^n f(x) = J_m(x), \quad x \in X'. \tag{4.6}$$

Next, we will show, by induction, that for each $n \in \mathbb{N}_0$

$$\begin{aligned} &\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x+\sigma(y)) - 2\mathcal{T}_m^n f(x), z\| \\ &\quad \le \alpha_m^n h_1(x,z) h_2(y,z), \quad x,y \in X', z \in Y. \end{aligned} \tag{4.7}$$

Since the case $n = 0$ is just (4.1), we fix $k \in \mathbb{N}$ and suppose that (4.7) holds for $n = k$. Then, for all $x, y \in X'$ and $z \in Y$, we have

$$\begin{aligned}
&\|\mathcal{T}_m^{k+1} f(x+y) + \mathcal{T}_m^{k+1} f(x+\sigma(y)) - 2\mathcal{T}_m^{k+1} f(x), z\| \\
&= \|\mathcal{T}_m\left(\mathcal{T}_m^k f(x+y)\right) + \mathcal{T}_m\left(\mathcal{T}_m^k f(x+\sigma(y))\right) - 2\mathcal{T}_m\left(\mathcal{T}_m^k f(x)\right), z\| \\
&= \|2\mathcal{T}_m^k f((m+1)(x+y)) - \mathcal{T}_m^k f((m+1)(x+y) - m\sigma(x+y)) \\
&\quad + 2\mathcal{T}_m^k f((m+1)(x+\sigma(y))) \\
&\quad - \mathcal{T}_m^k f(((m+1)(x+\sigma(y))) - m(x+\sigma(y))) - 4\mathcal{T}_m^k f((m+1)x) \\
&\quad + 2\mathcal{T}_m^k f((m+1)x - m\sigma(x)), z\| \\
&\le \max\{|2|\|\mathcal{T}_m^k f((m+1)(x+y)) \\
&\quad + \mathcal{T}_m^k f((m+1)(x+\sigma(y))) - 2\mathcal{T}_m^k f((m+1)x), z\|; \\
&\|\mathcal{T}_m^k f((m+1)(x+y) - m\sigma(x+y)) \\
&\quad + \mathcal{T}_m^k f((m+1)(x+\sigma(y)) - m\sigma(x+\sigma(y))) \\
&\quad - 2\mathcal{T}_m^k f((m+1)x - m\sigma(x)), z\|\} \\
&\le \max\{\|\mathcal{T}_m^k f((m+1)(x+y)) \\
&\quad + \mathcal{T}_m^k f((m+1)(x+\sigma(y))) - 2\mathcal{T}_m^k f((m+1)x), z\|; \\
&\|\mathcal{T}_m^k f((m+1)(x+y) - m\sigma(x+y))
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{T}_m^k f((m+1)(x+\sigma(y))-m\sigma(x+\sigma(y)))\\
&-2\mathcal{T}_m^k f((m+1)x-m\sigma(x)),z\|\}\\
\le\ & \alpha_m^k \max\{h_1((m+1)x,z)h_2((m+1)y,z);\\
& h_1((m+1)x-m\sigma(x),z)h_2((m+1)y-m\sigma(y),z)\}\\
\le\ & \alpha_m^k h_1(x,z)h_2(y,z)\max\{\lambda_1(m+1)\lambda_2(m+1);\lambda_1^\sigma(m)\lambda_2^\sigma(m)\}\\
=\ & \alpha_m^{k+1}h_1(x,z)h_2(y,z).
\end{aligned}
$$

Thus, we have shown that (4.7) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (4.7), we obtain, for each $m \in \mathcal{U}$, that

$$J_m(x+y)+J_m(x+\sigma(y))=2J_m(x), \quad x\in X'.$$

In this way, we find a sequence $\{J_m\}_{m\in\mathcal{U}}$ of a σ-Jensen functions on X' such that

$$
\begin{aligned}
\|f(x)-J_m(x),z\| &\le \sup_{n\in\mathbb{N}}\{\lambda_1(m+1)\lambda_2(-m)\alpha_m^n h_1(x,z)h_2(x,z)\}\\
&\le \lambda_1(m+1)\lambda_2(-m)h_1(x,z)h_2(x,z)\sup_{n\in\mathbb{N}}\{\alpha_m^n\},
\end{aligned}
$$

for all $x\in X'$, $z\in Y$, and $m\in\mathcal{U}$. It follows, with $m\to\infty$, that f is σ-Jensen on X'. □

By similar method, we prove the following theorem.

Theorem 4.2 *Let $h : X'\times Y \to \mathbb{R}_+$ be a function such that*

$$\mathcal{U} := \left\{n\in\mathbb{N} : \alpha_n = \max\left\{\lambda(n+1)\ ,\ \lambda^\sigma(n)\right\} < 1\right\},$$

where

$$
\begin{aligned}
\lambda(n) &:= \inf\{t\in\mathbb{R}_+ : h(nx,z)\le th(x,z),\quad x\in X',\ z\in Y\} \quad \text{and}\\
\lambda(n)^\sigma &:= \inf\{t\in\mathbb{R}_+ : h((n+1)x-n\sigma(x),z)\le th(x,z),\quad x\in X',\ z\in Y\}
\end{aligned}
$$

for all $n\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\lambda(n+1)+\lambda(-n)=0.$$

Suppose that $f : X'\to Y$ satisfies the inequality

$$\|f(x+y)+f(x+\sigma(y))-2f(x),z\|\le h(x,z)+h(y,z),\quad x,y\in X',\quad z\in Y. \tag{4.8}$$

Then f is a σ-Jensen on X'.

Proof Replacing x by $(m+1)x$ and y by $-mx$ where $m \in \mathcal{U}$ in (4.8), we get

$$\|2f((m+1)x) - f((m+1)x - m\sigma(x)) - f(x), z\| \leq h((m+1)x, z) + h(-mx, z),$$
$$x \in X', \; z \in Y. \tag{4.9}$$

For each $m \in \mathcal{U}$, we define the operator $\mathcal{T}_m : Y^{X'} \to Y^{X'}$ and the function $\varepsilon : X' \times Y \to \mathbb{R}_+$ by

$$\mathcal{T}_m \xi(x) := 2\xi((m+1)x) - \xi((m+1)x - m\sigma(x)), \quad \xi \in Y^{X'}, \; x \in X'.$$
$$\varepsilon_m(x, z) = h((m+1)x, z) + h(-mx, z) \quad x \in X', \quad z \in Y.$$

Then the inequality (4.9) takes the form

$$\|\mathcal{T}_m f(x) - f(x), z\| \leq \varepsilon_m(x, z), \quad x \in X', \; z \in Y.$$

Furthermore, for every $\xi, \mu \in Y^{X'}$, $x \in X'$, $z \in Y$, and $m \in \mathcal{U}$, we obtain

$$\begin{aligned}
&\|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x), z\| \\
&= \|2\xi((m+1)x) - \xi((m+1)x - m\sigma(x)) - 2\mu((m+1)x) \\
&\quad + \mu((m+1)x - m\sigma(x)), z\| \\
&\leq \max\big\{|2| \|\xi((m+1)x) - \mu((m+1)x), z\|, \\
&\quad \|\xi((m+1)x - m\sigma(x)) - \mu((m+1)x - m\sigma(x)), z\|\big\} \\
&\leq \max\big\{\|\xi((m+1)x) - \mu((m+1)x), z\|, \\
&\quad \|\xi((m+1)x - m\sigma(x)) - \mu((m+1)x - m\sigma(x)), z\|\big\}.
\end{aligned}$$

Take

$$\Lambda_m \delta(x, z) := \max\big\{\delta((m+1)x, z), \delta((m+1)x - m\sigma(x), z)\big\}, \quad \delta \in \mathbb{R}_+^{X' \times Y},$$
$$x \in X', z \in Y.$$

Therefore, for each $m \in \mathcal{U}$, the operator $\Lambda := \Lambda_m$ has the form described in (**H3**) with $k = 2$, $f_1(x) = (m+1)x$, $f_2(x) = (m+1)x - m\sigma(x)$, $L_1(x, z) = L_2(x, z) = 1$, $g_i = Id_Y$, $i = 1, 2$ for all $x \in X'$ and $z \in Y$. Observe that

$$\varepsilon_m(x, z) \leq \big(\lambda(m+1) + \lambda(-m)\big) h(x, z) \quad x \in X', \quad z \in \quad m \in \mathcal{U}. \tag{4.10}$$

By using mathematical induction we will show that for each $n \in \mathbb{N}_0$

$$\Lambda_m^n \varepsilon_m(x, z) \leq \Big(\lambda(m+1) + \lambda(-m)\Big) \alpha_m^n h(x, z), \quad x \in X' \; z \in Y, \tag{4.11}$$

for all $m \in \mathcal{U}$. From (4.10), we obtain that the inequality (4.11) holds for $n = 0$. Next, we will assume that (4.11) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x,z) &= \Lambda_m\big(\Lambda_m^k\varepsilon_m(x,z)\big)\\ &= \max\Big\{\Lambda_m^k\varepsilon_m\big((m+1)x,z\big)\ ,\ \Lambda_m^k\varepsilon_m\big((m+1)x-m\sigma(x),z\big)\Big\}\\ &\le \big(\lambda(m+1)+\lambda(-m)\big)\alpha_m^k\\ &\quad\times\max\{h((m+1)x,z)\ ,\ h((m+1)x-m\sigma(x),z)\}\\ &\le \big(\lambda(m+1)+\lambda(-m)\big)\alpha_m^k h(x,z)\ \max\big\{\lambda(m+1)\ ,\ \lambda^{\sigma}(m)\big\}\\ &\le \Big(\lambda(m+1)+\lambda(-m)\Big)\alpha_m^{k+1}h(x,z),\end{aligned}$$

for all $x \in X'$, $z \in Y$, and $m \in \mathcal{U}$.

This shows that (4.11) holds for $n = k + 1$. Now, we can conclude that the inequality (4.11) holds for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (4.11), we obtain

$$\lim_{n\to\infty}\Lambda^n\varepsilon_m(x,z)=0,$$

for all $x \in X'$, $z \in Y$ and all $m \in \mathcal{U}$. Hence, according to Theorem 3.1, there exists, for each $m \in \mathcal{U}$, a fixed point J_m of the operator $\mathcal{T}_m$ such that

$$\|f(x)-J_m(x),z\| \le \sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m(x,z), \tag{4.12}$$

for all $x \in X'$ and all $z \in Y$ and

$$\lim_{n\to\infty}\mathcal{T}_m^n f(x)=J_m(x),\quad x\in X'. \tag{4.13}$$

Next, we will show, by induction, that for each $n \in \mathbb{N}_0$

$$\|\mathcal{T}_m^n f(x+y)+\mathcal{T}_m^n f(x+\sigma(y))-2\mathcal{T}_m^n f(x),z\| \le \alpha_m^n\big(h(x,z)+h(y,z)\big), \tag{4.14}$$

for all $x, y \in X'$, $z \in Y$ and all $m \in \mathcal{U}$.

Since the case $n = 0$ is just (4.8), so we fix $k \in \mathbb{N}$ and suppose that (4.14) holds for $n = k$. Then, for all $x, y \in X'$ and $z \in Y$ we have

$$\begin{aligned}&\|\mathcal{T}_m^{k+1}f(x+y)+\mathcal{T}_m^{k+1}f(x+\sigma(y))-2\mathcal{T}_m^{k+1}f(x),z\|\\ &=\|\mathcal{T}_m\Big(\mathcal{T}_m^k f(x+y)\Big)+\mathcal{T}_m\Big(\mathcal{T}_m^k f(x+\sigma(y))\Big)-2\mathcal{T}_m\Big(\mathcal{T}_m^k f(x)\Big),z\|\\ &=\|2\mathcal{T}_m^k f((m+1)(x+y))-\mathcal{T}_m^k f((m+1)(x+y)-m\sigma(x+y))\\ &\quad+2\mathcal{T}_m^k f((m+1)(x+\sigma(y)))\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{T}_m^k f(((m+1)(x+\sigma(y))) - m(x+\sigma(y))) - 4\mathcal{T}_m^k f((m+1)x)\\
&+ 2\mathcal{T}_m^k f((m+1)x - m\sigma(x)), z\|\\
\le{}& \max\{|2|\|\mathcal{T}_m^k f((m+1)(x+y)) + \mathcal{T}_m^k f((m+1)(x+\sigma(y)))\\
&- 2\mathcal{T}_m^k f((m+1)x), z\|;\\
\|&\mathcal{T}_m^k f((m+1)(x+y) - m\sigma(x+y)) + \mathcal{T}_m^k f((m+1)(x+\sigma(y)) - m\sigma(x+\sigma(y)))\\
&- 2\mathcal{T}_m^k f((m+1)x - m\sigma(x)), z\|\}\\
\le{}& \max\{\|\mathcal{T}_m^k f((m+1)(x+y)) + \mathcal{T}_m^k f((m+1)(x+\sigma(y)))\\
&- 2\mathcal{T}_m^k f((m+1)x), z\|;\\
\|&\mathcal{T}_m^k f((m+1)(x+y) - m\sigma(x+y)) + \mathcal{T}_m^k f((m+1)(x+\sigma(y)) - m\sigma(x+\sigma(y)))\\
&- 2\mathcal{T}_m^k f((m+1)x - m\sigma(x)), z\|\}\\
\le{}& \alpha_m^k \max\{h((m+1)x, z) + h((m+1)y, z); h((m+1)x - m\sigma(x), z)\\
&+ h((m+1)y - m\sigma(y), z)\}\\
\le{}& \alpha_m^k \big(h(x,z) + h(y,z)\big) \max\{\lambda(m+1); \lambda^\sigma(m)\}\\
={}& \alpha_m^{k+1}\big(h(x,z) + h(y,z)\big).
\end{aligned}
$$

Thus, we have shown that (4.14) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (4.14), we obtain, for each $m \in \mathcal{U}$, that

$$J_m(x+y) + J_m(x+\sigma(y)) = 2J_m(x), \quad x \in X'.$$

In this way, we find a sequence $\{J_m\}_{m\in\mathcal{U}}$ of σ-Jensen functions on X' such that

$$\|f(x) - J_m(x), z\| \le \sup_{n\in\mathbb{N}}\{\big(\lambda(m+1) + \lambda(-m)\big)\alpha_m^n\big(h(x,z) + h(y,z)\big)\}, \quad x \in X', z \in Y.$$

It follows, with $m \to \infty$, that f is σ-Jensen on X'. □

Theorems 4.1 and 4.2 imply, as particular cases, the following two corollaries concerning the inhomogeneity of the σ-Jensen functional equation.

Corollary 4.3 *Let $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ be two functions, σ be a homomorphism of X, $G : X \times X \to Y$ be a function such that $G(0,0) = 0$, and $f : X \to Y$ be a function. Assume that f, G satisfy the inequality:*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y), z\| \le h_1(x,z)h_2(y,z), \tag{4.15}$$

for all $x, y \in X'$ and $z \in Y$. If the functional equation

$$f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y) = 0, \tag{4.16}$$

has a solution $f_0 : X \to Y$, then f is a solution of the functional equation (4.16) on X.

Proof Let be $\varphi : X \to Y$ be a function defined by $\varphi(x) = f(x) - f_0(x)$ for all $x \in X$, then for all $x, y \in X'$ and $z \in Y$ we have

$$\begin{aligned}
&\|\varphi(x+y) + \varphi(x+\sigma(y)) - 2\varphi(x), z\| \\
&= \|f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y) - f_0(x+y) \\
&- f_0(x+\sigma(y)) + 2f_0(x) + G(x,y), z\| \\
&= \|f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y), z\| \\
&\leq h_1(x,z)h_2(y,z),
\end{aligned}$$

thus φ is σ-Jensen on X'. Moreover, for all $x, y \in X'$ we get

$$\begin{aligned}
&f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y) \\
&= \varphi(x+y) + \varphi(x+\sigma(y)) - 2\varphi(x) + f_0(x+y) \\
&\quad + f_0(x+\sigma(y)) - 2f_0(x) - G(x,y) \\
&= 0,
\end{aligned}$$

which means that f is a solution of the functional equation (4.16) on X. □

With an analogous proof of Corollary 4.3, we present the following corollary.

Corollary 4.4 *Let $h : X' \times Y \to \mathbb{R}_+$ be two functions, σ be a homomorphism of X, $G : X \times X \to Y$ be a function such that $G(0,0) = 0$, and $f : X \to Y$ be a function. Assume that f, G satisfy the inequality:*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y), z\| \leq h(x,z) + h(y,z), \tag{4.17}$$

for all $x, y \in X'$ and $z \in Y$. If the functional equation

$$f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y) = 0, \tag{4.18}$$

has a solution $f_0 : X \to Y$, then f is a solution of the functional equation (4.18) on X.

5 Applications

In this section, we take $(X, \|.\|_X)$ as a normed space and $X' = X \setminus \{0\}$. Let $h_1(x,z) = c_1 \|x\|_X^p \|z,w\|^{r_1}$, $h_2(x,z) = c_2 \|x\|_X^q \|z,w\|^{r_2}$, and $h(x,z) = c\|x\|_X^p \|z,w\|^r$ for all $x \in X'$ and $z, w \in Y$ where $c_1, c_2, c \geq 0$, $r_1, r_2, r > 0$ and $p, q \in \mathbb{R}$.

In the case when σ is a homomorphism of X such that $\sigma = -id_X$ with X is a real or complex vector space, we get the Cauchy–Jensen functional equation

$$f(x+y) + f(x-y) = 2f(x) \quad x, y \in X.$$

From Theorem 4.1 and Theorem 4.2, we derive the following corollaries.

Corollary 5.1 *Let Y be a non-Archimedean 2-Banach space. Assume that a function $f : X' \to Y$ verifies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x), z\| \leq c\|x\|_X^p \|y\|_X^q \|z,w\|^r \tag{5.1}$$

for all $x, y \in X'$ and $z, w \in Y$ with $c = c_1 \times c_2 \geq 0$, $p+q < 0$ and $r = r_1 + r_2 > 0$. Then $f(x) = A(x) + a$, $x \in X'$, where A is an additive map $X' \to Y$ and $a \in Y$ is an arbitrary constant.

Proof For each $m \in \mathbb{N}$ we define $\lambda_1(m+1)$ and $\lambda_1^\sigma(m)$ as in Theorem 4.1

$$\begin{aligned}
\lambda_1(m+1) &= \inf\{t \in \mathbb{R}_+ : h_1((m+1)x, z) \leq t h_1(x,z)\} \qquad x, z \in X' \\
&= \inf\big\{t \in \mathbb{R}_+ : c_1\|(m+1)x\|_X^p \|z,w\|^{r_1} \\
&\leq tc_1 \|x\|_X^p \|z,w\|^{r_1}\big\} \qquad x \in X', z, w \in Y \\
&= \inf\big\{t \in \mathbb{R}_+ : |m+1|^p \|x\|_X^p \|z,w\|^{r_1} \\
&\leq t\|x\|_X^p \|z,w\|^{r_1}\big\} \qquad x \in X', z, w \in Y \\
&= |m+1|^p,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1^\sigma(m) &= \inf\{t \in \mathbb{R}_+ : h_1((m+1)x - m\sigma(x), z) \leq t h_1(x,z)\} \qquad x, z \in X' \\
&= \inf\big\{t \in \mathbb{R}_+ : c_1\|(2m+1)x\|_X^p \|z,w\|^{r_1} \\
&\leq tc_1 \|x\|_X^p \|z,w\|^{r_1}\big\} \qquad x \in X', z, w \in Y \\
&= \inf\big\{t \in \mathbb{R}_+ : |2m+1|^p \|x\|_X^p \|z,w\|^{r_1} \\
&\leq t\|x\|_X^p \|z,w\|^{r_1}\big\} \qquad x \in X', z, w \in Y \\
&= |2m+1|^p,
\end{aligned}$$

also, for $m \in \mathbb{N}$ we have $\lambda_2(-m) = |m+1|^q$ and $\lambda_2^\sigma(m) = |2m+1|^q$. Then for $m \in \mathbb{N}$ we obtain:

$$\begin{aligned}\alpha_m &= \max\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1^\sigma(m)\lambda_2^\sigma(m)\}\\ &= \max\{|m+1|^{p+q}\ ,\ |2m+1|^{p+q}\} < 1.\end{aligned}$$

On the other hand, since $p+q < 0$, then

$$\lim_{m\to\infty} \lambda_1(m+1)\lambda_2(-m) = \lim_{m\to\infty} |m+1|^{p+q} = 0.$$

Applying theorem 4.1, we get that f is a Cauchy–Jensen on X. □

By a similar proof, we have the following corollary:

Corollary 5.2 *Let Y be a non-Archimedean 2-Banach space. Assume that a function $f : X' \to Y$ verifies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x), z\| \le c(\|x\|_X^p + \|y\|_X^p)\,\|z, w\|^r \tag{5.2}$$

for all $x, y \in X'$ and $z, w \in Y$ with $c \ge 0$, $p < 0$ and $r > 0$. Then $f(x) = A(x) + a$, $x \in X'$, where A is an additive map $X' \to Y$ and $a \in Y$ is an arbitrary constant.

Corollary 5.3 *Let $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ be two functions, $G : X \times X \to Y$ be a function such that $G(0,0) = 0$, and $f : X \to Y$ be a function. Assume that f, G satisfy the inequality:*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - G(x,y), z\| \le c\,\|x\|_X^p\,\|y\|_X^q\,\|z, w\|^r\ ,\\ &\quad \text{or } c(\|x\|_X^s + \|y\|_X^s)\,\|z, w\|^r\end{aligned} \tag{5.3}$$

for all $x, y \in X'$ and $z, w \in Y$ with $c \ge 0$, $p+q < 0, s < 0$, and $r > 0$. If the functional equation

$$f(x+y) + f(x+\sigma(y)) - 2f(x) - G(x,y) = 0, \tag{5.4}$$

has a solution $f_0 : X \to Y$ then f is a solution of functional equation (5.4) on X.

In the following corollaries, we investigate some hyperstability results in a C^*-algebra. We know that a C^*-algebra X is a Banach algebra $(X, \|.\|_X)$ over a $\mathbb{C}$ equipped with an involution $*$ satisfying the C^*-identities

$$\left\|x^*x\right\|_X = \|x\|_X^2 \quad \text{and} \quad \left\|x^*\right\|_X = \|x\|_X\ , \quad x \in X.$$

Take the homomorphism σ of X as follows $\sigma(x) = x^*$, we get this functional equation

$$f(x+y)+f(x+y^*)=2f(x), \quad x,y\in X. \tag{5.5}$$

The solution of this equation is as follows $f(x)=A(x)+a, \quad x\in X$, where A is an additive map $X\to Y$ and $a\in Y$ is an arbitrary constant such that $A(x^*)=-A(x)$.

Corollary 5.4 *Let Y be a non-Archimedean 2-Banach space. Assume that a function $f:X'\to Y$ verifies the inequality*

$$\|f(x+y)+f(x+y^*)-2f(x),z\|\le c\,\|x\|_X^p\,\|y\|_X^q\,\|z,w\|^r\,, \tag{5.6}$$

for all $x,y\in X'$ and $z,w\in Y$ with $c=c_1\times c_2\ge 0$, $p+q<0$ and $r=r_1+r_2>0$. Then $f(x)=A(x)+a, \quad x\in X$, where A is an additive map $X\to Y$ and $a\in Y$ is an arbitrary constant such that $A(x^*)=-A(x)$.

Proof Reformulate (5.1) as

$$\|f(x+y)+f(x-y)-2f(x)-G(x,y),z\|\le c\,\|x\|_X^p\,\|y\|_X^q\,\|z,w\|^r\,,$$

where $G(x,y)=f(x-y)-f(x+y^*)$, $x,y\in X$. Observe that $G(0,0)=0$, and the functional equation (5.5) has a solution, so by corollary 5.3, we get the desired result. □

Corollary 5.5 *Let Y be a non-Archimedean 2-Banach space. Assume that a function $f:X'\to Y$ verifies the inequality*

$$\|f(x+y)+f(x+y)-2f(x),z\|\le c(\|x\|_X^p+\|y\|_X^p)\,\|z,w\|^r\,,$$

for all $x,y\in X'$ and $z,w\in Y$ with $c\ge 0$, $p<0$ and $r>0$. Then $f(x)=A(x)+a, \quad x\in X$, where A is an additive map $X\to Y$ and $a\in Y$ is an arbitrary constant such that $A(x^*)=-A(x)$.

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A Characterization for the Validity of the Hermite–Hadamard Inequality on a Simplex

Allal Guessab

Abstract We consider the d-dimensional Hermite–Hadamard inequality

$$\frac{1}{|S|}\int_S f(\boldsymbol{x})\,d\boldsymbol{x} \le Q^{\mathrm{tra}}(f) := \frac{1}{|\partial S|}\int\limits_{\partial S} f(\boldsymbol{x})d\gamma. \tag{1}$$

Here f is a convex function defined on a simplex $S \subset \mathbb{R}^d$, $(d \in \mathbb{N})$. We give necessary and sufficient conditions on S for the validity of (1). More specifically, we establish that (1) holds if and only if S is an equiareal simplex. We will give two proofs of this result:

- The first proof is based on Green's identity. Here, in addition to the convexity requirement, the C^1-regularity assumption is necessary.
- In the second proof, the convexity is only required.

A series of equivalent criteria for validity of (1) is simply reformulated in terms of coincidences of certain simplex centers.

1 Introduction, Motivation, and Problem Setting

The classical right-hand side of the Hermite–Hadamard inequality, with which we teach our calculus students to obtain a numerical approximation to definite integrals, provides an upper estimation for the integral of any convex function defined on a compact interval of real numbers. More precisely, it can be stated in the (maybe) more familiar form. If f is a convex function on $[a, b]$, then

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$$\frac{1}{b-a}\int_a^b f(x)\,dx \le Q^{\mathrm{tra}}(f) = \frac{1}{2}\,(f(a)+f(b))\,. \tag{2}$$

Hermite–Hadamard's inequality (2) has been generalized or modified in many other directions, we refer to the monograph of Dragomir and Pearce [8]. In the univariate case, this inequality is reasonably well-studied and understood. However, in higher dimensions, relatively little work has been done for a multivariate version of inequality (2); we refer to [16–18] and the further references given there. We now consider a simplex $S \subset \mathbb{R}^d (d \in \mathbb{N})$. By ∂S we denote the boundary of S, and $|S|$ denotes the d-dimensional volume of S, namely area in two dimensions and volume in high dimensions. The main purpose of this chapter is to derive a characterization of the simplices S, for which the d-dimensional Hermite–Hadamard inequality (1) is valid. It is shown that (1) holds if and only if S is an equiareal simplex. We will offer two proofs of this result, both relying on new characterizations of equiareal simplices. While in the first proof the C^1-regularity on the functions is needed for proving the validity of (1), the second one requires the convexity property only.

Some motivation may be helpful. The functional $Q^{\mathrm{tra}}(f)$, defined in (1), is a natural multivariate version of the classical trapezoidal rule, hence it can serve as a cubature formula (multidimensional integration formula) for the approximation of the exact value of the integral $\int_S f(\boldsymbol{x})\,d\boldsymbol{x}$. We call it the trapezoidal cubature formula. Let us recall here that most of classical numerical methods for approximation of a multivariate function (or integrals of it) use the values of function at some nodes. However, as described in [1, 16–18], in many practical problems, the available data is not restricted by function evaluations, but contains also a number of integrals over certain hyperplane sections, or more generally, over certain domains defined implicitly by an indicator function such as the level set function. In such cases, generalizations of the existing theory and algorithms of numerical integration are required. The motivation for discussing such fundamental issues arises in a variety of cases since the data obtained in measurements in number of applications contains the mean values of functions over some surfaces. This type of data is inherent to computer tomography and it is widely used in geology, radiology, medicine, etc. The mathematical foundation behind these techniques is the so-called Radon transform [33], which integrates a function over a set of spheres with a given set of centers. It is worth noting that the cubature formulas developed in this chapter can be also used to construct numerical algorithms for solving partial differential equations with the error control. Such methods of integration are needed in finite element methods and algorithms for solving inverse problems. Here we refer the interested reader to [5, 10, 31, 32]. The papers [19, 31] show the importance of the cubature formulas of this type in nonconforming finite elements.

This chapter, except for the introduction, is divided into five sections. In Sect. 2, we introduce some notations and present some of the essential properties of barycentric coordinates. We also provide a closed form formula for these coordinates. We state the definition of admissible simplices for (1) and give necessary conditions for a simplex to be valid for it. Under convexity assumption and C^1-regularity of the functions involved, Sect. 3 gives several characterization results for a simplex to be valid for (1). Section 4 solves the same problem, but this time under the convexity assumption only. Section 5, where, based on a general characterization result, see Theorem 6, we have established the explicit expressions of the best constants, which appear in the error estimates for our new cubature formulas, see Corollary 1. Finally, in Sect. 6, we give some numerical examples to illustrate the proposed methodology.

2 Auxiliary Results

In this section, we recall some definitions and introduce some notations which we shall use in the sequel. We then present some of the essential properties of barycentric coordinates. We provide a closed form formula for these coordinates and collect some necessary technical lemmas. Let σ^d be a simplex, say $\sigma^d = \operatorname{conv}[\boldsymbol{v}_0\boldsymbol{v}_1 \dots \boldsymbol{v}_d]$. A superscript for a simplex indicates the dimension, so, for example, σ^d is a d-simplex. The notation is used to indicate that σ^d is the convex hull of the $(d+1)$ affinely independent points (or position vectors) $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_d$ in $\mathbb{R}^d$. The points $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_d$ are the vertices. As a matter of convention, the $(d-1)$-faces usually are denoted by $\tau_0^{d-1}, \tau_1^{d-1}, \dots, \tau_d^{d-1}$ with the understanding that face τ_i^{d-1} is opposite to the vertex $\boldsymbol{v}_i$. Every face that is an $(d-1)$-simplex is called a boundary face. The union of the boundary faces is the boundary of σ^d, denoted by $\partial\sigma^d$.

When dealing with a simplex it is often convenient to use barycentric coordinates. For all $\boldsymbol{x} \in \sigma^d[\boldsymbol{v}_0\boldsymbol{v}_1 \dots \boldsymbol{v}_d]$ and all $i = 0, \dots, d$, we denote by $\lambda_i(\boldsymbol{x})$ the components of vector $\boldsymbol{x} - \boldsymbol{v}_0$ in basis $\{\boldsymbol{v}_1 - \boldsymbol{v}_0, \dots, \boldsymbol{v}_d - \boldsymbol{v}_0\}$, i.e.

$$\boldsymbol{x} - \boldsymbol{v}_0 = \sum_{i=1}^{d} \lambda_i(\boldsymbol{x})\,(\boldsymbol{v}_i - \boldsymbol{v}_0)\,. \tag{3}$$

To allow all the vertices of σ^d to play a symmetric role, we introduce the additional function

$$\lambda_0(\boldsymbol{x}) := 1 - \sum_{i=1}^{d} \lambda_i(\boldsymbol{x}). \tag{4}$$

A simple inspection of (3) and (4) reveals that coefficients $\lambda_i(\boldsymbol{x}),\ i = 0, \dots, d$, are uniquely determined, they are affine functions of $\boldsymbol{x}$, and they satisfy the following three properties for a point $\boldsymbol{x}$ inside σ^d:

$$\lambda_i(\boldsymbol{x}) \geq 0,\ i = 0, \dots, d,\ \text{(positivity)} \tag{5}$$

$$1 = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x}), \text{ (partition of unity)} \tag{6}$$

$$\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i, \text{ (linear precision).} \tag{7}$$

The uniquely determined coefficients $\lambda_i(\boldsymbol{x})$, $i = 0, \ldots, d$, are called the barycentric coordinates of $\boldsymbol{x}$ with respect to σ^d. Linear precision (7) implies that points inside σ^d can be represented by convex combination of the vertices. Note also that (7) will continue to hold for any $\boldsymbol{x} \in \mathbb{R}^d$. However, the positivity condition (5) is not satisfied in this case. As immediate consequence of (7), the barycentric coordinates on σ^d satisfy the "delta property" or "Lagrange property"

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij}, \ i, j = 0, \ldots, d,$$

where δ_{ij} denotes the Kronecker δ-symbol. The parametrization of σ^d with a convex combination of the vertices reads as follows:

$$\sigma^d = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i, \, with \ \lambda_i(\boldsymbol{x}) \geq 0, \ i = 0, \ldots, d, \ and \sum_{i=0}^{d} \lambda_i(\boldsymbol{x}) = 1 \right\}. \tag{8}$$

Any of its face τ_i^{d-1}, $i = 0, \ldots, d$, can be described as:

$$\tau_i^{d-1} = \left\{ \boldsymbol{x} \in \sigma^d, \lambda_i(\boldsymbol{x}) = 0 \right\}.$$

Henceforth, we will let $\boldsymbol{c}_{\sigma^d}$ denote the centroid of σ^d which is defined as

$$\boldsymbol{c}_{\sigma^d} = \frac{1}{|\sigma^d|} \int_{\sigma^d} \boldsymbol{x} \, d\boldsymbol{x}. \tag{9}$$

The centroid of a simplex can be thought of as the center of mass. It is just the average of the vertices (treated as equal point masses), that is,

$$\boldsymbol{c}_{\sigma^d} = \frac{1}{d+1} \sum_{i=0}^{d} \boldsymbol{v}_i. \tag{10}$$

Hence, the centroid is identified as the point in the simplex with all its barycentric coordinates equal, that is

$$\lambda_0(\boldsymbol{c}_{\sigma^d}) = \frac{1}{d+1}, \ldots, \lambda_d(\boldsymbol{c}_{\sigma^d}) = \frac{1}{d+1}. \tag{11}$$

We now give explicit expressions for the barycentric coordinates. Here we denote by $\boldsymbol{n}_i$ the outward unit normal vector to τ_i^{d-1}, for each i. We did not find any reference to this result; however, we give a simple and direct proof of it.

Proposition 1 *The barycentric coordinates are such that*

$$\lambda_i(\boldsymbol{x}) = \frac{d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big) - \big\langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{c}_{\sigma^d}\big\rangle}{(d+1)d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big)}, \tag{12}$$

for all $\boldsymbol{x} \in \sigma^d$ *and all* $i = 0, \ldots, d$.

Proof Let us first recall that for any $i = 0, \ldots, d$, the barycentric coordinate λ_i associated with the vertex $\boldsymbol{v}_i$ is the only affine function that satisfies Lagrange property at the vertices, that is

$$\lambda_i(\boldsymbol{v}_j) = \delta_i^j, \text{ the Kronecker delta, } j = 0, \ldots, d. \tag{13}$$

This property is equivalent to the following conditions:

$$\lambda_i(\boldsymbol{v}_j) = 0,\ j = 0, \ldots, d,\ j \neq i, \tag{14}$$

$$\lambda_i(\boldsymbol{c}_{\sigma^d}) = \frac{1}{d+1}, \tag{15}$$

for each i. Let h_i denote the function defined by

$$h_i(\boldsymbol{x}) = \frac{d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big) - \big\langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{c}_{\sigma^d}\big\rangle}{(d+1)d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big)}. \tag{16}$$

Clearly, h_i is affine, which obviously satisfies (15). To verify (14), let us denote the closest point of $\boldsymbol{c}_{\sigma^d}$ in τ_i^{d-1} by $cp_i(\boldsymbol{c}_{\sigma^d})$. Then, for any $j = 0, \ldots, d$, $j \neq i$, it holds

$$\begin{aligned} \big\langle \boldsymbol{n}_i, \boldsymbol{v}_j - \boldsymbol{c}_{\sigma^d}\big\rangle &= \big\langle \boldsymbol{n}_i, \boldsymbol{v}_j - cp_i(\boldsymbol{c}_{\sigma^d})\big\rangle + \big\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d}\big\rangle \\ &= \big\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d}\big\rangle \qquad (17) \\ &= d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big), \qquad (18) \end{aligned}$$

where in the second last equality we have used that $\boldsymbol{n}_i$ is orthogonal to τ_i^{d-1}. This means that condition (14) is also satisfied and hence uniqueness of the barycentric coordinates shows that $h_i = \lambda_i$, $i = 0, \ldots, d$.

Remark 1 Let g_i, $i = 0, \ldots, d$, be the functions defined on the simplex σ^d, by

$$g_i(\boldsymbol{x}) = \big\langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{c}_{\sigma^d}\big\rangle. \tag{19}$$

Then, σ^d and its faces τ_i^{d-1} can be, respectively, expressed in terms of functions g_i as follows:

$$\sigma^d = \left\{ \boldsymbol{x} \in \mathbb{R}^d, g_i(\boldsymbol{x}) \le d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big), i = 0, \ldots, d \right\}, \tag{20}$$

$$\tau_i^{d-1} = \left\{ \boldsymbol{x} \in \sigma^d, g_i(\boldsymbol{x}) = d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big) \right\}. \tag{21}$$

The insphere of a $d-$simplex σ^d is the sphere that is tangent to all $d+1$ facets $\tau_i^{d-1}, i = 0, \ldots, d$, of σ^d; its center is the incenter $\boldsymbol{c}_{in}$ and its radius is the inradius r_{in} of σ^d.
There is a simple geometric meaning of the barycentric coordinates. Given a point $\boldsymbol{x}$ in σ^d, let $\sigma_i^d(\boldsymbol{x})$ be the d-simplex obtained from σ^d by replacing $\boldsymbol{v}_i$ by $\boldsymbol{x}$. Then, it follows by the Cramer's rule for solving (6) and (7),

$$\lambda_i(\boldsymbol{x}) = \frac{\left|\sigma_i^d(\boldsymbol{x})\right|}{\left|\sigma^d\right|}, \ i = 0, \ldots, d. \tag{22}$$

Clearly, the λ_i as given in Eqs. (22) are nonnegative, and their sum is equal to 1, since the d-volume $\left|\sigma^d\right|$ of σ^d is represented by $\left|\sigma^d\right| = \sum_{i=0}^d \left|\sigma_i^d(\boldsymbol{x})\right|$. On the other hand, the volume of a simplex of height h over a base of $(d-1)$-dimensional volume B is Bh/d, then we have $\left|\sigma_i^d(\boldsymbol{x})\right| = \frac{d(\boldsymbol{x}, \tau_i^{d-1})}{d} \left|\tau_i^{d-1}\right|$. Therefore, the λ_i can be expressed in the equivalent form

$$\lambda_i(\boldsymbol{x}) = \frac{d\big(\boldsymbol{x}, \tau_i^{d-1}\big)}{r_{in}} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|}. \tag{23}$$

Using (7), it follows that every point in σ^d can be alternatively expressed as:

$$\boldsymbol{x} = \sum_{i=0}^{d} \frac{d\big(\boldsymbol{x}, \tau_i^{d-1}\big)}{r_{in}} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{v}_i. \tag{24}$$

Since $\boldsymbol{c}_{in}$ lies at equal distances r_{in} from the faces, it follows that the incenter is algebraically defined by

$$\boldsymbol{c}_{in} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{v}_i. \tag{25}$$

This representation will be frequently used.

For later reference we prove the following result, which is satisfied by any d-simplex.

Lemma 1 *Let σ^d be a d-simplex and let f be an affine function on σ^d. Then the following identity holds:*

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{d+1}\sum_{i=0}^{d}\frac{1}{\left|\tau_i^{d-1}\right|}\int\limits_{\tau_i^{d-1}} f(\boldsymbol{x})d\gamma_i. \tag{26}$$

Proof By an affine function f on σ^d, we mean a mapping $f : \boldsymbol{x} \to a + \langle \boldsymbol{b}, \boldsymbol{x}\rangle$, where $a \in \mathbb{R}$, $\boldsymbol{b}, \boldsymbol{x} \in \mathbb{R}^d$. We have $\nabla f(\boldsymbol{x}) = \boldsymbol{b}$, therefore the gradient is constant for any affine function, then the second integral on the left-hand side in (48) vanishes, and consequently Green's identity (48) simplifies to (26), showing that the required statement (26) holds for any affine function.

We now introduce the notion of admissible simplices.

Definition 1 A d-simplex σ^d is called admissible if it satisfies the Hermite–Hadamard inequality (1).

The following result will be useful in our subsequent analysis. A similar argument to the proof of Lemma 1 shows that:

Lemma 2 *Let σ^d be an admissible simplex and let f be an affine function on σ^d. Then the following identity holds:*

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} l(\boldsymbol{x})d\gamma. \tag{27}$$

We need the following simple lemma:

Lemma 3 *The following two statements are equivalent:*

(i) For any affine function $l : \sigma^d \to \mathbb{R}$, we have

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} l(\boldsymbol{x})d\gamma. \tag{28}$$

(ii) The centroid of σ^d and the centroid of its boundary coincide.

Proof Assume that (i) holds. Then, if we take in (28), $l(\boldsymbol{x}) = x_j$ for $j = 1, \ldots, d$, we get

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} \boldsymbol{x}\,d\boldsymbol{x} = \frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} \boldsymbol{x}\,d\gamma. \tag{29}$$

This shows that property (ii) holds. For the converse, assume that (ii) holds. Then, since for any affine function l, we have

$$l\left(\frac{1}{|\sigma^d|}\int_{\sigma^d} \boldsymbol{x}\, d\boldsymbol{x}\right) = \frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x}$$

$$l\left(\frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} \boldsymbol{x}\, d\gamma\right) = \frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} l(\boldsymbol{x})\, d\gamma,$$

it follows that (i) holds.

Remark 2 Using Lemmas 2 and 3, it is easy to see that a necessary condition for a simplex to be admissible is that its centroid and the centroid of its boundary coincide. We will see that for a very important family of simplices, it is also a sufficient condition.

It is not generally true that any simplex is admissible in the sense of definition 1. Next we present our counterexample, to show that there exists a simplex which is not admissible for (1). Consider the standard unit-simplex $\tilde{\sigma}^d$, that is the set of points $\boldsymbol{x} \in \mathbb{R}^d$ such that for each $i = 1, \ldots, d$, $x_i \geq 0$, and $x_1 + \ldots, x_d \leq 1$. There are several ways to compute the d-volume (d-dimensional volume) of $\tilde{\sigma}^d$ and the $(d-1)$-volumes of its faces. An elegant way to perform all these integrals is proposed in [22, section 18.10] and uses the classical Laplace transform. By using this technique, we can easily derive:

$$\left|\tilde{\sigma}^d\right| = \frac{1}{d!} \tag{30}$$

$$\left|\tilde{\tau}_0^{d-1}\right| = \frac{\sqrt{d}}{(d-1)!} \tag{31}$$

$$\left|\tilde{\tau}_i^{d-1}\right| = \frac{1}{(d-1)!}, \ (i = 1, \ldots, d). \tag{32}$$

As mentioned before, the barycentric coordinates of the centroid of $\tilde{\sigma}^d$ are

$$\lambda_0 = \frac{1}{d+1}, \ldots, \lambda_d = \frac{1}{d+1}. \tag{33}$$

However, using the identities (30), (31), (32), and (25), we can see that those of its incenter are

$$\mu_0 = \frac{\sqrt{d}}{d+\sqrt{d}}$$

$$\mu_i = \frac{1}{d+\sqrt{d}}, \ (i = 1, \ldots, d).$$

It is readily seen that the centroid and the incenter do not coincide with the exception of dimension one. The centroid of its boundary can be calculated according the formula (see Lemma 8):

$$\sum_{i=0}^{d} \frac{\left|\tilde{\tau}_i^{d-1}\right|}{\left|\partial\tilde{\sigma}^d\right|} \boldsymbol{c}_{\tilde{\tau}_i^{d-1}} = \frac{1}{d}\left(\boldsymbol{c}_{\tilde{\sigma}^d} - \boldsymbol{c}_{in}\right) + \boldsymbol{c}_{\tilde{\sigma}^d}. \tag{34}$$

Since the centroid and the incenter do not coincide, so the centroid and the centroid of the boundary do not coincide, too. Then, Remark 2 tells us that $\tilde{\sigma}^d$ is not a valid simplex for (1).

After seeing this result, the reader might naturally ask: Is the coincidence of these two centers also sufficient? The answer is yes, and we will give several equivalent conditions to this coincidence property.

3 The Regular Case

This section contains the main results of the chapter (Theorems 1, 2 and its converse 3, below), as well as some technical lemmas related to it. We will prove several results characterizing an equiareal simplex (see Lemma 9). The central question in this section is whether a d-simplex satisfies inequality (1) or not. Here, we deal with the case when in addition to the convexity requirement the C^1-regularity is also assumed. Hence, throughout this section we assume that the functions involved are continuously differentiable in σ^d. We need the following key lemma.

Lemma 4 *Let σ^d be non-degenerate simplex with faces τ_i^{d-1}, $i = 0, \ldots, d$. Then , the following identities hold:*

$$\left\langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \right\rangle = d(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}) \tag{35}$$

$$= \frac{d}{d+1} \frac{\left|\sigma^d\right|}{\left|\tau_i^{d-1}\right|}, \tag{36}$$

for any $\boldsymbol{x} \in \tau_i^{d-1}$.

Proof First let us observe that for any $i = 0, \ldots, d$, and any $\boldsymbol{x} \in \tau_i^{d-1}$, it holds

$$\left\langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \right\rangle = \left\langle \boldsymbol{n}_i, \boldsymbol{x} - cp_i(\boldsymbol{c}_{\sigma^d}) \right\rangle + \left\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d} \right\rangle \tag{37}$$

$$= \left\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d} \right\rangle \tag{38}$$

$$= d(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}), \tag{39}$$

where in the second last equality we have used that $cp_i(\boldsymbol{x})$ is the closest point of $\boldsymbol{c}_{\sigma^d}$ in τ_i^{d-1} and that $\boldsymbol{n}_i$ is orthogonal to τ_i^{d-1}. An alternative proof of the above formula can also be obtained from the characterization result of a face given in (21). Now we can prove the main inequality (36). Let us denote by σ_i^d the simplex obtained from σ^d by replacing $\boldsymbol{v}_i$ by $\boldsymbol{c}_{\sigma^d}$. We shall need to know the volume of σ_i^d as a function of σ^d. Observe first that the volume of σ_i^d is given by the familiar formula: $1/d$ times its height multiplied by the $(d-1)$-dimensional volume of its base. Hence, we have $\left|\sigma_i^d\right| = \frac{h_i}{d}\left|\tau_i^{d-1}\right|$ and $h_i = \left\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d}\right\rangle$, then by (39) we get

$$\left\langle \boldsymbol{n}_i, cp_i(\boldsymbol{c}_{\sigma^d}) - \boldsymbol{c}_{\sigma^d}\right\rangle = d\,\frac{\left|\sigma_i^d\right|}{\left|\tau_i^{d-1}\right|}. \tag{40}$$

The simplices σ_j^d, $j = 0, \ldots, d$, decompose σ^d and for all $j = 0, \ldots, d$, we have

$$\left|\sigma_j^d\right| = \frac{\left|\sigma^d\right|}{d+1}. \tag{41}$$

This is obvious for a regular d-simplex (determined by $(d+1)$ points arranged equidistantly in $\mathbb{R}^d$), but since an affine transformation maps the centroid of a simplex to the centroid of its image and preserves the ratio of volumes, we conclude that (41) also holds for an arbitrary simplex. Nevertheless we may give an alternative proof of identity (41) by simply using other expressions of barycentric coordinates, which will be provided later in (22). Then to obtain (41), it suffices to take $\boldsymbol{x} = \boldsymbol{c}_{\sigma^d}$ in (22). Finally, combining (41) with (17) and (18) gives directly the desired result.

We are now ready to prove the following theorem. The same approach was consistently used by B. Semisalov and the author in a series of papers in [16–18].

Theorem 1 *Given a non-degenerate simplex σ^d with faces $\tau_i^{d-1}, i = 0, \ldots, d$. Then, for any continuously differentiable function we have*

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x} \le Q_d^{tra}(f) := \frac{1}{d+1}\sum_{i=0}^{d}\frac{1}{\left|\tau_i^{d-1}\right|}\int_{\tau_i^{d-1}} f(\boldsymbol{x})d\gamma_i. \tag{42}$$

Proof Let us take any continuously differentiable function f. Let $\nabla f(\boldsymbol{x})$ denote the gradient of f at $\boldsymbol{x}$. Then, Green's identity tells us that

$$\int_{\sigma^d} f(\boldsymbol{x})\triangle u(\boldsymbol{x})\,d\boldsymbol{x} + \int_{\sigma^d}\langle \nabla f(\boldsymbol{x}), \nabla u(\boldsymbol{x})\rangle d\boldsymbol{x} = \sum_{i=0}^{d}\int_{\tau_i^{d-1}} f(\boldsymbol{x})\langle \boldsymbol{n}_i(\boldsymbol{x}), \nabla u(\boldsymbol{x})\rangle\,d\gamma_i, \tag{43}$$

where $d\gamma_i$ is the element of integration over τ_i^{d-1}, and

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator. Formula (43) is valid for any twice continuously differentiable function u. Here $\boldsymbol{n}_i(\boldsymbol{x})$ is the outward unit normal vector to the boundary τ_i^{d-1} at the point $\boldsymbol{x} \in \tau_i^{d-1}$. Hence, if we take

$$u(\boldsymbol{x}) = \frac{\left\| \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \right\|^2}{2},$$

then the following identity holds:

$$\nabla u(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{c}_{\sigma^d}.$$

With help of Lemma 4, (43) reduces to

$$\begin{aligned} d \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\sigma^d} \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \rangle d\boldsymbol{x} &= \sum_{i=0}^{d} \int_{\tau_i^{d-1}} \langle \boldsymbol{n}_i(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \rangle f(\boldsymbol{x}) d\gamma_i \\ &= \sum_{i=0}^{d} d(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}) \int_{\tau_i^{d-1}} f(\boldsymbol{x}) d\gamma_i \end{aligned} \quad (44)$$

This can be written in a more convenient form, as follows:

$$\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} + \frac{1}{d} \int_{\sigma^d} \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \rangle d\boldsymbol{x} = \sum_{i=0}^{d} \frac{d(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1})}{d} \int_{\tau_i^{d-1}} f(\boldsymbol{x}) d\gamma_i. \quad (45)$$

Let us observe that the centroid $\boldsymbol{c}_{\sigma^d}$ of σ^d satisfies

$$\boldsymbol{c}_{\sigma^d} = \frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} \boldsymbol{x}\, d\boldsymbol{x}, \text{ or } \int_{\sigma^d} \left(\boldsymbol{x} - \boldsymbol{c}_{\sigma^d}\right) d\boldsymbol{x} = \boldsymbol{0}. \quad (46)$$

From the above equation, we can clearly see that

$$\int_{\sigma^d} \langle \nabla f(\boldsymbol{c}_{\sigma^d}), \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \rangle d\boldsymbol{x} = \langle \nabla f(\boldsymbol{c}_{\sigma^d}), \int_{\sigma^d} \left(\boldsymbol{x} - \boldsymbol{c}_{\sigma^d}\right) d\boldsymbol{x} \rangle = 0. \quad (47)$$

Hence, by (47) Green's identity (45) can be rewritten as follows:

$$\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} + \frac{1}{d} \int_{\sigma^d} \langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{c}_{\sigma^d}), \boldsymbol{x} - \boldsymbol{c}_{\sigma^d} \rangle d\boldsymbol{x}$$

$$= \sum_{i=0}^{d} \frac{d(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1})}{d} \int_{\tau_i^{d-1}} f(\boldsymbol{x}) d\gamma_i. \tag{48}$$

Now, using the monotonicity of the gradient, we see that the second integral in the above identity is nonnegative for any continuously differentiable convex function on σ^d, then it follows

$$\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \sum_{i=0}^{d} \frac{d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big)}{d} \int_{\tau_i^{d-1}} f(\boldsymbol{x}) d\gamma_i,$$

which gives the desired result using again Lemma 4.

A d-simplex is said to be equiareal if its faces are of equal area.

Theorem 2 *Let σ^d be an equiareal d-simplex with faces τ_i^{d-1}, $i = 0, \ldots, d$. Then, σ^d is an admissible simplex. Moreover, for all $i = 0, \ldots, d$, it holds*

$$\frac{d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big)}{d} = \frac{\left|\sigma^d\right|}{\left|\partial\sigma^d\right|}. \tag{49}$$

Proof By Theorem 1 it suffices to show that (49) holds. But, by Lemma 4 it follows that

$$\frac{d\big(\boldsymbol{c}_{\sigma^d}, \tau_i^{d-1}\big)}{d} = \frac{\left|\sigma^d\right|}{(d+1)\left|\tau_i^{d-1}\right|}. \tag{50}$$

Any equiareal simplex has faces of equal area; therefore, the desired result (49) follows from the fact that $(d+1)\left|\tau_i^{d-1}\right| = \left|\partial\sigma^d\right|$.

We now move to the harder direction. The converse of Theorem 2 is also true, indeed we have:

Theorem 3 *If the simplex σ^d is admissible, then σ^d is an equiareal simplex.*

Theorem 2 and its converse Theorem 3 tell us that the d-dimensional Hermite–Hadamard inequality (1) holds if and only if the d-simplex S is equiareal. In order to prove Theorem 3, we need some auxiliary results that are actually of independent interest. We begin with a simple fact:

Lemma 5 *Let σ^d be a d-simplex with vertices $\boldsymbol{v}_i$, $i = 0, \ldots, d$. If l is an affine function on σ^d, then*

$$\frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+1} \sum_{i=0}^{d} l(\boldsymbol{v}_i) \tag{51}$$

$$= l(\boldsymbol{c}_{\sigma^d}). \tag{52}$$

Proof Take $l : \sigma^d \to \mathbb{R}$, any affine function. Then l can be expressed as $l(\boldsymbol{x}) = \langle u, \boldsymbol{x}\rangle + a$ for some $u \in \mathbb{R}^d$, and $a \in \mathbb{R}$. Hence

$$\begin{aligned}
\frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} &= \frac{1}{|\sigma^d|}\int_{\sigma^d} (\langle u, \boldsymbol{x}\rangle + a)\; d\boldsymbol{x} \\
&= \frac{1}{|\sigma^d|}\int_{\sigma^d} \langle u, \boldsymbol{x}\rangle\; d\boldsymbol{x} + a \\
&= \left\langle u, \frac{1}{|\sigma^d|}\int_{\sigma^d} \boldsymbol{x}\, d\boldsymbol{x}\right\rangle + a \\
&= \langle u, \boldsymbol{c}_{\sigma^d}\rangle + a,
\end{aligned}$$

where $\boldsymbol{c}_{\sigma^d}$ is the centroid of σ^d. Recall that, by the definition the latter is also simply the average of its vertices, then we must have

$$\begin{aligned}
\frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} &= \left\langle u, \frac{1}{d+1}\sum_{i=0}^{d} \boldsymbol{v}_i\right\rangle + a \\
&= \frac{1}{d+1}\sum_{i=0}^{d} \langle u, \boldsymbol{v}_i\rangle + a \\
&= \frac{1}{d+1}\sum_{i=0}^{d} (\langle u, \boldsymbol{v}_i\rangle + a) \\
&= \frac{1}{d+1}\sum_{i=0}^{d} l(\boldsymbol{v}_i).
\end{aligned}$$

This shows that identity (51) holds. On the other hand, we also have

$$\frac{1}{d+1}\sum_{i=0}^{d} l(\boldsymbol{v}_i) = l\left(\frac{1}{d+1}\sum_{i=0}^{d} \boldsymbol{v}_i\right) \tag{53}$$

$$= l(\boldsymbol{c}_{\sigma^d}). \tag{54}$$

This shows that identity (52) also holds.

As mentioned before, we note that special cases of Lemma 5 for $l_i(\boldsymbol{x}) := x_i, i = 1, \ldots, d$ yield that the coordinates of the centroid are the arithmetic mean of the coordinates of all the vertices of a simplex, see (10).

Every face of a d-simplex is an $(d-1)$-simplex, then Lemma 5 combined by the arguments used in its proof imply the following result.

Lemma 6 *Let σ^d be a d-simplex with faces $\tau_i^{d-1}, i = 0, \dots, d$. If l is an affine function on σ^d, then we have for all $i = 0, \dots, d$,*

$$\frac{1}{\left|\partial \tau_i^{d-1}\right|} \int_{\tau_i^{d-1}} l(\boldsymbol{x})\, d\gamma = \frac{1}{d}\left(\frac{d+1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} - l(\boldsymbol{v}_i)\right). \tag{55}$$

We also require the following lemma.

Lemma 7 *Let σ^d be a d-simplex with faces $\tau_i^{d-1}, i = 0, \dots, d$. If l is an affine function on σ^d, then*

$$\frac{1}{\left|\partial\sigma^d\right|} \int_{\partial\sigma^d} l(\boldsymbol{x})\, d\gamma - \frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d}\left(\frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i)\right). \tag{56}$$

Proof Multiplying equality (55) by $\left|\partial \tau_i^{d-1}\right|$ and summing on all the faces of σ^d, we get

$$\int_{\partial\sigma^d} l(\boldsymbol{x})\, d\gamma = \frac{1}{d}\left(\frac{(d+1)\left|\partial\sigma^d\right|}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{i=0}^{d} \left|\tau_i^{d-1}\right| l(\boldsymbol{v}_i)\right). \tag{57}$$

Then we can divide both sides by $\left|\partial\sigma^d\right|$ to get

$$\frac{1}{\left|\partial\sigma^d\right|} \int_{\partial\sigma^d} l(\boldsymbol{x})\, d\gamma = \frac{1}{d}\left(\frac{d+1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i)\right). \tag{58}$$

Finally, the proof of the lemma is completed by subtracting $\frac{1}{|\sigma^d|}\int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x}$ from both sides.

The following lemma establishes an important relationship between the centroid, the incenter, and the boundary center of a d-simplex.

Lemma 8 *Let σ^d be a d-simplex with faces and vertices, respectively, $\tau_i^{d-1}, i = 0, \dots, d$, and $\boldsymbol{v}_i, i = 0, \dots, d$. Let $\boldsymbol{c}_{\tau_i^{d-1}}$ denote the centroid of the face τ_i^{d-1} for each i. Then the following identity holds:*

$$\frac{1}{d}\left(\boldsymbol{c}_{\sigma^d} - \boldsymbol{c}_{in}\right) + \boldsymbol{c}_{\sigma^d} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{c}_{\tau_i^{d-1}}. \tag{59}$$

Proof Since the centroid of any face of σ^d is just the average of all vertices which forms it, then a simple calculation shows that

$$\sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{c}_{\tau_i^{d-1}} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \frac{1}{\left|\tau_i^{d-1}\right|} \int_{\tau_i^{d-1}} \boldsymbol{x}\, d\boldsymbol{x}$$
$$= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \frac{1}{d} \left(\sum_{j=0,}^{d} \boldsymbol{v}_j - \boldsymbol{v}_i \right) = \frac{1}{d} \sum_{j=0,}^{d} \boldsymbol{v}_j - \frac{1}{d} \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{v}_i,$$

and it follows that

$$\sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{c}_{\tau_i^{d-1}} = \frac{d+1}{d} \boldsymbol{c}_{\sigma^d} - \frac{1}{d} \boldsymbol{c}_{in}.$$

The desired result now follows since the last term in the right-hand side above is just $\frac{1}{d}\left(\boldsymbol{c}_{\sigma^d} - \boldsymbol{c}_{in}\right) + \boldsymbol{c}_{\sigma^d}$.

We are now ready to state several characterizations of an equiareal simplex.

Lemma 9 *Let σ^d be a d-simplex with faces and vertices, respectively, $\tau_i^{d-1}, i = 0, \ldots, d$, and $\boldsymbol{v}_i, i = 0, \ldots, d$. Let $\boldsymbol{c}_{\tau_i^{d-1}}$ denote the centroid of the face τ_i^{d-1} for each i. Then the following statements are equivalent:*

(i) σ^d is an equiareal simplex.
(ii) For every affine function l, we have

$$\frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i). \tag{60}$$

(iii) The centroid and the incenter of the simplex σ^d coincide.
(iv) The incenter of the simplex σ^d can be represented as $\boldsymbol{c}_{in} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{c}_{\tau_i^{d-1}}$.
(v) The centroid of σ^d and the centroid of its boundary coincide.
(vi) For every affine function l on σ^d, we have

$$l(\boldsymbol{c}_{in}) = \frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x}. \tag{61}$$

Proof Assume that σ^d is an equiareal d-simplex. Then, all its faces $\tau_i^{d-1}, i = 0, \ldots, d$, have equal area. Note that since $\partial\sigma^d = \sum_{i=0}^{d} \left|\tau_i^{d-1}\right|$, then we get

$$\frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} = \frac{1}{d+1} \tag{62}$$

for each i. Hence, (60) can be reduced to

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+1}\sum_{i=0}^{d} l(\boldsymbol{v}_i). \tag{63}$$

But, Lemma 5 says that the above identity is satisfied by any affine function. This shows that (ii) holds. Assume that (ii) is satisfied. Then, the special cases $l(\boldsymbol{x}) = x_j$ for $j = 1, \ldots, d$, yield that

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} \boldsymbol{x}\, d\boldsymbol{x} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|}\boldsymbol{v}_i. \tag{64}$$

But the left-hand term is the centroid, and the right-hand one is the incenter, we may conclude, therefore, that (iii) holds. Assume now that (iii) is satisfied. Since $\boldsymbol{c}_{\sigma^d} = \boldsymbol{c}_{in}$, it follows from Lemma 8 that (iv) holds. Assume now that (iv) is satisfied. To prove (v), we write

$$\begin{aligned}
l(\boldsymbol{c}_{in}) &= l\left(\sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|}\boldsymbol{v}_i\right) \\
&= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i) \\
&= l\left(\sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|}\boldsymbol{c}_{\tau_i^{d-1}}\right) \\
&= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l\left(\boldsymbol{c}_{\tau_i^{d-1}}\right),
\end{aligned}$$

it follows that

$$l(\boldsymbol{c}_{in}) = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l\left(\boldsymbol{v}_i\right)$$

$$= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l\left(\frac{1}{\left|\tau_i^{d-1}\right|}\int_{\tau_i^{d-1}} \boldsymbol{x}\, d\boldsymbol{x}\right)$$

$$= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \frac{1}{\left|\tau_i^{d-1}\right|}\int_{\tau_i^{d-1}} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x}$$

$$= \sum_{i=0}^{d} \frac{1}{\left|\partial\sigma^d\right|}\int_{\tau_i^{d-1}} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x}$$

$$= \frac{1}{\left|\partial\sigma^d\right|}\int_{\partial\sigma^d} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x}.$$

Hence

$$\frac{1}{\left|\partial\sigma^d\right|}\int_{\partial\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i), \tag{65}$$

and therefore by Lemma 7 we obtain

$$\frac{1}{\left|\partial\sigma^d\right|}\int_{\partial\sigma^d} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x} = \frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x}, \tag{66}$$

for any affine function. Thus, according to Lemma 3 we get

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} \boldsymbol{x}\, d\boldsymbol{x} = \frac{1}{\left|\partial\sigma^d\right|}\int\limits_{\partial\sigma^d} \boldsymbol{x}\, d\gamma, \tag{67}$$

and then property (v) holds. Assume (v) holds. Then, by Lemma 8, the centroid and the incenter coincide. Hence for any affine function l we derive

$$l(\boldsymbol{c}_{in}) = l(\boldsymbol{c}_{\sigma^d}).$$

By Lemma 5, it follows that (vi) holds. Assume that (vi) holds. Then, we have

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} l\left(\boldsymbol{x}\right)\, d\boldsymbol{x} = l(\boldsymbol{c}_{in}) \tag{68}$$

$$= \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i). \tag{69}$$

Applying Lemma 5 we get

$$\frac{1}{d+1}\sum_{i=0}^{d} l(\boldsymbol{v}_i) = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i). \tag{70}$$

In particular, for each $l(\boldsymbol{x}) = x_j$ for $j = 1, \ldots, d$, it is true that

$$\boldsymbol{c}_{\sigma^d} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} \boldsymbol{v}_i. \tag{71}$$

So, the uniqueness of barycentric coordinates implies

$$\frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} = \frac{1}{d+1}, \tag{72}$$

or equivalently

$$\left|\tau_i^{d-1}\right| = \frac{\left|\partial\sigma^d\right|}{d+1}, \ (i = 0, \ldots, d). \tag{73}$$

This says that all faces of σ^d are of equal area and hence σ^d is an equiareal simplex. Therefore, property (i) holds and the equivalence of the six statements follows.

Remark 3 The equivalence of (i) and (iii) was established by Edmonds et al. [9, Theorem 3.2]. However, the direct proof we give here is new (as far as we know).

With the help of these lemmas, we are now prepared to prove Theorem 3:

Proof of Theorem 3 Assume that σ^d is an admissible simplex. Then, Lemma 2 tells us

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} l(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{\left|\partial\sigma^d\right|}\int_{\partial\sigma^d} l(\boldsymbol{x})\,d\gamma, \tag{74}$$

for any affine function. By Lemma 7, this is equivalent to that for any affine function l it holds

$$\frac{1}{\left|\sigma^d\right|}\int_{\sigma^d} l(\boldsymbol{x})\,d\boldsymbol{x} = \sum_{i=0}^{d} \frac{\left|\tau_i^{d-1}\right|}{\left|\partial\sigma^d\right|} l(\boldsymbol{v}_i). \tag{75}$$

Finally, Lemma 9 implies that σ^d must be an equiareal simplex. This leads to the conclusion of the theorem.

It is important to note, apart convexity, in the proof of the sufficient condition, we did not use any regularity requirement on the functions in Theorem 3.

4 Characterization Without Using Any Regularity Condition

The challenge in this section is to show that our characterization remains valid if the convexity is only assumed. Hence, the main novelty is that we do not require here any regularity except convexity of the integrands. Our first result comes from the observation that the identity (56) proved in Lemma 7 can be equivalently rewritten as:

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x} - f(\boldsymbol{c}_{in}) = d\left(\frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} f(\boldsymbol{x})d\gamma_i - \frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x}\right), \tag{76}$$

which is valid for any affine function. We now extend this result to the case of convex functions. Therefore, Lemma 7 can be viewed as a special case of the following more general result:

Theorem 4 *Let σ^d be a d-simplex with incenter $\boldsymbol{c}_{in}$, inradius r_{in}, and faces $\tau_i^{d-1}, i = 0, \ldots, d$. Then, for any convex function we have*

$$\frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x} - f(\boldsymbol{c}_{in}) \leq d\left(\frac{1}{|\partial\sigma^d|}\int\limits_{\partial\sigma^d} f(\boldsymbol{x})d\gamma_i - \frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\,d\boldsymbol{x}\right). \tag{77}$$

Proof Let us denote by σ_i^d the simplex obtained from σ^d by replacing $\boldsymbol{v}_i$ by the incenter $\boldsymbol{c}_{in}$ of σ^d. Let us prove that for every σ_i^d we have

$$\int_{\sigma_i^d} f(\boldsymbol{x})\,d\boldsymbol{x} \leq \frac{r_{in}}{d+1}\int_{\tau_i^{d-1}} f(\boldsymbol{x})\,d\gamma_i + \frac{r_{in}}{d(d+1)}\left|\tau_i^{d-1}\right| f(\boldsymbol{c}_{in}), \ (i = 1, \ldots, d), \tag{78}$$

for each convex function defined on σ^d. Due to affine invariance of our considerations, it suffices to consider the following special situation: The face τ_i^{d-1} belongs to the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d, x_d = 0\}$, and $\boldsymbol{c}_{in} = (0, \ldots, r_{in})$. Then it is easily seen that the last coordinate of a point in σ_i^d varies from 0 to r_{in}, so it follows that

$$\int_{\sigma_i^d} f(\boldsymbol{x})\,d\boldsymbol{x} = \int_0^{r_{in}}\left(\int_{\sigma_{i,s}^d} f(\tilde{\boldsymbol{x}}, s)\,d\tilde{\boldsymbol{x}}\right)ds,$$

where $\sigma_{i,s}^d = \{\tilde{\boldsymbol{x}} \in \mathbb{R}^{d-1}, (\tilde{\boldsymbol{x}}, s) \in \sigma_i^d\}$. Let us observe that $\sigma_{i,s}^d$ can also be described as:

$$\sigma_{i,s}^d = \left\{ \frac{r_{in} - s}{r_{in}} \tilde{\boldsymbol{x}} + \frac{s}{r_{in}} \boldsymbol{c}_{in}, \tilde{\boldsymbol{x}} \in \tau_i^{d-1}, s \in [0, r_{in}] \right\}. \tag{79}$$

Now changing variables in the integral gives

$$\int_{\sigma_i^d} f(\boldsymbol{x})\, d\boldsymbol{x} = \int_0^{r_{in}} \frac{(r_{in} - s)^{d-1}}{r_{in}^{d-1}} \left(\int_{\tau_i^{d-1}} f\left(\frac{r_{in} - s}{r_{in}} \tilde{\boldsymbol{x}} + \frac{s}{r_{in}} \boldsymbol{c}_{in} \right) d\tilde{\boldsymbol{x}} \right) ds. \tag{80}$$

On the other hand, the convexity of f implies

$$\int_{\sigma_i^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \left(\int_0^{r_{in}} \frac{(r_{in} - s)^d}{r_{in}^d} ds \right) \int_{\tau_i^{d-1}} f(\tilde{\boldsymbol{x}}) d\tilde{\boldsymbol{x}} + \left(\int_0^{r_{in}} \frac{(r_{in} - s)^{d-1} s}{r_{in}^d} ds \right) \left| \tau_i^{d-1} \right| f(\boldsymbol{c}_{in}). \tag{81}$$

Inequality (78) now follows by simply combining (81) and the identities:

$$\int_0^{r_{in}} \frac{(r_{in} - s)^d}{r_{in}^d} ds = \frac{r_{in}}{d+1}$$

$$\int_0^{r_{in}} \frac{(r_{in} - s)^{d-1} s}{r_{in}^d} ds = \frac{r_{in}}{d(d+1)}.$$

Now, summing up all inequalities (78), we get

$$\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \frac{r_{in}}{d+1} \int_{\partial\sigma^d} f(\tilde{\boldsymbol{x}}) d\tilde{\boldsymbol{x}} + \frac{r_{in}}{d(d+1)} \left| \partial\sigma^d \right| f(\boldsymbol{c}_{in}). \tag{82}$$

In view of the fact that $\left|\sigma^d\right| = \frac{r}{d} \left|\partial\sigma^d\right|$, we derive

$$\frac{1}{\left|\sigma^d\right|} \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \frac{d}{d+1} \frac{1}{\left|\partial\sigma^d\right|} \int_{\partial\sigma^d} f(\tilde{\boldsymbol{x}}) d\tilde{\boldsymbol{x}} + \frac{1}{d+1} f(\boldsymbol{c}_{in}). \tag{83}$$

Multiply both sides by $d+1$ and subtracting $\frac{d}{|\sigma^d|} \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x}$ from both sides give the desired result.

Without using Theorem 1, where C^1-regularity is needed, we are now in position to give the following simple and direct proof for the characterization of the validity of (1).

Theorem 5 *Let σ^d be a non-degenerate d-simplex. Then the following statements are equivalent:*

(i) σ^d is an admissible simplex for (1) *on the set of convex functions.*
(ii) σ^d is an equiareal simplex.

Proof Assume that (i) holds. Then the centroid of σ^d and the centroid of its boundary coincide, see Remark 2. Hence, by Lemma 9, σ^d is an equiareal simplex. Assume that (ii) holds. Then, by Lemma 9 the incenter satisfies:

$$l(\boldsymbol{c}_{in}) = \frac{1}{|\sigma^d|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x}, \tag{84}$$

for every affine function l. Let us now take any convex function f on σ^d. By convexity of f, there exists an affine function l such that

$$f(\boldsymbol{c}_{in}) = l(\boldsymbol{c}_{in}), \tag{85}$$

$$f(\boldsymbol{x}) \geq l(\boldsymbol{x}),\ \boldsymbol{x} \in \sigma^d. \tag{86}$$

Integrating both sides of (86) directly yields

$$\begin{aligned}\frac{1}{|\sigma^d|} \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} &\geq \frac{1}{|\sigma^d|} \int_{\sigma^d} l(\boldsymbol{x})\, d\boldsymbol{x} \\ &= l(\boldsymbol{c}_{in}) \\ &= f(\boldsymbol{c}_{in}).\end{aligned}$$

Hence, the first term in (76) is nonnegative, then by Theorem 4, (i) holds for any convex function.

The reader can see that the above theorem can be reformulated in several equivalent forms. For example, using any other equivalent property provided by Lemma 9.

5 Best Cubature Error Bounds

5.1 Characterization of the New Extended Cubature Formulas

Theorem 1 says that every d-simplex satisfies:

$$\frac{1}{|\sigma^d|} \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq Q_d^{\mathrm{tra}}(f) := \frac{1}{d+1} \sum_{i=0}^{d} \frac{1}{\left|\tau_i^{d-1}\right|} \int_{\tau_i^{d-1}} f(\boldsymbol{x}) d\gamma_i, \tag{87}$$

for every continuously differentiable function $f : \sigma^d \to \mathbb{R}$. The application to the cubature formulas runs as follows. First observe that the functional $Q^{\mathrm{tra}}(f)$, which is a natural multivariate version of the classical trapezoidal rule, can serve as a cubature formula (multidimensional integration formula) for the approximation of the exact value of the integral $\frac{1}{|\sigma^d|} \int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x}$. We call it the trapezoidal cubature formula. In this section, best estimates are established for the integration error, defined for a given continuous function $f : \sigma^d \to \mathbb{R}$ as follows:

$$E[f] := Q^{\mathrm{tra}}(f) - \frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{88}$$

The approximation error of such cubature formulas is always nonnegative for convex functions. One interesting advantage of this property is to ensure a sharp error bound for the associated integration formula. The main result in this section is Theorem 6 that completely characterizes a cubature formula of the form (87).
Here, for a twice differentiable function $f : \sigma^d \to \mathbb{R}$, we say that f is continuously differentiable on σ^d if it is continuously differentiable on an open set containing σ^d.

Definition 2 A differentiable function $f : \sigma^d \to \mathbb{R}$ is said to have a Lipschitz continuous gradient, if there exists a constant $\rho(\nabla f)$, such that

$$\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\| \leq \rho(\nabla f)\, \|\boldsymbol{x} - \boldsymbol{y}\|\,,\ (\boldsymbol{x}, \boldsymbol{y} \in \sigma^d). \tag{89}$$

For any differentiable f with Lipschitz continuous gradient, there exists a smallest possible $\rho(\nabla f)$ such that (89) holds. The smallest constant $L(\nabla f) := Lip(\nabla f)$ satisfying inequality (89) is called the Lipschitz constant for ∇f. By $C^{1,1}(\sigma^d)$ we will denote the subclass of all functions f which are continuously differentiable on σ^d with Lipschitz continuous gradients.

The integration error of the trapezoidal cubature formula Q_d^{tra} when applied to the function $\|.\|^2$ plays an important role in our best error estimate. Indeed, the latter is characterized as follows:

Theorem 6 *Let Q^{tra} and E be defined respectively as in (87) and (88). Then, the two following statements are equivalent:*

(i) *For every convex function $g \in C^{1,1}(\sigma^d)$, we have*

$$E[g] \geq 0. \tag{90}$$

(ii) *For every $f \in C^{1,1}(\sigma^d)$ with $L(\nabla f)$-Lipschitz gradient, we have*

$$|E[f]| \leq E\left[\|.\|^2\right] \frac{L(\nabla f)}{2}. \tag{91}$$

Equality is attained for all functions of the form

$$f(\boldsymbol{x}) := a(\boldsymbol{x}) + c\|.\|^2, \tag{92}$$

where $c \in \mathbb{R}$ and $a(\cdot)$ is any affine function.

Proof First we prove that (i) implies (ii). Take f to be any continuous function from $C^{1,1}(\sigma^d)$ with Lipschitz constant $L(\nabla f)$, and define the two functions

$$g_{\pm} := \|.\|^2 \frac{L(\nabla f)}{2} \pm f.$$

According to [12, proposition 2.2], we know that both of these functions are convex and clearly belong to $C^{1,1}(\sigma^d)$. Hence, by applying (90) to $g_{\pm}$, we immediately deduce

$$E\left[\|.\|^2 \frac{L(\nabla f)}{2} \pm f\right] \geq 0,$$

or equivalently, by using the linearity of E,

$$-\left|E\left[\|.\|^2\right]\right| \frac{L(\nabla f)}{2} \leq E\left[f\right] \leq \left|E\left[\|.\|^2\right]\right| \frac{L(\nabla f)}{2}.$$

This is equivalent to the desired result (91).

For the statement on the occurrence of equality, it is enough to note that if E satisfies (90) for all convex functions from $C^{1,1}(\sigma^d)$ then it must vanish for affine functions.

Let us now prove that (ii) implies (i). It clearly follows from (91) that

$$E\left[\|.\|^2\right] \geq 0, \tag{93}$$

and that, for any $f \in C^{1,1}(\sigma^d)$,

$$E\left[\|.\|^2 \frac{L(\nabla f)}{2} - f\right] \geq 0. \tag{94}$$

Now, let us take an arbitrary convex function $g \in C^{1,1}(\sigma^d)$, and define

$$f := \frac{L(\nabla g)}{2} \|.\|^2 - g.$$

Then, by Guessab [12, proposition 2.2], we have

$$f \in C^{1,1}(\sigma^d) \text{ with } L(\nabla f) \leq L(\nabla g). \tag{95}$$

Furthermore, since

$$g = \frac{L(\nabla g)}{2} \|.\|^2 - f,$$

we obviously have

$$g = \left(\|.\|^2 \frac{L(\nabla f)}{2} - f\right) + \|.\|^2 \left(\frac{L(\nabla g)}{2} - \frac{L(\nabla f)}{2}\right),$$

and hence we arrive at

$$E[g] = E\left[\|.\|^2 \frac{L(\nabla f)}{2} - f\right] + E\left[\|.\|^2\right]\left(\frac{L(\nabla g)}{2} - \frac{L(\nabla f)}{2}\right).$$

Finally, (93), (94) together with (95) yield that (i) is valid. This shows the equivalence between these two statements.

Let us recall some important facts from numerical integration methods. In one dimensional case it suffices to define quadrature rules on a "standard" interval, as any other interval could be transformed to this standard interval. In the multidimensional case, it is convenient to have a standard (or reference) simplex $\hat{\sigma}^d$ spanned by the vertices $\boldsymbol{e}_0, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$. Here we use the notation $\boldsymbol{e}_0 = \boldsymbol{0}$ and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is the standard basis for $\mathbb{R}^d$, with the coordinates of $\boldsymbol{e}_j$ equal to 0 in all entries except for the jth entry, where the coordinate is equal to 1. Thus, the faces $\hat{\tau}_1^{d-1}, \hat{\tau}_2^{d-1}, \ldots, \hat{\tau}_d^{d-1}$ of $\hat{\sigma}^d$ lay in coordinate hyperplanes $x_i = 0, i = 1, \ldots, d$, respectively, while the facet $\hat{\tau}_0^{d-1}$ opposite to the origin lies in the hyperplane $x_1 + x_2 + \cdots + x_d = 1$. Then any simplex $\sigma^d = [\boldsymbol{v}_0\boldsymbol{v}_1 \ldots \boldsymbol{v}_d]$ can be as an image of $\hat{\sigma}^d$ through the affine transformation

$$\begin{aligned} B_{\sigma^d} : \hat{\sigma}^d &\to \sigma^d \\ \boldsymbol{x} &\mapsto B_{\sigma^d}(\boldsymbol{x}) = (1 - x_1 - \cdots - x_d)\boldsymbol{v}_0 + x_1\boldsymbol{v}_1 + \cdots + x_d\boldsymbol{v}_d, \end{aligned}$$

with the property $B_{\sigma^d}(\boldsymbol{e}_i) = \boldsymbol{v}_i, i = 0, \ldots, d$. Now, as the measure of $\hat{\sigma}^d$ is $\frac{1}{d!}$ then we have

$$\begin{aligned} \frac{1}{|\sigma^d|}\int_{\sigma^d} f(\boldsymbol{x})\, d\boldsymbol{x} &= \frac{1}{|\hat{\sigma}^d|}\int_{\hat{\sigma}^d} f \circ B_{\sigma^d}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= d!\int_{\hat{\sigma}^d} f \circ B_{\sigma^d}(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}$$

Hence, our task reduces to the problem of constructing the trapezoidal cubature formula over reference simplex $\hat{\sigma}^d$. In this latter, it takes the following form. For any continuously differentiable convex function f on $\hat{\sigma}^d$, it holds:

$$d!\int_{\hat{\sigma}^d} f(\boldsymbol{x})\, d\boldsymbol{x} \le \hat{Q}_d^{\text{tra}}(f) := \frac{1}{d+1}\sum_{i=0}^{d} \frac{1}{\left|\hat{\tau}_i^{d-1}\right|}\int_{\hat{\tau}_i^{d-1}} f(\boldsymbol{x}) d\gamma_i. \tag{96}$$

We now give a useful identity regarding an integration rule for products of barycentric coordinates over simplices. The following identity holds for all nonnegative integers $m_0, \ldots, m_d$, see [35]:

$$\int_{\sigma^d} \prod_{i=0}^{d} \lambda_i^{m_i}(\boldsymbol{x})\, d\boldsymbol{x} = \frac{|\Delta_\sigma| \prod_{i=0}^{d} m_i!}{(d+m_0+\cdots+m_d)!}, \tag{97}$$

where $|\Delta_\sigma| = \det(v_1 - v_0, \ldots, v_d - v_0) = d!\left|\sigma^d\right|$.

As a consequence of Theorem 6, the following explicit integration error is obtained.

Corollary 1 *Let $f \in C^{1,1}(\hat{\sigma}^d)$ with $L(\nabla f)$-Lipschitz gradient. Then, for the trapezoidal cubature formula $\hat{Q}_d^{tra}$, the following error estimate holds:*

$$\hat{E}[f] := \hat{Q}_d^{tra}(f) - \frac{1}{\left|\hat{\sigma}^d\right|}\int_{\hat{\sigma}^d} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \frac{dL(\nabla f)}{(d+2)(d+1)^2}. \tag{98}$$

Proof By Theorem 6, it suffices to determine the integration error associated with the function $\|.\|^2$. The following identities can be obtained directly from the general formula (97):

$$\hat{Q}_d^{\text{tra}}(x_j^2) = \frac{2}{(d+1)^2},\ (j = 1, \ldots, d), \tag{99}$$

$$\frac{1}{\left|\hat{\sigma}^d\right|}\int_{\hat{\sigma}^d} \|\boldsymbol{x}\|^2\, d\boldsymbol{x} = \frac{2d}{(d+1)(d+2)}. \tag{100}$$

The required result now follows from Theorem 6.

6 Numerical Experiments

In this section, we provide some numerical tests, which we perform in order to validate our theoretical predictions. Here, we limit our experiments to the case of a simplex; however, we refer to [17–19] for many experiments in more general domains. The properties of the method derived in this work are illustrated using the following two bivariate test functions:

$$f_1(x, y) = \exp(ax + by),\ a, b \in \mathbb{R},\ (x, y) \in \Omega = [0, 1]^2, \tag{101}$$

$$f_2(x, y) = \sin(ax + by),\ a, b \in \mathbb{R}^+,\ (x, y) \in \Omega = [0, 1]^2. \tag{102}$$

Both of them are infinitely differentiable functions having singularities (large gradient and rapid oscillations), when a or b is large enough. Note that $f_1(x, y)$ is convex and if $a + b > \pi$, $f_2(x, y)$ is convex and concave at the same time, i.e. there are subdomains of Ω where $f_2(x, y)$ is convex and concave in others, see Fig. 1.

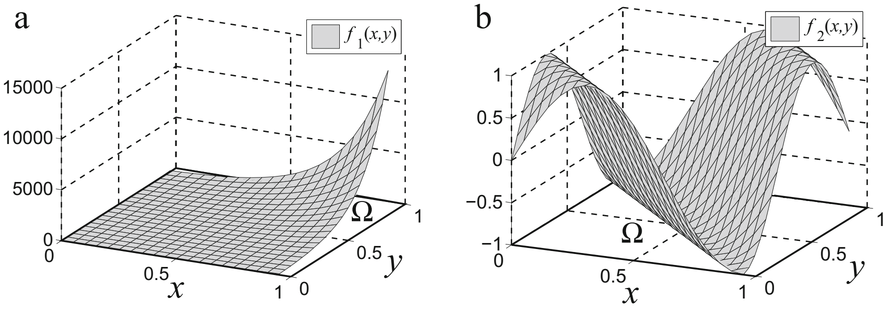

Fig. 1 Test functions $f_1(x, y)$ (**a**), $f_2(x, y)$ (**b**) for the case $a = b = 5$

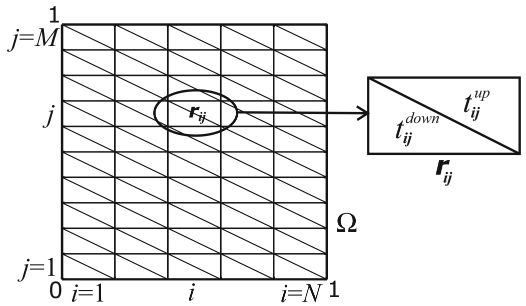

Fig. 2 Domain Ω and its NM-triangulation

We shall use the proposed methods $\hat{Q}_d^{\text{tra}}(f)$, as defined in (96) for computing numerical values of the integrals of f over Ω in the case $d = 2$ and $f = f_1(x, y)$ or $f = f_2(x, y)$. To this end we introduce simple uniform decomposition of Ω, see Fig. 2, dividing Ω into NM equal rectangles r_{ij}, $i = 1, \dots, N$, $j = 1, \dots, M$. Further usually we shall consider the case $N = M$. Each rectangle r_{ij} is divided in natural way into two right triangles t_{ij}^{up}, t_{ij}^{down}. This uniform decomposition of Ω we shall call NM-triangulation. Numerical values of integrals of f_1, f_2 over Ω will be computed by applying our approximate formulas in each of triangles t_{ij}^{up}, t_{ij}^{down} and by summing the results over all i, j.

Exact values of integrals are

$$I(f_1) = \frac{(\exp(a) - 1)(\exp(b) - 1)}{ab}, \ I(f_2) = \frac{\sin(a) + \sin(b) - \sin(a + b)}{ab}.$$

Let $E_{NM}^{\text{tra}}(f)$ be relative error of method $\hat{Q}_2^{\text{tra}}(f)$, obtained using NM-triangulation

$$E_{NM}^{\text{tra}}(f) = \frac{\hat{Q}_2^{\text{tra}}(f) - I(f)}{I(f)}, \ f \in \{f_1, f_2\}.$$

If $N = M$ we shall also use the notation $E_N^{\text{tra}}(f)$ or E_N^{tra}.

Table 1 Errors of integration obtained in tests for f_1 with $a = b = 1$, $N = M$

N	16	32	64	128	256	512
$E_N^{\mathrm{Tra}}(g)$	0.00011	2.71E–05	6.78E–06	1.7E–06	4.24E–07	1.06E–07

Table 2 Orders of convergence obtained in tests for f_1 with $a = b = 1$, $N = M$

N	16	32	64	128	256	512
$E_N^{\mathrm{Tra}}(g)$	2.00035	2.00009	2.00002	2.00001	2.00001	2

Table 3 Errors of integration obtained in tests for f_2 with $a = 10$, $b = 50$, $N = M$

N	16	32	64	128	256	512	1024
$E_N^{\mathrm{Tra}}(g)$	−0.27133	−0.05917	−0.01437	−0.00357	−0.00089	−0.00022	−5.56

Table 4 Orders of convergence obtained in tests for f_2 with $a = 10$, $b = 50$, $N = M$

N	16	32	64	128	256	512
$E_N^{\mathrm{Tra}}(g)$	7.79623	2.24357	2.05183	2.01249	2.0031	2.00077

It is necessary to note that while computing the integral of f_1 by formula $\hat{Q}_2^{\mathrm{tra}}(f_1)$, we have been checking the inequality (87). Numerical values of the integral often satisfied it.

In Tables 1, 2, 3 and 4 the relative errors and the orders of convergence obtained in tests for f_1 with $a = b = 1$ and for f_2 with $a = 10$, $b = 50$ are given. To compute the order of convergence of method $\hat{Q}_2^{\mathrm{tra}}(f)$ in case $N = M$, we used the formula

$$R_N^{\mathrm{tra}}(f) = \log_2 \frac{|\hat{Q}_2^{\mathrm{tra},2N}(f) - \hat{Q}_2^{\mathrm{tra},N}(f)|}{|\hat{Q}_2^{\mathrm{tra},N}(f) - \hat{Q}_2^{\mathrm{tra},N/2}(f)|}, \quad f \in \{f_1, f_2\},$$

where $\hat{Q}_2^{\mathrm{tra},N}(f)$ is the approximate value of integral of f, obtained using the trapezoidal cubature formula $\hat{Q}_2^{\mathrm{tra}}(f)$ and NN-triangulation of Ω.

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On the Stability of the Triangular Equilibrium Points in the Photogravitational R3BP with an Oblate Infinitesimal and Triaxial Primaries for the Binary Lalande 21258 System

Jessica Mrumun Gyegwe, Aguda Ekele Vincent, and Angela E. Perdiou

Abstract In the framework of the planar circular restricted three-body problem (R3BP), we explore the effects of oblateness of the infinitesimal mass body as well as radiation pressure and triaxiality of the two primaries on the position and stability of the triangular equilibrium points (TEPs). It is found that all the involved parameters affect the positions and stability of these points. Specifically, it has been shown that TEPs are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leqslant \mu \leqslant 1/2$, where μ_c denotes the critical mass parameter which depends on system's parameters. In addition, all the parameters of the bigger primary, except that of triaxiality, have destabilizing tendencies resulting in a decrease in the size of the region of stability. Finally, we justify the relevance of the model in astronomy by applying it to the binary Lalande 21258 system for which the equilibrium points have been seen to be unstable.

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1 Introduction

The restricted three-body problem (R3BP) has to do with three bodies or masses. Two of the bodies have tangible masses and are known as the primaries while

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the third body has a negligible mass (massless) and is often referred to as the infinitesimal body. The primaries move in circular orbit about a common barycentre, and their motion is not affected by that of the infinitesimal body (or test particle), whereas the motion of the primaries affects that of the test particle [17, 36]. This problem has also some simpler versions such as Hill's three-body problem which can be treated as a perturbed two-body problem and has many applications in the Sun–Earth–Satellite system (see for example, [6, 12, 13, 19, 37, 41], among others) or the Sitnikov problem in which the massless body moves along a straight line that is perpendicular to the orbital plane formed by the two primaries of equal masses, and this rectilinear motion can be used as generator of families of spatial periodic orbits for the classical R3BP or for its modifications (see, e.g., [2, 3, 21–23, 26, 42, 43] and references therein).

Five stationary points (Lagrangian points) exist where the force on the massless body is zero. This is because the true gravitational forces exerted by both primaries are just cancelled by the centrifugal force at these points. Three of the Lagrangian points, denoted by L_1, L_2, and L_3, lie on the line joining the primaries and are called collinear equilibrium points while the other two, denoted by L_4 and L_5, are called triangular equilibrium points (TEPs) as they form the third vertex of a triangle with the primaries. The TEPs are very important in the natural world since they have been found to be the home of asteroids in at least three systems. For instance, on January 25, 2010, the only Earth Trojan also known as 2010TK7 was discovered. The Earth Trojan is an asteroid that orbits the Sun in the vicinity of the Sun–Earth Lagrangian points L4 (leading 600) or L5 (trailing 600), making it to have the same type of Earth's orbit [8]. Also, in July, 2011 a total of four thousand nine hundred and seventeen (4917) Trojan Asteroids were found in the Sun–Jupiter systems, three thousand one hundred and sixty-eight (3168) around the L_4 point, and one thousand six hundred and forty-five (1645) around the L_5 point [34].

Many researchers over the years have made modifications to the classical R3BP. Some of the modifications made include the consideration of one or both primaries as being sources of radiation pressure and/or oblate spheroids and/or triaxial rigid bodies (see, e.g., [4, 5, 11, 35, 40], among others). Regarding the TEPs, Devi and Singh [7] found their location in the photogravitational circular R3BP with perturbations in the Coriolis and centrifugal forces. Their results showed that these points are affected by all the parameters involved. Singh and Begha [29] studied the existence and stability of equilibrium points under the influence of small perturbations in the Coriolis and centrifugal forces when the primary and secondary are triaxial and oblate spheroids, respectively. It was observed that the TEPs are stable for certain interval of the mass ratio while the collinear equilibrium points remain unstable. In the framework of the photogravitational version of the R3BP, Kumar and Sharma [15] considered the case where the primaries are oblate spheroids and explored both the positions and stability of TEPs. For the elliptic R3BP when the primary is an oblate spheroid and the secondary a source of radiation, Kalantonis et al. [14] studied the stability of TEPs and determined, both analytically and numerically, the transition curves which separate stable from unstable regions in the parametric space.

By modelling the primaries as triaxial rigid bodies as well as sources of radiation together with small perturbations in the Coriolis and centrifugal forces, Singh [28] examined the existence and stability of the equilibrium points in the R3BP. It was observed that the positions of the usual five (three collinear and two triangular) equilibrium points are affected by the radiation, triaxiality, and a small perturbation in the centrifugal force, but are unaffected by that of the Coriolis force. The collinear equilibrium points remained unstable while the TEPs are seen to be stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leqslant \mu \leqslant 1/2$, where μ_c is the critical mass ratio influenced by small perturbations in the Coriolis and centrifugal forces, radiation, and triaxiality. In the same vein, Singh and Simeon [32] explored the existence and linear stability of the TEPs in the framework of circular R3BP with the postulation that the primaries are triaxial rigid bodies, radiating in nature and are also under the influence of Poynting–Robertson (P–R) drag. Numerical simulations were made using the binary stars Kruger 60 (AB) and Archird. It was observed that the TEPs move towards the line joining the primaries in the direction of the bigger primary with increasing triaxiality, and the TEPs are unstable owing to the destabilizing influence of P–R drag. Also, Gao and Wang [10] considered the HD 191408 binary system in the case of triaxial and radiating primaries and presented bifurcation diagrams in the parameter space as well as they provided semi-analytical periodic solutions about the equilibrium points. Additionally, in the case where the primaries move in elliptical orbits around their common barycentre, Zahra et al. [39] investigated the location and stability of TEPs under the effects due to the triaxiality of the more massive primary, the oblateness of the less massive one as well as relativistic corrections.

By utilizing a different approach, Pathak and Elshaboury [20] within the framework of the R3BP when both primaries are triaxial rigid bodies constructed for different cases of Euler's angles, the locations of the TEPs, and the stability conditions of motion in the proximity of these points. The numerical solution was obtained by using a fourth order Runge–Kutta–Gill integrator. Similarly, Selim et al. [25] studied analytically the existence and the stability of the libration points in the R3BP, when the primaries are triaxial rigid bodies in the case when the Euler angles of the rotational motion are equal to $\theta_i = \pi/2$, $\psi_i = 0$, and $\Phi_i = \pi/2$, $i = 1, 2$. It was established that the locations and the stability of the TEPs change according to the effect of the triaxiality of the primaries. By taking into consideration the shape of the infinitesimal body, Narayan et al. [18] investigated the pulsating surfaces of zero velocity of the elliptic R3BP when the primaries are luminous oblate spheroids as well as a consideration of the effect of the oblateness of the infinitesimal. In the case of a triaxial rigid primary body, Saeed and Zotos [24] revealed numerically the way in which the triaxiality parameters affect the position and linear stability of the libration points.

In this paper, we examine the existence and stability of triangular equilibrium points in the restricted three-body problem when the primaries are modelled as triaxial rigid bodies as well as sources of radiation pressure while the infinitesimal body is taken to be an oblate spheroid. Particularly, in Sect. 2, the equations of motion of this mathematical model are presented while the existence and location of the TEPs are established in Sect. 3. In Sect. 4, we give an analysis of the dynamics around the TEPs by examining their stability. In Sect. 5, a numerical simulation is made by using the physical data of the binary Lalande 21258 system and in Sect. 6, we discuss the results and make the relevant conclusions.

2 Equations of Motion

The equations of motion in the barycentric, synodic, and dimensionless coordinate system Oxyz of the massless body (see, [27, 28]) are represented by:

$$\ddot{x} - 2n\dot{y} = \Omega_x, \qquad \ddot{y} + 2n\dot{x} = \Omega_y, \tag{1}$$

where Ω represents the pseudo-force (potential function) and is given by:

$$\begin{aligned}\Omega = \frac{n^2}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\Phi_1-\Phi_2)q_1}{2r_1^3} + \frac{\mu(2\Psi_1-\Psi_2)q_2}{2r_2^3} \\ - \frac{3(1-\mu)(\Phi_1-\Phi_2)q_1y^2}{2r_1^5} - \frac{3\mu(\Psi_1-\Psi_2)q_2y^2}{2r_2^5} + \frac{(1-\mu)A_3}{2r_1^3} + \frac{\mu A_3}{2r_2^3},\end{aligned} \tag{2}$$

while

$$r_1 = [(x-\mu)^2 + y^2]^{1/2}, \quad r_2 = [(x+1-\mu)^2 + y^2]^{1/2}, \tag{3}$$

are the distances of the third body from the primary and secondary body, respectively and $\mu \in (0, 1/2]$ (the mass parameter) is the ratio of the mass of smaller primary to the total mass of the primaries. The perturbed, due to the triaxiality of the primaries, mean motion n is given by the formula:

$$n = \sqrt{1 + \frac{3}{2}(2\Phi_1 - \Phi_2) + \frac{3}{2}(2\Psi_1 - \Psi_2)}, \tag{4}$$

with

$$\Phi_1 = \frac{\Xi_{t_1}^2 - \Xi_{t_3}^2}{5R^2}, \quad \Phi_2 = \frac{\Xi_{t_2}^2 - \Xi_{t_3}^2}{5R^2}, \quad \Psi_1 = \frac{\Xi_{t_1}'^2 - \Xi_{t_3}'^2}{5R^2}, \quad \Psi_2 = \frac{\Xi_{t_2}'^2 - \Xi_{t_3}'^2}{5R^2}, \tag{5}$$

characterizing the triaxiality of the primary and secondary body, respectively, with $\Phi_i,\ \Psi_i \ll 1$, $i = 1, 2$, where $\Xi_{t_1},\ \Xi_{t_2},\ \Xi_{t_3}$ are the semi-axes of the larger primary body, $\Xi_{t_1}',\ \Xi_{t_2}',\ \Xi_{t_3}'$ are the semi-axes of the smaller one, and R is the dimensional distance between the primaries. Also, $A_3 \ll 1$ is the oblateness coefficient of the infinitesimal body defined by:

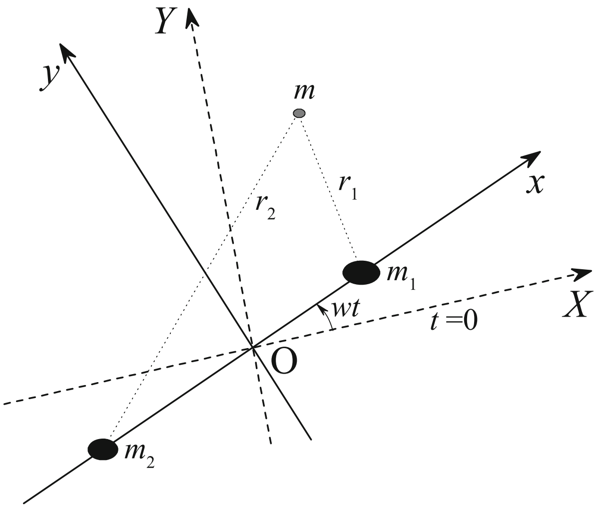

Fig. 1 The configuration of the rotating coordinate system Oxy for the photogravitational R3BP where m_1, m_2 are the triaxial primaries and m is the oblate infinitesimal body while OXY represents the inertial frame

$$A_3 = \frac{\Xi_{o_1}^2 - \Xi_{o_3}^2}{5R^2}, \tag{6}$$

where Ξ_{o_1}, Ξ_{o_2}, Ξ_{o_3} being the semi-axes of the infinitesimal body with $\Xi_{o_1} = \Xi_{o_2} > \Xi_{o_3}$, while q_i, $i = 1, 2$, stand for the radiation factors of the primary and secondary bodies respectively and are given by $F_{pi} = F_{gi}(1 - q_i)$, $i = 1, 2$, such that $0 < 1 - q_i = \delta_i \ll 1$ where F_{gi} and F_{pi} are the gravitational and radiation pressure forces, respectively. The energy (Jacobi) integral of this problem is given by the expression:

$$C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2), \tag{7}$$

where C is the Jacobian-like constant while $\dot{x}$ and $\dot{y}$ are the velocity components. The configuration of the rotating coordinate system for the problem is presented in Fig. 1.

3 Location of the Triangular Points

The equilibrium points (or Lagrangian points) are obtained when the acceleration and velocity of the infinitesimal body are zero. In other words, the equilibrium points are obtained when the infinitesimal body is experiencing a state of rest. These points are the solutions of the following system of non-linear algebraic equations:

$$\Omega_x = n^2 x - \frac{(1-\mu)(x-\mu)q_1}{r_1^3} - \frac{\mu(x-\mu+1)q_2}{r_2^3} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5}$$
$$-\frac{3A_3\mu(x-\mu+1)}{2r_2^5} - \frac{3(1-\mu)(x-\mu)(2\Phi_1-\Phi_2)q_1}{2r_1^5}$$
$$-\frac{3\mu(x-\mu+1)(2\Psi_1-\Psi_2)q_2}{2r_2^5}$$
$$+\frac{15(1-\mu)(x-\mu)(\Phi_1-\Phi_2)q_1y^2}{2r_1^7} + \frac{15\mu(x-\mu+1)(\Psi_1-\Psi_2)q_2y^2}{2r_2^7} = 0,$$
$$\Omega_y = \left[n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3A_3\mu}{2r_2^5}\right.$$
$$-\frac{3(1-\mu)(\Phi_1-\Phi_2)q_1}{r_1^5} - \frac{3(1-\mu)(2\Phi_1-\Phi_2)q_1}{2r_1^5} - \frac{3\mu(\Psi_1-\Psi_2)q_2}{r_2^5}$$
$$\left.-\frac{3\mu(2\Psi_1-\Psi_2)q_2}{2r_2^5} + \frac{15(1-\mu)(\Phi_1-\Phi_2)q_1y^2}{2r_1^7} + \frac{15\mu(\Psi_1-\Psi_2)q_2y^2}{2r_2^7}\right] y = 0, \tag{8}$$

for $y \neq 0$ while Ω_x and Ω_y are the partial derivatives of the potential function (2). If we take $\Phi_1 = \Phi_2 = \Psi_1 = \Psi_2 = 0 = A_3$ and $q_1 = q_2 = 1$ in Eqs. (8), we fall on the classical R3BP and the solutions yield $r_1 = r_2 = 1$ while from Eq. (4) we have $n = 1$. Now, with the presence of the perturbation parameters, i.e. $\Phi_1,\ \Phi_2,\ \Psi_1,\ \Psi_2,\ A_3 \neq 0$ and $q_1,\ q_2 \neq 1$, we assume that the solutions of System (8) are:

$$r_1 = 1 + \Theta_1, \qquad r_2 = 1 + \Theta_2, \tag{9}$$

where $\Theta_1,\ \Theta_2 \ll 1$. Next, we substitute the values of r_i, $i = 1, 2$, from Eqs. (9) into Eqs. (3) and solving for x and y retaining only linear terms in Θ_1 and Θ_2, we obtain:

$$x = \Theta_2 - \Theta_1 + \mu - \frac{1}{2}, \qquad y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\Theta_1 + \Theta_2)\right]. \tag{10}$$

In order to find the values of the small quantities Θ_1 and Θ_2, we make use of $r_{1,2}$, x and y from Eqs. (9) and (10), respectively, and n^2 from Eq. (4). These are then substituted into the two equations of System (8) appropriately, such that higher order terms in $\Phi_1,\ \Phi_2,\ \Psi_1,\ \Psi_2, q_1 = 1-\delta_1, q_2 = 1-\delta_2$ ($\delta_{1,2} \ll 1$), and A_3 are neglected. Thus, we get:

$$\Theta_1 = -\frac{\delta_1}{3} - \frac{11}{8}\Phi_1 + \frac{11}{8}\Phi_2 + \left[\frac{\mu}{2(1-\mu)} - 1\right]\Psi_1 + \left[\frac{1}{2} - \frac{\mu}{2(1-\mu)}\right]\Psi_2 + \frac{1}{2}A_3,$$
$$\Theta_2 = -\frac{\delta_2}{3} - \left[\frac{3}{2} - \frac{1}{2\mu}\right]\Phi_1 + \left(1 - \frac{1}{2\mu}\right)\Phi_2 - \frac{11}{8}\Psi_1 + \frac{11}{8}\Psi_2 + \frac{1}{2}A_3. \tag{11}$$

As a result, the coordinates of the TEPs are obtained, after substituting (11) into Eqs. (10), in the following form:

$$x_0 = \frac{1}{2}\left\{2\mu - 1 + \frac{2}{3}(1-q_1) - \frac{2}{3}(1-q_2) + \left(\frac{1}{\mu} - \frac{1}{4}\right)\Phi_1 - \left(\frac{1}{\mu} + \frac{3}{4}\right)\Phi_2 - \left[\frac{\mu}{1-\mu} + \frac{3}{4}\right]\Psi_1 + \left[\frac{\mu}{1-\mu} + \frac{7}{4}\right]\Psi_2\right\},$$

$$y_0 = \pm\frac{\sqrt{3}}{2}\left\{1 - \frac{2(1-q_1)}{9} - \frac{2(1-q_2)}{9} + \frac{1}{3}\left[\left(\frac{1}{\mu} - \frac{23}{4}\right)\Phi_1 + \left(\frac{19}{4} - \frac{1}{\mu}\right)\Phi_2 + \left[\frac{\mu}{1-\mu} - \frac{19}{4}\right]\Psi_1 - \left[\frac{\mu}{1-\mu} - \frac{15}{4}\right]\Psi_2 + 2A_3\right]\right\}. \tag{12}$$

Note here that for $A_3 = 0$ in (12), the positions of the triangular equilibrium points conform to those given by Singh in [28] when the perturbation in the centrifugal force is neglected there (case $\beta = 1$). It is seen that the pair of points $(x_0, \pm y_0)$ which correspond to the positions of the TEPs $L_{4,5}$, are affected by the triaxiality and radiation of the primaries as well as the oblateness of the infinitesimal body. Meanwhile, the point x_0 is independent of the oblateness of the infinitesimal body; it is only affected by triaxiality and radiation factors of the primaries whereas the point y_0 is affected by the oblateness of the infinitesimal body combined with triaxiality and radiation pressure of both primaries.

4 Analysis of the Dynamics Around the Triangular Equilibrium Points

We obtain the variational equations by first of all considering small displacements at the triangular equilibrium points $L(x_0, y_0)$. That is:

$$x = x_0 + \mathrm{X}, \qquad y = y_0 + \mathrm{Y}, \tag{13}$$

where X and Y are small displacements in (x_0, y_0). By substituting the last two equations in (1), we get:

$$\begin{aligned} \ddot{\mathrm{X}} - 2n\dot{\mathrm{Y}} &= \Omega^0_{xx}\mathrm{X} + \Omega^0_{xy}\mathrm{Y}, \\ \ddot{\mathrm{Y}} - 2n\dot{\mathrm{X}} &= \Omega^0_{xy}\mathrm{X} + \Omega^0_{yy}\mathrm{Y}, \end{aligned} \tag{14}$$

where the superscript “0” specifies that the partial derivatives are being evaluated at the triangular equilibrium points (x_0, y_0) given by (12).

In order to examine the stability of the TEPs, we make use of the characteristic equation which has been obtained from system (14) in the form:

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - (\Omega^0_{xy})^2 = 0, \tag{15}$$

where

$$\begin{aligned}\Omega^0_{xx} &= \frac{3}{4} - \left(\frac{1}{2} - \frac{3\mu}{2}\right)(1-q_1) + \left(1 - \frac{3\mu}{2}\right)(1-q_2) + \left(\frac{57}{16} + \frac{45\mu}{16} - \frac{3}{2\mu}\right)\Phi_1 \\ &+ \left(-\frac{3}{16} - \frac{93\mu}{16} + \frac{3}{2\mu}\right)\Phi_2 + \left(\frac{39}{8} - \frac{69\mu}{16} - \frac{3\mu^2}{2(1-\mu)}\right)\Psi_1 \\ &+ \left(-\frac{9}{2} + \frac{117\mu}{16} + \frac{3\mu^2}{2(1-\mu)}\right)\Psi_2,\end{aligned}$$

$$\begin{aligned}\Omega^0_{yy} &= \frac{9}{4} + 3A_3 + \left(\frac{1}{2} - \frac{3\mu}{2}\right)(1-q_1) - \left(1 - \frac{3\mu}{2}\right)(1-q_2) + \left(\frac{87}{16} - \frac{45\mu}{16} + \frac{3}{2\mu}\right)\Phi_1 \\ &+ \left(-\frac{21}{16} + \frac{45\mu}{16} - \frac{3}{2\mu}\right)\Phi_2 + \left(\frac{33}{8} + \frac{135\mu}{16} + \frac{45\mu^2 - 33\mu}{8(1-\mu)}\right)\Psi_1 \\ &+ \left(-\frac{135\mu}{16} + \frac{33\mu - 45\mu^2}{8(1-\mu)}\right)\Psi_2,\end{aligned}$$

$$\begin{aligned}\Omega^0_{xy} &= \sqrt{3}\left\{-\frac{3}{4} + \left(-\frac{1}{2} + \mu\right)A_3 + \frac{3\mu}{2} + \left(\frac{1}{6} + \frac{\mu}{6}\right)(1-q_1) + \left(\frac{\mu}{6} - \frac{1}{3}\right)(1-q_2)\right. \\ &+ \left(-\frac{47}{16} + \frac{89\mu}{16} + \frac{1}{2\mu}\right)\Phi_1 + \left(\frac{9}{16} - \frac{37\mu}{16} - \frac{1}{2\mu}\right)\Phi_2 \\ &\left.+ \left(-\frac{25}{8} + \frac{85\mu}{16} - \frac{\mu+\mu^2}{4(1-\mu)}\right)\Psi_1 + \left(\frac{9}{4} - \frac{33\mu}{16} + \frac{\mu+\mu^2}{4(1-\mu)}\right)\Psi_2\right\}.\end{aligned}$$

Simplifying Eq. (15), we have:

$$\Lambda^2 + \mathsf{Q}\Lambda + \mathsf{W} = 0, \tag{16}$$

where λ^2 has been replaced by Λ while:

$$\mathsf{Q} = 1 - 3A_3 + 3\Phi_1 + 3(\mu - \frac{3}{2})\Phi_2 + 3\Psi_1 - 3(\mu + \frac{1}{2})\Psi_2 > 0,$$

and

$$\begin{aligned}\mathsf{W} &= \mu(1-\mu)\left[\frac{27}{4} + 9A_3 + \frac{3}{2}(1-q_1) + \frac{3}{2}(1-q_2)\right] + \left(-\frac{45}{8} + \frac{891}{16}\mu - \frac{801}{16}\mu^2\right)\Phi_1 \\ &+ \left(\frac{45}{8} - \frac{423}{16}\mu + \frac{333}{16}\mu^2\right)\Phi_2 + \left(\frac{711}{16}\mu - \frac{801}{16}\mu^2\right)\Psi_1 - \left(\frac{243}{16}\mu - \frac{333}{16}\mu^2\right)\Psi_2.\end{aligned}$$

By solving Eq. (16) we get the roots:

$$\Lambda_{1,2} = \frac{1}{2}\left(-\mathsf{Q} \pm \sqrt{\mathsf{Q}^2 - 4\mathsf{W}}\right), \tag{17}$$

so, the roots of the characteristic polynomial (15) are $\lambda_1 = +\Lambda_1^{1/2}$, $\lambda_2 = -\Lambda_1^{1/2}$, $\lambda_3 = +\Lambda_2^{1/2}$, and $\lambda_4 = -\Lambda_2^{1/2}$. These roots depend on the value of the parameters $\mu,\ \Phi_1,\ \Phi_2,\ \Psi_1,\ \Psi_2$, and A_3. Explicitly, the discriminant is:

$$\begin{aligned}\Delta &= \mathsf{Q}^2 - 4\mathsf{W}\\ &= \mu^2\left[27 + 36A_3 + 6(1-q_1) + 6(1-q_2) + \frac{801}{4}(\Phi_1 + \Psi_1) - \frac{333}{4}(\Phi_2 + \Psi_2)\right]\\ &\quad -\mu\left[27 + 36A_3 + 6(1-q_1) + 6(1-q_2) + \frac{891}{4}\Phi_1 - \frac{447}{4}\Phi_2 + \frac{711}{4}\Psi_1 - \frac{219}{4}\Psi_2\right]\\ &\quad +1 - 6A_3 + \frac{57}{2}\Phi_1 - \frac{63}{2}\Phi_2 + 6\Psi_1 - 3\Psi_2.\end{aligned} \tag{18}$$

We note that, for the values of the mass parameter $\mu = 0$ and $\mu = \frac{1}{2}$, the discriminant is:

$$(\Delta)_{\mu=0} = 1 - 6A_3 + \frac{57}{2}\Phi_1 - \frac{63}{2}\Phi_2 + 6\Psi_1 - 3\Psi_2 > 0, \tag{19}$$

and

$$(\Delta)_{\mu=\frac{1}{2}} = \frac{-23}{4} - 15A_3 - \frac{3}{2}(1-q_1) - \frac{3}{2}(1-q_2) - \frac{525}{16}\Phi_1 + \frac{57}{16}\Phi_2 - \frac{525}{16}\Psi_1 + \frac{57}{16}\Psi_2 < 0, \tag{20}$$

respectively, where the parameters are very small quantities. Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=0.5}$ are of opposite signs, there is only one value of μ in the open interval $(0, \frac{1}{2})$ for which Δ vanishes and it is denoted as μ_c (the critical mass parameter).

The solution of the quadratic equation $\Delta = 0$ for μ gives the critical mass ratio value μ_c of the mass parameter, namely:

$$\mu_c = \mu_{\mathrm{B}} + \mu_{\mathrm{O}} + \mu_{\mathrm{R}} + \mu_{\mathrm{T}}, \tag{21}$$

with

$$\mu_{\mathrm{B}} = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right), \qquad \mu_{\mathrm{O}} = -\frac{22A_3}{9\sqrt{69}}, \qquad \mu_{\mathrm{R}} = -\frac{2}{27\sqrt{69}}[(1-q_1)+(1-q_2)],$$

and

$$\begin{aligned}\mu_{\mathrm{T}} = &\left(\frac{5}{12} + \frac{59}{18\sqrt{69}}\right)\Phi_1 - \left(\frac{19}{36} + \frac{85}{18\sqrt{69}}\right)\Phi_2\\ &- \left(\frac{5}{12} - \frac{59}{18\sqrt{69}}\right)\Psi_1 + \left(\frac{19}{36} - \frac{85}{18\sqrt{69}}\right)\Psi_2.\end{aligned}$$

Clearly, μ_c represents the combined effects of radiation and triaxiality of the primaries with oblateness of the infinitesimal body on the critical mass value of the restricted three-body problem. On ignoring A_3, μ_c confirms that obtained by Singh [28] in the absence of perturbations in Coriolis and centrifugal forces. Also by setting appropriate parameter(s) to zero, our results are in excellent agreement with those obtained previously.

Next, we consider, separately, the following three regions of values of μ:

(i) For the interval $0 \leqslant \mu < \mu_c$ we have $\Delta < 0$. Here, the values of $\lambda_{1,2,3,4}$, which are the roots of Eq. (17), are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are linearly stable.
(ii) For the interval $\mu_c < \mu \leqslant \frac{1}{2}$ we have $\Delta < 0$. Here, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
(iii) When $\mu = \mu_c$, $\Delta = 0$ the values of $\lambda_{1,2,3,4}$, which are the roots of Eq. (17) are the same. This induces instability of the triangular points.

Hence, the stability region for the TEPs is defined by:

$$0 < \mu < \mu_{\text{B}} - \frac{22A_3}{9\sqrt{69}} - \frac{2}{27\sqrt{69}}[(1-q_1)+(1-q_2)]\left(\frac{5}{12}+\frac{59}{18\sqrt{69}}\right)\Phi_1 - \left(\frac{19}{36}+\frac{85}{18\sqrt{69}}\right)\Phi_2 - \left(\frac{5}{12}-\frac{59}{18\sqrt{69}}\right)\Psi_1 + \left(\frac{19}{36}-\frac{85}{18\sqrt{69}}\right)\Psi_2, \tag{22}$$

where $\mu_{\text{B}} = 0.0385\ldots$ is the Routh's value.

5 Numerical Simulation

In this section, we compute and examine numerically and graphically the positions of the triangular points as well as the critical mass parameter and stability of the binary Lalande 21258 systems as presented in Table 1, for some assumed values of oblateness coefficient of the massless body and triaxiality of the primaries. The parameters M_A and M_B are the masses of Lalande 21258A and Lalande 21258B respectively, as compared to the mass of the Sun. The luminosity of the binary systems denoted by L_A and L_B, respectively, is obtained from the relation given by Mia and Kushvah in [16]:

$$\frac{L}{L_S} \approx \left(\frac{M}{M_S}\right)^{3.9}, \tag{23}$$

where L_S and M_S are the luminosity and mass of the Sun.

Radiation pressure has had a key effect on the formation of stars and shaping of clouds of dust and gases on a wide range of scales. The mass reduction factors are represented as $q_i = 1 - F_p/F_g$, $i = 1, 2$, where F_p and F_g are the radiation pressure and the gravitational attraction forces being exerted by the binary systems

Table 1 Physical parameter values for the binary Lalande 21258 system

M_A	M_B	μ	$L_A(L_S)$	$L_B(L_S)$	q_1	q_2	Φ_1	Φ_2	Ψ_1	Ψ_2	A_3
$0.48M_S$	$0.1M_S$	0.1724	0.00637	0.0000344	0.997269	0.999292	0.008	0.006	0.006	0.004	0.005

on objects around them or equivalently $q_i = 1 - \beta$, $i = 1, 2$, or on the basis of the Stefan–Boltzmann's law [30, 31, 38] as:

$$q_i = 1 - \frac{A\kappa L}{a\rho M}, \quad i = 1, 2, \tag{24}$$

where M, L, and κ are the mass, luminosity, and radiation pressure efficiency factor of a star, respectively. Also, a and ρ are the radius and density of the dust grain particles moving in the binary systems while $A = \frac{3}{16\pi cG}$ is a constant with c and G being the speed of light and Gravitational constant, correspondingly. The values of the luminosity and mass reduction factor q_i, $i = 1, 2$, have been obtained by computing in the $C.G.S.$ system of unit, using $L_s = 3.846 \times 10^{33}$ erg/s, $c = 3 \times 10^{10}$ cm/s, $G = 6.67384 \times 10^{-8}\,\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$, $M_s = 1.989 \times 10^{33}$ g, and $\kappa = 1$. Also, we have assumed the values for the radius and density of the dust grain particles as $a = 2 \times 10^{-2}$ cm and $\rho = 1.4\,\text{g/cm}^3$ [33, 38]. Arbitrary values are been used for the oblateness coefficient of the massless body A_3 and the triaxiality coefficients of the primary and secondary star, Φ_1, Φ_2, Ψ_1, and Ψ_2, as shown in Table 1.

We now proceed to numerically compute the positions of the TEPs and critical mass for Lalande 21258 binary system using the astrophysical parameters presented in Table 1. Results are presented for eight (8) different cases in Table 2. The considered cases are:

Case 1: The classical case, i.e. $\Phi_1 = \Phi_2 = \Psi_1 = \Psi_2 = A_3 = 0$, $q_1 = q_2 = 1$.
Case 2: Varying oblateness of the infinitesimal body.
Case 3: Varying Triaxiality of the primary only.
Case 4: Varying Triaxiality of the secondary only.
Case 5: Varying Triaxiality of both primaries only.
Case 6: Varying Triaxiality of the primary as well as oblateness of the infinitesimal body only.
Case 7: Varying Triaxiality of the secondary as well as oblateness of the infinitesimal body only.
Case 8: Varying Triaxiality of both primaries as well as oblateness of the infinitesimal body only.

In particular, the positions of TEPs and critical mass value for the binary Lalande 21258 system have been computed using the physical parameters, presented in Table 1, in Eqs. (12) and (21) for the eight aforementioned cases. It can be observed that each case produces two distinct TEPs with a critical mass value, indicating that every parameter under consideration, i.e., radiation pressure and triaxiality coefficient of both stars, as well as the oblateness of the massless body, has a significant effect on the positions of the TEPs and critical mass value.

The effects of the parameters involved in the positions of the TEPs are shown in Figs. 2, 3, and 4. In particular, Fig. 2 shows the positions of $L_{4,5}$ as a function of oblateness (case 2), triaxiality of the first and second primaries (cases 3 and 4), respectively, keeping the remaining parameters constant. It is observed that

Table 2 The effects of perturbations on the locations of the triangular equilibrium points and the critical mass value for Lalande 21258 system

Cases	Φ_1	Φ_2	Ψ_1	Ψ_2	A_3	q_1	q_2	$L_{4,5}(x \pm y)$	μ_c
1	0.0	0.0	0.0	0.0	0.0	1.0	1.0	(−0.3276000, ±0.8660254)	0.0385209
2	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.3233340, ±0.8656570)	0.0351647
					0.05			(−0.3233340, ±0.8887510)	0.0233937
					0.1			(−0.3233340, ±0.9176190)	0.0086798
3	0.01	0.003	0.006	0.004	0.01	0.997269	0.999292	(−0.3079570, ±0.8665960)	0.0400761
	0.03	0.015						(−0.2917550, ±0.8632490)	0.0431462
	0.05	0.03						(−0.2853790, ±0.8589910)	0.0429274
4	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.2880840, ±0.9024640)	0.0336991
			0.03	0.015				(−0.3240630, ±0.8454380)	0.0341873
			0.05	0.03				(−0.3189580, ±0.8345520)	0.0331352
5	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.3233340, ±0.8656570)	0.0351647
	0.03	0.02	0.015	0.01				(−0.3065690, ±0.8560670)	0.0372219
	0.05	0.035	0.03	0.02				(−0.2975890, ±0.8423680)	0.0362651
6	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.3233340, ±0.8656570)	0.0351647
	0.03	0.02			0.05			(−0.3081320, ±0.8848260)	0.0258938
	0.05	0.035			0.1			(−0.3017550, ±0.9094370)	0.0109612
7	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.3233340, ±0.8656570)	0.0351647
			0.015	0.01	0.05			(−0.3217710, ±0.8830860)	0.0229508
			0.03	0.02	0.1			(−0.3191670, ±0.9025110)	0.0074988
8	0.008	0.006	0.006	0.004	0.01	0.997269	0.999292	(−0.3233340, ±0.8656570)	0.0351647
	0.03	0.02	0.015	0.01	0.05			(−0.3065690, ±0.8791610)	0.0254509
	0.05	0.035	0.03	0.02	0.1			(−0.2975890, ±0.8943290)	0.0097802

increasing oblateness factor moves the TEPs farther from the line joining the primaries while increasing the triaxiality parameters brings the TEPs closer to the line connecting the primaries. We observe that the variational trend of the equilibria locations is similar to the scenario presented in Table 2. Figure 3 shows the positions of $L_{4,5}$ for combined effect of triaxiality of both primaries (case 5), triaxiality of the first primary and oblateness (case 6) and, triaxiality of the second primary and oblateness (case 7), respectively, keeping the remaining parameters constant. Obviously, with increasing triaxiality of both primaries and/or oblateness, the variational trend of the corresponding positions is similar to the scenario presented in Fig. 2. Figure 4 shows the combined effect of triaxiality of both primaries and oblateness. We observe that with increasing triaxiality of the primaries and oblateness the variational trend of the equilibria locations is similar to the scenario presented in Figs. 2 and 3. On the basis of numerical as well as graphical results, we note that the oblateness factor has greater influence on the motion of a binary system Lalande 21285 than the triaxiality.

The critical value of the mass parameter, given by Eq. (21), shows the effects of the various parameters on the size of the region of stability. It is seen from Table 3 that an increase of some parameters results in a decrease/increase in the size of the stability region. It is worth mentioning that the comprehensive effects of the perturbations have stabilizing tendencies. However, triaxiality of the smaller primary has tendency for instability. The four roots of characteristic equation (16) are presented in Table 3 for a wide range of the remaining parameters Φ_1, Φ_2, Ψ_1, Ψ_2, and A_3. It is clear from this table that, for a particular set of values of these parameters there exists at least a complex root with positive real part; consequently, the motion is unbounded and thus unstable.

6 Discussion and Conclusions

A modification of the photogravitational restricted three-body problem in which the two primaries are triaxial rigid bodies while the third body of negligible mass is an oblate spheroid was explored. This special modified version can be considered as a generalization of the classical restricted three-body problem in the sense that the shape of the involved bodies was taken into account as well as additional forces, except that of the gravitation, were also adapted.

In particular, the location of the triangular equilibrium points was obtained in semi-analytical form and it was found that their position is affected by all the involved parameters, i.e. by the triaxiality and radiation of the two primary bodies as well as the oblateness of the massless one. Specifically, it was shown that the relevant abscissa is independent of the oblateness coefficient while the corresponding ordinate is expressed through all the parameters of the system. The linear stability of the triangular equilibrium points was also investigated. Particularly, it was demonstrated that the corresponding equilibria are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leqslant \mu \leqslant 1/2$, where μ_c depends on all system's parameters and denotes the value of the critical mass parameter at which stability gives place to instability and vice versa. Moreover, it was observed that the parameters of the problem play significant role on the regions of stability since it was identified that the comprehensive effects of the perturbations have destabilizing tendencies. This can be verified from the results of Table 2 in which the critical mass is incorporated, and it is clear that the values of the critical mass approach zero with the addition of triaxiality and/or oblateness factors of the participating bodies. The model under consideration was applied to the binary system Lalande 21258 by using

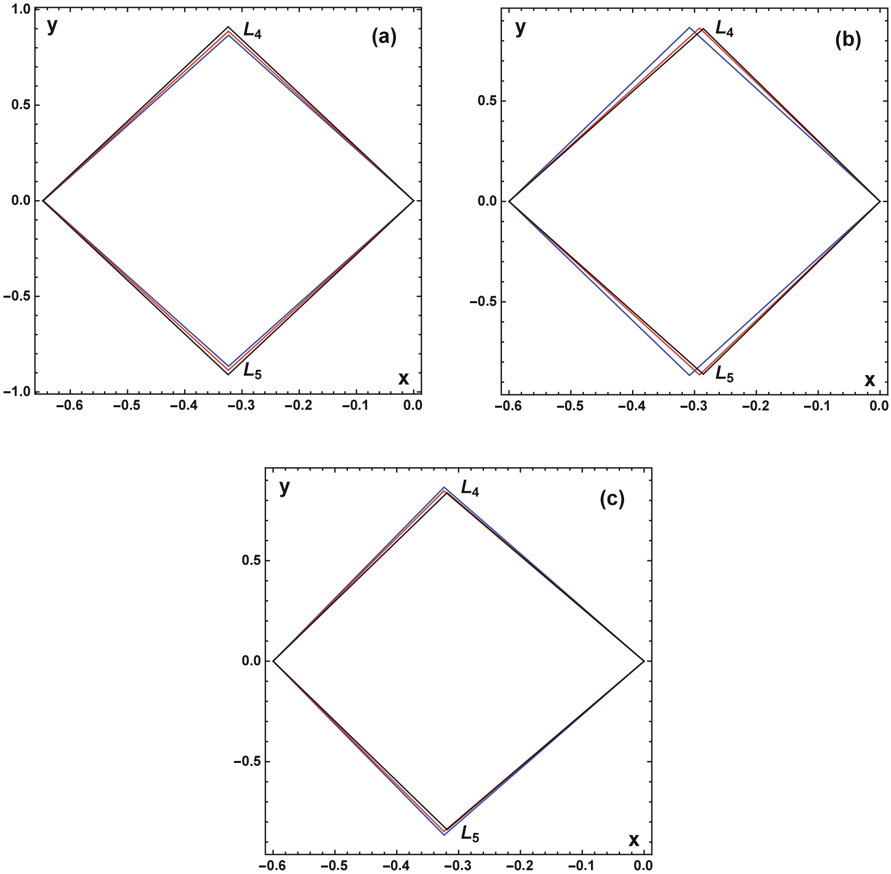

Fig. 2 (Colour figure online) Positions of $L_{4,5}$ of Lalande 21258 **(a)** for the oblateness parameter $A_3 = 0.01$ (blue), 0.05 (red), and 0.1 (black) with fixed values of the remaining parameters as given in the second case of Table 2 **(b)** for the triaxiality parameters of the first primary $(\Phi_1, \Phi_2) = (0.01, 0.003), (0.03, 0.015)$, and $(0.05, 0.03)$, denoted by colour code blue, red, and black, respectively, with fixed values of the remaining parameters as given in the third case of Table 2, **(c)** for the triaxiality of the second primary $(\Psi_1, \Psi_2) = (0.006, 0.004), (0.03, 0.015)$ and $(0.05, 0.03)$, denoted by colour code blue, red, and black, respectively, with fixed values of the remaining parameters as given in the fourth case of Table 2

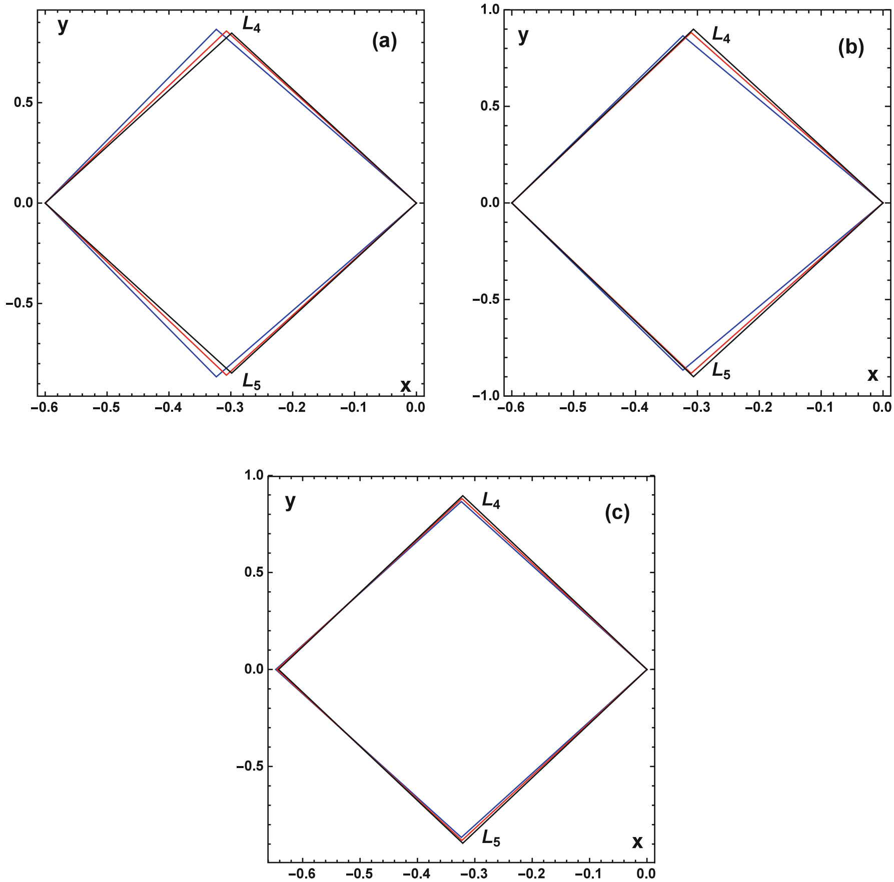

Fig. 3 (Colour figure online) Positions of $L_{4,5}$ of Lalande 21258 (**a**) for combined effects of triaxiality of both primaries $(\Phi_1, \Phi_2) = (0.008, 0.006), (0.03, 0.02), (0.05, 0.035)$ and $(\Psi_1, \Psi_2) = (0.006, 0.004), (0.015, 0.01), (0.03, 0.02)$, correspondingly, denoted by colour code blue, red, and black, respectively, with fixed value of $A_3 = 0.01$ (case 5 of Table 2), (**b**) for combined effects of triaxiality parameter of the first primary and oblateness $(\Phi_1, \Phi_2, A_3) = (0.008, 0.006, 0.01), (0.03, 0.02, 0.05)$, and $(0.05, 0.035, 0.1)$, correspondingly, denoted by blue, red, and black, respectively, with fixed values of $\Psi_1 = 0.006$, $\Psi_2 = 0.004$ (case 6 of Table 2), (**c**) for combined effects of triaxiality parameter of the second primary and oblateness $(\Psi_1, \Psi_2, A_3) = (0.006, 0.004, 0.01), (0.015, 0.01, 0.05)$, and $(0.03, 0.02, 0.1)$, correspondingly, denoted by blue, red, and black, respectively, with fixed values of $\Phi_1 = 0.008$, $\Phi_2 = 0.006$ (case 7 of Table 2)

its physical parameters. In this case, we found that the characteristic polynomial has no roots which are purely imaginary or complex with negative real parts; therefore, the triangular equilibrium points are unstable.

It is worth to mention here that, if we do not consider the oblateness of the massless body, i.e. $A_3 = 0$, our results will be in excellent agreement with those provided by Singh [28]. Furthermore, if the primary bodies do not also emit

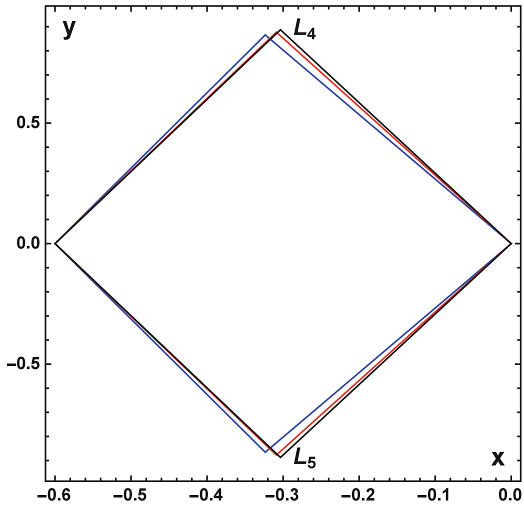

Fig. 4 (Colour figure online) Positions of $L_{4,5}$ of Lalande 21258 for combined effects of triaxiality of both primaries and oblateness (case 8 of Table 2): blue colour $\Phi_1 = 0.008$, $\Phi_2 = 0.006$, $\Psi_1 = 0.006$, $\Psi_2 = 0.004$, and $A_3 = 0.01$; red colour $\Phi_1 = 0.03$, $\Phi_2 = 0.02$, $\Psi_1 = 0.015$, $\Psi_2 = 0.01$, and $A_3 = 0.05$; black colour $\Phi_1 = 0.05$, $\Phi_2 = 0.035$, $\Psi_1 = 0.03$, $\Psi_2 = 0.02$, and $A_3 = 0.1$

Table 3 The eigenvalues $\lambda_{1,2}$, $\lambda_{3,4}$ of Eq. (16) for Lalande 21258 system

Φ_1	Φ_2	Ψ_1	Ψ_2	A_3	$\lambda_{1,2}$	$\lambda_{3,4}$
0.008	0.006	0.006	0.004	0.01	$-0.513933 \pm 0.868413i$	$0.513933 \pm 0.868413i$
0.03	0.02	0.015	0.01	0.05	$-0.574577 \pm 0.879048i$	$0.574577 \pm 0.879048i$
0.05	0.035	0.03	0.02	0.1	$-0.642373 \pm 0.890378i$	$0.642373 \pm 0.890378i$
0.09	0.05	0.04	0.03	0.15	$-0.684338 \pm 0.899161i$	$0.684338 \pm 0.899161i$

radiation, i.e. $q_1 = q_2 = 1$, $A_3 = 0$, then we will get the results obtained by Elshaboury et al. [9]. Also, in the presence of the Coriolis and centrifugal forces when the two primaries are oblate spheroids, namely $q_1 = q_2 = 1$, $\Phi_1 = \Phi_2$, $\Psi_1 = \Psi_2$, our results will tally with those given by Abouelmagd and Guirao [1]. Finally, in the absence of radiation forces when the smaller primary body is an oblate spheroid and if the Coriolis and centrifugal forces are taken also into account, i.e. $q_1 = q_2 = 1$, $A_3 = 0$, $\Psi_1 = \Psi_2$, our results are in agreement with those of Singh and Begha [29]. In a future correspondence, we intend to extend our results by studying the families of short and long periodic orbits emanating from triangular equilibrium points. Furthermore, a natural extension of our present work would be also to incorporate the Poynting–Robertson relativistic correction in the radiation force and study how this strengthened force affects the location and stability of the respective equilibria.

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Normalized Symmetric Differential Operators in the Open Unit Disk

Rabha W. Ibrahim

Abstract The symmetric differential operator SDO is a simplification functioning of the recognized ordinary derivative. The purpose of this effort is to provide a study of SDO connected with the geometric function theory. These differential operators indicate a generalization of well known differential operator including the Sàlàgean differential operator. Our contribution is to deliver two classes of symmetric differential operators in the open unit disk and to describe the further development of these operators by introducing convex linear symmetric operators. In addition, by acting these SDOs on the class of univalent functions, we display a set of sub-classes of analytic functions having geometric representation, such as starlikeness and convexity properties. Investigations in this direction lead to some applications in the univalent function theory of well known formulas, by defining and studying some sub-classes of analytic functions type Janowski function, bounded turning function subclass and convolution structures. Consequently, we define a linear combination differential operator involving the Sàlàgean differential operator and the Ruscheweyh derivative. The new operator is a generalization of the Lupus differential operator. Moreover, we aim to solve some special complex boundary problems for differential equations, spatially the class of Briot-Bouquet differential equations. All solutions are symmetric under the suggested SDOs. Additionally, by using the SDOs, we introduce a generalized class of Briot-Bouquet differential equations to deliver, what is called the symmetric Briot-Bouquet differential equations. We shall show that the upper solution is symmetric in the open unit disk by considering a set of examples of univalent functions.

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1 Introduction

The term Symmetry, from Greek, means arrangement in measurements, due proportion, organization. In free language, it mentions to a concept of harmonious and attractive proportion and equilibrium. In mathematics, “symmetry” has a more detailed definition, and typically utilized to discuss an object that is invariant via certain transformations; containing reflection, translation, rotation, or scaling. Even more, these two senses of symmetry" can occasionally be expressed distant; they are complexly connected, and henceforth is deliberated composed in this object. Research on the operator theory involving all types of functions (integration, differentiation, convolution, deference, diverse, and linear) has been attractive in all mathematical sciences. It can be seen in the fields of computing (self-adjoint operator or Hermitian operator), engineering studies (the peridynamic differential operator). The association of geometry and operator theory indicates an important recognition in theory of geometric functions that can be virtue in the open unit disk. This firm grows openly related to the reality between operators and geometric performance [1, 2].

In 1983, Sàlàgean announced his illustrious differential operator of normalized analytic functions in the open unit disk [3]. By employing this operator, many sub-classes of analytic functions are introduced. An extension in the space of the parameters of this operator is given by Al-Oboudi [4]. Later, these operators show important studies in the geometric function theory, by signifying diverse modules of operators and categories of univalent functions (see [5–10] for recent works). Our investigation is to express some categories of symmetric differential operators with integral by exploiting the idea of the symmetric derivative in a complex domain. This notion is a process covering the innovative derivative. Note that the symmetry representations in mathematical modeling stay open. For example, as an application in mathematical physics, it is critical to apply group analysis approaches. Such approaches allow systems for splitting solution structure using the symmetric group theory. Recently, a special type of SDOs delivered by Ibrahim and Jay [7] called a complex conformable differential operator. A situation of specific attention, mainly with an appreciation in the direction of applications in physics and engineering, is smearing the conformable calculus to smooth (analytic) functions. In this situation, the operator develops the term $\chi^{1-\wp} f'(\chi)$ (see [11]). The complex conformable calculus (CCC) that formulated in [7] indicated the term $\xi\varphi'(\xi)$, where ξ is a complex variable and φ is a complex valued analytic function. This term is suggested to normalize the CCC in order to study it in view of univalent function theory. In addition the CCC can be viewed as a generalization of the Ruscheweyh derivative and the Sàlàgean differential operator.

In this chapter, we deliver two types of symmetric differential operators in the open unit disk. We act these operators in some classes of univalent functions. Convolution classes of these operators are also suggested in the sequel. Moreover, we aim to solve some special complex boundary problems for differential equations, spatially the class of Briot-Bouquet differential equations.

2 Methodology

This section gives out the mathematical processing to deliver the suggested SDOs and complex conformable operator for some categories of smooth functions in the open unit disk $\cup = \{\xi \in \mathbb{C} : |\xi|1\}$. Let $\bigwedge$ be the category of smooth function elicited as pursue

$$\Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} \Upsilon_n \xi^n, \quad \xi \in \cup. \tag{1}$$

A function $\Upsilon \in \bigwedge$ is known as a starlike via the (0,0) (origin in $\cup$) if the linear slice combining the origin to all else point of Υ embedding completely in $\Upsilon(\xi : |\xi| < 1)$. The aim is that each point of $\Upsilon(\xi : |\xi| < 1)$ must be manifest via (0,0). A univalent function ($\Upsilon \in \textcircled{S}$) is said to be convex in $\cup$ if the linear slice combining two points of $\Upsilon(\xi : |\xi| < 1)$ stay completely in $\Upsilon(\xi : |\xi| < 1)$. We denote these categories by $\mathscr{S}^*$ and $\mathscr{C}$ for starlike and convex, respectively. In addition, suppose that the category $\mathscr{P}$ involves all functions Υ analytic in $\cup$ with a positive real part in $\cup$ achieving $\Upsilon(0) = 1$. Mathematically, $\Upsilon \in \mathscr{S}^*$ if and only if $\xi\, \Upsilon'(\xi)/\Upsilon(\xi) \in \mathscr{P}$ and $\Upsilon \in \mathscr{C}$ if and only if $1 + \xi\, \Upsilon''(\xi)/\Upsilon'(\xi) \in \mathscr{P}$ equivalently,

$$\Re(\xi\, \Upsilon'(\xi)/\Upsilon(\xi)) > 0,$$

for starlikeness and

$$1 + \Re(\xi\, \Upsilon''(\xi)/\Upsilon'(\xi)) > 0,$$

for convexity.

For two functions Υ_1 and Υ_2 belong to the category $\bigwedge$, are said to be subordinate, noting by $\Upsilon_1 \prec \Upsilon_2$, if we can find a Schwarz function $\top$ with $\top(0) = 0$ and $|\top(\xi)| < 1$ achieving $\Upsilon_1(\xi) = \Upsilon_2(\top(\xi))$, $\xi \in \cup$ (the detail can be located in [12]). Obviously, $\Upsilon_1(\xi) \prec \Upsilon_2(\xi)$ congregants with $\Upsilon_1(0) = \Upsilon_2(0)$ and $\Upsilon_1(\cup) \subset \Upsilon_2(\cup)$. We employ next facts, one can find it in [12].

Lemma 1 *Let* $\mathbf{a} \in \mathbb{C}$, *a positive integer* n *and* $\aleph[\mathbf{a}, n] = \{\Upsilon : \Upsilon(\xi) = \mathbf{a} + \mathbf{a}_n \xi^n + \mathbf{a}_{n+1}\xi^{n+1} + \ldots\}$.

(i) Suppose that $\ell \in \mathbb{R}$; *then* $\Re\Big(\Upsilon(\xi) + \ell\, \xi\, \Upsilon'(\xi)\Big) > 0 \Longrightarrow \Re\Big(\Upsilon(\xi)\Big) > 0$. *In addition, if* $\ell > 0$ *and* $\Upsilon \in \aleph[1, n]$, *then there occur some constants* $a > 0$ *and* $b > 0$ *with* $b = b(\ell, a, n)$ *where*

$$\Upsilon(\xi) + \ell\xi\, \Upsilon'(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b \Rightarrow \Upsilon(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^a.$$

(ii) *Assume that* $\eth \in [0, 1)$ *and* $\Upsilon \in \aleph[1, n]$; *then a constant* $k > 0$ *exists satisfying* $k = k(a, n)$ *so that*

$$\Re\Big(\Upsilon^2(\xi) + 2\Upsilon(\xi).\xi\Upsilon'(\xi)\Big) > \eth \Rightarrow \Re(\Upsilon(\xi)) > k.$$

(iii) *If* $\Upsilon \in \aleph[\mathbf{a}, n]$ *with* $\Re(\mathbf{a}) > 0$, *then* $\Re\Big(\Upsilon(\xi) + \xi\,\Upsilon'(\xi) + \xi^2\,\Upsilon''(\xi)\Big) > 0$ *or for* $\iota : \cup \to \mathbb{R}$ *with* $\Re\Big(\Upsilon(\xi) + \iota(\xi)\dfrac{\xi\,\Upsilon'(\xi)}{\Upsilon(\xi)}\Big) > 0$ *then* $\Re(\Upsilon(\xi)) > 0$.

Lemma 2 *Suppose that* $\hbar$ *is a convex function achieving* $\hbar(0) = \mathbf{a}$, *and assume that* $\Bbbk \in \mathbb{C} \setminus \{0\}$ *is a complex number satisfying* $\Re(\Bbbk) \geq 0$. *If* $\Upsilon \in \aleph[\mathbf{a}, n]$, *and*

$$\Upsilon(\xi) + (1/\Bbbk)\xi\,\Upsilon'(\xi) \prec \hbar(\xi), \quad \xi \in \cup,$$

then

$$\Upsilon(\xi) \prec \iota(\xi) \prec \hbar(\xi),$$

where

$$\iota(\xi) = \frac{\Bbbk}{n\xi^{\Bbbk/n}} \int_0^{\xi} \hbar(\tau)\tau^{\frac{\Bbbk}{(n-1)}} d\tau, \quad \xi \in \cup.$$

Lemma 3 ([13]) *Suppose that* $\Upsilon \in \bigwedge$ *and there occurs a positive constant* $0 < \upsilon \leq 1$. *If*

$$\frac{\xi\,\Upsilon'(\xi) - \xi}{\Upsilon(\xi)} \prec \frac{2\upsilon\xi}{1+\xi},$$

then

$$\frac{\Upsilon(\xi)}{\xi} \prec 1 + \upsilon\,\xi, \quad \xi \in \cup.$$

And the result is sharp.

2.1 Symmetric Differential Operators (SDO)

In this place, we deliver two SDOs in the open unit disk under the category $\bigwedge$.

For a function $\Upsilon \in \bigwedge$, we formulate the pursuing SDO

$$
\begin{aligned}
&\Delta_\alpha^0 \Upsilon(\xi) = \Upsilon(\xi) \\
&\Delta_\alpha^1 \Upsilon(\xi) = \left(\frac{\alpha}{\bar{\alpha}}\right)\xi\ \Upsilon'(\xi) - \left(1-\frac{\alpha}{\bar{\alpha}}\right)\xi\ \Upsilon'(-\xi) \\
&\vdots \\
&\Delta_\alpha^m \Upsilon(\xi) = \Delta_\alpha(\Delta_\alpha^{m-1}\Upsilon(\xi)) \\
&\qquad\qquad = \xi + \sum_{n=2}^{\infty}\left(n\left(\frac{\alpha}{\bar{\alpha}} - (1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)\right)^m \Upsilon_n\,\xi^n \\
&\qquad\qquad := \Upsilon(\xi) * \mathfrak{D}_\alpha(\xi),
\end{aligned}
\tag{2}
$$

where $\alpha \neq 0$ is a complex number and $*$ is the convolution product such that

$$
\mathfrak{D}_\alpha(\xi) := \xi + \sum_{n=2}^{\infty}\left(n\left(\frac{\alpha}{\bar{\alpha}} - (1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)\right)^m \xi^n.
$$

Certainly, when α is real (for example, $\alpha = 1$), we get the Sàlàgean differential operator [3]

$$
\mathscr{S}^m \Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} n^m\ \Upsilon_n\,\xi^n.
$$

We proceed to define a linear combination operator involving the SDO (2) and the Ruscheweyh derivative (RD). Let $\Upsilon \in \bigwedge$, then the RD satisfies the functional

$$
\mathscr{R}^m \Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} C_{m+n-1}^m\ \Upsilon_n\,\xi^n,
$$

where the term C_{m+n-1}^m is the combination coefficients. In this direction, we impose a new linear combination of $\mathscr{R}^m$ and Δ_α^m as follows:

$$
\begin{aligned}
\mathbf{P}_{\alpha,\lambda}^m \Upsilon(\xi) &= (1-\lambda)\mathscr{R}^m \Upsilon(\xi) + \lambda\Delta_\alpha^m \Upsilon(\xi) \\
&= \xi + \sum_{n=2}^{\infty}\left[(1-\lambda)C_{m+n-1}^m + \lambda\left(n\left(\frac{\alpha}{\bar{\alpha}} - (1-\frac{\alpha}{\bar{\alpha}})(-1)^n\right)\right)^m\right]\Upsilon_n\,\xi^n.
\end{aligned}
\tag{3}
$$

Remark 1

- $m = 0 \Longrightarrow \mathbf{P}_{\alpha,\lambda}^0 \Upsilon(\xi) = \Upsilon(\xi)$;
- α is real then $\mathbf{P}_{\alpha,\lambda}^m \Upsilon(\xi) = \mathscr{L}_\lambda^m \Upsilon(\xi)$; [14] (Lupas operator)
- $\lambda = 0 \Longrightarrow \mathbf{P}_{\alpha,0}^m \Upsilon(\xi) = \mathscr{R}^m \Upsilon(\xi)$;

- $\lambda = 1 \Longrightarrow \mathbf{P}^m_{\alpha,1} \curlyvee(\xi) = \mathscr{S}^k \curlyvee(\xi)$, provide that α is real;
- $\lambda = 1 \Longrightarrow \mathbf{P}^m_{\alpha,1} \curlyvee(\xi) = \Delta^m_\alpha \curlyvee(\xi)$.

Our study is based on the following classes of analytic functions involving the operators (2) and (3).

$$S^{*\alpha}_m(\hbar) = \Big\{\curlyvee \in \bigwedge : \frac{\xi(\Delta^m_\alpha \curlyvee(\xi))'}{\Delta^m_\alpha \curlyvee(\xi)} \prec \hbar(\xi),\ \hbar \in \mathscr{C}\Big\}.$$

Clearly, the subcategory $S^*_0(\hbar) = \mathscr{S}^*(\hbar)$.

Definition 1 For a function $\curlyvee \in \bigwedge$ is in the category $\mathbb{J}^b_\alpha(A, B, m)$ if and only if

$$1 + \frac{1}{\flat}\Big(\frac{2\Delta^{m+1}_\alpha \curlyvee(\xi)}{\Delta^m_\alpha \curlyvee(\xi) - \Delta^m_\alpha \curlyvee(-\xi)}\Big) \prec \frac{1 + A\xi}{1 + B\xi},$$

$$\Big(\xi \in \cup,\ -1 \le B < A \le 1,\ k = 1, 2, \ldots,\ \flat \in \mathbb{C} \setminus \{0\},\ \alpha \in \mathbb{C} \setminus \{0\}\Big).$$

The category $\mathbb{J}^b_\alpha(A, B, m)$ has the following special cases:

- $\alpha \in \mathbb{R} \rightarrow$ [15];
- $\alpha \in \mathbb{R}\ \&\ B = 0 \rightarrow$ [16];
- $\alpha \in \mathbb{R},\ A = 1, B = -1, \flat = 2 \rightarrow$ [17].

Moreover, we seek another category that includes the linear operator $\mathbf{P}^m_{\alpha,\lambda}$ as follows:

Definition 2 Consider the following data: $\epsilon \in [0, 1), \alpha \in \mathbb{C}\backslash\{0\}, \lambda \ge 0, \curlyvee \in \bigwedge$ and $m \in \mathbb{N}$ then the function $\curlyvee \in T_m(\alpha, \lambda, \epsilon)$ if and only if

$$\Re\Big(\big(\mathbf{P}^m_{\alpha,\lambda} \curlyvee(\xi)\big)'\Big) > \epsilon, \quad z \in U.$$

The category $T_m(\alpha, \lambda, \epsilon)$ represents to the generalization of the class of bounding turning analytic functions in $\cup$. For example, when $m = 0$ we have the usual category

$$\Re\Big(\big(\mathbf{P}^0_{\alpha,\lambda} \curlyvee(\xi)\big)'\Big) > \epsilon \rightarrow \Re\big(\curlyvee(\xi)\big)' > \epsilon.$$

Our aim is to study the above classes in different approaches.

3 Results

This place concerns about the outcomes that utilizing the above operators to get some geometric presentation.

Theorem 1 *For $\Upsilon \in \bigwedge$ and $\alpha \in \mathbb{C}\backslash\{0\}$, if one of the sequencing subordinations is valid*

- *The operator $\Delta_\alpha^m \Upsilon(\xi)$ in (2) is of bounded boundary rotation;*
- *Υ satisfies the subordination relation*

$$(\Delta_\alpha^m \Upsilon(\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^b, \quad b > 0,\ \xi \in \cup;$$

- *Υ fulfilled the inequality*

$$\Re\Big((\Delta_\alpha^m \Upsilon(\xi))' \frac{\Delta_\alpha^m \Upsilon(\xi)}{\xi}\Big) > \frac{\delta}{2}, \quad \delta \in [0,1),\ \xi \in \cup,$$

- *Υ admits the inequality*

$$\Re\Big(\xi \Delta_\alpha^m \Upsilon(\xi))'' - \Delta_\alpha^m \Upsilon(\xi))' + 2\frac{\Delta_\alpha^m \Upsilon(\xi))}{\xi}\Big) > 0,$$

- *Υ confesses the inequality*

$$\Re\Big(\frac{z\Delta_\alpha^m \Upsilon(\xi))'}{\Delta_\alpha^m \Upsilon(\xi))} + 2\frac{\Delta_\alpha^m \Upsilon(\xi)}{\xi}\Big) > 1,$$

then $\frac{\Delta_\alpha^m \Upsilon(\xi)}{\xi} \in \mathscr{P}(\epsilon)$ for some $\epsilon \in [0,1)$.

Proof Define a function ϱ as follows

$$\rho(\xi) = \frac{\Delta_\alpha^m \Upsilon(\xi)}{\xi} \Rightarrow \xi\rho'(\xi) + \rho(\xi) = (\Delta_\alpha^m \Upsilon(\xi))'. \tag{4}$$

By the first fact, $\Delta_\alpha^m \Upsilon(\xi)$ is of bounded boundary rotation, it implies that

$$\Re(\xi\, \varrho'(\xi) + \varrho(\xi)) > 0.$$

Therefore, according to Lemma 1(i), we attain $\Re(\varrho(\xi)) > 0$ which gets the first term of the theorem. In view of the second fact, we have the following subordination relation

$$(\Delta_\alpha^m \Upsilon(\xi))' = \xi\, \varrho'(\xi) + \varrho(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b.$$

Now, by employing again Lemma 1(i), there occurs a fixed constant $a > 0$ with $b = b(a)$ with the following property

$$\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^a .$$

This implies that $\Re(\Delta_\alpha^m \curlyvee(\xi)/\xi) > \epsilon$, for some $\epsilon \in [0, 1)$. Lastly, agree with the third relation to get

$$\Re\Big(\varrho^2(\xi) + 2\varrho(\xi).\xi\,\varrho'(\xi)\Big) = 2\Re\Big((\Delta_\alpha^m \curlyvee(\xi))'\frac{\Delta\alpha^k \curlyvee(\xi)}{\xi}\Big) > \delta. \tag{5}$$

According to Lemma 1(ii), there exists a positive fixed constant $\lambda > 0$ such that $\Re(\varrho(\xi)) > \lambda$, which yields $\varrho(\xi) = \frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi} \in \mathscr{P}(\epsilon)$ for some $\epsilon \in [0, 1)$. It indicates from (5) that $\Re\Big(\Delta_\alpha^m \curlyvee(\xi))'\Big) > 0$, consequently by Noshiro-Warschawski and Kaplan Theorems imply that $\Delta_\alpha^m \curlyvee(\xi)$ is univalent and of bounded boundary rotation in $\cup$. Now via the differentiating (4) and concluding the real case, we indicate that

$$\begin{aligned}
&\Re\Big(\varrho(\xi) + \xi\,\varrho'(\xi) + \xi^2\varrho''(\xi)\Big)\\
&= \Re\Big(\xi(\Delta_\alpha^m \curlyvee(\xi))'' - (\Delta_\alpha^m \curlyvee(\xi))' + 2\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi}\Big)\\
&> 0.
\end{aligned}$$

Thus, we finished the conclusion of Lemma 1(ii), which indicates the inequality

$$\Re(\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi}) > 0.$$

Taking the logarithmic differentiation (4) and indicating the real, we arrive at the following conclusion:

$$\begin{aligned}
&\Re\Big(\varrho(\xi) + \frac{\xi\,\varrho'(\xi)}{\varrho(\xi)} + \xi^2\varrho''(\xi)\Big)\\
&= \Re\Big(\frac{\xi(\Delta_\alpha^m \curlyvee(\xi))'}{\Delta_\alpha^m \curlyvee(\xi)} + 2\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi} - 1\Big)\\
&> 0.
\end{aligned}$$

A direct application of Lemma 1(iii), we get the positive real, i.e., $\Re(\frac{\Delta_\alpha^m \curlyvee(\xi)}{\xi}) > 0$. This completes the proof.

Theorem 2 *Suppose that* $\curlyvee \in \mathbb{J}_\alpha^\flat(A, B, m)$ *then for every function of the form*

$$\mathfrak{X}(\xi) = \frac{1}{2}[\curlyvee(\xi) - \curlyvee(-\xi)], \quad \xi \in \cup$$

agrees with the pursuing relation

$$1+\frac{1}{\flat}\Big(\frac{\Delta_\alpha^{m+1}\mathfrak{X}(\xi)}{\Delta_\alpha^m\mathfrak{X}(\xi)}-1\Big)\prec\frac{1+A\xi}{1+B\xi},$$

and

$$\Re\Big(\frac{\xi\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)}\Big)\geq\frac{1-\curlywedge^2}{1+\curlywedge^2},\quad |\xi|=\curlywedge<1,$$

$$\Big(\xi\in\cup,\ -1\leq B<A\leq 1,\ m=1,2,\ldots,\ \flat\in\mathbb{C}\setminus\{0\},\ \alpha\in\mathbb{C}\Big).$$

Proof Because the function $\curlyvee\in\mathbb{J}_\alpha^\flat(A,B,m)$ then there occurs a function $\wp\in\mathbb{J}(A,B)$ such that

$$\flat(\wp(\xi)-1)=\Big(\frac{2\Delta_\alpha^{m+1}\curlyvee(\xi)}{\Delta_\alpha^m\curlyvee(\xi)-\Delta_\alpha^m\curlyvee(-\xi)}\Big)$$

and

$$\flat(\wp(-\xi)-1)=\Big(\frac{-2\Delta_\alpha^{m+1}\curlyvee(-\xi)}{\Delta_\alpha^m\curlyvee(\xi)-\Delta_\alpha^m\curlyvee(-\xi)}\Big).$$

This implies that

$$1+\frac{1}{\flat}\Big(\frac{\Delta_\alpha^{m+1}\mathfrak{X}(\xi)}{\Delta_\alpha^m\mathfrak{X}(\xi)}-1\Big)=\frac{\wp(\xi)+\wp(-\xi)}{2}.$$

Also, since $\wp(\xi)\prec\frac{1+A\xi}{1+B\xi}$, where $\frac{1+A\xi}{1+B\xi}$ is univalent then by the concept of the subordination, we arrive at

$$1+\frac{1}{\flat}\Big(\frac{\Delta_\alpha^{m+1}\mathfrak{X}(\xi)}{\Delta_\alpha^m\mathfrak{X}(\xi)}-1\Big)\prec\frac{1+A\xi}{1+B\xi}.$$

But the function $\mathfrak{X}(\xi)$ is starlike in $\cup$, which means that

$$\frac{\xi\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)}\prec\frac{1-\xi^2}{1+\xi^2}$$

and there holds a Schwarz function $\top\in\cup$, $|\top(\xi)|\leq|\xi|<1$, $\top(0)=0$ such that

$$\Psi(\xi):=\frac{z\mathfrak{X}(\xi)'}{\mathfrak{X}(\xi)}\prec\frac{1-\top(\xi)^2}{1+\top(\xi)^2}$$

which implies that there exists ζ, $|\zeta| = \curlywedge < 1$ achieving

$$\mathsf{T}^2(\zeta) = \frac{1-\Psi(\zeta)}{1+\Psi(\zeta)}, \quad \zeta \in \cup.$$

A computation brings that

$$\Big|\frac{1-\Psi(\zeta)}{1+\Psi(\zeta)}\Big| = |\mathsf{T}(\zeta)|^2 \leq |\zeta|^2.$$

Thus, we conclude that

$$\Big|\Psi(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\Big|^2 \leq \frac{4|\zeta|^4}{(1-|\zeta|^4)^2}$$

or

$$\Big|\Psi(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\Big| \leq \frac{2|\zeta|^2}{(1-|\zeta|^4)}.$$

Consequently, we obtain

$$\Re(\Psi(\zeta)) \geq \frac{1-\curlywedge^2}{1+\curlywedge^2}, \quad |\zeta| = \curlywedge < 1.$$

Next results come directly from Theorem 2, which may be found in [15, 17], respectively.

Corollary 1 *Let α be a real number in Theorem 2. Then*

$$1 + \frac{1}{\flat}\Big(\frac{\Delta_\alpha^{m+1}\mathfrak{X}(\xi)}{\Delta_\alpha^m\mathfrak{X}(\xi)} - 1\Big) \prec \frac{1+A\xi}{1+B\xi}.$$

Corollary 2 *Let α be a real number and $m = 1$ in Theorem 2. Then*

$$1 + \frac{1}{\flat}\Big(\frac{\Delta_\alpha^{2}\mathfrak{X}(\xi)}{\Delta_\alpha^1\mathfrak{X}(\xi)} - 1\Big) \prec \frac{1+A\xi}{1+B\xi}.$$

Theorem 3 *Suppose that $\curlyvee \in T_m(\alpha, \lambda, \epsilon)$, and the convex analytic function g satisfying the integral equation*

$$F(\xi) = \frac{2+c}{\xi^{1+c}} \int_0^\xi \tau^c \curlyvee(\tau) d\tau, \quad \xi \in \cup$$

then the subordination

$$\left(\mathbf{P}^m_{\alpha,\lambda}\,\Upsilon\,(\xi)\right)' \prec g(\xi) + \frac{(\xi g'(\xi))}{2+c}, \quad c > 0,$$

implies the subordination

$$\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)' \prec g(\xi),$$

and this result is sharp.

Proof Here, we aim to utilize the result of Lemma 2. By the conclusion of $F(\xi)$, we acquire

$$\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)' + \frac{\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)''}{2+c} = \left(\mathbf{P}^m_{\alpha,\lambda}\,\Upsilon\,(\xi)\right)'.$$

Following the conditions of the theorem, we get

$$\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)' + \frac{\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)''}{2+c} \prec g(\xi) + \frac{(\xi\, g'(\xi))}{2+c}.$$

By assuming

$$\varrho(\xi) := \left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)',$$

we have

$$\varrho(\xi) + \frac{(\xi \varrho'(\xi))}{2+c} \prec g(\xi) + \frac{(\xi g'(\xi))}{2+c}.$$

According to Lemma 2, we attain

$$\left(\mathbf{P}^m_{\alpha,\lambda} F(\xi)\right)' \prec g(\xi),$$

and g is the best dominant.

Theorem 4 *Let g be convex such that $g(0) = 1$. If*

$$\left(\mathbf{P}^m_{\alpha,\lambda}\,\Upsilon\,(\xi)\right)' \prec g(\xi) + \xi g'(\xi), \quad \xi \in \cup,$$

then $\dfrac{\mathbf{P}^m_{\alpha,\lambda}\,\Upsilon\,(\xi)}{\xi} \prec g(\xi)$, *and this result is sharp.*

Proof Define the pursuing function

$$\varrho(\xi) := \frac{\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)}{\xi} \in \aleph[1, 1]. \tag{6}$$

A direct application of Lemma 1yields

$$\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi) = \xi\varrho(\xi) \Longrightarrow \Big(\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)\Big)' = \varrho(\xi) + \xi\varrho'(\xi).$$

Thus, we introduce the pursuing subordination:

$$\varrho(\xi) + \xi\varrho'(\xi) \prec g(\xi) + \xi g'(\xi).$$

Hence, we conclude that $\frac{\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)}{\xi} \prec g(\xi)$, and g is the best dominant.

Theorem 5 *If* $\curlyvee \in \bigwedge$ *fulfills the subordination*

$$(\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^b, \quad \xi \in \cup,\ b > 0,$$

then

$$\Re\Big(\frac{\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)}{\xi}\Big) > \epsilon$$

for some $\epsilon \in [0, 1)$.

Proof Define a function ϱ as in (6). Then, by subordination properties, we have

$$(\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi))' = \xi\varrho'(\xi) + \varrho(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^b.$$

With the help of Lemma 1(i), there occurs a constant $a > 0$ with $b = b(a)$ such that

$$\frac{\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^a.$$

This leads to real conclusion $\Re(\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)/\xi) > \epsilon$, for some $\epsilon \in [0, 1)$.

Theorem 6 *If* $\curlyvee \in \bigwedge$ *fulfills the real inequality*

$$\Re\Big((\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi))' \frac{\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi)}{\xi}\Big) > \Re(\frac{\alpha}{2}), \quad \xi \in \cup,\ \alpha \in \mathbb{C},$$

then $\mathbf{P}^m_{\alpha,\lambda} \curlyvee (\xi) \in T_m(\alpha, \lambda, \epsilon)$ *for some* $\epsilon \in [0, 1)$. *In addition, it is univalent and of bounded boundary rotation in* $\cup$.

Proof Formulate ϱ as in (6). A simple calculation gives

$$\Re\Big(\varrho^2(\xi)+2\varrho(\xi).\xi\varrho'(\xi)\Big)=2\Re\Big(\mathbf{P}^m_{\alpha,\lambda}\Upsilon(\xi))'\frac{\mathbf{P}^m_{\alpha,\lambda}\Upsilon(\xi)}{\xi}\Big)>\Re(\alpha). \tag{7}$$

By the advantage of Lemma 1(ii), there occurs a constant κ concerning on $\Re(\alpha)$ such that $\Re(\varrho(\xi))>\kappa$, which gives that $\Re(\varrho(\xi)\,)>\epsilon$ for some $\epsilon\in[0,1)$. It implies from (7) that $\Re\Big(\mathbf{P}^m_{\alpha,\lambda}\Upsilon(\xi))'\Big)>\epsilon$ and hence based on the idea of Noshiro-Warschawski and Kaplan Theorems, $\mathbf{P}^m_{\alpha,\lambda}\Upsilon(\xi)$ is univalent and of bounded boundary rotation in $\cup$.

Theorem 7 *The set $T_m(\alpha,\lambda,\epsilon)$ is convex.*

Proof Let $\Upsilon_i, i=1,2$ be two functions in the set $T_m(\alpha,\lambda,\epsilon)$ achieving the formulas $\Upsilon_1(\xi)=\xi+\sum_{n=2}^{\infty}a_n\xi^n$ and $\Upsilon_2(\xi)=\xi+\sum_{n=2}^{\infty}b_n\xi^n$, respectively. It is adequate to show that the linear combination function

$$G(\xi)=w_1\Upsilon_1(\xi)+w_2\Upsilon_2(\xi),\quad \xi\in\cup$$

belongs to $T_m(\alpha,\lambda,\epsilon)$, where $w_1>0$, $w_2>0$ and $w_1+w_2=1$. By the definition of $G(\xi)$, a computation yields that

$$G(\xi)=\xi+\sum_{n=2}^{\infty}(w_1a_n+w_2b_n)\xi^n$$

then under the formal $\mathbf{P}^m_{\alpha,\lambda}$, we obtain

$$\mathbf{P}^m_{\alpha,\lambda}G(\xi)=\xi+\sum_{n=2}^{\infty}(w_1a_n+w_2b_n)\times\Big[(1-\lambda)C^m_{m+n-1}+\lambda\Big(n\Big(\frac{\alpha}{\bar{\alpha}}-\Big(1-\frac{\alpha}{\bar{\alpha}}\Big)(-1)^n\Big)\Big)^m\Big]\xi^n.$$

By considering the derivative, we have

$$\begin{aligned}&\Re\Big\{(\mathbf{P}^m_{\alpha,\lambda}G(\xi))'\Big\}\\&=1+w_1\Re\Big\{\sum_{n=2}^{\infty}n\Big[(1-\lambda)C^m_{m+n-1}+\lambda\Big(n\Big(\frac{\alpha}{\bar{\alpha}}-\Big(1-\frac{\alpha}{\bar{\alpha}}\Big)(-1)^n\Big)\Big)^m\Big]a_n\xi^{n-1}\Big\}\\&\quad+w_2\Re\Big\{\sum_{n=2}^{\infty}n\Big[(1-\lambda)C^m_{m+n-1}+\lambda\Big(n\Big(\frac{\alpha}{\bar{\alpha}}-\Big(1-\frac{\alpha}{\bar{\alpha}}\Big)(-1)^n\Big)\Big)^m\Big]b_n\xi^{n-1}\Big\}\\&>1+w_1(\epsilon-1)+w_2(\epsilon-1)=\epsilon.\end{aligned}$$

This completes the proof.

Next consequence result of Theorem 7 can be found in [14].

Corollary 3 *Let α be a real number in Theorem 7. Then the set $T_m(\alpha, \lambda, \epsilon)$ is convex.*

4 Applications

A set of complex differential equations is an assembly of differential equations whose consequences are terms of a complex variable. Accumulating integrals encloses superior paths to proceed, which incomes singularities. This branch fact of the equations and its applications must investigate widely. The most important study in this direction is to establish the existence and uniqueness of solutions. There are diffident types of techniques including the utility of majorants and minorants (or subordination and superordination concepts) (see [12]). Investigation of ODEs in the complex domain suggests the detection of novel transcendental special functions, which currently called a Briot-Bouquet differential equation (BBDE)

$$\omega \Upsilon(\xi) + (1-\omega)\frac{\xi(\Upsilon(\xi))'}{\Upsilon(\xi)} = \hbar(\xi),$$

$$\Big(\hbar(0) = \Upsilon(0),\ \omega \in [0,1],\ \xi \in \cup,\ \Upsilon \in \bigwedge\Big).$$

In this place, we shall generalize the BBDE into a symmetric BBDE by using SDO. Numerous presentations of these comparisons in the geometric function model have recently achieved in [12].

Needham and McAllister [18] reflected a two-dimensional complex holomorphic dynamical system, pleasing the 2-D form

$$\xi_t = \Theta(\xi, \omega); \quad \omega_t = \Theta(\xi, w), \quad \xi, \omega \in \cup$$

and t is in any real interval. In detailed, they utilized the BB-singular point theory to find the existence and uniqueness of complex holomorphic result of the system in the neighborhood of an equilibrium point with two purely imaginary eigenvalues. Consequently, they recognized the existence of isochronous center relations in the neighborhood of the equilibrium point. Yuan et al. [19] achieved the meromorphic result of a class of 2D-BBDE. Development application of the BBDE seemed new, with unlike approaches (see [20]) in resolving (as a singular situation) the equation of electronic nano-shells (see [21]). Controlled by the situation effort of traditional shell theory, the transposition fields of the nano-shell take the dynamic system

$$\xi_t = \Theta(\xi, \omega) + \Theta_\theta(\xi, \omega); \quad \omega_t = \Theta(\xi, \omega) + \Theta_\theta(\bar{\xi}, \bar{\omega}), \quad \xi, \omega \in \cup,$$

where θ is the angles between ξ and ω and their conjugates.

Our purpose is to simplify this category of equation by utilizing the SDO and establish its resolutions by applying the subordination associations. By employing the (2), we have the generalized BBDE

$$\omega \curlyvee (\xi) + (1-\omega)\Big(\frac{\xi(\Delta_\alpha^m \curlyvee (\xi))'}{\Delta_\alpha^m \curlyvee (\xi)}\Big) = \hbar(\xi), \quad \hbar(0) = \curlyvee(0),\ \xi \in \cup. \tag{8}$$

The subordination settings and alteration bound for a session of SDO specified in the following formula. A trivial resolution of (8) is given when $\omega = 1$. Consequently, our vision is to carry out the situation, $\curlyvee \in \bigwedge$ and $\omega = 0$. We proceed to present the behavior of the solution of (8).

Theorem 8 *For $\curlyvee \in \bigwedge$, $\alpha \in [0, \infty)$ and $\hbar$ is univalent convex in $\cup$ if*

$$\Big(\frac{\xi(\Delta_\alpha^m \curlyvee (\xi))'}{\Delta_\alpha^m \curlyvee (\xi)}\Big) \prec \hbar(\xi), \quad \xi \in \cup, \tag{9}$$

then

$$\Delta_\alpha^m \curlyvee (\xi) \prec \xi \exp\Big(\int_0^\xi \frac{\hbar(\top(\xi))-1}{\ell} d\ell\Big),$$

where $\top$ is a Schwarz function in $\cup$. In addition, we have

$$|\xi| \exp\left(\int_0^1 \frac{\hbar(\top(-\sigma))-1}{\sigma} d\sigma\right) \le \left|\Delta_\alpha^m \curlyvee(\xi)\right| \le |\xi| \exp\left(\int_0^1 \frac{\hbar(\top(\sigma))-1}{\sigma} d\sigma\right).$$

Proof The subordination fact, in (9), implies that there occurs a Schwarz function $\top$ with the attaching inequality

$$\Big(\frac{\xi(\Delta_\alpha^m \curlyvee (\xi))'}{\Delta_\alpha^m \curlyvee (\xi)}\Big) = \hbar(\top(\xi)), \quad \xi \in \cup.$$

This yields the inequality

$$\Big(\frac{\xi(\Delta_\alpha^m \curlyvee (\xi))'}{\Delta_\alpha^m \curlyvee (\xi)}\Big) - \frac{1}{\xi} = \frac{\hbar(\top(\xi))-1}{\xi}.$$

By making the integrated operating, we attain the attaching equality

$$\log\left(\frac{\Delta_\alpha^m \curlyvee (\xi)}{\xi}\right) = \int_0^\xi \frac{\hbar(\top(\ell))-1}{\ell} d\ell. \tag{10}$$

Consequently, we have

$$\log \Delta_\alpha^m \curlyvee (\xi) = \left(\int_0^\xi \frac{\hbar(\top(\ell)) - 1}{\ell} d\ell \right) - \log(\xi). \tag{11}$$

A calculation brings the next subordination relation

$$\Delta_\alpha^m \curlyvee (\xi) \prec \xi \exp\Big(\int_0^\xi \frac{\hbar(\top(\ell)) - 1}{\ell} d\ell \Big).$$

Moreover, we indicate that the function $\hbar$ translates the disk $0 < |\xi| < \sigma \leq 1$ onto a domain, which is convex and symmetric via the real axis; on other words, we have

$$\hbar(-\sigma|\xi|) \leq \Re(\hbar(\top(\sigma\xi))) \leq \hbar(\sigma|\xi|), \quad \sigma \in (0, 1], |\xi| \neq \sigma,$$

which implies the inequalities:

$$\hbar(-\sigma) \leq \hbar(-\sigma|\xi|), \quad \hbar(\sigma|\xi|) \leq \hbar(\sigma)$$

and

$$\int_0^1 \frac{\hbar(\top(-\sigma|\xi|)) - 1}{\sigma} d\sigma \leq \Re\Big(\int_0^1 \frac{\hbar(\top(\sigma)) - 1}{\sigma} d\sigma \Big) \leq \int_0^1 \frac{\hbar(\top(\sigma|\xi|)) - 1}{\eta} d\sigma.$$

By employing the above inequality and Eq. (10), we arrive at

$$\int_0^1 \frac{\hbar(\top(-\sigma|\xi|)) - 1}{\sigma} d\sigma \leq \log \Big| \frac{\Delta_\alpha^m \curlyvee (\xi)}{\xi} \Big| \leq \int_0^1 \frac{\hbar(\top(\sigma|\xi|)) - 1}{\sigma} d\sigma.$$

This equivalence to the fact

$$\exp\Big(\int_0^1 \frac{\hbar(\top(-\sigma|\xi|)) - 1}{\sigma} d\sigma \Big) \leq \Big| \frac{\Delta_\alpha^m \curlyvee (\xi)}{\xi} \Big| \leq \exp\Big(\int_0^1 \frac{\hbar(\top(\sigma|\xi|)) - 1}{\sigma} d\sigma \Big).$$

This completes the proof.

We note that the condition of Theorem 8, which the BB formula subordinates by a convex univalent function $\hbar$, can be replaced by a general condition as follows:

Theorem 9 *Suppose that* $\curlyvee \in \bigwedge$, $\alpha \in [0, \infty)$ *and* $0 < \upsilon \leq 1$. *If*

$$\Big(\frac{\xi(\Delta_\alpha^m \curlyvee (\xi))' - \xi}{\Delta_\alpha^m \curlyvee (\xi)} \Big) \prec \frac{2\upsilon\xi}{1 + \xi}, \quad \xi \in \cup, \tag{12}$$

then

$$\Big| \frac{\Delta_\alpha^m \curlyvee (\xi)}{\xi} - 1 \Big| \leq \upsilon. \tag{13}$$

Moreover, if $\upsilon := \frac{1}{(1-r)^{\upsilon}}$, $0 < r < 1$, *for some positive constant* υ, *then*

$$\Big|\left(\frac{\Delta_{\alpha}^{m}\Upsilon(\xi)}{\xi}\right)'\Big| \leq \frac{\upsilon+1}{(1-r)^{\upsilon+1}}. \tag{14}$$

Proof In view of Lemma 3, we have the subordination inequality

$$\frac{\Delta_{\alpha}^{m}\Upsilon(\xi)}{\xi} \prec 1+\upsilon\xi.$$

Since the result is sharp, then directly, we obtain the inequality (13). Consequently, by ([22], lemma 5.1.3), we have the inequality (14).

5 Conclusion

In deduction, it is worth it to memo that the environment of the SDOs and their applications is a nonetheless to be completely discovered the area and it is estimated that the contemporary effort prompts the upcoming research on this subject. Furthermore, we delivered a relation between starlike class and the upper bound solution of BBDE by using the symmetric operator (Theorem 8). In the same manner of this result, we can use the linear operator $\mathbf{P}_{\alpha,\lambda}^{m}$ to present the same property. The linear operator indicates and extends many propositions involving the Sàlàgean differential operator and the Ruscheweyh derivative.

For future work, one can suggest the SDO (2) in different types of categories of analytic functions in $\cup$. These categories include the harmonic, meromorphic, and p-valent categories. Moreover, one can use any other geometric classes of univalent functions such as symmetric classes, spiral-like, close to convex, etc. classes. Different studies can also suggest by using (2), such as generalized various classes of differential equations describing the symmetry property, including wave equations and heat equations in a complex domain (space-time investigation).

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New Hermite–Hadamard Inequalities Concerning Twice Differentiable Generalized ψ-Convex Mappings via Conformable Fractional Integrals

Artion Kashuri and Rozana Liko

Abstract In this article, we first introduced a new class of generalized $((p_1, p_2); (\psi_1, \psi_2))$–convex mappings and an interesting lemma regarding Hermite–Hadamard type conformable fractional integral inequalities. By using the notion of generalized $((p_1, p_2); (\psi_1, \psi_2))$–convexity and lemma as an auxiliary result, some new estimates with respect to Hermite–Hadamard type integral inequalities associated with twice differentiable generalized $((p_1, p_2); (\psi_1, \psi_2))$–convex mappings via conformable fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article. At the end, some applications to special means are also given.

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1 Introduction

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1 *Let $f : I \subseteq \Re \longrightarrow \Re$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality (1.1) it is known as trapezium inequality.

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The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve, and extend the inequality (1.1) through various classes of convex functions interested readers are referred to [2–37, 39, 40, 43, 44, 47–51, 54, 57, 58]. Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.2 The Euler beta function is defined for $a, b > 0$ as

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{1.2}$$

Definition 1.3 The incomplete beta function is defined for $a, b > 0$ as

$$\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < x \le 1.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

Definition 1.4 Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x. \tag{1.3}$$

Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In the following, we give some definitions and properties of conformable fractional integrals which help to obtain main identity and results. Recently, some authors started to study on conformable fractional integrals. In [28], Khalil et al. defined the fractional integral of order $0 < \alpha \le 1$ only. In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 1.5 Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by

$$\left(I_{\alpha}^{a} f\right)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^{b}I_{\alpha}f\right)(t) = \frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots,$ and hence $\left(I_{\alpha}^{a}f\right)(t) = \left(J_{n+1}^{a}f\right)(t)$.

In [47], Set et al. established a generalization of Hermite–Hadamard type inequality for s-convex functions and gave some remarks to show the relationships with the classical and Riemann-Liouville fractional integrals inequality by using the given properties of conformable fractional integrals.

Theorem 1.6 *Let $f : [a,b] \longrightarrow \Re$ be a function with $0 \le a < b$, $s \in (0,1]$, and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right) \le \frac{1}{2^{s}(b-a)^{\alpha}}\left[\left(I_{\alpha}^{a}f\right)(b) + \left({}^{b}I_{\alpha}f\right)(a)\right]$$
$$\le \left[\frac{\beta(n+s+1,\alpha-n)+\beta(n+1,\alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^{s}},$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots$.

In [46–50], Set et al. established some results for some kind of inequalities via conformable fractional integrals.

Definition 1.7 ([56]) A set $S \subseteq \Re^{n}$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \Re^{n}$, if $x + t\eta(y,x) \in S$ for every $x, y \in S$ and $t \in [0,1]$.

The invex set S is also termed an η-connected set.

Definition 1.8 ([41]) Let $h : [0,1] \longrightarrow \Re$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \le h(1-t)f(x) + h(t)f(y) \tag{1.4}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.8, f becomes a preinvex function [45]. If the mapping $\eta(y,x) = y - x$ in Definition 1.8, then the non-negative function f reduces to h-convex mappings [53].

Definition 1.9 ([55]) Let $S \subseteq \Re^{n}$ be an invex set with respect to $\eta : S \times S \longrightarrow \Re^{n}$. A function $f : S \longrightarrow [0, +\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0,1]$, if for every $x, y \in S$ and $t \in [0,1]$,

$$f\big(x + t\eta(y,x)\big) \le (1-t)^{s}f(x) + t^{s}f(y). \tag{1.5}$$

Definition 1.10 ([42]) A function $f : K \longrightarrow \Re$ is said to be s-Godunova-Levin-Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y, x)\big) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \tag{1.6}$$

for each $x, y \in K$, $t \in (0, 1)$ and $s \in (0, 1]$.

Definition 1.11 ([14]) A non-negative function $f : I \subseteq \Re \longrightarrow [0, +\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 1.12 ([52]) Let $f : K \subseteq \Re \longrightarrow \Re$ be a non-negative function, a function $f : K \longrightarrow \Re$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.7}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.13 ([38]) A function $f : I \subseteq \Re \longrightarrow \Re$ is said to MT-convex functions, if it is non-negative and $\forall x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

Definition 1.14 ([43]) A function: $I \subseteq \Re \longrightarrow \Re$ is said to be $m - MT$-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0, 1)$, among $m \in (0, 1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.9}$$

For $m = 1$, Definition 1.14 reduces to Definition 1.13.

The concept of η-convex functions (at the beginning was named by χ-convex functions), considered in [19], has been introduced as the following.

Definition 1.15 Consider a convex set $I \subseteq \Re$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \Re$. A function $f : I \longrightarrow \Re$ is called convex with respect to η (briefly η-convex), if

$$f\big(\lambda x + (1-\lambda)y\big) \leq f(y) + \lambda\eta(f(x), f(y)), \tag{1.10}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every

$x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. For more results about η-convex functions readers are referred to [11, 12, 18, 19].

Definition 1.16 ([3]) Let $I \subseteq \Re$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \Re$. Consider $f : I \longrightarrow \Re$ and $\eta_2 : f(I) \times f(I) \longrightarrow \Re$. The function f is said to be (η_1, η_2)-convex if

$$f\big(x + \lambda\eta_1(y, x)\big) \leq f(x) + \lambda\eta_2(f(y), f(x)), \tag{1.11}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Motivated by above works and references therein, the main objective of this article is to establish some new estimates with respect to Hermite–Hadamard type integral inequalities using the notion of generalized $((p_1, p_2); (\psi_1, \psi_2))$-convexity and an interesting lemma as auxiliary result for conformable fractional integrals. It is pointed out that some new special cases will be deduced from main results of the article. At the end, some applications to special means will be obtained.

2 Main Results

The following definitions will be used in this section.

Definition 2.1 [15] A set $K \subseteq \Re$ is named as m-invex with respect to the mapping $\psi : K \times K \longrightarrow \Re$ for some fixed $m \in (0, 1]$, if $mx + t\psi(y, mx) \in K$ for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2 In Definition 2.1, under certain conditions, the mapping $\psi(y, mx)$ could reduce to $\psi(y, x)$. When $m = 1$, we get Definition 1.7.

We next introduce the concept of generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mappings.

Definition 2.3 Let $K \subseteq \Re$ be an open m-invex set with respect to the mapping $\psi_1 : \Re \times \Re \longrightarrow \Re$. Suppose $p_1, p_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\chi : I \longrightarrow \Re$ are continuous. Consider $f : K \longrightarrow (0, +\infty)$ and $\psi_2 : f(K) \times f(K) \longrightarrow [0, +\infty)$. The mapping f is said to be generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex, if

$$f\big(m\chi(x) + t\psi_1(\chi(y), m\chi(x))\big) \leq \big[mp_1(t)f^r(x) + p_2(t)\psi_2(f^r(y), f^r(x))\big]^{\frac{1}{r}} \tag{2.1}$$

holds for all $x, y \in I$, $r \in (0, 1]$, $t \in [0, 1]$ and some fixed $m \in (0, 1]$.

Remark 2.4 In Definition 2.3, if we choose $m = r = 1$, $p_1(t) = 1$, $p_2(t) = t$, $\psi_1(\chi(y), m\chi(x)) = \chi(y) - m\chi(x)$, $\psi_2(f^r(y), f^r(x)) = \eta(f^r(y), f^r(x))$ and $\chi(x) = x$, $\forall x \in I$, then we get Definition 1.15. Also, in Definition 2.3, if we choose $m = r = 1$, $p_1(t) = 1$, $p_2(t) = t$ and $\chi(x) = x$, $\forall x \in I$, then we get

Definition 1.16. Under some suitable choices as we done above, we can get also the Definitions 1.9 and 1.10.

Remark 2.5 For $r = 1$, let us discuss some special cases in Definition 2.3 as follows:

(I) Taking $p_1(t) = p_2(t) = 1$, then we get generalized $((m, P); (\psi_1, \psi_2))$-convex mappings.
(II) Taking $p_1(t) = h(1-t)$, $p_2(t) = h(t)$, then we get generalized $((m, h); (\psi_1, \psi_2))$-convex mappings.
(III) Taking $p_1(t) = (1-t)^s$, $p_2(t) = t^s$ for $s \in (0, 1]$, then we get generalized $((m, s); (\psi_1, \psi_2))$-Breckner-convex mappings.
(IV) Taking $p_1(t) = (1-t)^{-s}$, $p_2(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized $((m, s); (\psi_1, \psi_2))$-Godunova-Levin-Dragomir-convex mappings.
(V) Taking $p_1(t) = p_2(t) = t(1-t)$, then we get generalized $((m, tgs); (\psi_1, \psi_2))$-convex mappings.
(VI) Taking $p_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $p_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized $(m; (\psi_1, \psi_2)) - MT$-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let see the following example of a generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mapping which is not convex.

Example 2.6 Let us take $r = \frac{1}{2}$, $p_1(t) = t^l$, $p_2(t) = (1-t)^s$ for all $l, s \in (0, 1]$ and χ be an identity function. Consider the function $f : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$f(x) = \begin{cases} x, & 0 \le x \le 2; \\ 4, & x > 2. \end{cases}$$

Define two bifunctions $\psi_1 : [0, +\infty) \times [0, +\infty) \longrightarrow \Re$ and $\psi_2 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ by

$$\psi_1(x, y) = \begin{cases} -y, & 0 \le y \le 2; \\ x + y, & y > 2, \end{cases}$$

and

$$\psi_2(x, y) = \begin{cases} x + y, & x \le y; \\ 4(x + y), & x > y. \end{cases}$$

Then f is generalized $\left(\left(t^{\frac{l}{2}}, (1-t)^{\frac{s}{2}}\right); (\psi_1, \psi_2)\right)$-convex mapping. But f is not preinvex with respect to ψ_1 and also it is not convex (consider $x = 0$, $y = 3$ and $t \in (0, 1]$).

For establishing our main results regarding some new Hermite–Hadamard type integral inequalities associated with twice differentiable generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals, we need the following lemma.

Lemma 2.7 *Let $\chi : I \longrightarrow \Re$ be a continuous function. Suppose $K = [m\chi(a), m\chi(a) + \eta(\chi(b), m\chi(a))] \subseteq \Re$ be an open m-invex subset with respect to $\eta : K \times K \longrightarrow \Re$ for some fixed $m \in (0, 1]$, where $\eta(\chi(b), m\chi(a)) > 0$ and $a < b$. Assume that $f : K \longrightarrow \Re$ be a twice differentiable mapping on K° such that $f'' \in L_1(K)$. Then for $\alpha \in (n, n+1]$, where $n = 0, 1, 2, \dots,$ the following identity for conformable fractional integrals holds:*

$$
\begin{aligned}
&-\frac{\eta^{\alpha+1}(\chi(x), m\chi(a))f'(m\chi(a)) + \eta^{\alpha+1}(\chi(x), m\chi(b))f'(m\chi(b))}{\eta(\chi(b), m\chi(a))}\\
&\quad -\frac{(n+2-\alpha)(n+1)!}{\eta(\chi(b), m\chi(a))}\\
&\quad \times\Big[\Big({}^{(m\chi(a)+\eta(\chi(x),m\chi(a)))}I_\alpha f\Big)(m\chi(a)) + \Big({}^{(m\chi(b)+\eta(\chi(x),m\chi(b)))}I_\alpha f\Big)(m\chi(b))\Big]\\
&\quad =\frac{\eta^{\alpha+2}(\chi(x), m\chi(a))}{\eta(\chi(b), m\chi(a))} \qquad (2.2)\\
&\quad \times\int_0^1 \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] f''(m\chi(a) + t\eta(\chi(x), m\chi(a)))dt\\
&\quad +\frac{\eta^{\alpha+2}(\chi(x), m\chi(b))}{\eta(\chi(b), m\chi(a))}\\
&\quad \times\int_0^1 \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] f''(m\chi(b) + t\eta(\chi(x), m\chi(b)))dt.
\end{aligned}
$$

Proof A simple proof of the equality can be done by performing two integration by parts in the integrals and changing the variables. The details are left to the interested reader. □

Remark 2.8 In Lemma 2.7, if we choose $\alpha = n + 1$, where $n = 0, 1, 2, \dots,$ we get an identity for fractional integrals.

Throughout this paper, we denote

$$
\begin{aligned}
I_{f,\eta,\chi}(x; \alpha, n, m, a, b) &:= \frac{\eta^{\alpha+2}(\chi(x), m\chi(a))}{\eta(\chi(b), m\chi(a))} \qquad (2.3)\\
&\times\int_0^1 \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] f''(m\chi(a) + t\eta(\chi(x), m\chi(a)))dt\\
&+\frac{\eta^{\alpha+2}(\chi(x), m\chi(b))}{\eta(\chi(b), m\chi(a))}
\end{aligned}
$$

$$\times \int_0^1 \left[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\right] f''(m\chi(b) + t\eta(\chi(x), m\chi(b)))dt.$$

Using Lemma 2.7, we now state the following theorems for the corresponding version for power of second derivative.

Theorem 2.9 *Let $\alpha \in (n, n+1]$, where $n = 0, 1, 2, \ldots$, and $0 < r \leq 1$. Also, let $p_1, p_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\chi : I \longrightarrow \Re$ are continuous functions. Suppose $K = [m\chi(a), m\chi(a) + \psi_1(\chi(b), m\chi(a))] \subseteq \Re$ be an open m-invex subset with respect to $\psi_1 : \Re \times \Re \longrightarrow \Re$ for some fixed $m \in (0, 1]$, where $\psi_1(\chi(b), m\chi(a)) > 0$ and $a < b$. Assume that $f : K \longrightarrow (0, +\infty)$ be a twice differentiable mapping on K° such that $f'' \in L_1(K)$ and $\psi_2 : f(K) \times f(K) \longrightarrow [0, +\infty)$. If $\left(f''(x)\right)^q$ is generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mapping, where $q > 1$ and $p^{-1}+q^{-1} = 1$, then the following inequality for conformable fractional integrals holds:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.4}$$

$$\times \Bigg\{ |\psi_1(\chi(x), m\chi(a))|^{\alpha+2} \Big[m\left(f''(a)\right)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); r) \Big]^{\frac{1}{rq}}$$

$$+ |\psi_1(\chi(x), m\chi(b))|^{\alpha+2} \Big[m\left(f''(b)\right)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); r) \Big]^{\frac{1}{rq}} \Bigg\},$$

where

$$\delta(p, \alpha, n) := \int_0^1 \left[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\right]^p dt$$

and

$$I(p_i(t); r) := \int_0^1 p_i^{\frac{1}{r}}(t)dt, \quad \forall i = 1, 2.$$

Proof From Lemma 2.7, generalized $((p_1, p_2); (\psi_1, \psi_2))$-convexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{|\psi_1(\chi(x), m\chi(a))|^{\alpha+2}}{|\psi_1(\chi(b), m\chi(a))|}$$

$$\times \int_0^1 \left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right|\left|f''(m\chi(a)+t\psi_1(\chi(x),m\chi(a)))\right|dt$$

$$+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{|\psi_1(\chi(b),m\chi(a))|}$$

$$\times \int_0^1 \left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right|\left|f''(m\chi(b)+t\psi_1(\chi(x),m\chi(b)))\right|dt$$

$$\leq \frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\left(\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(f''(m\chi(a)+t\psi_1(\chi(x),m\chi(a)))\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\left(\int_0^1 \left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(f''(m\chi(b)+t\psi_1(\chi(x),m\chi(b)))\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\delta^{\frac{1}{p}}(p,\alpha,n)$$

$$\times\left[\int_0^1 \left[mh_1(t)(f''(a))^{rq}+p_2(t)\psi_2\left((f''(x))^{rq},(f''(a))^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}$$

$$+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\delta^{\frac{1}{p}}(p,\alpha,n)$$

$$\times\left[\int_0^1 \left[mh_1(t)(f''(b))^{rq}+p_2(t)\psi_2\left((f''(x))^{rq},(f''(b))^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}$$

$$\leq \frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\delta^{\frac{1}{p}}(p,\alpha,n)$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}(f''(a))^q p_1^{\frac{1}{r}}(t)dt\right)^r+\left(\int_0^1 \psi_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(a))^{rq}\right)p_2^{\frac{1}{r}}(t)dt\right)^r\right]^{\frac{1}{rq}}$$

$$+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\delta^{\frac{1}{p}}(p,\alpha,n)$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}(f''(b))^q p_1^{\frac{1}{r}}(t)dt\right)^r+\left(\int_0^1 \psi_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(b))^{rq}\right)p_2^{\frac{1}{r}}(t)dt\right)^r\right]^{\frac{1}{rq}}$$

$$=\frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\psi_1(\chi(b),m\chi(a))}$$

$$\times\Big\{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}\Big[m\left(f''(a)\right)^{rq}I^r(p_1(t);r)$$

$$+\psi_2\left((f''(x))^{rq},(f''(a))^{rq}\right)I^r(p_2(t);r)\Big]^{\frac{1}{rq}}$$

$$+ |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\Big[m\left(f''(b)\right)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); r)\Big]^{\frac{1}{rq}}\Big\}.$$

The proof of theorem 2.9 is completed. □

We point out some special cases of Theorem 2.9 as follows:

Corollary 2.10 *In Theorem 2.9, if we choose* $\alpha = n+1$ *where* $n = 0, 1, 2, \ldots,$ *we have the following Hermite–Hadamard type inequality for generalized* $((p_1, p_2); (\psi_1, \psi_2))$*-convex mappings via fractional integrals:*

$$\Big| -\frac{\psi_1^{n+2}(\chi(x), m\chi(a))f'(m\chi(a)) + \psi_1^{n+2}(\chi(x), m\chi(b))f'(m\chi(b))}{(n+2)\psi_1(\chi(b), m\chi(a))}$$

$$+\frac{\psi_1^{n+1}(\chi(x), m\chi(a))f(m\chi(a) + \psi_1(\chi(x), m\chi(a))) + \psi_1^{n+1}(\chi(x), m\chi(b))f(m\chi(b) + \psi_1(\chi(x), m\chi(b)))}{\psi_1(\chi(b), m\chi(a))}$$

$$-\frac{\Gamma(n+2)}{\psi_1(\chi(b), m\chi(a))}$$

$$\times\left[J^{n+1}_{(m\chi(a)+\psi_1(\chi(x),m\chi(a)))^-}f(m\chi(a)) + J^{n+1}_{(m\chi(b)+\psi_1(\chi(x),m\chi(b)))^-}f(m\chi(b))\right]\Big|$$

$$\le \frac{\delta^{\frac{1}{p}}(p, n+1, n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.5}$$

$$\times\Big\{|\psi_1(\chi(x), m\chi(a))|^{n+3}\Big[m\left(f''(a)\right)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); r)\Big]^{\frac{1}{rq}}$$

$$+\Big\{|\psi_1(\chi(x), m\chi(b))|^{n+3}\Big[m\left(f''(b)\right)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); r)\Big]^{\frac{1}{rq}}\Big\}.$$

Corollary 2.11 *In Theorem 2.9 for* $p = q = 2$*, we have the following Hermite–Hadamard type inequality for generalized* $((p_1, p_2); (\psi_1, \psi_2))$*-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \le \frac{\sqrt{\delta(2, \alpha, n)}}{\psi_1(\chi(b), m\chi(a))} \tag{2.6}$$

$$\times\Big\{|\psi_1(\chi(x), m\chi(a))|^{\alpha+2}\Big[m\left(f''(a)\right)^{2r} I^r(p_1(t); r)$$

$$+ \psi_2\left((f''(x))^{2r}, (f''(a))^{2r}\right) I^r(p_2(t); r)\Big]^{\frac{1}{2r}}$$

$$+\left\{|\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\Big[m\left(f''(b)\right)^{2r} I^r(p_1(t); r)\right.$$

$$\left.+\psi_2\left((f''(x))^{2r}, (f''(b))^{2r}\right) I^r(p_2(t); r)\Big]^{\frac{1}{2r}}\right\}.$$

Corollary 2.12 *In Theorem 2.9 for* $p_1(t) = p_2(t) = 1$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type inequality for generalized* $((m, P); (\psi_1, \psi_2))$*-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.7}$$

$$\times\left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.13 *In Theorem 2.9 for* $p_1(t) = h(1-t)$, $p_2(t) = h(t)$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type inequality for generalized* $((m, h); (\psi_1, \psi_2))$*-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\psi_1(\chi(b), m\chi(a))} I^{\frac{1}{q}}(h(t); r) \tag{2.8}$$

$$\times\left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.14 *In Corollary 2.13 for* $p_1(t) = (1-t)^s$, $p_2(t) = t^s$, *we get the following Hermite–Hadamard type inequality for generalized* $((m, s); (\psi_1, \psi_2))$*-Breckner-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\psi_1(\chi(b), m\chi(a))}\left(\frac{r}{r+s}\right)^{\frac{1}{q}} \tag{2.9}$$

$$\times\left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.15 *In Corollary 2.13 for* $p_1(t) = (1-t)^{-s}$, $p_2(t) = t^{-s}$ *and* $0 < s < r$, *we get the following Hermite–Hadamard type inequality for generalized* $((m, s); (\psi_1, \psi_2))$*-Godunova-Levin-Dragomir-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\psi_1(\chi(b), m\chi(a))}\left(\frac{r}{r-s}\right)^{\frac{1}{q}} \tag{2.10}$$

$$\times\left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.16 *In Theorem 2.9 for $p_1(t) = p_2(t) = t(1-t)$ and $f''(x) \leq L$, $\forall x \in I$, we get the following Hermite–Hadamard type inequality for generalized $((m, tgs); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x;\alpha,n,m,a,b)\right| \leq \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\psi_1(\chi(b),m\chi(a))}\beta^{\frac{1}{q}}\left(1+\frac{1}{r},1+\frac{1}{r}\right) \tag{2.11}$$
$$\times\left[|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}+|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}\right]\left[mL^{rq}+\psi_2\left(L^{rq},L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.17 *In Corollary 2.13 for $p_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $p_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $r \in \left(\frac{1}{2}, 1\right]$, we get the following Hermite–Hadamard type inequality for generalized $(m; (\psi_1, \psi_2)) - MT$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x;\alpha,n,m,a,b)\right| \leq \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\psi_1(\chi(b),m\chi(a))}\beta^{\frac{1}{q}}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right) \tag{2.12}$$
$$\times\left[|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}+|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}\right]\left[mL^{rq}+\psi_2\left(L^{rq},L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Theorem 2.18 *Let $\alpha \in (n, n+1]$, where $n = 0, 1, 2, \ldots,$ and $0 < r \leq 1$. Also, let $p_1, p_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\chi : I \longrightarrow \Re$ are continuous functions. Suppose $K = [m\chi(a), m\chi(a) + \psi_1(\chi(b), m\chi(a))] \subseteq \Re$ be an open m-invex subset with respect to $\psi_1 : \Re \times \Re \longrightarrow \Re$ for some fixed $m \in (0, 1]$, where $\psi_1(\chi(b), m\chi(a)) > 0$ and $a < b$. Assume that $f : K \longrightarrow (0, +\infty)$ be a twice differentiable mapping on K° such that $f'' \in L_1(K)$ and $\psi_2 : f(K) \times f(K) \longrightarrow [0, +\infty)$. If $\left(f''(x)\right)^q$ is generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mapping and $q \geq 1$, then the following inequality for conformable fractional integrals holds:*

$$\left|I_{f,\psi_1,\chi}(x;\alpha,n,m,a,b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\psi_1(\chi(b),m\chi(a))} \tag{2.13}$$
$$\times\left\{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}\left[m\left(f''(a)\right)^{rq} I^r(p_1(t);\alpha,n,r)\right.\right.$$
$$\left.+\psi_2\left((f''(x))^{rq},(f''(a))^{rq}\right)I^r(p_2(t);\alpha,n,r)\right]^{\frac{1}{rq}}$$
$$+|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}\left[m\left(f''(b)\right)^{rq} I^r(p_1(t);\alpha,n,r)\right.$$
$$\left.\left.+\psi_2\left((f''(x))^{rq},(f''(b))^{rq}\right)I^r(p_2(t);\alpha,n,r)\right]^{\frac{1}{rq}}\right\},$$

where

$$I(p_i(t);\alpha,n,r):=\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]p_i^{\frac{1}{r}}(t)dt,\quad \forall i=1,2.$$

Proof From Lemma 2.7, generalized $((p_1,p_2);(\psi_1,\psi_2))$-convexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|I_{f,\psi_1,\chi}(x;\alpha,n,m,a,b)\right|\le\frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{|\psi_1(\chi(b),m\chi(a))|}\\
&\times\int_0^1\left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right|\left|f''(m\chi(a)+t\psi_1(\chi(x),m\chi(a)))\right|dt\\
&+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{|\psi_1(\chi(b),m\chi(a))|}\\
&\times\int_0^1\left|\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right|\left|f''(m\chi(b)+t\psi_1(\chi(x),m\chi(b)))\right|dt\\
&\quad\le\frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\left(\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]\left(f''(m\chi(a)+t\psi_1(\chi(x),m\chi(a)))\right)^q dt\right)^{\frac{1}{q}}\\
&+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\left(\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]\left(f''(m\chi(b)+t\psi_1(\chi(x),m\chi(b)))\right)^q dt\right)^{\frac{1}{q}}\\
&\quad\le\frac{|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\beta^{1-\frac{1}{q}}(n+3,\alpha-n)\\
&\times\left[\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]\right.\\
&\times\left.\left[mh_1(t)(f''(a))^{rq}+p_2(t)\psi_2\left((f''(x))^{rq},(f''(a))^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}\\
&+\frac{|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b),m\chi(a))}\beta^{1-\frac{1}{q}}(n+3,\alpha-n)\\
&\times\left[\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]\right.\\
&\times\left.\left[mh_1(t)(f''(b))^{rq}+p_2(t)\psi_2\left((f''(x))^{rq},(f''(b))^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{|\psi_1(\chi(x), m\chi(a))|^{\alpha+2}}{\psi_1(\chi(b), m\chi(a))} \beta^{1-\frac{1}{q}}(n+3, \alpha-n)$$

$$\times \Bigg[\left(\int_0^1 m^{\frac{1}{r}} (f''(a))^q \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] p_1^{\frac{1}{r}}(t)dt \right)^r$$

$$+ \left(\int_0^1 \psi_2^{\frac{1}{r}} \left((f''(x))^{rq}, (f''(a))^{rq}\right) \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] p_2^{\frac{1}{r}}(t)dt \right)^r \Bigg]^{\frac{1}{rq}}$$

$$+ \frac{|\psi_1(\chi(x), m\chi(b))|^{\alpha+2}}{\psi_1(\chi(b), m\chi(a))} \beta^{1-\frac{1}{q}}(n+3, \alpha-n)$$

$$\times \Bigg[\left(\int_0^1 m^{\frac{1}{r}} (f''(b))^q \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] p_1^{\frac{1}{r}}(t)dt \right)^r$$

$$+ \left(\int_0^1 \psi_2^{\frac{1}{r}} \left((f''(x))^{rq}, (f''(b))^{rq}\right) \big[\beta(n+2, \alpha-n) - \beta_t(n+2, \alpha-n)\big] p_2^{\frac{1}{r}}(t)dt \right)^r \Bigg]^{\frac{1}{rq}}$$

$$= \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))}$$

$$\times \Bigg\{ |\psi_1(\chi(x), m\chi(a))|^{\alpha+2} \Big[m \left(f''(a)\right)^{rq} I^r(p_1(t); \alpha, n, r)$$

$$+ \psi_2 \left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); \alpha, n, r) \Big]^{\frac{1}{rq}}$$

$$+ |\psi_1(\chi(x), m\chi(b))|^{\alpha+2} \Big[m \left(f''(b)\right)^{rq} I^r(p_1(t); \alpha, n, r)$$

$$+ \psi_2 \left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); \alpha, n, r) \Big]^{\frac{1}{rq}} \Bigg\}.$$

The proof of theorem 2.18 is completed. □

We point out some special cases of Theorem 2.18 as follows:

Corollary 2.19 *In Theorem 2.18, if we choose $\alpha = n + 1$ where $n = 0, 1, 2, \ldots$, we have the following Hermite–Hadamard type inequality for generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mappings via fractional integrals:*

$$\Bigg| - \frac{\psi_1^{n+2}(\chi(x), m\chi(a)) f'(m\chi(a)) + \psi_1^{n+2}(\chi(x), m\chi(b)) f'(m\chi(b))}{(n+2)\psi_1(\chi(b), m\chi(a))}$$

$$+ \frac{\psi_1^{n+1}(\chi(x), m\chi(a)) f(m\chi(a) + \psi_1(\chi(x), m\chi(a))) + \psi_1^{n+1}(\chi(x), m\chi(b)) f(m\chi(b) + \psi_1(\chi(x), m\chi(b)))}{\psi_1(\chi(b), m\chi(a))}$$

$$- \frac{\Gamma(n+2)}{\psi_1(\chi(b), m\chi(a))}$$

$$\times \left[J^{n+1}_{(m\chi(a)+\psi_1(\chi(x), m\chi(a)))^-} f(m\chi(a)) + J^{n+1}_{(m\chi(b)+\psi_1(\chi(x), m\chi(b)))^-} f(m\chi(b)) \right] \Bigg|$$

$$\leq \left(\frac{1}{n+3} \right)^{1-\frac{1}{q}} \frac{1}{\psi_1(\chi(b), m\chi(a))} \tag{2.14}$$

$$\times\Bigg\{|\psi_1(\chi(x), m\chi(a))|^{n+3}\Big[m\left(f''(a)\right)^{rq} I^r(p_1(t); n+1, n, r)$$

$$+\psi_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); n+1, n, r)\Big]^{\frac{1}{rq}}$$

$$+|\psi_1(\chi(x), m\chi(b))|^{n+3}\Big[m\left(f''(b)\right)^{rq} I^r(p_1(t); n+1, n, r)$$

$$+\psi_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); n+1, n, r)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 2.20 *In Theorem 2.18 for $q = 1$, we have the following Hermite–Hadamard type inequality for generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{1}{\psi_1(\chi(b), m\chi(a))} \tag{2.15}$$

$$\times\Bigg\{|\psi_1(\chi(x), m\chi(a))|^{\alpha+2}\Big[m\left(f''(a)\right)^{r} I^r(p_1(t); \alpha, n, r)$$

$$+\psi_2\left((f''(x))^{r}, (f''(a))^{r}\right) I^r(p_2(t); \alpha, n, r)\Big]^{\frac{1}{r}}$$

$$+|\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\Big[m\left(f''(b)\right)^{r} I^r(p_1(t); \alpha, n, r)$$

$$+\psi_2\left((f''(x))^{r}, (f''(b))^{r}\right) I^r(p_2(t); \alpha, n, r)\Big]^{\frac{1}{r}}\Bigg\}.$$

Corollary 2.21 *In Theorem 2.18 for $p_1(t) = p_2(t) = 1$ and $f''(x) \leq L$, $\forall x \in I$, we get the following Hermite–Hadamard type inequality for generalized $((m, P); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\beta(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.16}$$

$$\times\left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.22 *In Theorem 2.18 for $p_1(t) = h(1-t)$, $p_2(t) = h(t)$ and $f''(x) \leq L$, $\forall x \in I$, we get the following Hermite–Hadamard type inequality for generalized $((m, h); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.17}$$

$$\times \left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]$$
$$\times \left[mL^{rq} I^r(h(1-t); \alpha, n, r) + \psi_2\left(L^{rq}, L^{rq}\right) I^r(h(t); \alpha, n, r)\right]^{\frac{1}{rq}}.$$

Corollary 2.23 *In Corollary 2.22 for $p_1(t) = (1-t)^s$, $p_2(t) = t^s$, we get the following Hermite–Hadamard type inequality for generalized $((m, s); (\psi_1, \psi_2))$-Breckner-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.18}$$
$$\times \left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]$$
$$\times \left[mL^{rq} I^r((1-t)^s; \alpha, n, r) + \psi_2\left(L^{rq}, L^{rq}\right) I^r(t^s; \alpha, n, r)\right]^{\frac{1}{rq}}.$$

Corollary 2.24 *In Corollary 2.22 for $p_1(t) = (1-t)^{-s}$, $p_2(t) = t^{-s}$, we get the following Hermite–Hadamard type inequality for generalized $((m, s); (\psi_1, \psi_2))$-Godunova-Levin-Dragomir-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))} \tag{2.19}$$
$$\times \left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]$$
$$\times \left[mL^{rq} I^r((1-t)^{-s}; \alpha, n, r) + \psi_2\left(L^{rq}, L^{rq}\right) I^r(t^{-s}; \alpha, n, r)\right]^{\frac{1}{rq}}.$$

Corollary 2.25 *In Theorem 2.18 for $p_1(t) = p_2(t) = t(1-t)$ and $f''(x) \leq L$, $\forall x \in I$, we get the following Hermite–Hadamard type inequality for generalized $((m, tgs); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x; \alpha, n, m, a, b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\psi_1(\chi(b), m\chi(a))} I^{\frac{1}{q}}(t(1-t); \alpha, n, r) \tag{2.20}$$
$$\times \left[|\psi_1(\chi(x), m\chi(a))|^{\alpha+2} + |\psi_1(\chi(x), m\chi(b))|^{\alpha+2}\right]\left[mL^{rq} + \psi_2\left(L^{rq}, L^{rq}\right)\right]^{\frac{1}{rq}}.$$

Corollary 2.26 *In Corollary 2.22 for $p_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $p_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we get the following Hermite–Hadamard type inequality for generalized $(m; (\psi_1, \psi_2)) - MT$-convex mappings via conformable fractional integrals:*

$$\left|I_{f,\psi_1,\chi}(x;\alpha,n,m,a,b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\psi_1(\chi(b),m\chi(a))} \tag{2.21}$$

$$\times\left[|\psi_1(\chi(x),m\chi(a))|^{\alpha+2}+|\psi_1(\chi(x),m\chi(b))|^{\alpha+2}\right]$$

$$\times\left[mL^{rq}I^r\left(\frac{\sqrt{1-t}}{2\sqrt{t}};\alpha,n,r\right)+\psi_2\left(L^{rq},L^{rq}\right)I^r\left(\frac{\sqrt{t}}{2\sqrt{1-t}};\alpha,n,r\right)\right]^{\frac{1}{rq}}.$$

Remark 2.27 For $\psi_2(f^r(y), f^r(x)) = f^r(y) - f^r(x)$, $\forall x, y \in I$ and $0 < r \leq 1$, by our Theorems 2.9 and 2.18 and their corresponding corollaries, respectively 2.10–2.17 and 2.19–2.26, we can get some new special Hermite–Hadamard type inequalities associated with generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mappings via conformable fractional integrals and fractional integrals. The details are left to the interested reader.

3 Applications to Special Means

Definition 3.1 ([4]) A function $M : \Re_+^2 \longrightarrow \Re_+$ is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
(2) Symmetry: $M(x, y) = M(y, x)$,
(3) Reflexivity: $M(x, x) = x$,
(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
(5) Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for different positive real numbers α and β.

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha+\beta}{2}.$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \Re \setminus \{-1, 0\}.$$

(8) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, \alpha_2, \cdots, \alpha_n \\ u_1, u_2, \cdots, u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}},$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \Re$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider a continuous function $\chi : I \longrightarrow \Re$, $\psi_1 : \Re \times \Re \longrightarrow \Re$, $\psi_2 : f(K) \times f(K) \longrightarrow [0, +\infty)$, and $\overline{M} := M(\chi(a), \chi(b)) : [\chi(a), \chi(a) + \psi_1(\chi(b), \chi(a))] \times [\chi(a), \chi(a) + \psi_1(\chi(b), \chi(a))] \longrightarrow \Re_+$, which is one of the above mentioned means. Therefore, one can obtain various inequalities using the results of Sect. 2 for these means as follows: Replace $\psi_1(\chi(y), \chi(x)) = M(\chi(x), \chi(y))$, $\forall x, y \in I$, for value $m = 1$ in (2.4) and (2.13), one can obtain the following interesting inequalities involving means:

$$\left|I_{f,M(\cdot,\cdot),\chi}(x; \alpha, n, 1, a, b)\right| \leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\overline{M}} \tag{3.1}$$
$$\times \left\{M^{\alpha+2}(\chi(x), \chi(a))\Big[\left(f''(a)\right)^{rq} I^r(p_1(t); r)\right.$$
$$+ \psi_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); r)\Big]^{\frac{1}{rq}}$$

$$+ M^{\alpha+2}(\chi(x), \chi(b))\Big[\big(f''(b)\big)^{rq} I^r(p_1(t); r)$$

$$+ \psi_2 \left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); r)\Big]^{\frac{1}{rq}} \Bigg\},$$

$$\left|I_{f,M(\cdot,\cdot),\chi}(x; \alpha, n, 1, a, b)\right| \leq \frac{\beta^{1-\frac{1}{q}}(n+3, \alpha-n)}{\overline{M}} \tag{3.2}$$

$$\times \Bigg\{ M^{\alpha+2}(\chi(x), \chi(a))\Big[\big(f''(a)\big)^{rq} I^r(p_1(t); \alpha, n, r)$$

$$+ \psi_2 \left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(p_2(t); \alpha, n, r)\Big]^{\frac{1}{rq}}$$

$$+ M^{\alpha+2}(\chi(x), \chi(b))\Big[\big(f''(b)\big)^{rq} I^r(p_1(t); \alpha, n, r)$$

$$+ \psi_2 \left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(p_2(t); \alpha, n, r)\Big]^{\frac{1}{rq}} \Bigg\}.$$

Letting $M(\chi(x), \chi(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in I$ in (3.1) and (3.2), we get inequalities involving means for a particular choices of twice differentiable generalized $((p_1, p_2); (\psi_1, \psi_2))$-convex mapping f. The details are left to the interested reader.

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Some New Fractional Inequalities Using n-Polynomials s-Type Convexity

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Abstract In the present paper, the authors establish a new version of the Hermite–Hadamard and Ostrowski type fractional integral inequalities for a class of n-polynomial s-type convex functions. Using our generalizations we are able to also deduce some already known results. We present two different techniques, for functions whose first and second derivatives in absolute value at certain powers are n-polynomial s-type convex by employing k-fractional integral operators. These techniques have yielded some interesting results. In the form of corollaries, some estimates of k-fractional integrals are obtained which contain bounds of RL-fractional integrals. We also obtain a refined bound of the Midpoint, Trapezoidal, and Simpson type inequalities for twice differentiable n-polynomial s-type convex functions.

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1 Introduction

Convexity has played a crucial role in the advancement of different areas of science and technology. Due to its robustness, convex functions and convex sets have been generalized and extended in various directions. The Hermite–Hadamard inequality is a fundamental inequality which has been extensively used in several problems

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of pure and applied Mathematics, cf. [5–7, 9, 10, 26]. The Hermite–Hadamard type inequality, cf. [16, 17], is stated as follows:

$$\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big) \leq \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathcal{D}(\ell)d\ell \leq \frac{\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)}{2}, \tag{1.1}$$

where $\mathcal{D}$ is an integrable real valued function over an interval $[\theta_1, \theta_2]$.

The inequality (1.1) and its various generalizations, refinements, extensions, and converses, etc. have many applications in different fields of science, such as electrical engineering, mathematical statistics, mathematical finance, information theory, coding, to mention just a few (cf. [9, 10, 26]). It has been proved that a function is convex if and only if it satisfies an integral inequality (1.1).

In [25], Ostrowski proved an inequality that establishes bounds of the integral average of a function $\mathcal{D}$ over an interval $[\theta_1, \theta_2]$ to its value $\mathcal{D}(\ell)$ at a point $\ell \in [\theta_1, \theta_2]$.

Theorem 1.1 *Let $\mathcal{J} \subset \Re$ and $\mathcal{D} : \mathcal{J} \to \Re$ be a differentiable function in $\mathcal{J}^\circ$ (the interior of $\mathcal{J}$) such that $\theta_1, \theta_2 \in \mathcal{J}^\circ$ with $\theta_1 < \theta_2$. If $|\mathcal{D}'(\imath)| \leq \mathcal{K}$, for all $\imath \in [\theta_1, \theta_2]$, then we have*

$$\Big|\mathcal{D}(\ell) - \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathcal{D}(\ell)d\ell\Big| \leq \mathcal{K}(\theta_2-\theta_1)\Bigg[\frac{1}{4}+\frac{\big(\ell-\frac{\theta_1+\theta_2}{2}\big)^2}{(\theta_2-\theta_1)^2}\Bigg], \quad \forall\, \ell \in [\theta_1, \theta_2]. \tag{1.2}$$

Ostrowski type inequalities have significant applications in numerical analysis as they provide error estimates for many quadrature rules. In recent years, such inequalities have been extended and generalized in various aspects, cf. [12–14, 28]. Moreover, fractional integral inequalities have several applications in scientific domains (cf. [1–31]). In [31], the authors have derived several generalizations for new Ostrowski type inequalities for generalized k-fractional integrals. Farid et al. [12] established Ostrowski type fractional integral inequalities for s-Godunova–Levin functions via k-fractional integrals (cf. [12–14, 27, 31]).

In the following, we present a class of functional variants for convex functions as well as several other generalizations. The novel technique that we establish here is useful to generate the Mandelbrot and Julia sets for quadratic and cubic polynomials with s-convexity (cf. [19, 22, 33]).

We discuss some connections between the class of convex functions and s-type convex functions.

Definition 1.1 Let $s \in [0, 1]$. We say that $\mathcal{D} : \mathcal{J} \to \Re$ is an s-type convex function on $\mathcal{J}$, if the inequality

$$\mathcal{D}\big(\imath x + (1-\imath)y\big) \leq \big[1 - s(1-\imath)\big]\mathcal{D}(x) + \big[1 - s\imath\big]\mathcal{D}(y) \tag{1.3}$$

holds for all $x, y \in \mathcal{J}$ and $\imath \in [0, 1]$.

Remark 1.1 *In Definition 1.1:*

(1) *If we choose* $s = 1$, *then we derive the classical convex function.*
(2) *If we choose* $s = 0$, *then we derive the definition of* P*-function, as in [11].*
(3) *If* $\mathcal{D}$ *is* s*-type convex on* $\mathcal{J}$, *then the codomain of* $\mathcal{D}$ *is* $[0, +\infty)$.
(4) *Indeed, let* x *be an arbitrary point on* $\mathcal{J}$, *then by the* s*-type convexity of* $\mathcal{D}$, *we have*

$$\mathcal{D}(\imath\theta + (1-\imath)x) \leq \big[1 - s(1-\imath)\big]\mathcal{D}(\theta) + \big[1 - s\imath\big]\mathcal{D}(x)$$

for all $\theta \in \mathcal{J}$ *and* $\imath \in [0, 1]$.

If we choose $\imath = 1$, *then we get*

$$\mathcal{D}(\theta) \leq \mathcal{D}(\theta) + (1-s)\mathcal{D}(x)$$

$$\Rightarrow \quad (1-s)\mathcal{D}(x) \geq 0 \quad \Rightarrow \quad \mathcal{D}(x) = 0.$$

Proposition 1.1 *Every nonnegative convex function is also* s*-type convex function.*

Proof The proof is evident, since

$$s(1-\imath) \leq (1-\imath) \text{ and } \imath \geq s\imath$$

for all $\imath \in [0, 1]$ and $s \in [0, 1]$. □

We present below the definition of n-polynomial s-type convex function.

Definition 1.2 Let $s \in [0, 1]$ with $n \in \mathbb{N}$. We say that $\mathcal{D} : \mathcal{J} \to \Re$ is a n-polynomial s-type convex function on $\mathcal{J}$, if the inequality

$$\mathcal{D}\big(\imath x + (1-\imath)y\big) \leq \frac{1}{n}\sum_{i=1}^{n}\big[1 - (s(1-\imath))^i\big]\mathcal{D}(x) + \frac{1}{n}\sum_{i=1}^{n}\big[1 - (s\imath)^i\big]\mathcal{D}(y) \tag{1.4}$$

holds for all $x, y \in \mathcal{J}$ and $\imath \in [0, 1]$.

Remark 1.2 *In Definition 1.2:*

(1) *If we choose* $s = 0$, *then we get* P*-functions, as in [11].*
(2) *If we choose* $s = 1$, *then we get Definition 2 of [32].*
(3) *If we choose* $n = s = 1$, *then we get Definition 1.1.*
(4) *If* $\mathcal{D}$ *is a polynomial* s*-convex function then the codomain of* $\mathcal{D}$ *is* $[0, +\infty)$.

Remark 1.3 *Every nonnegative* n*-polynomial convex function is also* n*-polynomial* s*-type convex function. Indeed*

$$\frac{1}{n}\sum_{i=1}^{n}\big[1 - (s(1-\imath))^i\big] \geq \frac{1}{n}\sum_{i=1}^{n}\big[1 - (1-\imath)^i\big]$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left[1-\imath^{i}\right]\leq\frac{1}{n}\sum_{i=1}^{n}\left[1-(s\imath)^{i}\right]$$

for all $\imath \in [0,1],\ n \in \mathbb{N}$ *and* $s \in [0,1]$.

We now demonstrate some essential ideas associated with the fractional integral which are mainly due to Mubeen et al. [24].

Let $\mathcal{D} \in L_1([\theta_1, \theta_2])$. Then k-fractional integrals of order $\alpha > 0$ with $k > 0$ are defined by

$$\mathcal{J}_{\theta_1^+}^{\alpha,k}\mathcal{D}(\ell)=\frac{1}{k\Gamma_k(\alpha)}\int_{\theta_1}^{\ell}(\ell-\xi)^{\frac{\alpha}{k}-1}\mathcal{D}(\xi)d\xi,\quad \ell>\theta_1 \tag{1.5}$$

and

$$\mathcal{J}_{\theta_2^-}^{\alpha,k}\mathcal{D}(\ell)=\frac{1}{k\Gamma_k(\alpha)}\int_{\ell}^{\theta_2}(\xi-\ell)^{\frac{\alpha}{k}-1}\mathcal{D}(\xi)d\xi,\ \ \ell<\theta_2, \tag{1.6}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function [8], defined as

$$\Gamma_k(\alpha)=\int_0^{+\infty}\imath^{\alpha-1}e^{-\frac{\imath^k}{k}}d\imath.$$

It holds that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha)$$

and for $k = 1$, the k-fractional integrals yield the well-known RL-fractional integrals.

In the following we recall the Euler beta function and hypergeometric functions, respectively,

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1\imath^{x-1}(1-\imath)^{y-1}d\imath,\ \ x,y>0$$

and

$${}_2\mathcal{F}_1(a,b,c,z)=\frac{1}{\beta(b,c-b)}\int_0^1\imath^{b-1}(1-\imath)^{c-b-1}(1-z\imath)^{-a}d\imath,\ \ 0<b<c,\ |z|<1,$$

for more details, see [18].

Motivated by the above results and literatures, the aim of this paper is to derive some interesting new integral inequalities including a Hermite–Hadamard type and Ostrowski type pertaining n-polynomial s-type convexity via k-fractional integral operator. By making use of the fractional operators, we will establish new estimates on generalizations whose first and second derivatives in absolute value at certain powers are n-polynomial s-type convex. Interestingly, special cases of the presented results are RL-fractional integral inequalities and quadrature rules.

2 Hermite–Hadamard Inequalities for *n*-Polynomial *s*-Type Convex Functions

In this section, we will state and prove some inequalities of Hermite–Hadamard type for n-polynomial s-type convex functions.

Theorem 2.1 *For $s \in [0,1]$, $\alpha, k > 0$ with $n \in \mathbb{N}$, assume that there is a positive function $\mathcal{D} : \mathcal{J} \to \Re$ with $\theta_2 > \theta_1$ and $\mathcal{D} \in L_1([\theta_1, \theta_2])$. If $\mathcal{D}$ is an n-polynomial s-type convex function on $\mathcal{J}$, then*

$$\left[\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\right]\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big) \le \frac{\Gamma_k(k+\alpha)}{(\theta_2-\theta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\theta_1^+}\mathcal{D}(\theta_2)+\mathcal{J}^{\alpha,k}_{\theta_2^-}\mathcal{D}(\theta_1)\right]$$
$$\le \frac{\big[\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)\big]}{n}\left[\sum_{i=1}^{n}\left[\frac{\alpha(2-s^i)+2ik}{(\alpha+ik)}-\frac{\alpha s^i}{k}\beta\Big(\frac{\alpha}{k}, i+1\Big)\right]\right]. \tag{2.1}$$

Proof By utilizing the n-polynomial s-type convexity of $\mathcal{D}$ on $\mathcal{J}$, we have

$$\mathcal{D}\Big(\frac{\ell_1+\ell_2}{2}\Big) \le \frac{1}{n}\sum_{i=1}^{n}\Big[1-\Big(\frac{s}{2}\Big)^i\Big]\Big[\mathcal{D}(\ell_1)+\mathcal{D}(\ell_2)\Big]. \tag{2.2}$$

Setting $\ell_1 = \imath\theta_2 + (1-\imath)\theta_1,\ \ \ell_2 = \imath\theta_1 + (1-\imath)\theta_2$, we get

$$\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big) \le \frac{1}{n}\sum_{i=1}^{n}\Big[1-\Big(\frac{s}{2}\Big)^i\Big]\Big[\mathcal{D}(\imath\theta_2+(1-\imath)\theta_1)+\mathcal{D}(\imath\theta_1+(1-\imath)\theta_2)\Big]. \tag{2.3}$$

Multiplying both sides of (2.3) by $\imath^{\frac{\alpha}{k}-1}$, then integrating with respect to $\imath$ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{k}{\alpha}\Big(\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\Big)\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big)\\
&\le \left[\int_0^1 \imath^{\frac{\alpha}{k}-1}\mathcal{D}(\imath\theta_2+(1-\imath)\theta_1)d\imath + \int_0^1 \imath^{\frac{\alpha}{k}-1}\mathcal{D}(\imath\theta_1+(1-\imath)\theta_2)d\imath\right]\\
&= \frac{1}{(\theta_2-\theta_1)^{\frac{\alpha}{k}}}\left[\int_{\theta_1}^{\theta_2}\Big(\frac{v-\theta_1}{\theta_2-\theta_1}\Big)^{\frac{\alpha}{k}-1}\mathcal{D}(v)dv + \int_{\theta_1}^{\theta_2}\Big(\frac{\theta_2-v}{\theta_2-\theta_1}\Big)^{\frac{\alpha}{k}-1}\mathcal{D}(v)dv\right]\\
&= \frac{k\Gamma_k(\alpha)}{(\theta_2-\theta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\theta_1^+}\mathcal{D}(\theta_2)+\mathcal{J}^{\alpha,k}_{\theta_2^-}\mathcal{D}(\theta_1)\right].
\end{aligned}$$

Then, we have

$$\Big(\frac{n(2-s)2^n}{2^n(2n-s(n+1))+s^{n+1}}\Big)\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big) \ \leq \frac{\Gamma_k(k+\alpha)}{(\theta_2-\theta_1)^{\frac{\alpha}{k}}}\Big[\mathcal{J}^{\alpha,k}_{\theta_1^+}\mathcal{D}(\theta_2)+\mathcal{J}^{\alpha,k}_{\theta_2^-}\mathcal{D}(\theta_1)\Big]$$

and the first inequality of (2.1) is obtained. For the proof of the second inequality in (2.1), we first note that, if $\mathcal{D}$ is n-polynomial s-type convex, we get

$$\mathcal{D}\Big(\imath\theta_2+(1-\imath)\theta_1\Big)\leq\frac{1}{n}\sum_{i=1}^{n}\big[1-(s\imath)^i\big]\mathcal{D}(\theta_1)+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^i\big]\mathcal{D}(\theta_2)$$

and

$$\mathcal{D}\Big(\imath\theta_1+(1-\imath)\theta_2\Big)\leq\frac{1}{n}\sum_{i=1}^{n}\big[1-s\imath^i\big]\mathcal{D}(\theta_2)+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^i\big]\mathcal{D}(\theta_1).$$

Adding the above inequalities, we obtain

$$\mathcal{D}\Big(\imath\theta_2+(1-\imath)\theta_1\Big)+\mathcal{D}\Big(\imath\theta_1+(1-\imath)\theta_2\Big)\leq\Big[\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)\Big]\Big[\frac{1}{n}\sum_{i=1}^{n}\big[1-s\imath\big]^i$$
$$+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^i\big]\Big]. \qquad (2.4)$$

Again, multiplying both sides of (2.4) by $\imath^{\frac{\alpha}{k}-1}$, and integrating the inequality with respect to $\imath$ over [0, 1], and then making the change of variable, we deduce that

$$\int_0^1\imath^{\frac{\alpha}{k}-1}\mathcal{D}\Big(\imath\theta_2+(1-\imath)\theta_1\Big)d\imath+\int_0^1\imath^{\frac{\alpha}{k}-1}\mathcal{D}\Big(\imath\theta_1+(1-\imath)\theta_2\Big)d\imath$$
$$\leq\big[\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)\big]\int_0^1\imath^{\frac{\alpha}{k}-1}\Big[\frac{1}{n}\sum_{i=1}^{n}\big[1-(s\imath)^i\big]+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^i\big]\Big]d\imath,$$

which completes the proof. □

Now, we derive some special cases of Theorem 2.1 as follows:

(I) Choosing $s=1$, we have:

Corollary 2.2 *Under the assumptions of Theorem 2.1, we obtain*

$$\Big(\frac{n2^n}{2^n(n-1)+1}\Big)\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big)\leq\frac{\Gamma_k(k+\alpha)}{(\theta_2-\theta_1)^{\frac{\alpha}{k}}}\Big[\mathcal{J}^{\alpha,k}_{\theta_1^+}\mathcal{D}(\theta_2)+\mathcal{J}^{\alpha,k}_{\theta_2^-}\mathcal{D}(\theta_1)\Big]$$

$$\leq \frac{\big[\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)\big]}{n}\times\sum_{i=1}^{n}\Big[\frac{\alpha+2ik)}{(\alpha+ik)}-\frac{\alpha}{k}\beta\Big(\frac{\alpha}{k},i+1\Big)\Big].$$

(II) Setting $s=1=k$, we get

Corollary 2.3 *Under the assumptions of Theorem 2.1, we get*

$$\Big(\frac{n2^n}{2^n(n-1)+1}\Big)\mathcal{D}\Big(\frac{\theta_1+\theta_2}{2}\Big)\leq\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^{\alpha}}\Big[\mathcal{J}^{\alpha}_{\theta_1^+}\mathcal{D}(\theta_2)+\mathcal{J}^{\alpha}_{\theta_2^-}\mathcal{D}(\theta_1)\Big]$$
$$\leq\frac{\big[\mathcal{D}(\theta_1)+\mathcal{D}(\theta_2)\big]}{n}\sum_{i=1}^{n}\Big[\frac{\alpha+2i}{(\alpha+i)}-\alpha\beta\big(\alpha,i+1\big)\Big].$$

Remark 2.1 If we set $s=1=k$ and $\alpha=1$, we obtain Theorem 4 of [32].

3 Ostrowski Type Inequalities for Differentiable Functions

In this section, we will prove some new estimates that refine Ostrowski type inequalities for functions whose first derivative in absolute value at certain power is higher order n-polynomial s-type convex.

Lemma 3.1 ([12]) *Consider a differentiable function $\mathcal{D}:\mathcal{J}\to\Re$ on $\mathcal{J}^{\circ}$(interior of $\mathcal{J}$) with $\theta_2>\theta_1$ such that $\mathcal{D}'\in L_1[\theta_1,\theta_2]$. Then*

$$\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell)-\frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\big[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\big] \tag{3.1}$$
$$=\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1\imath^{\frac{\alpha}{k}}\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)d\imath-\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}$$
$$\times\int_0^1\imath^{\frac{\alpha}{k}}\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)d\imath.$$

Using Lemma 3.1, the following results will be proved.

Theorem 3.2 *For $\alpha,k>0$ with $s\in[0,1]$, $n\in\mathbb{N}$ assume that there is a differentiable function $\mathcal{D}:\mathcal{J}\to\Re$ on $\mathcal{J}^{\circ}$(interior of $\mathcal{J}$) with $\theta_2>\theta_1$ such that $\mathcal{D}'\in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}'(\ell)|$ is an n-polynomial s-type convex function on $\mathcal{J}$ and $|\mathcal{D}'(\ell)|\leq\mathcal{K},\ \forall\,\ell\in[\theta_1,\theta_2]$, then*

$$\Bigg|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell)-\frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\big[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\big]\Bigg| \tag{3.2}$$

$$\leq \mathcal{K}\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\Big]\frac{1}{n}\sum_{i=1}^{n}\Big[\Big[\frac{\alpha(1-si)+k(i+1-si)}{(\alpha+k)((\alpha+(i+1)k))}\Big]$$

$$+\Big[\frac{k}{\alpha+k}-s^i\beta\Big(\frac{\alpha}{k}+1,i+1\Big)\Big]\Big].$$

Proof Using Lemma 3.1 and the fact that $|\mathcal{D}'|$ is an n-polynomial s-type convex on $\mathcal{J}$, we have

$$\Big|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell)-\frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\big[\mathcal{J}_{\ell^-}^{\alpha,k}\mathcal{D}(\theta_1)+\mathcal{J}_{\ell^+}^{\alpha,k}\mathcal{D}(\theta_2)\big]\Big|$$

$$\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)\Big|d\imath+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}$$

$$\times\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)\Big|d\imath$$

$$\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[\frac{1}{n}\sum_{i=1}^{n}\Big[1-(s\imath)^i\Big]|\mathcal{D}'(\theta_1)|$$

$$+\frac{1}{n}\sum_{i=1}^{n}\Big[1-(s(1-\imath))^i\Big]|\mathcal{D}'(\ell)|\Big]d\imath$$

$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[\frac{1}{n}\sum_{i=1}^{n}\Big[1-(s\imath)^i\Big]|\mathcal{D}'(\theta_2)|$$

$$+\frac{1}{n}\sum_{i=1}^{n}\Big[1-(s(1-\imath))^i\Big]|\mathcal{D}'(\ell)|\Big]d\imath$$

$$\leq \Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\Big]\frac{\mathcal{K}}{n}\sum_{i=1}^{n}\Big[\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s\imath)^i\Big]d\imath$$

$$+\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s(1-\imath))^i\Big]d\imath\Big]$$

$$=\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\Big]\frac{\mathcal{K}}{n}\sum_{i=1}^{n}\Big[\Big[\frac{\alpha(1-si)+k(i+1-si)}{(\alpha+k)((\alpha+(i+1)k))}\Big]$$

$$+\Big[\frac{k}{\alpha+k}-s^i\beta\Big(\frac{\alpha}{k+1}.i+1\Big)\Big]\Big],$$

where we have used the following identities,

$$\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s\imath)^i\Big]d\imath = \frac{\alpha(1-si)+k(i+1-si)}{(\alpha+k)((\alpha+(i+1)k))}$$

and

$$\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s(1-\imath))^i\Big]d\imath = \frac{k}{\alpha+k} - s^i\beta\Big(\frac{\alpha}{k}+1, i+1\Big).$$

This completes the proof. □

Now, we derive some special cases of Theorem 3.2 as follows:

(I) Taking $k = 1$, we have:

Corollary 3.3 *For* $\alpha > 0$ *with* $s \in [0, 1]$, $n \in \mathbb{N}$, *assume that there is a differentiable function* $\mathcal{D} : \mathcal{J} \to \Re$ *on* $\mathcal{J}^{\circ}$*(interior of* $\mathcal{J}$*) with* $\theta_2 > \theta_1$ *such that* $\mathcal{D}' \in L_1[\theta_1, \theta_2]$*. If* $|\mathcal{D}'(\ell)|$ *is n-polynomial s-type convex function on* $\mathcal{J}$ *and* $|\mathcal{D}'(\ell)| \leq \mathcal{K},\ \ell \in [\theta_1, \theta_2]$*, then*

$$\left|\frac{(\ell-\theta_1)^\alpha+(\theta_2-\ell)^\alpha}{\theta_2-\theta_1}\mathcal{D}(\ell) - \frac{\Gamma(\alpha+1)}{\theta_2-\theta_1}\Big[\mathcal{J}^\alpha_{\ell-}\mathcal{D}(\theta_1)+\mathcal{J}^\alpha_{\ell+}\mathcal{D}(\theta_2)\Big]\right|$$

$$\leq \mathcal{K}\Big[\frac{(\ell-\theta_1)^{\alpha+1}+(\theta_2-\ell)^{\alpha+1}}{\theta_2-\theta_1}\Big]\frac{1}{n}\sum_{i=1}^{n}\Bigg[\Big[\frac{\alpha(1-si)+(i+1-si)}{(\alpha+1)(\alpha+(i+1))}\Big]$$

$$+\Big[\frac{1}{\alpha+1} - s^i\beta\Big(\alpha+1, i+1\Big)\Big]\Bigg].$$

(II) Setting $k = 1 = \alpha$, we get:

Corollary 3.4 *For* $s \in [0, 1]$, $n \in \mathbb{N}$, *assume that there is a differentiable function* $\mathcal{D} : \mathcal{J} \to \Re$ *on* $\mathcal{J}^{\circ}$*(interior of* $\mathcal{J}$*) with* $\theta_2 > \theta_1$ *such that* $\mathcal{D}' \in L_1[\theta_1, \theta_2]$*. If* $|\mathcal{D}'(\ell)|$ *is n-polynomial s-type convex function on* $\mathcal{J}$ *and* $|\mathcal{D}'(\ell)| \leq \mathcal{K},\ \ell \in [\theta_1, \theta_2]$*, then*

$$\left|\mathcal{D}(\ell) - \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathcal{D}(\ell)d\ell\right| \leq \mathcal{K}\Big[\frac{(\ell-\theta_1)^2+(\theta_2-\ell)^2}{\theta_2-\theta_1}\Big]$$

$$\times\frac{1}{n}\sum_{i=1}^{n}\Big[\frac{4+9i+3i^2-4si^2-6si}{2(i+1)(i+2)}\Big].$$

Remark 3.1 *Setting* $k = 1 = \alpha$ *along with* $s = 1 = n$*, then we recapture inequality* (1.2).

Theorem 3.5 *For* $\alpha, k > 0$ *with* $s \in [0, 1]$, $n \in \mathbb{N}$, *assume that there is a differentiable function* $\mathcal{D} : \mathcal{J} \to \Re$ *on* $\mathcal{J}^{\circ}$*(interior of* $\mathcal{J}$*) with* $\theta_2 > \theta_1$ *such that* $\mathcal{D}' \in L_1[\theta_1, \theta_2]$*. If* $|\mathcal{D}'(\ell)|^q$*, where* $q \geq 1$*, is an n-polynomial s-type convex function on* $\mathcal{J}$ *and* $|\mathcal{D}'(\ell)| \leq \mathcal{K},\ \forall\, \ell \in [\theta_1, \theta_2]$*, then*

$$\begin{aligned}
&\left|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell)-\frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\left[\mathcal{J}_{\ell^-}^{\alpha,k}\mathcal{D}(\theta_1)+\mathcal{J}_{\ell^+}^{\alpha,k}\mathcal{D}(\theta_2)\right]\right|\\
&\quad\leq \mathcal{K}\Big(\frac{k}{\alpha+k}\Big)^{1-\frac{1}{q}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\right] \qquad (3.3)\\
&\qquad\times\left[\frac{1}{n}\sum_{i=1}^{n}\left(\left[\frac{\alpha(1-si)+k(i+1-si)}{(\alpha+k)(\alpha+(i+1)k)}\right]+\left[\frac{k}{\alpha+k}-s^i\beta\Big(\frac{\alpha}{k}+1,i+1\Big)\right]\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof Using Lemma 2.1, the fact that $|\mathcal{D}'|^q$ is n-polynomial s-type convex on $\mathcal{J}$ and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell)-\frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\left[\mathcal{J}_{\ell^-}^{\alpha,k}\mathcal{D}(\theta_1)+\mathcal{J}_{\ell^+}^{\alpha,k}\mathcal{D}(\theta_2)\right]\right|\\
&\quad\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)\Big|d\imath+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\\
&\qquad\times\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)\Big|d\imath\\
&\quad\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\Big(\int_0^1 \imath^{\frac{\alpha}{k}}d\imath\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)\Big|^q d\imath\Big)^{\frac{1}{q}}\\
&\qquad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\Big(\int_0^1 \imath^{\frac{\alpha}{k}}d\imath\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \imath^{\frac{\alpha}{k}}\Big|\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)\Big|^q d\imath\Big)^{\frac{1}{q}}\\
&\leq \Big(\frac{k}{\alpha+k}\Big)^{1-\frac{1}{q}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s\imath)^i\Big]d\imath\right)|\mathcal{D}'(\theta_1)|^q\right.\right.\\
&\qquad\left.+\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s(1-\imath))^i\Big]d\imath\right)|\mathcal{D}'(\ell)|^q\right)^{\frac{1}{q}}\\
&\qquad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s\imath)^i\Big]d\imath\right)|\mathcal{D}'(\theta_2)|^q\right.\\
&\qquad\left.\left.+\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \imath^{\frac{\alpha}{k}}\Big[1-(s(1-\imath))^i\Big]d\imath\right)|\mathcal{D}'(\ell)|^q\right)^{\frac{1}{q}}\right]\\
&\quad\leq \mathcal{K}\Big(\frac{k}{\alpha+k}\Big)^{1-\frac{1}{q}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\right]\\
&\qquad\times\left[\frac{1}{n}\sum_{i=1}^{n}\left(\left[\frac{\alpha(1-si)+k(i+1-si)}{(\alpha+k)((\alpha+(i+1)k))}\right]+\left[\frac{k}{\alpha+k}-s^i\beta\Big(\frac{\alpha}{k}+1,i+1\Big)\right]\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 3.2 *If we choose $q = 1$, then our Theorem 3.5 reduces to Theorem 3.2.*

Theorem 3.6 *For $\alpha, k > 0$ with $s \in [0, 1]$, $n \in \mathbb{N}$, assume that there is a differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}' \in L_1[\theta_1, \theta_2]$. If $|\mathcal{D}'(\ell)|^q$, $q > 1$ is n-polynomial s-type convex function on $\mathcal{J}$ and $|\mathcal{D}'(\ell)| \leq \mathcal{K}$, $\ell \in [\theta_1, \theta_2]$, then*

$$\left|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}} + (\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell) - \frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\left[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1) + \mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\right]\right| \tag{3.4}$$
$$\leq \mathcal{K}\left(\frac{k}{p\alpha+k}\right)^{\frac{1}{p}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1} + (\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\right]\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2(i+1-s^i)}{i+1}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, the fact that $|\mathcal{D}'|^q$ is n-polynomial s-type convex on $\mathcal{J}$ and Hölder's inequality, we have

$$\left|\frac{(\ell-\theta_1)^{\frac{\alpha}{k}} + (\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\mathcal{D}(\ell) - \frac{\Gamma_k(\alpha+k)}{\theta_2-\theta_1}\left[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1) + \mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\right]\right|$$
$$\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\int_0^1 \imath^{\frac{\alpha}{k}}\left|\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)\right|d\imath + \frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}$$
$$\times\int_0^1 \imath^{\frac{\alpha}{k}}\left|\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)\right|d\imath$$
$$\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\int_0^1 \imath^{\frac{p\alpha}{k}}d\imath\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{D}'(\imath\ell+(1-\imath)\theta_1)\right|^q d\imath\right)^{\frac{1}{q}}$$
$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\int_0^1 \imath^{\frac{p\alpha}{k}}d\imath\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{D}'(\imath\ell+(1-\imath)\theta_2)\right|^q d\imath\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{k}{p\alpha+k}\right)^{\frac{1}{p}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left[1-(s\imath)^i\right]d\imath\right)|\mathcal{D}'(\theta_1)|^q\right.\right.$$
$$\left.+\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left[1-(s(1-\imath))^i\right]d\imath\right)|\mathcal{D}'(\ell)|^q\right)^{\frac{1}{q}}$$
$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left[1-(s\imath)^i\right]d\imath\right)|\mathcal{D}'(\theta_2)|^q\right.$$

$$+\left(\frac{1}{n}\sum_{i=1}^{n}\int_0^1\left[1-(s(1-\imath))^i\right]d\imath\right)|\mathcal{D}'(\ell)|^q\Big)^{\frac{1}{q}}\Bigg]$$

$$\leq \mathcal{K}\left(\frac{k}{p\alpha+k}\right)^{\frac{1}{p}}\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+1}+(\theta_2-\ell)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\right]\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2(i+1-s^i)}{i+1}\right)^{\frac{1}{q}}.$$

This completes the proof. □

4 Other Results About Ostrowski Type Inequalities for Twice Differentiable Functions

In this section, the following useful fractional integral identity including the second-order derivative of $\mathcal{D}$ will be proved.

Lemma 4.1 ([28]) *For $\alpha, k > 0$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \subset \Re \to \Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1, \theta_2]$. Then the following identity holds:*

$$(1-\rho)\left[\frac{(\theta_2-\ell)^{\frac{\alpha}{k}}-(\ell-\theta_1)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}'(\ell)+\left(1+\frac{\alpha}{k}-\rho\right)\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}(\ell)$$
$$+\rho\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}\mathcal{D}(\theta_1)+(\theta_2-\ell)^{\frac{\alpha}{k}}\mathcal{D}(\theta_2)}{\theta_2-\theta_1}\right]-\frac{\Gamma_k(\alpha+2k)}{\theta_2-\theta_1}\left[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\right]$$
$$=\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1\imath(\rho-\imath^{\frac{\alpha}{k}})\mathcal{D}''(\imath\ell+(1-\imath)\theta_1)d\imath+\frac{(\theta_2-\imath)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}$$
$$\times\int_0^1\imath(\rho-\imath^{\frac{\alpha}{k}})\mathcal{D}''(\imath\ell+(1-\imath)\theta_2)d\imath \tag{4.1}$$

for all $\ell \in [\theta_1, \theta_2]$ and $\rho \in [0, 1]$.

For the simplicity of the notation, let

$$\Psi_{\mathcal{D}}\big(\rho,\alpha,k;\theta_1,\theta_2,\ell\big)=(1-\rho)\left[\frac{(\theta_2-\ell)^{\frac{\alpha}{k}}-(\ell-\theta_1)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}'(\ell)+\left(1+\frac{\alpha}{k}-\rho\right)$$
$$\times\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}(\ell)$$
$$+\rho\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}\mathcal{D}(\theta_1)+(\theta_2-\ell)^{\frac{\alpha}{k}}\mathcal{D}(\theta_2)}{\theta_2-\theta_1}\right]-\frac{\Gamma_k(\alpha+2k)}{\theta_2-\theta_1}$$
$$\times\left[\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_2)\right]. \tag{4.2}$$

Theorem 4.2 *For $\alpha, k > 0$ with $s \in [0,1]$, $n \in \mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^{\circ}$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1, \theta_2]$. If $|\mathcal{D}''(\ell)|$ is an n-polynomial s-type convex function on $\mathcal{J}$ and for all $\ell \in [\theta_1, \theta_2]$, then*

$$\begin{aligned}
&\big|\Psi_{\mathcal{D}}\big(\rho, \alpha, k; \theta_1, \theta_2, \ell\big)\big| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\mathcal{C}_1(\alpha, k; i, s)\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}|\mathcal{D}''(\theta_1)| + \frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}|\mathcal{D}''(\theta_2)|\right] \\
&\quad + \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}+(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\mathcal{C}_2(\alpha, k; i, s)|\mathcal{D}''(\ell)|,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{C}_1(\alpha, k; i, s) &= \int_0^1 \imath(\rho - \imath^{\frac{\alpha}{k}})\big[1-(s\imath)^i\big]d\imath \\
&= \left[\frac{\rho\alpha - 2k(1-\rho)}{2(\alpha+2k)} - s^i\,\frac{\rho\alpha - k(i+2)(1-\rho)}{(i+2)(\alpha+k(i+2))}\right]
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\mathcal{C}_2(\alpha, k; i, s) &= \int_0^1 \imath(\rho - \imath^{\frac{\alpha}{k}})\big[1-(s(1-\imath))^i\big]d\imath \\
&= \left[\frac{\rho\alpha - 2k(1-\rho)}{2(\alpha+2k)} - s^i\left[\frac{\rho}{(i+2)(i+1)} - \frac{\Gamma\big(\frac{\alpha}{k}+2\big)\Gamma(i+1)}{\Gamma\big(\frac{\alpha}{k}+i+3\big)}\right]\right].
\end{aligned} \tag{4.5}$$

Proof Using Lemma 4.1, the property of the modulus and the n-polynomial s-type convexity of $|\mathcal{D}'|$ on $\mathcal{J}$, we derive that

$$\begin{aligned}
&\Bigg|(1-\rho)\left[\frac{(\theta_2-\ell)^{\frac{\alpha}{k}} - (\ell-\theta_1)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}'(\ell) + \Big(1+\frac{\alpha}{k}-\rho\Big)\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}} + (\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\right]\mathcal{D}(\ell) \\
&\quad + \rho\left[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}\mathcal{D}(\theta_1) + (\theta_2-\ell)^{\frac{\alpha}{k}}\mathcal{D}(\theta_2)}{\theta_2-\theta_1}\right] - \frac{\Gamma_k(\alpha+2k)}{\theta_2-\theta_1} \\
&\quad \times\Big[\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_1) + \mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_2)\Big]\Bigg| \\
&\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1 \imath(\rho-\imath^{\frac{\alpha}{k}})\Big|\mathcal{D}''(\imath\ell+(1-\imath)\theta_1)\Big|d\imath + \frac{(\theta_2-\imath)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1} \\
&\quad \times\int_0^1 \imath(\rho-\imath^{\frac{\alpha}{k}})\Big|\mathcal{D}''(\imath\ell+(1-\imath)\theta_2)\Big|d\imath \\
&\leq \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1 \imath(\rho-\imath^{\frac{\alpha}{k}})\Bigg[\frac{1}{n}\sum_{i=1}^{n}(1-(s\imath)^i)|\mathcal{D}''(\theta_1)|
\end{aligned}$$

$$+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^{i}\big]|\mathcal{D}''(\ell)|\bigg]d\imath$$

$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1 \imath(\rho-\imath^{\frac{\alpha}{k}})\bigg[\frac{1}{n}\sum_{i=1}^{n}(1-(s\imath)^{i})|\mathcal{D}''(\theta_2)|$$

$$+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\imath))^{J}\big]|\mathcal{D}''(\ell)|\bigg]d\imath$$

$$=\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big[\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_1)|+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|\Big]$$

$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big[\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_2)|+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|\Big]$$

$$=\frac{1}{n}\sum_{i=1}^{n}\mathcal{C}_1(\alpha,k;i,s)\bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}|\mathcal{D}''(\theta_1)|+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}|\mathcal{D}''(\theta_2)|\bigg]$$

$$+\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}+(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|.$$

This completes the proof. □

Theorem 4.3 *For $\alpha, k > 0$ with $s \in [0,1]$, $n \in \mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^{\circ}$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}''(\ell)|^q$, $q \geq 1$ is an n-polynomial s-type convex function on $\mathcal{J}$, then*

$$|\Psi_{\mathcal{D}}\big(\rho,\alpha,k;\theta_1,\theta_2,\ell\big)|$$

$$\leq \mathcal{C}_3^{1-\frac{1}{q}}(\alpha,k,\rho)\bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big(\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_1)|^q+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|^q\Big)^{\frac{1}{q}}$$

$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big(\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_2)|^q+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|^q\Big)^{\frac{1}{q}}\bigg],$$

where

$$\begin{aligned}\mathcal{C}_3(\alpha,k,\rho) &= \int_0^1 \imath(\rho-\imath^{\frac{\alpha}{k}})d\imath\\ &= \frac{k\rho^{\frac{2k+\alpha}{\alpha}}}{\alpha}\bigg[\Gamma\Big(\frac{2k+\alpha}{\alpha}\Big)\,{}_2\mathcal{F}_1\Big(1,2;3+\frac{2k}{\alpha};1\Big)\\ &\quad+\beta\Big(2,-\frac{2k+\alpha}{\alpha}\Big)-\beta\Big(\rho,2,-\frac{2k+\alpha}{\alpha}\Big)\bigg],\end{aligned} \tag{4.6}$$

$\mathcal{C}_1(\alpha,k;i,s)$ and $\mathcal{C}_2(\alpha,k;i,s)$ are given in (4.4) and (4.5), respectively.

Proof Using Lemma 4.2, the property of modulus, the power mean inequality and the n-polynomial s-type convexity of $|\mathcal{D}'|$ on $\mathcal{J}$, we have

$$\begin{aligned}
&\Big|(1-\rho)\Big[\frac{(\theta_2-\ell)^{\frac{\alpha}{k}}-(\ell-\theta_1)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\Big]\mathcal{D}'(\ell)+(1+\frac{\alpha}{k}-\rho)\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\Big]\mathcal{D}(\ell)\\
&\quad+\rho\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}\mathcal{D}(\theta_1)+(\theta_2-\ell)^{\frac{\alpha}{k}}\mathcal{D}(\theta_2)}{\theta_2-\theta_1}\Big]-\frac{\Gamma_k(\alpha+2k)}{\theta_2-\theta_1}\Big[\mathcal{J}^{\alpha,k}_{\ell+}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell-}\mathcal{D}(\theta_2)\Big]\Big|\\
&\le\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_1)\Big|d\iota+\frac{(\theta_2-\iota)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1}\\
&\quad\times\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_2)\Big|d\iota\\
&\le\Big(\int_0^1\iota(\rho-\iota^{\frac{\alpha}{k}})d\iota\Big)^{1-\frac{1}{q}}\Bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1\iota(\rho-\iota^{\frac{\alpha}{k}})\Big[\frac{1}{n}\sum_{i=1}^{n}(1-(s\iota)^i)|\mathcal{D}''(\theta_1)|^q\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\iota))^i\big]|\mathcal{D}''(\ell)|^q\Big]d\iota\Big)^{\frac{1}{q}}\\
&\quad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1\iota(\rho-\iota^{\frac{\alpha}{k}})\Big[\frac{1}{n}\sum_{i=1}^{n}(1-(s\iota)^i)|\mathcal{D}''(\theta_2)|^q\\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\big[1-(s(1-\iota))^i\big]|\mathcal{D}''(\ell)|^q\Big]d\iota\Big)^{\frac{1}{q}}\Bigg]\\
&=\mathcal{C}_3^{1-\frac{1}{q}}(\alpha,k,\rho)\Bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big(\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_1)|^q+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|^q\Big)^{\frac{1}{q}}\\
&\quad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{n(\theta_2-\theta_1)}\sum_{i=1}^{n}\Big(\mathcal{C}_1(\alpha,k;i,s)|\mathcal{D}''(\theta_2)|^q+\mathcal{C}_2(\alpha,k;i,s)|\mathcal{D}''(\ell)|^q\Big)^{\frac{1}{q}}\Bigg].
\end{aligned}\tag{4.7}$$

This completes the proof. □

Theorem 4.4 *For $\alpha, k > 0$ with $s \in [0, 1]$, $n \in \mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1, \theta_2]$. If $|\mathcal{D}''(\ell)|^q$, $q > 1$ is an n-polynomial s-type convex function on $\mathcal{J}$, then*

$$\begin{aligned}
&|\Psi_{\mathcal{D}}\big(\rho,\alpha,k;\theta_1,\theta_2,\ell\big)|\\
&\le\mathcal{C}_3^{\frac{1}{p}}(p,\alpha,k,\rho)\Bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{(\theta_2-\theta_1)}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{i+1-s^i}{i+1}\Big)\Big[|\mathcal{D}''(\ell)|^q+|\mathcal{D}''(\theta_1)|^q\Big]\Big)^{\frac{1}{q}}\\
&\quad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{(\theta_2-\theta_1)}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{i+1-s^i}{i+1}\Big)\Big[|\mathcal{D}''(\ell)|^q+|\mathcal{D}''(\theta_2)|^q\Big]\Big)^{\frac{1}{q}}\Bigg],
\end{aligned}\tag{4.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{aligned}\mathcal{C}_3(p,\alpha,k,\rho) &= \int_0^1 \left(\iota(\rho-\iota^{\frac{\alpha}{k}})\right)^p d\iota \\ &= \frac{k\rho^{\frac{(1+p)k+\alpha p}{\alpha}}}{\alpha}\Big[\Gamma(1+p)\Gamma\Big(\frac{k(1+p)+\alpha}{\alpha}\Big)\ {}_2\mathcal{F}_1\Big(1,1+p,2+p+\frac{k(p+1)}{\alpha},1\Big) \\ &\quad +\beta\Big(1+p,-\frac{k(1+p)+\alpha p}{\alpha}\Big)-\beta\Big(\rho,1+p,-\frac{(1+p)k+\alpha p}{\alpha}\Big)\Big]. \qquad (4.9)\end{aligned}$$

Proof Using Lemma 4.2, the property of the modulus, Hölder's inequality and the n-polynomial s-type convexity of $|\mathcal{D}'|$ on $\mathcal{J}$, we have

$$\begin{aligned}&\Big|(1-\rho)\Big[\frac{(\theta_2-\ell)^{\frac{\alpha}{k}}-(\ell-\theta_1)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\Big]\mathcal{D}'(\ell)+(1+\frac{\alpha}{k}-\rho)\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}+(\theta_2-\ell)^{\frac{\alpha}{k}}}{\theta_2-\theta_1}\Big]\mathcal{D}(\ell) \\ &\quad +\rho\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}}\mathcal{D}(\theta_1)+(\theta_2-\ell)^{\frac{\alpha}{k}}\mathcal{D}(\theta_2)}{\theta_2-\theta_1}\Big]-\frac{\Gamma_k(\alpha+2k)}{\theta_2-\theta_1}\Big[\mathcal{J}^{\alpha,k}_{\ell^+}\mathcal{D}(\theta_1)+\mathcal{J}^{\alpha,k}_{\ell^-}\mathcal{D}(\theta_2)\Big]\Big| \\ &\le \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_1)\Big|d\iota+\frac{(\theta_2-\iota)^{\frac{\alpha}{k}+1}}{\theta_2-\theta_1} \\ &\quad\times\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_2)\Big|d\iota \\ &\le \frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|^p d\iota\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_1)\Big|^q d\iota\Big)^{\frac{1}{q}} \\ &\quad+\frac{(\theta_2-\iota)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1|\iota(\rho-\iota^{\frac{\alpha}{k}})|^p d\iota\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\mathcal{D}''(\iota\ell+(1-\iota)\theta_2)\Big|d\iota\Big)^{\frac{1}{q}} \\ &\le \mathcal{C}_3^{\frac{1}{p}}(p,\alpha,k;\rho)\Big[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1\Big[\frac{1}{n}\sum_{i=1}^n(1-(s\iota)^i)|\mathcal{D}''(\theta_1)|^q \\ &\quad+\frac{1}{n}\sum_{i=1}^n\big[1-(s(1-\iota))^i\big]|\mathcal{D}''(\ell)|^q\Big]d\iota\Big)^{\frac{1}{q}} \\ &\quad+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{\theta_2-\theta_1}\Big(\int_0^1\Big[\frac{1}{n}\sum_{i=1}^n(1-(s\iota)^i)|\mathcal{D}''(\theta_2)|^q \\ &\quad+\frac{1}{n}\sum_{i=1}^n\big[1-(s(1-\iota))^i\big]|\mathcal{D}''(\ell)|^q\Big]d\iota\Big)^{\frac{1}{q}}\Big]\end{aligned}$$

$$= \mathcal{C}_3^{\frac{1}{p}}(p,\alpha,k,\rho)\Bigg[\frac{(\ell-\theta_1)^{\frac{\alpha}{k}+2}}{(\theta_2-\theta_1)}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{i+1-s^i}{i+1}\Big)\Big[|\mathcal{D}''(\ell)|^q+|\mathcal{D}''(\theta_1)|^q\Big]\Bigg)^{\frac{1}{q}}$$

$$+\frac{(\theta_2-\ell)^{\frac{\alpha}{k}+2}}{(\theta_2-\theta_1)}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{i+1-s^i}{i+1}\Big)\Big[|\mathcal{D}''(\ell)|^q+|\mathcal{D}''(\theta_2)|^q\Big]\Bigg)^{\frac{1}{q}}\Bigg]. \tag{4.10}$$

This completes the proof. □

5 Corollaries

In this final section, we are interested to estimate k-fractional integrals which contain bounds of RL-fractional integrals that can be comprised as special cases of results of our previous section. Finally, we will obtain a refined bound of the Midpoint, Trapezoidal, and Simpson type inequalities for twice differentiable n-polynomial s-type convex functions.

By applying Theorem 4.2 we obtain the following results:

(I) For $\rho = 0$ and $\ell = \frac{\theta_1+\theta_2}{2}$, then

Corollary 5.1 *For $\alpha, k > 0$ with $s \in [0,1]$, $n \in \mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}''(\ell)|$ is an n-polynomial s-type convex function on $\mathcal{J}$ and for all $\ell \in [\theta_1,\theta_2]$, then*

$$\left|\Psi_{\mathcal{D}}\left(0,\alpha,k;\theta_1,\theta_2,\frac{\theta_1+\theta_2}{2}\right)\right|$$

$$\leq \left(\frac{(\theta_2-\theta_1)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}n}\right)\Bigg[\sum_{i=1}^{n}\mathcal{C}_1(\alpha,k;i,s)\big[|\mathcal{D}''(\theta_1)|+|\mathcal{D}''(\theta_2)|\big]$$

$$+\sum_{i=1}^{n}\mathcal{C}_2(\alpha,k;i,s)\left|\mathcal{D}''\left(\frac{\theta_1+\theta_2}{2}\right)\right|\Bigg],$$

where $\mathcal{C}_1(\alpha,k;i,s)$ and $\mathcal{C}_2(\alpha,k;i,s)$ are given in (4.4) and (4.5), respectively.

(II) For $\rho = 1$ and $\ell = \frac{\theta_1+\theta_2}{2}$, then one has:

Corollary 5.2 *For $\alpha, k > 0$ with $s \in [0,1]$, $n \in \mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D} : \mathcal{J} \to \Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2 > \theta_1$ such that $\mathcal{D}'' \in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}''(\ell)|$ is an n-polynomial s-type convex function on $\mathcal{J}$ and for all $\ell \in [\theta_1,\theta_2]$, then*

$$\left|\Psi_{\mathcal{D}}\left(1,\alpha,k;\theta_1,\theta_2,\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq\left(\frac{(\theta_2-\theta_1)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}n}\right)\Bigg[\sum_{i=1}^{n}\left[\frac{\alpha}{2(\alpha+2k)}-s^i\left(\frac{1}{(i+1)(i+2)}\right.\right.$$
$$\left.\left.-\frac{\Gamma\left(\frac{\alpha}{k}+2\right)\Gamma(i+1)}{\Gamma\left(\frac{\alpha}{k}+i+3\right)}\right)\right]\left|\mathcal{D}''\left(\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$+\sum_{i=1}^{n}\left[\frac{\alpha}{2(\alpha+2k)}-\frac{s^i\alpha}{(i+2)(\alpha+k(i+2))}\right]\big[|\mathcal{D}''(\theta_1)|+|\mathcal{D}''(\theta_2)|\big]\Bigg].$$

(III) For $\rho=\frac{1}{2}$ and $\ell=\frac{\theta_1+\theta_2}{2}$, then one has:

Corollary 5.3 *For $\alpha,k>0$ assume that there is a differentiable function $\mathcal{D}:\mathcal{J}\to\Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2>\theta_1$ such that $\mathcal{D}''\in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}''(\ell)|$ is an n-polynomial s-convex function on $\mathcal{J}$ and for all $\ell\in[\theta_1,\theta_2]$, then*

$$\left|\Psi_{\mathcal{D}}\left(\frac{1}{2},\alpha,k;\theta_1,\theta_2,\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq\left(\frac{(\theta_2-\theta_1)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}n}\right)\Bigg[\sum_{i=1}^{n}\left[\frac{\alpha-2k}{4(\alpha+2k)}-\frac{s^i\big[\alpha-k(i+2)\big]}{2(i+2)(\alpha+k(i+2))}\right]\big[|\mathcal{D}''(\theta_1)|+|\mathcal{D}''(\theta_2)|\big]$$
$$+\sum_{i=1}^{n}\left[\frac{\alpha-2k}{4(\alpha+2k)}-s^i\left(\frac{1}{2(i+1)(i+2)}-\frac{\Gamma\left(\frac{\alpha}{k}+2\right)\Gamma(i+1)}{\Gamma\left(\frac{\alpha}{k}+i+3\right)}\right)\right]\left|\mathcal{D}''\left(\frac{\theta_1+\theta_2}{2}\right)\right|\Bigg].$$

(IV) For $\rho=\frac{1}{3}$ and $\ell=\frac{\theta_1+\theta_2}{2}$, then one obtain that:

Corollary 5.4 *For $\alpha,k>0$ with $s\in[0,1]$, $n\in\mathbb{N}$, assume that there is a twice differentiable function $\mathcal{D}:\mathcal{J}\to\Re$ on $\mathcal{J}^\circ$(interior of $\mathcal{J}$) with $\theta_2>\theta_1$ such that $\mathcal{D}''\in L_1[\theta_1,\theta_2]$. If $|\mathcal{D}''(\ell)|$ is an n-polynomial s-type convex function on $\mathcal{J}$ and for all $\ell\in[\theta_1,\theta_2]$, then*

$$\left|\Psi_{\mathcal{D}}\left(\frac{1}{3},\alpha,k;\theta_1,\theta_2,\frac{\theta_1+\theta_2}{2}\right)\right|$$
$$\leq\left(\frac{(\theta_2-\theta_1)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}n}\right)\Bigg[\sum_{i=1}^{n}\left[\frac{\alpha-4k}{6(\alpha+2k)}-\frac{s^i\big[\alpha-2k(i+2)\big]}{3(i+2)(\alpha+k(i+2))}\right]\big[|\mathcal{D}''(\theta_1)|+|\mathcal{D}''(\theta_2)|\big]$$
$$+\sum_{i=1}^{n}\left[\frac{\alpha-4k}{6(\alpha+2k)}-s^i\left(\frac{1}{3(i+1)(i+2)}-\frac{\Gamma\left(\frac{\alpha}{k}+2\right)\Gamma(i+1)}{\Gamma\left(\frac{\alpha}{k}+i+3\right)}\right)\right]\left|\mathcal{D}''\left(\frac{\theta_1+\theta_2}{2}\right)\right|\Bigg].$$

Remark 5.1 Under the assumptions of Theorems 4.3 and 4.4 we can derive some interesting new results for some special values of ρ and ℓ. We omit here their proofs and the details are left to the interested readers.

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Hyperstability of Orthogonally 3-Lie Homomorphism: An Orthogonally Fixed Point Approach

Vahid Keshavarz and Sedigheh Jahedi

Abstract In this chapter, by using the orthogonally fixed point method, we prove the Hyers–Ulam stability and the hyperstability of orthogonally 3-Lie homomorphisms for additive ρ-functional equation in 3-Lie algebras. Indeed, we investigate the stability and the hyperstability of the system of functional equations

$$\begin{cases} f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) + f(x) + f(y)\right), \\ f([[u, v], w]) = [[f(u), f(v)], f(w)] \end{cases}$$

in 3-Lie algebras where $\rho \neq 1$ is a fixed real number.

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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Th. M. Rassias [3] for linear mappings by considering an unbounded Cauchy difference. A generalization of Rassias' theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias' approach. In 1996, G. Isac and Th. M. Rassias [5] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The stability problems of several functional equations have been extensively investigated by a number of authors (see [6–14]).

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There are several orthogonality notions on a real normed space such as Birkhoff–James, Boussouis, (semi-)inner product, Singer, Carlsson, unitary-Boussouis, Roberts, Pythagorean and Diminnie (see [15, 16]). Recently, Eshaghi Gordji et al. [17] introduced orthogonal sets and some corresponding concepts.

Definition 1.1 ([17])

(i) Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0; (\forall y; y \perp x_0) \;\; or \;\; (\forall y; x_0 \perp y),$$

then $(X, \perp)$ is called an orthogonal set (briefly, O-set).

(ii) Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n; x_n \perp x_{n+1}) \;\; or \;\; (\forall n; x_{n+1} \perp x_n).$$

(iii) If $(X, \perp)$ is an O-set and (X, d) is a metric space, then $(X, \perp, d)$ is an orthogonally metric space. A mapping $f : X \to X$ is a $\perp$-continuous in $x \in X$ if for each O-sequence $\{x_n\}_{n \in \mathbb{N}}$ in X with $x_n \to x$, $f(x_n) \to f(x)$. Obviously, every continuous mapping is $\perp$-continuous.

(iv) A Cauchy sequence $\{x_n\}$ in X is said to be a Cauchy orthogonally sequence (briefly, Cauchy O-sequence) if for all $n \in A$, $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$. An orthogonally metric space $(X, \perp, d)$ is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and the converse is not true in general.

(v) Let $(X, \perp, d)$ be an orthogonally metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is said to be orthogonality contraction with Lipschitz constant λ if for any x, y with $x \perp y$

$$d(fx, fy) \leq \lambda d(x, y).$$

Eshaghi Gordji et al. in [18] proved a fixed point theorem in O-sets and several authors worked on orthogonally fixed point (see [19–22]).

Theorem 1.2 ([18]) *Let $(X, d, \perp)$ be an O-complete generalized metric space. Let $T : X \to X$ be a $\perp$-preserving, $\perp$-continuous and $\perp - \lambda$-contraction. Let $x_0 \in X$ satisfy for all $y \in X$, $x_0 \perp y$ or for all $y \in X$, $y \perp x_0$, and consider the O-sequence of successive approximations with initial element x_0; x_0, $T(x_0)$, $T^2(x_0)$, $\ldots$, $T^n(x_0)$, $\ldots$. Then, either $d(T^n(x_0), T^{n+1}(x_0)) = \infty$ for all $n \geq 0$, or there exists a positive integer n_0 such that $d(T^n(x_0), T^{n+1}(x_0)) < \infty$ for all $n > n_0$. If the second alternative holds, then*

(i) the O-sequence of $\{T^n(x_0)\}$ is convergent to a fixed point x^* of T,
(ii) x^* is the unique fixed point of T in

$$X^* = \{y \in X : d(T^n(x_0), y) < \infty\}.$$

(iii) if $y \in X$, then

$$d(y, x^*) \leq \frac{1}{1-\lambda} d(y, T(y)).$$

Note that a Lie algebra is a Banach algebra endowed with the Lie product

$$[x, y] := \frac{(xy - yx)}{2}.$$

Similarly, a 3-Lie algebra is a Banach algebra endowed with the product

$$\Big[[x, y], z\Big] := \frac{[x, y]z - z[x, y]}{2},$$

for all $x, y, z \in A$.

Definition 1.3 Let A and B be two 3-Lie algebras. A mapping $H : A \to B$ is called an orthogonally 3-Lie homomorphism if

(i) H is a linear mapping;
(ii) for all $u, v, w \in A$ with $u \perp v, \quad u \perp w, \quad v \perp w,$

$$H([[u, v], w]) = [[H(u), H(v)], H(w)].$$

In this chapter, we investigate the stability and the hyperstability of the system of functional equations

$$\begin{cases} f(x+y) - f(x) - f(y) = \rho(2f(\frac{x+y}{2}) + f(x) + f(y)), \\ \\ f([[u, v], w]) = [[f(u), f(v)], f(w)] \end{cases}$$

in 3-Lie algebras where $\rho \neq 1$ is a fixed real number, by using the orthogonally fixed point method.

2 Main Results

Throughout this section, assume that A and B are two orthogonally 3-Lie algebras and $\rho \neq 1$ is a fixed real number. Denote

$$V_{\perp} := \{x, y, z \in A \quad | \quad x \perp y, \quad x \perp z, \quad y \perp z\},$$

$$\Delta_\rho f(x, y) := f(x+y) - f(x) - f(y) - \rho\Big(2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y)\Big), \tag{2.1}$$

and

$$\Re f([[u, v], w]) := f([[u, v], w] - [[f(u), f(v)], f(w)], \tag{2.2}$$

where $u, v, w \in V_\perp$.

Lemma 2.1 ([23]) *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.3}$$

for all $x, y \in X$, then f is additive.

In the following theorem, we prove the Hyers–Ulam stability of orthogonally 3-Lie homomorphism in orthogonally 3-Lie algebras.

Theorem 2.2 *Let $f : A \to B$ be a mapping, and let $\varphi : A^2 \to [0, \infty)$ and $\psi : A^3 \to [0, \infty)$ be two functions such that there exists an $L < 1$ with*

$$\|\Delta_\rho f(x, y)\| \leq \varphi(x, y) \tag{2.4}$$

and

$$\|\Re f([[[u, v], w])\| \leq \psi(u, v, w) \tag{2.5}$$

for all $x, y, u, v, w \in V_\perp$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(x, y) \tag{2.6}$$

$$\psi\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) \leq \frac{L}{2^3}\psi(u, v, w) \tag{2.7}$$

for all $x, y, u, v, w \in V_\perp$, then there exists a unique orthogonally 3-Lie homomorphism $\Im : A \to B$ such that

$$\|f(x) - \Im(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x) \tag{2.8}$$

for all $x \in A$.

Proof Inequalities (2.6) and (2.7) imply that

$$\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2}, \frac{y}{2}\right) = 0 \quad and \quad \lim_{n\to\infty} 2^{3n} \psi\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) = 0 \tag{2.9}$$

for all $x, y, u, v, w \in V_{\perp}$. Letting $x = y = u = v = w = 0$ in (2.9), we obtain that $\varphi(0, 0) = 0 \quad and \quad \psi(0, 0, 0) = 0$.

Consider the set

$$\Lambda = \left\{g : A \to B \mid g(0) = 0 \quad g(x) \perp 2g\left(\frac{x}{2}\right) \quad or \quad 2g\left(\frac{x}{2}\right) \perp g(x)\right\}.$$

For every $g, h \in \Lambda$, define,

$$d(g, h) = \inf\{k \in (0, \infty) \mid \|g(x) - h(x)\| \le k\varphi(x, x) \quad , x \in A\}.$$

Now, we put the orthogonality relation $\perp$ on Λ for all $g, h \in \Lambda$ as follows:

$$h \perp g \Leftrightarrow h(x) \perp g(x) \quad or \quad g(x) \perp h(x) \quad \forall x \in A.$$

It is easy to show that $(\Lambda, d, \perp)$ is an O-complete generalized metric space.

Consider the mapping $T : \Lambda \to \Lambda$ defined by

$$Tg(x) = 2g\left(\frac{x}{2}\right) \quad \forall x \in A.$$

Clearly, T is $\perp$-preserving. It follows that for all $g, h \in \Lambda$ with $g \perp h$ and $x \in A$,

$$\left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2k\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le Lk\varphi(x, x).$$

Hence, we see that

$$d(Tg, Th) \le Ld(g, h)$$

for all $g, h \in \Lambda$, that is, T is a strictly $\perp$-contractive self-mapping of Λ with the Lipschitz constant L. The function T is $\perp$-continuous. In fact, if $\{g_n\}$ is an O-sequence in Λ, which converges to $g \in \Lambda$, then for a given $\varepsilon > 0$ there exists $k > 0$ with $k < \varepsilon$ and $n \in \mathbb{N}$ such that

$$\|g_n(x) - g(x)\| \le k\varphi(x, x)$$

for all $x \in \mathfrak{A}$ and $n \in \mathbb{N}$.

By Theorem 1.2, there exists a mapping $\Im : A \to B$ such that

1. $\Im$ is a fixed point of T, i.e.,

$$\Im(x) = 2\Im\left(\frac{x}{2}\right) \quad \forall x \in A. \tag{2.10}$$

The mapping $\Im$ is a unique fixed point of T in the set

$$\Omega = \{g \in \Lambda : d(f, g) < \infty\}.$$

$\Im$ is a unique mapping satisfying (2.10) such that

$$\|f(x) - \Im(x)\| \leq k\varphi(x, x) \quad \forall x \in A$$

for some $k \in (o, \infty)$.

2. $d(T^n f, \Im) \to 0$ as $n \to \infty$. By Theorem 1.2, there exists a fixed point $\Im$ of T such that

$$\Im(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \quad \forall x \in A. \tag{2.11}$$

On the other hand, it follows from (2.4), (2.6) and (2.9)

$$\begin{aligned}\|\Delta_\rho \Im(x, y)\| &= \lim_{n\to\infty} 2^n \|\Delta_\rho f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\| \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= 0.\end{aligned}$$

Therefore, $\Delta_\rho \Im(x, y) = 0$ for all $x, y \in V_\perp$. By Lemma 2.1 and ([23], Theorem 2.2), $\Im$ is a unique additive. By the result in [18], $\Im$ is an orthogonal mapping. Then, by definition of $\Im$, (2.5), (2.7) and (2.9), we have

$$\begin{aligned}\|\Re\Im([[u, v], w])\| &= \lim_{n\to\infty} 2^{3n} \left\|\Re f\left(\left[\left[\frac{u}{2^n}, \frac{v}{2^n}\right], \frac{w}{2^n}\right]\right)\right\| \\ &\leq \lim_{n\to\infty} 2^{3n} \psi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \\ &= 0\end{aligned}$$

for all $u, v, w \in V_\perp$.

3. $d(f, \Im) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{1}{2(1-L)}\varphi(x, x)$ for all $x \in A$. This completes the proof.

□

The following result shows the Hyers–Ulam–Rassias stability of the orthogonally 3-Lie homomorphism additive ρ-functional equation (2.1).

Corollary 2.3 *Let $s \neq 1$ and θ be nonnegative real numbers. Suppose $f : A \to B$ is a mapping such that*

$$\|\Delta_\rho f(x, y)\| \leq \theta(\|x\|^s + \|y\|^s) \tag{2.12}$$

and

$$\|\Re f([[u, v], w])\| \le \theta(\|u\|^s \cdot \|v\|^s \cdot \|w\|^s) \tag{2.13}$$

for all $u, v, w \in V_\perp$. *Then, there exists a unique 3-Lie homomorphism* $\Im : A \to B$ *such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^s - 2}\|x\|^s, \qquad (for \quad s < 1) \tag{2.14}$$

and

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^s}\|x\|^s, \qquad (for \quad s > 1) \tag{2.15}$$

for all $x \in A$.

Proof The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) = \theta(\|x\|^s + \|y\|^s)$$

$$\psi(u, v, w) = \theta(\|u\|^s \cdot \|v\|^s \cdot \|w\|^s)$$

for all $x, y, u, v, w \in V_\perp$. Then, by choosing $L = 2^{s-1}$ in (2.14), $L = 2^{1-s}$ in (2.15), we get the desired result. □

Now, we will prove the hyperstability of orthogonally 3-Lie homomorphism in 3-Lie algebras in the following theorem.

Theorem 2.4 *Let* $f : A \to B$ *be a mapping and* $\varphi : A^5 \to [0, \infty)$ *be a function such that*

$$\|\Delta_\rho f(x, y) + \Re(u, v, w)\| \le \varphi(0, y, u, v, w) \tag{2.16}$$

for all $x, y, u, v, w \in V_\perp$. *If there exists a constant* $0 < L < 1$ *such that*

$$\varphi\left(0, \frac{y}{2}, \frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) \le \frac{L}{2}\varphi(0, y, u, v, w) \tag{2.17}$$

for all $x, y, u, v, w \in V_\perp$, *then f is an orthogonally 3-Lie homomorphism.*

Proof Put $y = u = v = w = 0$ in (2.16). So, we have

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \varphi(0, 0, 0, 0, 0) \tag{2.18}$$

for all $x \in A$. On the other hand, by (2.17), we obtain that

$$\lim_{n\to\infty} 2^n \varphi\left(0, \frac{y}{2}, \frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) = 0 \tag{2.19}$$

for all $y, u, v, w \in V_\perp$. But $\varphi(0,0,0,0,0) = 0$, thus by (2.16), $f(x) = \frac{1}{2} f(2x)$, and then for all $n \in \mathbb{N}$ and $x \in A$, we have

$$f(x) = \frac{1}{2^n} f(2^n x). \tag{2.20}$$

From (2.16) and (2.20), we have

$$\begin{aligned} \|\Delta_\rho f(x, y)\| &= \frac{1}{2^n} \|\Delta_\rho f(2^n x, 2^n y)\| \\ &\le \frac{1}{2^n} \varphi\left(0, 2^n y, 0, 0, 0\right) \\ &= 0 \end{aligned} \tag{2.21}$$

for all $x, y \in V_\perp$. Letting $n \to \infty$ in (2.21) and using (2.19), we have $\|\Delta_\rho f(x, y)\| = 0$ for all $x, y \in V_\perp$. On the other hand, we have

$$\begin{aligned} \|\Re f(u, v, w)\| &= \frac{1}{2^n} \|\Re f(2^n u, 2^n v, 2^n w)\| \\ &\le \frac{1}{2^n} \varphi\left(0, 0, 2^n u, 2^n v, 2^n w\right) \\ &= 0 \end{aligned} \tag{2.22}$$

for all $u, v, w \in V_\perp$. Hence, by letting $n \to \infty$ in (2.22) and using (2.19), we have

$$\Re f(a, b, c) = 0 \qquad \forall a, b, c \in V_\perp.$$

Therefore, f is an orthogonally 3-Lie homomorphism. □

Corollary 2.5 *Let θ and $s \neq 1$ be nonnegative real numbers. Suppose $f : A \to B$ is a mapping such that*

$$\|\Delta_\rho f(x, y) + \Re f(u, v, w)\| \le \theta(\|y\|^s + \|u\|^s + \|v\|^s + \|w\|^s) \tag{2.23}$$

for all $x, y, u, v, w \in V_\perp$. Then, f is an orthogonally 3-Lie homomorphism.

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Some New Inequalities for Fractional Integral Operators

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Abstract In this chapter, we introduce some new fractional integral operators and fractional area balance operators. The corresponding integral operator inequalities are established.

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1 Introduction

It is well known that fractional integral operator is one of the important operators in harmonic analysis with background of partial differential equations. In fact, the solution of the Laplace equation $\triangle g = f$ for good functions on $\mathbb{R}^n$ can be represented by using the fractional integral operators acting on f. Recently, different versions of fractional integral operators have been developed which are useful in the study of different classes of differential and integral equations. These fractional integral operators act as ready tools to study the classes of differential and integral equations. Hence, fractional integral inequalities are very important in the theory and applications of differential equations. Such inequalities are also of great importance in the mathematical modeling of the fractional boundary value problems (see e.g. [1–11] and the references cited therein). First, we recall the following definitions and some related results.

Definition 1 (cf.[1, 2]) Let $f \in L[a, b]$, then Riemann–Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_1(f, x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)dt, x > a, \tag{1}$$

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and

$$T_2(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b, \tag{2}$$

respectively, where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{3}$$

is the Gamma function and when $\alpha = 0$, $T_1(f,x) = T_2(f,x) = f(x)$.

Definition 2 (cf.[3]) Let $f \in L[a,b]$, then Riemann–Liouville k-fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$T_3(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)dt, , \ x > a, \tag{4}$$

and

$$T_4(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{(\alpha/k)-1} f(t)dt, \ x < b, \tag{5}$$

respectively, where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-(t^k/k)} dt, \ \alpha > 0, \tag{6}$$

is the k-Gamma function. Also, $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x)$, $\Gamma_k(\alpha) = k^{(\alpha/k)-1}\Gamma(\alpha/k)$ and $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$.

It is well known that the Mellin transform of the exponential function on $\exp -t^k/k$ is the k-Gamma function.

Definition 3 (cf.[4, 5]) Let $f \in L^{1,r}[a,b]$, $a \geq 0$, then the generalized Riemann–Liouville fractional integrals of f of order (α, r) are defined by

$$T_5(f,x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t)dt, \ x > a, \tag{7}$$

$$T_6(f,x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{\alpha-1} t^r f(t)dt, \ x < b, \tag{8}$$

and

$$T_7(f,x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{(\alpha/k)-1} t^r f(t)dt, \ x > a, \tag{9}$$

$$T_8(f,x) = \frac{(r+1)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+1} - x^{r+1})^{(\alpha/k)-1} t^r f(t)dt, \ \ x < b, \tag{10}$$

respectively, where $k, \alpha > 0$, $r \geq 0$, and $x \in [a, b]$.

In particular, if $r = 0$, then Definition 3 reduces to Definitions 1 and 2.

Theorem 1 ([6]) *Let* $f : [a, b] \to (0, \infty)$, $f \in L[a, b]$, $a \geq 0$. *If* f *is a convex function on* $[a, b]$, *then*

$$f(\frac{a+b}{2}) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[T_1(f,x) + T_2(f,x)] \leq \frac{1}{2}[f(a) + f(b)]. \tag{11}$$

Theorem 2 ([7]) *Let* $f : [a, b] \to [0, \infty)$, $f \in L[a, b]$, $a \geq 0$. *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *If* $|f^{'}|^q$ *is a convex function on* $[a, b]$, *then*

$$\begin{aligned} &|\frac{1}{2}(f(a) + f(b)) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[T_1(f,b) + T_2(f,a)]| \\ &\leq \frac{b-a}{2^{1/q}(p\alpha+1)^{1/p}}(|f^{'}(a)|^q + |f^{'}(b)|^q)^{1/q}. \end{aligned} \tag{12}$$

Definition 4 (cf.[8, 9]) Let f be a conformable integrable function on $[a, b] \subset [0, \infty)$. The right-sided and left-sided generalized conformable fractional integrals T_9 and T_{10} of f of order $\alpha > 0$ are defined by

$$T_9(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{x^{r+s} - t^{r+s}}{r+s}\right)^{\alpha-1} t^{r+s-1} f(t)dt, \ \ x > a, \tag{13}$$

and

$$T_{10}(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\frac{t^{r+s} - x^{r+s}}{r+s})^{\alpha-1} t^{r+s-1} f(t)dt \ \ x < b, \tag{14}$$

respectively, where $r, s \geq 0$, $r + s \neq 0$.

In particular, if $s = 1$, then T_9 and T_{10} reduce to T_5 and T_6, respectively.

Definition 5 (cf.[10, 11]) Let $f \in L[a, b]$, $g : [a, b] \to (0, \infty)$ be an increasing function, and $g^{'} \in C[a, b]$, $\alpha > 0$. Then, g-Riemann–Liouville fractional integrals of f with respect to the function g on $[a, b]$ are defined by

$$T_{11}(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x g^{'}(t)[g(x) - g(t)]^{\alpha-1} f(t)dt, \ \ x > a, \tag{15}$$

and

$$T_{12}(f,x)=\frac{1}{\Gamma(\alpha)}\int_x^b g^{'}(t)[g(t)-g(x)]^{\alpha-1}f(t)dt,\ \ x<b, \tag{16}$$

respectively.

Theorem 3 ([11]) *Let $\alpha\in(0,1), 0\le a<b$, $f,g:[a,b]\to(0,\infty)$, $f\in L[a,b]$, $g^{'}\in C[a,b]$, g be an increasing function. If f is an s-convex function on $[a,b]$. Then,*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\{T_{11}(f\circ g)(g^{-1}(b))$$
$$+T_{12}(f\circ g)(g^{-1}(a))\}\le\frac{\alpha}{\alpha+s}\left(3-\frac{1}{2^{\alpha+s}}\right)\frac{f(a)+f(b)}{2}. \tag{17}$$

In 2018, Dragomir [12] introduced the new notion of the area balance function.

Definition 6 ([12]) Let $f\in L[a,b]$, then the area balance function of f is defined by

$$T_{13}(f,x)=\frac{1}{2}\left\{\int_x^b f(t)dt-\int_a^x f(t)dt\right\}. \tag{18}$$

Theorem 4 ([12]) *Let $f\in AC[a,b]$. If $f^{'}(t)\ge 0\,a.e.\,t\in[a,b]$, then*

$$\left(\frac{a+b}{2}-x\right)f(x)\le T_{13}(f,x)$$
$$\le\frac{1}{2}[bf(b)+af(a)]-\frac{1}{2}[f(b)+f(a)]x, \tag{19}$$

for all $x\in[a,b]$.

Theorem 5 ([12]) *Let $f\in AC[a,b]$, $f^{'}\in BV[a,b]$, then for any $x\in[a,b]$,*

$$\left|T_{13}(f,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{1}{4}\left[f^{'}(a)+f^{'}(b)\right]\right.$$
$$\left.\times\left[(x-\frac{a+b}{2})^2+\frac{1}{4}(b-a)^2\right]\right|\le\frac{1}{4}\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]V_a^b(f^{'}),$$

and

$$\left|T_{13}(f,x)-\frac{1}{2}\left[af(a)+bf(b)\right]+\frac{1}{2}\left[f(a)+f(b)\right]x\right.$$
$$\left.+\frac{1}{4}\left[f^{'}(a)+f^{'}(b)\right]\times\left[(x-\frac{a+b}{2})^2+\frac{1}{4}(b-a)^2\right]\right|$$

$$\leq \frac{1}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] V_a^b(f^{'})$$

In this chapter, we introduce some new generalized fractional integral operators and fractional area balance operators in Sect. 2. The corresponding integral operator inequalities are established in Sects. 3 and 4.

2 Generalized Fractional Integral Operators and Fractional Area Balance Operators

Definition 7 Let $f \in L[a,b]$, $g : [a,b] \to (0,\infty)$ be an increasing function, and $g \in AC[a,b]$, $k, c, \alpha > 0$, $a \geq 0$. Then, the generalized fractional integral operator T_{14} with respect to the function g on $[a,b]$ is defined by

$$T_{14}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^b g^{'}(t)|g(x) - g(t)|^{(\alpha/k)-1} f(t)dt, \tag{20}$$

where $\Gamma_k(\alpha)$ is defined by (6).

Let

$$T_{15}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_a^x g^{'}(t)[g(x) - g(t)]^{(\alpha/k)-1} f(t)dt, \;\; x > a, \tag{21}$$

and

$$T_{16}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \int_x^b g^{'}(t)[g(t) - g(x)]^{(\alpha/k)-1} f(t)dt, \;\; x < b. \tag{22}$$

Then,

$$T_{14}(f,x) = T_{15}(f,x) + T_{16}(f,x). \tag{23}$$

In particular, if $c = k = 1$ in (23), then (23) reduces to

$$T_{14}(f,x) = T_{11}(f,x) + T_{12}(f,x). \tag{24}$$

If $c = (r+s)^{-\alpha}$, $g(t) = t^{r+s}$, $r, s \geq 0$, $r + s \neq 0$, $k = 1$ in (23), then (23) reduces to

$$T_{14}(f,x) = T_9(f,x) + T_{10}(f,x). \tag{25}$$

If $c = (r+1)^{-(\alpha/k)}$, $g(t) = t^{r+s}$, $r \geq 0$, in (23), then (23) reduces to

$$T_{14}(f,x) = T_7(f,x) + T_8(f,x). \tag{26}$$

If $s = 1$ in (25), then (25) reduces to

$$T_{14}(f,x) = T_5(f,x) + T_6(f,x). \tag{27}$$

If $r = 0$ in (26), then (26) reduces to

$$T_{14}(f,x) = T_3(f,x) + T_4(f,x). \tag{28}$$

If $k = 1$ in (28), then (28) reduces to

$$T_{14}(f,x) = T_1(f,x) + T_2(f,x). \tag{29}$$

We can also rewrite T_9 and T_{10} as

$$T_9(f,x) = \frac{(r+s)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+s} - t^{r+s})^{\alpha-1} t^{r+s-1} f(t)dt, \ \ x > a,$$

and

$$T_{10}(f,x) = \frac{(r+s)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{r+s} - x^{r+s})^{\alpha-1} t^{r+s-1} f(t)dt, \ \ x < b,$$

and then generalize them to

$$T_{17}(f,x) = \frac{(r+s)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_a^x (x^{r+s} - t^{r+s})^{(\alpha/k)-1} t^{r+s-1} f(t)dt, \ \ x > a, \tag{30}$$

and

$$T_{18}(f,x) = \frac{(r+s)^{1-(\alpha/k)}}{k\Gamma_k(\alpha)} \int_x^b (t^{r+s} - x^{r+s})^{(\alpha/k)-1} t^{r+s-1} f(t)dt, \ \ x < b. \tag{31}$$

If $c = (r+s)^{-(\alpha/k)}$, $g(t) = t^{r+s}$, $r, s \geq 0, r+s \neq 0$ in (23), then (23) reduces to

$$T_{14}(f,x) = T_{17}(f,x) + T_{18}(f,x). \tag{32}$$

Definition 8 ([13, 14]) Let $f \in L[a,b], a \geq 0$. The left-sided and right-sided Hadamard fractional integrals T_{20} and T_{21} of f of order $\alpha > 0$ are defined by

$$T_{20}(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\log x - \log t)^{\alpha-1} t^{-1} f(t)dt, \ \ x > a,$$

and

$$T_{21}(f,x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\log t - \log x)^{\alpha-1} t^{-1} f(t)dt, \;\; x < b,$$

respectively.

Theorem 6 (cf.[14]) *Let $f \in L[a,b], a \geq 0$. If f is a GA-convex function, then*

$$f(\sqrt{ab}) \leq \frac{\Gamma(\alpha+1)}{2\log^{\alpha}(b/a)}(T_{20}(f,b) + T_{21}(f,a)) \leq \frac{1}{2}(f(a) + f(b)).$$

We can generalize them to

$$T_{22}(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (\log x - \log t)^{(\alpha/k)-1} t^{-1} f(t)dt, \;\; x > a,$$

and

$$T_{23}(f,x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (\log t - \log x)^{(\alpha/k)-1} t^{-1} f(t)dt, \;\; x < b.$$

If $c = 1$, $g(t) = \log t$ in (23), then (23) reduces to

$$T_{14}(f,x) = T_{22}(f,x) + T_{23}(f,x).$$

In particular, if $k = 1$, then

$$T_{14}(f,x) = T_{20}(f,x) + T_{21}(f,x).$$

Definition 9 Under the assumptions of Definition 7, the fractional area balance operator T_{19} with respect to the function g on $[a,b]$ is defined by

$$T_{19}(f,x) = \frac{c}{k\Gamma_k(\alpha)} \left\{ \int_x^b g'(t)[g(t) - g(x)]^{(\alpha/k)-1} f(t)dt \right. \\ \left. - \int_a^x g'(t)[g(x) - g(t)]^{(\alpha/k)-1} f(t)dt \right\}, \tag{33}$$

where $\Gamma_k(\alpha)$ is defined by (6).

Using (21) and (22), we have

$$T_{19}(f,x) = T_{16}(f,x) - T_{15}(f,x). \tag{34}$$

In particular, if $c = (r+s)^{-(\alpha/k)}$, $g(t) = t^{r+s}$, $r, s \geq 0$, $r + s \neq 0$ in (34), then (34) reduces to

$$T_{19}(f,x) = T_{18}(f,x) - T_{17}(f,x). \tag{35}$$

If $g(t) = t, \alpha = k = 1, c = 1/2$ in (34), then T_{19} reduces to T_{13}. If $c = k = 1$ in (34), then (34) reduces to

$$T_{19}(f, x) = T_{12}(f, x) - T_{11}(f, x). \tag{36}$$

If $c = (r+s)^{-\alpha}$, $g(t) = t^{r+s}$, $r, s \geq 0$, $r + s \neq 0$, $k = 1$ in (34), then (34) reduces to

$$T_{19}(f, x) = T_{10}(f, x) - T_9(f, x). \tag{37}$$

If $c = (r+1)^{-(\alpha/k)}$, $g(t) = t^{r+1}$, $r \geq 0$ in (34), then (34) reduces to

$$T_{19}(f, x) = T_8(f, x) - T_7(f, x). \tag{38}$$

If $k = 1$ in (38), then (38) reduces to

$$T_{19}(f, x) = T_6(f, x) - T_5(f, x). \tag{39}$$

If $r = 0$ in (38), then (38) reduces to

$$T_{19}(f, x) = T_4(f, x) - T_3(f, x). \tag{40}$$

If $k = 1$ in (40), then (40) reduces to

$$T_{19}(f, x) = T_2(f, x) - T_1(f, x). \tag{41}$$

If $c = 1$ and $g(t) = \log t$ in (34), then (34) reduces to

$$T_{19}(f, x) = T_{23}(f, x) - T_{22}(f, x).$$

In particular, if $k = 1$, then

$$T_{19}(f, x) = T_{21}(f, x) - T_{20}(f, x).$$

Hence, Definitions 7 and 9 unified and generalized many known and new classes of fractional integral operators.

3 Some Inequalities for T_{19}

We require the following definition and lemmas to prove our main results.

Definition 10 ([1]) Let $[a, b] \subset [0, \infty)$,$h : [a, b] \to (0, \infty)$ be given function. A function $f : [a, b] \to [0, \infty)$ is called exponentially (β, s, s_1, s_2, h)-strongly convex if

$$f(tx_1 + (1-t)x_2) \leq \left\{ t^{ss_1} \left(\frac{f(x_1)}{e^{rx_1}} \right)^{\beta} + (1 - t^{s_2})^{s} (\frac{f(x_2)}{e^{rx_2}})^{\beta} \right\}^{1/\beta} - t(1-t)h(|x_1 - x_2|), \tag{42}$$

where $x_1, x_2 \in [a, b]$, $t, s, s_1, s_2 \in [0, 1]$, $r, \beta \in \mathbb{R}$, $\beta \neq 0$.

Lemma 1 *Let* $[a, b] \subset [0, \infty)$, $f \in L[a, b]$, $g : [a, b] \to [0, \infty)$ *be an increasing function, and* $g \in AC[a, b]$, $k, \alpha, c > 0$, *then*

$$T_{19}(f, x) = [T_{16}(1, x) - T_{15}(1, x)] f(x) + \left\{ \int_x^b G_{16}(1, t) f^{'}(t) dt + \int_a^x G_{15}(1, t) f^{'}(t) dt \right\}, \tag{43}$$

where $G_{15}(1, t)$ *and* $G_{16}(1, t)$ *are defined by*

$$G_{15}(1, t) = \frac{c}{k\Gamma_k(\alpha)} \int_a^t g^{'}(u)[g(x) - g(u)]^{(\alpha/k)-1} du, \tag{44}$$

and

$$G_{16}(1, t) = \frac{c}{k\Gamma_k(\alpha)} \int_t^b g^{'}(u)[g(u) - g(x)]^{(\alpha/k)-1} du, \tag{45}$$

and T_{19}, T_{15}, T_{16}, *and* $\Gamma_k(\alpha)$ *are defined by (33), (21), (22), and (6), respectively.*

Proof From (21) and (22), we have

$$T_{15}(1, t) = \frac{c}{k\Gamma_k(\alpha)} \int_a^t g^{'}(u)[g(t) - g(u)]^{(\alpha/k)-1} du,$$

and

$$T_{16}(1, t) = \frac{c}{k\Gamma_k(\alpha)} \int_t^b g^{'}(u)[g(u) - g(t)]^{(\alpha/k)-1} du.$$

Thus,

$$G_{15}(1, x) = T_{15}(1, x); G_{16}(1, x) = T_{16}(1, x).$$

Then, making use of integration by parts, we get

$$\int_a^x G_{15}(1,t)f'(t)dt = G_{15}(1,t)f(t)|_a^x$$
$$-\frac{c}{k\Gamma_k(\alpha)}\int_a^x g'(t)[g(x)-g(t)]^{(\alpha/k)-1}f(t)dt$$
$$= T_{15}(1,x)f(x) - T_{15}(f,x),$$

which leads to

$$T_{15}(f,x) = T_{15}(1,x)f(x) - \int_a^x G_{15}(1,t)f'(t)dt. \tag{46}$$

Similarly, we have

$$T_{16}(f,x) = T_{16}(1,x)f(x) + \int_x^b G_{16}(1,t)f'(t)dt. \tag{47}$$

Hence, (43) follows from (34), (46), and (47). The proof is completed.

Similarly, we get the following lemma:

Lemma 2 *Under the assumptions of Lemma 1, we have*

$$T_{14}(f,x) = [T_{16}(1,x) + T_{15}(1,x)]f(x)$$
$$+\int_x^b G_{16}(1,t)f'(t)dt - \int_a^x G_{15}(1,t)f'(t)dt. \tag{48}$$

Theorem 7 *Under the assumptions of Lemma 1, if* $|f'|^p$ *is exponentially* (β, s, s_1, s_2, h)*-strongly convex on* $[a,b]$*, and* $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{s}{\beta}+1>0$, *then*

$$|T_{19}(f,x) - [T_{16}(1,x) - T_{15}(1,x)]f(x)|$$
$$\le \left(\int_x^b |G_{16}(1,t)|^q dt\right)^{1/q}(b-x)^{1/p}$$
$$\times\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f'(b)|^p}{e^{rb}}\right.\right.$$
$$\left.\left.+\frac{1}{s_2}B\left(\frac{s}{\beta}+1,\frac{1}{s_2}\right)\frac{|f'(x)|^p}{e^{rx}}\right] - \frac{1}{6}h(b-x)\right\}^{1/p}$$
$$+\left(\int_a^x |G_{15}(1,t)|^q dt\right)^{1/q}(x-a)^{1/p}$$

$$\times \left\{ C_\beta \left[\frac{\beta}{ss_1+\beta} \frac{|f^{'}(x)|^p}{e^{rx}} \right.\right.$$
$$\left.\left. + \frac{1}{s_2} B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right) \frac{|f^{'}(a)|^p}{e^{ra}} \right] - \frac{1}{6} h(x-a) \right\}^{1/p}. \tag{49}$$

If $p = 1$, *then*

$$|T_{19}(f,x) - [T_{16}(1,x) - T_{15}(1,x)] f(x)|$$
$$\leq \|G_{16}\|_\infty (b-x) \Big\{ C_\beta \Big[\frac{\beta}{ss_1+\beta} \frac{|f^{'}(b)|}{e^{rb}}$$
$$+ \frac{1}{s_2} B\Big(\frac{s}{\beta}+1, \frac{1}{s_2}\Big) \frac{|f^{'}(x)|}{e^{rx}} \Big] - \frac{1}{6} h(b-x) \Big\}$$
$$+ \|G_{15}\|_\infty (x-a) \Big\{ C_\beta \Big[\frac{\beta}{ss_1+\beta} \frac{|f^{'}(x)|}{e^{rx}}$$
$$+ \frac{1}{s_2} B\Big(\frac{s}{\beta}+1, \frac{1}{s_2}\Big) \frac{|f^{'}(a)|}{e^{ra}} \Big] - \frac{1}{6} h(x-a) \Big\},$$

where

$$C_\beta = \begin{cases} 1 & \beta \geq 1, \\ 2^{(1/\beta)-1}, & 0 < \beta < 1, \end{cases} \tag{50}$$

and

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

is the Beta function.

Proof For $1 < p < \infty$, by using Lemma 1 and the Hölder inequality, we obtain

$$|T_{19}(f,x) - [T_{16}(1,x) - T_{15}(1,x)] f(x)|$$
$$\leq \int_x^b |G_{16}(1,t)| \times |f^{'}(t)| dt + \int_a^x |G_{15}(1,t)| \times |f^{'}(t)| dt$$
$$\leq \left(\int_x^b |G_{16}(1,t)|^q dt \right)^{1/q} \left(\int_x^b |f^{'}(t)|^p dt \right)^{1/p}$$
$$+ \left(\int_a^x |G_{15}(1,t)|^q dt \right)^{1/q} \left(\int_a^x |f^{'}(t)|^p dt \right)^{1/p}. \tag{51}$$

Setting $u = x + (b - x)t$ and using the exponentially (β, s, s_1, s_2, h)-strongly convexity of $|f^{'}|^p$ on $[a, b]$, we have

$$\int_x^b |f^{'}(u)|^p du = (b-x)\int_0^1 |f^{'}(tb+(1-t)x)|^p dt$$
$$\leq (b-x)\int_0^1 \left\{\left[t^{ss_1}\left(\frac{|f^{'}(b)|^p}{e^{rb}}\right)^{\beta} + (1-t^{s_2})^s\left(\frac{|f^{'}(x)|^p}{e^{rx}}\right)^{\beta}\right]^{1/\beta}\right.$$
$$\left. -t(1-t)h(b-x)\right\} dt$$
$$\leq (b-x)\left\{C_\beta \int_0^1 \left[t^{(ss_1)/\beta}\left(\frac{|f^{'}(b)|^p}{e^{rb}}\right) + (1-t^{s_2})^{s/\beta}\left(\frac{|f^{'}(x)|^p}{e^{rx}}\right)\right] dt\right.$$
$$\left. -h(b-x)\int_0^1 t(1-t)dt\right\}$$
$$= (b-x)\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(b)|^p}{e^{rb}}\right.\right.$$
$$\left.\left. +\frac{1}{s_2}B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right)\frac{|f^{'}(x)|^p}{e^{rx}}\right] - \frac{1}{6}h(b-x)\right\}. \quad (52)$$

By letting $u = a + (x - a)t$ and similar arguments, we get

$$\int_a^x |f^{'}(u)|^p du = (x-a)\int_0^1 |f^{'}(tx+(1-t)a)|^p dt$$
$$\leq (x-a)\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(x)|^p}{e^{rx}} + \frac{1}{s_2}B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right)\frac{|f^{'}(a)|^p}{e^{ra}}\right]\right.$$
$$\left. -\frac{1}{6}h(x-a)\right\}. \quad (53)$$

Hence, (49) follows from (51), (52), and (53). The case $p = 1$ can be treated analogously. The proof is completed.

By giving particular values to the parameters in Theorem 7, we get the corresponding integral inequalities for different fractional integral operators. Such as, taking $k = c = \beta = 1$ in Theorem 7, then T_{19} reduces to $T_{12} - T_{11}$. Thus, we get the following corollary:

Corollary 1 *Under the assumptions of Theorem 7, if* $k = c = \beta = 1$, *and* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}
&|T_{12}(f,x) - T_{11}(f,x) - \frac{1}{\alpha\Gamma(\alpha)}[(g(b)-g(x))^{\alpha} \\
&-(g(x)-g(a))^{\alpha}]f(x)| \leq \left(\int_x^b |G_{12}(1,t)|^q dt\right)^{1/q} (b-x)^{1/p} \\
&\times \left\{\frac{1}{ss_1+1}\frac{|f^{'}(b)|^p}{e^{rb}} + \frac{1}{s_2}B\left(s+1, \frac{1}{s_2}\right)\frac{|f^{'}(x)|^p}{e^{rx}} - \frac{1}{6}h(b-x)\right\}^{1/p} \\
&+\left(\int_a^x |G_{11}(1,t)|^q dt\right)^{1/q} (x-a)^{1/p} \\
&\times \left\{\frac{1}{ss_1+1}\frac{|f^{'}(x)|^p}{e^{rx}} + \frac{1}{s_2}B\left(s+1, \frac{1}{s_2}\right)\frac{|f^{'}(a)|^p}{e^{ra}} - \frac{1}{6}h(x-a)\right\}^{1/p}, \quad (54)
\end{aligned}$$

where $G_{11}(1,t)$ *and* $G_{12}(1,t)$ *are defined by*

$$G_{11}(1,t) = \frac{1}{\Gamma(\alpha)}\int_a^t g^{'}(u)[g(x)-g(u)]^{\alpha-1}du$$

and

$$G_{12}(1,t) = \frac{1}{\Gamma(\alpha)}\int_t^b g^{'}(u)[g(u)-g(x)]^{\alpha-1}du,$$

respectively.

Corollary 2 *Under the assumptions of Theorem 7, if* $g(t) = t, c = 1$, *then* T_{19} *reduces to* $T_4 - T_3$. *Thus, for* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *we get*

$$\begin{aligned}
&\left|T_4(f,x) - T_3(f,x) - \frac{1}{\alpha\Gamma_k(\alpha)}\left[(b-x)^{(\alpha/k)} - (x-a)^{\alpha/k}\right]f(x)\right| \\
&\leq \frac{1}{\alpha\Gamma_k(\alpha)}\left(\int_x^b |(b-x)^{\alpha/k} - (t-x)^{\alpha/k}|^q dt\right)^{1/q} (b-x)^{1/p} \\
&\times \left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(b)|^p}{e^{rb}} + \frac{1}{s_2}B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right)\frac{|f^{'}(x)|^p}{e^{rx}}\right]\right. \\
&\left.-\frac{1}{6}h(b-x)\right\}^{1/p}
\end{aligned}$$

$$+\frac{1}{\alpha\Gamma_k(\alpha)}\left(\int_a^x |(x-a)^{\alpha/k}-(x-t)^{\alpha/k}|^q dt\right)^{1/q}(x-a)^{1/p}$$
$$\times\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(x)|^p}{e^{rx}}+\frac{1}{s_2}B\left(\frac{s}{\beta}+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|^p}{e^{ra}}\right]\right.$$
$$\left.-\frac{1}{6}h(x-a)\right\}^{1/p}. \tag{55}$$

If $p=1$, *then*

$$|T_4(f,x)-T_3(f,x)-\frac{1}{\alpha\Gamma_k(\alpha)}[(b-x)^{(\alpha/k)}-(x-a)^{(\alpha/k)}]f(x)|$$
$$\leq\frac{1}{\alpha\Gamma_k(\alpha)}(b-x)^{1+(\alpha/k)}\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(b)|}{e^{rb}}+\frac{1}{s_2}B(\frac{s}{\beta}+1,\frac{1}{s_2})\frac{|f^{'}(x)|}{e^{rx}}\right]\right.$$
$$\left.-\frac{1}{6}h(b-x)\right\}$$
$$+\frac{1}{\alpha\Gamma_k(\alpha)}(x-a)^{1+(\alpha/k)}\left\{C_\beta\left[\frac{\beta}{ss_1+\beta}\frac{|f^{'}(x)|}{e^{rx}}+\frac{1}{s_2}B\left(\frac{s}{\beta}+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|}{e^{ra}}\right]\right.$$
$$\left.-\frac{1}{6}h(x-a)\right\}.$$

If $k=\beta=1$ in Corollary 2, then T_{19} reduces to T_2-T_1. Thus, we get the following corollary:

Corollary 3 *Under the assumptions of Corollary 2, let* $k=\beta=1$. *If* $1<p<\infty$, *then*

$$|T_2(f,x)-T_1(f,x)-\frac{1}{\alpha\Gamma(\alpha)}[(b-x)^{\alpha}-(x-a)^{\alpha}]f(x)|$$
$$\leq\frac{1}{\alpha\Gamma(\alpha)}\left(\int_x^b |(b-x)^{\alpha}-(t-x)^{\alpha}|^q dt\right)^{1/q}(b-x)^{1/p}$$
$$\times\left\{\frac{1}{ss_1+1}\left[\frac{|f^{'}(b)|^p}{e^{rb}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(x)|^p}{e^{rx}}\right]-\frac{1}{6}h(b-x)\right\}^{1/p}$$
$$+\frac{1}{\alpha\Gamma(\alpha)}\left(\int_a^x |(x-a)^{\alpha}-(x-t)^{\alpha}|^q dt\right)^{1/q}(x-a)^{1/p}$$
$$\times\left\{\frac{1}{ss_1+1}\left[\frac{|f^{'}(x)|^p}{e^{rx}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|^p}{e^{ra}}\right]-\frac{1}{6}h(x-a)\right\}^{1/p} \tag{56}$$

If $p=1$, *then*

$$\begin{aligned}&|T_2(f,x)-T_1(f,x)-\frac{1}{\alpha\Gamma(\alpha)}[(b-x)^{\alpha}-(x-a)^{\alpha}]f(x)|\\&\le\frac{1}{\alpha\Gamma(\alpha)}(b-x)^{1+\alpha}\left\{\frac{1}{ss_1+1}\frac{|f^{'}(b)|}{e^{rb}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(x)|}{e^{rx}}\right.\\&\left.-\frac{1}{6}h(b-x)\right\}\\&+\frac{1}{\alpha\Gamma(\alpha)}(x-a)^{1+\alpha}\left\{\frac{1}{ss_1+1}\frac{|f^{'}(x)|}{e^{rx}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|}{e^{ra}}\right.\\&\left.-\frac{1}{6}h(x-a)\right\}.\end{aligned}\tag{57}$$

Taking $c=1/2, \alpha=k=1$, and $g(t)=t$ in Theorem 7, then T_{19} reduces to the area balance function T_{13}. Thus, we get the following corollary:

Corollary 4 *Let $f^{'}\in L^p[a,b], a\ge 0$, and $|f^{'}|^p$ be exponentially (s,s_1,s_2,h)-strongly convex on $[a,b]$, where $1\le p<\infty, \frac{1}{p}+\frac{1}{q}=1$, and for $p=1$, define $q=\infty, \frac{1}{\infty}=0$.*

If $1<p<\infty$, then

$$\begin{aligned}&|T_{13}(f,x)-((b+a)/2-x)f(x)|\\&\le\frac{(b-x)^2}{2(q+1)^{1/q}}\left\{\frac{1}{ss_1+1}\frac{|f^{'}(b)|^p}{e^{rb}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(x)|^p}{e^{rx}}\right.\\&\left.-\frac{1}{6}h(b-x)\right\}^{1/p}\\&+\frac{(x-a)^2}{2(q+1)^{1/q}}\left\{\frac{1}{ss_1+1}\frac{|f^{'}(x)|^p}{e^{rx}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|^p}{e^{ra}}\right.\\&\left.-\frac{1}{6}h(x-a)\right\}^{1/p}.\end{aligned}\tag{58}$$

If $p=1$, then

$$\begin{aligned}&|T_{13}(f,x)-((b+a)/2-x)f(x)|\\&\le\frac{1}{2}(b-x)^2\left\{\frac{1}{ss_1+1}\frac{|f^{'}(b)|}{e^{rb}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(x)|}{e^{rx}}\right.\\&\left.-\frac{1}{6}h(b-x)\right\}\end{aligned}$$

$$+\frac{1}{2}(x-a)^2\left\{\frac{1}{ss_1+1}\frac{|f^{'}(x)|}{e^{rx}}+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)\frac{|f^{'}(a)|}{e^{ra}}\right.$$
$$\left.-\frac{1}{6}h(x-a)\right\}. \tag{59}$$

Let $r=h=0$ in (58) and (59). If $1<p<\infty$, then

$$|T_{13}(f,x)-((b+a)/2-x)f(x)|$$
$$\leq\frac{1}{2(q+1)^{1/q}}\left\{(b-x)^2\left[\frac{1}{ss_1+1}|f^{'}(b)|^p+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)|f^{'}(x)|^p\right]^{1/p}\right.$$
$$\left.+(x-a)^2\left[\frac{1}{ss_1+1}|f^{'}(x)|^p+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)|f^{'}(a)|^p\right]^{1/p}\right\}. \tag{60}$$

If $p=1$, then

$$|T_{13}(f,x)-((b+a)/2-x)f(x)|$$
$$\leq\frac{1}{2}\left\{(b-x)^2\left[\frac{1}{ss_1+1}|f^{'}(b)|+\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)|f^{'}(x)|\right]\right.$$
$$\left.+(x-a)^2\left[\frac{1}{ss_1+1}|f^{'}(x)|+|\frac{1}{s_2}B\left(s+1,\frac{1}{s_2}\right)f^{'}(a)|\right]\right\}. \tag{61}$$

By using the following identity (see [12])

$$T_{13}(f,x)=\frac{1}{2}\{bf(b)+af(a)-[f(b)+f(a)]x\}$$
$$-\frac{1}{2}\int_a^b|t-x|f^{'}(t)dt,\ \ x\in[a,b], \tag{62}$$

we get the following theorem:

Theorem 8 *Let $f\in AC[a,b], a\geq 0$, and $|f^{'}|$ be exponentially (β,s,s_1,s_2,h)-strongly convex on $[a,b]$, $\frac{s}{\beta}+1>0$, then*

$$|T_{13}(f,x)-\frac{1}{2}\{bf(b)+af(a)-[f(b)+f(a)]x\}|$$
$$\leq\frac{1}{2}\left\{(b-x)^2\times C_\beta\left[\frac{\beta}{ss_1+2\beta}\frac{|f^{'}(b)|}{e^{rb}}+\frac{1}{s_2}B\left(\frac{s}{\beta}+1,\frac{2}{s_2}\right)\frac{|f^{'}(x)|}{e^{rx}}\right]\right.$$
$$-\frac{1}{12}h(b-x)$$

$$+(x-a)^2 \times C_\beta \left[\frac{\beta^2}{(ss_1+\beta)(ss_1+2\beta)} \frac{|f^{'}(x)|}{e^{rx}} \right.$$
$$\left. +\frac{1}{s_2}\left(B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right) - B\left(\frac{s}{\beta}+1, \frac{2}{s_2}\right)\right) \frac{|f^{'}(a)|}{e^{ra}} \right]$$
$$\left. -\frac{1}{12}h(x-a) \right\}, \tag{63}$$

where C_β is defined by (50).

Proof From (62), we have

$$|T_{13}(f,x) - \frac{1}{2}\{bf(b)+af(a)-[f(a)+f(b)]x\}|$$
$$\leq \frac{1}{2}\int_a^b |t-x||f^{'}(t)|dt$$
$$= \frac{1}{2}\left\{\int_a^x (x-t)|f^{'}(t)|dt + \int_x^b (t-x)|f^{'}(t)|dt\right\}. \tag{64}$$

Let

$$I_1 = \int_a^x (x-u)|f^{'}(u)|du; \; I_2 = \int_x^b (u-x)|f^{'}(u)|du.$$

Setting $u = x + (b-x)t$ and using the exponentially (β, s, s_1, s_2, h)-strongly convexity of $|f^{'}|$ on $[a, b]$, we have

$$I_2 = \int_x^b (u-x)|f^{'}(u)|du = (b-x)^2 \int_0^1 t|f^{'}(tb+(1-t)x)|dt$$
$$\leq (b-x)^2 \int_0^1 t \left\{ \left[t^{ss_1}\left(\frac{|f^{'}(b)|}{e^{rb}}\right)^\beta + (1-t^{s_2})^s \left(\frac{|f^{'}(x)|}{e^{rx}}\right)^\beta \right]^{1/\beta} \right.$$
$$\left. -t(1-t)h(b-x)\right\} dt$$
$$\leq (b-x)^2 \left\{ C_\beta \int_0^1 \left[t^{(ss_1/\beta)+1}\left(\frac{|f^{'}(b)|}{e^{rb}}\right) + t(1-t^{s_2})^{s/\beta}\left(\frac{|f^{'}(x)|}{e^{rx}}\right)\right] dt \right.$$
$$\left. -h(b-x)\int_0^1 t^2(1-t)dt \right\}$$
$$= (b-x)^2 \left\{ C_\beta \left[\frac{\beta}{ss_1+2\beta}\frac{|f^{'}(b)|}{e^{rb}} + \frac{1}{s_2}B\left(\frac{s}{\beta}+1, \frac{2}{s_2}\right)\frac{|f^{'}(x)|}{e^{rx}} \right] \right.$$

$$-\frac{1}{12}h(b-x)\Bigg\}. \tag{65}$$

By letting $u = a + (x-a)t$, noting that

$$\int_0^1 (1-t)t^{\frac{ss_1}{\beta}}dt = \frac{\beta^2}{(ss_1+\beta)(ss_1+2\beta)},$$

and setting $u = 1 - t^{s_2}$, we have

$$\begin{aligned}
&\int_0^1 (1-t)(1-t^{s_2})^{s/\beta}dt \\
&= \frac{1}{s_2}\int_0^1 u^{s/\beta}(1-u)^{(1/s_2)-1}(1-(1-u)^{1/s_2})dt \\
&= \frac{1}{s_2}\left[B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right) - B\left(\frac{s}{\beta}+1, \frac{2}{s_2}\right)\right],
\end{aligned}$$

and similar arguments, we get

$$\begin{aligned}
I_1 &= \int_a^x (x-u)|f'(u)|du = (x-a)^2\int_0^1 (1-t)|f'(tx+(1-t)a)|dt \\
&\le (x-a)^2\Bigg\{C_\beta\Bigg[\left(\int_0^1 (1-t)t^{\frac{ss_1}{\beta}}dt\right)\frac{|f'(x)|}{e^{rx}} \\
&\quad + \left(\int_0^1 (1-t)(1-t^{s_2})^{s/\beta}dt\right)\frac{|f'(a)|}{e^{ra}}\Bigg] - h(x-a)\int_0^1 t(1-t)^2dt\Bigg\} \\
&\le (x-a)^2\Bigg\{C_\beta\Bigg[\frac{\beta^2}{(ss_1+\beta)(ss_1+2\beta)}\frac{|f'(x)|}{e^{rx}} \\
&\quad + \frac{1}{s_2}\left(B\left(\frac{s}{\beta}+1, \frac{1}{s_2}\right) - B\left(\frac{s}{\beta}+1, \frac{2}{s_2}\right)\right)\frac{|f'(a)|}{e^{ra}}\Bigg] - \frac{1}{12}h(x-a)\Bigg\}.
\end{aligned} \tag{66}$$

Hence, (63) follows from (64), (65), and (66). The proof is completed.

Taking $h = r = 0$ and $x = (a+b)/2$ in Theorem 8, we get the following corollary:

Corollary 5 *Let $f \in AC[a,b]$, $a \ge 0$, if $|f'|$ is (β, s, s_1, s_2)-convex on $[a,b]$, $\frac{s}{\beta} + 1 > 0$, and $x = (a+b)/2$, then*

$$\begin{aligned}
&|T_{13}(f, \frac{a+b}{2}) - \frac{1}{4}[f(b) - f(a)](b-a)| \\
&\le \frac{C_\beta(b-a)^2}{8}\Bigg\{\frac{\beta}{ss_1+2\beta}|f'(b)|
\end{aligned}$$

$$+\left(\frac{\beta^2}{(ss_1+\beta)(ss_1+2\beta)}+\frac{1}{s_2}B\left(\frac{s}{\beta}+1,\frac{2}{s_2}\right)\right)\left|f^{'}\left(\frac{a+b}{2}\right)\right|$$
$$+\frac{1}{s_2}\left(B\left(\frac{s}{\beta}+1,\frac{1}{s_2}\right)-B\left(\frac{s}{\beta}+1,\frac{2}{s_2}\right)\right)|f^{'}(a)|\Bigg\}. \tag{67}$$

4 Some Inequalities for Operator T_{14}

Theorem 9 *Under the assumptions of Lemma 1, let* $f^{'}\in L^p[a,b], a\geq 0, 1\leq p<\infty, \frac{1}{p}+\frac{1}{q}=1$, *and for* $p=1$, *define* $q=\infty, \frac{1}{\infty}=0$. *If* $1<p<\infty$, *then*

$$|T_{14}(f,x)-[T_{15}(1,x)+T_{16}(1,x)]f(x)|$$
$$\leq\left\{\left(\int_a^x|G_{15}(1,t)|^q dt\right)^{1/q}+\left(\int_x^b|G_{16}(1,t)|^q dt\right)^{1/q}\right\}\|f^{'}\|_p. \tag{68}$$

If $p=1$, *then*

$$|T_{14}(f,x)-[T_{15}(1,x)+T_{16}(1,x)]f(x)|\leq\|G\|_\infty\|f^{'}\|_1, \tag{69}$$

where $T_{14}, T_{15}, T_{16}, G_{15}$, *and* G_{16} *are defined by (20), (21), (22), (44), and (45), respectively, and*

$$G(t)=G_{16}(1,t)\varphi_{D_2}(t)-G_{15}(1,t)\varphi_{D_1}(t), \tag{70}$$

$D_1=[a,x]$, $D_2=[x,b]$, *and* φ_D *is the characteristic function of the set D, that is,*

$$\varphi_D(t)=\begin{cases}1, & t\in D,\\ 0, & t\in D^c.\end{cases}$$

Proof For $1<p<\infty$, by using Lemma 2, we obtain

$$|T_{14}(f,x)-[T_{16}(1,x)+T_{15}(1,x)]f(x)|$$
$$\leq\left|\int_x^b G_{16}(1,t)f^{'}(t)dt-\int_a^x G_{15}(1,t)f^{'}(t)dt\right|$$
$$=\left|\int_a^b[G_{16}(1,t)\varphi_{D_2}(t)-G_{15}(1,t)\varphi_{D_1}(t)]f^{'}(t)dt\right|. \tag{71}$$

From (70), we have

$$G(t) = \begin{cases} G_{16}(1,t), & t \in D_2, \\ -G_{15}(1,t), & t \in D_1. \end{cases}$$

This implies that

$$\int_a^b |G(t)|^q dt = \int_a^x |G_{15}(1,t)|^q dt. + \int_x^b |G_{16}(1,t)|^q dt$$

Using the Hölder inequality, we obtain

$$\begin{aligned} &|T_{14}(f,x) - [T_{16}(1,x) + T_{15}(1,x)]f(x)| \\ &\leq \left(\int_a^b |G(t)|^q dt\right)^{1/q} \left(\int_a^b |f^{'}(t)|^p dt\right)^{1/p} \\ &= \left\{\int_a^x |G_{15}(1,t)|^q dt + \int_x^b |G_{16}(1,t)|^q dt\right\}^{1/q} \|f^{'}\|_p \\ &\leq \left\{\left(\int_a^x |G_{15}(1,t)|^q dt\right)^{1/q} + \left(\int_x^b |G_{16}(1,t)|^q dt\right)^{1/q}\right\} \|f^{'}\|_p, \end{aligned}$$

and for $p = 1$, we have

$$|T_{14}(f,x) - [T_{15}(1,x) + T_{16}(1,x)]f(x)| \leq \|G\|_\infty \|f\|_1.$$

The proof is completed.

Taking $c = 1$ and $g(t) = t$ in Theorem 9, T_{14} reduces to $T_3 + T_4$. Thus, we get the following corollary:

Corollary 6 *Let $f^{'} \in L^p[a,b], a \geq 0, 1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, and for $p = 1$, define $q = \infty, \frac{1}{\infty} = 0$. If $1 < p < \infty$, then*

$$\begin{aligned} &|T_4(f,x) + T_3(f,x) - \frac{1}{\alpha\Gamma_k(\alpha)}[(b-x)^{\alpha/k} + (x-a)^{\alpha/k}]f(x)| \\ &\leq \frac{[kB(\frac{k}{\alpha}, q+1)]^{1/q}}{\alpha^{1+(1/q)}\Gamma_k(\alpha)} \left\{(x-a)^{\frac{\alpha}{k}+\frac{1}{q}} + (b-x)^{\frac{\alpha}{k}+\frac{1}{q}}\right\} \|f^{'}\|_p. \end{aligned} \tag{72}$$

If $p = 1$, then

$$\begin{aligned} &|T_4(f,x) + T_3(f,x) - \frac{1}{\alpha\Gamma_k(\alpha)}[(x-a)^{(\alpha/k)} + (b-x)^{(\alpha/k)}]f(x)| \\ &\leq \frac{1}{\alpha\Gamma_k(\alpha)} \left\{(x-a)^{(\alpha/k)} + (b-x)^{(\alpha/k)}\right\} \|f^{'}\|_1. \end{aligned} \tag{73}$$

Taking $k = 1$ in (72) and (73), respectively, we get

$$\begin{aligned}&|T_2(f,x)+T_1(f,x)-\frac{1}{\alpha\Gamma(\alpha)}[(x-a)^\alpha+(b-x)^\alpha]f(x)|\\&\le\frac{(B(\frac{1}{\alpha},q+1))^{1/q}}{\alpha^{1+(1/q)}\Gamma(\alpha)}\left\{(x-a)^{\alpha+(1/q)}+(b-x)^{\alpha+(1/q)}\right\}\|f^{'}\|_p,\end{aligned}\tag{74}$$

and

$$\begin{aligned}&|T_1(f,x)+T_2(f,x)-\frac{1}{\alpha\Gamma(\alpha)}[(x-a)^\alpha+(b-x)^\alpha]f(x)|\\&\le\frac{1}{\alpha\Gamma(\alpha)}\left\{(x-a)^\alpha+(b-x)^\alpha\right\}\|f^{'}\|_1.\end{aligned}\tag{75}$$

Taking $x = (a + b)/2$ in (72), (73), (74), and (75), respectively, we get

$$\begin{aligned}&\left|T_3\left(f,\frac{a+b}{2}\right)+T_4\left(f,\frac{a+b}{2}\right)-\frac{2^{1-(\alpha/k)}(b-a)^{(\alpha/k)}}{\alpha\Gamma_k(\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{2^{(1/p)-(\alpha/k)}(kB(\frac{k}{\alpha},q+1))^{1/q}(b-a)^{(\alpha/k)+(1/q)}}{\alpha^{1+(1/q)}\Gamma_k(\alpha)}\|f^{'}\|_p,\end{aligned}\tag{76}$$

$$\begin{aligned}&\left|T_4\left(f,\frac{a+b}{2}\right)+T_3\left(f,\frac{a+b}{2}\right)-\frac{2^{1-(\alpha/k)}(b-a)^{(\alpha/k)}}{\alpha\Gamma_k(\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{2^{1-(\alpha/k)}(b-a)^{\alpha/k}}{\alpha\Gamma_k(\alpha)}\|f^{'}\|_1,\end{aligned}\tag{77}$$

$$\begin{aligned}&\left|T_2\left(f,\frac{a+b}{2}\right)+T_1\left(f,\frac{a+b}{2}\right)-\frac{2^{1-\alpha}(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{2^{(1/p)-(\alpha)}\left(B\left(\frac{1}{\alpha},q+1\right)\right)^{1/q}}{\alpha^{1+(1/q)}\Gamma(\alpha)}(b-a)^{\alpha+(1/q)}\|f^{'}\|_p,\end{aligned}\tag{78}$$

and

$$\begin{aligned}&\left|T_1\left(f,\frac{a+b}{2}\right)+T_2\left(f,\frac{a+b}{2}\right)-\frac{2^{1-\alpha}}{\alpha\Gamma(\alpha)}(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{2^{1-\alpha}}{\alpha\Gamma(\alpha)}(b-a)^{\alpha}\|f^{'}\|_1.\end{aligned}\tag{79}$$

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New Generalized Convexity and Their Applications

Jichang Kuang

Abstract In this chapter, we introduce some very general new notions of Kg strongly convex functional in normed linear spaces. As their applications, new generalized Ostrowski type and perturbed Simpson type inequalities are established. We apply these inequalities to provide approximations for the integral of a real valued function.

Mathematics Subject Classification 26D15, 26A51

1 Introduction

A function $f : [a, b] \to \mathbb{R}$ is called convex (in the classical sense), if

$$f(\lambda x_1 + (1 - \lambda)x_2)) \leq \lambda f(x_1) + (1 - \lambda) f(x_2), \tag{1}$$

for all $x_1, x_2 \in [a, b], \lambda \in [0, 1]$. This classical inequality (1) plays an important role in analysis, optimization and in the theory of inequalities, and it has a huge literature dealing with its applications, various generalization, and refinements. Further, the convexity is one of the most fundamental and important notions in mathematics. The convexity has wide applications in many branches of pure and applied mathematics, many inequalities can be derived via the convexity theory. The convexity theory and its inequalities are fields of interest of numerous mathematicians and there are many paper, books, and monographs devoted to these fields and various applications (see e.g.[1–8]). In 2018, Awan, M.U. et al. introduced the new notion of exponentially convex function:

Definition 1 ([9]) A function $f : [a, b] \to \mathbb{R}$ is called exponentially convex if

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$$f(tx_1 + (1-t)x_2) \le t\frac{f(x_1)}{e^{rx_1}} + (1-t)\frac{f(x_2)}{e^{rx_2}}, \tag{2}$$

for $\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$ and $r \in \mathbb{R}$.

In particular, if $r = 0$, then (2) reduces to convex function (1). Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{3}$$

is known in the literature as the Hermite-Hadamard inequality (see, for instance, [1] and [10]). In fact, the inequality (3) holds if and only if f is a convex function. The Hermite-Hadamard inequality provides approximations for integral mean of a real valued function f. The concept of convex function was extended in many directions and frameworks due to its numerous applications in optimization, variational methods, geometry, and artificial intelligence. Hence, the inequality (3) has also been extended and generalized for different classes of generalized convex functions (see [1, 7, 8, 11, 12] and the references therein). In 2019, Mehreen and Anwar [10] extended the above Definition 1 by introducing the new notions of exponentially p-convex function and exponentially s-convex function in the second sense, respectively. In fact, they can be generalized uniformly as follows:

Definition 2 Let $[a, b] \subset (0, \infty)$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially (α, s)-convex if

$$f((tx_1^\alpha + (1-t)x_2^\alpha)^{1/\alpha}) \le t^s\frac{f(x_1)}{e^{rx_1}} + (1-t)^s\frac{f(x_2)}{e^{rx_2}}, \tag{4}$$

for $\forall x_1, x_2 \in [a, b]$,$\forall t \in [0, 1]$, $s \in (0, 1]$,$\alpha \ne 0$ and $r \in \mathbb{R}$.

In particular, if $s = 1$, then (4) reduces to exponentially α-convex function in [10]; if $s = 1, r = 0$, then (4) reduces to α-convex function in [10]; if $\alpha = 1$, then (4) reduces to exponentially s-convex function in [10]; if $r = 0$, $\alpha = 1$, then (4) reduces to s-convex function in [13].

Definition 3 Let $[a, b] \subset \mathbb{R} - \{0\}$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially harmonically s-convex, if

$$f\left(\frac{x_1x_2}{tx_2 + (1-t)x_1}\right) \le t^s\frac{f(x_1)}{e^{rx_1}} + (1-t)^s\frac{f(x_2)}{e^{rx_2}}, \tag{5}$$

for $\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$, $s \in (0, 1]$ and $r \in \mathbb{R}$.

If $s = 1$,$r = 0$, then (5) reduces to harmonically convex function in [14]. In 1966, Polyak [14] introduced the notion of strongly convex functions:

Definition 4 ([14]) A function $f : [a, b] \to \mathbb{R}$ is called strongly convex with modulus c if

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2, \tag{6}$$

$\forall x_1, x_2 \in [a, b], \forall t \in [0, 1], c > 0.$

Strongly convex functions have properties useful in optimization, mathematical economics, and other branches of pure and applied mathematics. Many properties and applications of them can be found in the literature (see, for instance, [2, 6, 7, 15], and the references therein). In 2016, Adamek [15] generalized (6) to the following

Definition 5 ([15]) A function $f : [a, b] \to \mathbb{R}$ is called h-strongly convex if

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) - t(1-t)h(x_1 - x_2), \tag{7}$$

$\forall x_1, x_2 \in [a, b], \forall t \in [0, 1]$, and $h : [a, b] \to [0, \infty)$.

In particular, if $h(x_1 - x_2) = c(x_1 - x_2)^2, c > 0$, then (7) reduces to (6). In what follows, $(X, \|\cdot\|)$ denotes the real normed linear spaces, D be a convex subset of X, $h : (0, 1) \to (0, \infty)$ is a given function and c be a positive constant. In 2019, the author [5] introduced a new class of generalized convex functionals, that is,

Definition 6 A functional $f : D \to (0, \infty)$ is called $(\alpha, \beta, \lambda, \lambda_0, t, \xi, h)$ convex if

$$f^{\beta}((\lambda\|x_1\|^{\alpha} + \lambda_0(1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) \le h(t^{\xi})f^{\beta}(\|x_1\|) + \lambda_0 h(1-t^{\xi})f^{\beta}(\|x_2\|), \tag{8}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0, 1]$, α, β are real numbers, and $\alpha, \beta \neq 0$.

In particular, if $D = (0, \infty)$, $\lambda_0 = 1$ in (8), that is, if a function $f : (0, \infty) \to (0, \infty)$ satisfies

$$f^{\beta}((\lambda x_1^{\alpha} + (1-\lambda)x_2^{\alpha})^{1/\alpha}) \le h(t^{\xi})f^{\beta}(x_1) + h(1-t^{\xi})f^{\beta}(x_2), \tag{9}$$

$\forall x_1, x_2 \in (0, \infty), \forall \lambda \in [0, 1]$, α, β are real numbers, and $\alpha, \beta \neq 0$, then f is said to be a $(\alpha, \beta, \lambda, t, \xi, h)$ convex function.

Definition 7 ([1]) Let M be the family of all mean of two positive numbers a, b. Given $M_1, M_2 \in M$. A function $f : (0, \infty) \to (0, \infty)$ is called (h_1, h_2, M_1, M_2)-convex, if

$$\begin{aligned} &f(tM_1(a, b) + (1-t)M_2(a, b)) \\ &\quad \le h_1(t)M_2(f(a), f(b)) + h_2(1-t)M_1(f(a), f(b)). \end{aligned} \tag{10}$$

If $h_1(t) = h_2(t) = t$ in (10), that is,

$$f(tM_1(a, b) + (1-t)M_2(a, b))$$

$$\leq tM_2(f(a), f(b)) + (1-t)M_1(f(a), f(b)), \tag{11}$$

then f is said to be a (t, M_1, M_2)-convex function. If $t = 1$ in (11), that is,

$$f(M_1(a, b)) \leq M_2(f(a), f(b)), \tag{12}$$

then f is said to be a (M_1, M_2)-convex function.

Theorem 1 ([1, 2]) *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, then for all $x \in [a, b]$,*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x)\right| \leq \left[\frac{1}{4} + \left(\frac{x-(a+b)/2}{b-a}\right)^2\right](b-a)\|f'\|_\infty. \tag{13}$$

The constant $\frac{1}{4}$ is the best possible.

This is well-known as Ostrowski inequality. Many authors have made generalizations to inequality (13). For more results and details, see [1–3, 16, 17] and the references therein. In [18], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 2 *Let $f \in BV[a, b]$, then for all $x \in [a, b]$*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f(x)\right| \leq \left[\frac{1}{2} + \left|\frac{x-(a+b)/2}{b-a}\right|\right] V_a^b(f). \tag{14}$$

The constant $1/2$ is the best possible.

In [19], Lerone et al. established the following generalized trapezoid inequalities for functions of bounded variation:

Theorem 3 *Let $f \in BV[a, b]$, then for all $x \in [a, b]$*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right|$$
$$\leq \left[\frac{1}{2} + \left|\frac{x-(a+b)/2}{b-a}\right|\right] V_a^b(f). \tag{15}$$

The constant $1/2$ is the best possible.

Theorem 4 ([20]) *Let $f \in BV[a, b]$, then for all $x \in [a, (a+b)/2]$,*

$$\left|\int_a^b f(u)du - (x-a)[f(a)+f(b)] - (a+b-2x)f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right] V_a^b(f). \tag{16}$$

The constant $1/4$ *is the best possible.*

Taking $x = (3a+b)/4$ in (16), we get

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{1}{4}V_a^b(f).$$

Taking $x = a$ in (16), we get the midpoint inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2}V_a^b(f).$$

Taking $x = (a+b)/2$ in (16), we get the trapezoid inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(a)+f(b)]\right| \leq \frac{1}{2}V_a^b(f).$$

Theorem 5 ([8]) *Let* $f^{''} \in BV[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}f^{''}\left(\frac{a+b}{2}\right)\right.$$
$$\left. - \frac{b-a}{96}\left[f^{''}(a)+f^{''}(b)\right]\right| \leq \frac{(b-a)^2}{96}V_a^b(f^{''}). \tag{17}$$

Theorem 6 ([21]) *Let* $f^{''} \in BV[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(a)+f(b)] + \frac{(b-a)^2}{8}\left[f^{'}(b)-f^{'}(a)\right]\right.$$
$$\left. - \frac{(b-a)^2}{48}\left[f^{''}(a)+f^{''}(b)\right]\right| \leq \frac{(b-a)^2}{48}V_a^b(f^{''}).$$

In this chapter, we generalized (8) to exponentially strongly convex functional, and introduce some very general new notions of Kg strongly convex functional in normed linear spaces in Sect. 2. It unifies and generalizes the many known and new classes of convex functions. As their applications, new generalized Ostrowski type inequalities are established in Sect. 3. We apply these inequalities to provide approximations for the integral of a real valued function. In Sect. 4, we give proofs of Theorem 7 and 8. In Sect. 5, perturbed Simpson type inequalities and some new approximations for the integral of a real valued function are also given.

2 Generalized Convexity in Normed Linear Spaces

In this section, we generalized (8) to exponentially strongly convex functional and introduce some very general new notions of Kg strongly convex functional in normed linear spaces.

Definition 8 Let $(X, \|\cdot\|)$ denotes the real normed linear spaces, D be a convex subset of X,$g, h_1, h_2 : (0, 1) \to (0, \infty)$ and $h : (0, \infty) \to (0, \infty)$ be given functions. A function $f : D \to [0, \infty)$ is called exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, \xi, t, h_1, g, h)$-strongly convex if

$$f\left(\left(\lambda\|x_1\|^\alpha + \lambda_1(1-\lambda)\|x_2\|^\alpha\right)^{1/\alpha}\right) \le \left\{h_1(t^\xi)\left(\frac{f(\|x_1\|)}{e^{r\|x_1\|}}\right)^\beta + \lambda_2 h_1(1-t^\xi)\left(\frac{f(\|x_2\|)}{e^{r\|x_2\|}}\right)^\beta\right\}^{1/\beta} - g(t)h(\|x_1-x_2\|), \tag{18}$$

where $x_1, x_2 \in D, \lambda, \lambda_1, \lambda_2, \xi, t \in [0, 1], r \in \mathbb{R}$,$\alpha, \beta$ are real numbers, and $\alpha, \beta \neq 0$.

In particular, if $g(t) = \lambda_0 h_2(t)h_2(1-t)$,$\lambda_0 \in [0, 1]$, that is,

$$f\left(\left(\lambda\|x_1\|^\alpha + \lambda_1(1-\lambda)\|x_2\|^\alpha\right)^{1/\alpha}\right) \le \left\{h_1(t^\xi)(\frac{f(\|x_1\|)}{e^{r\|x_1\|}})^\beta + \lambda_2 h_1(1-t^\xi)\left(\frac{f(\|x_2\|)}{e^{r\|x_2\|}}\right)^\beta\right\}^{1/\beta} - \lambda_0 h_2(t)h_2(1-t)h(\|x_1-x_2\|), \tag{19}$$

we say that f is an exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, \lambda_0, \xi, t, h_1, h_2, h)$-strongly convex functional. If $h_2(t) = t, \lambda_0 = 1$, then (19) reduces to exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, \xi, t, h_1, h)$-strongly convex functional in [6]. If $D \subset (0, \infty)$, $h_2(t) = t, \lambda_0 = 1$ in (19), that is, if a function $f : D \to (0, \infty)$ satisfies

$$f((\lambda x_1^\alpha + \lambda_1(1-\lambda)x_2^\alpha)^{1/\alpha}) \le \left\{h_1(t^\xi)\left(\frac{f(x_1)}{e^{rx_1}}\right)^\beta + \lambda_2 h_1(1-t^\xi)\left(\frac{f(x_2)}{e^{rx_2}}\right)^\beta\right\}^{1/\beta} - t(1-t)h(|x_1-x_2|), \tag{20}$$

where $x_1, x_2 \in D, \lambda, \lambda_1, \lambda_2, \xi, t \in [0, 1], r \in \mathbb{R}$,$\alpha, \beta$ are real numbers, and $\alpha, \beta \neq 0$, we say that f is an exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, \xi, t, h_1, h)$-strongly convex function. If $\xi = 1$ in (20), that is,

$$f\left(\left(\lambda x_1^\alpha+\lambda_1(1-\lambda)x_2^\alpha\right)^{1/\alpha}\right)\le\left\{h_1(t)\left(\frac{f(x_1)}{e^{rx_1}}\right)^\beta\right.$$

$$\left.+\lambda_2 h_1(1-t)\left(\frac{f(x_2)}{e^{rx_2}}\right)^\beta\right\}^{1/\beta}-t(1-t)h(|x_1-x_2|),\tag{21}$$

we say that f is an exponentially $(\alpha,\beta,\lambda,\lambda_1,\lambda_2,t,h_1,h)$-strongly convex function. If $\lambda_1=\lambda_2=1$, $\lambda=t$, $h_1(t)=t^s$, $0<|s|\le 1$ in (21), that is,

$$f\left(\left(tx_1^\alpha+(1-t)x_2^\alpha\right)^{1/\alpha}\right)\le\left\{t^s\left(\frac{f(x_1)}{e^{rx_1}}\right)^\beta\right.$$

$$\left.+(1-t)^s\left(\frac{f(x_2)}{e^{rx_2}}\right)^\beta\right\}^{1/\beta}-t(1-t)h(|x_1-x_2|),\tag{22}$$

we say that f is an exponentially (α,β,s,h)-strongly convex function. If $\alpha=1$ in (22), then f is an exponentially (β,s,h)-strongly convex function. If $\alpha=\beta=1$ in (22), then f is an exponentially (s,h)-strongly convex function. If $\alpha=\beta=s=1$ in (22), then f is an exponentially h-strongly convex function. If $\lambda_1=\lambda_2=1$, $\lambda=t$, $r=0$ in (20), that is,

$$f\left(\left(tx_1^\alpha+(1-t)x_2^\alpha\right)^{1/\alpha}\right)\le\left\{h_1(t^\xi)(f(x_1))^\beta\right.$$

$$\left.+h_1(1-t^\xi)(f(x_2))^\beta\right\}^{1/\beta}-t(1-t)h(|x_1-x_2|),\tag{23}$$

we say that f is a (α,β,ξ,h,h_1)-strongly convex function. If $\alpha=\beta=\xi=1$,$h_1(t)=t$, then (23) reduces to (7).

Definition 9 Let $D\subset X$, $g:[0,1]\to D$,$h:[0,\infty)\to(0,\infty)$ and $h_1,h_2:(0,1)\to(0,\infty)$ be given function. $K:(0,\infty)\times(0,\infty)\to(0,\infty)$. A functional $f:D\to\mathbb{R}$ is called Kg-strongly convex, if

$$f(g(\lambda))\le K(f(g(0)),f(g(1)))-\lambda_0h_2(t)h_2(1-t)h(\|g(0)-g(1)\|),\tag{24}$$

where $\lambda,\lambda_0,t\in[0,1]$.

If

$$f(g(\lambda))\ge K(f(g(0)),f(g(1)))+\lambda_0h_2(t)h_2(1-t)h(\|g(0)-g(1)\|),$$

we say that f is a Kg-strongly concave function. In particular, let $g(\lambda)=(\lambda\|x_2\|^\alpha+(1-\lambda)\|x_1\|^\alpha)^{1/\alpha}$, $\lambda,\xi,t\in[0,1]$,$r\in\mathbb{R}$,

$$K(x,y)=\{h_1(t^\xi)(e^{-r\|x_1\|}x)^\beta+h_1(1-t^\xi)(e^{-r\|x_2\|y})^\beta\}^{1/\beta},$$

then (24) reduces to

$$f\left(\left(\lambda\|x_2\|^\alpha+(1-\lambda)\|x_1\|^\alpha\right)^{1/\alpha}\right)\le\left\{h_1(t^\xi)\left(\frac{f(\|x_1\|)}{e^{r\|x_1\|}}\right)^\beta\right.$$

$$\left.+h_1(1-t^\xi)\left(\frac{f(\|x_2\|)}{e^{r\|x_2\|}}\right)^\beta\right\}^{1/\beta}-\lambda_0 h_2(t)h_2(1-t)h(|\|x_1\|-\|x_2\||).$$

If h is a decreasing function, then $h(|\|x_1\|-\|x_2\||)\ge h(\|x_1-x_2\|)$, thus (24) reduces to (19). If $g(\lambda)=M_1(q^\lambda\|x_1\|,q^{1-\lambda}\|x_2\|), 0<q,\lambda<1, K(x,y)=M_2(x,y)$, then (24) reduces to

$$\begin{aligned}&f(M_1(q^\lambda\|x_1\|,q^{1-\lambda}\|x_2\|))\le M_2(f(M_1(\|x_1\|,q\|x_2\|)),f(M_1(q\|x_1\|,\|x_2\|)))\\&\quad-\lambda_0 h_2(t)h_2(1-t)h(|M_1(\|x_1\|,q\|x_2\|)-M_1(q\|x_1\|,\|x_2\|)|).\end{aligned}\tag{25}$$

If $g(t)=tM_1(\|x_1\|,\|x_2\|)+(1-t)M_2(\|x_1\|,\|x_2\|), K(x,y)=h_1(t)x+h_1(1-t)y$, then (24) reduces to

$$\begin{aligned}&f(tM_1(\|x_1\|,\|x_2\|)+(1-t)M_2(\|x_1\|,\|x_2\|))\\&\quad\le h_1(t)f(M_2(\|x_1\|,\|x_2\|))+h_1(1-t)f(M_1(\|x_1\|,\|x_2\|))\\&\qquad-\lambda_0 h_2(t)h_2(1-t)h(|M_2(\|x_1\|,\|x_2\|)-M_1(\|x_1\|,\|x_2\|)|).\end{aligned}\tag{26}$$

In what follows, let $X=[0,\infty)$,$[a,b]\subset X$, then (24) reduces to

$$f(g(\lambda))\le K(f(g(0)),f(g(1)))-\lambda_0 h_2(t)h_2(1-t)h(|g(0)-g(1)|).\tag{27}$$

If $g(t)=((1-t)a^\alpha+tb^\alpha)^{1/\alpha}$, $K(x,y)=\{h_1(t^\xi)(e^{-ra}x)^\beta+h_1(1-t^\xi)(e^{-rb}y)^\beta\}^{1/\beta}$, then (27) reduces to

$$\begin{aligned}&f\left(\left((1-t)a^\alpha+tb^\alpha\right)^{1/\alpha}\right)\le\left\{h_1(t^\xi)(e^{-ra}f(a))^\beta+h_1(1-t^\xi)(e^{-rb}f(b))^\beta\right\}^{1/\beta}\\&\quad-\lambda_0 h_2(t)h_2(1-t)h(b-a).\end{aligned}\tag{28}$$

If $\lambda_0=1,\lambda_1=\lambda_2=1$,$h_2(t)=t$, then (28) reduces to (20). If $g(t)=ta+(1-t)b, K(x,y)=ty+(1-t)x,\lambda_0=0$, then (27) reduces to (1). If $g(t)=(1-t)a+tb, K(x,y)=x^{1-t}y^t$, then (27) reduces to AG-strongly convex function:

$$f((1-t)a+tb)\le\{f(a)\}^{1-t}\{f(b)\}^t-\lambda_0 h_2(t)h_2(1-t)h(b-a).$$

If $g(t)=a^{1-t}b^t, K(x,y)=tx+(1-t)y$, then (27) reduces to GA-strongly convex function:

$$f(a^{1-t}b^t) \leq tf(a) + (1-t)f(b) - \lambda_0 h_2(t)h_2(1-t)h(b-a).$$

If $g(t) = a^{1-t}b^t$, $K(x, y) = x^{1-t}y^t$, then (27) reduces to GG-strongly convex function:

$$f(a^{1-t}b^t) \leq \{f(a)\}^{1-t} f(b)^t - \lambda_0 h_2(t)h_2(1-t)h(b-a).$$

In particular, if $\lambda_0 = 0$, then the above inequality reduces to GG-convex function in [22]. If $g(t) = ((1-t)a^{-1} + tb^{-1})^{-1}$, $K(x, y) = ty + (1-t)x$, then (27) reduces to HA-strongly convex function:

$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq tf(b) + (1-t)f(a) - \lambda_0 h_2(t)h_2(1-t)h(b-a).$$

If $g(t) = ta + (1-t)b$, $K(x, y) = \max\{x, y\}$, then (27) reduces to strongly quasi-convex function:

$$f(ta + (1-t)b) \leq \max\{f(a), f(b)\} - \lambda_0 h_2(t)h_2(1-t)h(b-a).$$

If $g(t) = (1-t)a + tb$, $K(x, y) = (t^\alpha x^{-1} + (1-t)^\alpha y^{-1})^{-1}$, then (27) reduces to $\alpha - AH$-strongly convex function:

$$f((1-t)a + tb) \leq \frac{f(a)f(b)}{t^\alpha f(b) + (1-t)^\alpha f(a)} - \lambda_0 h_2(t)h_2(1-t)h(b-a).$$

If $g(t) = ((1-t)a^{-1} + tb^{-1})^{-1}$, $K(x, y) = (t^\alpha x^{-1} + (1-t)^\alpha y^{-1})^{-1}$, then (27) reduces to $\alpha - HH$ strongly convex function:

$$f\left(\left((1-t)a^{-1} + tb^{-1}\right)^{-1}\right) \leq \left\{t^\alpha (f(a))^{-1} + (1-t)^\alpha (f(b))^{-1}\right\}^{-1}$$
$$-\lambda_0 h_2(t)h_2(1-t)h(b-a).$$

In particular, if $\lambda_0 = 1$, then the above inequality reduces to $\alpha - HH$ convex function in [23]. If $g(t) = ((1-t)a^\alpha + tb^\alpha)^{1/\alpha}$, $K(x, y) = \{h_1(t^\xi)x^\beta + (1 - h_1(t^\xi))y^\beta\}^{1/\beta}$, then (27) reduces to

$$f\left(\left((1-t)a^\alpha + tb^\alpha\right)^{1/\alpha}\right) \leq \left\{h_1(t^\xi)(f(a))^\beta + (1 - h_1(t^\xi))(f(b))^\beta\right\}^{1/\beta}$$
$$-\lambda_0 h_2(t)h_2(1-t)h(b-a). \tag{29}$$

If $\alpha = \beta = 1, \lambda_0 = 0, \xi = 1$, then (29) reduces to modified h_1-convex function [11]:

$$f((1-t)a + tb) \leq h_1(t)f(a) + (1 - h_1(t))f(b).$$

Hence, Definitions 8 and 9 are very general notions of convex functions. They unified and generalized many known and new classes of convex functions.

3 Generalized Ostrowski Type Inequalities

In what follows, let

$$S_n(f, x) = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^k(x); \tag{30}$$

$AC[a, b]$ denotes the class of absolutely continuous functions on $[a, b]$, and

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt, \ \ \alpha, \beta > 0$$

is the Beta function.

Theorem 7 *Let* $[a, b] \subset (0, \infty)$, $f : [a, b] \to (0, \infty)$ *be a differentiable mapping such that* $f^{(n-1)} \in AC[a, b]$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *and for* $p = 1$, *define* $q = \infty$, $\frac{1}{\infty} = 0$. *If* $|f^{(n)}|^p$ *is exponentially* (β, s, h)*-strongly convex on* $[a, b]$, *thus, we get an approximation error estimate:*

$$\begin{aligned} &\left| \int_a^b f(u)du - S_n(f, x) \right| \\ &\quad \le \frac{(x-a)^{n+1}}{n!(nq+1)^{1/q}} \times \left\{ \frac{\beta C_\beta}{s+\beta} \left[\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}} \right] - \frac{1}{6} h(x-a) \right\}^{1/p} \\ &\qquad + \frac{(b-x)^{n+1}}{n!(nq+1)^{1/q}} \times \left\{ \frac{\beta C_\beta}{s+\beta} \left[\frac{|f^{(n)}(b)|^p}{e^{rb}} + \frac{|f^{(n)}(x)|^p}{e^{rx}} \right] - \frac{1}{6} h(b-x) \right\}^{1/p}, \end{aligned} \tag{31}$$

where

$$C_\beta = \begin{cases} 1, & \beta \ge 1, \\ 2^{(1/\beta)-1}, & 0 < \beta < 1. \end{cases} \tag{32}$$

Taking $\beta = 1$ in (31), we get

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!(nq+1)^{1/q}} \times \left\{\frac{1}{s+1}\left[\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right] - \frac{1}{6}h(x-a)\right\}^{1/p}$$
$$+ \frac{(b-x)^{n+1}}{n!(nq+1)^{1/q}} \times \left\{\frac{1}{s+1}\left[\frac{|f^{(n)}(b)|^p}{e^{rb}} + \frac{|f^{(n)}(x)|^p}{e^{rx}}\right] - \frac{1}{6}h(b-x)\right\}^{1/p}. \tag{33}$$

If $p = 1$ in (33), then

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!} \times \left\{\frac{1}{s+1}\left[\frac{|f^{(n)}(x)|}{e^{rx}} + \frac{|f^{(n)}(a)|}{e^{ra}}\right] - \frac{1}{6}h(x-a)\right\}$$
$$+ \frac{(b-x)^{n+1}}{n!} \times \left\{\frac{1}{s+1}\left[\frac{|f^{(n)}(b)|}{e^{rb}} + \frac{|f^{(n)}(x)|}{e^{rx}}\right] - \frac{1}{6}h(b-x)\right\}. \tag{34}$$

Let

$$M_{k,p} = \sup\left\{\frac{|f^{(k)}(x)|^p}{e^{rx}} : x \in [a,b]\right\}. \tag{35}$$

Taking $h = 0$ in (33), (34), respectively, we get

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\leq \frac{(2M_{n,p})^{1/p}}{n!(nq+1)^{1/q}(s+1)^{1/p}}\left[(x-a)^{n+1} + (b-x)^{n+1}\right]; \tag{36}$$

$$\left|\int_a^b f(u)du - S_n(f,x)\right| \leq \frac{2M_{n,1}}{n!(s+1)}\left[(x-a)^{n+1} + (b-x)^{n+1}\right]. \tag{37}$$

Taking $x = (a+b)/2$ in (33), (34), (36), and (37), we get some new midpoint type inequalities:

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right| \leq \frac{(b-a)^{n+1}}{2^{n+1}n!(nq+1)^{1/q}}$$

$$\times\left\{\left[\frac{1}{s+1}\left(\frac{|f^{(n)}((a+b)/2)|^p}{e^{r(a+b)/2}}+\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right.$$

$$\left.+\left[\frac{1}{s+1}\left(\frac{|f^{(n)}(b)|^p}{e^{rb}}+\frac{|f^{(n)}((a+b)/2)|^p}{e^{r(a+b)/2}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right\};\quad(38)$$

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right|$$
$$\le \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\frac{1}{s+1}\left[\frac{2|f^{(n)}((a+b)/2)|}{e^{r(a+b)/2}}+\frac{|f^{(n)}(a)|}{e^{ra}}+\frac{|f^{(n)}(b)|}{e^{rb}}\right]-\frac{1}{3}h\left(\frac{b-a}{2}\right)\right\};\quad(39)$$

$$\left|\int_a^b f(u)du - S_n\left(f,\frac{a+b}{2}\right)\right| \le \frac{(2M_{n,p})^{1/p}(b-a)^{n+1}}{2^n n!(nq+1)^{1/q}(s+1)^{1/p}};\quad(40)$$

$$\left|\int_a^b f(u)du - S_n\left(f,\frac{a+b}{2}\right)\right| \le \frac{M_{n,1}(b-a)^{n+1}}{2^{n-1}n!(s+1)}.\quad(41)$$

Taking $n=1$ in (33), (34), (36), (37), (38), (39), (40) and (41), respectively, we get some new versions of Ostrowski type and midpoint type inequalities:

$$\left|\int_a^b f(u)du-(b-a)f(x)\right|$$
$$\le \frac{(x-a)^2}{(q+1)^{1/q}}\times\left\{\frac{1}{s+1}\left[\frac{|f'(x)|^p}{e^{rx}}+\frac{|f'(a)|^p}{e^{ra}}\right]-\frac{1}{6}h(x-a)\right\}^{1/p}$$
$$+\frac{(b-x)^2}{(q+1)^{1/q}}\times\left\{\frac{1}{s+1}\left[\frac{|f'(b)|^p}{e^{rb}}+\frac{|f'(x)|^p}{e^{rx}}\right]-\frac{1}{6}h(b-x)\right\}^{1/p};\quad(42)$$

$$\left|\int_a^b f(u)du-(b-a)f(x)\right|$$
$$\le (x-a)^2\times\left\{\frac{1}{s+1}\left[\frac{|f'(x)|}{e^{rx}}+\frac{|f'(a)|}{e^{ra}}\right]-\frac{1}{6}h(x-a)\right\}$$

$$+(b-x)^2 \times \left\{ \frac{1}{s+1} \left[\frac{|f^{'}(b)|}{e^{rb}} + \frac{|f^{'}(x)|}{e^{rx}} \right] - \frac{1}{6} h(b-x) \right\}; \tag{43}$$

$$\left| \int_a^b f(u)du - (b-a)f(x) \right| \\ \leq \frac{(2M_{1,p})^{1/p}}{(q+1)^{1/q}(s+1)^{1/p}} \left[(x-a)^2 + (b-x)^2 \right]; \tag{44}$$

$$\left| \int_a^b f(u)du - (b-a)f(x) \right| \\ \leq \frac{2M_{1,1}}{s+1} [(x-a)^2 + (b-x)^2]; \tag{45}$$

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{4(q+1)^{1/q}} \\ \times \left\{ \left[\frac{1}{s+1} \left(\frac{|f^{'}((a+b)/2)|^p}{e^{r(a+b)/2}} + \frac{|f^{'}(a)|^p}{e^{ra}} \right) - \frac{1}{6} h\left(\frac{b-a}{2}\right) \right]^{1/p} \right. \\ \left. + \left[\frac{1}{s+1} \left(\frac{|f^{'}(b)|^p}{e^{rb}} + \frac{|f^{'}((a+b)/2)|^p}{e^{r(a+b)/2}} \right) - \frac{1}{6} h\left(\frac{b-a}{2}\right) \right]^{1/p} \right\}; \tag{46}$$

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{4} \left\{ \frac{1}{s+1} \left[\frac{2|f^{'}((a+b)/2)|}{e^{r(a+b)/2}} \right. \right. \\ \left. \left. + \frac{|f^{'}(a)|}{e^{ra}} + \frac{|f^{'}(b)|}{e^{rb}} \right] - \frac{1}{3} h\left(\frac{b-a}{2}\right) \right\}; \tag{47}$$

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f(\frac{a+b}{2}) \right| \leq \frac{(M_{1,p})^{1/p}(b-a)}{2^{1/q}(q+1)^{1/q}(s+1)^{1/p}}; \tag{48}$$

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f(\frac{a+b}{2}) \right| \leq \frac{M_{1,1}(b-a)}{s+1}. \tag{49}$$

If $r=0$, $h=0$, that is, $|f'|^p$ is s-convex on $[a,b]$, then

$$\left|\int_a^b f(u)du-(b-a)f(x)\right| \le \frac{(x-a)^2}{(q+1)^{1/q}(s+1)^{1/p}}(|f'(x)|^p+|f'(a)|^p)^{1/p} + \frac{(b-x)^2}{(q+1)^{1/q}(s+1)^{1/p}}(|f'(b)|^p+|f'(x)|^p)^{1/p};$$

$$\left|\int_a^b f(u)du-(b-a)f(x)\right| \le \frac{(x-a)^2}{s+1}(|f'(x)|+|f'(a)|)+\frac{(b-x)^2}{s+1}(|f'(b)|+|f'(x)|);$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4(q+1)^{1/q}(s+1)^{1/p}} \times\left\{(|f'((a+b)/2)|^p+|f'(a)|^p)^{1/p}+(|f'(b)|^p+|f'((a+b)/2)|^p)^{1/p}\right\};$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4(s+1)}\left\{|f'(a)|+|f'(b)|+2|f'((a+b)/2)|\right\}.$$

Taking $n=2$ in (38), (39), (40), and (41), respectively, and note that

$$\left|\int_a^b f(u)du-S_2(f,(a+b)/2)\right|=\left|\int_a^b f(u)du-(b-a)f((a+b)/2)\right|,$$

we get also some new versions of midpoint type inequalities:

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{16(2q+1)^{1/q}} \times\left\{\left[\frac{1}{s+1}\left(\frac{|f''((a+b)/2)|^p}{e^{r(a+b)/2}}+\frac{|f''(a)|^p}{e^{ra}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p} + \left[\frac{1}{s+1}\left(\frac{|f''(b)|^p}{e^{rb}}+\frac{|f''((a+b)/2)|^p}{e^{r(a+b)/2}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right\} \quad (50)$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{16}\left\{\frac{1}{s+1}\left[\frac{2|f^{''}((a+b)/2)|}{e^{r(a+b)/2}}\right.\right.$$
$$\left.\left.+\frac{|f^{''}(a)|}{e^{ra}}+\frac{|f^{''}(b)|}{e^{rb}}\right]-\frac{1}{3}h\left(\frac{b-a}{2}\right)\right\}; \tag{51}$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \le \frac{(2M_{2,p})^{1/p}(b-a)^2}{8(2q+1)^{1/q}(s+1)^{1/p}}; \tag{52}$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \le \frac{M_{2,1}(b-a)^2}{4(s+1)}. \tag{53}$$

If $r=0$, $h=0$, that is,$|f^{''}|^p$ is s-convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{16(2q+1)^{1/q}(s+1)^{1/p}}$$
$$\times\left\{(|f^{''}((a+b)/2)|^p+|f^{''}(a)|^p)^{1/p}+(|f^{''}(b)|^p+|f^{''}((a+b)/2)|^p)^{1/p}\right\};$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{16(s+1)}\left\{|f^{''}(a)|+|f^{''}(b)|+2|f^{''}((a+b)/2)|\right\}.$$

Theorem 8 *Under the assumptions of Theorem 7, we have another approximation error estimate:*

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\le \frac{(x-a)^{n+1}}{n!(n+1)^{1/q}}\left\{C_\beta\left[\frac{\beta}{s+\beta(n+1)}\frac{|f^{(n)}(x)|^p}{e^{rx}}\right.\right.$$
$$\left.\left.+B\left(n+1,\frac{s}{\beta}+1\right)\frac{|f^{(n)}(a)|^p}{e^{ra}}\right]-\frac{1}{(n+2)(n+3)}h(x-a)\right\}^{1/p}$$
$$+\frac{(b-x)^{n+1}}{n!(n+1)^{1/q}}\left\{C_\beta\left[B(n+1,\frac{s}{\beta}+1)\frac{|f^{(n)}(b)|^p}{e^{rb}}\right.\right.$$

$$+\frac{\beta}{s+\beta(n+1)}\frac{|f^{(n)}(x)|^p}{e^{rx}}\Bigg]-\frac{1}{(n+2)(n+3)}h(b-x)\Bigg\}^{1/p}, \tag{54}$$

where C_β is defined by (32).

Taking $\beta = 1$ in (54), we get

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\le \frac{(x-a)^{n+1}}{n!(n+1)^{1/q}}\Bigg\{\frac{1}{s+n+1}\frac{|f^{(n)}(x)|^p}{e^{rx}}$$
$$+B(n+1,s+1)\frac{|f^{(n)}(a)|^p}{e^{ra}}-\frac{1}{(n+2)(n+3)}h(x-a)\Bigg\}^{1/p}$$
$$+\frac{(b-x)^{n+1}}{n!(n+1)^{1/q}}\Bigg\{B(n+1,s+1)\frac{|f^{(n)}(b)|^p}{e^{rb}}$$
$$+\frac{1}{s+n+1}\frac{|f^{(n)}(x)|^p}{e^{rx}}-\frac{1}{(n+2)(n+3)h(b-x)}\Bigg\}^{1/p}, \tag{55}$$

If $p = 1$ in (55), then

$$\left|\int_a^b f(u)du - S_n(f,x)\right| \le \frac{(x-a)^{n+1}}{n!}\Bigg\{\frac{1}{s+n+1}\frac{|f^{(n)}(x)|}{e^{rx}}$$
$$+B(n+1,s+1)\frac{|f^{(n)}(a)|}{e^{ra}}-\frac{1}{(n+2)(n+3)}h(x-a)\Bigg\}$$
$$+\frac{(b-x)^{n+1}}{n!}\Bigg\{B(n+1,s+1)\frac{|f^{(n)}(b)|}{e^{rb}}$$
$$+\frac{1}{s+n+1}\frac{|f^{(n)}(x)|}{e^{rx}}-\frac{1}{(n+2)(n+3)}h(b-x)\Bigg\}. \tag{56}$$

Taking $h = 0$ in (55), (56), respectively, we get

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\le \frac{(M_{n,p})^{1/p}}{n!(n+1)^{1/q}}\left\{B(n+1,s+1)+\frac{1}{s+n+1}\right\}^{1/p}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]; \tag{57}$$

$$\left|\int_a^b f(u)du - S_n(f,x)\right|$$
$$\leq \frac{M_{n,1}}{n!}\left\{B(n+1,s+1)+\frac{1}{s+n+1}\right\}\left[(x-a)^{n+1}+(b-x)^{n+1}\right], \qquad (58)$$

where $M_{k,p}$ is defined by (35). Taking $x = (a+b)/2$ in (55), (56), (57), and (58), respectively, we get some new midpoint type inequalities:

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!(n+1)^{1/q}}\left\{\left[\frac{1}{s+n+1}\frac{|f^{(n)}((a+b)/2)|^p}{e^{r(a+b)/2}}\right.\right.$$
$$\left.+B(n+1,s+1)\frac{|f^{(n)}(a)|^p}{e^{ra}}-\frac{1}{(n+2)(n+3)}h(\frac{b-a}{2})\right]^{1/p}$$
$$+\left[B(n+1,s+1)\frac{|f^{(n)}(b)|^p}{e^{rb}}\right.$$
$$\left.\left.+\frac{1}{s+n+1}\frac{|f^{(n)}((a+b)/2)|^p}{e^{r(a+b)/2}}-\frac{1}{(n+2)(n+3)}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right\}; \qquad (59)$$

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\frac{2}{s+n+1}\frac{|f^{(n)}((a+b)/2)|}{e^{r(a+b)/2}}\right.$$
$$+B(n+1,s+1)\left(\frac{|f^{(n)}(a)|}{e^{ra}}+\frac{|f^{(n)}(b)|}{e^{rb}}\right)$$
$$\left.-\frac{2}{(n+2)(n+3)}h((b-a)/2)\right\}; \qquad (60)$$

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right|$$
$$\leq \frac{(M_{n,p})^{1/p}}{2^n n!(n+1)^{1/q}}\left\{B(n+1,s+1)+\frac{1}{s+n+1}\right\}^{1/p}(b-a)^{n+1}; \quad (61)$$

$$\left|\int_a^b f(u)du - S_n(f,(a+b)/2)\right|$$
$$\leq \frac{M_{n,1}}{2^n n!}\left\{B(n+1,s+1)+\frac{1}{s+n+1}\right\}(b-a)^{n+1}. \qquad (62)$$

Taking $n = 1$ in (55), (56), (57) and (58), respectively, we get some new versions of Ostrowski type inequalities:

$$\begin{aligned}&\left|\int_a^b f(u)du-(b-a)f(x)\right|\\&\le \frac{(x-a)^2}{2^{1/q}}\left\{\frac{1}{s+2}\left[\frac{|f^{'}(x)|^p}{e^{rx}}+\frac{1}{s+1}\frac{|f^{'}(a)|^p}{e^{ra}}\right]-\frac{1}{12}h(x-a)\right\}^{1/p}\\&+\frac{(b-x)^2}{2^{1/q}}\left\{\frac{1}{s+2}\left[\frac{1}{s+1}\frac{|f^{'}(b)|^p}{e^{rb}}+\frac{|f^{'}(x)|^p}{e^{rx}}\right]-\frac{1}{12}h(b-x)\right\}^{1/p}; \qquad (63)\end{aligned}$$

$$\begin{aligned}&\left|\int_a^b f(u)du-(b-a)f(x)\right|\\&\le (x-a)^2\left\{\frac{1}{s+2}\left[\frac{|f^{'}(x)|}{e^{rx}}+\frac{1}{s+1}\frac{|f^{'}(a)|}{e^{ra}}\right]-\frac{1}{12}h(x-a)\right\}\\&+(b-x)^2\left\{\frac{1}{s+2}\left[\frac{1}{s+1}\frac{|f^{'}(b)|}{e^{rb}}+\frac{|f^{'}(x)|}{e^{rx}}\right]-\frac{1}{12}h(b-x)\right\}; \qquad (64)\end{aligned}$$

$$\begin{aligned}&\left|\int_a^b f(u)du-(b-a)f(x)\right|\\&\le \frac{(M_{1,p})^{1/p}}{2^{1/q}(s+2)^{1/p}}\left\{1+\frac{1}{s+1}\right\}^{1/p}\left[(x-a)^2+(b-x)^2\right]; \qquad (65)\end{aligned}$$

$$\left|\int_a^b f(u)du-(b-a)f(x)\right|\le \frac{M_{1,1}}{s+2}\left\{1+\frac{1}{s+1}\right\}\left[(x-a)^2+(b-x)^2\right]. \qquad (66)$$

Taking $n = 2$ in (59), (60), (61), and (62), respectively, we get also some new versions of midpoint type inequalities:

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|\\&\le \frac{(b-a)^2}{16\times 3^{1/q}}\left\{\left[\frac{1}{s+3}\left(\frac{|f^{"}((a+b)/2)|^p}{e^{r(a+b)/2}}\right.\right.\right.\\&\left.\left.+\frac{2}{(s+1)(s+2)}\frac{|f^{"}(a)|^p}{e^{ra}}\right)-\frac{1}{20}h\left(\frac{b-a}{2}\right)\right]^{1/p}\end{aligned}$$

$$+\left[\frac{1}{s+3}\left(\frac{2}{(s+1)(s+2)}\frac{|f''(b)|^p}{e^{rb}}\right.\right.$$

$$\left.\left.+\frac{|f''((a+b)/2)|^p}{e^{r(a+b)/2}}\right)-\frac{1}{20}h\left(\frac{b-a}{2}\right)\right]^{1/p}\Bigg\}; \tag{67}$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)^2}{16}\left\{\frac{2}{s+3}\left[\frac{|f''((a+b)/2)|}{e^{r(a+b)/2}}\right.\right.$$

$$\left.\left.+\frac{1}{(s+1)(s+2)}\left(\frac{|f''(a)|}{e^{ra}}+\frac{|f''(b)|}{e^{rb}}\right)\right]-\frac{1}{10}h\left(\frac{b-a}{2}\right)\right\}; \tag{68}$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\frac{(M_{2,p})^{1/p}}{8\times 3^{1/q}(s+3)^{1/p}}\left\{1+\frac{2}{(s+1)(s+2)}\right\}^{1/p}(b-a)^2; \tag{69}$$

$$\left|\frac{1}{b-a}\int_a^b f(u)du-f\left(\frac{a+b}{2}\right)\right|$$

$$\le\frac{M_{2,1}}{8(s+3)}\left\{1+\frac{2}{(s+1)(s+2)}\right\}(b-a)^2. \tag{70}$$

4 Proofs of Theorems 7 and 8

We require the following Lemma to prove our results.

Lemma 1 ([16, 24]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}\in AC[a,b]$, then for all $x\in[a,b]$, we have*

$$\int_a^b f(u)du-S_n(f,x)=(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)dt, \tag{71}$$

where the kernel$K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!}, & t\in[a,x],\\ \frac{(t-b)^n}{n!}, & t\in[x,b],\end{cases}, \tag{72}$$

and $S_n(f,x)$ is defined by (30).

Proof of Theorem 7 By Lemma 1, we have

$$\left|\int_a^b f(u)du - S_n(f,x)\right| \leq \int_a^x \frac{(u-a)^n}{n!}|f^{(n)}(u)|du + \int_x^b \frac{(b-u)^n}{n!}|f^{(n)}(u)|du = I_1 + I_2. \tag{73}$$

Setting $u = a + (x-a)t$, and using the Höder inequality, we obtain

$$\begin{aligned} I_1 &= \frac{1}{n!}\int_a^x (u-a)^n |f^{(n)}(u)|du \\ &= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n |f^{(n)}(tx+(1-t)a)|dt \\ &\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{nq}dt\right)^{1/q}\left\{\int_0^1 |f^{(n)}(tx+(1-t)a)|^p dt\right\}^{1/p} \\ &= \frac{(x-a)^{n+1}}{n!(nq+1)^{1/q}}\left\{\int_0^1 |f^{(n)}(tx+(1-t)a)|^p dt\right\}^{1/p}. \end{aligned} \tag{74}$$

By using the exponentially (β, s, h)-strongly convexity of $|f^{(n)}|^p$ on $[a, b]$, we have

$$\begin{aligned} I_3 &= \int_0^1 |f^{(n)}(tx+(1-t)a)|^p dt \\ &\leq \int_0^1 \left\{\left[t^s\left(\frac{|f^{(n)}(x)|^p}{e^{rx}}\right)^\beta + (1-t)^s\left(\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)^\beta\right]^{1/\beta} - t(1-t)h(x-a)\right\}dt \\ &\leq C_\beta \int_0^1 \left[t^{s/\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}}\right) + (1-t)^{s/\beta}\left(\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)\right]dt - h(x-a)\int_0^1 t(1-t)dt \\ &= \frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a), \end{aligned} \tag{75}$$

where C_β is defined by (32). Hence,

$$I_1 \leq \frac{(x-a)^{n+1}}{n!(nq+1)^{1/q}}\left\{\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a)\right\}^{1/p}. \tag{76}$$

By letting $u = x + (b-x)t$ and similar arguments, we get

$$\begin{aligned} I_2 &= \frac{1}{n!}\int_x^b (b-u)^n |f^{(n)}(u)|du \\ &= \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n |f^{(n)}(tb+(1-t)x)|dt \end{aligned}$$

$$\leq \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^{nq}dt\right)^{1/q}\left\{|f^{(n)}(tb+(1-t)x)|^p dt\right\}^{1/p}$$

$$\leq \frac{(b-x)^{n+1}}{n!(nq+1)^{1/q}}\left\{\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(b)|^p}{e^{rb}}+\frac{|f^{(n)}(x)|^p}{e^{rx}}\right)-\frac{1}{6}h(b-x)\right\}^{1/p}. \quad (77)$$

A combination of (73), (76), and (77) gives the required result. The proof is completed.

Proof of Theorem 8 In (73), we have a different decomposition of the integrand for I_1 and I_2, and using the Hölder inequality. Setting $u = a + (x-a)t$, we obtain

$$\begin{aligned}
I_1 &= \frac{1}{n!}\int_a^x (u-a)^n |f^{(n)}(u)|du \\
&= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n |f^{(n)}(tx+(1-t)a)|dt \\
&\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1/q}\left\{\int_0^1 t^n |f^{(n)}(tx+(1-t)a)|^p dt\right\}^{1/p} \\
&= \frac{(x-a)^{n+1}}{n!(n+1)^{1/q}} \times \{I_4(x,a)\}^{1/p}, \qquad (78)
\end{aligned}$$

where

$$I_4 = \int_0^1 t^n |f^{(n)}(tx+(1-t)a)|^p dt. \quad (79)$$

By using the exponentially (β, s, h)-strongly convexity of $|f^{(n)}|^p$ on $[a, b]$, we have

$$\begin{aligned}
I_4 &= \int_0^1 t^n |f^{(n)}(tx+(1-t)a)|^p dt \\
&\leq \int_0^1 t^n \left\{\left[t^s\left(\frac{|f^{(n)}(x)|^p}{e^{rx}}\right)^\beta + (1-t)^s\left(\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)^\beta\right]^{1/\beta}\right. \\
&\qquad \left. -t(1-t)h(x-a)\right\} dt \\
&\leq \int_0^1 t^n \left\{C_\beta\left[t^{s/\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}}\right) + (1-t)^{s/\beta}\left(\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)\right]\right.
\end{aligned}$$

$$
\left. -t(1-t)h(x-a) \right\} dt
$$

$$
\le C_\beta \left[\frac{\beta}{s+\beta(n+1)} \frac{|f^{(n)}(x)|^p}{e^{rx}} + B\left(n+1, \frac{s}{\beta}+1\right) \frac{|f^{(n)}(a)|^p}{e^{ra}} \right]
$$

$$
- \frac{1}{(n+2)(n+3)} h(x-a). \tag{80}
$$

By letting $u = x + (b-x)t$ and similar arguments, we get

$$
\begin{aligned}
I_2 &= \frac{1}{n!} \int_x^b (b-u)^n |f^{(n)}(u)| du \\
&= \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n |f^{(n)}(tb+(1-t)x)| dt \\
&\le \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1/q} \left\{ \int_0^1 (1-t)^n |f^{(n)}(tb+(1-t)x)|^p dt \right\}^{1/p} \\
&= \frac{(b-x)^{n+1}}{n!(n+1)^{1/q}} \times \{I_5(b,x)\}^{1/p}.
\end{aligned} \tag{81}
$$

By letting $t_1 = 1-t$, we get

$$
\begin{aligned}
I_5(b,x) &= \int_0^1 (1-t)^n |f^{(n)}(tb+(1-t)x)|^p dt \\
&= \int_0^1 t_1^n |f^{(n)}(t_1 x + (1-t_1)b|^p dt_1 \\
&\le C_\beta \left[\frac{\beta}{s+\beta(n+1)} \frac{|f^{(n)}(x)|^p}{e^{rx}} + B(n+1, \frac{s}{\beta}+1) \frac{|f^{(n)}(b)|^p}{e^{rb}} \right] \\
&\quad - \frac{1}{(n+2)(n+3)} h(b-x).
\end{aligned} \tag{82}
$$

A combination of (73), (78), (80), (81), and (82) gives the required result. The proof is completed.

5 Perturbed Simpson Type Inequalities and Approximations

In what follows, let

$$\triangle(f,\lambda)=\frac{1}{(b-a)^2}\left\{\frac{1}{b-a}\int_a^b f(u)du-(1-\lambda)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}\right\}. \tag{83}$$

In particular,

$$\triangle\left(f,\frac{1}{3}\right)=\frac{1}{(b-a)^2}\left\{\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\right\}.$$

Theorem 9 ([25, 26]) *Let $f^{(4)}\in L^{\infty}[a,b]$, then the following Simpson's inequality holds:*

$$|\triangle\left(f,\frac{1}{3}\right)|\leq\frac{1}{2880}\|f^{(4)}\|_{\infty}.$$

Lemma 2 ([27]) *If $f^{''}\in L^1[a,b]$, then*

$$\triangle(f,\lambda)=\frac{1}{2}\int_0^1 K(t)f^{''}(ta+(1-t)b)dt, \tag{84}$$

where

$$K(t)=\begin{cases} t(t-\lambda), & t\in[0,1/2),\\ (1-t)(1-\lambda-t), & t\in[1/2,1].\end{cases}$$

Theorem 10 *Let $[a,b]\subset(0,\infty)$, $f:[a,b]\to(0,\infty)$ be a differentiable mapping such that $f^{'}\in AC[a,b]$. Let $|f^{''}|^p$ is exponentially (β,s,h)-strongly convex on $[a,b]$, $1<p,q_1,q_2<\infty$, $\frac{1}{p}+\frac{1}{q_1}+\frac{1}{q_2}=1$, $0\leq\lambda<1$. If $0\leq\lambda<1/2$, then*

$$\begin{aligned}|\triangle(f,\lambda)|\leq{}&\frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}}\left(\lambda^{q_2+1}+\left(\frac{1}{2}-\lambda\right)^{q_2+1}\right)^{1/q_2}\\&\times\left\{\left[\frac{\beta C_\beta}{s+\beta}\left(\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(a)|^p}{e^{ra}}+\left(1-\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)\right)-\frac{1}{12}h(b-a)\right]^{1/p}\right.\\&\left.+\left[\frac{\beta C_\beta}{s+\beta}\left(\left(1-\frac{1}{2^{1+(s/\beta)}}\right)\times\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)-\frac{1}{12}h(b-a)\right]^{1/p}\right\}.\end{aligned} \tag{85}$$

If $\frac{1}{2}\leq\lambda\leq 1$, then

$$\begin{aligned}|\triangle(f,\lambda)|\leq{}&\frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}}\left(\lambda^{q_2+1}-\left(\lambda-\frac{1}{2}\right)^{q_2+1}\right)^{1/q_2}\\&\times\left\{\left[\frac{\beta C_\beta}{s+\beta}\left(\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(a)|^p}{e^{ra}}+\left(1-\frac{1}{2^{1+(s/\beta)}}\right)\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)-\frac{1}{12}h(b-a)\right]^{1/p}\right.\end{aligned}$$

$$+\left[\frac{\beta C_\beta}{s+\beta}\left(\left(1-\frac{1}{2^{1+(s/\beta)}}\right)\times\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)-\frac{1}{12}h(b-a)\right]^{1/p}\Bigg\}, \tag{86}$$

where C_β is defined by (32).

Taking $\lambda = 1/3$ in (85), we get Simpson type inequality:

$$|\triangle\left(f,\frac{1}{3}\right)| \le \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}\times 3^{1+(1/q_2)}}\left(1+\frac{1}{2^{q_2}+1}\right)^{1/q_2}$$
$$\times\Bigg\{\left[\frac{\beta C_\beta}{s+\beta}\left(\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(a)|^p}{e^{ra}}+\left(1-\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)\right)-\frac{1}{12}h(b-a)\right]^{1/p}$$
$$+\left[\frac{\beta C_\beta}{s+\beta}\left(\left(1-\frac{1}{2^{1+(s/\beta)}}\right)\times\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{1}{2^{1+(s/\beta)}}\times\frac{|f^{''}(b)|^p}{e^{rb}}\right)-\frac{1}{12}h(b-a)\right]^{1/p}\Bigg\}.$$

Hence, we say that (85) and (86) are the perturbed Simpson type inequalities. Taking $\beta = 1, r = 0, h = 0$ in (85) and (86), respectively, that is, $|f^{''}|^p$ is s-convex on $[a, b]$, we get

$$|\triangle(f,\lambda)| \le \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}(s+1)^{1/p}}$$
$$\times\left(\lambda^{q_2+1}+\left(\frac{1}{2}-\lambda\right)^{q_2+1}\right)^{1/q_2}$$
$$\times\Bigg\{\left[\frac{1}{2^{1+s}}|f^{''}(a)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f^{''}(b)|^p\right]^{1/p}$$
$$+\left[\frac{1}{2^{1+s}}|f^{''}(b)|+\left(1-\frac{1}{2^{1+s}}\right)|f^{''}(a)|^p\right]^{1/p}\Bigg\}; \tag{87}$$

and

$$|\triangle(f,\lambda)| \le \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}(s+1)^{1/p}}$$
$$\times\left(\lambda^{q_2+1}-\left(\lambda-\frac{1}{2}\right)^{q_2+1}\right)^{1/q_2}$$
$$\times\Bigg\{\left[\frac{1}{2^{1+s}}|f^{''}(a)|^p+(1-\frac{1}{2^{1+s}})|f^{''}(b)|^p\right]^{1/p}$$
$$+\left[\frac{1}{2^{1+s}}|f^{''}(b)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f^{''}(a)|^p\right]^{1/p}\Bigg\}. \tag{88}$$

If we take $\lambda = 1/3$ in (87), then we get a Simpson type inequality:

$$|\triangle\left(f,\frac{1}{3}\right)| \le \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}\times 3^{1+(1/q_2)}}\left(1+\frac{1}{2^{q_2+1}}\right)^{1/q_2}$$
$$\times\left\{\left[\frac{1}{2^{1+s}}|f''(a)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f''(b)|^p\right]^{1/p}\right.$$
$$\left.+\left[\frac{1}{2^{1+s}}|f''(b)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f''(a)|^p\right]^{1/p}\right\}. \tag{89}$$

If we take $\lambda = 0$ in (87), then we get a midpoint type inequality:

$$|\triangle(f,0)| = \frac{1}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{1}{2^{3+(1/q_1)+(1/q_2)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}(s+1)^{1/p}}$$
$$\times\left\{\left[\frac{1}{2^{1+s}}|f''(a)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f''(b)|^p\right]^{1/p}\right.$$
$$\left.+\left[\frac{1}{2^{1+s}}|f''(b)|^p+\left(1-\frac{1}{2^{1+s}}\right)|f''(a)|^p\right]^{1/p}\right\}.$$

If we take $\lambda = 1$ in (88), then we get a trapezoid type inequality:

$$|\triangle(f,1)| = \frac{1}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(a)+f(b)}{2}\right|$$
$$\le \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}(s+1)^{1/p}}\left(1-\frac{1}{2^{q_2+1}}\right)^{1/q_2}$$
$$\times\left\{\left[\frac{1}{2^{1+s}}|f''(a)|^p+(1-\frac{1}{2^{1+s}})|f''(b)|^p\right]^{1/p}\right.$$
$$\left.+\left[\frac{1}{2^{1+s}}|f''(b)|^p+(1-\frac{1}{2^{1+s}})|f''(a)|^p\right]^{1/p}\right\}.$$

If we take $\lambda = 1/2$ in (87), then we get an averaged midpoint-trapezoid type inequality:

$$|\triangle\left(f,\frac{1}{2}\right)| = \frac{1}{(b-a)^2}\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{1}{2^{3+(1/q_1)+(1/q_2)}(q_1+1)^{1/q_1}(q_2+1)^{q_2}(s+1)^{1/p}}$$
$$\times \left\{ \left[\frac{1}{2^{1+s}} |f''(a)|^p + \left(1 - \frac{1}{2^{1+s}}\right) |f''(b)|^p \right]^{1/p} \right.$$
$$\left. + \left[\frac{1}{2^{1+s}} |f''(b)|^p + \left(1 - \frac{1}{2^{1+s}}\right) |f''(a)|^p \right]^{1/p} \right\}$$

Proof of Theorem 10 By Lemma 2, we have

$$|\triangle(f,\lambda)| \leq \frac{1}{2}\int_0^{1/2} t|t-\lambda||f''(ta+(1-t)b)|dt$$
$$+\frac{1}{2}\int_{1/2}^{1} (1-t)|1-\lambda-t||f''(ta+(1-t)b)|dt = I_1 + I_2. \quad (90)$$

Using the Hölder inequality, we obtain

$$I_1 \leq \frac{1}{2}\left(\int_0^{1/2} t^{q_1} dt\right)^{1/q_1} \left(\int_0^{1/2} |t-\lambda|^{q_2} dt\right)^{1/q_2}$$
$$\times \left(\int_0^{1/2} |f''(ta+(1-t)b)|^p dt\right)^{1/p}. \quad (91)$$

$$I_2 \leq \frac{1}{2}\left(\int_{1/2}^{1} |1-t|^{q_1} dt\right)^{1/q_1} \left(\int_{1/2}^{1} |1-\lambda-t|^{q_2} dt\right)^{1/q_2}$$
$$\times \left(\int_{1/2}^{1} |f''(ta+(1-t)b)|^p dt\right)^{1/p}. \quad (92)$$

If $0 \leq \lambda \leq 1/2$, then

$$\int_0^{1/2} t^{q_1} dt = \frac{1}{(q_1+1)2^{q_1+1}},$$
$$\int_0^{1/2} |t-\lambda|^{q_2} dt = \int_0^{\lambda} (\lambda-t)^{q_2} dt + \int_{\lambda}^{1/2} (t-\lambda)^{q_2} dt$$
$$= \frac{1}{q_2+1}\left(\lambda^{q_2+1} + \left(\frac{1}{2}-\lambda\right)^{q_2+1}\right).$$

By using the exponentially (β, s, h)-strongly convexity of $|f''|^p$ on $[a, b]$, we have

$$\int_0^{1/2} |f^{"}(ta+(1-t)b)|^p dt$$
$$\leq \int_0^{1/2} \left\{ \left[t^s \left(\frac{|f^{"}(a)|^p}{e^{ra}} \right)^\beta + (1-t)^s \left(\frac{|f^{"}(b)|^p}{e^{rb}} \right)^\beta \right]^{1/\beta} - t(1-t)h(b-a) \right\} dt$$
$$\leq \frac{\beta C_\beta}{s+\beta} \left\{ \frac{1}{2^{(s/\beta)+1}} \left(\frac{|f^{"}(a)|^p}{e^{ra}} \right) + (1 - \frac{1}{2^{(s/\beta)+1}}) \left(\frac{|f^{"}(b)|^p}{e^{rb}} \right) \right\} - \frac{1}{12} h(b-a).$$

Hence, we get

$$I_1 \leq \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}} \left(\lambda^{q_2+1} + \left(\frac{1}{2} - \lambda \right)^{q_2+1} \right)^{1/q_2}$$
$$\times \left\{ \frac{\beta C_\beta}{s+\beta} \left[\frac{1}{2^{1+(s/\beta)}} \times \frac{|f^{"}(a)|^p}{e^{ra}} + \left(1 - \frac{1}{2^{1+(s/\beta)}} \right) \right. \right.$$
$$\left. \left. \times \frac{|f^{"}(b)|^p}{e^{rb}} \right] - \frac{1}{12} h(b-a) \right\}^{1/p}. \tag{93}$$

$$I_2 \leq \frac{1}{2^{2+(1/q_1)}(q_1+1)^{1/q_1}(q_2+1)^{1/q_2}} \left(\lambda^{q_2+1} + \left(\frac{1}{2} - \lambda \right)^{q_2+1} \right)^{1/q_2}$$
$$\times \left\{ \frac{\beta C_\beta}{s+\beta} \left[\frac{1}{2^{1+(s/\beta)}} \times \frac{|f^{"}(b)|^p}{e^{rb}} + \left(1 - \frac{1}{2^{1+(s/\beta)}} \right) \right. \right.$$
$$\left. \left. \times \frac{|f^{"}(a)|^p}{e^{ra}} \right] - \frac{1}{12} h(b-a) \right\}^{1/p}. \tag{94}$$

It follows from(90)–(94) that (85) holds. When$1/2 \leq \lambda \leq 1$, by using similar arguments, we get (86). The proof is completed.

In what follows, let

$$\sigma_n(f,x) = \sum_{k=1}^{n-1} \frac{(n-k)}{k!} \frac{f^{k-1}(b)(x-b)^k - f^{k-1}(a)(x-a)^k}{b-a}. \tag{95}$$

The sum in (95) is zero when $n = 1$.

Lemma 3 ([28]) *Let $n \geq 1$ and $f : [a,b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)} \in AC[a,b]$, then for all $x \in [a,b]$, we have*

$$f(x) = \frac{n}{b-a} \int_a^b f(u)du + \sigma_n(f,x)$$

$$+\frac{1}{(n-1)!(b-a)}\int_a^b (x-u)^{n-1}K(u,x)f^{(n)}(u)du, \tag{96}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K(u,x) = \begin{cases} u-a, \ a \le u \le x \le b, \\ u-b, \ \ a \le x < u \le b, \end{cases},$$

and $\sigma_n(f,x)$ *is defined by (95).*

Theorem 11 *Under the assumptions of Theorem 7, we have*

$$\begin{aligned} &|f(x) - \frac{n}{b-a}\int_a^b f(u)du - \sigma_n(f,x)| \\ &\le \frac{[B(q+1,(n-1)q+1)]^{1/q}}{(n-1)!(b-a)} \\ &\times \left\{(x-a)^{n+1}\left[\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a)\right]^{1/p}\right. \\ &\left. +(b-x)^{n+1}[\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(b)|^p}{e^{rb}} + \frac{|f^{(n)}(x)|^p}{e^{rx}}\right) - \frac{1}{6}h(b-x)]^{1/p}\right\}. \end{aligned} \tag{97}$$

If $n=1, \beta=1$ in (97), then

$$\begin{aligned} &|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{1}{(b-a)(q+1)^{1/q}} \\ &\times \left\{(x-a)^2\left[\frac{1}{s+1}\left(\frac{|f'(x)|^p}{e^{rx}} + \frac{|f'(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a)\right]^{1/p}\right. \\ &\left. +(b-x)^2\left[\frac{1}{s+1}\left(\frac{|f'(b)|^p}{e^{rb}} + \frac{|f'(x)|^p}{e^{rx}}\right) - \frac{1}{6}h(b-x)\right]^{1/p}\right\}. \end{aligned} \tag{98}$$

If $p=1$ in (98), then

$$\begin{aligned} &|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{1}{b-a} \\ &\times \left\{(x-a)^2\left[\frac{1}{s+1}\left(\frac{|f'(x)|}{e^{rx}} + \frac{|f'(a)|}{e^{ra}}\right) - \frac{1}{6}h(x-a)\right]\right. \end{aligned}$$

$$+(b-x)^2\left[\frac{1}{s+1}\left(\frac{|f^{'}(b)|}{e^{rb}}+\frac{|f^{'}(x)|}{e^{rx}}\right)-\frac{1}{6}h(b-x)\right]\Bigg\}. \tag{99}$$

Taking $x=(a+b)/2$ in (98), we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{4(q+1)^{1/q}}\\&\times\left\{\left[\frac{1}{s+1}\left(\frac{|f^{'}((a+b)/2)|^p}{e^{r(a+b)/2}}+\frac{|f^{'}(a)|^p}{e^{ra}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right.\\&\left.+\left[\frac{1}{s+1}\left(\frac{|f^{'}(b)|^p}{e^{rb}}+\frac{|f^{'}((a+b)/2)|^p}{e^{r(a+b)/2}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right\}.\end{aligned} \tag{100}$$

If $p=1$ in (100), then

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\\&\leq\frac{(b-a)}{4(s+1)}\left\{\frac{|f^{'}(a)|}{e^{ra}}+\frac{2|f^{'}((a+b)/2)|}{e^{r(a+b)/2}}+\frac{|f^{'}(b)|}{e^{rb}}-\frac{1}{3}h\left(\frac{b-a}{2}\right)\right\}.\end{aligned} \tag{101}$$

If $n=2$, $\beta=1$ in (97), then

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left\{f(x)+\frac{f(a)(x-a)+f(b)(b-x)}{b-a}\right\}\right|\\&\leq\frac{[B(q+1,q+1)]^{1/q}}{2(b-a)}\\&\quad\times\left\{(x-a)^3\left[\frac{1}{s+1}\left(\frac{|f^{''}(x)|^p}{e^{rx}}+\frac{|f^{''}(a)|^p}{e^{ra}}\right)-\frac{1}{6}h(x-a)\right]^{1/p}\right.\\&\quad\left.+(b-x)^3\left[\frac{1}{s+1}\left(\frac{|f^{''}(b)|^p}{e^{rb}}+\frac{|f^{''}(x)|^p}{e^{rx}}\right)-\frac{1}{6}h(b-x)\right]^{1/p}\right\}.\end{aligned} \tag{102}$$

Taking $x=(a+b)/2$ in (102), then

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right\}\right|\\&\leq\frac{[B(q+1,q+1)]^{1/q}(b-a)^2}{16}\end{aligned}$$

$$\times\left\{\left[\frac{1}{s+1}\left(\frac{|f''((a+b)/2)|^p}{e^{r(a+b)/2}}+\frac{|f''(a)|^p}{e^{ra}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right.$$
$$\left.+\left[\frac{1}{s+1}\left(\frac{|f''(b)|^p}{e^{rb}}+\frac{|f''((a+b)/2)|^p}{e^{r(a+b)/2}}\right)-\frac{1}{6}h\left(\frac{b-a}{2}\right)\right]^{1/p}\right\}. \quad (103)$$

If $p=1$ in (103), then

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right\}\right|$$
$$\le\frac{(b-a)^2}{16(s+1)}\left\{\frac{|f''(a)|}{e^{ra}}+\frac{2|f''((a+b)/2)|}{e^{r(a+b)/2}}+\frac{|f''(b)|}{e^{rb}}-\frac{1}{3}h\left(\frac{b-a}{2}\right)\right\}. \quad (104)$$

If $r=0$, $h=0$ in (104), that is, $|f''|$ is s-convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{2}\left\{f(\frac{a+b}{2})+\frac{f(a)+f(b)}{2}\right\}\right|$$
$$\le\frac{(b-a)^2}{16(s+1)}\left\{|f''(a)|+2\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right\}.$$

Proof of Theorem 11 By Lemma 3, we have

$$\left|f(x)-\frac{n}{b-a}\int_a^b f(u)du-\sigma_n(f,x)\right|$$
$$=\frac{1}{(n-1)!(b-a)}\left|\int_a^x (x-u)^{n-1}(u-a)f^{(n)}(u)du\right.$$
$$\left.+\int_x^b (x-u)^{n-1}(u-b)f^{(n)}(u)du\right|$$
$$\le\frac{1}{(n-1)!(b-a)}\left\{\int_a^x (x-u)^{n-1}(u-a)|f^{(n)}(u)|du\right.$$
$$\left.+\int_x^b (u-x)^{n-1}(b-u)|f^{(n)}(u)|du\right\}. \quad (105)$$

Setting $u=a+(x-a)t$ and using the Hölder inequality, we obtain

$$\int_a^x (x-u)^{n-1}(u-a)|f^{(n)}(u)|du$$
$$=(x-a)^{n+1}\int_0^1 t(1-t)^{n-1}|f^{(n)}(tx+(1-t)a)|dt$$

$$\leq (x-a)^{n+1}\left\{B(q+1,(n-1)q+1)\right\}^{1/q} \times \left\{\int_0^1 |f^{(n)}(tx+(1-t)a)|^p dt\right\}^{1/p}. \tag{106}$$

By using the exponentially (β, s, h)-strongly convexity of $|f^{(n)}|^p$ on $[a, b]$, we have

$$\begin{aligned}
&\int_0^1 \left|f^{(n)}(tx+(1-t)a)\right|^p dt \\
&\leq \int_0^1 \left\{\left[t^s\left(\frac{|f^{(n)}(x)|^p}{e^{rx}}\right)^\beta + (1-t)^s\left(\frac{|f^{(n)}(a)|^p}{e^{ra}}\right)^\beta\right]^{1/\beta} - t(1-t)h(x-a)\right\} dt \\
&\leq \frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a),
\end{aligned} \tag{107}$$

where C_β is defined by (32). Hence,

$$\begin{aligned}
&\int_a^x (x-u)^{n-1}(u-a)|f^{(n)}(u)|du \\
&\leq (x-a)^{n+1}\left\{B(q+1,(n-1)q+1)\right\}^{1/q} \\
&\quad \times \left\{\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(x)|^p}{e^{rx}} + \frac{|f^{(n)}(a)|^p}{e^{ra}}\right) - \frac{1}{6}h(x-a)\right\}^{1/p}.
\end{aligned} \tag{108}$$

By letting $u = x + (b-x)t$ and similar arguments, we get

$$\begin{aligned}
&\int_x^b (u-x)^{n-1}(b-u)|f^{(n)}(u)|du \\
&= (b-x)^{n+1}\int_0^1 t^{n-1}(1-t)|f^{(n)}(tb+(1-t)x)|dt \\
&\leq (b-x)^{n+1}\left\{B((n-1)q+1,q+1)\right\}^{1/q} \\
&\quad \times \left\{\int_0^1 |f^{(n)}(tb+(1-t)x)|^p dt\right\}^{1/p} \\
&\leq (b-x)^{n+1}\left\{B((n-1)q+1,q+1)\right\}^{1/q} \\
&\quad \times \left\{\frac{\beta C_\beta}{s+\beta}\left(\frac{|f^{(n)}(b)|^p}{e^{rb}} + \frac{|f^{(n)}(x)|^p}{e^{rx}}\right) - \frac{1}{6}h(b-x)\right\}^{1/p}.
\end{aligned} \tag{109}$$

A combination of (105), (108), and (109) gives the required result. The proof is completed.

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Ternary Biderivations and Ternary Bihomorphisms in C^*-Ternary Algebras

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Abstract In (Park et al., Rocky Mountain J Math 49:593–607, 2019), Park introduced the following bi-additive s-functional inequality

$$\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \\ \leq \left\| s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|, \quad (1)$$

where s is a fixed nonzero complex number with $|s| < 1$. Using the fixed point method, we prove the Hyers–Ulam stability of ternary biderivations and ternary bihomomorphism in C^*-ternary algebras, associated with the bi-additive s-functional inequality (1).

1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [24] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$,

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does there exist a $\delta 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [14] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta$$

for all $x \in E$. In 1978, Rassias [23] proved the following theorem.

Theorem 1 ([23]) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{2}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{3}$$

for all $x \in E$. If $p < 0$, then (2) holds for all $x, y \neq 0$, and (3) holds for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbf{R}$ into E' is continuous in $t \in \mathbf{R}$ for each fixed $x \in E$, then T is $\mathbf{R}$-linear.

A generalization of the Rassias' theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function.

Theorem 2 ([11]) *Suppose $(G, +)$ is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \to \mathbf{R}$ satisfies*

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, *then there exists a unique mapping* $T : G \to E$ *such that* $T(x + y) = T(x) + T(y)$ *and* $\|f(x) - T(x)\| \le \tilde{\varphi}(x, x)$ *for all* $x, y \in G$.

Gilányi [12] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \le \|f(x + y)\| \tag{4}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y)$$

Fechner [10] and Gilányi [13] proved the Hyers–Ulam stability of the functional inequality (4). Park [18, 19] defined additive ρ-functional inequalities and proved the Hyers–Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [1, 8, 9]).

Using the result on fixed point given in [4, 7], Isac and Rassias [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 22]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbf{C}$-linear in the outer variables, conjugate $\mathbf{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [25]).

Let A and B be C^*-ternary algebras. A $\mathbf{C}$-linear mapping $G : A \to B$ is called a *ternary homomorphism* if

$$G([a, b, c]) = [G(a), G(b), G(c)]$$

for all $a, b, c \in A$. A $\mathbf{C}$-linear mapping $D : A \to A$ is called a *ternary derivation* if

$$D([a, b, c]) = [D(a), b, c] + [a, D(b), c] + [a, b, D(c)]$$

for all $a, b, c \in A$ (see [2, 17]).

Park [20] defined ternary bihomomorphisms and ternary biderivations in C^*-ternary algebras.

Definition 1 ([20, Definition 2]) Let A and B be C^*-ternary algebras. A $\mathbf{C}$-bilinear mapping $G : A \times A \to B$ is called a *ternary bihomomorphism* if

$$G([x, y, z], [w, w, w]) = [G(x, w), G(y, w), G(z, w)],$$

$$G([x, x, x], [y, z, w]) = [G(x, y), G(x, z), G(x, w)]$$

for all $x, y, z, w \in A$. A $\mathbf{C}$-bilinear mapping $D : A \times A \to A$ is called a *ternary biderivation* if

$$D([x, y, z], w) = [D(x, w), y, z] + [x, D(y, w^*), z] + [x, y, D(z, w)],$$
$$D(x, [y, z, w]) = [D(x, y), z, w] + [y, D(x^*, z), w] + [y, z, D(x, w)]$$

for all $x, y, z, w \in A$.

In this paper, we prove the Hyers–Ulam stability of ternary bihomomorphisms and ternary biderivations in C^*-ternary algebras by using the fixed point method.

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2 Ternary Bihomomorphisms in C^*-Ternary Algebras

Throughout this paper, assume that A and B are C^*-ternary algebras.

Lemma 1 ([3, Lemma 2.1]) *Let $g : X \times X \to Y$ be a mapping such that*

$$g(\lambda(x + y), \nu(z - w)) + g(\lambda(x - y), \nu(z + w)) = 2\lambda\nu g(x, z) - 2\lambda\nu g(y, w)$$

for all $\lambda, \nu \in S^1 := \{\eta \in \mathbf{C} : |\eta| = 1\}$ and all $x, y, z, w \in X$. Then $g : X \times X \to Y$ is $\mathbf{C}$-bilinear.

For a given mapping $g : A \times A \to B$, we define

$$E_{\lambda,\nu} g(x, y, z, w)$$
$$:= g(\lambda(x + y), \nu(z - w)) + g(\lambda(x - y), \nu(z + w)) - 2\lambda\nu g(x, z) + 2\lambda\nu g(y, w),$$

$$F_{\lambda,\nu} g(x, y, z, w)$$
$$:= g\left(\lambda\frac{x + y}{2}, \nu(z - w)\right) + g\left(\lambda\frac{x - y}{2}, \nu(z + w)\right) - 2\lambda\nu g(x, z) + 2\lambda\nu g(y, w)$$

for all $\lambda, \nu \in S^1$ and all $x, y, z, w \in A$.

Lemma 2 *Let $f : X \times X \to Y$ be a mapping such that*

$$\|E_{\lambda,\mu} f(x, y, z, w)\| \le \|s F_{\lambda,\mu} f(x, y, z, w)\|$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in X$. Then $f : X \times X \to Y$ is $\mathbf{C}$-bilinear.

Proof Let $\lambda = \mu = 1$. By [21, Lemma 2.1], the mapping $f : X \times X \to Y$ is bi-additive. So

$$\|D_{\lambda,\mu} f(x, y, z, w)\| \leq \|s E_{\lambda,\mu} f(x, y, z, w)\| \\ = \|s D_{\lambda,\mu} f(x, y, z, w)\|$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in X$. Thus $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in X$, since $|s| < 1$. By Lemma 1, the mapping $f : X \times X \to Y$ is **C**-bilinear.

We prove the Hyers–Ulam stability of ternary bihomomorphisms in C^*-ternary algebras.

Theorem 3 *Let $\varphi : A^4 \to [0, \infty)$ be a function such that there exists a $\kappa < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right) \leq \frac{\kappa}{64}\varphi(x, y, z, w) \leq \frac{\kappa}{4}\varphi(x, y, z, w) \tag{5}$$

for all $x, y, z, w \in X$. Let $f : A \times A \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and

$$\|E_{\lambda,\nu} f(x, y, z, w)\| \leq \|s F_{\lambda,\nu} f(x, y, z, w)\| + \varphi(x, y, z, w), \tag{6}$$

$$\|f([x, y, z], [w, w, w]) - [f(x, w), f(y, w), f(z, w)]\| \\ + \|f([x, x, x], [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\| \leq \varphi(x, y, z, w) \tag{7}$$

for all $\lambda, \nu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary bihomomorphism $H : A \times A \to B$ such that

$$\|f(x, z) - H(x, z)\| \leq \frac{\kappa}{1-\kappa}\left(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)\right) \tag{8}$$

for all $x, z \in A$.

Proof Letting $y = x$, $w = 0$, and $\lambda = \nu = 1$ in (6), we get

$$\|f(2x, z) - 2f(x, z)\| \leq \varphi(x, x, z, 0) \tag{9}$$

for all $x, z \in A$.

Letting $y = 0$, $w = z$, and $\lambda = \nu = 1$ in (6), we get

$$\|f(x, 2z) - 2f(x, z)\| \leq \varphi(x, 0, z, z)$$

and so

$$\|f(2x, 2z) - 2f(2x, z)\| \leq \varphi(2x, 0, z, z) \tag{10}$$

for all $x, z \in A$.

It follows from (9) and (10) that

$$\|f(2x, 2z) - 4f(x, z)\| \le 2\varphi(x, x, z, 0) + \varphi(2x, 0, z, z) \tag{11}$$

for all $x, z \in A$. Thus

$$\left\| f(x, z) - \frac{1}{4} f(2x, 2z) \right\| \le \frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)$$

for all $x, z \in A$.

Consider the set

$$S := \{h : A \times A \to B, \;\; h(x, 0) = h(0, z) = 0, \quad \forall x, z \in A\}$$

and introduce the generalized metric on S:

$$\begin{aligned} d(g, h) = \inf\{\beta \in \mathbf{R}_+ : \|g(x, z) - h(x, z)\| \\ \le \beta(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)), \;\; \forall x, z \in A\}, \end{aligned}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x, z) := 4g\left(\frac{x}{2}, \frac{z}{2}\right)$$

for all $x, z \in A$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x, z) - h(x, z)\| \le \varepsilon(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z))$$

for all $x, z \in A$. Since

$$\begin{aligned} \|Jg(x, z) - Jh(x, z)\| &= \left\|4g\left(\frac{x}{2}, \frac{z}{2}\right) - 4h\left(\frac{x}{2}, \frac{z}{2}\right)\right\| \\ &\le 4\varepsilon(\frac{1}{2}\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, 0\right) + \frac{1}{4}\varphi\left(x, 0, \frac{z}{2}, \frac{z}{2}\right)) \\ &\le 4\varepsilon\frac{\kappa}{4}(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)) \\ &= \kappa\varepsilon(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)) \end{aligned}$$

for all $x, z \in A$, $d(Jg, Jh) \le \kappa\varepsilon$. This means that

$$d(Jg, Jh) \le \kappa d(g, h)$$

for all $g, h \in S$.

It follows from (11) that

$$\begin{aligned}\left\| f(x,z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right)\right\| &\le 2\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, 0\right) + \varphi\left(x, 0, \frac{z}{2}, \frac{z}{2}\right) \\ &\le \kappa(\frac{1}{2}\varphi(x,x,z,0) + \frac{1}{4}\varphi(2x,0,z,z))\end{aligned}$$

for all $x, z \in A$. So $d(f, Jf) \le \kappa$.

By Theorem [4, Theorem 2.1], there exists a mapping $H : A \times A \to B$ satisfying the following:

1. H is a fixed point of J, i.e.,

$$H(x,z) = 4H\left(\frac{x}{2}, \frac{z}{2}\right) \tag{12}$$

for all $x, z \in A$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (12) such that there exists a $\beta \in (0, \infty)$ satisfying

$$\|f(x,z) - H(x,z)\| \le \beta(\frac{1}{2}\varphi(x,x,z,0) + \frac{1}{4}\varphi(2x,0,z,z))$$

for all $x, z \in A$;

2. $d(J^l f, H) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^l f\left(\frac{x}{2^l}, \frac{z}{2^l}\right) = H(x,z)$$

for all $x, z \in A$;

3. $d(f, H) \le \frac{1}{1-\kappa} d(f, Jf)$, which implies

$$\|f(x,z) - H(x,z)\| \le \frac{\kappa}{1-\kappa}(\frac{1}{2}\varphi(x,x,z,0) + \frac{1}{4}\varphi(2x,0,z,z))$$

for all $x \in A$.

It follows from (5) and (6) that

$$\left\|E_{\lambda,\nu}H(x,y,z,w)\right\| = \lim_{n\to\infty} 4^n \left\|E_{\lambda,\nu}f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right)\right\|$$

$$\leq \lim_{n\to\infty} 4^n \left\| sF_{\lambda,\nu} f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right)\right\| + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right)$$
$$= \left\| sF_{\lambda,\nu} H(x, y, z, w)\right\|$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$, since

$$\lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) \leq \lim_{n\to\infty} \frac{4^n \kappa^n}{4^n} \varphi(x, y, z, w) = 0$$

So

$$\|E_{\lambda,\nu} H(x, y, z, w)\| \leq \|sF_{\lambda,\nu} H(x, y, z, w)\|$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. By Lemma 2, the mapping $H : A \times A \to B$ is $\mathbf{C}$-bilinear. So there exists a unique $\mathbf{C}$-bilinear mapping $H : A \times A \to B$ satisfying (8).

It follows from (6) that

$$\|H([x, y, z], [w, w, w]) - [H(x, w), H(y, w), H(z, w)]\|$$
$$+\|H([x, x, x], [y, z, w]) - [H(x, y), H(x, z), H(x, w)]\|$$
$$\leq \lim_{n\to\infty} 64^n \left\| f\left(\frac{[x, y, z]}{8^n}, \frac{[w, w, w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), f\left(\frac{y}{2^n}, \frac{w}{2^n}\right), f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right]\right\|$$
$$+ \lim_{n\to\infty} 64^n \left\| f\left(\frac{[x, x, x]}{8^n}, \frac{[y, z, w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), f\left(\frac{x}{2^n}, \frac{z}{2^n}\right), f\left(\frac{x}{2^n}, \frac{w}{2^n}\right)\right]\right\|$$
$$\leq \lim_{n\to\infty} 64^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) \leq \lim_{n\to\infty} \frac{64^n \kappa^n}{64^n} \varphi(x, y, z, w) = 0$$

for all $x, y, z, w \in A$. So

$$H([x, y, z], [w, w, w]) = [H(x, w), H(y, w), H(z, w)],$$
$$H([x, x, x], [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in A$, as desired.

Corollary 1 *Let $r > 6$ and θ be nonnegative real numbers and $f : A \times A \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\|E_{\lambda,\nu} f(x, y, z, w)\| \leq \|sF_{\lambda,\nu} f(x, y, z, w)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \quad (13)$$

$$\|f([x, y, z], [w, w, w]) - [f(x, w), f(y, w), f(z, w)]\| \quad (14)$$
$$+\|f([x, x, x], [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\|$$
$$\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$$

for all $\lambda, \nu \in S^1$ *and all* $x, y, z, w \in A$. *Then there exists a unique ternary bihomomorphism* $H : A \times A \to B$ *such that*

$$\|f(x,z) - H(x,z)\| \leq \frac{(2^r + 4)\theta}{2^r - 4}\|x\|^r + \frac{4\theta}{2^r - 4}\|z\|^r$$

for all $x, z \in A$.

Proof The proof follows from Theorem 3 by taking $\varphi(x, y, z, w) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$ for all $x, y, z, w \in A$. Choosing $\kappa = 2^{2-r}$, we obtain the desired result.

Theorem 4 *Let* $\varphi : A^4 \to [0, \infty)$ *be a function such that there exists a* $\kappa < 1$ *with*

$$\varphi(x, y, z, w) \leq 4\kappa\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right) \tag{15}$$

for all $x, y, z, w \in A$. *Let* $f : A \times A \to B$ *be a mapping satisfying* $f(x, 0) = f(0, z) = 0$, *(6) and (6). Then there exists a unique ternary bihomomorphism* $H : A \times A \to B$ *such that*

$$\|f(x,z) - H(x,z)\| \leq \frac{1}{1-\kappa}\left(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)\right)$$

for all $x, z \in A$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := \frac{1}{4}g(2x, 2z)$$

for all $x, z \in A$.

It follows from (11) that

$$\left\|f(x,z) - \frac{1}{4}f(2x, 2z)\right\| \leq \frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 3.

Corollary 2 *Let* $r < 2$ *and* θ *be nonnegative real numbers and* $f : A \times A \to B$ *be a mapping satisfying* $f(x, 0) = f(0, z) = 0$, *(13) and (14). Then there exists a unique ternary bihomomorphism* $H : A \times A \to B$ *such that*

$$\|f(x,z) - H(x,z)\| \leq \frac{(4 + 2^r)\theta}{4 - 2^r}\|x\|^r + \frac{4\theta}{4 - 2^r}\|z\|^r$$

for all $x, z \in A$.

Proof The proof follows from Theorem 4 by taking $\varphi(x, y, z, w) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$ for all $x, y, z, w \in A$. Choosing $\kappa = 2^{r-2}$, we obtain the desired result.

3 Ternary Biderivations on C^*-Ternary Algebras Associated with the Bi-additive Functional Inequality (1)

In this section, we prove the Hyers–Ulam stability of the bi-additive s-functional inequality (1) in complex Banach spaces.

Theorem 5 *Let $\varphi : X^4 \to \mathbf{R}$ be a function satisfying (5) and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) + 2f(y, w)\| \\ &\le \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x, z) + 2f(y, w)\right)\right\| \\ &\quad + \varphi(x, y, z, w) \end{aligned} \tag{16}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $G : X^2 \to Y$ such that

$$\|f(x, z) - G(x, z)\| \le \frac{\kappa}{1-\kappa}\left(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)\right) \tag{17}$$

for all $x, z \in X$.

Proof Letting $y = x$ and $w = 0$ in (16), we get

$$\|f(2x, z) - 2f(x, z)\| \le \varphi(x, x, z, 0) \tag{18}$$

for all $x, z \in X$.

Letting $y = 0$ and $w = z$ in (16), we get

$$\|f(x, 2z) - 2f(x, z)\| \le \varphi(x, 0, z, z) \tag{19}$$

for all $x, z \in X$.

The rest of the proof is similar to the proof of Theorem 3.

Corollary 3 *Let $r > 2$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) + 2f(y, w)\| \tag{20}$$

$$\leq \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $G : X^2 \to Y$ such that

$$\|f(x,z) - G(x,z)\| \leq \frac{(2^r+4)\theta}{2^r-4}\|x\|^r + \frac{4\theta}{2^r-4}\|z\|^r \tag{21}$$

for all $x, z \in X$.

Theorem 6 *Let $\varphi : X^4 \to \mathbf{R}$ be a function satisfying (15) and $f : X^2 \to Y$ be a mapping satisfying (16) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $G : X^2 \to Y$ such that*

$$\|f(x,z) - G(x,z)\| \leq \frac{1}{1-\kappa}\left(\frac{1}{2}\varphi(x,x,z,0) + \frac{1}{4}\varphi(2x,0,z,z)\right) \tag{22}$$

for all $x, z \in X$.

Corollary 4 *Let $r < 2$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying (20) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $G : X^2 \to Y$ such that*

$$\|f(x,z) - G(x,z)\| \leq \frac{(4+2^r)\theta}{4-2^r}\|x\|^r + \frac{4\theta}{4-2^r}\|z\|^r \tag{23}$$

for all $x, z \in X$.

Now, we investigate ternary biderivations on C^*-ternary algebras associated with the bi-additive s-functional inequality (1).

From now on, assume that A is a C^*-ternary algebra.

Theorem 7 *Let $\varphi : A^4 \to \mathbf{R}$ be a function satisfying (5) and $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\|f(\lambda(x+y), \nu(z-w)) + f(\lambda(x-y), \nu(z+w)) - 2\lambda\nu f(x,z) + 2\lambda\nu f(y,w)\| \tag{24}$$
$$\leq \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| + \varphi(x,y,z,w)$$

for all $\lambda, \nu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x,z) - D(x,z)\| \leq \frac{\kappa}{1-\kappa}\left(\frac{1}{2}\varphi(x,x,z,0) + \frac{1}{4}\varphi(2x,0,z,z)\right) \tag{25}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ *satisfies* $f(2x, z) = 2f(x, z)$ *and*

$$\|f([x, y, z], w) - [f(x, w), y, z] - [x, f(y, w^*), z] - [x, y, f(z, w)]\| \leq \varphi(x, y, z, w), \tag{26}$$

$$\|f(x, [y, z, w]) - [f(x, y), z, w] - [y, f(x^*, z, w] - [y, z, f(x, w)]\| \leq \varphi(x, y, z, w) \tag{27}$$

for all $x, y, z, w \in A$*, then the* **C***-bilinear mapping* $D : A^2 \to A$ *is a ternary biderivation.*

Proof Let $\lambda = \nu = 1$ in (24). By Theorem 5, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (25) defined by

$$D(x, z) := \lim_{m\to\infty} 4^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right)$$

for all $x, z \in A$.

Letting $y = w = 0$ in (24), we get $f(\lambda x, \nu z) = \lambda\nu f(x, z)$ for all $x, z \in A$ and all $\lambda, \nu \in S^1$. By Lemma 2, the bi-additive mapping $D : A^2 \to A$ is **C**-bilinear.

It follows from (26) that

$$\begin{aligned}
&\|D([x, y, z], w) - [D(x, w), y, z] - [x, D(y, w^*), z] - [x, y, D(z, w)]\| \\
&= \lim_{n\to\infty} 16^n \left(\left\| \frac{1}{4^n} f\left(\frac{[x, y, z]}{2^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right]\right.\right. \\
&\quad \left.\left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right]\right\|\right) \\
&= \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{[x, y, z]}{8^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right]\right.\right. \\
&\quad \left.\left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right]\right\|\right) \\
&\leq \lim_{n\to\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) \leq \lim_{n\to\infty} \frac{16^n \kappa^n}{64^n} \varphi(x, y, z, w) = 0
\end{aligned}$$

for all $x, y, z, w \in A$. Thus

$$D([x, y, z], w) = [D(x, w), y, z] + [x, D(y, w^*), z] + [x, y, D(z, w)]$$

for all $x, y, z, w \in A$.

Similarly, one can show that

$$D(x, [y, z, w]) = [D(x, y), z, w] - [y, D(x^*, z, w] - [y, z, D(x, w)]$$

for all $x, y, z, w \in A$. So the $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ is a ternary biderivation.

Corollary 5 *Let $r > 6$ and θ be nonnegative real numbers and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x+y), \nu(z-w)) + f(\lambda(x-y), \nu(z+w)) - 2\lambda\nu f(x,z) + 2\lambda\nu f(y,w)\| \\ &\leq \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| \\ &+\theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \end{aligned} \tag{28}$$

for all $\lambda, \nu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x,z) - D(x,z)\| \leq \frac{(2^r+4)\theta}{2^r-4}\|x\|^r + \frac{4\theta}{2^r-4}\|z\|^r \tag{29}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\begin{aligned} &\|f([x,y,z], w) - [f(x,w), y, z] - [x, f(y, w^*), z] - [x, y, f(z,w)]\| \\ &\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|z\|^r), \end{aligned} \tag{30}$$

$$\begin{aligned} &\|f(x, [y,z,w]) - [f(x,y), z, w] - [y, f(x^*, z, w] - [y, z, f(x,w)]\| \\ &\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \end{aligned} \tag{31}$$

for all $x, y, z, w \in A$, then the $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ is a ternary biderivation.

Theorem 8 *Let $\varphi : A^4 \to \mathbf{R}$ be a function satisfying (15) and $f : A^2 \to A$ be a mapping satisfying (24) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{1}{1-\kappa}\left(\frac{1}{2}\varphi(x, x, z, 0) + \frac{1}{4}\varphi(2x, 0, z, z)\right) \tag{32}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ is a ternary biderivation.

Proof The proof is similar to the proof of Theorem 7.

Corollary 6 *Let $r < 2$ and θ be nonnegative real numbers and $f : A^2 \to A$ be a mapping satisfying (28) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x, z) - D(x, z)\| \le \frac{(4 + 2^r)\theta}{4 - 2^r}\|x\|^r + \frac{4\theta}{4 - 2^r}\|z\|^r \tag{33}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (30), (31) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the **C***-bilinear mapping $D : A^2 \to A$ is a ternary biderivation.*

4 Ternary Bihomomorphisms in C^*-Ternary Algebras Associated with the Bi-additive Functional Inequality (1)

In this section, we investigate ternary bihomomorphisms in C^*-ternary algebras associated with the bi-additive s-functional inequality (1).

Theorem 9 *Let $\varphi : A^4 \to$ **R** be a function satisfying (5) and $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and (24). Then there exists a unique* **C***-bilinear mapping $H : A^2 \to B$ satisfying (25), where D is replaced by H in (25).*

If, in addition, the mapping $f : A^2 \to B$ satisfies

$$\|f([x, y, z], [w, w, w]) - [f(x, w), f(y, w), f(z, w)]\| \le \varphi(x, y, z, w), \tag{34}$$

$$\|f([x, x, x], [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\| \le \varphi(x, y, z, w) \tag{35}$$

for all $x, y, z, w \in A$, then the **C***-bilinear mapping $H : A^2 \to B$ is a ternary bihomomorphism.*

Proof By the same reasoning as in the proof of Theorem 7, there is a unique **C**-bilinear mapping $H : A^2 \to B$, which is defined by

$$H(x, z) = \lim_{m\to\infty} 4^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right)$$

for all $x, z \in A$.

It follows from (34) that

$$\|H([x, y, z], [w, w, w]) - [H(x, w), H(y, w), H(z, w)]\|$$
$$= \lim_{n\to\infty} 64^n \left\| f\left(\frac{[x, y, z]}{8^n}, \frac{[w, w, w]}{8^n}\right)\right.$$

$$-\left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right), f\left(\frac{y}{2^n},\frac{w}{2^n}\right), f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\Big\|$$
$$\leq \lim_{n\to\infty} 64^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right) \leq \lim_{n\to\infty}\frac{64^n\kappa^n}{64^n}\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right) = 0$$

for all $x, y, z, w \in A$. Thus

$$H([x, y, z], [w, w, w]) = [H(x, w), H(y, w), H(z, w)]$$

for all $x, y, z, w \in A$.

Similarly, one can show that

$$H([x, x, x], [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in A$. Hence the **C**-bilinear mapping $H : A^2 \to B$ is a ternary bihomomorphism.

Corollary 7 *Let $r > 6$ and θ be nonnegative real numbers and $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$ and (28). Then there exists a unique* **C**-*bilinear mapping $H : A^2 \to B$ such that*

$$\|f(x, z) - H(x, z)\| \leq \frac{(2^r + 4)\theta}{2^r - 4}\|x\|^r + \frac{4\theta}{2^r - 4}\|z\|^r$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to B$ satisfies

$$\|f([x, y, z], [w, w, w]) - [f(x, w), f(y, w), f(z, w)]\|$$
$$\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r), \quad (36)$$
$$\|f([x, x, x], [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\|$$
$$\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \quad (37)$$

for all $x, y, z, w \in A$, then the **C**-*bilinear mapping $H : A^2 \to B$ is a ternary bihomomorphism.*

Theorem 10 *Let $\varphi : A^4 \to$* **R** *be a function satisfying (15) and let $f : A^2 \to B$ be a mapping satisfying (24) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C**-*bilinear mapping $H : A^2 \to B$ satisfying (32), where D is replaced by H in (32).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (34) and (35), then the **C**-*bilinear mapping $H : A^2 \to B$ is a ternary bihomomorphism.*

Proof The proof is similar to the proof of Theorem 9.

Corollary 8 *Let $r < 2$ and θ be nonnegative real numbers and $f : A^2 \to B$ be a mapping satisfying (28) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C**-*bilinear mapping $H : A^2 \to B$ such that*

$$\|f(x, z) - H(x, z)\| \leq \frac{(4 + 2^r)\theta}{4 - 2^r}\|x\|^r + \frac{4\theta}{4 - 2^r}\|z\|^r$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to B$ satisfies (36) and (37), then the **C**-*bilinear mapping $H : A^2 \to B$ is a ternary bihomomorphism.*

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Hyers–Ulam Stability of an Additive-Quadratic Functional Equation

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Abstract Using the fixed point method and the direct method, we prove the Hyers–Ulam stability of Lie biderivations and Lie bihomomorphisms in Lie Banach algebras, associated with the bi-additive functional inequality

$$\begin{aligned} &\| f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) \\ &\quad + f(x-y, z-w) - 4f(x, z)\| \\ &\leq \|s\,(2f\,(x+y, z-w) + 2f\,(x-y, z+w) - 4f(x, z) + 4f(y, w))\|\,, \end{aligned} \tag{1}$$

where s is a fixed nonzero complex number with $|s| < 1$.

1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy

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difference by a general control function in the spirit of Rassias' approach. Park [18–20] defined additive ρ-functional inequalities and proved the Hyers–Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. Various functional equations and functional inequalities have been extensively investigated by a number of authors (see [11–13, 16]).

We recall a fundamental result in fixed point theory.

Theorem 1 ([3, 6]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 7, 23]).

Maksa [14, 15] introduced and investigated biderivations and symmetric biderivations on rings. Öztürk and Sapanci [17], Vukman [26] and Yazarli [27] investigated some properties of symmetric biderivations on rings.

Definition 1 ([14, 15]) Let A be a ring. A bi-additive mapping $D : A \times A \to A$ is called a *symmetric biderivation* on A if D satisfies

$$D(xy, z) = D(x, z)y + xD(y, z),$$
$$D(x, y) = D(y, x)$$

for all $x, y, z \in A$.

In this paper, we introduce biderivations and bihomomorphisms in Lie Banach algebras.

Let A be a Lie Banach algebra. Suppose that a $\mathbf{C}$-bilinear mapping $D : A \times A \to A$ is a Lie derivation in each variable, i.e.,

$$D([x, y], z) = [D(x, z), y] + [x, D(y, z)],$$
$$D(x, [z, w]) = [D(x, z), w] + [z, D(x, w)]$$

for all $x, y, z, w \in A$. It is easy to show that

$$D([x, y], [z, w]) = [[D(x, z), w], y] + [[z, D(x, w)], y] + [x, [D(y, z), w]] + [x, [z, D(y, w)]]$$

for all $x, y, z, w \in A$.

Definition 2 Let A be a complex Lie Banach algebra. A **C**-bilinear mapping $D : A \times A \to A$ is called a *Lie biderivation* on A if D satisfies

$$D([x, y], [z, w]) = [[D(x, z), w], y] + [[z, D(x, w)], y] + [x, [D(y, z), w]] + [x, [z, D(y, w)]]$$

for all $x, y, z, w \in A$.

Definition 3 Let A and B be complex Lie Banach algebras. A **C**-bilinear mapping $H : A \times A \to B$ is called a *Lie bihomomorphism* if H satisfies

$$H([x, y], [z, w]) = [H(x, z), H(y, w)]$$

for all $x, y, z, w \in A$.

This paper is organized as follows: In Sects. 2 and 3, we prove the Hyers–Ulam stability of Lie biderivations and Lie bihomomorphisms in Lie Banach algebras associated with the bi-additive s-functional inequality (1) by using the direct method. In Sects. 4 and 5, we prove the Hyers–Ulam stability of Lie biderivations and Lie bihomomorphisms in Lie Banach algebras associated with the bi-additive s-functional inequality (1) by using the fixed point method.

Throughout this paper, let X be a complex normed space and Y be a complex Banach space. Let A and B be Lie Banach algebras. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2 Hyers–Ulam Stability of Lie biderivations on Lie Banach Algebras: Direct Method

We investigate the bi-additive s-functional inequality (1) in complex normed spaces.

Lemma 1 ([21, Lemma 2.1]) *If a mapping $f : X^2 \to Y$ satisfies $f(0, z) = f(x, 0) = 0$ and*

$$\begin{aligned}&\|f(x + y, z + w) + f(x + y, z - w) + f(x - y, z + w) \\ &\quad + f(x - y, z - w) - 4f(x, z)\| \\ &\leq \|s\,(2f\,(x + y, z - w) + 2f\,(x - y, z + w) - 4f(x, z) + 4f(y, w))\|\end{aligned}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

In [22], Park proved the Hyers–Ulam stability of the bi-additive s-functional inequality (1) in complex Banach spaces.

Theorem 2 ([22, Theorem 2.2]) *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2}$$

for all $x, y \in X$ and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and

$$\begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) \\ &\quad + f(x-y, z-w) - 4f(x, z)\| \end{aligned} \tag{3}$$

$$\leq \|s\,(2f\,(x+y, z-w) + 2f\,(x-y, z+w) - 4f(x, z) + 4f(y, w))\| + \varphi(x, y)\varphi(z, w)$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x, z) - P(x, z)\| \leq \frac{1}{4(1-|s|)}\Psi(x, x)\varphi(z, 0) \tag{4}$$

for all $x, z \in X$, where

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x, y \in X$.

Theorem 3 ([22, Theorem 2.2]) *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \tag{5}$$

for all $x, y \in X$ and $f : X^2 \to Y$ be a mapping satisfying (3) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x, z) - P(x, z)\| \leq \frac{1}{2(1-|s|)}\Psi(x, x)\varphi(z, 0) \tag{6}$$

for all $x, z \in X$.

Now, we investigate Lie biderivations on Lie Banach algebras associated with the bi-additive s-functional inequality (1).

Lemma 2 ([2, Lemma 2.1]) *Let $f : X^2 \to Y$ be a bi-additive mapping such that $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $x, z \in X$ and $\lambda, \mu \in S^1 := \{\nu \in \mathbf{C} \ : \ |\nu| = 1\}$. Then f is $\mathbf{C}$-bilinear.*

Theorem 4 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (2) with $X = A$ and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x+y), \mu(z+w)) + f(\lambda(x+y), \mu(z-w)) + f(\lambda(x-y), \mu(z+w)) \\ &\qquad + f(\lambda(x-y), \mu(z-w)) - 4\lambda\mu f(x, z)\| \\ &\quad \le \|s\,(2f\,(x+y, z-w) + 2f\,(x-y, z+w) - 4f(x, z) + 4f(y, w))\| \\ &\qquad + \varphi(x, y)\varphi(z, w) \end{aligned} \tag{7}$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $D : A^2 \to A$ satisfying (4) with $X = A$, where P is replaced by D in (4).

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\begin{aligned} &\|f([x, y], [z, w]) - [[f(x, z), w], y] - [[z, f(x, w)], y] \\ &\qquad - [x, [f(y, z), w]] - [x, z, f(y, w)]]\| \\ &\quad \le \varphi(x, y)\varphi(z, w) \end{aligned} \tag{8}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof Let $\lambda = \mu = 1$ in (7). By Theorem 2, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (4) defined by

$$D(x, z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

Letting $y = w = 0$ in (7), we get $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $x, z \in A$ and all $\lambda, \mu \in S^1$. By Lemma 2, the bi-additive mapping $D : A^2 \to A$ is $\mathbf{C}$-bilinear.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

It follows from (8) that

$$\begin{aligned} &\|D([x, y], [z, w]) - [[D(x, z), w], y] - [[z, D(x, w)], y] \\ &\qquad - [x, [D(y, z), w]] - [x, z, D(y, w)]]\| \\ &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x, y]}{2^n \cdot 2^n}, [z, w]\right) - \left[\left[f\left(\frac{x}{2^n}, z\right), w\right], \frac{y}{2^n}\right] - \left[\left[z, f\left(\frac{x}{2^n}, w\right)\right], \frac{y}{2^n}\right] \right. \\ &\quad \left. - \left[\frac{x}{2^n}, \left[f\left(\frac{y}{2^n}, z\right), w\right]\right] - \left[\frac{z}{2^n}, \left[z, f\left(\frac{y}{2^n}, w\right)\right]\right] \right\| \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\varphi(z, w) = 0 \end{aligned}$$

for all $x, y, z, w \in A$. Thus

$$\begin{aligned} D([x, y], [z, w]) &= [[D(x, z), w], y] + [[z, D(x, w)], y] \\ &+ [x, [D(y, z), w]] + [x, z, [D(y, w)]] \end{aligned}$$

for all $x, y, z, w \in A$. Hence the mapping $f : A^2 \to A$ is a Lie biderivation.

Corollary 1 *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x + y), \mu(z + w)) + f(\lambda(x + y), \mu(z - w)) + f(\lambda(x - y), \mu(z + w)) \\ &\quad + f(\lambda(x - y), \mu(z - w)) - 4\lambda\mu f(x, z)\| \qquad (9) \\ &\le \|s\,(2f\,(x + y, z - w) + 2f\,(x - y, z + w) - 4f(x, z) + 4f(y, w))\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned}$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique **C***-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x, z) - D(x, z)\| \le \frac{\theta}{(1 - |s|)(2^r - 2)}\|x\|^r\|z\|^r \qquad (10)$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\begin{aligned} &\|f([x, y], [z, w]) - [[f(x, z), w], y] - [[z, f(x, w)], y] \\ &\quad - [x, [f(y, z), w]] - [x, z, f(y, w)]]\| \\ &\le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \qquad (11) \end{aligned}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof The proof follows from Theorem 4 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$.

Theorem 5 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (5) with $X = A$ and $f : A^2 \to A$ be a mapping satisfying (7) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $D : A^2 \to A$ satisfying (6) with $X = A$.*

If, in addition, the mapping $f : A^2 \to A$ satisfies (8) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof The proof is similar to the proof of Theorem 4.

Corollary 2 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying (9) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z) - D(x,z)\| \le \frac{\theta}{(1-|s|)(2-2^r)}\|x\|^r\|z\|^r \tag{12}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (11) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof The proof follows from Theorem 5 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$.

3 Hyers–Ulam Stability of Lie Bihomomorphisms in Lie Banach Algebras: Direct Method

Now, we investigate Lie bihomomorphisms in Lie Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 6 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (2) with $X = A$ and $f : A^2 \to B$ be a mapping satisfying (7) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $H : A^2 \to B$ satisfying (4) with $X = A$ and $Y = B$, where P is replaced by H in (4).*

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f([x, y], [z, w]) - [f(x, z), f(y, w)]\| \le \varphi(x, y)\varphi(z, w) \tag{13}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Proof By the same reasoning as in the proof of Theorem 4, there is a unique $\mathbf{C}$-bilinear mapping $H : A^2 \to B$, which is defined by

$$H(x, z) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $H(x, z) = f(x, z)$ for all $x, z \in A$.

It follows from (13) that

$$\begin{aligned}&\|H([x, y], [z, w]) - [H(x, z), H(y, w)]\|\\ &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x, y]}{2^n \cdot 2^n}, [z, w]\right) - \left[f\left(\frac{x}{2^n}, z\right), f\left(\frac{y}{2^n}, w\right)\right]\right\|\\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\varphi(z, w) = 0\end{aligned}$$

for all $x, y, z, w \in A$. Thus

$$H([x, y], [z, w]) = [H(x, z), H(y, w)]$$

for all $x, y, z, w \in A$. Hence the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Corollary 3 *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (9) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $H : A^2 \to B$ satisfying (10) with $X = A$ and $Y = B$, where P is replaced by H in (10).*

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f([x, y], [z, w]) - [f(x, z), f(y, w)]\| \le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \quad (14)$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Theorem 7 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (5) with $X = A$ and $f : A^2 \to B$ be a mapping satisfying (7) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $H : A^2 \to B$ satisfying (6) with $X = A$ and $Y = B$, where P is replaced by H in (6).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (13) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Proof The proof is similar to the proof of Theorem 6.

Corollary 4 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (9) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $H : A^2 \to B$ satisfying (12) with $X = A$ and $Y = B$, where D is replaced by H in (12).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (14) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

4 Hyers–Ulam Stability of Lie Biderivations on Lie Banach Algebras: Fixed Point Method

Using the fixed point method, Park [22] proved the Hyers–Ulam stability of the bi-additive s-functional inequality (1) in complex Banach spaces.

Theorem 8 *[22, Theorem 4.1] Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x, y) \le \frac{L}{2}\varphi(x, y) \quad (15)$$

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying (3) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z)-P(x,z)\| \le \frac{L}{4(1-|s|)(1-L)}\varphi(x,x)\varphi(z,0) \tag{16}$$

for all $x, z \in X$.

Using the fixed point method, we prove the Hyers–Ulam stability of Lie biderivations on Lie Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 9 *Let* $\varphi : A^2 \to [0,\infty)$ *be a function satisfying (15) with* $A = X$ *and* $f : A^2 \to A$ *be a mapping satisfying (7) and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in A$. *Then there exists a unique* $\mathbf{C}$*-bilinear mapping* $D : A^2 \to A$ *satisfying (16) with* $X = A$.

If, in addition, the mapping $f : A^2 \to A$ *satisfies (8) and* $f(2x,z) = 2f(x,z)$ *for all* $x, z \in A$, *then the mapping* $f : A^2 \to A$ *is a Lie biderivation.*

Proof Let $\lambda = \mu = 1$ in (5). By Theorem 8, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (16) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

Letting $y = w = 0$ in (5), we get $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x, z \in A$ and all $\lambda, \mu \in S^1$. By Lemma 2, the bi-additive mapping $D : A^2 \to A$ is $\mathbf{C}$-bilinear.

The rest of the proof is similar to the proof of Theorem 4.

Corollary 5 *Let* $r > 2$ *and* θ *be nonnegative real numbers, and* $f : A^2 \to A$ *be a mapping satisfying (8) and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in A$. *Then there exists a unique* $\mathbf{C}$*-bilinear mapping* $D : A^2 \to A$ *satisfying (10).*

If, in addition, the mapping $f : A^2 \to A$ *satisfies (9), (10) and* $f(2x,z) = 2f(x,z)$ *for all* $x, z \in A$, *then the mapping* $f : A^2 \to A$ *is a Lie biderivation.*

Proof The proof follows from Theorem 9 by taking $L = 2^{1-r}$ and $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$.

Theorem 10 ([22, Theorem 4.4]) *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{17}$$

for all $x, y \in X$. *Let* $f : X^2 \to Y$ *be a mapping satisfying (4) and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in X$. *Then there exists a unique bi-additive mapping* $P : X^2 \to Y$ *such that*

$$\|f(x,z)-P(x,z)\| \le \frac{1}{4(1-|s|)(1-L)}\varphi(x,x)\varphi(z,0) \tag{18}$$

for all $x, z \in X$.

Theorem 11 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (17) with $X = A$ and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and (5). Then there exists a unique* **C***-bilinear mapping $D : A^2 \to A$ satisfying (18).*

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and (7), then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof The proof is similar to the proof of Theorem 9.

Corollary 6 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying (8) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $D : A^2 \to A$ satisfying (12).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (9), (10) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a Lie biderivation.

Proof The proof follows from Theorem 11 by taking $L = 2^{r-1}$ and $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$.

5 Hyers–Ulam Stability of Lie Bihomomorphisms in Lie Banach Algebras: Fixed Point Method

Using the fixed point method, we prove the Hyers–Ulam stability of Lie bihomomorphisms in Lie Banach algebras associated with the bi-additive s-functional inequality (1).

Theorem 12 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (15) with $X = A$ and $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$ and (5). Then there exists a unique* **C***-bilinear mapping $H : A^2 \to B$ satisfying (16) with $X = A$ and $Y = B$, where P is replaced by H in (16).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (13) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Proof By Theorem 9, there is a unique **C**-bilinear mapping $H : A^2 \to B$ satisfying (16) defined by

$$H(x, z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 6.

Corollary 7 *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (8) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $H : A^2 \to B$ satisfying (10).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (14) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Theorem 13 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (17) with $X = A$ and $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and (5). Then there exists a unique* **C***-bilinear mapping $H : A^2 \to B$ satisfying (18) with $X = A$ and $Y = B$, where P is replaced by H in (18).*

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$ and (13), then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

Proof The proof is similar to the proof of Theorem 12.

Corollary 8 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to B$ be a mapping satisfying (8) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique* **C***-bilinear mapping $H : A^2 \to B$ satisfying (12).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (14) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a Lie bihomomorphism.

6 Conclusions

Using the fixed point method and the direct method, we have proved the Hyers–Ulam stability of Lie biderivations and Lie bihomomorphisms in Lie Banach algebras, associated with the bi-additive functional inequality (1).

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Orthogonal Dirichlet Polynomials

Doron S. Lubinsky

Abstract Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers. Let w be a non-negative function, integrable on the real line. One can form orthogonal Dirichlet polynomials $\{\phi_n\}$ from linear combinations of $\left\{\lambda_j^{-it}\right\}_{j=1}^{n}$, satisfying the orthogonality relation

$$\int_{-\infty}^{\infty} \phi_n(t)\,\overline{\phi_m(t)}\,w(t)\,dt = \delta_{mn}.$$

Weights that have been considered include the arctan density $w(t) = \frac{1}{\pi(1+t^2)}$; rational function choices of w; $w(t) = e^{-t}$; and $w(t)$ constant on an interval symmetric about 0. We survey these results and discuss possible future directions.

1 Introduction

Throughout, let

$$\{\lambda_j\}_{j=1}^{\infty} \text{ be a sequence of distinct positive numbers.} \tag{1.1}$$

Given $m \geq 1$, a *Dirichlet polynomial of degree* $\leq m$ [17, 23] associated with this sequence of exponents has the form

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$$\sum_{n=1}^{m} a_n \lambda_n^{-it} = \sum_{n=1}^{m} a_n e^{-i(\log \lambda_n)t},$$

where $\{a_n\} \subset \mathbb{C}$. We denote the set of all such polynomials by $\mathcal{L}_m$.

The theory of almost-periodic functions [2, 3] is based on orthogonality in the mean:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \lambda_j^{-it} \overline{\lambda_k^{-it}} dt = \delta_{jk}.$$

Thus in an asymptotic sense, the "monomials" $\left\{\lambda_j^{-it}\right\}_{j\geq 1}$ are orthonormal polynomials. In the hope that a more standard orthogonality relation might have some advantages, the author [6] introduced Dirichlet orthogonal polynomials associated with the arctan density.

In the general case, one can consider a non-negative function w, integrable on the real line, and positive on a set of positive measure. The corresponding orthonormal polynomials $\phi_n \in \mathcal{L}_n$ have positive leading coefficient and satisfy

$$\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} w(t)\, dt = \delta_{mn}, \quad m, n \geq 1.$$

If we use as the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t) \overline{g(t)} w(t)\, dt$$

and assume $\int_{-\infty}^{\infty} w = 1$, then ϕ_n admits the representation

$$\phi_n(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1} A_n}} \times \det \begin{bmatrix} \lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\ 1 & \left(\lambda_1^{-it}, \lambda_2^{-it}\right) & \left(\lambda_1^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_1^{-it}, \lambda_n^{-it}\right) \\ \left(\lambda_2^{-it}, \lambda_1^{-it}\right) & 1 & \left(\lambda_2^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_2^{-it}, \lambda_n^{-it}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\lambda_{n-1}^{-it}, \lambda_1^{-it}\right) & \left(\lambda_{n-1}^{-it}, \lambda_2^{-it}\right) & \left(\lambda_{n-1}^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_{n-1}^{-it}, \lambda_n^{-it}\right) \end{bmatrix}, \tag{1.2}$$

where

$$A_n = \det \left[\left(\lambda_j^{-it}, \lambda_k^{-it}\right)\right]_{1\leq j,k\leq n}. \tag{1.3}$$

The leading coefficient of $\phi_n(x)$ is

$$\gamma_n = \sqrt{\frac{A_{n-1}}{A_n}}.$$

In analyzing orthonormal polynomials, the reproducing kernels

$$K_n(x, y) = \sum_{j=1}^{n} \phi_j(x) \overline{\phi_j(y)}$$

are useful. The nth *Christoffel function* is

$$1/K_n(x, x) = 1/\sum_{j=1}^{n} \left|\phi_j(x)\right|^2.$$

The extremal property

$$K_n(x, x) = \sup_{P \in \mathcal{L}_n} \frac{|P(x)|^2}{\int_{-\infty}^{\infty} |P|^2 w}$$

facilitates estimation of $K_n(x, x)$ and the Christoffel function. The extremal property is an easy consequence of the Cauchy-Schwarz inequality.

Examples of weights w for which some analysis has been undertaken are the arctan density

$$w(t) = \frac{1}{\pi\left(1 + t^2\right)}, \quad t \in \mathbb{R};$$

rational functions of special form; $w(t) = e^{-t}, t \in [0, \infty)$ and $w(t) = 1$ on $[-T, T]$, $T > 0$. We shall survey some of the results in Sects. 2–5. It seems of some interest to develop also a theory for general weights.

One reason for studying Dirichlet orthogonal polynomials is that they might offer some insight into the behavior of general Dirichlet polynomials, just as classical orthogonal polynomials are useful in analyzing algebraic polynomials $P(x) = \sum_{j=0}^{n} c_j x^j$. There is of course a vast literature on Dirichlet polynomials, with connections to Turán's formulation of the Lindelöf hypothesis, Hilbert's inequality, the large sieve of number theory, the Montgomery-Vaughn theory, and higher dimensional results such as the Vinogradov Mean Value Theorem. We cannot hope to review these here, but present a few results relevant to our topic:

The classical conjecture of Lindelöf asserts that given $\varepsilon > 0$, the Riemann ζ function admits the bound

$$|\zeta(s + it)| \leq C(\varepsilon)(2 + |t|)^{\varepsilon}$$

provided $s \geq \frac{1}{2}$ and $s+it$ lies outside a small disk centered on 1. Using a very simple argument, Turán showed in a 1962 paper [22] that this conjecture is equivalent to the estimate on a specific Dirichlet polynomial: given $\varepsilon > 0$, we have for all real t and $n \geq 1$,

$$\left| \sum_{j=1}^{n} (-1)^j \, j^{-it} \right| < C\left(\varepsilon\right) n^{\frac{1}{2}+\varepsilon} \left(2 + |t|\right)^{\varepsilon}.$$

Another classical connection, to Hilbert's inequality, involves the Montgomery-Vaughan refinement of the Mean Value Theorem. There are several versions, among them [13, 14], [15, p. 74, Corollary 2]

$$\int_0^T \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt = T \sum_{j=1}^{n} \left| a_j^2 \right| + 3\pi\theta \sum_{j=1}^{n} \left| a_j^2 \right| \delta_j^{-1}. \tag{1.4}$$

Here $T > 0$, and (in the notation here):

$$\delta_j = \min\left\{ \left| \log \lambda_j - \log \lambda_k \right| : k \neq j, k \leq n \right\},$$

while $|\theta| \leq 1$.

A much more recent result is Weber's Mean Value Theorem [24] when there are non-negative coefficients:

$$\int_0^T \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^{2q} dt \geq cT \left(\sum_{j=1}^{n} a_j^2 \right)^q.$$

Here we assume that q is a positive integer, all $a_j \geq 0$, while c is independent of $N, \left\{a_j\right\}, \left\{\lambda_j\right\}$.

This paper is organized as follows: in Sect. 2, we review results for the arctangent density. In Sect. 3, we consider the exponential weight and the connection to Müntz orthogonal polynomials. In Sect. 4, we look at rational weights, and in Sect. 5, we look at constant weights on $[-T, T]$.

2 The Arctangent Density

Let

$$w(t) = \frac{1}{\pi\left(1+t^2\right)}, \quad t \in \mathbb{R}.$$

We also assume that

$$1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

It was shown in [6] that $\phi_1 = 1$ and for $n \geq 2$,

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.$$

Here it is essential that the $\{\lambda_j\}$ are increasing, while it is intriguing that ϕ_n involves only the last two powers. The proof of course is elementary, and based on the following integral (itself a simple consequence of the residue theorem):

$$\int_{-\infty}^{\infty} \frac{e^{i\mu t}}{\pi\left(1+t^2\right)} dt = e^{-|\mu|}.$$

The nth reproducing kernel along the diagonal is given for real x by [6, p. 46]

$$K_n(x,x) = 1 + \sum_{n=1}^{m} \frac{1}{\lambda_n^2 - \lambda_{n-1}^2}\left[(\lambda_n - \lambda_{n-1})^2 + 4\lambda_{n-1}\lambda_n \sin^2\left(\frac{x}{2}\log\frac{\lambda_n}{\lambda_{n-1}}\right)\right].$$

Because of the simple explicit form, it is easy to do analysis. Thus one can check that

$$\sup_{t\in\mathbb{R}} |\phi_n(t)| = \sqrt{\frac{\lambda_n + \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}}$$

while

$$\sup_{t\in\mathbb{R}} \left|\phi_n'(t)\right| = \frac{\lambda_n \log\lambda_n + \lambda_{n-1}\log\lambda_{n-1}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.$$

The zeros of ϕ_n have the form $-i + \frac{2k\pi}{\log(\lambda_n/\lambda_{n-1})}$, $k \in \mathbb{Z}$.

If $\lambda_m \to \infty$, as $m \to \infty$, the reproducing kernel admits the asymptotic

$$\lim_{m\to\infty} \frac{1}{\log\lambda_m} K_m(x,x) = \frac{1+x^2}{2},$$

uniformly for x in compact subsets of the real line. The universality limit takes the form

$$\lim_{m\to\infty}\frac{1}{\log\lambda_m}K_m\left(x+\frac{\alpha}{\log\lambda_m},x+\frac{\beta}{\log\lambda_m}\right)=\frac{1+x^2}{2}e^{i(\beta-\alpha)/2}\mathbb{S}\left(\frac{\alpha-\beta}{2}\right),$$

where

$$\mathbb{S}(t)=\frac{\sin t}{t}$$

is the usual sinc kernel. The limit holds uniformly for x in compact subsets of $\mathbb{R}$ and α,β in compact subsets of $\mathbb{C}$. Markov-Bernstein inequalities for derivatives of Dirichlet polynomials were also established in [6].

Orthonormal expansions in the $\{\phi_n\}$ were also considered there, and in the follow up paper [7]. For example, it was shown using such orthonormal expansions that if

$$f(t)=\sum_{n=1}^{\infty}a_n\lambda_n^{-it}$$

where the coefficients are complex numbers, and $r>0$, then

$$\int_{-\infty}^{\infty}|f(rt)|^2\frac{dt}{\pi\left(1+t^2\right)}=\sum_{k=1}^{\infty}\left(\lambda_k^{2r}-\lambda_{k-1}^{2r}\right)\left|\sum_{n=k}^{\infty}\frac{a_n}{\lambda_n^r}\right|^2,$$

provided the series on the right-hand side converges. This was used to establish a number of inequalities of Hilbert/mean value type. If for example, $r>0$ and $\{a_k\}$ are non-negative numbers with $\left\{a_k/\lambda_k^r\right\}$ decreasing, then

$$F(t)=\sum_{n=1}^{\infty}(-1)^{n-1}a_n\lambda_n^{-it}$$

satisfies

$$\int_{-\infty}^{\infty}|F(rt)|^2\frac{dt}{\pi\left(1+t^2\right)}\le\sum_{n=1}^{\infty}a_n^2.$$

M. Weber used the orthonormal expansions above in studying Cauchy means of Dirichlet polynomials and series, with a more definitive version of the limits for orthonormal expansions than given in [6, 7]. For example, he proved that if q is a positive integer, and $\{a_n\}$ are complex, [26, p. 65, Proposition 1.4]

$$\lim_{s\to\infty}\int_{-\infty}^{\infty}\left|\sum_{j=1}^{\infty}a_j j^{-ist}\right|^{2q}\frac{dt}{\pi\left(1+t^2\right)}=\lim_{s\to\infty}\frac{1}{2s}\int_{-\infty}^{\infty}\left|\sum_{j=1}^{\infty}a_j j^{-it}\right|^{2q}dt,$$

provided the limit on the right exists. He established estimates such as [26, p. 65, Proposition 1.5]

$$\frac{1}{S}\int_0^S \left|\sum_{j=1}^{n} a_j j^{-it}\right|^{2q} dt \le \frac{2\pi}{\log 2} \sup_{S\le s\le 2S} \int_{-\infty}^{\infty} \left|\sum_{j=1}^{N} a_j j^{-ist}\right|^{2q} \frac{dt}{\pi\left(1+t^2\right)}.$$

Another application has been given by D. Dimitrov and W.D. Oliviera [5], to finding the Dirichlet polynomials that minimize

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|P\left(\frac{1}{p}+it\right)\right|^2 \frac{dt}{\frac{1}{p}+t^2}$$

among all Dirichlet polynomials of degree $\le n$ satisfying the interpolation conditions $P\left(\frac{1}{p}+it_j\right)=1$, at m distinct points $\left\{t_j\right\}_{j=1}^m$. See also [16].

3 Laguerre Weight

Let

$$w(t)=e^{-t}, \quad t\in[0,\infty),$$

so that our orthogonality relation becomes

$$\int_0^{\infty} \phi_n(t)\,\overline{\phi_m(t)}e^{-t}dt=\delta_{mn}. \tag{3.1}$$

In [8], it was shown that

$$\phi_n(t)=\frac{\Delta_n}{2\pi i}\int_{\Gamma} e^{-tz}R_n(t)\,dt,$$

where Γ is a simple closed positively oriented curve in the half plane $\mathrm{Re}\, z>-1$ that encloses $i\log\lambda_j$, $1\le j\le n$, while

$$R_n(z)=\frac{1}{z-i\log\lambda_n}\prod_{j=1}^{n-1}\left(1+\frac{1}{z-i\log\lambda_j}\right);$$

$$\Delta_n=\frac{D_n}{|D_n|};$$

and

$$D_n = \prod_{j=1}^{n-1} \left(1 + \left[i \log \frac{\lambda_j}{\lambda_n} \right]^{-1} \right).$$

For $x \in (0, \infty)$, there is the simplified form

$$\phi_n(x) = -\Delta_n \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} R_n(-\alpha + is)\, ds.$$

Here $\alpha \in (0, 1)$. It was shown there that

$$\phi_n(x) = \sum_{j=1}^{n} B_{nj} \lambda_j^{-ix}, \tag{3.2}$$

where

$$B_{nj} = \frac{\Delta_n}{i \log \frac{\lambda_j}{\lambda_n}} \prod_{k=1, k \neq j}^{n-1} \left(1 + \frac{1}{i \log \frac{\lambda_j}{\lambda_k}} \right).$$

In addition, formulae were given for ϕ_n' and Markov-Bernstein inequalities were established. Among the more interesting inequalities established are the bounds

$$e^{-x} \sum_{j=1}^{n} \left| \phi_j(x) \right|^2 \leq \sum_{j=1}^{n} \left| \phi_j(0) \right|^2 = n.$$

Moreover, the left-hand side is a decreasing function of $x \in [0, \infty)$. Similarly,

$$\begin{aligned} &e^{-x} \sum_{j=1}^{n} \left| \phi_j'(x) \right|^2 \\ &\leq \sum_{j=1}^{n} \left| \phi_j'(0) \right|^2 = \frac{n(n-1)(2n-1)}{6} + \sum_{j=1}^{n} \left(\log \lambda_j \right)^2, \end{aligned}$$

and the left-hand side is also a decreasing function of x.

As it turns out, many of the above results were not new, and subsumed by existing results on Müntz orthogonal polynomials. Suppose we make the substitution $x = e^{-t}$ in (3.1). We obtain

$$\int_0^1 \phi_n\left(\log \frac{1}{x}\right) \overline{\phi_m\left(\log \frac{1}{x}\right)} dx = \delta_{mn},$$

and

$$\phi_n\left(\log\frac{1}{x}\right)=\sum_{j=1}^{n}B_{nj}x^{-i\lambda_j}.$$

These are Müntz orthogonal polynomials that were explored in the Russian literature as far back as 1955—see [1, 21]. An excellent reference is the beautiful book of Borwein and Erdelyi [4, p. 125 ff.]. As the author knows much of that book well, he ought to have noticed the connection.

The treatment in [4] allows complex λ_j, so let us change notation: given complex ρ_j with Re $\rho_j > -\frac{1}{2}$, $j \geq 0$, define the nth Müntz-Legendre polynomial

$$L_n(x)=\frac{1}{2\pi i}\int_{\Gamma}\prod_{k=0}^{n-1}\frac{t+\overline{\varrho_k}+1}{t-\rho_k}\frac{x^t}{t-\rho_n}dt.$$

Here Γ is a simple closed positively oriented curve enclosing all the $\left\{\rho_j\right\}$. It can be shown that L_n is a linear combination of $\{x^{\rho_j}\}_{j=0}^{n}$ admitting the orthogonality relation

$$\int_0^1 L_n(x)\,\overline{L_m(x)}dx=\delta_{mn}\frac{1}{1+2\text{Re }\rho_n}.$$

Müntz orthogonal polynomials have been used in numerical quadrature [11, 12]. A thorough study of their asymptotics was undertaken by Ulfar Stefansson. See for example [18, 19].

4 Rational Weights

Since the formulae for the arctan density are so simple, it is natural to try generalize them to linear combinations of scaled arctan densities. Let

$$w(t)=\sum_{m=1}^{L}\frac{c_m}{\pi\left(1+(b_m t)^2\right)}, \tag{4.1}$$

where $L \geq 2$, the $\left\{c_j\right\}$ are real, and

$$1=b_1<b_2<\cdots<b_m. \tag{4.2}$$

One would also hope to preserve the simple structure for the arctan density. Some guidance is provided by expressing ϕ_n of Sect. 2, in the determinant form (1.2):

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} = -\frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \det \begin{bmatrix} \lambda_{n-1}^{-it} & \lambda_n^{-it} \\ \lambda_{n-1}^{-1} & \lambda_n^{-1} \end{bmatrix}.$$

By analogy, define for $n \geq L$,

$$\psi_n(t) = \det \begin{bmatrix} \lambda_{n-L}^{it} & \lambda_{n-L+1}^{it} & \cdots & \lambda_{n-1}^{it} & \lambda_n^{it} \\ \lambda_{n-L}^{-1/b_1} & \lambda_{n-L+1}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-L}^{-1/b_{L-1}} & \lambda_{n-L+1}^{-1/b_{L-1}} & \cdots & \lambda_{n-1}^{-1/b_{L-1}} & \lambda_n^{-1/b_{L-1}} \\ \lambda_{n-L}^{-1/b_L} & \lambda_{n-L+1}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L} \end{bmatrix}. \tag{4.3}$$

Observe that $\psi_n(t)$ is a linear combination of only $\left\{\lambda_j^{-it}\right\}_{n-L \leq j \leq n}$. Also define for a given fixed n, and $j \geq 1$, $1 \leq m \leq L$,

$$d_{jm} = \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_j^{it}}{\pi\left(1 + (b_m t)^2\right)} dt \tag{4.4}$$

and let B be the $(L-1) \times L$ matrix

$$B = \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \end{bmatrix} \tag{4.5}$$

and

$$D = \det \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ d_{n,1} & d_{n,2} & \cdots & d_{n,L} \end{bmatrix}. \tag{4.6}$$

In [9] we proved:

Proposition 4.1 *Let* $\mathbf{c} = [c_1\ c_2 \ldots c_L]^T$ *be taken as any non-trivial solution of* $B\mathbf{c} = \mathbf{0}$. *Let* w *be as in (4.1). Then for* $1 \leq j \leq n-1$,

$$\int_{-\infty}^{\infty} \psi_n(t) \lambda_j^{it} w(t)\, dt = 0. \tag{4.7}$$

*If D defined by (4.6) is non-*0*, then we can take*

$$w(t) = A \det \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ \frac{1}{\pi\left(1+(b_1 t)^2\right)} & \frac{1}{\pi\left(1+(b_2 t)^2\right)} & \cdots & \frac{1}{\pi\left(1+(b_L t)^2\right)} \end{bmatrix}, \tag{4.8}$$

for any $A \neq 0$*, while*

$$\int_{-\infty}^{\infty} \psi_n(t)\,\lambda_n^{it} w(t)\,dt = AD.$$

Only in the case $L = 2$, could we prove positivity of the weight, with appropriately chosen $0 < c_1 < c_2$. It seems a worthwhile project to investigate if for $L \geq 3$ that the weight can be chosen to be of one sign.

If one can prove positivity of w for arbitrary L, there is the hope that one can use such rational weights to approximate general weights in much the same way as Bernstein-Szegő weights are used in the theory of "algebraic" orthogonal polynomials [20]. However, this might be quite a reach, as there is at present no indication that even if we could prove positivity, that there is the wealth of detail and formulae that make Bernstein-Szegő weights such a valuable tool.

5 Legendre Weight

A natural choice for the weight is the Legendre weight $w = \text{Constant}$ on some interval or subset of the real line. In [10], we considered the normalized Legendre weight $w = \frac{1}{2T}$ on $[-T, T]$ for $T > 0$. To emphasize the dependence on $T > 0$, we denote the Dirichlet orthogonal polynomial by $\phi_{n,T}$, with positive leading coefficient $\gamma_{n,T}$, such that

$$\left(\phi_{n,T}, \phi_{m,T}\right)_T = \frac{1}{2T}\int_{-T}^{T} \phi_{n,T}(t)\,\overline{\phi_{m,T}(t)}dt = \delta_{mn}.$$

The nth reproducing kernel is

$$K_{n,T}(u, v) = \sum_{j=1}^{n} \phi_{j,T}(u)\,\overline{\phi_{j,T}(v)}.$$

Let, as above,

$$\mathbb{S}(u) = \frac{\sin u}{u}$$

denote the sinc kernel. From

$$\frac{1}{2T}\int_{-T}^{T} \left(\lambda_j/\lambda_k\right)^{-it} dt = \mathbb{S}\left(T \log\left(\lambda_j/\lambda_k\right)\right),$$

the determinantal representation (1.2) becomes

$$\phi_{n,T}(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1,T}A_{n,T}}}$$
$$\times \det \begin{bmatrix} \lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\ 1 & \mathbb{S}(T\log\lambda_1/\lambda_2) & \mathbb{S}(T\log\lambda_1/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_1/\lambda_n) \\ \mathbb{S}(T\log\lambda_2/\lambda_1) & 1 & \mathbb{S}(T\log\lambda_2/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_2/\lambda_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}(T\log\lambda_{n-1}/\lambda_1) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_2) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_{n-1}/\lambda_n) \end{bmatrix}.$$

The leading coefficient of $\phi_{n,T}(x)$ is $\gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}}$, where

$$A_{n,T} = \det\left[\mathbb{S}\left(T\log\lambda_j/\lambda_k\right)\right]_{1\le j,k\le n}. \tag{5.1}$$

It follows from the determinantal expression and the limit $\lim_{x\to\infty}\mathbb{S}(x) = 0$ that

$$\lim_{T\to\infty}\phi_{n,T}(x) = \lambda_n^{-ix}.$$

One motivation for considering the Legendre weight is the Montgomery-Vaughan mean value relation (1.4). It is to be hoped that a theory of orthogonal Dirichlet polynomials might contribute to this circle of ideas and to estimates involving Dirichlet polynomials. In this vein, write for $j \ge 1$, $T > 0$,

$$\lambda_j^{-it} = \sum_{k=1}^{j} c_{T,j,k}\phi_{k,T}(t).$$

Let

$$C_{T,n} = \begin{bmatrix} c_{T,1,1} & c_{T,2,1} & c_{T,3,1} & \cdots & c_{T,n,1} \\ 0 & c_{T,2,2} & c_{T,3,2} & \cdots & c_{T,n,2} \\ 0 & 0 & c_{T,3,3} & \cdots & c_{T,n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{T,n,n} \end{bmatrix}.$$

In [10], there is the simple observation that

$$\sup_{\{a_j\}} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt / \sum_{j=1}^{n} \left| a_j \right|^2 = \left\| C_{T,n} \right\|^2,$$

where the norm is the usual matrix norm induced by the Euclidean norm on $\mathbb{C}^n$. The Montgomery-Vaughan inequality shows that

$$\left\| C_{T,n} \right\|^2 = T + 3\pi\theta_0 / \min_{j \neq k} \left| \log \lambda_j - \log \lambda_k \right|,$$

where $|\theta_0| \leq 1$, but it would be of interest to use $\left\| C_{T,n} \right\|$ to study refinements in the other direction as $T \to \infty$. Of course this would require understanding how $\phi_{n,T}$ changes as T does. Some initial estimates were obtained in [10]:

Proposition 5.1 *Let $S > T$.*
(a)

$$\frac{1}{2T} \int_{-T}^{T} \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt \leq \frac{S}{T} - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2.$$

(b)

$$\frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left(\frac{S}{T} \right)^{1/2}.$$

(c)

$$K_{n,T}(x,x) + \left(\frac{S}{T} - 2 \right) K_{n,S}(x,x) \geq 0. \tag{5.2}$$

(d)

$$\frac{\partial}{\partial T} K_{n,T}(x,x) = \frac{1}{T} K_{n,T}(x,x) - \frac{1}{2T} \left(|K_n(x,T)|^2 + |K_n(x,-T)|^2 \right). \tag{5.3}$$

(e)

$$\frac{\partial \left(\ln \gamma_{n,T} \right)}{\partial T} = \frac{1}{2T} (1 - \left| \psi_{n,T}(T) \right|^2).$$

(f)

$$\frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} \left(n - K_{n,T}(T,T) \right).$$

(g)

$$\frac{\partial}{\partial T} c_{T,j,k} + \frac{1}{T} c_{T,j,k} = \frac{1}{2T} \left[\lambda_j^{-iT} \overline{\phi_{k,T}(T)} + \lambda_j^{iT} \phi_{k,T}(T) \right] + \frac{1}{2T} \int_{-T}^{T} \lambda_j^{-it} \frac{\partial}{\partial T} \overline{\phi_{k,T}(t)} dt.$$

6 Conclusions

The hope in studying Dirichlet orthogonal polynomials is that they might give new insights into estimates for Dirichlet polynomials such as mean value theorems. At this preliminary stage, this is little more than a hope. However, it seems of intrinsic interest to develop analogues of the analysis for ordinary orthogonal polynomials: estimates and asymptotics for the Christoffel functions, orthogonal polynomials, and reproducing kernels for general weights. The first step in such a direction would be explicit formulae for a significant set of special weights that can approximate others—perhaps something like the Bernstein-Szegő weight. As is clear from the above, even a more basic theory for special weights is incomplete.

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Generalizations and Improvements of Approximations of Some Analytic Functions: A Survey

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Abstract In this paper, we present a survey of some recent results concerning generalizations and improvements of approximations of some analytic functions including trigonometric, inverse trigonometric, polynomial, and irrational functions.

1 Introduction

Even though investigations and proving of inequalities involving trigonometric, inverse trigonometric, polynomial, and irrational functions have been attracting attention of scientists through the centuries, these topics remain in the focus of numerous studies of mathematicians and researchers in various fields [2, 8, 25].

In addition to theoretical significance, these inequalities are also effectively applied to various problems in fundamental sciences and many areas of engineering, such as electronics, mechanics, aeronautics, etc.

Proving of analytical inequalities as well as development of corresponding formal methods and procedures still represent important and very challenging tasks. In this paper, we present a brief survey of some of the authors' recent results in this field.

Our approach is based on using one-sided and double-sided Taylor's approximations, power series expansions of the corresponding functions, Cauchy's product of power series, characteristic of some special numbers (such as, for example, Bernoulli's numbers), Leibniz's criterion for alternating series, mathematical induction, many results connected with derivatives of functions, L'Hospital's rule for monotonicity, analysis and solving of recurrent relations, and many results connected with the localization of real zeros of polynomials.

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The obtained results present improvements of approximations of the corresponding functions. Also, in many cases these results enable generalizations by producing sequences of polynomial approximations, thus allowing arbitrary uniform precision.

Our method could be a basis for the development of a general methodology for proving various analytical inequalities. Also, it represents a contribution to the library of tools for automated proving of analytic inequalities.

2 Refinements and Generalizations of Some Inequalities Involving Inverse Trigonometric Functions

Inverse trigonometric functions have many applications in computer science and engineering. For example, numerous inequalities involving the *arcsine* and *arctangent* functions have been studied and effectively applied to problems in many areas of engineering and theoretical sciences such as electronics, telecommunications, optical fiber telecommunications, signal processing, machine learning, aeronautics, mechanics, etc.

In this section, we present some refinements and generalizations of Shafer–Fink's type and Shafer's type inequalities.

2.1 Shafer–Fink's Type Inequalities

Let us state Shafer–Fink's inequality [10, 25] :

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2+\sqrt{1-x^2}}, \quad \text{for } x \in [0,1].$$

In [15] we proposed and proved some new inequalities which present refinements and generalizations of inequalities stated in [6], related to Shafer–Fink's inequality for the inverse sine function.

The following results were obtained and proved by using the power series expansions of the corresponding functions, and by establishing and solving recurrent relations between the coefficients of the obtained power series.

Theorem 1 ([15], Theorem 1) *For $x \in [0,1]$, $n \in N$ and $k = 3$ or $k = \pi$ the following inequality holds*:

$$\sum_{m=0}^{n} D_k(m)x^{2m+1} \leq \arcsin x - \frac{kx}{2+\sqrt{1-x^2}},$$

where

$$D_k(m) = \frac{(2m)!}{(m!)^2(2m+1)2^{2m}} - \left(\frac{(-1)^m k}{3^{m+1}} + \sum_{i=0}^{m-1} \frac{k(-1)^{m-1-i}(2i)!}{3^{m-i} i!(i+1)! 2^{2i+1}}\right). \tag{1}$$

Theorem 2 ([34], Theorem 2.6) *For $x \in [0, 1]$, $n \in N$, $k = 3$ or $k = \pi$, and the sequence $\{D_k(m)\}_{m \in N_0, m \geq 2}$ defined by (1), the following double-sided inequalities hold true* :

$$\sum_{m=0}^{n} D_k(m) x^{2m+1} < \arcsin x - \frac{kx}{2 + \sqrt{1 - x^2}} < \sum_{m=0}^{n-1} D_k(m) x^{2m+1} + \left(\frac{\pi - k}{2} - \sum_{m=0}^{n-1} D_k(m)\right) x^{2n+1}.$$

Examples For $x \in \left(0, \frac{\pi}{2}\right)$, following inequalities hold for $n = 5$ and $n = 4$, respectively:

$$\frac{x^5}{180} + \frac{x^7}{189} + \frac{23x^9}{5184} + \frac{629x^{11}}{171072} < \arcsin x - \frac{3x}{2 + \sqrt{1 - x^2}} < \frac{x^5}{180} + \frac{x^7}{189} + \frac{23x^9}{5184} + \left(-\frac{274933}{181440} + \frac{\pi}{2}\right) x^{11},$$

$$\left(1 - \frac{\pi}{3}\right)x + \left(\frac{1}{6} - \frac{\pi}{18}\right)x^3 + \left(\frac{3}{40} - \frac{5\pi}{216}\right)x^5 + \left(\frac{5}{112} - \frac{17\pi}{1296}\right)x^7 + \left(\frac{35}{1152} - \frac{269\pi}{31104}\right)x^9 < \arcsin x - \frac{\pi x}{2 + \sqrt{1 - x^2}} < \left(1 - \frac{\pi}{3}\right)x + \left(\frac{1}{6} - \frac{\pi}{18}\right)x^3 + \left(\frac{3}{40} - \frac{5\pi}{216}\right)x^5 + \left(\frac{5}{112} - \frac{17\pi}{1296}\right)x^7 + \left(-\frac{2161}{1680} + \frac{551\pi}{1296}\right)x^9.$$

Theorem 3 ([15], Theorem 2) *If $n \in N$ and $n \geq 2$, then*

$$\frac{\sum_{m=2}^{n} E(m) x^{2m+1}}{2 + \sqrt{1 - x^2}} \leq \arcsin x - \frac{3x}{2 + \sqrt{1 - x^2}},$$

for every $x \in [0, 1]$, where

$$E(m) = \frac{m\,(2m-1)!}{(2m+1)2^{2m-2}(m!)^2} - \frac{2m\,2^{2m-2}((m-1)!)^2}{(2m+1)!},\quad m \in N,\ m \geq 2. \tag{2}$$

Using the WD theorem [39] we obtained the upper bounds for the previous inequality.

Theorem 4 ([34], Theorem 2.10) *For* $x \in [0, 1]$ *and the sequence* $\{E(m)\}_{m\in N,\, m\geq 2}$, *defined by (2), the following double-sided inequalities hold true*:

$$\frac{\displaystyle\sum_{m=2}^{n} E(m)x^{2m+1}}{2+\sqrt{1-x^2}} < \arcsin x - \frac{3x}{2+\sqrt{1-x^2}}$$
$$< \frac{\displaystyle\sum_{m=2}^{n-1} E(m)x^{2m+1} + \left(\pi - \sum_{m=0}^{n-1} E(m)\right)x^{2n+1}}{2+\sqrt{1-x^2}}.$$

Example From the above two theorems, for $n = 4$ and $x \in \left(0, \frac{\pi}{2}\right)$ the following inequalities hold:

$$\frac{\frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9}{2+\sqrt{1-x^2}} < \arcsin x - \frac{3x}{2+\sqrt{1-x^2}} < \frac{\frac{1}{60}x^5 + \frac{11}{840}x^7 + \left(\pi - \frac{509}{168}\right)x^9}{2+\sqrt{1-x^2}}.$$

2.2 *Shafer's Type Inequalities*

Let us state Shafer's inequality [37]:

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x, \quad \text{for } x > 0.$$

In [18] we proved some sharper refinements and generalizations of inequalities related to Shafer's inequality, stated in [27]. Note that in our proofs we utilize power series expansions, recurrence relations of power series coefficients, the Wilf–Zeilberger method [32], as well as Leibniz's criterion for alternating series.

Theorem 5 ([18], Theorem 9) *For the real analytic function*:

$$f(x) = \arctan x - \frac{3x}{2+\sqrt{1+x^2}}$$

the following inequalities hold for $k \in N$ *and* $x \in \left(0, \dfrac{\sqrt{3}}{2}\right]$:

$$\sum_{m=0}^{2k+1} (-1)^m C(m) x^{2m+1} < f(x) < \sum_{m=0}^{2k} (-1)^m C(m) x^{2m+1},$$

where $C(0) = C(1) = 0$, *and for* $m \geq 2$ *the following holds*:

$$C(m) = \frac{1}{2m+1} - \frac{4^{m-1}}{3^m}\left(1 - 8\sum_{i=2}^{m} \frac{(2i-2)!}{(i-1)!\, i!\, 2^{2i-1}}\left(\frac{3}{4}\right)^i\right).$$

Example For $k=3$ and $x \in \left(0, \sqrt{3}/2\right]$:

$$\frac{x^5}{180} - \frac{13x^7}{1512} + \frac{53x^9}{5184} - \frac{3791x^{11}}{342144} + \frac{55801x^{13}}{4852224} - \frac{130591x^{15}}{11197440} < \arctan x - \frac{3x}{1+2\sqrt{1+x^2}}$$
$$< \frac{x^5}{180} - \frac{13x^7}{1512} + \frac{53x^9}{5184} - \frac{3791x^{11}}{342144} + \frac{55801x^{13}}{4852224}.$$

Theorem 6 ([18], Theorem 10) *For every* $x \in (0, 1]$ *and* $k \in N$, *it is asserted that*:

$$\frac{3x + \sum_{m=2}^{2k+1} (-1)^m E(m) x^{2m+1}}{1 + 2\sqrt{1+x^2}} < \arctan x < \frac{3x + \sum_{m=2}^{2k} (-1)^m E(m) x^{2m+1}}{1 + 2\sqrt{1+x^2}},$$

where

$$E(m) = \frac{3}{2m+1} - \sum_{i=0}^{m-1} \frac{(2m-2i-2)!}{2^{(2m-2i-2)}\,(2i+1)\,(m-i-1)!\,(m-i)!}.$$

Example For $k = 2$ the following hold:

$$\frac{3x + \frac{1}{60}x^5 - \frac{17}{840}x^7 + \frac{139}{6720}x^9 - \frac{8947}{443520}x^{11}}{1 + 2\sqrt{1+x^2}} < \arctan x$$
$$< \frac{3x + \frac{1}{60}x^5 - \frac{17}{840}x^7 + \frac{139}{6720}x^9}{1 + 2\sqrt{1+x^2}}.$$

Theorem 7 ([18], Theorem 11) *For every $x \in (0, 1]$ and $k \in N$, it is asserted that*:

$$\sum_{m=1}^{2k-1} (-1)^m C(m) x^{2m+1} < \arctan x - \frac{2x}{1+\sqrt{1+x^2}} < \sum_{m=1}^{2k} (-1)^m C(m) x^{2m+1},$$

where

$$C(m) = \frac{1}{2m+1} - \frac{(2m-1)!!}{(m+1)!\, 2^m}.$$

Example For $k = 3$ the following inequalities hold:

$$-\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 + \frac{65}{1152}x^9 - \frac{281}{5632}x^{11} < \arctan x - \frac{2x}{1+\sqrt{1-x^2}}$$
$$< -\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 + \frac{65}{1152}x^9 - \frac{281}{5632}x^{11} + \frac{595}{13312}x^{13}.$$

3 Inequalities Containing the Sinc Function

It is well-known that inequalities with the sinc function, i.e. $\operatorname{sinc} x = \frac{\sin x}{x}$ $(x \neq 0)$, occur in various fields of mathematics and engineering such as difference equations and inequalities, Fourier analysis and its applications, theory of stability, theory of approximations, signal processing, optics, radio transmission, sound recording, etc.

3.1 Inequalities Related to Wilker–Cusa–Huygens's Inequalities

Following the idea to compare and replace functions with their corresponding power series to get more accurate approximations, and using WD theorem ([39], Theorem 2) as well as Leibniz's criterion for alternating series, we proposed and proved new inequalities which represent refinements and generalizations of the inequalities stated in [26] and related to Wilker–Cusa–Huygens's inequalities.

Theorem 8 ([20], Theorem 1)

(*i*) *For every* $x \in \left(0, \frac{\pi}{2}\right)$ *and every* $n \in N$, *we have*:

$$\sum_{k=2}^{2n} (-1)^k A(k)\, x^{2k} < \cos x - \left(\frac{\sin x}{x}\right)^3 < \sum_{k=2}^{2n+1} (-1)^k A(k)\, x^{2k},$$

where $A(k) = \dfrac{3^{2k+3} - 32k^3 - 96k^2 - 88k - 27}{4\,(2k+3)!}$.

(ii) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $m \in N$*, we have the following error estimation*:

$$\left|\, \cos x - \left(\frac{\sin x}{x}\right)^3 - \sum_{k=1}^{m} (-1)^k A(k)\, x^{2k} \,\right| < A(m+1)\, x^{2m+2}.$$

Example For $n = 2$ and every $x \in \left(0, \dfrac{\pi}{2}\right)$, the following inequalities hold:

$$-\frac{1}{15}x^4 + \frac{23}{1890}x^6 - \frac{41}{37800}x^8 < \cos x - \left(\frac{\sin x}{x}\right)^3$$
$$< -\frac{1}{15}x^4 + \frac{23}{1890}x^6 - \frac{41}{37800}x^8 + \frac{53}{831600}x^{10}.$$

Theorem 9 ([20], Theorem 2)

(i) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $n \in N$*, we have*:

$$\sum_{k=2}^{2n} (-1)^{k+1} B(k)\, x^{2k} < \frac{\sin x}{x} - \frac{\cos x + 2}{3} < \sum_{k=2}^{2n+1} (-1)^{k+1} B(k)\, x^{2k},$$

where $B(k) = \dfrac{2}{3}\, \dfrac{k-1}{(2\,k+1)!}$.

(ii) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $m \in N$*, we have the following error estimation*:

$$\left|\, \frac{\sin x}{x} - \frac{\cos x + 2}{3} - \sum_{k=0}^{m} (-1)^{k+1} B(k)\, x^{2k} \,\right| < B(m+1)\, x^{2m+2}.$$

Example For $n = 2$ and every $x \in \left(0, \dfrac{\pi}{2}\right)$, the following inequalities hold:

$$-\frac{1}{180}x^4 + \frac{1}{3780}x^6 - \frac{1}{181440}x^8 < \frac{\sin x}{x} - \frac{1}{3}\cos x - \frac{2}{3}$$
$$< -\frac{1}{180}x^4 + \frac{1}{3780}x^6 - \frac{1}{181440}x^8 + \frac{1}{14968800}x^{10}.$$

Theorem 10 ([20], Theorem 3)

(i) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $n \in N$, *we have*:

$$3 + \frac{1}{\cos x} \sum_{k=2}^{2n+1} (-1)^k \, C(k) \, x^{2k} \; < \; 2 \, \frac{\sin x}{x} + \frac{\tan \, x}{x}$$

$$< \; 3 \; + \; \frac{1}{\cos x} \sum_{k=2}^{2n} (-1)^k \, C(k) \; x^{2k},$$

where $C(k) = 2 \, \dfrac{4^k - 3\,k - 1}{(2\,k+1)!}$.

(ii) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $m \in N$, $m \geq 2$, *we have the following error estimation*:

$$\left| 2 \, \frac{\sin x}{x} + \frac{\tan \, x}{x} - \left(3 + \frac{1}{\cos \, x} \sum_{k=2}^{m} (-1)^{k+1} C(k) \, x^{2k} \right) \right| \; < \; C(m+1) \, \frac{x^{2m+2}}{\cos \, x}.$$

Example For $n = 2$ and every $x \in \left(0, \dfrac{\pi}{2}\right)$, the following inequalities hold:

$$2 + \frac{1}{\cos x} \left(\frac{3}{20} \, x^4 - \frac{3}{140} \, x^6 + \frac{3}{2240} \, x^8 - \frac{1}{19800} \, x^{10} \right) < 2 \, \frac{\sin x}{x} + \frac{\tan x}{x}$$

$$< 2 + \frac{1}{\cos x} \left(\frac{3}{20} \, x^4 - \frac{3}{140} \, x^6 + \frac{3}{2240} \, x^8 \right).$$

Theorem 11 ([20], Theorem 4)

(i) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $n \in N$, *we have*:

$$2 + \frac{1}{\cos x} \sum_{k=2}^{2n+1} (-1)^k D(k) \, x^{2k} < \left(\frac{\sin \, x}{x} \right)^2 + \frac{\tan x}{x}$$

$$< 2 + \frac{1}{\cos x} \sum_{k=2}^{2n} (-1)^k D(k) \, x^{2k},$$

where $D(k) = \dfrac{1}{4} \, \dfrac{-9 + 3^{2k+2} - 40\,k - 32\,k^2}{(2\,k+2)!}$.

(ii) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and every* $m \in N, m \geq 2$, *we have the following error estimation*:

$$\left|\left(\frac{\sin\ x}{x}\right)^2 + \frac{\tan x}{x} - \left(2 + \frac{1}{\cos\ x}\sum_{k=2}^{m}(-1)^{k+1}D(k)\,x^{2k}\right)\right| < D(m+1)\,\frac{x^{2m+2}}{\cos\ x}.$$

Example Let $x \in \left(0, \frac{\pi}{2}\right)$ and $n = 2$. The following inequalities are true:

$$2 + \frac{1}{\cos x}\left(\frac{8}{45}\,x^4 - \frac{4}{105}\,x^6 + \frac{19}{4725}\,x^8 - \frac{37}{133650}\,x^{10}\right) < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}$$
$$< 2 + \frac{1}{\cos x}\left(\frac{8}{45}\,x^4 - \frac{4}{105}\,x^6 + \frac{19}{4725}\,x^8\right).$$

Theorem 12 ([20], Theorem 5) *For every $x \in \left(0, \frac{\pi}{2}\right)$ and $m \in N$, $m \geq 2$, the following inequalities hold*:

$$2 + \sum_{k=2}^{m}\frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!}x^{2k} < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}$$
$$< 2 + \sum_{k=2}^{m-1}\frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!}x^{2k} + \left(\frac{2x}{\pi}\right)^{2n}\left(\frac{\pi^2}{4} - 2 - \sum_{k=2}^{m-1}\frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}\right),$$

where $\boldsymbol{B}_i$ are Bernoulli's numbers.

The above theorem allows for the approximation error to be estimated:

$$R_n\left(x\right) = \left(f\left(\frac{\pi}{2}\right) - 2 - \sum_{k=1}^{n}\frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}\right)\left(\frac{2x}{\pi}\right)^{2n}.$$

Example For $x \in \left(0, \frac{\pi}{2}\right)$ and $m = 2$, the following holds true:

$$2 + \frac{2}{45}x^4 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < 2 + \left(\frac{2}{\pi}\right)^4\left(-2 + \frac{\pi^2}{4}\right)x^4.$$

Theorem 13 ([20], Theorem 6) *For every $x \in \left(0, \frac{\pi}{2}\right)$ and $m \in N$, $m \geq 3$, the following inequality holds*:

$$4 + \sum_{k=1}^{m}\frac{3\,|\boldsymbol{B}_{2k}|\,(2^{2k}-2)+(-1)^k}{(2k)!}x^{2k} < 3\frac{x}{\sin x} + \cos x$$
$$< 4 + \sum_{k=1}^{m-1}\frac{3|\boldsymbol{B}_{2k}|\,(2^{2k}-2)+(-1)^k}{(2k)!}x^{2k}$$

$$+\left(\frac{2x}{\pi}\right)^{2m}\left(f\left(\frac{\pi}{2}\right)-4-\sum_{k=1}^{m-1}\frac{3|\boldsymbol{B}_{2k}|\,(2^{2k}-2)+(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}\right),$$

where $\boldsymbol{B}_i$ are Bernoulli's numbers.

Example For $m = 4$ and every $x \in \left(0, \frac{\pi}{2}\right)$ the following holds:

$$4+\frac{1}{10}x^4+\frac{1}{210}x^6+\frac{11}{16800}x^8 < 3\frac{x}{\sin x}+\cos x$$
$$< 4+\frac{1}{10}x^4+\frac{1}{210}x^6+\left(\frac{2}{\pi}\right)^8\left(-4+\frac{3\pi}{2}-\frac{\pi^4}{160}-\frac{\pi^6}{13440}\right)x^8.$$

3.2 Some Exponential Inequalities Related to the Sinc Function

Starting from Jordan's inequality [25]:

$$\frac{2}{\pi} \le \frac{\sin x}{x} \le 1, \quad 0 < x \le \frac{\pi}{2},$$

and continuing with the polynomial bounds [1, 9, 33], some exponential bounds have recently been considered [7, 30, 31].

In [12] we presented a new approach to proving some exponential inequalities connected with the sinc function. Using the power series expansions of the corresponding functions and some newly developed approximation techniques, we reduce exponential inequalities to the corresponding polynomial inequalities that are more easily analyzed and proved.

Theorem 14 ([12], Theorem 2.1) *Let function f_1 be defined in the interval $\left(0, \frac{\pi}{2}\right]$ by:*

$$f_1(x) = \left(1-\frac{4(\pi-2)}{\pi^3}\,x^2\right)^{\alpha_1 x^3+\beta_1 x^2+\gamma_1 x+\delta_1} - \frac{\sin x}{x},$$

and let the following conditions hold:

$$f_1(0+)=f_1'(0+)=f_1''(0+)=0,\ \ f_1\left(\frac{\pi}{2}\right)=f_1'\left(\frac{\pi}{2}\right)=f_1''\left(\frac{\pi}{2}\right)=0.$$

Then:

$$\alpha_1 = \frac{\left(-\pi^3+24\pi-48\right)\ln^2\frac{\pi}{2}-3\,(\pi-2)\left(3\pi^2-20\pi+36\right)\ln\frac{\pi}{2}+24(\pi-3)(\pi-2)^2}{3\,(\pi-2)\,\pi^3\ln^2\frac{\pi}{2}},$$
$$\beta_1 = \frac{(\pi^3-24\pi+48)\ln^2\frac{\pi}{2}+6(\pi-2)(\pi-4)^2\ln\frac{\pi}{2}-16(\pi-3)(\pi-2)^2}{2(\pi-2)\pi^2\ln^2\frac{\pi}{2}},$$
$$\gamma_1 = \frac{(-\pi^3+24\pi-48)\ln^2\frac{\pi}{2}+(3\pi-10)(\pi-6)(\pi-2)\ln\frac{\pi}{2}+8(\pi-3)(\pi-2)^2}{4\pi(\pi-2)\ln^2\frac{\pi}{2}},$$
$$\delta_1 = \frac{\pi^3}{24(\pi-2)}$$

and

$$f_1(x) > 0, \quad \textit{i.e.} \quad \frac{\sin x}{x} < \left(1-\frac{4(\pi-2)}{\pi^3}\,x^2\right)^{\alpha_1 x^3+\beta_1 x^2+\gamma_1 x+\delta_1}$$

for every $x \in \left(0, \frac{\pi}{2}\right]$.

Theorem 15 ([12], Theorem 2.2) *Let the function*

$$f_2(x)=\left(1-\frac{4(\pi-2)}{\pi^3}\,x^2\right)^{\alpha_2 x^3+\beta_2 x^2+\delta_2}-\frac{\sin x}{x},$$

for $x \in \left(0, \frac{\pi}{2}\right]$ *satisfy the following conditions*:

$$f_2(0+)=f_2'(0+)=f_2''(0+)=0, \quad f_2\left(\frac{\pi}{2}\right)=f_2'\left(\frac{\pi}{2}\right)=0.$$

Then:

$$\alpha_2 = -\frac{2}{3}\frac{12(\pi-2)(\pi-3)-(48-24\pi+\pi^3)\ln\frac{\pi}{2}}{\pi^3(\pi-2)\ln\frac{\pi}{2}},$$
$$\beta_2 = \frac{8(\pi-2)(\pi-3)-(48-24\pi+\pi^3)\ln\frac{\pi}{2}}{2\pi^2(\pi-2)\ln\frac{\pi}{2}},$$
$$\delta_2 = \frac{\pi^3}{24(\pi-2)}$$

and

$$f_2(x) < 0, \quad \textit{i.e.} \quad \left(1-\frac{4(\pi-2)}{\pi^3}\,x^2\right)^{\alpha_2 x^3+\beta_2 x^2+\delta_2} < \frac{\sin x}{x}$$

for every $x \in \left(0, \frac{\pi}{2}\right]$.

Theorem 16 ([12], Theorem 2.3) *Let the function*

$$f_3(x) = \left(1 - \frac{4(\pi - 2)}{\pi^3}\, x^2\right)^{\alpha_3 x^3 + \delta_3} - \frac{\sin x}{x}$$

for $x \in \left(0, \dfrac{\pi}{2}\right]$ *satisfy the following conditions*:

$$f_3(0+) = f_3'(0+) = f_3''(0+) = 0, \quad f_3\left(\frac{\pi}{2}\right) = 0.$$

Then:

$$\alpha_3 = -\frac{\pi^3 - 24\pi + 48}{3(\pi - 2)\pi^3},$$
$$\delta_3 = \frac{\pi^3}{24(\pi - 2)}.$$

and

$$f_3(x) < 0, \quad \text{i.e.} \quad \left(1 - \frac{4(\pi - 2)}{\pi^3}\, x^2\right)^{\alpha_3 x^3 + \delta_3} < \frac{\sin x}{x}$$

for every $x \in \left(0, \dfrac{\pi}{2}\right]$.

In [35] we proved some exponential inequalities, with constant exponents and with certain polynomial exponents, involving the sinc function. Also, we determined a relation between the cases of the constant and of the polynomial exponent.

Theorem 17 ([35], Theorem 5) *For every* $a \geq 2$ *and every* $x \in (0, \pi)$ *the following inequality holds true*:

$$\left(\frac{\sin x}{x}\right)^a \leq \cos^2 \frac{x}{2}.$$

Theorem 18 ([35], Theorem 6) *For every* $a \in \left(\dfrac{3}{2}, 2\right)$, *and every* $x \in (0, x_a]$, *where* $0 < x_a < \pi$, *the following inequality holds true*:

$$\left(\frac{\sin x}{x}\right)^a \leq \cos^2 \frac{x}{2}.$$

Theorem 19 ([35], Theorem 8) *For every* $x \in (0, 3.1)$ *the following double-sided inequality holds*:

$$\left(\frac{\sin x}{x}\right)^{p_1(x)} < \cos^2 \frac{x}{2} < \left(\frac{\sin x}{x}\right)^{p_2(x)},$$

where $p_1(x) = \dfrac{3}{2} + \dfrac{x^2}{2\pi^2}$ *and* $p_2(x) = \dfrac{3}{2} + \dfrac{x^2}{80}$.

Theorem 20 ([35], Theorem 10) *For every* $a \in \left(\frac{3}{2}, 2\right)$ *and every* $x \in (0, m_a)$, *where* $m_a = \sqrt{2\pi^2\left(a - \frac{3}{2}\right)}$, *the following double-sided inequality holds*:

$$\left(\frac{\sin x}{x}\right)^a < \left(\frac{\sin x}{x}\right)^{\frac{3}{2}+\frac{x^2}{2\pi^2}} < \cos^2\frac{x}{2}.$$

3.3 Wilker's Type Inequalities

Starting from Wilker's inequality [38]:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad \text{for } x \in \left(0, \frac{\pi}{2}\right),$$

using the power series expansions and WD theorem, we obtained some generalizations and sharpenings of certain inequalities from [29].

Theorem 21 ([34], Theorem 2.2) *For the function*

$$f(x) = \frac{1}{x} + \frac{\sin 2x}{2x^2} - 2\cot x - \frac{8x^3}{45} + \frac{8x^5}{945},$$

where $x \in \left(0, \frac{\pi}{2}\right)$, *the following sequence of inequalities holds*:

$$\sum_{k=0}^{m} c_k x^{2k+1} < f(x) < \sum_{k=0}^{m-1} c_k x^{2k+1} + \left(f\left(\frac{\pi}{2}\right) - \sum_{k=0}^{m-1} c_k \left(\frac{\pi}{2}\right)^{2k+1}\right)\left(\frac{2x}{\pi}\right)^{2m+1},$$

for $x \in \left(0, \frac{\pi}{2}\right)$, $m \in N$, $c_0 = c_1 = c_2 = 0$ *and for* $k \geq 3$:

$$c_k = \frac{2^{2k+2}\left((4k+6)|\boldsymbol{B}_{2k+2}| + (-1)^{k+1}\right)}{(2k+3)!},$$

where $\boldsymbol{B}_i$ *are Bernoulli's numbers.*

Example For $m = 4$, and $x \in \left(0, \frac{\pi}{2}\right)$ we have:

$$\frac{16\,x^7}{14175} + \frac{8\,x^9}{467775} < f(x) < \frac{16\,x^7}{14175} + \left(\frac{2}{\pi}\right)^9\left(\frac{2}{\pi} - \frac{\pi^3}{45} + \frac{\pi^5}{3780} - \frac{\pi^7}{113400}\right)x^9.$$

The above theorem allows for the approximation error to be estimated:

$$R_m(x) = \left(f\left(\frac{\pi}{2}\right) - \sum_{k=3}^{m} \frac{2^{2k+2}\big((4k+6)\,|\boldsymbol{B}_{2k+2}| + (-1)^{k+1}\big)}{(2k+3)!} \left(\frac{\pi}{2}\right)^{2k+1} \right) \left(\frac{2x}{\pi}\right)^{2m+1}.$$

4 Generalizations and Improvements of Some Inequalities Using the Double-Sided Taylor's Approximations

In [21] the double-sided Taylor's approximations were studied, and two new theorems were proved regarding the monotonicity of such approximations. Also, some new applications of the double-sided Taylor's approximations in the theory of analytic and trigonometric inequalities were presented.

First, let us introduce the notation. For a real function $f : (a, b) \longrightarrow \mathbb{R}$ such that there exist finite limits $f^{(k)}(a+) = \lim\limits_{x \to a+} f^{(k)}(x)$ and $f^{(k)}(b-) = \lim\limits_{x \to b-} f^{(k)}(x)$ for $k = 0, 1, \ldots, n$, $n \in \mathbb{N}_0$ the following polynomials:

$$T_n^{f,\,a+}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k \quad \text{and} \quad T_n^{f,\,b-}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k,$$

are called the *first Taylor's approximation in the right neighborhood of* a, and the *first Taylor's approximation in the left neighborhood of* b, respectively.

Also, for $n \in \mathbb{N}$, the following functions:

$$R_n^{f,\,a+}(x) = f(x) - T_{n-1}^{f,\,a+}(x) \quad \text{and} \quad R_n^{f,\,b-}(x) = f(x) - T_{n-1}^{f,\,b-}(x),$$

are called the *remainder of the first Taylor's approximation in the right neighborhood of* a, and the *remainder of the first Taylor's approximation in the left neighborhood of* b, respectively.

The following polynomials:

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) = \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n} R_n^{f,\,a+}(b-)(x-a)^n & : n \geq 1 \\ f(b-) & : n = 0, \end{cases}$$

and

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) = \begin{cases} T_{n-1}^{f,\,b-}(x) + \dfrac{1}{(a-b)^n} R_n^{f,\,b-}(a+)(x-b)^n & : n \geq 1 \\ f(a+) & : n = 0, \end{cases}$$

are called the *second Taylor's approximation in the right neighborhood of* a, and the *second Taylor's approximation in the left neighborhood of* b, respectively.

In [21] the following theorem was proved. This theorem concerns real analytic functions which are of special interest in proofs of analytic inequalities.

Theorem 22 ([21], Theorem 4) *Consider the real analytic functions* $f:(a,b)\longrightarrow\mathbb{R}$:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

where $c_k \in \mathbb{R}$ *and* $c_k \geq 0$ *for all* $k \in \mathbb{N}_0$*. Then,*

$$\begin{gathered} T_0^{f,\,a+}(x) \leq \cdots \leq T_n^{f,\,a+}(x) \leq T_{n+1}^{f,\,a+}(x) \leq \cdots \\ \cdots \leq f(x) \leq \cdots \\ \cdots \leq \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) \leq \mathbb{T}_n^{f;\,a+,\,b-}(x) \leq \cdots \leq \mathbb{T}_0^{f;\,a+,\,b-}(x), \end{gathered}$$

for all $x \in (a,b)$*. If* $c_k \in \mathbb{R}$ *and* $c_k \leq 0$ *for all* $k \in \mathbb{N}_0$*, then the reversed inequalities hold.*

In [21] we applied the above theorem on double-sided Taylor's approximations to the sequence of functions $h_n : \left(0, \dfrac{\pi}{2}\right) \longrightarrow R$, defined by:

$$h_n(x) = \frac{\tan x - T_{2n-1}^{\tan,\,0}(x)}{x^{2n}\tan x}, \quad n \in \mathbb{N}.$$

Using the change of variables and some algebraic transformations, we proved:

$$h_n(x) = \sum_{i=0}^{\infty}\sum_{j=1}^{n} \frac{2^{2(n+i+1)}(2^{2(n-j+1)}-1)\,|\boldsymbol{B}_{2(n-j+1)}|\,|\boldsymbol{B}_{2(i+j)}|}{(2(n-j+1))!\,(2(i+j))!}\,x^{2i}.$$

As the Taylor series expansions of the functions $h_n(x)$ satisfy the conditions of Theorem 22 we improved the results from [40], as follows:

Theorem 23 ([21], Theorem 8) *For $x \in \left(0, \dfrac{\pi}{2}\right)$ and $n \in \mathbb{N}$, we have*:

$$T_0^{h_n(x),\,0+}(x) = \frac{2^{2(n+1)}(2^{2(n+1)}-1)|\boldsymbol{B}_{2(n+1)}|}{(2n+2)!} <$$

$$< T_2^{h_n(x),\,0+}(x) < \cdots < T_{2m}^{h_n(x),\,0+}(x) < T_{2m+2}^{h_n(x),\,0+}(x) < \cdots$$

$$\cdots < h_n(x) < \cdots$$

$$\cdots < \mathbb{T}_{2m+2}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) < \mathbb{T}_{2m}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) < \cdots < \mathbb{T}_{2}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) <$$

$$< \mathbb{T}_{0}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) = \left(\frac{2}{\pi}\right)^{2n}.$$

Also, we obtained improvements of results from [28]:

Theorem 24 ([21], Theorem 10) *For every $x \in \left(0, \dfrac{\pi}{2}\right)$ and $m \in N$, $m \geq 2$, the following inequalities hold*:

$$T_1^{g,\,0+}\left(\frac{\pi}{2}-x\right) \leq \cdots \leq T_{2m-1}^{g,\,0+}\left(\frac{\pi}{2}-x\right) \leq T_{2m+1}^{g,\,0+}\left(\frac{\pi}{2}-x\right) \leq \cdots$$

$$\cdots \leq \tan x - \frac{4x}{\pi(2\pi - x)} \leq \cdots$$

$$\cdots \leq \mathbb{T}_{2m+1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right) \leq \mathbb{T}_{2m-1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right) \leq \cdots \leq \mathbb{T}_{1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right).$$

Example For $m = 1$, the following inequalities hold:

$$\mathbb{T}_1^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2}-x\right) = \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\pi}{2}-x\right) \leq \tan x - \frac{4x}{\pi(2\pi - x)}$$

$$\leq \frac{2}{\pi} - \frac{1}{3}\left(\frac{\pi}{2}-x\right) = T_1^{g,\,0}\left(\frac{\pi}{2}-x\right),$$

which further implies the following:

$$Q_1(x) < \mathbb{T}_1^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2}-x\right) \leq \tan x - \frac{4x}{\pi(2\pi - x)} \leq T_1^{g,\,0+}\left(\frac{\pi}{2}-x\right) = R_1(x),$$

for $x \in \left(0, \dfrac{\pi}{2}\right)$, where $Q_1(x) = \dfrac{2}{\pi} - \dfrac{1}{2}\left(\dfrac{\pi}{2}-x\right)$, and $R_1(x) = \dfrac{2}{\pi} - \dfrac{1}{3}\left(\dfrac{\pi}{2}-x\right)$ are the bounds from [9].

The following theorem gives some generalizations of the results obtained in Theorem 12.

Theorem 25 ([21], Theorem 12) *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and* $m \in N$, $m \geq 2$, *the following inequalities hold*:

$$T_0^{f,\,0+}(x) \leq \cdots \leq T_{2m}^{f,\,0+}(x) \leq T_{2m+2}^{f,\,0+}(x) \leq \cdots$$
$$\cdots \leq \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \leq \cdots$$
$$\cdots \leq \mathbb{T}_{2m+2}^{f;\,0+,\,\pi/2-}(x) \leq \mathbb{T}_{2m}^{f;\,0+,\,\pi/2-}(x) \leq \cdots \leq \mathbb{T}_{0}^{f;\,0+,\,\pi/2-}(x).$$

In [22], using the double-sided Taylor's approximations, we obtained improvements of some trigonometric inequalities proved in [36]. Our approach enabled generalizations of the considered inequalities and produced sequences of polynomial approximations of considered trigonometric functions.

Theorem 26 ([22], Theorem 3) *For the function*

$$f(x) = \frac{1}{x^2}\left(1 - \frac{\cos x}{\cos \frac{x}{2}}\right),\ \textit{for } x \in (0, \pi),\ \textit{and } f(0) = \frac{3}{8},$$

and any $c \in (0, \pi)$ *the following inequalities hold true*:

$$\frac{3}{8} = T_0^{f,\,0+}(x) \leq T_2^{f,\,0+}(x) \leq \cdots \leq T_{2n}^{f,\,0+}(x) \leq \cdots$$
$$\cdots \leq f(x) \leq \cdots$$
$$\leq \mathbb{T}_{2m}^{f;\,0+,\,c-}(x) \leq \cdots \leq \mathbb{T}_{2}^{f;\,0+,\,c-}(x) \leq \mathbb{T}_{0}^{f;\,0+,\,c-}(x) = \tfrac{1}{c^2}\left(1 - \frac{\cos c}{\cos \frac{c}{2}}\right)$$

for every $x \in (0, c)$, *where* $m, n \in \mathbb{N}_0$.

Example For $c = \pi/2$, i.e. for $x \in \left(0, \dfrac{\pi}{2}\right)$ the following inequalities hold:

$$\frac{3}{8} \leq T_2^{f,\,0+}(x) = \frac{3}{8} + \frac{1}{128}x^2 \leq f(x) \leq \mathbb{T}_{2}^{f;\,0+,\,\pi/2-}(x) = \frac{3}{8} + \left(\frac{16}{\pi^4} - \frac{3}{2\pi^2}\right)x^2 \leq \frac{4}{\pi^2}.$$

Now, let us consider functions:

$$g_1(x) = \begin{cases} \dfrac{1}{4} &, \quad x = 0, \\ \dfrac{\cosh\frac{x}{2} - \cos\frac{x}{2}}{x^2} &, \ x \in (0, \beta] \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} 0 &, \quad x = 0, \\ \dfrac{\cosh\frac{x}{2} + \cos\frac{x}{2} - 2}{x^2} &, \ x \in (0, \beta] \end{cases},$$

and $g(x) = g_1(x) - g_2(x)$, for $\beta \in (0, \pi)$.

We obtained the following results.

Theorem 27 ([22], Theorem 4) *For every $c \in (0,\pi)$ the following inequalities hold true*:

$$\frac{1}{4} = T_0^{g_1,\,0+}(x) \le \cdots \le T_{4n}^{g_1,\,0+}(x) \le T_{4n+4}^{g_1,\,0+}(x) \le \cdots$$
$$\cdots \le g_1(x) \le \cdots$$
$$\cdots \le \mathbb{T}_{4m+4}^{g_1;\,0+,\,c-}(x) \le \mathbb{T}_{4m}^{g_1;\,0+,\,c-}(x) \le \cdots \le \mathbb{T}_{0}^{g_1;\,0+,\,c-}(x) = g_1(c).$$

for all $x \in (0,c)$, where $m, n \in \mathbb{N}_0$.

Theorem 28 ([22], Theorem 5) *For every $c \in (0,\pi)$ the following inequalities hold true*:

$$\frac{1}{192}x^2 = T_2^{g_2,\,0+}(x) \le \cdots \le T_{4n+2}^{g_2,\,0+}(x) \le T_{4n+6}^{g_2,\,0+}(x) \le \cdots$$
$$\cdots \le g_2(x) \le \cdots$$
$$\cdots \le \mathbb{T}_{4m+6}^{g_2;\,0+,\,c-}(x) \le \mathbb{T}_{4m+2}^{g_2;\,0+,\,c-}(x) \le \cdots \le \mathbb{T}_{2}^{g_2;\,0+,\,c-}(x) = \frac{g_2(c)}{c^2}x^2$$

for all $x \in (0,c)$, where $m, n \in \mathbb{N}_0$.

Example For $c = \dfrac{\pi}{2}$ and $x \in (0,\pi/2)$, the following inequalities hold true:

$$\frac{1}{4} - \frac{4}{\pi^2}\, g_2\!\left(\frac{\pi}{2}\right) x^2 \le g(x) \le g_1\!\left(\frac{\pi}{2}\right) - \frac{1}{192}\, x^2$$

i.e.

$$\frac{1}{4} - \frac{16}{\pi^4}\left(\cosh\frac{\pi}{4} + \frac{\sqrt{2}}{2} - 2\right) x^2 \le g(x) \le \frac{4}{\pi^2}\left(\cosh\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right) - \frac{1}{192}\, x^2.$$

5 Conclusion

In this paper, we gave a brief survey of our recent results in the area of analytic inequalities. Presented results make a good basis for the systematic proving in this field. Developing general, automated-oriented methodology for proving of analytic inequalities is an area of our continuing interest and research [3–5, 11–21, 23, 24, 28, 29, 34, 35].

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Some Classes of Meir–Keeler Contractions

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Abstract In the present paper, we prove that $\mathcal{Z}$-contractions, weakly type contractions, and some type of F-contractions are actually Meir–Keeler contractions.

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1 Introduction

Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is called a *Banach contraction* on X if there exist $\lambda \in [0, 1)$ so that

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \text{for all } x, y \in X.$$

We say that T is a *Picard operator* if T has a unique fixed point x^* in X and for every $x \in X$, the sequence of successive approximations $\{x_n\}_{n\in\mathbb{N}}$, where $x_{n+1} = Tx_n, n \in \mathbb{N},\ x_0 = x$ converges to x^*. See [19].

S. Banach proved in [3] that if (X, d) is complete, then Banach contraction is a Picard operator.

After this result, a large number of generalizations were obtained. See, for example, the book of I.A. Rus et al. [18] and the articles [4] and [17].

Definition 1 We say that T is a Meir–Keeler contraction if given an $\varepsilon > 0$, there exists $\delta > 0$ such that

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$$\varepsilon \le d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

In [14] it is proved the following Theorem.

Theorem 1 *Let (X, d) be a complete metric space and T be a Meir–Keeler contraction. Then T is a Picard operator.*

Among the numerous papers regarding Meir–Keeler contractions, we mention the papers of W.-S. Du and Th.M. Rassias [5], J. Jachymski [9] and M. Turinici [23].

The fixed point theorems have various application in chemistry, biology, computer sciences, differential equations, existence of invariant subspaces of linear operators, Hyers-Ulam-Rassias stability, and much more. Because of the wide range of applicatios of the fixed point theory in various fields, many scientists work on developing new fixed points theorems: in [12], the autors introduce $\mathcal{Z}$-contractions and prove that are Picard operators, in [6, 17], is proved that weakly contractive mappings are Picard operators. See also [7] and [2]. In the following papers is proved, under different assumptions, that F-contractions are Picard operators: [15], [20] and [25].

In this paper, we prove that $\mathcal{Z}$-contractions, weakly type contractions, and some type of F-contractions are actually Meir–Keeler contractions.

2 $\mathcal{Z}$-Contractions Are Meir–Keeler Contractions

In [12], the authors introduced a new class of contractions, called $\mathcal{Z}$-contractions.

Definition 2 ([12]) A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions

(i) $\zeta(0, 0) = 0$;
(ii) $\zeta(t, s) < s - t$, for all $t, s > 0$;
(iii) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$

We denote by $\mathcal{Z}$ the set of all simulation functions.

Definition 3 ([12]) T is called a $\mathcal{Z}$-contraction if

$$\zeta(d(Tx, Ty), d(x, y)) \ge 0, \quad \text{for all } x, y \in X.$$

Also in [12], F. Khojasteh et al. prove the following Theorem.

Theorem 2 ([12]) *Let (X, d) be a complete metric space and T a $\mathcal{Z}$-contraction. Then T has a unique fixed point u and the Picard sequence $x_n = Tx_{n-1}$ converges to u.*

We prove that the above result is following from the next Theorem.

Theorem 3 *A $\mathcal{Z}$-contraction is a Meir–Keeler contraction.*

Proof Let $T : X \to X$ be a $\mathcal{Z}$-contraction. We suppose that T is not a Meir–Keeler. Then there is $\varepsilon_0 > 0$ so that for any $\delta > 0$ there is $x_\delta, y_\delta \in X$ so that

$$\varepsilon_0 \leq d(x_\delta, y_\delta) < \varepsilon_0 + \delta \tag{1}$$

and

$$d(Tx_\delta, Ty_\delta) \geq \varepsilon_0. \tag{2}$$

We take $\delta = \dfrac{1}{n}$, $n \geq 1$ natural number.

It follows that there are two sequences $\{x_n\}, \{y_n\} \subset X$ so that

$$d(x_n, y_n) \geq \varepsilon_0, \quad n \geq 1 \text{ and } \lim_{n\to\infty} d(x_n, y_n) = \varepsilon_0 \tag{3}$$

and

$$d(Tx_n, Ty_n) \geq \varepsilon_0. \tag{4}$$

Since T is a $\mathcal{Z}$-contraction, there is ζ a simulation function so that

$$0 \leq \zeta(d(Tx_n, Ty_n), d(x_n, y_n)). \tag{5}$$

From (4), (5) and the condition (ii) in Definition 2 it follows

$$0 \leq \zeta(d(Tx_n, Ty_n), d(x_n, y_n)) < d(x_n, y_n) - d(Tx_n, Ty_n) \leq d(x_n, y_n) - \varepsilon_0.$$

It follows

$$\lim_{n\to\infty} \zeta(d(Tx_n, Ty_n), d(x_n, y_n)) = 0. \tag{6}$$

But $\varepsilon_0 \leq d(Tx_n, Ty_n) < d(x_n, y_n) - \zeta(d(Tx_n, Ty_n), d(x_n, y_n))$ which implies that

$$\lim_{n\to\infty} d(Tx_n, Ty_n) = \varepsilon_0.$$

From condition (iii) in Definition 2 we have

$$\limsup_{n\to\infty} \zeta(d(Tx_n, Ty_n), d(x_n, y_n)) < 0$$

in contradiction with relation (6).

3 Weakly Type Contractions Are Meir–Keeler Contractions

A significant number of generalizations of the contraction principle were obtain over the years. We mention here the result of Alber and Guerre-Delabriere [1] in Hilbert Spaces. B.E. Rhoades [17] showed that their is also valid in complete metric spaces. Moreover, B.E. Rhoades obtain the result presented in the next Theorem, without using an additional condition on ϕ, namely $\lim_{t\to\infty} \phi(t) = \infty$.

Definition 4 ([17]) A mapping $T : X \to X$, where (X, d) is a metric space, is said to be *weakly contractive* if

$$d(Tx, Tx) \le d(x, y) - \phi(d(x, y)),$$

where $x, y \in X$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t)$ if and only if $t = 0$.

Theorem 4 ([17]) *If $T : X \to X$ is a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point.*

In 2008, P.N. Dutta and B.S. Choudhury [6] obtain a generalization of the result of the above result. The main results of [6] state as follows.

Theorem 5 ([6]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \phi(d(x, y)),$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(0)$ if and only if $t = 0$. Then T has a unique fixed point.

M. Eslamian and A. Abkar (see [7]) generalizes the main results from [6] and [17]. See also [2].

Theorem 6 *[7] Let (X, d) be a complete metric space and $T : X \to X$ be such that*

$$\psi(d(Tx, Ty)) \le \alpha(d(x, y)) - \beta(d(x, y)), \tag{7}$$

for all $x, y \in X$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \text{ if and only if } t = 0, \alpha(0) = \beta(0) = 0, \tag{8}$$

and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \text{ for all } t > 0. \tag{9}$$

Then T has a unique fixed point.

We consider the following generalization of contractions.

Definition 5 Let (X, d) be a metric space and $T : X \to X$ be such that

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y)) - \beta(d(x, y)),$$

for all $x, y \in X$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is nondecreasing, α is continuous, β is lower semi-continuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \text{ for all } t > 0.$$

Then we say that T is a *weakly type contraction.*

Theorem 7 *If T is a weakly type contraction on a complete metric space, then T is a Meir–Keeler contraction.*

Proof We suppose that T is not a Meir–Keeler contraction. Then there exists $\varepsilon_0 > 0$ and two sequences $\{x_n\}, \{y_n\} \subset X$ such that

$$\varepsilon_0 \leq d(x_n, y_n) < \varepsilon_0 + \frac{1}{n}$$

and

$$d(Tx_n, Ty_n) \geq \varepsilon_0, \quad n \geq 1.$$

We have

$$\psi(d(Tx_n, Ty_n)) \geq \psi(\varepsilon_0) \text{ and } \psi(d(Tx_n, Ty_n)) \leq \alpha(d(x_n, y_n)) - \beta(d(x_n, y_n)).$$

It follows

$$\psi(\varepsilon_0) \leq \alpha(d(x_n, y_n)) - \beta(d(x_n, y_n))$$

or

$$\beta(d(x_n, y_n)) \leq \alpha(d(x_n, y_n)) - \psi(\varepsilon_0).$$

Using the continuity of α, we have

$$\liminf_{n\to\infty} \beta(d(x_n, y_n)) \le \alpha(\varepsilon_0) - \psi(\varepsilon_0).$$

Since β lower semi-continuous, it follows $\beta(\varepsilon_0) \le \alpha(\varepsilon_0) - \psi(\varepsilon_0)$, in contradiction with the hypothesis.

By Theorems 1 and 7, we have the following Corollary.

Corollary 1 *Let (X, d) be a complete metric space and T be a weakly type contraction. Then T is a Picard operator.*

4 F-Contractions and Meir–Keeler Contractions

In 2012, D. Wardowski [25] generalized the Banach theorem for a new type of contractions.

We denote by $\mathcal{F}$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is strictly increasing;
(F_2) for each sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $\lim_{n\to\infty} t_n = 0$ if and only if

$$\lim_{n\to\infty} F(t_n) = -\infty;$$

(F_3) there exists $k \in (0, 1)$ such that $\lim_{n\to 0_+} t^k F(t) = 0$.

Definition 6 ([25]) Let (X, d) be a metric space and $T : X \to X$. The mapping T is called an F-contraction relative to $\mathcal{F}$ if there exists $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Tx, Ty)) \le F(d(x, y))$$

holds for any $x, y \in X$ with $d(Tx, Ty) > 0$.

Theorem 8 ([25]) *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T is a Picard operator.*

In 2013, Secelean [20] proved that the condition $(F2)$ can be replaced by an equivalent condition

$$(F_2') \ \inf F = -\infty.$$

See also [22].

In 2014, Piri and Kumam [15] introduced instead (F_3), the following condition:

$$(F_3') \ F \text{ is continuous on } (0, \infty).$$

We denote by $\mathcal{F}'$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the conditions (F_1), (F_2'), (F_3').

Theorem 9 ([15]) *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction relative to $\mathcal{F}'$. Then T is a Picard operator.*

For other results concerning F-contractions, see the recent papers [16] and [26] and their references.

We denote by $\mathcal{F}''$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the conditions (F_1') and (F_3''), where

$$(F_1')\ F \text{ is nondecreasing}$$

$$(F_3'')\ F \text{ is continuous at right.}$$

We also consider $\varphi : (0, \infty) \to [0, \infty)$ lower semi-continuous, $\varphi(t) > 0$, for $t > 0$. In the next Theorem, we improve the result of [15].

Theorem 10 *If T is a (φ, F)-contraction relative to $\mathcal{F}''$, i.e.:*

$$\varphi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$$

for $x, y \in X$ with $d(Tx, Ty) > 0$, then T is a Meir–Keeler contraction.

Proof We suppose that T is not a Meir–Keeler. Then there is $\varepsilon_0 > 0$ and two sequences $\{x_n\}, \{y_n\} \subset X$ such that

$$\varepsilon_0 \leq d(x_n, y_n) < \varepsilon_0 + \frac{1}{n}$$

and

$$d(Tx_n, Ty_n) \geq \varepsilon_0,\ n \geq 1.$$

We have

$$F(d(Tx_n, Ty_n)) \geq F(\varepsilon_0),\ n \geq 1$$

and

$$\varphi(d(x_n, y_n)) + F(d(Tx_n, Ty_n)) \leq F(d(x_n, y_n)).$$

Hence

$$\varphi(d(x_n, y_n)) + F(\varepsilon_0) \leq F(d(x_n, y_n)),\ n \geq 1.$$

We take $n \to \infty$ and we obtain

$$\varphi(\varepsilon_0) + F(\varepsilon_0) \leq F(\varepsilon_0), \text{ contradiction.}$$

Theorem 10 is also a generalization of a result of [21]. Our proof is more simple.

We denote by $\mathcal{F}_0$ the set of functions $F : (0, \infty) \to \mathbb{R}$ which are nondecreasing. In 2020, Popescu and Stan [16] proved, among other interesting results, the following:

Theorem 11 ([16]) *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that T is F-contraction relative to $\mathcal{F}_0$, i.e., there exist $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

holds for any $x, y \in X$ with $d(Tx, Ty) > 0$. Then T is a Picard operator.

Related to this result, it would be interesting to show that a F-contraction relative to $\mathcal{F}_0$ is Meir–Keeler contraction.

Comments

This paper is a more larger version of an old paper by the same authors: L.Găvruţa (maiden name of Laura Manolescu), P. Găvruţa and F. Khojasteh, two classes of Meir–Keeler contractions, https://arxiv.org/pdf/1405.5034.pdf. In this form, the paper was cited in [8, 10, 11, 13, 24, 27].

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Interpolation of the Zech's Logarithm: Explicit Forms

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Abstract Zech's logarithm is a function closely related to the Discrete Logarithm. It has applications in communications, cryptography, and computing. In this paper, we provide polynomial and exponential formulas for Zech's logarithm over prime fields.

1 Introduction

Let g be a fixed primitive element for $\mathbb{F}_q$, $q = p^n$, p prime. For every element h, of $\mathbb{F}_q^*$, the discrete logarithm of h, with base g, is the unique integer k, $0 \leq k \leq q-2$, satisfying $g^k = h$.

The discrete logarithm problem amounts to finding a quick method (efficient algorithm) for the computation of k given g and h. In the case that g and k are known, the computation of h can be done rapidly. (Discrete exponential function [18], [11, p. 399]. However, obtaining k from g and h does not appear to admit a fast algorithm [18].

Because of the various cryptographic applications (cf. [3, 4, 10, 15–17]), in the last decades there has been considerable research activity regarding both computational, as well as theoretical aspects of computing discrete logarithms in finite fields. (cf. [10, 16–18])

Zech's logarithm or Jacobi logarithm is another important function for computations in finite fields. It has applications in cryptography and coding theory.

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Let g be a primitive element of $\mathbb{F}_q$. The Zech logarithm is defined by $Z(n)$, $0 \leq n \leq q-2$, in the equation

$$1 + g^n = g^{Z(n)}.$$

This is related to the discrete logarithm, since

$$Z(n) = Log_g(g^n + 1).$$

In the case $1 + g^n = 0$ (e.g., the characteristic of the field is odd and $n = \frac{q-1}{2}$, therefore $g^n = g^{\frac{q-1}{2}} = -1$), we denote $Z(n) = \infty$ (in some books/papers $-\infty$) [5, p. 358], [6]. Obviously, $Z(n)$ is a permutation on the set

$$\{0, 1, \ldots, q-2\} \cup \{\infty\}.$$

Intuitively, the discrete logarithm and Zech's logarithm are related problems.

Zech's logarithms in conjunction with discrete logarithms are used for computations over relatively small finite fields, since Zech's logarithms can be pre-computed.

If the elements of $\mathbb{F}_q^*$ are represented as powers of a fixed primitive element g, then pre-computation of Zech's logarithms provides a quick way to add (or to subtract) elements, since

$$g^m + g^n = g^m(1 + g^{n-m}) = g^{m+Z(n-m)}, \text{ (cf. [9]).}$$

Conway showed that a table of Zech's logarithms is useful to perform addition in $GF(p^n)$, (see [2]).

Fast computations in Galois fields are substantial for decoding algorithms for error-correcting codes ([12, 13]).

Zech's logarithms have been used for the fast solution of quadratic, cubic, and quartic equations over $GF(2^n)$, (cf. [6]). The results are generalized for fields of characteristic p, $p > 2$, (cf. [7]). Fast computations via Zech's logarithms have been applied to methods and devices for generating pseudo-random sequences. Some of these have been patented, (cf. [8]).

As an application of Zech's logarithms we mention the construction of binary de Bruijn sequences (binary sequences of order 2^n in which each n-tuple occurs exactly once in one period of the sequence), (cf. [1, 19]). De Bruijn sequences are well known and well studied. They are applied to various areas, such as cryptography (pseudo-random number generators, key generators), coding theory, robotics, communications, bioinformatics, DNA coding, to name a few.

2 Lagrange and Exponential Interpolation

It is well known that in a finite field $\mathbb{F}_q$, every function $g : \mathbb{F}_q \to \mathbb{F}_q$ admits a polynomial representation. In addition, there exists a unique polynomial of degree $\leq q-1$ representing the function. The following proposition provides the coefficients of the polynomial in the case of prime fields (without loss of generality we assume $f(0)=0$).

Proposition 1 *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$, $f(0)=0$. Assume that the unique polynomial of degree $\leq p-1$ which represents f is $\sum_{i=1}^{p-1} c_i x^i$. Then,*

$$(c_1, \ldots, c_{p-1})^{\perp} = \boldsymbol{L} \cdot (f(1), \ldots, f(p-1))^{\perp}. \tag{1}$$

By $\boldsymbol{L}$ we denote the matrix

$$\boldsymbol{L} = (-j^{p-1-i}) = (-j^{-i}),\ 1 \leq i, j \leq p-1 ,$$

where i stands for the row and j stands for the column.

Proof It is profound that $\boldsymbol{L}$ is non-singular since it is a Vandermonde matrix. Define $\delta(y,c)$ as

$$\delta(y,c) := \begin{cases} 1 & y = c \\ 0 & y \neq c \end{cases} \text{ (Kronecker's } \delta) .$$

Furthermore,

$$\delta(y,c) = 1 - (y-c)^{p-1} = 1 - \sum_{i=0}^{p-1} (-c)^{p-1-i} \cdot y^i \cdot \binom{p-1}{i} .$$

However,

$$\binom{p-1}{i} = \frac{(p-1)\cdots(p-i)}{1 \cdot 2 \cdots i} = (-1)^i \text{ (mod p)} .$$

We obtain the following interpolation formula for $c \neq 0$:

$$\delta(y,c) = -\sum_{i=1}^{p-1} y^i c^{-i} . \tag{2}$$

Therefore,

$$f(x) = \sum_{j=1}^{p-1} f(j)\delta(x, j) = \sum_{j=1}^{p-1} f(j) \left(\sum_{i=1}^{p-1} x^i(-j^{-i}) \right)$$
$$= \sum_{i=1}^{p-1} x^i \left(\sum_{j=1}^{p-1} f(j)(-j^{-i}) \right) = \sum_{i=1}^{p-1} c_i x^i \ .$$

We derive that

$$c_i = \sum_{j=1}^{p-1} f(j)(-j^{-i}).$$

□

In [14] the discrete exponential function has been represented as a polynomial over $\mathbb{Z}_p$. For the non-zero elements w of the $\mathbb{Z}_p$ field, the definition

$$exp_a(w) := a^w, \ a \neq 0,$$

is given. Also, the polynomial formula

$$exp_a(x) = \sum_{i=1}^{p-1} b_i x^i$$

is given. The b_i's are given by the formulas:

$$b_i = -\sum_{j=1}^{p-1} j^{-i} \cdot a^j \ .$$

Proposition 2 *The discrete exponential functions* $\{exp_a\}_{a \in \mathbb{Z}_p^*}$ *form a basis for interpolations of functions* $f : \mathbb{Z}_p \to \mathbb{Z}_p$. *Every function* f *can be written as an exponential "polynomial"*

$$f(x) = \sum_{i=1}^{p-1} d_i \ exp_i(x), \quad \textit{for all} \quad x \neq 0 \ .$$

Also the coefficients d_i *are given from the equation*

$$(d_1, \ldots, d_{p-1})^{\perp} = \boldsymbol{L}^{\perp} \cdot (f(1), \ldots, f(p-1))^{\perp} \ . \tag{3}$$

Proof Consider the vector $v(k) = (k, k^2, \cdots, k^{p-1})^{\perp}$, $k = 1, \cdots, p-1$. Then,

$$\boldsymbol{L}^{\perp} v(k) = (-\sum_{j=1}^{p-1} \left(\frac{k}{1}\right)^j \cdots - \sum_{j=1}^{p-1} \left(\frac{k}{i}\right)^j \cdots - \sum_{j=1}^{p-1} \left(\frac{k}{p-1}\right)^j)^{\perp} .$$

The i-th (general) entry of the vector is

$$-\sum_{j=1}^{p-1} \left(\frac{k}{i}\right)^j = \delta(k, i),$$

from (2).

Since $\boldsymbol{L}^{\perp}$ is non-singular and the vectors

$$(\delta(k, 1), \cdots, \delta(k, p-1))^{\perp}, \; k = 1, \cdots, p-1 ,$$

consist a basis, then the vectors $v(k), \; k = 1, \cdots, p-1$ consist a basis. It is clear that $\boldsymbol{L}^{\perp}$ is the basis change matrix, therefore (3) follows. □

3 Main Computations

Consider the case of a field of prime order $\mathbb{Z}_p$, p odd prime, a a generator of $\mathbb{Z}_p^*$. The equation

$$a^{Z(x)} \equiv a^x + 1 \pmod{p}$$

defines Zech's logarithm, $x = 0, 1, \ldots, p-2, x \neq \frac{p-1}{2}$.
Therefore, Zech's logarithm can be treated as a function $f(x)$, where $f : \mathbb{Z}_p \to \mathbb{Z}_p$. Since

$$1 \equiv a^0 \equiv a^{p-1} \pmod{p} ,$$

we may equivalently assume that $x = 1, \ldots, p-1, x \neq \frac{p-1}{2}$.
The following interpolation formula can be derived:

$$f(x) = \sum_{z=1}^{p-1} z \cdot \delta(a^z, a^x + 1) .$$

However,

$$f(x) = \sum_{z=1}^{p-2} z \cdot \delta(a^x, a^z - 1) , \tag{4}$$

since for the $(p-1)$-th term, we have

$$(p-1)\cdot\delta(a^{p-1}, a^x+1) = (p-1)\cdot\delta(a^{p-1}-1, a^x)$$
$$= \delta(0, a^x) = 0, \text{ for all } x .$$

Also, for $x = \frac{p-1}{2}$,

$$a^x \text{ becomes } -1 \text{ and } \delta(a^x, a^z-1) = 0, \text{ for all } z.$$

The “conventional" Zech logarithm of $\frac{p-1}{2}$ becomes 0 instead of ∞.
Since $z \neq p-1$ (or $a^z - 1 \neq 0$), we derive from (2) that:

$$\delta(a^x, a^z-1) = -\sum_{k=1}^{p-1}\left(\frac{a^x}{a^z-1}\right)^k ,$$

therefore from (4) we obtain

$$f(x) = \sum_{z=1}^{p-2} z\left(\sum_{k=1}^{p-1}\frac{-a^{xk}}{(a^z-1)^k}\right)$$
$$= \sum_{k=1}^{p-1}\left(\sum_{z=1}^{p-2}\frac{-z}{(a^z-1)^k}a^{xk}\right) .$$

From the previous computations we obtain the following proposition.

Proposition 3 *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be the “conventional” Zech’s logarithm, that is the function which coincide with $Z(x)$, for $x \neq 0$, $x \neq \frac{p-1}{2}$ and $f(0) = f(\frac{p-1}{2}) =$* 0*. Then,*

$$f(x) = \sum_{k=1}^{p-1} d_k \cdot exp_{a^k}(x), \ x \neq 0 ,$$

where

$$d_k = \sum_{z=1}^{p-2}\frac{-z}{(a^z-1)^k} .$$

□

According to §2 and [16] the discrete exponential function has the form:

$$a^{kx} = (a^k)^x = exp_{a^k}(x) = \sum_{i=1}^{p-2} b_i x^i ,$$

where

$$b_i = -\sum_{j=1}^{p-1} j^{-i}(a^k)^j \ .$$

We derive a proposition for the Lagrange interpolation.

Proposition 4 *Under the same assumptions as in Proposition 3, it holds*

$$f(x) = \sum_{i=1}^{p-1} \left(\sum_{k=1}^{p-1} d_k b_i \right) x^i \ .$$

□

4 Examples

(I) We give examples for $p = 7$ and $p = 11$. In the case of $\mathbb{Z}_7$ we use the primitive element 3. The "conventional" Zech's logarithm is represented from the polynomial:

$$3x + x^2 + 2x^3 + 3x^4 + 3x^5 + 6x^6 \tag{5}$$

and from the exponential "polynomial"

$$6 \cdot exp_1(x) + 3 \cdot exp_2(x) + 4 \cdot exp_3(x) + 6 \cdot exp_4(x) + 5 \cdot exp_5(x) + 6 \cdot exp_6(x) \ . \tag{6}$$

(II) In the case of the $\mathbb{Z}_{11}$, we select 2 as primitive element. Then, the Zech's logarithm is given by

$$0x + 10x^2 + 7x^3 + 4x^4 + 7x^5 + 10x^6 + 7x^7 + 9x^8 + 10x^9 + 10x^{10} \ . \tag{7}$$

Also, the exponential polynomial is

$$\begin{aligned} &10 \cdot exp_1(x) + 9 \cdot exp_2(x) + 1 \cdot exp_3(x) + 2 \cdot exp_4(x) + 3 \cdot exp_5(x) \\ &\quad + 2 \cdot exp_6(x) + 0 \cdot exp_7(x) + 3 \cdot exp_8(x) + 1 \cdot exp_9(x) + 3 \cdot exp_{10}(x) \ . \end{aligned} \tag{8}$$

The number of terms in (5), (6), (7), and (8) is almost $p - 1$.

5 Conclusions

In general the expressions for the coefficients in Propositions 3 and 4 are complex; It is like in the case of discrete exponential function [14], the resulting polynomial formula is of theoretical and no computational interest. In addition to this, computing Zech's logarithms in large finite fields does not appear to admit a fast algorithm. Zech's logarithms are useful in general for pre-computation in small fields.

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Numerical Calculations on Multi-Photon Processes in Alkali Metal Vapors

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Abstract We present the theoretical framework and the approximations needed to numerically simulate the response of alkali metal atoms under multi-photon excitation. By applying the semi-classical approximation, we obtain a system of coupled ordinary and partial differential equations accounting both for the nonlinear dynamics of the atomic medium and the spatiotemporal evolution of the emitted fields. The case of two-photon excitation by a laser field with an additional one-photon coupling field is investigated by numerically solving the set of differential equations employing a self-consistent computational scheme. The computation of the emission intensities and atomic level populations and coherences is then possible.

1 Introduction

Systems of ordinary and partial differential equations have been extensively used in quantum physics and are considered fundamental in order to theoretically understand laser radiation—matter interaction and nonlinear optics. Nonlinear optics is the branch of physics that describes the behavior of light in nonlinear media, that is, media that respond nonlinearly to an applied electromagnetic field

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[1]. The nonlinearity is typically observed only at very high light intensities, such as those provided by lasers. In nonlinear optics, the superposition principle no longer holds. Alkali metal atoms have been extensively used as model systems due to their low-lying energy levels. Consequently, the excitation and experimental study of the nonlinear response of alkali metal atom systems using two-photon schemes is easily feasible using laser systems in the visible range of the spectrum. The theoretical description and computation of the nonlinear processes observed in experiments can be implemented using systems of differential equations and by applying either semi-classical approximations, where the atom is treated quantum mechanically but the participating fields classically, or fully quantum descriptions (quantum optics).

Resonant or near resonant multi-photon interaction of laser pulses with atomic systems and the induced nonlinear response in terms of generated radiation have been important research topics. Two-photon excitation, whereby nanosecond (ns) or femtosecond (fs) laser pulses are tuned near a two-photon resonance, has been extensively used to study the atomic system dynamics in a vapor cell. Well known nonlinear phenomena can be easily observed under two-photon excitation, such as the partially coherent amplified spontaneous emission (ASE), stimulated hyper Raman scattering (SHRS) and four-wave mixing emissions (FWM) [2–11]. Forward and backward propagating fields that are emitted axially or conically have also been recorded depending on the laser field detuning and propagation characteristics of the laser beam [12–14]. Excitation of alkali metal vapors have been proven a convenient methodology for the study of phase matching mechanisms, wave mixing emissions, multi-photon mechanisms, energy transfer between atomic states, efficient generation of laser radiation and ultrafast processes [15–27].

Internally generated radiations resulting from the two-photon excitation of alkali metal atoms have been shown to compete with the laser pulse to nonlinearly modify the response of the atomic system, specifically the emitted pulse shapes, the temporal evolution of the emitted pulses and the population distribution in atomic levels. In addition, destructive quantum interference (QI) can take place between laser photons and internally generated photons connecting the same levels modifying the nonlinear response of the system [3, 6, 12, 14, 28]. In addition, several different approaches for the realization of atomic memories in closed systems had been proposed over the past decades [29–32]. Atomic coherence and electromagnetically induced transparency (EIT) [3, 33–37], slow light propagation [38] and lasing without inversion (LWI) [39–43] have been also extensively studied. The theoretical study and experimental demonstration of the manipulation of quantum states between fields and atoms have made feasible the production of quantum memory devices that can efficiently delay or store the quantum states of light fields in order to write, store and "read-out" faithfully these states and the information they carry.

Optical free induction decay (OFID) [44] can become a useful method for studying light-matter interactions, in particular for probing dipole dephasing times in gases and solids. The interaction of an atom with two laser pulses, a pump and a coupling one, in a temporally counterintuitive order (the coupling precedes the pump) have also been considered as an effective method to enhance the nonlinearities of an atomic system [45–49]. In general, enhancement of the internally generated fields occurs if the arrival of a pump pulse follows the coherent coupling pulse or if they partially overlap. In the case of a three-level system, the coupling laser, either between the ground and the low excited state (V-type system) or between the two excited states (Λ-type system), creates a coherent superposition of the two states, resulting in enhancement of the parametric emissions driven by the pump laser connecting the ground state to the high excited one [48]. In addition, enhancing the nonlinearities via the use of resonant atomic transitions, has led to the investigation of FWM processes in a counterintuitive pulse sequence [50, 51], which results in the enhancement of the parametric emissions and additional flexibility in their temporal control. It was shown, by using a single pump field, that the response of the system is affected mainly by the pump intensity, the atomic density, and the elastic dephasing collision rates [52]. Finally, observed suppression of emissions due to QI effects, and ionization losses to the continuum (open atomic systems) in the case of focused laser pulses should also be taken into account in computations for a more complete description of the atomic system response [4, 53–60].

In this work, we review the theoretical framework and the approximations needed to simulate the atomic response of alkali metal atoms under two-photon excitation by a laser field. By applying the semi-classical approximation, where the atoms are treated quantum mechanically and the fields classically, we obtain a system of coupled ordinary and partial differential equations for the propagation of the emission fields in the nonlinear atomic medium. The calculation of the emission intensities and the atomic level populations and coherences is then possible after certain additional justifiable approximations are introduced.

2 Theoretical Modeling and Approximations

Two-photon excitation of alkali metal atoms is possible when the orbital angular momentum and parity of the initial $|1\rangle$ and final $|2\rangle$ atomic states satisfy certain selection rules. A typical configuration for the two-photon excitation should include many energy levels having lower energy than $|2\rangle$ (closed system) and possibly the continuum if ionization is taken into account due to absorption of an additional photon (open system). In order to simplify the model and the calculations, the most intense emissions and the associated energy levels are typically included. In this work, the four-level model for the simulation of the atomic system is similar to that presented in [59], where the transition $|1\rangle - |2\rangle$ is excited by two photons, while the de-excitation of atomic state $|2\rangle$ is possible through the lower energy atomic states $|3\rangle$ and $|4\rangle$. The external laser pulse waveform (pump field) used to provide the two photons for the excitation has an intensity which varies with time t. This is simulated in the model either as a secant hyperbolic function $F(t) = \text{sech}^2(t/t_c)$ for $t < 0$ and Gaussian function $F(t) = \exp(-t^2/t_c^2)$ for $t > 0$ or as a single Gaussian function, depending on the waveform characteristics of the experimental laser used to excite the two-photon transition (t_c is the temporal Full Width at Half Maximum

(FWHM) or pulse duration). In order to apply the model and compute the nonlinear response with realistic atomic parameters, potassium atoms are used and levels $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ correspond to the potassium atomic levels $4S_{1/2}$, $6S_{1/2}$, $4P_{3/2}$ and $5P_{3/2}$, respectively. Emissions are generated at one photon allowed transitions (electric dipole selection rules [61]) at frequencies ω_{24}, ω_{41}, ω_{23} and ω_{31} that correspond to the dipole allowed atomic transitions $|2\rangle \leftrightarrow |4\rangle$, $|4\rangle \leftrightarrow |1\rangle$, $|2\rangle \leftrightarrow |3\rangle$ and $|3\rangle \leftrightarrow |1\rangle$.

The semi-classical approximation is used for the interaction of the atom with the electromagnetic field of the laser pulse. This is adequate to simulate experimental results in the case of intense excitation laser fields, where the photon creation and annihilation operators used in a quantum mechanical description of the field can be replaced with the amplitude of the time-dependent classical field. In this case, the Hamiltonian for the interaction of the atom with the electromagnetic field is given by

$$H_I = \frac{1}{2m}[\vec{p} - e\vec{A}(\vec{r}, t)]^2 + e\phi(\vec{r}, t) \tag{1}$$

where $\vec{A}(\vec{r}, t)$ and $\phi(\vec{r}, t)$ are the vector and scalar potentials of the field. In the Coulomb gauge, $\phi(\vec{r}, t) = 0$, and in the dipole approximation, where $\vec{A}(\vec{r}, t) \approx \vec{A}(\vec{r}_0, t)$, we can ignore the spatial derivatives of the vector potential, and using the unitary transformation $|\psi(t)\rangle = \exp\left[\frac{ie\vec{r}}{\hbar} \cdot \vec{A}(\vec{r}, t)\right] |\chi(t)\rangle$ for the state vector and the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{2}$$

we can finally write the total Hamiltonian as $H = H_0 + H_I$, where $H_I = -e\vec{r} \cdot \vec{E}(\vec{r}_0, t)$ is the electric dipole interaction Hamiltonian and H_0 the atomic Hamiltonian.

The unitary transformation of the Hamiltonian from the Schrödinger picture to the interaction picture is effected by applying the unitary operator $U_0(t) = \exp[-\frac{i}{\hbar} H_0 t]$:

$$H^{(I)} = U_0^{\dagger} H^{(S)} U_0(t) \tag{3}$$

The free atom Hamiltonian H_0 can be written in the form $H_0 = \sum \hbar\omega_i |i\rangle\langle i|$ where $\hbar\omega_i$ is the energy of the $|i\rangle$ state. Finally, the unitary transformation leads to a Hamiltonian $H^{(I)}$ in the interaction picture [5, 59, 62] that has the form:

$$\begin{aligned} H^{(I)} =& - \hbar(\Omega_{12}^{(2)}|1\rangle\langle 2|e^{-i\Delta_{12}t} + \Omega_{14}|1\rangle\langle 4|e^{-i\Delta_{14}t} + \Omega_{13}|1\rangle\langle 3|e^{-i\Delta_{13}t} + \\ &+ \Omega_{23}|2\rangle\langle 3|e^{-i\Delta_{23}t} + \Omega_{24}|2\rangle\langle 4|e^{-i\Delta_{24}t}) + H.c. \end{aligned} \tag{4}$$

The two-photon Rabi frequency $\Omega_{12}^{(2)}$ is expressed as a linear function of the maximum laser intensity $I_{\max}$ [63]:

$$\Omega_{12}^{(2)}(t) = \frac{\mu_{12}^{(2)}}{c\epsilon_0\hbar} I_{\max} F(t) \tag{5}$$

The two-photon matrix element $\mu_{12}^{(2)}$ is calculated using an effective Green's function approach in the context of the single-channel quantum defect theory [64–67], a technique well established for the calculation of multi-photon matrix elements in alkali metal atoms. The contribution of all non-resonant virtual intermediate states, including the continuum as well, should be included in the calculations. However, taking into account only the contributions of states $|3\rangle$ and $|4\rangle$ is a good approximation in the proposed model. In deriving the density operator equations of motion, the non-resonant virtual atomic levels effectively contributing to the two-photon excitation are adiabatically eliminated. The internally generated radiations, with electric fields

$$E_{ij}(z,t) = \epsilon_{ij}(z,t)\exp[-i(\nu_{ij}t - k_{ij}z)]/2 + c.c. \tag{6}$$

are included in the model in the form of the single-photon Rabi frequencies Ω_{ij}, which are proportional to the complex amplitudes $\epsilon_{ij}(\zeta,t)$ of the emitted fields at transition $|i\rangle \leftrightarrow |j\rangle$. The detuning from the transition $|i\rangle \leftrightarrow |j\rangle$, is denoted as $\Delta_{ij} = \nu_{ij} - \omega_{ij}$, where ν_{ij} is the frequency of the generated field, with indices ij taking values from the set 1, 2, 3, 4 as appropriate. In the computations below, it is assumed that $\Delta_{12} = 0$ (two-photon detuning of the pump) and $\Delta_{ij} = 0$ (single-photon detunings).

In order to derive the equations for the atom, we apply the density operator formulation, where the density operator is defined as $\hat{\rho} = \sum_i a_i|i\rangle\langle i|$ with a_i being the probability of the system to be in the $|i\rangle$ state. Knowing the density operator matrix elements we can extract any information for the atomic system as it can be shown that for an observable A and its corresponding operator $\hat{A}$, the expectation value is $\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$. The time evolution of the density matrix is governed by the Schrödinger–von Neumann equation: $i\hbar\frac{\partial}{\partial t}\hat{\rho} = [\hat{H},\hat{\rho}]$, where $[\hat{H},\hat{\rho}] = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}$ is the commutator. By applying the rotating wave approximation (RWA) with the transformation $\rho_{ij} = \sigma_{ij}\exp(-i\omega_{ij}t)$, the following set of coupled ordinary differential equations is obtained:

$$\begin{aligned}\dot{\sigma}_{11} = i\big(&\Omega_{12}^{(2)}\sigma_{21} - \Omega_{21}^{(2)}\sigma_{12} + \Omega_{14}\sigma_{41} - \Omega_{41}\sigma_{14} + \\ &+ \Omega_{13}\sigma_{31} - \Omega_{31}\sigma_{13}\big) + \Gamma_{2R}\sigma_{22} + \Gamma_{31}\sigma_{33} + \Gamma_{41}\sigma_{44}\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{\sigma}_{22} = i(&\Omega_{21}^{(2)}\sigma_{12} - \Omega_{12}^{(2)}\sigma_{21} + \Omega_{24}\sigma_{42} - \Omega_{42}\sigma_{24} + \Omega_{23}\sigma_{32} - \Omega_{32}\sigma_{23}) \\ &- (\Gamma_{2R} + \Gamma_{23} + \Gamma_{24})\sigma_{22}\end{aligned} \tag{8}$$

$$\dot{\sigma}_{33} = i(\Omega_{32}\sigma_{23} - \Omega_{23}\sigma_{32} + \Omega_{31}\sigma_{13} - \Omega_{13}\sigma_{31}) - \Gamma_{31}\sigma_{33} + \Gamma_{23}\sigma_{22} \tag{9}$$

$$\dot{\sigma}_{44} = i\big(\Omega_{41}\sigma_{14} - \Omega_{14}\sigma_{41} + \Omega_{42}\sigma_{24} - \Omega_{24}\sigma_{42}\big)$$

$$- \varGamma_{41}\sigma_{44} + \varGamma_{24}\sigma_{22} \tag{10}$$

$$\dot{\sigma}_{12} = i\,(\varDelta_{12} + i\,(\gamma_{12} + \gamma_{\text{col}}))\,\sigma_{12} + i\,\varOmega_{12}^{(2)}(\sigma_{22} - \sigma_{11}) + \\ + i\big(\varOmega_{14}\sigma_{42} + \varOmega_{13}\sigma_{32} - \varOmega_{32}\sigma_{13} - \varOmega_{42}\sigma_{14}\big) \tag{11}$$

$$\dot{\sigma}_{13} = i\big(\varDelta_{13} + i(\gamma_{13} + \gamma_{\text{col}})\big)\sigma_{13} + i\,\varOmega_{13}(\sigma_{33} - \sigma_{11}) \\ + i\big(\varOmega_{41}\sigma_{43} + \varOmega_{12}\sigma_{23} - \varOmega_{23}\sigma_{12}\big) \tag{12}$$

$$\dot{\sigma}_{14} = i\,(\varDelta_{14} + i\,(\gamma_{14} + \gamma_{\text{col}}))\,\sigma_{14} + i\,\varOmega_{14}(\sigma_{44} - \sigma_{11}) + \\ + i(\varOmega_{12}^{(2)}\sigma_{24} + \varOmega_{13}\sigma_{34} - \varOmega_{24}\sigma_{12}) \tag{13}$$

$$\dot{\sigma}_{23} = -i\,(\varDelta_{12} - \varDelta_{13} - i\,(\gamma_{23} + \gamma_{\text{col}}))\,\sigma_{23} + i\big(\varOmega_{23}(\sigma_{33} - \sigma_{22}) + \\ + \varOmega_{21}^{(2)}\sigma_{13} + \varOmega_{24}\sigma_{43} - \varOmega_{13}\sigma_{21}\big) \tag{14}$$

$$\dot{\sigma}_{24} = -i\,(\varDelta_{12} - \varDelta_{14} - i\,(\gamma_{24} + \gamma_{\text{col}}))\,\sigma_{24} + \\ + i\left(\varOmega_{24}(\sigma_{44} - \sigma_{22}) + \varOmega_{21}^{(2)}\sigma_{14} + \varOmega_{23}\sigma_{34} - \varOmega_{14}\sigma_{21}\right) \tag{15}$$

$$\dot{\sigma}_{34} = -i\,(\varDelta_{14} - \varDelta_{13} - i\,(\gamma_{34} + \gamma_{\text{col}}))\,\sigma_{34} + i\,(\varOmega_{31}\sigma_{14} + \varOmega_{32}\sigma_{24} - \\ - \varOmega_{14}\sigma_{31} - \varOmega_{24}\sigma_{32}) \tag{16}$$

$$+ \text{c. c.}$$

The coherence decay rates of the four-level model system of potassium atom are phenomenologically added as γ_{12}, γ_{24}, γ_{41}, γ_{23}, γ_{31} and they are calculated by the formula $\gamma_{ij} = \sum \varGamma_{ij}/2$, $(i \neq j)$, where the decay constant $\varGamma_{ij}$ is the inverse lifetime (ns^{-1}) of transition $|i\rangle \leftrightarrow |j\rangle$ [5, 62, 68–71]. In addition, the contribution of collision dephasing rate γ_{col} is considered in the non-diagonal density matrix elements, simulating the elastic collisions of potassium atom with the buffer gas used in the experiments. The effective decay $\varGamma_{2R}$ in (7), is obtained from the contribution of the states $|4\rangle$, $|3\rangle$ and the intermediate ones $|3D_{3/2}\rangle$ and $|5S_{1/2}\rangle$, through which the atom decays from the state $|2\rangle$ to $|1\rangle$ [3, 5, 31].

In order to account for the generation of the internally generated fields and their propagation along the z axis, the Maxwell equations are used for the field amplitudes (or Rabi frequencies) within the slowly varying envelope approximation (SVEA). Transformed in the retarded time frame by the transformation $\tau = t - z/c$ and $z = \zeta$, they read as:

$$\frac{\partial}{\partial \zeta}\varOmega_{ij}(\zeta, \tau) = i\frac{k_{ij}}{4\epsilon_0\hbar}\mu_{ij}\,p_{ij}(\zeta, \tau)$$

where $p_{ij}(\zeta, \tau) = N\text{Tr}(\hat{\mu}\hat{\rho})$ is the quantum mechanical atomic polarization, N the atomic density of potassium, ϵ_0 the permittivity of free space, k_{ij} the wave-number for each transition and μ_{ij} the matrix element of the electric dipole operator for the corresponding single-photon transition [63]. The matrix elements of the transitions of interest are taken from [61]: $\mu_{24} = 10.7$ a.u., $\mu_{41} = -0.453$ a.u., $\mu_{23} = 1.07$ a.u. and $\mu_{31} = -5.13$ a.u., respectively. The two-photon matrix element of the pumping transition is calculated to be $\mu_{12}^{(2)} = -950$ a.u., where a.u. denotes atomic units [63]. Finally, the propagation equations for the internally generated Rabi frequencies in a co-propagating reference frame assumed the following form:

$$\frac{\partial}{\partial \zeta}\Omega_{24}(\zeta,\tau) = iN\,\frac{k_{24}}{2\epsilon_0 \hbar}\,\mu_{24}^2 \sigma_{24} \tag{17}$$

$$\frac{\partial}{\partial \zeta}\Omega_{41}(\zeta,\tau) = iN\,\frac{k_{41}}{2\epsilon_0 \hbar}\,\mu_{14}^2 \sigma_{41} \tag{18}$$

$$\frac{\partial}{\partial \zeta}\Omega_{23}(\zeta,\tau) = iN\,\frac{k_{23}}{2\epsilon_0 \hbar}\,\mu_{23}^2 \sigma_{23} \tag{19}$$

$$\frac{\partial}{\partial \zeta}\Omega_{31}(\zeta,\tau) = iN\,\frac{k_{31}}{2\epsilon_0 \hbar}\,\mu_{13}^2 \sigma_{31} \tag{20}$$

The set of coupled equations (7)–(20) are the Maxwell-Bloch equations of our system and can be numerically solved self-consistently obtaining the spatiotemporal dependence for the unknown quantities Ω_{ij} and σ_{ij}. The intensity I_{ij} of the generated emissions is calculated as $I_{ij} = \frac{2\hbar^2 \epsilon_0 c}{\mu_{ij}^2}\Omega_{ij}^2$.

In addition, transition $|4\rangle \leftrightarrow |1\rangle$ or $|3\rangle \leftrightarrow |1\rangle$ can be excited in our model by an external field in order to compute the characteristics of the system under a V-type coupling scheme (laser pump field excites the two-photon transition while an external coupling field is applied on the one-photon transition). The tunable external coupling field is considered to have maximum intensity I_{14}^c and the same waveform and duration as the pump laser field, in a pump-coupling excitation scheme. In the case of $|4\rangle \leftrightarrow |1\rangle$ coupling (V_{14} coupling scheme), the new coupling Rabi frequency denoted as $\Omega_{14}^c = \frac{\mu_{14}}{2\hbar}\sqrt{\frac{2}{c\epsilon_0}}\sqrt{|I_{14}^c|}F(\tau)$ is added in every term containing Ω_{14} replacing Ω_{14} with $(\Omega_{14}+\Omega_{14}^c)$ and Ω_{41} with $(\Omega_{41}+\Omega_{14}^c)$ in the set of Eqs. (7), (10), (11), (12), (13), (15), and (16), while the system interaction Hamiltonian describing the V-type coupling scheme takes the following form:

$$\begin{aligned} H^{(I)} = &-\hbar(\Omega_{12}^{(2)}|1\rangle\langle 2|e^{-i\Delta_{12}t} + (\Omega_{14}+\Omega_{14}^c)|1\rangle\langle 4|e^{-i\Delta_{14}t} + \Omega_{13}|1\rangle\langle 3|e^{-i\Delta_{13}t} + \\ &+ \Omega_{23}|2\rangle\langle 3|e^{-i\Delta_{23}t} + \Omega_{24}|2\rangle\langle 4|e^{-i\Delta_{24}t}) + H.c. \end{aligned} \tag{21}$$

We assume that both external fields resonantly excite the transitions of interest and as a consequence $\Delta_{12} = 0$ and $\Delta_{14}^c = 0$ (the latter is the coupling field detuning). The $|3\rangle \leftrightarrow |1\rangle$ external excitation (V_{13} coupling scheme) can be investigated in a similar way.

Furthermore, an external coupling laser field with maximum intensity I_{23}^c and the same waveform and duration can be used to excite the upper single-photon transition $|2\rangle \leftrightarrow |3\rangle$, in a Λ-type pump-coupling scheme. In this case, the Rabi frequency of the coupling field is defined as $\Omega_{23}^{c} = \frac{\mu_{23}}{2\hbar}\sqrt{\frac{2}{c\epsilon_0}}\sqrt{|I_{23}^{c}|}F(\tau)$, and both pump and coupling fields are assumed to resonantly excite the transitions of interest, so $\Delta_{12} = 0$ and $\Delta_{23}^{c} = 0$ (single-photon detuning of the coupling). The coupling Rabi frequency is also added in every term containing Ω_{23} and Ω_{32} in the set of Eqs. (8), (9), (11), (12), (14), (15) and (16) with the new Hamiltonian describing the Λ-type coupling scheme being:

$$H^{(I)} = -\hbar(\Omega_{12}^{(2)}|1\rangle\langle 2|e^{-i\Delta_{12}t} + \Omega_{14}|1\rangle\langle 4|e^{-i\Delta_{14}t} + \Omega_{13}|1\rangle\langle 3|e^{-i\Delta_{13}t} + \\ + (\Omega_{23} + \Omega_{23}^{c})|2\rangle\langle 3|e^{-i\Delta_{23}t} + \Omega_{24}|2\rangle\langle 4|e^{-i\Delta_{24}t}) + H.c. \qquad (22)$$

The enhancements observed for the internally generated emissions in the case of a V-type or a Λ-type coupling scheme are discussed in [69–71].

An important phenomenon that can interfere with the model described is the ionization process. In the previous discussion, the atomic system is presented as a closed system, meaning that the atoms are excited and participate in several processes due to the interaction with the electromagnetic fields, but they remain unaltered (no electrons are absorbed or lost) and finally, after a certain period of time, the atoms return to their original state $|1\rangle$. This means that the number of participating atoms in the model remains constant. However, ionization processes are possible due to the strong intensities of the electromagnetic fields used, for example, three pump laser photons can cause the extraction of an electron and the subsequent ionization of the atom. Since ionized atoms are different than the neutral atoms and the model becomes extremely complicated if ions are also taken into account, a different approach is needed. In order to take into consideration the ionization process, we assume that ions generated are extracted from the system and do not participate in the model. Consequently, we discuss the atomic model as an open system in which neutral atoms population decreases with time. The effect of the ionization process (transition to the continuum) was presented in [59] in the case of the potassium atom.

In order to include the transition to the continuum through the two-photon resonant, three-photon ionization mechanism [62], the Maxwell-Bloch equations have to be transformed. At first, the sum of the population derivatives is non-zero, in contrast to a closed four-level system, so Eq. (8) has to be modified by the addition of the term $-\Gamma_{ion} I_{max} F(\tau)\sigma_{22}$.

I_{max} is the pump laser peak intensity and Γ_{ion} is the ionization width (more information about the ionization rate in an open atomic system can be found in [59]). In addition, the term $-\Gamma_{ion} I_{max} F(\tau)\sigma_{mn}/2$ must be added in Eqs. (11), (14), and (15), since all transitions connected with state $|2\rangle$ are affected by the transition from the state to the continuum (quantified by the factor Γ_{ion}) due to the two-photon resonant, three-photon ionization process. Furthermore, in order to take into account ionization from state $|4\rangle$ (system loses from state $|4\rangle$ to the continuum) by the absorption of one laser photon, the term $-\Gamma'_{ion} I_{max} F(\tau)\sigma_{44}$ is also added in

Eq. (10), and the term $-\Gamma'_{ion} I_{max} F(\tau)\sigma_{mn}/2$ is also added in Eqs. (13), (15), and (16), which are related to the off-diagonal matrix elements.

For short laser pulses, such as in the femtosecond (fs) range, the propagation of both the pump and coupling laser fields in the medium has to be taken into account. For the coupling Rabi frequency we add another equation in the form of (17)–(20):

$$\frac{\partial}{\partial \zeta}\Omega_c(\zeta, \tau) = iN\frac{k_c}{2\epsilon_0\hbar}\mu_c^2\sigma_c \tag{23}$$

where c is to be replaced with the appropriate numbers of the coupled transitions. The two-photon field propagation is governed [12] by the equation

$$\frac{\partial}{\partial \zeta}\Omega_{12}^{(2)}(\zeta, \tau) = iN\frac{k_{12}}{2\epsilon_0\hbar}4K_{12}^{(2)}\sigma_{12}\Omega_{12}^{(2)} \tag{24}$$

where the second order coupling strength $K_{12}^{(2)} = \frac{1}{2\hbar}\sum_i \frac{\mu_{2i}\mu_{i1}}{\omega_{2i}-\omega_1}$ is calculated over all the virtual states between the states $|1\rangle$ and $|2\rangle$.

3 Results and Discussion

The set of differential equations (7)–(20) and (23), (24) is numerically solved employing a FORTRAN code. We calculate both the field and atomic variables by taking alternate steps in space and time along two grids of constant step size, one spatial, along the propagation axis, and one temporal, starting from known initial conditions for the atomic variables at each position and known boundary conditions for the field variables at each time [72]. Initially, the atoms are considered to be at the ground state for each ζ at $\tau = 0$, while the boundary conditions for the generated Rabi frequency Ω_{ij} at $\zeta_0 = 0$ correspond to the quantum noise level, which induces single-photon transitions by quantum fluctuations, with a typical value of $\Omega_{ij}(0, \tau) \propto \epsilon_0(0, \tau) = 10^{-4}$ V/cm [73].

To solve the first-order coupled differential equations, either with respect to time or with respect to position, we employ the fourth-order Runge–Kutta method of constant step size. This method is simple but sufficiently accurate and allows for explicit control of the step sizes to match the requirements of the physics problem and provide the necessary detail in the representation of the evolution of both the atomic and field variables. In our system both the duration of the pump pulse and the total propagation length are fixed. We have chosen to advance the set of variables in time at discrete positions and we typically study the outcome at the exit face of the vapor cell that allows us to compare directly with experimental results.

However, for short, sub-ps pulsed excitation a very small time step is needed to accurately describe the atomic and field evolution, even more so since the pump pulse should be also propagated. In typical computing platforms, the execution time of the code becomes prohibitively large, so the total propagation length was limited

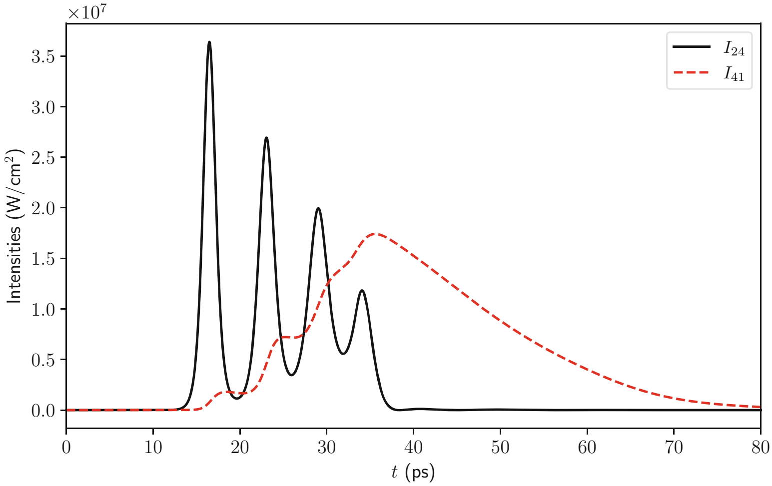

Fig. 1 Intensities versus time for the internally generated emissions at the $|2\rangle - |4\rangle$ and the $|4\rangle - |1\rangle$ transitions. The system parameters are: Laser intensity $I_{\max} = 45\,\mathrm{GW/cm^2}$, coupling intensity $I_{14}^{c} = 5\,\mathrm{W/cm^2}$, pulses FWHM $\tau_c = 40\,\mathrm{fs}$, pump-coupling temporal separation $\Delta t = -2\,\mathrm{ps}$ and atomic density $N = 4 \times 10^{15}\,\mathrm{cm^{-3}}$

to 1 cm. This length corresponds to typical vapor cell sizes used in experimental setups and, in principle, longer propagation lengths can be studied numerically given sufficient computational resources.

In the following computations the atomic system is assumed to be open in order to take into account ionization processes. Short pulse excitation (0.04 ps pulses) and a V-coupling scheme are applied, with the coupling field having the same pulse characteristics as the excitation pulse applied at the two-photon transition. In this case, emissions at ω_{24} and ω_{41} partially overlap temporally within the excitation pulse duration and are clearly synchronized as is evident in Fig. 1. Populations of state $|3\rangle$ and emissions at ω_{23} and ω_{31} remain in the noise regime for the parameters used in our model, so they are not shown in the following figures and discussion.

In Fig. 2, the populations of the atomic states are shown. It is evident that state $|3\rangle$ remains unpopulated at all times and that state $|2\rangle$ builds its maximum population during the short excitation pulse duration of 0.04 ps. The system assumes a steady state driven by short pulses of internally generated emissions and subsequently spontaneous decay that drives the population back to the ground state via a cascade of emissions. The time scale of spontaneous emission is far longer than the one depicted in the figures.

Further insight into the evolution of the internally generated emissions is provided by the study of the coherences, i.e., the off-diagonal matrix elements of the atomic density operator. In Fig. 3a and b the calculated time profiles of both σ_{24} and σ_{41} are depicted. Their time evolution correlates well with the calculated intensities for the corresponding emissions, shown in Fig. 1. In particular, the time

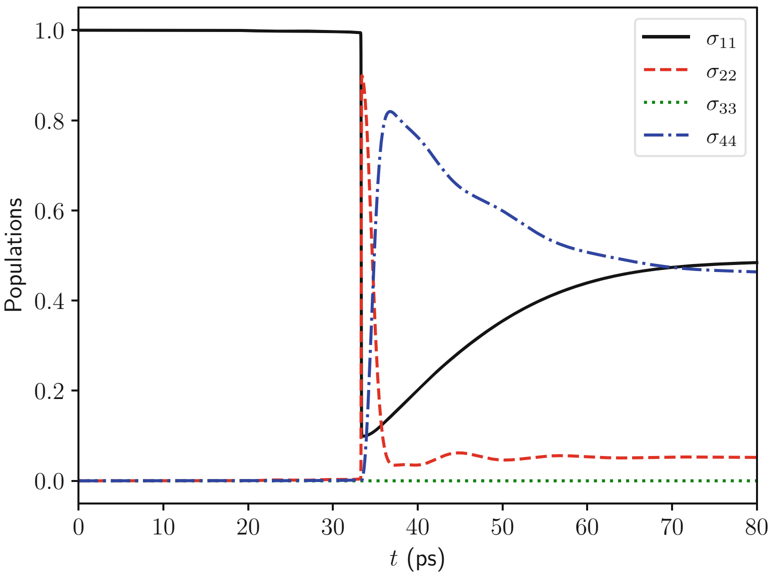

Fig. 2 Populations of the states $|1\rangle \leftrightarrow |4\rangle$ versus time. The system parameters are the same as in Fig. 1

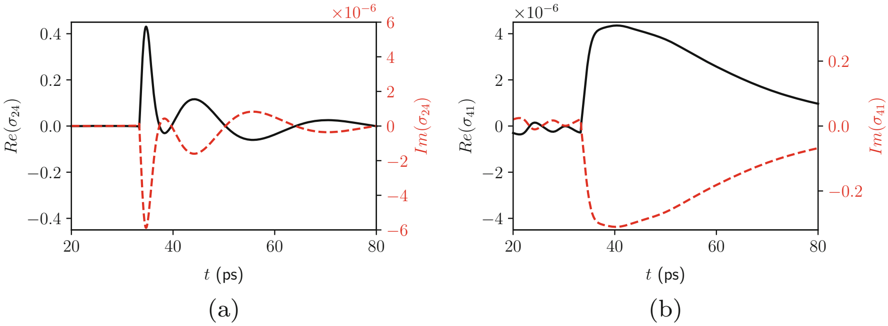

Fig. 3 Coherences of the $|2\rangle \leftrightarrow |4\rangle$ (**a**) and of the $|4\rangle \leftrightarrow |1\rangle$ (**b**) transition versus time. The system parameters are the same as in Fig. 1

evolution of the imaginary parts of both coherences provides an insight into the multi-peaked emission profile at ω_{24} and the gradual build-up and broad emission profile at ω_{41}.

The introduction of a weak coupling pulse, either in the $|1\rangle \leftrightarrow |3\rangle$ or the $|2\rangle \leftrightarrow |3\rangle$ transitions, transforms the system's dynamics, and significantly enhances the ω_{23} and ω_{31} radiations, while no significant population in state $|3\rangle$ is obtained. In Fig. 4 the coupling field connecting the $|1\rangle \leftrightarrow |3\rangle$ states (V_{13} coupling scheme) with maximum intensity $I^c_{13} = 1\,\text{kW/cm}^2$ enhance the emissions via state $|3\rangle$ (termed path-2 emissions) several orders of magnitude, while the emissions via state $|4\rangle$

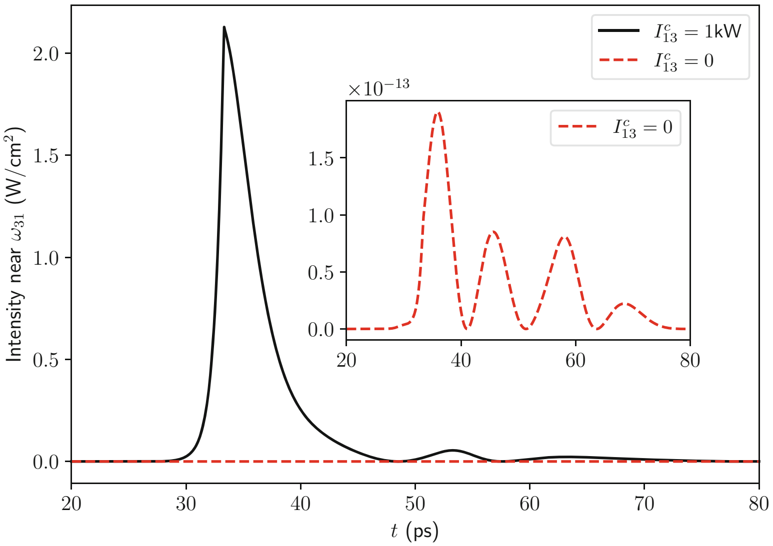

Fig. 4 Intensities of the ω_{31} emission with excitation intensity 45GW/cm^2 and a $|1\rangle$-$|3\rangle$ coupling field of $I^c_{13} = 1$kW/cm^2 (solid line) and without the coupling field (dashed line). The enhancement of the ω_{31} emission is 13 orders of magnitude

(termed path-1 emissions) are unaffected. Furthermore, the internally generated path-2 radiations are synchronous to the path-1 ones and to the excitation pulse, an indication of a parametric process. For a coupling pulse of strength comparable to the excitation pulse, the dynamics of the system is reversed and the energy is transferred through the path-2 emissions while the path-1 ones are negligible. The reliable numerical investigation of the system dynamics offers valuable insights for the efficient control of the emissions in the system, guiding future experimental work.

The relative temporal delay of the two pulses can be used to estimate the coherence relaxation time (CRT) of the atomic states. For the V_{14} or the V_{13} coupling schemes, the induced coherence by the coupling field, when it precedes the pump, enhances the ω_{41} or the ω_{31} emissions, so the exponential increase in the corresponding intensities, that can be accurately calculated as a function of time, can provide an estimate the CRT of the $|4\rangle$ or the $|3\rangle$ states, respectively. When the coupling pulse follows the pump, the effect on the ω_{41} or ω_{31} emissions is governed by the σ_{12} coherence and the calculated exponential decrease in the corresponding intensities provides an estimate of the CRT of the $|2\rangle$ state. The theoretical calculations are in good agreement with the experiment [70, 71] in the V_{14} system. The coupling field in a Λ configuration which connects the upper $|2\rangle$ state with the $|4\rangle$ or the $|3\rangle$ ones, does not induce coherence when it precedes the pump (negative temporal delay), a condition that was observed both in the experiment and in the theoretical calculations [62], where it is shown that for positive temporal delays the ω_{41} or ω_{31} radiation enhancement can provide an estimate of the CRT for the $|2\rangle$ state.

4 Conclusions

The semi-classical approximation, where the atoms are treated quantum mechanically and the fields classically, is employed in order to compute the atomic response of alkali metal atoms under different multi-photon processes. In the case of a four-level atomic system and two-photon excitation by a laser field, a system of coupled linear ordinary and partial differential equations is numerically solved self-consistently in order to compute the atomic parameters (populations and coherences) and the emission fields propagating in the nonlinear atomic medium. The numerical solution provides a comprehensive spatiotemporal description of the evolution of both the driven atomic system and the input and internally generated fields that afford direct comparison with experimental results.

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General Preinvex Functions and Variational-Like Inequalities

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Abstract In this paper, we define and introduce some new concepts of the higher order strongly general preinvex functions and higher order strongly monotone operators involving the arbitrary bifunction. Some new relationships among various concepts of higher order strongly general preinvex functions have been established. It is shown that the new parallelogram laws for Banach spaces can be obtained as applications of higher order strongly affine general preinvex functions, which is itself a novel application. It is proved that the optimality conditions of the higher order strongly general preinvex functions are characterized by a class of variational inequalities, which are called the higher order strongly general variational-like inequalities. An auxiliary principle technique is used to suggest an implicit method

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for solving strongly general variational-like inequalities. Convergence analysis of the proposed method is investigated using the pseudo-monotonicity of the operator. Some special cases are also discussed. Results obtained in this paper can be viewed as a refinement and improvement of previously known results.

1 Introduction

Convexity theory is a branch of Mathematics with a wide range of applications in industry, physical, social, and engineering sciences. Researches in this domain have established important and novel connections with all areas of pure and applied sciences. The general theory of convexity started soon after the introduction of differential and integral calculus by Newton and Leibniz, although some individual optimization problems had been investigated before that. It is worth mentioning that the first phase of the development of convexity was characterized by a combination of philosophical concepts, mathematical methods, and physical problems. Motivated by geometrical considerations, Euler deduced his first principle which is now referred to as Euler's differential equation for the determination of maximizing and minimizing arcs. By convexity, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry, and related areas.

We point out that the history of convexity comprises distinct stages. The basic search of solutions of variational problems, led through the work of Euler, Lagrange, Legendre, Jacobi, and many others, developed along the lines of differential and integral equations as well as functional analysis. The Hamiltonian-Jacobi theory represents a general framework for the mathematical description of the propagation of actions in nature and optimal modelling of control processes in daily life. Using the ideas and techniques of Hamiltonian-Jacobi theory in mechanics, Cartan introduced differential geometry and his exterior calculus in the calculus of variations. Many basic equations of mathematical physics result from variational problems. It is known that the gauge fields theories constitute a continuation of Einstein's concept of describing physical effects mathematically in terms of differential geometry. These theories play a fundamental role in the modern theory of elementary particles and are the right tool of building up a unified theory of elementary particles, which includes all kinds of known interactions. For example, the Weinberg-Salam theory unifies weak and electromagnetic interactions. It is also known that the variational formulation of field theories allows for a degree of unification absent in terms of differential equations. Variational principles play an important part in the existence and stability of solitons, which occur in almost every branch of physics.

Optimization came into being because of equilibrium problems arising in economics and transportation from the 1950s onwards. In recent years, several new generalizations of convex functions have been introduced using novel and innovative ideas to tackle difficult problems, which arise in various fields of pure and applied sciences. Mohsen et al. [16] as well as Noor and Noor [28] introduced the concept of higher order strongly convex functions and studied their properties. These results can be viewed as significant refinements of the results of Lin and Fukushima [13] and Alabdali et al. [1] for higher order strongly uniformly convex functions. Higher order strongly convex functions include strongly convex functions as a special case, which were introduced and studied by Polyak [30]. Karmardian[12] used strongly convex functions to discuss the unique existence of a solution of nonlinear complementarity problems. With appropriate choice of non-negative arbitrary functions, one can obtain various known classes of convex functions. For the properties of strongly convex functions and their variant forms, cf. Adamek [2], Awan et al. [3] Nikodem et al. [18] as well as Noor and Noor [25–29].

Hanson [11] introduced the concept of invex function for differentiable functions, which played significant role in mathematical programming. Ben-Israel and Mond [4] introduced the concept of invex set and preinvex functions. It is known that differentiable preinvex functions are invex functions. The converse also holds under certain conditions, cf. [15]. Noor [20] proved that the minimum of differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, known as variational-like inequalities. For recent developments in variational-like inequalities and invex equilibrium problems, cf. [18, 19, 27] and the references therein. Noor at el. [20, 21, 26, 29] investigated the properties of strongly preinvex functions and their variant forms.

In many problems, a set may not be convex. To overcome this, the underlying set can be made convex with respect to an arbitrary function. This fact motivated Noor [24] to introduce the concept of general convex sets and general convex functions involving an arbitrary function. Cristescu at al[8, 9] have investigated algebraic and topological properties of general convex sets defined by Noor [24] in order to deduce their shape. These general sets constitute a subclass of star-shaped sets, which have Youness [36] type convexity. A representation theorem based on extremal points is given for the class of bounded general convex sets. Results showing that this convexity is a frequent property in connection with a wide range of applications are given, cf. [8, 9]. Noor [24] has shown that the optimality conditions of the differentiable general convex functions can be characterized as a class of variational inequalities called general variational inequalities, the origin of which can be traced back to Stampacchia [32]. Noor and Noor [23, 25–29] introduced the higher order strongly general convex functions and studied their properties. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of general variational inequalities, cf. [17–22, 26–30, 32, 35, 38] and the references therein.

We would like to point out that preinvex functions and general convex functions are two different generalizations and extensions of convex functions in various directions. These types of functions have played a leading role in the development of various branches of pure and applied sciences. Inspired by the research work conducted in this field, we introduce and consider another class of non-convex functions with respect to the arbitrary non-negative bifunction. This class of non-convex functions is called the higher order strongly general preinvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. As a novel and innovative application of these higher order strongly affine general preinvex functions, we obtain the parallelogram-like laws for uniformly Banach spaces. We have shown that the minimum of a differentiable higher order strongly general preinvex function on the general invex set can be characterized by a class of variational-like inequalities. These results motivated us to consider the higher order strongly general variational-like inequalities. Due to the inherent nonlinearity, the projection method and its variant form cannot be used to suggest the iterative methods for solving these general variational-like inequalities. To overcome these drawbacks, we use the technique of the auxiliary principle (cf. [10, 14, 23, 29, 38]) to suggest an implicit method for solving general variational-like inequalities. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. As special cases, one can obtain various new and refined versions of known results. It is hoped that the ideas and techniques featured in this paper may stimulate further research in this field.

2 Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively. Let $F : K \to R$ be a continuous function and let $g : [0, \infty) \to R$ be a non-negative function.

Definition 1 ([9, 17]) The set K in H is said to be a convex set, if

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2 ([9, 17]) A function F is said to be a convex function, if

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

If the convex function F is differentiable, then $u \in K$ is the minimum of F if and only if $u \in K$ satisfies the inequality

$$\langle F'(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

which is called the variational inequality, introduced and studied by Stampacchia [32] in 1964. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, cf. [10, 19–23, 26, 29, 32, 38] and the references therein.

It is known that in many problems the underlying set may not be a convex set. To overcome this drawback, Noor [24] introduced the general convex set with respect to an arbitrary function.

Definition 3 ([24]) The set K_g in H is said to be a general convex set, if there exists an arbitrary function g, such that

$$(1-t)u + tg(v) \in K_g, \quad \forall u, v \in H : u, g(v) \in K_g, \; t \in [0, 1].$$

If $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true. Cristescu et al. [8] discussed various applications of the general convex sets related to the necessity of adjusting investment or development projects due to environmental or social reasons. They have discussed the applications of the general convex sets in the problem of modernizing the railway transport system and have investigated the shape properties of the general convex sets with respect to a projection.

For the properties and applications of the general convex sets, cf. Noor [24] and Cristescu et al. [8, 9]. It is worth mentioning that this general convex set is different than the g-convex set introduced by Youness [36].

For the sake of simplicity, we always assume that $\forall u, v \in H : u, g(v) \in K_g$, unless otherwise specified.

Definition 4 A function F is said to be a general convex (g-convex) function, if there exists an arbitrary non-negative function g, such that

$$F((1-t)u + tg(v)) \leq (1-t)F(u) + tF(g(v)), \quad \forall u, g(v) \in K_g, \quad t \in [0, 1].$$

The general convex functions were introduced by Noor [24]. Noor [24] proved that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K_g,$$

which are known as general variational inequalities, introduced and studied by Noor [24] in 2008.

Ben-Israel and Mond [4] introduced the concept of invex set and preinvex functions, which has inspired a great deal of interest of the applications of invex sets and preinvex functions in mathematical programming and optimization problems.

Definition 5 ([4]) The set K_η in H is said to be an invex set with respect to an arbitrary bifunction $\eta(\cdot, \cdot)$, if

$$u + t\eta(v, u) \in K_\eta, \quad \forall u, v \in K_\eta, \quad t \in [0, 1].$$

The invex set K_η is also called η-connected set. Note that the invex set with $\eta(v, u) = v - u$ is a convex set K, but the converse is not true. For example, the set $K_\eta = R - (-\frac{1}{2}, \frac{1}{2})$ is an invex set with respect to η, where

$$\eta(v,u)=\begin{cases} v-u, \text{ for } & v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, \text{ for } & v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_η is not a convex set.

In the sequel, K_η will be a nonempty closed invex set in H with respect to the bifunction $\eta(\cdot,\cdot)$, unless otherwise specified.

Clearly the general convex set and invex set are two different generalizations of the convex set and have important applications. It is natural to unify these concepts. This fact motivated us to introduce the following:

Definition 6 A set $K_{g\eta} \subset H$ is said to be a general invex set with respect to an arbitrary function g and bifunction $\eta(.,.)$, if

$$u + t\eta(g(v), u) \in K_{g\eta}, \quad \forall u, v \in H : u, g(v) \in K_{g\eta}, t \in [0,1],$$

which was introduced by Noor [22] in 2008.

Definition 7 The function F on the invex set $K_{g\eta}$ is said to be higher order strongly general preinvex with respect to the bifunction $\eta(\cdot,\cdot)$ and function g, if there exists a constant $\mu > 0$, such that

$$\begin{aligned} F(u + t\eta(g(v), u)) \leq & (1-t)F(u) + tF(g(v)) \\ & -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p, \\ & \forall u, g(v) \in K_{g\eta}, t \in [0,1], p > 1. \end{aligned}$$

The function F is said to be higher order strongly general preconcave if and only if $-F$ is a higher order strongly general preinvex function. Note that every higher order strongly general convex function is a higher order strongly general preinvex function, but the converse is not true. It is worth mentioning that for $g = I$, higher order strongly preinvex functions were introduced by Noor and Noor [29]. Awan et al. [3] derived the error estimates for higher order strongly preinvex functions and their variant forms.

I. If $\eta(v,u) = g(v) - u$, then the higher order strongly generalized preinvex function becomes higher order strongly general convex function, that is,

Definition 8 The function F on the general convex set K_g is said to be higher order strongly general convex with respect to a function g, if there exists a constant $\mu > 0$, such that

$$\begin{aligned} & F(u + t(g(v) - u)) \leq (1-t)F(u) + tF(g(v)) \\ & \quad -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - u\|^p, \forall u, g(v) \in K_g, t \in [0,1], \end{aligned}$$

which was introduced and studied by Noor and Noor [28].

For properties of the higher order strongly general convex functions in variational inequalities and equilibrium problems, cf. Noor [28].

II. If $\eta(g(v), u) = v - u$, then the higher order strongly general preinvex function becomes higher order strongly convex functions, that is,

Definition 9 The function F on the convex set K is said to be higher order strongly convex, if there exists a constant $\mu > 0$, such that

$$F(u + t(v - u)) \leq (1 - t)F(u) + tF(v) - \mu\{t^p(1 - t) + t(1 - t)^p\}\|v - u\|^p,$$
$$\forall u, g(v) \in K, t \in [0, 1],$$

which were introduced and studied by Mohsen et al. [16].

For properties of the higher order strongly convex functions in variational inequalities and equilibrium problems, cf. Noor [20–22, 25, 27, 28].

III. If $p = 2$, then Definition 7 becomes:

Definition 10 A function F is said to be strongly general preinvex with respect to the function g, if

$$F(u + t\eta(g(v), u)) \leq (1 - t)F(u) + tF(g(v)) - \mu t(1 - t)\|\eta(g(v), u)\|^2,$$
$$\forall u, g(v) \in K_{g\eta}, t \in [0, 1].$$

IV. If $\mu = 0$, then Definition 7 becomes:

Definition 11 A function F is said to be a general preinvex function with respect to the function g, if

$$F(u + t\eta(g(v), u)) \leq (1 - t)F(u) + tF(g(v)), \quad \forall u, g(v) \in K_{g\eta}, t \in [0, 1]. \tag{1}$$

Definition 12 The function F on the general invex set $K_{g\eta}$ is said to be higher order strongly general quasi-preinvex with respect to the bifunction $\eta(\cdot, \cdot)$ and a function g, if there exists a constant $\mu > 0$, such that

$$F(u+t\eta(g(v), u)) \leq \max\{F(u), F(g(v))\} - \mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v), u)\|^p,$$
$$\forall u, g(v) \in K_{g\eta}, t \in [0, 1], p > 1.$$

Definition 13 The function F on the general invex set $K_{g\eta}$ is said to be higher order strongly general log-preinvex with respect to the bifunction $\eta(\cdot, \cdot)$ and a function g, if there exists a constant $\mu > 0$, such that

$$F(u+t\eta(g(v), u)) \leq (F(u))^{1-t}(F(g(v)))^t - \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p,$$
$$\forall u, g(v) \in K_{g\eta}, t \in [0, 1], p > 1,$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$
\begin{aligned}
F(u + t\eta(g(v), u)) &\leq (F(u))^{1-t}(F(g(v)))^{t} \\
&\quad -\mu\{t^{p}(1-t) + t(1-t)^{p}\}\|\eta(g(v), u)\|^{p} \\
&\leq (1-t)F(u) + tF(g(v))) \\
&\quad -\mu\{t^{p}(1-t) + t(1-t)^{p}\}\|\eta(g(v), u)\|^{p} \\
&\leq \max\{F(u), F(g(v))\} \\
&\quad -\mu\{t^{p}(1-t) + t(1-t)^{p}\}\|\eta(g(v), u)\|^{p}, \ p \geq 1.
\end{aligned}
$$

This shows that every higher order strongly general log-preinvex function is a higher order strongly general preinvex function and every higher order strongly general preinvex function is a higher order strongly general quasi-preinvex function. However, the converse is not true.

Definition 14 The function F on the invex set $K_{g\eta}$ is said to be a higher order strongly affine function with respect to the bifunction $\eta(\cdot, \cdot)$ and a function g, if there exists a constant $\mu > 0$, such that

$$
\begin{aligned}
F(u + t\eta(g(v), u)) = (1-t)F(u) + tF(g(v)) - \mu\{t^{p}(1-t) \\
+t(1-t)^{p}\}\|\eta(g(v), u)\|^{p}, \\
\forall u, g(v) \in K_{g\eta}, t \in [0, 1].
\end{aligned}
$$

For $t = 1$, Definitions 7 and 14 reduce to the following condition:

Condition A

$$
F(u + \eta(g(v), u)) \leq F(g(v)), \quad \forall u, g(v) \in K_{g\eta}.
$$

Definition 15 The differentiable function F on the invex set $K_{g\eta}$ is said to be a higher order strongly general invex function with respect to the bifunction $\eta(\cdot, \cdot)$ and a function g, if there exists a constant $\mu > 0$ such that

$$
F(g(v)) - F(u) \geq \langle F'(u), \eta(g(v), u)\rangle + \mu\|\eta(g(v), u)||^{p}, \quad \forall u, v \in K_{g\eta},
$$

where $F'(u)$ is the differential of F at u.

It is noted that if $\mu = 0$ and $g = I$, then the Definition 10 reduces to the definition of the invex function as introduced by Hanson [11]. It is well known that the concepts of preinvex and invex functions have played a significant role in mathematical programming and optimization theory, cf. [4, 11, 15, 31, 34, 35, 37] and the references therein.

Remark 1 Note that if $\mu = 0$, then the Definitions 10–13 appear to be original.

Definition 16 An operator $T : K_{g\eta} \to H$ is said to be:

1. higher order strongly $g\eta$-monotone, if and only if there exists a constant $\alpha > 0$ such that

$$\langle Tu, \eta(g(v), u)\rangle + \langle Tv, \eta(u, g(v))\rangle \leq -\alpha\{\|\eta(g(v), u)\|^p + \|\eta(u, v)\|^p\},$$
$$u, g(v) \in K_{g\eta}.$$

2. η-monotone, if and only if

$$\langle Tu, \eta(g(v), u)\rangle + \langle Tv, \eta(u, g(v))\rangle \leq 0, \quad u, g(v) \in K_{g\eta}.$$

3. higher order strongly $g\eta$-pseudomonotone, if and only if there exists a constant $\nu > 0$ such that

$$\langle Tu, \eta(g(v), u)\rangle + \nu\|\eta(g(v), u)\|^p \geq 0 \Rightarrow -\langle Tv, \eta(u, g(v))\rangle \geq 0,$$
$$u, g(v) \in K_{g\eta}.$$

4. higher order strongly relaxed η-pseudomonotone, if and only if there exists a constant $\mu > 0$ such that

$$\langle Tu, \eta(g(v), u)\rangle \geq 0 \Rightarrow -\langle Tv, \eta(u, g(v)\rangle + \mu\|\eta(u, g(v))\|^p \geq 0,$$
$$u, g(v) \in K_{g\eta}.$$

5. strictly $g\eta$-monotone, if and only if

$$\langle Tu, \eta(g(v), u)\rangle + \langle Tv, \eta(u, g(v))\rangle < 0, \quad u, g(v) \in K_{g\eta}.$$

6. $g\eta$-pseudomonotone, if and only if

$$\langle Tu, \eta(g(v), u)\rangle \geq 0 \Rightarrow \langle Tv, \eta(u, g(v))\rangle \leq 0, \quad u, g(v) \in K_{g\eta}.$$

7. quasi $g\eta$-monotone, if and only if

$$\langle Tu, \eta(g(v), u)\rangle > 0 \Rightarrow \langle Tv, \eta(u, g(v))\rangle \leq 0, \quad u, g(v) \in K_{g\eta}.$$

8. strictly $g\eta$-pseudomonotone, if and only if

$$\langle Tu, \eta(g(v), u)\rangle \geq 0 \Rightarrow \langle Tv, \eta(u, g(v))\rangle < 0, \quad u, g(v) \in K_{g\eta}.$$

Definition 17 A differentiable function F on the invex set $K_{g\eta}$ is said to be a higher order strongly pseudo $g\eta$-invex function, if and only if there exists a constant $\mu > 0$ such that

$$\langle F'(u), \eta(v, u)\rangle + \mu\|\eta(v, u)\|^p \geq 0 \Rightarrow F(v) - F(u) \geq 0, \qquad \forall u, g(v) \in K_{g\eta}.$$

Definition 18 A differentiable function F on $K_{g\eta}$ is said to be a higher order strongly quasi-invex function, if and only if there exists a constant $\mu > 0$ such that

$$\begin{aligned} & F(v) \leq F(u) \\ & \Rightarrow \\ & \langle F'(u), \eta(v, u)\rangle + \mu\|\eta(u, v)\|^p \leq 0, \quad \forall u, g(v) \in K_{g\eta}, p > 1. \end{aligned}$$

Definition 19 The function F on the set $K_{g\eta}$ is said to be pseudo-invex, if

$$\langle F'(u), \eta(v, u)\rangle \geq 0 \Rightarrow F(v) \geq F(u), \qquad \forall u, g(v) \in K_{g\eta}.$$

Definition 20 The differentiable function F on the $K_{g\eta}$ is said to be a higher order strongly general quasi-invex function, if

$$F(v) \leq F(u) \Rightarrow \langle F'(u), \eta(v, u)\rangle \leq 0, \qquad \forall u, g(v) \in K_{g\eta}.$$

We also need the following assumption regarding the bifunction $\eta(., .)$, which can be viewed as a generalization of the condition of Mohan and Neogy [15].

Condition C
Let $\eta(\cdot, \cdot) : K_\eta \times K_\eta \rightarrow H$ satisfy the assumptions

$$\eta(u, u + t\eta(g(v), u)) = -t\eta(g(v), u)$$
$$\eta(g(v), u + t\eta(g(v), u)) = (1 - t)\eta(g(v), u), \quad \forall u, g(v) \in K_{g\eta}, t \in [0, 1].$$

Clearly for $t = 0$, we have $\eta(u, g(v)) = 0$, if and only if $u = g(v), \forall u, v \in K_{g\eta}$. One can easily show (see [11, 15]) that

$$\eta(u + t\eta(g(v), u), u) = t\eta(g(v), u), \forall u, v \in K_{g\eta}.$$

3 Main Results

In this section, we consider some basic properties of higher order strongly general preinvex functions on the general invex set $K_{g\eta}$.

Theorem 1 *Let F be a differentiable function on the invex set $K_{g\eta}$ in H and let the condition C hold true. Then a function F is a higher order strongly general preinvex function, if and only if F is a higher order strongly general invex function.*

Proof Let F be a higher order strongly general preinvex function on the invex set $K_{g\eta}$. Then

$$F(u+t\eta(g(v),u)) \leq (1-t)F(u)+tF(g(v))-\mu\{t^p(1-t) \\ +t(1-t)^p\}\|\eta(g(v),u)\|^p, \\ \forall u, g(v)\in K_{g\eta}, t\in[0,1], p>1,$$

which can be written as

$$F(g(v))-F(u) \geq \left\{\frac{F(u+t\eta(g(v),u))-F(u)}{t}\right\} \\ +\mu\{t^{p-1}(1-t)+(1-t)^p\}\|\eta(g(v),u)\|^p.$$

Taking the limit in the above inequality as $t\to 0$, we have

$$F(g(v))-F(u)\geq \langle F'(u),\eta(g(v),u))\rangle+\mu\|\eta(g(v),u)\|^p.$$

This shows that F is a higher order strongly general invex function.

Conversely, let F be a higher order strongly general invex function on the invex set $K_{g\eta}$. Then,

$$\forall u, g(v)\in K_{g\eta}, t\in[0,1],\ g(v_t)=u+t\eta(g(v),u)\in K_{g\eta}$$

and using the condition C, we have

$$\begin{aligned} &F(g(v))-F(u+t\eta(g(v),u)) \\ &\quad\geq \langle F'(u+t\eta(g(v),u)),\eta(g(v),u+t\eta(g(v),u))\rangle \\ &\qquad+\mu\|\eta(g(v),u+t\eta(g(v),u))\|^p \\ &\quad= (1-t)F'(u+t\eta(g(v),u)),\eta(g(v),u)\rangle+\mu(1-t)^p\|\eta(g(v),u)\|^p. \end{aligned} \tag{2}$$

In a similar way, we have

$$\begin{aligned} &F(u)-F(u+t\eta(g(v),u)) \\ &\quad\geq \langle F'(u+t\eta(g(v),u)),\eta(u,u+t\eta(g(v),u))\rangle+\mu\|\eta(u,u+t\eta(g(v),u))\|^p \\ &\quad= -tF'(u+t\eta(g(v),u)),\eta(g(v),u)\rangle+\mu t^p\|\eta(g(v),u)\|^p. \end{aligned} \tag{3}$$

Multiplying (2) by t and (3) by $(1-t)$ and adding the resultant, we have

$$F(u+t\eta(g(v),u)) \leq (1-t)F(u)+tF(g(v))-\{t^p(1-t) \\ +t(1-t)^p\}\|\eta(g(v),u)\|^p,$$

showing that F is a higher order strongly general preinvex function.

Theorem 2 *Let F be differentiable higher order strongly general preinvex function on the invex set $K_{g\eta}$. If F is a higher order strongly general invex function, then*

$$\langle F'(u),\eta(g(v),u))\rangle+\langle F'(g(v)),\eta(u,g(v))\rangle \\ \leq -\mu\{\|\eta(g(v),u)\|^p+\|\eta(u,g(v))\|^p\}, \forall u, g(v)\in K_{g\eta}. \quad (4)$$

Proof Let F be a higher order strongly general invex function on the general invex set $K_{g\eta}$. Then

$$F(g(v))-F(u) \geq \langle F'(u),\eta(g(v),u))\rangle \\ +\mu\|\eta(g(v),u)\|^p, \forall u, g(v)\in K_{g\eta}. \quad (5)$$

Interchanging the role of u and $g(v)$ in (5), we have

$$F(u)-F(g(v)) \geq \langle F'(g(v)),\eta(u,v)\rangle+\mu\|\eta(u,g(v))\|^p, \quad \forall u, g(v)\in K_{g\eta}. (6)$$

Adding (5) and (6), we have

$$\langle F'(u),\eta(g(v),u))\rangle+\langle F'(g(v)),\eta(u,g(v))\rangle \\ \leq -\mu\{\|\eta(g(v),u)\|^p+\|\eta(u,g(v))\|^p\}, \forall u, g(v)\in K_{g\eta}, \quad (7)$$

which shows that $F'(.)$ is a higher order strongly η-monotone operator.

We note that the converse of Theorem 2 is true only for $p=2$. However, we have:

Theorem 3 *If the differential $F'(.)$ is higher order strongly η-monotone, then*

$$F(g(v))-F(u) \geq \langle F'(u),\eta(g(v),u)\rangle \\ +\frac{2}{p}\mu\|\eta(g(v),u)\|^p.$$

Proof Let $F'(.)$ be higher order strongly η-monotone. From (7), we have

$$\langle F'(g(v)),\eta(u,g(v))\rangle \geq \langle F'(u),\eta(g(v),u))\rangle \\ -\mu\{\|\eta(g(v),u)\|^p+\|\eta(u,g(v))\|^p\}. \quad (8)$$

Since $K_{g\eta}$ is a general invex set, $\forall u, g(v) \in K_{g\eta}, t \in [0, 1]$,

$$g(v_t) = u + t\eta(g(v), u) \in K_{g\eta}.$$

Taking $g(v) = g(v_t)$ in (8) and using Condition C, we have

$$\begin{aligned}\langle F'(g(v_t)), \eta(u, u + t\eta(g(v), u))\rangle &\leq \langle F'(u), \eta(u + t\eta(g(v), u), u))\rangle \\ &\quad -\mu\{\|\eta(u + t\eta(g(v), u), u)\|^p \\ &\quad +\|\eta(u, u + t\eta(g(v), u)\|^p\} \\ &= -t\langle F'(u), \eta(g(v), u)\rangle - 2t^p\mu\|\eta(g(v), u)\|^p,\end{aligned}$$

which implies that

$$\langle F'(v_t), \eta(g(v), u)\rangle \geq \langle F'(u), \eta(g(v), u)\rangle + 2\mu t^{p-1}\|\eta(g(v), u)\|^p. \tag{9}$$

Let $\xi(t) = F(u + t\eta(g(v), u))$. Then, from (9), we have

$$\begin{aligned}\xi'(t) &= \langle F'(u + t\eta(g(v), u)), \eta(g(v), u)\rangle \\ &\geq \langle F'(u), \eta(g(v), u)\rangle + 2\mu t^{p-1}\|\eta(g(v), u)\|^p.\end{aligned} \tag{10}$$

Integrating (10) between 0 and 1, we have

$$\xi(1) - \xi(0) \geq \langle F'(u), \eta(g(v), u)\rangle + \frac{2}{p}\mu\|\eta(g(v), u)\|^p,$$

that is,

$$F(u + t\eta(g(v), u)) - F(u) \geq \langle F'(u), \eta(g(v), u)\rangle + \frac{2}{p}\mu\|\eta(g(v), u\|^p).$$

By using Condition A, we have

$$F(gv)) - F(u) \geq \langle F'(u), \eta(g(v), u)\rangle + \frac{2}{p}\mu\|\eta(g(v), u)\|^p,$$

which is the desired result.

We now give a necessary condition for a higher order strongly η-pseudo-invex function.

Theorem 4 *Let $F'(.)$ be a higher order strongly relaxed η-pseudomonotone operator and Conditions A and C hold true. Then F is a higher order strongly η-pseudo-invex function.*

Proof Let F' be higher order strongly relaxed η-pseudomonotone. Then, $\forall u, g(v) \in K_{g\eta}$,

$$\langle F'(u), \eta(g(v), u)\rangle \geq 0,$$

implies that

$$-\langle F'(g(v)), \eta(u, v)\rangle \geq \alpha\|\eta(u, g(v))\|^p. \tag{11}$$

Since K is an invex set, $\forall u, g(v) \in K_{g\eta}$, $t \in [0, 1]$,

$$g(v_t) = u + t\eta(g(v), u) \in K_\eta.$$

Taking $g(v) = g(v_t)$ in (11) and using Condition C, we have

$$-\langle F'(u + t\eta(g(v), u)), \eta(u, v)\rangle \geq t\alpha\|\eta(g(v), u)\|^p. \tag{12}$$

Let

$$\xi(t) = F(u + t\eta(g(v), u)), \quad \forall u, g(v) \in K_{g\eta}, t \in [0, 1].$$

Then, using (12), we have

$$\xi'(t) = \langle F'(u + t\eta(g(v), u)), \eta(u, v)\rangle \geq t\alpha\|\eta(g(v), u)\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq \frac{\alpha}{2}\|\eta(g(v), u)\|^p,$$

that is,

$$F(u + t\eta(g(v), u)) - F(u) \geq \frac{\alpha}{2}\|\eta(g(v), u)\|^p,$$

which implies, using Condition A,

$$F(g(v)) - F(u) \geq \frac{\alpha}{2}\|\eta(g(v), u)\|^p,$$

showing that F is a higher order strongly η-pseudo-invex function.

Theorem 5 *Let the differential $F'(u)$ of a differentiable higher order strongly general preinvex function $F(u)$ be Lipschitz continuous on the invex set $K_{g\eta}$ with a constant $\beta > 0$. Then*

$$F(u+\eta(g(v),u))-F(u) \leq \langle F'(u), \eta(g(v),u)\rangle+\frac{\beta}{2}\|\eta(g(v),u)\|^2, \quad u, g(v) \in K_{g\eta}.$$

Proof The proof follows from Noor and Noor [26].

Definition 21 The function F is said to be sharply higher order strongly general pseudo preinvex, if there exists a constant $\mu > 0$ such that

$$\langle F'(u), \eta(g(v),u)\rangle \geq 0$$
$$\Rightarrow$$
$$F(g(v)) \geq F(v+t\eta(g(v),u)) + \mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),u)\|^p,$$
$$\forall u, g(v) \in K_{g\eta}, t \in [0,1].$$

Theorem 6 *Let F be a higher order strongly sharply general pseudo preinvex function on $K_{g\eta}$ with a constant $\mu > 0$. Then*

$$-\langle F'(g(v)), \eta(g(v),u)\rangle \geq \mu\|\eta(g(v),u)\|^p, \quad \forall u, g(v) \in K_{g\eta}.$$

Proof Let F be a higher strongly sharply pseudo preinvex function on K_η. Then

$$F(g(v)) \geq F(v+t\eta(g(v),u)) + \mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v),u)\|^p,$$
$$\forall u, g(v) \in K_{g\eta}, t \in [0,1],$$

from which we have

$$\frac{F(g(v)+t\eta(g(v),u)) - F(g(v))}{t} + \mu\{t^{p-1}(1-t)+(1-t)^p\}\|\eta(g(v),u\|^p) \leq 0.$$

Taking the limit in the above inequality, as $t \to 0$, we have

$$-\langle F'(g(v)), \eta(g(v),u)\rangle \geq \mu\|\eta(g(v),u)\|^p,$$

which is the desired result.

Definition 22 A function F is said to be a pseudo preinvex function, if there exists a strictly positive bifunction $B(.,.)$, such that

$$F(g(v)) < F(u)$$
$$\Rightarrow$$
$$F(u+t\eta(g(v),u)) < F(u)+t(t-1)B(g(v),u), \forall u, g(v) \in K_{g\eta}, t \in [0,1].$$

Theorem 7 *If the function F is a higher order strongly general preinvex function such that $F(g(v)) < F(u)$, then F is a higher order strongly general pseudo preinvex function.*

Proof Since $F(v) < F(u)$ and F is higher order strongly preinvex function, then $\forall u, g(v) \in K_{g\eta}, \quad t \in [0, 1]$, we have

$$\begin{aligned} F(u + t\eta(g(v), u)) &\leq F(u) + t(F(g(v)) - F(u)) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p \\ &< F(u) + t(1-t)(F(g(v)) - F(u)) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p \\ &= F(u) + t(t-1)(F(u) - F(g(v))) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p \\ &< F(u) + t(t-1)B(u, g(v)) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p, \forall u, g(v) \in K_{g\eta}, \end{aligned}$$

where

$$D(u, g(v)) = F(u) - F(g(v)) > 0.$$

This shows that the function F is higher order strongly general pseudopreinvex.

4 Applications

In this section, we show that the characterizations of uniformly Banach spaces involving the notion of higher order strongly general invexity can be established.

Considering $F(u) = \|u\|^p$ in Definition 14, we have

$$\begin{aligned} \|u + t\eta(g(v), u)\|^p &= (1-t)\|u\|^p + t\|g(v)\|^p \\ &\quad - \mu\{t^p(1-t) + t(1-t)^p\}\|\eta(g(v), u)\|^p, \\ &\quad \forall u, g(v) \in K_{g\eta}, t \in [0, 1], p > 1. \end{aligned} \tag{13}$$

Setting $t = \frac{1}{2}$ in (13), we have

$$\|\frac{2u + \eta(g(v), u)}{2}\|^p + \mu\frac{1}{2^p}\|\eta(g(v), u)\|^p = \frac{1}{2}\|u\|^p + \frac{1}{2}\|g(v)\|^p, \quad \forall u, g(v) \in K_{g\eta}, \tag{14}$$

which is known as the parallelogram-like laws for the Banach spaces involving the bifunction $\eta(.,.)$ and the arbitrary function g.

If $\eta(g(v), u) = g(v) - u$, then (14) reduces to the parallelogram-like law as:

$$\|g(v) + u\|^p + \mu\|g(v) - u\|^p = 2^{p-1}\{\|u\|^p + \|g(v)\|^p\}, \forall u, v \in K_g, \quad (15)$$

which is called the parallelogram-like law and can be used to characterize the uniform Banach spaces.

If $g = I$,then (15) reduces to the parallelogram-like law as:

$$\|v + u\|^p + \mu\|v - u\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\}, \forall u, v \in K, \quad (16)$$

which is known as the parallelogram-like law for the uniform Banach spaces. Xu [33] obtained these characterizations of p-uniform convexity and q-uniform smoothness of a Banach space via the functionals $\|.\|^p$ and $\|.\|^q$, respectively. Bynum [5] and Chen et al. [6, 7] have studied the properties and applications of the parallelogram laws for the Banach spaces in prediction theory and applied sciences.

5 General Variational-Like Inequalities

In this section, we introduce and consider a new class of variational-like inequalities, which arises as an optimality condition of differentiable general preinvex functions, which is the main motivation of our next result.

Theorem 8 *Let F be a differentiable higher order strongly general preinvex function with modulus $\mu > 0$. If $u \in K_{g\eta}$ is the minimum of the function F, then*

$$F(g(v)) - F(u) \geq \mu\|\eta(g(v), u)\|^p, \quad \forall u, g(v) \in K_{g\eta}. \quad (17)$$

Proof *Let $u \in K_{g\eta}$ be a minimum of the function F. Then*

$$F(u) \leq F(g(v)), \forall g(v) \in K_{g\eta}. \quad (18)$$

Since K_η is an invex set, it follows that $\forall u, g(v) \in K_{g\eta}, \quad t \in [0, 1]$,

$$g(v_t) = u + t\eta(g(v), u) \in K_{g\eta}.$$

Setting $g(v) = g(v_t)$ in (18), we have

$$0 \leq \lim_{t\to 0}\{\frac{F(u + t\eta((g(v).u)) - F(u)}{t}\} = \langle F'(u), \eta(g(v), u)\rangle. \quad (19)$$

Since F is a differentiable higher order strongly general preinvex function, we obtain that

$$F(u+t\eta(g(v),u)) \leq F(u)+t(F(g(v))-F(u))$$
$$-\mu\{t^p(1-t)+t(1-t)^p\}\|\eta(g(v).u)\|^p, \forall u, g(v) \in K_{g\eta},$$

from which, using (19), we have

$$F(g(v))-F(u) \geq \lim_{t\to 0}\{\frac{F(u+t\eta(g(v),u))-F(u)}{t}\}+\mu\|\eta(g(v),u)\|^p$$
$$=\langle F'(u),\eta(g(v),u)\rangle+\mu\|\eta(g(v),u)\|^p,$$

which is the required result (17).

Remark We would like to mention that, if

$$\langle F'(u),\eta(g(v),u)\rangle+\mu\|\eta(g(v),u)\|^p \geq 0, \quad \forall u, g(v) \in K_{g\eta}, \tag{20}$$

then $u \in K_\eta$ is the minimum of the function F.

The inequality of the type (20) is called the higher order strongly general variational-like inequality. We now consider a more general variational-like inequality of which (20) is a special case.

For given two operators T, g, we consider the problem of finding $u \in u \in K_{g\eta}$ for a constant μ such that

$$\langle Tu,\eta(g(v),u)\rangle+\mu\|\eta(g(v),u)\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p>1, \tag{21}$$

which is called the higher order strongly general variational-like inequality.

We now discuss several special cases of the problem (21).

- **(i)** If $Tu = F'(g(u))$, then problem (21) is exactly the general variational-like inequality (20).
- **(ii)** If $\mu = 0$, then (21) is equivalent to finding $u \in K_{g\eta}$, such that

$$\langle Tu,\eta(g(v),u)\rangle \geq 0, \quad \forall g(v) \in K_{g\eta}, \tag{22}$$

  which is known as the general variational-like inequality.
- **(iii)** If $\eta(g(v(,u)) = g(v)-u$, then problem (21) reduces to the problem of finding $u \in K_g$ such that

$$\langle Tu, g(v)-u\rangle+\mu\|g(v)-u\|^p \geq 0, \quad \forall g(v) \in K_g, p>1, \tag{23}$$

which is called the higher order general variational inequality, introduced and studied by Noor and Noor [28].

For suitable and appropriate choice of the parameter μ and p, one can obtain several new and known classes of variational inequalities. We note that the projection method and its variant forms can be used to study the higher order strongly general variational inequalities (21) due to its inherent structure. This fact motivated us to consider the auxiliary principle technique, which is mainly due to Lions and Stampacchia [14], Glowinski et al. [10], as developed by Noor [20, 21, 23] and Noor et al. [28, 29]. We use this technique to suggest some iterative methods for solving the general variational-like inequalities (21).

For given $u \in K_{g\eta}$ satisfying (21), consider the problem of finding $w \in K_{g\eta}$, such that

$$\langle \rho T w, \eta(g(v), w)\rangle + \langle w - u, v - w\rangle + \nu\|\eta(g(v), w)\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1, \tag{24}$$

where $\rho > 0$ is a parameter. The problem (24) is called the auxiliary higher order strongly general variational-like inequality. It is clear that the relation (24) defines a mapping connecting the problems (21) and (24). We note that, if $w(u) = u$, then w is a solution of problem (21). This simple observation enables us to suggest an iterative method for solving (21).

Algorithm 1 *For given $u_0 \in K_{g\eta}$, find the approximate solution u_{n+1} by the scheme*

$$\begin{aligned} &\langle \rho T u_{n+1}, \eta(g(v), u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &+\nu\|\eta(g(v), u_{n+1})\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1. \end{aligned} \tag{25}$$

The Algorithm 1 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities, cf. [20, 21, 23, 29] and the reference therein.

If $\nu = 0$, then Algorithm 1 reduces to:

Algorithm 2 *For given $u_0 \in K_{g\eta}$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho T u_{n+1}, \eta(g(v), u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall g(v) \in K_{g\eta},$$

which appears to be new, even for solving the general variational-like inequalities (22).

In order to study the convergence analysis of Algorithm 1, we need the following concept.

Definition 23 The operator T is said to be pseudo $g\eta$-monotone with respect to $\mu\|\eta(g(v), u)\|^p$, $p > 1$, if

$$\langle \rho Tu, \eta(g(v), u)\rangle + \mu\|\eta(g(v), u)\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1,$$
$$\Longrightarrow$$
$$-\langle \rho Tv, \eta(v, g(u))\rangle - \mu\|\eta(g(v), v)\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1.$$

If $\mu = 0$, then Definition 23 reduces to:

Definition 24 The operator T is said to be pseudo g-monotone, if

$$\langle \rho Tu, \eta(g(v), u)\rangle \geq 0, \forall g(v) \in K_{g\eta}$$
$$\Longrightarrow$$
$$\langle \rho Tv, \eta(v, g(u))\rangle \geq 0, \forall g(v) \in K_{g\eta},$$

which appears to be new.

We now study the convergence analysis of Algorithm 1.

Theorem 9 *Let $u \in K_{g\eta}$ be a solution of (21) and u_{n+1} be the approximate solution obtained from Algorithm 1. If T is a pseudo $g\eta$- monotone operator, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{26}$$

Proof Let $u \in K_{g\eta}$ be a solution of (21). Then

$$\langle \rho Tu, \eta(g(v), u)\rangle + \mu\|\eta(g(v), u)\|^p \geq 0, \forall g(v) \in K_{g\eta},$$

implies that

$$-\langle \rho Tv, \eta(g(u), v)\rangle - \mu\|\eta(g(u), v)\|^p \geq 0, \forall g(v) \in K_{g\eta}. \tag{27}$$

Now taking $v = u_{n+1}$ in (27), we have

$$-\langle \rho Tu_{n+1}, \eta(u_{n+1}, g(u))\rangle - \mu\|\eta(u_{n+1}, g(u))\|^p \geq 0. \tag{28}$$

Taking $v = u$ in (25), we have

$$\langle \rho Tu_{n+1}, \eta(g(u), u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \nu\|\eta(g(u), u_{n+1})\|^p \geq 0. \tag{29}$$
$$\forall g(v) \in K, p > 1.$$

Combining (28) and (29), we have

$$\langle u_{n+1} - u_n, u_{n+1} - u\rangle \geq 0.$$

Using the inequality

$$2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H,$$

we obtain

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

which is the desired result (26).

Theorem 10 *Let the operator T be pseudo $g\eta$-monotone. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u \in K_{g\eta}$ is the exact solution (21), then*

$$\lim_{n\to\infty} u_n = u.$$

Proof Let $u \in K$ be a solution of (21). Then, from (26), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (26), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{30}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence u_n converge to

$$\hat{u} \in H.$$

Replacing u_n by u_{n_j} in (25), taking the limit $n_j \to 0$ and by (30), we have

$$\langle T\hat{u}, g(v) - \hat{u}\rangle + \mu\|g(v) - \hat{u}\|^p \geq 0, \quad \forall g(v) \in K_{g\eta}, p > 1.$$

This implies that $\hat{u} \in K_{g\eta}$ satisfies (21) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

In order to implement the implicit Algorithm 1, one uses the predictor-corrector technique. Consequently, Algorithm 1 is equivalent to the following iterative method for solving the general variational inequality (21).

Algorithm 3 *For a given $u_0 \in K_{g\eta}$, find the approximate solution u_{n+1} by the schemes*

$$\langle \rho T u_n, \eta(g(v), y_n)\rangle + \langle y_n - u_n v - y_n\rangle + \mu\|\eta(g(v), y_n)\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1$$

$$\langle \rho T y_n, \eta(g(v), y_n)\rangle + \langle u_n - y_n, v - y_n\rangle \mu\|\eta(g(v), u_n)\|^p \geq 0, \forall g(v) \in K_{g\eta}, p > 1.$$

Algorithm 3 is called the two-step method and appears to be new.

Using the auxiliary principle technique, one can suggest several iterative methods for solving the higher order strongly general variational inequalities and related optimization problems. We have only given some glimpse of the higher order strongly general variational inequalities. It is an interesting problem to explore the applications of such type of variational inequalities in various fields of pure and applied sciences.

Conclusion

In this paper, we have introduced and studied a new class of convex functions, which are called higher order strongly general preinvex functions. It is shown that several new classes of strongly convex functions can be obtained as special cases of these higher order strongly general preinvex functions. We have studied the basic properties of these functions. New parallelogram laws for uniformly Banach spaces have been derived as applications of the higher order strongly general preinvex functions. It is an open problem to study the applications of these parallelogram laws. We have also considered a new class of general variational-like inequalities. Using the auxiliary principle technique, an implicit iterative method is suggested for finding the approximate solution of general variational-like inequalities. Using the pseudo-monotonicity of the operator, convergence criteria are discussed. Some special cases are considered as applications of the main results.

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A Variational Formulation of Network Games with Random Utility Functions

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Abstract We consider a class of games played on networks in which the utility functions consist of both deterministic and random terms. In order to find the Nash equilibrium of the game we formulate the problem as a variational inequality in a probabilistic Lebesgue space which is solved numerically to provide approximations for the mean value of the random equilibrium. We also numerically compare the solution thus obtained, with the solution computed by solving the deterministic variational inequality derived by taking the expectation of the pseudo-gradient of the game with respect to the random parameters.

1 Introduction

Games played on networks are a class of non-cooperative games where players are considered as nodes of a graph, and direct connections between any two players are represented by arcs connecting them. A basic assumption is that the utility function of a given, arbitrary, player depends on his/her strategy, as well as on the strategies of his/her neighbors in the graph. Therefore, it seems natural that this setting has proved to be very useful in describing social or economic interactions among various types of agents. In this regard it is interesting to investigate the two classes of *games with strategic complements and substitutes*. Roughly speaking, in the first case, the incentive for a player to take an action increases when the number of his/her social contacts who take the action increases, while in the second case this monotonic relation is reversed. As is usual in game theory, equilibrium concepts are considered of paramount importance, and in this context, the study

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of Nash equilibria is investigated with respect to the algebraic or graph-theoretic properties of the network structure. This line of research was initiated with the seminal paper by Ballester et al. [1], who also used some centrality measures to assess the importance of the various players, along the same lines of Katz and Bonacich (see, e.g., [3]). The interested reader can find in the beautiful survey by Jackson and Zenou [7] an account of the main concepts about network games, along with a wealth of social and economic applications. Most of the scholars dealing with this topic tackle the corresponding problems with classic game-theoretical methods, such as best response analysis and fixed point theory. However, quite recently some authors utilized the variational inequality approach to provide a deep analysis of many aspects of these games and the interesting paper by Parise and Ozdaglar [12] provides a self-consistent treatment of many interesting developments. The fact that Nash equilibrium problems admit, under suitable hypotheses, an equivalent variational inequality formulation was recognized long time ago by Gabay and Moulin [6]. It seems, though, that this powerful tool has not been fully applied to the topic of network games. In this note we allow for the possibility that the utility functions also depend on a random parameter ω of an abstract sample space Ω, and then derive the corresponding parametric variational inequality. However, our objective here is to compute the mean value of the equilibrium, hence, we wish that the solution admits finite first and (possibly) second moments. In this regard, an integral variational inequality in the probabilistic Lebesgue space $L^2(\Omega, P)$ fits our requirements. This variational inequality is then transformed to the image space of the random variables involved so as to be numerically approximated. The theory of random (or stochastic) variational inequalities has been developed by various authors in the last fifteen years, with different methodologies. We follow here the so-called *L^p approach* and refer the interested reader to [5, 8, 9] for a detailed account of the theoretical framework and for several applications. For a description of different approaches, as well as for other interesting developments, the reader can see [15], where the authors also describe the so-called *expected value approach* which we compare with our approach by means of a worked out example.

The paper is organized as follows. In the following Sect. 2 we introduce the notation, and briefly outline the basic network game classes. Moreover, we define the random Nash equilibrium and the associated variational inequality. In Sect. 3, we describe in detail the linear-quadratic model, investigate the monotonicity property of the relevant operator, and introduce the associated integral variational inequality. In Sect. 4, we numerically solve a test problem. A short concluding section ends the paper.

2 Network Game Classes and Variational Inequality Approach

We begin this section by recalling a few concepts and definitions of graph theory that will be used in the sequel. We warn the reader that the terminology is not uniform in the related literature. Formally, a graph g is a pair of sets (V, E), where V is the set of nodes and E is the set of arcs, formed by pairs of nodes (v, w). Arcs which have the same end nodes are called parallel, while arcs of the form (v, v) are called loops. We consider here *simple* graphs, that is graphs with no parallel arcs or loops. In our setting, the players will be represented by the n nodes in the graph. Moreover, we consider here indirect graphs: the arcs (v, w) and (w, v) are the same. Two nodes v and w are adjacent if they are connected by an arc, i.e., if (v, w) is an arc. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i, v_j) is an arc, 0 otherwise. G is thus a symmetric and zero diagonal matrix. Given a node v, the nodes connected to v with an arc are called the *neighbors* of v and are grouped in the set $N_v(g)$. The number of elements of $N_v(g)$ is the *degree* of v.

We now proceed to specify the game that we will consider. For simplicity, the set of players will be denoted by $\{1, 2, \ldots, n\}$ instead of $\{v_1, v_2, \ldots, v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player i from the action of all the other players. Let (Ω, P) be a probability space. Each player i is endowed with a payoff function

$$u_i : \Omega \times A \to \mathbb{R}$$

that he/she wishes to maximize for almost every elementary event $\omega \in \Omega$, that is P-almost surely.

The notation $u_i(\omega, a, g)$ is often utilized when one wants to emphasize the influence of the graph structure. The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek a random vector $a^* : \Omega \to A$ such that for each $i \in \{1, \ldots, n\}$, and, P-a.s.:

$$u_i(a_i^*(\omega), a_{-i}^*(\omega)) \geq u_i(a_i, a_{-i}^*(\omega)), \qquad \forall\, a_i \in A_i. \tag{1}$$

A peculiarity of network games is that the vector a_{-i} is only made up of components a_j such that $j \in N_i(g)$, that is, j is a neighbor of i.

We mentioned in the introduction that it is convenient to consider two specific classes of games which allow a deeper investigation of the patterns of interactions among players. For any given player i it is interesting to distinguish how variations of the actions of player's i neighbors affect his/her marginal utility. In the case where the utility functions are twice continuously differentiable the following definitions clarify this point.

Definition 1 We say that the network game has the property of strategic substitutes if for each player i and P-a.s. the following condition holds:

$$\frac{\partial^2 u_i(\omega, a_i, a_{-i})}{\partial a_j \partial a_i} < 0, \qquad \forall (i, j) : g_{ij} = 1, \ \forall a \in A.$$

Definition 2 We say that the network game has the property of strategic complements if for each player i and P-a.s. the following condition holds:

$$\frac{\partial^2 u_i(\omega, a_i, a_{-i})}{\partial a_j \partial a_i} > 0, \qquad \forall (i, j) : g_{ij} = 1, \forall a \in A.$$

Let us notice that we are requiring that each of the two properties specified above holds for almost every $\omega \in \Omega$, i.e., we assume that the game class does not change according to the random variable.

For the subsequent development it is important to recall that if the u_i are continuously differentiable functions on A, the Nash equilibrium problem is equivalent to the variational inequality $VI(F, A)$: find $a^* \in A$ such that, P-a.s.

$$F(\omega, a^*(\omega))^\top (a - a^*(\omega)) \geq 0, \qquad \forall a \in A, \tag{2}$$

where

$$[F(\omega, a)]^\top := -\left(\frac{\partial u_1}{\partial a_1}(\omega, a), \dots, \frac{\partial u_n}{\partial a_n}(\omega, a)\right) \tag{3}$$

is also called the pseudo-gradient of the game, according to the terminology introduced by Rosen [14]. For an account of variational inequalities the interested reader can refer to [4, 10, 11]. We recall here some useful monotonicity properties.

Definition 3 $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[F(\omega, x) - F(\omega, y)]^\top (x - y) \geq 0, \qquad \forall x, y \in A, \ \forall \omega \in \Omega.$$

If the equality holds only when $x = y$, F is said to be strictly monotone.

A stronger type of monotonicity is given by the following

Definition 4 $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be β-strongly monotone on A iff, for every ω, we can find $\beta(\omega) > 0$:

$$[F(\omega, x) - F(\omega, y)]^\top (x - y) \geq \beta(\omega) \|x - y\|^2, \qquad \forall x, y \in A.$$

If we can find a β which does not depend on ω in the above definition, we say that F is strongly monotone, *uniformly* with respect to ω.

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the Jacobian matrix of the operator.

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [4, 10, 11]), which can be applied to our framework for each (or almost each) fixed ω.

Theorem 1 *If $K \subset \mathbb{R}^n$ is a compact convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(F, K)$ admits at least one solution. In the case that K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\|\to+\infty} \frac{[F(x) - F(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$. Furtheremore, if F is strictly monotone on K, then the solution is unique.

3 The Random Linear-Quadratic Model

In what follows A_i can be either $\mathbb{R}_+$ for any $i \in \{1, \dots, n\}$, or $[0, L_i]$, hence $A = \mathbb{R}^n_+$ or $[0, L_1] \times \dots \times [0, L_n]$. The payoff of player i is given by

$$u_i(\omega, a, g) = \alpha(\omega)a_i - \frac{1}{2}a_i^2 + \varphi(\omega)a_i \sum_{i=1}^{n} g_{ij}a_j - \gamma a_i \sum_{i=1}^{n} a_j, \tag{4}$$

where $\alpha(\omega), \varphi(\omega) > 0$, P-a.s. and γ is a positive real number. The term involving the adjacency matrix describes the local complementarities ($\varphi(\omega) > 0$), which means that the neighbors of each player contribute to positively enhance his/her strategy. On the other hand, the term involving γ has opposite sign, thus describing strategic substitutes and it is of global nature.

The pseudo-gradient's components of this game are easily computed as:

$$F_i(\omega, a, g) = (1 + \gamma)a_i - \varphi(\omega) \sum_{j=1}^{n} g_{ij}a_j - \gamma \sum_{j=1}^{n} a_j - \alpha(\omega) \qquad i \in \{1, \dots, n\},$$

which can be written in compact form as:

$$F(\omega, a, g) = [(1 + \gamma)I - \varphi(\omega)G + \gamma U]\,a - \alpha(\omega)\mathbf{1}, \tag{5}$$

where U is the $n \times n$ matrix whose entries are all equal to one and $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^n$.

We will seek random Nash equilibrium points by solving the following variational inequality: for each ω, find $a^*(\omega) \in A$, such that for all $a \in A$ and P-a.s. we have

$$[(1+\gamma)Ia^*(\omega) - \varphi(\omega)Ga^*(\omega)]^\top (a - a^*(\omega)) + [\gamma U\, a^*(\omega)]^\top (a - a^*(\omega)) \\ \geq \alpha(\omega)\mathbf{1}^\top (a - a^*(\omega)). \tag{6}$$

For the subsequent developments it is important to study the monotonicity properties of the operator F in the above variational inequality.

Lemma 1 *Let F be as in (5) and $\rho(G)$ be the spectral radius of G. For all ω such that $\varphi(\omega) < (1+\gamma)/\rho(G)$, F is strictly monotone. Moreover, if $\overline{\varphi}$ is a real number such that $0 < \overline{\varphi} < (1+\gamma)/\rho(G)$, then F is strongly monotone uniformly in the set $\{\omega : 0 < \varphi(\omega) \leq \overline{\varphi}\}$, in the sense that it exists $\beta > 0$ such that*

$$[F(\omega, a) - F(\omega, a')]^\top (a - a') \geq \beta \|a - a'\|^2,$$

for all $a, a' \in \mathbb{R}^n$ and for all ω such that $\varphi(\omega) \in (0, \overline{\varphi}]$.

Proof It is sufficient to study the linear part of F. Thus, let us consider the expression:

$$(1+\gamma)Ia - \varphi(\omega)Ga + \gamma U\, a$$

and notice that for every $\gamma > 0$ the matrix $\gamma\, U$ is positive semidefinite, thus defining a monotone operator. Because the sum of a strongly (strictly) monotone and a monotone operator gives a strongly (strictly) monotone operator, we seek conditions which ensure the strong monotonicity of $(1+\gamma)I - \varphi(\omega)G$. To this end, let us notice that G is a zero trace matrix, hence its largest eigenvalue is positive. Moreover, it can be proved that the largest eigenvalue of G coincides with its spectral radius $\rho(G)$. It follows that, for each ω, the minimum eigenvalue of $(1+\gamma)I - \varphi(\omega)G$ is given by $1 + \gamma - \varphi(\omega)\rho(G)$, which is positive whenever $\varphi(\omega)\rho(G) < 1 + \gamma$. Thus, for each ω such that $\varphi(\omega) \in (0, (1+\gamma)/\rho(G)]$, we get

$$a^\top [(1+\gamma)I - \varphi(\omega)G]a \geq [1 + \gamma - \varphi(\omega)\rho(G)]\, \|a\|^2.$$

Furthermore, let $\overline{\varphi}$ be a real number such that $0 < \overline{\varphi} < (1+\gamma)/\rho(G)$, and $\beta = 1 + \gamma - \overline{\varphi}\rho(G)$. We then obtain that:

$$a^\top [(1+\gamma)I - \varphi(\omega)G]a \geq \beta \|a\|^2$$

holds for any ω such that $0 < \varphi(\omega) \leq \overline{\varphi}$. □

We now proceed to provide an integral formulation of the variational inequality (6). Thus, we make the additional assumptions that the random variable α has finite second order moment, that is, $\alpha \in L^2(\Omega, P)$, while $\varphi \in L^\infty(\Omega, P)$, with

$0 < \underline{\varphi} \leq \varphi(\omega) \leq \overline{\varphi}$. We can now consider the variational inequality problem of finding $a^* \in L^2(\Omega, P)$, such that $a^*(\omega) \in A$, and $\forall a \in L^2(\Omega, P)$ such that $a(\omega) \in A$:

$$\int_\Omega \Big\{[(1+\gamma)Ia^*(\omega) - \varphi(\omega)Ga^*(\omega)]^\top (a - a^*(\omega)) + [\gamma\, U\, a^*(\omega)]^\top (a - a^*(\omega))\Big\}\, dP(\omega) \geq \int_\Omega \alpha(\omega)\mathbf{1}^\top (a - a^*(\omega)) dP(\omega). \tag{7}$$

Remark 1 The theoretical investigation of the above variational inequality requires tools from infinite dimensional functional analysis that are beyond the scope of this paper. The interested reader can see [10] or the papers cited in the introduction for more details. Here, we only mention that under the relevant assumption of uniform strong monotonicity of F we get the existence and uniqueness of the solution a^*.

We now transform the variational inequality (7) in the image space of the two random variables involved. To this end, let $y = \alpha(\omega)$, $z = \varphi(\omega)$, and $\mathbb{P}$ the probability induced by P on the image space of the two random variables. We thus have to consider the variational inequality problem of finding $a^* \in L^2(\mathbb{R}^2, \mathbb{P})$ such that $a^*(y, z) \in A$, and for each $a \in L^2(\mathbb{R}^2, \mathbb{P})$ with $a(y, z) \in A$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{\underline{\varphi}}^{\overline{\varphi}} \Big\{[(1+\gamma)Ia^*(y,z) - zGa^*(y,z)]^\top [a(y,z) - a^*(y,z)] \\ &\quad + [\gamma U a^*(y,z)]^\top [a(y,z) - a^*(y,z)]\Big\}\, d\mathbb{P}(y,z) \\ &\geq \int_{-\infty}^{\infty}\int_{\underline{\varphi}}^{\overline{\varphi}} y\mathbf{1}^\top [a(y,z) - a^*(y,z)]\,\}d\mathbb{P}(y,z). \end{aligned} \tag{8}$$

We denote by $E_{\mathbb{P}}[a^*(y, z)]$ the expected value of the solution with respect to the probability measure $\mathbb{P}$ on the image space of the random variables. The L^p theory of random variational inequalities provides an approximation procedure for the expected values and we refer again the interested reader to the references mentioned in the introduction for a thorough treatment of this matter. In the subsequent section we apply this approximation procedure to a worked out example. Moreover, we compare our results with the ones obtained by solving the deterministic variational inequality obtained by taking the expectation $E_{\mathbb{P}}[F(y, z)]$ of the pseudo-gradient with respect to the random variables involved. This second solution concept is known as the expected value approach and, in this case, leads in a straightforward manner to solving a finite dimensional variational inequality, since the expectation of the pseudo-gradient can be computed exactly. Nevertheless, as it will be illustrated by the numerical examples of the following section, the two approaches can give quite different results for certain parameter ranges.

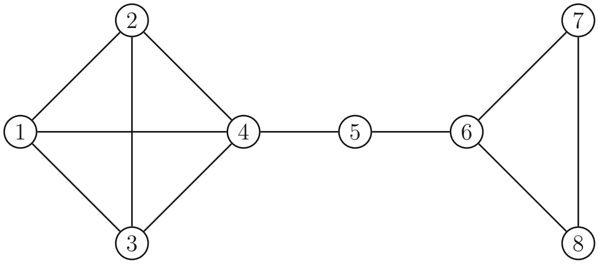

Fig. 1 Network topology of Example 1

4 Numerical Experiments

In this section, we show some preliminary numerical experiments for the random linear-quadratic network game described in Sect. 3.

Example 1 We consider the network shown in Fig. 1 (see also [2]) with 8 nodes (players). The spectral radius of the adjacency matrix G is $\rho(G) \simeq 3.1019$. We set the congestion parameter $\gamma = 0.1$ and the upper bounds $L_i = 5$ for any player $i = 1, \dots, 8$. We assume that the random variable $y = \alpha(\omega)$ varies in the interval $[1, 10]$ with either uniform distribution (denoted by $y \sim \mathscr{U}(1, 10)$) or truncated normal distribution with mean 5.5 and standard deviation 0.9 ($y \sim \mathscr{N}(5.5, 0.9)$), while the random variable $z = \varphi(\omega)$ varies in the interval $[0.01, 0.34]$ with either uniform distribution ($z \sim \mathscr{U}(0.01, 0.34)$) or truncated normal distribution with mean 0.175 and standard deviation 0.033 ($z \sim \mathscr{N}(0.175, 0.033)$). Notice that $(1 + \gamma)/\rho(G) \simeq 0.3546$, hence the assumption of Lemma 1 is satisfied and the operator F is uniformly strongly monotone.

The approximation procedure considers a uniform partition of both intervals $[1, 10]$ and $[0.01, 0.34]$ into N subintervals and solves N^2 finite dimensional variational inequalities for each N.

Table 1 reports in columns 2–6 the convergence of the mean values of the approximate solution obtained for different values of N, when the random variables y and z vary in the corresponding intervals with uniform distribution. Moreover, column 7 shows the solution given by the expected value approach, while the last column shows the percentage difference between columns 6 and 7. Notice that the difference between the approximate solution found by the L^p approach and the solution given by the expected value approach is significant, especially for the first 4 components.

Tables 2, 3, and 4 report the convergence of the mean values of the approximate solution and its comparison with the solution given by the expected value approach when y and z vary with different distributions. We remark that the difference between the approximate solution found by the L^p approach and the solution given

Table 1 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{U}(1, 10)$ and $z \sim \mathscr{U}(0.01, 0.34)$ and comparison with the solution given by the expected value approach (col. 7–8)

	N					Expected value	
Variables	32	64	128	256	512	approach sol.	Diff.
x_1	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_2	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_3	3.651	3.697	3.720	3.732	3.737	4.264	−12.34%
x_4	3.788	3.835	3.858	3.869	3.875	4.744	−18.33%
x_5	3.270	3.311	3.332	3.342	3.348	3.504	−4.45%
x_6	3.368	3.409	3.429	3.439	3.444	3.750	−8.14%
x_7	3.140	3.179	3.198	3.208	3.213	3.269	−1.71%
x_8	3.140	3.179	3.198	3.208	3.213	3.269	−1.71%

Table 2 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{N}(5.5, 0.9)$ and $z \sim \mathscr{U}(0.01, 0.34)$ and comparison with the solution given by the expected value approach (col. 7–8)

	N					Expected value	
Variables	32	64	128	256	512	approach sol.	Diff.
x_1	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_2	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_3	4.035	4.085	4.110	4.122	4.128	4.264	−3.18%
x_4	4.220	4.267	4.289	4.300	4.305	4.744	−9.25%
x_5	3.457	3.513	3.540	3.554	3.561	3.504	1.65%
x_6	3.678	3.739	3.770	3.785	3.792	3.750	1.14%
x_7	3.278	3.337	3.366	3.381	3.389	3.269	3.67%
x_8	3.278	3.337	3.366	3.381	3.389	3.269	3.67%

Table 3 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathscr{U}(1, 10)$ and $z \sim \mathscr{N}(0.175, 0.033)$ and comparison with the solution given by the expected value approach (col. 7–8)

	N					Expected value	
Variables	32	64	128	256	512	approach sol.	Diff.
x_1	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_2	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_3	3.628	3.675	3.698	3.710	3.715	4.264	−12.86%
x_4	3.803	3.850	3.873	3.884	3.890	4.744	−18.01%
x_5	3.259	3.301	3.322	3.333	3.338	3.504	−4.72%
x_6	3.408	3.450	3.470	3.480	3.485	3.750	−7.05%
x_7	3.156	3.195	3.215	3.225	3.230	3.269	−1.19%
x_8	3.156	3.195	3.215	3.225	3.230	3.269	−1.19%

Table 4 Convergence of the mean values of the approximate solution (col. 2–6) for $y \sim \mathcal{N}(5.5, 0.9)$ and $z \sim \mathcal{N}(0.175, 0.033)$ and comparison with the solution given by the expected value approach (col. 7–8)

Variables	N 32	64	128	256	512	Expected value approach sol.	Diff.
x_1	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_2	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_3	4.062	4.132	4.167	4.184	4.192	4.264	−1.68%
x_4	4.385	4.448	4.478	4.493	4.500	4.744	−5.15%
x_5	3.395	3.452	3.480	3.494	3.501	3.504	−0.07%
x_6	3.648	3.710	3.741	3.757	3.765	3.750	0.41%
x_7	3.207	3.259	3.286	3.299	3.306	3.269	1.15%
x_8	3.207	3.259	3.286	3.299	3.306	3.269	1.15%

by the expected value approach is rather small when both random variables y and z vary with truncated normal distribution.

5 Conclusions and Future Research Directions

In this chapter we investigated a model of network games with random utility functions by means of its reformulation as a variational inequality in a probabilistic Lebesgue space. We illustrated our methodology through a worked out example which was numerically solved in order to approximate the mean value of the unique random Nash equilibrium of the game. Furthermore the approximated mean value thus computed was compared with the Nash equilibrium which is obtained by solving a deterministic variational inequality derived by taking the expectation of the pseudo-gradient of the game. Future research work could be performed with nonlinear random utility functions. Another promising research perspective is the variational inequality formulation of generalized network games (with shared constraints), which was initiated in [13] and offers a wealth of potential theoretical developments and possible applications.

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Fixed Point Theory in Graph Metric Spaces

A. Petruşel and G. Petruşel

Abstract Let (X, d) be a metric space, G be a graph associated with X and $f : X \to X$ be an operator which satisfies two main assumptions:

(1) f is generalized G-monotone;
(2) f is a G-contraction with respect to d.

In the above framework, we will present sufficient conditions under which:

(i) f is a Picard operator;
(ii) the fixed point problem $x = f(x), x \in X$ is well-posed in the sense of Reich and Zaslavski;
(iii) the fixed point problem $x = f(x), x \in X$ has the Ulam-Hyers stability property;
(iv) f has the Ostrowski stability property;
(v) f satisfies to some Gronwall type inequalities.

Some open questions are presented.

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1 Introduction

Metric fixed point theory is an important branch of Nonlinear Analysis with a strong development in the last decades. For recent results and problems in metric fixed point theory see [1, 2, 8, 18, 27, 28, 30, 31, 34, 35, 38–40, 46],...

Metric fixed point theorems were often related to different other structures, in order to relax the contraction type conditions on the operator. Ordered structures are an example in this sense. As a consequence, many fixed point results in ordered metric spaces are proved in the last 15 years. The starting point of this research direction was the paper of Ran and Reurings, see [33]. For other results of this type see also [7, 12, 15, 19, 20, 22–24, 26, 32, 43, 47], ...

An effective extension of the above framework was given by Jachymski in [17], where the metric space is endowed with a graph structure. Several recent contributions in this context were given in the following works and the references therein: [3–5, 9, 10, 13, 21, 25, 29, 41, 42, 45], ...

The aim of this paper is to present a study of the fixed point equation $x = f(x)$, where $f : Z \subseteq X \to X$ is a given single-valued operator in a metric space (X, d) endowed with a graph $G := (V(G), E(G))$, under the following two main assumptions:

(1) f is generalized G-monotone;
(2) f is a G-contraction with respect to d.

Finally, some open problems are presented. Our results extend and generalize some results from [29].

2 Preliminaries

Let X be a nonempty set and $f : X \to X$ be an operator. Then, we will denote by $f^0 := 1_X,\ f^1 := f, \ldots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$ the iterate operators of f. By $I(f) := \{Y \subset X | f(Y) \subseteq Y\}$ we will denote the set of all nonempty invariant subsets of f and by $F_f := \{x \in X |\ x = f(x)\}$ we denote the fixed point set of f. Also, by $Graph(f) := \{(x, y) \in X \times X | f(x) = y\}$ we denote the graphic of f.

We recall now the following important concepts for the theory of Picard and weakly Picard operators.

Definition 1 Let (X, d) be a metric space. An operator $f : X \to X$ is, by definition, a Picard operator (briefly PO) if:

(i) $F_f = \{x^*\}$;
(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

For example, on a complete metric space (X, d) any operator $f : X \to X$ satisfying Banach's contraction condition with constant L, i.e., $L \in]0, 1[$ and

$$d(f(x), f(y)) \leq Ld(x, y), \text{ for every } (x, y) \in X \times X$$

is a PO.

Definition 2 Let (X, d) be a metric space. Then, $f : X \to X$ is called a weakly Picard operator (briefly WPO) if, for all $x \in X$, the sequence $(f^n(x))_{n\in N}$ converges and the limit (which may depend on x) is a fixed point of f.

For example, any continuous graphic contraction $f : X \to X$ on a complete metric space (X, d), (i.e., f satisfies Banach's contraction condition with constant L, for every pair $(x, y) \in Graph(f)$) is a WPO.

Notice that, if $f : X \to X$ is a WPO, then the following set retraction

$$f^\infty : X \to F_f, \quad f^\infty(x) := \lim_{n\to\infty} f^n(x)$$

is well defined.

The following abstract Gronwall type lemma takes place for WPOs.

Lemma 1 *Let $(X, d, \preceq)$ be an ordered metric space and $f : X \to X$ be an operator. We suppose:*

(a) f is a WPO;
(b) f is increasing with respect to $\preceq$.
Then, we have:

(i) the operator f^∞ is increasing;
(ii) $x \in X$, $x \preceq f(x)$ implies $x \preceq f^\infty(x)$;
(iii) $x \in X$, $x \succeq f(x)$ implies $x \succeq f^\infty(x)$.

In particular, if f is a PO and we denote by x_f^* its unique fixed point, then the above result takes place with $f^\infty(x) = x_f^*$, for each $x \in X$.

Another important concept is given below.

Definition 3 Let (X, d) be a metric space. Then, $f : X \to X$ is called a ψ-weakly Picard operator (briefly ψ-WPO) if f is a WPO, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, continuous in 0 with $\psi(0) = 0$, such that the following relation holds:

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In particular, if f is a PO and $x^* \in X$ denotes its unique fixed point, then f is said to be a ψ-Picard operator (briefly ψ-PO) if

$$d(x, x^*) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In both cases, if $\psi(t) := ct$, for every $t \in \mathbb{R}_+$ (for some $c > 0$), then f is called a c-WPO, respectively c-PO.

For example, on a complete metric space (X, d) any Banach contraction with constant L is a $\frac{1}{1-L}$-PO, while any continuous graphic contraction with constant L is a $\frac{1}{1-L}$-WPO.

We present now some concepts from stability theory (see [37, 38]).

Let (X, d) be a metric space and $f : X \to X$ be an operator. In this context we have the following notions.

Definition 4

(a) The fixed point problem $x = f(x)$ is well-posed in the sense of Reich and Zaslavski if $F_f = \{x^*\}$ and for any sequence $\{u_n\}$ in X with $d(u_n, f(u_n)) \to 0$ we have that $u_n \to x^*$ as $n \to \infty$;
(b) The operator f has the Ostrowski stability property if $F_f = \{x^*\}$ and for any sequence $\{y_n\}$ in X with $d(y_{n+1}, f(y_n)) \to 0$ we have that $y_n \to x^*$ as $n \to \infty$;
(c) The fixed point problem $x = f(x)$ is Ulam-Hyers stable if there exists $c > 0$ such that, for every $\epsilon > 0$ and any $z \in X$ with $d(z, f(z)) \leq \epsilon$, there exists $x^* \in F_f$ with $d(z, x^*) \leq c \cdot \epsilon$;
(d) The fixed point problem $x = f(x)$ is generalized Ulam-Hyers stable if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\epsilon > 0$ and any $z \in X$ with $d(z, f(z)) \leq \epsilon$, there exists $x^* \in F_f$ with $d(z, x^*) \leq \psi(\epsilon)$.

For example, in the case of contraction mappings, we have the following result.

Theorem 1 (Saturated Principle of Contraction, [38]) *Let (X, d) be a complete metric space and $f : X \to X$ be a Banach contraction with constant L. Then the following conclusions hold:*

(i) there exists $x^* \in X$ such that $F_f = F_{f^n} = \{x^*\}$;
(ii) f is a PO;
(iii) f is a $\frac{1}{1-L}$-PO;
(iv) the fixed point problem $x = f(x)$ is well-posed;
(v) the operator f has the Ostrowski property;
(vi) the operator f has the limit shadowing property;
(vii) the operator f has the shadowing property;
(viii) the fixed point equation $x = f(x)$ is Ulam-Hyers stable.

If (X, d) is a metric space, then let us consider a directed graph $G := (V(G), E(G))$, such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of the edges of the graph contains the diagonal $\Delta := \{(x, x) : x \in X\}$ of $X \times X$. Assume also that G has no parallel edges, which yields that one can identify G with the pair $(V(G), E(G))$.

The purpose of this paper is to present an extended study of the fixed point equation $x = f(x)$ in the case of a metric space endowed with and a directed graph.

3 Main Results

Let (X, d) be a metric space and $G := (V(G), E(G))$ be a directed graph. Throughout this paper we assume that the set $V(G)$ of its vertices coincides with X, the set $E(G)$ of the edges of the graph contains the diagonal $\Delta := \{(x, x) : x \in X\}$ of $X \times X$ and G has no parallel edges. We will say that G is associated with (X, d). If, additionally, for every sequence $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in X such that $(x_n, y_n) \in E(G)$ for every $n \in \mathbb{N}$, $x_n \to x$, $y_n \to y$ as $n \to \infty$, we have that $(x, y) \in E(G)$, then we say that (X, d, G) is a graph metric space.

Let $f : X \to X$ be an operator. We define $(f \times f)(x, y) := (f(x), f(y))$, for $(x, y) \in X \times X$ and we denote by $O_f(x) := \{f^n(x)|n \in \mathbb{N}\}$ the orbit of f at $x \in X$.

Definition 5 An operator $f : X \to X$ is called a Banach G-contraction with constant L (see Definition 2.1 in [17]) if:

(a) f is edge preserving, i.e., $E(G) \in I(f \times f)$;
(b) $L \in]0, 1[$ and the following implication holds:

$$(x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq Ld(x, y).$$

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}^*$ is a finite sequence $(x_n)_{n\in\{0,1,2,...,k\}}$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, \ldots, k\}$. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Notice that $\widetilde{G}$ can be view as a directed graph with the set $E(G)$ symmetric. If G^{-1} is the graph obtained from G by reversing the direction of edges, i.e.,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}, \tag{1}$$

then, we can write

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{2}$$

In the above context, if $f : X \to X$ is an operator, then we denote

$$X_G^f := \{x \in X : \ (x, f(x)) \in E(G)\}.$$

Observe that $X_{\tilde{G}}^f := \{x \in X : \ (x, f(x)) \in E(\tilde{G})\}$. Obvious, in general $X_G^f \subset X_{\tilde{G}}^f$, but for the case of a symmetric graph G we have equality between the above sets.

Definition 6 Let X be a nonempty set, $G := (V(G), E(G))$ be a directed graph and $f : X \to X$ be an operator. Then, f is called a generalized G-monotone operator if $(f \times f)(E(\widetilde{G})) \subset E(\widetilde{G})$, i.e., $E(\widetilde{G}) \in I(f \times f)$.

Notice that any edge preserving operator is generalized G-monotone, but the reverse implication, in general, does not hold.

Remark 1 Let $f : X \to X$ be an operator such that there exists a constant $L \in]0, 1[$ such that the following implication holds:

$$(b') \ (x, y) \in E(\widetilde{G}) \Rightarrow d(f(x), f(y)) \le Ld(x, y).$$

Notice that, due to the symmetry of the contraction assumption, the above condition (b') and condition (b) in Definition 5 are equivalent.

Remark 2 If $f : X \to X$ be an operator on a metric space (X, d) endowed with a directed graph G, then we observe that:

(a) $F_f \subset X_G^f$;
(b) If f is edge preserving, then:
   (i) $f(X_G^f) \subset X_G^f$;
   (ii) for $x \in X_G^f$ we have $O_f(x) \subset X_G^f$;
   (iii) if $x^* \in F_f$ and $X_{x^*}^G := \{x \in X | (x, x^*) \in E(G)\}$, then $X_{x^*}^G \in I(f)$, i.e., $f(X_{x^*}^G) \subset X_{x^*}^G$.
(c) If f is generalized G-monotone, then:
   (i) $f(X_{\widetilde{G}}^f) \subset X_{\widetilde{G}}^f$;
   (ii) for $x \in X_{\widetilde{G}}^f$ we have $O_f(x) \subset X_{\widetilde{G}}^f$;
   (iii) if $x^* \in F_f$, then $X_{x^*}^{\widetilde{G}} \in I(f)$, i.e., $f(X_{x^*}^{\widetilde{G}}) \subset X_{x^*}^{\widetilde{G}}$.

We give now some examples of generalized G-monotone operators.

Example 1

(1) Let $(X, \preceq)$ be an ordered set. Any monotone operator with respect to $\preceq$ (increasing or decreasing) $f : X \to X$ is a generalized G-monotone operator with respect to $G := (V(G) := X, E(G))$, where $E(G) := \{(x, y) \in X \times X | x \preceq y\}$ or $E(G) := \{(x, y) \in X \times X | y \preceq x\}$.
(2) Let X and Y be two nonempty sets endowed with a directed graph G_1 and G_2, respectively. Let $f : X \to X$ be a generalized G_1-monotone operator and $g : Y \to Y$ be a generalized G_2-monotone operator. Let $Z := X \cup Y$ be the disjoint union of the sets X and Y. We consider on Z the following directed graph $G := (V(G), E(G))$, where $V(G) = V(G_1) \cup V(G_2)$ and $E(G) := E(G_1) \cup E(G_2)$.

In the above conditions, the operator $h : Z \to Z$ defined by

$$h(z) = \begin{cases} f(z), & \text{if } z \in X \\ g(z), & \text{if } z \in Y \end{cases}$$

is generalized G-monotone and $Z^h_{\widetilde{G}} = X^f_{\widetilde{G_1}} \cup Y^g_{\widetilde{G_2}}$.

We introduce now a new class of sets, which will be used in our main theorems.

Definition 7 Let X be a nonempty set and $G := (V(G), E(G))$ be a directed graph. Then X is said to be a generalized G-directed set if for each pair of elements $x, y \in X$ there exists $z \in X$ such that (x, z) and (y, z) are in $E(\widetilde{G})$.

If G is a graph such that $E(G)$ is symmetric, then for $x \in X$ we denote

$$[x]_G := \{y \in X : \text{ there is a path in G from } x \text{ to } y\}.$$

Before our first main result, we recall the following known concept.

Definition 8 Let (X, d) be a metric space and $f : X \to X$ be an operator. Then, two elements $x, y \in X$ are called asymptotically equivalent if

$$d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty.$$

The following result is fundamental in our approach.

Theorem 2 *Let (X, d) be a metric space and $G := (V(G), E(G))$ be a directed graph associated with X. Let $f : X \to X$ be a generalized G-monotone operator. We suppose:*

(i) X is a generalized G-directed set;
(ii) if $(x, y) \in E(\widetilde{G})$, then x and y are asymptotically equivalent;
(iii) $X^f_{\widetilde{G}} \neq \emptyset$ and $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a WPO.

Then, $f : X \to X$ is a PO.

Proof Let $x \in X$ be arbitrarily chosen. Let $y \in X^f_{\widetilde{G}}$. For the pair $(x, y) \in X \times X$, by (i), there is $z \in X$ such that $(x, z), (y, z) \in E(\widetilde{G})$. By (ii) it follows that

$$d(f^n(x), f^n(z)) \to 0 \text{ and } d(f^n(y), f^n(z)) \to 0, \text{ and } n \to \infty.$$

By (iii) we have that $f^n(y) \to f^\infty(y) \in F_f$ as $n \to \infty$. Thus, $f^n(x) \to f^\infty(y)$ as $n \to \infty$, for every $x \in X$. If we denote $x^* := f^\infty(y) \in F_f$, then $f^n(x) \to x^*$ as $n \to \infty$, for every $x \in X$. If there exits $u \in F_f$ with $u \neq x^*$, then $u = f^n(u) \to x^*$, a contradiction. As a conclusion, f is a PO.

Remark 3 In particular, if $(X, d, \preceq)$ is an ordered metric space (see [29]) and we define a graph G by

$$V(G) = X, E(G) := \{(x, y) \in X \times X : x \preceq y\},$$

then $E(\widetilde{G}) = X_{\preceq}$ and the above result reduces to Lemma 4.1 in [29].

In what follows, we will propose some metric assumptions on f which assure that the following conditions are realized:

(i) $(x, y) \in E(\widetilde{G}) \Rightarrow$ x and y are asymptotically equivalent;
(ii) $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a WPO.

We recall from [17] that $f : X \to X$ is called orbital G-continuous on X if for all $x \in X$ and for any sequence $(n(i))_{i\in\mathbb{N}}$ of positive integers such that $(f^{n(i)}(x), f^{n(i)+1}(x)) \in E(G)$ for every $i \in \mathbb{N}$, the following implication holds

$$\lim_{i\to\infty} f^{n(i)}(x) = y \Rightarrow \lim_{i\to\infty} f^{n(i)+1}(x) = f(y).$$

If for all $x \in X$ and any sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_n, x_{n+1}) \in E(G)$ for every $n \in \mathbb{N}$, the following implication holds

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x),$$

then we say that f is G-continuous on X. Notice that "continuity" implies "G-continuity" implies "orbital G-continuity", see [17] for other details.

The next result is known as Cauchy-Toeplitz lemma, see, for example, [39]

Lemma 2 (Cauchy-Toeplitz Lemma) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n\geq 0} a_n$ is convergent and $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$ such that $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty} (\sum_{k=0}^{n} a_{n-k} b_k) = 0.$$

Now, from Theorem 2, we obtain the following useful result for applications.

Theorem 3 *Let (X, d) be a complete metric space and $G := (V(G), E(G))$ be a directed graph such that (X, d, G) is a graph metric space. Let $f : X \to X$ be a generalized G-monotone operator. We suppose:*

(i) $(X, \preceq)$ is a generalized G-directed set;
(ii) there exists $L \in]0, 1[$ such that

$$d(f(x), f(y)) \leq L d(x, y), \text{ for every } (x, y) \in E(G);$$

(iii) $X^f_{\widetilde{G}} \neq \emptyset$ and $f : X \to X$ is G-continuous.

Then, the following conclusions hold:

(1) $f : X \to X$ is a PO;
(2) $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a $\frac{1}{1-L}$-PO;

(3) $f : X^G_{x^*} \to X^G_{x^*}$ *is L-quasicontraction;*

(4) $f : X^G_{x^*} \to X^G_{x^*}$ *is a* $\frac{1}{1-L}$*-PO;*

(5) if $(y_n)_{n\in\mathbb{N}} \subset X^G_{x^*}$ *and* $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty$*, then* $y_n \to x^*$ *as* $n \to \infty$*, i.e., the fixed point problem is well-posed in the sense of Reich and Zaslavski for* $f|_{X^G_{x^*}}$*;*

(6) if $(y_n)_{n\in\mathbb{N}} \subset X^G_{x^*}$ *and* $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty$*, then* $y_n \to x^*$ *as* $n \to \infty$*, i.e.,* $f|_{X^G_{x^*}}$ *has the Ostrowski property.*

Proof Notice first that, by the symmetry of the metric assumption on f (see Remark 1), the condition (ii) is satisfied for all $(x, y) \in E(\widetilde{G})$.

(1) By (ii) it follows that for every $(x, y) \in E(\widetilde{G})$, the elements x and y are asymptotically equivalent. By the generalized G-monotonicity of f and (ii) it follows that $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a graphic L-contraction. Since $X^f_{\widetilde{G}}$ is closed in X, by the graphic contraction principle (see [30]) we have that $f^n(x) \to f^\infty(x)$ as $n \to \infty$, for each $x \in X^f_{\widetilde{G}}$. By the G-continuity of f, we get that $f^\infty(x) \in F_f$, i.e., $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a WPO. The first conclusion follows by Theorem 2.

(2) By (1) we have that $F_f = \{x^*\}$. Let $x \in X^f_{\widetilde{G}}$ be arbitrarily chosen. Since $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a graphic L-contraction, using Remark 1, for every $x \in X^f_{\widetilde{G}}$, we have that

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x)) \\ &\quad + d(f^{n+1}(x), x^*) \\ &\leq \frac{1}{1-L} d(x, f(x)) + d(f^{n+1}(x), x^*), \text{ for all } n \in \mathbb{N}^*. \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$d(x, x^*) \leq \frac{1}{1-L} d(x, f(x)), \text{ for each } x \in X^f_{\widetilde{G}}.$$

Thus $f : X^f_{\widetilde{G}} \to X^f_{\widetilde{G}}$ is a $\frac{1}{1-L}$-PO.

(3) Let $x \in X^G_{x^*}$. Then $d(f(x), x^*) = d(f(x), f(x^*)) \leq Ld(x, x^*)$. Thus, $f|_{X^G_{x^*}}$ is an L-quasicontraction.

(4) By (1) it follows that $f : X^G_{x^*} \to X^G_{x^*}$ is a PO. Then, for every $x \in X^G_{x^*}$, we have $d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + Ld(x, x^*)$. Thus

$$d(x, x^*) \leq \frac{1}{1-L} d(x, f(x)), \text{ for all } x \in X^G_{x^*}.$$

(5) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}$ such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. Then, for every $n \in \mathbb{N}$, we have

$$d(y_n, x^*) \leq d(y_n, f(y_n)) + d(f(y_n), x^*) \leq d(y_n, f(y_n)) + Ld(y_n, x^*).$$

Then,

$$d(y_n, x^*) \leq \frac{1}{1-L} d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

(6) Let $(y_n)_{n\in\mathbb{N}} \subset X_{x^*}^G$ such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. Then, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, f(y_n)) + d(f(y_n), x^*) \leq d(y_{n+1}, f(y_n)) + Ld(y_n, x^*) \\ &\leq d(y_{n+1}, f(y_n)) + Ld(y_n, f(y_{n-1})) + L^2 d(y_{n-1}, x^*) \leq \cdots \\ &\leq \sum_{k=0}^{n} L^{n-k} d(y_{k+1}, f(y_k)) + L^{n+1} d(y_0, x^*). \end{aligned}$$

The conclusion follows by Cauchy-Toeplitz Lemma.

Finally, the following Gronwall type lemma takes place for in a graph metric space.

Theorem 4 *Let (X, d, G) be a graph metric space and $f : X \to X$ be an operator. We suppose:*

(a) f is a WPO;
(b) f is edge preserving.
Then, we have:

(i) the operator f^∞ is edge preserving;
(ii) if, additionally, G has the following transitivity property:

$$(x, y), (y, z) \in E(G) \Rightarrow (x, z) \in E(G),$$

then the following implication holds

$$x \in X_{\widetilde{G}}^f \Rightarrow (x, f^\infty(x)) \in E(\widetilde{G}).$$

Proof

(i) Let $(x, y) \in E(G)$. By (b) we have $(f^n(x), f^n(y)) \in E(G)$, for every $n \in \mathbb{N}$. By (a) we know that $f^n(x) \to f^\infty(x)$ and $f^n(y) \to f^\infty(y)$ as $n \to \infty$. By the graph metric space condition we obtain $(f^\infty(x), f^\infty(x)) \in E(G)$.
(ii) Let $x \in X_{\widetilde{G}}^f$ be arbitrary. Then $(x, f(x)) \in E(G) \cup E(G^{-1})$. Assume first that $(x, f(x)) \in E(G)$. By (b) we obtain that $(f^n(x), f^{n+1}(x)) \in E(G)$, for every $n \in \mathbb{N}$. Thus, $(x, f^n(x)) \in E(G)$ for every $n \in \mathbb{N}$. By the graph metric space

condition we obtain $(x, f^\infty(x)) \in E(G)$. The case $(x, f(x)) \in E(G^{-1})$ is similar.

Remark 4 In particular, if f is a PO and we denote by x_f^* its unique fixed point, then the above result takes place with $f^\infty(x) = x_f^*$, for each $x \in X$.

Remark 5 If $(X, d, \preceq)$ be an ordered metric space and we define the graph G by $V(G) := X$ and $E(G) := \{(x, y) \in X \times X | x \preceq y\}$, then Theorem 4 reduces to Lemma 1.

Remark 6

(1) It is an open problem to prove the above result under the weaker condition of orbital G-continuity of f on X.
(2) Another open problem is to extend the above result by considering the case of a φ-contraction in a graph metric space, see [29].
(3) Finally, a nice research direction is to extend the above results to different generalized metric spaces endowed with a graph.

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Approximate Solution of Fredholm Integral and Integro-Differential Equations with Non-Separable Kernels

E. Providas

Abstract This chapter deals with the approximate solution of Fredholm integral equations and a type of integro-differential equations having non-separable kernels, as they appear in many applications. The procedure proposed consists of firstly approximating the non-separable kernel by a finite partial sum of a power series and then constructing the solution of the degenerate equation explicitly by a direct matrix method. The method, which is easily programmable in a computer algebra system, is explained and tested by solving several examples from the literature.

1 Introduction

Integral and integro-differential equations appear in many applications in sciences and engineering. Integral equations have been studied extensively and there is today accumulated knowledge which one can find in good treatises, see, for example, [6, 10, 16]. Integro-differential equations are a less researched topic and usually they occupy a separate chapter in integral equations text books [13, 15]. Integral and Integro-differential equations are usually solved by numerical methods, see, for example, the monograph [1]. Direct solution methods have also been used, as it can be seen in the above-mentioned references, in the cases where the kernels are degenerate. Recently, the author with his co-authors developed a direct matrix method for solving exactly integro-differential equations with separable kernels [7–9, 12]. However, in many engineering applications, such as nonlocal or gradient elasticity [4, 5, 11, 14] and hydrodynamics [2], integral and integro-differential equations emerge with non-separable kernels. The aim of this article is to propose a procedure by which the non-separable kernel is approximated by a degenerate one and then solving the integral or integro-differential equation explicitly by the direct matrix method above.

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In Sect. 2, we present a direct matrix method for obtaining in closed form the unique solution of the Fredholm integral equation

$$\mathcal{I}u(x) = u(x) - \int_a^b K(x,s)u(s)ds = f(x), \quad x \in [a,b], \tag{1}$$

where $\mathcal{I} : C[a,b] \to C[a,b]$ is a linear operator, $K(x,s)$ is a given kernel function which is assumed to be continuous on the closed square $Q(a,b) = \{(x,s) : a \leq x \leq b,\ a \leq s \leq b\}$ and separable, $f(x) \in C[a,b]$ is an input free function, and $u(x)$ is the unknown function describing the response of the system modeled by (1). Also, we propose a technique for establishing uniqueness and constructing in closed form the solution of the Fredholm integro-differential equation

$$\begin{aligned} Bu(x) &= \widehat{A}u(x) - \int_a^b K(x,s)\widehat{A}u(s)ds = f(x), \quad x \in [a,b], \\ D(B) &= D(\widehat{A}), \end{aligned} \tag{2}$$

where $\widehat{A} : C[a,b] \to C[a,b]$ is a bijective linear differential operator incorporating initial or boundary conditions, and $B : C[a,b] \to C[a,b]$ is a linear operator with $D(B) = D(\widehat{A})$. As an example of equations of this kind, we refer to the case of modeling the Euler-Bernoulli beams using Eringen's integral formulation [14].

In Sect. 3, we find approximate solutions to Fredholm integral equations with non-separable kernel functions $K(x,s)$. The approach we follow consists of representing $K(x,s)$ as a power series at a point and replacing $K(x,s)$ in the integral equation by the partial sum $K_n(x,s)$ of the power series. The resulting degenerate integral equation is then solved by the direct matrix method.

The same procedure is employed in Sect. 4 to acquire an approximate solution of Fredholm integro-differential equations with non-separable kernels.

Finally, some conclusions regarding the efficiency of the method proposed are quoted in Sect. 5.

2 Direct Matrix Methods

Let the integral equation (1) and assume that the kernel $K(x,s)$ is a separable function which has the specific form

$$K(x,s) = \sum_{k=1}^{n} g_k(x)h_k(s), \quad x, s \in [a,b], \tag{3}$$

where $g_k(x),\ h_k(s) \in C[a,b]$. Also, it is assumed without loss of generality that the sets of the functions $\{g_k(x)\}$ and $\{h_k(x)\}$ are linearly independent; otherwise, the number of functions should be lessened. Then the integral equation (1) becomes

$$\mathcal{I}u(x) = u(x) - \sum_{k=1}^{n} g_k(x) \int_a^b h_k(s)u(s)ds = f(x), \quad x \in [a,b]. \tag{4}$$

We introduce the vector of functions

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_n \end{pmatrix}, \quad g_k = g_k(x) \in C[a,b], \quad k = 1, 2, \dots, n, \tag{5}$$

and the vector of linear bounded functionals

$$\Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \\ \vdots \\ \Phi_n(u) \end{pmatrix}, \quad \Phi_k(u) = \int_a^b h_k(s)u(s)ds, \quad k = 1, 2, \dots, n, \tag{6}$$

and write Eq. (4) as

$$\mathcal{I}u = u - \mathbf{g}\Phi(u) = f, \tag{7}$$

where $f = f(x),\ u = u(x) \in C[a,b]$.

For the solution of (7), we state and prove the next theorem where use is made of the notations

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1) & \Phi_1(g_2) & \cdots & \Phi_1(g_n) \\ \Phi_2(g_1) & \Phi_2(g_2) & \cdots & \Phi_2(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_n(g_1) & \Phi_n(g_2) & \cdots & \Phi_n(g_n) \end{bmatrix}, \quad \Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_n(f) \end{pmatrix}, \tag{8}$$

$\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{0}$ the zero column vector. We note that

$$\Phi(\mathbf{gN}) = \Phi(\mathbf{g})\mathbf{N}, \tag{9}$$

where $\mathbf{N}$ is an $n \times m,\ m \in \mathbb{N}$, constant matrix. Finally, it is recalled that a linear operator $P : C[a,b] \to C[a,b]$ is said to be *correct* if P is bijective and its inverse P^{-1} is bounded on $C[a,b]$.

Theorem 1 *In $C[a,b]$, let the vectors $\mathbf{g}$ and Φ be defined as in (5) and (6), respectively, and $\mathcal{I} : C[a,b] \to C[a,b]$ be the linear operator*

$$\mathcal{I}u = u - \mathbf{g}\Phi(u). \tag{10}$$

Then the operator $\mathcal{I}$ is bijective on $C[a, b]$ if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{11}$$

and the unique solution of the integral equation $\mathcal{I}u = f$, for any $f \in C[a, b]$, is given by the formula

$$u = \mathcal{I}^{-1} f = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{12}$$

The operator $\mathcal{I}$ is correct.

Proof

(i) Let $\det \mathbf{W} \neq 0$ and $u \in \ker \mathcal{I}$. Then,

$$\mathcal{I}u = u - \mathbf{g}\Phi(u) = 0, \tag{13}$$

and by acting by the vector Φ on both sides of (13), we get

$$\Phi\left(u - \mathbf{g}\Phi(u)\right) = [\mathbf{I}_n - \Phi(\mathbf{g})]\,\Phi(u) = \mathbf{W}\Phi(u) = \mathbf{0}, \tag{14}$$

which implies that $\Phi(u) = \mathbf{0}$. Substitution into (13) yields $\mathcal{I}u = u = 0$, which means that the $\ker \mathcal{I} = \{0\}$ and hence the operator $\mathcal{I}$ is injective. Conversely, we prove that if $\mathcal{I}$ is an injective operator then $\det \mathbf{W} \neq 0$, or equivalently, if $\det \mathbf{W} = 0$, then $\mathcal{I}$ is not injective. Let $\det \mathbf{W} = 0$. Then there exists a nonzero vector $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_n)$ such that $\mathbf{W}\mathbf{c} = \mathbf{0}$. Let the element $u_0 = \mathbf{g}\mathbf{c}$ and note that $u_0 \neq 0$; otherwise, $u_0 = \mathbf{g}\mathbf{c} = 0$ implies $\mathbf{W}\mathbf{c} = [\mathbf{I}_n - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{c} - \Phi(\mathbf{g}\mathbf{c}) = \mathbf{c} = \mathbf{0}$. From Eq. (13), we get

$$\mathcal{I}u_0 = \mathbf{g}\mathbf{c} - \mathbf{g}\Phi(\mathbf{g})\mathbf{c} = \mathbf{g}[\mathbf{I}_n - \Phi(\mathbf{g})]\mathbf{c} = \mathbf{g}\mathbf{W}\mathbf{c} = \mathbf{g}\mathbf{0} = 0, \tag{15}$$

which means that $\ker \mathcal{I} \neq 0$ and so $\mathcal{I}$ is not injective.

By applying now the vector Φ on $\mathcal{I}u = f$, we have

$$[\mathbf{I}_n - \Phi(\mathbf{g})]\,\Phi(u) = \mathbf{W}\Phi(u) = \Phi(f). \tag{16}$$

Since $\det \mathbf{W} \neq 0$ it follows that $\Phi(u) = \mathbf{W}^{-1}\Phi(f)$ and hence

$$\mathcal{I}u = u - \mathbf{g}\mathbf{W}^{-1}\Phi(f) = f, \tag{17}$$

from where formula (12) is obtained. Moreover, since the input function $f \in C[a, b]$ is arbitrary, we have $R(\mathcal{I}) = C[a, b]$ which means that $\mathcal{I}$ is bijective.

Lastly, in (12) the functionals Φ_k are bounded on $C[a, b]$ and hence the operator $\mathcal{I}^{-1}$ is bounded. Thus, if the operator $\mathcal{I}$ is bijective then it is correct. □

Let now the mth order linear differential operator $A : C[a, b] \to C[a, b]$:

$$Au = a_m(x)\frac{d^m u}{dx^m} + a_{m-1}(x)\frac{d^{m-1}u}{dx^{m-1}} + \cdots + a_1(x)\frac{du}{dx} + a_0(x), \tag{18}$$

where the coefficients $a_i(x) \in C[a, b]$, $i = 0, \ldots, m$, $a_m(x) \neq 0$, and $\widehat{A} : C[a, b] \to C[a, b]$ be a restriction of A on $D(\widehat{A})$ by specifying initial or boundary conditions. We assume that $\widehat{A}$ is a bijective operator and that the inverse $\widehat{A}^{-1}$ is known. Further, let $K(x, s)$ be degenerate as in (3), and the vectors $\mathbf{g}$ and Φ be as in (5) and (6), respectively. Then the Fredholm integro-differential equation (2) can be put in the form

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u) = f, \quad D(B) = D(\widehat{A}). \tag{19}$$

The existence and uniqueness criteria and the solution of the integro-differential equation (19) are provided by the following theorem.

Theorem 2 *Let the restriction $\widehat{A} : C[a, b] \to C[a, b]$ be a bijective linear operator and $\widehat{A}^{-1}$ its inverse, the vectors $\mathbf{g}$ and Φ as in (5) and (6), respectively, and $B : C[a, b] \to C[a, b]$ the linear operator*

$$Bu = \widehat{A}u - \mathbf{g}\Phi(\widehat{A}u), \quad D(B) = D(\widehat{A}). \tag{20}$$

Then the following statements are true:

(i) The operator B is bijective on $C[a, b]$ if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{21}$$

and the unique solution to problem $Bu = f$, for any $f \in C[a, b]$, is given by the formula

$$u = B^{-1}f = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{22}$$

(ii) If in addition the inverse operator $\widehat{A}^{-1}$ is bounded on $C[a, b]$, then the operator B correct.

Proof

(i) Set $\widehat{A}u = y$, $y \in C[a, b]$, and express $Bu = f$ as

$$y - \mathbf{g}\Phi(y) = f. \tag{23}$$

This is an integral equation of the type (7). From Theorem 1 follows that Eq. (23) has a unique solution if and only if

$$\det \mathbf{W} = \det[\mathbf{I}_n - \Phi(\mathbf{g})] \neq 0, \tag{24}$$

and that its unique solution is given by

$$y = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{25}$$

Acting by the operator $\widehat{A}^{-1}$ on both sides of (25), we get

$$\widehat{A}^{-1}y = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f). \tag{26}$$

and hence

$$u = \widehat{A}^{-1}f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f), \tag{27}$$

which is the solution formula (22). Furthermore, since $f \in C[a,b]$ is arbitrary, we have $R(B) = C[a,b]$ which means that B is bijective.

(ii) Suppose that (21) is true and that the operator $\widehat{A}^{-1}$ is bounded on $C[a,b]$. Then by (i) the operator B is bijective and the unique solution to $Bu = f$ is given by (22). Additionally, in (22) the operator $\widehat{A}^{-1}$ and the functionals $\Phi_1, \ldots, \Phi_n$ are bounded on $C[a,b]$ and hence the operator B^{-1} is bounded too. Therefore the operator B is correct.

□

3 Approximate Solution of Integral Equations with Non-Separable Kernels

Let the integral equation (1) and suppose the kernel function $K(x,s)$ is non-separable, but it can be represented as a power series in s at a point s_0 such that

$$K(x,s) = \sum_{k=0}^{\infty} p_k(x)(s-s_0)^k, \tag{28}$$

where the functions $p_k(x)$ are continuous functions. We truncate this series and take the partial sum of the first $n+1$ terms, namely

$$K_n(x,s) = \sum_{k=1}^{n+1} p_{k-1}(x)(s-s_0)^{k-1}. \tag{29}$$

We replace the kernel $K(x,s)$ in (1) by (29) to obtain the degenerate Fredholm integral equation

$$\mathcal{I}_n\tilde{u}(x) = \tilde{u}(x) - \sum_{k=1}^{n+1} p_{k-1}(x)\int_a^b (s-s_0)^{k-1}\tilde{u}(s)ds = f(x), \quad x \in [a,b], \tag{30}$$

where $\mathcal{I}_n : C[a,b] \to C[a,b]$ is a linear operator. Further, we define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_{n+1} \end{pmatrix} = \begin{pmatrix} p_0(x) & p_1(x) & \dots & p_n(x) \end{pmatrix}, \tag{31}$$

and

$$\Phi(\tilde{u}) = \begin{pmatrix} \Phi_1(\tilde{u}) \\ \Phi_2(\tilde{u}) \\ \vdots \\ \Phi_{n+1}(\tilde{u}) \end{pmatrix}, \quad \Phi_k(\tilde{u}) = \int_a^b (s-s_0)^{k-1}\tilde{u}(s)ds, \quad k = 1, 2, \dots, n+1, \tag{32}$$

and write Eq. (30) in the compact form

$$\mathcal{I}_n \tilde{u} = \tilde{u} - \mathbf{g}\Phi(\tilde{u}) = f. \tag{33}$$

The solution $\tilde{u} = \mathcal{I}_n^{-1} f$ of (33) can be obtained by applying Theorem 1. This solution is an approximate solution to (1) having a non-separable kernel $K(x,s)$ which was expressed as in (29).

An estimation of the error $|u - \tilde{u}|$ can be found by using standard analysis techniques [6, 16]. A similar procedure would have resulted if we had used a power series in x or a double power series.

Example 1 Let us derive an approximate solution of the Fredholm integral equation of the second kind

$$u(x) - \int_0^{1/2} e^{-x^2 s^2} u(s)ds = f(x), \quad 0 \le x \le \frac{1}{2}, \tag{34}$$

for any $f(x) \in C[0, \frac{1}{2}]$. The kernel is non-separable and therefore we take its Taylor series expansion in the variable s (or in x) about the point 0, viz.

$$\begin{aligned} K(x,s) = e^{-x^2 s^2} &= 1 - x^2 s^2 + \frac{1}{2}x^4 s^4 - \frac{1}{6}x^6 s^6 \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} s^{2k}. \end{aligned}$$

By taking the partial sum

$$K_n(x,s) = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!} x^{2(k-1)} s^{2(k-1)},$$

and placing it in (34), we get the companion equation

$$\tilde{u}(x) - \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!} x^{2(k-1)} \int_0^{1/2} s^{2(k-1)} \tilde{u}(s) ds = f(x), \quad 0 \le x \le \frac{1}{2}. \tag{35}$$

We define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \dots & g_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & -x^2 & \dots & \frac{(-1)^n}{n!} x^{2n} \end{pmatrix},$$

and

$$\Phi(\tilde{u}(s)) = \begin{pmatrix} \Phi_1(\tilde{u}(s)) \\ \Phi_2(\tilde{u}(s)) \\ \vdots \\ \Phi_{n+1}(\tilde{u}(s)) \end{pmatrix} = \begin{pmatrix} \int_0^{1/2} \tilde{u}(s) ds \\ \int_0^{1/2} s^2 \tilde{u}(s) ds \\ \vdots \\ \int_0^{1/2} s^{2n} \tilde{u}(s) ds \end{pmatrix},$$

and write (35) as

$$\mathcal{I}_n \tilde{u}(x) = \tilde{u}(x) - \mathbf{g}(x) \Phi(\tilde{u}(s)) = f(x). \tag{36}$$

Then, we construct the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_1(1) & \Phi_1(-s^2) & \cdots & \Phi_1\left(\frac{(-1)^n}{n!} s^{2n}\right) \\ \Phi_2(1) & \Phi_2(-s^2) & \cdots & \Phi_2\left(\frac{(-1)^n}{n!} s^{2n}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(1) & \Phi_{n+1}(-s^2) & \cdots & \Phi_{n+1}\left(\frac{(-1)^n}{n!} s^{2n}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{1/2} ds & -\int_0^{1/2} s^2 ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{2n} ds \\ \int_0^{1/2} s^2 ds & -\int_0^{1/2} s^4 ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{2(n+1)} ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^{1/2} s^{2n} ds & -\int_0^{1/2} s^{2(n+1)} ds & \cdots & \frac{(-1)^n}{n!} \int_0^{1/2} s^{4n} ds \end{bmatrix},$$

and thus the matrix

$$\mathbf{W} = \mathbf{I_{n+1}} - \Phi(\mathbf{g}).$$

If det $\mathbf{W} \neq 0$, then Eq. (36) has exactly one solution. To obtain the solution, we put up the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_{n+1}(f) \end{pmatrix} = \begin{pmatrix} \int_0^{1/2} f(s)ds \\ \int_0^{1/2} s^2 f(s)ds \\ \vdots \\ \int_0^{1/2} s^{2n} f(s)ds \end{pmatrix},$$

and by Theorem 1 compute

$$\tilde{u} = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f).$$

Let $f(x) = 1$ [10]. Then for $n = 2$, $n = 4$, and $n = 6$, we have

$$K_2(x, s) = 1 - x^2 s^2 + \frac{1}{2}x^4 s^4,$$
$$K_4(x, s) = 1 - x^2 s^2 + \frac{1}{2}x^4 s^4 - \frac{1}{6}x^6 s^6 + \frac{1}{24}x^8 s^8,$$
$$K_6(x, s) = 1 - x^2 s^2 + \frac{1}{2}x^4 s^4 - \frac{1}{6}x^6 s^6 + \frac{1}{24}x^8 s^8 - \frac{1}{120}x^{10} s^{10} + \frac{1}{720}x^{12} s^{12},$$

and the approximate solutions

$$\tilde{u}_2 = 1.993199 - 0.082541x^2 + 0.006183x^4,$$
$$\tilde{u}_4 = 1.993198 - 0.082541x^2 + 0.006183x^4 - 0.000368x^6 + 0.000018x^8,$$
$$\tilde{u}_6 = 1.993198 - 0.082541x^2 + 0.006183x^4 - 0.000368x^6 + 0.000018x^8$$
$$-7.309486 \times 10^{-7}x^{10} + 2.576526 \times 10^{-8}x^{12},$$

respectively, where all coefficients have been rounded up to six decimal digits. The results are in very good agreement with those obtained in [10] where the same problem has been solved for $n = 2$.

Example 2 Consider the inhomogeneous Fredholm integral equation

$$u(x) - \frac{1}{2}\int_{-1}^{1} \sin\left(\frac{\pi s x}{2}\right) u(s)ds = f(x), \quad -1 \leq x \leq 1, \tag{37}$$

where $f(x) \in C[-1, 1]$. The kernel is non-separable, but it can be represented in Taylor series in x (or in s) about the point 0, namely

$$K(x,s) = \sin\left(\frac{\pi s x}{2}\right) = \frac{\pi s x}{2} - \frac{\pi^3 s^3 x^3}{48} + \frac{\pi^5 s^5 x^5}{3840} \cdots$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1} s^{2k+1} x^{2k+1}}{2^{2k+1}(2k+1)!}.$$

After replacing $K(x,s)$ in (37) with the partial sum

$$K_n(x,s) = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{\pi^{2k-1} s^{2k-1} x^{2k-1}}{2^{2k-1}(2k-1)!},$$

we get the auxiliary equation

$$\tilde{u}(x) - \frac{1}{2}\sum_{k=1}^{n+1} (-1)^{k-1} \frac{\pi^{2k-1} x^{2k-1}}{2^{2k-1}(2k-1)!} \int_{-1}^{1} s^{2k-1} \tilde{u}(s) ds = f(x), \quad -1 \le x \le 1. \tag{38}$$

We set up the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \ldots & g_{n+1}(x) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\pi x}{2} & -\frac{\pi^3 x^3}{48} & \ldots & (-1)^n \frac{\pi^{2n+1} x^{2n+1}}{2^{2n+1}(2n+1)!} \end{pmatrix},$$

and

$$\Phi(\tilde{u}(s)) = \begin{pmatrix} \Phi_1(\tilde{u}(s)) \\ \Phi_2(\tilde{u}(s)) \\ \vdots \\ \Phi_{n+1}(\tilde{u}(s)) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} s\tilde{u}(s)ds \\ \int_{-1}^{1} s^3 \tilde{u}(s)ds \\ \vdots \\ \int_{-1}^{1} s^{2n+1} \tilde{u}(s)ds \end{pmatrix},$$

and write (38) as

$$\mathcal{I}_n \tilde{u}(x) = \tilde{u}(x) - \mathbf{g}(x)\Phi(\tilde{u}(s)) = f(x). \tag{39}$$

Then, we form the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$

and compute the matrix

$$\mathbf{W} = \mathbf{I}_{n+1} - \Phi(\mathbf{g}).$$

If det $\mathbf{W} \neq 0$, then Eq. (39) admits exactly one solution. By setting up the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1(f) \\ \Phi_2(f) \\ \vdots \\ \Phi_{n+1}(f) \end{pmatrix},$$

we can determine the solution from Theorem 1, which is

$$\tilde{u} = f + \mathbf{g}\mathbf{W}^{-1}\Phi(f).$$

Let $f(x) = x^3$ [16], which is continuous in $[-1, 1]$. Then, for $n = 2$, $n = 4$, and $n = 6$, we get

$$K_2(x, s) = \frac{\pi s x}{2} - \frac{\pi^3 s^3 x^3}{48} + \frac{\pi^5 s^5 x^5}{3840},$$
$$K_4(x, s) = K_2(x, s) - \frac{\pi^7 s^7 x^7}{645120} + \frac{\pi^9 s^9 x^9}{185794560},$$
$$K_6(x, s) = K_4(x, s) - \frac{\pi^{11} s^{11} x^{11}}{81749606400} + \frac{\pi^{13} s^{13} x^{13}}{51011754393600},$$

and the approximate solutions

$$\tilde{u}_2(x) = 0.565621x + 0.847692x^3 + 0.014047x^5,$$
$$\tilde{u}_4(x) = 0.565421x + 0.847751x^3 + 0.014042x^5 - 0.000660x^7 + 0.000019x^9,$$
$$\tilde{u}_6(x) = 0.565421x + 0.847751x^3 + 0.014042x^5 - 0.000660x^7 + 0.000019x^9$$
$$-3.627733 \times 10^{-7}x^{11} + 5.024528 \times 10^{-9}x^{13},$$

respectively, where the coefficients have been rounded up to six decimal places. The same problem is solved in [16] for $n = 2$ using other techniques such as the resolvent kernel, Simpson's rule, or Gaussian quadrature. The results obtained here for $n = 2$ are identical with those reported in [16] with six decimal digits.

4 Approximate Solution of Integro-Differential Equations with Non-Separable Kernels

Let the integro-differential equation (2) with a kernel function $K(x, s)$ which is non-separable, but it can be expanded in a power series in x at a point x_0

$$K(x, s) = \sum_{k=0}^{\infty} h_k(s)(x - x_0)^k, \tag{40}$$

where the functions $h_k(s)$ are continuous functions. We consider the partial sum of the first $n+1$ terms

$$K_n(x,s) = \sum_{k=1}^{n+1} h_{k-1}(s)(x-x_0)^{k-1}, \tag{41}$$

and place it in (2) instead of $K(x,s)$. As a result, we obtain the degenerate Fredholm integro-differential equation

$$B_n\tilde{u}(x) = \widehat{A}\tilde{u}(x) - \sum_{k=1}^{n+1}(x-x_0)^{k-1}\int_a^b h_{k-1}(s)\widehat{A}\tilde{u}(s)ds = f(x), \quad x \in [a,b], \tag{42}$$

where $B_n : C[a,b] \to C[a,b]$ is a linear operator with $D(B_n) = D(\widehat{A})$. Define the vectors

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \dots & g_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & x-x_0 & \dots & (x-x_0)^n \end{pmatrix}, \tag{43}$$

and

$$\Phi(\widehat{A}\tilde{u}) = \begin{pmatrix} \Phi_1(\widehat{A}\tilde{u}) \\ \Phi_2(\widehat{A}\tilde{u}) \\ \vdots \\ \Phi_{n+1}(\widehat{A}\tilde{u}) \end{pmatrix}, \ \Phi_k(\widehat{A}\tilde{u}) = \int_a^b h_{k-1}(s)\widehat{A}\tilde{u}(s)ds, \ k = 1, 2, \dots, n+1, \tag{44}$$

and formulate Eq. (42) as

$$B_n\tilde{u} = \widehat{A}\tilde{u} - \mathbf{g}\Phi(\widehat{A}\tilde{u}) = f. \tag{45}$$

By using Theorem 2, we can compute the solution $\tilde{u} = B_n^{-1}f$ of (45), which is an approximate solution of Eq. (2) having the non-separable kernel $K(x,s)$ approximated by (41).

As before, an evaluation of the error $|u - \tilde{u}|$ can be found by using standard analysis techniques [6, 16]. A similar procedure results if one uses a power series in s or a double power series.

Example 3 Consider the Fredholm integro-differential equation

$$u'(x) - \int_0^1 e^{xs}u'(s)ds = f(x), \quad 0 \le x \le 1, \quad u(0) = 1, \tag{46}$$

for an input function $f(x) \in C[0,1]$. By means of $v(x) = u(x) - 1$, we can transform this equation to the following one with a homogeneous condition

$$v'(x) - \int_0^1 e^{xs} v'(s)ds = f(x), \quad 0 \le x \le 1, \quad v(0) = 0. \tag{47}$$

The kernel is non-separable, but it can be represented as Taylor series in x (or in s) about 0 as

$$K(x,s) = e^{xs} = 1 + sx + \frac{1}{2}s^2x^2 + \frac{1}{6}s^3x^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{s^k x^k}{k!}.$$

Let the partial sum

$$K_n(x,s) = \sum_{k=1}^{n+1} \frac{s^{k-1}x^{k-1}}{(k-1)!},$$

which when is placed in (47) instead of $K(x,s)$ yields the auxiliary equation

$$\tilde{v}'(x) - \sum_{k=1}^{n+1} x^{k-1} \int_0^1 \frac{s^{k-1}}{(k-1)!}\tilde{v}'(s)ds = f(x), \quad 0 \le x \le 1. \tag{48}$$

Take the operator $\widehat{A} : C[0,1] \to C[0,1]$ to be

$$\widehat{A}\tilde{v}(x) = \tilde{v}'(x), \quad D(\widehat{A}) = \{\tilde{v}(x) \in C^1[0,1] : \ \tilde{v}(0) = 0\},$$

which is bijective and its inverse is

$$\widehat{A}^{-1} f(x) = \int_0^x f(s)ds, \quad f(x) \in C[0,1].$$

Set up the vectors

$$\mathbf{g} = \begin{pmatrix} g_1(x) & g_2(x) & \ldots & g_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & \ldots & x^n \end{pmatrix},$$

and

$$\Phi(\widehat{A}\tilde{v}(s)) = \begin{pmatrix} \Phi_1 & (\widehat{A}\tilde{v}(s)) \\ \Phi_2 & (\widehat{A}\tilde{v}(s)) \\ & \vdots \\ \Phi_{n+1} & (\widehat{A}\tilde{v}(s)) \end{pmatrix} = \begin{pmatrix} \int_0^1 & \widehat{A}\tilde{v}(s)ds \\ \int_0^1 s & \widehat{A}\tilde{v}(s)ds \\ & \vdots \\ \int_0^1 \frac{s^n}{n!} & \widehat{A}\tilde{v}(s)ds \end{pmatrix},$$

and write (48) as

$$\widehat{A}\tilde{v}(x) - \mathbf{g}(x)\Phi(\widehat{A}\tilde{v}(s)) = f(x). \tag{49}$$

Form the matrix

$$\Phi(\mathbf{g}) = \begin{bmatrix} \Phi_1(g_1(s)) & \Phi_1(g_2(s)) & \cdots & \Phi_1(g_{n+1}(s)) \\ \Phi_2(g_1(s)) & \Phi_2(g_2(s)) & \cdots & \Phi_2(g_{n+1}(s)) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(g_1(s)) & \Phi_{n+1}(g_2(s)) & \cdots & \Phi_{n+1}(g_{n+1}(s)) \end{bmatrix}$$
$$= \begin{bmatrix} \Phi_1(1) & \Phi_1(s) & \cdots & \Phi_1(s^n) \\ \Phi_2(1) & \Phi_2(s) & \cdots & \Phi_2(s^n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n+1}(1) & \Phi_{n+1}(s) & \cdots & \Phi_{n+1}(s^n) \end{bmatrix}$$

and then the matrix

$$\mathbf{W} = \mathbf{I_{n+1}} - \Phi(\mathbf{g}).$$

If $\det \mathbf{W} \neq 0$, then Eq. (49) has exactly one solution. To obtain the solution, we construct the vector

$$\Phi(f) = \begin{pmatrix} \Phi_1 & (f) \\ \Phi_2 & (f) \\ & \vdots \\ \Phi_{n+1} & (f) \end{pmatrix} = \begin{pmatrix} \int_0^1 & f(s)ds \\ \int_0^1 & sf(s)ds \\ & \vdots \\ \int_0^1 & \frac{s^n}{n!} f(s)ds \end{pmatrix},$$

and by Theorem 2 compute

$$\tilde{v} = \widehat{A}^{-1} f + \widehat{A}^{-1}\mathbf{g}\mathbf{W}^{-1}\Phi(f) \quad \text{and then} \quad \tilde{u} = \tilde{v} + 1.$$

Let

$$f(x) = e^x + \frac{1 - e^{x+1}}{x+1}, \quad 0 \le x \le 1,$$

as in a comparable problem in [3]. Then Eq. (46) admits the exact solution $u(x) = e^x$. We take Taylor series expansions for both $K(x, s)$ and $f(x)$ in x around 0. For $n = 2$, we have

$$K_2(x,s) = 1 + sx + \frac{1}{2}s^2x^2, \quad f_2(x) = 2 - e - \frac{(e-3)x^2}{2},$$

and analogous expressions for $n = 4$ and $n = 8$. The corresponding solutions are as follows

$$\tilde{u}_2 = 1.0 + 1.184093x + 0.542764x^2 + 0.175518x^3,$$

$$\tilde{u}_4 = 1.0 + 1.005793x + 0.501306x^2 + 0.166926x^3 + 0.041710x^4 + 0.008340x^5,$$

$$\tilde{u}_8 = 1.0 + 1.000001x + 0.500000x^2 + 0.166667x^3 + 0.0416667x^4 + 0.008333x^5 + 0.001389x^6 + 0.000198x^7 + 0.000025x^8 + 0.000003x^9,$$

respectively, where all coefficients have been rounded up to six decimal digits. The results are of high accuracy and agree with the exact solution $u(x) = e^x$.

5 Conclusions

An efficient matrix procedure for solving Fredholm integral and integro-differential equations has been presented. The procedure involves the approximation of the non-separable kernel by a degenerate one, such as the partial sum of a power series, and the application of a direct matrix method to obtain the solution. We have programmed the method into Maxima computer algebra system and solved several example problems. In all cases the results obtained are of very high accuracy. The novelty and the main advantage of the method is the management of the computations involved and that it can be repeated many times with easiness and a large number of terms of the series.

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Ordinary, Super and Hyper Relators Can Be Used To Treat the Various Generalized Open Sets in a Unified Way

Themistocles M. Rassias and Árpád Száz

Abstract If $\mathscr{R}$ is a family of relations on X to Y, $\mathscr{U}$ is a family of relations on $\mathscr{P}(X)$ to Y, and $\mathscr{V}$ is a family of relations on $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, then we say that $\mathscr{R}$ is an ordinary relator, $\mathscr{U}$ is a super relator, and $\mathscr{V}$ is a hyper relator on X to Y.

We show that the $X = Y$, $\mathscr{U} = \{U\}$ and $\mathscr{V} = \{V\}$ particular case of the non-conventional three relator space $(X, Y)(\mathscr{R}, \mathscr{U}, \mathscr{V})$ can be used to treat, in a unified way, the various generalized open sets studied by a great number of topologists.

1 Motivations

If $\mathscr{T}$ is a family of subsets of a set X such that $\mathscr{T}$ is closed under finite intersections and arbitrary unions, then the family $\mathscr{T}$ is called a *topology* on X, and the ordered pair $X(\mathscr{T}) = (X, \mathscr{T})$ is called a *topological space*.

The members of $\mathscr{T}$ are called the *open subsets* of X. While, the members of $\mathscr{F} = \{A^c : A \in \mathscr{T}\}$, where $A^c = X \setminus A$, are called the *closed subsets* of X. Moreover, the members of $\mathscr{T} \cap \mathscr{F}$ are called the *clopen subsets* of X.

Since $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$, we necessarily have $\{\emptyset, X\} \subseteq \mathscr{T} \cap \mathscr{F}$. Therefore, if in particular $\mathscr{T} = \{\emptyset, X\}$, then $\mathscr{T}$ is called *minimal* [78] instead of indiscrete. While, if $\mathscr{T} \cap \mathscr{F} = \{\emptyset, X\}$, then $\mathscr{T}$ is called *connected* [119, p. 31].

For a subset A of $X(\mathscr{T})$, the sets $A^\circ = \text{int}(A) = \bigcup \mathscr{T} \cap \mathscr{P}(A)$,

$$A^- = \text{cl}(A) = \text{int}(A^c)^c \qquad \text{and} \qquad A^\dagger = \text{res}(A) = \text{cl}(A) \setminus A$$

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are called the *interior, closure and residue of* A, respectively.

Thus, $-$ is a *Kuratowski closure operation* on $\mathscr{P}(X)$. That is, $\emptyset^- = \emptyset$, and $-$ is *extensive, idempotent and additive* in the sense that, for any A, $B \subseteq X$, we have $A \subseteq A^-$, $A^{--} = A^-$ and $(A \cup B)^- = A^- \cup B^-$.

In particular, the members of the families

$$\mathscr{D} = \{ A \subseteq X : \ A^- = X \} \qquad \text{and} \qquad \mathscr{N} = \{ A \subseteq X : \ A^{-\circ} = \emptyset \}$$

are called the *dense and rare (or nowhere dense) subsets* of $X(\mathscr{T})$, respectively.

In 1922, a subset A of a closure space $X(-)$ was called *regular open* by Kuratowski [47] if $A = A^{-\circ}$. While, in 1937, a subset A of a topological space $X(\mathscr{T})$ was called regular open by Stone [83] if $A = B^\circ$ for some $B \in \mathscr{F}$.

The importance of regular open subsets of $X(\mathscr{T})$ lies mainly in the fact that their family forms a complete *Boolean algebra* [35, p. 66] with respect to the operations defined by $A' = A^{-c}$, $A \wedge B = A \cap B$ and $A \vee B = (A \cup B)''$.

In 1982, a subset A of $X(\mathscr{T})$ was called *preopen* by Mashhour et al. [58] if $A \subseteq A^{-\circ}$. However, by Dontchev [25], preopen sets, under different names, were much earlier studied by several mathematicians.

For instance, in 1964, Corson and Michael [12] called a subset A of $X(\mathscr{T})$ *locally dense* if it is a dense subset of some $V \in \mathscr{T}$ in the sense that $A \subseteq V \subseteq A^-$. Moreover, they noted that this property is equivalent to the inclusion $A \subseteq A^{-\circ}$.

This equivalence was later also stated by Jun et al. [41]. Moreover, Ganster [31] proved that A is preopen if and only if there exist $V \in \mathscr{T}$ and $B \in \mathscr{D}$ such that $A = V \cap B$. (See also Dontchev [25].)

In 1963, a subset A of $X(\mathscr{T})$ was called *semi-open* by Levine [52] if there exists $V \in \mathscr{T}$ such that $V \subseteq A \subseteq V^-$. First of all, he showed that the set A is semi-open if and only if $A \subseteq A^{\circ -}$.

Moreover, he also proved that if A is a semi-open subset of $X(\mathscr{T})$, then there exist $V \in \mathscr{T}$ and $B \in \mathscr{N}$ such that $A = V \cup B$ and $V \cap B = \emptyset$. In addition, he also noted that the converse statement is false.

Levine's statement closely resembles a famous stability theorem of Hyers [39] which says that an ε-approximately additive function of one Banach space to another is the sum of an additive function and an ε-small function.

Analogously to the paper of Hyers, Levine's paper has also attracted the interest of a surprisingly great number of mathematicians. For instance, by the Google Scholar, it has been cited by 3036 works.

Moreover, the above statement of Levine was improved by Dlaska et al. [24] who observed that a subset A of $X(\mathscr{T})$ is semi-open if and only if there exist $V \in \mathscr{T}$ and $B \subseteq V^\dagger$ such that $A = V \cup B$.

The latter observation was later reformulated, in a more convenient form, by Duszyński and Noiri [26] who noted that a subset A of $X(\mathscr{T})$ is semi-open if and only if there exists $B \subseteq A^{\circ\dagger}$ such that $A = A^\circ \cup B$.

In particular, in 1965 and 1971, Njåstad [64] and Isomichi [40], being not aware of the paper of Levine, studied semi-open sets under the names β*-sets* and *subcondensed sets*, respectively.

Moreover, Njåstad called a subset A of $X(\mathscr{T})$ to be an α-*set* if $A \subseteq A^{\circ-\circ}$. And, he proved that the set A is an α-set if and only if there exist $V \in \mathscr{T}$ and $B \in \mathscr{N}$ such that $A = V \setminus B$.

He also proved that A is an α-set if and only if its intersection with every β-set is a β–set. Thus, the family of all α-sets is a topology. The fact that the semi-open sets form only a generalized topology was already noticed by Levine.

A further important property of α-sets was established by Noiri [65] and Reilly and Wamanamurthy [77], in 1984 and 1985, respectively, who proved that a set is α-open if and only if it is both preopen and semi-open.

In 1983, the subset A was called β-*open* by Abd El-Monsef et al. [1] if $A \subseteq A^{-\circ-}$. Moreover, in 1986 Andrijević [2] used the term *semi-preopen* instead of β-open without knowing of [1].

Actually, Andrijević called a subset A of $X(\mathscr{T})$ to be semi-preopen if there exists a preopen subset V of $X(\mathscr{T})$ such that $V \subseteq A \subseteq V^{-}$. And, he showed that this is equivalent to the inclusion $A \subseteq A^{-\circ-}$.

Moreover, in 1996, a subset A of $X(\mathscr{T})$ was called *b-open* by Andrijević [3] if $A \subseteq A^{\circ-} \cup A^{-\circ}$. And, he proved that A is b-open if and only if there exist a preopen subset B and a semi-open subset C of $X(\mathscr{T})$ such that $A = B \cup C$.

In 1961, a subset A of a topological space $X(\mathscr{T})$ was said to have *property* Q by Levine [51] if $A^{\circ-} = A^{-\circ}$. He proved that A has property Q if and only if there exist $V \in \mathscr{T} \cap \mathscr{F}$ and $B \in \mathscr{N}$ such that $A = V \Delta B$. (See also [8, 11].)

While, in 1991, a subset A of $X(\mathscr{T})$ was called a δ-*set* by Chattopadhyay and Bandyopadhyay [9] if $A^{-\circ} \subseteq A^{\circ-}$. Moreover, in 2001, δ-open sets, under the name *quasi-open sets*, were more systematically studied by Császár [16, 17].

In 1992, Ganster et al. [32] already proved that A is a δ-set if and only if $A = V \cup N$ for some $V \in \mathscr{T}$ and $B \in \mathscr{N}$. Thus, δ-sets coincide with the *simply open sets* of Biswas [6] and Neubrunnová [63]. (See also [46, 61, 62].)

Actually, such sets were also first studied by Kuratowski [48, p. 69] in a more general framework. By his definition, a subset A of $X(\mathscr{T})$ has to be called *open modulo nowhere dense sets* if there exists $V \in \mathscr{T}$ such that $A \Delta V \in \mathscr{N}$.

In our former papers [75, 76], we have shown that the above definitions and several theorems on *generalized open sets* can be naturally extended not only to generalized topological and closure spaces, but also to *relator spaces* [85, 98].

Generalized neighbourhood, topological, closure and proximity spaces have mainly been studied by [7, 18, 21, 23, 48, 59, 79]. While, generalized uniformity spaces have formerly been also studied by [30, 60, 70, 120]. (See also [5, 10, 23].)

Now, extending some ideas of Gargouri and Rezgui [34], Premska [71, 72] and our former papers [73, 74], we shall show that the various generalized open sets can even be much better studied in some *non-conventional three relator spaces*.

2 Preliminaries

If $R \subseteq X \times Y$, then we say that R is a *relation or an ordinary relation* on X to Y. Namely, if U is a relation on $\mathscr{P}(X)$ to Y, and V is a relation on $\mathscr{P}(X)$ to

$\mathscr{P}(Y)$, then we may naturally say that U is a *super relation* and V is a *hyper relation* on X to Y.

For any $x \in X$ and $A \subseteq X$, the sets $R(x) = \{y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{x \in A} R(x)$ will be called the *images or neighbourhoods* of x and A under R, respectively. Thus, the sets $U(A)$ and $V(A)$ are also images or neighbourhoods of A under U and V, respectively.

For the ordinary relation R, we may naturally define a super relation $R^{\triangleright}$ on X to Y such that $R^{\triangleright}(A) = R[A]$ for all $A \subseteq X$. While, for the super relation U, we may also define an ordinary relation $U^{\triangleleft}$ on X to Y such that $U^{\triangleleft}(x) = U(\{x\})$ for all $x \in X$.

Namely, thus the maps $\triangleright$ and $\triangleleft$ form a *partial Galois connection* such that $R^{\triangleright} \subseteq U$ always implies $R \subseteq U^{\triangleleft}$, however for the converse implication we have to assume that U is *quasi-increasing* in the sense that $U(\{x\}) \subseteq U(A)$, i. e., $U^{\triangleleft}(x) \subseteq U(A)$ for all $x \in A \subseteq X$.

Actually, we always have $R^{\triangleright\triangleleft} = R$, and $R^{\triangleright}$ is always *union-preserving*. Thus, for instance, $U^{\circ} = U^{\triangleleft\triangleright}$ is also union-preserving. Moreover, we have $U = U^{\circ}$ if and only if U is union-preserving. Therefore, only the union-preserving super relations can be identified with the ordinary ones.

For the super relation U, by using appropriate complements, we may also naturally define a *dual super relation* $U^{\star}$ on X to Y such that $U^{\star}(A) = U(A^c)^c$ for all $A \subseteq X$. Thus, if in particular U is a *closure relation* on X, then $U^{\star}$ is an *interior relation* on X. A similar notation was formerly used by Császár [14].

Now, if $\mathscr{R}$ is a family of ordinary relations, $\mathscr{U}$ is a family of super relations and $\mathscr{V}$ is a family hyper relations on X to Y, then we may naturally say that $\mathscr{R}$ is an *ordinary relator*, $\mathscr{U}$ is a *super relator* and $\mathscr{V}$ is a *hyper relator* on X to Y. Such relators have already been briefly considered in [73, 113, 117].

Moreover, we may also naturally consider the *non-conventional three relator space* $(X, Y)(\mathscr{R}, \mathscr{U}, \mathscr{V})$, and its certain adjoint $(X, Y)(\mathscr{U}^{\triangleleft}, \mathscr{R}^{\triangleright}, \mathscr{V})$. However, practically it would be enough to consider only the particular case when $X = Y$, and moreover $\mathscr{R}$ is countable and both $\mathscr{U}$ and $\mathscr{V}$ are singletons.

Thus, for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, we may naturally define:

(1) $A \in \operatorname{Int}_{\mathscr{U}}(B)$ if $U(A) \subseteq B$ for some $U \in \mathscr{U}$;
(2) $A \in \operatorname{Cl}_{\mathscr{U}}(B)$ if $U(A) \cap B \neq \emptyset$ for all $U \in \mathscr{U}$;
(3) $x \in \operatorname{int}_{\mathscr{U}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathscr{U}}(B)$;
(4) $x \in \operatorname{cl}_{\mathscr{U}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathscr{U}}(B)$;
(5) $B \in \mathscr{E}_{\mathscr{U}}$ if $\operatorname{int}_{\mathscr{U}}(B) \neq \emptyset$;
(6) $B \in \mathscr{D}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(B) = X$.

Moreover, in the $X = Y$ particular case, for any $A \subseteq X$, we may also define:

(7) $A \in \tau_{\mathscr{U}}$ if $A \in \operatorname{Int}_{\mathscr{U}}(A)$;
(8) $A \in \bar{\tau}_{\mathscr{U}}$ if $A^c \notin \operatorname{Cl}_{\mathscr{U}}(A)$;
(9) $A \in \mathscr{T}_{\mathscr{U}}$ if $A \subseteq \operatorname{int}_{\mathscr{U}}(A)$;
(10) $A \in \mathscr{F}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(A) \subseteq A$;
(11) $A \in \mathscr{N}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(A) \notin \mathscr{E}_{\mathscr{U}}$;
(12) $A \in \mathscr{M}_{\mathscr{U}}$ if $\operatorname{int}_{\mathscr{U}}(A) \in \mathscr{D}_{\mathscr{U}}$.

Now, by using the super relator $\mathscr{R}^{\triangleright} = \{ R^{\triangleright} : \quad R \in \mathscr{R} \}$, for instance we may also naturally define $\operatorname{Int}_{\mathscr{R}} = \operatorname{Int}_{\mathscr{R}^{\triangleright}}$ and $\operatorname{int}_{\mathscr{R}} = \operatorname{int}_{\mathscr{R}^{\triangleright}}$. Thus, by using the super relator $\mathscr{U}^{\circ} = \{ U^{\circ} : \ U \in \mathscr{U} \}$ and the ordinary relator $\mathscr{U}^{\triangleleft} = \{ U^{\triangleleft} : \ U \in \mathscr{U} \}$, it can be easily shown that $\operatorname{Int}_{\mathscr{U}^{\circ}} = \operatorname{Int}_{\mathscr{U}^{\triangleleft}}$ and $\operatorname{int}_{\mathscr{U}} = \operatorname{int}_{\mathscr{U}^{\triangleleft}}$.

Moreover, in the $X = Y$ particular case, for instance we may also naturally define $\tau_{\mathscr{R}} = \tau_{\mathscr{R}^{\triangleright}}$ and $\mathscr{T}_{\mathscr{R}} = \mathscr{T}_{\mathscr{R}^{\triangleright}}$. Thus, by our former statements, we evidently have $\tau_{\mathscr{U}^{\circ}} = \tau_{\mathscr{U}^{\triangleleft}}$ and $\mathscr{T}_{\mathscr{U}} = \mathscr{T}_{\mathscr{U}^{\triangleleft}}$. Therefore, a great deal of the theory of ordinary relators cannot be generalized by using super relators.

In the present paper, we shall also show that

$$\tau_{\mathscr{U}} = \Big\{ A \subseteq X : \quad \exists \ U \in \mathscr{U} : \quad A \subseteq U^{\star}(A) \Big\} .$$

Thus, for the *dual super relator* $\mathscr{U}^{\star} = \{ \mathscr{U}^{\star} : \ U \in \mathscr{U} \}$, we can state that

$$\tau_{\mathscr{U}^{\star}} = \Big\{ A \subseteq X : \quad \exists \ U \in \mathscr{U} : \quad A \subseteq U(A) \Big\} .$$

The $\mathscr{U} = \{ U \}$ particular case of the latter statements already allows of a unification of several generalized open sets. Namely, if for instance

$$U(A) = \operatorname{cl}_{\mathscr{R}} \big(\operatorname{int}_{\mathscr{R}}(A) \big), \qquad \text{and thus} \qquad U^{\star}(A) = \operatorname{int}_{\mathscr{R}} \big(\operatorname{cl}_{\mathscr{R}}(A) \big)$$

for all $A \subseteq X$, then τ_U and $\tau_{U^{\star}}$ are just the families of all *topologically preopen and semi-open subsets* of the relator space $X(\mathscr{R})$ considered first in [75].

Now, for instance we may also naturally define

$$\mathscr{T}_{\mathscr{R}}^{k} = \mathscr{T}_{\mathscr{R}}^{k_{\mathscr{V}}} = \operatorname{cl}_{\mathscr{V}}(\mathscr{T}_{\mathscr{R}}) \qquad \text{and} \qquad \mathscr{T}_{\mathscr{R}}^{\ell} = \mathscr{T}_{\mathscr{R}}^{\ell_{\mathscr{V}}} = \operatorname{cl}_{\mathscr{V}^{-1}}(\mathscr{T}_{\mathscr{R}}),$$

where $\mathscr{V}^{-1} = \big\{ V^{-1} : \ V \in \mathscr{V} \big\}$. Namely, thus we can easily prove that

$$\mathscr{T}_{\mathscr{R}}^{\ell} = \{ B \subseteq X : \quad \forall \ V \in \mathscr{V} : \quad \exists \ A \in \mathscr{T}_{\mathscr{R}} : \quad B \in V(A) \big\}$$

and

$$\mathscr{T}_{\mathscr{R}}^{k} = \{ A \subseteq X : \quad \forall \ V \in \mathscr{V} : \quad \exists \ B \in \mathscr{T}_{\mathscr{R}} : \quad B \in V(A) \big\} .$$

Therefore, the $\mathscr{V} = \{ V \}$ particular case of the latter statements allow of a unification of several further generalized open sets. Namely, if for instance

$$V(A) = \Big\{ B \subseteq X : \quad A \subseteq B \subseteq \operatorname{cl}_{\mathscr{R}}(A) \Big\}$$

for all $A \subseteq X$, then $\mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\mathscr{T}_{\mathscr{R}}^{k_V}$ are just the families of all *topologically quasi-open and pseudo-open subsets* of the relator space $X(\mathscr{R})$ considered first in [75].

3 A Few Basic Facts on Relations

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. In particular, a relation on X to itself is called a *relation on* X. And, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation of* X.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, the latter view of the relation F would be quite unnatural for several purposes.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup\{F(x) : x \in A\}$ are called the *images or neighbourhoods* of x and A under F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x\,F\,y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ may be called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we may say that F is a *relation of* X *to* Y, or that F is a *non-partial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on* X. While, a function $*$ of X^2 to X is called a *binary operation on* X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function* of F if $f(x) \in F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be shown that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathscr{P}(Y)$ and Φ_F of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathscr{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ are more general objects than relations on X to Y. In [107, 113, 114], they were briefly called *corelations on* X *to* Y.

However, if R is a relation on X to Y, U is a relation on $\mathscr{P}(X)$ to Y, and V is a relation on $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, then it is better to say that R is an *ordinary relation*, U is a *super relation* and V is a *hyper relation* on X to Y [117].

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the images $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement* F^c and the *inverse* F^{-1} can be defined such that $F^c(x) = F(x)^c$ for all $x \in X$ and $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Namely, thus we also have $F^c = X \times Y \setminus F$ and $F^{-1} = \{(y, x) : (x, y) \in F\}$.

Moreover, if in addition G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, it can be easily shown that $(G \circ F)[A] = G\big[F[A]\big]$ also holds for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$ [105].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

Now, a relation R on X may be briefly defined to be *reflexive* on X if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For any relation R on X, we may also define $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, it can be shown that R^∞ is the smallest preorder relation on X containing R [36].

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is an important preorder on X [70]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [120].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \big(A^c\big)^2$ is already an equivalence relation on X. And, more generally if $\mathscr{A}$ is a *cover (partition)* of X, then $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$ is a tolerance (equivalence) relation on X.

As an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker–Lindgren relation* $R_{(A,B)} = A \times B \ \cap \ A^c \times Y$ [38]. Namely, thus we evidently have $R_A = R_{(A,A)}$.

The Pervin relations R_A and the Hunsaker–Lindgren relations $R_{(A,B)}$ were actually first used by Davis [23] and Császár [13, pp. 42 and 351] in some less explicit and convenient forms, respectively.

4 Some Basic Properties of Super Relations

Notation 1 *In this section, we shall assume that U is a super relation on X to Y.*

Remark 1 Thus, by our former definitions, U is actually an ordinary relation on $\mathscr{P}(X)$ to Y, i. e., it is an arbitrary subset of $\mathscr{P}(X) \times Y$.

Moreover, U can be identified with the set-valued function φ_U, defined by $\varphi_U(A) = U(A)$ for all $A \subseteq X$, which is a particular subset of $\mathscr{P}(X) \times \mathscr{P}(Y)$.

Thus, several properties of the super relation U can be easily defined with the help of the set-valued function φ_U. For instance, we may naturally introduce

Definition 1 The super relation U will be called

(1) *increasing* if $U(A) \subseteq U(B)$ for all $A \subseteq B \subseteq X$;
(2) *quasi-increasing* if $U\big(\{x\}\big) \subseteq U(A)$ for all $x \in A \subseteq X$;

(3) *union-preserving* if $U\big(\bigcup\mathscr{A}\big)=\bigcup_{A\in\mathscr{A}}U(A)$ for all $\mathscr{A}\subseteq\mathscr{P}(X)$.

Thus, we can at once state the following two theorems.

Theorem 1 *The following assertions are equivalent:*

(1) U *is quasi-increasing;*
(2) $\bigcup_{x\in A}U\big(\{x\}\big)\subseteq U(A)$ *for all* $A\subseteq X$.

Theorem 2 *The following assertions are equivalent:*

(1) U *is increasing;*
(2) $U\big(\bigcap\mathscr{A}\big)\subseteq\bigcap_{A\in\mathscr{A}}U(A)$ *for all* $\mathscr{A}\subseteq\mathscr{P}(X)$*;*
(3) $\bigcup_{A\in\mathscr{A}}U(A)\subseteq U\big(\bigcup\mathscr{A}\big)$ *for all* $\mathscr{A}\subseteq\mathscr{P}(X)$.

Proof If $A\subseteq B\subseteq X$, then by the $\mathscr{A}=\{A,\,B\}$ particular case of (3) we have $U(A)\subseteq U(A)\cup U(B)\subseteq U(A\cup B)=U(B)$, and thus (1) also holds.

Moreover, by using Definition 1 and Theorem 2, we can also easily prove

Theorem 3 *The following assertions are equivalent:*

(1) U *is union-preserving;*
(2) $U(A)=\bigcup_{x\in A}U\big(\{x\}\big)$ *for all* $A\subseteq X$.

Proof If (1) holds, then because of $A=\bigcup_{x\in A}\{x\}$ it is clear that also holds. While, if (2) holds, then we can at once see that U increasing. Thus, by Theorem 2, we have $\bigcup_{A\in\mathscr{A}}U(A)\subseteq U\big(\bigcup\mathscr{A}\big)$ for all $\mathscr{A}\subseteq\mathscr{P}(X)$. Therefore, to obtain (1), we need only prove the converse inclusion.

For this, note that if $\mathscr{A}\subseteq\mathscr{P}(X)$, then by (2) we have

$$U\Big(\bigcup\mathscr{A}\Big)=\bigcup_{x\in\bigcup\mathscr{A}}U\big(\{x\}\big).$$

Therefore, if $y\in U\big(\bigcup\mathscr{A}\big)$, then there exists $x\in\bigcup\mathscr{A}$ such that $y\in U\big(\{x\}\big)$. Thus, in particular there exists $A_0\in\mathscr{A}$ such that $x\in A_0$, and so $\{x\}\subseteq A_0$. Hence, by using the increasingness of U, we can already see that

$$y\in U\big(\{x\}\big)\subseteq U(A_0)\subseteq\bigcup_{A\in\mathscr{A}}U(A).$$

Therefore, $U\big(\bigcup\mathscr{A}\big)\subseteq\bigcup_{A\in\mathscr{A}}U(A)$ also holds.

Remark 2 In particular, a super relation U on X to itself may be simply called a super relation on X.

Thus, a super relation U on X may be called *extensive, intensive, involutive* and *idempotent* if $A\subseteq U(A)$, $U(A)\subseteq A$, $U\big(U(A)\big)=A$ and $U\big(U(A)\big)=U(A)$ for all $A\subseteq X$, respectively.

Moreover, an increasing involutive (idempotent) super relation may be called an *involution (projection) relation.* While, an extensive (intensive) projection relation may be called a *closure (interior) relation*.

5 Relationships Between Ordinary and Super Relations

Notation 2 *In this and the next two sections, we shall assume that* R *and* S *are ordinary relations, and* U *and* V *are super relations on* X *to* Y.

In [113], having in mind *Galois connections* [22, 104], we have introduced

Definition 2 For the ordinary relation R, we define a super relation $R^{\triangleright}$ on X to Y such that

$$R^{\triangleright}(A) = R\,[\,A\,] \quad \text{for all} \quad A \subseteq X.$$

While, for the super relation U, we define an ordinary relation $U^{\triangleleft}$ on X to Y such that

$$U^{\triangleleft}(x) = U\left(\{x\}\right) \quad \text{for all} \quad x \in X.$$

The appropriateness of the above definitions is apparent from the following two theorems whose proofs are included here only for the reader's convenience.

Theorem 4 $R^{\triangleright} \subseteq U$ *implies* $R \subseteq U^{\triangleleft}$.

Proof If $R^{\triangleright} \subseteq U$, then in particular we have

$$R\,(x) = R\big[\{x\}\big] = R^{\triangleright}\big(\{x\}\big) \subseteq U\big(\{x\}\big) = U^{\triangleleft}(x)$$

for all $x \in X$. Therefore, $R \subseteq U^{\triangleleft}$ also holds.

Remark 3 For the latter inclusion, we have only needed that $R^{\triangleright\triangleleft} \subseteq U^{\triangleleft}$. However, later we shall see that $R^{\triangleright\triangleleft} = R$, and thus $R \subseteq U^{\triangleleft}$ is actually equivalent to $R^{\triangleright\triangleleft} \subseteq U^{\triangleleft}$.

Theorem 5 *The following assertions are equivalent :*

(1) U *is quasi-increasing ;*
(2) $R \subseteq U^{\triangleleft}$ *implies* $R^{\triangleright} \subseteq U$ *for any relation* R *on* X *to* Y.

Proof If $R \subseteq U^{\triangleleft}$ and (1) holds, then by the corresponding definitions and Theorem 1, we have

$$R^{\triangleright}(A) = R\,[\,A\,] = \bigcup_{x \in A} R\,(x) \subseteq \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\{x\}\big) \subseteq U\,(A)$$

for all $A \subseteq X$. Therefore, $R^{\triangleright} \subseteq U$, and thus (2) also holds.

While, if (2) holds, then because of $U^{\triangleleft} \subseteq U^{\triangleleft}$ we have $U^{\triangleleft\triangleright} = (U^{\triangleleft})^{\triangleright} \subseteq U$. Therefore, for any $A \subseteq X$, we have $U^{\triangleleft\triangleright}(A) \subseteq U(A)$. Moreover, by using the corresponding definitions, we can see that

$$U^{\triangleleft\triangleright}(A) = \big(U^{\triangleleft}\big)^{\triangleright}(A) = U^{\triangleleft}[A] = \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\{x\}\big).$$

Therefore, $\bigcup_{x \in A} U\big(\{x\}\big) \subseteq U(A)$, and thus assertion (1) also holds.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 1 *If U is quasi-increasing, then*

$$R^{\triangleright} \subseteq U \iff R \subseteq U^{\triangleleft}.$$

Remark 4 This shows that the operations $\triangleright$ and $\triangleleft$ establish a *partial Galois connection* between the complete posets $\mathscr{P}(X \times Y)$ and $\mathscr{P}\big(\mathscr{P}(X) \times Y\big)$.

Therefore, we may also naturally introduce the following

Definition 3 The super relation

$$U^{\circ} = U^{\triangleleft\triangleright}$$

will be called the *Galois interior* of U.

Thus, by the proof of Theorem 5, we can at once state the following

Theorem 6 *For any $A \subseteq X$, we have*

$$U^{\circ}\big(A\big) = \bigcup_{x \in A} U\big(\{x\}\big).$$

Hence, it is clear that, in particular, we also have

Corollary 2 *For any $x \in X$, we have $U^{\circ}\big(\{x\}\big) = U\big(\{x\}\big)$.*

Example 1 If in particular $U(A) = A^c$ for all $A \subseteq X$, then for any $A \subseteq X$ we have

$$U^{\circ}(A) = \begin{cases} \emptyset & \text{if} \quad \operatorname{card}(A) = 0, \\ A^c & \text{if} \quad \operatorname{card}(A) = 1, \\ X & \text{if} \quad \operatorname{card}(A) > 1. \end{cases}$$

Namely, by Theorem 6 and De Morgan's law, we have

$$U^{\circ}\big(A\big) = \bigcup_{x \in A} U\big(\{x\}\big) = \bigcup_{x \in A} \{x\}^c = \Big(\bigcap_{x \in A} \{x\}\Big)^c,$$

whence the required equalities immediately follow.

6 Further Theorems on the Operations $\triangleright$, $\triangleleft$ and $\circ$

Several properties of the operations $\triangleright$, $\triangleleft$ and $\circ$ can be immediately derived from the general theory of *Galois and Pataki connections* [102, 104, 115].

However, because of the simplicity of Definition 2, it is now more convenient to use some direct proofs to establish the following four theorems.

Theorem 7 *The operations* $\triangleright$, $\triangleleft$ *and* $\circ$ *are increasing.*

Proof For instance, if $U \subseteq V$, then $U(A) \subseteq V(A)$ for all $A \subseteq X$. Thus, in particular we also have $U^{\triangleleft}(x) = U\big(\{x\}\big) \subseteq V\big(\{x\}\big) = V^{\triangleleft}(x)$ for all $X \in X$. Therefore, $U^{\triangleleft} \subseteq V^{\triangleleft}$ also holds.

Theorem 8 $R^{\triangleright}$ *is a union-preserving super relation on* X *to* Y *such that*

(1) $R^{\triangleright\triangleleft} = R$*;* *(2)* $R^{\triangleright\circ} = R^{\triangleright}$.

Proof By the corresponding definitions, we have

$$R^{\triangleright}(A) = R\,[\,A\,] = \bigcup_{x\in A} R\,(x) = \bigcup_{x\in A} R\big[\{x\}\big] = \bigcup_{x\in A} R^{\triangleright}\big(\{x\}\big)$$

for all $A \subseteq X$. Thus, by Theorem 3, the super relation $R^{\triangleright}$ is union-preserving.

Moreover, we can easily see that

$$R^{\triangleright\triangleleft}(x) = \big(R^{\triangleright}\big)^{\triangleleft}(x) = R^{\triangleright}\big(\{x\}\big) = R\,\big[\{x\}\big] = R(x)$$

for all $x \in X$. Thus, assertion (1) is also true.

Now, by using Definition 3 and assertion (1), we can also easily that

$$R^{\triangleright\circ} = \big(R^{\triangleright}\big)^{\circ} = \big(R^{\triangleright}\big)^{\triangleleft\triangleright} = \big(R^{\triangleright\triangleleft}\big)^{\triangleright} = R^{\triangleright}\,.$$

Corollary 3 *We have* $R \subseteq S$ *if and only if* $R^{\triangleright} \subseteq S^{\triangleright}$.

Theorem 9 U° *is a union-preserving super relation on* X *to* Y *such that*

(1) $U^{\circ\triangleleft} = U^{\triangleleft}$*;* *(2)* $U^{\circ\circ} = U^{\circ}$.

Proof From Definition 3, by using Theorem 8, we can see that U° is union-preserving and

$$U^{\circ\triangleleft} = \big(U^{\circ}\big)^{\triangleleft} = \big(U^{\triangleleft\triangleright}\big)^{\triangleleft} = \big(U^{\triangleleft}\big)^{\triangleright\triangleleft} = U^{\triangleleft}.$$

Assertion (1) is also an immediate consequence of Definition 2 and Corollary 2.

Moreover, by using Theorem 6 and its corollary, we can easily see that

$$U^{\circ\circ}(A) = \big(U^{\circ}\big)^{\circ}(A) = \bigcup_{x\in A} U^{\circ}\big(\{x\}\big) = \bigcup_{x\in A} U\big(\{x\}\big) = U^{\circ}(A)$$

for all $A \subseteq X$. Therefore, assertion (2) is also true.

Theorem 10 *The following assertions are equivalent:*

(1) $U^{\circ} = U$; *(2)* U *is union-preserving;*
(3) $U = R^{\triangleright}$ *for some relation* R *on* X *to* Y.

Proof If (2) holds, then by Theorems 6 and 3 we can see that

$$U^{\circ}(A) = \bigcup_{x \in A} U\big(\{x\}\big) = U(A)$$

for all $A \subseteq X$. Therefore, (1) also holds.

Now, since (1) trivially implies (3), we need only note that if (3) holds, then by Theorem 8 assertion (2) also holds.

Corollary 4 *If* U *and* V *are union-preserving, then* $U \subseteq V$ *if and only if* $U^{\triangleleft} \subseteq V^{\triangleleft}$.

Finally, we note that, by using our former results, the following four theorems can also be proved.

Theorem 11 *We have*

(1) $U \subseteq U^{\circ} \iff U(A) \subseteq U^{\triangleleft}[A]$ *for all* $A \subseteq X$;
(2) $U^{\circ} \subseteq U \iff U$ *is quasi-increasing* $\iff U^{\triangleleft}[A] \subseteq U(A)$ *for all* $A \subseteq X$;
(3) $U^{\circ} = U \iff U$ *is union-preserving* $\iff U(A) = U^{\triangleleft}[A]$ *for all* $A \subseteq X$.

Theorem 12 *We have*

(1) $U^{\circ} \subseteq V \implies U^{\circ} \subseteq V^{\circ} \iff U^{\triangleleft} \subseteq V^{\triangleleft}$;
(2) $U^{\circ} \subseteq V^{\circ} \implies U^{\circ} \subseteq V$ *if* V *is quasi-increasing;*
(3) $U \subseteq V \iff U^{\circ} \subseteq V^{\circ}$ *if* U *and* V *are union-preserving.*

Theorem 13 *If* $U = R^{\triangleright}$, *then*

(1) U *is a union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} = R$;
(2) U *is the smallest quasi-increasing super relation on* X *to* Y *such that* $R \subseteq U^{\triangleleft}$;
(3) U *is the largest union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} \subseteq R$.

Theorem 14 *If* $R = U^{\triangleleft}$, *then*

(1) $R^{\triangleright} \subseteq U$ *if and only if* U *is quasi-increasing;*
(2) $R^{\triangleright} = U$ *if and only if* U *is union-preserving;*
(3) If U *is quasi-increasing, then* R *is the largest relation on* X *to* Y *such that* $R^{\triangleright} \subseteq U$;
(4) if U *is union-preserving, then* R *is the smallest relation on* X *to* Y *such that* $U \subseteq R^{\triangleright}$.

7 Relationally Defined Inverses of Super Relations

Because of Remark 4, we may also naturally introduce the following

Definition 4 The super relation

$$U^{\mathbf{-1}} = U^{\lhd\,-1\,\rhd}$$

will be called the *relationally defined inverse* of U.

Remark 5 To feel the necessity of this bold inverse $U^{\mathbf{-1}}$, note that the ordinary inverse U^{-1} of U is not a super relation.

While, the ordinary inverse φ_U^{-1} of the associated set-valued function φ_U, which can be identified with U, is usually a hyper relation.

Now, using the corresponding definitions and Theorem 8, we can easily prove the following three theorems.

Theorem 15 *We have*

(1) $R^{\rhd\,\mathbf{-1}} = R^{-1\,\rhd}$; *(2)* $R^{\rhd\,\mathbf{-1}\,\lhd} = R^{-1}$.

Proof By Definition 4 and Theorem 8, we have

$$R^{\rhd\,\mathbf{-1}} = R^{\rhd\,\lhd\,-1\,\rhd} = R^{-1\,\rhd}, \quad \text{and thus also} \quad R^{\rhd\,\mathbf{-1}\,\lhd} = R^{-1\,\rhd\,\lhd} = R^{-1}.$$

Theorem 16 $U^{\mathbf{-1}}$ *is a union-preserving super relation on* Y *to* X *such that*

(1) $U^{\mathbf{-1}\,\lhd} = U^{\lhd\,-1}$; *(2)* $U^{\circ\,\mathbf{-1}} = U^{\mathbf{-1}}$.

Proof By Definitions 4 and 3 and Theorem 8 we have

$$U^{\mathbf{-1}\,\lhd} = U^{\lhd\,-1\,\rhd\,\lhd} = U^{\lhd\,-1} \qquad \text{and} \qquad U^{\circ\,\mathbf{-1}} = U^{\lhd\rhd\,\lhd\,-1\,\rhd} = U^{\lhd\,-1\,\rhd} = U^{\mathbf{-1}}.$$

Remark 6 Note that if $U^{\lhd}$ is symmetric, then $U^{\mathbf{-1}} = U^{\lhd\,-1\,\rhd} = U^{\lhd\rhd} = U^{\circ}$. Thus, if in addition U is union-preserving, then $U^{\mathbf{-1}} = U$.

In this respect, it is also worth noticing that if in particular U is as in Example 1, then $U^{\lhd}$ is symmetric. Thus, by the above observation, $U^{\mathbf{-1}} = U^{\circ}$.

Theorem 17 *We have*

$$\left(U^{\mathbf{-1}}\right)^{\mathbf{-1}} = U^{\circ}.$$

Proof By the corresponding definitions and Theorem 8, we have

$$\left(U^{\mathbf{-1}}\right)^{\mathbf{-1}} = U^{\lhd\,-1\,\rhd\,\lhd\,-1\,\rhd} = U^{\lhd\,-1\,-1\,\rhd} = U^{\lhd\rhd} = U^{\circ}.$$

Hence, by using Theorem 10, we can immediately derive

Corollary 5 *The following assertions are equivalent :*

(1) $U = \big(U^{-1}\big)^{-1}$; *(2)* U *is union-preserving.*

Moreover, as a counterpart of Theorem 6, we can also prove the following

Theorem 18 *For any* $B \subseteq Y$, *we have*

$$U^{-1}(B) = \big\{ x \in X : \quad U\big(\{x\}\big) \cap B \neq \emptyset \big\}.$$

Proof By the corresponding definitions, we have

$$U^{-1}(B) = \big(U^{\lhd -1 \rhd}\big)(B) = U^{\lhd -1}[B].$$

Moreover, it is clear that, for any $x \in X$, we have

$$x \in U^{\lhd -1}[B] \iff U^{\lhd}(x) \cap B \neq \emptyset \iff U\big(\{x\}\big) \cap B \neq \emptyset.$$

Therefore, the required equality is true.

Remark 7 From the above proof, by Theorem 16, we can also see that

$$U^{-1}(B) = U^{\lhd -1}[B] = U^{-1 \lhd}[B].$$

8 Functionally and Relationally Defined Compositions of Super Relations

Notation 3 *In this section, we shall assume that* R *is an ordinary relation and* U *is super relation on* X *to* Y.

Moreover, we shall also assume that S *is an ordinary relation and* V *is super relation on* Y *to* Z.

By the usual identification of U with φ_U, we may also naturally introduce

Definition 5 The super relation $V \circ U$, defined such that

$$(V \circ U)(A) = V\big(U(A)\big)$$

for all $A \subseteq X$, will be called the *functionally defined composition* of V and U.

Remark 8 Namely, thus we have

$$\varphi_{V \circ U}(A) = (V \circ U)(A) = V\big(U(A)\big) = \varphi_V\big(\varphi_U(A)\big) = (\varphi_V \circ \varphi_U)(A)$$

for all $A \subseteq X$, and thus $\varphi_{V \circ U} = \varphi_V \circ \varphi_U$.

The appropriateness of Definition 5 is also quite obvious from the following three simple theorems and their corollaries.

Theorem 19 *We have*

$$\big(S \circ R \big)^{\triangleright} = S^{\triangleright} \bullet R^{\triangleright} .$$

Corollary 6 *We have*

(1) $(S \circ U^{\triangleleft})^{\triangleright} = S^{\triangleright} \bullet U$ *if* U *is union-preserving ;*
(2) $(V^{\triangleleft} \circ R)^{\triangleright} = V \bullet R^{\triangleright}$ *if* V *is union-preserving.*

Theorem 20 *If* V *is union-preserving, then*

$$\big(V \bullet U \big)^{\triangleleft} = V^{\triangleleft} \circ U^{\triangleleft} .$$

Proof By the corresponding definitions and Theorem 10, we have

$$\begin{aligned}(V \bullet U)^{\triangleleft}(x) &= (V \bullet U)\big(\{x\}\big) = V\ \big(U\big(\{x\}\big)\big) = V\ \big(U^{\triangleleft}(x)\big) \\ &= V^{\circ}\big(U^{\triangleleft}(x)\big) = V^{\triangleleft\triangleright}\big(U^{\triangleleft}(x)\big) = V^{\triangleleft}\big[\,U^{\triangleleft}(x)\big] = \big(V^{\triangleleft} \circ U^{\triangleleft}\big)(x)\end{aligned}$$

for all $x \in X$. Therefore, the required equality is also true.

Corollary 7 *We have*

(1) $(S^{\triangleright} \bullet U)^{\triangleleft} = S \circ U^{\triangleleft}$ *;*
(2) $(V \bullet R^{\triangleright})^{\triangleleft} = V^{\triangleleft} \circ R$ *if* V *is union-preserving.*

Theorem 21 *If* V *is union-preserving, then*

$$(V \bullet U)^{\mathbf{-1}} = U^{\mathbf{-1}} \bullet V^{\mathbf{-1}} .$$

Proof By Definition 4 and Theorems 20, and 19, we have

$$\begin{aligned}(V \bullet U)^{\mathbf{-1}} &= (V \bullet U)^{\triangleleft\,-1\,\triangleright} = \big(V^{\triangleleft} \circ\ U^{\triangleleft}\big)^{-1\,\triangleright} \\ &= \Big(U^{\triangleleft\,-1} \circ V^{\triangleleft\,-1}\Big)^{\triangleright} = U^{\triangleleft\,-1\,\triangleright} \bullet V^{\triangleleft\,-1\,\triangleright} = U^{\mathbf{-1}} \bullet\ V^{\mathbf{-1}} .\end{aligned}$$

Corollary 8 *We have* $(S^{\triangleright} \bullet U)^{\mathbf{-1}} = U^{\mathbf{-1}} \bullet S^{-1\,\triangleright}$.

Remark 9 By using Definition 5, it can also be easily seen that the functionally defined composition of super relations is associative.

Now, analogously to Definition 4, we may also naturally introduce

Definition 6 The super relation

$$V \bullet U = \big(V^{\triangleleft} \circ\ U^{\triangleleft}\big)^{\triangleright}$$

will be called the *relationally defined composition* of V and U.

The appropriateness of this definition is apparent from the following theorems.

Theorem 22 *We have*

(1) $S^{\triangleright} \bullet R^{\triangleright} = \left(S \circ R\right)^{\triangleright}$ *(2)* $(S^{\triangleright} \bullet R^{\triangleright})^{\triangleleft} = S \circ R.$

Theorem 23 $V \bullet U$ *is a union-preserving super relation such that*

$$V \bullet U = V^{\circ} \circ U^{\circ}.$$

Proof From Definition 6, by Theorem 8, it is clear that $V \bullet U$ is a union-preserving. Moreover, by using Theorem 19 and Definition 3, we can see that

$$V \bullet U = \left(V^{\triangleleft} \circ U^{\triangleleft}\right)^{\triangleright} = V^{\triangleleft\triangleright} \circ U^{\triangleleft\triangleright} = V^{\circ} \circ U^{\circ}.$$

Thus, in particular, by Theorem 10, we can also state the following

Corollary 9 *If both* U *and* V *are union-preserving, then* $V \bullet U = V \circ U$.

Remark 10 From Theorem 23, by using Theorems 6 and 9, we can also infer that

$$\left(V \bullet U\right)(A) = \bigcup_{x \in A} \; \bigcup_{y \in U(\{x\})} V\left(\{y\}\right)$$

for all $A \subseteq X$.

Now, by using our former results, we can also prove the following

Theorem 24 *We have*

$$\left(V \bullet U\right)^{-1} = U^{-1} \circ V^{-1}.$$

Proof By Theorems 23, 9, 21, and 16, it is clear that

$$\left(V \bullet U\right)^{-1} = (V^{\circ} \circ U^{\circ})^{-1} = U^{\circ -1} \circ V^{\circ -1} = U^{-1} \circ V^{-1}.$$

Remark 11 Moreover, by using Theorem 23 and Remark 9, it can be easily seen that the relationally defined composition of super relations is also associative.

9 The Duals of Super and Hyper Relations

Having in mind the relationship between the usual closure and interior operations, we may naturally introduce the following

Definition 7 For a super relation U on X to Y, we define a *dual super relation* $U^{\star}$ on X to Y such that

$$U^{\star}(A) = U\,(A^c)^c$$

for all $A \subseteq X$.

Thus, we can easily prove the following four theorems.

Theorem 25 *If U and V are super relations on X to Y, then*

(1) $U = U^{\star\star}$ *;* *(2)* $U \subseteq V$ *implies* $V^{\star} \subseteq U^{\star}$.

Proof To prove (2), note that if $U \subseteq V$, then $U\,(A^c) \subseteq V\,(A^c)$, and thus

$$V^{\star}(A) = V\,(A^c)^c \subseteq U\,(A^c)^c = U^{\star}(A)$$

for all $A \subseteq X$. Therefore, $V^{\star} \subseteq U^{\star}$ also holds.

Theorem 26 *If U is a super relation on X to Y, then*

(1) $U^{\star}$ is increasing if and only if U is increasing ;
(2) $U^{\star}$ is union-preserving if and only if U is intersection-preserving ;
(3) $U^{\star}$ is intersection-preserving if and only if U is union-preserving.

Proof If for instance U is union-preserving, then by the corresponding definitions and De Morgan's law we have

$$U^{\star}\big(\bigcap\nolimits_{A\in\mathscr{A}} A\big) = U\left(\left(\bigcap_{A\in\mathscr{A}} A\right)^c\right)^c = U\left(\bigcup_{A\in\mathscr{A}} A^c\right)^c$$

$$= \left(\bigcup_{A\in\mathscr{A}} U\,(A^c)\right)^c = \bigcap_{A\in\mathscr{A}} U\,(A^c)^c = \bigcap_{A\in\mathscr{A}} U^{\star}(A)$$

for all $\mathscr{A} \subseteq \mathscr{P}(X)$. Therefore, $U^{\star}$ is intersection-preserving.

Thus, the "if part" of assertion (3) is true. Hence, since $U^{\star\star} = U$, it is clear that the "only if part" of assertion (2) is also true.

Theorem 27 *If U is a super relation on X to Y, then*

(1) $U^{\star}$ is intensive if and only if U is extensive ;
(2) $U^{\star}$ is extensive if and only if U is intensive ;
(4) $U^{\star}$ is involutive if and only if U is involutive ;
(5) $U^{\star}$ is idempotent if and only if U is idempotent.

Proof For instance if U is idempotent, then by the corresponding definitions

$$U^{\star}\big(U^{\star}(A)\big) = U^{\star}\big(U\,(A^c)^c\big) = U\,\big(U\,(A^c)\big)^c = U\,(A^c)^c = U^{\star}(A)$$

for all $A \subseteq X$. Therefore, $U^{\star}$ is also idempotent.

Thus, the "if part" of assertion (5) is true. Hence, since $U^{\star\star} = U$, it is clear that the "only if part" of assertion (5) is also true.

Theorem 28 *If U is a super relation on X to Y and V is a super relation on Y to Z, then*

(1) $(V \circ U)^\star = V^\star \circ U^\star$; *(2)* $(V \bullet U)^\star = V^{\circ\star} \circ U^{\circ\star}$.

Proof By Definitions 7 and 5, we have

$$(V \circ U)^\star(A) = (V \circ U)(A^c)^c = V\big(U(A^c)\big)^c$$
$$= V\big(U^\star(A)^c\big)^c = V^\star\big(U^\star(A)\big) = \big(V^\star \circ U^\star\big)(A)$$

for all $A \subseteq X$. Thus, assertion (1) is true.

Now, by using Theorem 23 and assertion (1), we can see that assertion (2) is also true.

Remark 12 Concerning the super relation $U^{\circ\star}$, by using Definition 7 and Theorem 6 we can only note that

$$U^{\circ\star}(A) = U^\circ(A^c)^c = \Big(\bigcup\nolimits_{x \in A^c} U\big(\{x\}\big)\Big)^c = \bigcap\nolimits_{x \in A^c} U\big(\{x\}\big)^c$$

for all $A \subseteq X$.

Analogously to Definition 7, we may also naturally introduce the following

Definition 8 For a hyper relation V on X to Y, we define two *dual hyper relations* $V^\star$ and $V^\bigstar$ on X to Y such that

$$V^\star(A) = V(A^c)^c = \mathscr{P}(Y) \setminus V(A^c)$$

and

$$V^\bigstar(A) = \big[V(A^c)\big]^c = \big\{B^c : \ B \in V(A^c)\big\}$$

for all $A \subseteq X$.

Remark 13 Thus, some properties of the hyper relations $V^\star$ and $V^\bigstar$ can also be easily derived from those of the hyper relation V.

Moreover, having in mind the derivations of small closures and interiors from the big ones [85, 91], we may also naturally introduce the following

Definition 9 For a hyper relation V on X to Y, we define a super relation $V^\triangleleft$ on X to Y such that

$$V^\triangleleft(A) = \big\{ y \in Y : \ \{y\} \in V(A) \big\}$$

for all $A \subseteq X$.

Remark 14 Thus, we may also naturally consider the ordinary relation $V^{\blacktriangleleft \triangleleft}$. In particular, we may define $\operatorname{int}_{\mathscr{R}} = \operatorname{Int}_{\mathscr{R}}^{\blacktriangleleft}$ and $\sigma_{\mathscr{R}} = \operatorname{int}_{\mathscr{R}}^{\triangleleft}$ for any relator $\mathscr{R}$.

In accordance with [113], we shall see that hyper relations can be derived from super, and thus also from ordinary relations and relators in several natural ways.

10 A Few Basic Facts on Relators

A family $\mathscr{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y, and the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. For the origins of this notion, see [85, 97], and the references in [85].

If in particular $\mathscr{R}$ is a relator on X to itself, then $\mathscr{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathscr{R})$ instead of $(X, X)(\mathscr{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [22, 109] and *uniform spaces* [30, 44]. However, they are insufficient for some important purposes. (See, [33, 97, 106, 108, 111, 118].)

A relator $\mathscr{R}$ on X to Y, or the relator space $(X, Y)(\mathscr{R})$, is called *simple* if $\mathscr{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [33] and [109], respectively.

Moreover, a relator $\mathscr{R}$ on X, or the relator space $X(\mathscr{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathscr{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance and equivalence relators*.

For instance, for a family $\mathscr{A}$ of subsets of X, the family $\mathscr{R}_{\mathscr{A}} = \{R_A : \ A \in \mathscr{A}\}$, where $R_A = A^2 \cup A^c \times X$, is an important preorder relator on X. Such relators were first used by Pervin [70] and Levine [55].

While, for a family $\mathscr{D}$ of *pseudo-metrics* on X, the family $\mathscr{R}_{\mathscr{D}} = \{B_r^d : \ r > 0, \ d \in \mathscr{D}\}$, where $B_r^d = \{(x, y) : \ d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [120].

Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathscr{R}_{\mathfrak{S}} = \{S_{\mathscr{A}} : \ \mathscr{A} \in \mathfrak{S}\}$, where $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$, is an important tolerance (equivalence) relator on X. Equivalence relators were first studied by Levine [54].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathscr{R}$ on X to Y we may naturally define $\mathscr{R}^{\star} = \big\{ R^{\star} : \ \ R \in \mathscr{R} \big\}$. However, this plausible notation may cause some confusions whenever, for instance, $\star = c$.

In particular, for any relator $\mathscr{R}$ on X, we may naturally define $\mathscr{R}^{\infty} = \big\{ R^{\infty} : R \in \mathscr{R} \big\}$. Moreover, we may also define $\mathscr{R}^{\partial} = \big\{ S \subseteq X^2 : \ \ S^{\infty} \in \mathscr{R} \big\}$. These operations were first introduced by Mala [56, 57] and Pataki [68, 69].

While, if $*$ is a binary operation for relations, then for any two relators $\mathscr{R}$ and $\mathscr{S}$ we may naturally define $\mathscr{R} * \mathscr{S} = \big\{ R * S : \ \ R \in \mathscr{R}, \ S \in \mathscr{S} \big\}$. However, this plausible notation may again cause some confusions whenever, for instance, $* = \cap$.

Therefore, in general we rather write $\mathscr{R} \wedge \mathscr{S} = \{ R \cap S : \ R \in \mathscr{R}, \ S \in \mathscr{S} \}$. Moreover, for instance, we also write $\mathscr{R} \triangle \mathscr{R}^{-1} = \{ R \cap R^{-1} : \ R \in \mathscr{R} \}$. Note that thus $\mathscr{R} \triangle \mathscr{R}^{-1}$ is a symmetric relator such that $\mathscr{R} \triangle \mathscr{R}^{-1} \subseteq \mathscr{R} \wedge \mathscr{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathscr{R}$ on X to Y, the value $\mathscr{R}^{\square} = \square(\mathscr{R})$ is a relator on X to Y (on Y to X).

For instance, c and -1 are *involution operations* for relators. While, ∞ and ∂ are *projection operations* for relators. Moreover, the operation $\square = c$, ∞ or ∂ is *inversion compatible* in the sense that $\mathscr{R}^{\square\,-1} = \mathscr{R}^{-1\,\square}$.

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathscr{R}$ on X to Y, the value $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}(\mathscr{R})$ is in a power set depending only on X and Y.

For instance, if $\operatorname{int}_{\mathscr{R}}(B) = \{x \in X : \ \exists\, R \in \mathscr{R} : \ R(x) \subseteq B\}$ for every relator $\mathscr{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathscr{R}) = \operatorname{int}_{\mathscr{R}}$, is a structure for relators such that $\mathfrak{F}(\mathscr{R}) \in \mathscr{P}(\mathscr{P}(Y) \times X)$.

Concerning structures and operations for relators, we can use the same terminology as in Definition 1 and Remark 2. Thus, by Theorem 3, the structure $\mathfrak{F}$ is *union-preserving* if and only if $\mathfrak{F}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathfrak{F}_R$ for every relator $\mathscr{R}$.

By using *Pataki connections* [68, 115], several closure operations can be derived from union-preserving structures. However, more generally, one can find first the *Galois adjoint* $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\square_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$ [101].

Now, for an operation $\square$ for relators, a relator $\mathscr{R}$ on X to Y may be naturally called $\square$-*fine* if $\mathscr{R}^{\square} = \mathscr{R}$. And, for some structure $\mathfrak{F}$ for relators, two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y may be naturally called $\mathfrak{F}$-*equivalent* if $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{S}}$.

Moreover, for a structure $\mathfrak{F}$ for relators, a relator $\mathscr{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$-*simple* if $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_R$ for some relation R on X to Y. Thus, in particular singleton relators have to be actually called *properly simple*.

Analogously to our former definition of an *ordinary relator* $\mathscr{R}$ on X to Y, a family $\mathscr{U}$ (resp. $\mathscr{V}$) of super relations (resp. hyper relations) on X to Y may now be naturally called a *super relator* (resp. *hyper relator*) on X to Y.

Note that thus $\mathscr{U}^{\triangleleft} = \{U^{\triangleleft} : \ U \in \mathscr{U}\}$ is a relator, while $\mathscr{R}^{\triangleright} = \{R^{\triangleright} : \ R \in \mathscr{R}\}$ is a super relator on X to Y. Thus, in addition to the *non-conventional birelator space* $(X, Y)(\mathscr{R}, \mathscr{U})$, we may also naturally consider its adjoint $(X, Y)(\mathscr{U}^{\triangleleft}, \mathscr{R}^{\triangleright})$.

Now, a super relator $\mathscr{U}$ on X to Y may, for instance, be naturally called *quasi-increasing* if each member of it is quasi-increasing. Moreover, a super relator $\mathscr{U}$ on X may for instance be called *reflexive* if the ordinary relator $\mathscr{U}^{\triangleleft}$ is reflexive.

Note that reflexivity, and several other properties need not be defined separately for hyper relators. Moreover, several structures for ordinary relators can be naturally derived from the corresponding structures for super relators.

11 Structures Derived from Super Relators

Notation 4 *In this and the next two sections, we shall assume that* $\mathscr{U}$ *is a super relator on* X *to* Y.

Definition 10 For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$ we define :

(1) $A \in \operatorname{Int}_{\mathscr{U}}(B)$ if $U(A) \subseteq B$ for some $U \in \mathscr{U}$;
(2) $A \in \operatorname{Cl}_{\mathscr{U}}(B)$ if $U(A) \cap B \neq \emptyset$ for all $U \in \mathscr{U}$;
(3) $x \in \operatorname{int}_{\mathscr{U}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathscr{U}}(B)$;
(4) $x \in \sigma_{\mathscr{U}}(y)$ if $x \in \operatorname{int}_{\mathscr{U}}\big(\{y\}\big)$;
(5) $x \in \operatorname{cl}_{\mathscr{U}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathscr{U}}(B)$;
(6) $x \in \rho_{\mathscr{U}}(y)$ if $x \in \operatorname{cl}_{\mathscr{U}}\big(\{y\}\big)$;
(7) $B \in \mathscr{E}_{\mathscr{U}}$ if $\operatorname{int}_{\mathscr{U}}(B) \neq \emptyset$;
(8) $B \in \mathscr{D}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(B) = X$.

Remark 15 The relations $\operatorname{Int}_{\mathscr{U}}$, $\operatorname{int}_{\mathscr{U}}$ and $\sigma_{\mathscr{U}}$ will be called *the proximal, topological and infinitesimal interiors* generated by $\mathscr{U}$, respectively. While, the members of the families, $\mathscr{E}_{\mathscr{U}}$ and $\mathscr{D}_{\mathscr{U}}$ will be called the *fat and dense subsets* of the super relator space $(X, Y)(\mathscr{U})$, respectively.

The origins of the relations $\operatorname{Cl}_{\mathscr{U}}$ and $\operatorname{Int}_{\mathscr{U}}$ go back to Efremović's proximity δ [27] and Smirnov's strong inclusion $\Subset$ [81], respectively. While, the convenient notations $\operatorname{Cl}_{\mathscr{U}}$ and $\operatorname{Int}_{\mathscr{U}}$, and family $\mathscr{E}_{\mathscr{U}}$, together with its dual $\mathscr{D}_{\mathscr{U}}$, was first explicitly used by the second author in [85, 89, 91, 100] for an ordinary relator.

The following theorem shows that the big closure and interior relations are equivalent tools in a super relator space.

Theorem 29 *For any* $B \subseteq Y$ *we have*

(1) $\operatorname{Cl}_{\mathscr{U}}(B) = \operatorname{Int}_{\mathscr{U}}(B^c)^c$; *(2)* $\operatorname{Int}_{\mathscr{U}}(B) = \operatorname{Cl}_{\mathscr{U}}(B^c)^c$.

Proof For any $A \subseteq X$, we have

$$A \in \operatorname{Cl}_{\mathscr{U}}(B) \iff \forall\, U \in \mathscr{U}:\ U(A) \cap B \neq \emptyset \iff$$
$$\forall\, U \in \mathscr{U}:\ U(A) \not\subseteq B^c \iff A \notin \operatorname{Int}_{\mathscr{U}}(B^c) \iff A \in \operatorname{Int}_{\mathscr{U}}(B^c)^c.$$

Therefore, assertion (1) is true. Now, assertion (2) can be derived from (1) by using complementations.

Remark 16 By using the notation $\mathscr{C}_Y(B) = B^c$, assertion (1) can be expressed in the more concise form that $\operatorname{Cl}_{\mathscr{U}} = \big(\operatorname{Int}_{\mathscr{U}} \circ \mathscr{C}_Y\big)^c$ or $\operatorname{Cl}_{\mathscr{U}} = \big(\operatorname{Int}_{\mathscr{U}}\big)^c \circ \mathscr{C}_Y$.

By using the definitions of the relations $\operatorname{Cl}_{\mathscr{U}}$ and $\operatorname{Int}_{\mathscr{U}}$, we can also easily prove the following two theorems.

Theorem 30 *We have*

(1) $\operatorname{Cl}_{\mathscr{U}}(\emptyset) = \emptyset$ *if and only if* $\mathscr{U} \neq \emptyset$ *;*
(2) $\operatorname{Cl}_{\mathscr{U}}(B_1) \subseteq \operatorname{Cl}_{\mathscr{U}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$.

Theorem 31 *We have*

(1) $\operatorname{Int}_{\mathscr{U}}(Y) = \mathscr{P}(X)$ *if and only if* $\mathscr{U} \neq \emptyset$ *;*
(2) $\operatorname{Int}_{\mathscr{U}}(B_1) \subseteq \operatorname{Int}_{\mathscr{U}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$.

Remark 17 Note that if in particular $\mathscr{U} = \emptyset$, then by the corresponding definitions we have $\operatorname{Int}_{\mathscr{U}}(B) = \emptyset$ and $\operatorname{Cl}_{\mathscr{U}}(B) = \mathscr{P}(X)$ for all $B \subseteq Y$.

Moreover, it is also worth noticing that $\operatorname{Int}_{\mathscr{U}}(\emptyset) = \emptyset$ $\big(\operatorname{Cl}_{\mathscr{U}}(Y) = \mathscr{P}(X) \big)$ if and only if $\mathscr{U}$ is *non-partial* in the sense that $U(A) \neq \emptyset$ for all $A \subseteq X$ and $U \in \mathscr{U}$.

Now, as an immediate consequence of the increasingness of the relations $\operatorname{Cl}_{\mathscr{U}}$ and $\operatorname{Int}_{\mathscr{U}}$, we can also state

Corollary 10 *The relations* $\operatorname{Cl}_{\mathscr{U}}^{-1}$ *and* $\operatorname{Int}_{\mathscr{U}}^{-1}$ *are ascending-valued.*

Moreover, in addition to Theorems 30 and 31, we can also easily prove

Theorem 32 *For any super relation* U *on* X *to* Y*,*

(1) Cl_U *is union-preserving ;* *(2)* Int_U *is intersection-preserving.*

Proof To prove (2), note that if $\mathscr{B} \subseteq \mathscr{P}(Y)$, then by the $\mathscr{U} = \{U\}$ particular case of Theorem 31 we have $\operatorname{Int}_U \big(\bigcap \mathscr{B} \big) \subseteq \operatorname{Int}_U(B)$ for all $B \in \mathscr{B}$. Therefore,

$$\operatorname{Int}_U \Big(\bigcap \mathscr{B} \Big) \subseteq \bigcap\nolimits_{B \in \mathscr{B}} \operatorname{Int}_U(B) .$$

Moreover, if $A \in \bigcap_{B \in \mathscr{B}} \operatorname{Int}_U(B)$, then $A \in \operatorname{Int}_U(B)$, and thus $U(A) \subseteq B$ for all $B \in \mathscr{B}$. Therefore, $U(A) \subseteq \bigcap \mathscr{B}$, and thus $A \in \operatorname{Int}_U \big(\bigcap \mathscr{B} \big)$. Therefore,

$$\bigcap\nolimits_{B \in \mathscr{B}} \operatorname{Int}_U(B) \subseteq \operatorname{Int}_U \Big(\bigcap \mathscr{B} \Big) ,$$

and thus the corresponding equality is also true.

Because of the definitions of the relations $\operatorname{Cl}_{\mathscr{U}}$ and $\operatorname{Int}_{\mathscr{U}}$, we can at once state

Theorem 33 *We have*

(1) $\operatorname{Cl}_{\mathscr{U}} = \bigcap_{U \in \mathscr{U}} \operatorname{Cl}_U$ *;* *(2)* $\operatorname{Int}_{\mathscr{U}} = \bigcup_{U \in \mathscr{U}} \operatorname{Int}_U$.

Corollary 11 *The mapping*

(1) $\mathscr{U} \mapsto \operatorname{Int}_{\mathscr{U}}$ *is union-preserving ;*
(2) $\mathscr{U} \mapsto \operatorname{Cl}_{\mathscr{U}}$ *is intersection-preserving.*

Proof Assertion (1) can be derived from assertion (2) of Theorem 33, by using Theorem 3. While, assertion (2) can be derived from assertion (1) by using Theorem 29.

Now, by calling the super relator $\mathcal{U}$ *quasi-increasing* if each of its member is quasi-increasing, we can also establish some important relationships between small and big closures and interiors.

Theorem 34 *If $\mathcal{U}$ is quasi-increasing, then for any $A \subseteq X$ and $B \subseteq Y$*

(1) $A \in \operatorname{Int}_{\mathcal{U}}(B)$ implies $A \subseteq \operatorname{int}_{\mathcal{U}}(B)$;
(2) $A \cap \operatorname{cl}_{\mathcal{U}}(B) \neq \emptyset$ implies $A \in \operatorname{Cl}_{\mathcal{U}}(B)$.

Proof For instance, if $A \in \operatorname{Int}_{\mathcal{U}}(B)$, then there exists $U \in \mathcal{U}$ such that $U(A) \subseteq B$. Thus, since U is quasi-increasing, we have $U(\{x\}) \subseteq U(A) \subseteq B$ for all $x \in A$. Hence, we can already see that $\{x\} \in \operatorname{Int}_{\mathcal{U}}(B)$, and thus also $x \in \operatorname{int}_{\mathcal{U}}(B)$ for all $x \in A$. Therefore, $A \subseteq \operatorname{int}_{\mathcal{U}}(B)$ also holds.

In addition to this theorem, it is also worth proving the following

Theorem 35 *For any union-preserving super relation U on X to Y, we have*

(1) $\operatorname{Int}_U(B) = \mathcal{P}\big(\operatorname{int}_U(B)\big)$; *(2) $\operatorname{Cl}_U(B) = \mathcal{P}\big(\operatorname{cl}_U(B)^c\big)^c$.*

Proof To prove (1), note that if $A \in \mathcal{P}\big(\operatorname{int}_U(B)\big)$, then $A \subseteq \operatorname{int}_U(B)$. Therefore, $x \in \operatorname{int}_U(B)$, and thus $\{x\} \in \operatorname{Int}_U(B)$ for all $x \in A$. This implies that $U(\{x\}) \subseteq A$ for all $x \in A$.

Moreover, since U is union-preserving, $U(A) = \bigcup_{x \in A} U(\{x\})$. Therefore, $U(A) \subseteq A$, and thus $A \in \operatorname{Int}_U(B)$. This proves that $\mathcal{P}\big(\operatorname{int}_U(B)\big) \subseteq \operatorname{Int}_U(B)$. The converse inclusion is immediate from Theorem 34.

While, to prove (2), note that, for any $A \subseteq X$, we have

$$\begin{aligned} A \cap \operatorname{cl}_U(B) \neq \emptyset \iff & A \not\subseteq \operatorname{cl}_U(B)^c \\ \iff & A \notin \mathcal{P}\big(\operatorname{cl}_U(B)^c\big) \iff A \in \mathcal{P}\big(\operatorname{cl}_U(B)^c\big)^c. \end{aligned}$$

Remark 18 Theorems 32 and 35 can be generalized by calling the super relator $\mathcal{U}$ *proximally simple* if $\operatorname{Cl}_{\mathcal{U}} = \operatorname{Cl}_U$ for some super relation U.

12 Basic Theorems on the Small Closure and Interior

By using Definition 10 and the corresponding results of Sect. 11, we can easily establish the following theorems.

Theorem 36 *For any $x \in X$ and $B \subseteq Y$, we have*

(1) $x \in \operatorname{int}_{\mathcal{U}}(B)$ if and only if $U^{\triangleleft}(x) \subseteq B$ for some $U \in \mathcal{U}$;
(2) $x \in \operatorname{cl}_{\mathcal{U}}(B)$ if and only if $U^{\triangleleft}(x) \cap B \neq \emptyset$ for all $U \in \mathcal{U}$.

Proof To prove (1), note that, by Definitions 10 and 2, for any $x \in X$ we have

$$x \in \operatorname{int}_{\mathcal{U}}(B) \iff \{x\} \in \operatorname{Int}_{\mathcal{U}}(B) \iff \exists\, U \in \mathcal{U}: \ U\big(\{x\}\big) \subseteq B \iff U^{\triangleleft}(x) \subseteq B\,.$$

Theorem 37 *For any* $B \subseteq Y$, *we have*

(1) $\mathrm{cl}_{\mathcal{U}}(B) = \mathrm{int}_{\mathcal{U}}(B^c)^c$ *;* *(2)* $\mathrm{int}_{\mathcal{U}}(B) = \mathrm{cl}_{\mathcal{U}}(B^c)^c$.

Remark 19 By using the notations

$$B^- = \mathrm{cl}_{\mathcal{U}}(B) \qquad \text{and} \qquad B^\circ = \mathrm{int}_{\mathcal{U}}(B),$$

assertion (1) can be expressed in the concise form that $- = c \circ c$ or $-c = c\,\circ$.

Theorem 38 *We have*

(1) $\mathrm{cl}_{\mathcal{U}}(B_1) \subseteq \mathrm{cl}_{\mathcal{U}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$*;*
(2) $\mathrm{cl}_{\mathcal{U}}(\emptyset) = \emptyset$ *if and only if either* $X = \emptyset$ *or* $\mathcal{U} \neq \emptyset$.

Theorem 39 *We have*

(1) $\mathrm{int}_{\mathcal{U}}(B_1) \subseteq \mathrm{int}_{\mathcal{U}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$*;*
(2) $\mathrm{int}_{\mathcal{U}}(Y) = X$ *if and only if either* $X = \emptyset$ *or* $\mathcal{U} \neq \emptyset$.

Remark 20 Note that if in particular $\mathcal{U} = \emptyset$, then by Remark 17 we have $\mathrm{int}_{\mathcal{U}}(B) = \emptyset$ and $\mathrm{cl}_{\mathcal{U}}(B) = X$ for all $B \subseteq Y$.

Moreover, it is also worth noticing that $\mathrm{int}_{\mathcal{U}}(\emptyset) = \emptyset$ $\big(\mathrm{cl}_{\mathcal{U}}(Y) = X\big)$ if and only if $\mathcal{U}$ is *quasi-non-partial* in the sense that $U^{\triangleleft}(x) \neq \emptyset$ for all $x \in X$ and $U \in \mathcal{U}$.

Theorem 40 *If* $A, B \subseteq Y$ *such that* $A \cap B = \emptyset$, *then*

$$\mathrm{cl}_{\mathcal{U}}(A) \cap \mathrm{int}_{\mathcal{U}}(B) = \emptyset.$$

Proof By using Theorems 38 and 37, we can see that

$$A \cap B = \emptyset \implies A \subseteq B^c \implies A^- \subseteq B^{c\,-} \implies A^- \subseteq B^{\circ\,c} \implies A^- \cap B^\circ = \emptyset.$$

Theorem 41 *For any super relation* U *on* X *to* Y,

(1) cl_U *is union-preserving ;* *(2)* int_U *is intersection-preserving.*

Theorem 42 *We have*

(1) $\mathrm{cl}_{\mathcal{U}} = \bigcap_{U \in \mathcal{U}} \mathrm{cl}_U$ *;* *(2)* $\mathrm{int}_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} \mathrm{int}_U$.

Corollary 12 *The mapping*

(1) $\mathcal{U} \mapsto \mathrm{int}_{\mathcal{U}}$ *is union-preserving;*
(2) $\mathcal{U} \mapsto \mathrm{cl}_{\mathcal{U}}$ *is intersection-preserving.*

Theorem 43 *For any* $B \subseteq Y$, *we have*

(1) $\mathrm{cl}_{\mathcal{U}}(B) = \bigcap_{U \in \mathcal{U}} U^{-1}(B)$ *;*
(2) $\mathrm{int}_{\mathcal{U}}(B) = \bigcup_{U \in \mathcal{U}} U^{-1\star}(B)$.

Proof By Theorems 36 and 18, for any $x \in X$ we have

$$x \in \mathrm{cl}_{\mathcal{U}}(B) \iff \iff \forall\ U \in \mathcal{U}: \ \ U\big(\{x\}\big) \cap B \neq \emptyset$$
$$\iff \forall\ U \in \mathcal{U}: \ \ x \in U^{-1}(B) \iff x \in \bigcap\nolimits_{U \in \mathcal{U}} U^{-1}(B).$$

Therefore, assertion (1) is true.

From assertion (1), by using Theorem 37 and Definition 7, we can see that assertion (2) is also true.

From the above theorem, by taking $\{y\}$ in place of B, we can derive

Corollary 13 *We have*

$$\rho_{\mathcal{U}} = \bigcap \mathcal{U}^{\triangleleft\,-1} = \big(\bigcap \mathcal{U}^{\triangleleft} \big)^{-1}.$$

By using Definition 10, we may also introduce some notions of *the boundary, residue and border of a set*. (See Kuratowski [47] and Elez and Papaz [29] for some similar definitions.)

Definition 11 For any $B \subseteq Y$, we define

(1) $\mathrm{bnd}_{\mathcal{U}}(B) = \mathrm{cl}_{\mathcal{U}}(B) \setminus \mathrm{int}_{\mathcal{U}}(B)$.

Moreover, if in particular $X = Y$, then for any $A \subseteq X$ we also define

(2) $\mathrm{res}_{\mathcal{U}}(A) = \mathrm{cl}_{\mathcal{U}}(A) \setminus A$; (3) $\mathrm{bor}_{\mathcal{U}}(A) = A \setminus \mathrm{int}_{\mathcal{U}}(A)$.

Remark 21 If in particular $\mathcal{U}$ is a *reflexive* in the sense that $x \in U^{\triangleleft}(x)$ for all $x \in X$ and $U \in \mathcal{U}$, then for any $A \subseteq X$ we have $\mathrm{int}_{\mathcal{U}}(A) \subseteq A \subseteq \mathrm{cl}_{\mathcal{U}}(A)$.

Therefore,

$$\mathrm{bnd}_{\mathcal{U}}(A) = \mathrm{res}_{\mathcal{U}}(A) \cup \mathrm{bor}_{\mathcal{U}}(A) = \mathrm{res}_{\mathcal{U}}(A) \cup \mathrm{res}_{\mathcal{U}}(A^{c}).$$

Namely, by using Definition 11 and Theorem 37, we can easily see that

$$\mathrm{res}_{\mathcal{U}}(A^{c}) = A^{c\,-} \setminus A^{c} = A^{c\,-} \cap A^{c\,c} = A^{\circ\,c} \cap A = A \setminus A^{\circ} = \mathrm{bor}_{\mathcal{U}}(A)\,.$$

Note that if $A \in \mathcal{T}_{\mathcal{U}}$ in the sense that $A \subseteq \mathrm{int}_{\mathcal{U}}(A)$, then $\mathrm{bor}_{\mathcal{U}}(A) = \emptyset$. Therefore, in this particular case, by the above equality, we can simply state that $\mathrm{bnd}_{\mathcal{U}}(A) = \mathrm{res}_{\mathcal{U}}(A)$.

13 Basic Theorems on Fat and Dense Sets

By using Definition 10 and the results of Sect. 12, we can easily establish the following theorems.

Theorem 44 *For any* $B \subseteq Y$, *we have*

(1) $B \in \mathscr{E}_{\mathscr{U}}$ *if and only if* $U^{\lhd}(x) \subseteq B$ *for some* $x \in X$ *and* $U \in \mathscr{U}$*;*
(2) $B \in \mathscr{D}_{\mathscr{U}}$ *if and only if* $U^{\lhd}(x) \cap B \neq \emptyset$ *for all* $x \in X$ *and* $U \in \mathscr{U}$*.*

Theorem 45 *For any* $B \subseteq Y$, *we have*

(1) $B \in \mathscr{D}_{\mathscr{U}}$ *if and only if* $X = U^{\lhd -1}[B]$ *for all* $U \in \mathscr{U}$*;*
(2) $B \in \mathscr{E}_{\mathscr{U}}$ *if and only if* $X \neq U^{\lhd -1}[B^c]$ *for some* $U \in \mathscr{U}$*.*

Theorem 46 *For any* $B \subseteq Y$ *we have*

(1) $B \in \mathscr{D}_{\mathscr{U}} \iff B^c \notin \mathscr{E}_{\mathscr{U}}$*;*
(2) $B \in \mathscr{E}_{\mathscr{U}} \iff B^c \notin \mathscr{D}_{\mathscr{U}}$*.*

Proof To prove (1), note that, by Definition 10 and Theorem 37, we have

$$B \in \mathscr{D}_{\mathscr{U}} \iff \operatorname{cl}_{\mathscr{U}}(B) = X \iff \operatorname{int}_{\mathscr{U}}(B^c)^c = X$$
$$\iff \operatorname{int}_{\mathscr{U}}(B^c) = \emptyset \iff B^c \notin \mathscr{E}_{\mathscr{U}}.$$

Remark 22 By Theorems 44 and 46, we can at once see that $U^{\lhd}(x) \in \mathscr{E}_{\mathscr{U}}$, and thus $U^{\lhd c}(x) = U^{\lhd}(x)^c \notin \mathscr{D}_{\mathscr{U}}$ for all $x \in X$ and $U \in \mathscr{U}$.

Theorem 47 *For any* $B \subseteq Y$ *we have*

(1) $B \in \mathscr{D}_{\mathscr{U}}$ *if and only if* $B \cap E \neq \emptyset$ *for all* $E \in \mathscr{E}_{\mathscr{U}}$*;*
(2) $B \in \mathscr{E}_{\mathscr{U}}$ *if and only if* $B \cap D \neq \emptyset$ *for all* $D \in \mathscr{D}_{\mathscr{U}}$*.*

Proof This theorem can, in principle, be derived from Theorem 46. However, it can be more easily proved with the help of Theorem 44.

Namely, if for instance $B \in \mathscr{D}_{\mathscr{U}}$, then for any $x \in X$ and $U \in \mathscr{U}$ we have $U^{\lhd}(x) \cap B \neq \emptyset$ Moreover, if $E \in \mathscr{E}_{\mathscr{U}}$, then there exists $x_0 \in X$ and $U_0 \in U$ such that $U_0^{\lhd}(x_0) \subseteq E$. Therefore, $\emptyset \neq U_0^{\lhd}(x_0) \cap B \subseteq E \cap B$, and thus $B \cap E \neq \emptyset$.

Theorem 48 *We have*

(1) $\emptyset \notin \mathscr{D}_{\mathscr{U}}$ *if and only if* $X \neq \emptyset$ *and* $\mathscr{U} \neq \emptyset$*;*
(2) $B \in \mathscr{D}_{\mathscr{U}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathscr{D}_{\mathscr{U}}$*.*

Theorem 49 *We have*

(1) $Y \in \mathscr{E}_{\mathscr{U}}$ *if and only if* $X \neq \emptyset$ *and* $\mathscr{U} \neq \emptyset$*;*
(2) $B \in \mathscr{E}_{\mathscr{U}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathscr{E}_{\mathscr{U}}$*.*

Remark 23 If $\mathscr{U}$ is super relator on X to Y such that $X \neq \emptyset$ and $\mathscr{U} \neq \emptyset$, then we shall say that $\mathscr{U}$ is a *non-degenerated.*

In addition to Theorems 48 and 49, it is also worth noticing that, by Remark 20, we have $\emptyset \notin \mathscr{E}_{\mathscr{U}}$ $\big(Y \in \mathscr{D}_{\mathscr{U}}\big)$ if and only if $\mathscr{U}$ is quasi-non-partial.

Moreover, it is also worth mentioning that if U is a super relation on X to Y, then the *stack* $\mathscr{E}_U$ has a base $\mathscr{B}$ with $\operatorname{card}(\mathscr{B}) \leq \operatorname{card}(X)$. (See Pataki [67].)

Theorem 50 *We have*

(1) $\mathscr{E}_{\mathscr{U}} = \bigcup_{U \in \mathscr{U}} \mathscr{E}_U$; *(2)* $\mathscr{D}_{\mathscr{U}} = \bigcap_{U \in \mathscr{U}} \mathscr{D}_U$.

Corollary 14 *The mapping*

(1) $\mathscr{R} \mapsto \mathscr{E}_{\mathscr{R}}$ *is union-preserving;*
(2) $\mathscr{R} \mapsto \mathscr{D}_{\mathscr{R}}$ *is intersection-preserving.*

Remark 24 Finally, we note that, by using the notation

$$\mathfrak{U}_{\mathscr{U}}(x) = \operatorname{int}_{\mathscr{U}}^{-1}(x) = \{ B \subseteq Y : \quad x \in \operatorname{int}_{\mathscr{U}}(B) \},$$

we can also prove that $\mathscr{E}_{\mathscr{U}} = \bigcup_{x \in X} \mathfrak{U}_{\mathscr{U}}(x)$.

14 Further Structures Derived from Super Relators

Notation 5 *In this and the next section, we shall already assume that* $\mathscr{U}$ *is a super relator on* X.

By using Definition 10, we may also naturally introduce the following

Definition 12 For any $A \subseteq X$, we define:

(1) $A \in \tau_{\mathscr{U}}$ if $A \in \operatorname{Int}_{\mathscr{U}}(A)$; (2) $A \in \tau\!\!\!-_{\mathscr{U}}$ if $A^c \notin \operatorname{Cl}_{\mathscr{U}}(A)$;
(3) $A \in \mathscr{T}_{\mathscr{U}}$ if $A \subseteq \operatorname{int}_{\mathscr{U}}(A)$; (4) $A \in \mathscr{F}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(A) \subseteq A$;
(5) $A \in \mathscr{N}_{\mathscr{U}}$ if $\operatorname{cl}_{\mathscr{U}}(A) \notin \mathscr{E}_{\mathscr{U}}$; (6) $A \in \mathscr{M}_{\mathscr{U}}$ if $\operatorname{int}_{\mathscr{U}}(A) \in \mathscr{D}_{\mathscr{U}}$.

Remark 25 The members of the families, $\tau_{\mathscr{U}}$ and $\mathscr{T}_{\mathscr{U}}$ and $\mathscr{N}_{\mathscr{U}}$ will be called the *proximally open, topologically open and rare (or nowhere dense) subsets* of the super relator space $X(\mathscr{U})$, respectively.

The family $\tau_{\mathscr{U}}$ was introduced by the second author in [89, 91] for an ordinary relator $\mathscr{U}$. While, the notation $\tau\!\!\!-_{\mathscr{U}}$ was suggested by János Kurdics who first noticed that connectedness is a particular case of well-chainedness [49, 50].

Analogously to well-chainedness and connectedness [69, 78], convergence and continuity [85, 110], completeness and compactness [93, 96], Lebesgue and Baire properties [87, 99], can also be most nicely treated in relator spaces.

By using the corresponding definitions and results of Sects. 11–13, we can easily establish the following theorems.

Theorem 51 *For any* $A \subseteq X$, *we have*

(1) $A \in \tau_{\mathscr{U}}$ *if and only if* $U(A) \subseteq A$ *for some* $U \in \mathscr{U}$; *(2)* $A \in \tau\!\!\!-_{\mathscr{U}}$ *if and only if* $A \cap U(A^c) = \emptyset$ *for some* $U \in \mathscr{U}$.

Theorem 52 *For any* $A \subseteq X$, *we have*

(1) $A \in \tau\!\!\!-_{\mathscr{U}} \iff A^c \in \tau_{\mathscr{U}}$; *(2)* $A \in \tau_{\mathscr{U}} \iff A^c \in \tau\!\!\!-_{\mathscr{U}}$.

Proof To prove (1), note that, by Definition 12 and Theorem 29, we have

$$A \in \tau\!\!\!-_{\mathscr{U}} \iff A^c \notin \mathrm{Cl}_{\mathscr{U}}(A) \iff A^c \notin \mathrm{Int}_{\mathscr{U}}(A^c)^c$$
$$\iff A^c \in \mathrm{Int}_{\mathscr{U}}(A^c) \iff A^c \in \tau_{\mathscr{U}}.$$

Theorem 53 *The following assertions are equivalent:*

(1) $\emptyset \in \tau\!\!\!-_{\mathscr{U}}$; *(2)* $X \in \tau_{\mathscr{U}}$; *(3)* $\mathscr{U} \neq \emptyset$; *(4)* $\tau_{\mathscr{U}} \neq \emptyset$; *(5)* $\tau\!\!\!-_{\mathscr{U}} \neq \emptyset$.

Theorem 54 *The following assertions are equivalent:*

(1) $\emptyset \in \tau_{\mathscr{U}}$; *(2)* $X \in \tau\!\!\!-_{\mathscr{U}}$; *(3)* $U(\emptyset) = \emptyset$ *for some* $U \in \mathscr{U}$.

Theorem 55 *We have*

(1) $\tau_{\mathscr{U}} = \bigcup_{U \in \mathscr{U}} \tau_U$; *(2)* $\tau\!\!\!-_{\mathscr{U}} = \bigcup_{U \in \mathscr{U}} \tau\!\!\!-_U$.

Corollary 15 *The mappings*

$$\mathscr{U} \mapsto \tau_{\mathscr{U}} \qquad \text{and} \qquad \mathscr{U} \mapsto \tau\!\!\!-_{\mathscr{U}}$$

are union-preserving.

Theorem 56 *If* $\mathscr{U}$ *is quasi-increasing, then*

(1) $\tau_{\mathscr{U}} \subseteq \mathscr{T}_{\mathscr{U}}$; *(2)* $\tau\!\!\!-_{\mathscr{U}} \subseteq \mathscr{F}_{\mathscr{U}}$.

Proof To prove (1), note that if $A \in \tau_{\mathscr{U}}$, then by Definition 12 we have $A \in \mathrm{Int}_{\mathscr{U}}(A)$. Hence, by using Theorem 34, we can infer that $A \subseteq \mathrm{int}_{\mathscr{U}}(A)$. Thus, by Definition 12, we also have $A \in \mathscr{T}_{\mathscr{U}}$.

Corollary 16 *If* U *is a union-preserving super relation on* X*, then*

(1) $\tau_U = \mathscr{T}_U$; *(2)* $\tau\!\!\!-_U = \mathscr{F}_U$.

Proof To prove the inclusion $\mathscr{T}_U \subseteq \tau_U$, note that if $A \in \mathscr{T}_{\mathscr{U}}$, then by Definition 12 we have $A \subseteq \mathrm{int}_U(A)$. Hence, by using Theorem 35, we can infer that $A \in \mathrm{Int}_U(A)$. Thus, by Definition 12, we also have $A \in \tau_U$.

In addition to this corollary, it is also worth proving the following

Theorem 57 *If* U *is an increasing super relation on* X*, then*

(1) $\tau\!\!\!-_U$ *is closed under arbitrary unions;*
(2) τ_U *is closed under arbitrary intersections.*

Proof If $\mathscr{A} \subseteq \tau_U$, then for any $A \in \mathscr{A}$ we have $A \in \tau_U$. Therefore, by Theorem 51, we have $U(A) \subseteq A$.

Hence, by using the increasingness of U, we can see that $U\left(\bigcap \mathscr{A}\right) \subseteq U(A) \subseteq A$ for all $A \in \mathscr{A}$, and thus $U\left(\bigcap \mathscr{A}\right) \subseteq \bigcap \mathscr{A}$. Therefore, by Theorem 51, we also have $\bigcap \mathscr{A} \in \tau_U$.

This proves assertion (2). Hence, by Theorem 52, assertion (1) follows.

Now as a useful consequence of the latter results, we can also state

Corollary 17 *If U is a union-preserving super relation on X, then the families τ_U and $\tau\!\!\!-_U$ are closed under arbitrary unions and intersections.*

Proof To prove the stated properties of τ_U, note that by Theorem 57 the family τ_U is closed under arbitrary intersections.

Moreover, by using Theorem 51, we can easily see that τ_U is also closed under arbitrary unions.

Remark 26 Thus, if U is a union-preserving super relation on X, then the families τ_U and $\tau\!\!\!-_U$ are *Alexandrov topologies* on X [4, 82].

15 Basic Theorems on Topologically Open Sets

Now, by using Definition 12 and the corresponding results of Sects. 12 and 13, we can easily establish the following theorems.

Theorem 58 *For any $A \subseteq X$, we have*

(1) $A \in \mathscr{T}_{\mathscr{U}}$ if and only if for each $x \in A$ there exists $U \in \mathscr{U}$ such that $U^{\triangleleft}(x) \subseteq A$;
(2) $A \in \mathscr{F}_{\mathscr{U}}$ if and only if for each $x \in A^c$ there exists $U \in \mathscr{U}$ such that $A \cap U^{\triangleleft}(x) = \emptyset$.

Theorem 59 *For any $A \subseteq X$, we have*

(1) $A \in \mathscr{F}_{\mathscr{R}} \iff A^c \in \mathscr{T}_{\mathscr{R}}$; (2) $A \in \mathscr{T}_{\mathscr{R}} \iff A^c \in \mathscr{F}_{\mathscr{R}}$.

Theorem 60 *If $A \subseteq X$ and $B \in \mathscr{T}_{\mathscr{U}}$ such that $A \cap B = \emptyset$, then*

$$\mathrm{cl}_{\mathscr{U}}(A) \cap B = \emptyset.$$

Proof If $A \cap B = \emptyset$, then Theorem 40 we have $A^- \cap B^\circ = \emptyset$. Hence, by Definition 12, we can see that $A^- \cap B = \emptyset$ also holds.

Remark 27 If $\mathscr{U}$ is reflexive, then $A \subseteq A^-$. Therefore, $A^- \cap B = \emptyset$ also implies $A \cap B = \emptyset$.

Theorem 61 *We have*

(1) $\mathscr{A} \subseteq \mathscr{F}_{\mathscr{U}}$ implies $\bigcap \mathscr{A} \in \mathscr{F}_{\mathscr{U}}$;
(2) $\emptyset \in \mathscr{F}_{\mathscr{U}}$ if and only if either $X = \emptyset$ or $\mathscr{U} \neq \emptyset$.

Theorem 62 *We have*

(1) $\mathscr{A} \subseteq \mathscr{T}_{\mathscr{U}}$ implies $\bigcup \mathscr{A} \in \mathscr{T}_{\mathscr{U}}$;
(2) $X \in \mathscr{T}_{\mathscr{U}}$ if and only if either $X = \emptyset$ or $\mathscr{U} \neq \emptyset$.

Remark 28 From the $\mathscr{A} = \emptyset$ particular cases of the latter two theorems, we can also see that $\emptyset \in \mathscr{T}_{\mathscr{U}}$ and $X \in \mathscr{F}_{\mathscr{U}}$ are always true.

Now, in contrast to Theorems 33, 42, 50, and 55, we can only prove

Theorem 63 *The mappings*

$$\mathscr{U} \mapsto \mathscr{T}_{\mathscr{U}} \qquad \text{and} \qquad \mathscr{U} \mapsto \mathscr{F}_{\mathscr{U}}$$

are increasing.

Corollary 18 *We have*

(1) $\bigcup_{U \in \mathscr{U}} \mathscr{T}_U \subseteq \mathscr{T}_{\mathscr{U}}$; *(2)* $\bigcup_{U \in \mathscr{U}} \mathscr{F}_U \subseteq \mathscr{F}_{\mathscr{U}}$.

The following example shows that the corresponding equalities need not be true.

Example 2 If $\operatorname{card}(X) > 2$ and $x_1, x_2 \in X$ such that $x_1 \neq x_2$, and

$$R_i = \{x_i\}^2 \cup \big(\{x_i\}^c\big)^2$$

for all $i = 1, 2$, then $\mathscr{R} = \{R_1, R_2\}$ is an equivalence relator on X such that for the associated super relator $\mathscr{R}^{\triangleright} = \{R_1^{\triangleright}, R_2^{\triangleright}\}$ we have

$$\{x_1, x_2\} \in \mathscr{T}_{\mathscr{R}^{\triangleright}} \setminus \big(\mathscr{T}_{R_1^{\triangleright}} \cup \mathscr{T}_{R_2^{\triangleright}}\big), \qquad \text{and thus} \qquad \mathscr{T}_{\mathscr{R}^{\triangleright}} \not\subseteq \mathscr{T}_{R_1^{\triangleright}} \cup \mathscr{T}_{R_2^{\triangleright}}.$$

By Theorem 8 and the definitions of the relations R_i, we have

$$R_1^{\triangleright\triangleleft}(x_1) = R_1(x_1) = \{x_1\},$$
$$R_1^{\triangleright\triangleleft}(x_2) = R_1(x_2) = \{x_1\}^c;$$
$$R_2^{\triangleright\triangleleft}(x_1) = R_2(x_1) = \{x_2\}^c,$$
$$R_2^{\triangleright\triangleleft}(x_2) = R_2(x_2) = \{x_2\}.$$

Hence, by Theorem 58, we can see that $\{x_1, x_2\} \in \mathscr{T}_{\mathscr{R}^{\triangleright}}$, but $\{x_1, x_2\} \notin \mathscr{T}_{\mathscr{R}^{\triangleright}{}_1}$ and $\{x_1, x_2\} \notin \mathscr{T}_{\mathscr{R}^{\triangleright}{}_2}$.

Finally, we note that the following theorems are also true.

Theorem 64 *We have*

(1) $\mathscr{T}_{\mathscr{U}} \setminus \{\emptyset\} \subseteq \mathscr{E}_{\mathscr{U}}$; *(2)* $\mathscr{D}_{\mathscr{U}} \cap \mathscr{F}_{\mathscr{U}} \subseteq \{X\}$.

Remark 29 Hence, by using global complementations, we can easily infer that $\mathscr{F}_{\mathscr{U}} \subseteq \big(\mathscr{D}_{\mathscr{U}}\big)^c \cup \{X\}$ and $\mathscr{D}_{\mathscr{U}} \subseteq \big(\mathscr{F}_{\mathscr{U}}\big)^c \cup \{X\}$.

Theorem 65 *For any* $A \subseteq X$ *we have*

(1) $A \in \mathscr{E}_{\mathscr{U}}$ *if* $V \subseteq A$ *for some* $V \in \mathscr{T}_{\mathscr{U}} \setminus \{\emptyset\}$;
(2) $A \in \mathscr{D}_{\mathscr{U}}$ *only if* $A \setminus W \neq \emptyset$ *for all* $W \in \mathscr{F}_{\mathscr{U}} \setminus \{X\}$.

Theorem 66 *For any* $A \subseteq X$, *we have*

(1) $A \in \mathscr{M}_{\mathscr{U}} \iff A^c \in \mathscr{N}_{\mathscr{U}}$;
(2) $A \in \mathscr{N}_{\mathscr{U}} \iff A^c \in \mathscr{M}_{\mathscr{U}}$.

Proof To prove (1), note that, by Definition 12 and Theorems 46 and 37,

$$A \in \mathscr{M}_{\mathscr{U}} \iff A^{\circ} \in \mathscr{D}_{\mathscr{U}} \iff A^{\circ c} \notin \mathscr{E}_{\mathscr{U}} \iff A^{c-} \notin \mathscr{E}_{\mathscr{U}} \iff A^c \in \mathscr{N}_{\mathscr{U}}.$$

16 Structures Derived from the Super Relator $\mathscr{U}^{\circ}$

From Theorem 36, by using Theorem 9, we can immediately derive

Theorem 67 *For any super relator* $\mathscr{U}$ *on* X *to* Y, *we have*

(1) $\mathrm{cl}_{\mathscr{U}^{\circ}} = \mathrm{cl}_{\mathscr{U}}$; *(2)* $\mathrm{int}_{\mathscr{U}^{\circ}} = \mathrm{int}_{\mathscr{U}}$.

Proof Note that, by Theorem 9, we have $U^{\circ \triangleleft} = U^{\triangleleft}$ for all $U \in \mathscr{U}$. Therefore, $\mathscr{U}^{\circ \triangleleft} = \mathscr{U}^{\triangleleft}$. Thus, Theorem 36 can be used to obtain the required equalities.

From this theorem, by Definition 10, it is clear that we also have

Corollary 19 *For any super relator* $\mathscr{U}$ *on* X *to* Y, *we have*

(1) $\mathscr{E}_{\mathscr{U}^{\circ}} = \mathscr{E}_{\mathscr{U}}$; *(2)* $\mathscr{D}_{\mathscr{U}^{\circ}} = \mathscr{D}_{\mathscr{U}}$.

Now, by Theorem 67 and Definition 12, we can also state the following

Corollary 20 *For any super relator* $\mathscr{U}$ *on* X, *we have*

(1) $\mathscr{T}_{\mathscr{U}^{\circ}} = \mathscr{T}_{\mathscr{U}}$; *(2)* $\mathscr{F}_{\mathscr{U}^{\circ}} = \mathscr{F}_{\mathscr{U}}$;
(3) $\mathscr{N}_{\mathscr{U}^{\circ}} = \mathscr{N}_{\mathscr{U}}$; *(4)* $\mathscr{M}_{\mathscr{U}^{\circ}} = \mathscr{M}_{\mathscr{U}}$.

The following example shows that, even for a super relation U, we may have $\mathrm{Int}_{U^{\circ}} \neq \mathrm{Int}_{U}$, and thus also $\mathrm{Cl}_{U^{\circ}} \neq \mathrm{Cl}_{U}$

Example 3 If $X = \{1, 2\}$ and U is super relation on X such that, for any $A \subseteq X$,

$$U(A) = A \quad \text{if} \quad A \neq X \qquad \text{and} \qquad U(A) = \{1\} \quad \text{if} \quad A = X,$$

then

(1) $\mathrm{Int}_U(\emptyset) = \{\emptyset\}$, $\mathrm{Int}_U(X) = \mathscr{P}(X)$,
$\mathrm{Int}_U\big(\{1\}\big) = \big\{\emptyset, \{1\}, X\big\}$, $\mathrm{Int}_U\big(\{2\}\big) = \big\{\emptyset, \{2\}\big\}$;
(2) $\mathrm{int}_U(A) = A$ for all $A \subseteq X$;
(3) $\tau_U = \mathscr{T}_{\mathscr{R}} = \mathscr{P}(X)$; (4), $\mathscr{E}_U = \mathscr{P}(X) \setminus \{\emptyset\}$; (5) $\mathscr{N}_U = \{\emptyset\}$;

and moreover

(6) $U^{\triangleleft}(x) = \{x\}$ for all $x \in X$; (7) $U^{\circ}(A) = A$ for all $A \subseteq X$;
(8) $\mathrm{Int}_{U^{\circ}}(A) = \mathscr{P}(A)$ for all $A \subseteq X$; (9) $\mathrm{int}_{U^{\circ}}(A) = A$ for all $A \subseteq X$;
(10) $\tau_{U^{\circ}} = \mathscr{T}_{U^{\circ}} = \mathscr{P}(X)$; (11) $\mathscr{E}_{U^{\circ}} = \mathscr{P}(X) \setminus \{\emptyset\}$; (12) $\mathscr{N}_{\mathscr{U}^{\circ}} = \{\emptyset\}$.

Remark 30 Concerning the above super relation U, it is also noteworthy that

(13) $\operatorname{Int}_U\big(\{1\}\big) \neq \operatorname{Int}_{\mathscr{V}}\big(\{1\}\big)$ for any quasi-increasing super relator $\mathscr{V}$ on X.

Namely, if $\mathscr{V}$ is a super relator on X such that $\operatorname{Int}_U\big(\{1\}\big) \subseteq \operatorname{Int}_{\mathscr{V}}\big(\{1\}\big)$, then because of $X \in \operatorname{Int}_U\big(\{1\}\big)$, we also have $X \in \operatorname{Int}_{\mathscr{V}}\big(\{1\}\big)$. Thus, there exists $V \in \mathscr{V}$ such that $V(X) \subseteq \{1\}$. Hence, if V is quasi-increasing, we can infer that $V\big(\{2\}\big) \subseteq \{1\}$, and thus $\{2\} \in \operatorname{Int}_{\mathscr{V}}\big(\{1\}\big)$ also holds. Therefore, $\operatorname{Int}_{\mathscr{V}}\big(\{1\}\big) \not\subseteq \operatorname{Int}_U\big(\{1\}\big)$.

Note that if in particular $\mathscr{R}$ is an ordinary relator on X, then by Theorem 8 the associated super relator $\mathscr{R}^{\triangleright} = \{R^{\triangleright} : \ R \in \mathscr{R}\}$ is union-preserving, and thus in particular it is increasing. Therefore, as an important particular case of assertion (13), we can also state that $\operatorname{Int}_U\big(\{1\}\big) \neq \operatorname{Int}_{\mathscr{R}^{\triangleright}}\big(\{1\}\big)$, and thus $\operatorname{Int}_U \neq \operatorname{Int}_{\mathscr{R}^{\triangleright}}$.

This shows that super relators are, in general, more powerful tools than ordinary relators.

17 Structures Derived from Ordinary Relators

Notation 6 *In this and the next two sections, we shall assume that $\mathscr{R}$ is an ordinary relator and $\mathscr{U}$ is a super relator on X to Y.*

By Theorem 8, the family $\mathscr{R}^{\triangleright} = \{R^{\triangleright} : \ R \in \mathscr{R}\}$ is a union-preserving super relator on X. Thus, in particular, we may naturally introduce the following

Definition 13 For any structure $\mathfrak{F}$ for super relators on X to Y, we define

$$\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{R}^{\triangleright}}.$$

By the corresponding definitions, we have $\mathscr{U}^{\circ} = \mathscr{U}^{\triangleleft\triangleright} = \big(\mathscr{U}^{\triangleleft}\big)^{\triangleright}$. Therefore, by Definition 13, we can at once state the following

Theorem 68 *We have*

(1) $\operatorname{Cl}_{\mathscr{U}^{\circ}} = \operatorname{Cl}_{\mathscr{U}^{\triangleleft}}$; *(2)* $\operatorname{Int}_{\mathscr{U}^{\circ}} = \operatorname{Int}_{\mathscr{U}^{\triangleleft}}$.

Hence, by Theorem 10, it is clear that in particular we also have

Corollary 21 *If $\mathscr{U}$ is union-preserving, then*

(1) $\operatorname{Cl}_{\mathscr{U}} = \operatorname{Cl}_{\mathscr{U}^{\triangleleft}}$; *(2)* $\operatorname{Int}_{\mathscr{U}} = \operatorname{Int}_{\mathscr{U}^{\triangleleft}}$.

However, it is now more important to note that, for any $R \in \mathscr{R}$ and $A \subseteq X$, we have $R^{\triangleright}(A) = R\,[\,A\,]$. Therefore, by Definitions 10 and 13, we have

Theorem 69 *For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$ we have*

(1) $A \in \operatorname{Int}_{\mathscr{R}}(B) \iff R\,[\,A\,] \subseteq B$ *for some* $R \in \mathscr{R}$;
(2) $A \in \operatorname{Cl}_{\mathscr{R}}(B) \iff R\,[\,A\,] \cap B \neq \emptyset$ *for all* $R \in \mathscr{R}$;
(3) $x \in \operatorname{int}_{\mathscr{R}}(B) \iff \{x\} \in \operatorname{Int}_{\mathscr{R}}(B)$;

(4) $x \in \sigma_{\mathscr{R}}(y) \iff x \in \operatorname{int}_{\mathscr{R}}\big(\{y\}\big)$;
(5) $x \in \operatorname{cl}_{\mathscr{R}}(B) \iff \{x\} \in \operatorname{Cl}_{\mathscr{R}}(B)$;
(6) $x \in \rho_{\mathscr{R}}(y) \iff x \in \operatorname{cl}_{\mathscr{R}}\big(\{y\}\big)$;
(7) $B \in \mathscr{E}_{\mathscr{R}} \iff \operatorname{int}_{\mathscr{R}}(B) \neq \emptyset$;
(8) $B \in \mathscr{D}_{\mathscr{R}} \iff \operatorname{cl}_{\mathscr{R}}(B) = X$.

By using this theorem and the results of Sect. 11, we can easily establish the subsequent theorems.

Theorem 70 *For any* $B \subseteq Y$ *we have*

(1) $\operatorname{Cl}_{\mathscr{R}}(B) = \operatorname{Int}_{\mathscr{R}}(B^c)^c$; *(2)* $\operatorname{Int}_{\mathscr{R}}(B) = \operatorname{Cl}_{\mathscr{R}}(B^c)^c$.

Theorem 71 *We have*

(1) $\operatorname{Cl}_{\mathscr{R}^{-1}} = \operatorname{Cl}_{\mathscr{R}}^{-1}$; *(2)* $\operatorname{Int}_{\mathscr{R}^{-1}} = \mathscr{C}_Y \circ \operatorname{Int}_{\mathscr{R}}^{-1} \circ \mathscr{C}_X$.

Proof To prove (1), note that, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$B \in \operatorname{Cl}_{\mathscr{R}^{-1}}(A) \iff \forall\ R \in \mathscr{R}:\ R^{-1}[B] \cap A \neq \emptyset \iff$$
$$\forall\ R \in \mathscr{R}:\ B \cap R[A] \neq \emptyset \iff A \in \operatorname{Cl}_{\mathscr{R}}(B) \iff B \in \operatorname{Cl}_{\mathscr{R}}^{-1}(A).$$

Remark 31 Hence, we can see that, despite their equivalence, closures are sometimes more convenient tools than interiors.

Moreover, this theorem, together with the next two theorems, also shows that ordinary relators are less general, but more flexible tools than super relators.

Recall that, for $U \in \mathscr{U}$, instead of its ordinary inverse U^{-1}, which is not a super relation, we had to consider its relationally generated inverse $U^{\mathbf{-1}} = U^{\triangleleft -1 \triangleright}$.

Theorem 72 *We have*

(1) $\operatorname{Cl}_{\mathscr{R}}(\emptyset) = \emptyset$, *resp.* $\operatorname{Cl}_{\mathscr{R}}^{-1}(\emptyset) = \emptyset$, *if and only if* $\mathscr{R} \neq \emptyset$;
(2) $\operatorname{Cl}_{\mathscr{R}}(B_1) \subseteq \operatorname{Cl}_{\mathscr{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$ *and* $\operatorname{Cl}_{\mathscr{R}}^{-1}(A_1) \subseteq \operatorname{Cl}_{\mathscr{R}}^{-1}(A_2)$ *if* $A_1 \subseteq A_2 \subseteq X$.

Remark 32 Note that if in particular $\mathscr{R} = \emptyset$, then we have $\operatorname{Cl}_{\mathscr{R}}(B) = \mathscr{P}(X)$ for all $B \subseteq X$.

Moreover, $\operatorname{Cl}_{\mathscr{R}}(Y) = \mathscr{P}(X) \setminus \{\emptyset\}$ if and only if $\mathscr{R}$ is non-partial, i. e., $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathscr{R}$.

Theorem 73 *We have*

(1) $\operatorname{Int}_{\mathscr{R}}(Y) = \mathscr{P}(X)$, *resp.* $\operatorname{Int}_{\mathscr{R}}^{-1}(\emptyset) = \mathscr{P}(Y)$, *if and only if* $\mathscr{R} \neq \emptyset$;
(2) $\operatorname{Int}_{\mathscr{R}}(B_1) \subseteq \operatorname{Int}_{\mathscr{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$ *and* $\operatorname{Int}_{\mathscr{R}}^{-1}(A_2) \subseteq \operatorname{Int}_{\mathscr{R}}^{-1}(A_1)$ *if* $A_1 \subseteq A_2 \subseteq X$.

Remark 33 Conversely, from [91], we can see that, for any such hyper relation Int on Y to X, there exists a nonvoid relator $\mathscr{R}$ on X to Y such that $\operatorname{Int} = \operatorname{Int}_{\mathscr{R}}$. Thus, generalized proximity relations should not be studied without generalized uniformities.

Theorem 74 *For an ordinary relation R on X to Y,*

(1) Cl_R *and* Cl_R^{-1} *are union-preserving ;*
(2) Int_R *and* Int_R^{-1} *are intersection-preserving.*

Theorem 75 *We have*

(1) $\mathrm{Cl}_{\mathscr{R}} = \bigcap_{R \in \mathscr{R}} \mathrm{Cl}_R ;$ *(2)* $\mathrm{Int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathrm{Int}_R .$

Corollary 22 *The mapping*

(1) $\mathscr{R} \mapsto \mathrm{Int}_{\mathscr{R}}$ *is union-preserving;*
(2) $\mathscr{R} \mapsto \mathrm{Cl}_{\mathscr{R}}$ *is intersection-preserving.*

Theorem 76 *For any $A \subseteq X$ and $B \subseteq Y$*

(1) $A \in \mathrm{Int}_{\mathscr{R}}(B)$ *implies* $A \subseteq \mathrm{int}_{\mathscr{R}}(B) ;$
(2) $A \cap \mathrm{cl}_{\mathscr{R}}(B) \neq \emptyset$ *implies* $A \in \mathrm{Cl}_{\mathscr{R}}(B).$

Theorem 77 *For an ordinary relation R on X to Y, we have*

(1) $\mathrm{Int}_R(B) = \mathscr{P}\big(\mathrm{int}_R(B)\big) ;$ *(2)* $\mathrm{Cl}_R(B) = \mathscr{P}\big(\mathrm{cl}_R(B)^c\big)^c .$

Remark 34 Theorems 74 and 77 can be generalized by calling the ordinary relator $\mathscr{R}$ *proximally simple* if $\mathrm{Cl}_{\mathscr{R}} = \mathrm{Cl}_R$ for some ordinary relation R.

18 Further Theorems on Small Closures and Interiors

By using Definition 13 and the results of Sect. 12, we can easily establish the following theorems.

Theorem 78 *For any $x \in X$ and $B \subseteq Y$, we have*

(1) $x \in \mathrm{int}_{\mathscr{R}}(B)$ *if and only if* $R(x) \subseteq B$ *for some* $R \in \mathscr{R} ;$
(2) $x \in \mathrm{cl}_{\mathscr{R}}(B)$ *if and only if* $R(x) \cap B \neq \emptyset$ *for all* $R \in \mathscr{R}.$

Theorem 79 *For any $B \subseteq Y$ we have*

(1) $\mathrm{cl}_{\mathscr{R}}(B) = \mathrm{int}_{\mathscr{R}}(B^c)^c ;$ *(2)* $\mathrm{int}_{\mathscr{R}}(B) = \mathrm{cl}_{\mathscr{R}}(B^c)^c.$

Theorem 80 *We have*

(1) $\mathrm{cl}_{\mathscr{R}}(B_1) \subseteq \mathrm{cl}_{\mathscr{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y;$
(2) $\mathrm{cl}_{\mathscr{R}}(\emptyset) = \emptyset$ *if and only if either* $X = \emptyset$ *or* $\mathscr{R} \neq \emptyset.$

Remark 35 If in particular, $\mathscr{R} = \emptyset$, then we have $\mathrm{cl}_{\mathscr{R}}(B) = X$, and thus $\mathrm{int}_{\mathscr{R}}(B) = \emptyset$ for all $B \subseteq Y$.

Moreover, it is also worth noticing that $\mathrm{cl}_{\mathscr{R}}(Y) = X$, and thus $\mathrm{int}_{\mathscr{R}}(\emptyset) = \emptyset$ if and only if $\mathscr{R}$ is non-partial.

Theorem 81 *We have*

(1) $\operatorname{int}_{\mathscr{R}}(B_1) \subseteq \operatorname{int}_{\mathscr{R}}(B_2)$ *if* $B_1 \subseteq B_2 \subseteq Y$;
(2) $\operatorname{int}_{\mathscr{R}}(X) = X$ *if and only if either* $X = \emptyset$ *or* $\mathscr{R} \neq \emptyset$.

Remark 36 Conversely, from [91], we can see that, for any such super relation int on Y to X, there exists a nonvoid relator $\mathscr{R}$ on X to Y such that $\operatorname{int} = \operatorname{int}_{\mathscr{R}}$. Thus, generalized closure relations should not also be studied without generalized uniformities.

Theorem 82 *For an ordinary relation* R *on* X *to* Y,

(1) cl_R *is union-preserving ;* *(2)* int_R *is intersection-preserving.*

Theorem 83 *We have*

(1) $\operatorname{cl}_{\mathscr{R}} = \bigcap_{R \in \mathscr{R}} \operatorname{cl}_R$; *(2)* $\operatorname{int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \operatorname{int}_R$.

Corollary 23 *The mapping*

(1) $\mathscr{R} \mapsto \operatorname{int}_{\mathscr{R}}$ *is union-preserving;* *(2)* $\mathscr{R} \mapsto \operatorname{cl}_{\mathscr{R}}$ *is intersection-preserving.*

Theorem 84 *For any* $B \subseteq Y$, *we have*

(1) $\operatorname{cl}_{\mathscr{R}}(B) = \bigcap_{R \in \mathscr{R}} R^{-1}[\,B\,]$; *(2)* $\operatorname{int}_{\mathscr{R}}(B) = \bigcup_{R \in \mathscr{R}} R^{-1}[\,B^c\,]^c$.

Corollary 24 *We have* $\rho_{\mathscr{R}} = \bigcap \mathscr{R}^{-1} = \big(\bigcap \mathscr{R}\big)^{-1}$.

Theorem 85 *If* R *is an ordinary relation on* X *to* Y, *then for any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*

$$A \subseteq \operatorname{int}_R(B) \quad \Longleftrightarrow \quad \operatorname{cl}_{R^{-1}}(A) \subseteq B.$$

Remark 37 This shows that the mappings

$$A \mapsto \operatorname{cl}_{R^{-1}}(A) \qquad \text{and} \qquad B \mapsto \operatorname{int}_R(B)$$

establish a Galois connection between the posets $\mathscr{P}(X)$ and $\mathscr{P}(Y)$.

The above important closure-interior Galois connection, used first in [112], is not independent from the well-known upper and lower bound one [104].

Now, by Definitions 11 and 13, we can also at once state

Theorem 86 *For any* $B \subseteq Y$, *we have*

(1) $\operatorname{bnd}_{\mathscr{R}}(B) = \operatorname{cl}_{\mathscr{R}}(B) \setminus \operatorname{int}_{\mathscr{R}}(B)$.

Moreover, if in particular $X = Y$, *then we also have*

(2) $\operatorname{res}_{\mathscr{R}}(A) = \operatorname{cl}_{\mathscr{R}}(A) \setminus A$; *(3)* $\operatorname{bor}_{\mathscr{R}}(A) = A \setminus \operatorname{int}_{\mathscr{R}}(A)$.

Thus, a counterpart of Remark 21 can also be established. However, it is now more important to note that, in addition to Theorem 68, we can also prove

Theorem 87 *We have*

(1) $\mathrm{cl}_{\mathscr{U}} = \mathrm{cl}_{\mathscr{U}^{\triangleleft}} = \mathrm{cl}_{\mathscr{U}^{\circ}}$ *;* *(2)* $\mathrm{int}_{\mathscr{U}} = \mathrm{int}_{\mathscr{U}^{\triangleleft}} = \mathrm{int}_{\mathscr{U}^{\circ}}$.

Proof If $x \in X$ and $B \subseteq Y$, then by using Theorems 36 and 78 we can see that

$$x \in \mathrm{cl}_{\mathscr{U}}(B) \iff \forall\ U \in \mathscr{U}:\ U^{\triangleleft}(x) \cap B \neq \emptyset \iff x \in \mathrm{cl}_{\mathscr{U}^{\triangleleft}}(B).$$

Therefore, $\mathrm{cl}_{\mathscr{U}}(B) = \mathrm{cl}_{\mathscr{U}^{\triangleleft}}(B)$ for all $B \subseteq Y$, and thus the first part of (1) is also true. The second part of (1) follows from Theorem 68.

Hence, it is clear that in particular we can also state

Corollary 25 *We have*

(1) $\mathscr{E}_{\mathscr{U}} = \mathscr{E}_{\mathscr{U}^{\triangleleft}}$ *;* *(2)* $\mathscr{D}_{\mathscr{U}} = \mathscr{D}_{\mathscr{U}^{\triangleleft}}$.

Remark 38 Theorem 87, and its various consequences, shows that a great deal of the theory of relator spaces cannot actually be generalized by using super relators instead of the ordinary ones.

19 Further Theorems on Fat and Dense Sets

By using Definition 10 and our former theorems of closures and interiors, we can also easily establish the following theorems.

Theorem 88 *For any* $B \subseteq Y$, *we have*

(1) $B \in \mathscr{E}_{\mathscr{R}}$ *if and only if* $R(x) \subseteq B$ *for some* $x \in X$ *and* $R \in \mathscr{R}$*;*
(2) $B \in \mathscr{D}_{\mathscr{R}}$ *if and only if* $R(x) \cap B \neq \emptyset$ *for all* $x \in X$ *and* $R \in \mathscr{R}$.

Theorem 89 *For any* $B \subseteq Y$, *we have*

(1) $B \in \mathscr{D}_{\mathscr{R}}$ *if and only if* $X = R^{-1}[\,B\,]$ *for all* $R \in \mathscr{R}$*;*
(2) $B \in \mathscr{E}_{\mathscr{R}}$ *if and only if* $X \neq R^{-1}[\,B^{c}\,]$ *for some* $R \in \mathscr{R}$.

Theorem 90 *For any* $B \subseteq Y$ *we have*

(1) $B \in \mathscr{D}_{\mathscr{R}} \iff B^{c} \notin \mathscr{E}_{\mathscr{R}}$*;* *(2)* $B \in \mathscr{E}_{\mathscr{R}} \iff B^{c} \notin \mathscr{D}_{\mathscr{R}}$.

Remark 39 By Theorems 88 and 90, we can also state that $R(x) \in \mathscr{E}_{\mathscr{R}}$, and thus $R(x)^{c} \notin \mathscr{D}_{\mathscr{R}}$ for all $x \in X$ and $R \in \mathscr{R}$.

Theorem 91 *For any* $B \subseteq Y$ *we have*

(1) $B \in \mathscr{D}_{\mathscr{R}}$ *if and only if* $B \cap E \neq \emptyset$ *for all* $E \in \mathscr{E}_{\mathscr{R}}$*;*
(2) $B \in \mathscr{E}_{\mathscr{R}}$ *if and only if* $B \cap D \neq \emptyset$ *for all* $D \in \mathscr{D}_{\mathscr{R}}$.

Theorem 92 *We have*

(1) $\emptyset \notin \mathscr{D}_{\mathscr{R}}$ *if and only if* $X \neq \emptyset$ *and* $\mathscr{R} \neq \emptyset$ *;*
(2) $B \in \mathscr{D}_{\mathscr{R}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathscr{D}_{\mathscr{R}}$*.*

Remark 40 In this respect, it is also worth noticing that $Y \in \mathscr{D}_{\mathscr{R}}$, and thus $\emptyset \notin \mathscr{E}_{\mathscr{R}}$ if and only if $\mathscr{R}$ is non-partial.

Theorem 93 *We have*

(1) $Y \in \mathscr{E}_{\mathscr{R}}$ *if and only if* $X \neq \emptyset$ *and* $\mathscr{R} \neq \emptyset$ *;*
(2) $B \in \mathscr{E}_{\mathscr{R}}$ *and* $B \subseteq C \subseteq Y$ *imply* $C \in \mathscr{E}_{\mathscr{R}}$*.*

Remark 41 Conversely, from [103], we can see that if $\mathscr{A}$ is a nonvoid, ascending family of subsets of a nonvoid set X, then there exists a nonvoid, preorder relator on X such that $\mathscr{A} = \mathscr{E}_{\mathscr{R}}$. Thus, stacks should not also be studied without generalized uniformities.

Theorem 94 *We have*

(1) $\mathscr{E}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathscr{E}_R$*;* *(2)* $\mathscr{D}_{\mathscr{R}} = \bigcap_{R \in \mathscr{R}} \mathscr{D}_R$*.*

Corollary 26 *The mapping*

(1) $\mathscr{R} \mapsto \mathscr{E}_{\mathscr{R}}$ *is union-preserving;* *(2)* $\mathscr{R} \mapsto \mathscr{D}_{\mathscr{R}}$ *is intersection-preserving.*

Remark 42 Now, by Remark 24 and Definition 13, we can also state that

$$\mathfrak{U}_{\mathscr{R}}(x) = \operatorname{int}_{\mathscr{R}}^{-1}(x) = \left\{ B \subseteq Y : \ x \in \operatorname{int}_{\mathscr{R}}(B) \right\}$$

and $\mathscr{E}_{\mathscr{R}} = \bigcup_{x \in X} \mathfrak{U}_{\mathscr{R}}(x)$.

20 Further Structures Derived from Ordinary Relators

Notation 7 *In this and the next section, we shall already assume that* $\mathscr{R}$ *is an ordinary relator on* X*.*

Because of Definitions 12 and 13, we can at once state the following:

Theorem 95 *For any* $A \subseteq X$*, we have*

(1) $A \in \tau_{\mathscr{R}} \iff A \in \operatorname{Int}_{\mathscr{R}}(A)$*;* *(2)* $A \in \tau_{\mathscr{R}} \iff A^c \notin \operatorname{Cl}_{\mathscr{R}}(A)$*;*
(3) $A \in \mathscr{T}_{\mathscr{R}} \iff A \subseteq \operatorname{int}_{\mathscr{R}}(A)$*;* *(4)* $A \in \mathscr{F}_{\mathscr{R}} \iff \operatorname{cl}_{\mathscr{R}}(A) \subseteq A$*;*
(5) $A \in \mathscr{N}_{\mathscr{R}} \iff \operatorname{cl}_{\mathscr{R}}(A) \notin \mathscr{E}_{\mathscr{R}}$*;* *(6)* $A \in \mathscr{M}_{\mathscr{R}} \iff \operatorname{int}_{\mathscr{R}}(A) \in \mathscr{D}_{\mathscr{R}}$.

Now, by using the corresponding results of Sect. 14, we can also easily establish the following theorems.

Theorem 96 *For any* $A \subseteq X$, *we have*

(1) $A \in \tau_{\mathscr{R}}$ *if and only if* $R\,[\,A\,] \subseteq A$ *for some* $R \in \mathscr{R}$;
(2) $A \in \bar{\tau}_{\mathscr{R}}$ *if and only if* $A \cap R\,[\,A^c\,] = \emptyset$ *for some* $R \in \mathscr{R}$.

Theorem 97 *For any* $A \subseteq X$, *we have*

(1) $A \in \bar{\tau}_{\mathscr{R}} \iff A^c \in \tau_{\mathscr{R}}$; *(2)* $A \in \tau_{\mathscr{R}} \iff A^c \in \bar{\tau}_{\mathscr{R}}$.

Theorem 98 *We have*

(1) $\bar{\tau}_{\mathscr{R}} = \tau_{\mathscr{R}^{-1}}$; *(2)* $\tau_{\mathscr{R}} = \bar{\tau}_{\mathscr{R}^{-1}}$.

Proof To prove (1), note that, by Theorems 95, 71, and 70, for any $A \subseteq X$ we have

$$A \in \bar{\tau}_{\mathscr{R}} \iff A^c \notin \mathrm{Cl}_{\mathscr{R}}(A) \iff A \notin \mathrm{Cl}_{\mathscr{R}}^{-1}(A^c) \iff$$
$$A \notin \mathrm{Cl}_{\mathscr{R}^{-1}}(A^c) \iff A \in \mathrm{Cl}_{\mathscr{R}^{-1}}(A^c)^c \iff A \in \mathrm{Int}_{\mathscr{R}^{-1}}(A) \iff A \in \tau_{\mathscr{R}^{-1}}\,.$$

Theorem 99 *The following assertions are equivalent:*

(1) $\mathscr{R} \neq \emptyset$; *(2)* $\tau_{\mathscr{R}} \neq \emptyset$; *(3)* $\bar{\tau}_{\mathscr{R}} \neq \emptyset$;

(4) $\emptyset \in \tau_{\mathscr{R}}$; *(5)* $X \in \tau_{\mathscr{R}}$; *(6)* $\emptyset \in \bar{\tau}_{\mathscr{R}}$; *(7)* $X \in \bar{\tau}_{\mathscr{R}}$.

Remark 43 Conversely, from [103], we can see that if $\mathscr{A}$ is a family of subsets of X containing $\emptyset$ and X, then there exists a nonvoid, preorder relator $\mathscr{R}$ on X such that $\mathscr{A} = \tau_{\mathscr{R}}$. Thus, minimal structures should not also be studied without generalized uniformities.

Theorem 100 *We have*

(1) $\tau_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \tau_R$; *(2)* $\bar{\tau}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \bar{\tau}_R$.

Corollary 27 *The mappings*

$$\mathscr{R} \mapsto \tau_{\mathscr{R}} \qquad \text{and} \qquad \mathscr{R} \mapsto \bar{\tau}_{\mathscr{R}}$$

are union-preserving.

Now, by using the corresponding results of Sect. 15, we can quite similarly establish the following theorems.

Theorem 101 *For any* $A \subseteq X$, *we have*

(1) $A \in \mathscr{T}_{\mathscr{R}}$ *if and only if for each* $x \in A$ *there exists* $R \in \mathscr{R}$ *such that* $R\,(x) \subseteq A$;
(2) $A \in \mathscr{F}_{\mathscr{R}}$ *if and only if for each* $x \in A^c$ *there exists* $R \in \mathscr{R}$ *such that* $A \cap R\,(x) = \emptyset$.

Theorem 102 *For any* $A \subseteq X$, *we have*

(1) $A \in \mathscr{F}_{\mathscr{R}} \iff A^c \in \mathscr{T}_{\mathscr{R}}$; *(2)* $A \in \mathscr{T}_{\mathscr{R}} \iff A^c \in \mathscr{F}_{\mathscr{R}}$.

Theorem 103 *If* $A \subseteq X$ *and* $V \in \mathscr{T}_{\mathscr{R}}$ *such that* $A \cap V = \emptyset$, *then*

$$\mathrm{cl}_{\mathscr{R}}(A) \cap V = \emptyset.$$

Remark 44 Note that if $\mathscr{R}$ is reflexive, then $A \subseteq A^-$ for all $A \subseteq X$. Therefore, $A^- \cap V = \emptyset$ trivially implies $A \cap V = \emptyset$ for all $A, V \subseteq X$.

Theorem 104 *The following assertions hold:*

(1) $\mathscr{A} \subseteq \mathscr{F}_{\mathscr{R}}$ *implies* $\bigcap \mathscr{A} \in \mathscr{F}_{\mathscr{R}}$;
(2) $\emptyset \in \mathscr{F}_{\mathscr{R}}$ *if and only if either* $X = \emptyset$ *or* $\mathscr{R} \neq \emptyset$.

Remark 45 From the $\mathscr{A} = \emptyset$ particular case of this theorem, we can see that $X \in \mathscr{F}_{\mathscr{R}}$, and thus $\emptyset \in \mathscr{T}_{\mathscr{R}}$ are always true.

Theorem 105 *The following assertions hold:*

(1) $\mathscr{A} \subseteq \mathscr{T}_{\mathscr{R}}$ *implies* $\bigcup \mathscr{A} \in \mathscr{T}_{\mathscr{R}}$;
(2) $X \in \mathscr{T}_{\mathscr{R}}$ *if and only if either* $X = \emptyset$ *or* $\mathscr{R} \neq \emptyset$.

Remark 46 Conversely, from [103], we can see that if $\mathscr{A}$ is a family of subsets of X such that $X \in \mathscr{A}$ and $\mathscr{A}$ is closed under arbitrary unions, then there exists a nonvoid, preorder relator $\mathscr{R}$ on X such that $\mathscr{A} = \mathscr{T}_{\mathscr{R}}$. Thus, generalized topologies should not also be studied without generalized uniformities.

Theorem 106 *The mappings*

$$\mathscr{R} \mapsto \mathscr{T}_{\mathscr{R}} \qquad \text{and} \qquad \mathscr{R} \mapsto \mathscr{F}_{\mathscr{R}}$$

are increasing.

Corollary 28 *We have*

(1) $\bigcup_{R \in \mathscr{R}} \mathscr{T}_R \subseteq \mathscr{T}_{\mathscr{R}}$; *(2)* $\bigcup_{R \in \mathscr{R}} \mathscr{F}_R \subseteq \mathscr{F}_{\mathscr{R}}$.

Remark 47 From Example 2, by Definition 13, we can see that the corresponding equalities need not be true.

In view of Theorems 75, 83, 94, and 100, this is a serious disadvantage of open sets on which Topology and Analysis have been mainly based on.

21 Further Theorems on Open and Fat Sets

From the corresponding results of the previous sections, by using Definition 13, we can also easily establish the following theorems.

Theorem 107 *We have*

(1) $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Corollary 29 *For an ordinary relation* R *on* X, *we have*

(1) $\tau_R = \mathcal{T}_R$; *(2)* $\tau\!\!\!-_R = \mathcal{F}_R$.

Theorem 108 *We have*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; *(2)* $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Theorem 109 *For any* $A \subseteq X$ *we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if* $V \subseteq A$ *for some* $V \in \mathcal{T}_R \setminus \{\emptyset\}$;
(2) $A \in \mathcal{D}_{\mathcal{R}}$ *only if* $A \setminus W \neq \emptyset$ *for all* $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$.

Theorem 110 *For any* $A \subseteq X$, *we have*

(1) $A \in \mathcal{M}_{\mathcal{R}} \iff A^c \in \mathcal{N}_{\mathcal{R}}$;
(2) $A \in \mathcal{N}_{\mathcal{R}} \iff A^c \in \mathcal{M}_{\mathcal{R}}$.

Remark 48 The fat sets are frequently more convenient tools than the topologically open ones. For instance, if $\leq$ is a relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are the families of all ascending and residual subsets of the goset $X(\leq)$, respectively.

Moreover, if in particular $X = \mathbb{R}$ and $R(x) = \{x - 1\} \cup [x, +\infty[$ for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R$ is quite a large family. Namely, the supersets of each $R(x)$ are also contained in $\mathcal{E}_R$.

Advantages of fat and dense sets over the open and closed ones were first stressed in [88]. However, their importance lies mainly in the following

Definition 14 If $\mathcal{R}$ is an ordinary relator on X to Y, and φ and ψ are functions of a super relator space $\Gamma(\mathcal{W})$ to X and Y, respectively, then by using the notation

$$(\varphi \boxtimes \psi)(\gamma) = \big(\varphi(\gamma),\ \psi(\gamma)\big)$$

for all $\gamma \in \Gamma$, we may also naturally define

(1) $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi \boxtimes \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{W}}$ for all $R \in \mathcal{R}$;
(2) $\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi \boxtimes \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{W}}$ for all $R \in \mathcal{R}$.

Moreover, for any $x \in X$, we may also naturally define :

(3) $x \in \lim_{\mathcal{R}}(\psi)$ if $x_\Gamma \in \operatorname{Lim}_{\mathcal{R}}(\psi)$;
(4) $x \in \operatorname{adh}_{\mathcal{R}}(\psi)$ if $x_\Gamma \in \operatorname{Adh}_{\mathcal{R}}(\psi)$,

where x_Γ is a function of Γ to X such that $x_\Gamma(\gamma) = x$ for all $\gamma \in \Gamma$.

Remark 49 Fortunately, the small limit and adherence relations are equivalent to the small closure and interior ones.

However, the big limit and adherence relations, suggested by Efremović and švarc [28], are usually stronger tools than the big closure and interior ones.

In this respect, we shall only mention here the following

Theorem 111 *If $\mathscr{R}$ is an ordinary relator on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$ the following assertions are equivalent :*

(1) $A \in \mathrm{Cl}_{\mathscr{R}}(B)$ *;*
(2) There exist functions φ and ψ of the poset $\mathscr{R}(\supseteq)$ to A and B, respectively, such that $\varphi \in \mathrm{Lim}_{\mathscr{R}}(\psi)$ *;*
(3) There exist functions φ and ψ of a super relator space $\Gamma(\mathscr{W})$ to A and B, respectively, such that $\varphi \in \mathrm{Lim}_{\mathscr{R}}(\psi)$.

Proof For instance, if (1) holds, then for each $R \in \mathscr{R}$, we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $\varphi(R) \in A$ and $\psi(R) \in B$ such that $\psi(R) \in R\big(\varphi(R)\big)$. Hence, we can already infer that $(\varphi \boxtimes \psi)(R) = \big(\varphi(R),\ \psi(R)\big) \in R$, and thus also $R \in (\varphi \boxtimes \psi)^{-1}[R]$.

Therefore, if $R \in \mathscr{R}$, then for any $S \in \mathscr{R}$, with $R \supseteq S$, we have

$$S \in (\varphi \boxtimes \psi)^{-1}[S] \subseteq (\varphi \otimes \psi)^{-1}[R].$$

This shows that $(\varphi \boxtimes \psi)^{-1}[R]$ is a fat subset of $\mathscr{R}(\supseteq)$, and thus $\varphi \in \mathrm{Lim}_{\mathscr{R}}(\psi)$.

Remark 50 Finally, we note that if $\mathscr{R}$ is a relator on X to Y, then according to [98], for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally define

(1) $A \in \mathrm{Lb}_{\mathscr{R}}(B)$ and $B \in \mathrm{Ub}_{\mathscr{R}}(A)$ if $A \times B \subseteq R$ for some $R \in \mathscr{R}$.

And, in the $X = Y$ particular case, for any $A \subseteq X$ we may also naturally define

(2) $\mathrm{Min}_{\mathscr{R}}(A) = \mathscr{P}(A) \cap \mathrm{Lb}_{\mathscr{R}}(A)$; (3) $\mathrm{Sup}_{\mathscr{R}}(A) = \mathrm{Min}_{\mathscr{R}}\big(\mathrm{Ub}_{\mathscr{R}}(A)\big)$.

However, the above algebraic structures are not independent of the former topological ones. Namely, by using appropriate complements, it can be shown that

$$\mathrm{Lb}_{\mathscr{R}} = \mathrm{Int}_{\mathscr{R}^c} \circ \mathscr{C}_Y \qquad \text{and} \qquad \mathrm{Int}_{\mathscr{R}} = \mathrm{Lb}_{\mathscr{R}^c} \circ \mathscr{C}_Y .$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other, by the above equalities, as the exponential and the trigonometric functions are so by the Euler formulas [84, p. 227].

22 Reflexive, Non-Partial and Non-Degenerated Relators

Definition 15 An ordinary relator $\mathscr{R}$ on X is called *reflexive* if each member R of $\mathscr{R}$ is a reflexive relation on X.

Remark 51 Thus, the following assertions are equivalent :

(1) $\mathscr{R}$ is reflexive ;
(2) $x \in R(x)$ for all $x \in X$ and $R \in \mathscr{R}$;
(3) $A \subseteq R[A]$ for all $A \subseteq X$ and $R \in \mathscr{R}$.

The importance of reflexive relators is also apparent from the following two obvious theorems.

Theorem 112 *For an ordinary relator* $\mathscr{R}$ *on* X, *the following assertions are equivalent :*

(1) $\rho_{\mathscr{R}}$ *is reflexive ;* *(2)* $\mathscr{R}$ *is reflexive ;*
(3) $A \subseteq \operatorname{cl}_{\mathscr{R}}(A)$ $\big(\operatorname{int}_{\mathscr{R}}(A) \subseteq A\big)$ *for all* $A \subseteq X$.

Proof To see the equivalence of (1) and (2), recall that $\rho_{\mathscr{R}} = \big(\bigcap \mathscr{R}\big)^{-1}$.

Remark 52 Therefore, if $\mathscr{R}$ is a reflexive ordinary relator on X, then for any $A \subseteq X$ we have $A \in \mathscr{T}_{\mathscr{R}}$ $(A \in \mathscr{F}_{\mathscr{R}})$ if and only if $A = \operatorname{int}_{\mathscr{R}}(A)$ $(A = \operatorname{cl}_{\mathscr{R}}(A))$.

Theorem 113 *For an ordinary relator* $\mathscr{R}$ *on* X, *the following assertions are equivalent :*

(1) $\mathscr{R}$ *is reflexive ;*
(2) $A \in \operatorname{Int}_{\mathscr{R}}(B)$ *implies* $A \subseteq B$ *for all* $A, B \subseteq X$;
(3) $A \cap B \neq \emptyset$ *implies* $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ *for all* $A, B \subseteq X$.

Remark 53 In addition to the above two theorems, it is also worth mentioning that if $\mathscr{R}$ is a reflexive ordinary relator on X, then

(1) $\operatorname{Int}_{\mathscr{R}}$ is transitive ;
(2) $B \in \operatorname{Cl}_{\mathscr{R}}(A)$ implies $\mathscr{P}(X) = \operatorname{Cl}_{\mathscr{R}}(A)^c \cup \operatorname{Cl}_{\mathscr{R}}^{-1}(B)$;
(3) $\operatorname{int}_{\mathscr{R}}\big(\operatorname{bor}_{\mathscr{R}}(A)\big) = \emptyset$ and $\operatorname{int}_{\mathscr{R}}\big(\operatorname{res}_{\mathscr{R}}(A)\big) = \emptyset$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $\operatorname{res}_{\mathscr{R}}(A) \in \mathscr{T}_{\mathscr{R}}$ if and only if $A \in \mathscr{F}_{\mathscr{R}}$.

In contrast to the reflexivity property of an ordinary relator $\mathscr{R}$ on X, we may naturally introduce a great abundance of important symmetry and transitivity properties of $\mathscr{R}$ [86, 89, 90].

However, it is now more important to note that, analogously to Definition 15, we may also naturally introduce the following

Definition 16 An ordinary relator $\mathscr{R}$ on X to Y is called *non-partial* if each member R of $\mathscr{R}$ is a non-partial relation on X to Y.

Remark 54 Thus, the following assertions are equivalent :

(1) $\mathscr{R}$ is non-partial ;
(2) $R^{-1}[Y] = X$ for all $R \in \mathscr{R}$;
(3) $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathscr{R}$.

The importance of non-partial relators is apparent from the following

Theorem 114 *For an ordinary relator $\mathscr{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathscr{R}$ *is non-partial;*

(2) $\emptyset \notin \mathscr{E}_{\mathscr{R}}$; *(3)* $\mathscr{D}_{\mathscr{R}} \neq \emptyset$; *(4)* $Y \in \mathscr{D}_{\mathscr{R}}$; *(5)* $\mathscr{E}_{\mathscr{R}} \neq \mathscr{P}(Y)$.

Sometimes, we also need the following localized form of Definition 16.

Definition 17 An ordinary relator $\mathscr{R}$ on X is called *locally non-partial* if for each $x \in X$ there exists $R \in \mathscr{R}$ such that for any $y \in R(x)$ and $S \in \mathscr{R}$ we have $S(y) \neq \emptyset$.

Remark 55 Thus, if either $X = \emptyset$ or $\mathscr{R}$ is nonvoid and non-partial, then $\mathscr{R}$ is locally non-partial.

Moreover, by using the corresponding definitions, we can also easily prove

Theorem 115 *For an ordinary relator $\mathscr{R}$ on X, the following assertions are equivalent:*

(1) $\mathscr{R}$ *is locally non-partial;* *(2)* $X = \operatorname{int}_{\mathscr{R}}\big(\operatorname{cl}_{\mathscr{R}}(X)\big)$.

Proof To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then for each $x \in X$ there exists $R \in \mathscr{R}$ such that for any $y \in R(x)$ and for any $S \in \mathscr{R}$ we have $S(y) \cap X = S(y) \neq \emptyset$, and thus $y \in \operatorname{cl}_{\mathscr{R}}(X)$.

Therefore, for each $x \in X$ there exists $R \in \mathscr{R}$ such that $R(x) \subseteq \operatorname{cl}_{\mathscr{R}}(X)$, and thus $x \in \operatorname{int}_{\mathscr{R}}\big(\operatorname{cl}_{\mathscr{R}}(X)\big)$. Hence, we can already see that $X \subseteq \operatorname{int}_{\mathscr{R}}\big(\operatorname{cl}_{\mathscr{R}}(X)\big)$, and thus (2) also holds.

Remark 56 Thus, the relator $\mathscr{R}$ is locally non-partial if and only if X is a *topologically regular open subset* of the relator space $X(\mathscr{R})$.

In addition to Definition 16, it is also worth introducing the following

Definition 18 An ordinary relator $\mathscr{R}$ on X to Y is called *non-degenerated* if both $X \neq \emptyset$ and $\mathscr{R} \neq \emptyset$.

Thus, analogously to Theorem 114, we can also easily establish the following

Theorem 116 *For an ordinary relator $\mathscr{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathscr{R}$ *is non-degenerated;*

(2) $\emptyset \notin \mathscr{D}_{\mathscr{R}}$; *(3)* $\mathscr{E}_{\mathscr{R}} \neq \emptyset$; *(4)* $Y \in \mathscr{E}_{\mathscr{R}}$; *(5)* $\mathscr{D}_{\mathscr{R}} \neq \mathscr{P}(Y)$.

Remark 57 In addition to Theorems 114 and 116, it is also worth mentioning that if the relator $\mathscr{R}$ is *paratopologically simple* in the sense that $\mathscr{E}_{\mathscr{R}} = \mathscr{E}_R$ for some relation R on X to Y, then the stack $\mathscr{E}_{\mathscr{R}}$ has a base $\mathscr{B}$ with $\operatorname{card}(\mathscr{B}) \leq \operatorname{card}(X)$. (See [67, Theorem 5.9] of Pataki.)

The existence of a non-paratopologically simple (actually finite equivalence) relator, proved first by Pataki [67, Example 5.11], shows that in our definitions of the relations $\operatorname{Lim}_{\mathscr{R}}$ and $\operatorname{Adh}_{\mathscr{R}}$ we cannot restrict ourselves to functions of gosets (generalized ordered sets) without some loss of generality.

23 Topological and Quasi-Topological Relators

Notation 8 *In this section, we shall again assume that* $\mathscr{R}$ *is an ordinary relator on* X.

The following improvement of [86, Definition 2.1] was first considered in [89]. (See [95] for a subsequent treatment.)

Definition 19 The ordinary relator $\mathscr{R}$ is called:

(1) *quasi-topological* if $x \in \operatorname{int}_{\mathscr{R}}\big(\operatorname{int}_{\mathscr{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathscr{R}$;
(2) *topological* if for any $x \in X$ and $R \in \mathscr{R}$ there exists $V \in \mathscr{T}_{\mathscr{R}}$ such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following four theorems.

Theorem 117 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is quasi-topological;*
(2) $\operatorname{int}_{\mathscr{R}}\big(R(x)\big) \in \mathscr{T}_{\mathscr{R}}$ *for all* $x \in X$ *and* $R \in \mathscr{R}$*;*
(3) $\operatorname{cl}_{\mathscr{R}}(A) \in \mathscr{F}_{\mathscr{R}}$ $\big(\operatorname{int}_{\mathscr{R}}(A) \in \mathscr{T}_{\mathscr{R}}\big)$ *for all* $A \subseteq X$.

Remark 58 Hence, we can see that the ordinary relator $\mathscr{R}$ is quasi-topological if and only if the super relation $\operatorname{cl}_{\mathscr{R}}$ is upper semi-idempotent ($\operatorname{int}_{\mathscr{R}}$ is lower semi-idempotent).

Theorem 118 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is topological;*
(2) $\mathscr{R}$ *is reflexive and quasi-topological.*

Remark 59 By Theorem 117, the relator $\mathscr{R}$ may be called *weakly (strongly) quasi-topological* if $\rho_{\mathscr{R}}(x) \in \mathscr{F}_{\mathscr{R}}$ $\big(R(x) \in \mathscr{T}_{\mathscr{R}}\big)$ for all $x \in X$ and $R \in \mathscr{R}$.

Moreover, by Theorem 118, the relator $\mathscr{R}$ may be called *weakly (strongly) topological* if it is reflexive and weakly (strongly) quasi-topological.

The following theorem shows that in a topological relator space $X(\mathscr{R})$, the relation $\operatorname{int}_{\mathscr{R}}$ and the family $\mathscr{T}_{\mathscr{R}}$ are equivalent tools.

Theorem 119 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is topological;*
(2) $\operatorname{int}_{\mathscr{R}}(A) = \bigcup \mathscr{T}_{\mathscr{R}} \cap \mathscr{P}(A)$ *for all* $A \subseteq X$*;*
(3) $\operatorname{cl}_{\mathscr{R}}(A) = \bigcap \mathscr{F}_{\mathscr{R}} \cap \mathscr{P}^{-1}(A)$ *for all* $A \subseteq X$.

Now, as an immediate consequence of Theorems 109 and 119, we can also state

Corollary 30 *If $\mathscr{R}$ is topological, then for any $A \subset X$, we have*

(1) $A \in \mathscr{E}_{\mathscr{R}}$ if and only if there exists $V \in \mathscr{T}_{\mathscr{R}} \setminus \{\emptyset\}$ such that $V \subseteq A$;
(2) $A \in \mathscr{D}_{\mathscr{R}}$ if and only if for all $W \in \mathscr{F}_{\mathscr{R}} \setminus \{X\}$ we have $A \setminus W \neq \emptyset$.

However, it is now more important to note that we can also prove the following

Theorem 120 *The following assertions are equivalent:*

(1) $\mathscr{R}$ is topological;
(2) $\mathscr{R}$ is topologically equivalent to a preorder relator.

Proof To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by Definition 19, for any $x \in X$ and $R \in \mathscr{R}$, there exists $V \in \mathscr{T}_{\mathscr{R}}$ such that $x \in V \subseteq R(x)$. Thus, by using the Pervin preorder relator

$$\mathscr{S} = \mathscr{R}_{\mathscr{T}_{\mathscr{R}}} = \big\{ R_V : \quad V \in \mathscr{T}_{\mathscr{R}} \big\}, \qquad \text{where} \qquad R_V = V^2 \cup V^c \times X,$$

we can show that $\operatorname{int}_{\mathscr{R}}(A) = \operatorname{int}_{\mathscr{S}}(A)$ for all $A \subseteq X$, and thus $\operatorname{int}_{\mathscr{R}} = \operatorname{int}_{\mathscr{S}}$.

Remark 60 The above theorem can also be proved by using the relators

$$\mathscr{R}^{\wedge} = \big\{ S \subseteq X^2 : \quad \forall\, x \in X : \quad x \in \operatorname{int}_{\mathscr{R}} \big(S(x) \big) \big\}$$

and $\mathscr{R}^{\wedge\infty} = \big\{ S^{\infty} : \quad S \in \mathscr{R}^{\wedge} \big\}$ considered mainly in [56, 57, 68, 92].

Note that some important operations for relators were already used by Kenyon [45], Nakano-Nakano [60] and the second author [85, 94].

In addition to Theorem 117, it is also worth proving the following

Theorem 121 *The following assertions are equivalent:*

(1) $\mathscr{R}$ is quasi-topological;
(2) $\mathscr{R} \subseteq \big(\mathscr{R}^{\wedge} \circ \mathscr{R}\big)^{\wedge}$; (3) $\mathscr{R}^{\wedge} \subseteq \big(\mathscr{R}^{\wedge} \circ \mathscr{R}^{\wedge}\big)^{\wedge}$.

Remark 61 By Száz [86], a relator $\mathscr{R}$ on X may be naturally called *topologically transitive* if, for each $x \in X$ and $R \in \mathscr{R}$ there exist $S, T \in \mathscr{R}$ such that $T\,[\,S\,(x)\,] \subseteq R\,(x)$.

This property can be reformulated in the concise form that $\mathscr{R} \subseteq \big(\mathscr{R} \circ \mathscr{R}\big)^{\wedge}$. Thus, the equivalence (1) and (3) can be expressed by saying that $\mathscr{R}$ is quasi-topological if and only if $\mathscr{R}^{\wedge}$ is topologically transitive.

In particular, we can easily prove the following

Theorem 122 *For an ordinary relation R on X, the following assertions are equivalent:*

(1) R is quasi-topological; (2) R is transitive.

Hence, it is clear that, even more specially, we can also state

Corollary 31 *An ordinary relation* R *on* X *is topological if and only if it is a preorder relation.*

Remark 62 Analogously to Definition 19, the relator $\mathscr{R}$ may be called *proximal* if for any $A \subseteq X$ and $R \in \mathscr{R}$ there exists $V \in \tau_{\mathscr{R}}$ such that $A \subseteq V \subseteq R[A]$.

Thus, in addition to the counterparts of Theorems 119 and 120, we can prove that $\mathscr{R}$ is topological if and only if its topological closure (refinement) $\mathscr{R}^{\wedge}$ is proximal.

24 A Few Basic Facts on Filtered Relators

Notation 9 *In this section, we shall assume that* $\mathscr{R}$ *is an ordinary relator on* X *to* Y.

The following definition was also first investigated in [86, 89].

Definition 20 The relator $\mathscr{R}$ is called

(1) *properly filtered* if for any $R, S \in \mathscr{R}$ we have $R \cap S \in \mathscr{R}$;
(2) *uniformly filtered* if for any $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T \subseteq R \cap S$;
(3) *proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T[A] \subseteq R[A] \cap S[A]$;
(4) *topologically filtered* if for any $x \in X$ and $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T(x) \subseteq R(x) \cap S(x)$.

Remark 63 By using the binary operation $\wedge$ and the basic closure operations on relators, the above properties can be reformulated in some more concise forms.

For instance, we can see that $\mathscr{R}$ is topologically filtered if and only if any one of the properties $\mathscr{R} \wedge \mathscr{R} \subseteq \mathscr{R}^{\wedge}$, $(\mathscr{R} \wedge \mathscr{R})^{\wedge} = \mathscr{R}^{\wedge}$ and $\mathscr{R}^{\wedge} \wedge \mathscr{R}^{\wedge} = \mathscr{R}^{\wedge}$ holds.

However, in general, we only have $(R \cap S)[A] \subseteq R[A] \cap S[A]$. Therefore, the corresponding proximal filteredness properties are, unfortunately, not equivalent.

Despite this, we can easily prove the following theorem which shows the appropriateness of the above proximal filteredness property.

Theorem 123 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is proximally filtered;*
(2) $\operatorname{Cl}_{\mathscr{R}}(A \cup B) = \operatorname{Cl}_{\mathscr{R}}(A) \cup \operatorname{Cl}_{\mathscr{R}}(B)$ *for all* $A, B \subseteq Y$;
(3) $\operatorname{Int}_{\mathscr{R}}(A \cap B) = \operatorname{Int}_{\mathscr{R}}(A) \cap \operatorname{Int}_{\mathscr{R}}(B)$ *for all* $A, B \subseteq Y$.

Proof To prove the implication (3) $\Longrightarrow$ (1), note that if $A \subseteq X$ and $R, S \in \mathscr{R}$, then by the definition of $\operatorname{Int}_{\mathscr{R}}$ we have $A \in \operatorname{Int}_{\mathscr{R}}\big(R[A]\big)$ and $A \in \operatorname{Int}_{\mathscr{R}}\big(S[A]\big)$. Therefore, if (3) holds, then we also have $A \in \operatorname{Int}_{\mathscr{R}}\big(R[A] \cap S[A]\big)$. Thus, by the definition of $\operatorname{Int}_{\mathscr{R}}$, there exists $T \in \mathscr{R}$ such that $T[A] \subseteq R[A] \cap S[A]$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 32 *If $\mathscr{R}$ is a proximally filtered relator on X, then the families $\tau\!\!\!-_{\mathscr{R}}$ and $\tau_{\mathscr{R}}$ are closed under binary unions and intersections, respectively.*

Analogously to Theorem 123, we can also easily prove the following

Theorem 124 *The following assertions are equivalent :*

(1) $\mathscr{R}$ *is topologically filtered ;*
(2) $\operatorname{cl}_{\mathscr{R}}(A \cup B) = \operatorname{cl}_{\mathscr{R}}(A) \cup \operatorname{cl}_{\mathscr{R}}(B)$ *for all* $A,\ B \subseteq Y;$
(3) $\operatorname{int}_{\mathscr{R}}(A \cap B) = \operatorname{int}_{\mathscr{R}}(A) \cap \operatorname{int}_{\mathscr{R}}(B)$ *for all* $A,\ B \subseteq Y.$

Thus, in particular, we can also state the following

Corollary 33 *If $\mathscr{R}$ is a topologically filtered relator on X, then the families $\mathscr{F}_{\mathscr{R}}$ and $\mathscr{T}_{\mathscr{R}}$ are closed under binary unions and intersections, respectively.*

The following example shows that, for a non-topological relator $\mathscr{R}$, the converse of the above corollary need not be true.

Example 4 If $X = \{1,\ 2,\ 3\}$ and R_i is relation on X, for each $i = 1,\ 2,$ such that

$$R_i(1) = \{1,\ i+1\} \qquad \text{and} \qquad R_i(2) = R_i(3) = \{2,\ 3\},$$

then $\mathscr{R} = \{R_1,\ R_2\}$ is a reflexive relator on X such that $\mathscr{T}_{\mathscr{R}}$ is closed under arbitrary intersections, but $\mathscr{R}$ is still not topologically filtered.

By the corresponding definitions, it is clear that $\mathscr{T}_{\mathscr{R}} = \{\emptyset,\ \{2,\ 3\},\ X\}$. Moreover, we can note that $R_i(1) \not\subseteq R_1(1) \cap R_2(1)$ for each $i = 1,\ 2$. Thus, the relator $\mathscr{R}$ is not topologically filtered.

In addition to Theorem 124, in [74] we have also proved the following

Theorem 125 *If $\mathscr{R}$ is a topologically filtered relator on X, $A,\ B \subseteq X$ and there exists $V \in \mathscr{T}_{\mathscr{R}} \cap \mathscr{F}_{\mathscr{R}}$ such that $A \subseteq V$ and $B \subseteq V^c$, then*

$$\operatorname{int}_{\mathscr{R}}(A \cup B) = \operatorname{int}_{\mathscr{R}}(A) \cup \operatorname{int}_{\mathscr{R}}(B)\,.$$

Remark 64 This statement is a straightforward generalization of [51, Lemma 7] of Levine.

More difficult conditions for the dual equality $(A \cap B)^- = A^- \cap B^-$ to hold were given by Gottschalk [37] and Jung and Nam [42, 43].

Concerning the latter problem, we can only prove here the following

Theorem 126 *If $\mathscr{R}$ is a nonvoid, reflexive relator on X such that*

$$\operatorname{cl}_{\mathscr{R}}(A \cap B) = \operatorname{cl}_{\mathscr{R}}(A) \cap \operatorname{cl}_{\mathscr{R}}(B)$$

for all $A,\ B \subseteq X$, then $\mathscr{T}_{\mathscr{R}} = \mathscr{P}(X)$.

Proof Namely, if this not the case, then there exists $A \subseteq X$ such that $A \notin \mathscr{T}_{\mathscr{R}}$, and thus $B = A^c \notin \mathscr{F}_{\mathscr{R}}$. Therefore, $B^- \not\subseteq B$, and thus there exists $x \in B^-$ such that $x \notin B$. Hence, by using the assumptions of the theorem, we can infer that $x \in \{x\}^- \cap B^- = \big(\{x\} \cap B\big)^- = \emptyset^- = \emptyset$, which is a contradiction.

Remark 65 If $\mathscr{T}_{\mathscr{R}} = \mathscr{P}(X)$, then we have $\mathscr{T}_{\mathscr{R}} = \mathscr{T}_{\Delta_X}$. Hence, by using a general theorem on quasi-topologically equivalent relators and the definitions of the operations $\wedge$ and ∞, we can infer that $\mathscr{R}^{\wedge\infty} = \{\Delta_X\}^{\wedge\infty} = \mathscr{P}(X^2)^{\infty}$.

25 A Few Basic Facts on Quasi-Filtered Relators

Notation 10 *In this and the next two sections, we shall already assume that $\mathscr{R}$ is an ordinary relator on X.*

Since $R \subseteq R^{\infty}$ for every relation R on X, in addition to Definition 20, we may also naturally introduce the following

Definition 21 The relator $\mathscr{R}$ is called

(1) *Quasi-uniformly filtered* if for any $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T \subseteq R^{\infty} \cap S^{\infty}$;
(2) *Quasi-proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T[A] \subseteq R^{\infty}[A] \cap S^{\infty}[A]$;
(3) *Quasi-topologically filtered* if for any $x \in X$ and $R, S \in \mathscr{R}^{\wedge}$ there exists $T \in \mathscr{R}$ such that $T(x) \subseteq R^{\infty}(x) \cap S^{\infty}(x)$.

Remark 66 Analogously to Remark 63, the above quasi-filteredness properties can also be reformulated in some more concise forms.

For instance, we can see that $\mathscr{R}$ is quasi-topologically filtered if and only if $\mathscr{R}^{\wedge\infty} \wedge \mathscr{R}^{\wedge\infty} \subseteq \mathscr{R}^{\wedge}$, $(\mathscr{R}^{\wedge\infty} \wedge \mathscr{R}^{\wedge\infty})^{\wedge\infty} = \mathscr{R}^{\wedge\infty}$ or $\mathscr{R}^{\wedge\infty} \wedge \mathscr{R}^{\wedge\infty} = \mathscr{R}^{\wedge\infty}$.

However, it is now more important to note that, by using some former results on relators, we can also prove the following two theorems which show the appropriateness of the above quasi-proximal and quasi-topological filteredness properties.

Theorem 127 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is a quasi-proximally filtered;*
(2) $\varepsilon_{\mathscr{R}}$ *is closed under binary unions;*
(3) $\tau_{\mathscr{R}}$ *is closed under binary intersections.*

Theorem 128 *The following assertions are equivalent:*

(1) $\mathscr{R}$ *is a quasi-topologically filtered;*
(2) $\mathscr{F}_{\mathscr{R}}$ *is closed under binary unions;*
(3) $\mathscr{T}_{\mathscr{R}}$ *is closed under binary intersections.*

Remark 67 In this respect it is also worth mentioning that if $\mathscr{R}$ is a relator on X to Y, then the family $\mathscr{E}_{\mathscr{R}}$ is closed under binary intersections if and only if $\mathscr{R}$ is *quasi-directed* in the sense that for any $x, y \in X$ and $R, S \in \mathscr{R}$ we have $R(x) \cap S(y) \in \mathscr{E}_{\mathscr{R}}$.

From the above two theorems, by using Corollaries 32 and 33, we can derive

Corollary 34 *If $\mathscr{R}$ is a proximally (topologically) filtered relator on X, then $\mathscr{R}$ is also quasi-proximally (quasi-topologically) filtered.*

Now, by using Theorem 127, we can also easily prove the following

Theorem 129 *If $\mathscr{R}$ is a quasi-proximally filtered, proximal relator on X, then $\mathscr{R}$ is proximally filtered.*

Proof Suppose that $A \subseteq X$ and $R, S \in \mathscr{R}$. Then, by Remark 62, there exist $U, V \in \tau_{\mathscr{R}}$ such that $A \subseteq U \subseteq R[A]$ and $A \subseteq V \subseteq S[A]$. Moreover, by Theorem 127, we can state that $U \cap V \in \tau_{\mathscr{R}}$. Therefore, by the definition of $\tau_{\mathscr{R}}$, there exists $T \in \mathscr{R}$ such that $T[U \cap V] \subseteq U \cap V$. Hence, we can already see that $T[A] \subseteq T[U \cap V] \subseteq U \cap V \subseteq R[A] \cap S[A]$. Thus, $\mathscr{R}$ is proximally filtered.

Moreover, by using Theorem 128, we can quite similarly prove the following

Theorem 130 *If $\mathscr{R}$ is a quasi-topologically filtered, topological relator on X, then $\mathscr{R}$ is topologically filtered.*

Remark 68 Our former Example 4 shows that even a quasi-proximally filtered, reflexive relator need not be topologically filtered.

Namely, if X and $\mathscr{R}$ are as in Example 4, then by the corresponding definitions it is clear that $\tau_{\mathscr{R}} = \{\emptyset, \{2, 3\}, X\}$, and thus by Theorem 127 the relator $\mathscr{R}$ is quasi-proximally filtered.

26 Some Further Theorems on Topologically Filtered Relators

The importance of topologically filtered relators is also apparent from

Theorem 131 *If $\mathscr{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

(1) $\mathrm{cl}_{\mathscr{R}}(A) \cap \mathrm{int}_{\mathscr{R}}(B) \subseteq \mathrm{cl}_{\mathscr{R}}(A \cap B)$*;*
(2) $\mathrm{int}_{\mathscr{R}}(A \cup B) \subseteq \mathrm{int}_{\mathscr{R}}(A) \cup \mathrm{cl}_{\mathscr{R}}(B)$.

Proof Assume that $x \in A^- \cap B^\circ$ and $R \in \mathscr{R}$. Then, since $x \in B^\circ$, there exists $S \in \mathscr{R}$ such that $S(x) \subseteq B$. Moreover, since $\mathscr{R}$ is topologically filtered, there exists $T \in \mathscr{R}$ such that $T(x) \subseteq R(x) \cap S(x)$. Furthermore, since $x \in A^-$, there exists $y \in A$ such that $y \in T(x)$. Hence, we can already infer that

$$y \in A \cap T(x) \subseteq A \cap S(x) \subseteq A \cap B \qquad \text{and} \qquad y \in T(x) \subseteq R(x).$$

Therefore, $R(x) \cap (A \cap B) \neq \emptyset$ for all $R \in \mathscr{R}$, and thus $x \in (A \cap B)^-$ also holds. This proves that $A^- \cap B^\circ \subseteq (A \cap B)^-$, and thus assertion (1) is true.

Now, by applying assertion (1) to the sets A^c and B^c and using De Morgan's laws and Theorem 79, we can easily see that assertion (2) is also true.

From this theorem, by using Theorem 95, we can immediately derive

Corollary 35 *If $\mathscr{R}$ is topologically filtered, then*

(1) $\operatorname{cl}_{\mathscr{R}}(A) \cap B \subseteq \operatorname{cl}_{\mathscr{R}}(A \cap B)$ *for all* $A \subseteq X$ *and* $B \in \mathscr{T}_{\mathscr{R}}$*;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) \subseteq \operatorname{int}_{\mathscr{R}}(A) \cup B$ *for all* $A \subseteq X$ *and* $B \in \mathscr{F}_{\mathscr{R}}$*.*

Remark 69 The important inclusion $A^- \cap B \subseteq (A \cap B)^-$, with B being open, was first revealed by Kuratowski [48, p. 45].

Later, Császár [14–17, 19, 20] and Sivagami [80] assumed it as an axiom for an increasing set-to-set function γ.

Now, as some improvements of the above theorem and its corollary, we can also prove the following theorem and its corollary.

Theorem 132 *If $\mathscr{R}$ is topologically filtered, then for any $A, B \subseteq X$ we have*

(1) $\operatorname{cl}_{\mathscr{R}}(A) \cap \operatorname{int}_{\mathscr{R}}(B) = \operatorname{cl}_{\mathscr{R}}(A \cap B) \cap \operatorname{int}_{\mathscr{R}}(B)$*;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) \cup \operatorname{cl}_{\mathscr{R}}(B) = \operatorname{int}_{\mathscr{R}}(A) \cup \operatorname{cl}_{\mathscr{R}}(B)$.

Proof To prove (1), note that, by Theorem 131, we have $A^- \cap B^\circ \subseteq (A \cap B)^-$, and thus also $A^- \cap B^\circ = A^- \cap B^\circ \cap B^\circ \subseteq (A \cap B)^- \cap B^\circ$.

On the other hand, by using Theorem 80, we can see that $(A \cap B)^- \subseteq A^-$, and thus also $(A \cap B)^- \cap B^\circ = (A \cap B)^- \cap B^\circ \cap B^\circ \subseteq A^- \cap B^\circ$.

Corollary 36 *If $\mathscr{R}$ is topologically filtered, then*

(1) $\operatorname{cl}_{\mathscr{R}}(A) \cap B = \operatorname{cl}_{\mathscr{R}}(A \cap B) \cap B$ *for all* $A \subseteq X$ *and* $B \in \mathscr{T}_{\mathscr{R}}$*;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) \cup B = \operatorname{int}_{\mathscr{R}}(A) \cup B$ *for all* $A \subseteq X$ *and* $B \in \mathscr{F}_{\mathscr{R}}$*.*

Proof To derive assertion (1) from that of Theorem 132, note that if $B \in \mathscr{T}_{\mathscr{R}}$, then by Theorem 95 we have $B \subseteq B^\circ$, and thus also $B^\circ \cap B = B$.

However, Theorem 132 and its corollary are less important than Theorem 131 and its corollary. Namely, for instance, by using Corollary 35 and our former theorems on topological relators, we can already prove the following

Theorem 133 *If $\mathscr{R}$ is topological and topologically filtered, then*

(1) $\operatorname{cl}_{\mathscr{R}}(A \cap B) = \operatorname{cl}_{\mathscr{R}}\big(\operatorname{cl}_{\mathscr{R}}(A) \cap B\big)$ *for all* $A \subseteq X$ *and* $B \in \mathscr{T}_{\mathscr{R}}$*;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) = \operatorname{int}_{\mathscr{R}}\big(\operatorname{int}_{\mathscr{R}}(A) \cup B\big)$ *for all* $A \subseteq X$ *and* $B \in \mathscr{F}_{\mathscr{R}}$*.*

Proof To prove (1), note that if $A \subseteq X$ and $B \in \mathscr{T}_{\mathscr{R}}$, then by Corollary 35 we have $A^- \cap B \subseteq (A \cap B)^-$. Hence, by using Theorem 117, we can infer that

$$(A^- \cap B)^- \subseteq (A \cap B)^{--} \subseteq (A \cap B)^-.$$

On the other hand, by Theorems 118 and 112, we have $A \subseteq A^-$, and thus also $A \cap B \subseteq A^- \cap B$. Hence, by Theorem 80, we can infer that $(A \cap B)^- \subseteq \big(A^- \cap B\big)^-$. Therefore, the corresponding equality is also true.

From this theorem, by using Theorem 69, we can immediately derive

Corollary 37 *If $\mathscr{R}$ is topological and topologically filtered, then*

(1) $\operatorname{cl}_{\mathscr{R}}(A \cap B) = \operatorname{cl}_{\mathscr{R}}(B)$ *for all* $A \in \mathscr{D}_{\mathscr{R}}$ *and* $B \in \mathscr{T}_{\mathscr{R}}$ *;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) = \operatorname{int}_{\mathscr{R}}(A)$ *for all* $A \in \mathscr{E}_{\mathscr{R}}^{c}$ *and* $B \in \mathscr{F}_{\mathscr{R}}$.

Now, by modifying an argument of Levine [53], we can also prove

Theorem 134 *If $\mathscr{R}$ is nonvoid and topological, and $A \subseteq X$, then*

(1) $\operatorname{cl}_{\mathscr{R}}(A \cap B) = \operatorname{cl}_{\mathscr{R}}(B)$ *for all* $B \in \mathscr{T}_{\mathscr{R}}$ *implies that* $A \in \mathscr{D}_{\mathscr{R}}$ *;*
(2) $\operatorname{int}_{\mathscr{R}}(A \cup B) = \operatorname{int}_{\mathscr{R}}(A)$ *for all* $B \in \mathscr{F}_{\mathscr{R}}$ *implies that* $A \notin \mathscr{E}_{\mathscr{R}}$.

Proof For instance, if $A \notin \mathscr{D}_{\mathscr{R}}$, then there exists $x \in X$ such that $x \notin A^-$. Thus, there exists $R \in \mathscr{R}$ such that $A \cap R(x) = \emptyset$. Moreover, since $\mathscr{R}$ is topological, there exists $B \in \mathscr{T}_{\mathscr{R}}$ such that $x \in B \subseteq R(x)$. Thus, we also have $A \cap B = \emptyset$.

Hence, by using the assumption of (1), we can infer that $B^- = (A \cap B)^- = \emptyset^- = \emptyset$. On the other hand, from $x \in B$, we can now infer that $x \in \{x\}^- \subseteq B^-$, and thus $B^- \neq \emptyset$. This contradiction proves (1).

Remark 70 If $\mathscr{R}$ is nonvoid and reflexive, and $A \subseteq X$ such that $\operatorname{cl}_{\mathscr{R}}\big(A \cap R(x)\big) = \operatorname{cl}_{\mathscr{R}}\big(R(x)\big)$ for all $x \in X$ and $R \in \mathscr{R}$, then we can even more easily prove that $A \in \mathscr{D}_{\mathscr{R}}$.

27 Some More Particular Theorems on Topologically Filtered Relators

The importance of Corollary 35 is also apparent from the following

Theorem 135 *If $\mathscr{R}$ is quasi-topological and topologically filtered, then for any $A, B \in \mathscr{N}_{\mathscr{R}}$ we have $A \cup B \in \mathscr{N}_{\mathscr{R}}$.*

Proof By Theorem 117, we have $B^- \in \mathscr{F}_{\mathscr{R}}$. Hence, by using Theorem 124, Corollary 35 and the definition of $\mathscr{N}_{\mathscr{R}}$, we can see that

$$(A \cup B)^{-\circ} = (A^- \cup B^-)^{\circ} \subseteq A^{-\circ} \cup B^- = \emptyset \cup B^- = B^-.$$

Moreover, by Theorem 117, we have $(A \cup B)^{-\circ} \in \mathscr{T}_{\mathscr{R}}$. Hence, by using the increasingness of $\circ$ and the definitions of $\mathscr{T}_{\mathscr{R}}$ and $\mathscr{N}_{\mathscr{R}}$, we can see that

$$(A \cup B)^{-\circ} \subseteq (A \cup B)^{-\circ\circ} \subseteq B^{-\circ} = \emptyset.$$

Therefore, $(A \cup B)^{-\circ} = \emptyset$, and thus $A \cup B \in \mathcal{N}_{\mathcal{R}}$ also holds.

Now, by using this theorem, we can also easily establish the following

Corollary 38 *If $\mathcal{R}$ is nonvoid, non-partial, quasi-topological and topologically filtered, then $\mathcal{N}_{\mathcal{R}}$ is an ideal on X.*

Proof By the definition of $\mathcal{N}_{\mathcal{R}}$ and the increasingness of $-\circ$, it is clear that $\mathcal{N}_{\mathcal{R}}$ is always descending. Moreover, since $\mathcal{R}$ is nonvoid and non-partial, we can also note that $\emptyset^{-\circ} = \emptyset^{\circ} = \emptyset$. Therefore, $\emptyset \in \mathcal{N}_{\mathcal{R}}$, and thus $\mathcal{N}_{\mathcal{R}} \neq \emptyset$. Furthermore, from Theorem 135, we know that $\mathcal{N}_{\mathcal{R}}$ is closed under pairwise unions.

Remark 71 Note that if $\mathcal{R}$ is locally non-partial, then by Theorem 115 we have $X^{-\circ} = X$. Therefore, if $X \neq \emptyset$, then we can also state that $X \notin \mathcal{N}_{\mathcal{R}}$, and thus $\mathcal{N}_{\mathcal{R}} \neq \mathcal{P}(X)$.

While, if $\mathcal{R}$ is quasi-topological and $A \in \mathcal{N}_{\mathcal{R}}$, then by using Theorem 117 and the increasingness of $\circ$ we can also see that $A^{--\circ} \subseteq A^{-\circ} = \emptyset$. Therefore, $A^{--\circ} = \emptyset$, and thus $A^{-} \in \mathcal{N}_{\mathcal{R}}$ also holds.

The importance of topologically filtered relators is also apparent from

Theorem 136 *If $\mathcal{R}$ is topological and topologically filtered, then for any $A \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}\,.$$

Proof By Theorem 102, we have $A^{c} \in \mathcal{F}_{\mathcal{R}}$. Moreover, by Theorem 117, we have $A^{-} \in \mathcal{F}_{\mathcal{R}}$. Hence, by using Corollary 33, we can see that

$$A^{\dagger} = A^{-} \setminus A = A^{-} \cap A^{c} \in \mathcal{F}_{\mathcal{R}}\,.$$

Moreover, by using Theorems 124, 37, 118, and 112, we can also see that

$$A^{\dagger\circ} = \big(A^{-} \cap A^{c}\big)^{\circ} = A^{-\circ} \cap A^{c\circ} = A^{-\circ} \cap A^{-c} \subseteq A^{-} \cap A^{-c} = \emptyset,$$

and thus $A^{\dagger\circ} = \emptyset$. Therefore, $A^{\dagger} \notin \mathcal{E}_{\mathcal{R}}$, and thus $A^{\dagger} \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

By this theorem, it is clear that in particular we also have

Corollary 39 *If $\mathcal{R}$ is topological and topologically filtered, then $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{N}_{\mathcal{R}}$ for all $A \in \mathcal{T}_{\mathcal{R}}$.*

Remark 72 Note that if $\mathcal{R}$ is topological and $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorems 95, 118, and 112 we have $A = A^{\circ}$. Therefore,

$$A^{\ddagger} = \operatorname{bnd}_{\mathcal{R}}(A) = A^{-} \setminus A^{\circ} = A^{-} \setminus A = A^{\dagger}.$$

Moreover, it is also worth noticing that in Theorem 136 and Corollary 39, it is enough to assume only that $\mathcal{R}$ is topological and quasi-topologically filtered. Namely, in this case, by Theorem 130, $\mathcal{R}$ is already topologically filtered.

28 Proximally Closed Sets in Super Relator Spaces

Notation 11 *In this and the next two sections, we shall assume that* $\mathscr{R}$ *is an ordinary relator and* $\mathscr{U}$ *is a super relator on* X.

The importance of the duals of super relations is also apparent from the following

Theorem 137 *We have*

$$\bar{\tau}_{\mathscr{U}} = \big\{ A \subseteq X : \quad \exists\ U \in \mathscr{U} : \quad A \subseteq U^{\star}(A) \big\}.$$

Proof By Theorems 52 and 51, we have

$$A \in \bar{\tau}_{\mathscr{U}} \iff A^c \in \tau_{\mathscr{U}} \iff \exists\ U \in \mathscr{U} : \quad U\,(A^c) \subseteq A^c\,.$$

Moreover, by the corresponding definitions, we can see that

$$U\,(A^c) \subseteq A^c \iff A \subseteq U\,(A^c)^c \iff A \subseteq U^{\star}(A)\,.$$

Therefore, we actually have

$$A \in \bar{\tau}_{\mathscr{U}} \iff \exists\ U \in \mathscr{U} : \quad A \subseteq U^{\star}(A)\,,$$

and thus the required equality is true.

From this theorem, by using the notation $\mathscr{U}^{\star} = \{U^{\star} : \ U \in \mathscr{U}\}$, we can obtain

Corollary 40 *We have*

$$\bar{\tau}_{\mathscr{U}^{\star}} = \big\{ A \subseteq X : \quad \exists\ U \in \mathscr{U} : \quad A \subseteq U(A) \big\}\,.$$

Thus, in particular, we can also state the following

Corollary 41 *For a super relation* U *on* X*, we have*

$$\bar{\tau}_{U} = \big\{ A \subseteq X : \quad A \subseteq U^{\star}(A) \big\} \qquad \textit{and} \qquad \bar{\tau}_{U^{\star}} = \big\{ A \subseteq X : \quad A \subseteq U(A) \big\}\,.$$

Remark 73 If U is a union-preserving super relation on X, then by Corollary 16 we have $\bar{\tau}_{U} = \mathscr{F}_{U}$.

While, if R is an ordinary relation on X, then by Theorem 98 and Corollary 29 we have $\bar{\tau}_{R} = \tau_{R^{-1}} = \mathscr{T}_{R^{-1}}$.

Example 5 If U is a super relation on X such that

$$U\,(A) = \operatorname{cl}_{\mathscr{R}}\big(\operatorname{int}_{\mathscr{R}}(A)\big)$$

for all $A \subseteq X$, then

$$\tau_U = \{ A \subseteq X : \quad A \subseteq \operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}}(A)) \}$$

and

$$\tau_{U^\star} = \{ A \subseteq X : \quad A \subseteq \operatorname{cl}_{\mathscr{R}} (\operatorname{int}_{\mathscr{R}}(A)) \} .$$

Namely, by Definition 7 and Theorem 79, we have

$$U^\star(A) = U(A^c)^c = \operatorname{cl}_{\mathscr{R}} (\operatorname{int}_{\mathscr{R}}(A^c))^c = A^{c \circ - c} = A^{- c - c} = A^{- \circ}$$
$$= \operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}}(A))$$

for all $A \subseteq X$.

Moreover, by Corollary 41, for instance we have

$$\tau_U = \{ A \subseteq X : \quad A \subseteq U^\star(A) \} = \{ A \subseteq X : \quad A \subseteq \operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}}(A)) \} .$$

Remark 74 Thus, if U is as in the above example, then τ_U and $\tau_{U^\star}$ are just the families $\mathscr{T}_{\mathscr{R}}^{p}$ and $\mathscr{T}_{\mathscr{R}}^{s}$ of all *topologically preopen and semi-open subsets* of the relator space $X(\mathscr{R})$ considered in [75].

Moreover, by using the important family

$$\mathscr{A}_{\mathscr{R}} = \{ A \subseteq X : \quad \operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}}(A)) \subseteq \operatorname{cl}_{\mathscr{R}} (\operatorname{int}_{\mathscr{R}}(A)) \}$$

considered in [74], we can note that $\tau_U \cap \mathscr{A}_{\mathscr{R}} \subseteq \tau_{U^\star}$.

29 Two Further Illustrative Examples and Two Further General Theorems

Analogously to Example 5, we can also establish the following two examples.

Example 6 If U is a super relation on X such that

$$U(A) = \operatorname{cl}_{\mathscr{R}} (\operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}}(A)))$$

for all $A \subseteq X$, then

$$\tau_U = \{ A \subseteq X : \quad A \subseteq \operatorname{int}_{\mathscr{R}} (\operatorname{cl}_{\mathscr{R}} (\operatorname{int}_{\mathscr{R}}(A))) \}$$

and

$$\tau_{U^\star} = \left\{ A \subseteq X : \quad A \subseteq \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A) \right) \right) \right\}.$$

Namely, by using Definition 7 and Theorem 79, we can see that

$$U^\star(A) = U(A^c)^c = \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A^c) \right) \right)^c = A^{c-\circ-c}$$
$$= A^{\circ c \circ - c} = A^{\circ - c - c} = A^{\circ - \circ} = \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A) \right) \right)$$

for all $A \subseteq X$. Hence, by Corollary 41, it is clear that the required equalities are true.

Remark 75 Thus, if U is as in the above example, then τ_U and $\tau_{U^\star}$ are just the families $\mathscr{T}_{\mathscr{R}}^{\alpha}$ and $\mathscr{T}_{\mathscr{R}}^{\beta}$ of all *topologically* α*-open and* β*-open subsets* of the relator space $X(\mathscr{R})$ considered in [75].

Example 7 If U is a super relation on X such that

$$U(A) = \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A) \right) \cup \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A) \right)$$

for all $A \subseteq X$, then

$$\tau_U = \left\{ A \subseteq X : \quad A \subseteq \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A) \right) \cap \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A) \right) \right\}$$

and

$$\tau_{U^\star} = \left\{ A \subseteq X : \quad A \subseteq \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A) \right) \cup \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A) \right) \right\}.$$

Namely, by using Definition 7 and Theorem 79, we can see that

$$U^\star(A) = U(A^c)^c = \left(\mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A^c) \right) \cup \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A^c) \right) \right)^c$$
$$= \left(A^{c \circ -} \cup A^{c - \circ} \right)^c = A^{c \circ - c} \cap A^{c - \circ c} = A^{c - \circ} \cap A^{\circ -}$$
$$= \mathrm{int}_{\mathscr{R}} \left(\mathrm{cl}_{\mathscr{R}}(A) \right) \cap \mathrm{cl}_{\mathscr{R}} \left(\mathrm{int}_{\mathscr{R}}(A) \right).$$

Hence, by Corollary 41, it is clear that the required equalities are true.

Remark 76 Thus, if U is as in the above example, then τ_U and $\tau_{U^\star}$ are just the families $\mathscr{T}_{\mathscr{R}}^{a}$ and $\mathscr{T}_{\mathscr{R}}^{b}$ of all *topologically* a*-open and* b*-open subsets* of the relator space $X(\mathscr{R})$ considered in [75].

To show that, in some important particular cases, the desirable inclusions $\mathscr{T}_{\mathscr{R}} \subseteq \tau_{\mathscr{U}}$ and $\mathscr{T}_{\mathscr{R}} \subseteq \tau_{\mathscr{U}^\star}$ are true; in addition to Theorem 137 and its corollaries, we must also prove the following two closely related theorems and their corollaries.

Theorem 138 *If for each* $A \in \mathscr{T}_{\mathscr{R}}$ *there exists* $U \in \mathscr{U}$ *such that* $\mathrm{int}_{\mathscr{R}}(A) \subseteq U(A)$, *then* $\mathscr{T}_{\mathscr{R}} \subseteq \tau_{\mathscr{U}^\star}$.

Proof If $A \in \mathscr{T}_{\mathcal{R}}$, then by Theorem 95, we have $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$. Moreover, by the assumption of the theorem, there exists $U \in \mathscr{U}$ such that $\operatorname{int}_{\mathcal{R}}(A) \subseteq U(A)$. Therefore, we also have $A \subseteq U(A)$. Hence, by Corollary 41, we can see that $A \in \tau_{U^\star}$. Thus, the required inclusion is also true.

Now, by this theorem, we can also state

Corollary 42 *If there exists* $U \in \mathscr{U}$ *such that* $\operatorname{int}_{\mathcal{R}} \subseteq U$, *then* $\mathscr{T}_{\mathcal{R}} \subseteq \tau_{\mathscr{U}^\star}$.

Hence, it is clear that in particular we also have

Corollary 43 *If* U *is a super relation on* X *such that* $\operatorname{int}_{\mathcal{R}} \subseteq U$, *then* $\mathscr{T}_{\mathcal{R}} \subseteq \tau_{U^\star}$.

Now, in addition to Theorem 138 and its corollaries, we can also easily establish the following theorem and its corollaries.

Theorem 139 *If for each* $A \in \mathscr{F}_{\mathcal{R}}$ *there exists* $U \in \mathscr{U}$ *such that* $U(A) \subseteq \operatorname{cl}_{\mathcal{R}}(A)$, *then* $\mathscr{T}_{\mathcal{R}} \subseteq \tau_{\mathscr{U}}$.

Proof If $A \in \mathscr{T}_{\mathcal{R}}$, then by Theorem 102 we have $A^c \in \mathscr{F}_{\mathcal{R}}$. Thus, by the assumption of the theorem, there exists $U \in \mathscr{U}$ such that $U(A^c) \subseteq \operatorname{cl}_{\mathcal{R}}(A^c)$. Hence, by using Theorem 79 and Definition 7, we can infer that

$$\operatorname{int}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A^c)^c \subseteq U(A^c)^c = U^\star(A).$$

Thus, Theorem 138 can be applied to the super relator $\mathscr{U}^\star$ to obtain the required inclusion.

Corollary 44 *If there exists* $U \in \mathscr{U}$ *such that* $U \subseteq \operatorname{cl}_{\mathcal{R}}$, *then* $\mathscr{T}_{\mathcal{R}} \subseteq \tau_{\mathscr{U}}$.

Corollary 45 *If* U *is a super relation on* X *such that* $U \subseteq \operatorname{cl}_{\mathcal{R}}$, *then* $\mathscr{T}_{\mathcal{R}} \subseteq \tau_U$.

Corollaries 43 and 45 allow us to easily establish the following

Example 8 If $\mathcal{R}$ is reflexive on X, then $\mathscr{T}_{\mathcal{R}} \subseteq \mathscr{T}_{\mathcal{R}}^{\kappa}$ for all $\kappa = p, s, \alpha, \beta, a, b$.
Namely, by Theorem 112, we have

$$\operatorname{int}_{\mathcal{R}}(A) \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(A)$$

for all $A \subseteq X$. Thus, even if U is as in Example 6, then we still have

$$\operatorname{int}_{\mathcal{R}}(A) \subseteq U(A) \subseteq \operatorname{cl}_{\mathcal{R}}(A)$$

for all $A \subseteq X$. Therefore, by Corollaries 43 and 45 and Remark 75 we have $\mathscr{T}_{\mathcal{R}} \subseteq \mathscr{T}_{\mathcal{R}}^{\kappa}$ for $\kappa = \alpha, \beta$.

30 Some Set-Theoretic Properties of the Families $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{U}^\star}$

By using Theorem 51 and the corresponding definitions, or Theorem 137 and its corollary, we can easily establish the following two theorems.

Theorem 140 *The following assertions are equivalent :*

(1) $\emptyset \in \mathcal{T}_{\mathcal{U}}$ *;* *(2)* $\emptyset \in \mathcal{T}_{\mathcal{U}^\star}$ *;*
(3) $\mathcal{U} \neq \emptyset$ *;* *(4)* $\mathcal{T}_{\mathcal{U}} \neq \emptyset$ *;* *(5)* $\mathcal{T}_{\mathcal{U}^\star} \neq \emptyset$*.*

Proof By Theorem 137, it is clear that assertions (1), (3) and (4) are equivalent. Hence, since $\mathcal{U}^\star \neq \emptyset$ if and only if $\mathcal{U} \neq \emptyset$, we can see that assertions (2), (3) and (5) are also equivalent.

Theorem 141 *The following assertions are true :*

(1) $X \in \mathcal{T}_{\mathcal{U}}$ *if and only if there exists* $U \in \mathcal{U}$ *such that* $U(\emptyset) = \emptyset$ *;*
(2) $X \in \mathcal{T}_{\mathcal{U}^\star}$ *if and only if there exists* $U \in \mathcal{U}$ *such that* $U(X) = X$*.*

Proof From Corollary 40, we can see that assertion (2) is true. Hence, since for any $U \in \mathcal{U}$ we have $U^\star(X) = X \iff U(X^c)^c = \emptyset^c \iff U(\emptyset) = \emptyset$, it is clear that assertion (1) is also true.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 46 *The following assertions are true :*

(1) $\mathcal{T}_{\mathcal{U}}$ *is a minimal structure on* X *if and only if there exists* $U \in \mathcal{U}$ *such that* $U(\emptyset) = \emptyset$ *;*
(2) $\mathcal{T}_{\mathcal{U}^\star}$ *is a minimal structure on* X *if and only if there exists* $U \in \mathcal{U}$ *such that* $U(X) = X$*.*

In addition to Theorems 140 and 141, we can also easily prove the following

Theorem 142 *If* U *is an increasing super relation on* X*, then the families* $\mathcal{T}_U$ *and* $\mathcal{T}_{U^\star}$ *are closed under arbitrary unions.*

Proof If $\mathcal{A} \subseteq \mathcal{T}_{U^\star}$, then by Corollary 40, for each $A \in \mathcal{A}$, we have $A \subseteq U(A)$. Hence, by using the increasingness of U, we can infer that

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} U(A) \subseteq \bigcup_{A \in \mathcal{A}} U\left(\bigcup \mathcal{A}\right) \subseteq U\left(\bigcup \mathcal{A}\right).$$

Therefore, by Corollary 40, we have $\bigcup \mathcal{A} \in \mathcal{T}_{U^\star}$. Thus, the family $\mathcal{T}_{U^\star}$ is closed under arbitrary unions.

Now, to prove the same assertion for the family $\mathcal{T}_U$, it is enough to note only that if U is increasing, then by Theorem 26 its dual $U^\star$ is also increasing.

Now, by Theorems 141 and 142, we can also state

Corollary 47 *If U is an increasing super relation on X, then*

(1) τ_U is a generalized topology on X if and only if $U(\emptyset)=\emptyset$;
(2) $\tau_{U^\star}$ is a generalized topology on X if and only if $U(X)=X$.

Remark 77 To apply the latter observations to the super relation U considered in Example 5, we can note that, by Theorems 38 and 39, the super relations $\mathrm{cl}_{\mathscr{R}}$ and $\mathrm{int}_{\mathscr{R}}$ are always increasing.

Moreover, by Theorem 78 and the corresponding definitions, we have

(1) $\mathrm{cl}_{\mathscr{R}}(\emptyset)=\emptyset \iff \mathrm{int}_{\mathscr{R}}(X)=X \iff X=\emptyset$ or $\mathscr{R}\neq\emptyset$;
(2) $\mathrm{int}_{\mathscr{R}}(\emptyset)=\emptyset \iff \mathrm{cl}_{\mathscr{R}}(X)=X \iff \mathscr{R}$ is non-partial.

Analogously to Theorem 142, we can also prove the following

Theorem 143 *If U is a union-preserving (resp. intersection-preserving) super relation on X, then the family τ_U $\big($resp. $\tau_{U^\star}\big)$ is closed under arbitrary unions and intersections.*

However, the latter theorem cannot be applied to the super relation U considered in Example 5.

Therefore, in addition to Theorem 142, it is more important to prove the following two closely related theorems.

Theorem 144 *If for any $A,\ B\subseteq X$ and $U\in\mathscr{U}$ we have*

$$U(A)\cap \mathrm{int}_{\mathscr{R}}(B)\subseteq U(A\cap B),$$

then for any $A\in\tau_{\mathscr{U}^\star}$ and $B\in\mathscr{T}_{\mathscr{R}}$ we have $A\cap B\in\tau_{\mathscr{U}^\star}$.

Proof If $A\in\tau_{\mathscr{U}^\star}$, then by Corollary 40 there exists $U\in\mathscr{U}$ such that $A\subseteq U(A)$. Moreover, if $B\in\mathscr{T}_{\mathscr{R}}$, then by Theorem 95, we have $B\subseteq \mathrm{int}_{\mathscr{R}}(B)$. Hence, by using the assumption of the theorem, we can see that

$$A\cap B\subseteq U(A)\cap \mathrm{int}_{\mathscr{R}}(B)\subseteq U(A\cap B).$$

Therefore, by Corollary 40, we also have $A\cap B\in\tau_{\mathscr{U}^\star}$.

Theorem 145 *If for any $A,\ B\subseteq X$ and $U\in\mathscr{U}$ we have*

$$U(A\cup B)\subseteq U(A)\cup \mathrm{cl}_{\mathscr{R}}(B),$$

then for any $A\in\tau_{\mathscr{U}}$ and $B\in\mathscr{T}_{\mathscr{R}}$ we have $A\cap B\in\tau_{\mathscr{U}}$.

Proof Now, by Definition 7 and the assumption of the theorem, for any $A,\ B\subseteq X$ and $U\in\mathscr{U}$ we have

$$U^\star(A)\cap B^\circ = U(A^c)^c\cap B^{c-c}=\big(U(A^c)\cup B^{c-}\big)^c$$

$$\subseteq U\left(A^{c} \cup B^{c}\right)^{c}=U\left(\left(A \cap B\right)^{c}\right)^{c}=U^{\star}(A \cap B)\,.$$

Therefore, Theorem 144 can be applied to the super relator $\mathscr{U}^*$.

Remark 78 The latter two theorems can already be applied to the super relation U considered in Example 5.

Namely, if the relator $\mathscr{R}$ is a topologically filtered, then by Theorem 131 we have

$$\mathrm{cl}_{\mathscr{R}}(A) \cap \mathrm{int}_{\mathscr{R}}(B) \subseteq \mathrm{cl}_{\mathscr{R}}(A \cap B) \qquad \text{and} \qquad \mathrm{int}_{\mathscr{R}}(A \cup B) \subseteq \mathrm{int}_{\mathscr{R}}(A) \cup \mathrm{cl}_{\mathscr{R}}(B)$$

for all $A,\ B \subseteq X$.

31 Topological Closures of Families of Sets

Notation 12 *In this and the next two sections, we shall assume that* $\mathscr{R}$ *is an ordinary relator on* X *and* $\mathscr{V}$ *is a hyper relator on* X *to* Y*, and moreover* $\mathscr{A} \subseteq \mathscr{P}(X)$ *and* $\mathscr{B} \subseteq \mathscr{P}(Y)$.

Definition 22 We define

$$\mathscr{B}^{k}=\mathscr{B}^{k_{\mathscr{V}}}=\mathrm{cl}_{\mathscr{V}}(\mathscr{B}) \qquad \text{and} \qquad \mathscr{A}^{\ell}=\mathscr{A}^{\ell_{\mathscr{V}}}=\mathrm{cl}_{\mathscr{V}^{-1}}(\mathscr{A})\,.$$

Thus, by Theorem 84, we can at once state the following

Theorem 146 *We have*

(1) $\mathscr{A}^{\ell}=\bigcap_{V \in \mathscr{V}} V[\mathscr{A}]$*;*
(2) $\mathscr{B}^{k}=\bigcap_{V \in \mathscr{V}} V^{-1}[\mathscr{B}]$.

From equality (1), we can immediately derive the following

Theorem 147 *For any* $B \subseteq Y$*, the following assertions are equivalent :*

(1) $B \in \mathscr{A}^{\ell}$*;*
(2) For each $V \in \mathscr{V}$ *we have* $B \in V[\mathscr{A}]$*;*
(3) For each $V \in \mathscr{V}$ *there exists* $A \in \mathscr{A}$ *such that* $B \in V(A)$.

While, from the equality $\mathscr{B}^{k}=\mathrm{cl}_{\mathscr{V}}(\mathscr{B})$, we can immediately derive

Theorem 148 *For any* $A \subseteq X$*, the following assertions are equivalent :*

(1) $A \in \mathscr{B}^{k}$*;*
(2) For each $V \in \mathscr{V}$ *we have* $V(A) \cap \mathscr{B} \neq \emptyset$*;*
(3) For each $V \in \mathscr{V}$ *there exists* $B \in \mathscr{B}$ *such that* $B \in V(A)$.

The importance of Definition 22 is apparent from the following

Example 9 If V is a hyper relation on X such that

$$V(A) = \{ B \subseteq X : \quad A \subseteq B \subseteq \operatorname{cl}_{\mathscr{R}}(A) \}$$

for all $A \subseteq X$, then for any $B \subseteq X$ we have

(1) $B \in \mathscr{A}^{\ell_V}$ if and only if there exists $A \in \mathscr{A}$ such that $A \subseteq B \subseteq \operatorname{cl}_{\mathscr{R}}(A)$;
(2) $B \in \mathscr{A}^{k_V}$ if and only if there exists $A \in \mathscr{A}$ such that $B \subseteq A \subseteq \operatorname{cl}_{\mathscr{R}}(B)$.

Namely, if for instance $B \in \mathscr{A}^{k_V}$, then by Theorem 148 there exists $A \in \mathscr{A}$ such that $A \in V(B)$. Hence, by the definition of V, we can infer that $B \subseteq A \subseteq \operatorname{cl}_{\mathscr{R}}(B)$.

Remark 79 The inclusion $A \subseteq \operatorname{cl}_{\mathscr{R}}(B)$ in a detailed form means only that for each $x \in A$ and $R \in \mathscr{R}$ there exists $y \in B$ such that $y \in R(x)$.

Thus, for instance, assertion (2) of Example 9 can be reformulated in the detailed form that $B \in \mathscr{A}^{k_V}$ if and only if there exists $A \in \mathscr{A}$ such that for each $x \in B$ we have $x \in A$, and for each $x \in A$ and $R \in \mathscr{R}$ there exists $y \in B$ such that $y \in R(x)$.

Remark 80 However, it is now more important to note that if V is in Example 9, then $\mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\mathscr{T}_{\mathscr{R}}^{k_V}$ are just the families $\mathscr{T}_{\mathscr{R}}^{q}$ and $\mathscr{T}_{\mathscr{R}}^{ps}$ of all *topologically quasi-open and pseudo-open subsets* of the relator space $X(\mathscr{R})$ considered in [75].

Remark 81 In [74, Section 32], concerning the families τ_U and $\mathscr{T}_{\mathscr{R}}^{k_V}$ mentioned in Remarks 74 and 80, we have proved that :

(1) $\mathscr{T}_{\mathscr{R}}^{k_V} \subseteq \tau_U$ is always true ;
(2) $\mathscr{T}_{\mathscr{R}} \subseteq \mathscr{T}_{\mathscr{R}}^{k_V} \subseteq \tau_U^{k_V}$ if $\mathscr{R}$ is reflexive ;
(3) $\tau_U^{k_V} \subseteq \tau_U$ if $\mathscr{R}$ is quasi-topological ;
(4) $\mathscr{T}_{\mathscr{R}}^{k_V} = \tau_U^{k_V} = \tau_U$ if $\mathscr{R}$ is topological.

Remark 82 In [74], we have proved that if $\mathscr{R}$ is topological, then $\mathscr{A} = \tau_U$ is actually the smallest subset of $\mathscr{P}(X)$ such that $\mathscr{T}_{\mathscr{R}} \subseteq \mathscr{A}$ and $\mathscr{A}^{k_V} \subseteq \mathscr{A}$.

Moreover, if $\mathscr{R}$ is topological and topologically filtered, then for any $B \subseteq X$ we have $B \in \tau_U$ if and only if there exist $A \in \mathscr{T}_{\mathscr{R}}$ and $D \in \mathscr{D}_{\mathscr{R}}$ such that $B = A \cap D$.

Remark 83 In this respect, it is curious that if $\mathscr{R}$ is topological and topologically filtered, then for any $B \in \tau_{U^\star}$ there exist $A \in \mathscr{T}_{\mathscr{R}}$ and $N \in \mathscr{N}_{\mathscr{R}}$ such that $B = A \cup N$ and $A \cap N = \emptyset$.

However, the converse statement need not be true. Moreover, the genuine characterizations of $\tau_{U^\star}$, established in [75, Section 24], do not require the relator $\mathscr{R}$ to be topologically filtered.

32 Some Basic Properties of the Operations k and ℓ

From the general properties of the induced topological closures or Theorems 147 and 148, we can easily establish several useful properties of the operations k and ℓ introduced in Definition 22.

For instance, by some general theorems on ordinary relators, we can at once state the following three theorems.

Theorem 149 *We have*

(1) $\emptyset^k = \emptyset$ *if and only if* $\mathscr{V} \neq \emptyset$ *;*
(2) $\mathscr{P}(Y)^k = \mathscr{P}(X)$ *if and only if* $\mathscr{V}$ *is non-partial.*

Theorem 150 *The following assertions hold :*

(1) k *is always increasing ;*
(2) k *is additive if and only if* $\mathscr{V}$ *is topologically filtered ;*
(3) k *is union-preserving if and only if* $\mathscr{V}$ *is topologically simple.*

Theorem 151 *In the* $X = Y$ *particular case, the following assertions hold :*

(1) k *is extensive if and only if* $\mathscr{V}$ *is reflexive ;*
(2) k *is upper quasi-idempotent if and only if* $\mathscr{V}$ *is quasi-topological.*

Example 10 If V is as in Example 9, then the following assertions are equivalent :

(1) V is reflexive ; (2) V is non-partial ; (3) $\mathscr{R}$ is reflexive.

To prove the implication (2) $\Longrightarrow$ (3), note that if (2) holds then for any $A \subseteq X$ we have $V(A) \neq \emptyset$. Therefore, there exists $B \subseteq X$ such that $B \in V(A)$. Hence, by the definition of V, we can infer that $A \subseteq B \subseteq \mathrm{cl}_{\mathscr{R}}(A)$. Therefore, for any $A \subseteq X$, we have $A \subseteq \mathrm{cl}_{\mathscr{R}}(A)$. Thus, by Theorem 112, assertion (3) also holds.

Example 11 If the relator $\mathscr{R}$ is quasi-topological, then the hyper relation V considered in Example 9 is transitive, and thus in particular it is also quasi-topological.

Namely, if $B \in V(A)$ and $C \in V(B)$, then by the definition of V, we have

$$A \subseteq B \subseteq \mathrm{cl}_{\mathscr{R}}(A) \qquad \text{and} \qquad B \subseteq C \subseteq \mathrm{cl}_{\mathscr{R}}(B)\,.$$

Thus, in particular $A \subseteq C$. Moreover, by Theorems 80 and 117, we also have

$$C \subseteq \mathrm{cl}_{\mathscr{R}}(B) \subseteq \mathrm{cl}_{\mathscr{R}}\big(\,\mathrm{cl}_{\mathscr{R}}(A)\big) \subseteq \mathrm{cl}_{\mathscr{R}}(A).$$

Hence, by the definition of V, we can already see that $C \in V(A)$. Therefore,

$$B \in V(A) \quad \text{and} \quad C \in V(B) \qquad \text{imply} \qquad C \in V(A)\,,$$

and thus V is transitive.

By Theorem 117, we can see that V is quasi-topological if and only if it is transitive. However, if V is quasi-topological, then it is certainly not true that $\mathscr{R}$ is also quasi-topological.

Remark 84 If in particular $\mathscr{R}$ is topological, then by Theorem 118 $\mathscr{R}$ is both reflexive and quasi-topological. Therefore, by Examples 10 and 11, the super relation considered in Example 9 is both reflexive and quasi-topological. Thus, by Theorems 150 and 151, we can state that k_V is a union-preserving closure relation on $\mathscr{P}(X)$.

To prove the same property of the relation ℓ_V, we have to note only that if V is a preorder, then V^{-1} is also a preorder. Thus, in particular V^{-1} is also reflexive and quasi-topological.

33 Some Set-Theoretic Properties of the Families $\mathscr{A}^{\ell}$ and $\mathscr{B}^{k}$

Definition 23 The hyper relator $\mathscr{V}$ will be called

(1) *empty-set-stable* if $\emptyset \in V(\emptyset)$ for all $V \in \mathscr{V}$;
(2) *ground-set-stable* if $Y \in V(X)$ for all $V \in \mathscr{V}$.

Thus, by using Theorems 147 and 148, we can easily prove the two theorems.

Theorem 152 *If $\mathscr{V}$ is empty-set-stable, then*

(1) $\emptyset \in \mathscr{A}$ implies $\emptyset \in \mathscr{A}^{\ell}$; *(2) $\emptyset \in \mathscr{B}$ implies $\emptyset \in \mathscr{B}^{k}$.*

Proof To prove (1), note that by Theorem 147 we have $\emptyset \in \mathscr{A}^{\ell}$ if and only if for each $V \in \mathscr{V}$ there exists $A \in \mathscr{A}$ such that $\emptyset \in V(A)$. Therefore, if $\emptyset \in \mathscr{A}$ and $\emptyset \in V(\emptyset)$ for all $V \in \mathscr{V}$, then $\emptyset \in \mathscr{A}^{\ell}$.

Theorem 153 *If $\mathscr{V}$ is ground-set-stable, then*

(1) $X \in \mathscr{A}$ implies $Y \in \mathscr{A}^{\ell}$; *(2) $Y \in \mathscr{B}$ implies $X \in \mathscr{B}^{k}$.*

Proof To prove (2), note that by Theorem 148 we have $X \in \mathscr{B}^{k}$ if and only if for each $V \in \mathscr{V}$ there exists $B \in \mathscr{B}$ such that $B \in V(X)$. Therefore, if $Y \in \mathscr{B}$ and $Y \in V(X)$ for all $V \in \mathscr{V}$, then $X \in \mathscr{B}^{k}$.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 48 *If $\mathscr{V}$ is both empty-set-stable and ground-set-stable, then*

(1) $\mathscr{A}^{\ell}$ is a minimal structure if $\mathscr{A}$ is a minimal structure;
(2) $\mathscr{B}^{k}$ is a minimal structure if $\mathscr{B}$ is a minimal structure.

The appropriateness of Definition 23 is also apparent from the following

Example 12 If V is as in Example 9, then

(1) V is empty-set-stable ;
(2) V is ground-set-stable if and only if $\mathscr{R}$ is non-partial.

To prove (2), note that by the corresponding definitions and Theorem 114 we have

$$X \in V(X) \iff X \subseteq X \subseteq \operatorname{cl}_{\mathscr{R}}(X)$$
$$\iff X = \operatorname{cl}_{\mathscr{R}}(X) \iff X \in \mathscr{D}_{\mathscr{R}} \iff \mathscr{R} \text{ is non-partial}.$$

Remark 85 Therefore, if V is as in Example 9, then

(1) $\emptyset \in \mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\emptyset \in \mathscr{T}_{\mathscr{R}}^{k_V}$ are always true ;
(2) $X \in \mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $X \in \mathscr{T}_{\mathscr{R}}^{k_V}$ if $\mathscr{R}$ is non-partial and either $X = \emptyset$ or $\mathscr{R} \neq \emptyset$.

To prove (2), note that if $\mathscr{R}$ is non-partial, then by Example 12 the relation V is ground-set-stable. While, if either $X = \emptyset$ or $\mathscr{R} \neq \emptyset$, then by Theorem 105 we have $X \in \mathscr{T}_{\mathscr{R}}$. Thus, Theorem 153 can be applied.

In addition to Definition 23, it is also worth introducing

Definition 24 The hyper relator $\mathscr{V}$ will be called *union-compatible* if $V \in \mathscr{V}$ and $B_i \in V(A_i)$ for all $i \in I$ imply that

$$\bigcup_{i \in I} B_i \in V\Big(\bigcup_{i \in I} A_i\Big).$$

Remark 86 The intersection-compatibility of $\mathscr{V}$ is to be defined quite similarly.

Thus, by letting $I = \emptyset$, we can see that if $\mathscr{V}$ is union-compatible (intersection-compatible), then $\mathscr{V}$ is in particular empty-set-stable (ground-set-stable).

Theorem 154 *If $\mathscr{V}$ is union-compatible, then*

(1) $\mathscr{A}^{\ell}$ is closed under unions if $\mathscr{A}$ is closed under unions ;
(2) $\mathscr{B}^{k}$ is closed under unions if $\mathscr{B}$ is closed under unions.

Proof To prove (1), note that if $B_i \in \mathscr{A}^{\ell}$ for all $i \in I$, and $V \in \mathscr{V}$, then by Theorem 147, for each $i \in I$, there exists $A_i \in \mathscr{A}$ such that $B_i \in V(A_i)$. Hence, if $\mathscr{A}$ is closed under unions, we can infer that $\bigcup_{i \in I} A_i \in \mathscr{A}$. Moreover, since $\mathscr{V}$ is union-compatible we can also state that $\bigcup_{i \in I} B_i \in V\big(\bigcup_{i \in I} A_i\big)$. Therefore, by Theorem 147, we also have $\bigcup_{i \in I} B_i \in \mathscr{A}^{\ell}$.

Now, as an immediate consequence of Theorems 153 and 154, we can also state

Corollary 49 *If $\mathscr{V}$ is both union-compatible and ground-set-stable, then*

(1) $\mathscr{A}^{\ell}$ is a generalized topology if $\mathscr{A}$ is a generalized topology ;
(2) $\mathscr{B}^{k}$ is a generalized topology if $\mathscr{B}$ is a generalized topology.

Example 13 If V is as in Example 9, then V is union-compatible.

Namely, if $B_i \in V(A_i)$ for all $i \in I$, then $A_i \subseteq B_i \subseteq \mathrm{cl}_{\mathscr{R}}(A_i)$ for all $i \in I$. Hence, by using the increasingness of $\mathrm{cl}_{\mathscr{R}}$, we can infer that

$$\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i \subseteq \bigcup_{i \in I} \mathrm{cl}_{\mathscr{R}}(B_i) \subseteq \mathrm{cl}_{\mathscr{R}}\Big(\bigcup_{i \in I} A_i\Big).$$

Thus, $\bigcup_{i \in I} B_i \in V\big(\bigcup_{i \in I} A_i\big)$ also holds.

Remark 87 Therefore, if V is as in Examples 9 then the families $\mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\mathscr{T}_{\mathscr{R}}^{k_V}$ are closed under unions. Thus, if in particular $\mathscr{R}$ is nonvoid and non-partial, then the above families are generalized topologies.

Namely, by Example 13, the relation V is union-compatible. Moreover, by Theorem 105, the family $\mathscr{T}_{\mathscr{R}}$ is closed under unions. Therefore, by Theorems 154 the families $\mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\mathscr{T}_{\mathscr{R}}^{k_V}$ are also closed under unions. Moreover, if $\mathscr{R}$ is nonvoid and non-partial, then by Remark 85 we also have $X \in \mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $X \in \mathscr{T}_{\mathscr{R}}^{k_V}$.

Now, in addition to Example 13, we can also easily establish

Example 14 If V is as in Example 9, then V need not even be finitely intersection-compatible.

Namely, if for instance $X = \mathbb{R}$ and

$$\mathscr{R} = \big\{ R_n : \ n \in \mathbb{N} \big\} \qquad \text{with} \qquad R_n = \big\{ (x, y) \in X^2 : \quad d(x, y) < n^{-1} \big\}$$

for all $n \in \mathbb{N}$, then by taking

$$A_1 = \mathbb{Q}, \qquad A_2 = \mathbb{Q}^c \qquad \text{and} \qquad B_1 = B_2 = X,$$

we can see that

$$A_i \subseteq B_i \subseteq \mathrm{cl}_{\mathscr{R}}(A_i), \qquad \text{and thus} \qquad B_i \in V(A_i)$$

for $i = 1, 2$. However,

$$B_1 \cap B_2 \not\subseteq \mathrm{cl}_{\mathscr{R}}(A_1 \cap A_2), \qquad \text{and thus} \qquad B_1 \cap B_2 \notin V(A_1 \cap A_2).$$

Remark 88 Therefore, it is not surprising that if X and $\mathscr{R}$ are as in Example 14 and V is as in Example 9, then the families $\mathscr{T}_{\mathscr{R}}^{\ell_V}$ and $\mathscr{T}_{\mathscr{R}}^{k_V}$ are not closed even under finite intersections.

Namely, if for instance

$$A = [0, 1], \qquad B = [1, 2] \qquad \text{and} \qquad C = \mathbb{Q}, \qquad D = \{1\} \cup \mathbb{Q}^c,$$

then we can easily see that

$$A, B \in \mathscr{T}_{\mathscr{R}}^{\ell_V}, \qquad C, D \in \mathscr{T}_{\mathscr{R}}^{k_V}, \qquad \text{but} \qquad A \cap B \notin \mathscr{T}_{\mathscr{R}}^{\ell_V}, \qquad C \cap D \notin \mathscr{T}_{\mathscr{R}}^{k_V}.$$

34 A Weak Intersection Property of the Families $\mathscr{A}^k$ and $\mathscr{A}^\ell$

Notation 13 *In this section, we shall assume that $\mathscr{R}$ is an ordinary relator and $\mathscr{V}$ is a hyper relator on X.*

Moreover, we shall assume that $\mathscr{V}$ is weakly intersection-compatible with respect to $\mathscr{R}$ in the sense that, for any $V \in \mathscr{V}$, $A, B \subseteq X$ and $C \in \mathscr{T}_{\mathscr{R}}$,

$$B \in V(A) \qquad \textit{implies} \qquad B \cap C \in V(A \cap C).$$

By using this weak intersection property, we can easily prove the following

Theorem 155 *If $\mathscr{A} \subseteq \mathscr{P}(X)$ and $B \in \mathscr{T}_{\mathscr{R}}$ such that $A \cap B \in \mathscr{A}$ for all $A \in \mathscr{A}$, then we also have*

(1) $A \cap B \in \mathscr{A}^k$ for all $A \in \mathscr{A}^k$; *(2) $A \cap B \in \mathscr{A}^\ell$ for all $A \in \mathscr{A}^\ell$.*

Proof If $A \in \mathscr{A}^k$ and $V \in \mathscr{V}$, then by Theorem 148 there exists $C \in \mathscr{A}$ such that $C \in V(A)$. Hence, by using the weak intersection-compatibility of $\mathscr{V}$, we can infer that

$$C \cap B \in V(A \cap B).$$

Moreover, by the assumption of the theorem, we also have $C \cap B \in \mathscr{A}$. Hence, by Theorem 148, we can see that $A \cap B \in \mathscr{A}^k$ also holds.

While, if $A \in \mathscr{A}^\ell$ and $V \in \mathscr{V}$, then by Theorem 147 there exists $C \in \mathscr{A}$ such that $A \in V(C)$. Hence, by using the weak intersection-compatibility of $\mathscr{V}$, we can infer that

$$A \cap B \in V(C \cap B).$$

Moreover, by the assumption of the theorem, we also have $C \cap B \in \mathscr{A}$. Hence, by Theorem 147, we can see that $A \cap B \in \mathscr{A}^k$ also holds.

Repeated applications of this theorem give the following

Corollary 50 *If $\mathscr{A}$ and B are as in Theorem 155, then $A \cap B \in \mathscr{A}^\kappa$ for all $A \in \mathscr{A}^\kappa$ with $\kappa = kk, \ell\ell, k\ell$ and ℓk.*

Proof For instance, by assertion (1) of Theorem 155, we have $A \cap B \in \mathscr{A}^k$ for all $A \in \mathscr{A}^k$. Hence, for instance, by applying assertion (2) of Theorem 155 to the family $\mathscr{A}^k$ instead of $\mathscr{A}$, we can infer that $A \cap B \in \mathscr{A}^{k\ell}$ for all $A \in \mathscr{A}^{k\ell}$.

From this corollary, we can immediately derive the following

Corollary 51 *If $\mathscr{R}$ is quasi-topologically filtered and $B \in \mathscr{T}_{\mathscr{R}}$, then $A \cap B \in \mathscr{T}_{\mathscr{R}}^\kappa$ for all $A \in \mathscr{T}^\kappa$ with $\kappa = kk, \ell\ell, k\ell$, and ℓk.*

Proof Now, by Theorem 128, we have $A \cap B \in \mathscr{T}_{\mathscr{R}}$ for all $A \in \mathscr{T}_{\mathscr{R}}$. Therefore, by Theorem 155, we also have $A \cap B \in \mathscr{T}_{\mathscr{R}}^{\kappa}$ for all $A \cap B \in \mathscr{T}^{\kappa}$ with $\kappa = \ell, k$.

The appropriateness of our present definition of weak intersection-compatibility is also apparent from the following

Example 15 If V is as in Example 9 and $\mathscr{R}$ is topologically filtered, then V is weakly intersection-compatible with respect to $\mathscr{R}$.

Namely, if $B \in V(A)$, then $A \subseteq B \subseteq \operatorname{cl}_{\mathscr{R}}(A)$. Hence, by using Corollary 35, we can infer that

$$A \cap C \subseteq B \cap C \subseteq \operatorname{cl}_{\mathscr{R}}(A) \cap C \subseteq \operatorname{cl}_{\mathscr{R}}(B \cap C)$$

for all $C \in \mathscr{T}_{\mathscr{R}}$. Therefore, $B \cap C \in V(A \cap C)$ also holds for all $C \in \mathscr{T}_{\mathscr{R}}$.

The fact that the condition $C \in \mathscr{T}_{\mathscr{R}}$, in the above proof, cannot be either omitted or replaced by $C \in \mathscr{F}_{\mathscr{R}}$ can be at once seen from the following

Example 16 If X and $\mathscr{R}$ are as in Example 14 and V is as in Example 9, then $\mathscr{R}$ is a properly filtered relator on X such that, for the sets

$$A = [\,0,\ 1\,[\qquad B = [\,0,\ 1\,] \qquad \text{and} \qquad C = \{1\},$$

we have $B \in V(A)$ and $B \cap C \notin V(A \cap C)$.

35 Some Further Theorems on the Operations ℓ and k

Notation 14 *In this section, we shall again assume that* $\mathscr{R}$ *is an ordinary relator and* $\mathscr{V}$ *is a hyper relator on* X.

Moreover, we shall assume that $\mathscr{V}$ *is strongly closure-compatible with respect to* $\mathscr{R}$ *in the sense that, for any* $V \in \mathscr{V}$ *and* $A, B \subseteq X$,

$$B \in V(A) \qquad \textit{implies} \qquad \operatorname{cl}_{\mathscr{R}}(A) \in V(B) \quad \textit{and} \quad \operatorname{cl}_{\mathscr{R}}(B) \in V\big(\operatorname{cl}_{\mathscr{R}}(A)\big).$$

Remark 89 From the above inclusions, we can infer that

$$\operatorname{cl}_{\mathscr{R}}(A) \in V(B) \subseteq V[\,V(A)\,] = (V \circ V)(A),$$
$$\operatorname{cl}_{\mathscr{R}}(B) \in V\big(\operatorname{cl}_{\mathscr{R}}(A)\big) \subseteq V[\,V(B)\,] = (V \circ V)(B),$$
$$\operatorname{cl}_{\mathscr{R}}(B) \in (V \circ V)(B) \subseteq (V \circ V)[\,V(A)\,] = (V \circ V \circ V)(A).$$

Therefore, if in particular V is transitive, then we can also state that

$$\operatorname{cl}_{\mathscr{R}}(A) \in V(A), \qquad \operatorname{cl}_{\mathscr{R}}(B) \in V(B) \qquad \text{and} \qquad \operatorname{cl}_{\mathscr{R}}(B) \in V(A).$$

The appropriateness of our present definition of strongly closure-compatibility is apparent from the following

Example 17 If V is as in Example 9, then V is strongly closure-compatible with respect to $\mathscr{R}$.

Namely, if $A, B \subseteq X$ such that $B \in V(A)$, then by using the definition of V and the notation $A^- = \text{cl}_{\mathscr{R}}(A)$, we can see that

$$A \subseteq B \subseteq A^- .$$

Hence, by using the increasingness of the operation $-$, we can already infer that

$$B \subseteq A^- \subseteq B^- \qquad \text{and} \qquad A^- \subseteq B^- \subseteq A^{--} .$$

Therefore, by the definition of V, we also have $A^- \in V(B)$ and $B^- \in V(A^-)$.

Remark 90 If $\mathscr{R}$ is topological, then from the inclusion $A^- \subseteq B^- \subseteq A^{--}$, by using Theorems 112 and 117, we can also infer that $A \subseteq B^- \subseteq A^-$, and thus $B^- \in V(A)$.

However, this seems to be a weaker statement than that can be obtained from Remark 89. Namely, in Example 11, to prove the transitivity of V it was enough to assume only that $\mathscr{R}$ is quasi-topological.

Now, by using the operations $k = k_{\mathscr{V}}$ and $\ell = \ell_{\mathscr{V}}$, and the elementwise closure $- = \text{cl}_{\mathscr{R}}$, we can prove the following

Theorem 156 *For any $\mathscr{A} \subseteq \mathscr{P}(X)$, we have*

(1) $\mathscr{A}^{\ell} \subseteq \mathscr{A}^{-k}$; *(2)* $\mathscr{A}^{k-} \subseteq \mathscr{A}^{\ell}$;
(3) $\mathscr{A}^{\ell -} \subseteq \mathscr{A}^{-\ell}$; *(4)* $\mathscr{A}^{k-} \subseteq \mathscr{A}^{-k}$.

Proof If $A \in \mathscr{A}^k$ and $V \in \mathscr{V}$, then by Theorem 148 there exists $B \in \mathscr{A}$ such that $B \in V(A)$. Hence, by using the second part of the strong closure-compatibility of $\mathscr{V}$, we can infer that $B^- \in V(A^-)$. Now, since $B^- \in \mathscr{A}^-$, by Theorem 148 we can see that $A^- \in \mathscr{A}^{-k}$. Therefore, $\mathscr{A}^{k-} \subseteq \mathscr{A}^{-k}$.

While, if $B \in \mathscr{A}^{\ell}$ and $V \in \mathscr{V}$, then by Theorem 147 there exists $A \in \mathscr{A}$ such that $B \in V(A)$. Hence, by using the first part of the strong closure-compatibility of $\mathscr{V}$, we can infer that $A^- \in V(B)$. Now, since $A^- \in \mathscr{A}^-$, by Theorem 148 we can see that $B \in \mathscr{A}^{-k}$. Therefore, $\mathscr{A}^{\ell} \subseteq \mathscr{A}^{-k}$.

Thus, we have proved assertions (1) and (4). The proof of assertions (2) and (3) is quite similar.

From this theorem, by using some basic properties of the operations k and ℓ, we can easily derive the following two corollaries.

Corollary 52 *If $\mathscr{V}^{-1}$ is quasi-topological, then for any $\mathscr{A} \subseteq \mathscr{P}(X)$ we have*

(1) $\mathscr{A}^{\ell k} \subseteq \mathscr{A}^{-k}$; *(2)* $\mathscr{A}^{k \ell} \subseteq \mathscr{A}^{-k}$.

Proof By using Theorems 156, 150, and 151, we can see that

$$\mathscr{A}^{\,\ell\, k} \subseteq \mathscr{A}^{-\,k\,k} \subseteq \mathscr{A}^{-\,k} \qquad \text{and} \qquad \mathscr{A}^{\,k\,\ell} \subseteq \mathscr{A}^{\,k\,-\,k} \subseteq \mathscr{A}^{-\,k\,k} \subseteq \mathscr{A}^{-\,k}\,.$$

Remark 91 Note that, by the proof of Theorem 156, assertion (1) does not also need the second part of the strong closure compatibility of $\mathscr{V}$. While, to prove assertion (2) both parts seem to be necessary.

Corollary 53 *If* $\mathscr{V}^{-1}$ *is quasi-topological, then for any* $\mathscr{A} \subseteq \mathscr{P}(X)$ *we have*

(1) $\mathscr{A}^{\ell\, k\,-} \subseteq \mathscr{A}^{\ell}$*;* *(2)* $\mathscr{A}^{k\,\ell\,-} \subseteq \mathscr{A}^{\ell}$.

Proof By using Theorems 156 and the counterparts of Theorem 150 and 151, we can see that

$$\mathscr{A}^{\ell k-} \subseteq \mathscr{A}^{\ell\ell} \subseteq \mathscr{A}^{\ell} \qquad \text{and} \qquad \mathscr{A}^{k\,\ell-} \subseteq \mathscr{A}^{k-\ell} \subseteq \mathscr{A}^{\ell\ell} \subseteq \mathscr{A}^{\ell}\,.$$

Remark 92 Note that if $\mathscr{V}$ is transitive, then $\mathscr{V}^{-1}$ is also transitive. Therefore, both $\mathscr{V}$ and $\mathscr{V}^{-1}$ are quasi-topological.

In this case, we can prove the assertions of the above two corollaries directly. Moreover, we can also easily prove the following

Theorem 157 *If* $\mathscr{V}$ *is transitive, then for any* $\mathscr{A} \subseteq \mathscr{P}(X)$ *we have*

(1) $\mathscr{A}^{\,k} \subseteq \mathscr{A}^{-\,k}$*;* *(2)* $\mathscr{A}^{\,\ell\,-} \subseteq \mathscr{A}^{\,\ell}$.

Proof If $A \in \mathscr{A}^{k}$ and $V \in \mathscr{V}$, then by Theorem 148 there exists $B \in \mathscr{A}$ such that $B \in V(A)$. Hence, by using Remark 89, we can infer that $B^{-} \in V(A)$. Now, since $B^{-} \in \mathscr{A}^{-}$, by Theorem 148 we can see that $A \in \mathscr{A}^{-\,k}$. Therefore, (1) is true.

While, if $B \in \mathscr{A}^{\ell}$ and $V \in \mathscr{V}$, then by Theorem 147 there exists $A \in \mathscr{A}$ such that $B \in V(A)$. Hence, by using Remark 89, we can infer that $B^{-} \in V(A)$. Now, by Theorem 147, we can see that $B^{-} \in \mathscr{A}^{\,\ell}$. Therefore, (2) is also true.

Remark 93 If $\mathscr{V}$ is reflexive, then by Theorem 151 and its counterpart, both k and ℓ are expansive. Therefore, in this case assertions (1) and (2) of Theorem 157 can be derived from assertions (2) of Corollaries 52 and 53.

36 A Further Theorem on Proximally Closed Sets

Notation 15 *In this section, we shall assume that* Φ*,* Ψ *and* U *are super relations, and* V *is a hyper relation on* X *such that*

$$V(A) = \big\{\, B \subseteq X : \quad \Phi(A) \subseteq B \subseteq \Psi(A) \,\big\}$$

for all $A \subseteq X$.

Thus, by using our former results, we can easily prove the following

Theorem 158 *We have*

(1) $\tau_{U^\star}^{k_V} \subseteq \tau_{U^\star \circ \Psi^\star}$ *if* Φ *is extensive and* U *increasing ;*
(2) $\tau_{U^\star}^{\ell_V} \subseteq \tau_{\Psi^\star \circ U^\star}$ *if* Φ *is extensive and both* Ψ *and* U *are increasing.*

Proof If $A \in \tau_{U^\star}^{k_V}$, then by Theorem 148 we can see that there exists $B \in \tau_{U^\star}$ such that $B \in V(A)$. Hence, by using Corollary 41 and the definition of V, we can infer that

$$B \subseteq U(B) \qquad \text{and} \qquad \Phi(A) \subseteq B \subseteq \Psi(A).$$

Thus, if Φ is extensive and U is increasing, then we can also state that

$$A \subseteq \Phi(A) \subseteq B \subseteq U(B) \subseteq U\big(\Psi(A)\big) = (U \circ \Psi)(A).$$

Hence, by using Corollary 41, we can infer that $A \in \tau_{(U \circ \Psi)^\star}$.

While, if $A \in \tau_{U^\star}^{\ell_V}$, then by Theorem 147 there exists $B \in \tau_{U^\star}$ such that $A \in V(B)$. Hence, by using Corollary 41 and the definition of V, we can infer that

$$B \subseteq U(B) \qquad \text{and} \qquad \Phi(B) \subseteq A \subseteq \Psi(B).$$

Thus, if Φ is extensive and U is increasing, then we can also state that

$$B \subseteq U(B) \subseteq U\big(\Phi(B)\big) \subseteq U(A.$$

Moreover, if Ψ is also increasing, then we can also state that

$$A \subseteq \Psi(B) \subseteq \Psi\big(U(A)\big) = (\Psi \circ U)(A).$$

Hence, by using Corollary 41, we can infer that $A \in \tau_{(\Psi \circ U)^\star}$.

The above arguments show that

(a) $\tau_{U^\star}^{k_V} \subseteq \tau_{(U \circ \Psi)^\star}$ if Φ is extensive and U is increasing ;
(b) $\tau_{U^\star}^{\ell_V} \subseteq \tau_{(\Psi \circ U)^\star}$ if Φ is extensive and both U and Ψ are increasing.

Thus, to complete the proof, it remains only to note only that, by Theorem 28, we have $(U \circ \Psi)^\star = U^\star \circ \Psi^\star$ and $(\Psi \circ U)^\star = \Psi^\star \circ U^\star$.

Remark 94 From the above theorem, by writing $U^\star$ and $\Psi^\star$ in place of U and Ψ, respectively, we can get some simpler assertions for the hyper relation W defined such that $W(A) = \{B \subseteq X : \ \Phi(A) \subseteq B \subseteq \Psi^\star(A)\}$ for all $A \subseteq X$.

However, it is now more important to note that, as an immediate consequence of Theorem 158, we can also state the following

Corollary 54 *If $\mathscr{R}$ is a relator on X, and moreover Φ is extensive and*

$$\Psi = \mathrm{cl}_{\mathscr{R}} \qquad \textit{and} \qquad U = \mathrm{int}_{\mathscr{R}}\,,$$

then

(1) $\mathscr{T}_{\mathscr{R}}^{k_V} \subseteq \tau_{\Psi \circ U}$ *;* *(2)* $\mathscr{T}_{\mathscr{R}}^{\ell_V} \subseteq \tau_{U \circ \Psi}$ *.*

Proof From Theorems 80 and 81, we know that Ψ and U are increasing. Moreover, by Theorem 79 and Definition 7, we have that $\Psi^{\star} = U$ and $U^{\star} = \Psi$.

Furthermore, by using Corollary 41 and Theorem 95, we can see that

$$\tau_{U^{\star}} = \left\{ A \subseteq X : \quad A \subseteq U(A) \right\} = \left\{ A \subseteq X : \quad A \subseteq \mathrm{int}_{\mathscr{R}}(A) \right\} = \mathscr{T}_{\mathscr{R}}\,.$$

Hence, by Theorem 158, we can already see that

$$\mathscr{T}_{\mathscr{R}}^{k_V} = \tau_{U^{\star}}^{k_V} \subseteq \tau_{U^{\star} \circ \Psi^{\star}} = \tau_{\Psi \circ U} \qquad \text{and} \qquad \mathscr{T}_{\mathscr{R}}^{\ell_V} = \tau_{U^{\star}}^{\ell_V} \subseteq \tau_{\Psi^{\star} \circ U^{\star}} = \tau_{U \circ \Psi}\,.$$

Remark 95 Note that, from the $\Phi = \Delta$ particular case of this corollary, we can already derive assertion (1) of Remark 81 and its dual.

Moreover, Theorem 158 is not a substantial generalization of Corollary 54. Namely, by [91], for an increasing super relation U on X, with $U(X) = X$, there exists a nonvoid relator $\mathscr{R}$ on X such that $U = \mathrm{int}_{\mathscr{R}}$.

Remark 96 Now, following O'Neil [66] and [116], we may also safely state that: Two or three relators are better than one.

Note that by using two ordinary relators $\mathscr{R}$ and $\mathscr{S}$, a super relator $\mathscr{U}$, and a hyper relator $\mathscr{V}$ on X, our present results can be greatly generalized.

Moreover, the non-conventional three relator space $X(\mathscr{R}, \mathscr{U}, \mathscr{V})$ can probably be also used for other purposes than generalizations of topologically open sets.

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Applications of Apostol-type Numbers and Polynomials: Approach to Techniques of Computation Algorithms in Approximation and Interpolation Functions

Yilmaz Simsek

Abstract The purpose of this chapter is to survey and make a compilation that covers many families of the special numbers and polynomials including the Apostol-Bernoulli numbers and polynomials, the Apostol-Euler numbers and polynomials, the Apostol-Genocchi numbers and polynomials, the Fubini numbers, the Stirling numbers, the Frobenius-Euler polynomials, and the others, blending new results for of the polynomials $W_n(x;\lambda)$, which were given in: *Y. Simsek, Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers, Math. Meth. Appl. Sci., 40 (2017), 2347–2361*. Many well-known results of these polynomials are given in this chapter. Using these known and new results, a large number of new formulas and new relations are created. Some well-known relations among the polynomials $W_n(x;\lambda)$, the Bernoulli and Euler polynomials of higher order, Apostol-type polynomials (Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Apostol-Genocchi polynomials, etc.) are given. It has been presented in new relations related to these polynomials. Some open problems are raised from the results for the polynomials $W_n(x;\lambda)$. Behaviors of the polynomials $W_n(x;\lambda)$ under integral transforms are also examined in this chapter. Firstly, Laplace transform of the polynomials $W_n(x;\lambda)$ is given. With the help of this transformation, new infinite series representations are found. Then, the behavior of the polynomials $W_n(x;\lambda)$ under the Melin transform is also given with help of the works Kucukoglu et al. (Quaest Math 42(4):465–478, 2019) and Simsek (AIP Conf Proc 1978:040012-1–040012-4, 2018). With the aid of this transformation, some relationships with the family of zeta functions are also blended in detail with the previously well-known results using values from negative integers. Since these results are known to be used frequently in both approximation theory, number theory, analysis of functions, and mathematical physics, these results can potentially be used in these scientific areas. In addition, it has been tried to give

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a detailed perspective on the applications of the polynomials $W_n(x;\lambda)$ with their generating functions in approximation theory. Firstly, with the help of computational algorithms, basic known information about numerical values and graphics of these polynomials are introduced. With the help of the algorithm given for these numerical values, the approach steps are tried to be given in detail. These details are then illustrated on graphics and shapes, so that the visual approach steps are made clearer. As a result, this chapter is compiled by blending, interpreting, and comparing the fundamental properties of the polynomials $W_n(x;\lambda)$ and the numbers $W_n(\lambda)$ with their generating functions and other special numbers and polynomials.

1 Preliminaries

The numbers are based on very ancient times. Therefore, there is a very close relationship between numbers and civilized communities. These civilized societies enabled the numbers to improve and to be used in daily life and trade. Each scientific development of human history has developed in proportion to numbers. As a result of the existence of different number sets outside of geometry, theory of analysis and functions, algebra and number theory, which are the main areas of mathematics, as well as other fields of physics and engineering, were provided and developed. Of course, it is not unusual to give the development and applications of numbers in this section. This development of numbers is based on similar development, especially in the presence and application of polynomials. When constructing any (special) family of polynomials, the problem of finding their coefficients depend on (special) numbers arises. This kind of problem requires difficult approaches and methods in order to discover explicit coefficient relations involving (special) numbers. The reason why (special) polynomials are important is that they can be applied easily in many areas, especially mathematics, physics, and engineering. Since algebraic operations with polynomials, derivative operation and integral operation are very easy, they are used in many other areas in mathematical modeling, approximation theory, modeling solution of real-world problems, and other problems. This section focuses on the investigation and survey of the polynomials $W_n(x;\lambda)$ and the numbers $W_n(\lambda)$ with generating functions. These polynomials are given relations with many families of special numbers and polynomials. Later on, their known relationships with the theory of approximation are examined and surveyed in detail. Because theory of approximation and its applications arise in all branches of engineering, physics, applied mathematics, and in many other disciplines. We believe that many results of the polynomials $W_n(x;\lambda)$ and the numbers $W_n(\lambda)$ with generating functions involving their approximations and numerical values may potentially be used in all branches of previous areas.

In recent years, we observe that techniques of the approximation are used frequently in approach theory, mathematics, statistics, probability theory, physics, engineering, economics, and other sciences. Recalling that, in mathematics and their applications, techniques of the approximation are related to how functions can best be approximated with simpler functions, and with quantitatively designating the errors presented thereby. There are various desire reasons for studying and investigating the approximation theory with their applications. Because approximation theory needs to represent functions in computer calculations to an interest in the theory of mathematics and other areas. Recently, approximation theory with its algorithms have been used in several areas of the sciences and also in many industrial and commercial areas.

In the work of Surana [122], he mentioned that all numerical methods are grouped in two categories as follows:

The first category: The numerical methods that do not involve any approximations. In such methods the calculated numerical solutions are exact solutions of the mathematical models within the accuracy of computations on the computer. Such methods refer to numerical methods or numerical methods without approximation.

The second category: Those methods in which the numerically calculated solution is always approximate. Such methods refer to methods of approximation or numerical methods with approximations. In such methods often one can progressively approach (converge to) the true solution but can never obtain precise theoretical solution.

This chapter presents an analysis of the polynomials $W_n(x;\lambda)$ including the Apostol-Bernoulli numbers and polynomials, the Apostol-Euler numbers and polynomials, the Apostol-Genocchi numbers and polynomials, the Fubini numbers and polynomials, and other special numbers and polynomials with their interpolation functions and their approximations and numerical values, and also asymptotic behavior.

This present chapter also deals with some fundamental properties of the polynomials $W_n^{(k)}(x,\lambda)$ and the numbers $W_n^{(k)}(\lambda)$, which was given by Kucukoglu and Simsek [42], with properties of their computational algorithms and their numerical methods with approximations of the polynomials by the help of rational functions. It is also providing the necessary materials about different family of special polynomials and numbers with their generating functions.

With the help of these computational algorithms arising from the recurrence formula for the numbers $W_n^{(k)}(\lambda)$, in this chapter, we calculate further numerical solutions, and also numerical methods with approximations of the polynomials $W_n^{(k)}(x,\lambda)$ with their related functions. By using these further numerical solutions, and also numerical methods with approximations of the polynomials $W_n^{(k)}(x,\lambda)$, we study on approximations of the functions $\mathcal{G}(\lambda, p, k)$ by the rational functions $W_n^{(k)}(\lambda)$.

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of positive integers. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of nonnegative integers. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+$ denote the set of positive real numbers. Let $\mathbb{C}$ be the set of the complex numbers.

Here, we begin by recalling some definitions and notations for the special numbers and polynomials including their generating functions as follows:

The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(k)}(x;w)$ of degree $n-1$ and order k are defined by the following generating function:

$$F_B(t,x;w;k)=\left(\frac{t}{we^t-1}\right)^k e^{xt}=\sum_{n=0}^{\infty}\mathscr{B}_n^{(k)}(x;w)\frac{t^n}{n!}, \tag{1}$$

where $\mathscr{B}_n^{(1)}(x;w)$ denotes the Apostol-Bernoulli polynomials (*cf.* [1]; and see also the references cited therein).

Assuming that log is the principal branch of the logarithm; that is, for $w \neq 0$,

$$\log(w)=Log\,|w|+i\arg(w),$$

with

$$\pi<\log(w)\leq\pi$$

and

$$Log 1=0.$$

The radius of convergence of the series for the function $F_B(t,x;w;1)$ in (1) is 2π when $w=1$ and $|\log(w)|$ when $w\neq 1$ (*cf.* [57, 61]). Here, main problem is to give comments on a relation between $\mathscr{B}_n(x;w)$ and $B_n(x)$ for suitable value of w. In [61], Navas et al. gave the following valuable comments:

For $\lambda=1$, $\mathscr{B}_n(x;w)$ reduces to the Bernoulli polynomials $B_n(x)$, that is

$$\mathscr{B}_n(x;1)=B_n(x).$$

There is a limiting relationship between $\mathscr{B}_n(x;w)$ and $B_n(x)$ as $w\to 1$, but it is not obtained easily. Another aspect of this discontinuity is that, although the polynomials $B_n(x)$ is monic of degree n, since $\mathscr{B}_0(w)=0$, for $w\neq 1$ the degree of polynomials $\mathscr{B}_n(x;w)$ is $n-1$ (*cf.* [1–121]).

It is time to give some of the special case of Eq. (1) as follows:

When $x=0$, these polynomials are reduced to the Apostol-Bernoulli numbers $\mathscr{B}_n^{(k)}(w)$ of order k with

$$\mathscr{B}_n^{(k)}(w)=\mathscr{B}_n^{(k)}(0;w)$$

(*cf.* [57]). Note that

$$\mathscr{B}_0^{(k)}(x;w)=0$$

and

$$\mathscr{B}_n^{(0)}(x;w)=x^n.$$

Note that the notation $\mathscr{B}_n^{(k)}(x; w)$ does not denote the k-th derivative of $\mathscr{B}_n(x; w)$.

Some well-known computation formulas for the Apostol-Bernoulli numbers $\mathscr{B}_n^{(k)}(w)$ of order k and the Apostol-Bernoulli polynomials $\mathscr{B}_n^{(k)}(x; w)$ of order k are given as follows:

$$\mathscr{B}_n^{(c+d)}(w) = \sum_{j=0}^{n} \binom{n}{j} \mathscr{B}_j^{(c)}(w) \mathscr{B}_{n-j}^{(d)}(w) \tag{2}$$

and

$$\mathscr{B}_n^{(c+d)}(x; w) = \sum_{v=0}^{n} \binom{n}{v} x^{n-v} \sum_{j=0}^{v} \binom{v}{j} \mathscr{B}_j^{(c)}(w) \mathscr{B}_{v-j}^{(d)}(w). \tag{3}$$

By using (1), we have

$$\sum_{n=0}^{\infty} \frac{(n)_k x^{n-k} t^n}{n!} = \sum_{b=0}^{k} (-1)^{k-b} \binom{k}{b} w^b \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathscr{B}_j^{(k)}(x; w) \frac{t^n}{n!},$$

where

$$(n)_k = n(n-1)(n-2)\dots(n-k+1),$$

with $(n)_0 = 1$.

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following relation:

Theorem 1 *Let $n \geq k \geq 0$. Then we have*

$$\sum_{b=0}^{k} (-1)^{k-b} \binom{k}{b} w^b \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathscr{B}_j^{(k)}(x; w) = (n)_k x^{n-k}.$$

It should be noted that

$$\mathscr{B}_n(x; w) = \mathscr{B}_n^{(1)}(x; w),$$

and

$$\mathscr{B}_n(w) = \mathscr{B}_n^{(1)}(w),$$

(*cf.* [1–121]).

Putting $k = 1$ in (1), we have a few well-known values of the Apostol-Bernoulli polynomials with the aid of (1) as follows:

$$\mathscr{B}_0(x;w)=0,$$

$$\mathscr{B}_1(x;w)=\frac{1}{w-1},$$

$$\mathscr{B}_2(x;w)=\frac{1}{w-1}x-\frac{2w}{(w-1)^2},$$

$$\mathscr{B}_3(x;w)=\frac{3}{w-1}x^2-\frac{6w}{(w-1)^2}x+\frac{3w(w+1)}{(w-1)^3},$$

$$\mathscr{B}_4(x;w)=\frac{4}{w-1}x^3-\frac{12w}{(w-1)^2}x^2+\frac{12w(w+1)}{(w-1)^3}x-\frac{4w\left(w^2+4w+1\right)}{(w-1)^4},$$

$$\mathscr{B}_5(x;w)=\frac{5}{w-1}x^4-\frac{20w}{(w-1)^2}x^3+\frac{30w(w+1)}{(w-1)^3}x^2-\frac{20w\left(w^2+4w+1\right)}{(w-1)^4}x$$
$$+\frac{5w\left(w^3+11w^2+11w+1\right)}{(w-1)^5},$$

and so on.

Putting $x=0$, since $\mathscr{B}_n(w)=\mathscr{B}_n(0;w)$, we also have a few well-known values of the Apostol-Bernoulli numbers:

$$\mathscr{B}_0(w)=0,$$

$$\mathscr{B}_1(w)=\frac{1}{w-1},$$

$$\mathscr{B}_2(w)=-\frac{2w}{(w-1)^2},$$

$$\mathscr{B}_3(w)=\frac{3w(w+1)}{(w-1)^3},$$

$$\mathscr{B}_4(w)=-\frac{4w\left(w^2+4w+1\right)}{(w-1)^4},$$

$$\mathscr{B}_5(w)=\frac{5w\left(w^3+11w^2+11w+1\right)}{(w-1)^5},$$

and so on.

Substituting some special values of x and $k=1$ into (1), we also have the following well-known results:

$$w\mathscr{B}_1(1;w)=1+\mathscr{B}_1(w)$$

and for $n\geq 2$,

$$w\mathscr{B}_n(1; w) = \mathscr{B}_n(w),$$

$$\mathscr{B}_n(x; w) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathscr{B}_j(w), \tag{4}$$

and

$$w\mathscr{B}_n(x+1; w) - \mathscr{B}_n(x; w) nx^{n-1} \tag{5}$$

(*cf.* [1]).

In [7], Bayad gave the Fourier series of the Apostol-Bernoulli polynomials. Let $w \in \mathbb{C} \setminus \{0\}$. For $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$. Then we have

$$\mathscr{B}_n(x; w) = \frac{-n!}{w^x (2\pi i)^n} \sum_{k \in \mathbb{Z}}^{\star} \frac{e^{2\pi i k x}}{\left(k - \frac{\log(w)}{2\pi i}\right)^n}, \tag{6}$$

where

$$\sum_{k \in \mathbb{Z}}^{\star} = \sum_{k \in \mathbb{Z} \setminus \{0\}}$$

if $w = 1$ and;

$$\sum_{k \in \mathbb{Z}}^{\star} = \sum_{k \in \mathbb{Z}}$$

if $w \neq 1$ (*cf.* [7]).

In the following theorem, asymptotic expansion for the Apostol-Bernoulli numbers was given by Navas et al. [61]:

Note that a set P is denoted poles of the function $F_B(t, x; w; 1)$. This set is given by

$$P = \{t = 2\pi i m - \log(w) : m \in \mathbb{Z}, t \in \mathbb{C}\}$$

when $i^2 = -1$, $w \neq 1$ and

$$P = \{t = 2\pi i m : m \in \mathbb{Z}, t \in \mathbb{C}\}$$

when $w = 1$. It is clear that $t = 0$ is a removable singularity of the function

$$F_B(t, x; 1; 1) = \frac{te^{tx}}{e^t - 1}$$

in the latter case.

Theorem 2 (cf. [61, Proposition 2]) *Given $w \in \mathbb{C}$, let P_1 be a finite subset of the set of poles P of the generating function (1) of the polynomials $\mathscr{B}_n(x; w)$ satisfying*

$$\max\{|a| : a \in P_1\} < \min\{|a| : a \in P \backslash P_1\} = \lambda.$$

For all integers $m \geq 2$, we have

$$\frac{\mathscr{B}_m(w)}{m!} = -\sum_{a \in P} \frac{1}{a^m} + O\left(\frac{1}{\lambda^m}\right),$$

where the constant implicit in the order term depends only on w and P.

Consequently, using the appropriate approximating sums over the sets P, the Fourier series of the polynomials $\mathscr{B}_m(x; w)$ at $x = 0$, which is given as follows:

$$\frac{\mathscr{B}_m(x; w)}{m!} = -\sum_{a \in P} \frac{e^{ax}}{a^m}$$

is an asymptotic expansion for the Apostol-Bernoulli numbers as $m \to \infty$ (*cf.* [7, 61, 104]).

By using (1), we now give some properties of the Bernoulli polynomials $B_n^{(k)}(x)$ of degree n and order k are defined by the following generating function:

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \tag{7}$$

such that

$$B_n^{(k)} = B_n^{(k)}(0),$$

where $B_n^{(k)}$ denotes the Bernoulli numbers of order k.

These polynomials can be computed by the following formula:

$$B_n^{(c+d)}(x) = \sum_{v=0}^{n} \binom{n}{v} x^{n-v} \sum_{j=0}^{v} \binom{v}{j} B_j^{(c)} B_{v-j}^{(d)} \tag{8}$$

(*cf.* [1, 21, 57, 59, 70, 114, 117]; and see also the references cited therein).

By differentiating both sides of (7) with respect to t, we have the following well-known recurrence formula for the polynomial $B_n^{(k)}(x)$:

$$B_n^{(k+1)}(x) = \left(1 - \frac{n}{k}\right) B_n^{(k)}(x) + k\left(\frac{x}{n} - 1\right) B_{n-1}^{(k)}(x) \tag{9}$$

(*cf*. [59, 117]; and see also the references cited therein).

Substituting $n = k$ into (9), we have

$$B_n^{(n+1)}(x) = (x-n)B_{n-1}^{(n)}(x).$$

Putting $x = n$ in the above equation, we have

$$B_n^{(n+1)}(n) = 0.$$

Substituting $x = 0$ into (9), we have

$$B_n^{(k+1)} = \left(1 - \frac{n}{k}\right) B_n^{(k)} - k B_{n-1}^{(k)} \tag{10}$$

(*cf*. [59, 117]; and see also the references cited therein).

Substituting $n = k$ into (10), we have

$$\begin{aligned} B_k^{(k+1)} &= -k B_{k-1}^{(k)} \\ &= k(k-1)B_{k-2}^{(k-1)} = \cdots = (-1)^k k! B_0 \\ &= (-1)^k k! \end{aligned}$$

see also [15].

It is time to give few values for the $B_n^{(k)}$ with the aid of the Eqs. (9) and (10):

$$\begin{aligned} B_0^{(k)} &= 1, \\ B_1^{(k)} &= -\frac{1}{2}k, \\ B_2^{(k)} &= \frac{1}{12}(3k^2 - k), \\ B_3^{(k)} &= -\frac{1}{8}(k^3 - k^2), \\ B_4^{(k)} &= \frac{1}{240}(15k^4 - 30k^3 + 5k^2 + 2k), \\ B_5^{(k)} &= -\frac{1}{96}(k^3 - k^2)(3k^2 - 7k - 2), \\ B_6^{(k)} &= \frac{1}{4032}(63k^6 - 315k^5 + 315k^4 + 91k^3 - 42k^2 - 16k), \ldots, \end{aligned}$$

and so on. Therefore

$$B_1^{(1)} = -\frac{1}{2}, B_2^{(2)} = \frac{5}{6}, B_3^{(3)} = -\frac{9}{4}, B_4^{(4)} = \frac{251}{30}, B_5^{(5)} = -\frac{475}{12}, B_6^{(6)} = \frac{19087}{84}$$

$$B_7^{(7)} = -\frac{36799}{24}, B_8^{(8)} = \frac{1070017}{90}, B_9^{(9)} = -\frac{2082753}{20}, B_{10}^{(10)} = \frac{134211265}{132}, \ldots$$

and so on (*cf.* [59]; and see also the references cited therein).

It can be easily calculated in other numbers with the help of the formulas given above. Since

$$B_k^{(k+1)} = (-1)^k k!,$$

we have

$$\begin{aligned}
B_0^{(1)} &= 0! = 1, \\
B_1^{(2)} &= -\frac{1}{2}2 = (-1)1! = -1, \\
B_2^{(3)} &= \frac{1}{12}(3.2^2 - 2) = 2! = 2, \\
B_3^{(4)} &= -\frac{1}{8}(3^3 - 3^2) = -3! = -6, \\
B_4^{(5)} &= \frac{1}{240}(15.4^4 - 30.4^3 + 5.4^2 + 2.4) = 4! = 24, \\
B_5^{(6)} &= -\frac{1}{96}(5^3 - 5^2)(3.5^2 - 7.5 - 2) = -5! = -120, \\
B_6^{(7)} &= \frac{1}{4032}(63.6^6 - 315.6^5 + 315.6^4 + 91.6^3 - 42.6^2 - 16.6) = 720, \ldots,
\end{aligned}$$

and so on.

Many computational formulas and relations of such numbers including the Bernoulli numbers of order k can also be found in different methods. In the literature, these can be used in other related formulas.

With the help of the well-known formula of the Riemann zeta function, asymptotic behavior of the Bernoulli polynomials is given as follows:

Dilcher [24] gave asymptotic behavior of the Bernoulli Polynomials, the Euler Polynomials, and the generalized Bernoulli Polynomials. He gave many novel properties of these asymptotic behavior for these polynomials. One of them is given as follows:

For all $w \in \mathbb{C}$, $m \in \mathbb{N}$ with $k = \left[\frac{m}{2}\right]$, $m \geq 2$, we have the following well-known relation:

$$\left| (-1)^k \frac{(2\pi)^m}{(2m)!} \sum_{j=0}^{m} \binom{m}{j} \frac{w^{n-j}}{2^j} - T_m(2\pi w) \right| < \frac{1}{2^m} e^{\frac{4\pi}{w}},$$

where $T_m(w)$, denotes the sections of the cosine and sine functions, is defined by

$$T_{2m}(w) = \sum_{j=0}^{m} (-1)^j \frac{w^{2j}}{(2j)!},$$

and

$$T_{2m+1}(w) = \sum_{j=0}^{m} (-1)^j \frac{w^{2j+1}}{(2j+1)!},$$

$[c]$ denotes the integral part of c ($\in \mathbb{R}$) (*cf.* [24, Theorem 1]). Since uniformly convergent on a compact subset to $\cos(2\pi w)$ if m is even an integer, and to $\sin(2\pi w)$ if m is an odd integer. Making replace $w + \frac{1}{2}$ by u, we arrive at the following well-known sequences converge uniformly on compact subsets of $\mathbb{C}$:

$$(-1)^{m-1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}(u) \to \sum_{j=0}^{\infty} (-1)^j \frac{(2\pi u)^{2j}}{(2j)!}$$

and

$$(-1)^{m-1} \frac{(2\pi)^{2m+1}}{2(2m+1)!} B_{2m+1}(u) \to \sum_{j=0}^{m} (-1)^j \frac{(2\pi u)^{2j+1}}{(2j+1)!}$$

(*cf.* [24, Corollary 1]; and for details, see also [60, 77, 88, 91, 104]; and the references cited therein).

The Apostol-Euler polynomials $\mathcal{E}_n^{(k)}(x;\lambda)$ of degree n and order k are defined by means of the following generating function:

$$F_{\mathcal{E}}(t, x; w; k) = \left(\frac{2}{we^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x; w) \frac{t^n}{n!}, \tag{11}$$

where $w \in \mathbb{C}$ and $|t| < |\log(-w)|$, so that, in the special case when $x = 0$, these polynomials are reduced to the Apostol-Euler numbers $\mathcal{E}_n^{(k)}(w)$ of higher order with

$$\mathcal{E}_n^{(k)}(w) = \mathcal{E}_n^{(k)}(0; w)$$

(*cf.* [53, 57]; and the references cited therein).

By using (11), we have

$$\sum_{n=0}^{\infty} \frac{2^k x^n t^n}{n!} = \sum_{b=0}^{k} \binom{k}{b} w^b \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathcal{E}_j^{(k)}(x; w) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following relation:

Theorem 3 *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\sum_{b=0}^{k} \binom{k}{b} w^b \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathscr{E}_j^{(k)}(x; w) = 2^k x^n.$$

It should be note that

$$\mathscr{E}_n(x; w) = \mathscr{E}_n^{(1)}(x; w),$$

$$\mathscr{E}_n(w) = \mathscr{E}_n^{(1)}(w)$$

and

$$E_n = \mathscr{E}_n(1),$$

where $\mathscr{E}_n(x; w)$, $\mathscr{E}_n(w)$, and E_n denote the Apostol-Euler polynomials, the Apostol-Euler numbers, and the classical Euler numbers of the first kind, respectively (*cf.* [1, 39, 53, 57, 70, 114, 115, 117, 119]; and see also the references cited therein).

In [7], Bayad also gave the Fourier series of the Apostol-Euler polynomials. Let $w \in \mathbb{C} \setminus \{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. Then we have

$$\mathscr{E}_n(x; w) = \frac{2(n)!}{w^x (2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}}^{\star\star} \frac{e^{2\pi i \left(k - \frac{1}{2}\right) x}}{\left(k - \frac{1}{2} - \frac{\log(w)}{2\pi i}\right)^{n+1}}, \tag{12}$$

where $\sum\limits_{k \in \mathbb{Z}}^{\star\star} = \sum\limits_{k \in \mathbb{Z} \setminus \{0\}}$ if $w = -1$ and $\sum\limits_{k \in \mathbb{Z}}^{\star\star} = \sum\limits_{k \in \mathbb{Z}}$ if $w \neq -1$ (*cf.* [7]).

Taking into account $\cos\left(\pi u - \frac{\pi}{2}\right) = \sin(\pi u)$ and $\sin\left(\pi u - \frac{\pi}{2}\right) = -\cos(\pi u)$, we arrive at the following well-known results for the Euler polynomials:

$$(-1)^m \frac{\pi^{2m+1}}{4(2m)!} E_{2m}(u) \to \sin(\pi u)$$

and

$$(-1)^{m+11} \frac{\pi^{2m+2}}{4(2m+1)!} E_{2m+1}(u) \to \cos(\pi u)$$

(*cf.* [24, Corollary 3]; and see also [60, 77, 88, 91, 104]; and the references cited therein).

In [39], Kim et al. considered the modification of the Apostol-Bernoulli polynomials and gave the following generating function for the w-Bernoulli polynomials $\mathfrak{B}_n(w;x)$:

$$\frac{\log(w)+t}{we^t-1}e^{tx}=\sum_{n=0}^{\infty}\mathfrak{B}_n(w;x)\frac{t^n}{n!}, \tag{13}$$

so that,

$$\mathfrak{B}_n(w)=\mathfrak{B}_n(w;0),$$

where $\mathfrak{B}_n(w)$ denotes the w-Bernoulli numbers and some values of these numbers are given as follows:

$$\mathfrak{B}_0(w)=\frac{\log(w)}{w-1},$$

$$\mathfrak{B}_1(w)=\frac{w-1-w\log(w)}{(w-1)^2},$$

and so on (*cf.* [31, 39, 97]; and see also the references cited therein).

Combining (13) with (1), we have

$$\left(\frac{\log(w)}{t}+1\right)F_B(t,0;w;1)=\frac{\log(w)+t}{we^t-1}.$$

By using the above functional equation, we get

$$\log(w)\sum_{n=0}^{\infty}\mathscr{B}_n(w)\frac{t^n}{n!}+\sum_{n=0}^{\infty}n\mathscr{B}_{n-1}(w)\frac{t^n}{n!}=\sum_{n=0}^{\infty}n\mathfrak{B}_{n-1}(w)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following explicit known representation for the numbers $\mathfrak{B}_n(w)$ and $\mathscr{B}_n(w)$:

$$\mathfrak{B}_{n-1}(w)=\frac{\log(w)}{n}\mathscr{B}_n(w)+\mathscr{B}_{n-1}(w),$$

where $n\in\mathbb{N}$.

In [39], Kim et al. derived a summation formula in connection with w-Bernoulli numbers and polynomials as follows (*cf.* [39, p. 9]):

$$\mathfrak{B}_l(w;k)-w^{-k}\mathfrak{B}_l(w)=l\sum_{n=0}^{k-1}w^{n-k}n^{l-1}. \tag{14}$$

Making use of the Eq. (14), we have the following well-known sums of powers of consecutive integers including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Genocchi numbers and polynomials

$$\sum_{j=1}^{m-1} j^n = \frac{B_{n+1}(m) - B_{n+1}}{n+1},$$

$$\sum_{j=1}^{m-1} (-1)^j j^n = \frac{E_n + (-1)^{m+1} E_n(m)}{2},$$

and

$$\sum_{j=1}^{m-1} (-1)^j j^n = \frac{G_{n+1} + (-1)^{m+1} G_{n+1}(m)}{2(n+1)},$$

where $n \in \mathbb{N}_0$ and $m \in \mathbb{N} \setminus \{1\}$ (*cf.* [30, 34, 69, 90, 110, 116, 120]).

Combining (1) and (11), we get

$$F_B(t, x; \lambda; k) = F_B\left(\frac{t}{2}, x; \lambda^{\frac{1}{2}}; k\right) F_{\mathscr{E}}\left(\frac{t}{2}, x; \lambda^{\frac{1}{2}}; k\right).$$

Note that there exist very kind of functional equation, including the Apostol-type numbers and polynomials, similar to the above functional equation (*cf.* [1–118]).

In this context, using the above equation, we have

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{(k)}(x; w) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathscr{B}_n^{(k)}(x; w^{\frac{1}{2}}) \frac{t^n}{2^n n!} \sum_{n=0}^{\infty} \mathscr{E}_n^{(k)}(x; w^{\frac{1}{2}}) \frac{t^n}{2^n n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{(k)}(x; w) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{b=0}^{n} \binom{n}{b} \mathscr{B}_b^{(k)}(x; w^{\frac{1}{2}}) \mathscr{E}_{n-b}^{(k)}(x; w^{\frac{1}{2}}) \frac{t^n}{2^n n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following presumably known relation:

$$\mathscr{B}_n^{(k)}(x; w) = \frac{1}{2^n} \sum_{b=0}^{n} \binom{n}{b} \mathscr{B}_b^{(k)}\left(x; w^{\frac{1}{2}}\right) \mathscr{E}_{n-b}^{(k)}\left(x; w^{\frac{1}{2}}\right). \tag{15}$$

The Apostol-Genocchi polynomials $\mathscr{G}_n^{(k)}(x; w)$ of degree n and order k are defined by the following generating function:

$$F_G(t, x; \lambda; k) = \left(\frac{2t}{we^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x; w)\frac{t^n}{n!}, \tag{16}$$

where $w \in \mathbb{C}$ and $|t| < |\log(-w)|$, so that, in the special case when $x = 0$, these polynomials are reduced to the Apostol-Genocchi numbers $\mathcal{G}_n^{(k)}(w)$ of higher order with

$$\mathcal{G}_n^{(k)}(w) = \mathcal{G}_n^{(k)}(0; w)$$

(*cf.* [55, 56, 58]; and see also the references cited therein).

By using (16), we have

$$\sum_{n=0}^{\infty} \frac{2^k (n)_k x^{n-k} t^n}{n!} = \sum_{b=0}^{k} \binom{k}{b} w^b \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathcal{G}_j^{(k)}(x; w)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following relation:

Theorem 4 *Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have*

$$\sum_{b=0}^{k} \binom{k}{b} w^b \sum_{j=0}^{n} \binom{n}{j} b^{n-j} \mathcal{G}_j^{(k)}(x; w) = 2^k (n)_k x^{n-k}.$$

It should be noted that

$$\mathcal{G}_n(x; w) = \mathcal{G}_n^{(1)}(x; w),$$

$$\mathcal{G}_n(w) = \mathcal{G}_n^{(1)}(w),$$

and

$$G_n = \mathcal{G}_n(1),$$

where $\mathcal{G}_n(x; w)$, $\mathcal{G}_n(w)$, and G_n denote the Apostol-Genocchi polynomials, the Apostol-Genocchi numbers and the Genocchi numbers, respectively (*cf.* [55, 56, 58, 70, 117]; and see also the references cited therein).

Combining (11) and (16), we have

$$F_G(t, x; w; k) = t^k F_{\mathcal{E}}(t, x; w; k).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} \mathscr{G}_n^{(k)}(x; w)\frac{t^n}{n!} = t^k \sum_{n=0}^{\infty} \mathscr{E}_n^{(k)}(x; w)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathscr{G}_n^{(k)}(x; w)\frac{t^n}{n!} = \sum_{n=0}^{\infty} (n)_k \mathscr{E}_{n-k}^{(k)}(x; w)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the following known relation:

$$\mathscr{G}_{n+k}^{(k)}(x; w) = (n+k)_k \mathscr{E}_n^{(k)}(x; w)$$

(*cf.* [1–121]).

Combining the above equation with (15) yields the following presumably known relation:

Theorem 5

$$\mathscr{B}_n^{(k)}(x; w) = \frac{1}{2^n} \sum_{b=0}^{n} \binom{n}{b} (n+k-b)_k \mathscr{B}_b^{(k)}\left(x; w^{\frac{1}{2}}\right) \mathscr{G}_{n+k-b}^{(k)}\left(x; w^{\frac{1}{2}}\right),$$

where assuming that $n + k \geq b$.

In [7], by using different method that of [54], Bayad also gave the following Fourier expansions of the Apostol-Genocchi polynomials:

Let $w \in \mathbb{C} \setminus \{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. The following equality holds true:

$$\mathscr{G}_n(x; w) = \frac{2\,(n)!}{w^x (2\pi i)^n} \sum_{k \in \mathbb{Z}}^{\star\star} \frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k - \frac{1}{2} - \frac{\log(w)}{2\pi i}\right)^n}, \tag{17}$$

(*cf.* [7]).

The Frobenius-Euler polynomials $\mathscr{H}_n^{(k)}(x|\lambda)$ of degree n and order k are defined by means of the following generating function:

$$F_H(t, x; \lambda; k) = \left(\frac{1-\lambda}{e^t - \lambda}\right)^k e^{xt} = \sum_{n=0}^{\infty} \mathscr{H}_n^{(k)}(x|\lambda)\frac{t^n}{n!}, \tag{18}$$

where $\lambda \in \mathbb{C}\backslash\{1\}$, for $x = 0$, the polynomials $\mathscr{H}_n^{(k)}(0|\lambda)$ are reduced to the Frobenius-Euler numbers $\mathscr{H}_n^{(k)}(\lambda)$ of higher order:

$$\mathscr{H}_n^{(k)}(\lambda) = \mathscr{H}_n^{(k)}(0|\lambda).$$

For $k = 1$, the numbers $\mathscr{H}_n^{(k)}(\lambda)$ are reduced to the Frobenius-Euler numbers (or Euler Frobenius numbers) $\mathscr{H}_n^{(1)}(\lambda)$:

$$H_n(\lambda) = \mathscr{H}_n^{(1)}(\lambda).$$

A relation between the numbers $\mathscr{H}_m^{(k)}(\lambda)$ and the polynomials $\mathscr{H}_n^{(k)}(x|\lambda)$ is given by

$$\mathscr{H}_n^{(k)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} \mathscr{H}_m^{(k)}(\lambda) \tag{19}$$

(*cf.* [35, 39, 40, 74, 76, 78–80, 86, 89, 93, 98, 103, 108, 111]; and see also the references cited therein).

The second kind Apostol-type Euler numbers of order $-k$ are defined by the following generating functions:

$$F_N(t; -k, \lambda) = \left(\frac{\lambda e^t + \lambda^{-1} e^{-t}}{2}\right)^k = \sum_{n=0}^{\infty} E_n^{*(-k)}(\lambda) \frac{t^n}{n!} \tag{20}$$

(*cf.* [99, 102]; see also the references cited therein).

By using (20), we have

$$\sum_{n=0}^{\infty} \frac{1}{2^k} \sum_{b=0}^{k} \binom{k}{b} \lambda^{2j-k} (2j-k)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{*(-k)}(\lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following known relation:

Theorem 6 *Let $n, k \in \mathbb{N}_0$. Then we have*

$$E_n^{*(-k)}(\lambda) = \frac{1}{2^k} \sum_{b=0}^{k} \binom{k}{b} \lambda^{2j-k} (2j-k)^n.$$

For the proof of the above theorem, see [99, 102], and see also the references cited therein).

The numbers $y_2(n, k; \lambda)$ are defined by the following generating function:

$$F_{y_2}(t,k;\lambda)=\frac{1}{(2k)!}\left(\lambda e^{t}+\lambda^{-1}e^{-t}+2\right)^{k}=\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}, \tag{21}$$

(*cf.* [102]).

By using (21), we have

$$F_{y_2}(t,k;\lambda)=\frac{1}{(2k)!}\sum_{b=0}^{k}\binom{k}{b}2^{k+b-j}F_N(t;-b,\lambda).$$

Using the above equation yields

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{(2k)!}\sum_{b=0}^{k}\binom{k}{b}2^{k+b-j}E_n^{*(-b)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation, we obtain the following known relation:

Theorem 7 *Let $n,k\in\mathbb{N}_0$. Then we have*

$$y_2(n,k;\lambda)=\frac{1}{(2k)!}\sum_{b=0}^{k}\binom{k}{b}2^{k+b-j}E_n^{*(-b)}(\lambda).$$

The λ-array polynomials $S_k^n(x;\lambda)$ (of degree n and order k) are given by the following generating function:

$$F_A(t,x,k;\lambda)=\frac{\left(\lambda e^{t}-1\right)^{k}}{k!}e^{xt}=\sum_{n=0}^{\infty}S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{22}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [10, 93]) which, for $\lambda=1$, yields the classical array polynomials $S_k^n(x)$, which are defined by the following explicit formula:

$$S_k^n(x)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n \tag{23}$$

with $S_0^0(x)=S_n^0(x)=1$, $S_0^n(x)=x^n$ and $S_0^0(x)=0$ when $k>n$. Moreover, for $x=0$, the λ-array polynomials $S_k^n(x;\lambda)$ reduce to the λ-Stirling numbers, that is:

$$S(n,k;\lambda)=S_k^n(0;\lambda), \tag{24}$$

(*cf.* [10, 58, 93, 115]). In the special case of $\lambda=1$, the λ-Stirling numbers reduce to the Stirling numbers of the second kind, that is

$$S_2(n,k) = S(n,k;1)$$

which are defined by

$$F_S(t,k) = \frac{\left(e^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k) \frac{t^n}{n!}, \tag{25}$$

and

$$x^n = \sum_{k=0}^{n} S_2(n,k) (x)_k, \tag{26}$$

(*cf.* [21, 117]; see also the references cited therein).

By combining (22) and (25), the following functional equation is obtained:

$$F_A(t,x,k;\lambda) = e^{xt} \sum_{b=0}^{k} (-1)^{k-b} \frac{1}{\binom{k}{b}(k-b)!} B_b^k(\lambda) F_S(t,b), \tag{27}$$

where $B_b^k(\lambda)$ denotes the Bernstein basis function, which are defined by means of the following well-known generating function:

$$\frac{(\lambda t)^m}{m!} e^{(1-\lambda)t} = \sum_{n=0}^{\infty} B_m^n(\lambda) \frac{t^n}{n!}, \tag{28}$$

where

$$B_m^n(\lambda) = \begin{cases} \binom{n}{m}\lambda^m (1-\lambda)^{n-m} & \text{if } 0 \le m \le n,\ n,m \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

The Bernstein polynomials, which are linear combination of the Bernstein basis function, are used to prove the Weierstrass approximation theorem that every real-valued continuous function on a real interval $[a,b]$ can be uniformly approximated by polynomial functions over $\mathbb{R}$ (*cf.* [51, 94, 96, 111]; see also the references cited therein).

By using (27), we get

$$\sum_{n=0}^{\infty} S_k^n(x;\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{b=0}^{k} (-1)^{k-b} \frac{B_b^k(\lambda)}{\binom{k}{b}(k-b)!} \sum_{j=0}^{n} \binom{n}{j} S_2(n,k) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the following theorem:

Theorem 8 *Let $n, k \in \mathbb{N}_0$. Then we have*

$$S_k^n(x;\lambda) = \sum_{b=0}^{k} (-1)^{k-b} \frac{B_b^k(\lambda)}{\binom{k}{b}(k-b)!} \sum_{j=0}^{n} \binom{n}{j} S_2(n,k).$$

The Stirling number of the first kind $S_1(n,k)$ are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)\, x^k \tag{29}$$

(*cf.* [17, 116, 117]; see also the references cited in each of these earlier works).

Note that the numbers of the first kind $S_1(n,k)$ are also defined by means of the following generating function:

$$\sum_{n=0}^{\infty} S_1(n,k) \frac{z^n}{n!} = \frac{(\log(1+z))^k}{k!},$$

where $k \in \mathbb{N}_0$ (*cf.* [17, 21, 116, 117]; and the references cited therein).

2 Apostol-Type Numbers and Polynomials with Their Properties and Relations

In [99], Simsek introduced the numbers $W_n(\lambda)$ and their higher order $W_n^{(k)}(\lambda)$, respectively, by the following generating functions:

$$F_w(t;\lambda) = \frac{1}{\lambda e^t + \lambda^{-1} e^{-t} + 2} = \sum_{n=0}^{\infty} W_n(\lambda) \frac{t^n}{n!}, \tag{30}$$

and

$$F_w(t;\lambda;k) = \frac{1}{\left(\lambda e^t + \lambda^{-1} e^{-t} + 2\right)^k} = \sum_{n=0}^{\infty} W_n^{(k)}(\lambda) \frac{t^n}{n!}, \tag{31}$$

where $n, k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [99]). In (30) and (31), we have the following restriction on the value of t: For $\lambda \in \mathbb{C}$ we have $|t| < |\log(-\lambda)|$ with $1^k = 1$.

By combining (30) with (11) with $k = 1$, we get

$$\sum_{n=0}^{\infty} W_n(\lambda) \frac{t^n}{n!} = \lambda \sum_{n=0}^{\infty} \mathcal{E}_n\left(\frac{1}{2};\lambda\right) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{E}_n\left(\frac{1}{2};\lambda\right) \frac{t^n}{n!}.$$

By using the Cauchy product rule for the series in the above equation we obtain

$$\sum_{n=0}^{\infty} W_n(\lambda)\frac{t^n}{n!} = \lambda \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathscr{E}_j\left(\frac{1}{2};\lambda\right)\mathscr{E}_{n-j}\left(\frac{1}{2};\lambda\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the following theorem:

Theorem 9 *Let* $n \in \mathbb{N}_0$. *Then we have*

$$W_n(\lambda) = \lambda \sum_{j=0}^{n}\binom{n}{j}\mathscr{E}_j\left(\frac{1}{2};\lambda\right)\mathscr{E}_{n-j}\left(\frac{1}{2};\lambda\right).$$

Combining (20) with (31), we have

$$1 = \sum_{b=0}^{k}\binom{k}{b}2^k \sum_{n=0}^{\infty} E_n^{*(-b)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} W_n^{(k)}(\lambda)\frac{t^n}{n!}.$$

Therefore

$$1 = \sum_{b=0}^{k}\binom{k}{b}2^k \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} E_{n-j}^{*(-b)}(\lambda) W_j^{(k)}(\lambda)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following result:

Theorem 10 *Let* $n, k \in \mathbb{N}_0$. *Then we have*

$$\sum_{b=0}^{k}\binom{k}{b}2^k \sum_{j=0}^{n}\binom{n}{j} E_{n-j}^{*(-b)}(\lambda) W_j^{(k)}(\lambda) = 0.$$

Substituting $\lambda = 1$ into (30), we obtain the following equation:

$$\frac{1}{e^t + e^{-t} + 2} = \sum_{n=0}^{\infty} W_n(1)\frac{t^n}{n!}.$$

Therefore

$$\frac{1}{4}\sum_{n=0}^{\infty} E_n^{(2)}(1)\frac{t^n}{n!} = \sum_{n=0}^{\infty} W_n(1)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain

$$W_n(1) = \frac{1}{4}E_n^{(2)}(1), \tag{32}$$

where

$$\begin{aligned} E_n^{(2)}(1) &= \sum_{j=0}^{n} \binom{n}{j} E_j^{(2)} \\ &= \sum_{j=0}^{n} \sum_{b=0}^{j} \binom{n}{j} \binom{j}{b} E_b E_{j-b}, \end{aligned}$$

where E_b denotes the Euler numbers of the first kind.

Substituting $\lambda = 1$ into (31), we obtain the following equation:

$$\frac{2^{2k} e^{kt}}{(e^t + 1)^{2k}} = \sum_{n=0}^{\infty} 2^{2k} W_n^{(k)}(1) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} E_n^{(k)}(k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2^{2k} W_n^{(k)}(1) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain

$$W_n^{(k)}(1) = \frac{1}{4^k} E_n^{(k)}(k), \tag{33}$$

where $E_n^{(k)}(k)$ denotes the Euler polynomials of the first kind of order k.

By using (31), we have

$$\frac{\lambda^k e^{kt}}{(\lambda e^t + 1)^{2k}} = \sum_{n=0}^{\infty} W_n^{(k)}(\lambda) \frac{t^n}{n!}.$$

Putting $\lambda = -1$ in the above equation, after some elementary calculations, we have

$$\sum_{m=0}^{\infty} B_m^{(2k)} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \binom{m}{2k} (2k)! \sum_{j=0}^{m-2k} (-1)^{m-j-k} \binom{m-2k}{j} k^{m-2k-j} W_j^{(k)}(-1) \frac{t^m}{m!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following result:

Theorem 11 *Let $m, k \in \mathbb{N}_0$ with $m \geq 2k$. Then we have*

$$B_m^{(2k)} = \binom{m}{2k}(2k)! \sum_{j=0}^{m-2k} (-1)^{m-j-k} \binom{m-2k}{j} k^{m-2k-j} W_j^{(k)}(-1). \tag{34}$$

With the aid of the Bernoulli numbers of order $2k$ and (34), we have

$$\sum_{j=0}^{1-2k} (-1)^{j-k} \binom{1-2k}{j} k^{1-2k-j} W_j^{(k)}(-1) = \frac{k}{6\binom{1}{2k}(2k)!},$$

$$\sum_{j=0}^{2-2k} (-1)^{j-k} \binom{2-2k}{j} k^{2-2k-j} W_j^{(k)}(-1) = \frac{k(6k-1)}{6\binom{2}{2k}(2k)!},$$

$$\sum_{j=0}^{3-2k} (-1)^{j-k} \binom{3-2k}{j} k^{3-2k-j} W_j^{(k)}(-1) = \frac{k^2(2k-1)}{2\binom{3}{2k}(2k)!},$$

$$\sum_{j=0}^{4-2k} (-1)^{j-k} \binom{4-2k}{j} k^{4-2k-j} W_j^{(k)}(-1) = \frac{k(120k^3-120k^2+10k+2)}{120\binom{4}{2k}(2k)!},$$

$$\sum_{j=0}^{5-2k} (-1)^{j-k} \binom{5-2k}{j} k^{5-2k-j} W_j^{(k)}(-1) = \frac{k^2(2k-1)(12k^2-14k-2)}{24\binom{5}{2k}(2k)!},$$

and

$$\sum_{j=0}^{6-2k} (-1)^{j-k} \binom{6-2k}{j} k^{6-2k-j} W_j^{(k)}(-1)$$
$$= \frac{k(2016k^5-4940k^4+2520k^3+364k^2-84k-16)}{2016\binom{6}{2k}(2k)!},$$

and so on. So the following some research problems can be given to the readers. Therefore, if we continue with the above processes, we can naturally raise the following *open question*:

Problem 1 Let $v, d \in \mathbb{N}$ and $a_0, a_1, a_2, \cdots, a_{v-1}, a_v \in \mathbb{Z}$.

$$\sum_{j=0}^{v-2k} (-1)^{j-k} \binom{v-2k}{j} k^{v-2k-j} W_j^{(k)}(-1) = \frac{P_v(k)}{d\binom{v}{2k}(2k)!},$$

where $P_v(k)$ is a polynomial in k of degree v. That is

$$P_v(k) = a_0 k^v + a_1 k^{v-1} + a_2 k^{v-2} + \cdots + a_{v-1} k + a_v.$$

Explore properties of the coefficients of the polynomial $P_v(k)$ with constant d? What can you say about the factors of the polynomial $P_v(k)$?

Theorem 12 (*cf.* [99]) *Let $n \in \mathbb{N}$. Then the numbers $W_n(\lambda)$ are given by the following recurrence relation:*

$$2W_n(\lambda) + \lambda \sum_{m=0}^{n} \binom{n}{m} W_m(\lambda) + \lambda^{-1} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} W_m(\lambda) = 0 \tag{35}$$

with the initial condition:

$$W_0(\lambda) = \frac{\lambda}{(\lambda+1)^2}.$$

Proof Applying the Umbral calculus convention to the Eq. (30), after some algebraic calculations with the aid of the Cauchy product rule for the related series, we have

$$1 = \lambda \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} W_m(\lambda) \frac{t^n}{n!} + \lambda^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} W_m(\lambda) \frac{t^n}{n!}$$
$$+2 \sum_{n=0}^{\infty} W_n(\lambda) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the assertion of Theorem 12.

For $n \in \mathbb{N}$, another form of the recurrence relation (35) is given by

$$W_n(\lambda) = W_0(\lambda) \sum_{m=0}^{n-1} \left((-1)^{n-m+1} \lambda^{-1} - \lambda \right) \binom{n}{m} W_m(\lambda), \tag{36}$$

(*cf.* [41, 42, 99]).

The well-known computation formulas of the numbers $W_n(\lambda)$, given in the Eq. (35) and (36), can be given by other methods and techniques. This formula can be given in the well-known division technique for the series below. Let us briefly give this technique:

We set

$$\frac{\sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!}}{\sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!}} = \sum_{n=0}^{\infty} C_n(\lambda)\frac{t^n}{n!},$$

where

$$C_0(\lambda) = \frac{A_0(\lambda)}{B_0(\lambda)}$$

with $B_0(\lambda) \neq 0$. Therefore,

$$\sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} C_n(\lambda)\frac{t^n}{n!}.$$

The Cauchy product of these two power series on the left side of the above equation is given by

$$\sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}B_j(\lambda)C_{n-j}(\lambda)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain

$$A_n(\lambda) = \sum_{j=0}^{n}\binom{n}{j}B_j(\lambda)C_{n-j}(\lambda).$$

Hence

$$A_n(\lambda) = C_n(\lambda)B_0(\lambda) + \sum_{j=1}^{n}\binom{n}{j}B_j(\lambda)C_{n-j}(\lambda).$$

Since $B_0(\lambda) \neq 0$, a known computation formula of the coefficients $C_n(\lambda)$ is given by

$$C_n(\lambda) = \frac{A_n(\lambda)}{B_0(\lambda)} - \frac{1}{B_0(\lambda)}\sum_{j=1}^{n}\binom{n}{j}B_j(\lambda)C_{n-j}(\lambda). \tag{37}$$

By applying (37) to (30), we get

$$\frac{1}{\lambda \sum_{n=0}^{\infty} \frac{t^n}{n!} + \lambda^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} + 2} = \sum_{n=0}^{\infty} W_n(\lambda) \frac{t^n}{n!}.$$

From the above equation, we obtain

$$A_0(\lambda) = 1$$

and for $n > 0$

$$A_n(\lambda) = 0.$$

$$B_0(\lambda) = \lambda + \lambda^{-1} + 2$$

and for $n > 0$

$$B_n(\lambda) = \frac{1}{n!} \left(\lambda + (-1)^n \lambda^{-1} \right).$$

Therefore

$$W_0(\lambda) = \frac{A_0(\lambda)}{B_0(\lambda)} = \frac{1}{\lambda + \lambda^{-1} + 2},$$

$$\begin{aligned} W_1(\lambda) &= \frac{A_1(\lambda)}{B_0(\lambda)} - \frac{1}{B_0(\lambda)} \sum_{j=1}^{1} \binom{n}{j} B_j(\lambda) C_{n-j}(\lambda) \\ &= -\frac{\lambda (\lambda - 1)}{(\lambda + 1)^3}. \end{aligned}$$

Consequently, combining (37) with the recurrence relation given by (35), first few values of the numbers $W_n(\lambda)$ are computed as follows:

$$W_2(\lambda) = \frac{\lambda \left(\lambda^2 - 4\lambda + 1 \right)}{(\lambda + 1)^4},$$

$$W_3(\lambda) = -\frac{\lambda \left(\lambda^3 - 11\lambda^2 + 11\lambda - 1 \right)}{(\lambda + 1)^5},$$

and so on (*cf*. [41, 42, 99]).

In [99], we gave the polynomials $W_n^{(k)}(x; \lambda)$ of degree n and order k by the following generating function:

$$G_w(t,x,k;\lambda) = e^{tx}F_w(t,k;\lambda) = \sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!}, \tag{38}$$

where $n, k \in \mathbb{N}_0$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, (*cf.* [99]).

The following relationship between the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ is given by the following theorem:

Theorem 13 (*cf.* [99]) *Let $n, k \in \mathbb{N}_0$. Then we have*

$$W_n^{(k)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} W_m^{(k)}(\lambda). \tag{39}$$

Substituting $x = 1$ into (39), we have

$$W_n^{(k)}(1;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(\lambda). \tag{40}$$

Notice that

$$W_n^{(k)}(\lambda) = W_n^{(k)}(0;\lambda), \tag{41}$$

with

$$W_n(\lambda) = W_n^{(1)}(\lambda)$$

and

$$W_n(x;\lambda) = W_n^{(1)}(x;\lambda)$$

(*cf.* [99]).

In [100], we gave several computation formulas for the numbers $W_n^{(k)}(\lambda)$.

Theorem 14 (*cf.* [100]) *Let $n \in \mathbb{N}_0$. Then we have*

$$W_n^{(c+d)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(c)}(\lambda) W_{n-m}^{(d)}(\lambda). \tag{42}$$

Proof *(cf. [100])* By using (31), one has the following functional equation:

$$F_w(t;\lambda;c+d) = F_w(t;\lambda;c)F_w(t;\lambda;d).$$

The above functional equation gives us the following series equations:

$$\sum_{n=0}^{\infty} W_n^{(c+d)}(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} W_n^{(c)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} W_n^{(d)}(\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} W_n^{(c+d)}(\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} W_m^{(c)}(\lambda) W_{n-m}^{(d)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the assertion of Theorem 14.

By using (42), we may compute the values of the numbers $W_n^{(k)}(\lambda)$. For instance, setting $c = d = 1$ in (42), we have

$$W_0^{(2)}(\lambda) = \frac{\lambda^2}{(\lambda+1)^4},$$

$$W_1^{(2)}(\lambda) = \frac{2\lambda^2(1-\lambda)}{(\lambda+1)^5},$$

and so on.

These numbers $W_n^{(k)}(\lambda)$ also satisfy the following computation formula (*cf*. [42, 100]):

$$W_n^{(k)}(\lambda) = \sum_{m=0}^{n}\binom{n}{m} W_m^{(k-1)}(\lambda) W_{n-m}(\lambda) \tag{43}$$

so that, by (43), a few values of the numbers $W_n^{(k)}(\lambda)$ are given as follows:

$$W_0^{(2)}(\lambda) = \frac{\lambda^2}{(\lambda+1)^4},$$

$$W_1^{(2)}(\lambda) = \frac{2\lambda^2(1-\lambda)}{(\lambda+1)^5},$$

$$W_2^{(2)}(\lambda) = \frac{4\lambda^2\left(\lambda^2-3\lambda+1\right)}{(\lambda+1)^6},$$

$$W_0^{(3)}(\lambda) = \frac{\lambda^3}{(\lambda+1)^6},$$

$$W_1^{(3)}(\lambda) = \frac{3\lambda^3(1-\lambda)}{(\lambda+1)^7},$$

$$W_2^{(3)}(\lambda) = \frac{3\lambda^3 \left(3\lambda^2 - 8\lambda + 3\right)}{(\lambda + 1)^8},$$

and so on.

By using the mathematical induction method with (43), we have

$$W_1^{(k)}(\lambda) = \frac{\lambda^k (1-\lambda)}{(\lambda+1)^{2k+1}} + \frac{\lambda}{(\lambda+1)^2} W_1^{(k-1)}(\lambda)$$

$$W_1^{(k-1)}(\lambda) = \frac{\lambda^{k-1} (1-\lambda)}{(\lambda+1)^{2k-1}} + \frac{\lambda}{(\lambda+1)^2} W_1^{(k-2)}(\lambda)$$

$$\vdots$$

$$W_1^{(2)}(\lambda) = \frac{\lambda^2 (1-\lambda)}{(\lambda+1)^5} + \frac{\lambda}{(\lambda+1)^2} W_1^{(1)}(\lambda).$$

By the above iteration steps, we get the following formula for the numbers $W_1^{(k)}(\lambda)$:

$$W_1^{(k)}(\lambda) = \frac{k\lambda^k (1-\lambda)}{(\lambda+1)^{2k+1}}. \tag{44}$$

Theorem 15 (*cf.* [100]) *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{c=0}^{n} \sum_{j=0}^{k} \sum_{v=0}^{j} \binom{n}{c} \binom{k}{j} \binom{j}{v} \frac{2^k \lambda^{2v}}{(2\lambda)^j} W_c^{(k)}(\lambda) \, (2v - j)^{n-c} = 0.$$

Relations among the Apostol-type numbers $W_n(\lambda)$, the Apostol-Bernoulli numbers, the Apostol-Euler numbers, and the Apostol-Genocchi numbers of order 2 are, respectively, given as follows:

$$W_n(\lambda) = \frac{\lambda}{4} \mathcal{E}_n^{(2)}(1; \lambda) \tag{45}$$

(*cf.* [99]),

$$W_{n-2}(-\lambda) = \frac{\lambda}{n\,(1-n)} \mathcal{B}_n^{(2)}(1; \lambda), \tag{46}$$

and

$$W_{n-2}(\lambda) = \frac{\lambda}{4n\,(n-1)} \mathcal{G}_n^{(2)}(1; \lambda), \tag{47}$$

(*cf.* [100]).

Generating functions, defined by Eqs. (30), (31), and (38), are associated with many well-known special number families and polynomial families. These relations will be given in the following sections with detailed and special notes.

Using (31), we get

$$\left(-\frac{1}{2}\right)^k \sum_{n=0}^{\infty} a_n^{(k)}(k)\frac{t^n}{n!} = \sum_{n=0}^{\infty} W_n^{(k)}\left(-\frac{1}{2}\right)\frac{t^n}{n!},$$

where $a_n^{(k)}(k)$ denotes the Fubini type polynomials which are defined by means of the following generating function:

$$\frac{2^k}{(2-e^t)^{2k}}e^{tx} = \sum_{n=0}^{\infty} a_n^{(k)}(x)\frac{t^n}{n!}$$

(*cf.* for details, see [32]). By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have

$$W_n^{(k)}\left(-\frac{1}{2}\right) = (-1)^k a_n^{(k)}(k). \tag{48}$$

When $k = 1$, we have

$$\frac{1}{2}a_n = w_g(n).$$

On the other hand in [33], Kilar and Simsek gave the following identity:

$$a_n^{(k)}(x) = (-1)^k W_n^{(k)}\left(x-k; -\frac{1}{2}\right).$$

Putting $x = k$ in the above equation, we also arrive at (48).

In works [21, 27, 32], we see that the numbers $w_g(n)$ are the nth ordered Bell numbers. These are evaluated by summation of the Stirling numbers of the second kind, which count the number of partitions of an n-element set into k nonempty subsets, expanded out into a double summation involving binomial coefficients (with the help of well-known formula expressing and representing the Stirling numbers of the first kind as a sum of binomial coefficients), or given by an infinite series.

$$w_g(n) = \sum_{m=0}^{n} m!S_2(n,m).$$

These numbers can be expressed by the ordered Bell numbers are the numbers in the first column of the infinite matrix

$$(2I - M)^{-1},$$

where I is the identity matrix and M is an infinite matrix form of Pascal's triangle. By using contour integration with the Cauchy residue theorem to the generating function for $w_g(n)$:

$$\frac{1}{2 - e^t} = \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!},$$

($|t| < \log 2$), the ordered Bell numbers $w_g(n)$, (or can be named the Fubini numbers, can be represented by the Cauchy's integral and the infinite series, respectively:

$$w_g(n) = \int_{\gamma} \frac{dz}{z^{n+1}(2 - e^z)}, \tag{49}$$

where γ is a closed loop enclosing the origin, but no other singularities of the integrand; and

$$w_g(n) = \frac{n!}{2} \sum_{m=-\infty}^{\infty} \frac{1}{(\log 2 + 2\pi i m)^{n+1}},$$

where $n \in \mathbb{N}$. Since $\log 2$ is less, these numbers are approximated as follows:

$$w_g(n) \approx \frac{n!}{2} \frac{1}{(\log 2)^{n+1}},$$

that is this approximation gives us the numbers $w_g(n)$ exceed the corresponding factorials by an exponential factor. Therefore, using (49) with the analytic continuation of the Riemann zeta function, Bailey [5] gave the following relation:

$$w_g(n) = \frac{n!}{2} \frac{1}{(\log 2)^{n+1}} + o(\Gamma(n)).$$

By using the above approach, we have

$$\lim_{n \to \infty} \frac{n w_g(n-1)}{w_g(n)} = \log 2$$

(*cf.* [5, 21, 113]).

Here we note that in the light of the theory of approach to the approximation error and other formulas given for the numbers $w_g(n)$ given above, similar properties involving the approximation error by considering all the singularities and the analytic continuation, on the complex plane, of the generating functions $F_w(t;\lambda)$ and $F_w(t;\lambda;k)$ of numbers $W_n(\lambda)$ and $W_n^{(k)}(\lambda)$ can now be examined. These stages of research and investigation are left to the readers.

The following theorem shows that the numbers $W_n(\lambda)$ are associated with the numbers $y_2(n,k;\lambda)$:

Theorem 16 (*cf.* [99]) *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n} \binom{n}{m} W_{n-m}^{(k)}(\lambda) y_2(m,k;\lambda) = 0.$$

Theorem 17 (*cf.* [99]) *Let $n, k \in \mathbb{N}_0$. Then we have*

$$W_n^{(k)}(x;\lambda) = \frac{1}{4^k} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \mathscr{E}_m^{(k)}\left(\frac{x}{2};\lambda\right) \mathscr{E}_{n-m}^{(k)}\left(-\frac{x}{2};\lambda^{-1}\right). \tag{50}$$

***Proof** (cf. [99])* By using (38) and (11), one has the following well-known functional equation:

$$2^{2k} G_w(t,x,k;\lambda) = F_{\mathscr{E}}\left(t,\frac{x}{2};k,\lambda\right) F_{\mathscr{E}}\left(-t,-\frac{x}{2};k,\lambda^{-1}\right).$$

By the help of (38) and (11) with the above equation, one has

$$\sum_{n=0}^{\infty} 2^{2k} W_n^{(k)}(x;\lambda) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \mathscr{E}_m^{(k)}\left(\frac{x}{2};\lambda\right) \mathscr{E}_{n-m}^{(k)}\left(-\frac{x}{2};\lambda^{-1}\right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the assertion of Theorem 17.

By using (21) and (31), we have

$$F_w(t;\lambda;-k) = (2k)! F_{y_2}(t,k;\lambda). \tag{51}$$

Using (51), we have the following relation between the numbers $W_n^{(-k)}(\lambda)$ and the numbers $y_2(n,k;\lambda)$:

Theorem 18 (*cf.* [99]) *Let* $n, k \in \mathbb{N}_0$. *Then we have*

$$W_n^{(-k)}(\lambda) = (2k)! y_2(n, k; \lambda). \tag{52}$$

Using (20) and (31), we have

$$F_N(t; -k, \lambda) = \sum_{m=0}^{k} \binom{k}{m} \frac{1}{2^m} F_w(t; \lambda; -m). \tag{53}$$

By applying the above equation, a relation between the numbers $E_n^{*(-k)}(\lambda)$ and $W_n^{(-k)}(\lambda)$ is given by the following theorem:

Theorem 19 (*cf.* [99]) *Let* $n, k \in \mathbb{N}_0$. *Then we have*

$$E_n^{*(-k)}(\lambda) = \sum_{m=0}^{k} \binom{k}{m} \frac{W_n^{(-m)}(\lambda)}{2^m}.$$

Remark 1 For a table including a few values of the numbers $W_n^{(-k)}(\lambda)$, the reader may glance at [99].

3 Identities and Derivative Formulas Arising from the Partial Differential Equations Including the Generating Functions for the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x; \lambda)$

In [41], by differentiating the generating functions for the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x; \lambda)$ with respect to their parameters, Kucukoglu and Simsek presented partial differential equations including these functions. By making use of these equations, they provided some formulas, relations, and identities including these numbers and polynomials and their derivatives. By using a collection of the generating functions for the Apostol-type numbers and polynomials of higher order and their functional equations, they also investigated the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x; \lambda)$ and their relationships with other well-known special numbers and polynomials including the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Frobenius-Euler numbers and polynomials of higher order, the λ-array polynomials, the λ-Stirling numbers, and the λ-Bernoulli numbers and polynomials.

Theorem 20 **(*cf.* [41])** *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n} \binom{n}{m} \left(\lambda^2 - (-1)^{n-m}\right) W_m^{(k+1)}(x;\lambda) = \lambda \frac{x W_n^{(k)}(x;\lambda) - W_{n+1}^{(k)}(x;\lambda)}{k}.$$

Proof *(cf. [41])* Differentiating both sides of (31) and (38) with respect to t yields, respectively, the following partial differential equations:

$$\frac{\partial}{\partial t}\left\{F_w(t;\lambda;k)\right\} = -k\left(\lambda e^t - \lambda^{-1}e^{-t}\right) F_w(t;\lambda;k+1), \tag{54}$$

and

$$\frac{\partial}{\partial t}\left\{G_w(t,x;\lambda;k)\right\} = xG_w(t,x;\lambda;k) - k\left(\lambda e^t - \lambda^{-1}e^{-t}\right) G_w(t,x;\lambda;k+1). \tag{55}$$

Combining (38) and (55) yields the following relation:

$$x\sum_{n=0}^{\infty} W_n^{(k)}(x;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} W_{n+1}^{(k)}(x;\lambda)\frac{t^n}{n!}$$
$$= k\sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m}\left(\lambda^2 - (-1)^{n-m}\right) W_m^{(k+1)}(x;\lambda)\frac{t^n}{n!}.$$

In order to complete proof of the assertion of Theorem 20, now it is time to compare the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation.

We now give some the special case of Theorem 20. When $x = 0$ with (41) yields the following result:

Corollary 1 **(*cf.* [41])** *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then we have*

$$W_{n+1}^{(k)}(\lambda) = \frac{k}{\lambda}\sum_{m=0}^{n}\left((-1)^{n-m} - \lambda^2\right)\binom{n}{m} W_m^{(k+1)}(\lambda).$$

When $x = 1$ with (41) yields the following result:

Corollary 2 *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n}\binom{n}{m}\left(1 - (-1)^{n-m}\right) W_m^{(k+1)}(1;1) = \frac{W_n^{(k)}(1;1) - W_{n+1}^{(k)}(1;1)}{k}. \tag{56}$$

Comparing (56) with (40), we have

$$\sum_{m=0}^{n}\binom{n}{m}\left(1-(-1)^{n-m}\right)\sum_{b=0}^{m}\binom{m}{b}W_b^{(k+1)}(1,1)$$
$$=\frac{1}{k}\sum_{b=0}^{n}\left(\binom{n}{b}-\binom{n+1}{b}\right)W_b^{(k)}(1,1)-\frac{1}{k}W_{n+1}^{(k)}(1;1).$$

Substituting the following the well-known the Pascal's rule (or Pascal's formula), which is a combinatorial identity about binomial coefficients, into the above equation

$$\binom{n-1}{b}+\binom{n-1}{b-1}=\binom{n}{b},$$

we get

$$\sum_{m=0}^{n}\binom{n}{m}\left(1-(-1)^{n-m}\right)\sum_{b=0}^{m}\binom{m}{b}W_b^{(k+1)}(1,1)$$
$$=\frac{1}{k}\sum_{b=0}^{n}\binom{n}{b-1}W_b^{(k)}(1,1)-\frac{1}{k}W_{n+1}^{(k)}(1;1),$$

where

$$\binom{n}{b}-\binom{n+1}{b}=\binom{n}{b-1}$$

and assuming that

$$\binom{n}{-1}=0.$$

Comparing the above equation with (40), we obtain

$$\frac{1}{4^{k+1}}\sum_{m=0}^{n}\binom{n}{m}\left(1-(-1)^{n-m}\right)\sum_{b=0}^{m}\binom{m}{b}W_b^{(k+1)}(1,1)$$
$$=\frac{1}{k}\sum_{b=1}^{n}\binom{n}{b-1}W_b^{(k)}(1,1)-\frac{1}{k}W_{n+1}^{(k)}(1;1).$$

Comparing the above equation with (33), we arrive at the following identity for the Euler polynomials of the first kind of higher order:

Theorem 21 *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n}\binom{n}{m}\left(1-(-1)^{n-m}\right)\sum_{b=0}^{m}\binom{m}{b}E_{n}^{(k+1)}(k+1) \tag{57}$$
$$=\frac{4}{k}\sum_{b=1}^{n}\binom{n}{b-1}E_{b}^{(k)}(k)-\frac{4}{k}E_{n+1}^{(k)}(k).$$

By using (57), we easily arrive at the following very interesting formulas:

Corollary 3 *Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then we have*

$$\frac{k}{2}\sum_{\substack{m=0\\ n-m\ odd}}^{n}\binom{n}{m}\sum_{b=0}^{m}\binom{m}{b}E_{n}^{(k+1)}(k+1)=\sum_{b=1}^{n}\binom{n}{b-1}E_{b}^{(k)}(k)-E_{n+1}^{(k)}(k),$$

otherwise

$$E_{n+1}^{(k)}(k)=\sum_{b=1}^{n}\binom{n}{b-1}E_{b}^{(k)}(k).$$

Theorem 22 (*cf.* [41])

$$W_{n+1}^{(k)}(\lambda)=k\left(W_{n}^{(k)}(\lambda)-2\lambda\sum_{m=0}^{n}\binom{n}{m}W_{m}^{(k+1)}(\lambda)-2W_{n}^{(k+1)}(\lambda)\right)$$

or, equivalently,

$$W_{n+1}^{(k)}(\lambda)=kW_{n}^{(k)}(\lambda)-2\lambda W_{n}^{(k+1)}(1;\lambda)-2W_{n}^{(k+1)}(\lambda).$$

Proof *(cf. [41])* By differentiating both sides of (31) with respect to t, we also have

$$\frac{\partial}{\partial t}\left\{F_{w}(t;\lambda;k)\right\}=kF_{w}(t;\lambda;k)-2k\left(\lambda e^{t}+1\right)F_{w}\left(t;\lambda;k+1\right).$$

Thus, we have

$$\sum_{n=0}^{\infty}W_{n+1}^{(k)}(\lambda)\frac{t^{n}}{n!}=k\sum_{n=0}^{\infty}\left(W_{n}^{(k)}(\lambda)-2\lambda\sum_{m=0}^{n}\binom{n}{m}W_{m}^{(k+1)}(\lambda)-2W_{n}^{(k+1)}(\lambda)\right)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the assertion of Theorem 22.

Theorem 23 **(*cf.* [41])**

$$\frac{\partial}{\partial\lambda}\{W_n^{(k)}(x;\lambda)\} = k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m}-1\right)W_m^{(k+1)}(x;\lambda).$$

Proof *(cf. [41])* Differentiating both sides of Eq. (38) with respect to λ yields the following partial differential equation:

$$\frac{\partial}{\partial\lambda}\{G_w(t,x;\lambda;k)\} = -k\left(e^t-\lambda^{-2}e^{-t}\right)G_w(t,x;\lambda;k+1). \tag{58}$$

From (58), we have

$$\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty}\left(\lambda^{-2}(-1)^n-1\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}W_n^{(k+1)}(x;\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial\lambda}W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m}-1\right)W_m^{(k+1)}(x;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the assertion of Theorem 23.

Combining the special case of Theorem 23 when $x = 0$ with (41) yields the following corollary:

Corollary 4 **(*cf.* [41])**

$$\frac{d}{d\lambda}\{W_n^{(k)}(\lambda)\} = k\sum_{m=0}^{n}\binom{n}{m}\left(\lambda^{-2}(-1)^{n-m}-1\right)W_m^{(k+1)}(\lambda).$$

Theorem 24 **(*cf.* [41])**

$$\frac{\partial}{\partial x}\{W_{n+1}^{(k)}(x;\lambda)\} = (n+1)W_n^{(k)}(x;\lambda). \tag{59}$$

Proof *(cf. [41])* Differentiating both sides of Eq. (38) with respect to x yields the following partial differential equation:

$$\frac{\partial}{\partial x}\{G_w(t,x;\lambda;k)\} = tG_w(t,x;\lambda;k).$$

Thus, we have

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} W_n^{(k)}(x;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} n W_{n-1}^{(k)}(x;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the assertion of Theorem 24.

By using (59), the following derivative formulas are obtained:

$$\frac{\partial^2}{\partial x^2}\{W_{n+1}^{(k)}(x;\lambda)\} = (n+1)n W_{n-1}^{(k)}(x;\lambda),$$

$$\frac{\partial^3}{\partial x^3}\{W_{n+1}^{(k)}(x;\lambda)\} = (n+1)n(n-1) W_{n-2}^{(k)}(x;\lambda),$$

therefore, with help of the mathematical induction method, for $d \geq 1$, we get

$$\frac{\partial^d}{\partial x^d}\{W_{n+1}^{(k)}(x;\lambda)\} = (n+1)_d W_{n-d+1}^{(k)}(x;\lambda).$$

Remark 2 (cf. [41]*)* According to Roman [73, Theorem 2.5.8, p. 27], an Appell sequence $s_n(x)$ satisfy the following Appell identity:

$$s_n(x+y) = \sum_{m=0}^{n} \binom{n}{m} s_m(x) y^{n-m}.$$

Hence, by (24), one can infer that the polynomials $W_n^{(k)}(x;\lambda)$ are Appell sequences and it is clear that the following identities hold for the polynomials $W_n^{(k)}(x;\lambda)$:

$$W_n^{(k)}(x+y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(x;\lambda) y^{n-m}$$

and

$$W_n^{(k)}(x+y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(y;\lambda) x^{n-m}. \tag{60}$$

Substituting $x = y = 1$ into (60), we have

$$W_n^{(k)}(2;\lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(1,\lambda). \tag{61}$$

Substituting $x = -1$ and $y = 1$ into (60), we have

$$W_n^{(k)}(\lambda) = \sum_{m=0}^{n} \sum_{b=0}^{m} (-1)^{n-m} \binom{n}{m} \binom{m}{b} W_m^{(k)}(\lambda).$$

Substituting $x = 1$ and $y = 0$ into (60), we have

$$W_n^{(k)}(1; \lambda) = \sum_{m=0}^{n} \binom{n}{m} W_m^{(k)}(\lambda). \tag{62}$$

Theorem 25 (*cf.* [41])

$$W_n^{(k)}(x+1; \lambda) - W_n^{(k)}(x-1; \lambda) = \sum_{m=0}^{n} \left(1 - (-1)^{n-m}\right) \binom{n}{m} W_m^{(k)}(x; \lambda). \tag{63}$$

Substituting $x = 1$ into (63), we have

$$W_n^{(k)}(2; \lambda) - W_n^{(k)}(\lambda) = \sum_{m=0}^{n} \left(1 - (-1)^{n-m}\right) \binom{n}{m} W_m^{(k)}(1; \lambda).$$

Combining the above equation with (61) and (62), the following result is obtained:

Corollary 5

$$W_n^{(k)}(\lambda) = \sum_{m=0}^{n} \binom{n}{m} \sum_{b=0}^{m} \binom{m}{b} W_b^{(k)}(\lambda) - \sum_{m=0}^{n} \left(1 - (-1)^{n-m}\right) \binom{n}{m} \sum_{b=0}^{m} \binom{m}{b} W_b^{(k)}(\lambda).$$

Substituting $x = 0$ into (63), the following result is obtained:

Corollary 6

$$W_n^{(k)}(-1; \lambda) = \sum_{b=0}^{n} \binom{n}{b} W_b^{(k)}(\lambda) - \sum_{m=0}^{n} \left(1 - (-1)^{n-m}\right) \binom{n}{m} W_m^{(k)}(\lambda).$$

Corollary 7

$$W_n^{(k)}(-1; \lambda) = \sum_{b=0}^{n} \binom{n}{b} W_b^{(k)}(\lambda) - \sum_{\substack{m=0 \\ n-m \text{ odd}}}^{n} \binom{n}{m} W_m^{(k)}(\lambda).$$

From (63), the following corollary is obtained:

Corollary 8 (*cf.* [41])

$$\frac{W_n^{(k)}(x+1;\lambda) - W_n^{(k)}(x-1;\lambda)}{2} = \sum_{\substack{m=0 \\ n-m \text{ odd}}}^{n} \binom{n}{m} W_m^{(k)}(x;\lambda). \tag{64}$$

4 Relations Among the Numbers $W_n^{(k)}(\lambda)$, the Polynomials $W_n^{(k)}(x;\lambda)$ and Other Well-Known Apostol-Type Special Numbers and Polynomials

In this section, relations among the numbers $W_n^{(k)}(\lambda)$, the polynomials $W_n^{(k)}(x;\lambda)$ and other well-known Apostol-type special numbers and polynomials are given. Moreover, we shall give just brief sketch of the proofs as the details are similar to those in [41].

4.1 Relations of the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Bernoulli Numbers and Polynomials of Higher Order

Here, with the aid of the techniques including generating functions and their functional equations, relations among the Apostol-Bernoulli numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ are given.

The relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Bernoulli polynomials of higher order is given by the following theorem:

Theorem 26 (*cf.* [41])

$$\mathcal{B}_{n+2k}^{(2k)}(x;\lambda) = \frac{(-1)^k (n+2k)_{2k}}{\lambda^k} W_n^{(k)}(x-k;-\lambda). \tag{65}$$

Proof *(cf. [41])* By combining (38) with (1), the following functional equation is obtained:

$$t^{-2k} F_B(t,x;\lambda;2k) = (-1)^k \lambda^{-k} G_w(t,x-k,k;-\lambda). \tag{66}$$

It follows from the above functional equation that

$$\sum_{n=0}^{\infty} \frac{1}{(n+2k)_{2k}} \mathscr{B}_{n+2k}^{(2k)}(x;\lambda)\frac{t^n}{n!} = (-1)^k \lambda^{-k} \sum_{n=0}^{\infty} W_n^{(k)}(x-k;-\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the assertion of Theorem 26.

Combining (39) with (65) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Apostol-Bernoulli polynomials of higher order by the following corollary:

Corollary 9 (*cf.* [41])

$$\mathscr{B}_{n+2k}^{(2k)}(x;\lambda) = \binom{n+2k}{2k}\frac{(2k)!}{\lambda}\sum_{m=0}^{n}\binom{n}{m} \times \sum_{j=0}^{n-m}(-1)^{n-m-j+k}\binom{n-m}{j}x^j k^{n-m-j} W_m^{(k)}(-\lambda). \tag{67}$$

Substituting $k=1$ into (65) yields the following corollary:

Corollary 10 (*cf.* [41])

$$\mathscr{B}_{n+2}^{(2)}(x;\lambda) = -\frac{(n+2)\,(n+1)}{\lambda} W_n(x-1;-\lambda). \tag{68}$$

Replacing x by k and λ by $-\lambda$ in (65) and using (41) yields the following corollary:

Corollary 11 (*cf.* [41])

$$\mathscr{B}_{n+2}^{(2k)}(k;-\lambda) = \frac{(n+2k)_{2k}}{\lambda^k} W_n^{(k)}(\lambda). \tag{69}$$

4.2 Relations of the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Euler Numbers and Polynomials of Higher Order

Here, with the aid of the techniques including generating functions and their functional equations, relations among the Apostol-Euler numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, are given.

The relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Euler polynomials of higher order is given by the following theorem:

Theorem 27 (*cf.* [41])

$$\mathcal{E}_n^{(2k)}(x;\lambda) = \left(\frac{4}{\lambda}\right)^k W_n^{(k)}(x-k;\lambda). \tag{70}$$

Using (39) and (70), one has the following result:

Corollary 12 (*cf.* [41])

$$\mathcal{E}_n^{(2k)}(x;\lambda) = \left(\frac{4}{\lambda}\right)^k \sum_{m=0}^{n}\sum_{j=0}^{n-m}(-k)^{n-m-j}\binom{n}{m}\binom{n-m}{j}x^j W_m^{(k)}(\lambda). \tag{71}$$

Equations (32)–(57) are also obtained by using (70). That is, substituting $k=1$ into (70) yields the following corollary:

Corollary 13 (*cf.* [41])

$$\mathcal{E}_n^{(2)}(x;\lambda) = \frac{4}{\lambda}W_n(x-1;\lambda). \tag{72}$$

Replacing x by k in (70) and using (41) yields the following corollary:

Corollary 14 (*cf.* [41])

$$\mathcal{E}_n^{(2k)}(k;\lambda) = \left(\frac{4}{\lambda}\right)^k W_n^{(k)}(\lambda). \tag{73}$$

4.3 Relations of the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x;\lambda)$ with the Apostol-Genocchi Numbers and Polynomials of Higher Order

Here, with the aid of the techniques including the generating functions and their functional equations, relations among the Apostol-Genocchi numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ are given.

The relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Apostol-Genocchi polynomials of higher order is given by the following theorem:

Theorem 28 **(*cf.* [41])**

$$\mathscr{G}_{n+2k}^{(2k)}(x;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k W_n^{(k)}(x-k;\lambda). \tag{74}$$

Combining (39) with (74) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Apostol-Genocchi polynomials of higher order by the following corollary:

Corollary 15 **(*cf.* [41])**

$$\mathscr{G}_{n+2k}^{(2k)}(x;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k \sum_{m=0}^{n}\binom{n}{m}(x-k)^{n-m}\, W_m^{(k)}(\lambda). \tag{75}$$

Putting $k = 1$ in (74), one has the following corollary:

Corollary 16 **(*cf.* [41])**

$$\mathscr{G}_{n+2}^{(2)}(x;\lambda) = \frac{4\,(n+2)\,(n+1)}{\lambda} W_n(x-1;\lambda). \tag{76}$$

Replacing x by k in (74) and using (41) yields the following corollary:

Corollary 17 **(*cf.* [41])**

$$\mathscr{G}_{n+2k}^{(2k)}(k;\lambda) = (n+2k)_{2k}\left(\frac{4}{\lambda}\right)^k W_n^{(k)}(\lambda). \tag{77}$$

4.4 *Relations of the Numbers* $\mathbf{W}_n^{(k)}(\boldsymbol{\lambda})$ *and the Polynomials* $\mathbf{W}_n^{(k)}(\boldsymbol{x};\boldsymbol{\lambda})$ *with the Frobenius-Euler Numbers and Polynomials of Higher Order*

Here, with the aid of the techniques including generating functions and their functional equations, relations among the Frobenius-Euler numbers and polynomials of higher order, the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ are given.

The relation between the polynomials $W_n^{(k)}(x;\lambda)$ and the Frobenius-Euler polynomials of higher order is given by the following theorem:

Theorem 29 **(*cf.* [41])**

$$\mathscr{H}_n^{(2k)}(x|\lambda^{-1}) = \frac{(-1)^k\,(\lambda-1)^{2k}}{\lambda^k} W_n^{(k)}(x-k;-\lambda). \tag{78}$$

Combining (39) with (78) yields a relation between the numbers $W_n^{(k)}(\lambda)$ and the Frobenius-Euler polynomials of higher order by the following corollary:

Corollary 18 (*cf.* [41])

$$\mathscr{H}_n^{(2k)}(x|\lambda^{-1}) = \frac{(-1)^k (\lambda-1)^{2k}}{\lambda^k} \sum_{m=0}^{n} \binom{n}{m} (x-k)^{n-m} W_m^{(k)}(-\lambda). \tag{79}$$

Substituting $k = 1$ into (78) yields the following corollary:

Corollary 19 (*cf.* [41])

$$\mathscr{H}_n^{(2)}(x|\lambda^{-1}) = -\frac{(\lambda-1)^2}{\lambda} W_n(x-1; -\lambda). \tag{80}$$

Replacing x by k in (78) and using (41) yields the following corollary:

Corollary 20 (*cf.* [41])

$$\mathscr{H}_n^{(2k)}(k|\lambda^{-1}) = \frac{(-1)^k (\lambda-1)^{2k}}{\lambda^k} W_n^{(k)}(-\lambda). \tag{81}$$

Putting $k = 1$ in (81), one has the following relation:

Corollary 21 (*cf.* [41])

$$\mathscr{H}_n^{(2)}(1|\lambda^{-1}) = -\frac{(\lambda-1)^2}{\lambda} W_n(-\lambda). \tag{82}$$

4.5 *Relations of the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x;\lambda)$ with the λ-array Polynomials, the λ-Stirling Numbers, and λ-Bernoulli Numbers and Polynomials*

Here, with the aid of techniques including generating functions and their functional equations, relations among the λ-array polynomials, the numbers $W_n^{(k)}(\lambda)$, the polynomials $W_n^{(k)}(x;\lambda)$, λ-array polynomials, the λ-Bernoulli numbers and polynomials, the λ-Stirling numbers are given.

Theorem 30 (*cf*. [41])

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(-x;\lambda\right)W_{n-m}^{(k)}(x;-\lambda)=\frac{(-1)^{k}\lambda^{k}k^{n}}{(2k)!}. \tag{83}$$

By (24), substituting $x = 0$ into (83) yields a relation between the λ-Stirling numbers and the numbers $W_n^{(k)}(\lambda)$ by the following corollary:

Corollary 22 (*cf*. [41])

$$\sum_{m=0}^{n}\binom{n}{m}S\left(m,2k;\lambda\right)W_{n-m}^{(k)}(-\lambda)=\frac{(-1)^{k}\lambda^{k}k^{n}}{(2k)!}. \tag{84}$$

Theorem 31 (*cf*. [41])

$$\sum_{k=0}^{v}\sum_{m=0}^{n}(-1)^{k}\left(2k\right)!\binom{n}{m}S_{2k}^{m}\left(-x;\lambda\right)W_{n-m}^{(k)}(x;-\lambda)$$
$$=\frac{\lambda^{v+1}B_{n+1}\left(\lambda;v+1\right)-B_{n+1}\left(\lambda\right)}{n+1}.$$

Theorem 32 (*cf*. [41])

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(x;\lambda\right)W_{n-m}^{(k)}(x;-\lambda)=\frac{(-1)^{k}\lambda^{k}}{(2k)!}\left(2x+k\right)^{n}.$$

Substituting $\lambda = 1$ into the above theorem, we have

$$\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}\left(x\right)W_{n-m}^{(k)}(x;-1)=\frac{(-1)^{k}}{(2k)!}\left(2x+k\right)^{n}.$$

Combining the above equation with (23), one has the following corollary:

Corollary 23 (*cf*. [41])

$$\sum_{m=0}^{n}\sum_{j=0}^{2k}(-1)^{j}\binom{n}{m}\binom{2k}{j}\left(x+j\right)^{m}W_{n-m}^{(k)}(x;-1)=(-1)^{k}\left(2x+k\right)^{n}.$$

5 Application of the Laplace Transform and Mellin Transformation to the Generating Function of Apostol-Type Polynomials and λ-array Polynomials

5.1 *Application of the Laplace Transform to the Generating Function for the Apostol-Type Polynomials $W_n^{(k)}(x;\lambda)$ and Array Type Polynomials*

Here, by applying the Laplace transform to the generating function for the Apostol-type polynomials $W_n^{(k)}(x;\lambda)$ and array type polynomials, infinite series representation involving these polynomials are introduced.

By (38) and (22), we have

$$F_A(-t,-x,2k;\lambda)\,G_w(-t,-x,k;-\lambda)=\frac{(-1)^k\lambda^k e^{-(-2x+k)t}}{(2k)!}.$$

After multiplying both sides of the above equation by t^c with $c\in\mathbb{N}_0$, the Laplace transform is applied to the resulting equation, the following result is achieved:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(-x;-\lambda)\int_0^{\infty}t^{n+c}e^{-2xt}dt$$
$$=\frac{(-1)^k\lambda^k}{(2k)!}\int_0^{\infty}t^ce^{-kt}dt,$$

where $x>0$, and $t>0$. After some elementary calculations, we get the following theorem:

Theorem 33 *Let $c\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n+c)!}{n!(2x)^{n+c+1}}\sum_{m=0}^{n}\binom{n}{m}S_{2k}^{m}(-x;\lambda)\,W_{n-m}^{(k)}(-x;-\lambda)=\frac{(-1)^k\lambda^kc!}{(2k)!k^{c+1}}.$$

6 Application of the Mellin Transformation to the Generating Function for the Apostol-Type Numbers $W_n(\lambda)$, the Numbers $W_n^{(k)}(\lambda)$, and the Polynomials $W_n^{(k)}(x;\lambda)$

6.1 *Interpolation Function Related to the Families of Zeta-Type Functions*

By applying the Mellin transformation to the generating functions of the Apostol-type numbers $W_n(\lambda)$, the numbers $W_n^{(k)}(\lambda)$, and the polynomials $W_n^{(k)}(x;\lambda)$, we give the families of zeta-type functions interpolating these numbers at the negative integer n. Moreover, some fundamental properties and applications of these functions are given.

6.2 Interpolation Function for the Apostol-Type Numbers $W_n(\lambda)$

In [100], we studied on the interpolation function for the Apostol-type numbers $W_n(\lambda)$. This function is related to the well-known families of zeta-type functions such as the Hurwitz-Lerch zeta function, the Hurwitz zeta function, and the Riemann zeta function.

Let $s \in \mathbb{C}$. $s = u + iv$ with $u > 1$. By applying Mellin transformation to (30), we have

$$\Gamma(s)\zeta_W(s;\lambda) = \int_0^\infty \frac{t^{s-1}}{\lambda e^{-t} + \lambda^{-1}e^t + 2}dt, \tag{85}$$

(*cf.* [100]). Assuming that $\left|\lambda e^t\right| < 1$. After some elementary calculations in Eq. (30) and in Eq. (85), we have the following interpolation function of the Apostol-type numbers $W_n(\lambda)$:

Let $\lambda, s \in \mathbb{C}$ with $|\lambda| < 1$ and $\Re(s) > 1$. Then an interpolation function, related to the families of zeta-type functions, for the Apostol-type numbers $W_n(\lambda)$ is defined by

$$\zeta_W(s;\lambda) = \sum_{m=0}^{\infty} \frac{(-2)_m \lambda^{m+1}}{m!\,(m+1)^s}, \tag{86}$$

(*cf.* [100]).

By using (86), we have the following Lerch transcendent type function:

$$\zeta_W(s;\lambda) = \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^{m+1}}{(m+1)^{s-1}}.$$

That is

$$\zeta_W(s;\lambda) = \lambda\Phi(-\lambda, s-1, 1),$$

where $\Phi(\lambda, s, q)$ denotes the Hurwitz-Lerch zeta function defined by

$$\Phi(\lambda, s, b) = \sum_{m=0}^{\infty} \frac{\lambda^m}{(m+b)^s}$$

$$\left(b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right),$$

in which

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \qquad (\mathbb{Z}^- := \{-1, -2, -3, \cdots\}).$$

(*cf.* [117]). When $\lambda = b = 1$, the function $\Phi(\lambda, s, b)$ reduces to the Riemann zeta function (or the Euler-Riemann zeta function):

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

(*cf.* [117]). The Riemann zeta function is a meromorphic function on $\mathbb{C}$. This function is also holomorphic on $\mathbb{C} \setminus \{1\}$. Notice that $s = 1$ is a simple pole of this function. Its residue at $s = 1$ is 1. The Riemann zeta function $\zeta(s)$ converges when (the real part of s)$>$ 1. This function plays many important roles not only in theory of analytic number theory but also in physics, in probability theory, in applied statistics, and in other related areas. This function has the following well-known results:

For $n \in \mathbb{N}$. Then we have

$$\zeta(-n) = -\frac{B_n}{n}$$

and the well-known formula of Euler:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \tag{87}$$

(*cf.* [117]).

Consequently, the zeta-type function $\zeta_W(s; \lambda)$ interpolates the polynomials $W_n(\lambda)$ for negative integer values of n.

By using the Cauchy residue theorem with aid of Hankel's contour in Eq. (85), by the principle of analytic continuation, we have the following result:

$$\zeta_W(-n; \lambda) = W_n(\lambda),$$

where n is a positive integer (*cf.* [100]).

6.3 Interpolation Functions for the Numbers $W_n^{(k)}(\lambda)$ and the Polynomials $W_n^{(k)}(x;\lambda)$

Interpolation function of the numbers $W_n(\lambda)$, the details of which are discussed in the previous section, similar to this interpolation function, by making similar discussions for the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$, their known interpolation functions can be presented here.

Recently, in [50], Kucukoglu, Simsek, and Srivastava defined a new family of Lerch-type zeta functions, which are interpolating a certain class of the Apostol-type numbers of higher order, $W_n^{(k)}(x;\lambda)$ and the Apostol-type polynomials of higher order, $W_n^{(k)}(x;\lambda)$. They also constructed Lerch-type zeta functions which interpolate the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ at negative integers. Here, we survey these interpolation functions.

Let $\lambda \in \mathbb{C}$. Assuming that $|\lambda| < 1$. Let $s \in \mathbb{C}$ with $s = u+iv$. Assuming that $u > 1$. Thus, the interpolation functions for the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x;\lambda)$ are defined, respectively, by

$$\zeta_w(s,k;\lambda) = \sum_{m=0}^{\infty}(-1)^m\binom{m+2k-1}{m}\frac{\lambda^{m+k}}{(m+k)^s}, \tag{88}$$

and

$$\zeta_w(s,x,k;\lambda) = \sum_{m=0}^{\infty}(-1)^m\binom{m+2k-1}{m}\frac{\lambda^{m+k}}{(x+m+k)^s}, \tag{89}$$

(*cf.* [50]).

Assuming that $|\lambda e^t| < 1$. Using (31), we have

$$\sum_{m=0}^{\infty}\binom{-2k}{m}\lambda^{m+k}e^{(m+k)t} = \sum_{n=0}^{\infty}W_n^{(k)}(\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{-2k}{m}\lambda^{m+k}(m+k)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}W_n^{(k)}(\lambda)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have

$$W_n^{(k)}(\lambda) = \sum_{n=0}^{\infty}\binom{-2k}{m}\lambda^{m+k}(m+k)^n, \tag{90}$$

where $|\lambda| < 1$.

By combining the principle of analytic continuation with (90), we have the following modification of the interpolation function for the numbers $W_n^{(k)}(\lambda)$:

$$\zeta_w^*(s,k;\lambda) = \sum_{n=0}^{\infty} \binom{-2k}{m} \frac{\lambda^{m+k}}{(m+k)^s}, \tag{91}$$

where $\lambda \in \mathbb{C}$ with $|\lambda| < 1$and $s \in \mathbb{C}$ with $s = u + iv$ and $u > 1$.

When replacing s by $-n$; ($n \in \mathbb{N}$) in the Eq. (89) and after that using the following well-known result:

$$W_n^{(k)}(x;\lambda) = \sum_{m=0}^{\infty} (-1)^m \binom{m+2k-1}{m} \lambda^{m+k} (x+m+k)^n, \tag{92}$$

where $|\lambda| < 1$ [*cf.* [50]], we have the following interpolation function for the polynomials $W_n^{(k)}(x;\lambda)$:

$$\zeta_w(-n,x,k;\lambda) = W_n^{(k)}(x;\lambda), \tag{93}$$

where $n \in \mathbb{N}$ (*cf.* [50, Theorem 5 and Corollary 1]). Putting $x = 0$ in (93), we have the following interpolation function for the numbers $W_n^{(k)}(\lambda)$:

$$\zeta_w(-n,k;\lambda) = W_n^{(k)}(\lambda), \tag{94}$$

(*cf.* [50, Theorem 5 and Corollary 1]).

Furthermore,

$$\zeta_w^*(-n,k;\lambda) = W_n^{(k)}(x;\lambda),$$

where $n \in \mathbb{N}$.

The other method that can be used to construct the interpolation functions of the polynomials $W_n^{(k)}(x;\lambda)$ is the application of the Mellin transformation to the generating function given by (38). Now, we shall give just a brief sketch as the details are similar to those in [50] as follows: Let $\lambda \in \mathbb{C}$ ($|\lambda| < 1$) and $s \in \mathbb{C}$. Then, by applying the Mellin transformation to the generating function given by (38), we have

$$\zeta_w(s,x,k;\lambda) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} G_w(-t,x,\lambda;k)\,dt, \tag{95}$$

where $\Re(s) > 1$; $k \in \mathbb{N}_0$. By using the principle of analytic continuation together with the Cauchy Residue Theorem to (95), and making use of the same method as that used earlier by Srivastava et al. [119, p. 254]), the Eq. (93) is obtained.

In [50], Kucukoglu et al. remarked that, by setting $k = 1$, (88) is reduced to (86). That is, we have

$$\zeta_W(s;\lambda) = \zeta_w(s,1;\lambda) = \sum_{m=0}^{\infty} (-1)^m \frac{\lambda^{m+1}}{(m+1)^{s-1}},$$

which is directly related to not only the Lerch transcendent function, but also the Hurwitz zeta function (see, for details [50, 100, 117]).

Some relations of the functions $\zeta_w(s,x,k;\lambda)$ and $\zeta_w(s,k;\lambda)$ were given in [50]. Some of them are listed as follows:

A relation between the function $\zeta_w(s,x,k;\lambda)$ and the Hurwitz-Lerch zeta function is given by (*cf.* [50]):

$$\zeta_w(s,x,1;\lambda) = \lambda\Phi(-\lambda,s-1,x+1) - x\lambda\Phi(-\lambda,s,x+1). \tag{96}$$

A relation between the function $\zeta_w(s,k;\lambda)$ and the polylogarithm function is given by (*cf.* [50]):

$$\zeta_w(s,1;-\lambda) = -Li_{s-1}(\lambda),$$

where $Li_s(z)$ stands for the polylogarithmic function defined by

$$Li_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \qquad \big(\Re(s) > 1\big)$$

(*cf.* [116, 117]; see also the references cited therein).

A relation between the function $\zeta_w(s,k;\lambda)$ and the Dirichlet eta function is given by (*cf.* [50]):

$$\zeta_w(s,1;1) = \eta(s-1),$$

where $\eta(s)$ stands for the Dirichlet eta function defined by

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s}$$

in which $\zeta(s)$ denotes the Riemann zeta function which is defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad \big(\Re(s) > 1\big)$$

(*cf.* [116, 117]; see also the references cited therein).

It is well-known that the Riemann zeta function can be derived by the Mellin transformation of the generating function for the Bernoulli numbers with the help of Hankel transformation. By the aid of the Mellin transformation of the Jacobi's theta function, the Riemann zeta function can also be derived as follows:

$$\vartheta(z)=\sum_{n\in\mathbb{Z}}e^{\pi izn^2},$$

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{2\Gamma\left(\frac{s}{2}\right)}\int_0^\infty(\vartheta(it)-1)t^{\frac{s}{2}-1}dt,$$

where $\Re(s)>1$. The above integral is also known as the representation of the completed zeta function (*cf*. [62]).

A relation between the function $\zeta_w(s,k;\lambda)$ and the Riemann zeta function is given by (*cf*. [50]):

$$\zeta_w(s,1;1)=\left(1-2^{2-s}\right)\zeta(s-1). \tag{97}$$

A relation between the function $\zeta_w(s,1;1)$ and the Bernoulli numbers is given as follows:

Substituting $s=2n+1$ (with $n\in\mathbb{N}_0$) into (97), after some elementary calculations, combining the last equation with (87), we arrive at the following result:

$$\zeta_w(2n+1,1;1)=(-1)^{n+1}\frac{(2\pi)^{2n}\left(1-2^{1-2n}\right)B_{2n}}{2(2n)!}, \tag{98}$$

where $n\in\mathbb{N}_0$. By using the following well-known relation between $\zeta(2k)$ and the Eisenstein series

$$G(z,2k)=\sum_{(0,0)\neq(m,n)\in\mathbb{Z}\times\mathbb{Z}}\frac{1}{(mz+n)^{2k}},$$

we have

$$\zeta(2k)=\frac{1}{2}G(z,2k)-\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{j=1}^{\infty}\sigma_{2k-1}(j)e^{2\pi ijz},$$

where

$$\sigma_m(j)=\sum_{d|j}d^m,$$

and

$$G(z, 2k) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(mz+n)^{2k}},$$

$2 \leq k \in \mathbb{N}$ and $z \in \mathbb{H} = \{z = u + iv \in C : v > 0\}$ (*cf.* [60, 62, 77, 88, 91]; and the references cited therein).

Combining the above equation with (87), we have

$$G(z, 2k) - 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{j=1}^{\infty} \sigma_{2k-1}(j) e^{2\pi i j z} = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{(2k)!}.$$

Setting $k = 1$ and $\lambda = -1$ in (89), the following functional equations are obtained (*cf.* [50]):

$$\zeta_w (s, 1; -1) = -\zeta(s-1) \qquad \left(\Re(s) > 1\right), \tag{99}$$

and

$$\zeta_w (s, x, 1; -1) = -\zeta(s-1, x+1) + x\zeta(s, x+1) \qquad \left(\Re(s) > 1\right), \tag{100}$$

where $\zeta(s, a)$ stands for the Hurwitz zeta function defined by (*cf.* [117]):

$$\zeta(s, a) = \sum_{m=1}^{\infty} \frac{1}{(m+a)^s}; \qquad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 1\right).$$

Rational approximations to the zeta function are given as follows:

In [6], for each $m \in \mathbb{N}$, Ball defined the following sums which converges locally uniformly to the Riemann zeta function for $\Re(s) > 0$:

$$F_m(s) = \sum_{v=0}^{m} \frac{a_{m,v}}{s+v-1} B_v$$

and

$$G_m(s) = \sum_{v=0}^{m} (-1)^v \frac{a_{m,v}}{s+v-1},$$

where

$$\prod_{v=1}^{m} \left(1 - \frac{t}{v}\right) = \sum_{v=0}^{m} (-1)^v a_{m,v} t^v.$$

Therefore

$$\frac{F_m(s)}{(s-1)G_m(s)} \to \zeta(s)$$

locally uniformly on the set $\{s \in \mathbb{C} : \Re(s) > 0\}$ with the obvious convention at $s = 1$ (*cf.* [6]).

From the above computation, one can easily arrive at the well-known results:

$$F_m(s) \approx h_m^{1-s}\Gamma(s)\zeta(s)$$

and

$$(s-1)G_m(s) \approx h_m^{1-s}\Gamma(s),$$

where h_m is the partial sum

$$\sum_{v=1}^{m} \frac{1}{v}$$

of the harmonic series or related to the harmonic numbers (*cf.* [6]).

By using the above method, rational approximations for the function $\zeta_w(s, k; \lambda)$ and $\zeta_W(s; \lambda)$ can be investigated. This problem may be dealt with the readers.

7 Functional Equations and Their Associated Raabe-Type Multiplication Formula for the Polynomials $W_n^{(k)}(x; \lambda)$

In order to give a classification for the family of special polynomials, Raabe formula or multiplication formula is used. For instance, in the theory of the normalized polynomials, this formula is very important because these polynomials satisfy the multiplication formula. The results, which are given here, can be used in the some following areas: the theory of multiplication formulas related to the periodic functions and the normalized polynomials occur in Franel's formula, in the theory of the Dedekind sums, and in the theory of the Hardy-Berndt sums, in the theory of the zeta and L-functions, and in the theory of periodic bounded variation, and others (*cf.* [81, 87, 88, 90, 91, 93]; see also the references cited therein).

In [50], by using the Chu-Vandermonde identity and interpolation function for the polynomials $W_n^{(k)}(x; \lambda)$, Kucukoglu and Simsek gave another family Raabe-type multiplication formula for the polynomials $W_n^{(k)}(x; \lambda)$, which is given as follows:

Theorem 34 (*cf.* [50]) *Let*

$$\Phi_i^*(s,x,k;\lambda)=\sum_{n=0}^{\infty}\binom{2k+n-1}{i}\frac{(-\lambda)^n}{(x+n+k)^s},$$

then the following Raabe-type multiplication formula holds true:

$$\begin{aligned}&\zeta_w(s,x,kd;\lambda)\\&=\sum_{j=0}^{d-1}\frac{(-1)^j\lambda^j}{d^s}\sum_{l=0}^{2kd-1}\binom{d+j-1}{2kd-1-l}\frac{1}{l!}\sum_{v=0}^{l}\sum_{i=0}^{v}S_1(l,v)\,S_2(v,i)\,d^v i!\\&\times\lambda^{dk}\Phi_i^*\left(s,\frac{x+j}{d},k;\lambda^d\right).\end{aligned}$$

Proof We shall give just a brief sketch of the proof as the details are similar to those in [50]. Using (89), we have

$$\zeta_w(s,x,k;\lambda)=\sum_{m=0}^{\infty}(-1)^m\binom{m+2k-1}{2k-1}\frac{\lambda^{m+k}}{(x+m+k)^s}\qquad(\Re(s)>1).$$

By substituting $k=kd$ and $m=nd+j$ ($j=0,1,\ldots,d-1;\ n\in\mathbb{N}_0$) into the above equation, and combining the final equation with the following well-known Chu-Vandermonde identity:

$$\binom{x+y}{n}=\sum_{j=0}^{n}\binom{x}{j}\binom{y}{n-j},$$

(*cf.* [17, 21]), we have

$$\begin{aligned}\zeta_w(s,x,kd;\lambda)=&\sum_{j=0}^{d-1}\frac{(-1)^j\lambda^j}{d^s}\sum_{n=0}^{\infty}(-1)^{nd}\sum_{l=0}^{2kd-1}\binom{d(2k+n-1)}{l}\\&\times\binom{d+j-1}{2kd-1-l}\frac{\left(\lambda^d\right)^{n+k}}{\left(\frac{x+j}{d}+n+k\right)^s}.\end{aligned}\tag{101}$$

Combining (101) with (29) and (26) yields the assertion of Theorem 34.

Remark 3 Let

$$c_{l,m}^{(k)}=\sum_{p=1}^{k}(-1)^{k-p}S_1(k,p)\,S_2(p,l)\,S_2(p,m).$$

Then, with the combination of the following identity (*cf.* [63]):

$$\binom{xy}{k} = \sum_{l,m=1}^{k} \frac{l!m!}{k!} c_{l,m}^{(k)} \binom{x}{l} \binom{y}{m}$$

with the Eq. (101), other forms of the Raabe-type multiplication formula given in Theorem 34 can be obtained (*cf.* [50]).

Theorem 35 (*cf.* [50]) *Let d be an odd positive integer. Then we have*

$$\zeta_w(s, x, 1; \lambda) = \sum_{j=0}^{d-1} \frac{(-\lambda)^j}{d^s} \left\{ d\zeta_w\left(s, \frac{x+j}{d}, 1; \lambda^d\right) + j\lambda^d \Phi(-\lambda^d, s, \frac{x+j}{d} + 1) \right\}. \tag{102}$$

It is well-known that the Hurwitz-Lerch zeta function is interpolation function of the Apostol-Bernoulli numbers because of the fact that

$$\Phi(\lambda, -m, x) = \sum_{n=0}^{\infty} \lambda^n (n+x)^m = -\frac{\mathscr{B}_{m+1}(x; \lambda)}{m+1} \tag{103}$$

(*cf.* [116, 117]; see also the references cited therein). Thus, the combination of (103) with Theorem 35 yields the following theorem:

Theorem 36 (*cf.* [50])

$$W_m(x; \lambda) = \sum_{j=0}^{d-1} (-\lambda)^j d^m \left\{ dW_m\left(\frac{x+j}{d}; \lambda^d\right) - \frac{j\lambda^d}{m+1} \mathscr{B}_{m+1}\left(\frac{x+j}{d} + 1; -\lambda^d\right) \right\}.$$

8 Some Special Power Series Including the Numbers of the Lyndon Words and Binomial Coefficients

In [42], by using the methods associated with zeta type functions interpolating the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x; \lambda)$, Kucukoglu and Simsek gave formulas for special power series representations involving the numbers counting Lyndon words and the numbers $W_n^{(k)}(\lambda)$ and the polynomials $W_n^{(k)}(x; \lambda)$.

Now, we continue with recalling some definitions and notations associated with the Lyndon words as follows:

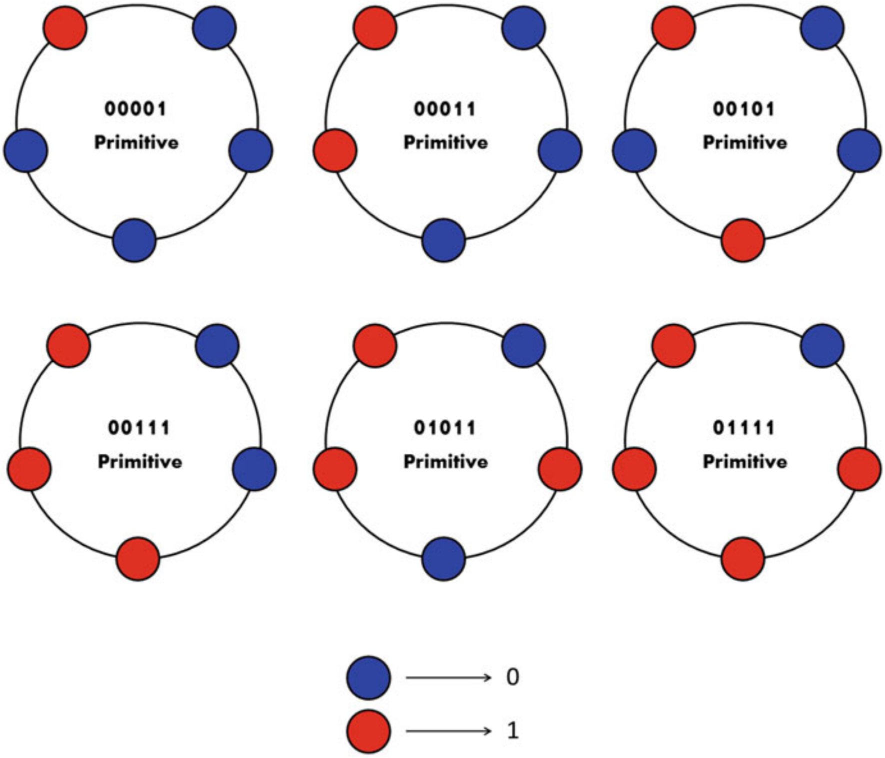

Fig. 1 Primitive necklaces, consisting of 5 beads of different 2 colors, each representing 2-ary Lyndon words of length 5

The k-ary Lyndon words of length n, represented by a primitive necklace consisting of n beads of different k colors, lexicographically smallest element of the set derived from all primitive words having length n over the k-letter alphabet. Here, primitive words means that a word cannot be written as a positive power of its subword. For instance, let us consider the set of alphabets as $\{0, 1\}$. All 2-ary Lyndon words of length 5 which are derived from this alphabet are given as follows: $\{00001, 00011, 00101, 00111, 01011, 01111\}$. It is clear that the elements of this finite set are primitive words (*cf.* [11, 22, 42, 43, 45, 47, 49, 52]; and see also the references cited therein).

Figure 1 illustrates the primitive necklaces, consisting of 5 beads of different 2 colors, each representing 2-ary Lyndon words of length 5.

For further primitive necklaces representatives of the Lyndon words of various lengths, the interested readers may glance at the works [11, 43, 45, 47–49, 123]; and also the references cited therein.

Let μ be the Möbius function (see, for details, [2]). Then, the computation formula that gives the numbers $L_k(n)$ of k-ary Lyndon words of length n is given as follows:

$$L_k\left(n\right)=\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)k^d, \tag{104}$$

where $\sum_{d|n}$ stands for the summation running over all positive divisors of the positive integer n (*cf.* [11, 22, 42, 43, 45, 47, 49, 52]).

Let $n\in\mathbb{N}$, $\lambda\in\mathbb{C}$ with $|\lambda|<1$. In [42], Kucukoglu and Simsek defined the function $\mathcal{G}(\lambda,n,k)$ arising from a power series involving the numbers $L_k(n)$ and the binomial coefficients as follows:

$$\mathcal{G}(\lambda,n,k)=\sum_{m=0}^{\infty}\binom{-2k}{m}L_{m+k}\left(n\right)\lambda^{m+k}, \tag{105}$$

(*cf.* [42]).

Remark 4 Because of the fact that

$$\binom{-2k}{m}=(-1)^m\binom{m+2k-1}{m},$$

the Eq. (105) can be rewritten as

$$\mathcal{G}(\lambda,n,k)=\sum_{m=0}^{\infty}(-1)^m\binom{m+2k-1}{m}L_{m+k}(n)\lambda^{m+k},$$

(*cf.* [42]).

Theorem 37 (*cf.* [42]) *Let $n\in\mathbb{N}$. Then we have*

$$\mathcal{G}(\lambda,n,k)=\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)W_d^{(k)}(\lambda). \tag{106}$$

Putting $k=1$ in (106), one has the following corollary:

Corollary 24 (*cf.* [42])

$$\mathcal{G}(\lambda,n,1)=\sum_{m=0}^{\infty}(-1)^m\left(m+1\right)L_{m+1}\left(n\right)\lambda^{m+1}$$

or equivalently

$$\mathcal{G}(\lambda,n,1)=\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)W_d(\lambda). \tag{107}$$

When we replace n by a prime number p in (106) gives the following corollary:

Corollary 25 (*cf.* [42]) *Let p be a prime number. Then we have*

$$\mathscr{G}(\lambda, p, k) = \frac{W_p^{(k)}(\lambda) - W_1^{(k)}(\lambda)}{p}. \tag{108}$$

By (44), we modify (108) as follows:

Corollary 26 *Let p be a prime number. Then we have*

$$p\mathscr{G}(\lambda, p, k) = \frac{(\lambda+1)^{2k+1} W_p^{(k)}(\lambda) + k\lambda^k (\lambda - 1)}{(\lambda+1)^{2k+1}}.$$

Remark 5 For some cases of (108) when $k = 1$ with $p = 2$ and $p = 3$, the interested readers may refer to [42]. Inspired by Kucukoglu and Simsek [42], we shall give a case of (108) as follows:

By setting $k = 3$ and $p = 2$, we get

$$\begin{aligned}\mathscr{G}(\lambda, 2, 3) &= \frac{W_2^{(3)}(\lambda) - W_1^{(3)}(\lambda)}{p} \\ &= \frac{3\lambda^3\left(2\lambda^2 - 4\lambda + 1\right)}{(\lambda+1)^8}.\end{aligned}$$

The polynomials $\mathscr{L}_n(x, m, k)$ of degree n are defined by (*cf.* [42]):

$$\mathscr{L}_n(x, m, k) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{j=0}^{d} \binom{d}{j} (m+k)^{d-j} x^j. \tag{109}$$

Replacing n by a prime number p in the Eq. (109) yields

$$\mathscr{L}_p(x, m, k) = \frac{(x+m+k)^p - (x+m+k)}{p},$$

(*cf.* [42, Corollary 3.7, p. 102]).

Let $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. In [42], Kucukoglu and Simsek also defined the function $\mathscr{H}(x; \lambda, n, k)$ arising from a power series involving $\mathscr{L}_n(x, m, k)$ as follows:

$$\mathscr{H}(x; \lambda, n, k) = \sum_{m=0}^{\infty} \binom{-2k}{m} \mathscr{L}_n(x, m, k)\, \lambda^{m+k} \tag{110}$$

(*cf.* [42, Definition 3.8, p. 103]).

Theorem 38 (*cf.* [42]) *The following identity holds true:*

$$\mathscr{H}(x;\lambda,n,k)=\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)W_{d}^{(k)}(x;\lambda). \tag{111}$$

Remark 6 Notice that putting $k=1$ in (111), one has

$$\mathscr{H}(x;\lambda,n,1)=\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)W_{d}(x;\lambda).$$

Putting $x=0$ in the above equation, we have

$$\mathscr{H}(0;\lambda,n,k)=\mathscr{G}(\lambda,n,k)\,,$$

(*cf.* [42, Remark 3.10, p. 103]).

9 Computational Algorithms Arising from the Recurrence Formula for the Numbers $W_n^{(k)}(\lambda)$

In order to provide numerical evaluations for the numbers $W_n(\lambda)$, the numbers $W_n^{(k)}(\lambda)$ and the function $\mathscr{G}(\lambda,n,k)$, Kucukoglu and Simsek [42] gave the following computational algorithms:

For the computation of the numbers $W_n(\lambda)$, the equation (36) allows to write Algorithm 1 as follows (see, for details, [42]):

Algorithm 1 Let $n\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. This algorithm will recursively return the numbers $W_n(\lambda)$ (*cf.* [42])

```
procedure W_APOSTOL_TYPE_NUM(n: nonnegative integer, λ)
    Begin
    Localvariablem : positive integer
    if n = 0 then
        return λ/power (λ + 1, 2)
    else
        return W_APOSTOL_TYPE_NUM (0, λ)
            ↪ *sum( ((1/λ) * power (−1, n − m + 1) − λ) * Binomial_Coef (n, m)
            ↪ * W_APOSTOL_TYPE_NUM(n − m, λ) , m, 1, n)
    end if
end procedure
```

In addition to the numerical values of the numbers $W_n(\lambda)$ obtained after implementing Algorithm 1 when $\lambda=\frac{1}{4}$ and $n\in\{1,2,3,4,5,6\}$ given in [42], by inspiring from [42], we shall give further numerical values of the numbers $W_n(\lambda)$ for the cases when $\lambda \in \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$ and $n \in \{0, 1, 2, 3, 4, 5, 6\}$ as follows:

$$W_0\left(\frac{1}{5}\right) = 0.1389, \quad W_0\left(\frac{1}{4}\right) = 0.16, \quad W_0\left(\frac{1}{3}\right) = 0.1875, \quad W_0\left(\frac{1}{2}\right) = 0.2222,$$

$$W_1\left(\frac{1}{5}\right) = 0.0926, \quad W_1\left(\frac{1}{4}\right) = 0.096, \quad W_1\left(\frac{1}{3}\right) = 0.0938, \quad W_1\left(\frac{1}{2}\right) = 0.0741,$$

$$W_2\left(\frac{1}{5}\right) = 0.0231, \quad W_2\left(\frac{1}{4}\right) = 0.0064, \quad W_2\left(\frac{1}{3}\right) = -0.0234, \quad W_2\left(\frac{1}{5}\right) = -0.0741,$$

$$W_3\left(\frac{1}{5}\right) = -0.0617, \quad W_3\left(\frac{1}{4}\right) = -0.0883, \quad W_3\left(\frac{1}{3}\right) = -0.1172, \quad W_3\left(\frac{1}{2}\right) = -0.1235,$$

$$W_4\left(\frac{1}{5}\right) = -0.1183, \quad W_4\left(\frac{1}{4}\right) = -0.1165, \quad W_4\left(\frac{1}{3}\right) = -0.0762, \quad W_4\left(\frac{1}{2}\right) = 0.0576,$$

$$W_5\left(\frac{1}{5}\right) = -0.0360, \quad W_5\left(\frac{1}{4}\right) = 0.0591, \quad W_5\left(\frac{1}{3}\right) = 0.2256, \quad W_5\left(\frac{1}{2}\right) = 0.4033,$$

$$W_6\left(\frac{1}{5}\right) = 0.3339, \quad W_6\left(\frac{1}{4}\right) = 0.5127, \quad W_6\left(\frac{1}{3}\right) = 0.6028, \quad W_6\left(\frac{1}{2}\right) = 0.1454.$$

For the computation of the numbers $W_n^{(k)}(\lambda)$, the recurrence relation given in (43) allows to write Algorithm 2 as follows (see, for details, [42]):

Algorithm 2 Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. This algorithm will recursively return the numbers $W_n^{(k)}(\lambda)$ with the aid of `W_APOSTOL_TYPE_NUM` procedure given by the Algorithm 1 (*cf.* [42])

---

procedure **`HIGHER_W_APOSTOL_TYPE_NUM`**(n: nonnegative integer, λ, k: positive integer)
  Begin
  Localvariable$m$: nonnegative integer
  if $k = 1$ **then**
    return `W_APOSTOL_TYPE_NUM`(n, λ)
  else
    return sum$\Big($$Binomial$_Coef $(n, m)*$`HIGHER_W_APOSTOL_TYPE_NUM` $(m, \lambda, k-1)$
      ↪ *`HIGHER_W_APOSTOL_TYPE_NUM`$(n-m, \lambda, 1)$, $m, 0, n\Big)$
  end if
end procedure

In addition to the numerical values of the numbers $W_n^{(k)}(\lambda)$ provided in [42], after implementing Algorithm 2, by inspiring from [42], we shall give further numerical values of the numbers $W_n^{(k)}(\lambda)$ for the cases when $k \in \{1, 2, 3\}$, $\lambda \in \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$ and $n \in \{0, 1, 2, 3, 4, 5, 6\}$ as follows:

$$W_0^{(1)}\left(\frac{1}{5}\right) = 0.1389, \quad W_0^{(1)}\left(\frac{1}{4}\right) = 0.16, \quad W_0^{(1)}\left(\frac{1}{3}\right) = 0.1875, \quad W_0^{(1)}\left(\frac{1}{2}\right) = 0.2222,$$

$$W_1^{(1)}\left(\frac{1}{5}\right)=0.0926,\quad W_1^{(1)}\left(\frac{1}{4}\right)=0.096,\quad W_1^{(1)}\left(\frac{1}{3}\right)=0.0938,\quad W_1^{(1)}\left(\frac{1}{2}\right)=0.0741,$$

$$W_2^{(1)}\left(\frac{1}{5}\right)=0.0231,\quad W_2^{(1)}\left(\frac{1}{4}\right)=0.0064,\quad W_2^{(1)}\left(\frac{1}{3}\right)=-0.0234,\quad W_2^{(1)}\left(\frac{1}{2}\right)=-0.0741,$$

$$W_3^{(1)}\left(\frac{1}{5}\right)=-0.0617,\quad W_3^{(1)}\left(\frac{1}{4}\right)=-0.0883,\quad W_3^{(1)}\left(\frac{1}{3}\right)=-0.1172,\quad W_3^{(1)}\left(\frac{1}{2}\right)=-0.1235,$$

$$W_4^{(1)}\left(\frac{1}{5}\right)=-0.1183,\quad W_4^{(1)}\left(\frac{1}{4}\right)=-0.1165,\quad W_4^{(1)}\left(\frac{1}{3}\right)=-0.0762,\quad W_4^{(1)}\left(\frac{1}{2}\right)=0.0576,$$

$$W_5^{(1)}\left(\frac{1}{5}\right)=-0.0360,\quad W_5^{(1)}\left(\frac{1}{4}\right)=0.0591,\quad W_5^{(1)}\left(\frac{1}{3}\right)=0.2256,\quad W_5^{(1)}\left(\frac{1}{2}\right)=0.4033,$$

$$W_6^{(1)}\left(\frac{1}{5}\right)=0.3339,\quad W_6^{(1)}\left(\frac{1}{4}\right)=0.5127,\quad W_6^{(1)}\left(\frac{1}{3}\right)=0.6028,\quad W_6^{(1)}\left(\frac{1}{2}\right)=0.1454,$$

$$W_0^{(2)}\left(\frac{1}{5}\right)=0.0193,\quad W_0^{(2)}\left(\frac{1}{4}\right)=0.0256,\quad W_0^{(2)}\left(\frac{1}{3}\right)=0.0352,\quad W_0^{(2)}\left(\frac{1}{2}\right)=0.0494,$$

$$W_1^{(2)}\left(\frac{1}{5}\right)=0.0257,\quad W_1^{(2)}\left(\frac{1}{4}\right)=0.0307,\quad W_1^{(2)}\left(\frac{1}{3}\right)=0.0352,\quad W_1^{(2)}\left(\frac{1}{2}\right)=0.0329,$$

$$W_2^{(2)}\left(\frac{1}{5}\right)=0.0236,\quad W_2^{(2)}\left(\frac{1}{4}\right)=0.0205,\quad W_2^{(2)}\left(\frac{1}{3}\right)=0.0088,\quad W_2^{(2)}\left(\frac{1}{2}\right)=-0.0219,$$

$$W_3^{(2)}\left(\frac{1}{5}\right)=-0.0043,\quad W_3^{(2)}\left(\frac{1}{4}\right)=-0.0246,\quad W_3^{(2)}\left(\frac{1}{3}\right)=-0.0571,\quad W_3^{(2)}\left(\frac{1}{2}\right)=-0.0878,$$

$$W_4^{(2)}\left(\frac{1}{5}\right)=-0.0754,\quad W_4^{(2)}\left(\frac{1}{4}\right)=-0.1049,\quad W_4^{(2)}\left(\frac{1}{3}\right)=-0.1132,\quad W_4^{(2)}\left(\frac{1}{2}\right)=-0.0146,$$

$$W_5^{(2)}\left(\frac{1}{5}\right)=-0.1481,\quad W_5^{(2)}\left(\frac{1}{4}\right)=-0.1042,\quad W_5^{(2)}\left(\frac{1}{3}\right)=0.0681,\quad W_5^{(2)}\left(\frac{1}{2}\right)=0.4048,$$

$$W_6^{(2)}\left(\frac{1}{5}\right)=0.0468,\quad W_6^{(2)}\left(\frac{1}{4}\right)=0.3658,\quad W_6^{(2)}\left(\frac{1}{3}\right)=0.8080,\quad W_6^{(2)}\left(\frac{1}{2}\right)=0.5999,$$

$$W_0^{(3)}\left(\frac{1}{5}\right)=0.0028,\quad W_0^{(3)}\left(\frac{1}{4}\right)=0.0041,\quad W_0^{(3)}\left(\frac{1}{3}\right)=0.0066,\quad W_0^{(3)}\left(\frac{1}{2}\right)=0.011,$$

$$W_1^{(3)}\left(\frac{1}{5}\right)=0.0054,\quad W_1^{(3)}\left(\frac{1}{4}\right)=0.0074,\quad W_1^{(3)}\left(\frac{1}{3}\right)=0.0099,\quad W_1^{(3)}\left(\frac{1}{2}\right)=0.011,$$

$$W_2^{(3)}\left(\frac{1}{5}\right)=0.0085,\quad W_2^{(3)}\left(\frac{1}{4}\right)=0.0093,\quad W_2^{(3)}\left(\frac{1}{3}\right)=0.0074,\quad W_2^{(3)}\left(\frac{1}{2}\right)=-0.0037,$$

$$W_3^{(3)}\left(\frac{1}{5}\right)=0.0065,\quad W_3^{(3)}(0.25)=0.0003,\quad W_3^{(3)}\left(\frac{1}{3}\right)=-0.0148,\quad W_3^{(3)}\left(\frac{1}{2}\right)=-0.0378,$$

$$W_4^{(3)}\left(\frac{1}{5}\right)=-0.0174,\quad W_4^{(3)}\left(\frac{1}{4}\right)=-0.0393,\quad W_4^{(3)}\left(\frac{1}{3}\right)=-0.0630,\quad W_4^{(3)}\left(\frac{1}{2}\right)=-0.0329,$$

$$W_5^{(3)}\left(\frac{1}{5}\right)=-0.0869,\quad W_5^{(3)}\left(\frac{1}{4}\right)=-0.1030,\quad W_5^{(3)}\left(\frac{1}{3}\right)=-0.0426,\quad W_5^{(3)}\left(\frac{1}{2}\right)=0.2061,$$

$$W_6^{(3)}\left(\frac{1}{5}\right)=-0.1376,\quad W_6^{(3)}\left(\frac{1}{4}\right)=0.0201,\quad W_6^{(3)}\left(\frac{1}{3}\right)=0.4222,\quad W_6^{(3)}\left(\frac{1}{2}\right)=0.6141.$$

For the computation of the functions $\mathscr{G}(\lambda, n, k)$, the formula given in (106) allows to write Algorithm 3 as follows (see, for details, [42]):

Algorithm 3 Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This algorithm will return $\mathscr{G}(\lambda, n, k)$ given by (106) with the help of HIGHER_W_APOSTOL_TYPE_NUM procedure given by the Algorithm 2 and the Möbius function denoted by Mobius_Func procedure (*cf.* [42])

```
procedure G_LYNDON_FUNC(λ, n:nonnegative integer, k: positive integer)
    Begin
    Local variableG ← 0
    for all positive divisors d of n do
        G ← G+Mobius_Func(n/d) *HIGHER_W_APOSTOL_TYPE_NUM(d, λ, k)
    end for
    return G
end procedure
```

In addition to the numerical values of the function $\mathscr{G}(\lambda, n, k)$ provided in [42], after implementing Algorithm 3, by inspiring from [42], we shall give further numerical values of the function $\mathscr{G}(\lambda, n, k)$ for the cases when $k \in \{1, 2, 3\}$, $\lambda \in \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{5}, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\}$ and $n \in \{2, 3, 5\}$ as follows:

$$\mathscr{G}\left(\frac{1}{5}, 2, 1\right) = -0.0347, \quad \mathscr{G}\left(\frac{1}{4}, 2, 1\right) = -0.0448, \quad \mathscr{G}\left(\frac{1}{3}, 2, 1\right) = -0.0586, \quad \mathscr{G}\left(\frac{1}{2}, 2, 1\right) = -0.0741,$$

$$\mathscr{G}\left(\frac{1}{5}, 3, 1\right) = -0.0514, \quad \mathscr{G}\left(\frac{1}{4}, 3, 1\right) = -0.0614, \quad \mathscr{G}\left(\frac{1}{3}, 3, 1\right) = -0.0703, \quad \mathscr{G}\left(\frac{1}{2}, 3, 1\right) = -0.0658,$$

$$\mathscr{G}\left(\frac{1}{5}, 5, 1\right) = -0.0257, \quad \mathscr{G}\left(\frac{1}{4}, 5, 1\right) = -0.0074, \quad \mathscr{G}\left(\frac{1}{3}, 5, 1\right) = 0.0264, \quad \mathscr{G}\left(\frac{1}{2}, 5, 1\right) = 0.0658,$$

$$\mathscr{G}\left(\frac{1}{5}, 2, 2\right) = -0.0011, \quad \mathscr{G}\left(\frac{1}{4}, 2, 2\right) = -0.0051, \quad \mathscr{G}\left(\frac{1}{3}, 2, 2\right) = -0.0132, \quad \mathscr{G}\left(\frac{1}{2}, 2, 2\right) = -0.0274,$$

$$\mathscr{G}\left(\frac{1}{5}, 3, 2\right) = -0.01, \quad \mathscr{G}\left(\frac{1}{4}, 3, 2\right) = -0.0184, \quad \mathscr{G}\left(\frac{1}{3}, 3, 2\right) = -0.0308, \quad \mathscr{G}\left(\frac{1}{2}, 3, 2\right) = -0.0402,$$

$$\mathscr{G}\left(\frac{1}{5}, 5, 2\right) = -0.0348, \quad \mathscr{G}\left(\frac{1}{4}, 5, 2\right) = -0.0270, \quad \mathscr{G}\left(\frac{1}{3}, 5, 2\right) = 0.0066, \quad \mathscr{G}\left(\frac{1}{2}, 5, 2\right) = 0.0744,$$

$$\mathscr{G}\left(\frac{1}{5}, 2, 3\right) = 0.002, \quad \mathscr{G}\left(\frac{1}{4}, 2, 3\right) = 0.001, \quad \mathscr{G}\left(\frac{1}{3}, 2, 3\right) = -0.0012, \quad \mathscr{G}\left(\frac{1}{2}, 2, 3\right) = -0.0073,$$

$$\mathscr{G}\left(\frac{1}{5}, 3, 3\right) = 0.0004, \quad \mathscr{G}\left(\frac{1}{4}, 3, 3\right) = -0.0024, \quad \mathscr{G}\left(\frac{1}{3}, 3, 3\right) = -0.0082, \quad \mathscr{G}\left(\frac{1}{2}, 3, 3\right) = -0.0163,$$

$$\mathscr{G}\left(\frac{1}{5}, 5, 3\right) = -0.0185, \quad \mathscr{G}\left(\frac{1}{4}, 5, 3\right) = -0.0221, \quad \mathscr{G}\left(\frac{1}{3}, 5, 3\right) = -0.0105, \quad \mathscr{G}\left(\frac{1}{2}, 5, 3\right) = 0.039.$$

$$\mathscr{G}\left(-\frac{1}{5}, 2, 1\right) = -0.2148, \quad \mathscr{G}\left(-\frac{1}{5}, 2, 2\right) = 0.354, \quad \mathscr{G}\left(-\frac{1}{5}, 2, 3\right) = -0.269,$$

$$\mathscr{G}\left(-\frac{1}{5},3,1\right)=-0.586,\quad \mathscr{G}\left(-\frac{1}{5},3,2\right)=1.2085,\quad \mathscr{G}\left(-\frac{1}{5},3,3\right)=-1.1673,$$

$$\mathscr{G}\left(-\frac{1}{5},5,1\right)=-5.0537,\quad \mathscr{G}\left(-\frac{1}{5},5,2\right)=17.6331,\quad \mathscr{G}\left(-\frac{1}{5},5,3\right)=-27.7319,$$

$$\mathscr{G}\left(-\frac{1}{4},2,1\right)=-0.4445,\quad \mathscr{G}\left(-\frac{1}{4},2,2\right)=0.9438,\quad \mathscr{G}\left(-\frac{1}{4},2,3\right)=-0.995,$$

$$\mathscr{G}\left(-\frac{1}{4},3,1\right)=-1.3169,\quad \mathscr{G}\left(-\frac{1}{4},3,2\right)=3.5848,\quad \mathscr{G}\left(-\frac{1}{4},3,3\right)=-4.8123,$$

$$\mathscr{G}\left(-\frac{1}{4},5,1\right)=-14.4856,\quad \mathscr{G}\left(-\frac{1}{4},5,2\right)=68.5018,\quad \mathscr{G}\left(-\frac{1}{4},5,3\right)=-149.0504,$$

$$\mathscr{G}\left(-\frac{1}{3},2,1\right)=-1.3125,\quad \mathscr{G}\left(-\frac{1}{3},2,2\right)=4.2188,\quad \mathscr{G}\left(-\frac{1}{3},2,3\right)=-7.2773,$$

$$\mathscr{G}\left(-\frac{1}{3},3,1\right)=-4.5,\quad \mathscr{G}\left(-\frac{1}{3},3,2\right)=19.125,\quad \mathscr{G}\left(-\frac{1}{3},3,3\right)=-42.1875,$$

$$\mathscr{G}\left(-\frac{1}{3},5,1\right)=-74.25,\quad \mathscr{G}\left(-\frac{1}{3},5,2\right)=563.625,\quad \mathscr{G}\left(-\frac{1}{3},5,3\right)=-2007.2813,$$

$$\mathscr{G}\left(-\frac{1}{2},2,1\right)=-10,\quad \mathscr{G}\left(-\frac{1}{2},2,2\right)=76,\quad \mathscr{G}\left(-\frac{1}{2},2,3\right)=-336,$$

$$\mathscr{G}\left(-\frac{1}{2},3,1\right)=-48,\quad \mathscr{G}\left(-\frac{1}{2},3,2\right)=504,\quad \mathscr{G}\left(-\frac{1}{2},3,3\right)=-2880,$$

$$\mathscr{G}\left(-\frac{1}{2},5,1\right)=-1872,\quad \mathscr{G}\left(-\frac{1}{2},5,2\right)=36072,\quad \mathscr{G}\left(-\frac{1}{2},5,3\right)=-331776.$$

10 Illustrations and Observations on Approximations of the Functions $\mathscr{G}(\lambda, p, k)$ by the Rational Functions $W_n^{(k)}(\lambda)$

In [42], Kucukoglu and Simsek simulated the numbers $W_n(\lambda)$ and $W_n^{(k)}(\lambda)$ and the functions $\mathscr{G}(\lambda, n, k)$ by their numerical evaluations and plots drawn by implementation of the computational algorithms of Algorithms 1, 2, and 3 mentioned in previous section. Furthermore, some illustrations and observations on approximations of the functions $\mathscr{G}(\lambda, p, k)$ by the rational functions $W_n^{(k)}(\lambda)$. Notice that this approach can provide an idea for reduction of the algorithmic complexity of one of the computational algorithms mentioned above.

Note that the numbers $W_n^{(k)}(\lambda)$ are rational functions of real variable λ. Thus, by inspiring from [42], we shall give some further plots of the rational functions $W_n^{(k)}(\lambda)$ in addition to Fig. 5.1 given by Kucukoglu and Simsek [42]. For this purpose, after implementing Algorithm 1 for the cases when $k = 1, 2, 3$, $n = 0, 1, \ldots, 6$ and $\lambda \in [0, 15]$, Figs. 2, 3, and 4 are obtained as follows:

The following figures illustrates the effects of k on the shape of the curve of the rational functions $W_n^{(k)}(\lambda)$.

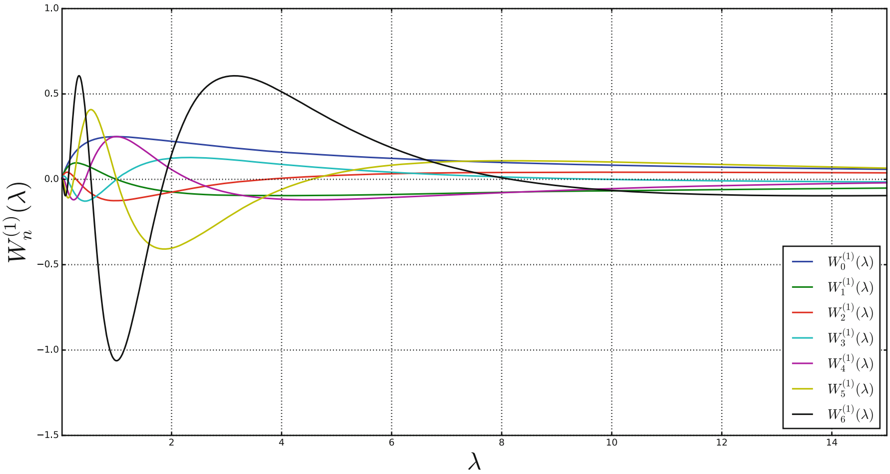

Fig. 2 Plots of the rational functions $W_n^{(k)}(\lambda)$ for the cases $k = 1$, $n \in \{0, 1, \ldots, 6\}$ and $\lambda \in [0, 15]$

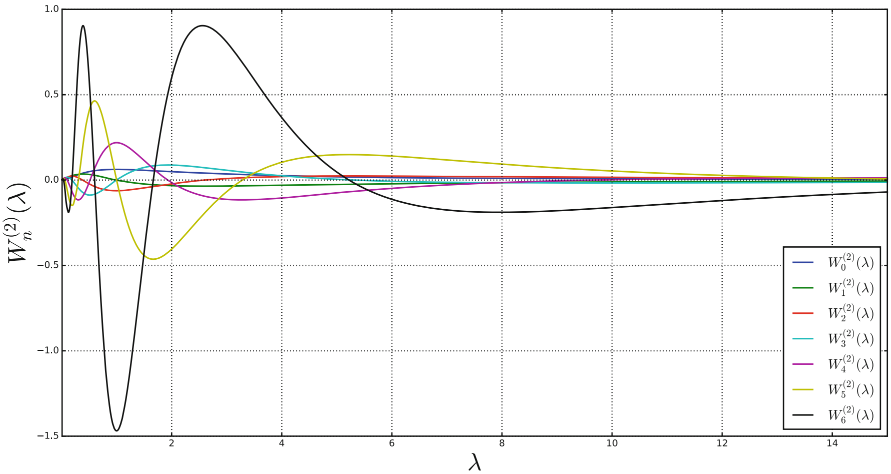

Fig. 3 Plots of the rational functions $W_n^{(k)}(\lambda)$ for the cases $k = 2$, $n \in \{0, 1, \ldots, 6\}$ and $\lambda \in [0, 15]$

Next, by inspiring from [42], we shall give some further plots of the functions $\mathcal{G}(\lambda, p, 1)$ in addition to Fig. 5.2 given by Kucukoglu and Simsek [42]. For this purpose, after implementing Algorithm 3 for the cases when $k = 1, 2, 3$, $p = 2, 3, 5, 7, 11, 13$ and $\lambda \in [0, 15]$, Figs. 5, 6, and 7 are obtained as follows:

The following figures illustrates the effects of k on the shape of the curve of the functions $\mathcal{G}(\lambda, p, 1)$.

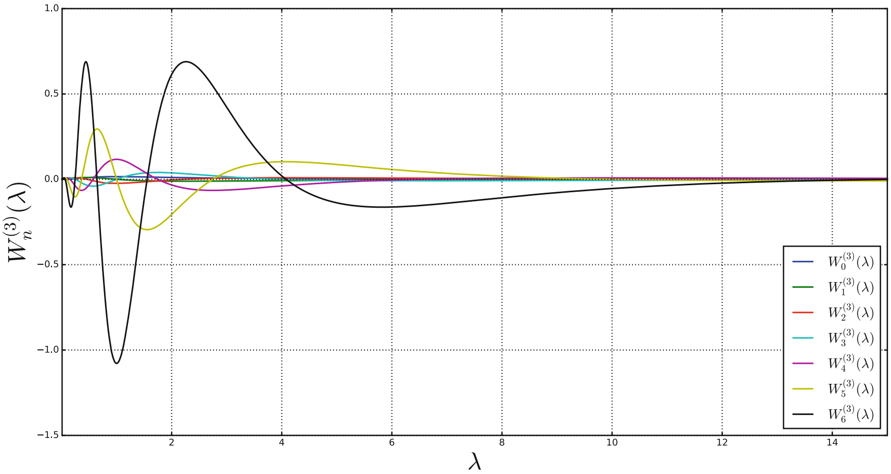

Fig. 4 Plots of the rational functions $W_n^{(k)}(\lambda)$ for the cases $k = 3$, $n \in \{0, 1, \ldots, 6\}$ and $\lambda \in [0, 15]$

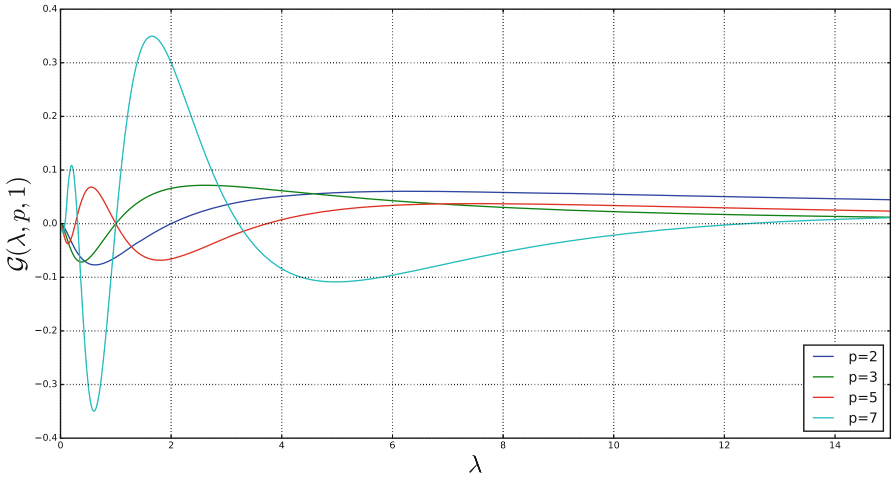

Fig. 5 Plots of the functions $\mathscr{G}(\lambda, p, k)$ for the cases $k = 1$, $p \in \{2, 3, 5, 7\}$ and $\lambda \in [0, 15]$

Next, by inspiring from [42], we shall give some illustrations and observations on approximations for the functions $\mathscr{G}(\lambda, p, k)$ by the rational functions $W_n^{(k)}(\lambda)$ with small error in order to present an approach for the reduction of the algorithmic complexity of Algorithm 3.

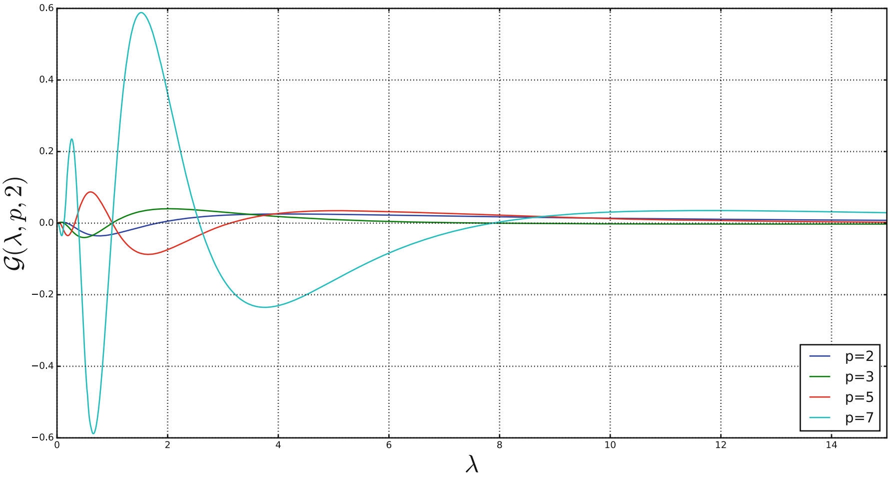

Fig. 6 Plots of the functions $\mathcal{G}(\lambda, p, k)$ for the cases $k = 2$, $p \in \{2, 3, 5, 7\}$ and $\lambda \in [0, 15]$

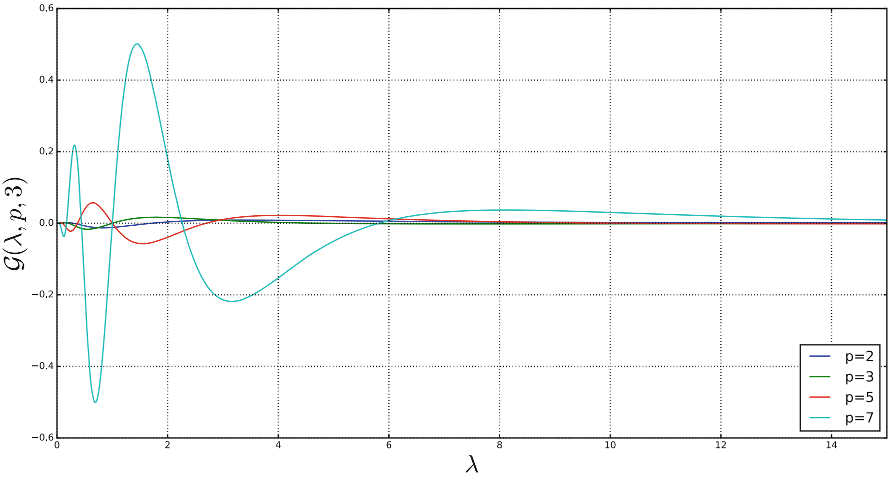

Fig. 7 Plots of the functions $\mathcal{G}(\lambda, p, k)$ for the cases $k = 3$, $p \in \{2, 3, 5, 7\}$ and $\lambda \in [0, 15]$

In order to give approximations for the functions $\mathcal{G}(\lambda, p, k)$ by the rational functions $\mathcal{G}_{Approx}(\lambda, p, k)$, we need to the following well-known simplest theorem which was given by Weierstrass in 1885:

Theorem 39 (*cf.* [23] (Weierstrass 1885)) *Each continuous real functions f on $[a, b]$ is uniformly approximable by algebraic polynomials: for each $\epsilon > 0$ there is some algebraic polynomials $Q_n(x)$ of degree $\leq n$ with*

$$|f(x) - Q_n(x)| \leq \epsilon,$$

$a \leq x \leq b$.

It is well-known that the functions $W_n^{(k)}(\lambda)$ are rational functions with variable λ. Here, assuming that $|\lambda| < 1$ and p be a prime number. Setting

$$\mathscr{G}_{Approx}(\lambda, p, k) = \frac{W_p^{(k)}(\lambda)}{p}.$$

After some elementary calculations with aid of the Weierstrass approximation theorem, we easily arrive at the following well-known result:

$$\left|\mathscr{G}(\lambda, p, k) - \mathscr{G}_{Approx}(\lambda, p, k)\right| \leq \varepsilon_p(k), \tag{112}$$

such that

$$\varepsilon_p(k) = k/p,$$

see, for details, [42].

It is time to give some plots using the above approximations values for $\varepsilon_p(k)$. Inspiring from [42], we shall give some further plots in order to illustrate approximations for the functions $\mathscr{G}(\lambda, p, k)$ by the rational functions $\mathscr{G}_{Approx}(\lambda, p, k)$, with an error less than $\varepsilon_p(k) = k/p$.

Due to Eq. (112), for sufficiently large p, that is, when $p \to \infty$, then $\varepsilon_p(k) \to 0$. Consequently, for sufficiently large p, the curves of the functions $\mathscr{G}(\lambda, p, k)$ and $\mathscr{G}_{Approx}(\lambda, p, k)$ tend to overlap. This indicates that by using the rational functions $\mathscr{G}_{Approx}(\lambda, p, k)$ instead of $\mathscr{G}(\lambda, p, k)$, Algorithm 3 can be implemented more efficiently for sufficiently large prime numbers.

For these special cases for $k \in \{1, 2, 3\}$, $p \in \{2, 3, 5, 7, 11\}$, and $\lambda \in [0, 15]$, the following figures (Figs. 8, 9, and 10) are plotted. In these related figures, the red curves are corresponding to the rational functions $\mathscr{G}_{Approx}(\lambda, p, k)$ while the black ones are corresponding to the functions $\mathscr{G}(\lambda, p, k)$.

Remark 7 The above evaluation on approximations for the functions arising from the special power series is given with the inspiration of [42, 48], and it is given by using similar techniques as ones in [42, 48]. In the special case when $k = 1$, the approach applied in this section is reduced to that of [42]. For evaluation on approximations performed with the Apostol-type numbers, the interested reader may glance at the aforementioned studies.

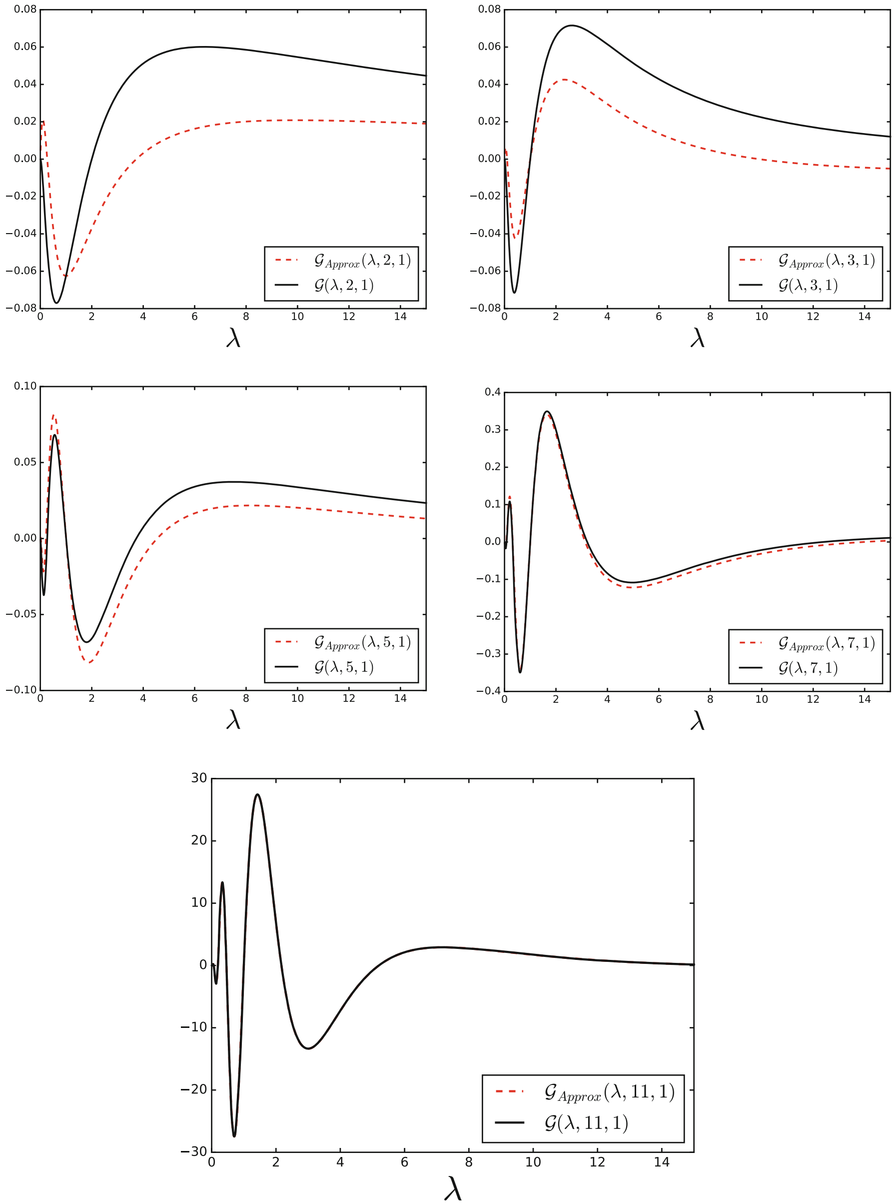

Fig. 8 Approximation of the function $\mathscr{G}(\lambda, p, k)$ by the rational function $\mathscr{G}_{Approx}(\lambda, p, k)$ for the cases: (**a**) $p = 2$, $\varepsilon_2(1) = 1/2$; (**b**) $p = 3$, $\varepsilon_3(1) = 1/3$; (**c**) $p = 5$, $\varepsilon_5(1) = 1/5$; (**d**) $p = 7$, $\varepsilon_7(1) = 1/7$; (**e**) $p = 11$, $\varepsilon_{11}(1) = 1/11$ with $k = 1$ and $\lambda \in [0, 15]$

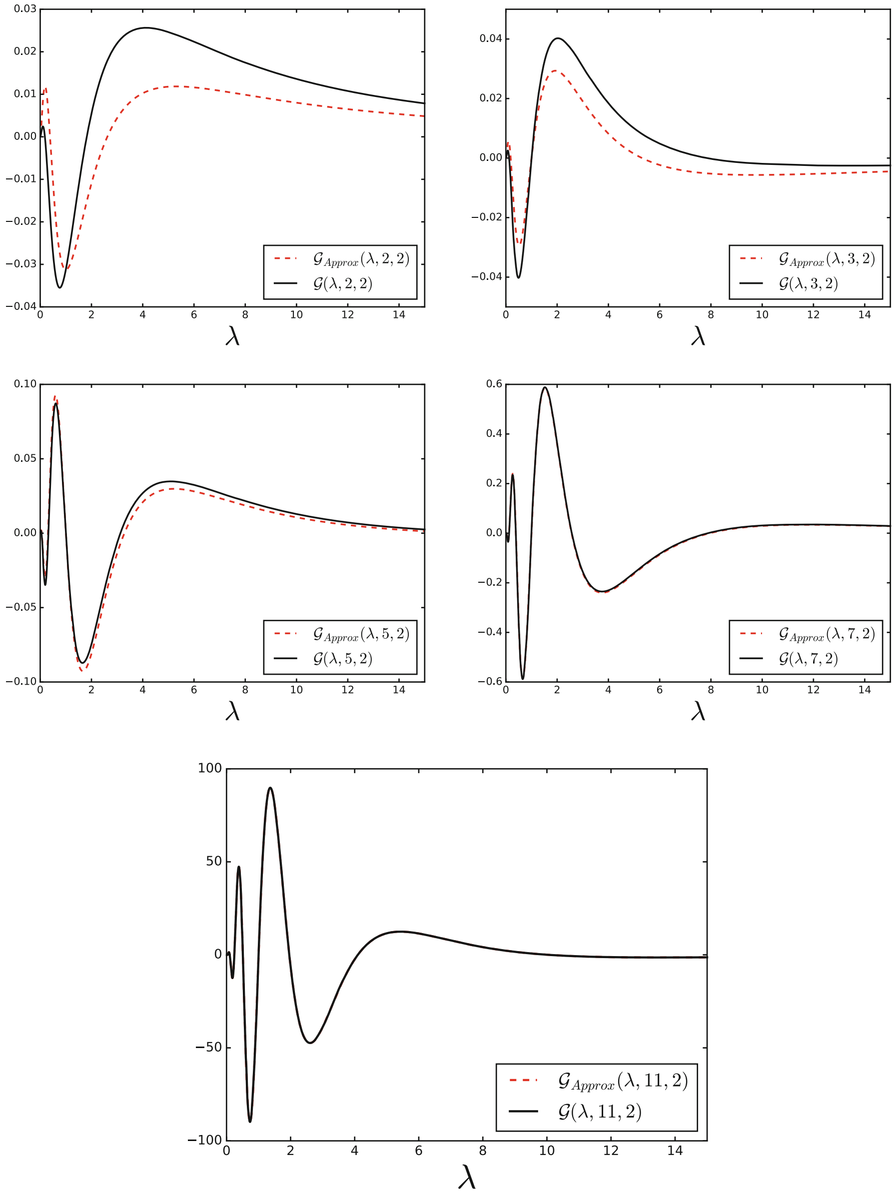

Fig. 9 Approximation of the function $\mathscr{G}(\lambda, p, k)$ by the rational function $\mathscr{G}_{Approx}(\lambda, p, k)$ for the cases: (**a**) $p = 2$, $\varepsilon_2(2) = 1$; (**b**) $p = 3$, $\varepsilon_3(2) = 2/3$; (**c**) $p = 5$, $\varepsilon_5(2) = 2/5$; (d) $p = 7$, $\varepsilon_7(2) = 2/7$; (e) $p = 11$, $\varepsilon_{11}(2) = 2/11$ with $k = 2$ and $\lambda \in [0, 15]$

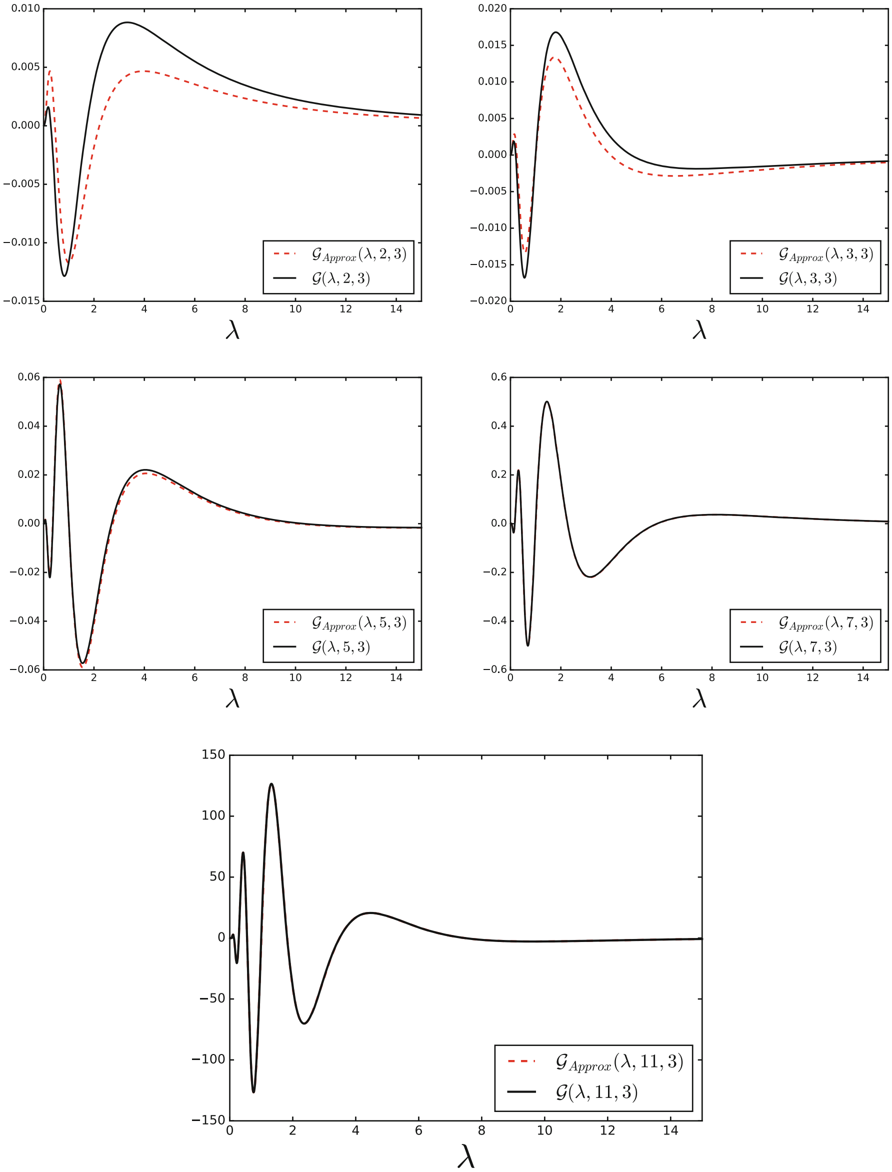

Fig. 10 Approximation of the function $\mathscr{G}(\lambda, p, k)$ by the rational function $\mathscr{G}_{Approx}(\lambda, p, k)$ for the cases: (**a**) $p = 2$, $\varepsilon_2(3) = 3/2$; (**b**) $p = 3$, $\varepsilon_3(3) = 1$; (**c**) $p = 5$, $\varepsilon_5(3) = 3/5$; (**d**) $p = 7$, $\varepsilon_7(3) = 3/7$; (**e**) $p = 11$, $\varepsilon_{11}(3) = 3/11$ with $k = 3$ and $\lambda \in [0, 15]$

11 Further Remarks and Observation on the Bernstein Polynomials and Their Approximations

The Bernstein polynomials, introduced about 115 years ago as a means to constructively prove the ability of polynomials to approximate any continuous function, to any desired accuracy, over a prescribed interval. Their slow convergence rate, and the lack of digital computers to efficiently construct them, caused the Bernstein polynomials to lie dormant in the theory rather than practice of approximation for the better part of a century. On the other hand, the Bernstein polynomials found its true vocation not only in approximation of functions by polynomials, but also in exploiting computers to interactively design (vector-valued) polynomial functions, that is parametric curves and surfaces.

We now give the following classes of functions on $f \in C[0,1]$ and approximable by polynomials, which are taken from the work of Devore and Lorentz [23] involving Problems 5.4–5.6.

If the function f is continuous on $[0,\infty]$ and has limit zero for $x \to +\infty$, then

$$\lim_{x\to\infty} S_y(x) = f(x)$$

uniformly for $0 \leq x < \infty$ where $S_y(x)$ is defined by

$$S_y(x) = \sum_{j=0}^{\infty} \frac{(xy)^j}{j!} f\left(\frac{j}{y}\right) e^{-yx}, \tag{113}$$

where $y > 0$.

By combining (28) with (113), we obtain

$$S_y(\lambda) = e^{-y} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} B_j^n(\lambda) f\left(\frac{j}{y}\right) \frac{y^n}{n!}.$$

Since $B_j^n(x) = 0$ for $j > n$, then after some elementary calculations with the aid of the Cauchy product for series, we get

$$S_y(\lambda) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \sum_{j=0}^{l} B_j^l(\lambda) f\left(\frac{j}{y}\right) \frac{y^n}{n!}.$$

Observe that

$$\sum_{j=0}^{l} B_j^l(\lambda) f\left(\frac{j}{y}\right)$$

associated with the Bernstein polynomials of f for $y \in \mathbb{N}$. These well-known polynomials are defined as follows:

$$B_l(f, \lambda) = \sum_{j=0}^{l} B_j^l(\lambda) f\left(\frac{j}{l}\right).$$

If $f > 0$, $B_l(f, \lambda)$ is a bounded operator of norm 1 for $\lambda \in [0, 1]$. It is easy to see that for $f \in C[0, 1]$, $B_l(f, \lambda) \to f, n \to \infty$ (*cf.* [23]).

On the other hand for $f \in C[0, 1]$ with $f(0) = f(1) = 0$, then a family of polynomials with integer coefficients

$$P_{l-1}(x) = \sum_{j=0}^{l-1} B_j^l(\lambda) f\left(\frac{j}{l}\right)$$

convergence uniformly to the related function $f(x)$. The function $f \in C[0, 1]$ is approximal by polynomials with integer coefficients if and only if $f(0), f(1) \in \mathbb{Z}$ (*cf.* [23]).

Therefore, there are various applications of the operator $B_l(f, \lambda)$ and $S_y(\lambda)$ in the theory of approximation of operators, in theory of special polynomials and in the other areas.

Acknowledgments This chapter is dedicated to the soul of my beloved mother.

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Spectra of Signed Graphs

Irene Triantafillou

Abstract A signed graph is a graph that has a sign assigned to each of its edges. Signed graphs were introduced by Harary in 1953 in relation to certain problems in social psychology, and the matroids of signed graphs were first introduced by Zaslavsky in 1982. The investigation of the spectra of signed graphs has gained much attention in recent years by various authors. In this chapter, we focus on some of the most important results related to the eigenvalues of the adjacency and the Laplacian matrices of signed graphs.

1 Introduction and Preliminaries

Let $G = (V(G), E(G))$ be a simple graph with nonempty vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix*, $A(G)$, of a graph G on n vertices is defined as the $n \times n$ symmetric matrix whose entries a_{ij} are $a_{ij} = 1$ if vertex v_i is adjacent to vertex v_j, and $a_{ij} = 0$ otherwise. The *degree* of a vertex v, $deg(v)$, is the number of edges incident to v, and the *degree matrix* of a graph of order n is the diagonal matrix $D(G) = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$. The *Laplacian matrix*, $L(G)$, of a graph G is the matrix defined as $L(G) = D(G) - A(G)$. Both the adjacency and the Laplacian matrices are among the most studied matrices in spectral graph theory as they provide useful information about the graph (e.g. the number of the graph's edges, the number of its connected components, etc.) and have various applications (see e.g. [13, 36]).

A *signed graph*, $\Gamma = (G, \sigma)$, is a pair of an unsigned graph $G = (V(G), E(G))$, called the *underlying graph* and a mapping $\sigma : E(G) \to \{+1, -1\}$ called the *sign function* or *signature*. The edges to which +1 (respectively, -1) is assigned are called *positive edges* (respectively, *negative edges*) and a signed graph is said to be *all-positive* (respectively, *all-negative*) if all of its edges are positive (respectively,

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negative). The adjacency matrix of a signed graph, $A(\Gamma) = (a^{\sigma}_{ij})$, is the matrix with entries $a^{\sigma}_{ij} = \sigma(ij)a_{ij}$, where $A(G) = (a_{ij})$. In an analogous manner, the Laplacian matrix of a signed graph, $\Gamma = (G, \sigma)$, is defined as $L(\Gamma) = D(G) - A(\Gamma)$. The spectrum of a signed graph is the set of the eigenvalues of its adjacency matrix together with their multiplicities. The set of eigenvalues (with their multiplicities) of the Laplacian matrix is called the Laplacian spectrum. In this chapter, we refer to the eigenvalues of the adjacency matrix as the eigenvalues of the signed graph and we refer to the eigenvalues of the Laplacian matrix as the Laplacian eigenvalues.

Signed graphs were first introduced by Harary in [25] to address several problems in social psychology. Harary proposed the concept of the signed graph as a way to describe the relation between people (vertices) being friendly (positive edges) or unfriendly or hostile (negative edges). Since then, there has been an extensive research on the applications of signed graphs [29, 30, 40, 45]. Zaslavsky introduced the matroids of signed graphs in [44], in which a matrix-tree theorem for signed graphs was also given (see also [11]). The spectrum of signed graphs is an area that has gained much attention recently by many authors [3, 7, 8, 20, 26, 27, 37, 43].

A path on n vertices in a signed graph is called *positive* (respectively, *negative*) if the product of its edge signs ($\prod_{i=1}^{n-1} \sigma(e_i)$) is positive (respectively, negative). Equivalently, if the number of negative edges in a path is even (odd), then the path is positive (negative). As the cycle is a closed path, the sign of a cycle in a signed graph is also defined as the product of its edge signs. We call a cycle *balanced* or *positive* if the number of its negative edges is even, otherwise we call the cycle *unbalanced* or *negative*. A signed graph, Γ, is called *balanced* if all of its cycles are positive. In [25], Harary gave a necessary and sufficient condition for a signed graph Γ to be balanced: there exists a bipartition of the vertex set V into subsets X and Y (one of which may be empty) such that all edges between the same subsets are positive and all edges that have one end point in X and one in Y are negative. We will focus on some of the most well-known results on the balance of signed graphs in the following sections.

Another important concept in the study of signed graphs is the concept of switching. Let $\Gamma = (G, \sigma)$ be a signed graph and $\theta : V \to \{+1, -1\}$ a sign function. *Switching* of the signed graph, Γ, by θ means changing the signature σ to $\sigma_\theta(uv) = \theta(u)\sigma(uv)\theta(v)$ ($uv \in E(\Gamma)$), while leaving the underlying graph, G, unchanged. The new signed graph formed is denoted by $\Gamma^\theta = (G, \sigma^\theta)$, and it is called the *switching equivalent* of Γ. We write $\Gamma^\theta \sim \Gamma$. The two signed graphs Γ^θ and Γ share many invariants, such as the set of positive cycles, and switching of a signed graph also preserves its spectrum [26, 46]. We say that the $n \times n$ matrices, M_1 and M_2, are *signature similar* if there exists a diagonal matrix $S = diag(s_1, s_2, \ldots, s_n)$, where $s_i = \pm 1$, such that $M_2 = SM_1S^{-1}$. Note that $S^{-1} = S$ and that two signature similar matrices have the same eigenvalues.

In this chapter, we denote the *path*, *cycle*, and *complete graph* with n vertices by P_n, C_n, and K_n, respectively.

2 Adjacency Matrix of Signed Graphs

In this section, we focus on some of the most important results regarding the adjacency matrix, $A(\Gamma)$, of a signed graph. As already mentioned, switching does not change the spectrum of the adjacency matrix [46], a direct result from the following proposition.

Proposition 1 ([46]) *Let Γ_1 and Γ_2 be two signed graphs of the same order. Γ_1 and Γ_2 are switching equivalent if and only if $A(\Gamma_2) = S^{-1}A(\Gamma_1)S$ for some diagonal matrix S with entries ± 1 in its diagonal.*

A balanced graph can be switched to an all-positive graph, that is, $\Gamma \sim (G, +)$, [44]. The following results were obtained for balanced graphs in regard to their adjacency matrices.

Proposition 2 ([44]) *Let $\Gamma = (G, \sigma)$ be a signed graph. Γ is balanced if and only if there exists a diagonal matrix, S, with entries ± 1 such that $SA(\Gamma)S = A(G)$.*

Proposition 3 ([1]) *Let $\Gamma = (G, \sigma)$ be a signed graph. Then, Γ is balanced if and only if Γ and G have the same set of eigenvalues (counting multiplicities).*

Another well-known result about the adjacency matrix of a signed graph was proved in [46].

Theorem 1 ([46]) *The (i, j)-entry of $A^k(\Gamma)$ is $A^k(\Gamma) = w_{ij}^+(k) - w_{ij}^-(k)$, where $w_{ij}^+(k)$ and $w_{ij}^-(k)$ denote the number of positive and negative walks (walks with even and odd negative edges), respectively.*

A corollary of the above theorem regarding the trace of the adjacency matrices A2(Γ) and A3(Γ) was given in [7, 46].

Corollary 1 ([7, 46]) *If Γ is a signed graph of order n, then*

(i) $trace(A^2(\Gamma)) = trace(diag(deg(v_1), deg(v_2), \ldots, deg(v_n)))$;
(ii) $trace(A^3(\Gamma)) = 6(t^+ - t^-)$, *where t^+ and t^- denote the number of positive and negative triangles, respectively.*

The Sachs theorem [12] for calculating the coefficients of the characteristic polynomial of the adjacency matrix of a graph is written for signed graphs as follows:

Theorem 2 ([8]) *Let Γ be a graph of order n and with characteristic polynomial $\phi_G(x) = |xI - A| = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$. Then,*

$$a_i = \sum_{U \in \mathscr{U}_i} (-1)^{p(U)} 2^{|c(U)|} \sigma(U),$$

for $j = 1, 2, \ldots, n$, where $\mathscr{U}_i$ consists of K_2 edges and cycles (also known as basic figures) of order i, $p(U)$ denotes the number of components of U, $c(U)$ is the set of all cycles of U, and $\sigma(U) = \prod_{C \in c(U)} \sigma(C)$.

Schwenk's formulas after the deletion of a vertex and an edge, respectively, can be written for signed graphs.

Theorem 3 ([7, 23]) *Let Γ be a signed graph. For any vertex $v \in G$ and edge $e \in E$, it holds*

$$\phi(\Gamma, x) = x\phi(\Gamma - v, x) - \sum_{u \sim v} \phi(\Gamma - u - v, x) - 2 \sum_{C \in \mathcal{C}_v} \sigma(C)\phi(\Gamma - C, x),$$

$$\phi(\Gamma, x) = \phi(\Gamma - e, x) - \phi(\Gamma - u - v, x) - 2 \sum_{C \in \mathcal{C}_e} \sigma(C)\phi(\Gamma - C, x),$$

where $\mathcal{C}_a$ denotes the set of cycles passing through a.

Interlacing also holds for signed graphs.

Theorem 4 *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices, and let $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \ldots \geq \lambda_n(\Gamma)$ be the eigenvalues of its adjacency matrix in non-increasing order. Then,*

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma - v) \geq \lambda_2(\Gamma) \geq \lambda_2(\Gamma - v) \geq \ldots \geq \lambda_{n-1}(\Gamma - v) \geq \lambda_n(\Gamma),$$

where $\Gamma - v$ is the induced subgraph of Γ after vertex v is deleted.

3 Laplacian Spectra of Signed Graphs

The Laplacian matrix $L(\Gamma) = D(G) - A(\Gamma)$ is a real symmetric matrix that is positive semi-definite ($L(\Gamma) = B(\Gamma)B^T(\Gamma)$, where $B(\Gamma)$ is the incidence matrix of the signed graph). It is easy to see that if the signed graph is all-positive, $\Gamma = (G, +)$, then its Laplacian matrix $L(\Gamma)$ coincides with $L(G)$, and if it is all-negative, $\Gamma = (G, -)$, then it coincides with matrix $Q(G) = D(G) + A(G)$, also known as the signless Laplacian matrix. In this section, we focus on some of the most important results regarding the eigenvalues of the Laplacian matrix of signed graphs.

The matrix-tree theorem for signed graph was generalized by Chaiken [11] and Zaslavsky [44], respectively. In [8], Belardo provided a formula for the coefficients of the Laplacian polynomial of signed graphs based on signed TU-subgraphs. A *signed TU-subgraph* of a signed graph Γ is a signed subgraph whose components are trees or unicyclic graphs that are unbalanced (the unique cycle has sign -1). If H is a signed TU-subgraph, then $H = T_1 \cup T_2 \cup \ldots \cup T_r \cup U_1 \cup U_2 \cup \ldots \cup U_s$, where T_i's and U_j's denote trees and unbalanced unicyclic graphs, respectively. The weight of the signed TU-subgraph H is defined as $w(H) = 4^s \prod_{i=1}^{r} |T_i|$.

Theorem 5 ([8]) *Let Γ be a signed graph. Let $\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1}x + b_n$ be the Laplacian characteristic polynomial of Γ. Then,*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} w(H)$$

for $i = 1, 2, \ldots, n$, where $\mathcal{H}_i$ denotes the set of signed TU-subgraphs of Γ containing i edges.

The following is a well-known result regarding the balance of a signed graph.

Proposition 4 ([27]) *Let $\Gamma = (G, \sigma)$ be a connected signed graph and $L(\Gamma)$ its Laplacian matrix. Then, Γ is balanced if and only if $det(L(\Gamma)) = 0$.*

For switching equivalent graphs, the following proposition holds.

Proposition 5 ([27]) *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be signed graph having the same underlying graph. Then, $\Gamma_1 \sim \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.*

For signed graphs, edge interlacing for the Laplacian spectra is written as follows.

Theorem 6 ([7, 27]) *Let $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \ldots \geq \mu_n(\Gamma)$ be the Laplacian eigenvalues of the signed graph $\Gamma = (G, \sigma)$ in non-increasing order and $\Gamma - e$ the signed graph obtained by the deletion of edge e from Γ. Then,*

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \ldots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e).$$

3.1 Largest Laplacian Eigenvalue

Let $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \ldots \geq \mu_n(\Gamma)$ be the Laplacian eigenvalues of the signed graph $\Gamma = (G, \sigma)$ in non-increasing order, where $\mu_1(\Gamma)$ denotes the largest Laplacian eigenvalue.

Proposition 6 ([27]) *Let $\Gamma = (G, \sigma)$ be a connected signed graph on n vertices. Then, $\mu_1(\Gamma) \leq \mu_1(-\Gamma)$, where $-\Gamma$ is an all-negative graph. Equality holds if and only if $(G, \sigma) \sim (G, -)$.*

In the last 50 years, there has been an extensive research on the Laplacian matrix of a graph and several important results relating the eigenvalues of the Laplacian matrix of an unsigned graph and various graph parameters have been found [5, 18, 31, 35, 36]. In [27], some of these results regarding the largest Laplacian eigenvalue have been generalized for signed graphs.

Theorem 7 ([27]) *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices. Then, $\mu_1(\Gamma) \leq 2(n-1)$. Equality holds if and only if Γ is switching equivalent to the complete graph K_n with all negative edges.*

Theorem 8 ([27]) *Let $\Gamma = (G, \sigma)$ be a connected signed graph. Then,*

(i) $\mu_1(\Gamma) \leq max\{deg(u) + deg(v) : uv \in E\}$.
(ii) $\mu_1(\Gamma) \leq max\{deg(u) + m(u) : u \in V\}$.
(iii) $\mu_1(\Gamma) \leq max\{\frac{(deg(u)(deg(u)+m(u))+deg(v)(deg(v)+m(v))}{deg(u)+deg(v)} : uv \in E\}$,

where $m(v)$ is the 2-degree of vertex v, that is, $m(v) = \frac{1}{deg(v)} \sum_{uv \in E} deg(u)$.
Equality holds if and only if $(G, \sigma) \sim (G, -)$ and G is regular bipartite or semiregular bipartite.

In the same paper, a lower bound for the largest Laplacian eigenvalue was also provided.

Theorem 9 ([27]) *Let $\Gamma = (G, \sigma)$ be a signed graph. Then, $\mu_1(\Gamma) \geq max\{deg(v) + 1, v \in V(G)\}$.*

We close this subsection, with a theorem that identifies certain values of the largest Laplacian eigenvalue that do not exceed 4 with the structure of the signed graph.

Theorem 10 ([7]) *Let $\Gamma = (G, \sigma)$ be a connected signed graph. Then, the following hold:*

(i) $\mu_1(\Gamma) = 0$ *if and only if* $\Gamma = K_1$.
(ii) $\mu_1(\Gamma) = 2$ *if and only if* $\Gamma = K_2$.
(iii) $\mu_1(\Gamma) = 3$ *if and only if* $\Gamma \in \{P_3, (K_3, +)\}$.
(iv) $3 < \mu_1(\Gamma) < 4$ *if and only if* $\Gamma \in \{P_n (n \geq 4), (C_{2n}, \overline{\sigma}), (C_{2n+1}, +)(n \geq 2)\}$.
(v) $\mu_1(\Gamma) = 4$ *if and only if* $\Gamma \in \{(C_{2n}, +), (C_{2n+1}, \overline{\sigma})(n \geq 2), K_{1,3}, (K_{1,3}^+, +), (K_4^-, +), (K_4, +)\}$,

where $\overline{\sigma}$ denotes the unbalanced cycle and $K_{1,3}^+$ and K_4^- are obtained from $K_{1,3}$ and K_4 by adding and deleting an edge, respectively.

3.2 Least Laplacian Eigenvalue

The Laplacian matrix is positive semi-definite, and therefore its Laplacian eigenvalues are $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$. A very well-known result about the least Laplacian eigenvalue μ_n is the following lemma.

Lemma 1 *Let $\Gamma = (G, \sigma)$ be a signed graph. Then, Γ is balanced if and only if its least Laplacian eigenvalue $\mu_n = 0$.*

The least eigenvalue μ_n also measures how "far" the signed graph is from being balanced. Similarly to the algebraic connectivity [18] and the algebraic bipartiteness

[14] of a graph, the least Laplacian eigenvalue of a signed graph has been called the algebraic frustration. The frustration number and the frustration index are two parameters that have been shown to bound μ_n. The *frustration number*, $\nu(\Gamma)$, is the minimum number of vertices deleted from the graph such that the new formed graph is balanced, and the *frustration index*, $\epsilon(\Gamma)$, is the minimum number of edges deleted so that the new graph is balanced. An upper bound for the least Laplacian eigenvalue in relation to the frustration number of a signed graph was given in the following theorem.

Theorem 11 ([6]) *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices. Then,*

$$\mu_n(\Gamma) \leq \nu(\Gamma) \leq \epsilon(\Gamma).$$

The frustration index and the least Laplacian eigenvalue were also linked in [34].

Theorem 12 ([34]) *Let $\Gamma = (G, \sigma)$ be a signed graph on n vertices, and let $D(G)$ be the largest vertex degree of G. Then,*

$$\frac{n}{4}\mu_n(\Gamma) \leq \epsilon(\Gamma) \leq \frac{n}{\sqrt{2}}\sqrt{\mu_n(\Gamma)(2D(G) - \mu_n(\Gamma))}.$$

For unbalanced blocks (connected signed graphs without cut vertices), a lower bound was provided for the least Laplacian eigenvalue of a signed graph in regard to the length of its longest negative cycle.

Theorem 13 ([42]) *Let $\Gamma = (G, \sigma)$ be an unbalanced block on n vertices, and let l_u denote the length of the longest negative cycle of Γ. Then,*

$$\mu_n(\Gamma) > \frac{4}{l_u n}.$$

We close this subsection with an upper bound for μ_n in relation to the degrees of two adjacent vertices.

Proposition 7 ([26]) *Let u and v be adjacent vertices of a signed graph Γ on n vertices. Then, $\mu_n(\Gamma) \leq \frac{1}{2}(deg(u) + deg(v) - 2)$.*

4 Spectra and Signed Graph Structure

In this section, we focus on some results examining the spectra of a signed graph in relation to its structure.

4.1 Paths and Cycles

For certain families of signed graphs, such as the signed paths and cycles, the spectra of the adjacency and the Laplacian matrix have been calculated.

Theorem 14 ([12]) *The eigenvalues of the adjacency matrix of a signed path, Pn, are given by* $\lambda_j = 2cos\frac{\pi j}{n+1}$, $j = 1, 2, \ldots, n$.

Theorem 15 ([19]) *The Laplacian eigenvalues of a signed path, Pn, are given by* $\mu_j = 2(1 + cos\frac{\pi j}{n})$, $j = 1, 2, \ldots, n$.

As already mentioned, the signed cycle can be either positive or negative (balanced or unbalanced). The respective eigenvalues in each case are given in the following theorem.

Theorem 16 ([28, 41]) *The eigenvalues of a signed cycle, Cn, with r negative edges are given by*

$$\lambda_j = 2cos\frac{(2j - [r])\pi}{n},$$

where [r] = 0 if r is even and [r] = 1 if r is odd, and $j = 1, 2, \ldots, n$.

From the above theorem, it is easy to calculate the Laplacian eigenvalues of the singed graph.

Theorem 17 ([19, 36]) *The Laplacian eigenvalues of a signed cycle, Cn, with r negative edges and [r] defined as in the above theorem are given by*

$$\mu_j = 2 - 2cos\frac{(2j - [r])\pi}{n},$$

where $j = 1, 2, \ldots, n$.

It is well known that the unsigned paths and cycles are determined by their spectrum. This is not true for all cases of signed paths and cycles as they admit cospectral non-isomorphic graphs. The signed paths P_n that are determined by their spectrum were identified in regard to their order n.

Theorem 18 ([2]) *The signed path P_n is determined by its spectrum if and only if*

(i) n *is even and* $n \neq 8, 14$;
(ii) $n \equiv 1(mod4)$ *and* $n \notin \{13, 17, 29\}$;
(iii) $n \equiv 3(mod4)$ *and* $n = 3$.

Similar results were obtained for signed cycles.

Proposition 8 ([7]) *Let* $(C_{2n}, +)$ *be a balanced cycle of even order, and let* $2n = 2^{t+1}r$*, where t and r are positive integers and r is odd. If* $r \geq 3$*, then* $(C_{2^{t+1}r}, +)$

is L-cospectral with $(C_{2^s r}, +) \cup_{i=s}^{t} (C_{2^i r}, \overline{\sigma})$*, where* $0 \le s \le t$*. If* $r = 1$*, then* $(C_{2^{t+1}}, +)$ *is L-cospectral with* $(C_{2^s}, +) \cup_{i=s}^{t} (C_{2^i}, \overline{\sigma})$*), where* $2 \le s \le t$*.*

More results on the spectral characterizations of signed cycles can be found in [3]. For L-cospectral graphs, the following theorem has been proved.

Theorem 19 ([7]) *Let* $\Gamma = (G, \sigma)$ *and* $\Lambda = (H, \sigma')$ *be two L-cospectral signed graphs. Then,*

(i) Γ *and* Λ *have the same number of vertices and edges.*
(ii) Γ *and* Λ *have the same number of balanced components.*
(iii) Γ *and* Λ *have the same Laplacian spectral moments.*
(iv) Γ *and* Λ *have the same sum of squares of degrees,* $\sum_{i=1}^{n} d_G(v_i)^2 = \sum_{i=1}^{n} d_H(v_i)^2$.
(v) $6(t_\Gamma^- - t_\Gamma^+) + \sum_{i=1}^{n} d_G(v_i)^3 = 6(t_\Lambda^- - t_\Lambda^+) + \sum_{i=1}^{n} d_H(v_i)^3$,

where t^+ *and* t^- *denote the number of positive and negative triangles, respectively.*

4.2 Signed Unicyclic Graphs

A class of graphs that has been shown to be determined by the spectrum of its adjacency and its Laplacian matrix is known as the lollipop graph [10, 24]. The *lollipop graph* is the graph obtained by appending a cycle to a pendant vertex of a path. Recently, the next result was obtained for signed lollipop graphs (while the case for the adjacency matrix remains open).

Theorem 20 ([7]) *The signed lollipop graph is determined by the spectrum of its Laplacian matrix.*

Unicyclic graphs have been researched extensively in regard to the relation between their spectrum and the graph structure [15, 32, 47]. A parameter that has been considered in these studies is their nullity. The *nullity*, $\eta(G)$, of a graph is the multiplicity of its zero eigenvalue. It is well known that for a graph of order n it holds that $0 \le \eta(G) \le n - 2$. Recently, certain results on the nullity and rank of unsigned graphs have been generalized for the signed case. In [16], it was shown that a signed unicyclic graph of order n has nullity $n-2$ and $n-3$ if and only if it is the balanced cycle C_4 and cycle C_3, respectively. Signed unicyclic graphs of nullity $n-4$ and $n-5$ were also identified. Other results on the nullity and rank of signed graphs can be found, for example, in [17] and [33].

4.3 Signed Graphs with Two Distinct Eigenvalues

It is well known that the complete graph K_n is the only graph with two distinct eigenvalues in its spectrum, but this is not true for signed graphs. The characterization of signed graphs with just two eigenvalues has gained much attention recently (see [21, 37, 39, 43]). Some interesting results that describe the properties of such graphs are given in the following theorems.

Theorem 21 ([37]) *Let $\Gamma = (G, \sigma)$ be a signed graph with exactly two distinct eigenvalues. Then, its underlying graph G is regular.*

Theorem 22 ([21]) *Let $\Gamma = (G, \sigma)$ be a signed graph, and let G be triangle-free and a k-regular graph. Γ has two distinct eigenvalues if and only if the number of positive paths and the number of negative paths of length two between each pair of non-adjacent vertices are equal, in which case $A(\Gamma)^2 = kI_n$.*

Several signed graphs with two distinct eigenvalues have been identified until now: graphs of order at most 10, graphs with negative eigenvalue greater than 2, or graphs that belong to the class of signed line graphs [43]. However, the problem of characterizing all signed graphs with two (or few) eigenvalues remains open.

4.4 Graphs with Symmetric Spectrum

We close this section with a note on signed graphs with symmetric spectrum. A well-known result is that the spectrum of an unsigned graph is symmetric if and only if the graph is bipartite. That is not the case for signed graphs. All signed bipartite graphs have symmetric spectrum, but there are signed graphs that are not bipartite yet they have a symmetric spectrum. A *sign-symmetric graph* is a signed graph $\Gamma = (G, \sigma)$ that is switching isomorphic to $-\Gamma = (G, -\sigma)$. The following theorem is known for sign-symmetric graphs.

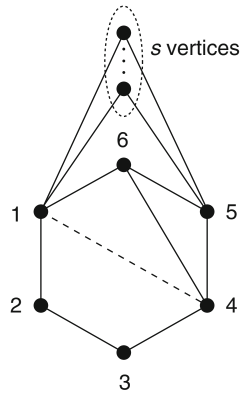
Theorem 23 *If $\Gamma = (G, \sigma)$ is a sign-symmetric graph, then Γ has a symmetric spectrum.*

It is well known that the Seidel matrix of a graph G of order n is the adjacency matrix of a signed complete graph Γ of the same order where the edges of G form all the negative edges in Γ. In [4], a family of signed complete graphs having a symmetric spectrum was constructed as follows.

Theorem 24 ([4]) *Let n be an even positive integer, and let V_1 and V_2 be two disjoint sets of size $\frac{n}{2}$. Let G be an arbitrary graph with the vertex set V_1. Construct the complement of G, G^c, with the vertex set V_2. Assume that $\Gamma = (K_n, \sigma)$ is a signed complete graph in which $E(G) \cup E(G^c)$ is the set of negative edges. Then, the spectrum of Γ is symmetric.*

Recently, the problem of finding non-complete connected signed graphs with symmetric spectrum that are not sign-symmetric was posed in [9]. Several such families of not sign-symmetric graphs were constructed in [38] and [22]. An example is the graph given in the figure below.

Fig. 1 Signed graph Γ_s



Theorem 25 *For* $s \geq 0$*, the graph* Γ_s *has a symmetric spectrum, but it is not sign-symmetric (Fig. 1, [22]).*

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Perturbed Geometric Contractions in Ordered Metric Spaces

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Abstract A geometric extension is given for the perturbed contraction principle in Aydi et al. [Abstr. Appl. Anal., Volume 2013, Article ID 312479].

AMS Subject Classification 47H10 (Primary), 54H25 (Secondary)

1 Introduction

Let X be a nonempty set. Call the subset Y of X *almost singleton* (in short, *asingleton*) when $y_1, y_2 \in Y \Longrightarrow y_1 = y_2$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$.

Take a *metric* $d : X \times X \to R_+ := [0, \infty[$ over X; the couple (X, d) is then referred to as a *metric space*. Furthermore, take a selfmap $T \in \mathscr{F}(X)$. [Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathscr{F}(A)$ in place of $\mathscr{F}(A, A)$]. Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. These points are to be determined in the context below, comparable with the one in Rus [37, Ch 2, Sect 2.2]:

(pic-0) We say that T is *fix-asingleton*, when $\text{Fix}(T)$ is an asingleton; likewise, we say that T is *fix-singleton* when $\text{Fix}(T)$ is a singleton

(pic-1) We say that $x \in X$ is a *Picard point* (modulo $(d; T)$) if the iterative sequence $(T^n x; n \geq 0)$ is d-Cauchy; when this property holds for all $x \in X$, we say that T is a *Picard operator* (modulo d)

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(pic-2) We say that $x \in X$ is a *strong Picard point* (modulo $(d; T)$) if the iterative sequence $(T^n x; n \geq 0)$ is d-convergent and $\lim_n (T^n x) \in \mathrm{Fix}(T)$; when this property holds for all $x \in X$, we say that T is a *strong Picard operator* (modulo d).

The basic result in this area (referred to as *Banach contraction principle*; in short, (B-cp)) may be stated as follows. Call $T : X \to X$, (d, μ)-*contractive* (where $\mu \geq 0$), provided

(con) $d(Tx, Ty) \leq \mu d(x, y), \forall x, y \in X.$

Theorem 1 *Assume that T is (d, μ)-contractive, for some $\mu \in [0, 1[$. In addition, let X be d-complete. Then,*

(11-a) *T is fix-singleton:* $\mathrm{Fix}(T) = \{z\}$, *for some $z \in X$*
(11-b) *T is strong Picard (modulo d):* $\lim_n T^n x = z$, *for each $x \in X$.*

This result, obtained in 1922 by Banach [3], found some basic applications to the operator equations theory. Consequently, a multitude of extensions for (B-cp) were proposed. The most general ones have the *implicit relational* form

(s-i-con) $(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \in \mathscr{M}$, for all $x, y \in X$, $x \nabla y$,

where $\mathscr{M} \subseteq R_+^6$ is a (nonempty) subset, and ∇ is a *relation* over X. In particular, when $\mathscr{M}$ is the zero-section of a certain function $F : R_+^6 \to R$, the implicit contractive condition above has the familiar form:

(f-i-con) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \leq 0$, for all $x, y \in X$, $x \nabla y$.

For the explicit trivial relation case of it, characterized as

(f-e-con) $d(Tx, Ty) \leq G(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y))$, for all $x, y \in X$

(where $G : R_+^5 \to R_+$ is a function), some consistent lists of such contractions may be found in the survey papers by Rhoades [35] or Collaco and E Silva [12]; these, in particular, include a lot of outstanding results in the area due to Boyd and Wong [6], Reich [34], and Matkowski [25]. Likewise, for the implicit setting above, certain technical aspects have been considered by Leader [24] and Turinici [41]. On the other hand, in the case of ∇ being a *(partial) order* on X, some early results were obtained in the 1986 papers by Turinici [43, 44]; two decades later, these results have been rediscovered—at the level of Banach contractive maps—by Ran and Reurings [33]; see also Nieto and Rodriguez-Lopez [32]. Furthermore, an extension—to the same framework—of Leader's contribution was performed in Agarwal et al. [1]; and, since then, the number of such papers increased rapidly. Finally, the case of ∇ being *amorphous* (i.e., it has no regularity properties at all) has been discussed (via graph techniques) in Jachymski [19] and (from a general perspective) by Samet and Turinici [38].

A basic particular case of the implicit contractive property above is

(2s-i-con) $(d(Tx, Ty), d(x, y)) \in \mathscr{M}$, for all $x, y \in X$, $x\nabla y$,

where $\mathscr{M} \subseteq R_+^2$ is a (nonempty) subset. The classical example in this direction (again over the trivial relation setting) is due to Meir and Keeler [28]; further refinements of the method were proposed by Matkowski [27] and Cirić [10]. Having these precise, it is our aim in the following to propose a *perturbation* enlargement of these results, which, in particular, includes the old (metrical) constructions due to Khan et al. [23] and Berinde [4] as well as the recent ones introduced by Aydi et al. [2]. Further aspects will be delineated elsewhere.

2 Dependent Choice Principles

Throughout this exposition, the axiomatic system in use is Zermelo-Fraenkel's (abbreviated (ZF)), as described by Cohen [11, Ch 2]. The notations and basic facts to be considered are standard; some important ones are discussed below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any (nonempty) part $\mathscr{R} \subseteq X \times X$; then, $(X, \mathscr{R})$ will be referred to as a *relational structure*. Note that $\mathscr{R}$ may be regarded as a mapping between X and $\exp[X]$ (= the class of all subsets in X). In fact, let us simplify the string $(x, y) \in \mathscr{R}$ as $x\mathscr{R}y$, and put

$X(x, \mathscr{R}) = \{y \in X; x\mathscr{R}y\}$ (the *section* of $\mathscr{R}$ through x), $x \in X$;

then, the desired mapping representation is $(\mathscr{R}(x) = X(x, \mathscr{R}); x \in X)$. A basic example of such object is

$\mathscr{I} = \{(x, x); x \in X\}$ [the *identical relation* over X].

Given the relations $\mathscr{R}$, $\mathscr{S}$ over X, define their *product* $\mathscr{R} \circ \mathscr{S}$ as

$(x, z) \in \mathscr{R} \circ \mathscr{S}$, if there exists $y \in X$ with $(x, y) \in \mathscr{R}$, $(y, z) \in \mathscr{S}$.

Also, for each relation $\mathscr{R}$ in X, denote

$\mathscr{R}^{-1} = \{(x, y) \in X \times X; (y, x) \in \mathscr{R}\}$ (the *inverse* of $\mathscr{R}$).

Finally, given the relations $\mathscr{R}$ and $\mathscr{S}$ on X, let us say that $\mathscr{R}$ is *coarser* than $\mathscr{S}$ (or, equivalently, $\mathscr{S}$ is *finer* than $\mathscr{R}$), provided

$\mathscr{R} \subseteq \mathscr{S}$; i.e., $x\mathscr{R}y$ implies $x\mathscr{S}y$.

Given a relation $\mathscr{R}$ on X, the following properties are to be discussed here:

(P1) $\mathscr{R}$ is *reflexive*: $\mathscr{I} \subseteq \mathscr{R}$.
(P2) $\mathscr{R}$ is *irreflexive*: $\mathscr{I} \cap \mathscr{R} = \emptyset$.
(P3) $\mathscr{R}$ is *transitive*: $\mathscr{R} \circ \mathscr{R} \subseteq \mathscr{R}$.
(P4) $\mathscr{R}$ is *symmetric*: $\mathscr{R}^{-1} = \mathscr{R}$.
(P5) $\mathscr{R}$ is *antisymmetric*: $\mathscr{R}^{-1} \cap \mathscr{R} \subseteq \mathscr{I}$.

This yields the classes of relations to be used; the following ones are important for our developments:

(C0) $\mathscr{R}$ is *amorphous* (i.e., it has no properties at all).
(C1) $\mathscr{R}$ is a *quasi-order* (reflexive and transitive).
(C2) $\mathscr{R}$ is a *strict order* (irreflexive and transitive).
(C3) $\mathscr{R}$ is an *equivalence* (reflexive, transitive, and symmetric).
(C4) $\mathscr{R}$ is a *(partial) order* (reflexive, transitive, and antisymmetric).
(C5) $\mathscr{R}$ is the *trivial* relation (i.e., $\mathscr{R} = X \times X$).

(B) A basic example of relational structure is to be constructed as below. Let

$$N = \{0, 1, 2, \ldots\}, \text{ where } 0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, \ldots,$$

denote the set of *natural* numbers. Technically speaking, the basic (algebraic and order) structures over N may be obtained by means of the *(immediate) successor* function $\mathrm{suc} : N \to N$ and the following Peano properties (deductible in our axiomatic system (ZF)):

(pea-1) $(0 \in N$ and$)$ $0 \notin \mathrm{suc}(N)$.
(pea-2) suc(.) is injective ($\mathrm{suc}(n) = \mathrm{suc}(m)$ implies $n = m$).
(pea-3) if $M \subseteq N$ fulfills $[0 \in M]$ and $[\mathrm{suc}(M) \subseteq M]$, then $M = N$.

(Note that, in the absence of our axiomatic setting, these properties become the well-known Peano axioms, as described in Halmos [16, Ch 12]; we do not give details). In fact, starting from these properties, one may construct, in a recurrent way, an *addition* $(a, b) \mapsto a + b$ over N, according to

$$(\forall m \in N)\colon m + 0 = m;\ m + \mathrm{suc}(n) = \mathrm{suc}(m + n).$$

This, in turn, makes possible the introduction of a (partial) order $(\le)$ over N, as

$$(m, n \in N)\colon m \le n \text{ iff } m + p = n, \text{ for some } p \in N.$$

Concerning the properties of this structure, the most important one writes

$(N, \le)$ is *well ordered*: any (nonempty) subset of N has a first element.

Denote, for simplicity,

$$N(r, \le) = \{n \in N; r \le n\} = \{r, r + 1, \ldots, \}, r \ge 0,$$
$$N(r, >) = \{n \in N; r > n\} = \{0, \ldots, r - 1\}, r \ge 1;$$

the latter one is referred to as the *initial interval* (in N) induced by r. Any set P with $N \sim P$ (in the sense, there exists a bijection from N to P) will be referred to as *effectively denumerable*. In addition, given some natural number $n \ge 1$, any (nonempty) set Q with $N(n, >) \sim Q$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. As a combination of these, we say that the (nonempty) set Y is (at most) *denumerable* iff it is either effectively denumerable or finite.

Having these precise, let the notion of *sequence* (in X) be used to designate any mapping $x : N \to X$. For simplicity reasons, it will be useful to denote it as $(x(n); n \ge 0)$ or $(x_n; n \ge 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is *strictly ascending* (hence, $i(n) \to \infty$ as $n \to \infty$)

will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Note that, under such a convention, the relation "subsequence of" is transitive; i.e.,

(z_n)=subsequence of (y_n) and (y_n)=subsequence of (x_n)
imply (z_n)=subsequence of (x_n).

(C) Remember that an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated (AC)); which, in a convenient manner, may be written as

(AC) For each couple (J, X) of nonempty sets and each function
$F : J \to \exp(X)$, there exists a (selective) function
$f : J \to X$, with $f(\nu) \in F(\nu)$, for each $\nu \in J$.

(Here, $\exp(X)$ stands for the class of all nonempty elements in $\exp[X]$). Sometimes, when the ambient set X is endowed with denumerable type structures, the existence of such a selective function (over $J = N$) may be determined by using a weaker form of (AC), referred to as *Dependent Choice* principle (in short, (DC)). Call the relation $\mathscr{R}$ over X *proper* when

$(X(x, \mathscr{R}) =)\mathscr{R}(x)$ is nonempty, for each $x \in X$.

Then, $\mathscr{R}$ is to be viewed as a mapping between X and $\exp(X)$, and the couple $(X, \mathscr{R})$ will be referred to as a *proper relational structure*. Furthermore, given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; \mathscr{R})$-*iterative*, provided

$x_0 = a$, and $x_n \mathscr{R} x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}(x_n)$), for all n.

Proposition 1 *Let the relational structure $(X, \mathscr{R})$ be proper. Then, for each $a \in X$, there is at least an $(a; \mathscr{R})$-iterative sequence in X.*

This principle—proposed, independently, by Bernays [5] and Tarski [40]—is deductible from (AC), but not conversely; cf. Wolk [48]. Moreover, by the developments in Moskhovakis [30, Ch 8] and Schechter [39, Ch 6], the *reduced system* (ZF-AC+DC) is comprehensive enough so as to cover the "usual" mathematics; see also Moore [29, Appendix 2].

Let $(\mathscr{R}_n; n \geq 0)$ be a sequence of relations on X. Given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; (\mathscr{R}_n; n \geq 0))$-*iterative*, provided

$x_0 = a$, and $x_n \mathscr{R}_n x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}_n(x_n)$), for all n.

The following *Diagonal Dependent Choice* principle (in short, (DDC)) is available.

Proposition 2 *Let $(\mathscr{R}_n; n \geq 0)$ be a sequence of proper relations on X. Then, for each $a \in X$, there exists at least one $(a; (\mathscr{R}_n; n \geq 0))$-iterative sequence in X.*

Clearly, (DDC) includes (DC), to which it reduces when $(\mathscr{R}_n; n \geq 0)$ is constant. The reciprocal of this is also true. In fact, letting the premises of (DDC) hold, put $P = N \times X$, and let $\mathscr{S}$ be the relation over P introduced as

$$\mathscr{S}(i, x) = \{i + 1\} \times \mathscr{R}_i(x), \ (i, x) \in P.$$

It will suffice applying (DC) to $(P, \mathscr{S})$ and $b := (0, a) \in P$ to get the conclusion in our statement; we do not give details.

Summing up, (DDC) is provable in (ZF-AC+DC). This is valid as well for its variant, referred to as *Selected Dependent Choice* principle (in short, (SDC)).

Proposition 3 *Let the map* $F : N \to \exp(X)$ *and the relation* $\mathscr{R}$ *over X fulfill*

$$(\forall n \in N)\text{: } \mathscr{R}(x) \cap F(n + 1) \neq \emptyset, \text{ for all } x \in F(n).$$

Then, for each $a \in F(0)$*, there exists a sequence* $(x(n); n \geq 0)$ *in X, with*

$$x(0) = a, \ x(n) \in F(n), \ x(n + 1) \in \mathscr{R}(x(n)), \forall n.$$

As before, (SDC) $\Longrightarrow$ (DC) ($\Longleftrightarrow$ (DDC)); just take $(F(n) = X; n \geq 0)$. But, the reciprocal is also true, in the sense (DDC) $\Longrightarrow$ (SDC). This follows from the following proposition reasoning below:

Proof of Proposition 3 Let the premises of (SDC) be true. Define a sequence of relations $(\mathscr{R}_n; n \geq 0)$ over X, as: for each $n \geq 0$,

$$\mathscr{R}_n(x) = \mathscr{R}(x) \cap F(n + 1), \text{ if } x \in F(n),$$
$$\mathscr{R}_n(x) = \{x\}, \text{ otherwise } (x \in X \setminus F(n)).$$

Clearly, $\mathscr{R}_n$ is proper, for all $n \geq 0$. So, by (DDC), it follows that for the starting $a \in F(0)$, there exists an $(a, (\mathscr{R}_n; n \geq 0))$-iterative sequence $(x(n); n \geq 0)$ in X. Combining with the very definition above, one derives that conclusion in the statement is holding.

In particular, when $\mathscr{R} = X \times X$, the regularity condition imposed in (SDC) holds. The corresponding variant of the underlying statement is just (AC(N)) (= the *Denumerable Axiom of Choice*). Precisely, we have the following statement.

Proposition 4 *Let* $F : N \to \exp(X)$ *be a function. Then, for each* $a \in F(0)$*, there exists a function* $f : N \to X$ *with* $f(0) = a$ *and* $f(n) \in F(n)$*,* $\forall n \in N$.

As a consequence of the above facts, (DC) $\Longrightarrow$ (AC(N)) in (ZF-AC). A direct verification of this is obtainable by taking $Q = N \times X$ and introducing the relation $\mathscr{S}$ over it, according to

$$\mathscr{S}(n, x) = \{n + 1\} \times F(n + 1), \ n \in N, x \in X;$$

we do not give details. The reciprocal of the written inclusion is not true; see, for instance, Moskhovakis [30, Ch 8, Sect 8.25].

3 Conv-Cauchy Structures

Let X be a nonempty set, and $\mathscr{S}(X)$ stands for the class of all sequences (x_n) in X. By a (sequential) *convergence structure* on X, we mean any part $\mathscr{C}$ of $\mathscr{S}(X) \times X$, with the properties (cf. Kasahara [22]):

(conv-1) $\mathscr{C}$ is *hereditary*:
$((x_n); x) \in \mathscr{C} \Longrightarrow ((y_n); x) \in \mathscr{C}$, for each subsequence (y_n) of (x_n)
(conv-2) $\mathscr{C}$ is *reflexive*: for each $u \in X$,
the constant sequence $(x_n = u; n \geq 0)$ fulfills $((x_n); u) \in \mathscr{C}$.

For each sequence (x_n) in $\mathscr{S}(X)$ and each $x \in X$, we write $((x_n); x) \in \mathscr{C}$ as $x_n \overset{\mathscr{C}}{\longrightarrow} x$; this reads

(x_n), $\mathscr{C}$-converges to x (also referred to as: x is the *$\mathscr{C}$-limit* of (x_n)).

The set of all such x is denoted $\mathscr{C} - \lim_n(x_n)$; when it is nonempty, we say that (x_n) is *$\mathscr{C}$-convergent*. The following condition is to be optionally considered here:

(conv-3) $\mathscr{C}$ is *separated*:
$\mathscr{C} - \lim_n(x_n)$ is an asingleton, for each sequence (x_n);

when it holds, $x_n \overset{\mathscr{C}}{\longrightarrow} z$ will also be written as $\mathscr{C} - \lim_n(x_n) = z$.

Furthermore, by a (sequential) *Cauchy structure* on X, we shall mean any part $\mathscr{H}$ of $\mathscr{S}(X)$ with (cf. Turinici [45]):

(Cauchy-1) $\mathscr{H}$ is *hereditary*:
$(x_n) \in \mathscr{H} \Longrightarrow (y_n) \in \mathscr{H}$, for each subsequence (y_n) of (x_n).
(Cauchy-2) $\mathscr{H}$ is *reflexive*: for each $u \in X$,
the constant sequence $(x_n = u; n \geq 0)$ fulfills $(x_n) \in \mathscr{H}$.

Each element of $\mathscr{H}$ will be referred to as a *$\mathscr{H}$-Cauchy* sequence in X.

Finally, given the couple $(\mathscr{C}, \mathscr{H})$ as before, we shall say that it is a *conv-Cauchy structure* on X. The optional conditions about the conv-Cauchy structure $(\mathscr{C}, \mathscr{H})$ to be considered here are

(CC-1) $(\mathscr{C}, \mathscr{H})$ is *regular*: each $\mathscr{C}$-convergent sequence is $\mathscr{H}$-Cauchy.
(CC-2) $(\mathscr{C}, \mathscr{H})$ is *complete*: each $\mathscr{H}$-Cauchy sequence is $\mathscr{C}$-convergent.

A standard way of introducing such structures is the *(pseudo)metrical* one. By a *pseudometric* over X, we shall mean any map $d : X \times X \to R_+$. Given such an object, the following properties are to be optionally used:

(ref) d is *reflexive*: $x = y$ implies $d(x, y) = 0$.
(tri) d is *triangular*: $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.
(suf) d is *sufficient*: $d(x, y) = 0$ implies $x = y$.
(sym) d is *symmetric*: $d(x, y) = d(y, x)$, $\forall x, y \in X$.

This yields the classes of pseudometrics to be used:

(p1) d is *r-pseudometric* (reflexive).

(p2) d is *t-pseudometric* (triangular).
(p3) d is *rs-pseudometric* (reflexive and sufficient).
(p4) d is *almost semimetric* (reflexive and triangular).
(p5) d is *semimetric* (reflexive, triangular, and symmetric).
(p6) d is *almost metric* [or *quasi-metric*] (reflexive, triangular, and sufficient).
(p7) d is *metric* (reflexive, triangular, sufficient, and symmetric).

Let in the following $d(.,.)$ be a reflexive pseudometric (in short, *r-pseudometric*) on X; in this case, (X, d) is called an *r-pseudometric space*.

Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n) *d-converges* to x (written as $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

$\forall \varepsilon > 0, \exists i = i(\varepsilon)\colon\ i \leq n \Longrightarrow d(x_n, x) < \varepsilon.$

By this very definition, we have the hereditary and reflexive properties:

(d-conv-1) $(\xrightarrow{d})$ is hereditary:
$x_n \xrightarrow{d} x$ implies $y_n \xrightarrow{d} x$, for each subsequence (y_n) of (x_n).
(d-conv-2) $(\xrightarrow{d})$ is reflexive: for each $u \in X$,
the constant sequence $(x_n = u; n \geq 0)$ fulfills $x_n \xrightarrow{d} u$.

As a consequence, $(\xrightarrow{d})$ is a sequential convergence on X. The set of all such limit points of (x_n) will be denoted $\lim_n(x_n)$; if it is nonempty, then (x_n) is called *d-convergent*. Finally, note that $(\xrightarrow{d})$ is not separated, in general. However, this property holds, provided (in addition)

(sym) d is triangular, sufficient, and symmetric (hence, a metric on X).

The following Lipschitz property of certain pseudometrics with respect to their variables will be useful in applications.

Proposition 5 *Suppose that $d(.,.)$ is a semimetric. Then,*

(31-1) *the mapping $(x, y) \mapsto d(x, y)$ is d-Lipschitz, in the sense*

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v), \text{ for all } (x, y), (u, v) \in X \times X$$

(31-2) *$d(.,.)$ is (sequentially) continuous in its variables:*
$x_n \xrightarrow{d} x$ *and* $y_n \xrightarrow{d} y$ *imply* $d(x_n, y_n) \to d(x, y)$.

Proof

(i) By the semimetric properties, we have for each (x, y) and (u, v) in $X \times X$

$d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$;
wherefrom $d(x, y) - d(u, v) \leq d(x, u) + d(y, v)$;
$d(u, v) \leq d(u, x) + d(x, y) + d(y, v)$;
wherefrom $d(u, v) - d(x, y) \leq d(x, u) + d(y, v)$,

and, from this, all is clear.

(ii) Evident, by the preceding stage.

Furthermore, call the sequence (x_n) (in X) *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty, m < n$; i.e.,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon): \; j \le m < n \Longrightarrow d(x_m, x_n) < \varepsilon;$$

the class of all these will be denoted as $Cauchy(d)$. As before, we have the hereditary and reflexive properties

(d-Cauchy-1) $Cauchy(d)$ is hereditary: (x_n) is d-Cauchy implies (y_n) is d-Cauchy, for each subsequence (y_n) of (x_n).
(d-Cauchy-2) $Cauchy(d)$ is reflexive: for each $u \in X$, the constant sequence $(x_n = u; n \ge 0)$ is d-Cauchy;

hence, $Cauchy(d)$ is a Cauchy structure on X.

Finally, the couple $((\stackrel{d}{\longrightarrow}), Cauchy(d))$ will be referred to as a *conv-Cauchy structure* on X generated by d. Note that, by the imposed (upon d) conditions, this conv-Cauchy structure is not (regular or complete), in general. But, when d is sufficient, triangular, and symmetric (hence, a metric), the regularity condition holds.

Concerning this combined structure over the ambient r-pseudometric space (X, d), the following question is of interest. Remember that, by definition, any subsequence of a d-convergent sequence is also d-convergent, with the same limit. Suppose now that a certain subsequence of a sequence is d-convergent; then, we may ask of to what extent it is true that the sequence itself is d-convergent (with the same limit). A positive answer to this may be given if d is, in addition, triangular (hence, an almost semimetric).

Proposition 6 *Given the almost semimetric space (X, d), let the d-Cauchy sequence $(x_n; n \ge 0)$ in X and the point $v \in X$ be taken according to*

$$y_n \stackrel{d}{\longrightarrow} v, \text{ for some subsequence } (y_n = x_{k(n)}; n \ge 0) \text{ of } (x_n; n \ge 0).$$

Then, necessarily, $x_n \stackrel{d}{\longrightarrow} v$ as $n \to \infty$.

Proof Let $\varepsilon > 0$ be arbitrary fixed. From the d-Cauchy property, there exists some rank $m(\varepsilon) \ge 0$, such that

$$m(\varepsilon) \le i \le j \Longrightarrow d(x_i, x_j) < \varepsilon/2.$$

On the other hand, by the subsequential convergence property, we have that, for the same $\varepsilon > 0$, there exists some $n(\varepsilon) \ge m(\varepsilon)$, such that

$$n(\varepsilon) \le n \Longrightarrow d(y_n, v) = d(x_{k(n)}, v) < \varepsilon/2.$$

Finally, as $(k(n); n \ge 0)$ is strictly ascending, we must have

$$k(n) \ge n, \text{ for all } n \ge 0; \text{ hence, } k(n) \ge n(\varepsilon), \text{ for all } n \ge n(\varepsilon).$$

Combining these, we have, for each $n \ge n(\varepsilon)(\ge m(\varepsilon))$,

$d(x_n, v) \le d(x_n, x_{k(n)}) + d(x_{k(n)}, v) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$

and the conclusion follows.

Returning to the general case, some (weaker than Cauchy) properties of sequences in our pseudometric structure may be introduced as below. Call the sequence (x_n) in X

(d-asy) *d-asymptotic*, if $\lim_n d(x_n, x_{n+1}) = 0$,
(d-to-asy) *d-total-asymptotic*, if $\lim_n d(x_n, x_{n+i}) = 0, \forall i \in N(1, \le)$.

Clearly, for each sequence (x_n) in X

d-Cauchy $\Longrightarrow$ d-total-asymptotic $\Longrightarrow$ d-asymptotic;

but, the converse is not in general true.

An appropriate setting to discuss the converse relationship between these asymptotic concepts is the triangular one. In this direction, we have the following statement:

Proposition 7 *Supposed that d is an almost semimetric on X. Then, for each sequence (x_n) in X,*

(x_n) is d-asymptotic iff (x_n) is d-total-asymptotic.

Proof Clearly, it will suffice verifying the left to right inclusion. Let $i \in N(1, \le)$ be arbitrary fixed. By the triangular inequality,

$(d(x_n, x_{n+i}) \le \rho_n + \ldots + \rho_{n+i-1}, \forall n)$, where $(\rho_n := d(x_n, x_{n+1}); n \ge 0)$.

By the imposed hypothesis, the right member of this relation tends to zero as $n \to \infty$; wherefrom, (x_n) is d-total-asymptotic.

We close this section with a few remarks involving convergent real sequences. For each sequence (r_n) in R and each element $r \in R$, denote

$r_n \to r+$ (resp., $r_n \to r-$), when $r_n \to r$ and $[r_n > r$ (resp., $r_n < r$), $\forall n]$.

Proposition 8 *Let the sequence $(r_n; n \ge 0)$ in R and the number $\varepsilon \in R$ be such that $r_n \to \varepsilon+$. Then, there exists a subsequence $(r_n^* := r_{i(n)}; n \ge 0)$ of $(r_n; n \ge 0)$, with*

$(r_n^*; n \ge 0)$ is strictly descending and $r_n^* \to \varepsilon+$.

Proof Put $i(0) = 0$. As $\varepsilon < r_{i(0)}$ and $r_n \to \varepsilon+$, we have that

$A(i(0)) := \{n > i(0); r_n < r_{i(0)}\}$ is not empty;
hence, $i(1) := \min(A(i(0)))$ is an element of it, and $r_{i(1)} < r_{i(0)}$.

Likewise, as $\varepsilon < r_{i(1)}$ and $r_n \to \varepsilon+$, we have that

$A(i(1)) := \{n > i(1); r_n < r_{i(1)}\}$ is not empty;
hence, $i(2) := \min(A(i(1)))$ is an element of it, and $r_{i(2)} < r_{i(1)}$.

This procedure may continue indefinitely and yields (without any choice technique) a strictly ascending rank sequence $(i(n); n \geq 0)$ (hence, $i(n) \to \infty$ as $n \to \infty$) for which the attached subsequence $(r_n^* := r_{i(n)}; n \geq 0)$ of $(r_n; n \geq 0)$ fulfills

$r_{n+1}^* < r_n^*$, for all n; hence, (r_n^*) is (strictly) descending.

On the other hand, by this very subsequence property,

$(r_n^* > \varepsilon, \forall n)$, and $\lim_n r_n^* = \lim_n r_n = \varepsilon$.

Putting these together, we get the desired fact.

A bidimensional counterpart of these facts may be given along the lines below. Let $\pi(t, s)$ (where $t, s \in R$) be a logical property involving pairs or real numbers. Given the couple of real sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$, call the subsequences $(t_n^*; n \geq 0)$ of (t_n) and $(s_n^*; n \geq 0)$ of (s_n), *compatible* when

$(t_n^* = t_{i(n)} n \geq 0)$, and $(s_n^* = s_{i(n)}; n \geq 0)$,
for the same strictly ascending rank sequence $(i(n); n \geq 0)$.

Proposition 9 *Let the couple of real sequences* $(t_n; n \geq 0)$, $(s_n; n \geq 0)$ *and the pair of real numbers* (a, b) *be such that*

$t_n \to a+$, $s_n \to b+$ *as* $n \to \infty$ *and* ($\pi(t_n, s_n)$ *is true*, $\forall n$).

There exists then a couple of subsequences $(t_n^*; n \geq 0)$ *of* $(t_n; n \geq 0)$ *and* $(s_n^*; n \geq 0)$ *of* $(s_n; n \geq 0)$, *respectively, with*

(35-1) $(t_n^*; n \geq 0)$ *and* $(s_n^*; n \geq 0)$ *are strictly descending and compatible.*
(35-2) $t_n^* \to a+$, $s_n^* \to b+$, *as* $n \to \infty$, *and* $\pi(t_n^*, s_n^*)$ *holds, for all* n.

Proof By the preceding statement, (t_n) admits a subsequence $(T_n := t_{i(n)}; n \geq 0)$, with the properties

$(T_n; n \geq 0)$ is strictly descending, and $(T_n \to a+$, as $n \to \infty)$.

Denote $(S_n := s_{i(n)}; n \geq 0)$; clearly,

$(S_n; n \geq 0)$ is a subsequence of $(s_n; n \geq 0)$ with $S_n \to b+$ as $n \to \infty$.

Moreover, by this very construction, $\pi(T_n, S_n)$ holds, for all n. Again by the statement above, there exists a subsequence $(s_n^* := S_{j(n)} = s_{i(j(n))}; n \geq 0)$ of $(S_n; n \geq 0)$ (hence, of $(s_n; n \geq 0)$ as well), with

$(s_n^*; n \geq 0)$ is strictly descending, and $(s_n^* \to b+$, as $n \to \infty)$.

Denote further $(t_n^* := T_{j(n)} = t_{i(j(n))}; n \geq 0)$; this is a subsequence of $(T_n; n \geq 0)$ (hence, of $(t_n; n \geq 0)$ as well), with

$(t_n^*; n \geq 0)$ is strictly descending, and $(t_n^* \to a+$, as $n \to \infty)$;

Finally, by this very construction (and a previous relation), $\pi(t_n^*, s_n^*)$ holds, for all n. Summing up, the couple of subsequences $(t_n^*; n \geq 0)$ and $(s_n^*; n \geq 0)$ have all needed properties, and the conclusion follows.

Note that further extensions of this result are possible, in the framework of quasi-metric spaces, taken as in Hitzler [17, Ch 1, Sect 1.2]; we shall discuss them in a separate paper.

4 Meir–Keeler Relations

Let $\Omega \subseteq R^0_+ \times R^0_+$ be a relation over R^0_+; as a rule, we write $(t, s) \in \Omega$ as $t\Omega s$. The starting global property to be considered upon this object is

(u-diag) Ω is *upper diagonal*: $t\Omega s$ implies $t < s$.

Denote the class of all upper diagonal relations as $\mathrm{udiag}(R^0_+)$. Our exposition below is essentially related to this basic condition.

To begin with, let us consider the global properties

(1-decr) Ω is *first variable decreasing*:
$t_1, t_2, s \in R^0_+$, $t_1 \geq t_2$, and $t_1\Omega s$ imply $t_2\Omega s$.
(2-incr) Ω is *second variable increasing*:
$t, s_1, s_2 \in R^0_+$, $s_1 \leq s_2$, and $t\Omega s_1$ imply $t\Omega s_2$.

Then, define the sequential condition below (for upper diagonal relations):

(M-ad) Ω in *Matkowski admissible*:
$(t_n; n \geq 0)$ in R^0_+ and $(t_{n+1}\Omega t_n, \forall n)$ imply $\lim_n t_n = 0$.

To discuss it, the following geometric conditions over $\mathrm{udiag}(R^0_+)$ are in effect:

(g-mk) Ω has the *geometric Meir–Keeler property*:
$\forall \varepsilon > 0, \exists \delta > 0$: $t\Omega s, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \leq \varepsilon$.
(g-bila-s) Ω is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t, s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow (t, s) \notin \Omega$.
(g-left-s) Ω is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow (t, \beta) \notin \Omega$.

The former of these local conditions—related to the developments in Meir and Keeler [28]—is strongly related to the Matkowski admissible property we just introduced. Precisely, the following basic fact is available.

Theorem 2 *Under these conditions, one has in (ZF-AC+DC):*

(41-a) (for each $\Omega \in \mathrm{udiag}(R^0_+)$),
Ω is geometric Meir–Keeler implies Ω is Matkowski admissible.
(41-b) (for each first variable decreasing $\Omega \in \mathrm{udiag}(R^0_+)$),
Ω is Matkowski admissible implies Ω is geometric Meir–Keeler.

Hence, summing up,

(41-c) (for each first variable decreasing $\Omega \in \mathrm{udiag}(R^0_+)$),
Ω is geometric Meir–Keeler iff Ω is Matkowski admissible.

Proof Three basic stages must be passed.

(i) Suppose that $\Omega \in \text{udiag}(R^0_+)$ is geometric Meir–Keeler; we have to establish that Ω is Matkowski admissible. Let $(t_n; n \geq 0)$ be a sequence in R^0_+, fulfilling $(t_{n+1}\Omega t_n$, for all $n)$. By the upper diagonal property, we get

$(t_{n+1} < t_n$, for all $n)$; i.e., (t_n) is strictly descending.

As a consequence, $\tau := \lim_n t_n$ exists in R_+, with, in addition, $t_n > \tau$, $\forall n$. Assume by contradiction that $\tau > 0$, and let $\sigma > 0$ be the number assured by the geometric Meir–Keeler property. By definition, there exists an index $n(\sigma)$, with

$(t_{n+1}\Omega t_n$ and) $\tau < t_n < \tau + \sigma$, for all $n \geq n(\sigma)$.

This, by the quoted property, gives (for the same ranks)

$\tau < t_{n+1} \leq \tau$, a contradiction.

Hence, necessarily, $\tau = 0$, and the conclusion follows.

(ii) Suppose that the first variable decreasing $\Omega \in \text{udiag}(R^0_+)$ is Matkowski admissible; we have to establish that Ω is geometric Meir–Keeler. Suppose by contradiction that this is not true; that is (for some $\varepsilon > 0$)

$H(\delta) := \{(t, s) \in \Omega; \varepsilon < s < \varepsilon + \delta, t > \varepsilon\}$ is nonempty, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R^0_+, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $((t_n, s_n); n \geq 0)$ in $R^0_+ \times R^0_+$, so as

$(\forall n)$: (t_n, s_n) is an element of $H(\delta_n)$; or, equivalently (by definition and upper diagonal property) $(t_n\Omega s_n$ and) $\varepsilon < t_n < s_n < \varepsilon + \delta_n$.

Note that, as a direct consequence,

$(t_n\Omega s_n$, for all $n)$, and $t_n \to \varepsilon+$, $s_n \to \varepsilon+$, as $n \to \infty$.

Put $i(0) = 0$. As $\varepsilon < t_{i(0)}$ and $s_n \to \varepsilon+$ as $n \to \infty$, we have that

$A(i(0)) := \{n > i(0); s_n < t_{i(0)}\}$ is not empty;
hence, $i(1) := \min(A(i(0)))$ is an element of it, and $s_{i(1)} < t_{i(0)}$;
wherefrom, $s_{i(1)}\Omega s_{i(0)}$ (as Ω is first variable decreasing).

Likewise, as $\varepsilon < t_{i(1)}$ and $s_n \to \varepsilon+$ as $n \to \infty$, we have that

$A(i(1)) := \{n > i(1); s_n < t_{i(1)}\}$ is not empty;
hence, $i(2) := \min(A(i(1)))$ is an element of it, and $s_{i(2)} < t_{i(1)}$;
wherefrom, $s_{i(2)}\Omega s_{i(1)}$ (as Ω is first variable decreasing).

This procedure may continue indefinitely and yields (without any choice technique) a strictly ascending rank sequence $(i(n); n \geq 0)$ in N for which the attached subsequence $(r_n := s_{i(n)}; n \geq 0)$ of $(s_n; n \geq 0)$ fulfills

$r_{n+1}\Omega r_n$, for all n; whence $r_n \to 0$ (as Ω is Matkowski admissible).

On the other hand, by our subsequence property,

$(r_n > \varepsilon, \forall n)$ and $\lim_n r_n = \lim_n s_n = \varepsilon$; that is, $r_n \to \varepsilon+$.

The obtained relation is in contradiction with the previous one. Hence, the working condition cannot be true, and we are done.

(iii) Evident, by the above.

In the following, equivalent (sequential) conditions are given for the properties appearing in our (geometric) concepts above. Given the upper diagonal relation Ω over R^0_+, let us introduce the (asymptotic type) conventions

(a-mk) Ω is *asymptotic Meir–Keeler*:
there are no strictly descending sequences (t_n) and (s_n) in R^0_+ and no elements ε in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

(a-bila-s) Ω is *asymptotic bilateral separable*:
there are no sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R^0_+ and no elements $\beta \in R^0_+$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \beta, s_n \to \beta)$.

(a-left-s) Ω is *asymptotic left separable*:
there are no strictly ascending sequences (t_n) in R^0_+
and no elements β in R^0_+, with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta-)$.

Remark 1 The relationships between our first and second concepts are described as

(for each upper diagonal relation $\Omega \subseteq R^0_+ \times R^0_+$)
Ω is asymptotic bilateral separable implies Ω is asymptotic Meir–Keeler.

In fact, let the upper diagonal relation $\Omega \subseteq R^0_+ \times R^0_+$ be asymptotic bilateral separable, and assume by contradiction that Ω is not asymptotic Meir–Keeler:

there exist strictly descending sequences (t_n) and (s_n) in R^0_+ and elements ε in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$; hence $(t_n \to \varepsilon, s_n \to \varepsilon)$.

This tells us that Ω is not asymptotic bilateral separable; in contradiction with the working hypothesis; and the assertion follows.

Passing to the relationships between the asymptotic concepts and their geometric counterparts, we have the result below.

Theorem 3 *The following generic relationships are valid (for an arbitrary upper diagonal relation $\Omega \subseteq R^0_+ \times R^0_+$), in the reduced system (ZF-AC+DC):*

(42-a) geometric Meir–Keeler is equivalent with asymptotic Meir-Keeler.
(42-b) geometric bilateral separable is equivalent with asymptotic bilateral separable.
(42-c) geometric left separable is equivalent with asymptotic left separable.

Proof There are three steps to be passed.

(i-1) Let $\Omega \in \mathrm{udiag}(R_+^0)$ be a geometric Meir–Keeler relation; but—contrary to the conclusion—assume that Ω does not have the asymptotic Meir–Keeler property:

there are two strictly descending sequences (t_n) and (s_n) in R_+^0 and an element ε in R_+^0, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

Let $\delta > 0$ be the number given by the geometric Meir–Keeler property of Ω. By definition, there exists a (common) rank $n(\delta)$, such that

$n \geq n(\delta)$ implies $\varepsilon < t_n < \varepsilon + \delta, \varepsilon < s_n < \varepsilon + \delta$.

From the second relation, we must have (by the hypothesis about Ω) $t_n \leq \varepsilon$, for all $n \geq n(\delta)$. This, however, contradicts the first relation above. Hence, Ω is asymptotic Meir–Keeler, as asserted.

(i-2) Let $\Omega \in \mathrm{udiag}(R_+^0)$ be an asymptotic Meir–Keeler relation; but—contrary to the conclusion—assume that Ω does not have the geometric Meir–Keeler property; that is (for some $\varepsilon > 0$)

$H(\delta) := \{(t, s) \in \Omega; \varepsilon < s < \varepsilon + \delta,\ t > \varepsilon\} \neq \emptyset$, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R_+^0, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $((t_n, s_n); n \geq 0)$ in $R_+^0 \times R_+^0$, so as

$(\forall n)$: (t_n, s_n) is an element of $H(\delta_n)$; or, equivalently (by definition and upper diagonal property) $((t_n, s_n) \in \Omega$ and) $\varepsilon < t_n < s_n < \varepsilon + \delta_n$.

Note that, as a direct consequence,

$(t_n \Omega s_n$, for all $n)$, and $t_n \to \varepsilon+, s_n \to \varepsilon+$, as $n \to \infty$.

By a previous result, there exist a compatible couple of subsequences $(t_n^* := t_{i(n)}; n \geq 0)$ of $(t_n; n \geq 0)$ and $(s_n^* := s_{i(n)}; n \geq 0)$ of $(s_n; n \geq 0)$, with

$(t_n^* \Omega s_n^*, \forall n)$; (t_n^*), (s_n^*) are strictly descending; $t_n^* \to \varepsilon+$ and $s_n^* \to \varepsilon+$.

This, however, is in contradiction with respect to the posed hypothesis upon Ω; wherefrom, our assertion follows.

(ii-1) Let $\Omega \in \mathrm{udiag}(R_+^0)$ be a geometric bilateral separable relation; we have to establish that Ω is asymptotic bilateral separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is,

there are two sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R_+^0 and an element $\beta \in R_+^0$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \beta, s_n \to \beta)$.

Let $\gamma \in]0, \beta[$ be the number given by the geometric bilateral separable property of Ω. By definition, there exists a (common) index $k = k(\gamma)$, such that

$(\forall n)$: $\beta - \gamma < T_n := t_{n+k} < \beta + \gamma$, $\beta - \gamma < S_n := s_{n+k} < \beta + \gamma$.

This, along with $[T_n \Omega S_n, \forall n]$, contradicts the geometric bilateral separable property of Ω. Hence, Ω is asymptotic bilateral separable.

(ii-2) Let $\Omega \in \text{udiag}(R^0_+)$ be an asymptotic bilateral separable relation; we have to establish that Ω is geometric bilateral separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is (for some $\beta > 0$)

$K(\gamma) := \{(t, s) \in \Omega; t, s \in]\beta - \gamma, \beta + \gamma[\} \neq \emptyset$, for each $\gamma \in]0, \beta[$.

Taking a strictly descending sequence $(\gamma_n; n \geq 0)$ in $]0, \beta[$ with $\gamma_n \to 0+$, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $((t_n, s_n); n \geq 0)$ in Ω, so as

$(\forall n)$: (t_n, s_n) is an element of $K(\gamma_n)$; or, equivalently (by the very definition above) $(t_n, s_n) \in \Omega$ and $t_n, s_n \in]\beta - \gamma_n, \beta + \gamma_n[$.

By the second half of this last relation, we must have $(t_n \to \beta, s_n \to \beta)$, and this, along with the first half of the same, contradicts the imposed hypothesis. Hence, necessarily, Ω is geometric bilateral separable.

(iii-1) Let $\Omega \in \text{udiag}(R^0_+)$ be a geometric left separable relation; we have to establish that Ω is asymptotic left separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is,

there exist a strictly ascending sequence $(t_n; n \geq 0)$ and an element $\beta \in R^0_+$,
with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta-)$.

Let $\gamma \in]0, \beta[$ be the number given by the geometric left separable property of Ω. By definition, there exists a rank $h = h(\gamma)$, such that

$(\forall n)$: $\beta - \gamma < T_n := t_{n+h} < \beta$.

This, along with $[T_n \Omega \beta, \forall n]$, contradicts the geometric left separable property of Ω. Hence, Ω is asymptotic left separable.

(iii-2) Let $\Omega \in \text{udiag}(R^0_+)$ be an asymptotic left separable relation; we have to establish that Ω is geometric left separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is (for some $\beta > 0$)

$L(\gamma) := \{t \in]\beta - \gamma, \beta[; (t, \beta) \in \Omega\} \neq \emptyset$, for each $\gamma \in]0, \beta[$.

Taking a strictly descending sequence $(\gamma_n; n \geq 0)$ in $]0, \beta[$ with $\gamma_n \to 0+$, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(t_n; n \geq 0)$ in R^0_+, so as

$(\forall n)$: t_n is an element of $L(\gamma_n)$; or, equivalently (by the very definition above) $\beta - \gamma_n < t_n < \beta$, and $(t_n, \beta) \in \Omega$.

By the first half of this last relation, $t_n \to \beta-$, and this, along with an auxiliary fact, tells us that there exists a subsequence $(t_n^* := t_{i(n)}; n \geq 0)$ of $(t_n; n \geq 0)$, with

$(t_n^*; n \geq 0)$ is strictly ascending, and $t_n^* \to \beta-$.

On the other hand, by the second half of our underlying relation, $[(t_n^*, \beta) \in \Omega, \forall n]$. Putting these together yields a contradiction with the asymptotic left separable property of Ω. Hence, Ω is geometric left separable, as claimed.

In the following, some basic examples of (upper diagonal) Matkowski admissible and geometric Meir–Keeler relations are given. The general scheme of constructing these may be described along the lines below.

Let $R(\pm\infty) := R \cup \{-\infty, \infty\}$ stand for the set of all *extended real numbers*. For each relation Ω over R_+^0, let us associate a function $\xi : R_+^0 \times R_+^0 \to R(\pm\infty)$, as

$\xi(t, s) = 0$, if $(t, s) \in \Omega$; $\xi(t, s) = -\infty$, if $(t, s) \notin \Omega$.

It will be referred to as the *function* generated by Ω; clearly,

$(t, s) \in \Omega$ iff $\xi(t, s) \geq 0$.

Conversely, given a function $\xi : R_+^0 \times R_+^0 \to R(\pm\infty)$, we may associate it a relation Ω over R_+^0 as

$\Omega = \{(t, s) \in R_+^0 \times R_+^0; \xi(t, s) \geq 0\}$ (in short, $\Omega = [\xi \geq 0]$),
referred to as *the positive section* of ξ.

Note that the correspondence between the function ξ and its associated relation $[\xi \geq 0]$ is not injective, because, for the function $\eta := \lambda\xi$ (where $\lambda > 0$), its associated relation $[\eta \geq 0]$ is identical with the relation $[\xi \geq 0]$ attached to ξ.

Now, call the function $\xi : R_+^0 \times R_+^0 \to R(\pm\infty)$, *upper diagonal* provided

(u-diag) $\xi(t, s) \geq 0$ implies $t < s$.

All subsequent constructions are being considered within this setting. The former of these concerns the sequential condition for upper diagonal functions:

(M-ad) ξ in *Matkowski admissible*:
$(t_n; n \geq 0)$ in R_+^0 and $(\xi(t_{n+1}, t_n) \geq 0, \forall n)$ imply $\lim_n t_n = 0$.

In a strong connection with this, the second group of such objects involves the geometric properties for upper diagonal functions:

(g-mk) ξ is *geometric Meir–Keeler*:
$\forall \varepsilon > 0, \exists \delta > 0$: $\xi(t, s) \geq 0, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \leq \varepsilon$;
(g-bila-s) ξ is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t, s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow \xi(t, s) < 0$;
(g-left-s) ξ is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow \xi(t, \beta) < 0$;

as well as the asymptotic versions of these. The relationships between the geometric Meir–Keeler condition and the Matkowski one attached to upper diagonal functions are nothing else than a simple translation of the previous ones involving upper diagonal relations; this is also valid for the relationships between the geometric and asymptotic concepts attached to upper diagonal functions.

Summing up, the duality principles below are holding:

(DP-1) any concept (like the ones above) about (upper diagonal) relations over R^0_+ may be written as a concept about (upper diagonal) functions in the class $\mathscr{F}(R^0_+ \times R^0_+, R(\pm\infty))$.

(DP-2) any concept (like the ones above) about (upper diagonal) functions in the class $\mathscr{F}(R^0_+ \times R^0_+, R(\pm\infty))$ may be written as a concept about (upper diagonal) relations over R^0_+.

For the rest of our exposition, it will be convenient working with relations over R^0_+ and not with functions in $\mathscr{F}(R^0_+ \times R^0_+, R(\pm\infty))$; this, however, is nothing but a methodology question.

We may now pass to the description of some basic objects in this area.

Part-Case (I) Let $\mathscr{F}(re)(R^0_+, R)$ be the family of all $\varphi \in \mathscr{F}(R^0_+, R)$, with

φ is *regressive*: $\varphi(t) < t$, for all $t \in R^0_+$.

For each $\varphi \in \mathscr{F}(re)(R^0_+, R)$, let us introduce the geometric property

(MK-a) φ is *Meir–Keeler admissible*:
$\forall \varepsilon > 0, \exists \delta > 0$, such that $(\varepsilon < s < \varepsilon + \delta)$ implies $\varphi(s) \le \varepsilon$;

suggested—essentially—by the classical developments in Meir and Keeler [28]. To get concrete circumstances under which it holds, let us consider the triple of sequential conditions

(M-a) φ is *Matkowski admissible*:
for each $(t_n; n \ge 0)$ in R^0_+ with $(t_{n+1} \le \varphi(t_n), \forall n)$, we have $\lim_n t_n = 0$.

(Nd-a) φ is *nondiagonal admissible*:
there are no strictly descending sequences $(t_n; n \ge 0)$ in R^0_+
and no elements ε in R^0_+ with $t_n \to \varepsilon+, \varphi(t_n) \to \varepsilon+$.

(The former convention is taken from Matkowski [25, 26], but the latter seems to be new). Here, for the sequence $(r_n; n \ge 0)$ in R and the point $r \in R$, we denoted

$r_n \to r+$ if $r_n \to r$ and $r_n > r$, for all $n \ge 0$.

Theorem 4 *For each $\varphi \in \mathscr{F}(re)(R^0_+, R)$, we have in (ZF-AC+DC)*

Meir–Keeler admissible $\Longrightarrow$ Matkowski admissible $\Longrightarrow$ nondiagonal admissible $\Longrightarrow$ Meir–Keeler admissible.

Hence, the Meir–Keeler admissible, Matkowski admissible, and nondiagonal admissible properties are equivalent over $\mathscr{F}(re)(R^0_+, R)$.

Proof

(i) Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Meir–Keeler admissible; we claim that it is Matkowski admissible. Let $(s_n; n \geq 0)$ be a sequence in R^0_+ with the property $(s_{n+1} \leq \varphi(s_n); n \geq 0)$. Clearly, (s_n) is strictly descending in R^0_+; hence, $\sigma := \lim_n s_n$ exists in R_+. Suppose by contradiction that $\sigma > 0$, and let $\rho > 0$ be given by the Meir–Keeler admissible property of φ; that is,

$$\sigma < t < \sigma + \rho \text{ implies } \varphi(t) \leq \sigma.$$

By the above convergence relations, there exists some rank $n(\rho)$, such that

$$n \geq n(\rho) \text{ implies } \sigma < s_n < \sigma + \rho.$$

But then, we get (for the same ranks)

$$\sigma < s_{n+1} \leq \varphi(s_n) < s_n < \sigma + \rho.$$

The obtained relations are in contradiction with the Meir–Keeler admissible property. Hence, $\sigma = 0$, and the assertion follows.

(ii) Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Matkowski admissible; we assert that φ is nondiagonal admissible. For, if φ is not endowed with such a property, there must be a strictly descending sequence $(t_n; n \geq 0)$ in R^0_+ and an $\varepsilon > 0$, such that

$$t_n \to \varepsilon+ \text{ and } \varphi(t_n) \to \varepsilon+, \text{ as } n \to \infty.$$

Put $i(0) = 0$. As $\varepsilon < \varphi(t_{i(0)})$ and $t_n \to \varepsilon+$, we have that

$$A(i(0)) := \{n > i(0); t_n < \varphi(t_{i(0)})\} \text{ is not empty;}$$
$$\text{hence, } i(1) := \min(A(i(0))) \text{ is an element of it, and } t_{i(1)} < \varphi(t_{i(0)}).$$

Likewise, as $\varepsilon < \varphi(t_{i(1)})$ and $t_n \to \varepsilon+$, we have that

$$A(i(1)) := \{n > i(1); t_n < \varphi(t_{i(1)})\} \text{ is not empty;}$$
$$\text{hence, } i(2) := \min(A(i(1))) \text{ is an element of it, and } t_{i(2)} < \varphi(t_{i(1)}).$$

This procedure may continue indefinitely and yields (without any choice technique) a strictly ascending rank sequence $(i(n); n \geq 0)$ for which the attached subsequence $(s_n := t_{i(n)}; n \geq 0)$ of $(t_n; n \geq 0)$ fulfills

$$s_{n+1} < \varphi(s_n)(< s_n), \text{ for all } n.$$

On the other hand, by this very subsequence property,

$$(s_n > \varepsilon, \forall n) \text{ and } \lim_n s_n = \lim_n t_n = \varepsilon.$$

The obtained relations are in contradiction with the Matkowski property of φ; hence, the working condition cannot be true, and we are done.

(iii) Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is nondiagonal admissible; we show that φ is Meir–Keeler admissible. For, otherwise, one has (for some $\varepsilon > 0$)

$$H(\delta) := \{t \in R^0_+; \varepsilon < t < \varepsilon + \delta, \varphi(t) > \varepsilon\} \text{ is not empty, for each } \delta > 0.$$

Taking a strictly descending sequence $(\delta_n; n \geq 0)$ in R^0_+ with $\delta_n \to 0$, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(t_n; n \geq 0)$ in R^0_+, so as

$(t_n \in H(\delta_n), \forall n)$; or, equivalently (by definition, and φ=regressive)
$(\varepsilon < \varphi(t_n) < t_n < \varepsilon + \delta_n, \forall n)$; hence, $\varphi(t_n) \to \varepsilon+$ and $t_n \to \varepsilon+$.

By a previous result, there exists a subsequence $(r_n := t_{i(n)})$ of (t_n), such that

(r_n) is strictly descending and $r_n \to \varepsilon+$; hence, necessarily, $\varphi(r_n) \to \varepsilon+$.

But, this last relation is in contradiction with the nondiagonal admissible property of our function. Hence, the assertion follows, and we are done.

A basic particular case of these developments may be described as below. Let $\mathscr{F}(re, in)(R^0_+, R)$ stand for the class of all $\varphi \in \mathscr{F}(re)(R^0_+, R)$, with

φ is increasing on R^0_+ $(0 < t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2))$.

Clearly, for each $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$, its Matkowski admissible property reads

(M-adm) $(\forall t > 0)$: $\lim_n \varphi^n(t) = 0$, as long as $(\varphi^n(t); n \geq 0)$ exists.

Here, as usual, we denoted for each $t > 0$

$\varphi^0(t) = t, \varphi^1(t) = \varphi(t), \ldots, \varphi^{n+1}(t) = \varphi(\varphi^n(t)), n \geq 1$.

Note that such a construction may be non-effective; for example,

$\varphi^2(t) = \varphi(\varphi(t))$ is undefined whenever $\varphi(t) \leq 0$.

Remark 2 By a preceding result, we have, in (ZF-AC+DC),

(for each $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$):
Meir-Keeler admissible is equivalent with Matkowski admissible.

However, for technical reasons, we will provide an argument for the second half of it (see also Jachymski [18]).

Assume that $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$ is Matkowski admissible; we want to establish that it is Meir–Keeler admissible. If this property fails, then (for some $\gamma > 0$)

$\forall \beta > 0, \exists t \in]\gamma, \gamma + \beta[$, such that $\varphi(t) > \gamma$.

Combining with the increasing property of φ, one gets

$(\forall t > \gamma)$: $\varphi(t) > \gamma$ [whence (by induction): $\varphi^n(t) > \gamma$, for each n].

Fixing some $t > \gamma$ and passing to limit as $n \to \infty$, one derives $0 \geq \gamma$, a contradiction, hence the claim.

Some important examples of such functions may be given along the lines below. For any $\varphi \in \mathscr{F}(re)(R^0_+, R)$ and any $s \in R^0_+$, put

$\Lambda^+\varphi(s) = \inf_{0<\varepsilon<s} \Phi(s+)(\varepsilon)$; where $\Phi(s+)(\varepsilon) = \sup \varphi(]s, s+\varepsilon[)$
$\Lambda^\pm\varphi(s) = \inf_{0<\varepsilon<s} \Phi(s\pm)(\varepsilon)$; where $\Phi(s\pm)(\varepsilon) = \sup \varphi(]s-\varepsilon, s+\varepsilon[)$.

From the regressive property of φ, these limit quantities fulfill

$(-\infty \leq)\ \Lambda^+\varphi(s) \leq \Lambda^\pm\varphi(s) \leq s,\ \forall s \in R^0_+,$

but the case of such limits having infinite values cannot be avoided.

The following auxiliary fact will be useful.

Proposition 10 *For $\varphi \in \mathscr{F}(re)(R^0_+, R)$ and $s \in R^0_+$, we have in (ZF-AC+DC)*

(41-1) $\limsup_n(\varphi(t_n)) \leq \Lambda^+\varphi(s)$, *for each sequence (t_n) in R^0_+ with $t_n \to s+$;*
(41-2) $\limsup_n(\varphi(t_n)) \leq \Lambda^\pm\varphi(s)$, *for each sequence (t_n) in R^0_+ with $t_n \to s$;*
(41-3) there exists a strictly descending sequence (r_n) in R^0_+ with $r_n \to s+$ and $\varphi(r_n) \to \Lambda^+\varphi(s)$;
(41-4) there exists a sequence (r_n) in R^0_+ with $r_n \to s$ and $\varphi(r_n) \to \Lambda^\pm\varphi(s)$.

Proof

(i) Given $\varepsilon \in]0, s[$, there exists a rank $p(\varepsilon) \geq 0$ such that $s < t_n < s+\varepsilon$, for all $n \geq p(\varepsilon)$; hence,

$\limsup_n(\varphi(t_n)) \leq \sup\{\varphi(t_n); n \geq p(\varepsilon)\} \leq \Phi(s+)(\varepsilon).$

It suffices taking the infimum over ε in this relation to get the desired fact.

(ii) The argument is very similar with the preceding one; so, it will be omitted.

(iii) Denote for simplicity

$\alpha = \Lambda^+\varphi(s)$; hence, $\alpha = \inf_{0<\varepsilon<s} \Phi(s+)(\varepsilon)$, and $-\infty \leq \alpha \leq s$.

Then, define $(\beta_n := \Phi(s+)(2^{-n-1}s); n \geq 0)$; it is a sequence in R, because

$(\forall n)$: $-\infty < \varphi(t) < t < s + 2^{-n-1}s$, for each $t \in]s, s+2^{-n-1}s[$.

Moreover, we have by definition that

(β_n) is descending $(\beta_n \geq \alpha,\ \forall n)$, $\inf_n \beta_n = \alpha$; hence, $\lim_n \beta_n = \alpha$.

Furthermore, denote

$(\gamma_n = \beta_n - 3^{-n}; n \geq 0)$; hence, $\gamma_n < \beta_n, \forall n$; $\lim_n \gamma_n = \lim_n \beta_n = \alpha$.

From the supremum definition,

$H_n := \{t \in]s, s+2^{-n-1}s[; \varphi(t) > \gamma_n\} \neq \emptyset$, for all $n \geq 0$.

This, along with Denumerable Axiom of Choice (deductible in (ZF-AC+DC)), yields a sequence (t_n) with

$(\forall n)$: $t_n \in H_n$; that is, $t_n \in]s, s+2^{-n-1}s[$, $\varphi(t_n) > \gamma_n$,
as well as (by definition), $\varphi(t_n) \leq \beta_n$;

so, putting these together, $[t_n \to s+$ and $\varphi(t_n) \to \alpha]$. By a previous result, there exists a subsequence $(r_n := t_{i(n)})$ of (t_n), with

(r_n) is strictly descending and $r_n \to \varepsilon+$; hence, $\varphi(r_n) \to \alpha$.

In other words, the obtained sequence $(r_n; n \geq 0)$ has all desired properties.

(iv) The argument is very similar with the preceding one; so, it will be omitted.

Call $\varphi \in \mathscr{F}(re)(R^0_+, R)$, *Boyd-Wong admissible* [6] if

(bw-adm) $\Lambda^+\varphi(s) < s$, for all $s > 0$.

In particular, $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible provided it is *upper semicontinuous at the right* on R^0_+:

$\Lambda^+\varphi(s) \leq \varphi(s)$, for each $s \in R^0_+$.

This, e.g., is fulfilled when φ is *continuous at the right* on R^0_+; for, in such a case,

$\Lambda^+\varphi(s) = \varphi(s)$, for each $s \in R^0_+$.

On the other hand, $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible when

φ is *strongly Boyd–Wong admissible*: $\Lambda^\pm\varphi(s) < s$, $\forall s \in R^0_+$.

A basic particular case to be discussed concerns the class $\mathscr{F}(re, in)(R^0_+, R)$. For each $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$, denote

$\varphi(s+0) := \lim_{t\to s+} \varphi(t)$, $s \in R^0_+$ (the *right limit* of φ at s).

Clearly, this limit always exists; moreover, by the involved definitions,

(for each $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$): $\Lambda^+\varphi(s) = \varphi(s+0)$, $s \in R^0_+$;

and this yields the useful characterization formula:

(for each $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$):
φ is Boyd–Wong admissible iff $\varphi(s+0) < s$, for all $s > 0$.

A related functional property may be introduced as below. Let $\varphi \in \mathscr{F}(re)(R^0_+, R)$ be a function; we call it *Geraghty admissible* [15], provided

$(t_n; n \geq 0)$= sequence in R^0_+ and $\varphi(t_n)/t_n \to 1$ imply $t_n \to 0$.

Proposition 11 *The following assertions hold over* $\mathscr{F}(re)(R^0_+, R)$*:*

(42-1) each Geraghty admissible function is Boyd–Wong admissible;
(42-2) there exist Boyd–Wong admissible functions that are not Geraghty admissible;
(42-3) each Boyd–Wong admissible function is Meir–Keeler admissible; or, equivalently, Matkowski admissible;
(42-4) there are Meir–Keeler (or, equivalently, Matkowski) admissible functions that are not Boyd–Wong admissible.

Proof

(i) Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Geraghty admissible; we have to establish that it is Boyd–Wong admissible. Suppose not: there exists some $s \in R^0_+$ with $\Lambda^+\varphi(s) = s$. Combining with a preceding fact, there must be a strictly descending sequence $(r_n; n \geq 0)$ in R^0_+, with

$$r_n \to s+ \text{ and } \varphi(r_n) \to s; \text{ whence } \varphi(r_n)/r_n \to 1;$$

i.e., φ is not Geraghty admissible. The obtained contradiction proves our claim.

(ii) Let us consider the function

$$\varphi \in \mathscr{F}(re)(R^0_+, R)\colon \varphi(t) = t(1 - e^{-t}), t > 0.$$

Clearly, φ is continuous, hence, Boyd–Wong admissible. On the other hand, taking the sequence $(t_n = n + 1; n \geq 0)$ in R^0_+, we have

$$\varphi(t_n)/t_n \to 1, \text{ and } t_n \to \infty; \text{ hence, } \varphi \text{ is not Geraghty admissible.}$$

(iii) (cf. Boyd and Wong [6]) Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible; we have to establish that it is Meir–Keeler (or, equivalently, Matkowski) admissible. Fix $\gamma > 0$; hence, $\Lambda^+\varphi(\gamma) < \gamma$. By definition, there exists $\beta > 0$ with

$$\gamma < t < \gamma + \beta \text{ implies } \varphi(t) < \gamma, \text{ proving that } \varphi \text{ is Meir–Keeler admissible.}$$

(iv) (see also Turinici [42]) Let us consider the function $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$, according to (for some $r > 0$)

$$(\varphi(t) = t/2, \text{ if } t \leq r), (\varphi(t) = r, \text{ if } t > r).$$

Clearly, φ is Matkowski admissible; or, equivalently, Meir–Keeler admissible. On the other hand,

$$(\Lambda^+\varphi(r) =)\varphi(r + 0) = r; \text{ whence, } \varphi \text{ is not Boyd–Wong admissible;}$$

and this proves our claim.

Having these precise, take a function $\chi \in \mathscr{F}(re)(R^0_+, R)$ and define the associated relation $\Omega := \Omega[\chi]$ over R^0_+, as

$$(t, s \in R^0_+)\colon (t, s) \in \Omega \text{ iff } t \leq \chi(s).$$

Clearly, Ω is upper diagonal. In fact, let $t, s \in R^0_+$ be such that $t\Omega s$; i.e., $t \leq \chi(s)$. As χ is regressive, one has $\chi(s) < s$, and this yields $t < s$, whence the conclusion follows. Further properties of this relation are deductible from

Proposition 12 *Let the function $\chi \in \mathscr{F}(re)(R^0_+, R)$ be given and $\Omega := \Omega[\chi]$ stand for the associated upper diagonal relation over R^0_+. Then,*

(43-1) Ω is first variable decreasing and geometric/asymptotic left separable;

(43-2) Ω is geometric/asymptotic Meir–Keeler when the starting function χ is Meir–Keeler admissible (or, equivalently, Matkowski admissible);

(43-3) Ω is geometric/asymptotic bilateral separable (hence, necessarily, geometric/asymptotic Meir–Keeler) when χ is strongly Boyd–Wong admissible.

Proof

(i) The first half is clear, and the second half is a direct consequence of

$$\Omega^{-1}(s) =]0, \chi(s)] \text{ (and } \chi(s) < s\text{), for each } s \in R_+^0.$$

(ii) Let $\varepsilon > 0$ be given and $\delta > 0$ be the number associated with it, via Meir–Keeler admissible property for χ. Given $t, s \in R_+^0$ with $t\Omega s$, $\varepsilon < s < \varepsilon + \delta$, we have $[t \le \chi(s), \varepsilon < s < \varepsilon + \delta]$. This, according to the underlying property of χ, gives $\chi(s) \le \varepsilon$ [hence, $t \le \varepsilon$]; wherefrom, Ω has the geometric Meir–Keeler property.

(iii) Suppose, by absurd, that Ω is not asymptotic bilateral separable:

there are sequences $(t_n; n \ge 0)$ and $(s_n; n \ge 0)$ in R_+^0 and elements $\beta \in R_+^0$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \beta, s_n \to \beta)$.

By the definition of our relation,

$$(t_n \le \chi(s_n), \forall n), \text{ and } t_n \to \beta, s_n \to \beta.$$

Passing to lim sup as $n \to \infty$ yields (by a previous result) $\beta \le \Lambda^{\pm}\chi(\beta) < \beta$, a contradiction, and this proves our assertion.

Part-Case (II) Let (ψ, φ) be a couple of functions over $\mathscr{F}(R_+^0, R)$, with

(norm) (ψ, φ) is *normal*:
ψ is increasing and φ is *strictly positive* $[\varphi(t) > 0, \forall t > 0]$.

(This concept may be related to the one introduced by Rhoades [36]; see also Dutta and Choudhury [14]). Then, define the relation $\Omega = \Omega[\psi, \varphi]$ in $\exp(R_+^0 \times R_+^0)$, as

$$(t, s) \in \Omega \text{ iff } \psi(t) \le \psi(s) - \varphi(s).$$

We claim that, necessarily, Ω is upper diagonal. In fact, let $t, s \in R_+^0$ be such that

$$(t, s) \in \Omega; \text{ i.e., } \psi(t) \le \psi(s) - \varphi(s).$$

By the strict positivity of φ, one gets $\psi(t) < \psi(s)$; and this, along with the increasing property of ψ, shows that $t < s$; whence the conclusion follows.

Further properties of this relation are available under certain supplementary conditions about the normal couple (ψ, φ), like below:

(as-pos) φ is *asymptotic positive*:
for each strictly descending sequence $(t_n; n \ge 0)$ in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon+$, we must have $\limsup_n(\varphi(t_n)) > 0$.

(bd-osc) (ψ, φ) is *limit-bounded oscillating*:
for each sequence $(t_n; n \ge 0)$ in R_+^0 and each $\beta > 0$ with $t_n \to \beta$, we have $\limsup_n(\varphi(t_n)) > \psi(\beta + 0) - \psi(\beta - 0)$.

(bd-le-osc) (ψ, φ) is *bounded left oscillating*:
for each $\beta > 0$, we have $\varphi(\beta) > \psi(\beta) - \psi(\beta - 0)$.

The following inclusion is clear

(for each normal couple (ψ, φ)):
(ψ, φ) is limit-bounded oscillating implies φ is asymptotic positive.

On the other hand, sufficient conditions under which the asymptotic property holds are obtainable (under the same normality setting) via

(φ=increasing or continuous) implies φ=asymptotic positive.

In fact, let the strictly descending sequence $(t_n; n \geq 0)$ in R_+^0 and the number $\varepsilon > 0$ be such that $t_n \to \varepsilon+$. When φ=increasing, we have (by normality)

$\varphi(t_n) \geq \varphi(\varepsilon) > 0, \forall n$; whence $\limsup_n(\varphi(t_n)) \geq \varphi(\varepsilon) > 0$.

On the other hand, when φ=continuous, the same normality condition yields

$\limsup_n(\varphi(t_n)) = \lim_n(\varphi(t_n)) = \varphi(\varepsilon) > 0$, and the conclusion follows.

Proposition 13 *Let (ψ, φ) be a normal couple of functions over $\mathscr{F}(R_+^0, R)$ and $\Omega := \Omega[\psi, \varphi]$ be the associated upper diagonal relation. Then,*

(44-1) if φ is asymptotic positive, then the associated relation Ω is asymptotic/geometric Meir-Keeler;
(44-2) if (ψ, φ) is limit-bounded oscillating, then Ω is asymptotic/geometric bilateral separable (hence, asymptotic/geometric Meir–Keeler as well);
(44-3) if (ψ, φ) is bounded left oscillating, then the associated relation Ω is asymptotic/geometric left separable.

Proof

(i) Suppose by contradiction that Ω is not asymptotic Meir–Keeler:

there exist strictly descending sequences (t_n) and (s_n) in R_+^0
and elements ε in R_+^0 with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

By the former of these, we get

$(0 <)\varphi(s_n) \leq \psi(s_n) - \psi(t_n), \forall n.$

Passing to limit as $n \to \infty$, and noting that $\lim_n \psi(s_n) = \lim_n \psi(t_n) = \psi(\varepsilon + 0)$, one gets $\lim_n \varphi(t_n) = 0$, in contradiction with the asymptotic positivity of φ. So, necessarily, Ω has the asymptotic Meir–Keeler property, as claimed.

(ii) Suppose by contradiction that Ω is not asymptotic bilateral separable; i.e.,

there exist sequences (t_n) and (s_n) in R_+^0 and elements β in R_+^0,
with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \beta, s_n \to \beta)$.

By the former of these, we get

$(0 <)\varphi(s_n) \leq \psi(s_n) - \psi(t_n), \forall n.$

Passing to lim sup as $n \to \infty$ yields $\limsup_n \varphi(s_n) \leq \psi(\beta + 0) - \psi(\beta - 0)$, in contradiction with (ψ, φ) being limit-bounded oscillating. This tells us that Ω is asymptotic bilateral separable, as claimed.

(iii) Suppose by contradiction that Ω is not asymptotic left separable:

there exist strictly ascending sequences (t_n) in R_+^0
and elements β in R_+^0, with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta-)$.

By the former of these, we get

$$\psi(t_n) \leq \psi(\beta) - \varphi(\beta), \forall n.$$

Passing to lim sup as $n \to \infty$ yields (as ψ=increasing)

$$\psi(\beta - 0) \leq \psi(\beta) - \varphi(\beta); \text{ that is, } \varphi(\beta) \leq \psi(\beta) - \psi(\beta - 0),$$

in contradiction with (ψ, φ) being bounded left oscillating. This tells us that Ω is asymptotic left separable, as claimed.

In the following, some basic (and useful) particular choices for the couple (ψ, φ) above are to be discussed.

Part-Case (II-a) The construction in the preceding step (involving a certain $\chi \in \mathscr{F}(re)(R_+^0, R)$) is nothing else than a particular case of this one, corresponding to

$$\psi(t) = t, \varphi(t) = t - \chi(t), t \in R_+^0.$$

Part-Case (II-b) Let $\lambda : R_+^0 \to]1, \infty[$ and $\mu : R_+^0 \to]0, 1[$ be a couple of functions, with λ=increasing. Define a relation $\Omega := \Omega[[\lambda, \mu]]$ over R_+^0 as

$$t\Omega s \text{ iff } \lambda(t) \leq [\lambda(s)]^{\mu(s)}.$$

This will be referred to as the *Jleli–Samet relation* attached to $\lambda(.)$ and $\mu(.)$. (The proposed convention comes from the developments in Jleli and Samet [21], corresponding to $\mu(.)$=constant). By a direct calculation, it is evident that

$$t\Omega s \text{ iff } t\Omega[\psi, \varphi]s; \text{ where } \psi(t) = \log[\log(\lambda(t))], \varphi(t) = -\log(\mu(t)), t > 0.$$

Hence, this construction is entirely reducible to the standard one in this series.

Part-Case (II-c) Let the couple (ψ, α) over $\mathscr{F}(R_+^0, R)$ be admissible; i.e.,

(admi-1) $\psi(.)$ is increasing, right continuous, and strictly positive.
(admi-2) $-\alpha(.)$ is right lsc on R_+^0, and $\gamma := \psi - \alpha$ is strictly positive.

Proposition 14 *Let the functions (ψ, α) be as before. Then,*

(45-1) *The couple (ψ, γ) (where $\gamma = \psi - \alpha$) is a normal couple over $\mathscr{F}(R_+^0, R)$, with γ=asymptotic positive.*
(45-2) *The associated to (ψ, γ) relation*

$$t\Omega s \textit{ iff } \psi(t) \leq \psi(s) - \gamma(s) \textit{ (that is, } \psi(t) \leq \alpha(s))$$

is upper diagonal and asymptotic/geometric Meir–Keeler.

Proof

(i) By definition, ψ is increasing and γ is strictly positive.
(ii) Suppose by contradiction that $\gamma(.)$ is not asymptotic positive: there exist $\varepsilon > 0$ and a strictly descending sequence (t_n) in R_+^0, with

$t_n \to \varepsilon+$ and $\limsup_n(\gamma(t_n)) = 0$; whence, $\lim_n(\gamma(t_n)) = 0$.

The last relation gives

$\lim_n(-\alpha(t_n)) = -\psi(\varepsilon)$ (as ψ is right continuous).

Combining with $-\alpha(.)$ being right lsc on R_+^0 yields (by this limit process)

$-\alpha(\varepsilon) \le -\psi(\varepsilon)$; or, equivalently, $\gamma(\varepsilon) \le 0$,

in contradiction with the strict positivity of γ. Hence, our working assumption is not acceptable, and the claim follows.
(iii) Evident, by our previous facts.

Part-Case (II-d) Let $\psi \in \mathscr{F}(R_+^0, R)$ and $\Delta \in \mathscr{F}(R)$ be a couple of functions. The following regularity condition involving these objects will be considered here

(BV-c) (ψ, Δ) is a *Bari–Vetro couple*:
ψ is increasing and Δ is regressive ($\Delta(r) < r$, for all $r \in R$).

In this case, by definition,

$\varphi(t) := \psi(t) - \Delta(\psi(t)) > 0$, for all $t > 0$,

so that (ψ, φ) is a normal couple of functions over $\mathscr{F}(R_+^0, R)$. Let $\Omega := \Omega[\psi, \Delta]$ be the (associated) *Bari–Vetro relation* over R_+^0, introduced as

$t\Omega s$ iff $\psi(t) \le \Delta(\psi(s))$.

(This convention is related to the developments in Di Bari and Vetro [13]). From (BV-c), Ω is an upper diagonal relation over R_+^0. It is natural then to ask under which extra assumptions about our data we have that Ω is an asymptotic Meir–Keeler relation. The simplest one may be written as

(a-reg) Δ is *asymptotic regressive*:
for each descending sequence (r_n) in R and each $\alpha \in R$ with $r_n \to \alpha$,
we have that $\liminf_n \Delta(r_n) < \alpha$.

Note that, by the non-strict character of the descending property above, one has

Δ is asymptotic regressive implies Δ is regressive.

Proposition 15 *Let the functions ($\psi \in \mathscr{F}(R_+^0, R)$, $\Delta \in \mathscr{F}(R)$) be such that*

(ψ, Δ) is an asymptotic Bari–Vetro couple; i.e.,
ψ is increasing and Δ is asymptotic regressive.

Then,
(46-1) *the above defined function φ is asymptotic positive;*

(46-2) *the associated relation Ω is upper diagonal and asymptotic Meir–Keeler (hence, geometric Meir–Keeler).*

Proof

(i) Let the strictly descending sequence $(t_n; n \geq 0)$ in R_+^0 and the number $\varepsilon > 0$ be such that $t_n \to \varepsilon+$; we must derive that $\limsup_n(\varphi(t_n)) > 0$. Denote

$$(r_n = \psi(t_n), n \geq 0); \alpha = \psi(\varepsilon + 0).$$

By the imposed conditions (and ψ=increasing)

(r_n) is descending and $r_n \to \alpha$ as $n \to \infty$.

In this case,

$$\limsup_n \varphi(t_n) = \limsup_n[r_n - \Delta(r_n)] = \alpha - \liminf_n \Delta(r_n) > 0,$$

hence the claim.

(ii) The assertion follows at once from (ψ, φ) being a normal couple with (φ=asymptotic positive) and a previous remark involving these objects.

In particular, when ψ and Δ are continuous, our theorem reduces to the one in Jachymski [20].

5 Statement of the Problem

Let X be a nonempty set and $d : X \times X \to R_+$ be a *metric* over X; then, (X, d) will be called a *metric space*. Furthermore, let $(\leq)$ be a *quasi-order* on X; the triple $(X, d, \leq)$ will be referred to as a *quasi-ordered metric space*. Let $(<)$ stand for the relation

$x < y$ iff $x \leq y$ and $x \neq y$ [clearly, $(<)$ is irreflexive].

Finally, call the subset Y of X $(\leq)$-*asingleton* if $[y_1, y_2 \in Y, y_1 \leq y_2]$ imply $y_1 = y_2$, and $(\leq)$-*singleton* if, in addition, Y is nonempty.

(A) Take some $T \in \mathscr{F}(X)$, and assume in the following that

(s-pro) T is *semi-progressive* $(X(T, \leq) := \{x \in X; x \leq Tx\} \neq \emptyset)$;
(incr) T is *increasing* ($x \leq y$ implies $Tx \leq Ty$).

We are interested in establishing sufficient conditions under which $Fix(T) \neq \emptyset$. The basic directions for getting this are described in our list below, comparable with the one proposed by Turinici [46]:

(pic-0) We say that T is *fix-*$(\leq)$*-asingleton*, when Fix(T) is a $(\leq)$-asingleton; likewise, T is called *fix-*$(\leq)$*-singleton* when Fix(T) is a $(\leq)$-singleton.

(pic-1) We say that $x \in X(T, \leq)$ is a *Picard point* (modulo $(d, \leq; T)$) if the iterative sequence $(T^n x; n \geq 0)$ is d-Cauchy; when this property holds for all $x \in X(T, \leq)$, then T is called a *Picard operator* (modulo $(d, \leq)$).

(pic-2) We say that $x \in X(T, \leq)$ is a *strong Picard point* (modulo $(d, \leq; T)$) if the iterative sequence $(T^n x; n \geq 0)$ is d-convergent and $\lim_n(T^n x) \in \mathrm{Fix}(T)$; when this property holds for all $x \in X(T, \leq)$, then T is called a *strong Picard operator* (modulo $(d, \leq)$).

(pic-3) We say that $x \in X(T, \leq)$ is a *Bellman Picard point* (modulo $(d, \leq; T)$) if the iterative sequence $(T^n x; n \geq 0)$ is d-convergent, $\lim_n(T^n x) \in \mathrm{Fix}(T)$, and $T^n x \leq \lim_n(T^n x)$, $\forall n$; when this property holds for all $x \in X(T, \leq)$, then T is called a *Bellman Picard operator* (modulo $(d, \leq)$).

The regularity conditions for such properties are being founded on *ascending orbital full* concepts (in short, (a-o-f)-concepts). Call the sequence $(z_n; n \geq 0)$ in X

ascending, if $z_i \leq z_j$ whenever $i \leq j$;
T-*orbital*, if $(z_n = T^n x; n \geq 0)$, for some $x \in X$; and
full, when $n \mapsto z_n$ is injective ($i \neq j$ implies $z_i \neq z_j$);

the intersection of these notions yields the precise ones.

(reg-1) Call X *(a-o-f,d)-complete*, provided (for each (a-o-f)-sequence) d-Cauchy $\Longrightarrow$ d-convergent.

(reg-2) We say that T is *(a-o-f,d)-continuous*, if ((z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$) imply $Tz_n \xrightarrow{d} Tz$.

(reg-3) Call $(\leq)$, *(a-o-f,d)-selfclosed* when ((z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$) imply ($z_n \leq z$, for each n).

When some of the ascending, orbital, and full properties are ignored, these conventions may be written in terms of remaining concepts; we do not give details.

(B) As an essential completion of these facts, we have to discuss the contractive type conditions to be used. Some preliminaries are needed. Let us introduce the mappings: for each $x, y \in X$,

$$P_0(x, y) = d(Tx, Ty),\ L_1(x, y) = \min\{d(x, y), d(Tx, Ty)\},$$
$$L(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, y), d(Tx, Ty)\},$$
$$M_1(x, y) = \max\{d(x, Tx), d(y, Ty)\},\ M(x, y) = \mathrm{diam}\{x, Tx, y, Ty\}.$$

Here, for each (nonempty) subset Z in X, we put

$$\mathrm{diam}(Z) = \sup\{d(x, y); x, y \in Z\} \text{ (the } \textit{diameter} \text{ of } Z).$$

Let P be a generic map in $\mathscr{F}(X \times X, R)$. For an easy reference, we give the list of *normality conditions* to be optionally fulfilled by P.

(I) The first group of conditions—stated with the aid of some other mapping K in $\mathscr{F}(X \times X, R)$—is of *positive boundedness* type:

(posi) (P, K) is *positive*: $x \leq y$, $K(x, y) > 0$ implies $P(x, y) > 0$.
(bd) (P, K) is *bounded*: $x \leq y$ implies $P(x, y) \leq K(x, y)$.
(fix-bd) (P, K) is *fix bounded*: $(x, y \in \mathrm{Fix}(T), x \leq y)$ imply $P(x, y) \leq K(x, y)$.

Technically speaking, this has the role of handing the contractive condition in different stages of the proof.

(II) The second group of conditions is of *telescopic-boundedness* type:

(n-tele) P is *telescopic null*: $x \le Tx$ implies $P(x, Tx) = 0$.
(t-bd) P is *telescopic bounded*: $x \le Tx$ implies $P(x, Tx) \le M_1(x, Tx)$.

This group may be viewed as a complement of the preceding one. It has the role of getting the d-Cauchy property of a d-asymptotic iterative sequence and/or making applicable the contractive property for the limit of a d-convergent iterative sequence.

(III) The third group of conditions is of *orbitally small* and *orbitally Cauchy* type. Remember that the sequence (x_n) is d-*asymptotic*, provided

$\lim_n d(x_n, x_{n+1}) = 0$; or, equivalently, $\lim_n d(x_n, x_{n+i}) = 0, \forall i \ge 1$.

Given the d-asymptotic sequence (x_n) in X and the number $\gamma > 0$, denote

$n(\gamma)$=the minimal index h with $n \ge h \Longrightarrow (d(x_n, x_{n+1}), d(x_n, x_{n+2}) < \gamma)$.

This will be referred to as the *asymptotic rank* of γ. Clearly,

$0 < \gamma_1 \le \gamma_2 \Longrightarrow n(\gamma_1) \ge n(\gamma_2)$ (i.e., $\gamma \mapsto n(\gamma)$ is decreasing).

We may now state the announced properties:

(o-sm) P is *orbitally small*: for each d-asymptotic (a-o-f)-sequence $(x_n = T^n x_0; n \ge 0)$ in $X(T, \le)$ and each couple (ε, δ) with $\varepsilon > \delta > 0$, there exists $\gamma = \gamma(\varepsilon, \delta) \in]0, \delta/6[$ (and the attached asymptotic rank $n(\gamma)$), such that for each $j \ge 2$ and each $k \ge n(\gamma)$ with $d(x_m, x_{m+i}) < \varepsilon + \delta/2$ for $(m \ge k, i \in \{1, \dots, j\})$, we have $P(x_n, x_{n+j}) < \varepsilon + \delta$, whenever $(n \ge k, d(x_n, x_{n+j+1}) \ge \varepsilon + \delta/2)$.
(o-C) P is *orbitally Cauchy*: for each d-asymptotic (a-o-f)-sequence $(x_n = T^n x_0; n \ge 0)$ in $X(T, \le)$, we have $P_n \to 0$ as $n \to \infty$, where $(P_n := \sup\{P(x_n, x_{n+i}); i \ge 1\}; n \ge 0)$.

These have the essential role of deducing the d-Cauchy property for the d-asymptotic iterative sequences to be considered.

Concerning the former concept, the following practical criterion is to be noted.

Proposition 16 *Under the above conventions,*

(51-1) if the couple (P, M) is bounded, then P is orbitally small;
(51-2) if the maps $P_1, P_2 : X \times X \to R_+$ are orbitally small, then $P_3 := \max\{P_1, P_2\}$ is orbitally small.

Proof

(i) Let the d-asymptotic (a-o-f)-sequence $(x_n = T^n x_0; n \ge 0)$ in $X(T, \le)$ and the couple (ε, δ) with $\varepsilon > \delta > 0$ be given. Furthermore, take some $\gamma \in]0, \delta/6[$, and let $n(\gamma)$ stand for the attached asymptotic rank. We claim that for each $j \geq 2$ and each $k \geq n(\gamma)$ with $d(x_m, x_{m+i}) < \varepsilon + \delta/2$ for $(m \geq k, i \in \{1, \ldots, j\})$, one derives $P(x_n, x_{n+j}) < \varepsilon + \delta$, for each $n \geq k$;

and this will complete the argument. In fact, let $n \geq k$ be arbitrary fixed. By the working hypothesis above, we have

$$d(x_n, x_{n+j}), d(x_{n+1}, x_{n+j}), d(x_{n+1}, x_{n+j+1}) < \varepsilon + \delta/2,$$

and, by the very definition of our index $n(\gamma)$,

$$d(x_n, x_{n+1}), d(x_{n+j}, x_{n+j+1}) < \gamma < \delta/2 < \varepsilon + \delta/2.$$

Finally, taking the triangular inequality into account, one gets (by the choice of γ)

$$d(x_n, x_{n+j+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+j+1}) < \gamma + \varepsilon + \delta/2 < \varepsilon + \delta.$$

Putting these together yields (via (P, M)=bounded)

$P(x_n, x_{n+j}) \leq M(x_n, x_{n+j}) < \varepsilon + \delta$, and our claim follows.

(ii) Given (ε, δ) with $\varepsilon > \delta > 0$, let $\gamma_1 \in]0, \delta/6[$ (with the associated asymptotic rank $n(\gamma_1)$) and $\gamma_2 \in]0, \delta/6[$ (with the associated asymptotic rank $n(\gamma_2)$) be assured by the orbitally small property of P_1 and P_2, respectively. Then, let us put

$$\gamma_3 = \min\{\gamma_1, \gamma_2\} \text{ (hence, } n(\gamma_3) \geq \max\{n(\gamma_1), n(\gamma_2)\}\text{)};$$

we claim that the desired property of P_3 is fulfilled with respect to the obtained pair $(\gamma_3, n(\gamma_3))$. In fact, let $j \geq 2$ and $k \geq n(\gamma_3)$ be such that

(hyp) $d(x_m, x_{m+i}) < \varepsilon + \delta/2$ for $(m \geq k, i \in \{1, \ldots, j\})$;

we have to establish that

(con) $P_3(x_n, x_{n+j}) < \varepsilon + \delta$, whenever $(n \geq k, d(x_n, x_{n+j+1}) \geq \varepsilon + \delta/2)$.

To verify this, note that by (hyp), one gets for each $h \in \{1, 2\}$

$(k \geq n(\gamma_h)$ and) $d(x_m, x_{m+i}) < \varepsilon + \delta/2$ for $(m \geq k, i \in \{1, \ldots, j\})$.

So, letting $n \geq k(\geq n(\gamma_3))$ be as in the premise of (con), we have (by the admitted properties of P_1 and P_2)

$P_h(x_n, x_{n+j}) < \varepsilon + \delta$, $h \in \{1, 2\}$, whence $P_3(x_n, x_{n+j}) < \varepsilon + \delta$,

and the conclusion follows.

(IV) The fourth group of conditions is of *asymptotic* type:

(o-conv) P is *orbitally convergent*:
for each (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, and each $z \in X$ with
$x_n \xrightarrow{d} z$, $(x_n < z, \forall n)$, and $d(z, Tz) > 0$, we have $P(x_n, z) \to 0$.

(o-sg-asy) P is *orbitally singular asymptotic*: for each (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n < z, \forall n)$, and $d(z, Tz) > 0$, we have $\liminf_n P(x_n, z) < d(z, Tz)$.

(o-reg-asy) P is *orbitally regular asymptotic*: for each (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n < z, \forall n)$, and $d(z, Tz) > 0$, we have $P(x_n, z) \to d(z, Tz)$

(o-sr-asy) P is *orbitally strongly regular asymptotic*: for each (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n < z, \forall n)$, and $d(z, Tz) > 0$, we have $P(x_n, z) \to\to d(z, Tz)$.

Here, given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, we denoted

$r_n \to\to r$, if there exists a subsequence $(s_n = r_{i(n)}; n \geq 0)$ of $(r_n; n \geq 0)$ such that $[s_n = r, \forall n \geq 0]$.

Technically speaking, these have the role of deducing the fixed point property of the limit of our iterative sequence in the final stage of the proof.

(C) Having these precise, we may now pass to the contraction requirement upon our data. Letting (P, G, H) be a triple of maps over $\mathscr{F}(X \times X, R_+)$, denote

$Q_0 = P_0 - G$, $Q = P - H$ (where P_0 is taken as before).

Let us say that (Q_0, Q) is *Meir–Keeler* $(d, \leq)$*-contractive*, in case

(mk-1) $[x \leq y, Q(x, y) > 0]$ imply $Q_0(x, y) < Q(x, y)$, referred to as (Q_0, Q) is *strictly nonexpansive* (modulo $(d, \leq)$).

(mk-2) for each $\varepsilon > 0$, there exists $\delta > 0$, such that $(x \leq y, \varepsilon < Q(x, y) < \varepsilon + \delta) \Longrightarrow Q_0(x, y) \leq \varepsilon$, expressed as (Q_0, Q) has the *Meir–Keeler property* (modulo $(d, \leq)$).

Note that, by the former of these, the Meir–Keeler property gives

(mk-3) for each $\varepsilon > 0$, there exists $\delta > 0$, such that $(x \leq y, 0 < Q(x, y) < \varepsilon + \delta) \Longrightarrow Q_0(x, y) \leq \varepsilon$, referred to as (Q_0, Q) has the *extended Meir–Keeler property* (modulo $(d, \leq)$).

In particular, if $G = H = 0$, $P = d$, this convention is comparable with the one in Meir and Keeler [28], subsequently refined by Matkowski [27] and Cirić [10].

A geometric version of the above concept may be given along the lines below. Remember that the relation $\Omega \in \exp(R^0_+ \times R^0_+)$ is called *upper diagonal*, if

(u-diag) $(t, s) \in \Omega$ implies $t < s$;

the class of all these will be denoted as $\text{udiag}(R^0_+)$. Then, along the class $\text{udiag}(R^0_+)$, define the geometric concepts

(g-mk) Ω has the *geometric Meir–Keeler property*:
$\forall \varepsilon > 0, \exists \delta > 0$: $t\Omega s, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \leq \varepsilon$;
(g-bila-s) Ω is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t, s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow (t, s) \notin \Omega$;
(g-left-s) Ω is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow (t, \beta) \notin \Omega$;

as well as the asymptotic ones

(a-mk) Ω has the *asymptotic Meir–Keeler property*:
there are no strictly descending sequences (t_n) and (s_n) in R^0_+ and no elements ε in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$;
(a-bila-s) Ω is *asymptotic bilateral separable*:
there are no sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R^0_+ and no elements β in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \beta, s_n \to \beta)$;
(a-left-s) Ω is *asymptotic left separable*:
there are no strictly ascending sequences (t_n) in R^0_+
and no elements β in R^0_+, with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta-)$.

Note that, by a previous auxiliary fact,

(g-eq-a) each geometric notion is equivalent with its asymptotic counterpart.

Given $\Omega \in \exp(R^0_+ \times R^0_+)$, let us say that (Q_0, Q) is $(d, \leq; \Omega)$-*contractive*, provided

(Om-con) $(Q_0(x, y), Q(x, y)) \in \Omega$,
whenever $(x \leq y, Q_0(x, y) > 0, Q(x, y) > 0)$.

Proposition 17 *Suppose that the couple (Q_0, Q) is $(d, \leq; \Omega)$-contractive where the relation $\Omega \in \exp(R^0_+ \times R^0_+)$ is upper diagonal and geometric Meir–Keeler. Then, (Q_0, Q) is Meir–Keeler $(d, \leq)$-contractive.*

Proof

(i) Let $x, y \in X$ be such that $x \leq y$, $Q(x, y) > 0$. If $Q_0(x, y) \leq 0$, then $Q_0(x, y) < Q(x, y)$. Suppose now that $Q_0(x, y) > 0$. As a consequence of this, $(t, s) \in \Omega$, where $t := Q_0(x, y)$, $s := Q(x, y)$. Combining with the upper diagonal property of Ω, one gets $t < s$; i.e., $Q_0(x, y) < Q(x, y)$. Summing up, (Q_0, Q) is strictly nonexpansive (modulo $(d, \leq)$).
(ii) Let $\varepsilon > 0$ be arbitrary fixed and $\delta > 0$ be the number assured by the geometric Meir–Keeler property for Ω. Furthermore, let $x, y \in X$ be such that

$$x \leq y \text{ and } \varepsilon < Q(x, y) < \varepsilon + \delta; \text{ hence, } \varepsilon < s < \varepsilon + \delta, \text{ where } s := Q(x, y).$$

If $Q_0(x, y) \leq 0$, then $Q_0(x, y) < \varepsilon$. Suppose now that $Q_0(x, y) > 0$. By definition, we must have $(t, s) \in \Omega$, where $t := Q_0(x, y)$, and this, along with $\varepsilon < s < \varepsilon + \delta$ and the geometric Meir–Keeler property for Ω, gives $t \leq \varepsilon$; i.e., $Q_0(x, y) \leq \varepsilon$, whence (Q_0, Q) has the Meir–Keeler property (modulo $(d, \leq)$). Putting these together, it follows that (Q_0, Q) is Meir–Keeler $(d, \leq)$-contractive, and we are done.

In the following, a kind of reciprocal for this result is formulated. Given the triple of maps $P, G, H \in \mathscr{F}(X \times X, R_+)$, let $\Omega := \Omega[d, \leq; Q_0; Q]$ stand for the associated relation over R_+^0:

$\Omega = \{(Q_0(x, y), Q(x, y)); x \leq y, Q_0(x, y), Q(x, y) > 0\}$;
or, in other words,
$(t, s) \in \Omega$ iff $t = Q_0(x, y), s = Q(x, y)$, where $x \leq y, Q_0(x, y), Q(x, y) > 0$.

Proposition 18 *Under these conventions, we have*

(53-1) *If (Q_0, Q) is Meir–Keeler $(d, \leq)$-contractive, then the attached relation $\Omega := \Omega[d, \leq; Q_0; Q]$ over R_+^0 is upper diagonal and geometric Meir–Keeler*

(53-2) *(Q_0, Q) is Meir–Keeler $(d, \leq)$-contractive if and only if the attached relation $\Omega := \Omega[d, \leq; Q_0; Q]$ over R_+^0 is upper diagonal and geometric Meir–Keeler.*

Proof

(i) Suppose that (Q_0, Q) is Meir–Keeler $(d, \leq)$-contractive; we have to establish that $\Omega := \Omega[d, \leq; Q_0; Q]$ is upper diagonal and geometric Meir–Keeler. There are two steps to be passed.

(i-1) Let $(t, s) \in R_+^0 \times R_+^0$ be such that $(t, s) \in \Omega$; hence (by definition),

$t = Q_0(x, y), s = Q(x, y),$ where $x \leq y, Q_0(x, y) > 0, Q(x, y) > 0$.

From the strict nonexpansive property of (Q_0, Q), we must have

$Q_0(x, y) < Q(x, y)$; or, equivalently, $t < s$,

which shows that Ω is upper diagonal.

(i-2) Let $\varepsilon > 0$ be arbitrary fixed and $\delta > 0$ be the number associated by the Meir–Keeler property for (Q_0, Q). Furthermore, let $(t, s) \in R_+^0 \times R_+^0$ be taken as

$(t, s) \in \Omega$ and $\varepsilon < s < \varepsilon + \delta$.

From the former of these, we have

$t = Q_0(x, y), s = Q(x, y)$, where $x \leq y, Q_0(x, y) > 0, Q(x, y) > 0$;
so, combining with the latter, $x \leq y$, and $\varepsilon < Q(x, y) < \varepsilon + \delta$.

By the Meir–Keeler property for T, we get

$Q_0(x, y) \leq \varepsilon$; i.e., under our notation, $t \leq \varepsilon$,

so that Ω has the geometric Meir–Keeler property.

(ii) Suppose that the associated relation $\Omega := \Omega[d, \leq; Q_0; Q]$ over R_+^0 is upper diagonal and geometric Meir–Keeler. By the very definition of this object, (Q_0, Q) is $(d, \leq; \Omega)$-contractive. Combining with the preceding result, one derives that (Q_0, Q) appears as Meir–Keeler $(d, \leq)$-contractive, and the conclusion follows.

As a consequence of this, it follows that the Meir–Keeler $(d, \leq)$-contractive property of (Q_0, Q) is finally reducible to the upper diagonal and geometric Meir–Keeler properties for the associated relation $\Omega[d, \leq; Q_0; Q]$. Concerning this aspect, remember that various examples of such objects were provided in a previous place. Some other aspects will be treated a bit further.

6 Main Result

Let $(X, d, \leq)$ be a quasi-ordered metric space and $T \in \mathscr{F}(X)$ be a selfmap of X, supposed to be semi-progressive and increasing. The general directions under which the problem of determining fixed points of T is to be solved were already made precise; moreover, the (sufficient) regularity conditions and metrical contractive properties of the same were settled.

The main result of this exposition (referred to as the *perturbed Meir–Keeler theorem on ordered metric spaces*; in short, (MK-pert-oms)) may be stated as below.

Theorem 5 *Assume that the couples of mappings* (P_0, P) *and* (G, H) *over* $\mathscr{F}(X \times X, R_+)$ *and the relation* $\Omega \in \exp(R_+^0 \times R_+^0)$ *are taken so as*

(61-i) $(P_0 - G, P - H)$ *is* $(d, \leq; \Omega)$*-contractive;*
(61-ii) P *is telescopic bounded and orbitally small;*
(61-iii) G *is telescopic null, orbitally Cauchy, and orbitally convergent;*
(61-iv) $(P - H, L)$ *is positive, and* H *is telescopic null;*
(61-v) Ω *is upper diagonal and geometric/asymptotic Meir–Keeler.*

In addition, let X be (a-o-f,d)-complete. Then,

(61-a) T *is a strong Picard operator (modulo* $(d, \leq)$*), provided (in addition)* T *is (a-o-f,d)-continuous;*
(61-b) T *is a Bellman Picard operator (modulo* $(d, \leq)$*), if (in addition)* $(\leq)$ *is (a-o-f,d)-selfclosed, and one of the extra conditions holds;*

(61-b1) P *is orbitally singular asymptotic;*
(61-b2) P *is orbitally regular asymptotic and* Ω *is endowed with the geometric/asymptotic bilateral separable property;*
(61-b3) $P - H$ *is orbitally strongly regular asymptotic and* Ω *is endowed with the geometric/asymptotic left separable property;*

(61-c) T *is fix-*$(\leq)$*-asingleton, provided (in addition)* $(P - H, L_1)$ *is positive and* $(P - H, P_0 - G)$ *is fix bounded.*

Proof There are several steps to be passed.
Part 0. Let us start with the last assertion in this statement. Take a couple of points (z_1, z_2) over $\text{Fix}(T)$ with $z_1 \leq z_2$, and suppose, by contradiction, that $z_1 \neq z_2$; hence, $z_1 < z_2$, $\delta := d(z_1, z_2) > 0$. By definition,

$L_1(z_1, z_2) = \delta > 0$; whence, $(P - H)(z_1, z_2) > 0$, if we remember that $(P-H, L_1)$ is positive. By the fix bounded property,

(rela-1) $(0<)(P-H)(z_1,z_2) \le (P_0-G)(z_1,z_2)$; whence, $(P_0-G)(z_1,z_2)>0$.

Consequently, the contractive condition is applicable to (z_1,z_2) and gives

(rela-2) $((P_0-G)(z_1,z_2),(P-H)(z_1,z_2)) \in \Omega$,
so that $(P_0-G)(z_1,z_2)<(P-H)(z_1,z_2)$ (in view of Ω=upper diagonal).

Since the obtained relations (rela-1) and (rela-2) are contradictory, it results that our working assumption is not acceptable, so that $z_1=z_2$.

Having these precise, take some $x_0 \in X(T,\le)$, and put $(x_n=T^n x_0; n \ge 0)$; clearly, this is an ascending orbital sequence. If $x_n=x_{n+1}$ for some $n \ge 0$, we are done; so, without loss, one may assume that the semi-full condition holds

(s-full) $(\forall n)$: $x_n \ne x_{n+1}$; hence, $x_n<x_{n+1}$, $\rho_n:=d(x_n,x_{n+1})>0$.

Part 1 We firstly assert that the following relations hold:

(iter) $(\forall n)$: $(\rho_{n+1}, P(x_n,x_{n+1})) \in \Omega$, $\rho_{n+1}<P(x_n,x_{n+1}) \le \rho_n$.

In fact, let $n \ge 0$ be arbitrary fixed. For the moment, we must have

$P_0(x_n,x_{n+1})=(P_0-G)(x_n,x_{n+1})=\rho_{n+1}>0$
(as G is taken as telescopic null).

In addition (by the semi-full property),

$L(x_n,x_{n+1})=\min\{\rho_n,\rho_{n+1}\}>0$; hence, $P(x_n,x_{n+1})=(P-H)(x_n,x_{n+1})>0$
(as $(P-H,L)$ is positive, and H is telescopic null).

Putting these together yields, by the contractive condition (and Ω=upper diagonal),

$(\forall n)$: $(\rho_{n+1},P(x_n,x_{n+1})) \in \Omega$, and $\rho_{n+1}<P(x_n,x_{n+1})$.

On the other hand, as P is telescopic bounded, we must have

$P(x_n,x_{n+1}) \le M_1(x_n,x_{n+1})=\max\{\rho_n,\rho_{n+1}\}$.

Combining with the preceding relation gives, for each $n \ge 0$,

$\rho_{n+1}<\max\{\rho_n,\rho_{n+1}\}$; wherefrom, $\rho_{n+1}<\rho_n$, $M_1(x_n,x_{n+1})=\rho_n$,

and the claim follows.

Part 2 From the preceding part, one derives $(\rho_{n+1}<\rho_n, \forall n)$, so that the sequence $(\rho_n; n \ge 0)$ is strictly descending; wherefrom, $\rho:=\lim_n \rho_n$ exists as an element of R_+. Assume by contradiction that $\rho>0$, and let $\sigma>0$ be the number given by the Meir–Keeler property of Ω. By definition, there exists a rank $n(\sigma)$ with

$n \ge n(\sigma)$ implies $\rho<(\rho_{n+1}<)\rho_n<\rho+\sigma$.

On the other hand, taking (iter) into account,

$(\forall n)$: $(0<)\ \rho_{n+1}<P(x_n,x_{n+1}) \le \rho_n$; so, combining with the above,
$n \ge n(\sigma)$ implies $(\rho_{n+1},P(x_n,x_{n+1})) \in \Omega$, and $\rho<P(x_n,x_{n+1})<\rho+\sigma$.

By the underlying Meir–Keeler property, we then get

$(\forall n \geq n(\sigma))$: $(\rho < \rho_{n+1}$ and$)$ $\rho_{n+1} \leq \rho$,

a contradiction. Hence, $\rho = 0$, so that

$\rho_n := d(x_n, x_{n+1}) = d(x_n, Tx_n) \to 0$, as $n \to \infty$;
or, in other words, $(x_n; n \geq 0)$ is d-asymptotic.

Part 3 Suppose that

there exist $i, j \in N$ such that $i < j$, $x_i = x_j$.

By the very meaning of our iterative process, we have $x_{i+1} = x_{j+1}$; hence, $\rho_i = \rho_j$, in contradiction with the strict descending property of $(\rho_n; n \geq 0)$. Hence, our working hypothesis cannot hold; wherefrom,

(x_n) is a full sequence ($i < j$ implies $x_i \neq x_j$; hence, $x_i < x_j$, $d(x_i, x_j) > 0$).

Part 4 Summing up, the iterative sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$ is ascending, orbital, full, and d-asymptotic. We now establish that $(x_n; n \geq 0)$ is d-Cauchy. Let $\varepsilon > 0$ be given and $\delta > 0$ be assured by the Meir–Keeler property of Ω; without loss, one may assume that $\delta < \varepsilon$. Furthermore, given the couple (ε, δ) as before, let the number $\gamma \in]0, \delta/6[$ [and the associated asymptotic rank $n(\gamma)$] be given via P=orbitally small. Finally, take some rank $m(\gamma) > n(\gamma)$ according to

(rela-G) $\quad n \geq m(\gamma)$ implies $G_n := \sup\{G(x_n, x_{n+i}); i \geq 1\} < \delta/6$
(possible, as G is supposed to be orbitally Cauchy).

We claim, via ordinary induction, that

(d-C;i) $\quad d(x_n, x_{n+i}) < \varepsilon + \delta/2$, for each $n \geq m(\gamma)$,

holds, for all $i \geq 1$; from this, the d-Cauchy property of (x_n) follows. The case $i \in \{1, 2\}$ is evident, via $m(\gamma) > n(\gamma)$ and

(d-asy) $\quad (\forall i \in \{1, 2\})$: $d(x_n, x_{n+i}) < \gamma < \delta/6 < \delta/2$, for each $n \geq n(\gamma)$.

Suppose that (d-C;i) holds for all $i \in \{1, \ldots, j\}$, where $j \geq 2$; we must establish that

(d-C;j+1) $\quad d(x_n, x_{n+j+1}) < \varepsilon + \delta/2$, for all $n \geq m(\gamma)$

holds too. Suppose by contradiction that (d-C;j+1) would be false:

$C(\varepsilon, \delta) := \{n \in N(m(\gamma), \leq); d(x_n, x_{n+j+1}) \geq \varepsilon + \delta/2\}$ is nonempty,

and put $n = \min C(\varepsilon, \delta)$. By (d-asy) and triangular inequality,

$d(x_{n+1}, x_{n+j+1}) \geq d(x_n, x_{n+j+1}) - d(x_n, x_{n+1}) \geq \varepsilon + \delta/2 - \delta/6 = \varepsilon + \delta/3$;

wherefrom, in view of (rela-G),

$(P_0 - G)(x_n, x_{n+j}) = d(x_{n+1}, x_{n+j+1}) - G(x_n, x_{n+j}) \geq \varepsilon + \delta/3 - G_n > \varepsilon > 0$.

On the other hand, by the full property of (x_n),

$x_n < x_{n+j}$ and $L(x_n, x_{n+j}) > 0$, whence $(P - H)(x_n, x_{n+j}) > 0$

if we remember that $(P - H, L)$ is positive. Putting these together, one derives that the contraction property holds

$((P_0 - G)(x_n, x_{n+j}), (P - H)(x_n, x_{n+j})) \in \Omega$.

Moreover, by the choice of our data and P=orbitally small,

$(0 <)(P - H)(x_n, x_{n+j}) \leq P(x_n, x_{n+j}) < \varepsilon + \delta$.

Combining with the underlying Meir–Keeler property of Ω gives

$d(x_{n+1}, x_{n+j+1}) - G(x_n, x_{n+j}) = (P_0 - G)(x_n, x_{n+j}) \leq \varepsilon$;

wherefrom, again taking (rela-G) into account,

$d(x_{n+1}, x_{n+j+1}) \leq \varepsilon + G(x_n, x_{n+j}) \leq \varepsilon + G_n < \varepsilon + \delta/6$.

Combining with the triangular inequality gives, finally,

$d(x_n, x_{n+j+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+j+1}) < \varepsilon + \delta/6 + \delta/6 < \varepsilon + \delta/2$,

in contradiction with the choice of $n \in C(\varepsilon, \delta)$. Hence, the precise inductive relation holds; wherefrom, $(x_n; n \geq 0)$ is d-Cauchy, as claimed.

Part 5 As X is (a-o-f,d)-complete,

$x_n \xrightarrow{d} z$ as $n \to \infty$, for some (uniquely determined) $z \in X$.

There are several cases to discuss.

Case 5a Suppose that T is (a-o-f,d)-continuous. Then, $y_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$. On the other hand, $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; whence $y_n \xrightarrow{d} z$, and this yields (as d is separated), $z = Tz$.

Case 5b Suppose that $(\leq)$ is (a-o-f,d)-selfclosed. For the moment,

$(x_n \leq z, \forall n)$; hence, $(Tx_n \leq Tz, \forall n)$, as T=increasing.

We show that $b := d(z, Tz) > 0$ yields a contradiction.

From the d-convergence relation (and a metrical property of $d(.,.)$)

$d(x_n, z),\ d(Tx_n, z) \to 0$, $d(x_n, Tx_n) \to 0$, as $n \to \infty$;
$d(x_n, Tz), d(Tx_n, Tz) \to b$, as $n \to \infty$.

On the other hand, by the full property of $(x_n; n \geq 0)$,

$E := \{n \in N; (x_{n+1} =)Tx_n = Tz\}$ is an asingleton,

so that the following separation property holds:

(sepa) $\exists h = h(z)$: $n \geq h \Longrightarrow x_n \neq z, Tx_n \neq Tz$; hence, $x_n < z, Tx_n < Tz$.

Without loss of generality, one may assume that $h = 0$ in this relation; that is, (sepa-0) $(\forall n)$: $x_n \neq z$, $Tx_n \neq Tz$; hence, $x_n < z$, $Tx_n < Tz$.

For, otherwise, passing to the subsequence

$(u_n = x_{h+n}; n \geq 0)$ (hence, $(u_n = T^n u_0; n \geq 0)$, where $u_0 = x_h$),

this property holds, as well as the remaining ones

$(u_n; n \geq 0)$ is an (a-o-f)-sequence with $u_n \xrightarrow{d} z$ as $n \to \infty$.

Having these precise, we have, as a direct consequence,

$(\forall n)$: $L(x_n, z) > 0$; hence, $(P - H)(x_n, z) > 0$,

in view of $(P - H, L)$ being positive. Moreover,

(lim-G) $\lim_n G(x_n, z) = 0$ (as G=orbitally convergent),
so that (by the above) $\lim_n (P_0 - G)(x_n, z) = b > 0$.

This, by definition, tells us that

(posi) $(\exists k \geq 0)$: $n \geq k \Longrightarrow (P_0 - G)(x_n, z) = d(Tx_n, Tz) - G(x_n, z) > 0$.

Again without loss of generality, one may assume that $k = 0$ in this relation; that is,

(posi-0) $(\forall n)$: $(P_0 - G)(x_n, z) = d(Tx_n, Tz) - G(x_n, z) > 0$.

[The argument was already developed; so, we do not repeat it]. As a consequence, the geometric contractive condition is applicable to (x_n, z), for all n, and gives

(contra) $(\forall n)$: $((P_0 - G)(x_n, z), (P - H)(x_n, z)) \in \Omega$; whence,
$(P_0 - G)(x_n, z) < (P - H)(x_n, z)$ (as Ω=upper diagonal).

There are several sub-cases to be analyzed.

Alter 1 Assume that P is orbitally singular asymptotic. Passing to lim inf as $n \to \infty$ in the second part of relation (contra) above gives (via $H \geq 0$)

$b = \liminf_n (P_0 - G)(x_n, z) \leq \liminf_n (P - H)(x_n, z) \leq \liminf_n P(x_n, z)$.

This, however, contradicts the very choice of P. Hence, necessarily, $b = 0$ [i.e., $z = Tz$], and the conclusion follows.

Alter 2 Suppose that P is orbitally regular asymptotic, and Ω is geometric/asymptotic bilateral separable. By (contra) (the second half) and $(H \geq 0)$,

$(\forall n)$: $(P_0 - G)(x_n, z) < (P - H)(x_n, z) \leq P(x_n, z)$; whence,
$\lim_n (P - H)(x_n, z) = b$ [by (lim-G) and the choice of P].

This, again via (lim-G) and (contra), cannot be in agreement with the bilateral separable property of Ω. Hence, necessarily, $b = 0$; i.e., $z = Tz$.

Alter 3 Suppose that $P - H$ is orbitally strongly regular asymptotic, and Ω is geometric/asymptotic left separable. By the very definition above, there exists a subsequence $(y_n := x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$, with (according to (lim-G))

($\beta_n := (P_0 - G)(y_n, z) \to b$ as $n \to \infty$ and) $(P - H)(y_n, z) = b$, for all n.

By the contractive property (contra), we get

$(\beta_n, b) \in \Omega$ (hence, $\beta_n < b$), $\forall n$); as well as (by the above) $\beta_n \to b-$ as $n \to \infty$.

From a previous auxiliary fact, there exists a subsequence $(\gamma_n := \beta_{i(n)}; n \geq 0)$ of $(\beta_n; n \geq 0)$, with

(γ_n) is strictly ascending, $(\gamma_n, b) \in \Omega$ (hence, $\gamma_n < b$), $\forall n$),
with, in addition, $\lim_n \gamma = b$; whence, $\gamma_n \to b-$ as $n \to \infty$.

This, however, is not compatible with the left separated property of Ω. Hence, $b = 0$; i.e., $z = Tz$, and the conclusion follows. The proof is thereby complete.

Note that multivalued enlargements of these facts are possible, under the lines in Nadler [31], and the obtained facts extend some related statement in Choudhury and Metiya [9]; we shall discuss these in a separate paper.

7 Particular Cases

Let $(X, d, \leq)$ be a quasi-ordered metric space. Furthermore, let T be a selfmap of X, supposed to be semi-progressive and increasing. As precise, we have to determine appropriate conditions under which $\text{Fix}(T)$ is nonempty. The specific directions under which this problem is to be solved were already listed. Sufficient conditions for getting such properties are being founded on the ascending orbital full concepts we just introduced. Finally, the specific contractive properties to be used have been described, and the main result incorporating all these is the already formulated one. It is our aim in the sequel to derive some particular cases of it, with a technical relevance. Remember that we defined the (basic) maps [for $x, y \in X$]

$P_0(x, y) = d(Tx, Ty)$, $L_1(x, y) = \min\{d(x, y), d(Tx, Ty)\}$,
$L(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, y), d(Tx, Ty)\}$,
$M_1(x, y) = \max\{d(x, Tx), d(y, Ty)\}$, $M(x, y) = \text{diam}\{x, Tx, y, Ty\}$.

Furthermore, let us complete this with the family of functions [for $x, y \in X$]

$A_0(x, y) = \max\{d(x, Tx), d(y, Ty)\}$, $A_1(x, y) = d(Tx, Ty)$,
$A_2(x, y) = (1 - \xi)d(x, Tx) + \xi d(y, Ty)$ (where $0 \leq \xi < 1$),
$A_3(x, y) = (1 - \eta)d(x, y) + \eta d(Tx, Ty)$ (where $0 \leq \eta < 1$)
$A_4(x, y) = (1/2)[d(x, Ty) + d(Tx, y)]$.

For a final completion, let us introduce the diagonal type subset of R_+^2

$\Delta = \{(\lambda, \mu) \in R_+ \times R_+^0; \lambda \leq \mu\}$.

This set is decomposed in two (mutually disjoint) parts, expressed as

$\Delta_s = \{(\lambda, \mu) \in \Delta; \lambda < \mu\}$ (the *singular* part),

$\Delta_r = \{(\lambda, \mu) \in \Delta; \lambda = \mu\} = \{(\nu, \nu); \nu \in R_+^0\}$ (the *regular* part).

For each $(\lambda, \mu) \in \Delta$, let us introduce the map $B := B[\lambda, \mu] : X \times X \to R_+$, as

$B(x, y) = d(y, Ty)[\lambda + d(x, Tx)]/[\mu + d(x, y)], x, y \in X.$

Furthermore, let us define the *singular* and *regular* maps

A_5=one of the maps $B[\lambda, \mu]$ with $(\lambda, \mu) \in \Delta_s$;
A_6=one of the maps $B[\lambda, \mu]$ with $(\lambda, \mu) \in \Delta_r$
(or, equivalently,
A_6=one of the maps $B[\nu, \nu]$ with $\nu \in R_+^0$);

the reason of such a splitting will become clear a bit further. Finally, for each $(\alpha, \beta) \in \Delta$, let us introduce the map $C := C[\alpha, \beta] : X \times X \to R_+$, as

$C(x, y) = d(x, Tx)[\alpha + d(y, Ty)]/[\beta + d(Tx, Ty)], x, y \in X.$

Then, let us define

A_7=one of the maps $C[\alpha, \beta]$ with $(\alpha, \beta) \in \Delta$.

Fix in the following $\xi, \eta \in [0, 1[$, $(\lambda, \mu) \in \Delta_s$, $(\nu, \nu) \in \Delta_r$, $(\alpha, \beta) \in \Delta$, and (according to the previous conventions) denote

$Z = \{0, 1, 2, 3, 4, 5, 6, 7\}$; hence, $\text{card}(\exp(Z)) = 2^8 - 1 = 255$.

For each subset $\Theta \in \exp(Z)$, let $E(\Theta) \in \mathscr{F}(X \times X, R_+)$ be the mapping

$E(\Theta)(x, y) = \max\{A_i(x, y); i \in \Theta\}, x, y \in X.$

The maps $P : X \times X \to R_+$ to be considered are of the form

$P = E(\Theta)$; where $\Theta \in \exp(Z)$.

So, it remains to establish which maps in this family are compatible with the conditions required by the variant of our main result, characterized as

(H-zero) $H(x, y) = 0$, for $x, y \in X$ (hence, H is telescopic null).

A technical motivation of this restriction is due to the fact that, unfortunately, the strong orbitally regular asymptotic property of $P - H = E(\Theta) - H$ is pretty difficult to be assured for $H \neq 0$. So, for a complete translation of our main result to this particular case, the only option to be considered is to take $H = 0$ in the sequel. But, for an incomplete translation of the same—founded on this alternative being ignored—the case of $H \neq 0$ is technically acceptable.

To begin with, define the set-family

$Z^* = \{\Theta \in \exp(Z); \{0, 1, 2, 3, 5, 6, 7\} \cap \Theta \neq \emptyset\}$;

as we will see, all conditions to be discussed are essentially depending on it.

(I) A first set of answers refers to the positive and boundedness conditions (expressed in terms of a certain map $K \in \mathscr{F}(X \times X, R_+)$):

(pos) (P, K) is *positive*: $x \leq y$ and $K(x, y) > 0$ imply $P(x, y) > 0$;
(bd) (P, K) is bounded: $x \leq y$ implies $P(x, y) \leq K(x, y)$;
(fix-bd) (P, K) is fix bounded: $x, y \in \text{Fix}(T)$ and $x \leq y$ imply $P(x, y) \leq K(x, y)$.

Proposition 19 *The following assertions hold for the arbitrary fixed $\Theta \in Z^*$:*

(71-1) $(E(\Theta), L)$ *is positive and* $((E(\Theta), L_1)$ *is positive if* $\{1, 3\} \cap \Theta \neq \emptyset)$.
(71-2) $(E(\Theta), M)$ *is bounded when* $\Theta \subseteq \{0, 1, 2, 3, 4\}$.
(71-3) $(E(\Theta), M)$ *is fix bounded.*

Proof

(i) Let $x, y \in X$ be such that $x \leq y$, $L(x, y) > 0$. Then, $A_i(x, y) > 0$, for all $i \in \{0, \dots, 7\}$, $i \neq 4$, and the conclusion follows, by the choice of Θ. Likewise, let $x, y \in X$ be such that $x \leq y$, $L_1(x, y) > 0$. Then, $A_i(x, y) > 0$, $i \in \{1, 3\}$, and we are done.
(ii) By definition, we have $A_i(x, y) \leq M(x, y)$, $\forall x, y \in X$, $\forall i \in \{0, 1, 2, 3, 4\}$, and this, along with $\Theta \subseteq \{0, 1, 2, 3, 4\}$, gives the stated conclusion.
(iii) Let $x, y \in \text{Fix}(T)$ be such that $x \leq y$. By definition,

$A_i(x, y) = d(x, y) = M(x, y)$, $i \in \{1, 3, 4\}$; $A_j(x, y) = 0 \leq M(x, y)$, $j \in \{0, 2\}$,
$B[\lambda, \mu](x, y) = 0 \leq M(x, y)$, $(\lambda, \mu) \in \Delta$; $C[\alpha, \beta](x, y) = 0 \leq M(x, y)$, $(\alpha, \beta) \in \Delta$;

wherefrom, combining with the maps A_5 or A_6 having the form $B[\lambda, \mu]$ and the map A_7 having the form $C[\alpha, \beta]$, we are done.

(II) The second answer refers to the telescopic boundedness condition

(t-bd) P is *telescopic bounded*: $x \leq Tx$ implies $P(x, Tx) \leq M_1(x, Tx)$.

Proposition 20 *All maps $P = E(\Theta)$, where $\Theta \in Z^*$ are telescopic bounded.*

Proof Firstly, we prove that any function $P = E(\{i\}) = A_i$ where $i \in Z$ has such a property. Given the arbitrary point $x \in X(T, \leq)$, we have

$A_0(x, Tx) = M_1(x, Tx)$, $A_1(x, Tx) = d(Tx, T^2x) \leq M_1(x, Tx)$,
$A_2(x, Tx) \leq (1 - \xi + \xi)M_1(x, Tx) = M_1(x, Tx)$,
$A_3(x, Tx) \leq (1 - \eta + \eta)M_1(x, Tx) = M_1(x, Tx)$,
$A_4(x, Tx) = (1/2)d(x, T^2x) \leq (1/2)[d(x, Tx) + d(Tx, T^2x)] \leq M_1(x, Tx)$;
$B[\lambda, \mu](x, Tx) = d(Tx, T^2x)[\lambda + d(x, Tx)]/[\mu + d(x, Tx)] \leq (Tx, T^2x) \leq M_1(x, Tx)$, $C[\alpha, \beta](x, Tx) = d(x, Tx)[\alpha + d(Tx, T^2x)]/[\beta + d(Tx, T^2x)] \leq d(x, Tx) \leq M_1(x, Tx)$,

and this, along with the maps A_5 or A_6 having the form $B[\lambda, \mu]$ and the map A_7 having the form $C[\alpha, \beta]$, proves the claim. Secondly, the property in question is transferable to $E(\Theta) = \max\{A_i; i \in \Theta\}$, and from this, all is clear.

(III) The next answer refers to the orbitally small property.

Proposition 21 *All maps $P = E(\Theta)$, where $\Theta \in Z^*$ are orbitally small.*

Proof It will suffice to establish that all maps $Q \in \{A_i; i \in Z\}$ have such a property. There are two cases to be discussed.

Case 1 $Q \in \{A_0, A_1, A_2, A_3, A_4\}$. By definition, we have $A_i(x, y) \leq M(x, y)$, for all $x, y \in X$, and all $i \in \{0, 1, 2, 3, 4\}$; i.e., the couples (Q, M), where $Q \in \{A_0, A_1, A_2, A_3, A_4\}$ are bounded; this, along with a previous auxiliary fact, assures us that the underlying maps are orbitally small.

Case 2 $Q \in \{A_5, A_6, A_7\}$; or, equivalently, $Q \in \{B, C\}$, where $B = B[\lambda, \mu]$, for some $(\lambda, \mu) \in \Delta$, and $C = C[\alpha, \beta]$, for some $(\alpha, \beta) \in \Delta$. Take the d-asymptotic (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, as well as the couple (ε, δ) with $\varepsilon > \delta > 0$. Furthermore, let $\gamma \in]0, \delta/6[$ be arbitrary for the moment and $n(\gamma)$ be the attached asymptotic rank. Finally, let $j \geq 2$, $k \geq n(\gamma)$, $n \geq k$ fulfill

$$d(x_m, x_{m+i}) < \varepsilon + \delta/2 \text{ for } (m \geq k, i \in \{1, \ldots, j\}); d(x_n, x_{n+j+1}) \geq \varepsilon + \delta/2;$$

we intend to show that, by an appropriate choice of γ, one derives $Q(x_n, x_{n+j}) < \varepsilon + \delta$. Denote, as usual, $(\rho_n = d(x_n, x_{n+1}); n \geq 0)$. By these hypotheses, we have

$$d(x_n, x_{n+j+1}) \leq d(x_n, x_{n+j}) + \rho_{n+j} < \varepsilon + \delta/2 + \gamma.$$

On the other hand, the triangular inequality (and $n \geq k \geq n(\gamma)$) gives

$$d(x_n, x_{n+j}) \geq d(x_n, x_{n+j+1}) - \rho_{n+j} \geq \varepsilon + \delta/2 - \gamma,$$
$$d(x_{n+1}, x_{n+j+1}) \geq d(x_n, x_{n+j+1}) - \rho_n \geq \varepsilon + \delta/2 - \gamma.$$

In this case, by definition,

(ev-1) $B(x_n, x_{n+j}) = \rho_{n+j}[\lambda + \rho_n]/[\mu + d(x_n, x_{n+j})] \leq$
$\rho_{n+j}[\lambda + \rho_n]/[\mu + \varepsilon + \delta/2 - \gamma] < \gamma[\lambda + \gamma]/[\mu + \varepsilon + \delta/2 - \gamma]$,
(ev-2) $C(x_n, x_{n+j}) = \rho_n[\alpha + \rho_{n+j}]/[\beta + d(x_{n+1}, x_{n+j+1})] \leq$
$\rho_n[\alpha + \rho_{n+j}]/[\beta + \varepsilon + \delta/2 - \gamma] < \gamma[\alpha + \gamma]/[\beta + \varepsilon + \delta/2 - \gamma]$.

Denote, for $0 < \gamma < \delta/6$,

$$\Phi(\gamma) = \gamma[\lambda + \gamma]/[\mu + \varepsilon + \delta/2 - \gamma], \Psi(\gamma) = \gamma[\alpha + \gamma]/[\beta + \varepsilon + \delta/2 - \gamma].$$

By the above evaluations, we have

$$B(x_n, x_{n+j}) < \Phi(\gamma), C(x_n, x_{n+j}) < \Psi(\gamma) \text{ (for all such } \gamma).$$

On the other hand,

$$\lim_{\gamma \to 0+} \Phi(\gamma) = 0 < \varepsilon + \delta, \lim_{\gamma \to 0+} \Psi(\gamma) = 0 < \varepsilon + \delta.$$

This tells us that, if $\gamma \in]0, \delta/6[$ is small enough, we have

$$B(x_n, x_{n+j}) < \Phi(\gamma) < \varepsilon + \delta; C(x_n, x_{n+j}) < \Psi(\gamma) < \varepsilon + \delta.$$

Putting these together yields the desired conclusion involving the maps $(A_i; i \in Z)$.

Case 3 The final conclusion relative to the maps $E(\Theta)$ is now clear [by a previous auxiliary fact] via $E(\Theta) = \max\{A_i; i \in \Theta\}$ and all maps A_i where $i \in \Theta$ being endowed with the orbitally small property.

(IV) Concerning the orbital asymptotic properties, the situation is a little bit complicated. Precisely, the following synthetic answer is available.

Proposition 22 *The following are valid, for the subset $\Theta \in Z^*$:*

(74-1) *Each (admissible) map $P = E(\Theta)$, where $\{0, 1, 6\} \cap \Theta = \emptyset$ is orbitally singular asymptotic.*
(74-2) *Each (admissible) map $P = E(\Theta)$, where $\{0, 1, 6\} \cap \Theta \neq \emptyset$, is orbitally regular asymptotic.*
(74-3) *Each (admissible) map $P = E(\Theta)$, where $\{1, 6\} \cap \Theta = \emptyset$ and $0 \in \Theta$, is orbitally strongly regular asymptotic.*

Proof Let the (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$ and the point $z \in X$ be such that

$$x_n \xrightarrow{d} z, (x_n < z \text{ for all } n), \text{ and } b := d(z, Tz) > 0.$$

From the convergence relation (and taking a metrical property of $d(.,.)$ into account), one gets, as $n \to \infty$,

$$d(x_n, z),\ d(Tx_n, z) \to 0, d(x_n, Tx_n) \to 0, d(x_n, Tz), d(Tx_n, Tz) \to b.$$

This, by definition, gives (as $n \to \infty$)

$A_0(x_n, z) \to b$, $A_1(x_n, z) \to b$, $A_2(x_n, z) \to \xi b < b$, $A_3(x_n, z) \to \eta b < b$, $A_4(x_n, z) \to b/2 < b$; wherefrom, any map $Q \in \{A_2, A_3, A_4\}$ is orbitally singular asymptotic and any map $Q \in \{A_0, A_1\}$ is orbitally regular asymptotic.

Moreover, again by definition, we have (putting $(\rho_n := d(x_n, x_{n+1}); n \geq 0)$) for each $(\lambda, \mu) \in \Delta$, $(\alpha, \beta) \in \Delta$, and all n,

$$B[\lambda, \mu](x_n, z) = b[\lambda + \rho_n]/[\mu + d(x_n, z)],$$
$$C[\alpha, \beta](x_n, z) = \rho_n[\alpha + b]/[\beta + d(x_{n+1}, Tz)].$$

This, by a limit process, gives

$\lim_n A_5(x_n, z) = b\lambda/\mu < b$, $\lim_n A_6(x_n, z) = b\nu/\nu = b$, $\lim_n A_7(x_n, z) = 0 < b$, which tells us that any map $Q \in \{A_5, A_7\}$ is orbitally singular asymptotic and the map $Q = A_6$ is orbitally regular asymptotic.

Finally, the same convergence properties of the sequences $(A_i(x_n, z); n \geq 0)$, where $i \in \{2, 3, 4, 5, 7\}$, tell us that, for a sufficiently large $n(z) \geq 0$, we must have

$$(\text{for all } n \geq n(z)): A_0(x_n, z) = b, \text{ and } A_i(x_n, z) < b, \forall i \in \{2, 3, 4, 5, 7\},$$

which, in particular, tells us that A_0 is orbitally strongly regular asymptotic.

By the above discussion, it is clear that our conclusion follows.

(V) Finally, let us make some remarks about the remaining properties appearing there. Call $J : X \times X \to R_+$, *regular* provided

(reg) J is telescopic null, orbitally Cauchy, and orbitally convergent.

Two examples are of interest; namely, for some $U, W \in \mathscr{F}_1(R_+)$, $V \in \mathscr{F}_2(R_+)$,

(ex-1) $J(U)(x, y) = U(\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\})$, $x, y \in X$,
(ex-2) $J(W, V)(x, y) = W(d(x, Tx)d(y, Ty))V(d(x, Ty)d(Tx, y))$, $x, y \in X$.

Here, for simplicity, we denoted

$\mathscr{F}_1(R_+) = \{U \in \mathscr{F}(R_+); U\text{=increasing and } U(t) \to 0 = U(0) \text{ as } t \to 0\}$,
$\mathscr{F}_2(R_+) = \{V \in \mathscr{F}(R_+); V(0) = 0 \text{ and } V_\infty := \sup V(R_+) < \infty\}$.

Proposition 23 *We have, for each* $J \in \{J(U), J(W, V)\}$,

(75-1) *J is necessarily regular (see above).*
(75-2) *$(E(\Theta), P_0 - J)$ is fix bounded, for each $\Theta \in Z^*$.*

Proof

(i) Clearly, J is telescopic null; since, for each $x \in X$,

$J(U)(x, Tx) = U(0) = 0;$
$J(W, V)(x, Tx) = W(d(x, Tx)d(Tx, T^2x))V(0) = 0.$

(ii) On the other hand, J is orbitally Cauchy; i.e.,

for each d-asymptotic (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, we have $J_n \to 0$ as $n \to \infty$, where $(J_n := \sup\{J(x_n, x_{n+i}); i \geq 1\}; n \geq 0)$.

In fact, let (x_n) be as in this premise. By definition, we have

$\rho_n := d(x_n, x_{n+1}) \to 0$; whence, $\tau_n := \sup\{\rho_n, \rho_{n+1}, \ldots\} \to 0$.

This yields, in a direct way,

$(J = J(U))$: $J(x_n, x_{n+i}) \leq U(\rho_n), \forall n \geq 0, \forall i \geq 1$, whence $J_n \to 0$.
$(J = J(W, V))$: $J(x_n, x_{n+i}) \leq V_\infty W(\tau_n^2), \forall n \geq 0, \forall i \geq 1$, whence $J_n \to 0$.

(iii) Finally, J is orbitally convergent, in the sense,

(o-conv) for each (a-o-f)-sequence $(x_n = T^n x_0; n \geq 0)$ in $X(T, \leq)$, and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n < z, \forall n)$, and $b := d(z, Tz) > 0$, we have $J(x_n, z) \to 0$.

In fact, let (x_n) and z be as in this premise. Then (cf. the above notation),

$(J = J(U))$: $J(x_n, z) \leq U(\rho_n), \forall n$, whence $J(x_n, z) \to 0$.
$(J = J(W, V))$: $J(x_n, z) \leq V_\infty W(\rho_n b), \forall n$; so, $J(x_n, z) \to 0$.

(iv) The required property means

$$x, y \in \text{Fix}(T), x \leq y \text{ imply } E(\Theta)(x, y) \leq P_0(x, y) - J(x, y);$$

or, equivalently (via $J(x, y) = 0, x, y \in \text{Fix}(T)$)

$$x, y \in \text{Fix}(T), x \leq y \text{ imply } E(\Theta)(x, y) \leq P_0(x, y) = M(x, y);$$
i.e., $(E(\Theta), M)$ is fix bounded.

But, according to a previous fact, this last property is fulfilled by any map $E(\Theta)$; wherefrom, all is clear.

By these developments, it results that $G \in \{J(U), J(W, V)\}$ fulfills all conditions in the main result where the regularity condition appears. On the other hand, $H \in \{J(U), J(W, V)\}$ fulfills the positivity condition in the main result involving $P = E(\Theta)$, under restrictive conditions upon Θ, U, W, and V. Precisely, we have the following statement:

Proposition 24 *The following are valid, for a given* $\Theta \in Z^*$*,*

(76-1) the couple $(E(\Theta) - J(U), L)$ *is positive, whenever* $\{0, 2\} \cap \Theta \neq \emptyset$ *and* $U(t) < t$*, for all* $t > 0$.
(76-2) $(E(\Theta) - J(W, V), L)$ *is positive, under* $0 \in \Theta$ *and* $V_\infty W(t^2) < t$*,* $\forall t > 0$.

Proof Let $x, y \in X$ be such that

$$x \leq y, L(x, y) > 0; \text{ hence, } \min\{d(x, Tx), d(y, Ty)\} > 0.$$

Under these conditions, one gets

(pp-1) $J(U)(x, y) \leq U(\min\{d(x, Tx), d(y, Ty)\}) <$
$\min\{d(x, Tx), d(y, Ty)\} \leq \min\{A_0(x, y), A_2(x, y)\} \leq E(\Theta)(x, y)$;
(pp-2) $J(W, V)(x, y) \leq V_\infty W(d(x, Tx)d(y, Ty)) \leq$
$V_\infty W(A_0^2(x, y)) < A_0(x, y) \leq E(\Theta)(x, y)$;

and the assertion follows.

In other words, the couple of maps $(E(\Theta), H)$, where the subset $\Theta \in Z^*$ and the map $H \in \{J(U), J(W, V)\}$ are taken as before, may be operational for the existence part of our main result that excludes the orbitally strongly regular asymptotic alternative. But, if we want that a complete translation of our main result be reached (so as to include the orbitally strongly regular asymptotic alternative as well), the only option to be considered is $H = 0$, because the underlying property of $E(\Theta) - H$ is holding only if this choice is effective.

Now, by simply combining this with our main result, one gets the following *rational* type fixed point statement (referred to as *Rational Function Meir–Keeler theorem*; in short, (MK-f-ra)).

Theorem 6 *Assume that the couple* $(P_0 - G, E(\Theta))$ *is* $(d, \leq; \Omega)$*-contractive, for some regular map* $G : X \times X \to R_+$*, some subset* $\Theta \in Z^*$ *and some relation* $\Omega \in \exp(R_+^0 \times R_+^0)$*, with*

Ω *is upper diagonal and geometric/asymptotic Meir–Keeler.*

In addition, let X be (a-o-f,d)-complete. Then,

(71-a) *T is a strong Picard operator (modulo $(d, \leq)$), provided (in addition) T is (a-o-f,d)-continuous;*

(71-b) *T is a Bellman Picard operator (modulo $(d, \leq)$), provided $(\leq)$ is (a-o-f,d)-selfclosed, and one of the following groups of extra conditions is holding;*

(71-b1) $\{0, 1, 6) \cap \Theta = \emptyset$*;*
(71-b2) $\{0, 1, 6\} \cap \Theta \neq \emptyset$*, and Ω is geometric/asymptotic bilateral separable;*
(71-b3) $\{1, 6\} \cap \Theta = \emptyset$*, $0 \in \Theta$, and Ω is geometric/asymptotic left separable.*

It remains now to expose some examples of relations Ω with such properties.

Ex-I) Letting $\chi \in \mathscr{F}(re)(R_+^0, R)$, consider the optional extra conditions:

(MK-adm) χ is *Meir–Keeler admissible*:
$\forall\gamma > 0, \exists\beta > 0, (\forall t)$: $\gamma < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$
(BW-s-adm) χ is *strongly Boyd–Wong admissible*: $\Lambda^{\pm}\chi(s) < s$, for all $s > 0$.

Given the functional couple (P, G) over $\mathscr{F}(X \times X, R_+)$, let us say that $(P_0 - G, P)$ is $(d, \leq; \chi)$*-contractive*, if

$(P_0 - G)(x, y) \leq \chi(P(x, y)), \forall x, y \in X, x \leq y, P(x, y) > 0.$

As a direct consequence of the above result, we get the following *rational* type fixed point statement (referred to as *Rational Function Boyd–Wong theorem*; in short, (BW-f-ra)).

Theorem 7 *Assume that $(P_0 - G, E(\Theta))$ is $(d, \leq; \chi)$-contractive, for some regular mapping $G : X \times X \to R_+$, some subset $\Theta \in Z^*$ and some function $\chi \in \mathscr{F}(re)(R_+^0, R)$ with the Meir–Keeler admissible property. In addition, let X be (a-o-f,d)-complete. Then, the following conclusions hold:*

(72-a) *T is a strong Picard operator (modulo $(d, \leq)$), provided (in addition) T is (a-o-f,d)-continuous;*

(72-b) *T is a Bellman Picard operator (modulo $(d, \leq)$), provided $(\leq)$ is (a-o-f,d)-selfclosed, and one of the following groups of extra conditions is holding:*

(72-b1) $\{1, 6\} \cap \Theta = \emptyset$*;*
(72-b2) $\{1, 6\} \cap \Theta \neq \emptyset$ *and χ is strongly Boyd–Wong admissible.*

Proof Let $\Omega := \Omega[\chi]$ stand for the relation over R_+^0

$(t, s \in R_+^0)$: $(t, s) \in \Omega$ iff $t \leq \chi(s)$ [clearly, Ω is upper diagonal].

By a series of auxiliary facts involving these objects, we have

(p-1) Ω is Meir–Keeler admissible when $\chi \in \mathscr{F}(re)(R_+^0, R)$ is endowed with the Meir–Keeler admissible property;

(p-2) Ω is geometric/asymptotic bilateral separable when $\chi \in \mathscr{F}(re)(R_+^0, R)$ is strongly Boyd–Wong admissible;

(p-3) Ω is geometric/asymptotic left separable;

and, from this, all is clear.

(Ex-II) Let (ψ, φ) be a pair of functions over $\mathscr{F}(R_+^0, R)$, fulfilling

(norm) (ψ, φ) is *normal*:
ψ is increasing and φ is *strictly positive* ($\varphi(R_+^0) \subseteq R_+^0$).

The following extra conditions upon this normal couple are to be considered:

(as-pos) φ is *asymptotic positive*:
for each strictly descending sequence $(t_n; n \geq 0)$ in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon+$, we must have $\limsup_n(\varphi(t_n)) > 0$.
(bd-osc) (ψ, φ) is *limit-bounded oscillating*:
for each sequence $(t_n; n \geq 0)$ in R_+^0 and each $\beta > 0$ with $t_n \to \beta$, we have $\limsup_n(\varphi(t_n)) > \psi(\beta + 0) - \psi(\beta - 0)$.
(bd-le-osc) (ψ, φ) is *bounded left oscillating*:
for each $\beta > 0$, we have $\varphi(\beta) > \psi(\beta) - \psi(\beta - 0)$.

Given the couple of maps (P, G) over $\mathscr{F}(X \times X, R_+)$ and the (normal) couple (ψ, φ) over $\mathscr{F}(R_+^0, R)$, let us say that $(P_0 - G, P)$ is $(d, \leq; \psi, \varphi)$-*contractive* if

$$\psi((P_0 - G)(x, y)) \leq \psi(P(x, y)) - \varphi(P(x, y)),$$
$$\forall x, y \in X, x \leq y, (P_0 - G)(x, y) > 0, P(x, y) > 0.$$

As another direct consequence of the above result, we get the following *rational* type fixed point statement (referred to as *Rational Function Rhoades theorem*; in short, (R-f-ra)).

Theorem 8 *Assume that $(P_0 - G, E(\Theta))$ is $(d, \leq; \psi, \varphi)$-contractive, for some regular mapping $G : X \times X \to R_+$, some subset $\Theta \in Z^*$ and some normal couple (ψ, φ) over $\mathscr{F}(R_+^0, R)$, with φ=asymptotic positive. In addition, let X be (a-o-f,d)-complete. Then,*

(73-a) *T is a strong Picard operator (modulo $(d, \leq)$), provided (in addition) T is (a-o-f,d)-continuous;*
(73-b) *T is a Bellman Picard operator (modulo $(d, \leq)$), provided $(\leq)$ is (a-o-f,d)-selfclosed, and one of the following groups of extra conditions is holding:*
(73-b1) $\{0, 1, 6\} \cap \Theta = \emptyset$;
(73-b2) $\{0, 1, 6\} \cap \Theta \neq \emptyset$ *and (ψ, φ) is limit-bounded oscillating;*
(73-b3) $\{1, 6\} \cap \Theta = \emptyset$, $0 \in \Theta$, *and (ψ, φ) is bounded left oscillating.*

Proof Let $\Omega := \Omega[\psi, \varphi]$ stand for the relation over R_+^0

$(t, s \in R_+^0)$: $(t, s) \in \Omega$ iff $\psi(t) \leq \psi(s) - \varphi(s)$ (clearly, Ω is upper diagonal).

By a series of auxiliary facts involving these objects, we have

(q-1) Ω is Meir–Keeler admissible when φ is asymptotic positive;
(q-2) Ω is geometric/asymptotic bilateral separable when the couple (ψ, φ) is limit-bounded oscillating;
(q-3) Ω is geometric/asymptotic left separable, when the couple (ψ, φ) is bounded left oscillating;

and, from this, all is clear.

Some particular cases of this result may be described as follows.

Case 1 Suppose that $(\leq)$ is the trivial quasi-order on X and $G = 0$. Then, the Rational Function Boyd–Wong theorem (BW-f-ra) includes in a direct way the basic statements due to Boyd and Wong [6], Matkowski [25], and Leader [24], and the Rational Function Rhoades theorem (R-f-ra) extends some contributions in Dutta and Choudhury [14].

Case 2 Suppose that $(\leq)$ is a (partial) order and $G = 0$. Then, the Rational Function Boyd–Wong theorem (BW-f-ra) includes the related statements in Agarwal et al. [1] and Cabrera et al. [7], and the Rational Function Rhoades theorem (R-f-ra) extends some related facts in Yadava et al., [49]; see also Turinici [47].

Case 3 Suppose that $(\leq)$ is again a (partial) order. Then, under $G = J(U)$, the Rational Function Boyd–Wong theorem (BW-f-ra) includes the related statement in Aydi et al [2]. The variant of this result with $H = J(W, V)$ seems to be new.

Finally, it is worth noting that, by the used techniques, our particular fixed point statement does not include the one in Chandok et al. [8]. However, if one starts from a certain refinement of our developments, this inclusion holds; further aspects will be considered elsewhere.

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On $G^{(\sigma,h)}$-Convexity of the Functions and Applications to Hermite-Hadamard's Inequality

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Abstract The aim of this chapter is to introduce the notion of $G^{(\sigma,h)}$-convex functions a generalized exponentially (σ, h)-convex functions. We show that for suitable choices of real function $h(.)$, the class of $G^{(\sigma,h)}$-convex functions reduces to some other new classes of G^{σ}-convex functions. We also show that for $G = \exp$, we have another new class which is called as $G^{(\sigma,h)}$-convex function. For the applications of this class we derive some new variants of Hermite-Hadamard's inequality using the class of $G^{(\sigma,h)}$-convex functions. In the last section, we define the class of strongly $G^{(\sigma,h)}$-convexity. We also derive a new Hermite-Hadamard like inequality involving strongly $G^{(\sigma,h)}$-convexity. Several new special cases which can be deduced from the main results of the chapter are also discussed.

1 Introduction and Preliminaries

An interval I is said to be a σ-convex set, if

$$M_{\sigma}(u, v; \mu) = [\mu u^{\sigma} + (1 - \mu)v^{\sigma}]^{\frac{1}{\sigma}} \in I, \quad \forall u, v \in I, \mu \in [0, 1], \sigma \neq 0.$$

And function $X : I \to \mathbb{R}$ is said to be σ-convex function, if

$$X(M_{\sigma}(u, v; \mu)) \leq \mu X(u) + (1 - \mu)X(v), \ \forall u, v \in I, \mu \in [0, 1].$$

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Fang and Shi [11] generalized the notion of σ-convex functions and introduced the notion of (σ, h)-convex functions.

Definition 1 ([11]) Let $h : (0, 1) \to \mathbb{R}$ be a real function. We say that $X : I \to \mathbb{R}$ be a (σ, h)-convex functions, if

$$X(M_\sigma(u, v; \mu)) \leq h(\mu)X(u) + h(1-\mu)X(v),\ \forall u, v \in I, \mu \in (0, 1).$$

Note that for $\sigma = 1$, the class of (σ, h)-convex functions reduces to the class of h-convex functions which was introduced and studied by Varosanec [20]. We can recapture the class of harmonic convex functions [14] from the class of (σ, h)-convex functions by taking $\sigma = -1$. We can also get other classes of σ-convexity by taking suitable choices of function $h(.)$. For some recent studies on (σ, h)-convex functions and its generalizations, see [2, 15, 17, 18].
Dragomir and Gomm introduced the class of exponential convex functions as:

Definition 2 ([7]) A real valued function $X : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex function, if

$$\mathbf{e}^{X(\mu a_1+(1-\mu)a_2)} \leq \mu \mathbf{e}^{X(a_1)} + (1-\mu)\mathbf{e}^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in [0, 1].$$

Polyak [19] introduced the notion of strongly convex functions and discussed its basic properties. For some recent investigations on strong convexity property of functions, see [1–3].
Hermite-Hadamard's inequality is one of the most studied inequality involving convex functions. This result provides us a necessary and sufficient condition for a function to be convex. It reads as:

Theorem 1 *Let $X : I = [a_1, a_2] \to \mathbb{R}$ be a convex function. If $X \in [a_1, a_2]$, then*

$$X\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} X(u)\mathrm{d}u \leq \frac{X(a_1) + X(a_2)}{2}.$$

Fang and Shi [11] also proved several variants of classical inequalities including Hermite-Hadamard's inequality utilizing the notion of (σ, h)-convex functions. Dragomir and Gomm [7] obtained a variant of Hermite-Hadamard's inequality using exponentially convex function. For some more details on convexity, its generalizations and related inequalities interested readers are referred to [1, 3–5, 8–10, 12, 13, 19, 21].
Research work going on in the field of inequalities involving convexity and its generalizations leads us to give a new generalization of exponential convex functions. This new class is named as $\mathrm{G}^{(\sigma,h)}$-convex functions. As special cases of this class we also define some other classes of exponentially convexity. We also derive some new variants of Hermite-Hadamard's inequality using the class of $\mathrm{G}^{(\sigma,h)}$-convex functions and discuss some new special cases of the obtained result.

In the last section, we define the notion of strongly $G^{(\sigma,h)}$-convex functions and derive a new version of Hermite-Hadamard's inequality. This is the main motivation of this chapter.

2 Results and Discussions

2.1 $G^{(\sigma,h)}$-Convexity

In this section, we first define the class of $G^{(\sigma,h)}$-convex functions and discuss some other types of exponentially convex functions. After that we derive some new variants of Hermite-Hadamard's inequality using the class of $G^{(\sigma,h)}$-convex functions.

Definition 3 Let $h : (0, 1) \to \mathbb{R}$ be a real function. We say that a function $X : I \to \mathbb{R}$ is said to be an $G^{(\sigma,h)}$-convex function with $G \geq 1$, if

$$G^{X\left([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}}\right)} \leq h(\mu)G^{X(a_1)} + h(1-\mu)G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in (0, 1).$$

Note that if we take $h(\mu) = \mu$ in Definition 3, then we have the class of G^{σ}-convex function.

Definition 4 A function $X : I \to \mathbb{R}$is said to be an G^{σ}-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq \mu G^{X(a_1)} + (1-\mu)G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in [0, 1].$$

If we take $h(\mu) = \mu^s$ where $s \in (0, 1]$ in Definition 3, then we have the class of Breckner type of $G^{(\sigma,s)}$-convex function.

Definition 5 A function $X : I \to \mathbb{R}$ is said to be Breckner type of $G^{(\sigma,s)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq \mu^s G^{X(a_1)} + (1-\mu)^s G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in [0, 1].$$

If we take $h(\mu) = \mu^{-s}$ where $s \in [0, 1]$ in Definition 3, then we have the class of Godunova-Levin type of $G^{(\sigma,s)}$-convex function.

Definition 6 A function $X : I \to \mathbb{R}$ is said to be Godunova-Levin type of $G^{(\sigma,s)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq \mu^{-s} G^{X(a_1)} + (1-\mu)^{-s} G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in (0, 1).$$

If we take $h(\mu) = 1$ in Definition 3, then we have the class of $G^{(\sigma,P)}$-convex function.

Definition 7 A function $X : I \to \mathbb{R}$ is said to be $G^{(\sigma,P)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq G^{X(a_1)} + G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in (0,1).$$

If we take $h(\mu) = \mu(1-\mu)$ in Definition 3, then we have the class of $G^{(\sigma,tgs)}$-convex function.

Definition 8 A function $X : I \to \mathbb{R}$ is said to be $G^{(\sigma,tgs)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq \mu(1-\mu)[G^{X(a_1)} + G^{X(a_2)}], \quad \forall a_1, a_2 \in I, \mu \in (0,1).$$

Note that if we take $G = 1$, then we have the class of (σ, h)-convex functions [11]. We would like to point out here that if we take $\sigma = 1$, then we have different classes G^h-convex functions. Also if we take $\sigma = -1$, then we have some new classes of harmonically G^h-convex functions. For example, if we take $\sigma = -1$ in Definition 3, then we have the class of harmonically G^h-convex function.

Definition 9 Let $h : (0,1) \to \mathbb{R}$ be a real function. We say that a function $X : I \setminus \{0\} \to \mathbb{R}$ is said to be a harmonically G^h-convex function, if

$$G^{X\left(\frac{a_1 a_2}{(1-\mu)a_1+\mu a_2}\right)} \leq h(\mu)G^{X(a_1)} + h(1-\mu)G^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in (0,1).$$

Now for different suitable choices of real function $h(.)$, we have some other new classes of harmonically G^h-convex function. We left the details for interested readers.
Another interesting special case of Definition 3 is taking $G = \exp$, then we have the class of $\mathbf{e}^{\sigma,h}$-convex functions, which is defined as:

Definition 10 Let $h : (0,1) \to \mathbb{R}$ be a real function. We say that a function $X : I \to \mathbb{R}$ is said to be an $\mathbf{e}^{(\sigma,h)}$-convex function, if

$$\mathbf{e}^{X([\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}]^{\frac{1}{\sigma}})} \leq h(\mu)\mathbf{e}^{X(a_1)} + h(1-\mu)\mathbf{e}^{X(a_2)}, \quad \forall a_1, a_2 \in I, \mu \in (0,1).$$

We now derive new variants of Hermite-Hadamard's inequality using the Definition 3.

Theorem 2 *Let $h : (0,1) \to \mathbb{R}$ be a real function. If $X : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is an $G^{(\sigma,h)}$-convex function and $X \in L[a_1, a_2]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} \leq \frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma} \int_{a_1}^{a_2} u^{\sigma-1} G^{X(u)} du$$

$$\leq [G^{X(a_1)} + G^{X(a_2)}] \int_0^1 h(\mu) d\mu.$$

Proof Since it is given that X is an $G^{(\sigma,h)}$-convex function, then

$$G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} \leq h\left(\frac{1}{2}\right)\left[G^{X([\mu a_1{}^\sigma + (1-\mu) a_2{}^\sigma]^{\frac{1}{\sigma}})} + G^{X\left([(1-\mu) a_1{}^\sigma + \mu a_2{}^\sigma]^{\frac{1}{\sigma}}\right)}\right].$$

Integrating both sides of above inequality, we have

$$\int_0^1 G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} d\mu$$

$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 G^{X\left([\mu a_1{}^\sigma + (1-\mu) a_2{}^\sigma]^{\frac{1}{\sigma}}\right)} d\mu + \int_0^1 G^{X\left([(1-\mu) a_1{}^\sigma + \mu a_2{}^\sigma]^{\frac{1}{\sigma}}\right)} d\mu\right].$$

This implies

$$\frac{1}{2h\left(\frac{1}{2}\right)} G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} \leq \frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma} \int_{a_1}^{a_2} u^{\sigma-1} G^{X(u)} du. \tag{1}$$

Also

$$G^{X([\mu a_1{}^\sigma + (1-\mu) a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq h(\mu) G^{X(a_1)} + h(1-\mu) G^{X(a_2)}.$$

Integrating both sides of above inequality, we have

$$\frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma} \int_{a_1}^{a_2} u^{\sigma-1} G^{X(u)} du \leq [G^{X(a_1)} + G^{X(a_2)}] \int_0^1 h(\mu) d\mu. \tag{2}$$

On summation of (1) and (2), we have the required results.

We now discuss some special cases of Theorem 2.

I. If $h(\mu) = \mu$ in Theorem 2, then we have a new result for G^σ-convex function, which reads as:

Corollary 1 *Under the assumptions of Theorem 2, if X is* G^{σ}*-convex function, then*

$$\mathrm{G}^{X}\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{d}u \leq \frac{\mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}}{2}.$$

II. If $h(\mu)=\mu^{s}$ in Theorem 2, then we have a new result for Breckner type of $\mathrm{G}^{(\sigma,s)}$-convex function, which reads as:

Corollary 2 *Under the assumptions of Theorem 2, if X is Breckner type of* $\mathrm{G}^{(\sigma,s)}$*-convex function, then*

$$2^{s-1}\mathrm{G}^{X}\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{d}u \leq \frac{\mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}}{s+1}.$$

III. If $h(\mu)=\mu^{-s}$ in Theorem 2, then we have a new result for Godunova-Levin type of $\mathrm{G}^{(\sigma,s)}$-convex function, which reads as:

Corollary 3 *Under the assumptions of Theorem 2, if X is Godunova-Levin type of* $\mathrm{G}^{(\sigma,s)}$*-convex function, then*

$$\frac{1}{2^{s+1}}\mathrm{G}^{X}\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{d}u \leq \frac{\mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}}{1-s}.$$

IV. If $h(\mu)=1$ in Theorem 2, then we have a new result for $\mathrm{G}^{(\sigma,P)}$-convex function, which reads as:

Corollary 4 *Under the assumptions of Theorem 2, if X is* $\mathrm{G}^{(\sigma,P)}$*-convex function, then*

$$\frac{1}{2}\mathrm{G}^{X}\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{d}u \leq \mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}.$$

V. If $h(\mu)=\mu(1-\mu)$ in Theorem 2, then we have a new result for $\mathrm{G}^{(\sigma,tgs)}$-convex function, which reads as:

Corollary 5 *Under the assumptions of Theorem 2, if X is* $\mathrm{G}^{(\sigma,tgs)}$*-convex function, then*

$$2G^{X\left(\left[\frac{a_1{}^\sigma+a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} \leq \frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma}\int_{a_1}^{a_2} u^{\sigma-1}G^{X(u)}du \leq \frac{G^{X(a_1)}+G^{X(a_2)}}{6}.$$

VI. Now if we take $\sigma = -1$ in Theorem 2, then we have a new result for the class of harmonically $G^{(\sigma,h)}$-convex functions.

Corollary 6 *Let* $h : (0, 1) \to \mathbb{R}$ *be a real function. If* $X : I = [a_1, a_2] \subset \mathbb{R}_+ \to \mathbb{R}$ *is an harmonically* $G^{(\sigma,h)}$*-convex function and* $X \in L[a_1, a_2]$*, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}G^{X(u)}du \leq [G^{X(a_1)}+G^{X(a_2)}]\int_0^1 h(\mu)d\mu.$$

VII. If we take $h(\mu) = \mu$ in Corollary 6, we have the result for harmonically $\exp -\sigma$-convex function.

Corollary 7 *Under the assumptions of Corollary 6, if X is harmonically* $\exp -\sigma$*-convex function, then*

$$G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}G^{X(u)}du \leq \frac{G^{X(a_1)}+G^{X(a_2)}}{2}.$$

VIII. If we take $h(\mu) = \mu^s$ in Corollary 6, we have the result for Breckner type of harmonically $G^{(\sigma,s)}$-convex function.

Corollary 8 *Under the assumptions of Corollary 6, if X is Breckner type of harmonically* $G^{(\sigma,s)}$*-convex function, then*

$$2^{s-1}G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}G^{X(u)}du \leq \frac{G^{X(a_1)}+G^{X(a_2)}}{s+1}.$$

IX. If we take $h(\mu) = \mu^{-s}$ in Corollary 6, we have the result for Godunova-Levin type of harmonically $G^{(\sigma,s)}$-convex function.

Corollary 9 *Under the assumptions of Corollary 6, if X is Godunova-Levin type of harmonically* $G^{(\sigma,s)}$*-convex function, then*

$$\frac{1}{2^{s+1}}G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}G^{X(u)}du \leq \frac{G^{X(a_1)}+G^{X(a_2)}}{1-s}.$$

X. If we take $h(\mu) = 1$ in Corollary 6, we have the result for harmonically $G^{(\sigma,P)}$-convex function.

Corollary 10 *Under the assumptions of Corollary 6, if X is harmonically* $\mathrm{G}^{(\sigma,P)}$*-convex function, then*

$$\frac{1}{2}\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}\mathrm{G}^{X(u)}\mathrm{d}u \leq \mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}.$$

XI. If we take $h(\mu)=\mu(1-\mu)$ in Corollary 6, we have the result for harmonically $\mathrm{G}^{(\sigma,tgs)}$-convex function.

Corollary 11 *Under the assumptions of Corollary 6, if X is harmonically* $\mathrm{G}^{(\sigma,tgs)}$*-convex function, then*

$$2\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)} \leq \frac{a_1a_2}{a_2-a_1}\int_{a_1}^{a_2} u^{-2}\mathrm{G}^{X(u)}\mathrm{d}u \leq \frac{\mathrm{G}^{X(a_1)}+\mathrm{G}^{X(a_2)}}{6}.$$

We now drive a new variant of Hermite-Hadamard's inequality utilizing product of two $\mathrm{G}^{(\sigma,h)}$-convex functions. First of all for the sake of simplicity, we let $M(a_1,a_2) := \mathrm{G}^{X(a_1)}\mathrm{G}^{Y(a_1)}+\mathrm{G}^{X(a_2)}\mathrm{G}^{Y(a_2)}$ and $N(a_1,a_2) := \mathrm{G}^{X(a_1)}\mathrm{G}^{Y(a_2)}+\mathrm{G}^{X(a_2)}\mathrm{G}^{Y(a_1)}$.

Theorem 3 *Let $h_1,h_2:(0,1)\to\mathbb{R}$ be two real functions, $X:I=[a_1,a_2]\subset\mathbb{R}\to\mathbb{R}$ be an* $\mathrm{G}^{(\sigma,h_1)}$*-convex function, $Y:I=[a_1,a_2]\subset\mathbb{R}\to\mathbb{R}$ be an* $\mathrm{G}^{(\sigma,h_2)}$*-convex function and $XY\in L[a_1,a_2]$, then*

$$\begin{aligned}&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathrm{G}^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\\
&\quad-\left[M(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu\right]\\
&\leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u\\
&\leq M(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu.\end{aligned}$$

Proof Since X and Y are $G^{(\sigma,h_1)}$ and $G^{(\sigma,h_2)}$ convex functions, respectively, so we have

$$
\begin{aligned}
&G^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\\
&= G^{X\left(\left[\frac{\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}}{2}+\frac{(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[\frac{\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}}{2}+\frac{(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\\
&\le h_1\left(\frac{1}{2}\right)\left[G^{X\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}+G^{X\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\times h_2\left(\frac{1}{2}\right)\left[G^{Y\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}+G^{X\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[G^{X\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[G^{X\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[h_1(\mu)G^{X(a_1)}+h_1(1-\mu)G^{X(a_2)}\right]\\
&\quad\times\left[h_2(1-\mu)G^{Y(a_1)}+h_2(\mu)G^{Y(a_2)}\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[h_1(1-\mu)G^{X(a_1)}+h_1(\mu)G^{X(a_2)}\right]\\
&\quad\times\left[h_2(\mu)G^{Y(a_1)}+h_2(1-\mu)G^{Y(a_2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[G^{X\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[G^{X\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}G^{Y\left(\left[(1-\mu)a_1{}^{\sigma}+\mu a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\{h_1(\mu)h_2(1-\mu)+h_1(1-\mu)h_2(\mu)\}M(a_1,a_2)\right]\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\{h_1(\mu)h_2(\mu)+h_1(1-\mu)h_2(1-\mu)\}N(a_1,a_2)\right].
\end{aligned}
$$

After integrating with respect to μ on [0, 1], we have

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathrm{G}^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}$$
$$-\left[M(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu\right]$$
$$\leq\frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u. \tag{3}$$

Also since X and Y are $\mathrm{G}^{(\sigma,h_1)}$ and $\mathrm{G}^{(\sigma,h_2)}$ convex functions, respectively, so we have

$$\mathrm{G}^{X\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\mu a_1{}^{\sigma}+(1-\mu)a_2{}^{\sigma}\right]^{\frac{1}{\sigma}}\right)}$$
$$\leq h_1(\mu)h_2(\mu)\mathrm{G}^{X(a_1)}\mathrm{G}^{Y(a_1)}+h_1(1-\mu)h_2(\mu)\mathrm{G}^{X(a_2)}\mathrm{G}^{Y(a_1)}$$
$$+h_1(\mu)h_2(1-\mu)\mathrm{G}^{X(a_1)}\mathrm{G}^{Y(a_2)}+h_1(1-\mu)h_2(1-\mu)\mathrm{G}^{X(a_2)}\mathrm{G}^{Y(a_2)}.$$

Integrating both sides of above inequality with respect to μ on [0, 1] yields

$$\frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u$$
$$\leq M(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu. \tag{4}$$

Summing up inequalities (3) and (4) completes the proof.

We now discuss some special cases of Theorem 3.

I. If $h_1(\mu)=\mu=h_2(\mu)$ in Theorem 3, then we have a new result for $\exp-\sigma$-convex function, which reads as:

Corollary 12 *Under the assumptions of Theorem 3, if X and Y are* $\exp-\sigma$*-convex function, then*

$$2\mathrm{G}^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)}-\left[\frac{1}{6}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2)\right]$$

$$\leq \frac{\sigma}{a_2{}^{\sigma} - a_1{}^{\sigma}} \int_{a_1}^{a_2} u^{\sigma-1} \mathrm{G}^{X(u)} \mathrm{G}^{Y(u)} \mathrm{d}u \leq \frac{1}{3} M(a_1, a_2) + \frac{1}{6} N(a_1, a_2).$$

II. If $h_1(\mu) = \mu^s = h_2(\mu)$ in Theorem 3, then we have a new result for Breckner type of $\mathrm{G}^{(\sigma,s)}$-convex function, which reads as:

Corollary 13 *Under the assumptions of Theorem 3, if X and Y are Breckner type of* $\mathrm{G}^{(\sigma,s)}$*-convex function, then*

$$\begin{aligned}
&\frac{1}{2^{1-2s}} \mathrm{G}^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)} \mathrm{G}^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)} \\
&\quad - \left[B(s+1, s+1) M(a_1, a_2) + \frac{1}{2s+1} N(a_1, a_2) \right] \\
&\quad \leq \frac{\sigma}{a_2{}^{\sigma} - a_1{}^{\sigma}} \int_{a_1}^{a_2} u^{\sigma-1} \mathrm{G}^{X(u)} \mathrm{G}^{Y(u)} \mathrm{d}u \\
&\quad \leq \frac{1}{2s+1} M(a_1, a_2) + B(s+1, s+1) N(a_1, a_2).
\end{aligned}$$

III. If $h_1(\mu) = \mu^{-s} = h_2(\mu)$ in Theorem 3, then we have a new result for Godunova-Levin type of $\mathrm{G}^{(\sigma,s)}$-convex function, which reads as:

Corollary 14 *Under the assumptions of Theorem 3, if X and Y are Godunova-Levin type of* $\mathrm{G}^{(\sigma,s)}$*-convex function, then*

$$\begin{aligned}
&\frac{1}{2^{1+2s}} \mathrm{G}^{X\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)} \mathrm{G}^{Y\left(\left[\frac{a_1{}^{\sigma}+a_2{}^{\sigma}}{2}\right]^{\frac{1}{\sigma}}\right)} \\
&\quad - \left[B(1-s, 1-s) M(a_1, a_2) + \frac{1}{1-2s} N(a_1, a_2) \right] \\
&\quad \leq \frac{\sigma}{a_2{}^{\sigma} - a_1{}^{\sigma}} \int_{a_1}^{a_2} u^{\sigma-1} \mathrm{G}^{X(u)} \mathrm{G}^{Y(u)} \mathrm{d}u \\
&\quad \leq \frac{1}{1-2s} M(a_1, a_2) + B(1-s, 1-s) N(a_1, a_2).
\end{aligned}$$

IV. If $h_1(\mu) = 1 = h_2(\mu)$ in Theorem 3, then we have a new result for $\mathrm{G}^{(\sigma,P)}$-convex function, which reads as:

Corollary 15 *Under the assumptions of Theorem 3, if X and Y are* $\mathrm{G}^{(\sigma,P)}$*-convex function, then*

$$\frac{1}{2}\mathrm{G}^{X\left(\left[\frac{a_1^\sigma+a_2^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\frac{a_1^\sigma+a_2^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)}-[M(a_1,a_2)+N(a_1,a_2)]$$
$$\leq\frac{\sigma}{a_2^\sigma-a_1^\sigma}\int_{a_1}^{a_2}u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u\leq M(a_1,a_2)+N(a_1,a_2).$$

V. If $h_1(\mu)=\mu(1-\mu)=h_2(\mu)$ in Theorem 3, then we have a new result for $\mathrm{G}^{(\sigma,tgs)}$-convex function, which reads as:

Corollary 16 *Under the assumptions of Theorem 3, if X and Y are* $\mathrm{G}^{(\sigma,tgs)}$*-convex function, then*

$$8\mathrm{G}^{X\left(\left[\frac{a_1^\sigma+a_2^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)}\mathrm{G}^{Y\left(\left[\frac{a_1^\sigma+a_2^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)}-\frac{1}{30}[M(a_1,a_2)+N(a_1,a_2)]$$
$$\leq\frac{\sigma}{a_2^\sigma-a_1^\sigma}\int_{a_1}^{a_2}u^{\sigma-1}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u\leq\frac{1}{30}[M(a_1,a_2)+N(a_1,a_2)].$$

VI. Now if we take $\sigma=-1$, then we have the result for harmonically $\mathrm{G}^{(\sigma,h)}$-convex functions.

Corollary 17 *Under the assumptions of Theorem 3, if* $X:I=[a_1,a_2]\subset(0,\infty)\to\mathbb{R}$ *is an harmonically* $\mathrm{G}^{(\sigma,h_1)}$*-convex function and harmonically* $Y:I=[a_1,a_2]\subset(0,\infty)\to\mathbb{R}$ *is an harmonically* $\mathrm{G}^{(\sigma,h_2)}$*-convex function and* $XY\in L[a_1,a_2]$*, then we have*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}\mathrm{G}^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)}$$
$$-\left[M(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu\right]$$
$$\leq\frac{\sigma}{a_2^\sigma-a_1^\sigma}\int_{a_1}^{a_2}u^{-2}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u$$
$$\leq M(a_1,a_2)\int_0^1 h_1(\mu)h_2(\mu)\mathrm{d}\mu+N(a_1,a_2)\int_0^1 h_1(\mu)h_2(1-\mu)\mathrm{d}\mu.$$

VII. If we take $h_1(\mu)=\mu=h_2(\mu)$ in Corollary 17, then we have result for harmonically exp-convex function.

Corollary 18 *Under the assumptions of Corollary 17, if* $X, Y : I = [a_1, a_2] \subset (0, \infty) \to \mathbb{R}$ *are harmonically* $\exp -\sigma$*-convex functions and* $XY \in L[a_1, a_2]$*, then we have*

$$2\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}\mathrm{G}^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)}\left[\frac{1}{6}M(a_1,a_2)+\frac{1}{3}N(a_1,a_2)\right]$$

$$\leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{-2}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u \leq \frac{1}{3}M(a_1,a_2)+\frac{1}{6}N(a_1,a_2).$$

VIII. If $h_1(\mu) = \mu^s = h_2(\mu)$ in Corollary 17, then we have a new result for Breckner type of harmonically $\exp -s$-convex function, which reads as:

Corollary 19 *Under the assumptions of Corollary 17, if X and Y are Breckner type of harmonically* $\exp -s$*-convex function, then*

$$\frac{1}{2^{1-2s}}\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}\mathrm{G}^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)} - \left[B(s+1,s+1)M(a_1,a_2)+\frac{1}{2s+1}N(a_1,a_2)\right]$$

$$\leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{-2}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u$$

$$\leq \frac{1}{2s+1}M(a_1,a_2)+B(s+1,s+1)N(a_1,a_2).$$

IX. If $h_1(\mu) = \mu^{-s} = h_2(\mu)$ in Corollary 17, then we have a new result for Godunova-Levin type of harmonically $\exp -s$-convex function, which reads as:

Corollary 20 *Under the assumptions of Corollary 17, if X and Y are Godunova-Levin type of harmonically* $\mathrm{G}^{(\sigma,s)}$*-convex function, then*

$$\frac{1}{2^{1+2s}}\mathrm{G}^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}\mathrm{G}^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)} - \left[B(1-s,1-s)M(a_1,a_2)+\frac{1}{1-2s}N(a_1,a_2)\right]$$

$$\leq \frac{\sigma}{a_2{}^{\sigma}-a_1{}^{\sigma}}\int_{a_1}^{a_2} u^{-2}\mathrm{G}^{X(u)}\mathrm{G}^{Y(u)}\mathrm{d}u$$

$$\leq \frac{1}{1-2s}M(a_1,a_2)+B(1-s,1-s)N(a_1,a_2).$$

X. If $h_1(\mu) = 1 = h_2(\mu)$ in Corollary 17, then we have a new result for harmonically $\exp -P$-convex function, which reads as:

Corollary 21 *Under the assumptions of Theorem 3, if X and Y are harmonically* $\exp -P$*-convex function, then*

$$\frac{1}{2}G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}G^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)} - [M(a_1,a_2)+N(a_1,a_2)] \leq \frac{\sigma}{{a_2}^{\sigma}-{a_1}^{\sigma}}\int_{a_1}^{a_2} u^{-2}G^{X(u)}G^{Y(u)}\mathrm{d}u \leq M(a_1,a_2)+N(a_1,a_2).$$

XI. If $h_1(\mu) = \mu(1-\mu) = h_2(\mu)$ in Corollary 17, then we have a new result for harmonically $\exp -tgs$-convex function, which reads as:

Corollary 22 *Under the assumptions of Corollary 17, if X and Y are harmonically* $\exp -tgs$*-convex function, then*

$$8G^{X\left(\frac{2a_1a_2}{a_1+a_2}\right)}G^{Y\left(\frac{2a_1a_2}{a_1+a_2}\right)} - \frac{1}{30}[M(a_1,a_2)+N(a_1,a_2)] \leq \frac{\sigma}{{a_2}^{\sigma}-{a_1}^{\sigma}}\int_{a_1}^{a_2} u^{-2}G^{X(u)}G^{Y(u)}\mathrm{d}u \leq \frac{1}{30}\left[M(a_1,a_2)+N(a_1,a_2)\right].$$

2.2 *Strongly* $G^{(\sigma,h)}$*-Convexity*

We now define the class of strongly $G^{(\sigma,h)}$-convex functions.

Definition 11 Let $h : (0,1) \to \mathbb{R}$ be a real function. We say that a function $X : I \to \mathbb{R}$ is said to be strongly $G^{(\sigma,h)}$-convex function with $G \geq 1$, if

$$G^{X([\mu {a_1}^{\sigma}+(1-\mu){a_2}^{\sigma}]^{\frac{1}{\sigma}})} \leq h(\mu)G^{X(a_1)} + h(1-\mu)G^{X(a_1)} - \mu\mu(1-\mu)({a_2}^{\sigma}-{a_1}^{\sigma})^2,$$
$$\forall a_1, a_2 \in I, \mu \in (0,1), \mu > 0.$$

Now if we take $h(\mu) = \mu$ in Definition 11, then we have the class of strongly G^{σ}-convex function.

Definition 12 A function $X : I \to \mathbb{R}$ is said to be strongly G^{σ}-convex function with $G \geq 1$, if

$$G^{X([\mu {a_1}^{\sigma}+(1-\mu){a_2}^{\sigma}]^{\frac{1}{\sigma}})} \leq \mu G^{X(a_1)} + (1-\mu)G^{X(a_1)} - \mu\mu(1-\mu)({a_2}^{\sigma}-{a_1}^{\sigma})^2,$$
$$\forall a_1, a_2 \in I, \mu \in [0,1], \mu > 0.$$

If we take $h(\mu) = \mu^s$ where $s \in (0, 1]$ in Definition 11, then we have the class of Breckner type of strongly $G^{(\sigma,s)}$-convex function.

Definition 13 A function $X : I \to \mathbb{R}$is said to be Breckner type of strongly $G^{(\sigma,s)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq \mu^s G^{X(a_1)} + (1-\mu)^s G^{X(a_1)} - \mu\mu(1-\mu)(a_2{}^\sigma - a_1{}^\sigma)^2,$$
$$\forall a_1, a_2 \in I, \mu \in [0, 1], \mu > 0.$$

If we take $h(\mu) = \mu^{-s}$ where $s \in [0, 1]$ in Definition 11, then we have the class of Godunova-Levin type of strongly $G^{(\sigma,s)}$-convex function.

Definition 14 A function $X : I \to \mathbb{R}$is said to be Godunova-Levin type of strongly $G^{(\sigma,s)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq \mu^{-s} G^{X(a_1)} + (1-\mu)^{-s} G^{X(a_1)} - \mu\mu(1-\mu)(a_2{}^\sigma - a_1{}^\sigma)^2,$$
$$\forall a_1, a_2 \in I, \mu \in (0, 1), \mu > 0.$$

If we take $h(\mu) = 1$ in Definition 11, then we have the class of strongly $G^{(\sigma,P)}$-convex function.

Definition 15 A function $X : I \to \mathbb{R}$ is said to be strongly $G^{(\sigma,P)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq G^{X(a_1)} + G^{X(a_1)} - \mu\mu(1-\mu)(a_2{}^\sigma - a_1{}^\sigma)^2,$$
$$\forall a_1, a_2 \in I, \mu \in [0, 1], \mu > 0.$$

If we take $h(\mu) = \mu(1-\mu)$ in Definition 11, then we have the class of strongly $G^{(\sigma,tgs)}$-convex function.

Definition 16 A function $X : I \to \mathbb{R}$ is said to be strongly $G^{(\sigma,tgs)}$-convex function with $G \geq 1$, if

$$G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq \mu(1-\mu)[G^{X(a_1)} + G^{X(a_1)}] - \mu\mu(1-\mu)(a_2{}^\sigma - a_1{}^\sigma)^2,$$
$$\forall a_1, a_2 \in I, \mu \in (0, 1), \mu > 0.$$

Now similarly as we have discussed in the previous section, we also point out here that if we take $\sigma = 1$, then we have different classes strongly G^h-convex functions. And if we take $\sigma = -1$, then we have new classes of harmonically strongly G^h-convex functions. For example, if we take $\sigma = -1$ in definition 11, then we have the class of harmonically strongly G^h-convex function.

Definition 17 Let $h : (0, 1) \to \mathbb{R}$ be a real function. We say that a function $X : I \setminus \{0\} \to \mathbb{R}$ is said to be an harmonically strongly G^h-convex function, if

$$G^{X\left(\frac{a_1a_2}{(1-\mu)a_1+\mu a_2}\right)} \leq h(\mu)G^{X(a_1)} + h(1-\mu)G^{X(a_1)} - \mu\mu(1-\mu)(a_2{}^{-1} - a_1{}^{-1})^2,$$
$$\forall a_1, a_2 \in I, \mu \in (0,1), \mu > 0.$$

For different suitable choices of the real function $h(.)$, we have some other new classes of harmonically strongly G^h-convex function. Also note that if we take $\mu = 0$, then we recapture all of the definitions of previous section from these newly introduced definitions.

Theorem 4 *Let $h : (0,1) \to \mathbb{R}$ be a real function. If $X : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is strongly $G^{(\sigma,h)}$-convex function and $X \in L[a_1, a_2]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} + \frac{\mu}{12}(a_2{}^\sigma - a_1{}^\sigma)^2\right]$$
$$\leq \frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma}\int_{a_1}^{a_2} u^{\sigma-1}G^{X(u)}\mathrm{d}u \leq [G^{X(a_1)} + G^{X(a_2)}]$$
$$\times \int_0^1 h(\mu)\mathrm{d}\mu - \frac{\mu}{6}(a_2{}^\sigma - a_1{}^\sigma)^2.$$

Proof Since it is given that X is an $G^{(\sigma,h)}$-convex function, then

$$G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)}$$
$$\leq h\left(\frac{1}{2}\right)\left[G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} + G^{X([(1-\mu)a_1{}^\sigma + \mu a_2{}^\sigma]^{\frac{1}{\sigma}})}\right]$$
$$- \frac{\mu}{4}(1-2\mu)^2(a_2{}^\sigma - a_1{}^\sigma)^2.$$

Integrating both sides of above inequality with respect to $\mu \in [0,1]$, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[G^{X\left(\left[\frac{a_1{}^\sigma + a_2{}^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)} + \frac{\mu}{12}(a_2{}^\sigma - a_1{}^\sigma)^2\right] \leq \frac{\sigma}{a_2{}^\sigma - a_1{}^\sigma}\int_{a_1}^{a_2} u^{\sigma-1}G^{X(u)}\mathrm{d}u. \tag{5}$$

Also

$$G^{X([\mu a_1{}^\sigma + (1-\mu)a_2{}^\sigma]^{\frac{1}{\sigma}})} \leq h(\mu)G^{X(a_1)} + h(1-\mu)G^{X(a_2)} - \mu\mu(1-\mu)(a_2{}^\sigma - a_1{}^\sigma)^2.$$

Integrating both sides of above inequality with respect to μ on [0, 1], we have

$$\frac{\sigma}{a_2{}^{\sigma} - a_1{}^{\sigma}} \int_{a_1}^{a_2} u^{\sigma-1} G^{X(u)} du \leq [G^{X(a_1)} + G^{X(a_2)}] \int_0^1 h(\mu) d\mu - \frac{\mu}{6}(a_2{}^{\sigma} - a_1{}^{\sigma})^2. \tag{6}$$

On summation of (5) and (6), we have the required results.

Remark 1 We would like to point out here that for different suitable choices of the function $h(.)$, we have several other new versions of Hermite-Hadamard like inequalities. Also for $\sigma = -1$ we get the results for harmonically strongly G^h-convex function. We left the details for interested readers. Also note that if we take $G = \exp$, then we have the results for $\mathbf{e}^{(\sigma,h)}$-convex functions.

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