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Editors

Extended Abstracts EuroComb 2021

European Conference on
Combinatorics, Graph Theory and
Applications



Trends in Mathematics

Research Perspectives CRM Barcelona

Volume 14

Managing Editor

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Juanjo Rué · Oriol Serra
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European Conference on Combinatorics,
Graph Theory and Applications

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ISSN 2297-0215

Trends in Mathematics

ISSN 2509-7407

Research Perspectives CRM Barcelona

ISBN 978-3-030-83822-5

<https://doi.org/10.1007/978-3-030-83823-2>

ISSN 2297-024X (electronic)

ISSN 2509-7415 (electronic)

ISBN 978-3-030-83823-2 (eBook)

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Preface to the Abstracts Volume of EuroComb21

This volume includes the collection of extended abstracts that were presented at the European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB'21), held online and organized by Universitat Politècnica de Catalunya from September 6 to September 10, 2021.

The EUROCOMB conferences are organized biannually. The series was started in Barcelona 2001 and continued with Prague 2003, Berlin 2005, Seville 2007, Bordeaux 2009, Budapest 2011, Pisa 2013, Bergen 2015, Viena 2017, and Bratislava 2019. In 2021 and for the first time in history, EUROCOMB will be held online due to the worldwide effects of COVID-19.

Combinatorics is a central topic in mathematics with countless applications in other disciplines such as theoretical physics, life and social sciences, and engineering. Most notably, combinatorial methods have played an essential role in the theoretical analysis of algorithms. This interaction has been recently recognized with the 2021 Abel Prize awarded to Lázlo Lovász and Avi Wigderson for their contributions on discrete mathematics and theoretical computer science.

EUROCOMB is the reference European conference in combinatorics and one of the main events worldwide in the area. Since EUROCOMB'03 in Prague, the European Prize in combinatorics is awarded to recognize groundbreaking contributions in combinatorics, discrete mathematics, and their applications by young European researchers not older than 35. It is supported by DIMATIA, by the local organizers and by private sources.

EUROCOMB'21 was organized by members of the Universitat Politècnica de Catalunya, Barcelona, Spain. A total of 177 contributions were submitted from which the program committee selected 135 to be presented at the conference. We would like to highlight the excellent quality of most of the submitted abstracts. In addition to the contributed presentations, the conference hosted ten plenary talks delivered by top researchers in the area on a variety of topics in extremal, probabilistic and structural combinatorics, and theoretical computer science.

- Julia Böttcher (LSE, London)
- Josep Díaz (UPC, Barcelona)

- Louis Esperet (G-SCOP, Grenoble)
- Christian Krattenthaler (U. Wien)
- Sergey Norin (McGill, Montreal)
- Will Perkins (UIC, Chicago)
- Marcin Pilipczuk (U. of Warsaw)
- Lisa Sauerman (IAS, Princeton)
- Eva Tardos (Cornell, Ithaca)
- David Wood (Monash, Melbourne)

In this edition, a special session has been organized to honor the memory of Robin Thomas and his contributions to the area. The session included the plenary talk by Sergey Norin and invited talks by Dan Král', Zdenek Dvořák, and Luke Postle.

The program committee members were

- Maria Axenovich (KIT, Karlsruhe)
- Agnes Backhausz (Eötvös Loránd U., Budapest)
- Marthe Bonamy (LABRI, Bordeaux)
- Michael Drmota (TUWien)
- Zdenek Dvořák (Charles University, Prague)
- Stefan Felsner (TU Berlin)
- Ervin Gyori (Alfred Rényi Institute, Budapest)
- Dan Král' (Masaryk University, Brno)
- Bojan Mohar (SFU, Vancouver and University of Ljubljana)
- Rob Morris (IMPA, Rio de Janeiro)
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- Patrice Ossona de Mendez (CNRS and EHESS, Paris)
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- Oleg Pikhurko (Warwick)
- Andrzej Ruciński (UAM, Póznán and Emory)
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- Martin Škoviera (Comenius U., Bratislava)
- Jozef Skokan (LSE, London)
- Maya Stein (U. de Chile, Santiago de Chile)
- Benjamin Sudakov (ETH, Zurich)
- Xuding Zhu (Zhejiang Normal U., Jinhua)

We thank all participants of EUROCOMB'21, all invited speakers, and all members of the program committee for their generous commitment to the scientific success of the conference, especially during challenging times.

Jaroslav Nešetřil
 Guillem Perarnau
 Juanjo Rué
 Oriol Serra

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Size of Local Finite Field Kakeya Sets

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Abstract. Let \mathbb{F} be a finite field consisting of q elements and let $n \geq 1$ be an integer. In this paper, we study the size of local Kakeya sets with respect to subsets of \mathbb{F}^n and obtain upper and lower bounds for the minimum size of a (local) Kakeya set with respect to an arbitrary set $\mathcal{T} \subseteq \mathbb{F}^n$.

Keywords: Local Kakeya sets · Minimum size · Probabilistic method

1 Introduction

The study of finite field Kakeya sets is of interest from both theoretical and application perspectives. Letting \mathbb{F} be a finite field containing q elements and $n \geq 1$ be an integer, Wolff [1] used counting arguments and planes to estimate that the minimum size of a global Kakeya set covering all vectors in \mathbb{F}^n grows at least as $q^{n/2}$. Later Dvir [2] used polynomial methods to obtain sharper bounds (of the form $C \cdot q^n$) on the minimum size of global Kakeya sets and for further improvements in the multiplicative constant C , we refer to Saraf and Sudan [3].

In this paper, we are interested in studying local Kakeya sets with respect to *subsets* of \mathbb{F}^n . Specifically, in Theorem 1, we obtain upper and lower bounds for the minimum size of a Kakeya set with respect to a subset $\mathcal{T} \subseteq \mathbb{F}^n$.

The paper is organized as follows. In Sect. 2, we describe local Kakeya sets and state and prove our main result (Theorem 1) regarding the minimum size of a local Kakeya set.

2 Local Kakeya Sets

Let \mathbb{F} be a finite field containing q elements and for $n \geq 1$ let \mathbb{F}^n be the set of all n -tuple vectors with entries belonging to \mathbb{F} .

We say that a set $\mathcal{K} \subseteq \mathbb{F}^n$ is a Kakeya set *with respect to* the vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}^n$ if there exists $\mathbf{y} = \mathbf{y}(\mathbf{x}) \in \mathbb{F}^n$ such that the line

$$L(\mathbf{x}, \mathbf{y}) := \bigcup_{a \in \mathbb{F}} \{\mathbf{y} + a \cdot \mathbf{x}\} \subseteq \mathcal{K}, \tag{2.1}$$

where $a \cdot \mathbf{x} := (ax_1, \dots, ax_n)$. For a set $\mathcal{T} \subseteq \mathbb{F}^n$, we say that $\mathcal{K} \subseteq \mathbb{F}^n$ is a Kakeya set with respect to \mathcal{T} if \mathcal{K} is a Kakeya set with respect to every vector $\mathbf{x} \in \mathcal{T}$.

The following result describes the minimum size of local Kakeya sets.

Theorem 1. Let $\mathcal{T} \subseteq \mathbb{F}^n$ be any set with cardinality $\#\mathcal{T}$ an integer multiple of $q - 1$ and let $\theta(\mathcal{T})$ be the minimum size of a Kakeya set with respect to \mathcal{T} . We then have that

$$q\sqrt{M} + \min\left(0, q - \sqrt{M}\right) \leq \theta(\mathcal{T}) \leq q + q^n \left(1 - \left(1 - \frac{1}{q^{n-1}}\right)^{M-1}\right) \quad (2.2)$$

where $M := \frac{\#\mathcal{T}}{q-1}$.

For example suppose $M = \epsilon \cdot \left(\frac{q^n - 1}{q - 1}\right)$ for some $0 < \epsilon \leq 1$. From the lower bound in (2.2), we then get that $\theta(\mathcal{T})$ grows at least of the order of $q^{n/2}$. Similarly, using the fact that $1 - x \geq e^{-x-x^2}$ for $0 < x \leq \frac{1}{2}$, we get that

$$\left(1 - \frac{1}{q^{n-1}}\right)^{M-1} \geq \exp\left(-\frac{M-1}{q^{n-1}}\left(1 + \frac{1}{q^{n-1}}\right)\right) \geq e^{-\Delta},$$

where $\Delta := \frac{q\epsilon}{q-1}\left(1 + \frac{1}{q^{n-1}}\right)$. From (2.2) we then get that

$$\theta(\mathcal{T}) \leq q + q^n(1 - e^{-\Delta}).$$

In what follows we prove the lower bound and the upper bound in Theorem 1 in that order.

Proof of Lower Bound in Theorem 1

The proof of the lower bound consists of two steps. In the first step, we extract a subset \mathcal{N} of \mathcal{T} containing vectors that are non-equivalent. In the next step, we then use a high incidence counting argument similar to Wolff (1999) and estimate the number of vectors in a Kakeya set \mathcal{K} with respect to \mathcal{N} .

Step 1: Say that vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}^n$ are *equivalent* if $\mathbf{x}_1 = a \cdot \mathbf{x}_2$ for some $a \in \mathbb{F} \setminus \{0\}$. We first extract a subset of vectors in \mathcal{T} that are pairwise non-equivalent. Pick a vector $\mathbf{x}_1 \in \mathcal{N}$ and throw away all the vectors in \mathcal{T} , that are equivalent to \mathbf{x}_1 . Next, pick a vector \mathbf{x}_2 in the remaining set and again throw away the vectors that are equivalent to \mathbf{x}_2 . Since we throw away at most $q - 1$ vectors in each step, after r steps, we are left with a set of size at least $\#\mathcal{T} - r(q - 1)$. Thus the procedure continues for

$$M = \frac{\#\mathcal{T}}{q-1} \quad (2.3)$$

steps, assuming henceforth that M is an integer.

Let $\mathcal{N} = \{\mathbf{x}_1, \dots, \mathbf{x}_M\} \subseteq \mathcal{T}$ be a set of size M and let \mathcal{K} be a Kakeya set with respect to \mathcal{N} , of minimum size. By definition (see (2.1)), there are vectors $\mathbf{y}_1, \dots, \mathbf{y}_M$ in \mathbb{F}^n such that the line $L(\mathbf{x}_i, \mathbf{y}_i) \subseteq \mathcal{K}$ for each $1 \leq i \leq M$. Moreover, since \mathcal{K} is of minimum size, we must have that

$$\mathcal{K} = \bigcup_{j=1}^M \{L(\mathbf{x}_i, \mathbf{y}_i)\}. \quad (2.4)$$

Step 2: To estimate the number of distinct vectors in \mathcal{K} , suppose first that each vector in \mathcal{K} belongs to at most t of the lines in $\{L(\mathbf{x}_i, \mathbf{y}_i)\}_{1 \leq i \leq M}$. Since each line $L(\mathbf{x}_i, \mathbf{y}_i)$ contains q vectors, we get that the total number of vectors in \mathcal{K} is bounded below by

$$\#\mathcal{K} \geq \frac{Mq}{t}. \quad (2.5)$$

Suppose now that there exists a vector \mathbf{v} in \mathcal{K} that belongs to at least $t+1$ lines $L_1, \dots, L_{t+1} \subseteq \bigcup_{1 \leq i \leq M} \{L(\mathbf{x}_i, \mathbf{y}_i)\}$. Because the vectors $\{\mathbf{x}_i\}_{1 \leq i \leq M}$ are not equivalent, any two lines $L(\mathbf{x}_i, \mathbf{y}_i)$ and $L(\mathbf{x}_j, \mathbf{y}_j)$ must have at most one point of intersection. To see this is true suppose there were scalars $a_1 \neq a_2$ and $b_1 \neq b_2$ in \mathbb{F} such that

$$\mathbf{y}_1 + a_i \cdot \mathbf{x}_1 = \mathbf{y}_2 + b_i \cdot \mathbf{x}_2 \text{ for } i = 1, 2.$$

Subtracting the equations we would then get $(a_1 - a_2) \cdot \mathbf{x}_1 = (b_1 - b_2) \cdot \mathbf{x}_2$, contradicting the fact that \mathbf{x}_1 and \mathbf{x}_2 are not equivalent.

From the above paragraph, we get that any two lines in $\{L_i\}_{1 \leq i \leq t+1}$ have exactly one point of intersection, the vector \mathbf{v} . Since each line L_i contains q vectors, the total number of vectors in $\{L_i\}_{1 \leq i \leq t+1}$ equals $(q-1)(t+1) + 1$, all of which must be in \mathcal{K} . From (2.5) we therefore get that

$$\#\mathcal{K} \geq \min\left(\frac{Mq}{t}, (q-1)(t+1) + 1\right)$$

and setting $t = \sqrt{M}$, we get

$$\#\mathcal{K} \geq q\sqrt{M} + \min(0, q - \sqrt{M}).$$

From the expression for M in (2.3), we then get (2.2). ■

Proof of Upper Bound in Theorem 1

We use the probabilistic method. Let $\{\mathbf{x}_1, \dots, \mathbf{x}_M\}$ be the set of non-equivalent vectors obtained in Step 1 in the proof of the lower bound with $M = \frac{\#\mathcal{T}}{q-1}$ (see (2.3)). Let $\mathbf{Y}_1, \dots, \mathbf{Y}_M$ be independently and uniformly randomly chosen from \mathbb{F}^n and for $1 \leq i \leq M$, set

$$\mathcal{S}_i := \bigcup_{j=1}^i \{L(\mathbf{x}_j, \mathbf{Y}_j)\},$$

where $L(\mathbf{x}, \mathbf{y})$ is the line containing the vectors \mathbf{x} and \mathbf{y} as defined in (2.1).

By construction, the set \mathcal{S}_M forms a Kakeya set with respect to \mathcal{T} . To estimate the expected size of \mathcal{S}_M , we use recursion. For $1 \leq i \leq M$, let $\theta_i := \mathbb{E}\#\mathcal{S}_i$ be the expected size of \mathcal{S}_i . Given \mathcal{S}_{i-1} , the probability that a vector chosen from \mathbb{F}^n , uniformly randomly and independent of \mathcal{S}_{i-1} , belongs to the set \mathcal{S}_{i-1} is given by $p_i := \frac{\#\mathcal{S}_{i-1}}{q^n}$. Therefore

$$\mathbb{E}\# \left(L(\mathbf{x}_i, \mathbf{Y}_i) \cap \mathcal{S}_{i-1} \right) = q \cdot \mathbb{E} \left(\frac{\#\mathcal{S}_{i-1}}{q^n} \right)$$

and so

$$\theta_i = \theta_{i-1} + q \left(1 - \frac{\theta_{i-1}}{q^n} \right) = \theta_{i-1} \left(1 - \frac{1}{q^{n-1}} \right) + q. \quad (2.6)$$

Letting $a = 1 - \frac{1}{q^{n-1}}$ and using (2.6) recursively, we get

$$\theta_i = a^{i-1} \cdot \theta_1 + q \cdot (1 + a + \dots + a^{i-2}) = a^{i-1} \cdot \theta_1 + \frac{q(1 - a^{i-1})}{1 - a}.$$

Using $\theta_1 = q$, we then get that

$$\begin{aligned} \theta_M &= a^{M-1} \cdot q + q^n \left(1 - \left(1 - \frac{1}{q^{n-1}} \right)^{M-1} \right) \\ &\leq q + q^n \left(1 - \left(1 - \frac{1}{q^{n-1}} \right)^{M-1} \right). \end{aligned}$$

This implies that there exists a Kakeya set with respect to \mathcal{T} of size at most θ_M . ■

Acknowledgement. I thank Professors V. Guruswami, V. Arvind, C. R. Subramanian and the referees for crucial comments that led to an improvement of the paper. I also thank IMSc for my fellowships.

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Coloring of Graphs Avoiding Bicolored Paths of a Fixed Length

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Abstract. The problem of finding the minimum number of colors to color a graph properly without containing any bicolored copy of a fixed family of subgraphs has been widely studied. Most well-known examples are star coloring and acyclic coloring of graphs (Grünbaum, 1973) where bicolored copies of P_4 and cycles are not allowed, respectively. We introduce a variation of these problems and study proper coloring of graphs not containing a bicolored path of a fixed length and provide general bounds for all graphs. A P_k -coloring of an undirected graph G is a proper vertex coloring of G such that there is no bicolored copy of P_k in G , and the minimum number of colors needed for a P_k -coloring of G is called the P_k -chromatic number of G , denoted by $s_k(G)$. We provide bounds on $s_k(G)$ for all graphs, in particular, proving that for any graph G with maximum degree $d \geq 2$, and $k \geq 4$, $s_k(G) \leq \lceil 6\sqrt{10}d^{\frac{k-1}{k-2}} \rceil$. Moreover, we find the exact values for the P_k -chromatic number of the products of some cycles and paths for $k = 5, 6$.

Keywords: Graphs · Acyclic coloring · Star coloring

1 Introduction

The proper coloring problem on graphs seeks to find colorings on vertices with minimum number of colors such that no two neighbors receive the same color. There have been studies introducing additional conditions to proper coloring, such as also forbidding 2-colored copies of some particular graphs. In particular, *star coloring* problem on a graph G asks to find the minimum number of colors in a proper coloring forbidding a 2-colored P_4 , called the star-chromatic number $\chi_s(G)$ [10]. Similarly, *acyclic chromatic number* of a graph G , $a(G)$, is the minimum number of colors used in a proper coloring not having any 2-colored cycle, also called acyclic coloring of G [10]. Both, the star coloring and acyclic coloring problems are shown to be NP-complete in [2] and [15], respectively.

These two problems have been studied widely on many different families of graphs such as product of graphs, particularly grids and hypercubes. In this paper, we introduce a variation of these problems and study proper coloring of

graphs not containing a bicolored (2-colored) path of a fixed length and provide general bounds for all graphs. The P_k -coloring of an undirected graph G , where $k \geq 4$, is a proper vertex coloring of G such that there is no bicolored copy of P_k in G , and the minimum number of colors needed for a P_k -coloring of G is called the P_k -chromatic number of G , denoted by $s_k(G)$. A special case of this coloring is the star-coloring, when $k = 4$, introduced by Grünbaum [10]. Hence, $\chi_S(G) = s_4(G)$ and all of the bounds on $s_k(G)$ in Sect. 2 apply to star chromatic number using $k = 4$.

If a graph does not contain a bicolored P_k , then it does not contain any bicolored cycle from the family $\mathcal{C}_k = \{C_i : i \geq k\}$. Thus, as the star coloring problem is a strengthening of the acyclic coloring problem, a P_k -coloring is also a coloring avoiding a bicolored member from \mathcal{C}_k . We call such a coloring, a \mathcal{C}_k -coloring, where the minimum number of colors needed for such a coloring of a graph G is called \mathcal{C}_k -chromatic number of G , denoted by $a_k(G)$. By this definition, we have $a_3(G) = a(G)$. In Sect. 2, we provide a lower bound for the \mathcal{C}_k -chromatic number of graphs as well.

Our results comprise lower bounds on these colorings and an upper bound for general graphs. Moreover, some exact results are presented. In Sect. 2, we provide lower bounds on $s_k(G)$ and $a_k(G)$ for any graph G . Moreover, we show that for any graph G with maximum degree $d \geq 2$, and $k \geq 4$, $s_k(G) = O(d^{\frac{k-1}{k-2}})$. Finally, in Sect. 3, we present exact results on the P_5 -coloring and P_6 -coloring for the products of some paths and cycles.

1.1 Related Work

Acyclic coloring was also introduced in 1973 by Grünbaum [10] who proved that a graph with maximum degree 3 has an acyclic coloring with 4 colors.

The following bounds obtained in [3] are the best available asymptotic bounds for the acyclic chromatic number, that are obtained using the probabilistic method.

$$\Omega\left(\frac{d^{\frac{4}{3}}}{(\log d)^{\frac{1}{3}}}\right) = a(G) = O(d^{\frac{4}{3}}).$$

Recently, there have been some improvements in the constant factor of the upper bound in [6, 9, 16], by using the entropy compression method. Similar results for the star chromatic number of graphs are obtained in [8], showing $\chi_s(G) \leq \lceil 20d^{3/2} \rceil$ for any graph G with maximum degree d .

We observe that the method in [6] is also used in finding a general upper bound for P_k -coloring of graphs, when k is even. This coloring is called *star k coloring*, where a proper coloring of the vertices is obtained avoiding a bicolored P_{2k} . In [6], it is shown that every graph with maximum degree Δ has a star k coloring with at most $c_k k^{\frac{1}{k-1}} \Delta^{\frac{2k-1}{2k-2}} + \Delta$ colors, where c_k is a function of k . Our result presented in Sect. 2 improves this result and generalizes Fertin et al.'s result in [8] to P_k -coloring of graphs for $k \geq 4$.

The star chromatic number and acyclic chromatic number of products of graphs have been studied widely as well. In [8], various bounds on the star chromatic number of some graph families such as hypercube, grid, tori are obtained,

providing exact values for 2-dimensional grids, trees, complete bipartite graphs, cycles, outerplanar graphs. More recent results on the acyclic coloring of grid and tori can be found in [1] and [11]. Similarly, the acyclic chromatic number of the grid and hypercube is studied in [7]. Moreover, [12–14] investigate the acyclic chromatic number for products of trees, products of cycles and Hamming graphs. For some graphs, finding the exact values of these chromatic numbers has been a longstanding problem, such as the hypercube.

2 General Bounds

We obtain lower bounds on $s_k(G)$ and $a_k(G)$ by using the theorem of Erdős and Gallai below.

Theorem 1 [4]. *For a graph G on n vertices, if the number of edges is more than*

1. $\frac{1}{2}(k-2)n$, then G contains P_k as a subgraph,
2. $\frac{1}{2}(k-1)(n-1)$, then G contains a member of \mathcal{C}_k as a subgraph,

for any P_k with $k \geq 2$, and for any \mathcal{C}_k with $k \geq 3$.

As also observed in [8] for star coloring, the subgraphs induced by any two color classes in a P_k -coloring are P_k -free. Using this observation together with Theorem 1, we obtain the results in Theorems 2 and 3.

Theorem 2. *For any graph $G = (V, E)$, let $|V| = n$ and $|E| = m$. Then, $s_k(G) \geq \frac{2m}{n(k-2)} + 1$, for any $k \geq 3$.*

Theorem 3. *For any graph $G = (V, E)$, let $|V| = n$, $|E| = m$ and $\Delta = 4n(n-1) - \frac{16m}{k-1} + 1$. Then, $a_k(G) \geq \frac{1}{2}(2n+1 - \sqrt{\Delta})$, for any $k \geq 3$.*

We obtain an upper bound on the P_k -chromatic number of any graph on n vertices and maximum degree d . Our proof relies on Lovasz Local Lemma, for which we provide some preliminary details as follows. An event A_i is *mutually independent* of a set of events $\{B_i \mid i = 1, 2, \dots, n\}$ if for any subset \mathcal{B} of events or their complements contained in $\{B_i\}$, we have $Pr[A_i \mid \mathcal{B}] = Pr[A_i]$. Let $\{A_1, A_2, \dots, A_n\}$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, 2, \dots, n\}$ is called a *dependency graph* for the events A_1, A_2, \dots, A_n if for each i , $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j \mid (i, j) \notin E\}$.

Theorem 4 (General Lovasz Local Lemma) [5]. *Suppose that $H = (V, E)$ is a dependency graph for the events A_1, A_2, \dots, A_n and suppose there are real numbers y_1, y_2, \dots, y_n such that $0 \leq y_i \leq 1$ and*

$$Pr[A_i] \leq y_i \prod_{(i,j) \in E} (1 - y_j) \quad (1)$$

for all $1 \leq i \leq n$. Then $Pr[\bigwedge_{i=1}^n A_i] \geq \prod_{i=1}^n (1 - y_i)$. In particular, with positive probability no event A_i holds.

We use Theorem 4 in the proof of the following upper bound.

Theorem 5. *Let G be any graph with maximum degree d . Then $s_k(G) \leq \lceil 6\sqrt{10}d^{\frac{k-1}{k-2}} \rceil$, for any $k \geq 4$ and $d \geq 2$.*

Proof. Assume that $x = \lceil ad^{\frac{k-1}{k-2}} \rceil$ and $a = 6\sqrt{10}$. Let $f : V \mapsto \{1, 2, \dots, x\}$ be a random vertex coloring of G , where for each vertex $v \in V$, the color $f(v) \in \{1, 2, \dots, x\}$ is chosen uniformly at random. It suffices to show that with positive probability f does not produce a bicolored P_k .

Below are the types of probabilistic events that are not allowed:

- Type I: For each pair of adjacent vertices u and v of G , let $A_{u,v}$ be the event that $f(u) = f(v)$.
- Type II: For each P_k called P , let A_P be the event that P is colored properly with two colors.

By definition of our coloring, none of these events are allowed to occur. We construct a dependency graph H , where the vertices are the events of Types I and II, and use Theorem 4 to show that with positive probability none of these events occur. For two vertices A_1 and A_2 to be adjacent in H , the subgraphs corresponding to these events should have common vertices in G . The dependency graph of the events is called H , where the vertices are the union of the events. We call a vertex of H of *Type i* if it corresponds to an event of Type i . For any vertex v in G , there are at most

- d pairs $\{u, v\}$ associated with an event of Type I, and
- $\frac{k+1}{2}d^{k-1}$ copies of P_k containing v , associated with an event of Type II (Table 1).

Table 1. The $(i, j)^{th}$ entry showing an upper bound on the number of vertices of type j that are adjacent to a vertex of type i in H .

	I	II
I	$2d$	$(k+1)d^{k-1}$
II	kd	$\frac{k}{2}(k+1)d^{k-1}$

The probabilities of the events are

- $Pr(A_{u,v}) = \frac{1}{x}$ for an event of type I, and
- $Pr(A_P) = \frac{1}{x^{k-2}}$ for an event of type II.

To apply Theorem 4, we choose the values of y_i 's accordingly so that (1) is satisfied:

$$y_1 = \frac{1}{3d}, \quad y_2 = \frac{1}{2(k+1)d^{k-1}}.$$

3 Coloring of Products of Paths and Cycles

The *cartesian product* of two graphs $G = (V, E)$ and $G' = (V', E')$ is shown by $G \square G'$ and its vertex set is $V \times V'$. For any vertices $x, y \in V$ and $x', y' \in V'$, there is an edge between (x, y) and (x', y') in $G \square G'$ if and only if either $x = y$ and $x'y' \in E'$ or $x' = y'$ and $xy \in E$. For simplicity, we let $G(n, m)$ denote the product $P_n \square P_m$.

Theorem 6

$$s_5(P_3 \square P_3) = s_5(C_3 \square C_3) = s_5(C_3 \square C_4) = s_5(C_4 \square C_4) = 4.$$

To prove this theorem, we start by showing that $s_5(P_3 \square P_3) \geq 4$. Since $C_3 \square C_3$, $C_3 \square C_4$ and $C_4 \square C_4$ contain $P_3 \square P_3$ as a subgraph, this shows that at least 4 colors are needed to color these graphs. Such a coloring can be obtained as in (2) by taking the first three or four rows/columns depending on the change in the grid dimension.

$$\begin{array}{ccc}
 a & b & c \\
 c & a & b \\
 b & c & a
 \end{array}
 \qquad
 \begin{array}{cccc}
 1 & 2 & 3 & 4 \\
 2 & 1 & 4 & 3 \\
 3 & 4 & 1 & 2 \\
 4 & 3 & 2 & 1
 \end{array}
 \qquad (2)$$

Theorem 7. $s_5(G(n, m)) = 4$ for all $n, m \geq 3$.

Proof. Note that $4 = s_5(G(3, 3)) \leq s_5(G(n, m))$ for all $m, n \geq 3$. Since there exists some integer k for which $3k \geq n, m$ and $G(n, m)$ is a subgraph of $G(3k, 3k)$, $s_5(G(n, m)) \leq s_5(G(3k, 3k))$ for some k . Hence, we show that $s_5(G(3k, 3k)) = 4$. In Theorem 6, a P_5 -coloring of $C_3 \square C_3$ is given by the upper left corner of the coloring in (2) by using 4 colors. By repeating this coloring of $C_3 \square C_3$ k times in $3k$ rows, we obtain a coloring of $G(3k, 3)$. Then repeating this colored $G(3k, 3)$ k times in $3k$ columns, we obtain a P_5 -coloring of $G(3k, 3k)$ using 4 colors. There exists no bicolored P_5 in this coloring.

In the following, we generalize the previous cases by making use of the well-known result below.

Theorem 8 (Sylvester, [17]). *If $r, s > 1$ are relatively prime integers, then there exist $\alpha, \beta \in \mathbb{N}$ such that $t = \alpha r + \beta s$ for all $t \geq (r - 1)(s - 1)$.*

Theorem 9. *Let $p, q \geq 3$ and $p, q \neq 5$. Then $s_5(C_p \square C_q) = 4$.*

Proof. The lower bound follows from Theorem 6. By Theorem 8, p and q can be written as a linear combination of 3 and 4 using nonnegative coefficients. By using this, we are able to tile the $p \times q$ -grid of $C_p \square C_q$ using these blocks of 3×3 , 3×4 , 4×3 , and 4×4 grids. Recall that the coloring pattern in (2) also provides a P_5 -coloring of smaller grids listed above by using the upper left portion for the required size. Therefore, using these coloring patterns on the smaller blocks of the tiling yields a P_5 -coloring of $C_p \square C_q$.

Corollary 1. *Let $i, j \geq 3$ and $i, j \neq 5$. Then, $s_5(P_i \square C_j) = 4$.*

Proof. Since $P_i \square P_j$ is a subgraph of $P_i \square C_j$, Theorem 7 gives the lower bound. By Theorem 9, we have equality.

The ideas used above can be generalized to P_6 -coloring of graphs. We are able to show the following result by using the fact $s_6(G(4, 4)) \leq s_5(G(4, 4)) = 4$ and by proving that three colors are not enough for a P_6 -coloring of $G(4, 4)$.

Theorem 10. $s_6(G(4, 4)) = 4$.

Together, with Theorem 10 and $s_6(G(n, m)) \leq s_5(G(n, m)) = 4$, we have the following.

Corollary 2. $s_6(G(n, m)) = 4$ for all $n, m \geq 4$.

Similarly, Theorem 9 and Corollary 2 imply the following result.

Corollary 3. $s_6(C_m \square C_n) = 4$ for all $m, n \geq 4$ and $m, n \neq 5$.

Acknowledgements. The research of the second author was supported in part by the BAGEP Award of the Science Academy.

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Nested Cycles with No Geometric Crossings

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Abstract. In 1975, Erdős asked the following question: what is the smallest function $f(n)$ for which all graphs with n vertices and $f(n)$ edges contain two edge-disjoint cycles C_1 and C_2 , such that the vertex set of C_2 is a subset of the vertex set of C_1 and their cyclic orderings of the vertices respect each other? We prove the optimal linear bound $f(n) = O(n)$ using sublinear expanders.

Keywords: Graph theory · Cycles · Sublinear expander

1 Introduction

Cycles C_1, \dots, C_k in a graph G are said to be *nested cycles* if the vertex set of C_{i+1} is a subset of the vertex set of C_i for each $i \in [k-1]$. If, in addition, their edge sets are disjoint, we say they are *edge-disjoint nested cycles*. In 1975, Erdős [5] conjectured that there is a constant c such that graphs with n vertices and at least cn edges must contain two edge-disjoint nested cycles. Bollobás [1] proved the conjecture and asked for extension to k edge-disjoint nested cycles. This was confirmed later in 1996 by Chen, Erdős and Staton [2], who showed that $O_k(n)$ many edges forces k edge-disjoint nested cycles.

A stronger conjecture of Erdős that also appeared in [5] is that there exists a constant C such that graphs with n vertices and at least Cn edges must contain two edge-disjoint nested cycles such that, geometrically, the edges of the inner

J.K. was supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and by the KAIX Challenge program of KAIST Advanced Institute for Science-X. Y.K. was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1A6A3A04005963). H.L. was supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

cycle do not cross each other, in other words, if $C_1 = v_1 \dots v_{\ell_1}$, then C_2 has no two edges $v_i v_{i'}$ and $v_j v_{j'}$ with $i < j < i' < j'$. In this case, C_1 and C_2 are said to be two *nested cycles without crossings*. Here we prove this conjecture.

Theorem 1. *There exists a constant $C > 0$ such that every graph G with average degree at least C has two nested cycles without crossings.*

Our proof utilise a notion of sublinear expanders, which plays an important role in some recent resolutions of long-standing conjectures, see e.g. [7, 8, 10, 11].

1.1 Notation

For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. If we claim that a result holds for $0 < a \ll b, c \ll d < 1$, it means that there exist positive functions f, g such that the result holds as long as $a < f(b, c)$ and $b < g(d)$ and $c < g(d)$. We will not compute these functions explicitly. In many cases, we treat large numbers as if they are integers, by omitting floors and ceilings if it does not affect the argument. We write \log for the base- e logarithm.

Given a graph G , denote its average degree $2e(G)/|G|$ by $d(G)$. Let $F \subseteq G$ and H be graphs, and $U \subseteq V(G)$. We write $G[U] \subseteq G$ for the induced subgraph of G on vertex set U . Denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and write $G - U$ for the induced subgraph $G[V(G) \setminus U]$, and $G \setminus F$ for the spanning subgraph of G obtained from removing the edge set of F . For a set of vertices $X \subseteq V(G)$ and $i \in \mathbb{N}$, denote

$$N^i(X) := \{u \in V(G) : \text{the distance in } G \text{ between } X \text{ and } u \text{ is exactly } i\},$$

and write $N^0(X) = X$, $N(X) := N^1(X)$, and for $i \in \mathbb{N} \cup \{0\}$, let $B^i(X) = \bigcup_{j=0}^i N^j(X)$ be the ball of radius i around X . For a path P , we write $\ell(P)$ for its length, which is the number of edges in the path.

1.2 Sublinear Expander

Our proof makes use of the sublinear expander introduced by Komlós and Szemerédi [9]. We shall use the following extension from [7].

Definition 1. *Let $\varepsilon_1 > 0$ and $k \in \mathbb{N}$. A graph G is an (ε_1, k) -expander if for all $X \subset V(G)$ with $k/2 \leq |X| \leq |G|/2$, and any subgraph $F \subseteq G$ with $e(F) \leq d(G) \cdot \varepsilon(|X|)|X|$, we have*

$$|N_{G \setminus F}(X)| \geq \varepsilon(|X|) \cdot |X|,$$

where

$$\varepsilon(x) = \varepsilon(x, \varepsilon_1, k) = \begin{cases} 0 & \text{if } x < k/5, \\ \varepsilon_1 / \log^2(15x/k) & \text{if } x \geq k/5. \end{cases}$$

We invoke [7, Lemma 3.2], which asserts that every graph contains an expander subgraph with almost the same average degree, to reduce Theorem 1 to an expander. That is, it suffices to show that any n -vertex expander with sufficiently large constant average degree contains two nested cycles without crossings. One of the main tools we use is the following lemma ([9, Corollary 2.3]), which allows us to link two sets with a short path avoiding a small set.

Lemma 1. *Let $\varepsilon_1, k > 0$. If G is an n -vertex (ε_1, k) -expander, then any two vertex sets X_1, X_2 , each of size at least $x \geq k$, are of distance at most $m = \frac{1}{\varepsilon_1} \log^3(15n/k)$ apart. This remains true even after deleting $\varepsilon(x) \cdot x/4$ vertices from G .*

1.3 Auxiliary Definitions and Results

Definition 2. *For $\lambda > 0$ and $k \in \mathbb{N}$, we say that a vertex set U in a graph G is (λ, k) -thin around A if, for each $i \in \mathbb{N}$,*

$$|N_G(B_{G-U}^{i-1}(A)) \cap U| \leq \lambda i^k.$$

We will use the following two results. The first one (which essentially follows from [7, Proposition 3.5]) shows that the rate of expansion for every small set is almost exponential in a robust expander even after deleting a thin set around it. The second one [11, Lemma 3.12] ensures the existence of a linear size vertex set with polylogarithmic diameter in G while avoiding an arbitrary set of size $o(n/\log^2 n)$.

Proposition 1. *Let $0 < 1/d \ll \varepsilon_1 \ll 1/\lambda, 1/k$ and $1 \leq r \leq \log n$. Suppose G is an n -vertex $(\varepsilon_1, \varepsilon_1 d)$ -expander with $\delta(G) \geq d$, and X, Y are sets of vertices with $|Y| \leq \frac{1}{4}\varepsilon(|X|) \cdot |X|$. Let W be a (λ, k) -thin set around X in $G - Y$. Then, for each $1 \leq r \leq \log n$, we have*

$$|B_{G-W-Y}^r(X)| \geq \exp(r^{1/4}).$$

Lemma 2. *Let $0 < 1/d \ll \varepsilon_1 < 1$ and let G be an n -vertex $(\varepsilon_1, \varepsilon_1 d)$ -expander with $\delta(G) \geq d$. For any $W \subseteq V(G)$ with $|W| \leq \varepsilon_1 n / 100 \log^2 n$, there is a set $B \subseteq G - W$ with size at least $n/25$ and diameter at most $100\varepsilon_1^{-1} \log^3 n$.*

2 Proof of Theorem 1

To prove Theorem 1, we embed the desired nested cycles by linking the arms of a kraken iteratively to get the outer cycle so that the cyclic orderings of the vertices of both cycles respect each other.

Definition 3. *For $k, m, s \in \mathbb{N}$, a graph K is a (k, m, s) -kraken if it contains a cycle C with vertices v_1, \dots, v_k , vertices $u_{i,1}, u_{i,2} \in V(G) \setminus V(C)$, $i \in [k]$, and subgraphs $A = \bigcup_{i=1}^k (A_{i,1} \cup A_{i,2})$ and $R = \bigcup_{i=1}^k (R_{i,1} \cup R_{i,2})$, where*

- $\{A_{i,j} : i \in [k], j \in [2]\}$ is a collection of pairwise disjoint sets of size s lying in $V(G) \setminus V(C)$ with $u_{i,j} \in A_{i,j}$, each with diameter at most m .
- $\{R_{i,j} : i \in [k], j \in [2]\}$ is a collection of pairwise internally vertex disjoint paths such that $R_{i,j}$ is a $v_i, u_{i,j}$ -path of length at most $10m$ with internal vertices disjoint from $V(C) \cup (V(A) \setminus V(A_{i,j}))$.

We usually write a kraken as a tuple $(C, A_{i,j}, R_{i,j}, u_{i,j})$, $i \in [k], j \in [2]$. The following lemma finds a kraken in any expander with average degree at least some large constant.

Lemma 3. *Let $0 < 1/d \ll \varepsilon_1 < 1$ and let G be an n -vertex $(\varepsilon_1, \varepsilon_1 d)$ -expander with $\delta(G) \geq d$. Let L be the set of vertices in G with degree at least $\log^{100} n$ and let $m = 100\varepsilon_1^{-1} \log^3 n$. Then, there exists a $(k, m, \log^{10} n)$ -kraken $(C, A_{i,j}, R_{i,j}, u_{i,j})$, $i \in [k], j \in [2]$, in G for some $k \leq \log n$ such that*

- either $\{u_{i,j} : i \in [k], j \in [2]\} \subseteq L$;
- or $|L| \leq 2 \log n$ and any distinct $u_{i,j}, u_{i',j'} \notin L$ are a distance at least $\sqrt{\log n}$ apart in $G - L$.

Proof (Theorem 1). Assume Lemma 3 is true. Let L be the set of high degree vertices and $K = (C, A_{i,j}, R_{i,j}, u_{i,j})$, $i \in [k], j \in [2]$ be the kraken as in Lemma 3. We will embed, for each $i \in \mathbb{Z}_k$, a $u_{i,2}, u_{i+1,1}$ -path P_i of length at most $30m$, such that all paths P_i are internally pairwise disjoint. Such paths P_i , $i \in [k]$, together with C and $R_{i,j}$, $i \in [k], j \in [2]$, form the desired nested cycles without crossings.

If the first alternative in Lemma 3 occurs, i.e., all $u_{i,j} \in L$, $i \in [k], j \in [2]$, then we can iteratively find the desired paths P_i , $i \in [k]$, by linking $N(u_{i,2})$ and $N(u_{i+1,1})$ avoiding previously built paths and K , using Lemma 1. Indeed, the number of vertices to avoid is at most $|V(K)| + k \cdot 30m \leq \log^{20} n$, which is much smaller than the degree of vertices in L .

We may then assume that $|L| \leq 2 \log n$ and distinct $u_{i,j}, u_{i',j'} \notin L$ are at distance at least $\sqrt{\log n}$ apart in $G' = G - L$. Let $V' \subseteq V(C)$ be the set of vertices not linked to vertices in L , i.e., $V' = \{v_i \in V(C) : \{u_{i,1}, u_{i,2}\} \not\subseteq L\}$.

For each $v_i \in V'$ and $j \in [2]$, write $Y_{i,j} = (\cup_{i' \in [k], j' \in [2]} R_{i',j'} \setminus \{u_{i,j}\}) \cup V(C)$. Note that $|Y_{i,j}| \leq \log^5 n$ and recall that $|A_{i,j}| = \log^{10} n$. Applying Proposition 1 with $(X, Y, W)_1 = (A_{i,j}, Y_{i,j} \cup L, \emptyset)$, we can expand $A_{i,j}$ in G' avoiding $Y_{i,j}$ to get $A_{i,j}^* := B_{G' - Y_{i,j}}^r(A_{i,j})$ of size at least $\log^{30} n$, where $r = (\log \log n)^{10}$. Moreover, as for distinct $v_i, v_{i'} \in V'$ and $j, j' \in [2]$, $u_{i,j}$ and $u_{i',j'}$ are at distance at least $\sqrt{\log n}$ apart in G' , $A_{i,j}^*$ and $A_{i',j'}^*$ are disjoint.

Finally, for all $v_i \in V(C) \setminus V'$ and $j \in [2]$, as the corresponding $u_{i,j}$ lie in L , we can choose pairwise disjoint $A_{i,j}^* \subseteq N(u_{i,j}) \setminus (\cup_{v_{i'} \in V', j' \in [2]} A_{i',j'}^*)$, each of size $\log^{30} n$. For each $i \in \mathbb{Z}_k$, link $A_{i,2}^*$ and $A_{i+1,1}^*$ in G to get a path Q_i with length at most m using Lemma 1, avoiding previously built path Q_j and K . The desired $u_{i,2}, u_{i+1,1}$ -path P_i can be obtained by extending Q_i in $A_{i,2}^* \cup A_{i+1,1}^*$.

This concludes the proof.

Proof (Proof sketch of Lemma 3). Let C be a shortest cycle in G with vertices v_1, \dots, v_k . Note that $k \leq 2 \log_d n \leq \log n$ due to $\delta(G) \geq d$. We distinguish two cases depending on the size of L , the set of high degree vertices.

Case 1: Suppose $|L| \geq 2k$, then there are enough vertices to join each vertex of C to two in L . Let \mathcal{P} be a maximal collection of paths in G from $V(C)$ to L such that:

- each $v \in V(C)$ is linked to at most 2 vertices in L ;
- all paths are pairwise disjoint outside of $V(C)$ with internal vertices in $V(G) \setminus (V(C) \cup L)$;
- each path has length at most $10m$.

Subject to $|\mathcal{P}|$ being maximal, let $\ell(\mathcal{P}) := \sum_{P \in \mathcal{P}} \ell(P)$ be minimised.

Suppose, for contradiction, that there is some $v \in V(C)$ which is in less than 2 paths in \mathcal{P} , then $|\mathcal{P}| < 2 \log n$. Let $U = V(C) \cup V(\mathcal{P}) \setminus \{v\}$, hence $|U| \leq 50\varepsilon_1^{-1} \log^4 n$.

A key part of the proof is to show that U is $(10, 2)$ -thin around v in G . This allows us to use Proposition 1 with $(X, Y, W)_1 = (\{v\}, \emptyset, U)$ to get the desired expansion

$$|B_{G-U}^r(v)| \geq \exp(r^{\frac{1}{4}}) = \log^{100} n. \quad (1)$$

Finally, since $|\mathcal{P}| < 2|C| \leq |L|$, we can choose a vertex $w \in L \setminus V(\mathcal{P})$. As $w \in L$, $\deg(w) \geq \log^{100} n$, and so by Lemma 1, we can connect $B_{G-U}^r(v)$ and $N_G(w)$ with a path of length at most m in $G - U$, which extends in $B_{G-U}^r(v) \cup \{w\}$ to an v, w -path in $G - U$ with length at most $10m$, contradicting the maximality of \mathcal{P} . Therefore, each vertex in $V(C)$ is in exactly 2 paths in \mathcal{P} , yielding the first alternative.

Case 2: Suppose $|L| < 2k$. Taking a maximal collection of paths \mathcal{P} from $V(C)$ to L as in Case 1, then the argument in Case 1 shows that, due to the maximality of \mathcal{P} , every vertex in L is linked to a path in \mathcal{P} , i.e., $L \subseteq V(\mathcal{P})$. Note that $|V(\mathcal{P})| \leq 2k \cdot 10m \leq \log^5 n$. Relabelling if necessary, let $k' \leq k$ be such that $v_1, \dots, v_{k'}$ are the vertices in C that are not linked in \mathcal{P} to two vertices in L . Let

$$V' = \{v_1, \dots, v_{k'}\} \quad \text{and} \quad G' := G - (V(\mathcal{P}) \setminus V') \subseteq G - L.$$

Then, by the definition of L , $\Delta(G') \leq \log^{100} n$.

Using G' has bounded maximum degree, we show that there are sets $B_i \subseteq V(G')$, $i \in [n^{1/8}]$, each of diameter at most m and size $n^{1/8}$, that are at distance at least $4\sqrt{\log n}$ from each other and from $V(C)$ in G' . Now, for each $i \in [n^{1/8}]$, let $B'_i \subseteq B_i$ be a connected subset of size $\log^{10} n$, and set $\mathcal{B} = \{B'_i\}_{i \in [n^{1/8}]}$. Let \mathcal{P}' be a maximal collection of paths in G' from V' to $V(\mathcal{B})$ such that

- each $v \in V'$ is in at most two paths of \mathcal{P}' and each set in \mathcal{B} is linked to at most one path;
- all paths are pairwise disjoint outside of V' with internal vertices in $V(G') \setminus V(\mathcal{B})$;
- the length of each path is at most $10m$.

Subject to $|\mathcal{P}'|$ being maximal, let $\ell(\mathcal{P}') := \sum_{P \in \mathcal{P}'} \ell(P)$ be minimised.

Now we proceed in the same way as in Case 1: we suppose, for contradiction, that there is some $v \in V'$ which is in less than two paths in \mathcal{P}' , and consider the subcollection $\mathcal{B}' \subseteq \mathcal{B}$ of sets linked to some path in \mathcal{P}' . Taking $U' := V(C) \cup V(\mathcal{P}) \cup V(\mathcal{P}') \setminus \{v\}$, we see that $|U'| + |V(\mathcal{B}')| < |\mathcal{B}|$, so there is a set $B \in \mathcal{B}$ disjoint from $U' \cup V(\mathcal{B}')$.

Finally, we show that U' is $(10, 2)$ -thin around v in G , so that we can expand v in $G - U' \subseteq G'$. And, since $B_{G-U'}^r(v)$ is disjoint from $B \cup V(\mathcal{B}')$, and $U' \cup V(\mathcal{B}')$ is much smaller than B and $B_{G-U'}^r(v)$, we can find a path of length at most m in $G - U' - V(\mathcal{B}')$ between B and $B_{G-U'}^r(v)$. Extend this path to v and trim B down to a set of size $\log^{10} n$, which gives one more $V', V(\mathcal{B})$ -path, contradicting the maximality of \mathcal{P}' . This yields the second alternative.

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Cut Vertices in Random Planar Graphs

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Abstract. We denote by $P(n, m)$ a graph chosen uniformly at random from the class of all vertex-labelled planar graphs on vertex set $\{1, \dots, n\}$ with $m = m(n)$ edges. We determine the asymptotic number of cut vertices in $P(n, m)$ in the sparse regime. For comparison, we also derive the asymptotic number of cut vertices in the Erdős-Rényi random graph $G(n, m)$.

Keywords: Random graphs · Random planar graphs · Cut vertices

1 Introduction and Main Results

The Erdős-Rényi random graph $G(n, m)$ is a graph chosen uniformly at random from the class of all vertex-labelled graphs on vertex set $[n] := \{1, \dots, n\}$ with $m = m(n)$ edges. Many exciting results on $G(n, m)$ and on the closely related binomial random graph $G(n, p)$ can be found in literature (see e.g. [1]). In the last decades various models of random graphs have been introduced by imposing additional constraints. Prominent examples of such models are random planar graphs and related objects (see e.g. [4–6, 8]).

Throughout this extended abstract, all asymptotics are taken as $n \rightarrow \infty$ and we say that an event holds with high probability (*whp* for short) if it holds with probability tending to 1 as $n \rightarrow \infty$. Given a graph H we denote by $V(H)$ its vertex set, by $v(H)$ the number of vertices, by $e(H)$ the number of edges, and by $d_H(v)$ the degree of a vertex v in H . We call a vertex $v \in V(H)$ a *cut vertex* in H if deleting v (and its incident edges) from H increases the number of components in H . We denote by $cv(H)$ the number of cut vertices in H divided by $v(H)$.

Let $P(n, m)$ be the random planar graph, i.e. a graph chosen uniformly at random from the class of all vertex-labelled planar graphs on vertex set $[n]$ with $m = m(n)$ edges, and $G = G(n, m)$ the Erdős-Rényi random graph. In this extended abstract, we determine the asymptotic behaviour of $cv(P)$ and $cv(G)$, i.e. the fraction of cut vertices in P and G respectively, revealing their coincidence if $2m/n \rightarrow d \in [0, 1]$ and stark difference otherwise. We note that Drmota, Noy, and Stuffer [3] studied the number of cut vertices in a random *planar map* (i.e. a connected planar graph embedded in the plane) with given number of edges.

To state our main results on $cv(P)$ we distinguish two cases depending on how large the average degree $2m/n$ is. Our first case is when $2m/n \rightarrow d \in [0, 1]$.

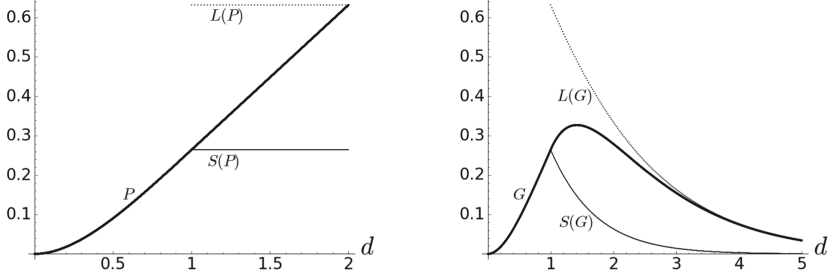


Fig. 1. Fraction of cut vertices in $P = P(n, m)$ and $G = G(n, m)$, where $m = m(n)$ is such that $2m/n \rightarrow d$.

Theorem 1. *Let $P = P(n, m)$ and $m = m(n) \leq n/2 + O(n^{2/3})$ such that $2m/n \rightarrow d \in [0, 1]$. Then whp $\text{cv}(P) = 1 - (d+1)e^{-d} + o(1)$.*

Kang and Łuczak [6] showed that if $m \geq n/2 + \omega(n^{2/3})$, then whp the largest component of $P(n, m)$ is significantly larger than each of the other components. Therefore, we compute not only the fraction of cut vertices in P , but also in the largest component $L(P)$ of P and the remaining part $S(P) := P \setminus L(P)$.

Theorem 2. *Let $P = P(n, m)$, $L = L(P)$ be the largest component of P , and $S = S(P) = P \setminus L$. Assume $m = m(n)$ is such that $n/2 + \omega(n^{2/3}) \leq m \leq n + o(n(\log n)^{-2/3})$ and $2m/n \rightarrow d \in [1, 2]$. Then whp $\text{cv}(L) = 1 - e^{-1} + o(1)$, $\text{cv}(S) = 1 - 2e^{-1} + o(1)$, and $\text{cv}(P) = 1 - (3-d)e^{-1} + o(1)$.*

The ‘technical’ assumption $m \leq n + o(n(\log n)^{-2/3})$ in Theorem 2 is inherited from a result in [8], which we will use in our proofs. In light of Theorems 1 and 2, a natural question is whether $\text{cv}(P)$ behaves like $\text{cv}(G)$ or very differently from it. We will provide an answer to this question in Theorems 3 and 4.

Theorem 3. *Let $G = G(n, m)$ and $m = m(n) \leq n/2 + O(n^{2/3})$ such that $2m/n \rightarrow d \in [0, 1]$. Then whp $\text{cv}(G) = 1 - (d+1)e^{-d} + o(1)$.*

To state our result on $\text{cv}(G)$ when $2m/n \rightarrow d \in [1, \infty)$, we let β_d denote the unique positive solution of the equation $1 - x = e^{-dx}$ for $d > 1$ and we set $\beta_d := 0$ for all $d \in [0, 1]$. In fact, β_d is the survival probability of a Galton-Watson process with offspring distribution $\text{Po}(d)$.

Theorem 4. *Let $G = G(n, m)$, $L = L(G)$ be the largest component of G , and $S = S(G) = G \setminus L$. Assume $m = m(n) \geq n/2 + \omega(n^{2/3})$ is such that $2m/n \rightarrow d \geq 1$. Then whp $\text{cv}(L) = 1 - e^{-d(1-\beta_d)} + o(1)$, $\text{cv}(S) = 1 - (d + e^{d\beta_d})e^{-d} + o(1)$, and $\text{cv}(G) = 1 - (d + e^{d\beta_d} - d\beta_d)e^{-d} + o(1)$.*

Theorems 1 and 3 show that $\text{cv}(P)$ and $\text{cv}(G)$ coincide asymptotically as long as $m \leq n/2 + O(n^{2/3})$. However, Theorems 2 and 4 reveal a completely

different behaviour of $\text{cv}(P)$ and $\text{cv}(G)$ beyond this region (see also Fig. 1). We could not find the results of Theorems 3 and 4 in literature and therefore, we provide sketches of their proofs in Sect. 2. In Sect. 3, we show Theorems 1 and 2.

2 Cut Vertices in the Erdős-Rényi Random Graph

To prove Theorems 3 and 4 on the number of cut vertices in the Erdős-Rényi random graph $G = G(n, m)$, we will use the following facts (see e.g. [1]).

Theorem 5. *Let $k \in \mathbb{N}$, $G = G(n, m)$, $L = L(G)$ be the largest component of G , and $m = m(n)$ be such that $2m/n \rightarrow d \geq 0$. Then whp (i) $v(L) = (\beta_d + o(1))n$, (ii) the second largest component of G has $o(n)$ vertices, and (iii) $|\{v \in V(G) \mid d_G(v) = k\}| = (e^{-d}d^k/k! + o(1))n$.*

Sketch Proofs of Theorems 3 and 4. We estimate the number $X^{(k)}$ of cut vertices in G with $d_G(v) = k$ for $k \geq 2$. Let $X_v^{(k)} = 1$ if $v \in V(G)$ is a cut vertex and $d_G(v) = k$, otherwise we set $X_v^{(k)} = 0$. We construct G conditioned on the event $d_G(v) = k$: We choose a random graph G' on vertex set $[n] \setminus \{v\}$ with $m - k$ edges and then pick independently a set $N_v \subseteq [n] \setminus \{v\}$ for the k neighbours of v . Now v is not a cut vertex if and only if all k vertices of N_v lie in the same component of G' . As G' is distributed like $G(n - 1, m - k)$, Theorem 5 implies that whp the largest component of G' has $(\beta_d + o(1))n$ vertices, while all other components have $o(n)$ vertices. Together with Theorem 5 this shows $\mathbb{P}[X_v^{(k)} = 1] = e^{-d}d^k/k! \cdot (1 - \beta_d^k) + o(1)$. Similarly, we obtain $\mathbb{P}[X_v^{(k)} = X_w^{(k)} = 1] = (e^{-d}d^k/k! \cdot (1 - \beta_d^k))^2 + o(1)$ for $v \neq w$. Hence, whp $X^{(k)} = (e^{-d}d^k(1 - \beta_d^k)/k! + o(1))n$ by the first and second moment method. Due to Theorem 5 there are $o(n)$ vertices $v \in V(G)$ with $d_G(v) = \omega(1)$. Thus, whp $\text{cv}(G) = \left(\sum_{k \geq 2} X^{(k)} + o(n)\right)/n = \sum_{k \geq 2} (e^{-d}d^k(1 - \beta_d^k)/k!) + o(1) = 1 - (d + e^{d\beta_d} - d\beta_d)e^{-d} + o(1)$, which proves the statements on $\text{cv}(G)$. Similarly, we show the assertions on $\text{cv}(L)$ and $\text{cv}(S)$. \square

3 Cut Vertices in the Random Planar Graph

3.1 Proof of Theorem 1

Theorem 1 is a direct consequence of the following well-known fact from [2].

Theorem 6 ([2]). *Let $G = G(n, m)$ and $m = m(n) \leq n/2 + O(n^{2/3})$. Then we have $\liminf_{n \rightarrow \infty} \mathbb{P}[G \text{ has no components with at least two cycles}] > 0$.*

Proof of Theorem 1. Due to Theorem 6 we have $\liminf_{n \rightarrow \infty} \mathbb{P}[G \text{ is planar}] > 0$. Therefore, each property that holds whp in $G(n, m)$ is also true whp in $P(n, m)$ as long as $m \leq n/2 + O(n^{2/3})$. Hence, Theorem 1 follows by Theorem 3. \square

3.2 Graph Decomposition and Conditional Random Graphs

To show Theorem 2 we use the graph decomposition as in [8]. Given a graph H , the *complex part* $Q(H)$ of H is the union of all components with at least two cycles. The remaining part $U(H) := H \setminus Q(H)$ is called the *non-complex part* and we define $n_U(H) := v(U(H))$ and $m_U(H) := e(U(H))$. The *core* $C(H)$ is the maximal subgraph of $Q(H)$ with minimum degree at least two. We note that $Q(H)$ arises from $C(H)$ by replacing each vertex by a rooted tree. The following results of Kang, Möbhammer, and Sprüssel [8] will be useful.

Theorem 7 ([8]). *Let $P = P(n, m)$, $Q = Q(P)$ the complex part of P , $C = C(P)$ the core, $L = L(P)$ the largest component, and $S = S(P) = P \setminus L$. Moreover, let $U = U(P)$ be the non-complex part of P , $n_U = v(U)$, and $m_U = e(U)$. Assume $m = m(n)$ is such that $n/2 + \omega(n^{2/3}) \leq m \leq n + o(n(\log n)^{-2/3})$ and $2m/n \rightarrow d \in [1, 2]$. Then whp (i) $v(C) = o(v(Q))$, (ii) $v(L) = (d - 1 + o(1))n$, (iii) $n_U = \omega(1)$, (iv) $m_U = n_U/2 + O(hn_U^{2/3})$ for each function $h = h(n) = o(1)$, (v) $|V(L)\Delta V(Q)| = o(v(L))$, and (vi) $|V(S)\Delta V(U)| = o(v(S))$.*

Theorem 7(v) and (vi) imply that whp $cv(L) = cv(Q) + o(1)$ and $cv(S) = cv(U) + o(1)$. Furthermore, Theorem 7(ii) allows us to compute $cv(P)$ once $cv(L)$ and $cv(S)$ are known. Thus, it suffices to determine $cv(Q)$ and $cv(U)$, which we will do in Sects. 3.3 and 3.4, respectively. We will compute $cv(Q)$ and $cv(U)$ conditioned that P satisfies some properties. Then we will use the following definition and lemma from [7] to deduce $cv(Q)$ and $cv(U)$.

Definition 1 ([7, Definition 3.1]). *Given a class \mathcal{A} of graphs let $\mathcal{A}(n)$ be the subclass containing those graphs with vertex set $[n]$. Let S be a set and $\Phi : \mathcal{A} \rightarrow S$ a function. We say that a sequence $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ is feasible for (\mathcal{A}, Φ) if for each $n \in \mathbb{N}$ there is a graph $H \in \mathcal{A}(n)$ such that $\Phi(H) = a_n$. Furthermore, for each $n \in \mathbb{N}$ we write $(A | \mathbf{a})(n)$ for a graph chosen uniformly at random from the set $\{H \in \mathcal{A}(n) : \Phi(H) = a_n\}$. We will often omit the dependence on n and write $A | \mathbf{a}$ instead of $(A | \mathbf{a})(n)$.*

Lemma 1 ([7, Lemma 3.2]). *Let \mathcal{A} be a class of graphs, $A = A(n)$ a graph chosen uniformly at random from $\mathcal{A}(n)$, S a set, $\Phi : \mathcal{A} \rightarrow S$ a function, and \mathcal{R} a graph property, i.e. \mathcal{R} is a set of graphs. If for every sequence $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ that is feasible for (\mathcal{A}, Φ) whp $A | \mathbf{a} \in \mathcal{R}$, then we have whp $A \in \mathcal{R}$.*

3.3 Cut Vertices in the Complex Part

Using the concept of ‘conditional’ random graphs (see Definition 1 and Lemma 1) we will determine $cv(Q(P))$ in this section. For a given core C and $q \in \mathbb{N}$ we denote by $Q(C, q)$ a graph chosen uniformly at random from the class of all complex parts which have C as its core and vertex set $[q]$. We observe that $Q(C, q)$ is distributed like $Q(P)$ conditioned on the event $C(P) = C$ and $v(Q(P)) = q$. Furthermore, $Q(C, q)$ can be constructed by choosing a random forest F on

vertex set $[q]$ with $v(C)$ many tree components such that the vertices from C lie all in different tree components. Then we obtain $Q(C, q)$ by replacing each vertex v in C by the tree component of F which is rooted at v . Therefore, we estimate $\text{cv}(F)$ in Lemma 2 and then deduce $\text{cv}(Q(C, q))$ in Lemma 3. To that end, we denote by $F(n, t)$ a forest chosen uniformly at random from the class of all forests on vertex set $[n]$ having exactly t trees as components such that the vertices $1, \dots, t$ lie all in different tree components.

Lemma 2. *Let $t = t(n) = o(n)$ and $F = F(n, t)$ be the random forest. Then we have whp $\text{cv}(F) = 1 - e^{-1} + o(1)$.*

Proof. A vertex $v \in [n] \setminus [t]$ is a cut vertex in F if and only if $d_F(v) \neq 1$. We set $X_v = 1$ if $d_F(v) = 1$ and $X_v = 0$ otherwise. As $t = o(n)$, it suffices to prove whp $\sum_{v \in [n] \setminus [t]} X_v = (e^{-1} + o(1))n$. Each realisation H of F with $d_H(v) = 1$ can be constructed by first choosing a forest H' on vertex set $[n] \setminus \{v\}$ with t tree components such that the vertices in $[t]$ lie all in different tree components. Then we obtain H by picking a vertex $v' \in [n] \setminus \{v\}$ and adding the edge vv' in H' . This implies $\mathbb{P}[X_v = 1] = t(n-1)^{n-t-2} \cdot (n-1) / (tn^{n-t-1}) = e^{-1} + o(1)$. Similarly, we obtain $\mathbb{P}[X_v = X_w = 1] = t(n-2)^{n-t-3} \cdot (n-2)^2 / (tn^{n-t-1}) = e^{-2} + o(1)$ for all $v \neq w$. The statement follows by the first and second moment method. \square

Lemma 3. *For each $n \in \mathbb{N}$, let $C = C(n)$ be a core, $q = q(n) \in \mathbb{N}$, and $Q = Q(C, q)$ be the random complex part with core C and vertex set $[q]$. If $v(C) = o(q)$, then whp $\text{cv}(Q) = 1 - e^{-1} + o(1)$.*

Proof. W.l.o.g. we assume $V(C) = [v(C)]$. We construct Q by picking a random forest $F = F(q, v(C))$ and replacing each vertex v in C by the tree component of F which is rooted at v . A vertex $v \in [q] \setminus [v(C)]$ is a cut vertex in Q if and only if it is a cut vertex in F . Together with the fact $v(C) = o(q)$ it implies whp $\text{cv}(Q) = \text{cv}(F) + o(1)$. Hence, whp $\text{cv}(Q) = 1 - e^{-1} + o(1)$ by Lemma 2. \square

Finally, we use Lemma 1 to transfer the result on the fraction of cut vertices in $Q(C, q)$ from Lemma 3 to the complex part $Q(P)$ of P .

Lemma 4. *Let $P = P(n, m)$ and $Q = Q(P)$ be the complex part of P . Assume $m = m(n)$ is such that $n/2 + \omega(n^{2/3}) \leq m \leq n + o(n(\log n)^{-2/3})$ and $2m/n \rightarrow d \in [1, 2]$. Then whp $\text{cv}(Q) = 1 - e^{-1} + o(1)$.*

Proof. To use Lemma 1, let $\mathcal{A}(n)$ be the class of planar graphs H with vertex set $[n]$ and m edges satisfying $v(C(H)) = o(v(Q(H)))$. By Theorem 7(i) we have whp $P \in \mathcal{A} := \cup_{n \in \mathbb{N}} \mathcal{A}(n)$. Let Φ be such that $\Phi(H) = (C(H), v(Q(H)))$ for each $H \in \mathcal{A}$, $A = A(n)$ be a graph chosen uniformly at random from $\mathcal{A}(n)$, and $\mathbf{a} = (C_n, q_n)_{n \in \mathbb{N}}$ be a sequence that is feasible for (\mathcal{A}, Φ) . We note that $Q(A | \mathbf{a})$ is distributed like the random complex part $Q(C_n, q_n)$. Thus, we get by Lemma 3 that whp $\text{cv}(Q(A | \mathbf{a})) = 1 - e^{-1} + o(1)$. Combining it with Lemma 1 yields whp $\text{cv}(Q(A)) = 1 - e^{-1} + o(1)$. This shows the statement, since whp $P \in \mathcal{A}$. \square

3.4 Cut Vertices in the Non-complex Part

For given $n, m \in \mathbb{N}$ let $U(n, m)$ be a graph chosen uniformly at random from all graphs with vertex set $[n]$ and m edges in which every component has at most one cycle. First we will determine $\text{cv}(U(n, m))$ and then deduce $\text{cv}(U(P))$.

Lemma 5. *Let $U = U(n, m)$ and $m = m(n) \leq n/2 + O(n^{2/3})$ such that $2m/n \rightarrow d \in [0, 1]$. Then whp $\text{cv}(U) = 1 - (d+1)e^{-d} + o(1)$.*

Proof. The assertion follows by combining Theorems 3 and 6. \square

Lemma 6. *Let $P = P(n, m)$ and $U = U(P)$ be the non-complex part of P . Assume $m = m(n)$ is such that $n/2 + \omega(n^{2/3}) \leq m \leq n + o(n(\log n)^{-2/3})$ and $2m/n \rightarrow d \in [1, 2]$. Then whp $\text{cv}(U) = 1 - 2e^{-1} + o(1)$.*

Proof. To use Lemma 1, let $\mathcal{A}(n)$ be the class of planar graphs H with vertex set $[n]$ and m edges satisfying $n_U(H) = \omega(1)$ and $m_U(H) = n_U(H)/2 + O(n_U(H)^{2/3})$. By Theorem 7(iii) and (iv) we can choose the implicit constants in these equations such that $P \in \mathcal{A} := \cup_{n \in \mathbb{N}} \mathcal{A}(n)$ with a probability of at least $1 - \delta$, where $\delta > 0$ is a given constant. Let Φ be such that $\Phi(H) = (n_U(H), m_U(H))$ for each $H \in \mathcal{A}$, $A = A(n)$ be a graph chosen uniformly at random from $\mathcal{A}(n)$ and $\mathbf{a} = (\nu_n, \mu_n)_{n \in \mathbb{N}}$ be a sequence that is feasible for (\mathcal{A}, Φ) . By Lemma 5 we get whp $\text{cv}(U(A | \mathbf{a})) = 1 - 2e^{-1} + o(1)$, as $U(A | \mathbf{a})$ is distributed like $U(\nu_n, \mu_n)$. Together with Lemma 1 this implies whp $\text{cv}(U(A)) = 1 - 2e^{-1} + o(1)$. Using $\mathbb{P}[P \in \mathcal{A}] > 1 - \delta$, we obtain that $\text{cv}(U(P)) = 1 - 2e^{-1} + o(1)$ holds with a probability of at least $1 - 2\delta$. The statement follows, as $\delta > 0$ was arbitrary. \square

3.5 Proof of Theorem 2

Let $Q = Q(P)$ be the complex part of P . By Theorem 7(v) we have whp $\text{cv}(L) = \text{cv}(Q) + o(1)$ and Lemma 4 states that whp $\text{cv}(Q) = 1 - e^{-1} + o(1)$. Thus, whp $\text{cv}(L) = 1 - e^{-1} + o(1)$. Due to Theorem 7(vi) and Lemma 6 we have whp $\text{cv}(S) = \text{cv}(U) + o(1) = 1 - 2e^{-1} + o(1)$, where $U = U(P)$ is the non-complex part of P . Finally, by Theorem 7(ii) we have whp $v(L) = (d - 1 + o(1))n$. Thus, we get whp $\text{cv}(P) = (\text{cv}(L)v(L) + \text{cv}(S)v(S))/n = 1 - (3 - d)e^{-1} + o(1)$. \square

Funding. Supported by Austrian Science Fund (FWF): W1230.

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Extremal Density for Sparse Minors and Subdivisions

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Abstract. We prove an asymptotically tight bound on the extremal density guaranteeing subdivisions of bounded-degree bipartite graphs with a mild separability condition. As corollaries, we answer several questions of Reed and Wood. Among others, $(1 + o(1))t^2$ average degree is sufficient to force the $t \times t$ grid as a topological minor; $(3/2 + o(1))t$ average degree forces *every* t -vertex planar graph as a minor, furthermore, surprisingly, the value is the same for t -vertex graphs embeddable on any fixed surface; average degree $(2 + o(1))t$ forces *every* t -vertex graph in *any* nontrivial minor-closed family as a minor. All these constants are best possible.

Keywords: Graph minors · Subdivisions · Extremal function · Average degree · Sparse graphs

1 Introduction

Classical extremal graph theory studies sufficient conditions forcing the appearance of substructures. A seminal result of this type is the Erdős–Stone–Simonovits theorem [4, 5], determining the asymptotics of the average degree needed for subgraph containment. We are interested here in the analogous problem of average degree conditions forcing H as a minor. A graph H is a *minor* of G , denoted by $G \succ H$, if it can be obtained from G by vertex deletions, edge deletions and contractions.

The study of this problem has a long history. An initial motivation was Hadwiger’s conjecture that every graph of chromatic number t has K_t as a minor, which is a far-reaching generalisation of the four-colour theorem. Since every graph of chromatic number k contains a subgraph of average degree at least $k - 1$, a natural angle of attack is to find bounds on the average degree which will guarantee a K_t -minor. The first upper bounds for general t were given by Mader

J.H. and H.L. were supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1. J.K. was supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and by the KAIX Challenge program of KAIST Advanced Institute for Science-X.

[13, 14]. In celebrated work of Kostochka [10] and, independently, Thomason [20], it was improved to the best possible bound $\Theta(t\sqrt{\log t})$, Thomason subsequently determining the optimal constant [21].

For a general graph H , denote

$$d_{\succ}(H) := \inf\{c : d(G) \geq c \Rightarrow G \succ H\}.$$

Myers and Thomason [16] determined this function when H is polynomially dense, showing that again $d_{\succ}(H) = \Theta(|H|\sqrt{\log|H|})$ and determining the optimal constant in terms of H . However, for sparse graphs their results only give $d_{\succ}(H) = o(|H|\sqrt{\log|H|})$, similar to the way that the Erdős–Stone–Simonovits theorem gives a degenerate bound for bipartite subgraphs, and so it is natural to ask for stronger bounds in this regime.

Reed and Wood [17] gave improved bounds for sparser graphs, and in particular showed that if H has bounded average degree then $d_{\succ}(H) = \Theta(|H|)$. They asked several interesting questions about the precise asymptotics in this regime. Among sparse graphs, grids play a central role in graph minor theory, and Reed and Wood raised the question of determining $d_{\succ}(\mathbb{G}_{t,t})$, where $\mathbb{G}_{t,t}$ is the $t \times t$ grid. That is, what is the minimum $\beta > 0$ such that every graph with average degree at least βt^2 contains $\mathbb{G}_{t,t}$ as a minor. Trivially $\beta \geq 1$ in order for the graph to have enough vertices, and their results give a bound of $\beta \leq 6.929$.

This question provides the motivating example for our results. However, we shall focus on a special class of minors: subdivisions or topological minors. A *subdivision* of H is a graph obtained from subdividing edges of H to pairwise internally disjoint paths. The name of topological minor comes from its key role in topological graph theory. A cornerstone result in this area is Kuratowski's theorem from 1930 that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. Again it is natural to ask what average degree will force K_t as a topological minor, and we define analogously

$$d_{\top}(H) := \inf\{c : d(G) \geq c \Rightarrow G \text{ contains } H \text{ as a topological minor}\}.$$

Clearly, for any H , $d_{\succ}(H) \leq d_{\top}(H)$. However, there can be a considerable gap between the two quantities; Komlós and Szemerédi [9] and, independently, Bollobás and Thomason [2] showed that $d_{\top}(K_t) = \Theta(t^2)$, meaning that clique topological minors are much harder to guarantee than clique minors. Furthermore, the optimal constant is still unknown in this case, and in general much less is known for bounds on average degree guaranteeing sparse graphs as topological minors.

1.1 Main Result

Our main result offers the asymptotics of the average degree needed to force subdivisions of a natural class of sparse bipartite graphs, showing that a necessary bound is already sufficient. It reads as follows.

Theorem 1. *For given $\varepsilon > 0$ and $\Delta \in \mathbb{N}$, there exist α_0 and d_0 satisfying the following for all $0 < \alpha < \alpha_0$ and $d \geq d_0$. If H is an α -separable bipartite graph with at most $(1 - \varepsilon)d$ vertices and $\Delta(H) \leq \Delta$, and G is a graph with average degree at least d , then G contains a subdivision of H .*

Here a graph H is α -separable if there exists a set S of at most $\alpha|H|$ vertices such that every component of $H - S$ has at most $\alpha|H|$ vertices. Graphs in many well-known classes are $o(1)$ -separable. For example, large graphs in any nontrivial minor-closed family are $o(1)$ -separable [1, 15].

As an immediate corollary, our main result answers the above question of Reed and Wood in a strong sense by showing that any $\beta > 1$ is sufficient to force the k -dimensional grid $G_{t,\dots,t}^k$ not only as a minor but as a topological minor, and so

$$d_{\top}(G_{t,\dots,t}^k) = d_{>}(G_{t,\dots,t}^k) = (1 + o_t(1))t^k.$$

We remark that the optimal constant 1 in Theorem 1 is no longer sufficient if H is not bipartite. Indeed, if e.g. H is the disjoint union of triangles, then the Corrádi–Hajnal theorem [3] implies that $d_{>}(H) = \frac{4}{3}|H| - 2$.

2 Applications

Reed and Wood [17] raised several other interesting questions on the average degree needed to force certain sparse graphs as minors. In particular, they asked the following.

- What is the least constant $c > 0$ such that every graph with average degree at least ct contains every planar graph with t vertices as a minor?
- What is the least function g_1 such that every graph with average degree at least $g_1(k) \cdot t$ contains every graph with t vertices and treewidth at most k as a minor?
- What is the least function g_2 such that every graph with average degree at least $g_2(k) \cdot t$ contains every K_k -minor-free graph with t vertices as a minor?

In applying our results to answer these questions, there are two obstacles to overcome. First, the graph classes considered have bounded average degree, but our main result only covers graphs of bounded maximum degree. Secondly, and more significantly, these classes include non-bipartite graphs. Both issues may be overcome by first constructing a suitable graph H' containing the target graph H as a minor, ensuring that H' is bipartite with bounded average degree but still inherits a suitable separability condition from the original target graph. We then find a subdivision of H' in the host graph. In order to ensure H' has bounded degree it cannot necessarily be a subdivision of H , and so this procedure gives H as a minor, but not necessarily a topological minor.

Passing from a bounded average degree H to a bounded degree graph only requires the addition of $o(t)$ vertices, whereas ensuring that H' is bipartite typically changes the constant required, in a way that depends on the precise class of graphs involved. Thus we obtain a range of different constants for different

classes; nevertheless, many of these constants are optimal. In the following results we use the notation

$$d_{\succ}(\mathcal{F}, t) := \inf\{c : d(G) \geq c \Rightarrow G \succ H, \forall H \in \mathcal{F} \text{ with } |H| \leq t\}$$

for a graph family \mathcal{F} . We answer the first question above in a strong sense, giving the optimal constant and showing that the answer is the same for graphs which may be drawn on any fixed surface.

Theorem 2. *Writing \mathcal{F}_g for the class of graphs with genus at most g , we have $d_{\succ}(\mathcal{F}_g, t) = (3/2 + o(1))t$.*

Many other important classes of graphs are naturally closed under taking minors. The seminal graph minor theorem of Robertson and Seymour (proved in a sequence of papers culminating in [18]) shows that every minor-closed family can be characterised by a finite list of minimal forbidden minors. For example, the linklessly-embeddable graphs are defined by a minimal family of seven forbidden minors, including K_6 and the Petersen graph [19]. We can extend the proof of Theorem 2 to minor-closed families more generally; in fact our results also apply to classes of polynomial expansion, which are not necessarily minor-closed. For each $k \in \mathbb{N}$, define $\alpha_k(G) := \max\{|U| : U \subseteq V(G), \chi(G[U]) = k\}$. So $\alpha_1(G)$ is the usual independence number and $\alpha_2(G)$ is the maximum size of the union of two independent sets.

Theorem 3. *Let \mathcal{F} be a nontrivial minor-closed family, or, more generally, a class of polynomial expansion. For each $F \in \mathcal{F}$ with t vertices, we have*

$$2t - 2\alpha(F) - O(1) \leq d_{\succ}(F) \leq 2t - \alpha_2(F) + o(t).$$

Theorem 3 yields the following consequences, for all of which the constants are best possible (note that the last example is not a minor-closed class).

- The class \mathcal{T}_k of treewidth at most k satisfies $d_{\succ}(\mathcal{T}_k, t) = (\frac{2k}{k+1} + o(1))t$; in particular, $g_1(k) = 2 - o_k(1)$.
- $g_2(k) = 2 - o_k(1)$.
- For any nontrivial minor closed family \mathcal{F} , we have $d_{\succ}(\mathcal{F}, t) \leq (2 + o(1))t$.
- The class \mathcal{L} of linklessly embeddable graphs satisfies $d_{\succ}(\mathcal{L}, t) = (8/5 + o(1))t$.
- The class \mathcal{P}_1 of 1-planar graphs satisfies $d_{\succ}(\mathcal{P}_1, t) = (5/3 + o_t(1))t$.

While for some families we are able to show that the upper and lower bounds from Theorem 3 match, giving the precise constant, in others this is not clear. In particular, for the K_k -minor-free graphs Hadwiger's conjecture would imply matching bounds.

3 Outline of the Proof

Our proof utilises both pseudorandomness from Szemerédi's regularity lemma and expansions for sparse graphs. The particular expander that we shall make

use of is an extension of the one introduced by Komlós and Szemerédi [8,9], which has played an important role in some recent developments on sparse graph embedding problems, see e.g. [7,11,12].

To prove Theorem 1, we first pass to a robust sublinear expander subgraph without losing much on the average degree. Depending on the density of this expander, we use different approaches. Roughly speaking, when the expander has positive edge density, we will utilise pseudorandomness via the machinery of the graph regularity lemma and the blow-up lemma, and otherwise we exploit its sublinear expansion property. Full proofs may be found in [6].

3.1 Embeddings in Dense Graphs

The regularity lemma essentially partitions our graph G into a bounded number of parts, in which the bipartite subgraphs induced by most of the pairs of parts behave pseudorandomly. The information of this partition is then stored to a (weighted) fixed-size so-called reduced graph R which inherits the density of G . We seek to embed H in G using the blow-up lemma, which boils down to finding a ‘balanced’ bounded-degree homomorphic image of H in R . This is where the additional separable assumption on H kicks in, enabling us to cut H into small pieces to offer suitable ‘balanced’ homomorphic images. If the reduced graph R is not bipartite, the density of R inherited from G is just large enough to guarantee an odd cycle in R long enough to serve as our bounded-degree homomorphic image of H . However, an even cycle of the same length would not be sufficient, since H could be an extremely asymmetric bipartite graph. To overcome this problem, when R is bipartite we make use of a ‘sun’ structure. This is a bipartite graph consisting of a cycle with some additional leaves, which help in balancing out any asymmetry of H .

3.2 Embeddings in Robust Expanders with Medium Density

The robust sublinear expansion underpins all of our constructions of H -subdivisions when the graph G is no longer dense. At a high level, in G , we anchor on some carefully chosen vertices and embed paths between anchors (corresponding to the edge set of H) one at a time. As these paths in the subdivision need to be internally vertex disjoint, to realise this greedy approach we will need to build a path avoiding a certain set of vertices. This set of vertices to avoid contains previous paths that we have already found and often some small set of ‘fragile’ vertices that we wish to keep free.

To carry out such robust connections, we use the small-diameter property of sublinear expanders. We aim to anchor at vertices with large ‘boundary’ compared to the total size of all paths needed, that is, being able to access many vertices within short distance. If there are d vertices of sufficiently high degree, we can anchor on them. Assuming this is not the case essentially enables us to view G as if it is a ‘relatively regular’ graph. We now use a web structure in which each core vertex is connected by a tree to a large ‘exterior’. Using the relative regularity of G , together with the fact that it is not too sparse, we can pull

out many reasonably large stars and link them up to find webs. We then anchor on their core vertices and connect pairs via the exteriors of the corresponding webs, while being careful to avoid the fragile centre parts of other webs.

3.3 Embeddings in Sparse Robust Expanders

The method of building and connecting webs breaks down if the expander is too sparse, and we need to use other structures in this case.

For the easier problem of finding minors, it suffices to find d large balls and link them up by internally disjoint paths according to the structure of H ; contracting each ball gives H as a minor. In order to be able to find the paths, we ensure the balls are sufficiently far apart that any given pair of balls can be expanded to very large size, avoiding all others, and then connect the pairs one by one.

Coming back to embedding H -subdivisions, we shall follow a similar general strategy. However, an immediate obstacle we encounter is that we need to be able to lead a constant number of paths arriving at each ball disjointly to the anchor vertex. In other words, each anchor vertex has to expand even after removing a constant number of disjoint paths starting from itself. Our expansion property is simply too weak for this.

We therefore use a new structure we call a ‘nakji’. Each nakji consists of several ‘legs’, which are balls pairwise far apart, linked to a central well-connected ‘head’. This structure is designed precisely to circumvent the above problem by doing everything in reverse order. Basically, instead of looking for anchor vertices that expand robustly, we rather anchor on nakjis and link them via their legs first and then extend the paths from the legs in each nakji’s head using connectivity. The remaining task is then to find many nakjis. This is done essentially by linking small subexpanders within G , after removing the few high-degree vertices.

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Enumerating Descents on Quasi-Stirling Permutations and Plane Trees

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Abstract. Gessel and Stanley introduced Stirling permutations to give a combinatorial interpretation of certain polynomials related to Stirling numbers. A natural extension of these permutations are quasi-Stirling permutations, which are in bijection with labeled rooted plane trees, and can be viewed as labeled noncrossing matchings. They were recently introduced by Archer et al., who conjectured that there are $(n + 1)^{n-1}$ quasi-Stirling permutations of size n having n descents. Here we prove this conjecture. More generally, we enumerate quasi-Stirling permutations, as well as a one-parameter family that generalizes them, by the number of descents, giving an implicit equation for their generating function in terms of that of Eulerian polynomials. We also show that many of the properties of descents on usual permutations and on Stirling permutations have an analogue for quasi-Stirling permutations.

Keywords: Quasi-Stirling · Stirling permutation · Descent · Plane tree

1 Introduction

1.1 Stirling Permutations

Stirling permutations were introduced by Gessel and Stanley in [7]. Denoted by \mathcal{Q}_n , they are defined as those permutations $\pi_1\pi_2\dots\pi_{2n}$ of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ satisfying that, if $i < j < k$ and $\pi_i = \pi_k$, then $\pi_j > \pi_i$. This condition can be described as avoiding the pattern 212. In general, given two sequences of positive integers π and σ , we say that π avoids σ if there is no subsequence of π whose entries are in the same relative order as those of σ . There is an extensive literature on Stirling permutations and their generalizations [3, 4, 8, 9, 12].

With the notation $[r] = \{1, 2, \dots, r\}$, let $i \in [r]$ be a *descent* of $\pi = \pi_1\pi_2\dots\pi_r$ if $\pi_i > \pi_{i+1}$ or $i = r$, and let $\text{des}(\pi)$ denote the number of descents of π . This definition agrees with [3, 7, 9], while other papers [2] do not consider $i = r$ to be a descent. We also consider *ascents*, which are indices $i \in \{0, \dots, r - 1\}$ such that $\pi_i < \pi_{i+1}$ or $i = 0$, and *plateaus*, which are indices $i \in [r - 1]$ such that

$\pi_i = \pi_{i+1}$. Let $\text{asc}(\pi)$ and $\text{plat}(\pi)$ denote the number of ascents and the number of plateaus of π , respectively.

Denoting by \mathcal{S}_n the set of permutations of $[n]$, the polynomials

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} \tag{1}$$

are called *Eulerian polynomials*. It is well known (see e.g. [13, Prop. 1.4.4]) that

$$\sum_{m \geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}. \tag{2}$$

Let $S(n, m)$ be the Stirling numbers of the second kind, which count partitions of an n -element set into m blocks. Gessel and Stanley [7] showed that, when replacing the coefficients in the left-hand side of Eq. (2) by these numbers, then the role of the Eulerian polynomials is played by the *Stirling polynomials* $Q_n(t) = \sum_{\pi \in \mathcal{Q}_n} t^{\text{des}(\pi)}$.

Theorem 1 ([7])

$$\sum_{m \geq 0} S(m+n, m) t^m = \frac{Q_n(t)}{(1-t)^{2n+1}}.$$

1.2 Quasi-Stirling Permutations

In [2], Archer et al. introduce the set $\overline{\mathcal{Q}}_n$ of *quasi-Stirling* permutations. These are permutations $\pi_1 \pi_2 \dots \pi_{2n}$ of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ avoiding 1212 and 2121, which means that there do not exist $i < j < k < \ell$ such that $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$. Thinking of π as a labeled matching of $[2n]$, by placing an arc between with label k between i with j if $\pi_i = \pi_j = k$, the avoidance requirement is equivalent to the matching being *noncrossing*¹. By definition, $\mathcal{Q}_n \subseteq \overline{\mathcal{Q}}_n$.

Archer et al. [2] note that $|\overline{\mathcal{Q}}_n| = n! C_n = \frac{(2n)!}{(n+1)!}$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number. They also compute the number of permutations in $\overline{\mathcal{Q}}_n$ avoiding some sets of patterns of length 3, and they enumerate quasi-Stirling permutations by the number of plateaus. They pose the open problem of enumerating quasi-Stirling permutations by the number of descents, and they conjecture that the number of $\pi \in \overline{\mathcal{Q}}_n$ with $\text{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

1.3 A Bijection to Plane Trees

Quasi-Stirling permutations are in bijection with labeled plane rooted trees, in much the same way that Stirling permutations are in bijection with increasing trees. Denote by \mathcal{T}_n the set of edge-labeled plane (i.e., ordered) rooted trees with

¹ The closely related set of labeled *nonnesting* matchings corresponds to permutations that avoid 1221 and 2112. The distribution of des on these permutations, which is different from its distribution on $\overline{\mathcal{Q}}_n$, is the topic of a forthcoming preprint [1].

n edges. Each edge of such a tree receives a unique label from $[n]$. The *root* is a distinguished vertex of the tree, which we place at the top. The *children* of a vertex i are the neighbors of i that are not in the path from i to the root; the neighbor of i in the path to the root (if i is not the root) is called the *parent* of i . The children of i are placed below i , and the left-to-right order in which they are placed matters.

Disregarding the labels, it is well known that the number of unlabeled plane rooted trees with n edges is C_n . Since there are $n!$ ways to label the edges of a particular tree, it follows that $|\mathcal{T}_n| = n! C_n$.

Denote by $\mathcal{I}_n \subseteq \mathcal{T}_n$ be the subset of those trees whose labels along any path from the root to a leaf are increasing. Elements of \mathcal{I}_n are called increasing edge-labeled plane trees, or simply increasing trees when there is no confusion.

A simple bijection between \mathcal{T}_n and $\overline{\mathcal{Q}}_n$ was introduced by Koganov [10, Thm. 3]. Archer et al. [2] showed that this bijection naturally extends to a bijection φ between \mathcal{T}_n and $\overline{\mathcal{Q}}_n$. Both bijections can be described as follows. Given a tree $T \in \mathcal{T}_n$, traverse its edges by following a depth-first walk from left to right (i.e., counterclockwise); that is, start at the root, go to the leftmost child and explore that branch recursively, return to the root, then continue to the next child, and so on. Recording the labels of the edges as they are traversed gives a permutation $\varphi(T) \in \overline{\mathcal{Q}}_n$; see Fig. 1 for an example. Note that each edge is traversed twice, once in each direction. As shown in [2], the map $\varphi : \mathcal{T}_n \rightarrow \overline{\mathcal{Q}}_n$ is a bijection. Additionally, the image of the subset of increasing trees is precisely the set of Stirling permutations, and so φ induces a bijection between \mathcal{I}_n and \mathcal{Q}_n , which is the map described in [10].

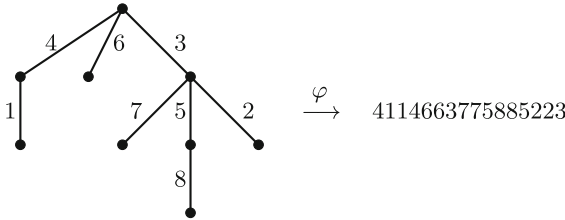


Fig. 1. An example of the bijection $\varphi : \mathcal{T}_n \rightarrow \overline{\mathcal{Q}}_n$.

2 Descents on Quasi-Stirling Permutations

In order to enumerate quasi-Stirling permutations by the number of descents, we first describe how descents are transformed by the bijection φ . Define the number of *cyclic descents* of a sequence of positive integers $\pi = \pi_1 \pi_2 \dots \pi_r$ to be

$$\text{cdes}(\pi) = |\{i \in [r] : \pi_i > \pi_{i+1}\}|,$$

with the convention $\pi_{r+1} := \pi_1$.

Let $T \in \mathcal{T}_n$, and let v a vertex of T . If v is not the root, define $\text{cdes}(v)$ to be the number of cyclic descents of the sequence obtained by listing the labels of the edges incident to v in counterclockwise order (note that the starting point is irrelevant). Equivalently, if the label of the edge between v and its parent is ℓ , and the labels of the edges between v and its children are a_1, a_2, \dots, a_d from left to right, then $\text{cdes}(v) = \text{cdes}(\ell a_1 \dots a_d)$. If v is the root of T , define $\text{cdes}(v)$ to be the number of descents of the sequence obtained by listing the labels of the edges incident to v from left to right, that is, $\text{cdes}(v) = \text{des}(a_1 \dots a_d)$ with the above notation. Finally, define the number of cyclic descents of T to be $\text{cdes}(T) = \sum_v \text{cdes}(v)$, where the sum ranges over all the vertices v of T .

Lemma 1. *The bijection $\varphi : \mathcal{T}_n \rightarrow \overline{\mathcal{Q}}_n$ has the following property: if $T \in \mathcal{T}_n$ and $\pi = \varphi(T) \in \overline{\mathcal{Q}}_n$, then $\text{des}(\pi) = \text{cdes}(T)$.*

It follows from Lemma 1 that the maximum value of $\text{des}(\pi)$ for $\pi \in \overline{\mathcal{Q}}_n$ is n . Our next result counts how many permutations attain this upper bound.

Theorem 2. *The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\text{des}(\pi) = n$ is equal to $(n + 1)^{n-1}$.*

Let

$$A(t, z) = \sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1 - t}{1 - te^{(1-t)z}}$$

be the well-known [13, Prop. 1.4.5] exponential generating function (EGF for short) of the Eulerian polynomials, defined in Eq. (2).

In analogy to $A_n(t)$ and $Q_n(t)$, define the *quasi-Stirling polynomials* and their EGF:

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\text{des}(\pi)}, \quad \overline{Q}(t, z) = \sum_{n \geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$

The following equation describes $\overline{Q}(t, z)$, and allows us to compute $\overline{Q}_n(t)$. We use $[z^n]F(z)$ to denote the coefficient of z^n in the generating function $F(z)$.

Theorem 3. *The EGF of quasi-Stirling permutations by the number of descents satisfies the implicit equation $\overline{Q}(t, z) = A(t, z\overline{Q}(t, z))$, that is,*

$$\overline{Q}(t, z) = \frac{1 - t}{1 - te^{(1-t)z\overline{Q}(t, z)}}.$$

In particular, extracting its coefficients using Lagrange inversion,

$$\overline{Q}_n(t) = \frac{n!}{n + 1} [z^n] A(t, z)^{n+1}.$$

Gessel and Stanley’s main result from [7] (stated above as Theorem 1) is the analogue for Stirling polynomials of Eq. (2). As a consequence of Theorem 3, we obtain the following analogue for quasi-Stirling polynomials.

Theorem 4

$$\sum_{m \geq 0} \frac{m^n}{n + 1} \binom{m + n}{m} t^m = \frac{\overline{Q}_n(t)}{(1 - t)^{2n+1}}.$$

3 Properties of Quasi-Stirling Polynomials

Bóna proves in [3, Cor. 1] that Stirling permutations in \mathcal{Q}_n have, on average, $(2n+1)/3$ ascents, $(2n+1)/3$ descents, and $(2n+1)/3$ plateaus. From Theorem 3, we can derive the following analogue for quasi-Stirling permutations.

Corollary 1. *Let $n \geq 1$. On average, elements of $\overline{\mathcal{Q}}_n$ have $(3n+1)/4$ ascents, $(3n+1)/4$ descents, and $(n+1)/2$ plateaus.*

It is well-known result of Frobenius that the roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive. In [3, Thm. 1], Bóna proves the analogous result for the Stirling polynomials $Q_n(t)$, although their real-rootedness had already been shown by Brenti [4, Thm. 6.6.3] in more generality. We can prove that quasi-Stirling polynomials $\overline{Q}_n(t)$ also have this property.

Theorem 5. *For every $n \geq 1$, the polynomials $\overline{Q}_n(t)$ have real, distinct, and nonpositive roots. Thus, their coefficients are unimodal and log-concave.*

Theorem 6. *The distribution of the number of descents (resp. ascents, plateaus) on elements of $\overline{\mathcal{Q}}_n$ converges to a normal distribution as $n \rightarrow \infty$.*

4 Generalization to k -Quasi-Stirling Permutations

Here we refine the results from Sect. 2 to track the joint distribution of asc, des and plat, and we generalize them to a one-parameter family of permutations.

Gessel and Stanley [7] proposed an extension of Stirling permutations by considering permutations of the multiset containing k copies of each element in $[n]$, while avoiding the pattern 212. These permutations, often called k -Stirling permutations, have been studied in [4, 9, 12].

In analogy to these, we define k -quasi-Stirling permutations as permutations with k copies of each element in $[n]$ that avoid the patterns 1212 and 2121. Denote this set by $\overline{\mathcal{Q}}_n^k$. Note that $\overline{\mathcal{Q}}_n^1 = \mathcal{S}_n$ and $\overline{\mathcal{Q}}_n^2 = \overline{\mathcal{Q}}_n$. Viewing such permutations as ordered set partitions into blocks of size k , the avoidance requirement is equivalent to the partition being *noncrossing*.

We have constructed bijections between k -quasi-Stirling permutations and two different kinds of decorated trees. The first one extends a bijection of Gessel [6] and Janson, Kuba and Panholzer [9, Thm. 1] between k -Stirling permutations and so-called $(k+1)$ -ary increasing trees. The second extends a construction of Kuba and Panholzer [11, Thm. 2.2]. These bijections allow us to easily count the number of k -quasi-Stirling permutations:

Theorem 7. *For $n \geq 1$ and $k \geq 1$,*

$$|\overline{\mathcal{Q}}_n^k| = \frac{(kn)!}{((k-1)n+1)!} = n! C_{n,k}, \quad \text{where} \quad C_{n,k} = \frac{1}{(k-1)n+1} \binom{kn}{n}.$$

Next we state our most general result, proved using our new bijections to decorated trees. Define the refined k -quasi-Stirling polynomials and their EGF:

$$\overline{P}_n^{(k)}(q, t, u) = \sum_{\pi \in \overline{\mathcal{Q}}_n^k} q^{\text{asc}(\pi)} t^{\text{des}(\pi)} u^{\text{plat}(\pi)}, \quad \overline{P}^{(k)}(q, t, u; z) = \sum_{n \geq 0} \overline{P}_n^{(k)}(q, t, u) \frac{z^n}{n!}.$$

Theorem 8. *Fix $k \geq 1$. The EGF of k -quasi-Stirling permutations by the number of ascents, the number of descents, and the number of plateaus satisfies the implicit equation*

$$\overline{P}^{(k)}(q, t, u; z) = 1 - q + \frac{q(q - t)}{q - t \exp\left((q - t)z(\overline{P}^{(k)}(q, t, u; z) - 1 + u)^{k-1}\right)}.$$

In particular, for $n \geq 1$, its coefficients satisfy

$$\overline{P}_n^{(k)}(q, t, u) = \frac{n!}{(k - 1)n + 1} [z^n] \left(u - q + \frac{q(q - t)}{q - te^{(q-t)z}} \right)^{(k-1)n+1}.$$

As a consequence of Theorem 8, we obtain the following generalization of Theorem 2. One can show that the maximum number of descents that a permutation in $\overline{\mathcal{Q}}_n^k$ can have is n .

Corollary 2. *The number of $\pi \in \overline{\mathcal{Q}}_n^k$ with $\text{des}(\pi) = n$ is $((k - 1)n + 1)^{n-1}$.*

The proofs of the results in this extended abstract can be found in [5].

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Hamiltonicity of Randomly Perturbed Graphs

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Abstract. The theory of randomly perturbed graphs deals with the properties of graphs obtained as the union of a deterministic graph H and a random graph G . We study Hamiltonicity in two distinct settings. In both of them, we assume H is some deterministic graph with minimum degree at least αn , for some α (possibly depending on n). We first consider the case when G is a random geometric graph, and obtain an asymptotically optimal result. We then consider the case when G is a random regular graph, and obtain different results depending on the regularity.

Keywords: Randomly perturbed graphs · Random regular graphs · Random geometric graphs · Hamiltonicity

1 Introduction

The theory of randomly perturbed graphs serves to bridge between two classical areas of combinatorics, namely the area of extremal combinatorics and the area of random graphs. In the case of Hamiltonicity, for instance, a classical result of Dirac asserts that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. As for random graphs, another classical result (originally due to Koršunov) states that the random graph $G_{n,p}$ is Hamiltonian asymptotically almost surely (abbreviated as a.a.s.; we say that a sequence of events $\{\mathcal{E}_i\}_{i \in \mathbb{N}}$ holds a.a.s. if $\mathbb{P}[\mathcal{E}_i] \rightarrow 1$ as $i \rightarrow \infty$) whenever $p \geq (1 + \varepsilon) \log n/n$, while a.a.s. it is not even connected if $p \leq (1 - \varepsilon) \log n/n$ (here, $G_{n,p}$ stands for the binomial random graph on n vertices, where each of the possible edges is added to the graph with probability p , independently of all other edges). Bohman, Frieze and Martin [2] were the first to consider the union of a deterministic graph and a random graph, and they proved the following result.

Theorem 1 ([2]). *For every $\alpha \in (0, 1/2)$, there exists a constant C such that, if H is an n -vertex graph with $\delta(H) \geq \alpha n$ and $p \geq C/n$, then a.a.s. $H \cup G_{n,p}$ is Hamiltonian.*

Observe that the union allows to reduce the degree of H below the threshold given by Dirac's theorem, and to also obtain a logarithmic-factor improvement on the threshold for Hamiltonicity in $G_{n,p}$.

After this seminal result, Hamiltonicity has been considered in randomly perturbed directed graphs [2, 9], hypergraphs [8, 9, 11] and subgraphs of the hypercube [3], with significant improvements on the probability threshold in all cases. Many properties other than Hamiltonicity have been considered as well (see, e.g., the references in [7]), similarly improving probability thresholds. However, all of these results rely on the random structure being binomial, and no results are known when G follows a different distribution. Here, we begin the study of graphs perturbed by random graphs which are not binomial. In particular, we consider random geometric and random regular graphs. All our results extend to pancyclicity, but for simplicity here we only refer to Hamiltonicity.

2 Hamiltonicity of Graphs Perturbed by a Random Geometric Graph

A d -dimensional random geometric graph $G^d(n, r)$, where r is a positive real number, is a graph with vertex set $V := [n]$ and edge set E defined as follows. Let X_1, \dots, X_n be n independent uniform random variables on $[0, 1]^d$. Then, let $E := \{\{i, j\} : \|X_i - X_j\| \leq r\}$, where $\|\cdot\|$ denotes the Euclidean norm. (The results in this section extend to any ℓ_p norm, but we consider $p = 2$ here for simplicity.)

Hamiltonicity is fairly well-understood in random geometric graphs. In particular, for all $d \geq 1$ there exists a constant c_d such that, for all $\varepsilon > 0$, if $r \geq (1 + \varepsilon)(c_d \log n/n)^{1/d}$, then a.a.s. $G^d(n, r)$ is Hamiltonian, and if $r \leq (1 - \varepsilon)(c_d \log n/n)^{1/d}$, then a.a.s. $G^d(n, r)$ is not Hamiltonian [1, 5, 12]. The main result of the first author in this context is the following, which can be seen as an analogue of Theorem 1 for random geometric graphs.

Theorem 2 ([6]). *For every integer $d \geq 1$ and $\alpha \in (0, 1/2)$, there exists a constant C such that the following holds. Let H be an n -vertex graph with $\delta(H) \geq \alpha n$, and let $r \geq (C/n)^{1/d}$. Then, a.a.s. $H \cup G^d(n, r)$ is Hamiltonian.*

Proof (sketch). We choose some constant C , which depends on α and d and is sufficiently large so that all subsequent claims hold. We assume that $r = (C/n)^{1/d}$ (which suffices since, by increasing r , we create a sequence of nested graphs).

Partition the hypercube $[0, 1]^d$ into smaller cubes of side y , which we call *cells*. Two cells c_1 and c_2 are *friends* if there is a cell c_3 whose boundary intersects those of both c_1 and c_2 . The value of y is chosen so that, if c_1 and c_2 are friends, then all $x_1 \in c_1$, $x_2 \in c_2$ satisfy $\|x_1 - x_2\| \leq r$ (i.e., all vertices in c_1 will be adjacent to all vertices in c_2). In particular, $y = \Theta(r)$ and there are $\Theta(n)$ cells.

Consider the variables X_1, \dots, X_n that assign positions in $[0, 1]^d$ to the vertices of H . A cell is called *dense* if it contains at least $2 \cdot 5^d$ vertices of H , and *sparse* otherwise. Furthermore, for each pair of (not necessarily distinct) vertices

$u, v \in V(H)$, we call a cell $\{u, v\}$ -dense if it contains two distinct vertices w and x such that $uw, vx \in E(H)$, and $\{u, v\}$ -sparse otherwise. By Azuma's inequality (and adjusting the value of C), we can show that a.a.s. the proportion of sparse cells is an arbitrarily small constant, and similarly the proportion of $\{u, v\}$ -sparse cells is an arbitrarily small constant, for all pairs of vertices $u, v \in V(H)$.

We now define an auxiliary graph Γ , whose vertices are the dense cells, and where two cells are adjacent whenever they are friends. This definition ensures Γ has bounded degrees ($\Delta(\Gamma) < 5^d$). Since there are “few” sparse cells, we can show Γ has “few” connected components (at most δn , where δ is an arbitrarily small constant). These two properties are crucial for the rest of the proof.

The main strategy for the proof now is as follows. We will use the auxiliary graph Γ to construct a set of cycles, each of which covers all the vertices in the cells that form a component of Γ . Then, using the fact that there are few $\{u, v\}$ -sparse cells, we will find suitable edges of H that can be added to the cycles in such a way that we obtain a unique cycle on the same vertex set as the original cycles. To complete the proof, we can again use the fact that there are few $\{u, v\}$ -sparse cells to incorporate all vertices in sparse cells into the cycle, again using some edges of H for this purpose.

In practice, however, in order to guarantee that this process works, it is better to proceed backwards. Indeed, first we are going to find paths which can incorporate all vertices in sparse cells. For each sparse cell c (which contains at least one vertex of H), we choose two vertices u and v in the cell (distinct whenever possible) and choose a $\{u, v\}$ -dense cell c' which is also dense. By the definition of $\{u, v\}$ -dense, we can find a path in $G^d(n, r)$ with endpoints in c' which contains all vertices in c . Moreover, since there are “few” sparse cells, we can do this in such a way that all the cells c' are distinct.

We define the edges that will be used to connect the different components similarly. Indeed, for each component of Γ , we choose a cell c that lies in this component and two vertices $u, v \in V(H)$ that lie in this cell. Then, we choose an arbitrary $\{u, v\}$ -dense cell c' which lies in a different component of Γ . Using the fact that Γ has few components, we can make sure that all the cells c and c' that are picked are distinct, and distinct from those we picked for sparse cells.

For each dense cell c or c' that we have picked in the previous steps, we have chosen exactly two edges of H incident to two vertices in this cell, say, w and x . Now, assume we can find a set of cycles, each of which covers all vertices in the cells of a component of Γ , with the added property that, for all cells c and c' that we picked in the previous steps, we can ensure that the vertices w and x form an edge in the cycle. Then, we can replace these edges by the corresponding edges of H and paths containing vertices in sparse cells to obtain a Hamilton cycle.

To complete the proof, it suffices to find such a set of cycles. For each component of Γ , we first find a spanning tree T of the component. Then we consider a traversal of the tree that goes through every edge of the tree twice and returns to the starting vertex (e.g., by a depth-first search). To build the cycle, we pick a vertex in the first cell of the traversal. Then, for each subsequent cell of the traversal, we pick an arbitrary vertex in the cell which has not been covered yet.

If the previous cell will be visited again at some point in the traversal, we add an edge joining the newly chosen vertex to the last vertex of the path we have built so far. Otherwise, we build a path that connects the two desired vertices while including all vertices in the previous cell which had not been covered yet. Here, we make sure that, if the current cell contains a pair of vertices w and x as above, we incorporate wx to the path. The same approach allows to close this path into a cycle at the end. This step can be carried out because $\Delta(T) < 5^d$, so no cell will appear in the traversal more than 5^d times, and since all these cells are dense, they contain sufficiently many vertices. It is also here that we use the fact that all cells c or c' are distinct. \square

3 Hamiltonicity of Graphs Perturbed by a Random Regular Graph

A graph is regular if all its vertices have the same degree, and a random d -regular graph on n vertices $G_{n,d}$ is a graph chosen uniformly at random from the set of all such graphs. It is well known by now that, for each $3 \leq d \leq n-1$, a.a.s. $G_{n,d}$ is Hamiltonian [4, 10, 13, 14]. We thus focus on randomly perturbed graphs for the two remaining values of d , and show that each case behaves quite differently.

Theorem 3 ([7]). *Let $\alpha = \omega((\log n/n)^{1/4})$. Let H be an n -vertex graph with $\delta(H) \geq \alpha n$. Then, a.a.s. $H \cup G_{n,2}$ is Hamiltonian.*

Theorem 4 ([7]). *For all $\varepsilon > 0$, if $\alpha := (1 + \varepsilon)(\sqrt{2} - 1)$ and H is an n -vertex graph with $\delta(H) \geq \alpha n$, then a.a.s. $H \cup G_{n,1}$ is Hamiltonian.*

Theorem 4 is best possible. Indeed, for every $\alpha < \sqrt{2} - 1$, there exist graphs H with $\delta(H) \geq \alpha n$ such that $H \cup G_{n,1}$ is not a.a.s. Hamiltonian. The main extremal construction for this lower bound is a complete unbalanced bipartite graph. One key feature of this example is that H does not contain a very large matching. Indeed, when we further impose that H contains an (almost) perfect matching, we can obtain the following result analogous to Theorem 3.

Theorem 5 ([7]). *Let $\alpha = \omega((\log n/n)^{1/4})$. Assume that H is an n -vertex graph with $\delta(H) \geq \alpha n$ which contains a matching M which covers $n - o(\alpha^2 n)$ vertices. Then, a.a.s. $H \cup G_{n,1}$ is Hamiltonian.*

Theorems 3 and 5 have analogous proofs. Here, for simplicity, we sketch the proof of the slightly weaker Proposition 1. The proofs of Theorems 3 and 5 follow the same ideas, although with some modifications which are needed to deal with smaller degrees.

Proposition 1. *Let $\alpha = \omega((\log n/n)^{1/6})$. Let H be an n -vertex graph with $\delta(H) \geq \alpha n$. Then, a.a.s. $H \cup G_{n,2}$ is Hamiltonian.*

Proof (sketch). Note that $G = G_{n,2}$ is, by definition, a union of vertex disjoint cycles. We first show that the following two properties hold a.a.s.

- (P1) G contains at most $\log^2 n$ cycles.
(P2) For every pair of vertices $(x, y) \in V(H) \times V(H)$, the set $N_H(x)$ contains at least $\alpha^3 n/4$ vertices z such that $N_G(z) \subseteq N_H(y)$.

In order to prove these properties, we rely on the configuration model (this is a well-studied model for generating random regular graphs). By considering a particular order in which to choose edges in this model, we can define a random variable that counts the number of components and is easy to analyse. In particular, we can derive (P1) from the expectation of this variable, together with Markov's inequality. The second property does not follow quite so easily, but can be obtained by making use of Azuma's inequality. We remark that the bound on α in the statement is in place so that (P2) holds.

We can now follow an iterative process to obtain a Hamilton cycle. First, choose an arbitrary cycle of G and remove one of its edges, so that we are left with a graph which consists of a path and several cycles, all vertex disjoint. In each step of the process, we will obtain a longer path by incorporating all the vertices of (at least) one of the cycles. In order to do so, we will rely on (P2) to replace some edges of G by some edges of H . We proceed as follows.

While the graph we are considering has at least two components, we let x be an endpoint of the path and let y be an arbitrary vertex outside the path. Then, we can choose some vertex z which satisfies the property described in (P2). By this property, we can construct a new path that contains all vertices of the original path and all vertices of the component containing y . There are three cases to consider here (z lies in the path, z lies in the same component as y , or z lies in a distinct component), but a quick analysis of each shows that this is possible. By repeating this process, in at most $\log^2 n$ steps (see (P1)) we end up with a graph which consists of a Hamilton path. We then apply (P2) once more, with x and y being both endpoints of the path, to obtain a Hamilton cycle.

We remark that, while following the iterative process above, the number of vertices z as described in (P2) that we can use will decrease; crucially, since (P1) means we only need to iterate at most $\log^2 n$ times, we can guarantee that at least one choice for z remains in each step. \square

The proof of Theorem 4 follows similar ideas, but is a lot more technical, in part due to the fact that we obtain the optimal bound on α ; thus, in the sketch that we provide, we will be more vague than in the previous sketches. More precisely, we rely on some structural properties of graphs with linear minimum degree which do not contain an almost spanning matching (otherwise, the proof is covered by Theorem 5). These properties are captured by the following lemma.

Lemma 1 ([7]). *Let $1/n \ll \beta < \alpha/2 < 1/4$. Let H be an n -vertex graph with $\delta(H) \geq \alpha n$ which does not contain a matching of size greater than $(n - \sqrt{n})/2$. Then, the vertex set of H can be partitioned into sets $A \dot{\cup} B_1 \dot{\cup} B_2 \dot{\cup} C_1 \dot{\cup} C_2 \dot{\cup} R$ in such a way that the following hold:*

- (H1) $|A| \leq 12\beta^{-2}$;
(H2) $|B_1| = |B_2|$, $H[B_1, B_2]$ contains a perfect matching and, for every $v \in B_2 \cup R$, we have $e_H(v, B_1) \geq (\alpha - 2\beta)n$, and

(H3) $|C_1| = |C_2|$ and either C_1 is empty or $G[C_1, C_2]$ contains a perfect matching; furthermore, for all $v \in C_2$ we have $e_H(v, B_2 \cup R) \leq \beta^{-1} + 1$.

Proof (Sketch of the Proof of Theorem 4). We first consider a partition of H as described by Lemma 1. Among other properties, for all $v \in R \cup B_2 \cup C_2$ we can show that the number of edges between v and B_1 is large (at least $n/5$), and each pair of vertices $x, y \in R \cup B_2$ have linearly many common neighbours in B_1 .

Consider the matching M which is the union of those spanned by B_1 and B_2 , and C_1 and C_2 . We first prove that $M \cup G_{n,1}$ a.a.s. satisfies certain important properties (namely, it contains at most $\log^2 n$ cycles, few edges belong to both M and $G_{n,1}$, and it satisfies an edge distribution property somewhat akin to (P2) from the proof of Proposition 1). Furthermore, we can show that a.a.s., for each pair of vertices $x, y \in R \cup B_2$, their common neighbourhood in B_1 spans many edges of $G_{n,1}$: more than $|R|/2$, which is roughly the number of paths in $M \cup G_{n,1}$. This is a crucial property in showing the optimal bound on α .

For the rest of the proof, we show that, by using edges of H , the paths and cycles of $M \cup G_{n,1}$ can be modified into a set of paths all whose endpoints lie in $R \cup B_2$. Then, using the properties that we have shown for these vertices (mainly about the number of edges in the common neighbourhood of any pair of such vertices), we can iteratively combine the paths into a single spanning path, which we eventually turn into a cycle. \square

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The Largest Hole in Sparse Random Graphs

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Abstract. We show that for $d \geq d_0(\epsilon)$, with high probability, the size of a largest induced cycle in the random graph $G(n, d/n)$ is $(2 \pm \epsilon) \frac{n}{d} \log d$. This settles a long-standing open problem in random graph theory.

Keywords: Random graph · Induced path · Hole · Second moment method

1 Introduction

Let $G(n, p)$ denote the binomial random graph on n vertices, where each edge is included independently with probability p . We are concerned here with *induced* subgraphs of $G(n, p)$, specifically trees, forests, paths and cycles.

The study of induced trees in $G(n, p)$ was initiated by Erdős and Palka [7] in the 80s. Among other things, they showed that for constant p , with high probability (**whp**) the size of a largest induced tree in $G(n, p)$ is asymptotically equal to $2 \log_q(np)$, where $q = \frac{1}{1-p}$. The obtained value coincides asymptotically with the *independence number* of $G(n, p)$, the study of which dates back even further to the work of Bollobás and Erdős [2], Grimmett and McDiarmid [13] and Matula [18].

As a natural continuation of their work, Erdős and Palka [7] posed the problem of determining the size of a largest induced tree in *sparse* random graphs, when $p = d/n$ for some fixed constant d . More precisely, they conjectured that for every $d > 1$ there exists $c(d) > 0$ such that **whp** $G(n, p)$ contains an induced tree of order at least $c(d) \cdot n$. This problem was settled independently in the late 80s by Fernandez de la Vega [8], Frieze and Jackson [12], Kučera and Rödl [15] as well as Łuczak and Palka [17]. In particular, Fernandez de la Vega [8] showed that one can take $c(d) \sim \frac{\log d}{d}$, and a simple first moment calculation reveals that this is tight within a factor of 2.

Two natural questions arise from there. First, one might wonder whether it is possible to find not only some *arbitrary* induced tree, but a *specific* one, say a long induced path. Indeed, Frieze and Jackson [11] in a separate paper showed that **whp** there is an induced path of length $\tilde{c}(d) \cdot n$. Two weaknesses of this

result were that their proof only worked for sufficiently large d , and that the value obtained for $\tilde{c}(d)$ was far away from the optimal one. Later, Łuczak [16] and Suen [21] independently remedied this situation twofold. They proved that an induced path of length linear in n exists for all $d > 1$, showing that the conjecture of Erdős and Palka holds even for induced paths. Moreover, they showed that one can take $\tilde{c}(d) \sim \frac{\log d}{d}$ as in the case of arbitrary trees.

A second obvious question is to determine the size of a largest induced tree (and path) more precisely. The aforementioned results were proved by analysing the behaviour of certain constructive algorithms which produce large induced trees and paths. The value $\frac{\log d}{d}$ seems to constitute a natural barrier for such approaches. On the other hand, recall that in the dense case, the size of a largest induced tree coincides asymptotically with the independence number. In 1990, Frieze [10] showed that the first moment bound $\sim 2^{\frac{n}{d}} \log d$ is tight for the independence number, also in the sparse case. His proof is based on the profound observation that the second moment method can be used even in situations where it apparently does not work, if one can combine it with a strong concentration inequality. Finally, in 1996, Fernandez de la Vega [9] observed that the earlier achievements around induced trees can be combined with Frieze’s breakthrough to prove that the size of a largest induced tree is indeed $\sim 2^{\frac{n}{d}} \log d$. This complements the result of Erdős and Palka [7] in the dense case. (When $p = o_n(1)$, we have $2 \log_q(np) \sim 2^{\frac{n}{d}} \log d$.)

Fernandez de la Vega [9] also posed the natural problem of improving the Łuczak–Suen bound [16, 21] for induced paths, for which his approach was “apparently helpless”. Despite the widely held belief (see [3, 6] for instance) that the upper bound $\sim 2^{\frac{n}{d}} \log d$ obtained via the first moment method is tight, the implicit constant 1 has not been improved in the last 30 years.

2 Long Induced Paths

Our main result is the following, which settles this problem and gives an asymptotically optimal result for the size of a largest induced path in $G(n, p)$.

Theorem 1. *For any $\epsilon > 0$ there is d_0 such that **whp** $G(n, p)$ contains an induced path of length at least $(2 - \epsilon)^{\frac{n}{d}} \log d$ whenever $d_0 \leq d = pn = o(n)$.*

For the sake of generality, we state our result for a wide range of functions $d = d(n)$. However, we remark that the most interesting case is when d is a sufficiently large constant. In fact, for dense graphs, when $d \geq n^{1/2} \log^2 n$, more precise results are already known (cf. [6, 19]).

Some of the earlier results [6, 11, 16] are phrased in terms of induced cycles (*holes*). Using a simple sprinkling argument, one can see that aiming for a cycle instead of a path does not make the problem any harder.

We briefly explain our strategy. Roughly speaking, the idea is to find a long induced path in two steps. First, we find many disjoint paths of some chosen length L , such that the subgraph consisting of their union is induced. To achieve this, we generalize a recent result of Cooley, Draganić, Kang and Sudakov [3]

who obtained large induced matchings. We will discuss this further in Sect. 3. Assuming now we can find such an induced linear forest F , the aim is to connect almost all of the small paths into one long induced path, using a few additional vertices. As a “reservoir” for these connecting vertices, we find a large independent set I which is disjoint from F . To model the connecting step, we give each path in F a direction, and define an auxiliary digraph whose vertices are the paths, and two paths (P_1, P_2) form an edge if there exists a “connecting” vertex $a \in I$ that has some edge to the last ϵL vertices of P_1 and some edge to the first ϵL vertices of P_2 , but no edge to the rest of F . Our goal is to find an almost spanning path in this auxiliary digraph. Observe that this will provide us with a path in $G(n, p)$ of length roughly $|F|$. The intuition is that the auxiliary digraph behaves quite randomly, which gives us hope that, even though it is very sparse, we can find an almost spanning path (see e.g. [14]).

Crucially, we do not perform this connecting step in the whole random graph. This is because ensuring that the new connecting vertices are only connected to two vertices of F is too costly, making the auxiliary digraph so sparse that it is impossible to find an almost spanning path. Instead, we use a sprinkling argument, meaning that we view $G(n, p)$ as the union of two independent random graphs G_1 and G_2 , where the edge probability of G_2 is much smaller than p . We then reveal the random choices in several stages. When finding F and I as above, we make sure that there are no G_1 -edges between F and I . Then, in the final connecting step, it remains to expose the G_2 -edges between F and I , with the advantage that now the edge probability is much smaller.

3 Induced Forests with Small Components

As outlined above, in the first step of our argument, we seek an induced linear forest whose components are reasonably long paths. For this, we generalize a recent result of Cooley, Draganić, Kang and Sudakov [3]. They proved that **whp** $G(n, p)$ contains an induced matching with $\sim 2 \log_q(np)$ vertices, which is asymptotically best possible. They also anticipated that using a similar approach one can probably obtain induced forests with larger, but bounded components. As a by-product, we confirm this. To state our result, we need the following definition. For a given graph T , a T -*matching* is a graph whose components are all isomorphic to T . Hence, a K_2 -matching is simply a matching, and the following for $T = K_2$ implies the main result of [3].

Theorem 2. *For any $\epsilon > 0$ and tree T , there exists $d_0 > 0$ such that **whp** the order of the largest induced T -matching in $G(n, p)$ is $(2 \pm \epsilon) \log_q(np)$, where $q = \frac{1}{1-p}$, whenever $\frac{d_0}{n} \leq p \leq 0.99$.*

We use the same approach as in [3], which goes back to the work of Frieze [10] (see also [1, 20]). The basic idea is as follows. Suppose we have a random variable X and want to show that **whp**, $X \geq b - t$, where b is some “target” value and t a small error. For many natural variables, we know that X is “concentrated”, say $\mathbb{P}[|X - \mathbb{E}[X]| \geq t/2] < \rho$ for some small ρ . This is the case for instance

when X is determined by many independent random choices, each of which has a small effect. However, it might be difficult to estimate $\mathbb{E}[X]$ well enough. But if we know in addition that $\mathbb{P}[X \geq b] \geq \rho$, then we can combine both estimates to $\mathbb{P}[X \geq b] > \mathbb{P}[X \geq \mathbb{E}[X] + t/2]$, which clearly implies that $b \leq \mathbb{E}[X] + t/2$. Applying now the other side of the concentration inequality, we infer $\mathbb{P}[X \leq b - t] \leq \mathbb{P}[X \leq \mathbb{E}[X] - t/2] < \rho$, as desired.

In our case, say X is the maximum order of an induced T -matching in $G(n, p)$. Since adding or deleting edges at any one vertex can create or destroy at most one component, we know that X is $|T|$ -Lipschitz and hence concentrated. Using the above approach, it remains to complement this with a lower bound on the probability that $X \geq b$. Introduce a new random variable Y which is the number of induced T -matchings of order b (a multiple of $|T|$). Then we have $X \geq b$ if and only if $Y > 0$. The main technical work is to obtain a lower bound for the probability of the latter event using the second moment method. We note that by applying the second moment method to labelled copies (instead of unlabelled copies as in [3]) the proof becomes shorter even in the case of matchings. More crucially, it turns out that one can even find induced forests where the component sizes can grow as a function of d , which we need in the proof of Theorem 1 (specifically, we need $L \gg \log d$). This is provided by the following auxiliary result.

Lemma 1. *For any $\epsilon > 0$, there exists $d_0 > 0$ such that **whp** $G(n, p)$ contains an induced linear forest of order at least $(2 - \epsilon)p^{-1} \log(np)$ and component paths of order $d^{1/2} / \log^4 d$, whenever $d_0 \leq d = np \leq n^{1/2} \log^2 n$.*

4 Concluding Remarks

In [3] it is conjectured that one should not only be able to find an induced path of size $\sim \frac{2n}{d} \log d$, but any given bounded degree tree. For dense graphs, when $d = \omega(n^{1/2} \log n)$, this follows from the second moment method (see [5]). On the contrary, the sparse case seems to be more difficult, mainly because the vanilla second moment method does not work. However, Dani and Moore [4] demonstrated that one can actually make the second moment method work, at least for independent sets, by considering a *weighted* version. This even gives a more precise result than the classical one due to Frieze [10]. It would be interesting to find out whether this method can be adapted to induced trees.

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On 13-Crossing-Critical Graphs with Arbitrarily Large Degrees

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Abstract. A surprising result of Bokal et al. proved that the exact minimum value of c such that c -crossing-critical graphs do *not* have bounded maximum degree is $c = 13$. The key to the result is an inductive construction of a family of 13-crossing-critical graphs with many vertices of arbitrarily high degrees. While the inductive part of the construction is rather easy, it all relies on the fact that a certain 17-vertex base graph has the crossing number 13, which was originally verified only by a machine-readable computer proof. We now provide a relatively short self-contained computer-free proof.

Keywords: Graph · Crossing number · Crossing-critical families

1 Introduction

The *crossing number* $cr(G)$ of a graph G is the minimum number of (pairwise) edge crossings in a drawing of G in the plane. To resolve ambiguity, we consider drawings of graphs such that no edge passes through another vertex and no three edges intersect in a common point which is not their end. A *crossing* is then an intersection point of two edges that is not a vertex, and we always assume a finite number of crossings. A graph G is *c -crossing-critical* if $cr(G) \geq c$, but for every edge e of G we have $cr(G - e) < c$ (i.e., the crossing number drops down in every proper subgraph).

There are two 1-crossing-critical graphs up to subdivisions, K_5 and $K_{3,3}$, but for every $c \geq 2$ there exists an infinite number of c -crossing-critical graphs, see Kochol [5]. A natural interesting question about c -crossing-critical graphs, first asked by Richter in 2013, is whether they have maximum degree bounded in c . A surprising negative answer has been implicitly confirmed by Dvořák and Mohar [4] in 2010, but no explicit examples were known until a significantly more recent exhaustive work of Bokal et al. [1] in 2019:

Theorem 1 (Bokal, Dvořák, Hliněný, Leaños, Mohar and Wiedera [1])

a) For each $1 \leq c \leq 12$, there exists a constant D_c such that every c -crossing-critical graph has vertex degrees at most D_c .

Supported by research grant GAČR 20-04567S of the Czech Science Foundation.

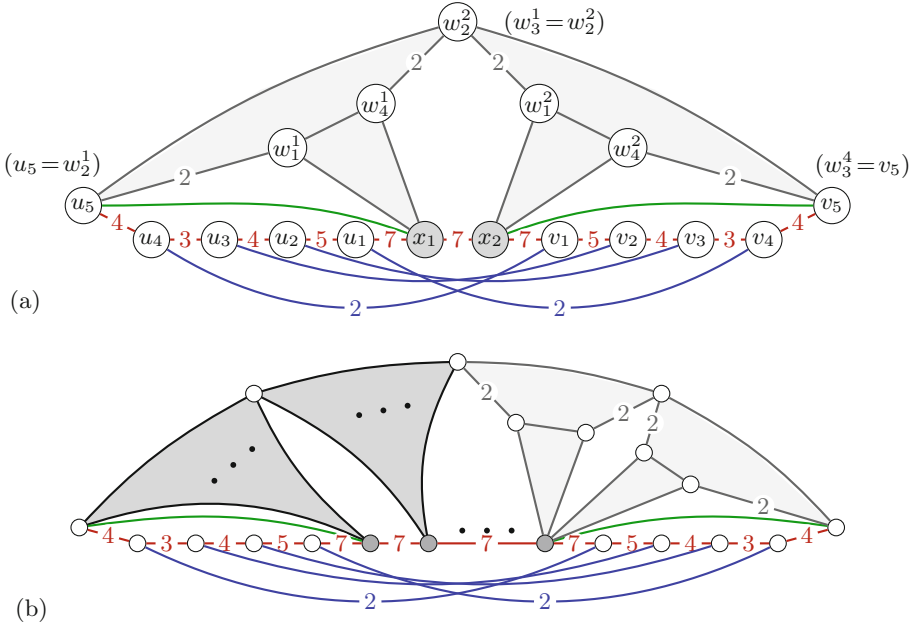


Fig. 1. The inductive construction of 13-crossing-critical graphs from Theorem 1 (note that all 13 depicted crossings are only between the blue edges). The edge labels in the picture represent the number of parallel edges between their end vertices (e.g., there are 7 parallel edges between x_1 and x_2). Figure (a) defines the *base graph* G_{13} of the construction, and (b) outlines the general construction which arbitrarily duplicates the two “wedge” shaped gray subgraphs of G_{13} and the gray vertices x_1, x_2 .

b) For each $c \geq 13$ and every integers m, d , one can construct a c -crossing-critical graph which has more than m vertices of degree at least d .

Due to space restrictions, we have to refer the readers to [1, 3] for a more detailed general discussion of crossing-critical graphs and their properties.

The (now improved) critical construction of Theorem 1(b) is outlined for $c = 13$ in Fig. 1. Figure 1(a) defines the 17-vertex 13-crossing-critical (multi)graph G_{13} which is the base graph of the full inductive construction. One can see in [1] that the proof of Theorem 1(b) follows straightforwardly (using induction) from the fact that $cr(G_{13}) \geq 13$. However, for the latter fact only a machine-readable computer proof is provided in [1]; the proof is based on an ILP branch-and-cut-and-price routine [2] with about a thousand cases of up to hundreds of constraints each. Our goal is to provide a much simpler handwritten proof.

Theorem 2 (a computer-free alternative to Theorem 1(b)). $cr(G_{13}) \geq 13$.

2 Analyzing Optimal Drawings of G_{13}

We divide the proof of Theorem 2 into two steps. We call *red* the edges of G_{13} (Fig. 1(a)) which form the path with multiple edges on $(u_5, u_4, u_3, u_2, u_1, x_1, x_2, v_1, v_2, v_3, v_4, v_5)$, and we call *blue* the edges $\{u_i, v_j\}$ where $i, j \in \{1, 2, 3, 4\}$.

- (1) We will first show that there is an *optimal drawing* (i.e., one minimizing the number of crossings) of G_{13} such that no red edge crosses a red or a blue edge. Note that blue–blue crossings are still allowed (and likely to occur).
- (2) While considering drawings as in the first point, we will focus only on selected crossings (roughly, those involving a blue edge), and prove at least 13 of them, or at least 12 with the remaining drawing still being non-planar.

We start with some basic facts about the crossing number.

Proposition 1 (folklore). *a) If D is an optimal drawing of a graph G , then two edges do not cross more than once, and not at all if sharing a common end. b) If e and f are parallel edges in G (i.e., e, f have the same end vertices), then there is an optimal drawing of G in which e and f are drawn “closely together”, meaning that they cross the same other edges in the same order.*

In view of Proposition 1(b), we adopt the following view of multiple edges: If the vertices u and v are joined by p parallel edges, we view all p of them as one edge f of weight p . If (multiple) edges f and g of weights p and q cross each other, then their crossing naturally contributes the amount of $p \cdot q$ to the total number of crossings. With help of the previous, we now finish the first step:

Lemma 1. *There exists an optimal drawing of the graph G_{13} in which no red edge crosses a red or a blue edge, or $cr(G_{13}) \geq 13$.*

Proof. Let D be an optimal drawing of G_{13} with less than 13 crossings. By Proposition 1, if two red edges cross in D , then it can only be that an edge from $\{u_5u_4, u_4u_3, u_3u_2\}$ crosses an edge from $\{v_5v_4, v_4v_3, v_3v_2\}$, which gives $3 \cdot 3 = 9$ or $3 \cdot 4 = 12$ crossings. Consider the edge-disjoint cycles $C_1 = (u_5, u_4, u_3, u_2, u_1, x_1)$ and $C_2 = (u_5, u_4, u_3, u_2, u_1, x_1, w_1^1)$, and symmetric $C'_1 = (v_5, \dots, v_1, x_1)$, $C'_2 = (v_5, \dots, v_1, x_1, w_1^1)$. Since C_1, C_2 transversely cross C'_1, C'_2 , they must cross a second time by the Jordan curve theorem, giving additional $2 \cdot 2 = 4$ crossings. Hence D had at least $9 + 4 = 13$ crossings, and $cr(G_{13}) \geq 13$.

It remains to get rid of possible red–blue crossings in D . Assume first that the red edge x_1x_2 (weight 7) is crossed by a blue edge e . If e is of weight 2, then D has 14 crossings. Hence, up to symmetry, $e = u_3v_2$. We redraw e tightly along the path $P = (u_3, u_4, v_1, v_2)$, saving 7 crossings on x_1x_2 , and newly crossing u_4v_1 (possibly) and p edges which cross P in D . If $p \geq 3$, there are already $7 + p \cdot 2 \geq 13$ crossings in D . Otherwise, redrawing e makes only $p + 2 \leq 4$ new crossings, contradicting optimality of D .

In the rest, we iteratively remove possible red–blue crossing without increasing the total number of crossings, until we reach a desired optimal drawing of G_{13} .

Assume in D that a red edge v_3v_4 (or, symmetrically u_4u_3) is crossed by blue e , and this is the blue crossing on v_3v_4 closest to v_3 . In this case, e has one end $w \in \{v_1, v_2\}$. Instead of crossing v_3v_4 (saving 3 crossings), we redraw part of e tightly along the path (v_3, v_2, v_1) to the end w , while possibly crossing blue v_3u_2, v_2u_3 (if $w = v_1$), and other r edges which in D cross red v_3v_2 or v_2v_1 . If $r \geq 3$, we already had $r \cdot 4 + 3 \geq 15$ crossings in D . If $r \geq 2$ and $e = v_1u_4$ (weight 2), we had $r \cdot 4 + 3 \cdot 2 \geq 14$ crossings in D . Otherwise, redrawn e crosses at most 3 new edges, and so we have no more crossings than in D .

Finally, if the red edge v_4v_5 is crossed by blue e in D , then we “pull” the end v_5 along v_4v_5 towards v_4 across e . This replaces the crossing of e with v_4v_5 of weight 4 by 4 crossings with the non-red edges of v_5 , so again no more crossings than in D . If red v_3v_2 (v_2v_1 , or v_1x_2) is crossed by blue e in D , we similarly “pull” along the red edges v_3 towards v_2 (v_2, v_3 towards v_1 , or v_1, v_2, v_3 towards x_2). This replaces the original crossing of e with red by crossings of e with v_3v_4 and v_3u_2 (plus v_2u_3 or plus v_2u_3, v_1u_4), but the total number of crossings stays the same as in D . Then we redraw the crossing of e and v_3v_4 as above. \square

3 Counting Selected Crossings in a Drawing of G_{13}

In the second step, we introduce two additional sorts of edges of G_{13} . The edges u_5x_1 and v_5x_2 are called *green*, and all remaining edges of G_{13} . Let G_0 denote the subgraph of G_{13} formed by all red and gray edges and the incident vertices. Let R denote the (multi)path of all red edges.

Lemma 2. *Let D be an optimal drawing of G_{13} as claimed by Lemma 1. If the subdrawing of G_0 within D is planar, then D has at least 13 crossings.*

Proof (a sketch). There are only two non-equivalent planar drawings of G_0 , as in Fig. 2. We picture them with the red path R drawn as a horizontal line. We call a blue edge of G_{13} *bottom* if it is attached to R from below at both ends, and *top* if attached from above at both ends. A blue edge is *switching* if it is neither top nor bottom. Note that we have only crossings involving a green edge, or crossings of a blue edge with a blue or gray edge.

If G_0 is drawn as in Fig. 2(a), by the Jordan curve theorem, we deduce:

- (I) If a blue edge e is bottom (top), and a blue edge $e' \neq e$ attaches to R from below (from above) at its end which is between the ends of e on R , then e and e' cross. In particular, two bottom (two top) blue edges always cross.
- (II) A top (switching) blue edge crosses at least 4 (at least 3) gray edges.
- (III) If there is weight k of top (or switching) blue edges and weight ℓ of bottom blue edges, then each (or at least one) green edge must cross $\min(k, \ell) \leq 3$ blue or red edges.

Say, if all blue edges are bottom, they pairwise give desired $\binom{6}{2} - 2 = 13$ crossings by (I). If one of blue u_4v_1, u_1v_4 is top (or switching) and all other blue edges are bottom, we get at least $5 + 2 \cdot 4 + 4 = 17$ crossings (or $5 + 2 \cdot 3 + 2 = 13$

crossings) by the claims (I), (II) and (III) in this order. If one of u_3v_2, u_2v_3 is top (or switching) and all other blue are bottom, we similarly get $8 + 4 + 2 = 14$ crossings (or $10 + 3 + 1 = 14$ crossings) by (I),(II),(III), and so on. One may solve all other cases using the same arguments, and we omit the details here.

Consider the drawing of G_0 as in Fig. 2(b). Now a top or bottom blue edge must cross at least 2 gray edges, and a switching blue edge at least 3 gray edges. Hence we get 13 crossings, unless all blue edges are top or bottom, giving $6 \cdot 2 = 12$ blue-gray crossings. Though, then we get another blue crossing as in (I). \square

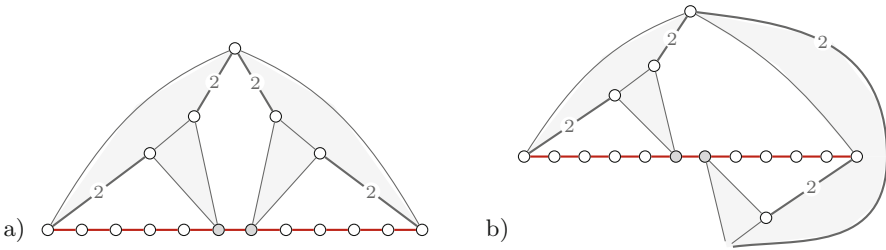


Fig. 2. Schematically, the two non-equivalent planar drawings of the subgraph G_0 (red) and gray) of G_{13} . This is used in the proof of Lemma 2.

We now focus on the following selected crossings in a drawing of G_{13} : a *refined crossing* is one in which a blue edge crosses a gray or green edge, or two blue edges cross each other.

Lemma 3. *Let D be a drawing of G_{13} as claimed by Lemma 1. If no red edge is crossed in D , then D contains 12 refined crossings or at least 13 crossings.*

Proof (a sketch). We again picture the red path R as a straight horizontal line, and use terms *top/bottom/switching* for blue edges as in the proof of Lemma 2. Then we consider the following six pairwise edge-disjoint gray and green paths: P_1 (of length 2) and P_2 (of length 6) join u_5 to v_5 via w_2^2 , Q_1 and Q'_1 are formed by the edges u_5x_1 and v_5x_2 , and Q_2, Q'_2 (of length 2) join u_5 to x_1 via w_1^1 and v_5 to x_2 via w_4^2 . Using the Jordan curve theorem (cf. Fig. 1), we deduce:

- (I) The same claim as (I) in the proof of Lemma 2 applies here.
- (II') If a blue edge e is switching, then e adds a refined crossing on each of P_1 and P_2 .
- (III') If the sum of weights of the blue edges attached to u_1, u_2, u_3, u_4 on R from below is k (and $6 - k$ from above), then each of Q_1, Q_2 (symmetrically, Q'_1, Q'_2) carries at least $\min(k, 6 - k)$ crossings with blue edges in D .

Now it remains to examine all combinations of the blue edges being top/bottom/switching, and in each one use the above claims to argue that D contains at least 12 pairwise distinct refined crossings. E.g., if u_3v_2 is top, u_2v_3 is switching

and both u_1v_4, u_4v_1 are bottom, then we get by (I') $2 \cdot 2 = 4$ crossings, by (II') additional 2 crossings and by (III') additional $2 \cdot (1 + 2) = 6$ crossings. We omit the remaining routine technical details here. \square

We can finish our self-contained computer-free proof of $cr(G_{13}) \geq 13$:

Proof (of Theorem 2). Let D be an optimal drawing of G_{13} satisfying the conclusion of Lemma 1. Thanks to Lemma 2, we may assume that the subdrawing of G_0 within D is not planar. If the red edges of G_0 are not crossed, we have a crossing of two gray edges of G_0 in D and 12 more refined crossings by Lemma 3, altogether 13 crossings.

If some red edge is crossed in a point x by a gray or green edge g (since g is not blue by Lemma 1), we redraw g as follows: Up to symmetry, let x be closer (or equal) to v_5 than to u_5 on R . We cut the drawing of g at x and route both parts of g closely along their side of the drawing of R , until we rejoin them at the end v_5 . This redrawing D' saves $\ell \in \{3, 4, 5, 7\}$ crossings of g at x , and creates at most $\ell - 1$ new refined crossings only between g and the blue edges ending on R between x and v_5 , as one can check in Fig. 1. (We possibly repeat the same procedure for other crossings of red edges in D .)

We now apply Lemma 3 to D' , finding at least 12 refined crossings. Hence, D contained at least $12 - (\ell - 1)$ refined crossings and ℓ other crossings at x , summing again to 13. \square

4 Conclusion

With the computer-free proof of Theorem 2, we have completed a thorough study of high and possibly unbounded vertex degrees in c -crossing-critical graphs. Still, there is an unanswered secondary question about what exact high degrees can be achieved, say, for $c = 13$. Looking at the construction in Fig. 1, we see that the two vertices x_1 and x_2 can attain any odd degree at least 17, and possible additional vertices on the red path between x_1 and x_2 (in the expanded construction) can attain any combination of even degrees at least 16, independently of each other. A slight modification of this construction of 13-crossing-critical graphs allowing for any combination of arbitrarily many even *and* odd degrees is the subject of our continuing research.

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Local Convergence of Random Planar Graphs

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Abstract. We describe the asymptotic local shape of a graph drawn uniformly at random from all connected simple planar graphs with n labelled vertices. We establish a novel uniform infinite planar graph (UIPG) as quenched limit in the local topology as n tends to infinity. We also establish such limits for random 2-connected planar graphs and maps as their number of edges tends to infinity. Our approach encompasses a new probabilistic view on the Tutte decomposition. This allows us to follow the path along the decomposition of connectivity from planar maps to planar graphs in a uniformed way, basing each step on condensation phenomena for random walks under subexponentiality, and Gibbs partitions. Using large deviation results, we recover the asymptotic formula by Giménez and Noy (2009) for the number of planar graphs.

Keywords: Planar graphs · Local convergence

A graph is planar if it may be drawn in the plane such that edges intersect only at endpoints. The reader may consult the book by [23] for details of graph embeddings on surfaces. We are interested in properties of the graph P_n selected uniformly at random among all simple connected planar graphs with vertices labelled from 1 to n . Here the term *simple* refers to the absence of loops and multiple edges.

Properties of the random graph P_n have received increasing attention in recent literature [3, 5, 10, 11, 16, 21, 27]. We refer the reader to the comprehensive survey by [24] for a detailed account. Our main theorem shows that P_n admits a local limit.¹

Theorem 1. *The uniform n -vertex connected planar graph P_n rooted at a uniformly selected vertex v_n admits a distributional limit \hat{P} in the local topology. We call \hat{P} the uniform infinite planar graph (UIPG). The regular conditional law $\mathfrak{L}((P_n, v_n) \mid P_n)$ satisfies*

$$\mathfrak{L}((P_n, v_n) \mid P_n) \xrightarrow{p} \mathfrak{L}(\hat{P}). \quad (1)$$

¹ The full version of the present extended abstract is available on the arXiv [29].

The UIPG is a random infinite planar graph that is connected, locally finite and has a root vertex. $\mathfrak{L}(\hat{\mathbb{P}})$ refers to its distribution, and $\mathfrak{L}((\mathbb{P}_n, v_n) \mid \mathbb{P}_n)$ denotes the uniform measure on the n vertex-rooted versions of the graph \mathbb{P}_n . The so-called quenched convergence of random probability measures with respect to the local topology in (1) means that for any integer $r \geq 1$ and any finite rooted connected planar graph H the proportion of vertices in \mathbb{P}_n whose r -neighbourhood is isomorphic to H concentrates around the probability for the r -neighbourhood of the root of $\hat{\mathbb{P}}$ to be isomorphic to H . Here the r -neighbourhood of a vertex refers to the subgraph induced by all vertices with graph distances at most r from that vertex.

The root degree of the UIPG follows the asymptotic degree distribution of \mathbb{P}_n established by [10] and [27]. We may also prove a version of this theorem (with a different limit object) where v_n is chosen according to the stationary distribution instead. That is, when v_n instead assumes a vertex of \mathbb{P}_n with probability proportional to its degree. By a celebrated result of [18, Thm. 1.1], this implies that the UIPG is almost surely recurrent.

Several milestones in the proof of our main result are of independent interest. For example, we prove local limits for 2-connected planar graphs and non-separable planar maps. That is, planar graphs and maps without cutvertices.

Theorem 2. *Let $v_n^{\mathbb{B}}$ denote a uniformly selected vertex of the uniform 2-connected planar graph \mathbb{B}_n with n edges. There is a uniform infinite planar graph $\hat{\mathbb{B}}$ with*

$$\mathfrak{L}((\mathbb{B}_n, v_n^{\mathbb{B}}) \mid \mathbb{B}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathbb{B}}). \tag{2}$$

We call $\hat{\mathbb{B}}$ the uniform infinite 2-connected planar graph (UI2PG).

There is a natural coupling where $\hat{\mathbb{P}}$ is obtained from $\hat{\mathbb{B}}$ by attaching independent and identically Boltzmann distributed connected vertex-marked planar graphs at the non-root vertices of $\hat{\mathbb{B}}$, and a Boltzmann distributed doubly vertex-marked connected planar graph at the root of $\hat{\mathbb{B}}$.

Theorem 3. *Let $v_n^{\mathbb{V}}$ denote a uniformly selected corner of the random non-separable planar map \mathbb{V}_n with n edges. There is uniform infinite planar map $\hat{\mathbb{V}}$ with*

$$\mathfrak{L}((\mathbb{V}_n, v_n^{\mathbb{V}}) \mid \mathbb{V}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathbb{V}}). \tag{3}$$

We call $\hat{\mathbb{V}}$ the uniform infinite 2-connected planar map (UI2PM).

The degree distribution of the non-separable case has been studied by [12]. We note that the local limit of a uniform random planar map (that is not required to be non-separable) with n edges is known and called the uniform infinite planar map (UIPM), see [4, 7, 22, 28] (and also [2, 19]). Our results entail that the UIPM may be constructed from $\hat{\mathbb{V}}$ by attaching independent identically Boltzmann distributed planar maps at each non-root corner, and a Boltzmann distributed doubly corner rooted planar map at its root-corner.

The methods we develop in order to prove our main result encompass a novel probabilistic view on the Tutte decomposition of these objects. We do not prove or build upon local convergence of uniform 3-connected planar maps and graphs with n edges. This highly relevant result was established by [1] using a different approach. As a further mayor application we recover a celebrated result in enumerative combinatorics by Giménez and Noy:

Theorem 4 ([16, Thm. 1]). *The number p_n of labelled simple planar graphs with n vertices satisfies the asymptotic expression*

$$p_n \sim c_G \rho_C^{-n} n^{-7/2} n!, \quad (4)$$

for some constants $c_G, \rho_C > 0$ admitting expressions in terms of generating series.

Giménez and Noy obtained this breakthrough result (resolving a history of rougher estimates by [3, 8, 15, 26]) by performing analytic integration and extending results by [3] on the number of 2-connected graphs. An approach employing “combinatorial integration” was given by [6]. We reprove Eq. (4) by different methods, without any integration step at all, deducing the asymptotic number of connected graphs from the number of 2-connected graphs using results for the big-jump domain by [9, Cor. 2.1] and properties of subexponential probability distributions, see [14]. We emphasize that the approach by Giménez and Noy additionally yields singular expansions for the involved generating series, and our proof does not. Hence the methods of [16] yield stronger results, and the methods employed here work under weaker assumptions.

Our main motivation for enhancing the toolset of enumerative combinatorics is that important problems in the field remain open, in particular the enumeration of unlabelled planar graphs. We believe that the approach we introduce here is promising for tackling this problem.

Theorem 1 has applications concerning subgraph count asymptotics. By a general result of [20, Lem. 4.3] and using the asymptotic degree distribution of \mathbb{P}_n established by [10], it follows that:

Corollary 1. *For any finite connected graph H the number $\text{emb}(H, \mathbb{P}_n)$ of occurrences of H in \mathbb{P}_n as a subgraph satisfies*

$$\frac{\text{emb}(H, \mathbb{P}_n)}{n} \xrightarrow{p} \mathbb{E}[\text{emb}^\bullet(H^\bullet, \hat{\mathbb{P}})]. \quad (5)$$

Here H^\bullet denotes any fixed vertex rooted version of H , and $\text{emb}^\bullet(H^\bullet, \hat{\mathbb{P}})$ counts the number of root-preserving embeddings of H^\bullet into $\hat{\mathbb{P}}$.

The study of the number of *pendant* copies (or *appearances*) of a fixed graph in \mathbb{P}_n was initiated by [21], and a normal central limit theorem was established by [17, Sec. 4.3]. The difficulty of studying $\text{emb}(H, \mathbb{P}_n)$ stems from the fact that it requires us to look inside the giant 2-connected component of \mathbb{P}_n , whereas pendant copies lie with high probability in the components attached to it. It is

natural to conjecture convergence to a normal limit law for the fluctuations of $\text{emb}(H, \mathbb{P}_n)$ around $n\mathbb{E}[\text{emb}^\bullet(H^\bullet, \hat{\mathbb{P}})]$ at the scale \sqrt{n} . Such a result has recently been established for the number of triangles in random cubic planar graphs by [25], and the number of double triangles in random planar maps by [13]. In light of [20], it would be interesting to know whether such a central limit theorem may be established in a way that applies to general sequences of random graphs that are locally convergent in some strengthened sense.

1 Summary of Proof

Full proofs of all results are given on the arXiv version [29] of the present manuscript, which has been submitted to a peer-reviewed journal. Here we give a rough overview by summarizing the main steps in the proof of Theorem 1, without going into any details.

A quenched local limit for the random planar map M_n^t with n edges and weight $t > 0$ at vertices was established in [31]. We pass this convergence down to a quenched limit for the non-separable core $\mathcal{V}(M_n^t)$. That is, the largest non-separable submap of the map M_n^t . For this, we employ a quenched version of an inductive argument discovered by [30, Thm. 6.59]. The idea is that we have full information about the components attached to the core. The neighbourhood of a uniformly selected corner of M_n^t gets patched together from a connected component containing it, a neighbourhood in the core, and neighbourhoods in components attached to the core neighbourhood. Expressing this yields a recursive equation, which by an inductive arguments allows us to prove convergence of $\mathcal{V}(M_n^t)$. It is important to note that $\mathcal{V}(M_n^t)$ has a random size, hence a priori properties of $\mathcal{V}(M_n^t)$ do not carry over automatically to properties of the random non-separable planar map V_n^t with n edges and weight t at vertices.

A planar network is a planar map with two distinguished vertices called the south and north pole, such that adding an edge between the two poles yields a non-separable network. Without this extra edge, the network may or may not be non-separable. Both cases are admissible. A planar network is called non-serial, if it cannot be expressed as a series composition of two networks, where the north pole of the first gets identified with the south pole of second. We reduce the study $\mathcal{V}(M_n^t)$ of the non-separable core to the study of non-serial networks by showing that series networks admit a giant non-series component, with precise limits for the small fragments.

We proceed to establish a novel fully recursive tree-like combinatorial encoding for non-serial networks in terms of a complex construct that we call \mathcal{R} -networks. This allows us to generate a non-serial network by starting with a random network \bar{R} where one edge is marked as “terminal”. The process proceeds recursively by substituting non-terminal edges by independent copies of \bar{R} until only terminal edges are left. This allows us to deduce a local limit theorem for the number of edges in a giant \mathcal{R} -core $\bar{\mathcal{R}}(M_n^t)$ of the \mathcal{V} -core $\mathcal{V}(M_n^t)$, and implies that the network $\mathcal{V}(M_n^t)$ behaves like a network obtained from the \mathcal{R} -core $\bar{\mathcal{R}}(M_n^t)$ by substituting all but a negligible number of edges by independent copies of the Boltzmann distributed \bar{R} -network \bar{R} . If we choose any fixed

number of corners independently and uniformly at random, the corresponding $\bar{\mathcal{R}}$ -components containing them will follow size-biased distributions by the famous waiting time paradox. This gives us full information on the $\bar{\mathcal{R}}$ -components in the vicinity of these components. Since we substitute at edges, the resulting recursive equation for the probability of neighbourhoods in $\mathcal{V}(M_n^t)$ to have a fixed shape do *not* allow for the same inductive argument as before. The reason for this problem is that the event for a radius r neighbourhood in $\mathcal{V}(M_n^t)$ to have a fixed shape with k edges may correspond to configurations with more than k edges in an r -neighbourhood in $\bar{\mathcal{R}}(M_n^t)$, since components of edges between vertices of distance r from the center do not always contribute to the r -neighbourhood in $\mathcal{V}(M_n^t)$. We solve his problem by abstraction, working with a more general convergence determining family of events (instead of shapes of neighbourhoods we look at shapes of what we call *communities*) that allows the induction step to work.

Having arrived at a quenched local limit for the $\bar{\mathcal{R}}$ -core, we deduce convergence of what we call the $\bar{\mathcal{O}}$ -core $\bar{\mathcal{O}}(M_n^t)$ and is a randomly sized map obtained from a 3-connected planar map by blowing up edges into paths. As we have a local limit theorem at hand for the number of edges of $\bar{\mathcal{O}}(M_n^t)$, we may transfer properties of $\bar{\mathcal{O}}(M_n^t)$ to other randomly sized $\bar{\mathcal{O}}$ -networks satisfying a similar local limit theorem (but with possibly different constants). For example, we may define similarly the $\bar{\mathcal{O}}$ -core $\bar{\mathcal{O}}(V_n^t)$ of V_n^t . The quenched convergence of $\bar{\mathcal{O}}(M_n^t)$ transfers to quenched local convergence of $\bar{\mathcal{O}}(V_n^t)$. The arguments we used to pass convergence from $\mathcal{V}(M_n^t)$ to $\bar{\mathcal{O}}(M_n^t)$ may be reversed to pass convergence from $\bar{\mathcal{O}}(V_n^t)$ to V_n^t , yielding a quenched local limit for V_n^t .

Whitney's theorem ensures that we may group $\bar{\mathcal{O}}$ -maps into pairs such that each pair corresponds to a unique graph. We call such graphs \mathcal{O} -graphs. $\bar{\mathcal{O}}$ -networks and \mathcal{O} -graphs form the link between planar maps and planar graphs in our proof.

Networks that encode 2-connected planar graphs differ from the networks that encode 2-connected planar maps, since we do not allow multiple edges and do not care about the order in parallel compositions. But guided by the fully recursive decomposition for non-serial networks encoding maps, we establish a somewhat more technical novel fully recursive decomposition for non-serial networks encoding graphs. The price we have to pay is that this decomposition is no longer isomorphism preserving. This constitutes no issue or downside for the present work, which concerns itself exclusively with *labelled* planar graphs. However, future applications to random *unlabelled* planar graphs will require careful consideration and further study of how this step affects the symmetries. The decomposition allows us to argue analogously as for planar maps. Hence quenched local convergence of the random 2-connected planar graph B_n^t with n edges and weight t at vertices follows from the corresponding convergence of a giant \mathcal{O} -core $\mathcal{O}(B_n^t)$, obtained via a transfer from the core $\bar{\mathcal{O}}(M_n^t)$.

If we condition the 2-connected core of the random planar graph P_n to have a fixed number m of edges, we do not obtain the uniform distribution on the 2-connected planar graphs with m edges. This effect does *not* go away as n and

m tend to infinity. Instead, a result by [17] shows that the 2-connected core has a vanishing total variational distance from a mixture of $(\mathcal{B}_n^t)_{n \geq 1}$ for the special case of vertex weight $t = \rho_{\mathcal{B}}$, the radius of convergence of the generating series for 2-connected planar graphs. This allows us to deduce quenched local convergence of the 2-connected core $\mathcal{B}(\mathcal{P}_n)$ of the random connected planar graph \mathcal{P}_n . A quenched extension of a result by [30, Thm. 6.39] then yields quenched convergence of \mathcal{P}_n , completing the proof of Theorem 1.

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Some Results on the Laplacian Spectra of Token Graphs

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Abstract. We study the Laplacian spectrum of token graphs, also called symmetric powers of graphs. The k -token graph $F_k(G)$ of a graph G is the graph whose vertices are the k -subsets of vertices from G , two of which being adjacent whenever their symmetric difference is a pair of adjacent vertices in G . In this work, we give a relationship between the Laplacian spectra of any two token graphs of a given graph. In particular, we show that, for any integers h and k such that $1 \leq h \leq k \leq \frac{n}{2}$, the Laplacian spectrum of $F_h(G)$ is contained in the Laplacian spectrum of $F_k(G)$. Besides, we obtain a relationship between the spectra of the k -token graph of G and the k -token graph of its complement \bar{G} . This generalizes a well-known property for Laplacian eigenvalues of graphs to token graphs.

The research of C. Dalfo and M. A. Fiol has been partially supported by AGAUR from the Catalan Government under project 2017SGR1087 and by MICINN from the Spanish Government under project PGC2018-095471-B-I00. The research of C. Dalfo has also been supported by MICINN from the Spanish Government under project MTM2017-83271-R. The research of C. Huemer was supported by PID2019-104129GB-I00/ AEI/ 10.13039/501100011033 and Gen. Cat. DGR 2017SGR1336. F. J. Zaragoza Martínez acknowledges the support of the National Council of Science and Technology (Conacyt) and its National System of Researchers (SNI).



This work has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

Keywords: Token graph · Laplacian spectrum · Algebraic connectivity · Binomial matrix · Adjacency spectrum · Doubled odd graph · Doubled Johnson graph · Complement graph

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \leq k \leq n$, the k -token graph $F_k(G)$ of G is the graph whose vertex set $V(F_k(G))$ consists of the $\binom{n}{k}$ k -subsets of vertices of G , and two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A \triangle B$ is a pair $\{a, b\}$ such that $a \in A$, $b \in B$, and $\{a, b\} \in E(G)$; see Fig. 1 for an example. Note that if $k = 1$, then $F_1(G) \cong G$; and if G is the complete graph K_n , then $F_k(K_n) \cong J(n, k)$, where $J(n, k)$ denotes a Johnson graph [5]. The naming token graph comes from an observation by Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [5], that vertices of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G , where two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. Such graphs are also called *symmetric k -th power of a graph* by Audenaert, Godsil, Royle, and Rudolph [2]; and *n -tuple vertex graphs* in Alavi, Lick, and Liu [1]. The token graphs have some applications in physics. For instance, a connection between symmetric powers of graphs and the exchange of Hamiltonian operators in quantum mechanics is given in [2]. They have also been considered in relation to the graph isomorphism problem, see Rudolph [8].

In this work, we focus on the Laplacian spectrum of $F_k(G)$ for any value of k . Recall that the Laplacian matrix $\mathbf{L}(G)$ of a graph G is $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$, where $\mathbf{A}(G)$ is the adjacency matrix of G , and $\mathbf{D}(G)$ is the diagonal matrix whose non-zero entries are the vertex degrees of G . For a d -regular graph G , each eigenvalue λ of $\mathbf{L}(G)$ corresponds to an eigenvalue μ of $\mathbf{A}(G)$ via the relation $\lambda = d - \mu$. In [3], Carballosa, Fabila-Monroy, Leaños, and Rivera proved that, for $1 < k < n - 1$, the k -token graph $F_k(G)$ is regular only if G is the complete graph K_n or its complement, or if $k = n/2$ and G is the star graph $K_{1, n-1}$ or its complement. Then, for most graphs, we cannot directly infer the Laplacian spectrum of $F_k(G)$ from the adjacency spectrum of $F_k(G)$. In fact, when considering the adjacency spectrum, we find graphs G whose spectrum is not contained in the spectrum of $F_k(G)$; see Rudolph [8]. Surprisingly, for the Laplacian spectrum, this holds and it is our first result.

This extended abstract is organized as follows. In Sect. 2, we show that the Laplacian spectrum of a graph G is contained in the Laplacian spectrum of its k -token graph $F_k(G)$. Besides, with the use of a new $(n; k)$ -binomial matrix, we give the relationship between the Laplacian spectrum of a graph G and that of its k -token graph. In Sect. 3, we show that an eigenvalue of a k -token graph is also an eigenvalue of the $(k + 1)$ -token graph for $1 \leq k < n/2$. Besides, we define another matrix, called $(n; h, k)$ -binomial matrix. With the use of this matrix, it

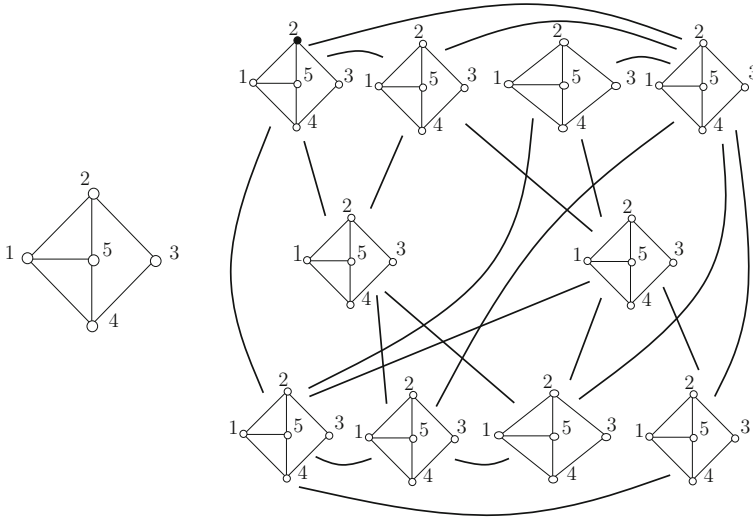


Fig. 1. A graph G (left) and its 2-token graph $F_2(G)$ (right). The Laplacian spectrum of G is $\{0, 2, 3, 4, 5\}$. The Laplacian spectrum of $F_2(G)$ is $\{0, 2, 3^2, 4, 5^3, 7, 8\}$.

is shown that, for any integers h and k such that $1 \leq h \leq k \leq \frac{n}{2}$, the Laplacian spectrum of $F_h(G)$ is contained in the spectrum of $F_k(G)$. Finally, in Sect. 4, we obtain a relationship between the Laplacian spectra of the k -token graph of G and the k -token graph of its complement \overline{G} . This generalizes a well-known property for Laplacian eigenvalues of graphs to token graphs.

For more information, not included in this extended abstract, see [4].

2 The Laplacian Spectra of Token Graphs

Let us first introduce some notation used throughout the paper. Given a graph $G = (V, E)$, we indicate with $a \sim b$ that a and b are adjacent in G . As usual, the transpose of a matrix M is denoted by M^\top , the identity matrix by I , the all-1 vector $(1, \dots, 1)^\top$ by $\mathbf{1}$, the all-1 (universal) matrix by \mathbf{J} , and the all-0 vector and all-0 matrix by $\mathbf{0}$ and \mathbf{O} , respectively. Let $[n] := \{1, \dots, n\}$. Let $\binom{[n]}{k}$ denote the set of k -subsets of $[n]$, the set of vertices of the k -token graph.

Our first theorem deals with the Laplacian spectrum of a graph G and its k -token graph $F_k(G)$.

Theorem 1. *Let G be a graph and $F_k(G)$ its k -token graph. Then, the Laplacian spectrum (eigenvalues and their multiplicities) of G is contained in the Laplacian spectrum of $F_k(G)$.*

Theorem 1 has a direct proof using token movements, and it can also be obtained using the $(n; k)$ binomial matrix B , defined in the following. Given some integers n and k (with $k \in [n]$), we define the $(n; k)$ -binomial matrix B .

This is a $\binom{n}{k} \times n$ matrix whose rows are the characteristic vectors of the k -subsets of $[n]$ in a given order. Thus, if the i -th k -subset is A , then

$$(\mathbf{B})_{ij} = \begin{cases} 1 & \text{if } j \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *The $(n; k)$ -binomial matrix \mathbf{B} satisfies*

$$\mathbf{B}^\top \mathbf{B} = \binom{n-2}{k-1} \mathbf{I} + \binom{n-2}{k-2} \mathbf{J}.$$

Let G be a graph with n vertices and, for $k \leq \frac{n}{2}$, let $F_k = F_k(G)$ be its k -token graph. The following result gives the relationship between the corresponding Laplacian matrices, \mathbf{L}_1 and \mathbf{L}_k .

Theorem 2. *Given a graph G and its k -token graph F_k , with corresponding Laplacian matrices \mathbf{L}_1 and \mathbf{L}_k , and $(n; k)$ -binomial matrix \mathbf{B} , the following holds:*

$$\mathbf{B}^\top \mathbf{L}_k \mathbf{B} = \binom{n-2}{k-1} \mathbf{L}_1. \tag{1}$$

Corollary 1. *Given a graph G , with $G \cong F_1$, and its k -token graph F_k , with corresponding Laplacian matrices \mathbf{L}_1 and \mathbf{L}_k , and $(n; k)$ -binomial matrix \mathbf{B} , the following implications hold:*

- (i) *If \mathbf{v} is a λ -eigenvector of \mathbf{L}_1 , then $\mathbf{B}\mathbf{v}$ is a λ -eigenvector of \mathbf{L}_k .*
- (ii) *If \mathbf{w} is a λ -eigenvector of \mathbf{L}_k and $\mathbf{B}^\top \mathbf{w} \neq \mathbf{0}$, then $\mathbf{B}^\top \mathbf{w}$ is a λ -eigenvector of \mathbf{L}_1 .*

Corollary 2. (i) *The Laplacian spectrum of \mathbf{L}_1 is contained in the Laplacian spectrum of \mathbf{L}_k .*
(ii) *Every eigenvalue λ of \mathbf{L}_k , having eigenvector \mathbf{w} such that $\mathbf{B}^\top \mathbf{w} \neq \mathbf{0}$, is a λ -eigenvector of \mathbf{L}_1 .*

3 A More General Result

In this section, we show a stronger result. Namely, for any $1 \leq k < n/2$, the Laplacian spectrum of the k -token graph $F_k(G)$ of a graph G is contained in the Laplacian spectrum of its $(k + 1)$ -token graph $F_{k+1}(G)$.

A ‘local analysis’, based on token movements, is used to show that every eigenvalue of $F_k(G)$ is also an eigenvalue of $F_{k+1}(G)$.

Theorem 3. *Let G be a graph on n vertices. Let h, k be integers such that $1 \leq h \leq k \leq n/2$. If λ is an eigenvalue of $F_h(G)$, then λ is an eigenvalue of $F_k(G)$.*

All previous results can be seen as consequences of the following matricial formulation. First, we define, for some integers n , k_1 , and k_2 (with $1 \leq k_1 < k_2 < n$), the $(n; k_2, k_1)$ -binomial matrix $\mathbf{B} = \mathbf{B}(n; k_2, k_1)$. This is a $\binom{n}{k_2} \times \binom{n}{k_1}$ $(0, 1)$ -matrix, whose rows are indexed by the k_2 -subsets $A \subset [n]$, and its columns are indexed by the k_1 -subsets $X \subset [n]$. The entries of \mathbf{B} are

$$(\mathbf{B})_{AX} = \begin{cases} 1 & \text{if } X \subset A, \\ 0 & \text{otherwise.} \end{cases}$$

The transpose of $\mathbf{B} = \mathbf{B}(n; k_2, k_1)$ is known as the *set-inclusion matrix*, denoted by $W_{k_1, k_2}(n)$ (see, for instance, Godsil [7]).

Lemma 2. *The matrix \mathbf{B} satisfies the following simple properties.*

- (i) *The number of 1's of each column of \mathbf{B} is $\binom{n-k_1}{k_2-k_1}$.*
- (ii) *The number of common 1's of any two columns of \mathbf{B} , corresponding to k_2 -subsets of $[n]$ whose intersection has $k_1 - 1$ elements, is $\binom{n-k_1-1}{k_2-k_1-1}$.*

The new matrix \mathbf{B} allows us to give the following result that can be seen as a generalization of Theorem 2 (see also Corollary 3).

Theorem 4. *Let G be a graph on $n = |V|$ vertices, with k_1 - and k_2 -token graphs $F_{k_1}(G)$ and $F_{k_2}(G)$, where $1 \leq k_1 \leq k_2 \leq n$. Let \mathbf{L}_{k_1} and \mathbf{L}_{k_2} be the respective Laplacian matrices, and \mathbf{B} the $(n; k_2, k_1)$ -binomial matrix. Then, the following holds:*

$$\mathbf{B}\mathbf{L}_{k_1} = \mathbf{L}_{k_2}\mathbf{B}. \tag{2}$$

Let us now see some consequences of this theorem. First, we get again Theorem 2.

Corollary 3. *With the same notation as before, the following holds.*

- (i) *For every k_1, k_2 with $1 \leq k_1 \leq k_2 \leq n$,*

$$\mathbf{B}^\top \mathbf{L}_{k_2} \mathbf{B} = \mathbf{B}^\top \mathbf{B} \mathbf{L}_{k_1}. \tag{3}$$

- (ii) *For $k_1 = 1$ and $k_2 = k$,*

$$\mathbf{B}^\top \mathbf{L}_k \mathbf{B} = \binom{n-2}{k-1} \mathbf{L}_1.$$

Since $F_k(G) \cong F_{n-k}(G)$ assume, without loss of generality, that $1 \leq k_1 \leq k_2 \leq \frac{n}{2}$. Then, we have a generalization of Corollary 3.

Corollary 4. *For any integers h, k such that $1 \leq h \leq k \leq \frac{n}{2}$, let \mathbf{B} be the $(n; k, h)$ -binomial matrix. Then, the eigenvalues and eigenvectors of the Laplacian matrices of the token graphs F_h and F_k are related in the following way.*

- (i) *If \mathbf{v} is a λ -eigenvector of \mathbf{L}_h , then $\mathbf{B}\mathbf{v}$ is a λ -eigenvector of \mathbf{L}_k . Moreover, the linear independence of the different eigenvectors is preserved. (That is, the spectrum of \mathbf{L}_h is contained in the spectrum of \mathbf{L}_k .)*

(ii) If \mathbf{w} is a λ -eigenvector of \mathbf{L}_k and $\mathbf{B}^\top \mathbf{w} \neq \mathbf{0}$, then $\mathbf{B}^\top \mathbf{w}$ is a λ -eigenvector of \mathbf{L}_h . Moreover, all the eigenvalues, including multiplicities, of \mathbf{L}_h are obtained (that is, one eigenvalue each time that the above non-zero condition is fulfilled).

In our context, Theorem 4 allows us to obtain the Laplacian matrix of F_h in terms of the Laplacian matrix of F_k , provided that we know the binomial matrix $\mathbf{B}(n; k, h)$ with its rows and columns in the right order (that is, the same order as the columns of \mathbf{L}_h and \mathbf{L}_k , respectively). Indeed, in this case, (3) with $k_1 = h$ and $k_2 = k$ leads to

$$\mathbf{L}_h = (\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{L}_k \mathbf{B}. \tag{4}$$

Notice that $\mathbf{B}^\top \mathbf{B}$ is a Gram matrix of the columns of \mathbf{B} , which are linearly independent vectors and, hence, $\mathbf{B}^\top \mathbf{B}$ is invertible.

Following with the simplified notation $k_1 = h$ and $k_2 = k$, the result of Theorem 4 can also be written in terms of the adjacency matrices \mathbf{A}_h and \mathbf{A}_k of F_h and F_k , respectively. Then, we get

$$\mathbf{A}_k \mathbf{B} - \mathbf{B} \mathbf{A}_h = \mathbf{D}_k \mathbf{B} - \mathbf{B} \mathbf{D}_h, \tag{5}$$

where \mathbf{D}_h and \mathbf{D}_k are the diagonal matrices with non-zero entries the degrees of the vertices of F_h and F_k , respectively. Some consequences of this are obtained when both F_h and F_k are regular.

Corollary 5. *Assume that a graph $G \equiv F_1$ and its k -token graph F_k are d_1 -regular and d_k -regular graphs, respectively. Let \mathbf{B} be the $(n; k, 1)$ -binomial matrix. Let \mathbf{A} and \mathbf{A}_k be the respective adjacency matrices of G and F_k . If \mathbf{v} is a λ -eigenvector of \mathbf{A} , then $\mathbf{B}\mathbf{v}$ is a μ -eigenvector of \mathbf{A}_k , where $\mu = (d_k - d_1 + \lambda)$.*

4 A Graph, Its Complement, and Their Token Graphs

Let us consider a graph G and its complement \overline{G} , with respective Laplacian matrices \mathbf{L} and $\overline{\mathbf{L}}$. We already know that the eigenvalues of G are closely related to the eigenvalues of \overline{G} , since $\mathbf{L} + \overline{\mathbf{L}} = n\mathbf{I} - \mathbf{J}$. Hence, the same relationship holds for the algebraic connectivity, see Fiedler [6].

Observe that the k -token graph of \overline{G} is the complement of the k -token graph of G with respect to the Johnson graph $J(n, k)$ (the k -token graph of K_n), see Carballosa, Fabila-Monroy, Leaños, and Rivera [3, Prop. 3]. Then, it is natural to ask whether a similar relationship holds between the Laplacian spectrum of the k -token graph of G and the Laplacian spectrum of the k -token graph of $\overline{G} = K_n - G$. In this section, we show that, indeed, this is the case.

Our result is a consequence of the following property.

Lemma 3. *Given a graph G and its complement \overline{G} , the Laplacian matrices $\mathbf{L} = \mathbf{L}(F_k(G))$ and $\overline{\mathbf{L}} = \mathbf{L}(F_k(\overline{G}))$ of their k -token graphs commute: $\mathbf{L}\overline{\mathbf{L}} = \overline{\mathbf{L}}\mathbf{L}$.*

Theorem 5. *Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and let \overline{G} be its complement. For a given k , with $1 \leq k < n - 1$, let us consider the token graphs $F_k(G)$ and $F_k(\overline{G})$. Then, the Laplacian spectrum of $F_k(\overline{G})$ is the complement of the Laplacian spectrum of $F_k(G)$ with respect to the Laplacian spectrum of the Johnson graph $J(n, k) = F_k(K_n)$. That is, every eigenvalue λ_J of $J(n, k)$ is the sum of one eigenvalue $\lambda_{F_k(G)}$ of $F_k(G)$ and one eigenvalue $\lambda_{F_k(\overline{G})}$ of $F_k(\overline{G})$, where each $\lambda_{F_k(G)}$ and each $\lambda_{F_k(\overline{G})}$ is used once:*

$$\lambda_{F_k(G)} + \lambda_{F_k(\overline{G})} = \lambda_J.$$

Theorem 5 leads to the following consequence.

Corollary 6. *Let G be a graph such that its complement \overline{G} has c connected components. Then, for $1 \leq k \leq n - 1$, the k -token graph $F_k(G)$ has at least c integer eigenvalues. If each of the c components of \overline{G} has at least k vertices, then $F_k(G)$ has at least $\binom{c+k-1}{k}$ integer eigenvalues.*

Acknowledgements. We thank Carlos Hidalgo-Toscano for valuable discussions.

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On the Number of Compositions of Two Polycubes

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Abstract. We provide almost tight bounds on the minimum and maximum possible numbers of compositions of two polycubes, either when each is of size n , or when their total size is $2n$, in two and higher dimensions. We also provide an efficient algorithm (with some trade-off between time and space) for computing the number of composition two given polyominoes (or polycubes) have.

Keywords: Lattice animals · Polyominoes · Polycubes · Compositions

1 Introduction

A d -dimensional *polycube* (*polyomino* if $d = 2$) is a connected set of cells on the cubical lattice \mathbb{Z}^d , where connectivity is through $(d-1)$ -dimensional faces. Polycubes and “animals” of other lattices play for more than half a century an important role in enumerative combinatorics [4] as well as in statistical physics [3].

The *size* (volume, or area in the plane) of a polycube is the number of d -dimensional cells it contains. A *composition* of two d -dimensional polycubes is the placement of one of them relative to the other, such that they touch each other (sharing one or more $(d-1)$ -dimensional faces) but do not overlap, so that the union of their cell sets is a valid (connected) polycube. The number of compositions of polycubes of certain sizes plays an important role in proving bounds on the growth constant of polycubes. For example, an incorrect upper

Work on this paper by the first author has been supported in part by FWF Grant P28466. Work on this paper by the second and third authors has been supported in part by ISF Grant 575/15 and BSF Grant 2017684.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 71–77, 2021.
https://doi.org/10.1007/978-3-030-83823-2_12

bound on the maximum possible compositions of polyominoes [1] was used for claiming an erroneous upper bound on the growth constant of polyominoes. A corrected version of the argument [2] was used for obtaining an upper bound on the growth constant of polyiamonds (edge-connected sets of cells on the regular planar triangular lattice). The main question which we ask is:

Question 1: Given two polycubes **of total size $2n$** , how many different compositions (up to translations) do they have?

Alternatively, we can ask a similar question but in a more restricted setting:

Question 2: Given two polycubes, **each of size n** , how many different compositions (up to translations) do they have?

Obviously, the set of pairs of polycubes, each being of size n , is a subset of pairs of polycubes of total size $2n$. Hence, any lower (resp., upper) bound on the minimum (resp., maximum) number of compositions of polycubes in Question 1 also carries over to Question 2, and any upper (resp., lower) bound on the minimum (resp., maximum) number of compositions of polycubes in Question 2 also carries over to Question 1. In fact, all our bounds apply to both versions of the question. In addition, any specific example provides both an upper bound on the minimum and a lower bound on the maximum of the respective number of compositions. We summarize our results in Table 1.

Table 1. The number of compositions of two polycubes of total size $2n$.

Number of compositions	Two dimensions		$d \geq 3$ dimensions	
	Lower bound	Upper bound	Lower bound	Upper bound
Minimum	$\Theta(n^{1/2})$		$2n^{1-1/d}$	$O(2^d dn^{1-1/d})$
Maximum	$n^2/2^{O(\log^{1/2} n)}$	$O(n^2)$	$\Theta(dn^2)$	

We also provide an efficient algorithm for computing the number of composition two given polyominoes (or polycubes) have. A few possible implementations of the required data structures suggest a trade-off between the running time and the required memory.

2 Two Dimensions

2.1 Minimum Number of Compositions

Theorem 1

- (i) Any pair of polyominoes of sizes n_1 and n_2 have $\Omega((n_1+n_2)^{1/2})$ compositions.
- (ii) For every two numbers $n_1 \geq 1, n_2 \geq 1$, there is a pair of polyominoes of sizes n_1 and n_2 with $\Theta((n_1 + n_2)^{1/2})$ compositions.

Proof. Let $n = n_1 + n_2$, and consider a pair of polyominoes P_1, P_2 of sizes n_1 and n_2 . Assume without loss of generality that $n_1 \geq n_2$, that is, $n_1 \geq n/2$. Assume, also without loss of generality, that the width (x -span) of P_1 is greater than (or equal to) the height (y -span) of P_1 . Hence, the width of P_1 is at least $n_1^{1/2}$. Then, P_2 may touch P_1 from below or above in different ways at least twice this width: Simply put P_2 below (or above) P_1 so that the left column of P_2 is aligned with the i th column of P_1 (for $1 \leq i \leq n_1^{1/2}$) and translate P_2 upward (or downward) until it touches P_1 . Hence, we have a least $2n_1^{1/2} \geq (2n)^{1/2}$ compositions.

To see that this lower bound is tight, we take polyominoes that fit in a square with side lengths $k_1 = \lceil n_1^{1/2} \rceil$ and $k_2 = \lceil n_2^{1/2} \rceil$. We form P_1 and P_2 by filling the respective squares row-wise until they have the desired size. P_1 and P_2 can be composed in at most $2(2k_1 - 1 + 2k_2 - 1) = 4(2((n_1 + n_2)^{1/2} + 1) - 1) < 8(n_1 + n_2)^{1/2} + 4$ ways.

The following is a trivial corollary of Theorem 1.

Corollary 1. *Any pair of polyominoes of total size $2n$ have $\Omega(n^{1/2})$ compositions. This lower bound is attainable. □*

2.2 Maximum Number of Compositions

In this section, we find bounds on the maximum number of compositions of two polyominoes of size n . First, we show a (quite trivial) upper bound of $O(n^2)$. Next, we show that it is “almost tight” by constructing an example that yields a lower bound of $\Omega(n^{2-\epsilon})$, for any $\epsilon > 0$.

Upper Bound

Observation 2. *Any pair of polyominoes of sizes n_1, n_2 has $O(n_1 n_2)$ compositions.*

Proof. Let n_1, n_2 be the sizes of polyominoes P_1, P_2 , resp. Every cell of P_1 can touch every cell of P_2 in at most 4 ways, yielding $4n_1 n_2$ as a trivial upper bound on the number of compositions.

Lower Bound

Theorem 3. *For every $n \geq 1$, there are two polyominoes, each of size at most n , that have at least $\frac{n^2}{2^{8 \cdot \sqrt{\log_2 n}}}$ compositions.*

The detailed proof of Theorem 3 is provided in the full version of the paper. In this extended abstract, we only note that the bound is obtained by a careful analysis of a construction of two “combs” whose first three levels are shown in Fig. 1.

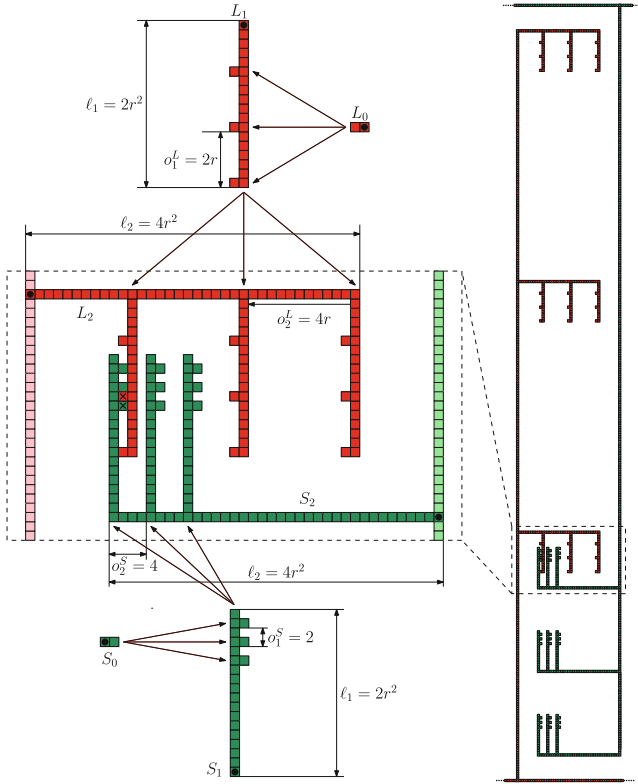


Fig. 1. A recursive construction of two “combs” for the proof of Theorem 3.

3 Higher Dimensions

3.1 Minimum Number of Compositions

Lower Bound

Theorem 4. *All pairs of d -dimensional polycubes of total size $2n$ have at least $2n^{1-1/d}$ compositions.*

Proof. The proof is similar to that of Theorem 1. Consider a pair of polycubes P_1, P_2 of total size $2n$. Assume, without loss of generality, that P_1 is the larger of the two polycubes, that is, the size (d -dimensional volume) of P_1 is at least n . Let V_i ($1 \leq i \leq d$) be the $(d-1)$ -dimensional volume of the projection of P_1 orthogonal to the x_i axis. An isoperimetric inequality of Loomis and Whitney [5] ensures that $\prod_{i=1}^d V_i \geq n^{d-1}$. Let $V_k \geq n^{1-1/d}$ be largest among V_1, \dots, V_d . Then, there are at least $2V_k \geq 2n^{1-1/d}$ different ways how P_2 may touch P_1 . The polycube P_1 has V_k “columns” in the x_k direction. Pick one “column” of P_2 and align it with each “column” of P_1 , putting it either “below”

or “above” P_1 along direction x_k , and find the unique translation along x_k by which they touch for the first time while being translated one towards the other.

Upper Bound

Theorem 5. *There exist pairs of d -dimensional polycubes, of total size $2n$, that have $O(2^d dn^{1-1/d})$ compositions.*

Proof. Figure 2 shows a composition of two copies of a d -dimensional hypercube P of size $k \times k \times \dots \times k$. The cube is made of n cells, hence, its sidelength is $k = n^{1/d}$. Two copies of P can slide towards each other in $2d$ directions until they touch. Obviously, there are no other compositions since no hypercube can penetrate into the bounding box of the other. Once we decide which facets of the hypercube touch each other, this can be done in $(2k-1)^{d-1}$ ways. Indeed, in each of the $d-1$ dimensions orthogonal to the sliding direction, there are $2k-1$ possible offsets of one hypercube relative to the other. Overall, the total number of compositions in this example is $(2d)(2k-1)^{d-1} = 2d(2n^{1/d} - 1)^{d-1} = \Theta(2^d dn^{1-1/d})$.

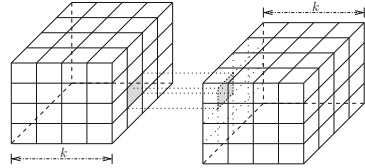


Fig. 2. A composition of two hypercubes

3.2 Maximum Number of Compositions in $d \geq 3$ Dimensions

Theorem 6. *Let $d \geq 3$. All pairs of d -dimensional polycubes of total size $2n$ have $O(dn^2)$ compositions. For $d \geq 3$, the upper bound is attainable: There exist pairs of d -dimensional polycubes of total size $2n$ with $\Omega(dn^2)$ compositions.*

Proof. Similarly to two dimensions, any two polycubes P_1, P_2 of total size $2n$ have $O(dn^2)$ compositions. Indeed, let $n_1 = |P_1|$ and $n_2 = |P_2|$, where $n_1 + n_2 = 2n$. Then, every cell of P_1 can touch every cell of P_2 in at most $2d$ ways, yielding $2dn_1n_2 \leq 2dn^2$ as an upper bound on the number of compositions.

The upper bound is attained asymptotically by two nonparallel “sticks” of size n , as shown in Fig. 3(a). Each stick has two extreme $(d-1)$ -D facets (orthogonal to the direction of the stick), plus $2(d-1)n$ middle facets. The number of compositions that involve only middle facets is $2(d-2)n^2 = \Omega(dn^2)$, see Fig. 3(b): Indeed, for each of the $d-2$ directions which are not parallel to one of the sticks, there are $2n^2$ different choices for letting two middle facets of the sticks touch. We can ignore the $4n$ compositions that involve an extreme facet (see Fig. 3(c)).

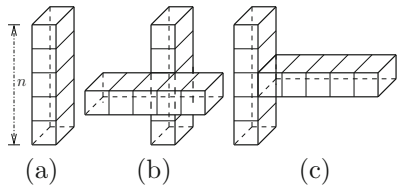


Fig. 3. Compositions of sticks

Note the difference, for the maximum number of compositions, between two and higher dimensions. In $d > 2$ dimensions, two of the dimensions (those along which the sticks in the proof of Theorem 6 are aligned) restrict the compositions of the sticks, but the existence of more dimensions allows every pair of cells, one of each polycube, to have compositions which manifest themselves through this pair only. This is not the case in two dimensions, a fact that makes the proof of Theorem 3 much more complicated.

4 Counting Compositions and Distribution Analysis

Finally, we refer to counting how many compositions a pair of polyominoes or polycubes have.

Theorem 7

(i) Given two polyominoes, each of size at most n , the number of compositions they have can be computed in $\Theta(n^2)$ time and $\Theta(n^2)$ space.

(ii) Given two d -dimensional polycubes, each of size at most n , the number of compositions they have can be computed in $O(d^2n^2)$ time and $O(dn^3)$ space, or $O(d^2n^2 \log n)$ time and $O(d^2n^2)$ space, or $O(d^2n^2)$ expected time and $O(d^2n^2)$ space.

We give the proof of Theorem 7 in the full version of the paper. The provided algorithms assume the unit-cost model of computation, in which numbers in the range $[-n, n]$ can be accessed and be subject to arithmetic operations in $O(1)$ time.

In the full version of the paper, we also present some empirical data concerning the *distribution* of $\text{NC}(n_1, n_2)$, the number of compositions of all pairs of polyominoes of sizes n_1, n_2 . The data suggest that the average value of $\text{NC}(n, n)$ for two random polyominoes grows linearly with n . With the available data for $3 \leq n \leq 14$, a linear regression gives the relation $\text{NC}(n, n) \approx 2.19n + 4.97$. Available data of $\text{NC}(n_1, n_2)$, for several values of a constant $n_1 + n_2$, were fitted to various discrete distributions. The best fit was found with the negative-binomial distribution.

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Halin's End Degree Conjecture

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Abstract. An end of a graph G is an equivalence class of rays, where two rays are equivalent if there are infinitely many vertex-disjoint paths between them in G . The degree of an end is the maximum cardinality of a collection of pairwise disjoint rays in this equivalence class.

An old question by Halin asks whether the end degree can be characterised in terms of typical ray configurations. Halin conjectured that it can – in a very strong form which would generalise his famous *grid theorem*. In particular, every end of regular uncountable degree κ would contain a *star of rays*, i.e. a configuration consisting of a central ray R and κ neighbouring rays $(R_i : i < \kappa)$ all disjoint from each other and each R_i sending a family of infinitely many disjoint paths to R so that paths from distinct families only meet in R .

We show that Halin's conjecture fails for end degree \aleph_1 , holds for end degree $\aleph_2, \aleph_3, \dots, \aleph_\omega$, fails for $\aleph_{\omega+1}$, and is undecidable (in ZFC) for the next $\aleph_{\omega+n}$ with $n \in \mathbb{N}, n \geq 2$.

Keywords: Infinite graph · Ends · End degree · Ray graph

1 Introduction

1.1 Halin's End Degree Conjecture

An *end* of a graph G is an equivalence class of rays, where two rays of G are *equivalent* if there are infinitely many vertex-disjoint paths between them in G . The *degree* $\deg(\varepsilon)$ of an end ε is the maximum cardinality of a collection of pairwise disjoint rays in ε , see Halin [10].

However, for many purposes a degree-witnessing collection $\mathcal{R} \subseteq \varepsilon$ on its own forgets significant information about the end, as it tells us nothing about how G links up the rays in \mathcal{R} ; in fact $G[\bigcup \mathcal{R}]$ is usually disconnected. Naturally, this raises the question of whether one can describe typical configurations in which G must link up the disjoint rays in some degree-witnessing subset of a pre-specified end.

Observing that prototypes of ends of any prescribed degree are given by the Cartesian product of a sufficiently large connected graph with a ray (see e.g. Fig. 1), Halin [10] suggested a way how to make question precise by introducing the notion of a 'ray graph', as follows.

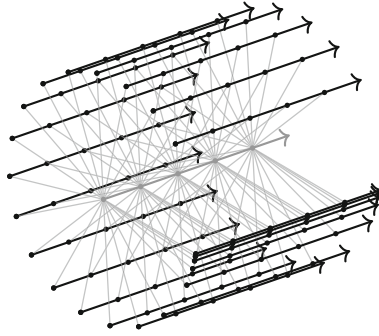


Fig. 1. The Cartesian product of a star and a ray.

Given a set \mathcal{R} of disjoint equivalent rays in a graph G , we call a graph H with vertex set \mathcal{R} a *ray graph* in G if there exists a set \mathcal{P} of independent \mathcal{R} -paths (independent paths with precisely their endvertices on rays from \mathcal{R}) in G such that for each edge RS of H there are infinitely many disjoint R - S paths in \mathcal{P} . Given an end ε in a graph G , a *ray graph for ε* is a connected ray graph in G on a degree-witnessing subset of ε . The precise formulation of the question reads as follows:

Does every graph contain ray graphs for all its ends?

For ends of finite degree it is straightforward to answer the question in the affirmative. For ends of countably infinite degree the answer is positive too, but only elaborate constructions are known. These constructions by Halin [10, Satz 4] and by Diestel [4,6] show that in this case the ray graph itself can always be chosen as a ray:

Theorem 1 (Halin's grid theorem). *Every graph with an end of infinite degree contains a subdivision of the hexagonal quarter grid whose rays belong to that end.*

For ends of uncountable degree, however, the question is a 20-year-old open conjecture that Halin stated in his legacy collection of problems [11, Conjecture 6.1]:

Conjecture 1 (Halin's Conjecture). Every graph contains ray graphs for all its ends.

In our paper [7], we settle Halin's conjecture: partly positively, partly negatively, with the answer essentially only depending on the degree of the end in question.

1.2 Our Results

If the degree in question is \aleph_1 , then any ray graph for such an end contains a vertex of degree \aleph_1 , which together with its neighbours already forms a ray

graph for the end in question, namely an ‘ \aleph_1 -star of rays’. Thus, finding in G a ray graph for an end of degree \aleph_1 reduces to finding such a star of rays. Already this case has remained open.

Let $\text{HC}(\kappa)$ be the statement that Halin’s conjecture holds for all ends of degree κ in any graph. As our first main result, we show that

$$\text{HC}(\aleph_1) \text{ fails.}$$

So, Halin’s conjecture is not true after all. But we do not stop here, for the question whether $\text{HC}(\kappa)$ holds remains open for end degrees $\kappa > \aleph_1$. And surprisingly, we show that $\text{HC}(\aleph_2)$ holds. In fact, we show more generally that

$$\text{HC}(\aleph_n) \text{ holds for all } n \text{ with } 2 \leq n \leq \omega.$$

Interestingly, this includes the first singular uncountable cardinal \aleph_ω . Having established these results, it came as a surprise to us that

$$\text{HC}(\aleph_{\omega+1}) \text{ fails.}$$

How does this pattern continue? It turns out that from this point onward, set-theoretic considerations start playing a role. Indeed

$$\text{HC}(\aleph_{\omega+n}) \text{ is undecidable for all } n \text{ with } 2 \leq n \leq \omega, \text{ while } \text{HC}(\aleph_{\omega \cdot 2+1}) \text{ fails.}$$

The following theorem decides Halin’s conjecture for all end degrees:

Theorem 2. *The following two assertions about $\text{HC}(\kappa)$ are provable in ZFC:*

- (1) $\text{HC}(\aleph_n)$ holds for all $2 \leq n \leq \omega$,
- (2) $\text{HC}(\kappa)$ fails for all κ with $\text{cf}(\kappa) \in \{\mu^+ : \text{cf}(\mu) = \omega\}$; in particular, $\text{HC}(\aleph_1)$ fails.

Furthermore, the following assertions about $\text{HC}(\kappa)$ are consistent:

- (3) Under GCH, $\text{HC}(\kappa)$ holds for all cardinals not excluded by (2).
- (4) However, for all κ with $\text{cf}(\kappa) \in \{\aleph_\alpha : \omega < \alpha < \omega_1\}$ it is consistent with ZFC+CH that $\text{HC}(\kappa)$ fails, and similarly also for all κ strictly greater than the least cardinal μ with $\mu = \aleph_\mu$.

1.3 Proof Sketch

All details can be found in [7].

The first step behind our affirmative results for $\text{HC}(\kappa)$ is the observation that it suffices to find some countable set of vertices U for which there is a set \mathcal{R} of κ many rays in ε , all disjoint from each other and from U , such that each $R \in \mathcal{R}$ comes with an infinite family \mathcal{P}_R of disjoint R - U paths which, for distinct R and R' , meet only in their endpoints in U : then it is not difficult to find a ray R^* that contains enough of U to include the endpoints of almost all path families \mathcal{P}_R , yielding a κ -star of rays on $\{R^*\} \cup \mathcal{R}'$ for some suitable $\mathcal{R}' \subseteq \mathcal{R}$.

While it may be hard to identify a countable such set U directly, for κ of cofinality at least \aleph_2 there is a neat greedy approach inspired by [2, 15] to finding a similar set U' of cardinality just $< \kappa$ rather than \aleph_0 . Let us illustrate this approach in the case of $\kappa = \aleph_2$: Starting from an arbitrary ray R_0 in ε , does $U_0 = V(R_0)$ already do the job? That is to say, are there \aleph_2 disjoint rays in ε that are independently attached to U_0 as above? If so, we have achieved our goal. If not, take a maximal set of disjoint rays \mathcal{R}_0 in ε all whose rays are independently attached to U_0 as above, and define U_1 to be the union of U_0 together with the vertices from all the rays in \mathcal{R}_0 and all their selected paths to U_0 . Then $|U_1| \leq \aleph_1$. Does U_1 do the job? If so, we have achieved our goal. If not, continue as above. We claim that when continuing transfinitely and building sets $U_0 \subsetneq U_1 \subsetneq \dots \subsetneq U_\omega \subsetneq U_{\omega+1} \subsetneq \dots$, we will achieve our goal at some countable ordinal $< \omega_1$. For suppose not. Then $U'' := \bigcup \{U_i : i < \omega_1\}$ meets all the rays in ε . Indeed, any ray R from ε outside of U'' could be joined to U'' by an infinite family of disjoint R - U'' paths. But then their countably many endvertices already belong to some U_i for $i < \omega_1$, contradicting the maximality of \mathcal{R}_i in the definition of U_{i+1} . Hence, $|U''| = \aleph_2$. For cofinality reasons there is a first index $j = i + 1$ with $|U_j| = \aleph_2$. Now $U' = U_i$ is as required.

Having identified a $< \kappa$ -sized set U' together with κ disjoint rays all independently attached to it, we aim to restrict U' to a countable set U while keeping κ many rays attached to U . For $\kappa = \aleph_2$ this is straightforward, since if U' is written as an increasing \aleph_1 -union of countable sets, one of them already contains all the endpoints of the path systems for some \aleph_2 -sized subcollection $\mathcal{R}' \subseteq \mathcal{R}$. Take this countable set as the set U originally sought. This completes the proof of $\text{HC}(\aleph_2)$.

What about general cardinalities κ ? The above strategy can fail in two different ways: First, if $\text{cf}(\kappa) = \aleph_1$, the greedy approach may not terminate: for example, it may well be possible that $|U'| = \aleph_1$ while $|U_i| = \aleph_0$ for all $i < \omega_1$. And indeed, we will show that rays in ends of degree \aleph_1 may be ‘arranged like an Aronszajn tree’, witnessing the failure of $\text{HC}(\aleph_1)$. This idea can be captured as follows: For an Aronszajn tree T , consider first a disjoint family of rays $\{R_t : t \in T\}$ indexed by the nodes of the tree. If t is a successor of s in T , add an infinite matching between the rays R_t and R_s . And if t is a limit, pick a cofinal ω -sequence $t_0 < t_1 < \dots < t$ of nodes below t , and add an edge from the n th vertex of R_t to the n th vertex of R_{t_n} , for all $n \in \mathbb{N}$. If these cofinal sequences $t_0 < t_1 < \dots < t$ below each limit t are chosen carefully (for this we rely on a trick by Diestel, Leader and Todorčević from [5]), the resulting graph, which we call the *ray inflation of T* , is one-ended of degree \aleph_1 but contains no \aleph_1 -star of rays. This refutes $\text{HC}(\aleph_1)$.

What about the remaining cardinals κ with $\text{cf}(\kappa) = \aleph_1$? Also there, counterexamples to $\text{HC}(\kappa)$ exist, and we have a machinery that produces a multitude of such examples: Any counterexample for $\text{HC}(\kappa)$ for regular κ may be turned canonically into a counterexample for $\text{HC}(\lambda)$ for all λ with $\text{cf}(\lambda) = \kappa$.

The second way in which our above strategy can fail is that even if our greedy algorithm terminates and provides a $< \kappa$ -sized U' to which there are κ disjoint

rays independently attached, it may not be possible to further reduce U' as earlier to some countable subset U . And indeed, using our idea of ray inflations of order trees, also this constellation can be exploited to construct counterexamples to Halin's conjecture. However, the trees that work now are quite different from the earlier Aronszajn trees: Generalising the concept of *binary trees with tops* introduced by Diestel and Leader in [5], we consider the class of λ -regular trees with tops, where λ is any singular cardinal of countable cofinality. These are order trees of height $\omega + 1$ in which every point of finite height has exactly λ successors, and above some $\kappa > \lambda$ many selected branches we add further nodes to the tree at height ω , called *tops*.

There is a reason why we take λ to be singular of countable cofinality: Just like the binary tree has uncountably many branches, these λ 's are the only other cardinals for which an uncountable regular tree is guaranteed to have strictly more than λ branches. And just like the precise number of branches of the binary tree is not determined in ZFC alone (it is 2^{\aleph_0} , which may be \aleph_1 if CH holds, or may be arbitrarily large), also the precise number of branches of the λ -regular tree is λ^+ if GCH holds, but it also may be much larger.

Now the starting point for our consistent counterexamples of (4) in Theorem 2 are simply models of ZFC + CH in which the two λ -regular trees for $\lambda = \aleph_\omega$ and λ equal to the first fixed point of the \aleph -function have a lot more branches than nodes. In these cases, any λ -regular tree with tops gives rise to counterexamples for Halin's conjecture. What happens if one looks for ZFC-counterexamples, not just consistent ones? With significantly more effort, and building on the concept of a *scale* from Shelah's pcf-theory from [16], we find that for any singular λ of countable cofinality one can directly select a suitable set of λ^+ many branches so that the λ -regular tree with corresponding tops gives rise to the counterexamples for Halin's conjecture, settling the remaining cases of (2) in Theorem 2.

1.4 Open Problems

We suspect that (1) and (2) in Theorem 2 capture all the cases of Halin's conjecture that can be proved or disproved in ZFC alone. This is certainly true up to \aleph_{ω_1} , as for each $\kappa \leq \aleph_{\omega_1}$ our main Theorem 2 provides either a ZFC or an independence result regarding the truth value of $\text{HC}(\kappa)$. While for all remaining cardinals assertion (3) of Theorem 2 gives consistent affirmative results, we do not know whether any of these can be established in ZFC.

Question 1. Is Halin's conjecture true for any $\kappa > \aleph_{\omega_1}$?

Question 2. Is it true that any end of degree \aleph_1 either contains an \aleph_1 -star of rays or a subdivision of an Aronszajn tree of rays?

Question 3. Is it true that for every cardinal κ there is $f(\kappa) \geq \kappa$, such that every end ε of degree $f(\kappa)$ contains a connected ray graph of size κ ?

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Coloring Circle Arrangements: New 4-Chromatic Planar Graphs

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Abstract. Felsner, Hurtado, Noy and Streinu (2000) conjectured that arrangement graphs of simple great-circle arrangements have chromatic number at most 3. This paper is motivated by the conjecture.

We show that the conjecture holds in the special case when the arrangement is Δ -saturated, i.e., arrangements where one color class of the 2-coloring of faces consists of triangles only. Moreover, we extend Δ -saturated arrangements with certain properties to a family of arrangements which are 4-chromatic. The construction has similarities with Koester's (1985) crowning construction.

We also investigate fractional colorings. We show that every arrangement \mathcal{A} of pairwise intersecting pseudocircles is “close” to being 3-colorable; more precisely $\chi_f(\mathcal{A}) \leq 3 + O(\frac{1}{n})$ where n is the number of pseudocircles. Furthermore, we construct an infinite family of 4-edge-critical 4-regular planar graphs which are fractionally 3-colorable. This disproves the conjecture of Gimbel, Kündgen, Li and Thomassen (2019) that every 4-chromatic planar graph has fractional chromatic number strictly greater than 3.

Keywords: Arrangement of pseudolines and pseudocircles · Triangle-saturated · Chromatic number · Fractional coloring · Critical graph

M.-K. Chiu was supported by ERC StG 757609. S. Felsner and M. Scheucher were supported by DFG Grant FE 340/12-1. M. Scheucher was supported by the internal research funding “Post-Doc-Funding” from Technische Universität Berlin. R. Steiner was supported by DFG-GRK 2434. B. Vogtenhuber was supported by the FWF project I 3340-N35. This work was initiated at a workshop of the collaborative DACH project *Arrangements and Drawings* in Malchow, Mecklenburg-Vorpommern. We thank the organizers and all the participants for the inspiring atmosphere.

1 Introduction

An arrangement of pseudocircles is a family of simple closed curves on the sphere or in the plane such that each pair of curves intersects at most twice. Similarly, an arrangement of pseudolines is a family of x -monotone curves such that every pair of curves intersects exactly once. An arrangement is *simple* if no three pseudolines/pseudocircles intersect in a common point and *intersecting* if every pair of pseudolines/pseudocircles intersects. Given an arrangement of pseudolines/pseudocircles, the *arrangement graph* is the planar graph obtained by placing vertices at the intersection points of the arrangement and thereby subdividing the pseudolines/pseudocircles into edges.

A (*proper*) *coloring* of a graph assigns a color to each vertex such that no two adjacent vertices have the same color. The *chromatic number* χ is the smallest number of colors needed for a proper coloring. The famous 4-color theorem and also Brook's theorem imply the 4-colorability of planar graphs with maximum degree 4. This motivates the question: which arrangement graphs need 4 colors in any proper coloring?

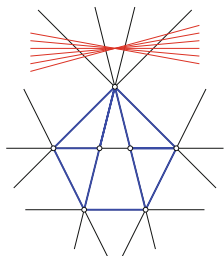


Fig. 1. A 4-chromatic non-simple line arrangement. The red sub-arrangement not intersecting the Moser spindle (highlighted blue) can be chosen arbitrarily.

There exist arbitrarily large non-simple line arrangements that require 4 colors. For example, the construction depicted in Fig. 1 contains the Moser spindle as a subgraph and hence cannot be 3-colored. Using an inverse central (gnomonic) projection, which maps lines to great-circles, one gets non-simple arrangements of great-circles with $\chi = 4$. Koester [12] presented a simple arrangement of 7 circles with $\chi = 4$ in which all but one pair of circles intersect, see Fig. 3(c). Moreover, there are simple intersecting arrangements that require 4 colors. We invite the reader to verify this property for the example depicted in Fig. 2.

In 2000, Felsner, Hurtado, Noy and Streinu [3] (cf. [4]) studied arrangement graphs of pseudoline and pseudocircle arrangements. They have results regarding connectivity, Hamiltonicity, and colorability of those graphs. In this work, they also stated the following conjecture:

Conjecture 1 (Felsner et al. [3,4]). The arrangement graph of every simple arrangement of great-circles is 3-colorable.

While the conjecture is fairly well known (cf. [10,14,18] and [19, Chapter 17.7]) there has been little progress in the last 20 years. Aichholzer, Aurenhammer, and Krasser verified the conjecture for up to 11 great-circles [13, Chapter 4.6.4].

Results and Outline. In Sect. 2 we show that Conjecture 1 holds for Δ -saturated arrangements of pseudocircles, i.e., arrangements where one color class of the

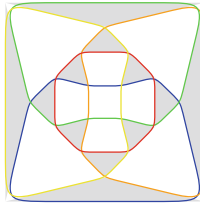


Fig. 2. A simple intersecting arrangement of 5 pseudocircles with $\chi = 4$ and $\chi_f = 3$.

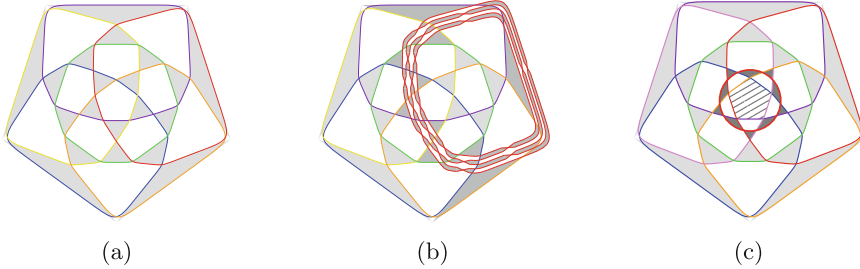


Fig. 3. (a) A Δ -saturated arrangement \mathcal{A} of 6 great-circles. (b) The doubling method applied to \mathcal{A} . The red pseudocircle is replaced by a cyclic arrangement. Triangular cells are shaded gray. (c) The corona extension of \mathcal{A} at its central pentagonal face. This arrangement is Koester’s [11] example of a 4-edge-critical 4-regular planar graph.

2-coloring of faces consists of triangles only. In Sect. 3 we extend our study of Δ -saturated arrangements and present an infinite family of arrangements which require 4 colors. The construction generalizes Koester’s [12] arrangement of 7 circles which requires 4 colors; see Fig. 3(c). Moreover, we believe that the construction results in infinitely many 4-vertex-critical¹ arrangement graphs. Koester [12] obtained his example using a “crowning” operation, which actually yields infinite families of 4-edge-critical 4-regular planar graphs. However, except for the 7 circles example these graphs are not arrangement graphs.

In Sect. 4 we investigate the fractional chromatic number χ_f of arrangement graphs. This variant of the chromatic number is the objective value of the linear relaxation of the ILP formulation for the chromatic number. We show that intersecting arrangements of pseudocircles are “close” to being 3-colorable by proving that $\chi_f(\mathcal{A}) \leq 3 + O(\frac{1}{n})$ where n is the number of pseudocircles of \mathcal{A} . In Sect. 5, we present an example of a 4-edge-critical arrangement graph which is fractionally 3-colorable. The example is the basis for constructing an infinite family of 4-regular planar graphs which are 4-edge-critical and fractionally 3-colorable. This disproves Conjecture 3.2 from Gimbel, Kündgen, Li and Thomassen [7] that every 4-chromatic planar graph has fractional chromatic number strictly greater

¹ A k -chromatic graph is k -vertex-critical (k -edge-critical, resp.) if the removal of every vertex (edge, resp.) decreases the chromatic number.

than 3. In Sect. 6 we report on our computational data, mention some new observations related to Conjecture 1, and present strengthened versions of the conjecture.

Due to space constraints, some proofs are deferred to the full version of this work.

2 \triangle -Saturated Arrangements are 3-Colorable

The maximum number of triangles in arrangements of pseudolines and pseudocircles has been studied intensively, see e.g. [2, 8, 15] and [6]. By recursively applying the “doubling method”, Harborth [9] and also [2, 15] proved the existence of infinite families of \triangle -saturated arrangements of pseudolines. Similarly, a doubling construction for arrangements of (great-)pseudocircles yields infinitely many \triangle -saturated arrangements of (great-)pseudocircles. Figures 3(a) and 3(b) illustrate the doubling method applied to an arrangement of great-pseudocircles. It will be relevant later that arrangements obtained via doubling contain pentagonal cells. Note that for $n \equiv 2 \pmod{3}$ there is no \triangle -saturated intersecting pseudocircle arrangement because the number of edges of the arrangement graph is not divisible by 3.

Theorem 1. *Every \triangle -saturated arrangement \mathcal{A} of pseudocircles is 3-colorable.*

Proof. Let H be a graph whose vertices correspond to the triangles of \mathcal{A} and whose edges correspond to pairs of triangles sharing a vertex of \mathcal{A} . This graph H is planar and 3-regular. Moreover, since the arrangement graph of \mathcal{A} is 2-connected, H is bridgeless. Now Tait’s theorem, a well known equivalent of the 4-color theorem, asserts that H is 3-edge-colorable, see e.g. [1] or [17]. The edges of H correspond bijectively to the vertices of the arrangement \mathcal{A} and, since adjacent vertices of \mathcal{A} are incident to a common triangle, the corresponding edges of H share a vertex. This shows that the graph of \mathcal{A} is 3-colorable.

3 Constructing 4-Chromatic Arrangement Graphs

In this section, we describe an operation that extends any \triangle -saturated intersecting arrangement of pseudocircles with a pentagonal cell (which is 3-colorable by Theorem 1) to a 4-chromatic arrangement of pseudocircles by inserting one additional pseudocircle.

The Corona Extension: We start with a \triangle -saturated arrangement of pseudocircles which contains a pentagonal cell \diamond . By definition, in the 2-coloring of the faces one of the two color classes consists of triangles only; see e.g. the arrangement from Fig. 3(a). Since the arrangement is \triangle -saturated, the pentagonal cell \diamond is surrounded by triangular cells. As illustrated in Fig. 3(c) we can now insert an additional pseudocircle close to \diamond . This newly inserted pseudocircle intersects only the 5 pseudocircles which bound \diamond , and in the so-obtained arrangement one

of the two dual color classes consists of triangles plus the pentagon \diamond . It is interesting to note that the arrangement depicted in Fig. 3(c) is precisely Koester’s arrangement [11, 12].

The following proposition plays a central role in this section.

Proposition 1. *The corona extension of a Δ -saturated arrangement of pseudocircles with a pentagonal cell \diamond is 4-chromatic.*

The proof is based on the observation that after the corona extension the inequality $3\alpha < |V|$ holds.

By applying the corona extension to members of the infinite family of Δ -saturated arrangements with pentagonal cells (cf. Sect. 2), we obtain an infinite family of arrangements that are not 3-colorable.

Theorem 2. *There exists an infinite family of 4-chromatic arrangements of pseudocircles.*

Koester [12] defines a related construction which he calls *crowning* and constructs his example by two-fold crowning of a graph on 10 vertices. He also uses crowning to generate an infinite family of 4-edge-critical 4-regular graphs. In the full version of our paper, we present sufficient conditions to obtain a 4-vertex-critical arrangement via the corona extension. We conclude this section with the following conjecture:

Conjecture 2. There exists an infinite family of arrangement graphs of arrangements of pseudocircles that are 4-vertex-critical.

4 Fractional Colorings

In this section, we investigate fractional colorings of arrangements. A *b-fold coloring* of a graph G with m colors is an assignment of a set of b colors from $\{1, \dots, m\}$ to each vertex of G such that the color sets of any two adjacent vertices are disjoint. The *b-fold chromatic number* $\chi_b(G)$ is the minimum m such that G admits a b -fold coloring with m colors. The *fractional chromatic number* of G is $\chi_f(G) := \lim_{b \rightarrow \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}$. With α being the independence number and ω being the clique number, it holds that:

$$\max \left\{ \frac{|V|}{\alpha(G)}, \omega(G) \right\} \leq \chi_f(G) \leq \frac{\chi_b(G)}{b} \leq \chi(G). \tag{1}$$

Theorem 3. *Let G be the arrangement graph of an intersecting arrangement \mathcal{A} of n pseudocircles, then $\chi_f(G) \leq 3 + \frac{6}{n-2}$.*

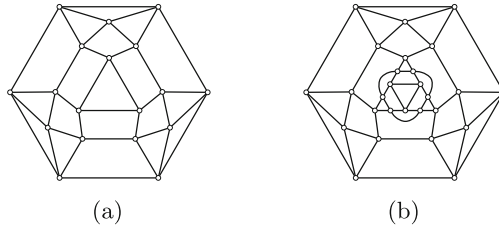


Fig. 4. (a) A 4-edge-critical 4-regular 18-vertex planar graph with $\chi = 4$ and $\chi_f = 3$. and (b) the crowning extension at its center triangular face.

Proof (Sketch of the proof). Let C be a pseudocircle of \mathcal{A} . After removing all vertices along C from the arrangement graph G we obtain a graph which has two connected components A (vertices in the interior of C) and B (vertices in the exterior). Let C' be a small circle contained in one of the faces of A , the Sweeping Lemma of Snoeyink and Hershberger [16] asserts that there is a continuous transformation of C' into C which traverses each vertex of A precisely once. In particular, when a vertex is traversed, at most two of its neighbors have been traversed before. Hence, we obtain a 3-coloring of the vertices of A by greedily coloring vertices in the order in which they occur during the sweep. An analogous argument applies to B . Taking such a partial 3-coloring of G for each of the n pseudocircles of \mathcal{A} , we obtain for each vertex a set of $n - 2$ colors, i.e., an $(n - 2)$ -fold coloring of G . The total number of colors used is $3n$. The statement now follows from inequality (1).

5 Fractionally 3-Colorable 4-Edge-Critical Planar Graphs

From our computational data (cf. [5]), we observed that some of the arrangements such as the 20 vertex graph depicted in Fig. 2 have $\chi = 4$ and $\chi_f = 3$, and therefore disprove Conjecture 3.2 by Gimbel et al. [7]. Moreover, we determined that there are precisely 17 4-regular 18-vertex planar graphs with $\chi = 4$ and $\chi_f = 3$, which are minimal in the sense that there are no 4-regular graphs on $n \leq 17$ vertices with $\chi = 4$ and $\chi_f = 3$. Each of these 17 graphs is 4-vertex-critical and the one depicted in Fig. 4(a) is even 4-edge-critical.

Starting with a triangular face in the 4-edge-critical 4-regular graph depicted in Fig. 4(a) and repeatedly applying the Koester's crowning operation [12] as illustrated in Fig. 4(b) (which by definition preserves the existence of a facial triangle), we deduce the following theorem.

Theorem 4. *There exists an infinite family of 4-edge-critical 4-regular planar graphs G with fractional chromatic number $\chi_f(G) = 3$.*

6 Discussion

With Theorem 1 we confirmed Conjecture 1 for Δ -saturated great-pseudocircle arrangements. While this is a very small subclass of great-pseudocircle arrange-

ments, it is reasonable to think of it as a “hard” class for 3-coloring. The rationale for such thoughts is that triangles restrict the freedom of extending partial colorings. Our computational data indicates that sufficiently large intersecting pseudocircle arrangements that are *diamond-free*, i.e., no two triangles of the arrangement share an edge, are also 3-colorable. Computations also suggest that sufficiently large great-pseudocircle arrangements have *antipodal colorings*, i.e., 3-colorings where antipodal points have the same color. Based on the experimental data we propose the following strengthened variants of Conjecture 1.

Conjecture 3. The following three statements hold.

- (a) Every diamond-free intersecting arrangement of $n \geq 6$ pseudocircles is 3-colorable.
- (b) Every intersecting arrangement of sufficiently many pseudocircles is 3-colorable.
- (c) Every arrangement of $n \geq 7$ great-pseudocircles has an antipodal 3-coloring.

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A Short Proof of Euler–Poincaré Formula

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Abstract. “ $V - E + F = 2$ ”, the famous Euler’s polyhedral formula, has a natural generalization to convex polytopes in every finite dimension, also known as the Euler–Poincaré Formula. We provide another short inductive combinatorial proof of the general formula. Our proof is self-contained and it does not use shellability of polytopes.

Keywords: Euler–Poincaré formula · Polytopes · Discharging

1 Introduction

In this paper we follow the standard terminology of polytopes theory, such as Ziegler [7]. We consider *convex polytopes*, defined as a convex hull of finitely many points, in the d -dimensional Euclidean space for an arbitrary $d \in \mathbb{N}$, $d \geq 1$. We shortly say a polytope to mean a convex polytope. A landmark discovery in the history of combinatorial investigation of polytopes was famous Euler’s formula, stating that for any 3-dimensional polytope with v vertices, e edges and f faces, $v - e + f = 2$ holds. This finding was later generalized, in every dimension d , to what is nowadays known as (generalized) Euler’s relation or Euler–Poincaré formula, as follows.

For instance, in dimension $d = 1$ we have $v = 2$, which can be rewritten as $v - 1 = 1$, and in dimension $d = 2$ we have got $v - e = 0$ or $v - e + 1 = 1$. Similarly, the $d = 3$ case can be rewritten as $v - e + f - 1 = 1$. Note that the ‘1’ left of ‘=’ stands in these expressions for the polytope itself. In general, the following holds:

Theorem 1 (“Euler–Poincaré formula”; Schläfli [5] 1852). *Let P be a convex polytope in \mathbb{R}^d , and denote by f^c , $c \in \{0, 1, \dots, d\}$, the numbers of faces of P of dimension c . Then*

$$f^0 - f^1 + f^2 - \dots + (-1)^d f^d = 1. \quad (1)$$

We refer to classical textbooks of Grünbaum [3] and Ziegler [7] for a closer discussion of the interesting history of this formula and of the difficulties associated with its proof. Here we just briefly remark that all the historical attempts

Supported by research grant GAČR 20-04567S of the Czech Science Foundation.

to prove the formula in a combinatorial way, starting from Schläfli, implicitly assumed validity of a special property called *shellability* of a polytope. However, the shellability of any polytope was formally established only in 1971 by Bruggesser and Mani [1].

We provide a new short and self-contained inductive combinatorial proof of (1) which does not assume shellability of polytopes.

2 New Combinatorial Proof

Our proof of Theorem 1 proceeds by induction on the dimension $d \geq 1$. Note that validity of (1) is trivial for $d = 1, 2$, and hence it is enough to show the following:

Lemma 1. *Let $k \geq 2$ and P be a polytope of dimension $k + 1$. Assume that (1) holds for any polytope of dimension $d \in \{k - 1, k\}$. Then (1) holds for P (with $d = k + 1$).*

Proof. Recall that f^c , $c \in \{0, 1, \dots, k + 1\}$, denote the numbers of faces of P of dimension c . The only (improper) face of dimension $k + 1$ is P itself, and the faces of dimension k are the *facets* of P . Our goal is to prove

$$f^0 - f^1 + f^2 - \dots + (-1)^{k-1} f^{k-1} + (-1)^k f^k + (-1)^{k+1} f^{k+1} = 1,$$

or equivalently, since $f^{k+1} = 1$,

$$f^0 - f^1 + f^2 - \dots + (-1)^{k-1} f^{k-1} = 1 + (-1)^k (1 - f^k). \tag{2}$$

We choose arbitrary two facets T_1, T_2 of P (distinct, but not necessarily disjoint) and two points $t_1 \in T_1$ and $t_2 \in T_2$ in their relative interior, such that the straight line $q = \overline{t_1 t_2}$ passing through t_1, t_2 is in a general position with respect to P . In particular, we demand that *no* nontrivial line segment lying in a face of P of dimension $c \leq k - 1$ is coplanar with q . We also denote by T_3, \dots, T_{f^k} the remaining facets of P , in any order.

In the proof we use a *discharging* argument, an advanced variant of the double-counting method in combinatorics. To every face F of P of dimension $0 \leq c \leq k - 1$, we assign *charge* of value $(-1)^c$ (the facets start with no charge). Hence the *total charge* initially assigned to all faces of P equals the left-hand side of (2).

Now we *discharge* all the assigned charge from those faces to the facets of P (which initially have no charge). The discharging rule is only one and very simple. Consider a facet T_i of P , $1 \leq i \leq f^k$. Let $t_i \in q$ denote the unique point which is the intersection of the line q with the support hyperplane of T_i . This is a sound definition of t_i according to a general position of q , and it is consistent with the choice of t_1, t_2 above. Consider further any proper face F of T_i (so F is a face of P as well and is of dimension $0 \leq c \leq k - 1$), and choose a fixed

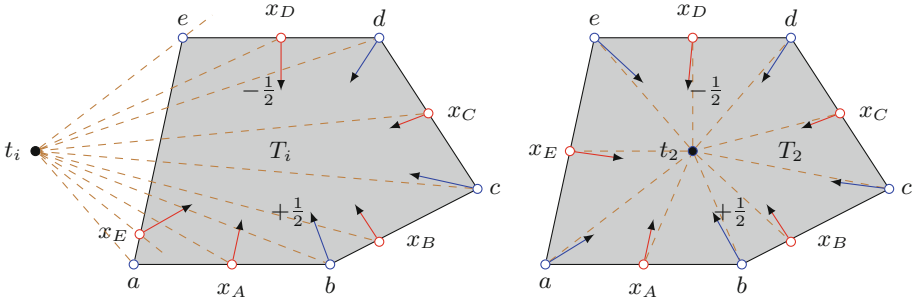


Fig. 1. Proof of Lemma 1: a facet in a 3-dimensional polytope P ($k = 2$). Each vertex of P initially gets charge of 1 and each edge -1 . Consider, e.g., a facet T_i of P which is a pentagon with vertices a, b, c, d, e and sides (edges) A, B, C, D, E in order. Let t_i be the point in which the plane of T_i intersects the line q (see in the proof). On the left of the picture ($t_i \notin T_i$, for $i \geq 3$), we have that the vertices b, c, d send charge of $\frac{1}{2}$ to T_i by the rule (3), while a, e are not sending to T_i . On the right ($t_i \in T_i$, $i = 1, 2$), all the vertices a, b, c, d, e send charge of $\frac{1}{2}$ to T_i . In both cases, every side A, B, C, D, E sends charge of $-\frac{1}{2}$ to T_i . Consequently, on the left T_i ends up with charge -1 (compare to (5)), while on the right with charge 0 (cf. (4)).

point x_F in the relative interior of F (note that $x_v = v$ if v is a vertex of P). Our discharging rule reads (see in Fig. 1):

The face F sends half of its initial charge, i.e. $\frac{1}{2}(-1)^c$, to the facet T_i (3) if, and only if, the straight line passing through x_F and t_i intersects the relative interior of T_i .

Note that we will be finished if we prove that, after applying the discharging rule, (i) every face of P of dimension $\leq k - 1$ ends up with charge 0, and (ii) the total charge of the facets of P sums up to the right-hand side of (2).

For the task (i), consider any face F of P of dimension $c \leq k - 1$ and the point x_F chosen in F above. Let L denote the plane determined by the line $q = \overline{t_1 t_2}$ and the point $x_F \notin q$. Then $N := P \cap L$ is a convex polygon. See Fig. 2. We claim that x_F must be a vertex of N : indeed, if x_F belonged to a relative interior of a side A_0 of N , then $A_0 \subseteq F$ and A_0 would be coplanar with q , contradicting our assumption of a general position of q . Consequently, x_F is incident to two sides A_1, A_2 of N , and there exist facets T_{i_1}, T_{i_2} of P , $1 \leq i_1 \neq i_2 \leq f^k$, such that $A_j = T_{i_j} \cap L$ for $j = 1, 2$. Observe that the support line of A_j intersects q precisely in t_{i_j} (which has been defined as the intersection of the support hyperplane of T_{i_j} with q).

Moreover, since A_j is coplanar with q , by our assumption of a general position of q it cannot happen that A_j is contained in a face of dimension $\leq k - 1$. Consequently, A_j (except its ends) belongs to the relative interior of T_{i_j} , and T_{i_j} is a unique such face for A_j . Hence, taking this argument for $j = 1, 2$, we see that F sends away by (3) exactly two halves of its initial charge, ending up with charge 0.

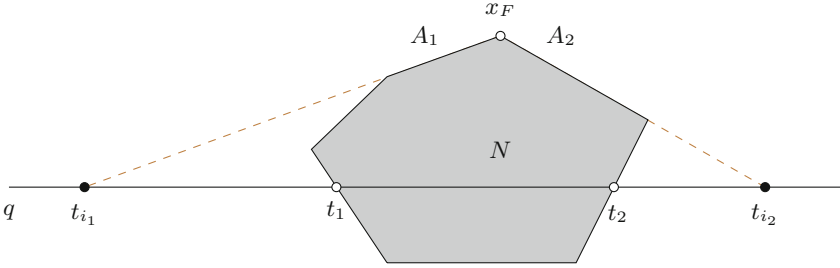


Fig. 2. Proof of Lemma 1: A polygon N which is the intersection of the polytope P with the plane spanning q and a point x_F of a face F . The two sides A_1, A_2 of N incident to x_F determine the two unique facets T_{i_1}, T_{i_2} of P that F sends charge to.

For the task (ii), let f_i^c , where $c \in \{0, 1, \dots, k\}$ and $i \in \{1, \dots, f^k\}$, denote the number of faces of T_i of dimension c . We first look at the two special facets T_i , $i = 1, 2$ (Fig. 1 right). Since $t_i \in T_i$ in this case, by (3) T_i receives charge from every of its proper faces. Using (1) for T_i , which is of dimension k , we thus get that the total charge T_i ends up with, is

$$\frac{1}{2} (f_i^0 - f_i^1 + \dots + (-1)^{k-1} f_i^{k-1}) = \frac{1}{2} (1 - (-1)^k f_i^k) = \frac{1}{2} (1 - (-1)^k). \quad (4)$$

Second, consider a facet T_i where $i \geq 3$. Let H_i be the support hyperplane of T_i . Then $\{t_i\} = H_i \cap q$ and $t_i \notin T_i$. We restrict ourselves to the affine space formed by H_i , and denote by S_i a projection of T_i from the point t_i onto a suitable hyperplane within H_i . Since t_i is in a general position with respect to T_i (which is implied by a general position of q), the following holds: every proper face of S_i is the image of an equivalent face of T_i (of the same dimension!). Furthermore, by convexity, a face F of T_i has no image among the faces of S_i if, and only if, the line through x_F and t_i intersects the relative interior of T_i . See also Fig. 1 left.

Consequently, as directed by (3), T_i receives charge precisely from those of its faces F which do not have an image among the proper faces of S_i (in particular, T_i receives charge from all of its faces of dimension $k - 1$). Denote by g_i^c the number of faces of S_i of dimension $c \leq k - 1$, and notice that $f_i^k = g_i^{k-1} = 1$. Hence, precisely, T_i receives $\frac{1}{2}(-1)^{k-1}$ of charge from each of its f_i^{k-1} faces of dimension $k - 1$, and $\frac{1}{2}(-1)^c$ from $f_i^c - g_i^c$ of its faces of dimension $0 \leq c \leq k - 2$. Summing together, and using (1) for T_i (of dimension k) and for S_i (of dimension $k - 1$), we get

$$\begin{aligned} \frac{1}{2}(-1)^{k-1} f_i^{k-1} + \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c (f_i^c - g_i^c) &= \frac{1}{2} \sum_{c=0}^{k-1} (-1)^c f_i^c - \frac{1}{2} \sum_{c=0}^{k-2} (-1)^c g_i^c \\ &= \frac{1}{2} (1 - (-1)^k f_i^k) - \frac{1}{2} (1 - (-1)^{k-1} g_i^{k-1}) = -(-1)^k. \end{aligned} \quad (5)$$

Since the total charge is not changed (only redistributed), we get that (the left-hand side of) (2) must equal the sum of (4) over $i = 1, 2$ and of (5) over $i = 3, \dots, f^k$, leading to

$$\begin{aligned} f^0 - f^1 + f^2 - \dots + (-1)^{k-1} f^{k-1} &= 2 \cdot \frac{1}{2} (1 - (-1)^k) - (f^k - 2) \cdot (-1)^k \\ &= 1 + (-1)^k (1 - f^k), \end{aligned}$$

and thus finishing the proof of (2) for P .

3 Final Remarks

We have shown a full proof of the Euler–Poincaré formula (1) with only simple, combinatorial and elementary geometric arguments. Our proof has been in parts inspired by a proof of basic Euler’s formula via angles [2, “Proof 8: Sum of Angles”], and by Welzl’s probabilistic proof [6] of Gram’s equation. Although, the resulting exposition of the proof does not resemble either of those; in fact, it looks like a generalization of the basic discharging proof [2, “Proof 6: Electrical Charge”], but that was not our way to the result. Lastly, we remark that the underlying idea of our proof can be expressed also in an alternative, more geometric way, such as the exposition in the preprint [4].

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Approximate MDS Property of Linear Codes

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Abstract. In this paper, we study the weight spectrum of linear codes with *super-linear* field size and use the probabilistic method to show that for nearly all such codes, the corresponding weight spectrum is very close to that of a maximum distance separable (MDS) code.

Keywords: Linear codes · Super-linear field size · Approximate MDS property

1 Introduction

MDS codes have the largest possible minimum distance since they meet the Singleton bound with equality (Huffman and Pless [1]) and many properties of the weight spectrum of MDS codes are known. For example, the weight spectrum of an MDS code is unique (Tolhuizen [2], MacWilliams and Sloane [3]) and any MDS code with length n and dimension k has precisely k distinct non-zero weights $n, n - 1, \dots, n - k + 1$ (Ezerman et al. [4]). In this paper, we study the weight spectrum of linear codes that are not necessarily MDS but are equipped with a field size that grows super-linear in the code length. We use the probabilistic method and weight concentration properties to show that such codes closely resemble MDS codes in terms of the weight spectrum. The paper is organized as follows: In the next Sect. 2, we state and prove our main result regarding the approximate MDS property of linear codes with super-linear field size.

2 Approximate MDS Property of Linear Codes

Let q be a power of a prime number and let \mathbb{F}_q be the finite field containing q elements. For integers $n \geq k \geq 1$, a subset $\mathcal{C} \subset \mathbb{F}_q^n$ of cardinality q^k is defined to be an $(n, k)_q$ -code. A vector subspace of \mathbb{F}_q^n of dimension k is defined to be a *linear* code and is also said to be an $[n, k]_q$ -code. Elements of \mathcal{C} are called codewords or simply words.

For two words $\mathbf{c} = (c_1, \dots, c_n)$ and $\mathbf{d} = (d_1, \dots, d_n)$ in \mathbb{F}_q^n , we define the *Hamming distance* between \mathbf{c} and \mathbf{d} to be $d_H(\mathbf{c}, \mathbf{d}) = \sum_{i=1}^n \mathbb{1}(c_i \neq d_i)$, where $\mathbb{1}(\cdot)$ refers to the indicator function. The Hamming weight of \mathbf{c} is the number of non-zero entries in \mathbf{c} . All distances and weights in this paper are Hamming and so we

suppress the term Hamming throughout. We define the minimum distance $d_H(\mathcal{C})$ of the code \mathcal{C} to be the minimum distance between any two codewords of \mathcal{C} .

From the Singleton bound, we know that $d_H(\mathcal{C}) \leq n - k + 1$ and if $q \geq n - 1$ is a power of prime, there are $[n, k]_q$ -codes that achieve the Singleton bound. Such codes are called maximum distance separable (MDS) codes (pp. 71, (Huffman and Pless [1]) and the MDS conjecture asserts that $q = n - 1$ is essentially the minimum required field size to construct MDS codes (see for example (Alderson [5]), for a precise formulation).

In our main result of this paper, we show that nearly all linear codes with super-linear field size behave approximately like MDS codes. We begin with a couple of definitions. Let \mathcal{C} be a linear $[n, k]_q$ -code and suppose for $1 \leq w \leq n$, the code \mathcal{C} contains A_w codewords of weight w . We define the n -tuple (A_1, \dots, A_n) to be the *weight spectrum* of \mathcal{C} . The weight spectrum of an $[n, k]_q$ -MDS code \mathcal{C} with $n \leq q$ is as follows (Theorem 6, pp. 320–321, MacWilliams and Sloane [3]):

- (p1) For $1 \leq w \leq n - k$, the number of codewords of weight w is $\lambda_w := 0$.
- (p2) For each $D := n - k + 1 \leq w \leq n$, the number of codewords of weight w equals

$$\lambda_w := \binom{n}{w} (q - 1) \sum_{j=0}^{w-D} (-1)^j \binom{w-1}{j} q^{w-D-j}. \tag{2.1}$$

It is well-known (Ezerman et al. [4]) that if $n \leq q$ then $\lambda_w > 0$ for each $n - k + 1 \leq w \leq n$.

The following result shows that nearly all linear codes with super-linear field size have a weight spectra closely resembling that of an MDS code.

Theorem 1. *For integer $n \geq 4$ let $k = k(n)$ be an integer and $q = q(n)$ be a power of a prime number satisfying*

$$\frac{1}{\sqrt{\log n}} \leq \frac{k(n)}{n} \leq 1 - \frac{1}{\sqrt{\log n}} \text{ and } \frac{q(n)}{n} \rightarrow \infty \tag{2.2}$$

as $n \rightarrow \infty$. Let \mathcal{P} be the set of all $[n, k]_q$ -codes and let $\mathcal{Q} \subseteq \mathcal{P}$ be the set of all codes satisfying the following properties:

- (a1) *There exists no word of weight w for any $1 \leq w \leq n - k - \frac{2n}{\log q}$.*
- (a2) *For each $n - k + 5 \leq w \leq n$, the number of codewords of weight w equals lies between $\lambda_w \left(1 - \frac{3n}{q}\right)$ and $\lambda_w \left(1 + \frac{3n}{q}\right)$.*

For all n large we have that

$$\#\mathcal{Q} \geq \#\mathcal{P} \cdot \left(1 - \frac{18q}{n^2}\right). \tag{2.3}$$

From (2.2) we have that $\frac{n}{\log q} = o(k)$, $\frac{n}{q} = o(1)$ and so in addition if we have that $\frac{q}{n^2} = o(1)$, then comparing with (a1) – (a2), we see that nearly all linear codes behave approximately like an MDS code.

In the following subsection, we derive a couple of preliminary estimates used in the proof of Theorem 1 and in the next subsection, we prove Theorem 1.

Preliminary Estimates

We use the probabilistic method to prove Theorem 1. Let \mathbf{G} be a random $k \times n$ matrix with entries i.i.d. uniform in \mathbb{F}_q . We prove Theorem 1 by estimating the weights of the words generated by the code $\mathcal{G} := \{\mathbf{x} \cdot \mathbf{G}\}_{\mathbf{x} \in \mathbb{F}_q^k}$. All vectors throughout are row vectors.

We collect auxiliary results used in the proof of Theorem 1, in the following Lemma. For $1 \leq w \leq n$ let \mathcal{C}_w be the set of all words in \mathbb{F}_q^n with weight w . The following result estimates the number of words of a given weight present in a linear code.

Lemma 1. *We have:*

(a) For $1 \leq w \leq n$ let N_w be the set of words of the random code \mathcal{G} present in \mathcal{C}_w . We have that the mean and the variance satisfy

$$\mu_w := \mathbb{E}N_w = \binom{n}{w} \cdot (q-1)^w \cdot \frac{q^k - 1}{q^n} \text{ and } \text{var}(N_w) \leq (2q+1) \cdot \mu_w, \quad (2.4)$$

respectively.

(b) Let λ_w and μ_w be as in (2.1) and (2.4), respectively. If $q \geq n$ then for each $n - k + 1 \leq w \leq n$ we have that

$$\left(1 - \frac{1}{q}\right) \cdot \left(1 - \frac{w-1}{q}\right) \leq \frac{\lambda_w}{\mu_w} \leq \frac{1}{1 - \frac{w}{q}}. \quad (2.5)$$

From (2.4) in part (a), we get the intuitive result that the number of words of weight w in a linear code is concentrated around its mean. From part (b) we see that if q is much larger than n , then $\frac{w}{q} = o(1)$ and so λ_w is approximately equal to μ_w .

Proof of Lemma 1(a): We first obtain the expression for μ_w . For any fixed non-zero vector $\mathbf{x} \in \mathbb{F}_q^k$, the random vector $\mathbf{x} \cdot \mathbf{G}$ is uniform in \mathbb{F}_q^n and so for any vector $\mathbf{y} \in \mathcal{C}_w$ we have that $\mathbb{P}(\mathbf{x} \cdot \mathbf{G} = \mathbf{y}) = \frac{1}{q^n}$. The relation for μ_w in (2.4) then follows from the fact that the number of words of weight w equals $\#\mathcal{C}_w = \binom{n}{w} \cdot (q-1)^w$ and the fact that there are $q^k - 1$ non-zero vectors in \mathbb{F}_q^k . To estimate the variance of N_w we write $N_w = \sum_{\mathbf{x} \in \mathbb{F}_q^k \setminus \{0\}} \mathbb{1}(A(\mathbf{x}))$ where $A(\mathbf{x})$ is the event that the vector $\mathbf{x} \cdot \mathbf{G} \in \mathcal{C}_w$. We then get that

$$\text{var}(N_w) = \sum_{\mathbf{x}} \Delta(\mathbf{x}) + \sum_{\mathbf{x}_1 \neq \mathbf{x}_2} \beta(\mathbf{x}_1, \mathbf{x}_2) \quad (2.6)$$

where $0 \leq \Delta(\mathbf{x}) := \mathbb{P}(A(\mathbf{x})) - \mathbb{P}^2(A(\mathbf{x})) \leq \mathbb{P}(A(\mathbf{x}))$ and

$$\begin{aligned} |\beta(\mathbf{x}_1, \mathbf{x}_2)| &:= \left| \mathbb{P}\left(A(\mathbf{x}_1) \cap A(\mathbf{x}_2)\right) - \mathbb{P}(A(\mathbf{x}_1))\mathbb{P}(A(\mathbf{x}_2)) \right| \\ &\leq \mathbb{P}\left(A(\mathbf{x}_1) \cap A(\mathbf{x}_2)\right) + \mathbb{P}(A(\mathbf{x}_1))\mathbb{P}(A(\mathbf{x}_2)) \leq 2\mathbb{P}(A(\mathbf{x}_1)). \end{aligned} \quad (2.7)$$

It is well-known that if \mathbf{x}_1 is not a multiple of \mathbf{x}_2 then the events $A(\mathbf{x}_1)$ and $A(\mathbf{x}_2)$ are independent (see for example, Chapter 7, Problem P.7.18, pp.

175, (Zamir [6]). Therefore for each \mathbf{x}_1 there are at most q values of \mathbf{x}_2 for which $\beta(\mathbf{x}_1, \mathbf{x}_2) \neq 0$. Thus $\text{var}(N_w) \leq \sum_{\mathbf{x}} \mathbb{P}(A(\mathbf{x})) + 2q \sum_{\mathbf{x}_1} \mathbb{P}(A(\mathbf{x}_1)) = (2q + 1)\mu_w$ and this proves the variance estimate in (2.4). ■

Proof of Lemma 1(b): We begin by showing that if $n - k + 1 \leq w \leq n$ then

$$\left(1 - \frac{1}{q}\right) \cdot \left(1 - \frac{w-1}{q}\right) \leq \frac{\lambda_w}{\binom{\binom{n}{w} \cdot q^w}{q^{n-k}}} \leq 1 - \frac{1}{q}. \quad (2.8)$$

To prove the upper bound in (2.8) we write $\lambda_w = \binom{\binom{n}{w} \cdot q^w}{q^{n-k}} \cdot \left(1 - \frac{1}{q}\right) \cdot \theta(w)$ where $\theta(w) := \sum_{j=0}^{w-D} (-1)^j \binom{w-1}{j} \cdot \frac{1}{q^j}$. Expanding $\theta(w)$ and regrouping we get $\theta(w) = 1 - (t_1(w) + t_3(w) + t_5(w) + \dots)$ where

$$t_j(w) := \binom{w-1}{j} \cdot \frac{1}{q^j} - \binom{w-1}{j+1} \cdot \frac{1}{q^{j+1}} \quad (2.9)$$

for all j if $w - D$ is odd. If $w - D$ is even, then an analogous expansion holds with the distinction that the final $t_j(w)$ term is simply $\binom{w-1}{j} \cdot \frac{1}{q^j}$. For simplicity we assume below that $w - D$ is odd and get $t_j(w) = \binom{w-1}{j} \cdot \frac{1}{q^j} \left(1 - \frac{r_j(w)}{q}\right)$ where $r_j(w) := \binom{w-1}{j+1} = \frac{w}{j+1} - 1$. Thus $|r_j(w)| \leq n$ and since $q \geq n$ we get that $t_j(w) \geq 0$. This implies that $\theta(w) \leq 1$ and so we get the upper bound in (2.8).

For the lower bound in (2.8) we write $\theta(w) = 1 - \frac{w-1}{q} + t_2(w) + t_4(w) + \dots$ and use $t_j(w) \geq 0$ to get that $\theta(w) \geq 1 - \frac{w-1}{q}$. This proves the lower bound in (2.8). Finally to prove (2.5), we write $\mu_w = \frac{\binom{n}{w} q^w}{q^{n-k}} \cdot \left(1 - \frac{1}{q}\right)^w \cdot \left(1 - \frac{1}{q^k}\right)$ and use $\left(1 - \frac{1}{q}\right)^w \geq 1 - \frac{w}{q}$ and $1 - \frac{1}{q^k} \geq 1 - \frac{1}{q}$ to get that

$$\left(1 - \frac{1}{q}\right) \cdot \left(1 - \frac{w}{q}\right) \leq \frac{\mu_w}{\frac{\binom{n}{w} q^w}{q^{n-k}}} \leq 1. \quad (2.10)$$

Together with (2.8) we then get (2.5). ■

Proof of Theorem 1

We first estimate the probability of occurrence of property (a1). Recalling that $\mu_w = \mathbb{E}N_w$ is the expected number of words of weight w in the random code \mathcal{G} (see Lemma 1) and using Stirling's approximation we have that

$$\mu_w \leq \frac{\binom{n}{w} q^w}{q^{n-k}} \leq 4en \cdot q^{\frac{nH(\frac{w}{n})}{\log q}} \cdot \frac{q^w}{q^{n-k}} \leq 4en \cdot q^{\frac{n}{\log q}} \cdot \frac{q^w}{q^{n-k}}. \quad (2.11)$$

Setting $w_{low} := n - k - \frac{2n}{\log q}$, we see for all $1 \leq w \leq w_{low}$ that $\mu_w = \mathbb{E}N_w \leq \frac{4en}{q^{\frac{n}{\log q}}} = 4en \cdot e^{-n}$. Therefore if F_{low} is the event that the property (a1) in the statement of the Theorem holds, then we get by the union bound that

$$\mathbb{P}(F_{low}^c) = \mathbb{P}\left(\bigcup_{1 \leq w \leq w_{low}} \{N_w \geq 1\}\right) \leq \sum_{w=1}^{w_{low}} \mathbb{E}N_w \leq w_{low} \cdot 4en \cdot e^{-n} \leq 4en^2 \cdot e^{-n} \quad (2.12)$$

since $w_{low} \leq n$.

Next we study property (a2) for weights $w \geq w_{up} = n - k + 5$. First we show that

$$\mu_w \geq \frac{\binom{n}{w} q^w}{q^{n-k}} \cdot \left(1 - \frac{w}{q}\right) \cdot \left(1 - \frac{1}{q^k}\right) \geq \frac{n^5}{4} \quad (2.13)$$

for all n large. The first bound in (2.13) follows from (2.10). For $w \geq n - k + 5$ we have that $\frac{\binom{n}{w} q^w}{q^{n-k}} \geq \binom{n}{w} \cdot q^5 \geq n^5$ since $q \geq n$ for all n large, by (2.2). Also from (2.2) we see that $1 - \frac{w}{q} \geq 1 - \frac{n}{q} \geq \frac{1}{2}$ and $1 - \frac{1}{q^k} \geq \frac{1}{2}$ for all n large. This proves the final bound in (2.13).

From Chebychev's inequality, the variance estimate in (2.4) and the above estimate (2.13), we therefore get that

$$\mathbb{P}\left(|N_w - \mu_w| \geq \frac{\mu_w}{n}\right) \leq \frac{n^2 \text{var}(N_w)}{\mu_w^2} \leq \frac{n^2(2q+1)}{\mu_w} \leq \frac{4(2q+1)}{n^3}. \quad (2.14)$$

If $\mu_w(1 - \frac{1}{n}) \leq N_w \leq \mu_w(1 + \frac{1}{n})$, then using the bounds (2.5), we see that

$$\lambda_w \left(1 - \frac{w}{q}\right) \cdot \left(1 - \frac{1}{n}\right) \leq N_w \leq \lambda_w \frac{\left(1 + \frac{1}{n}\right)}{\left(1 - \frac{1}{q}\right) \cdot \left(1 - \frac{w-1}{q}\right)}$$

and using (2.2) and the fact that $w \leq n$ we get that

$$N_w \leq \lambda_w \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{2}{q}\right) \cdot \left(1 + \frac{2n}{q}\right) \leq \lambda_w \left(1 + \frac{3n}{q}\right)$$

for all n large. Similarly we also get that $N_w \geq \lambda_w \left(1 - \frac{3n}{q}\right)$ for all n large.

Therefore if F_{up} denotes the event that property (a2) in the statement of the Theorem holds, then from (2.14) and the union bound we get that $\mathbb{P}(F_{up}^c)$ is bounded above by $\sum_{w=w_{up}}^n \mathbb{P}\left(|N_w - \mu_w| \geq \frac{\mu_w}{n}\right) \leq \frac{4(2q+1)}{n^2}$. Thus $\mathbb{P}(F_{up}) \geq 1 - \frac{4(2q+1)}{n^2}$ and combining this with (2.12) we get that

$$\mathbb{P}(F_{up} \cap F_{low}) \geq 1 - 4en^2 \cdot e^{-n} - \frac{4(2q+1)}{n^2} \geq 1 - \frac{9q}{n^2} \quad (2.15)$$

for all n large using the fact that $q \geq n$ (see statement of Theorem). If F_{full} denotes the event that the matrix \mathbf{G} has full rank then we show below that

$$\mathbb{P}(F_{full}) \geq 1 - \frac{2}{q^{n-k}} \geq \frac{1}{2} \quad (2.16)$$

since $q \geq 2$ and $n - k \geq 2$ (see statement of Theorem).

The ratio of the sets \mathcal{Q} and \mathcal{P} defined in the statement of the Theorem is therefore simply

$$\frac{\#\mathcal{Q}}{\#\mathcal{P}} = \frac{\mathbb{P}(F_{full} \cap F_{up} \cap F_{low})}{\mathbb{P}(F_{full})} \geq 1 - \frac{\mathbb{P}(F_{up}^c \cup F_{low}^c)}{\mathbb{P}(F_{full})} \quad (2.17)$$

using $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) - \mathbb{P}(B^c)$ with $A = F_{full}$ and $B = F_{up} \cap F_{low}$. Plugging (2.15) and (2.16) into (2.17) we get that $\frac{\#\mathcal{Q}}{\#\mathcal{P}} \geq 1 - \frac{18q}{n^2}$ and this proves (2.3).

It remains to prove (2.16). Let $\mathbf{V}_i, 1 \leq i \leq k$ be the independent and identically distributed (i.i.d.) vectors chosen uniformly randomly from \mathbb{F}_q^n that form the rows of the matrix \mathbf{G} . For $1 \leq i \leq k$ let E_i be the event that the vectors $\mathbf{V}_j, 1 \leq j \leq i$ are linearly independent so that $\mathbb{P}(E_1) = 1$. For $i \geq 2$, we note that the event $E_i = \bigcap_{1 \leq j \leq i} E_j$ and write

$$\mathbb{P}(E_i) = \mathbb{P}\left(\bigcap_{1 \leq j \leq i} E_j\right) = \mathbb{E}(\mathbb{1}(E_{i-1}) \cdot \mathbb{P}(E_i | \mathbf{V}_j, 1 \leq j \leq i-1)). \quad (2.18)$$

If E_{i-1} occurs, the size of the space spanned by the vectors $\mathbf{V}_j, 1 \leq j \leq i-1$ is q^{i-1} and so the event E_i occurs if and only if we choose \mathbf{V}_i from amongst the remaining $q^n - q^{i-1}$ vectors. Therefore from (2.18) we get that $\mathbb{P}(E_i) = \left(\frac{q^n - q^{i-1}}{q^n}\right) \mathbb{P}(E_{i-1})$ and continuing iteratively, we get that

$$\mathbb{P}(E_k) = \prod_{j=1}^{k-1} \left(1 - \frac{q^j}{q^n}\right) \geq 1 - \frac{1}{q^n} \sum_{j=1}^{k-1} q^j = 1 - \frac{q^k - 1}{q^n(q-1)} \geq 1 - \frac{2}{q^{n-k}}, \quad (2.19)$$

using the fact that $q \geq 2$ and so $\frac{q^k - 1}{q^n(q-1)} \leq \frac{q^k}{q^n(q-1)} \leq \frac{2}{q^{n-k}}$. This proves (2.16). ■

Acknowledgement. I thank Professors V. Guruswami, C. R. Subramanian and the referees for crucial comments that led to an improvement of the paper. I also thank IMSc for my fellowships.

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A SAT Attack on Higher Dimensional Erdős–Szekeres Numbers

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Abstract. A famous result by Erdős and Szekeres (1935) asserts that, for every $k, d \in \mathbb{N}$, there is a smallest integer $n = g^{(d)}(k)$, such that every set of at least n points in \mathbb{R}^d in general position contains a k -gon, i.e., a subset of k points which is in convex position. We present a SAT model for higher dimensional point sets which is based on chirotopes, and use modern SAT solvers to investigate Erdős–Szekeres numbers in dimensions $d = 3, 4, 5$. We show $g^{(3)}(7) \leq 13$, $g^{(4)}(8) \leq 13$, and $g^{(5)}(9) \leq 13$, which are the first improvements for decades. For the setting of k -holes (i.e., k -gons with no other points in the convex hull), where $h^{(d)}(k)$ denotes the minimum number n such that every set of at least n points in \mathbb{R}^d in general position contains a k -hole, we show $h^{(3)}(7) \leq 14$, $h^{(4)}(8) \leq 13$, and $h^{(5)}(9) \leq 13$. Moreover, all obtained bounds are sharp in the setting of chirotopes and we conjecture them to be sharp also in the original setting of point sets.

Keywords: Erdős–Szekeres Theorem · Higher dimensional point set · Chirotope · Boolean satisfiability (SAT) · Computer-assisted proof

1 Introduction

The classical Erdős–Szekeres Theorem [7] asserts that every sufficiently large point set in the plane *in general position* (i.e., no three points on a common line) contains a k -gon (i.e., a subset of k points in convex position).

Theorem 1 ([7], The Erdős–Szekeres Theorem). *For every integer $k \geq 3$, there is a smallest integer $n = g^{(2)}(k)$ such that every set of at least n points in general position in the plane contains a k -gon.*

Erdős and Szekeres showed that $g^{(2)}(k) \leq \binom{2k-4}{k-2} + 1$ [7] and constructed point sets of size 2^{k-2} without k -gons [8], which they conjectured to be extremal. There were several improvements of the upper bound in the past decades, each of magnitude $4^{k-o(k)}$, and in 2016, Suk showed $g^{(2)}(k) \leq 2^{k+o(k)}$ [24]. Shortly after, Holmsen et al. [12] slightly improved the error term in the exponent. The

M. Scheucher was partially supported by the internal research funding “Post-Doc-Funding” from Technische Universität Berlin.

lower bound $g^{(2)}(k) \geq 2^{k-2} + 1$ is known to be sharp for $k \leq 6$. The value $g^{(2)}(4) = 5$ was determined by Klein in 1935, $g^{(2)}(5) = 9$ was determined by Makai (cf. [14]), and $g^{(2)}(6) = 17$ was shown by Szekeres and Peters [25] using heavy computer assistance. While their computer program uses thousands of CPU hours, we have developed a SAT framework [23] which allows to verify this result within only 2 CPU hours, and an independent verification of their result using SAT solvers was done by Marić [18].

1.1 Planar k -Holes

In the 1970's, Erdős [6] asked whether every sufficiently large point set contains a k -hole, that is, a k -gon with the additional property that no other point lies in its convex hull. In the same vein as $g^{(2)}(k)$, we denote by $h^{(2)}(k)$ the smallest integer such that every set of at least $h^{(2)}(k)$ points in general position in the plane contains a k -hole. This variant differs significantly from the original setting as there exist arbitrarily large point sets without 7-holes [13]. While Harborth [11] showed $h^{(2)}(5) = 10$, the existence of 6-holes remained open until 2006, when Gerken [10] and Nicolás [20] independently showed that sufficiently large point sets contain 6-holes. Today the best bounds are $30 \leq h^{(2)}(6) \leq 463$ [17, 21].

1.2 Higher Dimensions

The notions *general position* (no $d + 1$ points in a common hyperplane), *k -gon* (a set of k points in convex position), and *k -hole* (a k -gon with no other points in the convex hull) naturally generalize to higher dimensions, and so does the Erdős–Szekeres Theorem. We denote by $g^{(d)}(k)$ and $h^{(d)}(k)$ the minimum number of points in \mathbb{R}^d in general position that guarantee the existence of k -gon and k -hole, respectively. In contrast to the planar case, the asymptotic behavior of the higher dimensional Erdős–Szekeres numbers $g^{(d)}(k)$ remains unknown for dimension $d \geq 3$. While a dimension-reduction argument by Károlyi [15] combined with Suk's bound [24] shows

$$g^{(d)}(k) \leq g^{(d-1)}(k-1) + 1 \leq \dots \leq g^{(2)}(k-d+2) + d-2 \leq 2^{(k-d)+o(k-d)}$$

for $k \geq d \geq 3$, the currently best asymptotic lower bound is $g^{(d)}(k) = \Omega(c^{d-\sqrt{k}})$ with $c = c(d) > 1$ is witnessed by a construction by Károlyi and Valtr [16].

1.3 Higher Dimensional Holes

Since Valtr [27] gave a construction for any dimension d without $d^{d+o(d)}$ -holes, generalizing the idea of Horton [13], the central open problem about higher dimensional holes is to determine the largest value $k = H(d)$ such that every sufficiently large set in d -space contains a k -hole. Note that with this notation we have $H(2) = 6$ because $h^{(2)}(6) < \infty$ [10, 20] and $h^{(2)}(7) = \infty$ [13]. Very

recently Bukh et al. [5] presented a construction without 2^{7d} -holes, which further improves Valtr’s bound and shows $H(d) < 2^{7d}$. On the other hand, the dimension-reduction argument by Károlyi [15] also applies to k -holes, and hence

$$h^{(d)}(k) \leq h^{(d-1)}(k-1) + 1 \leq \dots \leq h^{(2)}(k-d+2) + d - 2.$$

This inequality together with $h^{(2)}(6) < \infty$ implies that $h^{(d)}(d+4) < \infty$ and hence $H(d) \geq d+4$. However, already in dimension 3 the gap between the upper and the lower bound of $H(3)$ remains huge: while there are arbitrarily large sets without 23-holes [27], already the existence of 8-holes remains unknown ($7 \leq H(3) \leq 22$).

1.4 Precise Values

As discussed before, for the planar k -gons $g^{(2)}(5) = 9, g^{(2)}(6) = 17, h^{(2)}(5) = 10$, and $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ are known. For planar k -holes, $h^{(2)}(5) = 9, 30 \leq h^{(2)}(6) \leq 463$, and $h^{(2)}(k) = \infty$ for $k \geq 7$.

While the values $g^{(d)}(k) = h^{(d)}(k) = k$ for $k \leq d+1$ and $g^{(d)}(d+2) = h^{(d)}(d+2) = d+3$ are easy to determine (cf. [3]), Bisztriczky et al. [2, 3, 19] showed $g^{(d)}(k) = h^{(d)}(k) = 2k - d - 1$ for $d+2 \leq k \leq \frac{3d}{2} + 1$. This, in particular, determines the values for $(k, d) = (3, 5), (4, 6), (4, 7), (5, 7), (5, 8)$ and shows $H(d) \geq \lfloor \frac{3d}{2} \rfloor + 1$. For $k > \frac{3d}{2} + 1$ and $d \geq 3$, Bisztriczky and Soltan [3] moreover determined the values $g^{(3)}(6) = h^{(3)}(6) = 9$. Tables 1 and 2 summarize the currently best bounds for k -gons and k -holes in small dimensions.

Table 1. Known values and bounds for $g^{(d)}(k)$. Entries marked with a star (*) are new. Entries left blank can be upper-bounded by the estimate $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ [26] and the dimension-reduction argument from [15].

	$k = 4$	5	6	7	8	9	10	11
$d = 2$	5	9	17					
3	4	6	9	$\leq 13^*$				
4	4	5	7	9	$\leq 13^*$			
5	4	5	6	8	10	$\leq 13^*$		
6	4	5	6	7	9	11	13	

Table 2. Known values and bounds for $h^{(d)}(k)$. Entries marked with a star (*) are new.

	$k = 4$	5	6	7	8	9	10	11
$d = 2$	5	10	30..463	∞	∞	∞	∞	∞
3	4	6	9	$\leq 14^*$?	?	?	?
4	4	5	7	9	$\leq 13^*$?	?	?
5	4	5	6	8	10	$\leq 13^*$?	?
6	4	5	6	7	9	11	13	?

1.5 Our Results

In this article we show the following upper bounds on higher dimensional Erdős–Szekeres numbers and the k -holes variant in dimensions 3, 4 and 5, which we moreover conjecture to be sharp.

Theorem 2. *It holds $g^{(3)}(7) \leq 13$, $h^{(3)}(7) \leq 14$, $g^{(4)}(8) \leq h^{(4)}(8) \leq 13$, and $g^{(5)}(9) \leq h^{(5)}(9) \leq 13$.*

For the proof of Theorem 2, we generalize our SAT framework from [23] to higher dimensional point sets. Our framework for dimensions $d = 3, 4, 5$ is based on chirotopes of rank $r = d + 1$, and we use the SAT solver CaDiCaL [1] to prove unsatisfiability. Moreover, CaDiCaL can generate unsatisfiability proofs which then can be verified by a proof checking tool such as DRAT-trim [28].

2 Preliminaries

Let $\{p_1, \dots, p_n\}$ be a set of n labeled points in \mathbb{R}^d in general position with coordinates $p_i = (x_{i1}, \dots, x_{id})$. We assign to each $(d + 1)$ -tuple i_0, \dots, i_d a sign to indicate whether the $d + 1$ corresponding points p_{i_0}, \dots, p_{i_d} are positively or negatively oriented. Formally, we define $\chi : \{1, \dots, n\}^d \rightarrow \{-1, 0, +1\}$ with

$$\chi(i_0, \dots, i_d) = \text{sgn det} \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_{i_0} & p_{i_1} & \dots & p_{i_d} \end{pmatrix} = \text{sgn det} \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_{i_0 1} & x_{i_1 1} & \dots & x_{i_d 1} \\ \vdots & \vdots & & \vdots \\ x_{i_0 d} & x_{i_1 d} & \dots & x_{i_d d} \end{pmatrix}.$$

It is well known that this mapping χ is a *chirotope* of rank $r = d + 1$ (cf. [4, Definition 3.5.3]). Moreover, since we are only interested in point sets in general position in this article, we only consider *non-degenerate* chirotopes, that is, $\chi(a_1, \dots, a_r) \neq 0$ holds for any r distinct indices a_1, \dots, a_r . The following theorem fully characterizes non-degenerate chirotopes:

Theorem 3. (cf. [4, Theorem 3.6.2]). *A map $\chi : \{1, \dots, n\}^r \rightarrow \{-1, 0, +1\}$ is a non-degenerate chirotope of rank r if the following two properties are fulfilled:*

(i) *for every permutation σ on any r distinct indices $a_1, \dots, a_r \in \{1, \dots, n\}$,*

$$\chi(a_{\sigma(1)}, \dots, a_{\sigma(r)}) = \text{sgn}(\sigma) \cdot \chi(a_1, \dots, a_r) \neq 0;$$

(ii) *for any $a_1, \dots, a_r, b_1, b_2 \in \{1, \dots, n\}$,*

$$\begin{aligned} \text{if } & \chi(b_1, a_2, \dots, a_r) \cdot \chi(a_1, b_2, b_3, \dots, b_r) \geq 0 \\ \text{and } & \chi(b_2, a_2, \dots, a_r) \cdot \chi(b_1, a_1, b_3, \dots, b_r) \geq 0 \\ \text{then } & \chi(a_1, a_2, \dots, a_r) \cdot \chi(b_1, b_2, b_3, \dots, b_r) \geq 0. \end{aligned}$$

2.1 Gons and Holes

Carathéodory’s theorem asserts that a d -dimensional point set is in convex position if and only if all $(d + 2)$ -element subsets are in convex position. Now that a point $p_{i_{d+1}}$ lies in the convex hull of $\{p_{i_0}, \dots, p_{i_d}\}$ if and only if $\chi(i_0, \dots, i_d) = \chi(i_0, \dots, i_{j-1}, i_{d+1}, i_j, \dots, i_d)$ holds for every $j \in \{0, \dots, d\}$, we can fully axiomize k -gons and k -holes solely using the information of the chirotope, that is, the relative position of the points. (The explicit coordinates do not play a role.)

3 The SAT Framework

For the proof of Theorem 2, we proceed as following: To show $g^{(d)}(k) \leq n$ (or $h^{(d)}(k) \leq n$, resp.), assume towards a contradiction that there exists a set S of n points in \mathbb{R}^d in general position, which does not contain any k -gon (or k -hole, resp.). The point set S induces a chirotope χ of rank $d + 1$, which can be encoded using n^{d+1} Boolean variables. The chirotope χ fulfills the $\Theta(n^{d+3})$ conditions from Theorem 3, which we can encode as clauses.

Next, we introduce auxiliary variables for all $i_0, \dots, i_{d+1} \in \{1, \dots, n\}$ to indicate whether the point $p_{i_{d+1}}$ lies in the convex hull of $\{p_{i_0}, \dots, p_{i_d}\}$. As discussed in Sect. 1.2, the values of these auxiliary variables are fully determined by the chirotope (variables). Using these n^{d+2} auxiliary variables we can formulate $\binom{n}{k}$ clauses, each involving k^{d+2} literals, to assert that there are no k -gons in S : Among every subset $X \subset S$ of size $|X| = k$ there is at least one point $p \in X$ which is contained in the convex hull of $d + 1$ points of $X \setminus \{p\}$. (To assert that there are no k -holes in S , we can proceed in a similar manner: Among every subset $X \subset S$ of size $|X| = k$ there is at least one point $p \in S$ which is contained in the convex hull of $d + 1$ points of $X \setminus \{p\}$.)

Altogether, we can now create a Boolean satisfiability instance that is satisfiable if and only if there exists a rank $d + 1$ chirotope on n elements without k -gons (or k -holes, resp.). If the instance is provable unsatisfiable, no such chirotope (and hence no point set S) exists, and we have $g^{(d)}(k) \leq n$ (or $h^{(d)}(k) \leq n$, resp.).

3.1 Running Times and Resources

All our computations were performed on single CPUs. However, since some computations (especially for verifying the unsatisfiability certificates) required more resources than available on standard computers/laptops, we made use of the computing cluster from the Institute of Mathematics at TU Berlin.

- $g^{(3)}(7) \leq 13$: The size of the instance is about 245 MB and CaDiCaL managed to prove unsatisfiability in about 2 CPU days. Moreover, the unsatisfiability certificate created by CaDiCaL is about 39 GB and the DRAT-trim verification took about 1 CPU day.

- $h^{(3)}(7) \leq 14$: The size of the instance is about 433 MB and **CaDiCaL** (with parameter `--unsat`) managed to prove unsatisfiability in about 19 CPU days.
- $h^{(4)}(8) \leq 13$: The size of the instance is about 955 MB and **CaDiCaL** managed to prove unsatisfiability in about 7 CPU days.
- $h^{(5)}(9) \leq 13$: The size of the instance is about 4.2 GB and **CaDiCaL** managed to prove unsatisfiability in about 3 CPU days. Moreover, the unsatisfiability certificate created by **CaDiCaL** is about 117 GB and the **DRAT-trim** verification took about 3 CPU days.

The python program for creating the instances and further technical information is available on our supplemental website [22].

4 Discussion

Unfortunately, the unsatisfiability certificates for $h^{(3)}(7) \leq 14$ and $h^{(4)}(8) \leq 13$, respectively, created by **CaDiCaL** grew too big to be verifiable with our available resources. However, it might be possible to further optimize the SAT model to make the solver terminate faster (cf. [23]) so that one obtains smaller certificates.

In the course of our investigations we found chirotopes that witness that all bounds from Theorem 2 are sharp in the more general setting of chirotopes. However, since we have not yet succeeded in finding realizations of those chirotopes, we can only conjecture that all bounds from Theorem 2 are also sharp in the original setting, but we are looking forward to implementing further computer tools so that we can address all those realizability issues. It is worth noting that finding realizable witnesses is a notoriously hard and challenging task because (i) only $2^{\Theta(n \log n)}$ of the $2^{\Theta(n^d)}$ rank $d + 1$ chirotopes are realizable by point sets and (ii) the problem of deciding realizability is ETR-complete in general (cf. Chapters 7.4 and 8.7 in [4]).

Concerning the existence of 8-holes in 3-space: while we managed to find a rank 4 chirotope on 18 elements without 8-holes within only a few CPU hours, the solver did not terminate for months on the instance $h^{(3)}(8) \stackrel{?}{\leq} 19$. We see this as a strong evidence that sufficiently large sets in 3-space (possibly already 19 points suffice) contain 8-holes.

Last but not least we want to mention that our SAT framework can also be used to tackle other problems on higher dimensional point sets. For example, by slightly adapting our SAT framework, we managed to answer a Tverberg-type question by Fulek et al. (cf. Sect. 3.2 in [9]).

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Improved Bounds on the Cop Number of a Graph Drawn on a Surface

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Abstract. It is known that the cop number $c(G)$ of a connected graph G can be bounded as a function of the genus of the graph $g(G)$. It is conjectured by Schröder that $c(G) \leq g(G) + 3$. Recently, by relating this problem to a topological game, the authors, together with Bowler and Pitz, gave the current best known bound that $c(G) \leq \frac{4g(G)}{3} + \frac{10}{3}$. Combining some of these ideas with some techniques introduced by Schröder we improve this bound and show that $c(g) \leq (1 + o(1))(3 - \sqrt{3})g \approx 1.268g$.

Keywords: Cops and Robbers · Graph searching · Genus

1 Introduction

The game of *cops and robbers* was introduced independently by Nowakowski and Winkler [7] and Quillot [9]. The game is a pursuit game played on a connected graph $G = (V, E)$ by two players, one player controlling a set of $k \geq 1$ *cops* and the other controlling a *robber*. Initially, the first player chooses a starting configuration $(c_1, c_2, \dots, c_k) \in V^k$ for the cops and then the second player chooses a starting vertex $r \in V$ for the robber. The game then consists of alternating moves, one move by the cops and then a subsequent move by the robber. For a cop move, each cop may move to a vertex adjacent to his current location, or stay still, and the same goes for a subsequent move of the robber. Note that each cop may change his position in a move, and that multiple cops may occupy the same vertex. The first player wins if at some time there is a cop on the same vertex as the robber; otherwise the robber wins. We define the *cop number* $c(G)$ of a graph G to be the smallest number of cops k such that the first player has a winning strategy in this game.

The problem of bounding the cop number of a graph in terms of other graph invariants has been well studied, for a full introduction to the area see the book of Bonato and Nowakowski [3]. In particular there has been much interest in the relationship between the cop number of a graph and the topological properties of the graph, see for example the recent survey of Bonato and Mohar [2]. One of the first results in this direction is a result of Aigner and Fromme [1], who showed that every planar graph has cop number at most three.

Theorem 1. *Every planar graph G has $c(G) \leq 3$.*

More generally, given a graph G , let us write $g(G)$ for the *genus* of G , that is, the smallest k such that G can be drawn on an orientable surface of genus k without crossing edges. For $g \in \mathbb{N}$ we define

$$c(g) := \max\{c(G) \mid g(G) = g\}.$$

Hence, in this notation Theorem 1 says that $c(0) \leq 3$. Quillot [10] used similar methods to show that $c(g) \leq 2g + 3$, and these methods were then further refined by Schröder to show that $c(g) \leq \lfloor \frac{3g}{2} \rfloor + 3$.

A key part of Aigner and Fromme's strategy was the notion of *guarding* a vertex set. We say that a cop *guards* a set C of vertices of G if whenever the robber moves to a vertex of C , he is caught by that cop on the next move. We call $C \subseteq V(G)$ *guardable* if there is a strategy for a single cop c in which, after finitely many steps, c guards C . An important step in their proof was then the following lemma.

Lemma 1 ([1], Lemma 4). *For $x, y \in V(G)$, the vertex set of any geodesic path from x to y is guardable.*

This lemma has been turned out to be very fundamental in the study of the cops and robbers game, and in particular it is essential to all known upper bounds on the cop number of graphs drawn on surfaces. Broadly, the strategy of all of these proofs is to find collections of geodesic paths (initially in G , but later in some subgraph, to which the cops have restricted the robber) such that, if we delete these paths, then each component of the remaining graph has strictly smaller genus than before. By assigning a cop to guard each of these paths eventually the robber is restricted to a planar subgraph of G , in which three more cops can then catch him by Theorem 1. Perhaps the best bound we could hope for using such a strategy would be to always find a single path whose deletion leaves only components of smaller genus and, motivated by this Schröder conjectured that the following is true

Conjecture 2 (Schröder [8]). *For every $g \in \mathbb{N}$, we have*

$$c(g) \leq g + 3.$$

We note that whilst $c(0) = 3$ by Aigner and Fromme's result, the only other value of c which is known is $c(1) = 3$, which was shown recently by the second author [5], and it seems unlikely that the bound in Conjecture 2 is tight for any $g > 1$. In fact, very little seems to be known about lower bounds for the cop number of graphs drawn on a surface, with the only lower bound we could find in the literature coming from the survey paper of Bonato and Mohar [2] who give the following lower bound, which is due to a random construction of Mohar.

Theorem 3 (Mohar [6]).

$$c(g) \geq g^{\frac{1}{2} - o(1)}.$$

One ample source of geodesic paths in a graph is to pick a pair of vertices x and y and consider any shortest path between x and y . Using this, one way to give a strategy for the cops would be to split the cops strategy into distinct periods. In period t the cops pick some pair of vertices x_t and y_t , informed perhaps by the current state of the game and the previous moves, and then in the next period assign a cop to guard some shortest path P_t between x and y (in $G \setminus \bigcup_{i=1}^{t-1} P_i$), which he can achieve after a finite number of moves. Then, in the next period, we know that robber is restricted to some component of $G \setminus \bigcup_{i=1}^t P_i$. For example, Quillot's strategy can be phrased in this manner. Let us call such a strategy a *naive strategy*.

It might seem like a potential issue that we don't have much control over what the shortest paths between x and y will look like, however if we consider these paths as arcs living on the surface that G is drawn in, then at least in a topological sense, up to a homeomorphism of this surface, there will only be a small number of possibilities for what this arc can look like. By analysing carefully the possible types of arcs that can arise in this fashion, the authors together with Bowler and Pitz [4] were able to improve on Schröder's bound.

Theorem 4. *For every $g \in \mathbb{N}$, we have*

$$c(g) \leq \frac{4g}{3} + \frac{10}{3}.$$

Their proof related the cops and robbers game on a graph drawn on a surface S to a purely topological 'Waiter-Client' game on the surface S , where one player chooses two points in the boundary of the surface and the other chooses an arc between these points in the surface, cuts along this arc and then discards all but one of the resulting components. The first player wants to eventually get to a planar surface using as few cuts as possible, and they showed that he has a strategy to do so using $\frac{4}{3}g + O(1)$ cuts. However, they also showed that this strategy is essentially optimal, in that the second player has a strategy to ensure that at least $\frac{4}{3}g + O(1)$ cuts are used. This strongly suggests that the bound of Theorem 4 is the best one can hope for using a naive strategy.

However, whilst Schröder's strategy still uses Lemma 1 in an essential manner, there are some ingredients to Schröder's proof which improve on these naive strategies. By using some of these ideas to complement the ideas from [4] we are able to give an improvement to Theorem 4.

Theorem 5. *For every $g \in \mathbb{N}$, we have*

$$c(g) \leq (3 - \sqrt{3} + o(1))g.$$

It seems however that this is not the limit of these methods, and attempting to improve this bound leads to the following question, which seems to be interesting in its own right.

Question 6. *Let n and g be integers and let G be a drawing of K_n on the surface S of genus g . What is the size of the largest matching in G whose edge set is topologically connected in S ?*

If we could show the existence of a matching of size tending to infinity with n when $n = \epsilon g$ for some small $\epsilon > 0$, then this should be sufficient to improve the bound in Theorem 5 to $c(g) \leq (\frac{5}{4} + o(1))g$.

2 Sketch of Proof

To motivate our strategy, let us sketch Schröder's proof. To this end it will be useful to think of the graph G as being drawn on a fixed surface S . Given a path or cycle W in G there is a corresponding arc A_W in S . If we 'cut along'¹ the arc A_W in S we get a new surface $\text{Cut}(S, A_W)$ and there is a new graph $\text{Cut}(G, W)$ on this surface, where there might now be multiple distinct 'copies' of the vertices and edges in W on the surface, but there will only be a single copy of each other edge of G , and in particular each edge of G incident with a vertex v of W will only be incident with a single copy of v in $\text{Cut}(G, W)$.

If a collection of cops is currently guarding W in G , we can imagine that the remaining cops and robber are playing on $\text{Cut}(G, W)$, although in fact the robber in the actual game is weaker, he can't use vertices of W , and the cops in the actual game are stronger, they can 'teleport' between copies of the same vertex. In this way we can, during the play of the game, keep track of a sequence of graphs drawn on a sequence of surfaces where during a particular period of play we will be focusing on what we'll refer to as the *active* graph and surface. After assigning some cops to guard some geodesic paths in the active graph, we will cut up the active surface along the corresponding arc. The robber will be restricted to some component of this cut up graph, and we will take that component, and the component of the cut up surface containing it, to be the new active graph and component. Note that there may be multiple copies of the vertices of the paths that are being guarded by cops in the earlier graphs appearing in the active graph, and also there may say some paths where none of the copies of the vertices in these paths appear in the active graph, in which case it is no longer necessary that this cop keeps guarding his path.

Translated into this setting, there are then perhaps two ingredients to the proof of Schröder: He first shows that using two cops c_1 and c_2 it is possible to guard a subgraph whose deletion reduces the genus of the active surface and such that, in the new active surface S , c_1 and c_2 are only guarding vertices which appear in some cycle which lies in the boundary of S or *boundary cycle*.

He then shows that, given a boundary cycle C , either a single cop can guard a walk between two vertices in C whose deletion reduces the genus of the active surface, or using two cops c_3 and c_4 it is possible to restrict the robber to some part of the active surface where c_3 and c_4 are only guarding vertices which appear in some cycle which lies in the boundary of this part, and the only vertices of C appearing in this part are guarded by c_3 and c_4 .

¹ Imagine your surface being made out of rubber and cutting the surface precisely along the middle of the edges, so that a part of each edge remains on each side of the cut.

At this point the cops c_1 and c_2 are redundant, and so we can ‘relabel’ the cops c_3 and c_4 as c_1 and c_2 and repeat the previous argument. Eventually, since the graph is finite, and at each stage we are restricting the robber’s territory, we end up using a single cop to reduce the genus of the active surface, and so in all we’ve used three cops (as well as potentially two extra cops, but only for a finite period) to reduce the genus of the active surface by two.

By induction we can conclude that $c(G) \leq \frac{3}{2}g + O(1)$, and at this point Schröder carefully analyses the start and end of this induction to minimise this constant.

Broadly our strategy follows that of Schröder, but we begin by using a variant of his argument to first guard a subgraph using $2k$ cops which reduces the genus of the active surface by k and such that these cops are only guarding vertices which appear on boundary cycle. Whilst this may seem very inefficient, using $2k$ cops to reduce the genus by k , we will be able to leverage the existence of a boundary cycle C on which many different cops are guarding vertices to make more efficient moves.

At this point we consider a maximal sequence of nested paths P_1, P_2, \dots, P_a between points on C such that sequentially deleting these paths reduces the genus of the active surface by a .

If a is large, then we have found a sequence of very efficient genus reducing moves for the cops. However, if a is small, then C will split into a small number of boundary cycles in the new active surface, and we can use similar ideas as in Schröder’s argument to replace the $2k + a$ cops guarding vertices on these cycles with a smaller number of cops, and to make a further genus reducing cut in each of these cycles.

In both cases this represents an improvement over the asymptotic rate of reducing the genus by three using four cops, and by considering the worst choice of a for a given k it can be shown that a rate arbitrarily close to that given in Theorem 5 can be achieved in this fashion.

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Stability of Extremal Connected Hypergraphs Avoiding Berge-Paths

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Abstract. A Berge-path of length k in a hypergraph \mathcal{H} is a sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ of distinct vertices and hyperedges with $v_{i+1} \in e_i, e_{i+1}$ for all $i \in [k]$. Füredi, Kostochka and Luo, and independently Győri, Salia and Zamora determined the maximum number of hyperedges in an n -vertex, connected, r -uniform hypergraph that does not contain a Berge-path of length k provided k is large enough compared to r . They also determined the unique extremal hypergraph \mathcal{H}_1 .

We prove a stability version of this result by presenting another construction \mathcal{H}_2 and showing that any n -vertex, connected, r -uniform hypergraph without a Berge-path of length k , that contains more than $|\mathcal{H}_2|$ hyperedges must be a subhypergraph of the extremal hypergraph \mathcal{H}_1 , provided k is large enough compared to r .

Keywords: Extremal hypergraph theory · Berge-paths · Connectivity

1 Introduction

This work is an extended abstract of a manuscript [9].

In extremal graph theory, the Turán number $ex(n, G)$ of a graph G is the maximum number of edges that an n -vertex graph can have without containing G as a subgraph. If a class \mathcal{G} of graphs is forbidden, then the Turán number is denoted by $ex(n, \mathcal{G})$. The asymptotic behavior of the function $ex(n, G)$ is well-understood if G is not bipartite. However, much less is known if G is bipartite (see the survey [8]). One of the simplest classes of bipartite graphs is that of paths. Let P_k and C_k denote the path and the cycle with k edges and let $\mathcal{C}_{\geq k}$ denote the class of cycles of length at least k .

Erdős and Gallai [3] proved that for any $n \geq k \geq 1$, the Turán number satisfies $ex(n, P_k) \leq \frac{(k-1)n}{2}$. They obtained this result by first showing that for any $n \geq k \geq 3$, $ex(n, \mathcal{C}_{\geq k}) \leq \frac{(k-1)(n-1)}{2}$. The bounds are sharp for paths, if k divides n , and sharp for cycles, if $k-1$ divides $n-1$. These are shown by the example of n/k pairwise disjoint k -cliques for the path P_k , and adding an extra

vertex joined by an edge to every other vertex for the class $\mathcal{C}_{\geq k+2}$ of cycles. Later, Faudree and Schelp [4] gave the exact value of $ex(n, P_k)$ for every n .

Observe that the extremal construction for the path is not connected. Kopylov [14] and independently Balister, Győri, Lehel, and Schelp [1] determined the maximum number of edges $ex^{conn}(n, P_k)$ that an n -vertex connected graph can have without containing a path of length k . The stability version of these results was proved by Füredi, Kostochka and Verstraëte [7]. To state their result, we need to define the following class of graphs.

Definition 1. For $n \geq k$ and $\frac{k}{2} > a \geq 1$ we define the graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three disjoint parts A, B and L such that $|A| = a$, $|B| = k - 2a$ and $|L| = n - k + a$. The edge set of $H_{n,k,a}$ consists of all the edges between L and A and also all the edges in $A \cup B$. Let us denote the number of edges in $H_{n,k,a}$ by $|H_{n,k,a}|$.

Theorem 1 (Füredi, Kostochka, Verstraëte [7], Theorem 1.6). Let $t \geq 2$, $n \geq 3t - 1$ and $k \in \{2t, 2t + 1\}$. Suppose we have a n -vertex connected P_k -free graph G with more edges than $|H_{n+1,k+1,t-1}| - n$. Then we have either

- $k = 2t$, $k \neq 6$ and G is a subgraph of $H_{n,k,t-1}$, or
- $k = 2t + 1$ or $k = 6$, and $G \setminus A$ is a star forest for $A \subseteq V(G)$ of size at most $t - 1$.

The Turán numbers for hypergraphs $ex_r(n, \mathcal{H})$, $ex_r(n, \mathcal{H})$ can be defined analogously for r -uniform hypergraphs \mathcal{H} and classes \mathcal{H} of r -uniform hypergraphs.

Definition 2. A Berge-path of length t is an alternating sequence of $t + 1$ distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \dots, e_t, v_{t+1}$ such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$. The vertices v_1, v_2, \dots, v_{t+1} are called defining vertices and the hyperedges e_1, e_2, \dots, e_t are called defining hyperedges of the Berge-path. We denote the set of all Berge-paths of length t by \mathcal{BP}_t .

The study of the Turán numbers $ex_r(n, \mathcal{BP}_k)$ was initiated by Győri, Katona and Lemons [10], who determined the quantity in almost every case. Later Davoodi, Győri, Methuku and Tompkins [2] settled the missing case $r = k + 1$. For results on the maximum number of hyperedges in r -uniform hypergraphs not containing Berge-cycles longer than k see [5, 11] and the references therein.

Analogously to graphs, a hypergraph is *connected*, if for any two of its vertices, there is a Berge-path containing both vertices. The connected Turán numbers for an r -uniform hypergraph \mathcal{H} and class of r -uniform hypergraphs \mathcal{H} can be defined analogously, they are denoted by the functions $ex_r^{conn}(n, \mathcal{H})$ and $ex_r^{conn}(n, \mathcal{H})$, respectively. In order to introduce our contributions, we need the following definition.

Definition 3. For integers $n, a \geq 1$ and $b_1, \dots, b_t \geq 2$ with $n \geq 2a + \sum_{i=1}^t b_i$ let us denote by $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ the following r -uniform hypergraph.

- Let the vertex set of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be $A \cup L \cup \bigcup_{i=1}^t B_i$, where A, B_1, B_2, \dots, B_t and L are pairwise disjoint sets of sizes $|A| = a, |B_i| = b_i$ ($i = 1, 2, \dots, t$) and $|L| = n - a - \sum_{i=1}^t b_i$.
- Let the hyperedges of $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ be

$$\binom{A}{r} \cup \bigcup_{i=1}^t \binom{A \cup B_i}{r} \cup \left\{ \{c\} \cup A' : c \in L, A' \in \binom{A}{r-1} \right\}.$$

Note that, if $a \leq a'$ and $b_i \leq b'_i$ for all $i = 1, 2, \dots, t$, then $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is a subhypergraph of $\mathcal{H}_{n,a',b'_1,b'_2,\dots,b'_t}$. Finally, the length of the longest path in $\mathcal{H}_{n,a,b_1,b_2,\dots,b_t}$ is $2a - t + \sum_{i=1}^t b_i$ if $t \leq a + 1$, and $a - 1 + \sum_{i=1}^{a+1} b_i$ if $t > a + 1$ and the b_i 's are in non-increasing order.

With a slight abuse of notation, we define $\mathcal{H}_{n,a}^+$ to be a hypergraph obtained from $\mathcal{H}_{n,a}$ by adding an arbitrary hyperedge. Hyperedges containing at least $r - 1$ vertices from A are already in $\mathcal{H}_{n,a}$, therefore there are $r - 1$ pairwise different hypergraphs that we denote by $\mathcal{H}_{n,a}^+$ depending on the number of vertices from A in the extra hyperedge. Observe that the length of the longest path in $\mathcal{H}_{n,a}^+$ is one larger than in $\mathcal{H}_{n,a}$, in particular, if k is even, then $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$ does not contain a Berge-path of length k .

The first attempt to determine the largest number of hyperedges in connected r -uniform hypergraphs without a Berge-path of length k can be found in [12], where the asymptotics of the extremal function was determined. The Turán number of Berge-paths in connected hypergraphs was determined by Füredi, Kostochka and Luo [6] for $k \geq 4r \geq 12$ and n large enough. Independently in a different range it was also given by Győri, Salia and Zamora [13], who also proved the uniqueness of the extremal structure. To state their result, let us introduce the following notation: for a hypergraph \mathcal{H} we denote by $|\mathcal{H}|$ the numbers of hyperedges in \mathcal{H} .

Theorem 2 (Győri, Salia, Zamora, [13]). For all integers k, r with $k \geq 2r + 13 \geq 18$ there exists $n_{k,r}$ such that if $n > n_{k,r}$, then we have

$$\begin{aligned} - ex_r^{conn}(n, \mathcal{BP}_k) &= |\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}| = (n - \frac{k-1}{2}) \binom{\frac{k-1}{2}}{r-1} + \binom{\frac{k-1}{2}}{r}, \text{ if } k \text{ is odd, and} \\ - ex_r^{conn}(n, \mathcal{BP}_k) &= |\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}| = (n - \frac{k-2}{2}) \binom{\frac{k-2}{2}}{r-1} + \binom{\frac{k-2}{2}}{r} + \binom{\frac{k-2}{2}}{r-2}, \text{ if } k \text{ is even.} \end{aligned}$$

Depending on the parity of k , the unique extremal hypergraph is $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}$ or $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$.

Our main result provides a stability version (and thus a strengthening) of Theorem 2 and also an extension of Theorem 1 for uniformity at least 3.

First we state it for hypergraphs with minimum degree at least 2, and then in full generality. In the proof, the hypergraphs $\mathcal{H}_{n, \frac{k-3}{2}, 3}$ and $\mathcal{H}_{n, \frac{k-3}{2}, 2, 2}$ will play a crucial role in case k is odd, while if k is even, then the hypergraphs

$\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}$, $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3, 2}$ and $\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 2, 2, 2}$ will be of importance, note that all of them are n -vertex, maximal, \mathcal{BP}_k -free hypergraphs. In both cases, the hypergraph listed first contains the largest number of hyperedges. This number gives the lower bound in the following theorem.

Theorem 3. *For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2 + \varepsilon)r + q$, $n \geq n_{k,r}$ and \mathcal{H} is a connected n -vertex, r -uniform hypergraph with minimum degree at least 2, without a Berge-path of length k , then we have the following.*

- If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}| = (n - \frac{k+3}{2})\binom{\frac{k-3}{2}}{r-1} + \binom{\frac{k+3}{2}}{r}$, then \mathcal{H} is a subhypergraph of $\mathcal{H}_{n, \frac{k-1}{2}}$.
- If k is even and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}| = (n - \lfloor \frac{k+5}{2} \rfloor)\binom{\lfloor \frac{k-3}{2} \rfloor}{r-1} + \binom{\lfloor \frac{k+5}{2} \rfloor}{r}$, then \mathcal{H} is a subhypergraph of $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H}_{n, \lfloor \frac{k-1}{2} \rfloor}^+$.

Let $\mathcal{H}'_{n', a, b_1, b_2, \dots, b_t}$ be the class of hypergraphs that can be obtained from $\mathcal{H}_{n, a, b_1, b_2, \dots, b_t}$ for some $n \leq n'$ by adding hyperedges of the form $A'_j \cup D_j$, where the D_j 's partition $[n'] \setminus [n]$, all D_j 's are of size at least 2 and $A'_j \subseteq A$ for all j . Let us define $\mathcal{H}'_{n', \lfloor \frac{k-1}{2} \rfloor}$ analogously.

Theorem 4. *For any $\varepsilon > 0$ there exist integers $q = q_\varepsilon$ and $n_{k,r}$ such that if $r \geq 3$, $k \geq (2 + \varepsilon)r + q$, $n \geq n_{k,r}$ and \mathcal{H} is a connected n -vertex, r -uniform hypergraph without a Berge-path of length k , then we have the following.*

- If k is odd and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|$, then \mathcal{H} is a subhypergraph of some $\mathcal{H}' \in \mathcal{H}'_{n, \frac{k-1}{2}}$.
- If k is even and $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|$, then \mathcal{H} is a subhypergraph of some $\mathcal{H}' \in \mathcal{H}'_{n, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H}'_{n, \lfloor \frac{k-1}{2} \rfloor}^+$.

2 A Sketch of the Proof

We start the proof of Theorem 3 with a technical but crucial lemma.

Lemma 1. *Let \mathcal{H} be a connected r -uniform hypergraph with minimum degree at least 2 and with longest Berge-path and Berge-cycle of length $\ell - 1$. Let C be a Berge-cycle of length $\ell - 1$ in \mathcal{H} , with defining vertices $V = \{v_1, v_2, \dots, v_{\ell-1}\}$ and defining edges $\mathcal{E}(C) = \{e_1, e_2, \dots, e_{\ell-1}\}$ with $v_i, v_{i+1} \in e_i$ (modulo $\ell - 1$). Then, we have*

- (i) every hyperedge $h \in \mathcal{H} \setminus C$ contains at most one vertex from $V(\mathcal{H}) \setminus V$.
- (ii) If u, v are not necessarily distinct vertices from $V(\mathcal{H}) \setminus V$, then there cannot exist distinct hyperedges $h_1, h_2 \in \mathcal{H} \setminus C$ and an index i with $v, v_i \in h_1$ and $u, v_{i+1} \in h_2$.
- (iii) If there exists a vertex $v \in V(\mathcal{H}) \setminus V$ and there exist different hyperedges $h_1, h_2 \in \mathcal{H} \setminus C$ with $v, v_{i-1} \in h_1$ and $v, v_{i+1} \in h_2$, then there exists a cycle of length $\ell - 1$ not containing v_i as a defining vertex.

We say that an r -uniform hypergraph \mathcal{H} has the *set degree condition*, if for any set X of vertices with $|X| \leq k/2$, we have $|E(X)| \geq |X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$, i.e., the number of those hyperedges that are incident to some vertex in X is at least $|X| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$. We first prove Theorem 3 for such hypergraphs, see manuscript [9], this is an important part of the proof but we omit this part in this extended abstract. We first state an useful proposition without a proof.

Proposition 1. *Let $\ell - 1$ be the length of the longest Berge-path in \mathcal{H} . Then $\ell \geq k - 3$ and \mathcal{H} contains a Berge-cycle of length $\ell - 1$.*

Proof (Proof of Theorem 3 and Theorem 4). Let \mathcal{H} be a connected n -vertex r -uniform hypergraph without a Berge-path of length k , and suppose that if k is odd, then $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 3}|$ while if k is even, then $|\mathcal{H}| > |\mathcal{H}_{n, \lfloor \frac{k-3}{2} \rfloor, 4}|$. We obtain a subhypergraph \mathcal{H}' of \mathcal{H} using a standard greedy process: as long as there exists a set S of vertices with $|S| \leq k/2$ such that $|E(S)| < |S| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$, we remove S from \mathcal{H} and all hyperedges in $E(S)$. Let \mathcal{H}' denote the subhypergraph at the end of this process.

Proposition 2. *There exists a threshold $n''_{k,r}$, such that if $|V(\mathcal{H})| \geq n''_{k,r}$, then \mathcal{H}' is connected and contains at least $n'_{k,r}$ vertices.*

Proof. To see that \mathcal{H}' is connected, observe that every component of \mathcal{H}' possesses the set degree condition. Therefore Proposition 1 yields that every component contains a cycle of length at least $k - 4$. Therefore, as \mathcal{H} is connected, \mathcal{H} contains a Berge-path with at least $2k - 8$ vertices from two different components of \mathcal{H}' , a contradiction as $k \geq 9$.

Suppose to the contrary that \mathcal{H}' has less than $n'_{k,r}$ vertices. Observe that, by definition of the process, $|\mathcal{E}(\mathcal{H}')| - |V(\mathcal{H}')| \binom{\lfloor \frac{k-3}{2} \rfloor}{r-1}$ strictly increases at every removal of some set X of at most k vertices. Therefore if $n > n'_{k,r} + k \binom{n'_{k,r}}{r} = n''_{k,r}$ and $|V(\mathcal{H}')| < n'_{k,r}$, then at the end we would have more hyperedges than those in the complete r -uniform hypergraph on $|v(\mathcal{H}')|$ vertices, a contradiction.

By Proposition 2 and the statement for hypergraphs with the set degree property, we know that \mathcal{H}' has $n_1 \geq n'_{k,r}$ vertices, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1, \lfloor \frac{k-1}{2} \rfloor}$ if k is odd, and $\mathcal{H}' \subseteq \mathcal{H}_{n_1, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H}^+_{n_1, \lfloor \frac{k-1}{2} \rfloor}$ if k is even. Then for any hyperedge $h \in \mathcal{E}(\mathcal{H}) \setminus \mathcal{E}(\mathcal{H}')$ that contain at least one vertex from $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least two, we can apply Lemma 1 (i) to obtain that all such h must meet the A of \mathcal{H}' in $r - 1$ vertices. This shows that if the minimum degree of \mathcal{H} is at least 2, then $\mathcal{H} \subseteq \mathcal{H}_{n_2, \lfloor \frac{k-1}{2} \rfloor}$ if k is odd, and $\mathcal{H} \subseteq \mathcal{H}_{n_2, \lfloor \frac{k-1}{2} \rfloor, 2}$ or $\mathcal{H} \subseteq \mathcal{H}^+_{n_2, \lfloor \frac{k-1}{2} \rfloor}$ if k is even, where $n_2 \leq n$ is the number of vertices that are contained in a hyperedge of \mathcal{H} that is either in \mathcal{H}' or has a vertex in $V(\mathcal{H}) \setminus V(\mathcal{H}')$ with degree at least 2. This finishes the proof of Theorem 3.

Finally, consider the hyperedges that contain the remaining $n - n_2$ vertices. As all these vertices are of degree 1, they are partitioned by these edges. For such a hyperedge h let D_h denote the subset of such vertices. Observe that for

such a hyperedge h , we have that $h \setminus D_h \subseteq A$. Indeed if $v \in h \setminus (D_h \cup A)$, then there exists a cycle C of length $k - 1$ in \mathcal{H}' not containing v . Thus there is a path of length at least k starting at an arbitrary $d \in D_h$, continuing with h, v , and having $k - 1$ more vertices as it goes around C with defining hyperedges and vertices. This contradicts Proposition 1 and finishes the proof of Theorem 4.

Acknowledgement. Research partially sponsored by the National Research, Development and Innovation Office – NKFIH under the grants K 116769, K 132696, KH 130371, SNN 129364 and FK 132060. Research of Gerbner and Vizer was supported by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences. Patkós acknowledges the financial support from the Ministry of Educational and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926. Research of Salia was partially supported by the Shota Rustaveli National Science Foundation of Georgia SRNSFG, grant number DI-18-118. Research of Vizer was supported by the New National Excellence Program under the grant number ÚNKP-20-5-BME-45.

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On Tangencies Among Planar Curves with an Application to Coloring L-Shapes

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Abstract. We prove that there are $O(n)$ tangencies among any set of n red and blue planar curves in which every pair of curves intersects at most once and no two curves of the same color intersect. If every pair of curves may intersect more than once, then it is known that the number of tangencies could be super-linear. However, we show that a linear upper bound still holds if we replace tangencies by pairwise disjoint connecting curves that all intersect a certain face of the arrangement of red and blue curves.

The latter result has an application for the following problem studied by Keller, Rok and Smorodinsky [Disc. Comput. Geom. (2020)] in the context of *conflict-free coloring of string graphs*: what is the minimum number of colors that is always sufficient to color the members of any family of n *grounded L-shapes* such that among the L-shapes intersected by any L-shape there is one with a unique color? They showed that $O(\log^3 n)$ colors are always sufficient and that $\Omega(\log n)$ colors are sometimes necessary. We improve their upper bound to $O(\log^2 n)$.

Keywords: Curves · Tangencies · Delaunay-graph · Geometric hypergraph · Conflict-free coloring · L-shapes

1 Introduction

The *intersection graph* of a collection of geometric shapes is the graph whose vertex set consists of the shapes, and whose edge set consists of pairs of shapes with a non-empty intersection. Various aspects of such graphs have been studied vastly over the years. For example, by a celebrated result of Koebe [11] planar graphs are exactly the intersection graphs of interior-disjoint disks in the plane. Another example which is more recent and more related to our topic is a result by Pawlik et al. [17], who showed that intersection graphs of planar segments are not χ -bounded, that is, their chromatic number cannot be upper-bounded by a function of their clique number.

We will mainly consider (not necessarily closed) planar curves (Jordan arcs). A family of curves is *t-intersecting* if every pair of curves intersects in at most t points. We say that two curves *touch* each other at a *touching (tangency) point* p if both of them contain p in their interior, p is their only intersection point and it is not a crossing point.¹

According to a nice conjecture of Pach [13] the number of tangencies among a 1-intersecting family \mathcal{S} of n curves should be $O(n)$ if every pair of curves intersects. Györgyi et al. [8] proved this conjecture in the special case where there are constantly many faces of the arrangement of \mathcal{S} such that every curve in \mathcal{S} has one of its endpoints inside one of these faces. Here we prove the following variant.

Theorem 1. *Let \mathcal{S} be a 1-intersecting set of red and blue curves such that no two curves of the same color intersect. Then the number of tangencies among the curves in \mathcal{S} is $O(n)$.*

Note that it is trivial to construct an example with $\Omega(n)$ tangencies. Theorem 1 does not hold if a pair of curves in \mathcal{S} may intersect twice. Indeed, Pach et al. [16] considered the following problem: what is the maximum number $f(n)$ of tangencies among n x -monotone red and blue curves where no two curves of the same color intersect. They showed that $\Omega(n \log n) \leq f(n) \leq O(n \log^2 n)$, where their lower bound construction is 2-intersecting (but not 1-intersecting).

The number of tangencies within a 1-intersecting set of n (x -monotone) curves can be $\Omega(n^{4/3})$: it is not hard to obtain this bound using the famous construction of Erdős of n points and n lines that determine that many point-line incidences [14]. An almost matching upper bound of $O(n^{4/3} \log^{2/3} n)$ follows from a result of Pach and Sharir [15].²

Connecting Curves. Instead of considering touching points among curves in a family of curves \mathcal{S} , we may consider pairs of disjoint curves that are intersected by a curve c from a different family of curves \mathcal{C} , such that c does not intersect any other curve. Indeed, each touching point of two curves from \mathcal{S} can be replaced by a new, short curve that connects the two previously touching curves that become disjoint by redrawing one of them near the touching point. Conversely, if \mathcal{C} consists of curves that connect *disjoint* curves from \mathcal{S} , then each connecting curve in \mathcal{C} can be replaced by a touching point between the corresponding curves by redrawing one of these two curves. Therefore, studying touching points or such connecting curves are equivalent problems. This gives the following reformulation of Theorem 1.

Theorem 2. *Let \mathcal{S} be a set of 1-intersecting n red and blue curves such that no two curves of the same color intersect. Suppose that \mathcal{C} is a set of pairwise disjoint*

¹ p is a *crossing point* of two curves if there is a small disk D centered at p which contains no other intersection point of these curves, each curve intersects the boundary of D at exactly two points and in the cyclic order of these four points no two consecutive points belong to the same curve.

² A more careful analysis yields the upper bound $O(n^{4/3} \log^{1/3} n)$.

curves such that each of them intersects exactly a distinct pair of disjoint curves from \mathcal{S} . Then $|\mathcal{C}| = O(n)$.

If \mathcal{S} and \mathcal{C} are two families of curves, then we say that \mathcal{C} is *grounded* with respect to \mathcal{S} if there is a connected region of $\mathbb{R}^2 \setminus \mathcal{S}$ that contains at least one point of every curve in \mathcal{C} . If \mathcal{C} is grounded with respect to \mathcal{S} , then we can drop the assumption that \mathcal{S} is 1-intersecting and prove the following variant.

Theorem 3. *Let \mathcal{S} be a set of n red and blue curves such that no two curves of the same color intersect. Suppose that \mathcal{C} is a set of pairwise disjoint curves grounded with respect to \mathcal{S} , such that each of them intersects exactly a distinct pair of curves from \mathcal{S} . Then $|\mathcal{C}| = O(n)$.*

Note that we have also dropped the assumption that a red curve and a blue curve which are connected by a curve in \mathcal{C} are disjoint. Therefore it is essential that \mathcal{C} is grounded with respect to \mathcal{S} , for otherwise we might have $|\mathcal{C}| = \Omega(n^2)$. Indeed, let \mathcal{S} be a (1-intersecting) set of $n/2$ horizontal segments and $n/2$ vertical segments such that every horizontal segment and every vertical segment intersect. Then each such pair can be connected by a curve in \mathcal{C} very close to their intersection point. Hence $|\mathcal{C}| = n^2/4$.

Clearly, instead of *curves*, Theorem 3 could also be stated with a red and a blue family of disjoint *shapes*, with no requirement at all about the shapes, except that each of them is connected.

A corollary of Theorem 3 improves a result of Keller, Rok and Smorodinsky [10] about conflict-free colorings of *L*-shapes.

Coloring L-shapes. An *L-shape* consists of a vertical line-segment and a horizontal line-segment such that the left endpoint of the horizontal segment coincides with the bottom endpoint of the vertical segment (as in the letter ‘L’, hence the name). Whenever we consider a family L-shapes, we always assume that no pair of them have overlapping segments, that is, they have at most one intersection point.

A family of L-shapes is *grounded* if there is a horizontal line that contains the top point of each L-shape in the family. A (grounded) *L-graph* is a graph that can be represented as the intersection graph of a family of (grounded) L-shapes. Gonçalves et al. [7] proved that every planar graph is an L-graph. The line graph of every planar graph is also known to be an L-graph [6]. As for grounded L-graphs, McGuinness [12] proved that they are χ -bounded. Jelínek and Töpfer [9] characterized grounded L-graphs in terms of vertex ordering with forbidden patterns.

Keller, Rok and Smorodinsky [10] studied *conflict-free* colorings of string graphs³ and in particular of grounded L-graphs. A coloring of the vertices of a hypergraph is *conflict-free*, if every hyperedge contains a vertex whose color is not assigned to any of the other vertices of the hyperedge. The minimum

³ A *string graph* is the intersection graph of curves in the plane.

number of colors in a conflict-free coloring of a hypergraph \mathcal{H} is denoted by $\chi_{\text{CF}}(\mathcal{H})$. There is a vast literature on conflict-free coloring of hypergraphs that stem from geometric settings due to its application to frequencies assignment in wireless networks and its connection to *cover-decomposability* and other coloring problems—see the survey of Smorodinsky [18], the webpage [1] and the references within.

One natural way of defining a hypergraph $\mathcal{H}(\mathcal{S})$ with respect to a family of geometric shapes \mathcal{S} is as follows: the vertex set is \mathcal{S} and for every shape $S \in \mathcal{S}$ there is a hyperedge that consists of all the members of $\mathcal{S} \setminus \{S\}$ whose intersection with S is non-empty.⁴ This is the so-called *punctured* or *open neighborhood* hypergraph of the intersection graph. Similarly, for two families of shapes, \mathcal{S} and \mathcal{F} , the hypergraph $\mathcal{H}(\mathcal{S}, \mathcal{F})$ has \mathcal{S} as its vertex set and has a hyperedge for every $F \in \mathcal{F}$ which consists of all the members of $\mathcal{S} \setminus \{F\}$ whose intersection with F is non-empty. Hence, $\mathcal{H}(\mathcal{S}) = \mathcal{H}(\mathcal{S}, \mathcal{S})$.

Using Theorem 3 we can improve the following result.

Theorem 4 (Keller, Rok and Smorodinsky [10]). $\chi_{\text{CF}}(\mathcal{H}(\mathcal{L})) = O(\log^3 n)$ for every set \mathcal{L} of n grounded L-shapes. Furthermore, for every n there exists a set \mathcal{L} of n grounded L-shapes such that $\chi_{\text{CF}}(\mathcal{H}(\mathcal{L})) = \Omega(\log n)$.

In order to obtain this result and many other results considering (conflict-free) coloring of hypergraphs it is often enough to consider the chromatic number of a sub-hypergraph consisting of hyperedges of size two, that is, the *Delaunay graph*. For two families of geometric shapes \mathcal{S} and \mathcal{F} , the *Delaunay graph* of \mathcal{S} and \mathcal{F} , denoted by $\mathcal{D}(\mathcal{S}, \mathcal{F})$, is the graph whose vertex set is \mathcal{S} and whose edge set consists of pairs of vertices such that there is a member of \mathcal{F} that intersects exactly the shapes that correspond to these two vertices and no other shape. Note that if \mathcal{S} is a set of planar points and \mathcal{F} is the family of all disks, then $\mathcal{D}(\mathcal{S}, \mathcal{F})$ is the standard Delaunay graph of the point set \mathcal{S} .

A key ingredient in the proof of Theorem 4 in [10] is the following lemma.

Lemma 5 ([10, Proposition 3.9]). Let $\mathcal{L} \cup \mathcal{I}$ be a set of grounded L-shapes such that $|\mathcal{L}| = n$ and the L-shapes in \mathcal{I} are pairwise disjoint. Then $\mathcal{D}(\mathcal{L}, \mathcal{I})$ has $O(n \log n)$ edges.

Theorem 3 clearly implies Lemma 5, since every family of n L-shapes consists of n pairwise disjoint horizontal (red) segments and n pairwise disjoint vertical (blue) segments. Thus, Theorem 3 is a twofold improvement of Lemma 5: we consider a more general setting and prove a better upper bound. Furthermore, our proof is simpler than the proof of Lemma 5 in [10] (especially a weaker version of Theorem 3 which we prove separately).

By replacing Lemma 5 with Theorem 3 (or its weaker version) in the proof of Theorem 4 we obtain a better upper bound for the number of colors that suffice to conflict-free color n grounded L-shapes.

⁴ If two shapes give rise to the same hyperedge, then this hyperedge appears only once in the hypergraph.

Theorem 6. *Let \mathcal{L} be a set of n grounded L -shapes. Then it is possible to color every L -shape in \mathcal{L} with one of $O(\log^2 n)$ colors such that for each $\ell \in \mathcal{L}$ there is an L -shape with a unique color among the L -shapes whose intersection with ℓ is non-empty.*

The upper bound on the number of edges in the Delaunay graph also implies upper bounds for the number of hyperedges of size at most k , the chromatic number of the hypergraph and its VC-dimension.⁵ This was already shown, e.g., for *pseudo-disks* [3, 4], however, the same arguments apply in general. We summarize these facts in the following statement.

Theorem 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n -vertex hypergraph. Suppose that there exist absolute constants $c, c' \geq 0$ such that for every $\mathcal{V}' \subseteq \mathcal{V}$ the Delaunay graph of the sub-hypergraph⁶ induced by \mathcal{V}' has at most $c|\mathcal{V}'| - c'$ edges, then:*

- (i) *The chromatic number of \mathcal{H} is at most $2c + 1$ (at most $2c$ if $c' > 0$);*
- (ii) *The VC-dimension of \mathcal{H} is at most $2c + 1$ (at most $2c$ if $c' > 0$); and*
- (iii) *\mathcal{H} has $O(k^{d-1}n)$ hyperedges of size at most k where d is the VC-dimension of \mathcal{H} .*

Using a result of Chan et al. [5] this has another consequence about finding hitting sets. We follow the usage of this result as in [3] (see therein the definition of the minimum weight hitting set problem):

Theorem 8. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. Suppose that there exist absolute constants $c, c' \geq 0$ such that for every $\mathcal{V}' \subseteq \mathcal{V}$ the Delaunay graph of the sub-hypergraph induced by \mathcal{V}' has at most $c|\mathcal{V}'| - c'$ edges. Then there exists a randomized polynomial-time $O(1)$ -approximation algorithm for the minimum weight hitting set problem for \mathcal{H} .*

Thus, we have:

Corollary 9. *Let \mathcal{S} be a set of n red and blue curves, such that no two curves of the same color intersect and let \mathcal{C} be another set of pairwise disjoint curves which is grounded with respect to \mathcal{S} . Then the chromatic number of the intersection hypergraph $\mathcal{H}(\mathcal{S}, \mathcal{C})$ and its VC-dimension are bounded by a constant, and for every k the number of hyperedges of size at most k in $\mathcal{H}(\mathcal{S}, \mathcal{C})$ is $k^{O(1)}n$. Also there exists a randomized polynomial-time $O(1)$ -approximation algorithm for the minimum weight hitting set problem for \mathcal{H} .*

Note that the upper bounds on the chromatic number and VC-dimension can be deduced easily in this case without using Theorem 7.

Due to space constraints the proofs are omitted from this extended abstract. The interested reader can find them in the full version [2].

⁵ Recall that the VC-dimension of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the size of its largest subset of vertices $\mathcal{V}' \subseteq \mathcal{V}$ that can be *shattered*, that is, for every subset $\mathcal{V}'' \subseteq \mathcal{V}'$ there exists a hyperedge $h \in \mathcal{E}$ such that $h \cap \mathcal{V}' = \mathcal{V}''$.

⁶ The hyperedges of this sub-hypergraph are the non-empty subsets in $\{h \cap \mathcal{V}' \mid h \in \mathcal{E}\}$; this is sometimes called the *trace*, or the restriction to \mathcal{V}' .

Acknowledgement. We thank Chaya Keller for pointing out that Theorem 7 implies Theorem 8. Research of the second and third author supported by the Lendület program of the Hungarian Academy of Sciences, under the grant LP2017-19/2017. Research of the second author supported by the National Research, Development and Innovation Office – NKFIH under the grant K 132696.

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Semi-random Process Without Replacement

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Abstract. We introduce and study a semi-random multigraph process, which forms a no-replacement variant of the process that was introduced in [3]. The process starts with an empty graph on the vertex set $[n]$. For every positive integers q and $1 \leq r \leq n$, in the $((q-1)n+r)$ th round of the process, the decision-maker, called *Builder*, is offered the vertex $\pi_q(r)$, where π_1, π_2, \dots is a sequence of permutations in S_n , chosen independently and uniformly at random. Builder then chooses an additional vertex (according to a strategy of his choice) and connects it by an edge to $\pi_q(r)$.

For several natural graph properties, such as k -connectivity, minimum degree at least k , and building a given spanning graph (labeled or unlabeled), we determine the typical number of rounds Builder needs in order to construct a graph having the desired property. Along the way we introduce and analyze two urn models which may also have independent interest.

Keywords: Random process · Games on graphs

1 Introduction

This is an extended abstract to the paper [4], which contains all the proofs.

In this paper we introduce and analyze a general semi-random multigraph process, arising from an interplay between a sequence of random choices on the one hand, and a strategy of our choice on the other. It is a no-replacement variant of the process which was proposed by Peleg Michaeli, analyzed in [3], and further studied in [2] and [5]. The process starts with an empty graph on the vertex set $[n]$. Let π_1, π_2, \dots be a sequence of permutations in S_n , chosen independently and uniformly at random. For every positive integer k , in the k th round of the process, the decision-maker, called *Builder*, is offered the vertex $v_k := \pi_q(r)$, where q and r are the unique integers satisfying $(q-1)n+r = k$ and $1 \leq r \leq n$. Builder then irrevocably chooses an additional vertex u_k and adds the edge $u_k v_k$ to his (multi)graph, with the possibility of creating multiple

D. Hefetz—Research supported by ISF grant 822/18.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 129–135, 2021.
https://doi.org/10.1007/978-3-030-83823-2_21

edges (in fact, we will make an effort to avoid multiple edges; allowing them is a technical aid which ensures that Builder always has a legal edge to claim). The algorithm that Builder uses in order to add edges throughout this process is referred to as Builder's *strategy*.

Given a positive integer n and a family \mathcal{F} of labeled graphs on the vertex set $[n]$, we consider the one-player game in which Builder's goal is to build a multigraph with vertex set $[n]$ that contains, as a (spanning) subgraph, some graph from \mathcal{F} , as quickly as possible; we denote this game by $(\mathcal{F}, n)_{\text{lab}}$. In the case that the family \mathcal{F} consists of a single graph G , we will use the abbreviation $(G, n)_{\text{lab}}$ for $(\{G\}, n)_{\text{lab}}$. We also consider the one-player game in which Builder's goal is to build a multigraph with vertex set $[n]$ that contains a subgraph which is *isomorphic* to some graph from \mathcal{F} , as quickly as possible; we denote this game by (\mathcal{F}, n) . Note that

$$(\mathcal{F}, n) = (\mathcal{F}_{\text{iso}}, n)_{\text{lab}}, \quad (1)$$

where \mathcal{F}_{iso} is the family of all labeled graphs on the vertex set $[n]$ which are isomorphic to some graph from \mathcal{F} . The general problem discussed in this paper is that of determining the typical number of rounds Builder needs in order to construct such a multigraph under optimal play.

For the labeled game $(\mathcal{F}, n)_{\text{lab}}$ and a strategy \mathcal{S} of Builder, let $\tau(\mathcal{S})$ denote the total number of rounds played until Builder's graph first contains some graph from \mathcal{F} , assuming he plays according to \mathcal{S} . For completeness, if no such integer exists, we define $\tau(\mathcal{S})$ to be $+\infty$. Note that $\tau(\mathcal{S})$ is a random variable. Let $p_{\mathcal{S}}$ be the non-decreasing function from the set \mathbb{N} of non-negative integers to the interval $[0, 1]$ defined by $p_{\mathcal{S}}(k) = \Pr(\tau(\mathcal{S}) \leq k)$ for every non-negative integer k . For every non-negative integer k , let $p_{(\mathcal{F}, n)_{\text{lab}}}(k)$ be the maximum of $p_{\mathcal{S}}(k)$, taken over all possible strategies \mathcal{S} for $(\mathcal{F}, n)_{\text{lab}}$. Clearly, $p_{(\mathcal{F}, n)_{\text{lab}}}$ is a non-decreasing function from \mathbb{N} to $[0, 1]$; hence there exists a random variable $\tau_{\text{lab}}(\mathcal{F}, n)$ taking values in $\mathbb{N} \cup \{+\infty\}$ such that $\Pr(\tau_{\text{lab}}(\mathcal{F}, n) \leq k) = p_{(\mathcal{F}, n)_{\text{lab}}}(k)$ for every non-negative integer k . Note that if there is an optimal strategy \mathcal{S}' for the labeled game $(\mathcal{F}, n)_{\text{lab}}$ (i.e., such that for any strategy \mathcal{S} for $(\mathcal{F}, n)_{\text{lab}}$ it holds that $p_{\mathcal{S}}(k) \geq p_{\mathcal{S}'}(k)$ for every k), then we may take $\tau_{\text{lab}}(\mathcal{F}, n)$ to be $\tau(\mathcal{S}')$.

For the unlabeled game (\mathcal{F}, n) we define $\tau(\mathcal{F}, n)$ in an analogous manner, or by using (1), namely $\tau(\mathcal{F}, n) = \tau_{\text{lab}}(\mathcal{F}_{\text{iso}}, n)$. Since, obviously, $p_{(\mathcal{F}, n)}(k) \geq p_{(\mathcal{F}, n)_{\text{lab}}}(k)$ for every k , we may assume (by coupling) that $\tau(\mathcal{F}, n) \leq \tau_{\text{lab}}(\mathcal{F}, n)$.

2 Results

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function satisfying $f(n) = \omega(\sqrt{n})$.

2.1 General Bounds

The following two results are very simple but very widely applicable. Together with their many corollaries they form a good indication of what is interesting to prove in relation to the no-replacement semi-random process.

Proposition 1. *Let G be a graph on the vertex set $[n]$. If there exists an orientation D of the edges of G such that $d_D^+(u) \leq d$ for every $u \in [n]$, then $\tau(G, n) \leq \tau_{\text{lab}}(G, n) \leq dn$.*

Proposition 2. *Let G be a graph on the vertex set $[n]$. Let d be the largest integer such that in every orientation of the edges of G there exists a vertex of out-degree at least d . Then*

$$\tau_{\text{lab}}(G, n) \geq \tau(G, n) \geq \max\{(d - 1)n + 1, e(G)\}.$$

Remark 1. It is well-known (see Lemma 3.1 in [1]) that a graph G admits an orientation in which the out-degree of every vertex is at most d if and only if $d \geq L(G)$, where

$$L(G) := \max \left\{ \frac{e(H)}{v(H)} : \emptyset \neq H \subseteq G \right\}.$$

Corollary 1. *Let G be a $2d$ -regular graph on the vertex set $[n]$. Then,*

$$\tau(G, n) = \tau_{\text{lab}}(G, n) = dn.$$

Corollary 2. *Let G be a d -degenerate graph on the vertex set $[n]$. Then,*

$$e(G) \leq \tau(G, n) \leq \tau_{\text{lab}}(G, n) \leq dn.$$

In particular, in the special case where $e(G) = dn$, it holds that $\tau(G, n) = \tau_{\text{lab}}(G, n) = dn$. Another special case is when T is a tree, and then $n - 1 \leq \tau(T, n) \leq \tau_{\text{lab}}(T, n) \leq n$.

Corollary 3. *Let G be an arbitrary balanced¹ graph with m edges on the vertex set $[n]$. Then $m \leq \tau(G, n) \leq \tau_{\text{lab}}(G, n) \leq \lceil m/n \rceil n$. In particular, if m/n is an integer, then $\tau(G, n) = \tau_{\text{lab}}(G, n) = m$.*

Corollary 4. *Let $G \sim G(n, p)$, where $p = p(n) \geq (1 + o(1)) \ln n/n$ and let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function satisfying $g(n) = \omega(n\sqrt{p(n)})$. Then w.h.p.*

$$n^2p/2 - g(n) \leq \tau(G, n) \leq \tau_{\text{lab}}(G, n) \leq n^2p/2 + \sqrt{n^3p \ln n} + n.$$

It follows from all of the aforementioned corollaries that some properties which may still be interesting to study are the construction of odd regular graphs and the construction of a (not predetermined) graph from an interesting family, such as graphs of minimum degree k or k -connected graphs (where k is odd).

2.2 Minimum Degree

Let $\mathcal{D}_d = \mathcal{D}_d(n)$ be the family of n -vertex simple graphs with minimum degree at least d . Note that $\tau(\mathcal{D}_d, n) = \tau_{\text{lab}}(\mathcal{D}_d, n)$ for every d and every n .

¹ a graph G is balanced if $e(G)/v(G) = \max\{e(H)/v(H) : \emptyset \neq H \subseteq G\}$.

Theorem 1. *Let $d \leq n - 1$ be a positive integer.*

1. *If d is even, then $\tau(\mathcal{D}_d, n) = dn/2$;*
2. *If d is odd, then w.h.p. it holds that*

$$(d + 1 - 2/e)n/2 - f(n) \leq \tau(\mathcal{D}_d, n) \leq (d + 1 - 2/e)n/2 + f(n) + d,$$

where the upper bound holds under the additional assumption that $d = o(n)$.

2.3 Building Regular Graphs

Let G be a d -regular graph on n vertices. If d is even, then $\tau(G, n) = \tau_{\text{lab}}(G, n) = dn/2$ holds by Corollary 1. In the case $d = 1$, where G is a perfect matching, we prove the following result.

Theorem 2. *Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function satisfying $g(n) = \omega(n^{3/4})$, let n be an even integer and let G be a perfect matching on the vertex set $[n]$. Then w.h.p. it holds that*

$$n - g(n) \leq \tau(G, n) \leq \tau_{\text{lab}}(G, n) = n - \Theta(\sqrt{n}).$$

Finally, for odd $d > 1$, by Proposition 1 and by the second part of Theorem 1, it holds w.h.p. that

$$(d + 1 - 2/e - o(1))n/2 \leq \tau(\mathcal{D}_d, n) \leq \tau(G, n) \leq \tau_{\text{lab}}(G, n) \leq (d + 1)n/2. \quad (2)$$

The following result shows that the upper bound in (2) is asymptotically tight for $\tau_{\text{lab}}(G, n)$.

Theorem 3. *Let n be an even integer and let $1 < d < n$ be an odd integer. Let G be a d -regular graph on the vertex set $[n]$. Then w.h.p. $\tau_{\text{lab}}(G, n) \geq (d + 1)n/2 - f(n)$.*

For the complete graph, the game (K_n, n) is obviously the same as the game $(K_n, n)_{\text{lab}}$. Hence, Theorem 3 yields an asymptotically tight lower bound for $\tau(K_n, n)$ as well.

Corollary 5. *Let n be an even integer. Then w.h.p. $\tau(K_n, n) \geq n^2/2 - f(n)$.*

2.4 Trees

Recall that $n - 1 \leq \tau(T, n) \leq \tau_{\text{lab}}(T, n) \leq n$ holds by Corollary 2 for every tree T on n vertices. The remaining interesting question is to determine $\Pr(\tau(T, n) = n - 1)$ and $\Pr(\tau_{\text{lab}}(T, n) = n - 1)$ for every tree T . We make the following modest step in this direction.

Proposition 3. *Let $n \geq 2$ be an integer and let T be a tree on the vertex set $[n]$.*

1. *If T is a path, then $\tau(T, n) = n - 1$ and $\Pr(\tau_{\text{lab}}(T, n) = n - 1) = \Theta(1/n)$;*

- 2. If $\tau(T, n) = n - 1$, then T is a path;
- 3. If T is a star, then $\Pr(\tau(T, n) = n - 1) = \frac{1}{n-1} \left(1 + \sum_{k=1}^{n-2} \frac{1}{k}\right) = (1 + o(1)) \frac{\log n}{n}$
 and $\Pr(\tau_{\text{lab}}(T, n) = n - 1) = \frac{1}{n} \left(1 + \sum_{k=1}^{n-1} \frac{1}{k}\right) = (1 + o(1)) \frac{\log n}{n}$. In particular, the difference between $\Pr(\tau(T, n) = n - 1)$ and $\Pr(\tau_{\text{lab}}(T, n) = n - 1)$ is $\frac{1}{n(n-1)} \sum_{k=1}^{n-2} \frac{1}{k} = (1 + o(1)) \frac{\log n}{n^2}$.

It is interesting to note that, as can be seen from Proposition 3, $\Pr(\tau(T, n) = n - 1)$ and $\Pr(\tau_{\text{lab}}(T, n) = n - 1)$ are “very close” when T is a star but are “very far” when T is a path.

2.5 Edge-Connectivity

For every positive integer k , let $\mathcal{C}_k = \mathcal{C}_k(n)$ denote the family of all k -edge-connected n -vertex graphs. Since there are k -vertex-connected k -regular graphs for every $k \geq 2$, it follows by Corollary 1 that $\tau(\mathcal{C}_k, n) = kn/2$ for every positive even integer k and every sufficiently large n . Moreover, $\tau(\mathcal{C}_1, n) = n - 1$. Indeed, the lower bound is trivial and the upper bound holds since $\tau(\mathcal{C}_1, n) \leq \tau(P_n, n) = n - 1$, where the equality holds by the first part of Proposition 3. Finally, for odd $k > 1$, by the second part of Theorem 1, it holds w.h.p. that

$$(k + 1 - 2/e)n/2 - f(n) \leq \tau(\mathcal{D}_k, n) \leq \tau(\mathcal{C}_k, n) \leq \tau(\mathcal{C}_{k+1}, n) = (k + 1)n/2. \tag{3}$$

The following result shows that the lower bound in (3) is asymptotically tight for $\tau(\mathcal{C}_k, n)$ when k is not too small or too large.

Theorem 4. *Let $n \geq 12$ be an integer and let $5 \leq k = o(n)$ be an odd integer. Then w.h.p.*

$$\tau(\mathcal{C}_k, n) \leq (k + 1 - 2/e)n/2 + f(n) + k.$$

3 Urn Models

In the proofs of Theorems 1 and 2 we use the analysis of two urn models, which are somewhat reminiscent of Polya’s urn model [6], and may have independent interest.

First Urn Model. We start with an even number n of white balls in an urn. In each round, as long as there is at least one white ball in the urn, we remove one ball from the urn, chosen uniformly at random, and then if the removed ball was white, we replace one remaining white ball by one black ball. Let T be the number of rounds until this process terminates (i.e., until there are no white balls left in the urn); clearly $T \leq n$.

Proposition 4. *Let $\alpha(n)$ be a positive integer smaller than n . Then*

$$\Pr(T < n - \alpha(n)) < \frac{n^3}{(\alpha(n))^4}.$$

Second Urn Model. We start with n white balls in an urn. In each round, as long as there is at least one white ball in the urn, we remove one ball from the urn, chosen uniformly at random, and then if the urn still contains at least one white ball, we replace one white ball with one black ball. Let T be the number of rounds until the process terminates (i.e., until there are no white balls left in the urn); clearly $T \leq n - 1$.

Proposition 5. *Let $\alpha(n) < \lfloor(1 - 1/e)n\rfloor$ be a positive integer (in particular, $n \geq 4$). Then*

$$\Pr(T < \lfloor(1 - 1/e)n\rfloor - \alpha(n)) < \frac{6n}{(\alpha(n))^2}$$

and

$$\Pr\left(T > \lfloor(1 - 1/e)n\rfloor + 36\alpha(n) + \frac{12n}{(\alpha(n))^2}\right) < \frac{6n}{(\alpha(n))^2}.$$

4 Concluding Remarks and Open Problems

We suggest a few related open problems for future research.

Labeled vs. Unlabeled. As noted in the introduction, $\tau(\mathcal{F}, n) \leq \tau_{\text{lab}}(\mathcal{F}, n)$ holds for every family \mathcal{F} of n -vertex graphs. We have proved that $\tau(\mathcal{F}, n) = \tau_{\text{lab}}(\mathcal{F}, n)$ for several such families (e.g., when \mathcal{F} consists of a single regular graph of even degree). We have also proved that $\tau_{\text{lab}}(\mathcal{F}, n) - \tau(\mathcal{F}, n) = o(n)$ for several other families (e.g., for perfect matchings). It would be interesting to decide whether there exists a (natural) family \mathcal{F} of n -vertex graphs such that $\tau_{\text{lab}}(\mathcal{F}, n) - \tau(\mathcal{F}, n) = \Omega(n)$. In particular, it would be interesting to decide whether there exists an n -vertex regular graph of odd degree G such that $\tau_{\text{lab}}(G, n) - \tau(G, n) = \Omega(n)$; recall that we have proved that $\tau_{\text{lab}}(G, n) - \tau(G, n) \leq (1/e + o(1))n$ holds for all such graphs.

Trees. As noted in Sect. 2.4, the most natural and interesting question concerning trees is to determine $\Pr(\tau(T, n) = n - 1)$ and $\Pr(\tau_{\text{lab}}(T, n) = n - 1)$ for every tree T . We have proved some partial related results. In particular, we have shown that $\Pr(\tau(T, n) = n - 1) = 1$ if and only if $T \cong P_n$. This implies that P_n is the “best” tree in the sense that $\Pr(\tau(T, n) = n - 1) < \Pr(\tau(P_n, n) = n - 1)$ for every n -vertex tree $T \neq P_n$. We believe that the star $K_{1, n-1}$ is the “worst” tree. That is, that $\Pr(\tau(T, n) = n - 1) > \Pr(\tau(K_{1, n-1}, n) = n - 1)$ holds for every n -vertex tree $T \neq K_{1, n-1}$. As we saw, the situation is reversed for labeled trees, that is, $\Pr(\tau_{\text{lab}}(P_n, n) = n - 1) < \Pr(\tau_{\text{lab}}(K_{1, n-1}, n) = n - 1)$. It would be interesting to determine whether these are the extremal cases also for the labeled version of the game, that is, whether $\Pr(\tau_{\text{lab}}(P_n, n) = n - 1) < \Pr(\tau_{\text{lab}}(T, n) = n - 1) < \Pr(\tau_{\text{lab}}(K_{1, n-1}, n) = n - 1)$ for every n -vertex tree $T \notin \{P_n, K_{1, n-1}\}$.

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The Intersection Spectrum of 3-Chromatic Intersecting Hypergraphs

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Abstract. For a hypergraph H , define its intersection spectrum $I(H)$ as the set of all intersection sizes $|E \cap F|$ of distinct edges $E, F \in E(H)$. In their seminal paper from 1973 which introduced the local lemma, Erdős and Lovász asked: how large must the intersection spectrum of a k -uniform 3-chromatic intersecting hypergraph be? They showed that such a hypergraph must have at least three intersection sizes, and conjectured that the size of the intersection spectrum tends to infinity with k . Despite the problem being reiterated several times over the years by Erdős and other researchers, the lower bound of three intersection sizes has remarkably withstood any improvement until now. In this paper, we prove the Erdős–Lovász conjecture in a strong form by showing that there are at least $k^{1/2-o(1)}$ intersection sizes. In this extended abstract we sketch a simpler argument which gives slightly weaker bound of $k^{1/3-o(1)}$. Our proof consists of a delicate interplay between Ramsey type arguments and a density increment approach.

Keywords: Intersecting hypergraphs · Property B · Intersection spectrum

1 Introduction

A family \mathcal{F} of sets is said to have *property B* if there exists a set X which properly intersects every set of the family, that is, $\emptyset \neq F \cap X \neq F$ for all $F \in \mathcal{F}$. The term was coined in the 1930s by Miller [21, 22] in honor of Felix Bernstein. In 1908, Bernstein [5] proved that for any transfinite cardinal number κ , any family \mathcal{F} of cardinality at most κ , whose sets have cardinality at least κ , has property B. In the 60s, Erdős and Hajnal [12] revived the study of property B, and initiated its investigation for finite set systems, or hypergraphs. A hypergraph H consists of a *vertex set* $V(H)$ and an *edge set* $E(H)$, where every edge is a subset of the vertex set. As usual, H is *k -uniform* if every edge has size k . A hypergraph is *r -colorable*

Research supported in part by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation.

Research supported in part by SNSF grant 200021_196965.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 136–141, 2021.
https://doi.org/10.1007/978-3-030-83823-2_22

if its vertices can be colored with r colors such that no edge is monochromatic. Note that a hypergraph has property B if and only if it is 2-colorable.

The famous problem of Erdős and Hajnal is to determine $m(k)$, the minimum number of edges in a k -uniform hypergraph which is not 2-colorable. This can be viewed as an analogue of Bernstein’s result for finite cardinals. Clearly, one has $m(k) \leq \binom{2k-1}{k}$, since the family of all k -subsets of a given set of size $2k - 1$ does not have property B. On the other hand, Erdős [8] soon observed that $m(k) \geq 2^{k-1}$. Indeed, if a hypergraph has less than 2^{k-1} edges, then the expected number of monochromatic edges in a random 2-coloring is less than 1, hence a proper 2-coloring exists. Thanks to the effort of many researchers, see survey [23] and references within, the best known bounds are now

$$\Omega\left(\sqrt{k/\log k}\right) \leq m(k)/2^k \leq O(k^2), \tag{1}$$

proofs of which are now textbook examples of the probabilistic method [2]. Improving either of these bounds would be of immense interest.

The 2-colorability problem for hypergraphs has inspired a great amount of research over the last half century, with many deep results proved and methods developed. One outstanding example is the Lovász local lemma, originally employed by Erdős and Lovász [13] to show that a k -uniform hypergraph is 2-colorable if every edge intersects at most 2^{k-3} other edges. In addition to the Lovász local lemma, the seminal paper of Erdős and Lovász [13] from 1973 left behind a whole legacy of problems and results on the 2-colorability problem. Some of their problems were solved relatively soon [3,4], others took decades [14,17,18] or are still the subject of ongoing research.

At the heart of some particularly notorious problems are intersecting hypergraphs. In an *intersecting* hypergraph (Erdős and Lovász called them *cliques*), any two edges intersect in at least one vertex. The study of intersecting families is a rich topic in itself, which has brought forth many important results such as the Erdős–Ko–Rado theorem. We refer the interested reader to [15]. With regards to colorability, the intersecting property imposes strong restrictions. For instance, it is easy to see that any intersecting hypergraph has chromatic number at most 3. Hence, the 3-chromatic ones are exactly those which do not have property B. On the other hand, every 3-chromatic intersecting hypergraph is “critical” in the sense that deleting just one edge makes it 2-colorable. These and other reasons (explained below) make the 2-colorability problem for intersecting hypergraphs very interesting. It motivated Erdős and Lovász to initiate the study of 3-chromatic intersecting hypergraphs, proving some fundamental results and raising tantalizing questions.

Analogously to $m(k)$, define $\tilde{m}(k)$ as the minimum number of edges in a k -uniform intersecting hypergraph which is not 2-colorable. The problem of estimating $\tilde{m}(k)$ seems much harder. While for non-intersecting hypergraphs, we know at least that $\lim_{k \rightarrow \infty} \sqrt[k]{m(k)} = 2$, no such result is in sight for $\tilde{m}(k)$. Clearly, the lower bound in (1) also holds for $\tilde{m}(k)$. However, the best known upper bound for $\tilde{m}(k)$ is exponentially worse. For any k which is a power of 3, an iterative construction based on the Fano plane yields a k -uniform 3-chromatic

intersecting hypergraph with $7^{\frac{k-1}{2}}$ edges (see [1, 13]). Perhaps the main obstacle to improving this bound is that the probabilistic method does not seem applicable for intersecting hypergraphs. Erdős and Lovász also asked for the minimum number of edges in a k -uniform intersecting hypergraph with cover number k , which can be viewed as a relaxation of $\tilde{m}(k)$ since any k -uniform 3-chromatic intersecting hypergraph has cover number k . This problem was famously solved by Kahn [18].

In addition to the size of 3-chromatic intersecting hypergraphs, Erdős and Lovász also studied their “intersection spectrum”. For a hypergraph H , define $I(H)$ as the set of all intersection sizes $|E \cap F|$ of distinct edges $E, F \in E(H)$. A folklore observation is that if a hypergraph is not 2-colorable, then there must be two edges which intersect in exactly one vertex, that is, $1 \in I(H)$. A very natural question is what else we can say about the intersection spectrum of a non-2-colorable hypergraph. In general, hypergraphs can be non-2-colorable even if their only intersection sizes are 0 and 1. There are various basic constructions for this, see e.g. [20]. For instance, consider K_N^{k-1} , the complete $(k-1)$ -uniform hypergraph on N vertices. For N large enough, any 2-coloring of the edges will contain a monochromatic clique on k vertices by Ramsey’s theorem. Let H be the hypergraph with $V(H) = E(K_N^{k-1})$ whose edges correspond to the k -cliques of K_N^{k-1} . Then H is a k -uniform non-2-colorable hypergraph with $I(H) = \{0, 1\}$.

Erdős and Lovász observed that the situation changes drastically for intersecting hypergraphs. In the aforementioned construction of the iterated Fano plane, the intersection spectrum consists of all odd numbers (between 1 and $k-1$). In particular, the *maximal intersection size* is $k-2$, and the *number of intersection sizes* is $(k-1)/2$. Astonishingly, not a single example (of a k -uniform 3-chromatic intersecting hypergraph) is known where these quantities are any smaller. Intrigued by this, Erdős and Lovász studied the corresponding lower bounds. Concerning the maximal intersection size, they (and also Shelah) could prove that $\max I(H) = \Omega(k/\log k)$ for any k -uniform 3-chromatic intersecting hypergraph H . This is in stark contrast to non-intersecting hypergraphs where we can have $\max I(H) = 1$ as discussed. In fact, Erdős and Lovász conjectured that a linear bound should hold, or perhaps even $k - O(1)$. Erdős [11] later offered \$100 for settling this question.

Finally, consider the number of intersection sizes. As already noted, we always have $1 \in I(H)$. Moreover, the above result on $\max I(H)$ adds another intersection size for sufficiently large k . Hence, 3-chromatic intersecting hypergraphs have a small intersection size, namely 1, and a relatively big intersection size. Recall that general non-2-colorable hypergraphs might only have two intersection sizes. However, Erdős and Lovász were able to show that intersecting hypergraphs must have at least one more. Using a theorem of Deza [7] on sunflowers, they proved that $i(k) \geq 3$ for sufficiently large k , where $i(k)$ is the minimum of $|I(H)|$ over all k -uniform 3-chromatic intersecting hypergraphs. They also remarked that they “cannot even prove” that $i(k)$ tends to infinity. This is particularly striking in view of the best known upper bound being $(k-1)/2$.

Conjecture 1 (Erdős and Lovász, 1973). $i(k) \rightarrow \infty$ as $k \rightarrow \infty$.

Despite the fact that over the years this problem has been reiterated many times by Erdős and other researchers [6, 9–11, 23], remarkably, the lower bound of three intersection sizes has withstood any improvement until now. In this paper, we prove Conjecture 1 in the following strong form.

Theorem 2. *The intersection spectrum of a k -uniform 3-chromatic intersecting hypergraph has size at least $\Omega(k^{1/2}/\log k)$.*

2 Preliminaries

We list here a number of auxilliary results which we use in our argument.

Proposition 1. *Let \mathcal{A} be a k -uniform and \mathcal{B} a k' -uniform hypergraph on the same vertex set and with the same number of edges ℓ . Then*

$$\sum_{\{A,A'\} \subseteq E(\mathcal{A})} |A \cap A'| + \sum_{\{B,B'\} \subseteq E(\mathcal{B})} |B \cap B'| \geq \sum_{A \in E(\mathcal{A}), B \in E(\mathcal{B})} |A \cap B| - \ell(k+k')/2.$$

It will be convenient for us to introduce the following averaging functions. Given a hypergraph H and disjoint subsets $S, T \subseteq E(H)$ we define

$$\lambda_S := \frac{1}{\binom{|S|}{2}} \sum_{\{e,f\} \subseteq S} |e \cap f| \quad \text{and} \quad \lambda_{S,T} := \frac{1}{|S||T|} \sum_{e \in S, f \in T} |e \cap f|.$$

Lemma 1. *Let S, T be disjoint collections of ℓ edges of a k -uniform hypergraph H , with the property that there are x vertices of H which all belong to every edge in S and none of them belong to any edge in T . Then*

$$\frac{\lambda_S + \lambda_T}{2} \geq \lambda_{S,T} + \frac{x}{2} - \frac{k}{\ell - 1}.$$

Proposition 2. *Let H be a k -uniform 3-chromatic and intersecting hypergraph, and $X \subseteq V(H)$. Then for any $0 \leq i \leq k - |X|$, there exists a set $X_i \subseteq V(H)$ of size $|X| + i$ such that at least a k^{-i} proportion of the edges containing X also contain X_i .*

3 Proof Ideas

Here we present our proof ideas by sketching a slightly simpler argument which shows $|I(H)| \geq k^{1/3 - o(1)}$.

Let H be a k -uniform, 3-chromatic and intersecting hypergraph. Let $\lambda_1 < \dots < \lambda_r$ denote the distinct intersection sizes in H , so $r = |I(H)|$. A natural way to approach our problem is to define a coloring of the complete graph with vertex set $E(H)$ where an edge is colored according to the size of the intersection of its endpoints. We will refer to this coloring as the *intersection coloring*. The well-known argument for bounding Ramsey numbers actually gives us more than

just a monochromatic clique. If we repeatedly take out an arbitrary edge of H and only keep its majority color neighbors, we keep at least a proportion of $1/r$ of the edges per iteration. If we repeat rt many times we can find a set X consisting of t edges that we took out which had the same majority color, so in particular X is a monochromatic clique in the intersection coloring. Furthermore, we know that the size of the set of remaining edges Y has lost at most a factor of r^{rt} compared to the original number of edges. In addition, the complete bipartite graph between X and Y is also monochromatic in the same color as X .

In particular, this provides us with a pair (X, Y) of disjoint subsets of edges of H with the property that any two edges in X as well as any pair of edges one in X and one in Y intersect in exactly λ_i vertices, for some λ_i . We call such a pair a λ_i -pair. Note that if we choose $|X| = t \approx k^{1/3}$ and assume $r \leq O(k^{1/3}/\log k)$ (otherwise we are done) then $|Y| \geq |E(H)|/r^{rt} \geq |E(H)|/k^{O(k^{2/3})}$.

Our strategy will be to show that given a λ_i -pair one can find a λ_j -pair, for some $j > i$, whose set X still has size t and the size of Y shrinks by at most a factor of $k^{O(k^{2/3})}$. Since by Erdős' result mentioned in the introduction we know $|E(H)| \geq 2^{k-1}$, we can repeat this procedure at least $\Omega(k^{1/3}/\log k)$ times to conclude there are at least this many different intersection sizes and complete the proof.

To do this, let (X, Y) be a λ_i -pair with $|X| = t \approx k^{1/3}$. We can apply Lemma 1 to X and any t -subset $Y' \subseteq Y$ to conclude that $\lambda_{Y'} \geq \lambda_i - 2k^{2/3}$. So in particular, the average intersection size in any subset of Y of size at least t cannot be much lower than λ_i . Next we take an arbitrary edge U in X and consider the intersections of edges in Y with U . We will separate between two cases depending on the structure of Y .

In the first case, many of the edges in Y have almost the same intersection with U . In this case we will find a collection of at least $|Y|/k^{O(k^{2/3})}$ edges in Y which contain the same set of vertices of size at least $\lambda_i - x$, where $x \approx 10k^{2/3}$. Then we apply Proposition 2 (with $i = x+1$) to obtain a subset of edges of size at least $|Y|/k^{O(k^{2/3})}$ in which any pair of edges intersects in more than λ_i vertices. Applying once again the Ramsey argument, this time within this collection of edges, we find a λ_j -pair in which we only lost another factor of $k^{O(k^{2/3})}$ in terms of size of Y . Since all intersection sizes are larger than λ_i we know that $j > i$, so we found our desired new pair.

In the second case, the intersections of edges in Y with U are "spread out". Then we can find two disjoint subsets $A, B \subseteq Y$ both of size $|Y|/k^{O(k^{2/3})}$ with the property that there is a set of x vertices $W \subseteq U$ which belongs to every edge of A and is disjoint from all edges in B . By applying the Ramsey argument to the collection A and to the collection B we either find a desired λ_j -pair with $j > i$ or we find a t -subset $S \subseteq A$ and a t -subset $T \subseteq B$ such that all pairwise intersections inside S and T have size at most λ_i . In particular, $\lambda_S, \lambda_T \leq \lambda_i$. We now apply Lemma 1 to S and T , knowing that the $x = 10k^{2/3}$ vertices in W belong to every edge in S and none belong to any edge in T . This will give us $\lambda_{S,T} \leq \lambda_i - 4k^{2/3}$. Combining these three inequalities we

obtain $\lambda_{S \cup T} < \lambda_i - 2k^{2/3}$, which contradicts our lower bound on the average intersection size among subsets of Y and completes the argument.

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On Ordered Ramsey Numbers of Tripartite 3-Uniform Hypergraphs

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Abstract. For $k \geq 2$, an *ordered k -uniform hypergraph* $\mathcal{H} = (H, <)$ is a k -uniform hypergraph H together with a fixed linear ordering $<$ of its vertex set. The *ordered Ramsey number* $\bar{R}(\mathcal{H}, \mathcal{G})$ of two ordered k -uniform hypergraphs \mathcal{H} and \mathcal{G} is the smallest $N \in \mathbb{N}$ such that every red-blue coloring of the hyperedges of the ordered complete k -uniform hypergraph $\mathcal{K}_N^{(k)}$ contains a blue copy of \mathcal{H} or a red copy of \mathcal{G} .

The ordered Ramsey numbers are quite extensively studied for ordered graphs, but little is known about ordered hypergraphs of higher uniformity. We provide some of the first nontrivial estimates on ordered Ramsey numbers of ordered 3-uniform hypergraphs. In particular, we prove that for all $d, n \in \mathbb{N}$ and for every ordered 3-uniform hypergraph \mathcal{H} on n vertices with maximum degree d and with interval chromatic number 3 there is an $\varepsilon = \varepsilon(d) > 0$ such that $\bar{R}(\mathcal{H}, \mathcal{H}) \leq 2^{O(n^{2-\varepsilon})}$.

Keywords: Ordered graph · Ramsey number · Tripartite · Uniform

1 Introduction

For an integer $k \geq 2$ and a k -uniform hypergraph H , the *Ramsey number* $R(H)$ is the minimum $N \in \mathbb{N}$ such that every 2-coloring of the hyperedges of the complete k -uniform hypergraph $K_N^{(k)}$ on N vertices contains a monochromatic subhypergraph isomorphic to H . Estimating Ramsey numbers is a notoriously difficult problem. Despite many efforts in the last 70 years, no tight bounds are known even for the complete graph K_n on n vertices. Apart from some smaller term improvements, essentially the best known bounds are $2^{n/2} \leq R(K_n) \leq 2^{2n}$ by Erdős [12] and by Erdős and Szekeres [15]. The Ramsey numbers $R(K_n^{(k)})$ are even less understood for $k \geq 3$. For example, it is only known that

$$2^{\Omega(n^2)} \leq R(K_n^{(3)}) \leq 2^{2^{O(n)}}, \tag{1}$$

as shown by Erdős and Rado [14]. A famous conjecture of Erdős states that there is a constant $c > 0$ such that $R(K_n^{(3)}) \geq 2^{2^{cn}}$.

Recently a variant of Ramsey numbers has been introduced for hypergraphs with a fixed order on their vertex sets [2, 5]. For an integer $k \geq 2$, an *ordered k -uniform hypergraph* \mathcal{H} is a pair $(H, <)$ consisting of a k -uniform hypergraph H and a linear ordering $<$ of its vertex set. An ordered k -uniform hypergraph $\mathcal{H}_1 = (H_1, <_1)$ is an *ordered subhypergraph* of another ordered k -uniform hypergraph $\mathcal{H}_2 = (H_2, <_2)$, written $\mathcal{H}_1 \subseteq \mathcal{H}_2$, if H_1 is a subhypergraph of H_2 and $<_1$ is a suborder of $<_2$. Two ordered hypergraphs \mathcal{H}_1 and \mathcal{H}_2 are *isomorphic* if there is an isomorphism between their underlying hypergraphs that preserves the vertex orderings of \mathcal{H}_1 and \mathcal{H}_2 . Note that, up to isomorphism, there is a unique ordered complete k -uniform hypergraph $\mathcal{K}_n^{(k)}$ on n vertices.

The *ordered Ramsey number* $\overline{R}(\mathcal{H}, \mathcal{G})$ of two ordered k -uniform hypergraphs \mathcal{H} and \mathcal{G} is the smallest $N \in \mathbb{N}$ such that every coloring of the hyperedges of $\mathcal{K}_N^{(k)}$ by colors red and blue contains a blue ordered subhypergraph isomorphic to \mathcal{H} or a red ordered subhypergraph isomorphic to \mathcal{G} . In the *diagonal case* $\mathcal{H} = \mathcal{G}$, we just write $\overline{R}(\mathcal{G})$ instead of $\overline{R}(\mathcal{G}, \mathcal{G})$.

The ordered Ramsey numbers are known to be finite and it is easy to see that they grow at least as fast as the standard Ramsey numbers. Studying ordered Ramsey numbers has attracted a lot of attention lately (see the survey by Conlon, Fox, and Sudakov [10]), as there are various motivations coming from the field of discrete geometry. It is known that ordered Ramsey numbers can behave quite differently than the standard Ramsey numbers, especially for sparse ordered graphs [2, 3, 5]. However, so far, the ordered Ramsey numbers have been studied mostly for ordered graphs only and very little is known about ordered Ramsey numbers of ordered k -uniform hypergraphs with $k \geq 3$.

We focus on 3-uniform hypergraphs and we prove some new bounds on the ordered Ramsey numbers of ordered tripartite 3-uniform hypergraphs.

1.1 Preliminaries

For an ordered k -uniform hypergraph $\mathcal{H} = (H, <)$ and two subsets U and V of vertices of \mathcal{H} , we say that U and V are *consecutive* if all vertices from U precede all vertices of V in $<$. An *interval* in \mathcal{H} is a subset I of vertices of \mathcal{H} such that for all vertices u, v, w of \mathcal{H} with $u < v < w$ and $u, w \in I$ we have $v \in I$.

For integers $k \geq 2$ and $\chi \geq k$, we use $K_\chi^{(k)}(n)$ to denote the *complete k -uniform χ -partite hypergraph*, that is, the vertex set of $K_\chi^{(k)}(n)$ is partitioned into χ sets of size n and every k -tuple with at most one vertex in each of these parts forms a hyperedge. The ordering of $K_\chi^{(k)}(n)$, in which the color classes form consecutive intervals, is denoted by $\mathcal{K}_\chi^{(k)}(n)$. We use $\mathcal{K}_{n,n}$ to denote $\mathcal{K}_2^{(2)}(n)$.

The *degree* of a vertex v in a hypergraph H is the number of hyperedges of H that contain v . For $d \in \mathbb{N}$, a k -uniform hypergraph H is *d -degenerate* if there is an ordering $v_1 \prec \dots \prec v_t$ of vertices of H such that each v_i is contained in at most d hyperedges of H that contain a vertex from v_1, \dots, v_{i-1} . We use $[n]$ to denote the set $\{1, \dots, n\}$. We omit floor and ceiling signs whenever they are not crucial and we use \log and \ln to denote base 2 logarithm and the natural logarithm, respectively.

1.2 Previous Results

The ordered Ramsey numbers of k -uniform ordered hypergraphs with $k \geq 3$ remain quite unexplored. Only the ordered Ramsey numbers of so-called monotone hyperpaths are well understood due to their close connections to the famous Erdős–Szekeres Theorem [15]; see [1, 17, 20]. A *monotone hyperpath* $\mathcal{P}_n^{(k)}$ on n vertices is an ordered k -uniform hypergraph where the hyperedges are formed by k -tuples of consecutive vertices. Note that the maximum degree of a k -uniform monotone hyperpath is at most k . Moshkovitz and Shapira [20] showed that $\overline{R}(\mathcal{P}_n^{(k)}) = \text{tow}_{k-1}((2 - o(1))n)$ for $k \geq 3$, where tow_h is the *tower function* of height h defined as $\text{tow}_1(x) = x$ and $\text{tow}_h(x) = 2^{\text{tow}_{h-1}(x)}$ for $h \geq 2$.

Thus even for 3-uniform hypergraphs \mathcal{H} with bounded maximum degree the numbers $\overline{R}(\mathcal{H})$ can grow very fast. We get an exponential lower bound on $\overline{R}(\mathcal{H})$ even for 3-uniform ordered hypergraphs \mathcal{H} with maximum degree 3. A similar result is known for ordered graphs, as for arbitrarily large values of n there are ordered graphs \mathcal{M}_n with n vertices and maximum degree 1 such that $\overline{R}(\mathcal{M}_n) \geq n^{\Omega(\log n / \log \log n)}$ [2, 5]. This superpolynomial growth rate is in sharp contrast with the situation for unordered hypergraphs, where the Ramsey number $R(H)$ of every k -uniform hypergraph H with bounded k and with bounded maximum degree is at most linear in the number of vertices of H [4, 6, 11, 18, 21].

Therefore, in order to obtain smaller upper bounds on the ordered Ramsey numbers, it is necessary to bound other parameter besides the maximum degree. A natural choice is so-called interval chromatic number, which can be understood as an analogue of the chromatic number due to a variant of the Erdős–Stone–Simonovits theorem for ordered graphs proved by Pach and Tardos [23]. The *interval chromatic number* $\chi_{<}(\mathcal{H})$ of an ordered k -uniform hypergraph \mathcal{H} is the minimum number of intervals the vertex set of \mathcal{H} can be partitioned into so that each hyperedge of \mathcal{H} has at most one vertex in each interval.

For ordered graphs, bounding both parameters indeed helps, as the number $\overline{R}(\mathcal{G})$ of every ordered graph \mathcal{G} with bounded maximum degree d and bounded interval chromatic number χ is at most polynomial in the number of vertices [2, 5]. Since $\mathcal{G} \subseteq \mathcal{K}_\chi^{(2)}(n)$, this result follows from the following stronger estimate proved by Conlon, Fox, Lee, and Sudakov [5]: for all $d, \chi \in \mathbb{N}$, every d -degenerate ordered graph \mathcal{G} on n vertices with interval chromatic number χ satisfies

$$\overline{R}(\mathcal{G}, \mathcal{K}_\chi^{(2)}(n)) \leq n^{32d \log \chi}. \quad (2)$$

A natural question is whether we can also get some good upper bounds on ordered Ramsey numbers of similarly restricted classes of ordered hypergraphs. If the interval chromatic number is bounded, then we can use a result of Conlon, Fox, and Sudakov [8], who showed that, for all positive integers $\chi \geq 3$ and n , $R(K_\chi^{(3)}(n)) \leq 2^{2^R n^2}$, where $R = R(K_{\chi-1})$. Since every ordering of $K_\chi^{(3)}(\chi n)$ contains an ordered subhypergraph isomorphic to $K_\chi^{(3)}(n)$ and every ordered 3-uniform hypergraph on n vertices with interval chromatic number χ is an ordered subhypergraph of $K_\chi^{(3)}(n)$, we obtain the following bound.

Corollary 1. *For all positive integers $\chi \geq 3$ and n , every ordered 3-uniform hypergraph \mathcal{H} on n vertices with interval chromatic number χ satisfies $\overline{R}(\mathcal{H}) \leq 2^{2^{2R}\chi^2 n^2}$, where $R = R(K_{\chi-1})$. In particular, if the interval chromatic number χ of \mathcal{H} is fixed, we have $\overline{R}(\mathcal{H}) \leq 2^{O(n^2)}$.*

Note that the last bound is asymptotically tight for dense ordered hypergraphs, as a standard probabilistic argument shows that $\overline{R}(\mathcal{H}) \geq 2^{\Omega(n^2)}$ for every ordered 3-uniform hypergraph \mathcal{H} on n vertices with $\Omega(n^3)$ hyperedges. In particular, we get $\overline{R}(\mathcal{K}_3^{(3)}(n)) \geq 2^{\Omega(n^2)}$.

2 Our Results

Since the bounds on the ordered Ramsey numbers from Corollary 1 are asymptotically tight for dense ordered hypergraphs with bounded interval chromatic number, we consider the sparse case with bounded maximum degree and interval chromatic number. The situation for ordered hypergraphs seems to be more difficult than for ordered graphs, so we focus on the first nontrivial case, which is for ordered 3-uniform hypergraphs with interval chromatic number 3.

Assuming the maximum degree of an ordered hypergraph \mathcal{H} with $\chi_{<}(\mathcal{H}) = 3$ is sufficiently small, we obtain a better upper bound on $\overline{R}(\mathcal{H})$ than the estimate $2^{O(n^2)}$ we would get from Corollary 1. We can prove an estimate with a subquadratic exponent even in the more general setting $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n))$, where, additionally, the interval chromatic number of \mathcal{H} is arbitrary.

Theorem 1. *Let \mathcal{H} be an ordered 3-uniform hypergraph on t vertices with maximum degree d and let s be a positive integer. Then there are constants $C = C(d)$ and $c > 0$ such that $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(s)) \leq t \cdot 2^{C(s^{2-1/(1+cd^2)})}$. In particular, for $s = t = n$ and bounded d , we get the estimate*

$$\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n)) \leq 2^{O(n^{2-1/(1+cd^2)})}. \tag{3}$$

The main idea of the proof of Theorem 1 is based on an embedding lemma from [9]. We prove a variant of this lemma, which works for ordered hypergraphs, does not consider induced copies, and uses the assumption that the maximum degree of \mathcal{H} is bounded instead of assuming that the number of vertices of \mathcal{H} is fixed. Since every ordered 3-uniform hypergraph \mathcal{H} on n vertices with $\chi_{<}(\mathcal{H}) = 3$ is an ordered subhypergraph of $\mathcal{K}_3^{(3)}(n)$, we obtain the following corollary.

Corollary 2. *Let \mathcal{H} be an ordered 3-uniform hypergraph on n vertices with maximum degree d and with interval chromatic number 3. Then there exists an $\varepsilon = \varepsilon(d) > 0$ such that $\overline{R}(\mathcal{H}) \leq 2^{O(n^{2-\varepsilon})}$.*

It might seem wasteful to use Theorem 1 in order to obtain Corollary 2, as the ordered hypergraph \mathcal{H} is much sparser than $\mathcal{K}_3^{(3)}(n)$. However, as noted in [5], this intuition is wrong already for ordered graphs, as there are ordered

matchings \mathcal{M} on n vertices with $\chi_{<}(\mathcal{M}) = 2$ and ordered graphs \mathcal{G} on $N = 2^{n^c}$ vertices for some constant $c > 0$ such that \mathcal{G} has edge density at least $1 - n^{-c}$ and does not contain \mathcal{M} as an ordered subgraph. In fact, the best known upper bounds on $\overline{R}(\mathcal{G})$ for n -vertex ordered graphs \mathcal{G} with bounded maximum degree and $\chi_{<}(\mathcal{G}) = \chi$ are derived from the bound (2) on $\overline{R}(\mathcal{G}, \mathcal{K}_\chi^{(2)}(n))$.

The upper bound (3) is quite close to the truth, as even when \mathcal{H} is fixed we get a superexponential lower bound on $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n))$. We recently learned that Fox and He (Theorem 1.3 in [16]) independently proved the same lower bound for the unordered Ramsey number and that their result implies Theorem 2. However, we leave this result here as our proof is much simpler.

Theorem 2. *For every $t \geq 4$ and every positive integer n , there is an ordered 3-uniform hypergraph \mathcal{H} on t vertices such that $\overline{R}(\mathcal{H}, \mathcal{K}_3^{(3)}(n)) \geq 2^{\Omega(n \log n)}$.*

We do not have any nontrivial lower bound in the diagonal case $\overline{R}(\mathcal{H})$ for \mathcal{H} with bounded maximum degree and $\chi_{<}(\mathcal{H}) = 3$. Even for ordered graphs \mathcal{G} with bounded maximum degree d and $\chi_{<}(\mathcal{G}) = 2$ the best known lower and upper bounds on $\overline{R}(\mathcal{G})$ are of order $\Omega((n/\log n)^2)$ [3] and $n^{O(d)}$ [2, 5], respectively.

Concerning k -uniform hypergraphs with $k > 3$, the following result is based on a modification of the proof from [7, Proposition 6.3] and gives an estimate on ordered Ramsey numbers of ordered k -uniform hypergraphs with bounded interval chromatic number. In particular, this estimate shows that we do not have a tower-type growth rate for $\overline{R}(\mathcal{H})$ once the uniformity and the interval chromatic number of \mathcal{H} are bounded.

Proposition 1. *Let χ, k be integers with $\chi \geq k \geq 2$ and let \mathcal{H} be an ordered k -uniform hypergraph on n vertices with interval chromatic number χ . Then there is a constant c such that $\overline{R}(\mathcal{H}) \leq 2^{R^{\chi(x-1)(cxn)^{x-1}}}$, where $R = R(K_\chi^{(k)})$. In particular, if the uniformity k and the interval chromatic number χ of \mathcal{H} are fixed, we have $\overline{R}(\mathcal{H}) \leq 2^{O(n^{\chi-1})}$.*

Our understanding of the ordered Ramsey numbers of ordered hypergraphs is still very limited. Many interesting open problem arose during our study and we would like to draw attention to some of them in the full version.

Acknowledgment. The first author was supported by the grant no. 18-13685Y of the Czech Science Foundation (GAČR) and by the Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004). This article is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115).

The second author was supported by the Hungarian National Research, Development and Innovation Office – NKFIH under the grant SNN 129364, KH 130371 and FK 132060 by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and by the New National Excellence Program under the grant number ÚNKP-20-5-BME-45.

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Hypergraphs with Minimum Positive Uniform Turán Density

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Abstract. Reiher, Rödl and Schacht [J. London Math. Soc. 97 (2018), 77–97] showed that the uniform Turán density of every 3-uniform hypergraph is either 0 or at least $1/27$, and asked whether there exist 3-uniform hypergraphs with uniform Turán density equal or arbitrarily close to $1/27$. We construct 3-uniform hypergraphs with uniform Turán density equal to $1/27$.

Keywords: Extremal combinatorics · Turán density · Uniform Turán density

1 Introduction

Determining the minimum density of a (large) combinatorial structure that guarantees the existence of a given (small) substructure is a classical extremal combinatorics problem, which can be traced to the work of Turán [21] in the early 1940s. The *Turán density* of a k -uniform hypergraph H , which is denoted by $\pi(H)$, is the infimum over all d such that every sufficiently large host k -uniform hypergraph with edge density at least d contains H as a subhypergraph. It can be shown [9] that the Turán density of H is equal to the limit of the maximum density of a k -uniform n -vertex H -free hypergraph (n tends to infinity); in particular, Katona, Nemetz and Simonovits [9] showed that this sequence of maximum densities is non-increasing and so the limit always exists.

The Turán density of a complete graph K_r of order r is equal to $\frac{r-2}{r-1}$ as determined by Turán himself, and Erdős and Stone [5] showed that the Turán density of any r -chromatic graph H is equal to $\frac{r-2}{r-1}$ (also see [3]). The situation is more complex already for 3-uniform hypergraphs, in particular, determining the Turán density of a complete 3-uniform 4-vertex hypergraph $K_4^{(3)}$ is a major open problem, and likewise determining the Turán density of $K_4^{(3)-}$, i.e., $K_4^{(3)}$ with an edge removed, is a challenging open problem [1, 6, 12] despite some recent progress obtained using the flag algebra method of Razborov [11]; also see the survey [10] for further details.

It is well-known that H -free graphs with density close to the Turán density $\pi(H)$ are close to $(r-1)$ -partite complete graphs [7, 20], i.e., the edges in such graphs are distributed in a highly non-uniform way. The same applies to conjectured extremal constructions in the setting of 3-uniform hypergraphs [6]. We study the notion of uniform Turán density of hypergraphs, which requires the edges of the host hypergraph to be distributed uniformly. This notion was suggested by Erdős and Sós [2, 4] in the 1980s and there is a large amount of recent progress in relation to this notion and to some of its variants [8, 14–18], also see the survey [13].

The following result of Reiher, Rödl and Schacht [15] is the starting point of our work: the uniform Turán density of every 3-uniform hypergraph is either zero or at least $1/27$. Reiher et al. [15] asked whether there exist 3-uniform hypergraphs with the uniform Turán density equal or arbitrarily close to $1/27$. We answer this question in the affirmative by giving a sufficient condition for a 3-uniform hypergraph to have the uniform Turán density equal to $1/27$ and by constructing examples of 3-uniform hypergraphs satisfying this condition.

2 Main Result

We start with introducing the notation needed to state our results precisely. The ε -linear density of an n -vertex hypergraph H is the minimum density of an induced subhypergraph of H with at least εn vertices. The *uniform Turán density* of a hypergraph H_0 is the infimum over all d such that for every $\varepsilon > 0$, every sufficiently large hypergraph H with ε -linear density at least d contains H_0 . An equivalent definition, which is used by Reiher, Rödl and Schacht [14–18] reads as follows: an n -vertex k -uniform hypergraph H is (d, ε) -dense if every subset W of its vertices induces at least $d \binom{|W|}{k} - \varepsilon n^k$ edges, and the uniform Turán density of a hypergraph H_0 is the supremum over all d such that for every $\varepsilon > 0$, there exist arbitrarily large H_0 -free (d, ε) -dense hypergraphs. It is easy to show that the two definitions are equivalent.

The notion of the uniform Turán density is trivial for graphs as the uniform Turán density of every graph is equal to zero. However, the situation is much more complex already for 3-uniform hypergraphs. As we have already mentioned, the uniform Turán density of $K_4^{(3)-}$ has been determined only recently [8, 17]. Determining the uniform Turán density of $K_4^{(3)}$ is a challenging open problem though it is believed that the 35-year-old construction of Rödl [19] showing that the uniform Turán density of $K_4^{(3)}$ is at least $1/2$ is optimal [13].

Reiher, Rödl and Schacht [15] gave a simple characterization of 3-uniform hypergraphs with the uniform Turán density equal to zero, which we now present. Let H be a 3-uniform hypergraph with n vertices. We say that an ordering v_1, \dots, v_n of its vertices is *vanishing* if the set of pairs (i, j) , $1 \leq i < j \leq n$, can be partitioned to sets L , T and R such that every edge $\{v_i, v_j, v_k\}$ of H , where $i < j < k$, satisfies that $(i, j) \in L$, $(i, k) \in T$ and $(j, k) \in R$. The pairs that belong to L , T and R are referred to as *left*, *top* and *right*, respectively.

The characterization of 3-uniform hypergraphs with the uniform Turán density equal to zero is the following.

Theorem 1 (Reiher, Rödl and Schacht [15]). *Let H be a 3-uniform hypergraph. The uniform Turán density of H is zero if and only if H has a vanishing ordering of its vertices.*

If a 3-uniform hypergraph H has no vanishing ordering, then the uniform Turán density of H is at least $1/27$ because of the following construction from [15]. Fix a 3-uniform hypergraph H with no vanishing ordering and construct a random n -vertex 3-uniform hypergraph H_n as follows: let v_1, \dots, v_n be the vertices of H_n , randomly partition their pairs to sets L, T and R , and include $\{v_i, v_j, v_k\}$, $1 \leq i < j < k \leq n$, as an edge of H_n if $(i, j) \in L$, $(i, k) \in T$ and $(j, k) \in R$. Observe that H cannot be a subhypergraph of H_n (as H has no vanishing ordering). On the other hand, for every $\varepsilon > 0$ and $\delta > 0$, there exists n_0 such that the density of every subset of at least εn vertices of H_n for $n \geq n_0$ is at least $1/27 - \delta$ with positive probability. It follows that the uniform Turán density of H is at least $1/27$ as claimed. Hence, Theorem 1 implies the following.

Corollary 1. *The uniform Turán density of every 3-uniform hypergraph is either zero or at least $1/27$.*

Reiher, Rödl and Schacht [15] asked whether there exist 3-uniform hypergraphs with the uniform Turán density equal or arbitrarily close to $1/27$. The following theorem, which is our main result, gives a sufficient condition on a 3-uniform hypergraph to have the uniform Turán density equal to $1/27$.

Theorem 2. *Let H_0 be an n -vertex 3-uniform hypergraph that*

- *has no vanishing ordering of its vertices,*
- *can be partitioned to two spanning subhypergraphs H_1 and H_2 such that there exists an ordering of the vertices that is vanishing both for H_1 and H_2 and if e_1 is an edge of H_1 and e_2 is an edge of H_2 such that $|e_1 \cap e_2| = 2$, then the pair $e_1 \cap e_2$ is right with respect to H_1 and left with respect to H_2 , and*
- *can be partitioned to two spanning subhypergraphs H'_1 and H'_2 such that there exists an ordering of the vertices that is vanishing both for H'_1 and H'_2 and if e_1 is an edge of H'_1 and e_2 is an edge of H'_2 such that $|e_1 \cap e_2| = 2$, then the pair $e_1 \cap e_2$ is top with respect to H'_1 and left with respect to H'_2 .*

The uniform Turán density of H_0 is equal to $1/27$.

In Sect. 3, we present a 7-vertex 9-edge hypergraph (Theorem 3) and an infinite family of hypergraphs (Theorem 4), whose smallest element has 8 vertices and 9 edges, that have the properties given in Theorem 2. We remark that it is possible to show that the uniform Turán density of every 3-uniform hypergraph with at most 6 vertices is either zero or at least $1/8$.

We next present some steps of the proof of Theorem 2 and the main lemma needed for the proof. Using [13, Theorem 3.3], we consider so-called reduced

hypergraphs instead of general hypergraphs; these are hypergraphs that represent a regularity partition of a 3-uniform hypergraph. If a regularity partition of a 3-uniform hypergraph has N parts, the *reduced hypergraph* has a vertex set partitioned to sets $V_{ii'}$ corresponding to pairs of parts of the regularity partition, i.e., $i, i' \in [N]$. Three such sets corresponding to the three pairs in the same triple of parts form a *triad*, i.e., a triad is formed by sets $V_{ii'}$, $V_{ii''}$ and $V_{i'i''}$ for some $i, i', i'' \in [N]$. Edges of a reduced hypergraph may involve only vertices from different sets of the same triad. The condition of the uniform density of a hypergraph translates to a condition on the density of each triad.

We say that an ℓ -vertex 3-uniform hypergraph H_0 *embeds* in a reduced hypergraph H_R if it is possible to associate each vertex of H_0 with one of the parts of the reduced hypergraph H_R , say that k_1, \dots, k_n are indices of the associated parts, and to choose from each set $V_{k_i k_{i'}}$, $i, i' \in [\ell]$, a vertex $v_{ii'}$ in a way that if the vertices of an edge of H_0 are associated with the parts $k_i, k_{i'}$ and $k_{i''}$, then the vertices $v_{ii'}$, $v_{ii''}$ and $v_{i'i''}$ form an edge in H_R . Theorem 3.3 in [13] asserts that if H_0 embeds in every sufficiently large reduced hypergraph with the minimum density of its triad at least d , then the uniform Turán density of H_0 is at most d .

The following lemma is the main tool to prove Theorem 2. The *reverse* of a reduced hypergraph is the reduced hypergraph obtained by indexing the parts in the opposite order, i.e., the i -th part becomes the $(N + 1 - i)$ -th part if there are N parts in total.

Lemma 1. *For every real $\delta > 0$ and positive integer n , there exists an integer N such that the following holds. If H_R is a reduced hypergraph with N parts and with density of each triad at least $1/27 + \delta$, then the following is true for H_R or for the reverse of H_R . There exist indices k_1, \dots, k_n and vertices $\alpha_{ii'}, \beta_{ii'}, \gamma_{ii'}, \beta'_{ii'}, \gamma'_{ii'} \in V_{k_i k_{i'}}$ for all $1 \leq i < i' \leq n$ such that*

– $\{\alpha_{ii'}, \beta_{ii'}, \gamma_{ii'}\}$ is an edge for all $1 \leq i < i' < i'' \leq n$, and

at least one of the following holds:

– $\{\beta_{ii'}, \beta'_{ii'}, \gamma'_{ii'}\}$ is an edge for all $1 \leq i < i' < i'' \leq n$, or

– $\{\gamma_{ii'}, \beta'_{ii'}, \gamma'_{ii'}\}$ is an edge for all $1 \leq i < i' < i'' \leq n$.

3 Hypergraphs with Uniform Turán Density Equal to $1/27$

In this section, we give examples of hypergraphs that satisfy the assumption of Theorem 2 and so their uniform Turán density is equal to $1/27$. We have verified by a computer that there is no such hypergraph with six or fewer vertices; in fact, we are able to show that every 3-uniform hypergraph with six or fewer vertices has Turán density either equal to zero or at least $1/8$.

As the first example of a 3-uniform hypergraph with uniform Turán density equal to $1/27$, we present a hypergraph with seven vertices. This hypergraph has a non-trivial group of automorphisms, which correspond to a vertical mirror symmetry in Fig. 1 where the hypergraph is visualized.

Theorem 3. *Let H be a 3-uniform hypergraph with seven vertices a, \dots, g and the following 9 edges: $abc, ade, bcd, bcf, cde, def, abg, cdg$ and efg . The uniform Turán density of H is equal to $1/27$.*

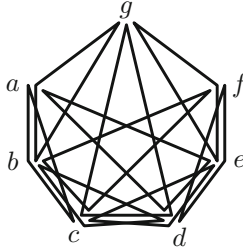


Fig. 1. The 3-uniform hypergraph H described in the statement of Theorem 3 (the edges correspond to the drawn triangles).

As our second example, we present a family of 3-uniform hypergraphs, which enjoys three cyclic symmetries (by mapping the vertices c_i, d_i and e_i to each other in a cyclic way); the smallest hypergraph in the family has eight vertices and nine edges.

Theorem 4. *For a positive integer k , let H^k be a 3-uniform hypergraph with $5+3k$ vertices $a, b, c_0, \dots, c_k, d_0, \dots, d_k, e_0, \dots, e_k$ and the following $3(k+2)$ edges:*

$$\begin{aligned}
 &abc_0, bc_0c_1, c_0c_1c_2, \dots, c_{k-2}c_{k-1}c_k, c_{k-1}c_kd_k, \\
 &abd_0, bd_0d_1, d_0d_1d_2, \dots, d_{k-2}d_{k-1}d_k, d_{k-1}d_ke_k, \\
 &abe_0, be_0e_1, e_0e_1e_2, \dots, e_{k-2}e_{k-1}e_k, e_{k-1}e_kc_k.
 \end{aligned}$$

The uniform Turán density of H^k is equal to $1/27$.

4 Conclusion

We conclude with the following open problem.

Problem 1. Does there exist $c > 1/27$ such that the uniform Turán density of every 3-uniform hypergraph is either equal to zero, equal to $1/27$ or at least c ?

Acknowledgements. The authors would like to thank Jacob Cooper for his comments on the topics covered in this paper.

This work has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 648509). This publication reflects only its authors’ view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains. The authors were also supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University.

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Rainbow Cliques in Randomly Perturbed Dense Graphs

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Abstract. For two graphs G and H , write $G \xrightarrow{\text{rbw}} H$ if G has the property that every *proper* colouring of its edges yields a *rainbow* copy of H . We study the thresholds for such so-called *anti-Ramsey* properties in randomly perturbed dense graphs, which are unions of the form $G \cup \mathbb{G}(n, p)$, where G is an n -vertex graph with edge-density at least d , and d is a constant that does not depend on n .

We determine the threshold for the property $G \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_s$ for every s . We show that for $s \geq 9$ the threshold is $n^{-1/m_2(K_{\lceil s/2 \rceil})}$; in fact, our 1-statement is a supersaturation result. This turns out to (almost) be the threshold for $s = 8$ as well, but for every $4 \leq s \leq 7$, the threshold is lower and is different for each $4 \leq s \leq 7$.

Moreover, we prove that for every $\ell \geq 2$ the threshold for the property $G \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_{2\ell-1}$ is n^{-2} ; in particular, the threshold does not depend on the length of the cycle $C_{2\ell-1}$. It is worth mentioning that for even cycles, or more generally for any fixed bipartite graph, no random edges are needed at all.

Keywords: Random graphs · Anti-Ramsey · Randomly perturbed graphs

1 Introduction

A *random perturbation* of a fixed n -vertex graph G , denoted by $G \cup \mathbb{G}(n, p)$, is a distribution over the supergraphs of G with the latter generated through the addition of random edges sampled from the binomial random graph of edge-density p , namely $\mathbb{G}(n, p)$. The fixed graph G being *perturbed* or *augmented* in this manner is referred to as the *seed* of the *perturbation* $G \cup \mathbb{G}(n, p)$.

The above model was introduced by Bohman, Frieze, and Martin [6], who allowed the seed G to range over the family of n -vertex graphs with minimum degree at least δn , denoted by $\mathcal{G}_{\delta,n}$. In particular, they discovered the phenomenon that for every $\delta > 0$, there exists a constant $C(\delta) > 0$ such that $G \cup \mathbb{G}(n, p)$ a.a.s. admits a Hamilton cycle, whenever $p := p(n) \geq C(\delta)/n$ and $G \in \mathcal{G}_{\delta,n}$. Their bound on p undershoots the threshold for Hamiltonicity in

$\mathbb{G}(n, p)$ by a logarithmic factor. The notation $\mathcal{G}_{\delta, n} \cup \mathbb{G}(n, p)$ then suggests itself to mean the collection of perturbations arising from the members of $\mathcal{G}_{\delta, n}$ for a prescribed $\delta > 0$.

Several strands of results regarding the properties of randomly perturbed (hyper)graphs can be found in the literature. One prominent such strand can be seen as an extension of the aforementioned result of [6]. Indeed, the emergence of various spanning configurations in randomly perturbed (hyper)graphs was studied, for example, in [3, 5, 7, 8, 11, 12, 15, 16, 22].

Another prominent line of research regarding random perturbations concerns Ramsey properties of $\mathcal{G}_{d, n} \cup \mathbb{G}(n, p)$, where here $\mathcal{G}_{d, n}$ stands for the family of n -vertex graphs with edge-density at least $d > 0$, and d is a constant. This strand stems from the work of Krivelevich, Sudakov, and Tetali [17] and is heavily influenced by the now fairly mature body of results regarding the thresholds of various Ramsey properties in random graphs see, e.g. [21, 26–28].

Krivelevich, Sudakov, and Tetali [17], amongst other things, proved that for every real $d > 0$, integer $t \geq 3$, and graph $G \in \mathcal{G}_{d, n}$, the perturbation $G \cup \mathbb{G}(n, p)$ a.a.s. satisfies the property $G \cup \mathbb{G}(n, p) \rightarrow (K_3, K_t)$, whenever $p := p(n) = \omega(n^{-2/(t-1)})$; moreover, this bound on p is asymptotically best possible. Here, the notation $G \rightarrow (H_1, \dots, H_r)$ is used to denote that G has the *asymmetric* Ramsey property asserting that any r -edge-colouring of G admits a colour $i \in [r]$ such that H_i appears with all its edges assigned the colour i .

Recently, the aforementioned result of Krivelevich, Sudakov, and Tetali [17] has been significantly extended by Das and Treglown [10] and also by Powierski [25]. In particular, there is now a significant body of results pertaining to the property $G \cup \mathbb{G}(n, p) \rightarrow (K_r, K_s)$ for any pair of integers $r, s \geq 3$, whenever $G \in \mathcal{G}_{d, n}$ for constant $d > 0$. Further in this direction, the work of Das, Morris, and Treglown [9] extends the results of Kreuter [14] pertaining to *vertex Ramsey* properties of random graphs into the perturbed model.

A subgraph $H \subseteq G$ is said to be *rainbow* with respect to an edge colouring ψ , if any two of its edges are assigned different colours under ψ . An edge-colouring ψ of a graph G is said to be *proper* if incident edges are assigned distinct colours under ψ . We write $G \xrightarrow{\text{rbw}} H$, if G has the property that every proper colouring of its edges admits a rainbow copy of H . The first to consider the emergence of small fixed rainbow configurations in random graphs with respect to proper colourings were Rödl and Tuza [29]. The systematic study of the emergence of general rainbow fixed graphs in random graphs with respect to proper colourings was initiated by Kohayakawa, Kostadinidis and Mota [18, 19].

In [18] it is proved that for every graph H , there exists a constant $C > 0$ such that $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$, whenever $p \geq Cn^{-1/m_2(H)}$, where here $m_2(H)$ denotes the *maximum 2-density* of H , see e.g. [13]. Nenadov, Person, Škorić, and Steger [24] proved, amongst other things, that for $H \cong C_\ell$ with $\ell \geq 7$, and for $H \cong K_r$ with $r \geq 19$, $n^{-1/m_2(H)}$ is, in fact, the threshold for the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$. Barros, Cavalari, Mota, and Parczyk [4] extended the result of [24] for cycles, proving that the threshold of the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_\ell$ remains $n^{-1/m_2(C_\ell)}$ also when $\ell \geq 5$. Kohayakawa, Mota, Parczyk, and Schnitzer [20] extended the

result of [24] for complete graphs, proving that the threshold of $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_r$ remains $n^{-1/m_2(K_r)}$ also when $r \geq 5$.

For C_4 and K_4 the situation is different. The threshold for the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_4$ is $n^{-3/4} = o(n^{-1/m_2(C_4)})$, as proved by Mota [23]. For the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$, the threshold is $n^{-7/15} = o(n^{-1/m_2(K_4)})$ as proved by Kohayakawa, Mota, Parczyk, and Schnitzer [20]. More generally, Kohayakawa, Kostadinidis and Mota [19] proved that there are infinitely many graphs H for which the threshold for the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ is significantly smaller than $n^{-1/m_2(H)}$.

Lastly, properly edge-coloured triangles are rainbow. Hence, the thresholds for the properties $K_3 \subseteq \mathbb{G}(n, p)$ and $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_3$ coincide so that n^{-1} is the threshold for the latter.

1.1 Our Results

For a real $d > 0$, we say that $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. satisfies a graph property \mathcal{P} , if $\lim_{n \rightarrow \infty} \mathbb{P}[G_n \cup \mathbb{G}(n, p) \in \mathcal{P}] = 1$ holds for every sequence $\{G_n\}_{n \in \mathbb{N}}$ satisfying $G_n \in \mathcal{G}_{d,n}$ for every $n \in \mathbb{N}$. We say that $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. does not satisfy \mathcal{P} , if $\lim_{n \rightarrow \infty} \mathbb{P}[G_n \cup \mathbb{G}(n, p) \in \mathcal{P}] = 0$ holds for at least one sequence $\{G_n\}_{n \in \mathbb{N}}$ satisfying $G_n \in \mathcal{G}_{d,n}$ for every $n \in \mathbb{N}$. Throughout, we suppress this sequence-based terminology and write more concisely that $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. satisfies (or does not) a certain property. In particular, given a fixed graph H , we write that a.a.s. $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ to mean that for every sequence $\{G_n\}_{n \in \mathbb{N}}$, satisfying $G_n \in \mathcal{G}_{d,n}$ for every $n \in \mathbb{N}$, the property $G_n \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ holds asymptotically almost surely. On the other hand, we write that a.a.s. $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \not\xrightarrow{\text{rbw}} H$ to mean that there exists a sequence $\{G_n\}_{n \in \mathbb{N}}$, satisfying $G_n \in \mathcal{G}_{d,n}$ for every $n \in \mathbb{N}$, for which a.a.s. $G_n \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ does not hold.

A sequence $\hat{p} := \hat{p}(n)$ is said to form a *threshold* for the property \mathcal{P} in the perturbed model, if $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. satisfies \mathcal{P} whenever $p = \omega(\hat{p})$, and if $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. does not satisfy \mathcal{P} whenever $p = o(\hat{p})$.

For every real $d > 0$ and every pair of integers $s, t \geq 1$, every sufficiently large graph $G \in \mathcal{G}_{d,n}$ satisfies $G \xrightarrow{\text{rbw}} K_{s,t}$; in fact, every proper colouring of G supersaturates G with $\Omega(n^{s+t})$ rainbow copies of $K_{s,t}$. Consequently, the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_{s,t}$ is trivial as no random perturbation is needed for it to be satisfied. The emergence of rainbow copies of non-bipartite prescribed graphs may then be of interest. For odd cycles (including K_3), we prove the following.

Proposition 1. *For every integer $\ell \geq 2$, and every real $0 < d \leq 1/2$, the threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_{2\ell-1}$ is n^{-2} .*

Unlike the threshold for the property $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_\ell$, established in [4, 24], the threshold for the counterpart property in the perturbed model is independent of the length of the cycle.

Our main result concerns the thresholds for the emergence of rainbow complete graphs in properly coloured randomly perturbed dense graphs. From the results of [20, 24], one easily deduces that if $r \geq 5$ and $p = o(n^{-1/m_2(K_r)})$, then a.a.s. there exists a proper edge-colouring of $\mathbb{G}(n, p)$ admitting no rainbow copy of K_r . Consequently, given a real number $0 < d \leq 1/2$ and an n -vertex bipartite graph G of edge-density d , a.a.s. there exists a proper edge-colouring of $G \cup \mathbb{G}(n, p)$ admitting no rainbow copy of K_{2r-1} , provided that $p = o(n^{-1/m_2(K_r)})$. We conclude that $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_{2r}$ and $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_{2r-1}$ hold a.a.s. whenever $p = o(n^{-1/m_2(K_r)})$.

For every $r \geq 5$, we prove a matching upper bound for the above construction. Our main result reads as follows.

Theorem 1. *Let a real number $0 < d \leq 1/2$ and an integer $r \geq 5$ be given. Then, the threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_{2r}$ is $n^{-1/m_2(K_r)}$. In fact, $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ a.a.s. has the property that every proper colouring of its edges gives rise to $\Omega\left(p^{\binom{r}{2}} n^{2r}\right)$ rainbow copies of K_{2r} , whenever $p = \omega(n^{-1/m_2(K_r)})$.*

The following result is an immediate consequence of Theorem 1 and of the aforementioned lower bound.

Corollary 1. *Let a real number $0 < d \leq 1/2$ and an integer $r \geq 5$ be given. Then, the threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_{2r-1}$ is $n^{-1/m_2(K_r)}$.*

Theorem 1 and Corollary 1 establish that for sufficiently large complete graphs, i.e., K_s with $s \geq 9$, the threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_s$ is governed by a single parameter, namely, $m_2(K_{\lceil s/2 \rceil})$. This turns out to be true (almost, at least) for $s = 8$ as well, but proving it requires new ideas. For $4 \leq s \leq 7$, this is not the case; here, for each value of s in this range, the threshold is different. Using completely different methods, we prove the following.

Theorem 2. *Let $0 < d \leq 1/2$ be given.*

1. *The threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$ is $n^{-5/4}$*
2. *The threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_5$ is n^{-1} .*
3. *The threshold for the property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_7$ is $n^{-7/15}$.*

For K_6 and K_8 , we can “almost” determine the thresholds.

Theorem 3. *Let $0 < d \leq 1/2$ be given.*

1. *The property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_6$ holds a.a.s. whenever $p = \omega(n^{-2/3})$.*
2. *For every constant $\varepsilon > 0$ it holds that a.a.s. $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_6$ whenever $p := p(n) = n^{-(2/3+\varepsilon)}$.*

Theorem 4. *Let $0 < d \leq 1/2$ be given.*

1. The property $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_8$ holds a.a.s. whenever $p = \omega(n^{-2/5})$.
2. For every constant $\varepsilon > 0$ it holds that a.a.s. $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_8$ whenever $p := p(n) = n^{-(2/5+\varepsilon)}$.

Proofs of all of our results can be found in [1, 2].

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Universal Singular Exponents in Catalytic Variable Equations

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Abstract. Catalytic equations appear in several combinatorial applications, most notably in the enumeration of lattice paths and in the enumeration of planar maps. The main purpose of this paper is to show that the asymptotic estimate for the coefficients of the solutions of (so-called) positive catalytic equations has a universal asymptotic behavior. In particular, this provides a rationale why the number of maps of size n in various planar map classes grows asymptotically like $c \cdot n^{-5/2} \gamma^n$, for suitable positive constants c and γ . Essentially we have to distinguish between linear catalytic equations (where the subexponential growth is $n^{-3/2}$) and non-linear catalytic equations (where we have $n^{-5/2}$ as in planar maps). The Proofs are based on a delicate analysis of systems of polynomial equations and singularity analysis and are omitted for lack of space. positive. (Supported by the Ministerio de Economía y Competitividad grant MTM2017-82166-P, and by the Special Research Program F50 *Algorithmic and Enumerative Combinatorics* of the Austrian Science Fund. Research supported by the Austrian Science Foundation FWF, project F 50-02.).

Keywords: Catalytic equations · Asymptotic estimates · Planar maps · Central limit theorem

1 Introduction and Statement of Results

A planar map is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is rooted if an edge e is distinguished and directed. This edge is called the root-edge. The initial vertex v of this (directed) root-edge is then the root-vertex. The face to the right of e is called the root-face and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps (up to homeomorphisms) is a classical subject, initiated by Tutte in the 1960's [7, 8]. Tutte (and Brown) introduced the

technique now called “the quadratic method” in order to compute the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!} 3^n. \tag{1}$$

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc. Using the previous formula, Stirling’s estimate gives $M_n \sim (2/\sqrt{\pi}) \cdot n^{-5/2} 12^n$. In all cases where a “natural” condition is imposed on maps, the asymptotic estimates turn out to be of this kind:

$$c \cdot n^{-5/2} \gamma^n.$$

The constants c and γ depend on the class under consideration, but one gets systematically an $n^{-5/2}$ term in the estimate.

This phenomenon is discussed by Banderier et al. [1]: ‘This generic asymptotic form is “universal” in so far as it is valid for all known “natural families of maps”.’ The goal of this paper is to provide to some extent an explanation for this universal phenomenon, based on a detailed analysis of functional equations for generating functions with a catalytic variable. Let us mention that the critical exponent $-5/2$ has been ‘explained’ previously in at least two different ways. In the physics literature using matrix integrals [3], and using bijections between classes of planar maps and ‘decorated trees’ [6].

Let us recall the basic technique for counting planar maps. Let $M_{n,k}$ be the number of maps with n edges and in which the degree of the root-face is equal to k . Let $M(z, u) = \sum_{n,k} M_{n,k} u^k z^n$ be the associated generating function. As shown by Tutte [8], $M(z, u)$ satisfies the quadratic equation

$$M(z, u) = 1 + zu^2M(z, u)^2 + uz \frac{uM(z, u) - M(z, 1)}{u - 1}. \tag{2}$$

In this context the variable u is usually called a “catalytic variable” (see [2]). It is needed to formulate and solve the equation, but afterwards can be ignored if we are just interested in the univariate generating function $M(z, 1) = \sum_n M_n z^n$ of $M_n = \sum_k M_{n,k}$. It turns out that

$$M(z, 1) = \sum_{n \geq 0} M_n z^n = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \dots, \tag{3}$$

from which we can deduce the explicit form (1). The remarkable fact here is the singular part $(1 - 12z)^{3/2}$ that reflects the asymptotic behavior $c \cdot n^{-5/2} 12^n$ of M_n .

A general approach to equations of the form (2) was carried out by Bousquet-Mélou and Jehanne [2]. First one rewrites (2) into the form

$$P(M(z, u), M_1(z), z, u) = 0,$$

where $P(x_0, x_1, z, u)$ is a polynomial and $M_1(z)$ abbreviates $M(z, 0)$ or $M(z, 1)$. Next one searches for functions $f(z)$, $y(z)$ and $u(z)$ with¹

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0, \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0, \\ P_u(f(z), y(z), z, u(z)) &= 0. \end{aligned} \tag{4}$$

The idea is to bind u and z in the function $G(z, u) = P_{x_0}(M(z, u), M_1(z), z, u)$ so that $G(z, u(z)) = 0$ for a proper function $u(z)$. By taking the derivative of $P(M(z, u), M_1(z), z, u)$ with respect to u one has

$$P_{x_0}(M(z, u), M_1(z), z, u)M_u(z, u) + P_u(M(z, u), M_1(z), z, u). \tag{5}$$

Thus, if $G(z, u(z)) = P_{x_0}(M(z, u(z)), M_1(z), z, u(z)) = 0$ then we also have the relation $P_u(M(z, u(z)), M_1(z), z, u(z)) = 0$. This leads to the system (4) for $f(z) = M(z, u(z))$, $y(z) = M_1(z)$ and $u(z)$.

At this moment it is not completely clear that this procedure gives the correct solution. To show that this is the case we can argue as follows. Bousquet-Mélou and Jehanne [2] considered in particular equations of the form²

$$\boxed{M(z, u) = F_0(u) + zQ\left(M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u\right)}, \tag{6}$$

where $F_0(u)$ and $Q(\alpha_0, \alpha_1, z, u)$ are polynomials, that is

$$P(x_0, x_1, z, u) = F_0(u) + zQ(x_0, (x_0 - x_1)/u, z, u) - x_0,$$

and showed that there is a unique power series solution $M(z, u)$, and that it is also an algebraic function. The Eq. (5) is now (if we multiply by u)

$$\begin{aligned} zuQ_{\alpha_0}\left(M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u\right) \\ + zQ_{\alpha_1}\left(M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u\right) - u = 0. \end{aligned} \tag{7}$$

Clearly this equation has a power series solutions $u(z)$ with $u(0) = 0$. Thus, the power series $f(z) = M(z, u(z))$, $y(z) = M(z, 0)$, $u(z)$ solve the system (4).

In the context of this paper we always assume that F_0 and Q have non-negative coefficients. This is natural since Eq. (6) can be seen as a translation of a recursive combinatorial description of maps or other combinatorial objects. This also implies that $M(z, u)$ has non-negative coefficients, since the Eq. (6) can be written as an infinite system of equations for the functions $M_j(z) = [u^j]M(z, u)$ with non-negative coefficients on the right hand side.

Proofs are omitted for lack of space; see [5] for full proofs.

¹ We denote by $P_x = \frac{\partial P}{\partial x}$ the partial derivative of the function P with respect to the variable x .

² Actually Bousquet-Mélou and Jehanne [2] considered more general functional equations that contain also higher differences.

1.1 The Linear Case

We consider the first case, where Q is linear in α_0 and α_1 , so we can write (6) as

$$M(z, u) = Q_0(z, u) + zM(z, u)Q_1(z, u) + z \frac{M(z, u) - M(z, 0)}{u} Q_2(z, u). \tag{8}$$

Here we are in the framework of the so-called *kernel method*. We rewrite (8) as

$$M(z, u)(u - zuQ_1(z, u) - zQ_2(z, u)) = uQ_0(z, u) - zM(z, 0)Q_2(z, u), \tag{9}$$

where

$$K(z, u) = u - zuQ_1(z, u) - zQ_2(z, u)$$

is the *kernel*. The idea of the kernel method is to bind u and z so that $K(z, u) = 0$, that is, one considers a function $u = u(z)$ such that $K(z, u(z)) = 0$. Then the left hand side of (9) cancels and $M(z, 0)$ can be calculated from the right hand side by setting $u = u(z)$. Of course, the kernel equation is precisely the Eq. (7). The kernel method is just a special case of the general procedure of Bousquet-Mélou and Jehanne [2].

Proposition 1. *Suppose that $Q_0, Q_1,$ and Q_2 are polynomials in z and u with non-negative coefficients and let $M(z, u)$ be the power series solution of (8). Furthermore let $u(z)$ be the power series solution of the equation*

$$u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z)), \quad \text{with } u(0) = 0.$$

Then $M(z, 0)$ is given by

$$M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))}.$$

Disregarding some exceptional cases (not included here for lack of space; see [5] for details), $M(z, 0)$ has universally a dominant square root singularity as our first main theorem states. We recall that $M(z, 0)$ is an algebraic function and has, thus, a Puiseux expansion around its (dominant) singularity.

Theorem 2. *Suppose that $Q_0, Q_1,$ and Q_2 are polynomials in z and u with non-negative coefficients such that $Q_1(z, u)$ depends on u and such that $Q_2(z, 0) \neq 0$.*

Let $M(z, u)$ be the power series solution of (8) and let $z_0 > 0$ be the radius of convergence of $M(z, 0)$. Then the local Puiseux expansion of $M(z, 0)$ at z_0 is given by

$$M(z, 0) = a_0 + a_1(1 - z/z_0)^{1/2} + a_2(1 - z/z_0) + \dots, \tag{10}$$

where $a_0 > 0$ and $a_1 < 0$. Furthermore, there exist $b \geq 1$, a non-empty set $J \subseteq \{0, 1, \dots, b - 1\}$ of residue classes modulo b and constants $c_j > 0$ such that for $j \in J$

$$M_n = [z^n]M(z, 0) = c_j n^{-3/2} z_0^{-n} \left(1 + O\left(\frac{1}{n}\right) \right), \quad (n \equiv j \pmod{b}, n \rightarrow \infty) \tag{11}$$

and $M_n = 0$ for $n \equiv j \pmod{b}$ with $j \notin J$ if Q_1 depends on u or $M_n = O((z_0(1 + \eta))^{-n})$ for some $\eta > 0$ and $n \equiv j \pmod{b}$ with $j \notin J$ if Q_1 does not depend on u .

1.2 The Non-linear Case

In the non-linear case the situation is more involved. Here we find the solution function $M(z, 0)$ in the following way.

Proposition 3. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1 , that is, $Q_{\alpha_1} \neq 0$, and let $M(z, u)$ be the power series solution of (6). Furthermore we assume that Q is not linear in α_0 and α_1 , that is, $Q_{\alpha_0\alpha_0} \neq 0$, or $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$.*

Let $f(z), u(z), w(z)$ be the power series solution of the system of equations

$$\begin{aligned} f(z) &= F_0(u(z)) + zQ(f(z), w(z), z, u(z)), \\ u(z) &= zu(z)Q_{\alpha_0}(f(z), w(z), z, u(z)) + zQ_{\alpha_1}(f(z), w(z), z, u(z)), \\ w(z) &= F'_0(u(z)) + zQ_u(f(z), w(z), z, u(z)) + zw(z)Q_{\alpha_0}(f(z), w(z), z, u(z)). \end{aligned} \tag{12}$$

with $f(0) = F_0(0), u(0) = 0, w(0) = F'_0(0)$. Then

$$M(z, 0) = f(z) - w(z)u(z).$$

We again recall that $M(z, 0)$ is an algebraic function and has, thus, a Puiseux expansion around its singularities.

Theorem 4. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1 , that is, $Q_{\alpha_1} \neq 0$ and let $M(z, u)$ be the power series solution of (6). Furthermore, we assume that Q is not linear in α_0 and α_1 , that is, $Q_{\alpha_0\alpha_0} \neq 0$ or $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$. We assume additionally that $Q_{\alpha_0u} \neq 0$.*

Let $z_0 > 0$ denote the radius of convergence of $M(z, 0)$. Then the local Puiseux expansion of $M(z, 0)$ around z_0 is given by

$$M(z, 0) = a_0 + a_2(1 - z/z_0) + a_3(1 - z/z_0)^{3/2} + O((1 - z/z_0)^2), \tag{13}$$

where $a_0 > 0$ and $a_3 > 0$.

Furthermore there exists $b \geq 1$ and a residue class a modulo b such that

$$M_n = [z^n]M(z, 0) = cn^{-5/2}z_0^{-n} \left(1 + O\left(\frac{1}{n}\right) \right), \quad (n \equiv a \pmod b, n \rightarrow \infty) \tag{14}$$

for some constant $c > 0$, and $M_n = 0$ for $n \not\equiv a \pmod b$.

2 Some Examples

Natural examples for the linear case come from the enumeration of lattice paths. We consider paths on \mathbb{N}^2 starting from the coordinate point $(0, 0)$ (or from $(0, t), t \in \mathbb{N}$) and allowed to move only to the right (up, straight or down), but forbid going below the x -axis $y = 0$ at each step. Define a step set $\mathcal{S} = \{(a_1, b_1), (a_2, b_2), \dots, (a_s, b_s) \mid (a_j, b_j) \in \mathbb{N} \times \mathbb{Z}\}$, and let $f_{n,k}$ be the number of paths ending at point (n, k) , where each step is in \mathcal{S} . The associated generating function is then defined as

$$F(z, u) = \sum_{n,k \geq 0} f_{n,k} z^n u^k.$$

Example 1 (*Motzkin Paths*). We start from $(0, 0)$ with step set $\mathcal{S} = \{(1, 1), (1, 0), (1, -1)\}$. The functional equation of its associated generating function is as follows:

$$F(z, u) = 1 + z \left(u + 1 + \frac{1}{u} \right) F(z, u) - \frac{z}{u} F(z, 0) = 1 + z(u + 1)F(z, u) + z \frac{F(z, u) - F(z, 0)}{u},$$

which in the notation of (8) corresponds to

$$Q_0(z, u) = 1, \quad Q_1(z, u) = u + 1, \quad \text{and} \quad Q_2(z, u) = 1.$$

We let $u(z)$ be the power series solution of the equation

$$u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z)) = z + zu(z)(1 + u(z)),$$

that is,

$$u(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}.$$

Then $F(z, 0)$ is given by

$$F(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))} = \frac{1}{1 - z(1 + u(z))} = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2},$$

and

$$M_n = f_{n,0} = [z^n]F(z, 0) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n - 2k)!k!(k + 1)!} \sim \frac{3\sqrt{3}}{2\sqrt{\pi}} n^{-3/2} 3^n.$$

Natural examples for the non-linear case come from the enumeration of planar maps. The starting point is the classical example of all planar maps [8].

Example 2 (*Planar maps*). Let $M(z, u)$ be the generating function of planar maps with n edges and in which the degree of the root-face is equal k [8]. We have already mentioned that $M(z, u)$ satisfies the non-linear catalytic equation (2). In order to apply Proposition 3 and Theorem 4 we use the substitution $u \rightarrow u + 1$ and obtain

$$M(z, u + 1) = 1 + z(u + 1) \left((u + 1)M(z, u + 1)^2 + M(z, u + 1) + \frac{M(z, u + 1) - M(z, 1 + 0)}{u} \right),$$

that is, we have $F_0(u) = 1$, and $Q(\alpha_0, \alpha_1, z, u) = (u + 1)^2 \alpha_0^2 + (u + 1) \alpha_0 + (u + 1) \alpha_1$. Here $Q_{\alpha_1} = u + 1 \neq 0$, $Q_{\alpha_0, u} \neq 0$, and $Q_{\alpha_0, \alpha_0} \neq 0$, so that Theorem 4 fully applies. Of course this is in accordance with

$$M(z, 1) = \sum_{n \geq 0} M_n z^n = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2},$$

and

$$M_n = [z^n]M(z, 1) \sim \frac{2}{\sqrt{\pi}}n^{-5/2}12^n.$$

Example 3 (Planar triangulations). Let $T(z, u)$ be the generating function for planar triangulations, which satisfies (see [4, 7])

$$T(z, u) = (1 - uT(z, u)) + (z + u)T(z, u)^2 + z(1 - uT(z, u))\frac{T(z, u) - T(z, 0)}{u}.$$

Here $T_{n,k} = [z^n u^k]T(z, u)$ denotes the number of near-triangulations, that is, all finite faces are triangles, with n internal vertices and $k + 3$ external vertices. In order to get rid of the negative sign we set $\tilde{T}(z, u) = T(z, u)/(1 - uT(z, u))$ and we obtain

$$\tilde{T}(z, u) = 1 + u\tilde{T}(z, u) + z(1 + \tilde{T}(z, u))\frac{\tilde{T}(z, u) - \tilde{T}(z, 0)}{u}.$$

Again this is not precisely of the form (6) but our methods apply once more. Note that $\tilde{T}(z, 0) = T(z, 0)$. We finally get for the number $T_{n,0}$ or triangulations

$$T_{n,0} = [z^n]T(z, 0) \sim \frac{8\sqrt{6}}{27\sqrt{\pi}}n^{-5/2}\left(\frac{256}{27}\right)^n.$$

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The Expected Number of Perfect Matchings in Cubic Planar Graphs

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Abstract. A well-known conjecture by Lovász and Plummer from the 1970s asserting that a bridgeless cubic graph has exponentially many perfect matchings was solved in the affirmative by Esperet et al. (Adv. Math. 2011). On the other hand, Chudnovsky and Seymour (Combinatorica 2012) proved the conjecture for the special case of cubic *planar* graphs. In our work we consider random bridgeless cubic planar graphs with the uniform distribution on graphs with n vertices. Under this model we show that the expected number of perfect matchings in *labeled* bridgeless cubic planar graphs is asymptotically $c\gamma^n$, where $c > 0$ and $\gamma \sim 1.14196$ is an explicit algebraic number. We also compute the expected number of perfect matchings in (non necessarily bridgeless) cubic planar graphs and provide lower bounds for *unlabeled* graphs. Our starting point is a correspondence between counting perfect matchings in rooted cubic planar maps and the partition function of the Ising model in rooted triangulations. (Supported by the Ministerio de Economía y Competitividad grant MTM2017-82166-P, and by the Special Research Program F50 *Algorithmic and Enumerative Combinatorics* of the Austrian Science Fund.).

Keywords: Perfect matching · Random planar graph · Ising model

1 Introduction

In the 1970s Lovász and Plummer conjectured that a bridgeless cubic graph has exponentially many perfect matchings. The conjecture was solved in the affirmative by Esperet, Kardoš, King, Král and Norine [4], and independently for cubic planar graphs by Chudnovsky and Seymour [3]. The lower bound from [4] is $2^{n/3656} \approx 1.0002^n$. It is natural to expect that a typical bridgeless cubic graph has more perfect matchings than those guaranteed by this lower bound.

Our main result gives estimates on the expected number of perfect matchings, both for labeled and unlabeled cubic planar graphs. The model we consider is the uniform distribution on graphs with n vertices.

Theorem 1. *Let X_n be the number of perfect matchings in a random (with the uniform distribution) labeled bridgeless cubic planar graph with $2n$ vertices. Then*

$$\mathbf{E}(X_n) \sim b\gamma^n,$$

where $b > 0$ is a constant and $\gamma \approx 1.14196$ is an explicit algebraic number. If X_n^u is the same random variable defined on unlabeled bridgeless cubic planar graphs, then

$$\mathbf{E}(X_n^u) \geq 1.119^n.$$

We obtain a similar result for general, non necessarily bridgeless, cubic planar graphs.

Theorem 2. *Let Y_n be the number of perfect matchings in a random (with the uniform distribution) labeled cubic planar graph with $2n$ vertices. Then*

$$\mathbf{E}(Y_n) \sim c\delta^n,$$

where $c > 0$ is a constant and $\delta \approx 1.14157$ is an explicit algebraic number. If Y_n^u is the same random variable defined on unlabeled cubic planar graphs, then

$$\mathbf{E}(Y_n^u) \geq 1.109^n.$$

2 Preliminaries

A map is a planar multigraph with a specific embedding in the plane. All maps considered in this paper are rooted, that is, an edge is marked and given a direction. A map is *simple* if it has no loops or multiple edges. It is 2-connected if it has no loops or cut vertices, and 3-connected if it has no 2-cuts or multiple edges. A map is cubic if it is 3-regular, and it is a triangulation if every face has degree 3. By duality, cubic maps are in bijection with triangulations. And since duality preserves 2- and 3-connectivity, k -connected cubic maps are in bijection with k -connected triangulations, for $k = 2, 3$. Notice that a general triangulation can have loops and multiple edges, and that a simple triangulation is necessarily 3-connected. The size of a cubic map is defined as the number of faces minus 2, a convention that simplifies the algebraic computations.

We need the generating function of 3-connected cubic maps, which is related to the generating function $T(z)$ of simple triangulations. The latter was obtained by Tutte [8] and is an algebraic function given by

$$T(z) = U(z)(1 - 2U(z)), \tag{1}$$

where $z = U(z)(1 - U(z))^3$, and z marks the number of vertices minus two. As shown in [8], the unique singularity of T , coming from a branch point, is located at $\tau = 27/256$ and $T(\tau) = 1/8$. The singular expansion of $T(z)$ near τ is

$$T(z) = \frac{1}{8} - \frac{3}{16}Z^2 + \frac{\sqrt{6}}{24}Z^3 + O(Z^4), \tag{2}$$

where $Z = \sqrt{1 - z/\tau}$. Notice that τ is a finite singularity, since $T(\tau) = 1/8 < \infty$.

The generating function $M_3(z)$ of 3-connected cubic maps, where z marks the number of faces minus 2 is equal to

$$M_3(z) = T(z) - z. \tag{3}$$

This follows directly from the duality between cubic maps and triangulations, which exchanges vertices and faces.

Adapting directly the proof from [6] for cubic planar graphs, one finds that cubic maps are partitioned into five subclasses, as defined below, and where st denotes the root edge of a cubic map M .

- \mathcal{L} (Loop). The root edge is a loop.
- \mathcal{I} (Isthmus). The root edge is an isthmus (an alternative name for a bridge).
- \mathcal{S} (Series). $M - st$ is connected but not 2-connected.
- \mathcal{P} (Parallel). $M - st$ is 2-connected but $M - \{s, t\}$ is not connected.
- \mathcal{H} (polyHedral). M is obtained from a 3-connected cubic map by possibly replacing each non-root edge with a cubic map whose root edge is not an isthmus.

3 The Ising Model on Rooted Triangulations and Perfect Matchings in Cubic Maps

Given a graph G , its Ising partition function is defined as follows. Given a 2-coloring, not necessarily proper, $c: V(T) \rightarrow \{1, 2\}$ of the vertices of G , let $m(c)$ be the number of monochromatic edges in the coloring. Then define

$$p_G(u) = \sum_{c: V(T) \rightarrow \{1,2\}} u^{m(c)}.$$

The same definition applies for rooted maps, using the fact that in a rooted map the vertices are distinguishable.

Suppose G is a triangulation with $2n$ faces. Since in a 2-coloring every face of T has at least one monochromatic edge, the number of monochromatic edges is at least n . The lower bound can be achieved taking the dual edge-set of a perfect matching in a cubic map. We show next that perfect matchings of a cubic map M with $2n$ vertices are in bijection with 2-colorings of the dual triangulation M^* with exactly n monochromatic edges, in which the color of the root vertex is fixed.

Lemma 1. *Let M be a rooted cubic map and $T = M^*$ its dual triangulation. There is a bijection between perfect matchings of M and 2-colorings of T with exactly n monochromatic edges in which the color of the root vertex of T is fixed.*

The generation function of the Ising partition of triangulations is defined as

$$P(z, u) = \sum_{T \in \mathcal{T}} p_T(u) z^n,$$

where \mathcal{T} is the class of rooted triangulations and the variable z marks the number of vertices minus 2. An expression for P was obtained by Bernardi and Bousquet-Mélou [1] in the wider context of counting q -colorings of maps with respect to monochromatic edges, which is equivalent to computing the q -Potts partition function. It is the algebraic function $Q_3(2, \nu, t)$ in [1, Theorem 23]. Here the parameter 2 refers to the number of colors, t marks edges and ν marks monochromatic edges. Extracting the coefficient $[\nu^n]Q_3(2, \nu, t)$ we obtain a generating function which is equivalent to the generating function $M(z)$ of rooted cubic maps with a distinguished perfect matching, where z marks faces minus 2. After a simple algebraic manipulation we obtain:

Lemma 2. *The generating function $M = M(z)$ counting rooted cubic maps with a distinguished perfect matching satisfies the quadratic equation*

$$72 M^2 z^2 + (216 z^2 - 36 z + 1) M + 162 z^2 - 6 z = 0. \tag{4}$$

where the variable z marks the number of faces minus two.

The former quadratic equation has a non-negative solution

$$M(z) = \frac{-1 + 36z - 216z^2 + (1 - 24z)^{3/2}}{144z^2}.$$

Expanding the binomial series one obtains the closed formula

$$[z^n]M(z) = 3 \cdot 6^n \frac{\binom{2n}{n}}{(n+2)(n+1)}, \tag{5}$$

a formula which can be proved combinatorially [7].

4 From the Ising Model to 3-Connected Cubic Graphs

We use the decomposition of cubic graphs as in [2] and [6], and the following observation. We say that a class \mathcal{N} of rooted maps is closed under rerooting if whenever a map N is in \mathcal{N} , so is any map obtained from N by forgetting the root edge and choosing a different one.

Lemma 3. *Let \mathcal{N} be a class of cubic maps closed under rerooting with a distinguished perfect matching. Let \mathcal{N}_1 be the maps in \mathcal{N} whose root edge belongs to the perfect matching, and \mathcal{N}_0 those whose root edge does not belong to the perfect matching. Let $N_i(z)$ be the associated generating functions. Then $N_0(z) = 2N_1(z)$.*

The previous lemma applies in particular to the class of all cubic maps and to the class of 3-connected cubic maps.

Lemma 4. *The following system of equations holds and has a unique solution in power series with non-negative coefficients.*

$$\begin{aligned}
 M_0(z) &= D_0(z), & M_1(z) &= D_1(z) + I(z), \\
 D_0(z) &= L(z) + S_0(z) + P_0(z) + H_0(z), & D_1(z) &= S_1(z) + P_1(z) + H_1(z) \\
 I(z) &= \frac{L(z)^2}{4z}, & L(z) &= 2z(1 + D_0(z)) \\
 S_1(z) &= D_1(z)(D_1(z) - S_1(z)), & S_0(z) &= D_0(z)(D_0(z) - S_0(z)) \\
 P_1(z) &= z(1 + D_0(z))^2, & P_0(z) &= 2z(1 + D_0(z))(1 + D_1(z)), \\
 H_1(z) &= \frac{T_1(z(1 + D_1(z))(1 + D_0(z))^2)}{1 + D_1}, & H_0(z) &= \frac{T_0(z(1 + D_1(z))(1 + D_0(z))^2)}{1 + D_0}.
 \end{aligned}
 \tag{6}$$

We sketch the justification of the former equations, starting with an observation. An edge e is replaced with a map whose root edge is in a perfect matching if and only if the two new edges resulting from the subdivision and replacement of e belong to the resulting perfect matching. The equation for $I(z)$ is because an isthmus map is composed of two loop maps; division by 4 takes into account the possible rootings of the two loops. The situation for $L(z)$, $S_i(z)$, $P_i(z)$ and $H_1(z)$ are rather straightforward. The equations for H_i can be detailed as follows: in a cubic map with $2n$ vertices there are n edges in a perfect matching and $2n$ not in it, hence the term $(1 + D_1(z))(1 + D_0(z))^2$ in the substitution.

By elimination we obtain $T_1(z)$ and $T_0(z) = 2T_1(z)$. The equation defining T_1 is

$$\begin{aligned}
 &T_1^6 + (24z + 16)T_1^5 + (60z^2 + 92z + 25)T_1^4 + (80z^3 + 208z^2 + 96z + 19)T_1^3 \\
 &+ (60z^4 + 232z^3 + 150z^2 + 12z + 7)T_1^2 + (24z^5 + 128z^4 + 112z^3 + z^2 - 16z + 1)T_1 \\
 &+ 4z^6 + 28z^5 + 33z^4 + 12z^3 - z^2 = 0.
 \end{aligned}$$

5 From 3-Connected Cubic Maps to Cubic Planar Graphs

A cubic *network* is a connected cubic planar multigraph G with an ordered pair of adjacent vertices (s, t) such that the graph obtained by removing one of the edges between s and t is connected and simple. We notice that st can be a simple edge, a loop or be part of a double edge, but cannot be an isthmus. The oriented edge st is called the *root* of the network, and s, t are called the *poles*. Replacement in networks is defined as for maps. We let \mathcal{D} be the class of cubic networks. The classes \mathcal{I} , \mathcal{L} , \mathcal{S} , \mathcal{P} and \mathcal{H} have the same meaning as for maps, and so do the subindices 0 and 1. We let \mathcal{C} be the class of connected cubic planar graphs (always with a distinguished perfect matching), with associated generating function $C(x)$, and $C^\bullet(x) = xC'(x)$ be the generating functions of those graphs rooted at a vertex. We also let $G(x)$ be the generating function of arbitrary (non-necessarily connected) cubic planar graphs.

Whitney’s theorem claims that a 3-connected planar graph has exactly two embeddings in the sphere up to homeomorphism. Thus counting 3-connected planar graphs rooted at a directed edge amounts to counting 3-connected maps,

up to a factor 2. Below we use the notation $T_i(x)$ for the exponential generating functions of 3-connected cubic planar graphs rooted at a directed edge, similarly to maps.

Lemma 5. *The following system of equations holds and has a unique solution in power series with non-negative coefficients.*

$$\begin{aligned}
 D_0 &= L + S_0 + P_0 + H_0, & D_1 &= S_1 + P_1 + H_1 \\
 I &= \frac{L^2}{x^2}, & L &= \frac{x^2}{2}(D_0 - L) \\
 S_1 &= D_1(D_1 - S_1), & S_0 &= D_0(D_0 - S_0) \\
 P_1 &= x^2D_0 + \frac{x^2}{2}D_0^2, & P_0 &= x^2(D_0 + D_1) + x^2D_0D_1 \\
 H_1 &= \frac{T_1(x^2(1 + D_1)(1 + D_0^2))}{2(1 + D_1)}, & H_0 &= \frac{T_0(x^2(1 + D_1)(1 + D_0^2))}{2(1 + D_0)}.
 \end{aligned} \tag{7}$$

Moreover, we have

$$3C^\bullet = I + D_0 + D_1 - L - L^2 - x^2(D_0 + D_1) - x^2D.$$

6 Proofs of the Main Results

Proof of Theorem 2. We first need to find the dominant singularity of $C(x)$, which is the same as that of $D_0(x)$, $D_1(x)$ and then $D(x)$. It is obtained by first computing the minimal polynomial for $D(x)$ and then its discriminant $\Delta(x)$. After discarding several factors of $\Delta(x)$ for combinatorial reasons (as in [6]), the relevant factor of $\Delta(x)$ turns out to be

$$904x^8 + 7232x^6 - 11833x^4 - 45362x^2 + 3616,$$

whose smallest positive root is equal to $\sigma \approx 0.27964$. After routinely checking the conditions of [6, Lemma 15], we conclude that σ is the only positive dominant singularity and that $D(x)$ admits an expansion near σ of the form

$$D(x) = d_0 + d_2X^2 + d_3X^3 + O(X^4), \quad X = \sqrt{1 - x/\sigma}.$$

And the same hold for $D_0(x)$ and $D_1(x)$. But also for $L(x)$ and $I(x)$, using their definitions given in terms of $D_0(x)$ in Lemma 5. There is a second singularity $-\sigma$ with a similar singular expansion and, as explained in [6], the contributions of $\pm\sigma$ are added.

From there, and using again Lemma 5 we can compute the singular expansion of $C^\bullet(x) = xC'(x)$, and by integration, that of $C(x)$. For arbitrary cubic planar graphs, we use the exponential formula $G(x) = e^{C(x)}$, which encodes the fact that a graph is an unordered set of connected graphs. The transfer theorem finally gives

$$G_n = [x^n]G(x) \approx c_1n^{-7/2}\sigma^{-n}n!. \tag{8}$$

To obtain the expected value of X_n we have to divide G_n by the number g_n of labeled cubic planar graphs, which as shown in [2, 6] is asymptotically $g_n \sim c_0n^{-7/2}\rho^{-n}n!$, where $c_0 > 0$ and $\rho \approx 0.31923$ is the smallest positive root of

$$729x^{12} + 17496x^{10} + 148716x^8 + 513216x^6 - 7293760x^4 + 279936x^2 + 46656 = 0.$$

And we obtain the claimed result by setting $c = c_1/c_0$ and $\delta = \rho/\sigma$. Furthermore, since σ and ρ are algebraic numbers, so is δ (actually of degree 48).

For the second part of the statement we argue as follows. Since a graph with n vertices has at most $n!$ automorphisms, the number of unlabeled graphs in a class is at least the number of labeled graphs divided by $n!$. It follows that the number U_n of unlabeled cubic planar graphs with a distinguished perfect matching is at least $G_n/n!$, where G_n is given in (8).

No precise estimate is known for the number u_n of unlabeled cubic planar graphs, but it can be upper bounded by the number C_n of simple rooted cubic planar maps, because a planar graph has at least one embedding in the plane. These maps have already been counted in [5] and the estimate $C_n \sim c_s \cdot n^{-5/2} \alpha^{-n}$, where $\alpha \sim 0.3102$, follows from [5, Corollary 3.2]. The relation between α and the value x_0 given in [5] is $\alpha = x_0^{1/2}$; this is due to the fact that we count cubic maps according to faces whereas in [5] they are counted according to vertices, and a map with $n + 2$ faces has $2n$ vertices. Disregarding subexponential terms, we have $U_n \geq \sigma^{-n}$ and $u_n \leq \alpha^{-n}$. The last result holds as claimed since $\alpha/\sigma \approx 1.109$. \square

Proof of Theorem 1. The proof follows the same scheme as that of Theorem 2 and is omitted. One just needs to adapt the system (7) to bridgeless cubic planar graphs by removing the generating functions $I(z)$ and $L(z)$, and follow a similar procedure.

7 Concluding Remarks

A natural open question is to prove some kind of concentration result for the number of perfect matchings in cubic planar graphs. But already computing the variance seems out of reach with our techniques, since for computing the second moment we would need to consider maps or graphs with a pair of distinguished perfect matchings, and this does not seem feasible using the connection with the Ising model on triangulations.

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Gallai-Ramsey Number for Complete Graphs

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Abstract. Given a graph H , the k -colored Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum integer n such that every k -coloring of the edges of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of H . Fox et al. [J. Fox, A. Grinshpun, and J. Pach. The Erdős-Hajnal conjecture for rainbow triangles. *J. Combin. Theory Ser. B*, 111:75-125, 2015.] conjectured the values of the Gallai Ramsey numbers for complete graphs. Recently, this conjecture has been verified for the first open case, when $H = K_4$.

In this paper we attack the next case, when $H = K_5$. Surprisingly it turns out, that the validity of the conjecture depends upon the (yet unknown) value of the Ramsey number $R(5, 5)$. It is known that $43 \leq R(5, 5) \leq 48$ and conjectured that $R(5, 5) = 43$ [B.D. McKay and S.P. Radziszowski. Subgraph counting identities and Ramsey numbers. *J. Combin. Theory Ser. B*, 69:193-209, 1997]. If $44 \leq R(5, 5) \leq 48$, then Fox et al.'s conjecture is true and we present a complete proof. If, however, $R(5, 5) = 43$, then Fox et al.'s conjecture is false, meaning that exactly one of these conjectures is true while the other is false. For the case when $R(5, 5) = 43$, we show lower and upper bounds for the Gallai Ramsey number $gr_k(K_3 : K_5)$.

Keywords: Ramsey numbers · Gallai-Ramsey numbers · Gallai coloring · McKay-Radziszowski conjecture · Fox-Grinshpun-Pach conjecture

1 Introduction

Given a graph G and a positive integer k , the k -color Ramsey number $R_k(G)$ is the minimum number of vertices n such that every k -coloring of the edges of K_N for $N \geq n$ must contain a monochromatic copy of G . We refer to [12] for a dynamic survey of known Ramsey numbers. As a restricted version of the Ramsey number, the k -color Gallai-Ramsey number $gr_k(K_3 : G)$ is defined to be the minimum integer n such that every k -coloring of the edges of K_N for $N \geq n$ must contain either a rainbow triangle or a monochromatic copy of G . We refer to [3] for a dynamic survey of known Gallai-Ramsey numbers.

In particular, the following was recently conjectured for complete graphs. Here $R(p, p) = R(K_p, K_p)$ denotes the classical Ramsey number.

Conjecture 1 [2]. For $k \geq 1$ and $p \geq 3$,

$$gr_k(K_3 : K_p) = (R(p, p) - 1)^{k/2} + 1, \text{ if } k \text{ is even,}$$

$$gr_k(K_3 : K_p) = (p - 1)(R(p, p) - 1)^{(k-1)/2} + 1, \text{ if } k \text{ is odd.}$$

The case where $p = 3$ was actually verified in 1983 by Chung and Graham [1]. A simplified proof was given by Gyárfás et al. [6].

Theorem 1 [1]. For $k \geq 1$,

$$gr_k(K_3 : K_3) = 5^{k/2} + 1, \text{ if } k \text{ is even,}$$

$$gr_k(K_3 : K_3) = 2 \cdot 5^{(k-1)/2} + 1, \text{ if } k \text{ is odd.}$$

The next case, where $p = 4$, was proven in [7].

Theorem 2. For $k \geq 1$,

$$gr_k(K_3 : K_4) = 17^{k/2} + 1, \text{ if } k \text{ is even,}$$

$$gr_k(K_3 : K_4) = 3 \cdot 17^{(k-1)/2} + 1, \text{ if } k \text{ is odd.}$$

Our main result is to essentially prove Conjecture 1 in the case where $p = 5$. This result is particularly interesting since $R(K_5, K_5)$ is still not known. Let $R = R(K_5, K_5) - 1$ and note that the known bounds on this Ramsey number give us $42 \leq R \leq 47$.

Theorem 3. For any integer $k \geq 2$,

$$gr_k(K_3 : K_5) = R^{k/2} + 1, \text{ if } k \text{ is even,}$$

$$gr_k(K_3 : K_5) = 4 \cdot R^{(k-1)/2} + 1, \text{ if } k \text{ is odd}$$

unless $R = 42$, in which case we have

$$gr_k(K_3 : K_5) = 43, \text{ if } k = 2,$$

$$gr_k(K_3 : K_5) = 42^{k/2} + 1 \leq gr_k(K_3 : K_5) \leq 43^{k/2} + 1, \text{ if } k \geq 4 \text{ is even,}$$

$$gr_k(K_3 : K_5) = 169 \cdot 42^{(k-3)/2} + 1 \leq gr_k(K_3 : K_5) \leq 4 \cdot 43^{(k-1)/2} + 1, \text{ if } k \geq 3 \text{ is odd.}$$

Theorem 3 will be proven in Sect. 4. Note that if $R = 42$, then Theorem 3 implies that Conjecture 1 is false.

Also recall the following well known conjecture about the sharp value for the 2-color Ramsey number of K_5 .

Conjecture 2 [11]. $R(K_5, K_5) = 43$.

By Theorem 3, it turns out that exactly one of Conjecture 1 or Conjecture 2 is true and the other is false.

In order to prove Theorem 3, we actually prove a more refined version, stated in Theorem 4. Note that Theorem 3 follows from Theorem 4 by setting $r = k$, $s = 0$ and $t = 0$.

Since we will generally be working only with K_5 or K_4 or K_3 , for three integers r, s, t we use the following shorthand notation:

$$gr_k(K_3 : rK_5, sK_4, tK_t) = gr_k(K_3 : K_5, K_5, \dots, K_5, K_4, \dots, K_4, K_3, \dots, K_3),$$

where we look for K_5 in any of the first r colors or K_4 in any of the s middle colors or K_3 in any of the last t colors.

To simplify the notation, we let c_1 denote the case where r, s, t are all even, c_2 denote the case where r, s are both even and t is odd, and so on for c_3, \dots, c_{11} . For nonnegative integers r, s, t , let $k = r + s + t$. Then we define

$$\begin{aligned} gr_k(K_3 : rK_5, sK_4, tK_3) = & \\ 1. R^{r/2} \cdot 17^{s/2} \cdot 5^{t/2} + 1, & \text{ if } r, s, t \text{ are even, } (c_1) \\ 2. R^{r/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1, & \text{ if } r, s \text{ are even, and } t \text{ is odd, } (c_2) \\ 3. R^{r/2} \cdot 17^{(s-1)/2} + 1, & \text{ if } r \text{ is even, } s \text{ is odd, and } t = 0, (c_3) \\ 4. R^{(r-1)/2} + 1, & \text{ if } r \text{ is odd, and } s = t = 0, (c_4) \\ 8. R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1, & \text{ if } r \text{ is even, and } s, t \text{ are odd, } (c_5) \\ 13. R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1, & \text{ if } r, t \text{ are odd, and } s \text{ is even, } (c_6) \\ 16. R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-2)/2} + 1, & \text{ if } r, t \text{ are even, } t \geq 2, \text{ and } s \text{ is odd, } (c_7) \\ 24. R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{t/2} + 1, & \text{ if } r, s \text{ are odd, and } t \text{ is even, } (c_8) \\ 26. R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-2)/2} + 1, & \text{ if } r \text{ is odd, } s \text{ is even, } t \geq 2 \text{ is even, } (c_9) \\ 48. R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1, & \text{ if } r, s, t \text{ are odd, } (c_{10}) \\ 72. R^{(r-1)/2} \cdot 17^{(s-2)/2} + 1, & \text{ if } r \text{ is odd, } t = 0, \text{ and } s \geq 2 \text{ is even. } (c_{11}) \end{aligned}$$

For ease of notation, let $g(r, s, t)$ be the value of $gr_k(K_3 : rK_5, sK_4, tK_3)$ claimed above. Also, for each i with $1 \leq i \leq 11$, let $g_i(r, s, t) = g(r, s, t) - 1$ in the case where (c_i) holds. Now we can state Theorem 4.

Theorem 4. For nonnegative integers r, s, t , let $k = r + s + t$. Then

$$gr_k(K_3 : rK_5, sK_4, tK_3) = g(r, s, t).$$

2 Preliminaries

In this section, we recall some known results and provide several helpful lemmas that will be used in the proof. First we state the main tool for looking at colored complete graphs with no rainbow triangle.

Theorem 5 [4]. *In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai-partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In light of this result, a colored complete graph with no rainbow triangle is called a *Gallai coloring* and the partition resulting from Theorem 5 is called a *Gallai partition*. Let D be the reduced graph of the Gallai partition, with vertices w_i corresponding to parts G_i of the partition.

Next recall some useful Ramsey numbers.

Theorem 6 [5].

$$R(K_3, K_5) = 14.$$

Theorem 7 [10].

$$R(K_4, K_5) = 25.$$

Also recall a general lower bound for Gallai-Ramsey numbers, a special case of the main result in [8]. We will present a more refined construction later for the purpose of proving Theorem 4.

Lemma 1 [8]. *For a complete graph H of order n and an integer $k \geq 2$, we have*

$$gr_k(K_3 : H) \geq (R(H, H) - 1)^{k/2} + 1, \text{ if } k \text{ is even,}$$

$$gr_k(K_3 : H) \geq (n - 1) \cdot (R(H, H) - 1)^{(k-1)/2} + 1, \text{ if } k \text{ is odd.}$$

3 Three Colors

In this section, we discuss a lower bound example that leads to a counterexample to either Conjecture 1 or Conjecture 2.

Lemma 2. *There exists a 3-colored copy of K_{169} which contains no rainbow triangle and no monochromatic copy of K_5 .*

Proof. Let G_{rb} be a sharpness example on 13 vertices for the Ramsey number $R(K_3, K_5) = 14$ say using colors red and blue respectively. Such an example as G_{rb} is 4-regular in red and 8-regular in blue. Similarly, let G_{rg} be a copy of the same graph with all blue edges replaced by green edges. We construct the desired graph G by making 13 copies of each vertex in G_{rb} and for each set of copies (corresponding to a vertex), insert a copy of G_{rg} . If an edge uv in G_{rb} is red (respectively blue), then all edges in G between the two inserted copies of G_{rg} corresponding to u and v are colored red (respectively blue). Then G contains no rainbow triangle by construction but also contains no monochromatic K_5 . Since $|G| = 169$, this provides the desired example.

Note that if $R(K_5, K_5) = 43$ so $R = 42$, then Conjecture 1 claims that $gr_3(K_3 : K_5) = 169$ but this example refutes this claim. On the other hand, if $R(K_5 : K_5) > 43$, then the conjecture holds for K_5 , as proven in Sect. 4 below.

4 Proof of Theorem 4 (and Theorem 3)

Note that the lower bound for Theorem 3 follows from Lemma 1 and was also presented in [2] but the lower bound for Theorem 4 must be more detailed. Here we give a sketch of proof for Theorem 4. A full proof for Theorem 4, which is much longer and quite technically, is given in [9].

Proof. (sketch) For the lower bounds, use the following constructions. For all constructions, we start with an i -colored base graph G_i (constructed below) and inductively suppose we have constructed an i -colored graph G_i containing no rainbow triangle and no appropriately colored monochromatic cliques. For each two unused colors requiring a K_5 , we construct G_{i+2} by making R copies of G_i , adding all edges in between the copies to form a blow-up of a sharpness example for $r(K_5, K_5)$ on R vertices. For each two unused colors requiring a K_4 , we construct G_{i+2} by making 17 copies of G_i , adding all edges in between the copies to form a blow-up of a sharpness example for $r(K_4, K_4)$ on 17 vertices. For each two unused colors requiring a K_3 , we construct G_{i+2} by making 5 copies of G_i , adding all edges in between the copies to form a blow-up of the sharpness example for $r(K_3, K_3)$ on 5 vertices.

The base graphs for this construction are constructed by case as follows.

- For Case (c_1), the base graph G_0 is a single vertex.
- For Case (c_2), the base graph G_1 is a monochromatic copy of K_2 .
- For Case (c_3), the base graph G_1 is a monochromatic copy of K_3 .
- For Case (c_4), the base graph G_1 is a monochromatic K_4 .
- For Case (c_5), the base graph G_2 is a sharpness example on 8 vertices for $r(K_3, K_4) = 9$.
- For Case (c_6), the base graph G_2 is a sharpness example on 13 vertices for $r(K_3, K_5) = 14$.
- For Case (c_7), the base graph G_3 is two copies of a sharpness example for $r(K_3, K_4) = 9$ with all edges in between the copies having a third color.
- For Case (c_8), the base graph G_2 is a sharpness example on 24 vertices for $r(K_4, K_5) = 25$.
- For Case (c_9), the base graph G_3 is two copies of a sharpness example on 13 vertices for $r(K_3, K_5) = 14$.
- For Case (c_{10}), the base graph G_3 is two copies of a sharpness example on 24 vertices for $r(K_4, K_5) = 25$ with all edges in between the copies having a third color.
- For Case (c_{11}), the base graph G_3 is three copies of a sharpness example on 24 vertices for $r(K_4, K_5) = 25$ with all edges in between the copies having a third color.

These base graphs and the corresponding completed constructions contain no rainbow triangle and no appropriately colored monochromatic cliques.

For the upper bound, let G be a Gallai coloring of K_n where n is given in the statement. We prove this result by induction on $3r + 2s + t$, meaning that it suffices to either reduce the order of a desired monochromatic subgraph or

eliminate a color. Let $k = r + s + t$. Then the case $k = 1$ is trivial, and the case $k = 2$ follows from classical Ramsey numbers $R(3, 3) = 6$, $R(3, 4) = 9$, $R(3, 5) = 14$, $R(4, 4) = 18$, $R(4, 5) = 25$, and $R = R(5, 5) - 1$. Now let $k \geq 3$ and suppose that Theorem 4 holds for all $r' + s' + t' < r + s + t$.

Consider a Gallai partition of G and let q be the number of parts in this partition. Choose such a partition so that q is minimized. Let red and blue be the colors on the edges between the parts. Now the colors red and blue can occur among the first r colors, among the s middle colors, or among the t last colors. This leads to the six main cases that comprise the (lengthy) remainder the proof.

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On the Dichromatic Number of Surfaces

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Abstract. In this paper, we give bounds on the dichromatic number $\vec{\chi}(\Sigma)$ of a surface Σ , which is the maximum dichromatic number of an oriented graph embeddable on Σ . We determine the asymptotic behaviour of $\vec{\chi}(\Sigma)$ by showing that there exist constants a_1 and a_2 such that, $a_1 \frac{\sqrt{-c}}{\log(-c)} \leq \vec{\chi}(\Sigma) \leq a_2 \frac{\sqrt{-c}}{\log(-c)}$ for every surface Σ with Euler characteristic $c \leq -2$. We then give more explicit bounds for some surfaces with high Euler characteristic. In particular, we show that the dichromatic numbers of the projective plane \mathbb{N}_1 , the Klein bottle \mathbb{N}_2 , the torus \mathbb{S}_1 , and Dyck's surface \mathbb{N}_3 are all equal to 3, and that the dichromatic numbers of the 5-torus \mathbb{S}_5 and the 10-cross surface \mathbb{N}_{10} are equal to 4.

Keywords: Dichromatic number · Planar graphs · Graphs on surfaces

1 Introduction

All surfaces considered in this paper are closed.

A graph is **embeddable** on a surface Σ if its vertices can be mapped onto distinct points of Σ and its edges onto simple curves of Σ joining the points onto which its endvertices are mapped, so that two edge curves do not intersect except in their common extremity. A **face** of an embedding \tilde{G} of a graph G is a component of $\Sigma \setminus \tilde{G}$. Recall that an important theorem of the topology of surfaces, known as the Classification Theorem for Surfaces, states that every surface is homeomorphic to either the k -torus – a sphere with k -handles \mathbb{S}_k or the k -cross surface – a sphere with k -cross-caps \mathbb{N}_k . The surface $\mathbb{S}_0 = \mathbb{N}_0$ is the sphere, and the surfaces $\mathbb{S}_1, \mathbb{S}_2, \mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3$ are also called the *torus*, the *double torus*, the *projective plane*, the *Klein bottle*, and *Dyck's surface*, respectively. The **Euler characteristic** of a surface homeomorphic to \mathbb{S}_k is $2 - 2k$ and of a surface homeomorphic to \mathbb{N}_k it is $2 - k$. We denote the Euler characteristic of a surface Σ by $c(\Sigma)$.

Let G be a graph. We denote by $n(G)$ its number of vertices, and by $m(G)$ its number of edges. If G is embedded in a surface Σ , then we denote by $f(G)$ the number of faces of the embedding. Euler's Formula relates the numbers of vertices, edges and faces of a (connected) graph embedded in a surface.

Theorem 1. EULER’S FORMULA

Let G be a connected graph embedded on a surface Σ . Then

$$n(G) - m(G) + f(G) \geq c(\Sigma).$$

We denote by $\text{Ad}(G) = 2m/n$ the average degree of a graph G . Euler’s formula implies that graphs on surfaces have bounded average degree.

Theorem 2. A connected graph G embeddable on a surface Σ satisfies:

$$m(G) \leq 3n(G) - 3c(\Sigma) \quad \text{and} \quad \text{Ad}(G) \leq 6 - \frac{6c(\Sigma)}{n(G)}.$$

Moreover, there is equality if and only if G is a triangulation.

A k -colouring of a graph G is a partition of the vertex set of G into k disjoint **stable sets** (i.e. sets of pairwise non-adjacent vertices). A graph is k -colourable if it has a k -colouring. The **chromatic number** of a graph G , denoted by $\chi(G)$, is the least integer k such that G is k -colourable, and the **chromatic number** of a surface Σ , denoted by $\chi(\Sigma)$, is the least integer k such that every graph embeddable on Σ is k -colourable. Determining the chromatic number of surfaces attracted lots of attention, with its most important instance being the Four Colour Conjecture, which was eventually proved by Appel and Haken [2]. The chromatic numbers of the other surfaces were established earlier. Franklin [4] showed that the Klein bottle has chromatic number 6, and combined results of Heawood [6] and Ringel and Youngs [12] imply that if Σ is a surface different from the Klein bottle \mathbb{N}_2 with Euler characteristic c , then $\chi(\Sigma) \leq H(c) = \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$.

In 1982, Neumann Lara [10] introduced the notion of directed colouring or dicolouring. A k -dicolouring of a digraph is a partition of its vertex set into k subsets inducing acyclic subdigraphs. A digraph is k -dicolourable if it has a k -dicolouring. The **dichromatic number** of a digraph D , denoted by $\overrightarrow{\chi}(D)$, is the least integer k such that D is k -dicolourable.

Let G be an undirected graph. The **bidirected graph** \overleftrightarrow{G} is the digraph obtained from G by replacing each edge by a **digon**, that is a pair of oppositely directed arcs between the same end-vertices. Observe that $\chi(G) = \overrightarrow{\chi}(\overleftrightarrow{G})$ since any two adjacent vertices in \overleftrightarrow{G} induce a directed cycle of length 2.

It is thus natural to consider **oriented graphs**, which are digraphs with no digons. Oriented graphs may be also seen as the digraphs which can be obtained from (simple) graphs by orienting every edge, that is replacing each edge by exactly one of the two possible arcs between its end-vertices. If \overrightarrow{G} is obtained from G by orienting its edges, we say that G is the **underlying graph** of \overrightarrow{G} . It is easy to show that oriented planar graphs are 3-dicolourable and Neumann Lara [10] proposed the following conjecture.

Conjecture 1 (Neumann Lara [10]). Every oriented planar graph is 2-dicolourable.

This conjecture is part of an active field of research. It has been verified for planar oriented graphs on at most 26 vertices [7] and holds for planar digraphs of with no directed cycle of length 3 [9].

In this paper, we study the dichromatic number of surfaces. The **dichromatic number** of a surface Σ , denoted by $\vec{\chi}(\Sigma)$, is the least integer k such that every oriented graph embeddable on Σ is k -dicolourable. We first establish asymptotic bounds on the dichromatic number of surfaces.

Theorem 3. *There exist two positive constants a_1 and a_2 such that, for every surface Σ with Euler characteristic $c \leq -2$, we have*

$$a_1 \frac{\sqrt{-c}}{\log(-c)} \leq \vec{\chi}(\Sigma) \leq a_2 \frac{\sqrt{-c}}{\log(-c)}$$

Due to lack of space, we do not include the proof of this theorem. Like every other proofs missing in this paper, it can be found in the long version of the paper [1].

We then estimate the exact value of the dichromatic number of surfaces close to the sphere. Table 1 summarizes the main results.

Table 1. Bounds on the dichromatic number of some surfaces.

Σ	$c(\Sigma)$	Bounds for $\vec{\chi}(\Sigma)$	Reference
Sphere $N_0 = S_0$	2	$2 \leq \vec{\chi} \leq 3$	Neumann Lara [10]
N_1, N_2, S_1, N_3	$\in \{1, 0, -1\}$	$\vec{\chi} = 3$	Theorem 5
$S_2, N_4, N_5, S_3, N_6, N_7, S_4, N_8, N_9$	$\in \{-2, \dots, -7\}$	$3 \leq \vec{\chi} \leq 4$	Theorems 5 and 6
S_5, N_{10}	-8	$\vec{\chi} = 4$	Theorem 6

In order to prove that the dichromatic number of a surface Σ is at most k , we shall prove that there is no $(k + 1)$ -dicritical digraph embeddable in Σ . A digraph D is $(k + 1)$ -**dicritical** if $\vec{\chi}(D) = k + 1$ and $\vec{\chi}(H) \leq k$ for every proper subdigraph H of D . Kostochka and Stiebitz [8] prove the following.

Theorem 4 (Kostochka and Stiebitz [8]). *Let \vec{G} be a 4-dicritical oriented graph then $3m(\vec{G}) \geq 10n(\vec{G}) - 4$. Moreover, if \vec{G} is embeddable in a surface with Euler characteristic c , then $n(\vec{G}) \leq 4 - 9c$.*

2 The Dichromatic Number of N_1, N_2, N_3 , and S_1

Theorem 5. $\vec{\chi}(N_1) = \vec{\chi}(N_2) = \vec{\chi}(N_3) = \vec{\chi}(S_1) = 3$.

Proof. $K_7 \setminus e$, the complete graph on seven vertices minus an edge, is embeddable in every surface other than the projective plane and the sphere. Neumann-Lara [11] proved that this graph has an orientation with dichromatic number 3. Hence $\vec{\chi}(N_2), \vec{\chi}(N_3), \vec{\chi}(S_1) \geq 3$.

The complete graph on 6 vertices K_6 can be embedded as a **triangulation** of the projective plane, that is an embedding of K_6 in the projective plane such that all faces are triangles. Let T be the orientation of K_6 displayed on the left of Fig. 1. Let \vec{G} be the oriented graph obtained from T by adding in each gray triangular face (which is a transitive tournament on three vertices with source s and sink t), the gadget graph depicted on the left of Fig. 1. Observe that in any 2-dicolouring of the gadget graph, the vertices of the outer face do not have all the same colour.

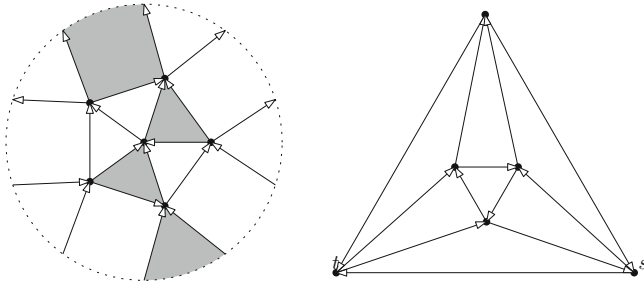


Fig. 1. Left: an orientation T of K_6 on the projective plane. Right: the gadget graph.

Assume now for a contradiction that \vec{G} admits a 2-dicolouring. Observe that either we have a monochromatic directed triangle in T or one of the gray triangles is monochromatic. But then the 2-dicolouring cannot be extended to the gadget inside this transitive tournament by the above observation. Hence \vec{G} is not 2-dicolourable. Hence $3 \leq \vec{\chi}(\mathbb{N}_1)$.

Suppose for a contradiction that there exists a 4-dicritical oriented graph \vec{G} embeddable on \mathbb{N}_3 . By Theorem 4, it has at most 13 vertices (because $c(\mathbb{N}_3) = -1$).

If G is not a triangulation of \mathbb{N}_3 , then, by Theorem 2, $m(\vec{G}) \leq 3n(\vec{G}) + 2$, that is $3m(\vec{G}) \leq 9n(\vec{G}) + 6$. But $3m(\vec{G}) \geq 10n(\vec{G}) - 4$ by Theorem 4. Hence $n(\vec{G}) \leq 10$. But Neumann Lara [11] proved that every oriented graph of order at most 10 is 3-dicolourable. This is a contradiction.

So \vec{G} is a triangulation of \mathbb{N}_3 . By Theorem 4 and the above mentioned result of Neumann-Lara, $11 \leq n(\vec{G}) \leq 13$. Then, an exhaustive enumeration of the triangulations of order 11, 12 and 13 shows that there is no 4-dicritical oriented graph in \mathbb{N}_3 .

Since every oriented graph embeddable in \mathbb{N}_1 , \mathbb{N}_2 , or \mathbb{S}_1 is also embeddable in \mathbb{N}_3 , we get the result. \square

2.1 The Dichromatic Number of \mathbb{S}_5 and \mathbb{N}_{10}

Theorem 6. $\vec{\chi}(\mathbb{S}_5) = \vec{\chi}(\mathbb{N}_{10}) = 4$.

Proof. The complete graph on 11 vertices is embeddable on \mathbb{S}_5 and \mathbb{N}_{10} and Neumann-Lara [11] showed an orientation of this graph with dichromatic number 4. Therefore $\overrightarrow{\chi}(\mathbb{S}_5), \overrightarrow{\chi}(\mathbb{N}_{10}) \geq 4$.

It remains to prove that every oriented graph embeddable on \mathbb{S}_5 or \mathbb{N}_{10} is 4-dicolourable. We now sketch this proof. The entire proof may be found in [1].

Assume for a contradiction that there is a 5-dicritical oriented graph \overrightarrow{G} of order n which is embedded in \mathbb{S}_5 or \mathbb{N}_{10} .

Let T be the subdigraph induced by the vertices of degree 8 (i.e. in-degree 4 and out-degree 4). Set $H = \overrightarrow{G} - T$, $n_8 = n(T)$ and let $m(H, T)$ be the number of arcs with one end-vertex in H and the other in T . A result of Bang-Jensen et al. [3] implies that T is a **directed cactus**, that is an oriented graph in which each block is a single arc or a directed cycle. In particular, T is 2-dicolourable. Therefore H is not 2-dicolourable. In particular, one can prove that $m(H) \geq 20$.

Euler’s Formula yields $8n_8 + 9(n - n_8) + \sum_{v \in V(H)} (d(v) - 9) = 2m(\overrightarrow{G}) \leq 6n + 48$ and so:

$$n_8 \geq 3(n - 16) + \sum_{v \in V(H)} (d(v) - 9) \geq 3(n - 16) \tag{1}$$

On the other hand, we have $\sum_{v \in V(T)} d(v) = 8n_8 = 2m(T) + m(H, T)$ and $m(\overrightarrow{G}) = m(H) + m(H, T) + m(T)$. We deduce

$$m(H) = m(\overrightarrow{G}) + m(T) - 8n_8 \tag{2}$$

Since T is a directed cactus, we have $m(T) \leq \frac{3}{2}(n_8 - 1)$. Thus $20 \leq m(H) \leq m(\overrightarrow{G}) + \frac{3}{2}(n_8 - 1) - 8n_8$. Hence $13n_8 \leq 2m(\overrightarrow{G}) - 43$. With Eq. (1) and Euler’s formula, it implies

$$3(n - 16) \leq n_8 \leq \frac{2m(\overrightarrow{G}) - 43}{13} \leq \frac{6n + 5}{13} \tag{3}$$

After simplifying, we get $n \leq 19$. Moreover, one can easily prove that every oriented graph of order at most 15 is 4-dicolourable, thus $n \geq 16$. We then distinguish few cases depending on the number n of vertices. We only sketch here the proof for $n = 19$. The details of this case and the proof of the other cases can be found in [1].

Case $n = 19$: By Eq. (3), we have $9 \leq n_8 \leq \frac{119}{13}$ and so $n_8 = 9$.

Assume first that $m(T) = \frac{3}{2}(n_8 - 1) = 12$. By as T is a directed cactus, T is connected and each block of T is a directed triangle. So T is Eulerian, i.e. $d_T^+(v) = d_T^-(v)$ for all $v \in V(T)$.

Since $n_8 = 9$, we have $n(H) = 10$. So, by a result of Neumann-Lara [11], H admits a 3-dicolouring ϕ with colour set $\{1, 2, 3\}$. Since all blocks of T are directed triangles, T contains a vertex v such that $d_T^+(v) = d_T^-(v) = 1$. So v has 3 out-neighbours in H . Let v_1, v_2 be two of these out-neighbours. Let us recolour v_1 and v_2 by setting $\phi(v_1) = \phi(v_2) = 4$ (since there is no digon, the resulting colouring is still proper). We then define for every vertex x of T :

$$L(x) = \{1, 2, 3, 4\} \setminus \phi(N^+(x) \cap V(H))$$

Observe that an L -colouring of T extends the 4-colouring of H into a 4-colouring of G , so T is not L -colourable. Observe that $|L(x)| \geq 4 - (4 - d_T^+(x)) = \max\{d_T^+(x), d_T^-(x)\}$ because T is Eulerian. Moreover, since v_1 and v_2 are both coloured 4, $|L(v)| \geq 2 = \max\{d_T^+(x), d_T^-(x)\} + 1$. So T is L -dicolourable by a theorem of Harutyunyan and Mohar [5], a contradiction.

Therefore we have $m(T) \leq 11$. By Euler's Formula, $m(\vec{G}) \leq 3n + 24$, and by Eq. (2) $m(H) = m(\vec{G}) - 8n_8 + m(T)$. Hence $m(H) \leq 20$. But H is not 2-dicolourable, so it contains a 3-dicritical oriented subgraph \tilde{H} , and $m(\tilde{H}) \leq 20$. One can show that there is a unique such 3-dicritical oriented graph with at most 20 arcs: it has 7 vertices and 20 arcs. Hence $n(\tilde{H}) = 7$, $m(\tilde{H}) = m(H) = 20$ and H is the disjoint union of \tilde{H} and a stable set S' of size 3. Observe that each vertex of S' has degree at least 9, which implies that they are adjacent to every vertex of T and have degree exactly 9.

Now, $m(\tilde{H}) < m(K_7)$, so there are two non-adjacent vertices x, y in \tilde{H} . Thus $S = S' \cup \{x, y\}$ is a stable set of order 5 in H . Moreover, since it is a directed cactus, T has an acyclic subdigraph A of order 6. Pick $v \in V(T) \setminus V(A)$. The subdigraph B of \vec{G} induced by $S \cup \{v\}$ is acyclic and has order 6. Let $G' = \vec{G} - (A \cup B)$. Observe that G' has order $19 - 6 - 6 = 7$. Recall that Neumann-Lara [11] showed that oriented graphs on at most 6 vertices are 2-dicolourable.

Let $w \in V(G') \cap V(T)$.

- If $|N(w) \cap V(A)| \leq 1$, then the subdigraph A' induced by $V(A) \cup \{w\}$ is acyclic. Hence G can be partitioned into two acyclic subdigraphs A' and B and $G - A' \cup B$ which has order 6 and so is 2-dicolourable. Thus \vec{G} is 4-dicolourable, a contradiction.
- If $|N(w) \cap V(A)| \geq 2$, then as w is adjacent to all vertices of S' , we have $d_{G'}(w) \leq 8 - 2 - 3 = 3$. Now, $G' - \{w\}$ is 2-dicolourable, and since $d_{G'}(w) = 3$, G' is also 2-dicolourable, and thus G is 4-dicolourable, a contradiction. \square

Acknowledgements. We particularly thank Raphael Steiner for pointing us to the argument for the upper bound in Theorem 3. This research was supported by projects Digraphs: ANR-19-CE48-0013-01, GATO: ANR-16-CE40-0009-01, ALCOIN: PID2019-104844GB-I00, RYC-2017-22701 and ALGORIDAM: ANR-19-CE48-0016

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Supersaturation, Counting, and Randomness in Forbidden Subset Problems

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Abstract. In the area of forbidden subposet problems we look for the largest possible size $La(n, P)$ of a family $\mathcal{F} \subseteq 2^{[n]}$ that does not contain a forbidden inclusion pattern described by P . The main conjecture of the area states that for any finite poset P there exists an integer $e(P)$ such that $La(n, P) = (e(P) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

In this paper, we formulate three strengthenings of this conjecture and prove them for some specific classes of posets.

Keywords: Extremal set theory · Forbidden subposet problem · Supersaturation

Extremal set theory starts with the seminal result of Sperner [15] that was generalized by Erdős [6] as follows: if a family $\mathcal{F} \subseteq 2^{[n]}$ of sets does not contain a nested sequence $F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{k+1}$ (such nested sequences are called *chains of length $k + 1$* or $(k + 1)$ -chains for short), then its size cannot exceed that of the union of k middle levels of $2^{[n]}$, i.e., $|\mathcal{F}| \leq \sum_{i=1}^k \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}$. This theorem has many applications and several of its variants have been investigated.

In the early 80's, Katona and Tarján [9] introduced the following general framework to study set families avoiding some fixed inclusion patterns: we say that a subfamily \mathcal{G} of \mathcal{F} is a (*non-induced*) *copy* of a poset (P, \leq) in \mathcal{F} , if there exists a bijection $i : P \rightarrow \mathcal{G}$ such that if $p, q \in P$ with $p \leq q$, then $i(p) \subseteq i(q)$. If i satisfies the property that for $p, q \in P$ we have $p \leq q$ if and only if $i(p) \subseteq i(q)$, then \mathcal{G} is called an *induced copy* of P in \mathcal{F} . If \mathcal{F} does not contain any (induced) copy of P , the \mathcal{F} is said to be (*induced*) P -free. The largest possible size of a (n induced) P -free family $\mathcal{F} \subseteq 2^{[n]}$ is denoted by $La(n, P)$ ($La^*(n, P)$). Let P_k denote the k -chain, then the result of Erdős mentioned above determines $La(n, P_{k+1})$. These parameters have attracted the attention of many researchers, and there are widely believed conjectures in the area (see Conjecture 1) that appeared first in [2] and [8], giving the asymptotics of $La(n, P)$ and $La^*(n, P)$.

Let $e(P)$ denote the maximum integer m such that for any $i \leq n$, the family $\binom{[n]}{i+1} \cup \binom{[n]}{i+2} \cup \dots \cup \binom{[n]}{i+m}$ is P -free. Similarly, let $e^*(P)$ denote the maximum

integer m such that for any $i \leq n$, the family $\binom{[n]}{i+1} \cup \binom{[n]}{i+2} \cup \dots \cup \binom{[n]}{i+m}$ is induced P -free.

Conjecture 1.

- (i) $La(n, P) = (e(P) + o(1)) \binom{n}{\lfloor n/2 \rfloor}$.
- (ii) $La^*(n, P) = (e^*(P) + o(1)) \binom{n}{\lfloor n/2 \rfloor}$.

Conjecture 1 has been verified for several classes of posets, but is still open in general. For more results on the $La(n, P)$ function, see Chap. 7 of [7], and see other chapters for more background on the generalizations considered in this paper.

After determining (the asymptotics of) the extremal size and the structure of the extremal families, one may continue in several directions. *Stability* results state that all P -free families having almost extremal size must be very similar in structure to the middle $e(P)$ levels of $2^{[n]}$. *Supersaturation* problems ask for the minimum number of copies of P that a family $\mathcal{F} \subseteq 2^{[n]}$ of size $La(n, P) + E$ may contain. This is clearly at least E , but usually one can say much more. *Counting* problems ask to determine the number of P -free families in $2^{[n]}$. As any subfamily of a P -free family is P -free, therefore the number of P -free families is at least $2^{La(n, P)}$. The question is how many more such families there are. Finally, one can address *random versions* of the forbidden subposet problem. Let $\mathcal{P}(n, p)$ denote the probability space of all subfamilies of $2^{[n]}$ such that for any $F \subseteq [n]$, the probability that F belongs to $\mathcal{P}(n, p)$ is p , independently of any other set F' . What is the size of the largest P -free subfamily of $\mathcal{P}(n, p)$ with high probability¹? Clearly, for $p = 1$, this is $La(n, P)$. For other values of p , an obvious construction is to take a P -free subfamily of $2^{[n]}$, and then the sets that are in $\mathcal{P}(n, p)$ form a P -free family. Taking the $e(P)$ middle levels shows that the size of the largest P -free family in $\mathcal{P}(n, P)$ is at least $p(e(P) + o(1)) \binom{n}{\lfloor n/2 \rfloor}$ w.h.p.. For what values of p does this formula give the asymptotically correct answer?

In this paper, we will consider supersaturation, counting and random versions of the forbidden subposet problem, mostly focusing on supersaturation results. We will propose three strengthenings of Conjecture 1 and prove them for some classes of posets. In the remainder of the introduction, we state our results and also what was known before.

The supersaturation version of Sperner's problem is to determine the minimum number of pairs $F \subsetneq F'$ over all subfamilies of $2^{[n]}$ of given size. We say that a family \mathcal{F} is *centered* if it consists of the sets closest to $n/2$. More precisely, if $F \in \mathcal{F}$ and $\|G\| - n/2 < \|F\| - n/2$ imply $G \in \mathcal{F}$. Kleitman [10] proved that among families of cardinality m , centered ones contain the smallest number of copies of P_2 . He conjectured that the same holds for any P_k . This was decades later confirmed by Samotij [14]. The following is a consequence of the result of Samotij. We will only use it with $k = 2$, i.e. the result of Kleitman.

¹ We say that a sequence of events $E_1, E_2, \dots, E_n, \dots$ holds with high probability (or w.h.p., in short) if $\mathbb{P}(E_n)$ tends to 1 as n tends to infinity.

Theorem 1. *For any k, t with $k - 1 \leq t$ and $\varepsilon > 0$ there exists $n_{k,t,\varepsilon}$ such that if $n \geq n_{k,t,\varepsilon}$, then any family $\mathcal{F} \subseteq 2^{[n]}$ of size at least $(t + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\varepsilon \frac{n^t}{2^{t+1}} \binom{n}{\lfloor n/2 \rfloor}$ chains of length k .*

We will investigate the number of copies of P created when the number of additional sets compared to a largest P -free family is proportional to the size of the middle level $\binom{n}{\lfloor n/2 \rfloor}$. Let $M(n, P)$ denote the number of copies of P in the $e(P) + 1$ middle levels of $2^{[n]}$, and let $M^*(n, P)$ denote the number of induced copies of P in the $e^*(P) + 1$ middle levels of $2^{[n]}$. The *Hasse diagram* of a poset P is the directed graph with vertex set P and for $p, q \in P$, (pq) is an arc in the Hasse diagram if $p < q$ and there does not exist $z \in P$ with $p < z < q$. We say that P is *connected*, if its Hasse diagram (as a digraph) is weakly connected, i.e., we cannot partition its vertices into two sets such that there is no arc between those sets. The undirected Hasse diagram is the undirected graph obtained from the Hasse diagram by removing orientations of all arcs.

Proposition 1. *For any connected poset P on at least two elements there exist positive integers $x(P)$ and $x^*(P)$ such that $M(n, P) = \Theta\left(n^{x(P)} \binom{n}{\lfloor n/2 \rfloor}\right)$ and $M^*(n, P) = \Theta\left(n^{x^*(P)} \binom{n}{\lfloor n/2 \rfloor}\right)$ hold.*

Now we can state the first generalization of Conjecture 1.

Conjecture 2.

- (i) For every poset P and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{F} \subseteq 2^{[n]}$ is of size at least $(e(P) + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$, then \mathcal{F} contains at least $\delta \cdot M(n, P)$ many copies of P .
- (ii) For every poset P and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{F} \subseteq 2^{[n]}$ is of size at least $(e^*(P) + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$, then \mathcal{F} contains at least $\delta \cdot M^*(n, P)$ many induced copies of P .

We will prove Conjecture 2 for several classes of tree posets. A poset T is a *tree poset*, if its undirected Hasse diagram is a tree. The *height* $h(P)$ of poset P is the length of the longest chain in P . Note that for any tree poset T of height 2, we have $x(T) = x^*(T) = |T| - 1$.

Theorem 2. *Let T be any height 2 tree poset of $t + 1$ elements. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that for any $n \geq n_0$ any family $\mathcal{F} \subseteq 2^{[n]}$ of size $|\mathcal{F}| \geq (1 + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\delta n^t \binom{n}{\lfloor n/2 \rfloor}$ copies of T .*

For two elements x, y of the poset P , we write $x \prec y$ if $x <_P y$ and there does not exist any $z \in P$ with $x <_P z <_P y$. We say that a tree poset T is *upward (downward) monotone*, if for any $x \in T$ there exists at most 1 element $y \in T$ with $y \prec x$ ($x \prec y$). A tree poset is called *monotone*, if it is either upward or downward monotone.

Theorem 3. *For any monotone tree poset T and $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that for any $n \geq n_0$ any family $\mathcal{F} \subseteq 2^{[n]}$ of size $|\mathcal{F}| \geq (h(T) - 1 + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\delta n^{x(T)} \binom{n}{\lfloor n/2 \rfloor}$ copies of T .*

The complete multipartite poset $K_{r_1, r_2, \dots, r_\ell}$ is a poset on $\sum_{i=1}^\ell r_i$ elements $a_{i,j}$ with $i = 1, 2, \dots, \ell, j = 1, 2, \dots, r_i$ such that $a_{i,j} < a_{i',j'}$ if and only if $i < i'$. The poset $K_{1,r}$ is usually denoted by \vee_r , and the poset $K_{r,1}$ is denoted by \wedge_r . The poset $K_{s,1,t}$ is a tree poset with $x(K_{s,1,t}) = x^*(K_{s,1,t}) = s + t$.

Theorem 4. *For any $s, t \in \mathbb{N}$ and $\varepsilon > 0$ there exist $n_0 = n_{\varepsilon, s, t}$ and $\delta > 0$ such that any $\mathcal{F} \subseteq 2^{[n]}$ of size at least $(2 + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ with $n \geq n_0$ contains at least $\delta n^{s+t} \binom{n}{\lfloor n/2 \rfloor}$ induced copies of $K_{s,1,t}$.*

We will consider the supersaturation problem for the generalized diamond D_s , i.e., the poset on $s + 2$ elements with $a < b_1, b_2, \dots, b_s < c$. For any integer $s \geq 2$, let us define $m_s = \lceil \log_2(s + 2) \rceil$ and $m_s^* = \min\{m : s \leq \binom{m}{\lfloor m/2 \rfloor}\}$. Clearly, for any integer $s \geq 2$, we have $e(D_s) = x(D_s) = m_s$ and $e^*(D_s) = x^*(D_s) = m_s^*$. The next theorem establishes a lower bound that is less by a factor of \sqrt{n} than what Conjecture 2 states for diamond posets D_s for infinitely many s .

Theorem 5

- (i) *If $s \in [2^{m_s-1} - 1, 2^{m_s} - \binom{m_s}{\lfloor m_s/2 \rfloor} - 1]$, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geq (m_s + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\delta \cdot n^{m_s-0.5} \binom{n}{\lfloor n/2 \rfloor}$ copies of D_s .*
- (ii) *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geq (4 + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\delta \cdot n^{3.5} \binom{n}{\lfloor n/2 \rfloor}$ induced copies of D_4 .*
- (iii) *For any constant c with $1/2 < c < 1$ there exists an integer s_c such that if $s \geq s_c$ and $s \leq c \binom{m_s^*}{\lfloor m_s^*/2 \rfloor}$, then the following holds: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geq (m_s^* + \varepsilon) \binom{n}{\lfloor n/2 \rfloor}$ contains at least $\delta \cdot n^{m_s^*-0.5} \binom{n}{\lfloor n/2 \rfloor}$ induced copies of D_s .*

Let us elaborate on the statement of Theorem 5. Part (i) partitions the integers according to powers of 2 and states that for every integer k , and for most of the integers s in the interval $[2^k - 1, 2^{k+1} - 2]$, the poset D_s possesses this weak supersaturation property. By “most of the integers” we mean that the ratio of integers for which the statement holds and the length of the interval tends to 1 as k tends to infinity. The smallest value of s that (i) applies to is $s = 3$ with $m_s = 3$ as then $3 \in [2^{3-1} - 1, 2^3 - \binom{3}{\lfloor 3/2 \rfloor} - 1] = [3, 4]$. Part (ii) is about supersaturation of induced copies of D_4 . Part (iii) is similar to (i) but again about induced copies of D_s . This time positive integers are partitioned into intervals according to the sequence $\{\binom{k}{\lfloor k/2 \rfloor}\}_{k=1}^\infty$, namely $\{[\binom{k}{\lfloor k/2 \rfloor} + 1, \binom{k+1}{\lfloor (k+1)/2 \rfloor}]\}_{k=1}^\infty$. As k tends to infinity, the ratio of right and left endpoints tends to 2. Part (iii) states that as k tends to infinity, those integers s in the initial segment of the k th interval

for which D_s has the claimed supersaturation property, take up larger and larger ratio of the interval.

Let us turn our attention to counting (induced) P -free families. As we mentioned earlier, every subfamily of a P -free family is P -free, therefore $2^{La(n,P)} \geq 2^{(e(P)+o(1))\binom{n}{\lfloor n/2 \rfloor}}$ is a lower bound on the number of such families. Determining the number of P_2 -free families has attracted a lot of attention. The upper bound $2^{(1+o(1))\binom{n}{\lfloor n/2 \rfloor}}$, asymptotically matching in the exponent the trivial lower bound was obtained by Kleitman [11]. After several improvements, Korshunov [12] determined asymptotically the number of P_2 -free families.

- Conjecture 3.* (i) The number of P -free families in $2^{[n]}$ is $2^{(e(P)+o(1))\binom{n}{\lfloor n/2 \rfloor}}$.
 (ii) The number of induced P -free families in $2^{[n]}$ is $2^{(e^*(P)+o(1))\binom{n}{\lfloor n/2 \rfloor}}$.

Theorem 6. (i) The number of induced \vee_{r+1} -free families is $2^{(1+o(1))\binom{n}{\lfloor n/2 \rfloor}}$.
 (ii) The number of induced $K_{s,1,t}$ -free families in $2^{[n]}$ is $2^{(2+o(1))\binom{n}{\lfloor n/2 \rfloor}}$.

As every height 2 poset P is a non-induced subposet of $K_{\lfloor P \rfloor, 1, \lfloor P \rfloor}$, Conjecture 3 (i) is an immediate consequence of Theorem 6 for those height 2 posets P for which $e(P) = 2$.

Finally, we turn to random versions of forbidden subposet problems. The probabilistic version of Sperner’s theorem was proved by Balogh, Mycroft, and Treglown [1] and Collares and Morris [3–5], independently. It states that if $p = \omega(1/n)$, then the largest antichain in $\mathcal{P}(n, p)$ is of size $(1 + o(1))p\binom{n}{\lfloor n/2 \rfloor}$ w.h.p.. This is sharp in the sense that if $p = o(1/n)$ then the asymptotics is different. Note that as any k -Sperner family is the union of k antichains, the analogous statement holds for k -Sperner families in $\mathcal{P}(n, p)$. Both papers used the container method. Hogenson in her PhD thesis [16] adapted the method of Balogh, Mycroft, and Treglown to obtain the same results for non-induced \vee_r -free families.

Let us state a general proposition that gives a range of p when one can have a P -free family in $\mathcal{P}(n, p)$ that is larger than $p(e(P) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

Proposition 2. For any finite connected poset P , the following statements hold.

- (i) If $p = o(n^{-\frac{x(P)}{|P|-1}})$ and $p\binom{n}{\lfloor n/2 \rfloor} \rightarrow \infty$, then the largest P -free family in $\mathcal{P}(n, p)$ has size at least $(e(P) + 1 - o(1))p\binom{n}{\lfloor n/2 \rfloor}$ w.h.p..
 (ii) If $p = o(n^{-\frac{x^*(P)}{|P|-1}})$ and $p\binom{n}{\lfloor n/2 \rfloor} \rightarrow \infty$, then the largest induced P -free family in $\mathcal{P}(n, p)$ has size at least $(e^*(P) + 1 - o(1))p\binom{n}{\lfloor n/2 \rfloor}$ w.h.p..

If \mathcal{M}_p does not contain a subposet P' of P , then it is P -free, thus we have the following.

Corollary 1. For any finite poset P , let $d(P) = \min \frac{x(P')}{|P'|-1}$, where P' runs through all connected subposets P' of P with $e(P) = e(P')$. Similarly, let $d^*(P) = \min \frac{x^*(P')}{|P'|-1}$, where P' runs through all connected subposets P' of P with $e^*(P) = e^*(P')$. Then the following statements hold.

- (i) If $p = o(n^{-d(P)})$, then the largest P -free family in $\mathcal{P}(n, p)$ has size at least $(e(P) + 1 - o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p.
- (ii) If $p = o(n^{-d^*(P)})$, then the largest induced P -free family in $\mathcal{P}(n, p)$ has size $(e^*(P) + 1 - o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p.

We conjecture that the bounds above are sharp.

Conjecture 4. For any finite connected poset P the following statements hold.

- (i) If $p = \omega(n^{-d(P)})$, then the largest P -free family in $\mathcal{P}(n, p)$ has size $(e(P) + o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p..
- (ii) If $p = \omega(n^{-d^*(P)})$, then the largest induced P -free family in $\mathcal{P}(n, p)$ has size $(e^*(P) + o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p..

The following theorem verifies Conjecture 4 for the posets \vee_{r+1} and $K_{s,1,t}$.

Theorem 7. *If $p = \omega(1/n)$, then the following are true.*

- (i) *For any integer $r \geq 0$, the largest induced \vee_{r+1} -free family in $\mathcal{P}(n, p)$ has size $(1 + o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p..*
- (ii) *For any pair $s, t \geq 1$ of integers, the largest induced $K_{s,1,t}$ -free family in $\mathcal{P}(n, p)$ has size $(2 + o(1))p \binom{n}{\lfloor n/2 \rfloor}$ w.h.p..*

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Path Decompositions of Tournaments

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Abstract. In 1976, Alspach, Mason, and Pullman conjectured that any tournament T of even order can be decomposed into exactly $\text{ex}(T)$ paths, where $\text{ex}(T) = \frac{1}{2} \sum_{v \in V(T)} |d_T^+(v) - d_T^-(v)|$. We prove this conjecture for all sufficiently large tournaments. We also prove an asymptotically optimal result for tournaments of odd order.

Keywords: Tournaments · Decompositions · Paths

1 Introduction

Path and cycle decomposition problems have a long history. For example, the Walecki construction [9], which goes back to the 19th century, gives a decomposition of the complete graph of odd order into Hamilton cycles (see also [2]). A version of this for (regular) directed tournaments was conjectured by Kelly in 1968 and proved for large tournaments in [6]. Beautiful open problems in the area include the Erdős–Gallai conjecture which asks for a decomposition of any graph into linearly many cycles and edges. The best bounds for this are due to Conlon, Fox, and Sudakov [5]. Another famous example is the linear arboricity conjecture, which asks for a decomposition of a d -regular graph into $\lceil \frac{d+1}{2} \rceil$ linear forests. The latter was resolved asymptotically by Alon [1] and the best current bounds are due to Lang and Postle [7].

1.1 Background

The problem of decomposing digraphs into paths was first explored by Alspach and Pullman [4], who provided sharp bounds for the minimum number of paths needed in path decompositions of digraphs. (Throughout this paper, in a digraph, for any two vertices $u \neq v$, we allow a directed edge uv from u to v as well as a directed edge vu from v to u , whereas in an oriented graph we allow at most one directed edge between any two distinct vertices.) Given a digraph D , define the *path number* of D , denoted by $\text{pn}(D)$, as the minimum integer k such that D can be decomposed into k paths. Alspach and Pullman [4] proved that, for any oriented graph D on n vertices, $\text{pn}(D) \leq \frac{n^2}{4}$, with equality holding for transitive

tournaments. O'Brien [10] showed that the same bound holds for digraphs on at least 4 vertices.

The path number of digraphs can be bounded below by the following quantity. Let D be a digraph and $v \in V(D)$. Define the *excess* at v as $\text{ex}_D(v) = d_D^+(v) - d_D^-(v)$. Let $\text{ex}_D^+(v) = \max\{0, \text{ex}_D(v)\}$ and $\text{ex}_D^-(v) = \max\{0, -\text{ex}_D(v)\}$ be the *positive excess* and *negative excess* at v , respectively. Then, as observed in [4], if $d_D^+(v) > d_D^-(v)$, then a path decomposition of D contains at least $d_D^+(v) - d_D^-(v) = \text{ex}_D^+(v)$ paths starting at v . Similarly, a path decomposition will contain at least $\text{ex}_D^-(v)$ paths ending at v . Thus, the *excess* of D , defined as

$$\text{ex}(D) = \sum_{v \in V(D)} \text{ex}_D^+(v) = \sum_{v \in V(D)} \text{ex}_D^-(v) = \frac{1}{2} \sum_{v \in V(D)} |\text{ex}_D(v)|,$$

provides a natural lower bound for the path number of D , i.e. any digraph D satisfies $\text{pn}(D) \geq \text{ex}(D)$. It was shown in [4] that equality is satisfied for acyclic digraphs. A digraph satisfying $\text{pn}(D) = \text{ex}(D)$ is called *consistent*. Clearly, not all digraphs are consistent (e.g. regular digraphs have excess 0). However, Alspach, Mason, and Pullman [3] conjectured in 1976 that tournaments of even order are consistent.

Conjecture 1 ([3]). *Any tournament T of even order satisfies $\text{pn}(T) = \text{ex}(T)$.*

Note that the results of Alspach and Pullman [4] mentioned above imply that Conjecture 1 holds for tournaments of excess $\frac{n^2}{4}$. Moreover, as observed by Lo, Patel, Skokan, and Talbot [8], Conjecture 1 for tournaments of excess $\frac{n}{2}$ is equivalent to Kelly's conjecture on Hamilton decompositions of regular tournaments. Recently, Conjecture 1 was verified in [8] for sufficiently large tournaments of sufficiently large excess. Moreover, they extended this result to tournaments of odd order n whose excess is at least $n^2 - \frac{1}{18}$.

Theorem 2 ([8]). *There exist $C, n_0 \in \mathbb{N}$ such that the following holds. If tournament T on $n \geq n_0$ vertices such that (i) n is even and $\text{ex}(T) \geq Cn$, or (ii) $\text{ex}(T) \geq n^2 - \frac{1}{18}$, then $\text{pn}(T) = \text{ex}(T)$*

1.2 New Results

Building on the results and methods of [6,8], we prove Conjecture 1 for large tournaments.

Theorem 3. *There exists $n_0 \in \mathbb{N}$ such that any tournament T of even order $n \geq n_0$ satisfies $\text{pn}(T) = \text{ex}(T)$.*

In fact, our methods are more general and allow us to determine the path number of most tournaments of odd order, whose behaviour turns out to be more complex. As mentioned above, not every digraph is consistent.

Let D be a digraph. Let $\Delta^0(D)$ denote the largest semidegree of D , that is $\Delta^0(D) = \max\{d^+(v), d^-(v) \mid v \in V(D)\}$. Note that $\Delta^0(D)$ is a natural lower

bound for $\text{pn}(D)$ as every vertex $v \in V(D)$ must be in at least $\max\{d^+(v), d^-(v)\}$ paths. This leads to the notation of the *modified excess* of a digraph D , which is defined as

$$\tilde{\text{ex}}(D) = \max\{\text{ex}(D), \Delta^0(D)\}.$$

This provides a natural lower bound for the path number of any digraph D . Namely, any digraph D satisfies $\text{pn}(D) \geq \tilde{\text{ex}}(D)$.

Observe that, by Theorem 2(ii), equality holds for large tournaments of excess at least $n^{2-\frac{1}{8}}$. However, note that equality does not hold for regular digraphs. Indeed, by considering the number of edges, one can show that any path decomposition of an r -regular digraph will contain at least $r + 1$ paths. Thus, any regular digraph satisfies $\text{pn}(D) \geq \tilde{\text{ex}}(D) + 1$. Denote by \mathcal{T}_{reg} the class of regular tournaments. Alspach, Mason, and Pullman [3] conjectured that equality holds for regular tournaments.

There also exist non-regular tournaments T for which $\text{pn}(T) > \tilde{\text{ex}}(T)$. Indeed, let $\mathcal{T}_{\text{apex}}$ be the set of tournaments T on $n \geq 5$ vertices for which there exists a partition $V(T) = V_0 \cup \{v_+\} \cup \{v_-\}$ such that $T[V_0]$ is a regular tournament on $n-2$ vertices (and so n is odd), $N_T^+(v_+) = V_0 = N_T^-(v_-)$, $N_T^-(v_+) = \{v_-\}$, and $N_T^+(v_-) = \{v_+\}$. We show that any sufficiently large tournament $T \in \mathcal{T}_{\text{apex}} \cup \mathcal{T}_{\text{reg}}$ satisfies $\text{pn}(T) = \tilde{\text{ex}}(T) + 1$.

Theorem 4. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \in \mathcal{T}_{\text{apex}} \cup \mathcal{T}_{\text{reg}}$ on $n \geq n_0$ vertices satisfies $\text{pn}(T) = \tilde{\text{ex}}(T) + 1$.*

We conjecture that these tournaments are the only ones with $\text{pn}(T) \neq \tilde{\text{ex}}(T)$.

Conjecture 5. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \notin \mathcal{T}_{\text{apex}} \cup \mathcal{T}_{\text{reg}}$ on $n \geq n_0$ vertices satisfies $\text{pn}(T) = \tilde{\text{ex}}(T)$.*

We prove an approximate version of this conjecture (see Theorem 7). Moreover, in Theorem 6, we prove Conjecture 5 exactly unless n is odd and T is extremely close to being a regular tournament (in the sense that the number of vertices of nonzero excess is $o(n)$, the excess of each vertex is $o(n)$, and the total excess is $\frac{n}{2} \pm o(n)$).

Theorem 6. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. If $T \notin \mathcal{T}_{\text{apex}} \cup \mathcal{T}_{\text{reg}}$ is a tournament on $n \geq n_0$ vertices such that (i) $\tilde{\text{ex}}(T) \geq \frac{n}{2} + \beta n$, or (ii) $|\{v \in V(T) \mid \text{ex}_T^\pm(v) > 0\}| + \tilde{\text{ex}}(T) - \text{ex}(T) \geq \beta n$, then $\text{pn}(T) = \tilde{\text{ex}}(T)$.*

Using of the fact that $\tilde{\text{ex}}(T) = \text{ex}(T)$ for even n , one can derive Theorem 3 (i.e. the exact solution when n is even) from Theorem 6. We also derive an approximate version of Conjecture 5 from Theorem 6.

Corollary 7. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that any tournament T on $n \geq n_0$ vertices satisfies $\text{pn}(T) \leq \tilde{\text{ex}}(T) + \beta n$.*

Note that Theorem 6(ii) corresponds to the case where linearly many different vertices can be used as endpoints of paths in an optimal decomposition. Indeed, let T be a tournament and \mathcal{P} be a path decomposition of T . Then, as mentioned

above, each $v \in V(T)$ must be the starting point of at least $\text{ex}_T^+(v)$ paths in \mathcal{P} . Thus, for any tournament T , $|\{v \in V(T) \mid \text{ex}_T^+(v) > 0\}| + \tilde{\text{ex}}(T) - \text{ex}(T)$ is the maximum number of distinct vertices which can be a starting point of a path in a decomposition of T of size $\tilde{\text{ex}}(T)$ and similarly for $|\{v \in V(T) \mid \text{ex}_T^-(v) > 0\}| + \tilde{\text{ex}}(T) - \text{ex}(T)$ and the ending points of paths.

2 Proof Overview

2.1 Robust Outexpanders

Our proof of Theorem 6 will be based on the concept of robust outexpanders. Roughly speaking, a digraph D is called a *robust outexpander* if, for any set $S \subseteq V(D)$ which is neither too small nor too large, there exist significantly more than $|S|$ vertices with many inneighbours in S . Any (almost) regular tournament is a robust outexpander and we will use that this property is inherited by random subdigraphs. The main result of [6] states that any regular robust outexpander of linear degree has a Hamilton decomposition. We can apply this to obtain an optimal path decomposition in the following setting. Let D be a digraph on n vertices, $0 < \eta < 1$, and suppose that $X^+ \cup X^- \cup X^0$ is a partition of $V(D)$ such that $|X^+| = |X^-| = \eta n$ and for each $v \in V(D)$,

$$d_D^+(v) = \begin{cases} \eta n - 1 & \text{if } v \in X^-, \\ \eta n & \text{otherwise,} \end{cases} \text{ and } d_D^-(v) = \begin{cases} \eta n - 1 & \text{if } v \in X^+, \\ \eta n & \text{otherwise.} \end{cases} \quad (1)$$

Then the digraph D' obtained from D by adding a new vertex v with $N_{D'}^\pm(v) = X^\pm$ is ηn -regular. Thus if D is a robust outexpander, then there exists a decomposition of D' into Hamilton cycles. This induces a decomposition \mathcal{P} of D into ηn Hamilton paths, where each vertex in X^+ is the starting point of exactly one path in \mathcal{P} and each vertex in X^- is the ending point of exactly one path in \mathcal{P} . Our main strategy will be to reduce our tournaments to a digraph of the above form. This will be achieved as follows.

2.2 Simplified Approach for Well Behaved Tournaments

Let $0 < \frac{1}{n_0} \ll \varepsilon \ll \gamma \ll \eta \ll \beta$ and T a tournament on $n \geq n_0$ vertices. Note that by Theorem 2, we may assume that $\tilde{\text{ex}}(T) \leq \varepsilon^2 n^2$. Moreover, for simplicity, we first also assume that each $v \in V(T)$ satisfies $|\text{ex}_T(v)| \leq \varepsilon n$ (i.e. T is almost regular), $\tilde{\text{ex}}(T) = \text{ex}(T)$, and both $|\{v \in V(T) \mid \text{ex}_T^\pm(v) > 0\}| \geq \eta n$.

Firstly, since T is almost regular, it is a robust outexpander and so we can fix a random spanning subdigraph $\Gamma \subseteq T$ of density γ such that Γ is a robust outexpander and $T \setminus \Gamma$ is almost regular. The digraph Γ will serve two purposes. Firstly, its robust outexpansion properties will be used to construct an approximate path decomposition of T . Secondly, provided few edges of Γ are used throughout this approximate decomposition, it will guarantee that the left-over (consisting of all of those edges of T not covered by the approximate path

decomposition) is still a robust outexpander, as required to complete our decomposition of T in the way described in Sect. 2.1.

Fix $X^\pm \subseteq \{v \in V(T) \mid \text{ex}_T^\pm(v) > 0\}$ of size ηn and let $X^0 = V(T) \setminus (X^+ \cup X^-)$. Our goal is then to find an approximate path decomposition \mathcal{P} of T such that $|\mathcal{P}| = \tilde{\text{ex}}(T) - \eta n$ and the leftover $D = T \setminus \bigcup \mathcal{P}$ satisfies the degree conditions in (1). Thus, it suffices that \mathcal{P} satisfies the following. Each $v \in V(T)$ is the starting point of exactly $\text{ex}_T^+(v) - \mathbb{1}(v \in X^+)$ paths, the ending point of exactly $\text{ex}_T^-(v) - \mathbb{1}(v \in X^-)$ paths and the internal vertex of exactly $\frac{(n-1) - |\text{ex}_T(v)|}{2} - \eta n + \mathbb{1}(v \in X^+ \cup X^-)$ paths in \mathcal{P} .

Recall that, by assumption, T is almost regular. Thus, in a nutshell, we need to construct edge-disjoint paths with specific endpoints and such that each vertex is covered by about $(\frac{1}{2} - \eta)n$ paths. To ensure the latter, we will in fact approximately decompose T into about $(\frac{1}{2} - \eta)n$ spanning sets of internally vertex-disjoint paths. To ensure the former, we will start by constructing $(\frac{1}{2} - \eta)n$ auxiliary digraphs on $V(T)$ such that, for each $v \in V(T)$, the total number of edges starting (and ending) at v is the number of paths that we want to start (and end, respectively) at v . These auxiliary digraphs will be called layouts. Then, it will be enough to construct, for each layout L , a spanning set of paths \mathcal{P}_L , called a *spanning configuration of shape L* , such that each path $P \in \mathcal{P}_L$ corresponds to some edge $e \in E(L)$ and the starting and ending points of P equal those of e .

These spanning configurations will be constructed one by one as follows. At each stage, given a layout L , fix an edge $uv \in E(L)$. Then, using the robust outexpanding properties of (the remainder of) Γ , find short internally vertex-disjoint paths with endpoints corresponding to the endpoints of the edges in $L \setminus \{uv\}$. Denote by \mathcal{P}'_L the set containing these paths. Then, it only remains to construct a path from u to v spanning $V(T) \setminus V(\mathcal{P}'_L)$. We achieve this as follows.

Let D' and Γ' be obtained from (the remainders of) $(T \setminus \Gamma) - V(\mathcal{P}'_L)$ and $\Gamma - V(\mathcal{P}'_L)$ by merging the vertices u and v into a new vertex w such that $N^+(w) = N^+(u)$ and $N^-(w) = N^-(v)$. Observe that a Hamilton cycle of $D' \cup \Gamma'$ corresponds to a path from u to v of T which spans $V(T) \setminus V(\mathcal{P}'_L)$. Of course, one can simply use the fact that Γ' is a robust expander to find a Hamilton cycle. However, if we proceed in this way, then the robust outexpanding property of Γ' might be destroyed before constructing all the desired spanning configurations. So instead we construct a Hamilton cycle with only few edges in Γ' as follows. Using the fact that $T \setminus \Gamma$ is almost regular, we first find an almost spanning linear forest F in D' which has few components. Then we use the robust outexpanding properties of Γ' to tie up F into a Hamilton cycle of $D' \cup \Gamma'$.

2.3 General Tournaments

For a general tournament T , we adapt the above argument as follows. Let W be the set of vertices $v \in V(T)$ such that $|\text{ex}_T(v)| > \varepsilon n$. If $W \neq \emptyset$, then T is no longer almost regular and we cannot proceed as above. However, since $\text{ex}(T) \leq \varepsilon^2 n^2$, $|W|$ is small. Thus, we can start with a cleaning procedure which efficiently decreases the excess and degree at W by taking out few edge-disjoint paths.

Then, we apply the above argument to (the remainder of) $T - W$. We incorporate all remaining edges at W in the approximate decomposition by generalising the concept of a layout introduced above.

If $|\{v \in V(T) \mid \text{ex}_T^+(v) > 0\}| < \eta n$ but $\tilde{\text{ex}}(T) = \text{ex}(T)$, say, then we cannot choose $X^+ \subseteq \{v \in V(T) \mid \text{ex}_T^+(v) > 0\}$ of size ηn . We circumvent this problem as follows. Select a small set of vertices W_A such that $\sum_{v \in W_A} \text{ex}_T^+(v) \geq \eta n$ and let A be a set of ηn edges such that the following hold. Each edge in A starts in W_A and ends in $V(T) \setminus W_A$. Moreover, each $v \in W_A$ is the starting point of at most $\text{ex}_T^+(v)$ edges in A and each $v \in V(T) \setminus W_A$ is the ending point of at most one edge in A . We will call A an *absorbing set of starting edges*. Then, let the ending points of the edges in A play the role of X^+ and add the vertices in W_A to W so that, at the end of the approximate decomposition, the only remaining edges at W_A are the edges in A . Thus, in the final decomposition step, we can use the edges in A to extend the paths starting at X^+ into paths starting in W_A . If $|\{v \in V(T) \mid \text{ex}_T^-(v) > 0\}| < \eta n$, then we proceed analogously.

If $\tilde{\text{ex}}(T) > \text{ex}(T)$, then not all paths will “correspond” to some excess. For simplicity, we will choose which additional endpoints to use at the beginning and artificially add excess to those vertices. This then enables us to proceed as if $\text{ex}(T) = \tilde{\text{ex}}(T)$. More precisely, we will choose a set $U^* \subseteq \{v \in V(T) \mid \text{ex}_T(v) = 0\}$ of size $\tilde{\text{ex}}(T) - \text{ex}(T)$ and we will treat the vertices in U^* in the same way as we treat those with $\text{ex}_T^\pm(v) = 1$. Note that selecting additional endpoints in this way maximises the number of distinct endpoints, which will enable us to choose $X^\pm \subseteq \{v \in V(T) \mid \text{ex}_T^\pm(v) > 0\} \cup U^*$ when $|\{v \in V(T) \mid \text{ex}_T^\pm(v) > 0\}| + \tilde{\text{ex}}(T) - \text{ex}(T) \geq \eta n$ and use absorbing edges otherwise, i.e. if condition (ii) fails in Theorem 6.

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Sorting by Shuffling Methods and a Queue

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Abstract. We consider sorting by a queue that can apply a permutation from a given set over its content. This gives us a sorting device \mathbb{Q}_Σ corresponding to any shuffling method Σ since every such method is associated with a set of permutations. Two variations of these devices are considered - \mathbb{Q}'_Σ and $\mathbb{Q}_\Sigma^{\text{pop}}$. These require the entire content of the device to be unloaded after a permutation is applied or unloaded by each pop operation, respectively.

First, we show that sorting by a deque is equivalent to sorting by a queue that can reverse its content. Next, we focus on sorting by *cuts*, which has a significance in genome rearrangements and has a natural interpretation. We prove that the set of permutations that one can sort by using $\mathbb{Q}'_{\text{cuts}}$ is the set of the 321-avoiding separable permutations. We give lower and upper bounds to the maximum number of times the device must be used to sort a permutation.

Furthermore, we give a formula for the number of n -permutations that one can sort by using \mathbb{Q}'_Σ , for any shuffling method Σ , such that the permutations associated with it are irreducible. The rest of the work is dedicated to a surprising conjecture inspired by Diaconis and Graham which states that one can sort the same number of permutations of any given size by using the devices $\mathbb{Q}_{\text{In-sh}}^{\text{pop}}$ and $\mathbb{Q}_{\text{Monge}}^{\text{pop}}$, corresponding to the popular *In-shuffle* and *Monge* shuffling methods.

Keywords: Sorting · Shuffling · Separable permutation · Pattern avoidance

1 Introduction and Definitions

A main line of research on the applications of permutation patterns in computer science is related to sorting of permutations using different sorting devices e.g., stacks, queues, deques and their modifications. Knuth [9, Chapter 2.2.1] was the first one to consider sorting by these classical data structures. The book [8, Chapter 2] lists several other articles dedicated to the topic.

A completely different, yet connected, line of research investigates shuffling methods for a given deck of cards or respectively for a given permutation. A shuffling method could be any procedure that will lead to a uniformly shuffled deck after applying the shuffling method multiple times. Diaconis et al. [3, Section 2.3] give an overview of the previous work related to shuffling.

In this work, we relate the areas of sorting devices and shuffling methods by considering sorting by special type of queues, called *shuffle queues*, which can rearrange their content by applying permutations in a given collection over it. We call any such collection of permutations a *shuffling method* and we focus on collections associated with some methods that are popular in the literature. Only a few previous works investigate sorting by modifications of a queue [1, 5]. Shuffle queues are a natural such modification since a sorting device is a machine whose sole function is to re-order its input data. These new devices lead to some surprising enumerative results and raise interesting combinatorial questions. More motivational points are described in Sect. 1.2.

1.1 Notation

A permutation of size n is a bijective map from $[n] = \{1, 2, \dots, n\}$ to itself. Permutations will be presented in one-line notation. Other standard definitions related to permutation patterns that will be used can be found in [2]. The reverse of $\pi = \pi_1 \cdots \pi_n$ will be denoted by $\pi^r := \pi_n \cdots \pi_1$. The empty sequence will be denoted by ε . For a sequence of distinct numbers s , denote by $Im(s)$ the set of elements of s and let $Im(s_1, \dots, s_t) = \bigcup_{k=1}^t Im(s_k)$. Let

$$T_n^3 = \{(s_1, s_2, s_3) \mid Im(s_i) \cap Im(s_j) = \emptyset, Im(s_1, s_2, s_3) = [n]\}$$

be the set of triples of sequences having sets of elements forming a partition of $[n]$. We will call the elements of T_n^3 *configurations*.

A sorting device \mathbb{D} is a tool that transforms a given input permutation π by following a particular algorithm which could be deterministic or non-deterministic. The result is an output permutation π' . During the execution of the algorithm, every device \mathbb{D} has a given configuration $(s_{inp}, s_{dev}, s_{out})$, comprised of three sequences (strings) corresponding to the current string in the input, the device and the output, respectively. Thus, we have a sequence of configurations beginning with $(\pi, \varepsilon, \varepsilon)$ and ending with $(\varepsilon, \varepsilon, \pi')$. This sequence will be called an *iteration* of \mathbb{D} over the input π . Denote by $\mathbb{D}(\pi)$ the set of possible output permutations, when using a device \mathbb{D} on input π . If id_n denotes the identity permutation of size n , then let $S_n(\mathbb{D}) := \{\pi \mid \pi \in S_n, id_n \in \mathbb{D}(\pi)\}$ be the set of the permutations sortable with \mathbb{D} . Furthermore, let $p_n(\mathbb{D}) := |S_n(\mathbb{D})|$.

A shuffling method transforms a given input permutation by multiplying it by another permutation, according to a given distribution over S_n . We will ignore this distribution and assume that every given shuffling method Σ is defined by a family of subsets of permutations $F_\Sigma = \{\Pi_n^\Sigma \subseteq S_n \mid n = 1, 2, \dots\}$. Each subset contains the possible multipliers of the input permutation for the corresponding value of n . We will call the collection of these subsets the *permutation family* of the method Σ . Below, we give the permutation family for the shuffling by cuts, which is a main focus of the present work.

Example 1. Shuffling by cuts:

$$\text{for all } n \geq 2 : \Pi_n^{\text{cuts}} = \{k(k+1) \dots n12 \dots (k-1) \mid k \in \{2, \dots, n\}\}. \quad (1)$$

For a given shuffling method Σ , we consider a non-deterministic sorting device \mathbb{Q}_Σ for which at any given step one can apply up to three possible operations over the current configuration $s = (s_{inp}, s_{dev}, s_{out})$:

1. Push

Move the first element x of $s_{inp} = xs'_{inp}$ at the end of s_{dev} .

2. Pop

Move the first element y of $s_{dev} = ys'_{dev}$ at the end of s_{out} .

3. Shuffle

Pick a permutation $\sigma \in \Pi_\Sigma^m$ and apply it over s_{dev} , where $|s_{dev}| = m$.

We will call the device \mathbb{Q}_Σ a *shuffle queue*. This work focuses on two natural variations of the devices \mathbb{Q}_Σ that will be called shuffle queues of type (i) and type (ii). They are obtained after imposing two additional restrictions:

- (i) The entire content of the device must be unloaded after each shuffle. Denote the corresponding sorting device by \mathbb{Q}'_Σ .
- (ii) The entire content of the device must be unloaded by each pop operation. Denote the corresponding sorting device by $\mathbb{Q}_\Sigma^{\text{POP}}$. This is the pop-version of the device \mathbb{Q}_Σ in analogy to the pop version of the stack-sorting device. We will also call them *pop shuffle queues*.

1.2 Motivation

We show that sorting by a deque is equivalent to sorting by a simple shuffle queue (see Sect. 2). In addition, sorting by \mathbb{Q}_{cuts} has a simple interpretation in terms of railway switching networks, similar to those given by Knuth in [9] for stack, queue and deque. It is not difficult to show that every permutation can be sorted by \mathbb{Q}_{cuts} . Thus, it is reasonable to ask which permutations can be sorted by cuts (and by other methods) if we consider the two natural restrictions defining shuffle queues of types (i) and (ii). We also formulate an unexpected conjecture involving shuffle queues of type (ii) (see Sect. 5). Furthermore, sorting by cuts is an important problem connected to genome rearrangements and an object of extensive study from the algorithms community [7].

2 Shuffle Queues Equivalent to Deque and Stack

Definition 1. *The sorting devices \mathbb{U} and \mathbb{V} are equivalent if for every $n \geq 1$,*

$$S_n(\mathbb{U}) = S_n(\mathbb{V}).$$

We will denote that by writing $\mathbb{U} \cong \mathbb{V}$.

Sorting by a deque turns out to be equivalent to sorting by the shuffle queue of a very simple shuffling method that can just reverse its content. A deque will be denoted by \mathbb{Deq} .

Theorem 1. $\mathbb{D}_{\text{eq}} \cong \mathbb{Q}_{\text{rev}}$, where

$$\Pi_{\text{rev}}^n = \{n(n-1) \cdots 21\}, \text{ for all } n \geq 2.$$

We were not able to find this surprisingly simple fact in the existing literature. Perhaps it can be used to make progress on the long-standing problem of finding the number of permutations sortable by a deque [10, A182216]. A reasonable next question is whether there exists a shuffle queue that is equivalent to a stack, denoted by $\mathbb{S}t$. We show that such a queue does not exist.

Theorem 2. *There is no shuffling method Σ , such that $\mathbb{S}t \cong \mathbb{Q}_{\Sigma}$.*

3 Sorting by Cuts

One of the simplest known shuffling methods is shuffling with cuts. Its permutation family is given by Eq. (1). Consider the device $\mathbb{Q}'_{\text{cuts}}$. Below is shown one possible iteration of $\mathbb{Q}'_{\text{cuts}}$ and the corresponding operations when sorting the permutation 213564. Each configuration is written in the column form $\begin{pmatrix} s_{\text{inp}} \\ s_{\text{dev}} \\ s_{\text{out}} \end{pmatrix}$.

Example 2. Iteration of $\mathbb{Q}'_{\text{cuts}}$ over 213645:

$$\begin{pmatrix} 213645 \\ \varepsilon \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 13645 \\ 2 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 3645 \\ 21 \\ \varepsilon \end{pmatrix} \xrightarrow[\text{+unload}]{\text{shuffle (cut)}} \begin{pmatrix} 3645 \\ \varepsilon \\ 12 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 645 \\ 3 \\ 12 \end{pmatrix}$$

$$\xrightarrow{\text{pop}} \begin{pmatrix} 645 \\ \varepsilon \\ 123 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 45 \\ 6 \\ 123 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 5 \\ 64 \\ 123 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ 645 \\ 123 \end{pmatrix} \xrightarrow[\text{+unload}]{\text{shuffle (cut)}} \begin{pmatrix} \varepsilon \\ \varepsilon \\ 123456 \end{pmatrix}$$

Note that we unload the entire content of the device after each shuffle operation. The fact that the set of the cut-sortable permutations $S_n(\mathbb{Q}'_{\text{cuts}})$ is a permutation class follows from the observation in the proof of [1, Proposition 1]. With the next theorem, we find this class.

Theorem 3. *The permutations sortable by $\mathbb{Q}'_{\text{cuts}}$ are the 321-avoiding separable permutations [10, A034943]; i.e.,*

$$S_n(\mathbb{Q}'_{\text{cuts}}) = Av_n(321, 2413, 3142).$$

The class of separable permutations have an important recursive description and are enumerated by the Schröder numbers [8, Chapter 2.2.5].

The next fact gives an alternative way to find the total number of sortable permutations when using cuts and generalizes Theorem 3. By IP_n , we denote the set of the irreducible permutations of size n , i.e., those not fixing $[1..j]$ for any $0 < j < n$ ([10, A003319]).

Theorem 4. *If $\Pi_{\Sigma}^k \subseteq IP_k$ for every $k \geq 2$ and $b_k := |\Pi_{\Sigma}^k|$, then*

$$p_n(\mathbb{Q}'_{\Sigma}) = 1 + \sum_{\substack{k_1 + \dots + k_l = n - u \\ k_i \geq 2, u \geq 0}} \binom{u + l}{l} \prod_{j=1}^l b_{k_j}.$$

An analogue of this formula for pop shuffle queues was also established.

4 Permutations of Higher Cost

Not all $\pi \in S_n$ are sortable by $\mathbb{Q}'_{\text{cuts}}$. However, one can use this device several times in a row by using the output after one iteration as an input to the next iteration. Denote the set of permutations that one can obtain after k iterations of $\mathbb{Q}'_{\text{cuts}}$ over a permutation $\pi \in S_n$ by $(\mathbb{Q}'_{\text{cuts}})^k(\pi)$.

Definition 2 (cost of permutation). *The cost of π is the minimum number of iterations needed to sort π using the device $\mathbb{Q}'_{\text{cuts}}$, i.e.,*

$$\text{cost}(\pi) := \min\{m \mid id_n \in (\mathbb{Q}'_{\text{cuts}})^m(\pi)\}.$$

Theorem 3 gives a characterization of the set of permutations of cost 1. In general, how big can be $\text{cost}(\pi)$, for $\pi \in S_n$? It is not difficult to obtain that $\text{cost}(\pi) \leq n$. This upper bound is improved significantly with the theorem given below.

Theorem 5. *$\text{cost}(\pi) \leq \lceil \frac{n}{2} \rceil$, for every $\pi \in S_n$, where $n \geq 1$.*

To prove Theorem 5, we show that one can always transform the input permutation, via a few cuts, to one that contains a segment of consecutive elements. This segment can be treated as a single element, which allows us to apply induction. The transformation is obtained by using a combination of a few additional observations that we prove separately.

Theorem 5 is a non-trivial upper bound for the *cost* function which is tight since there exist permutations of cost $\lceil \frac{n}{2} \rceil$. The best absolute lower bound is obviously 0 since $\text{cost}(id_n) = 0$, for every n . Let $M_n := \max_{\pi \in S_n} \text{cost}(\pi)$. Theorem 5 gives us that $M_n \leq \lceil \frac{n}{2} \rceil$. We also prove the following lower bound by showing that the reverse identity id_n^r always require at least $\lceil \log_2 n \rceil$ iterations to be sorted.

Theorem 6. *$M_n \geq \lceil \log_2 n \rceil$, for each $n \geq 2$.*

The bounds established with Theorem 5 and Theorem 6 are analogues of the bounds obtained in [6] for the maximal number of cuts needed to sort a permutation. We finish this section by showing that the permutations in S_n can be paired up in terms of cost, when using $\mathbb{Q}'_{\text{cuts}}$. For a permutation $\pi = \pi_1 \cdots \pi_n$, let $\bar{\pi}$ denote the complement permutation, defined by $\bar{\pi}_i = n + 1 - \pi_i$. Set $\pi^* = \overline{\bar{\pi}} = (\bar{\pi})^r$.

Theorem 7. *For any permutation π , $\text{cost}(\pi) = \text{cost}(\pi^*)$.*

5 A Conjecture on Two Pop Shuffle Queues

In this subsection, we conjecture and investigate a possible connection between the pop shuffle queues for two of the most popular shuffling methods, namely the *In-shuffle* and the *Monge* shuffling methods.

When using the *In-shuffle* method, the deck is divided into two halves and the cards in them are interleaved perfectly, i.e., the first card is coming from one of the halves, the second from the other half and so on. The original top card becomes second from top. Some of the mathematical properties of the In-shuffles are discussed in [4]. The permutation family of the In-shuffle method is:

$$\text{for all } n \geq 2 : \Pi_{\text{In-sh}}^n = \begin{cases} \{(k+1)1(k+2)2 \cdots (2k)k\}, & \text{if } n = 2k, \text{ and} \\ \{(k+1)1(k+2)2 \cdots (2k)k(2k+1)\}, & \text{if } n = 2k+1. \end{cases}$$

The Monge shuffle is carried out by successively putting cards over and under. The top card is taken into the other hand, the next is placed above, the third below these two cards and so on. The permutation family of the Monge shuffling method is:

$$\text{for all } n \geq 2 : \Pi_{\text{Monge}}^n = \{\cdots 642135 \cdots\}.$$

We make the following conjecture.

Conjecture 1. For every $n \geq 1$,

$$p_n(\mathbb{Q}_{\text{In-sh}}^{\text{POP}}) = p_n(\mathbb{Q}_{\text{Monge}}^{\text{POP}}).$$

First, we prove that Conjecture 1 holds, if one has to use a single pop operation. Let $p_n^1(\mathbb{Q}_{\Sigma}^{\text{POP}})$ be the number of permutations of size n sortable by $\mathbb{Q}_{\Sigma}^{\text{POP}}$ using only one pop operation.

Theorem 8. For every $n \geq 3$,

$$p_n^1(\mathbb{Q}_{\text{In-sh}}^{\text{POP}}) = p_n^1(\mathbb{Q}_{\text{Monge}}^{\text{POP}}) = a_{n-2},$$

where $a_1 = 2$, $a_2 = 4$ and $a_n = 3a_{n-2}$, for $n \geq 3$ (sequence A068911 in [10]).

We also give recurrence relations for the number of permutations in $S_n(\mathbb{Q}_{\text{Monge}}^{\text{POP}})$ that end with n and that do not end with n . Similar inequalities were established for these two subsets of $S_n(\mathbb{Q}_{\text{In-sh}}^{\text{POP}})$. By using Theorem 8 and these recurrence relations and inequalities, we obtain an important necessary condition for Conjecture 1 to hold. These facts also allowed us to check with a computer that the conjecture holds for $n < 20$.

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Circular Coloring of Signed Bipartite Planar Graphs

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Abstract. In this work, we study the notion of circular coloring of signed graphs which is a refinement of 0-free $2k$ -coloring of signed graphs. The main question is that given a positive integer ℓ , what is the smallest even value $f(\ell)$ such that for every signed bipartite (simple) planar graph (G, σ) of negative-girth at least $f(\ell)$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1}$. We answer this question when ℓ is small: $f(2) = 4$, $f(3) = 6$ and $f(4) = 8$. The results fit into the framework of the bipartite analogue of the Jaeger-Zhang conjecture.

Keywords: Circular coloring · Homomorphism · Signed bipartite graphs

1 Introduction

The theory of graph homomorphism is a natural generalization of the notion of proper coloring of graphs. It's well-known that the C_{2k+1} -coloring problem captures the $(2k+1)$ -coloring problem via a basic graph operation: Given a graph G , let G' be the graph obtained from G by subdividing each edge into a path of length $2k-1$. Then G' admits a homomorphism to C_{2k+1} if and only if G is properly $(2k+1)$ -colorable (see [6]). Moreover, a graph admits a homomorphism to C_{2k+1} if and only if its circular chromatic number is at most $\frac{2k+1}{k}$.

A famous question relevant to the C_{2k+1} -coloring of planar graphs is the Jaeger-Zhang conjecture (introduced in [17] and studied in [1, 5, 8, 16] among others):

Conjecture 1. Every planar graph G of odd-girth at least $4k+1$ admits a homomorphism to C_{2k+1} , or equivalently, $\chi_c(G) \leq \frac{2k+1}{k}$.

Using the notion of circular coloring of signed graphs, we explore the theory for (negative) even cycles. A *signed graph* (G, σ) is a graph G (allowing loops and multi-edges) together with an assignment $\sigma : E(G) \rightarrow \{+, -\}$. The *sign of a closed walk* is the product of signs of all its edges (allowing repetition). Given a signed graph (G, σ) and a vertex v of (G, σ) , a *switching at v* is to switch the signs of all the edges incident to v . We say a signed graph (G, σ') is *switching equivalent* to (G, σ) if it is obtained from (G, σ) by a series of switchings at vertices. It's

proven in [15] that two signed graphs (G, σ_1) and (G, σ_2) are switching equivalent if and only if they have the same set of negative cycles.

A *(switching) homomorphism* of a signed graph (G, σ) to (H, π) is a mapping of $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacencies, the incidences and the signs of closed walks are preserved. When there exists a homomorphism of (G, σ) to (H, π) , we write $(G, \sigma) \rightarrow (H, \pi)$. A homomorphism of (G, σ) to (H, π) is said to be *edge-sign preserving* if it, furthermore, preserves the signs of the edges. When there exists an edge-sign preserving homomorphism of (G, σ) to (H, π) , we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$. The connection between these two kinds of homomorphism is established as follows: Given two signed graphs (G, σ) and (H, π) , $(G, \sigma) \rightarrow (H, \pi)$ if and only if there exists an equivalent signature σ' of σ such that $(G, \sigma') \xrightarrow{s.p.} (H, \pi)$.

Observe that the parity of the lengths and the signs of closed walks are preserved by a homomorphism. Given a signed graph (G, σ) and an element $ij \in \mathbb{Z}_2^2$, we define $g_{ij}(G, \sigma)$ to be the length of a shortest closed walk whose number of negative edges modulo 2 is i and whose length modulo 2 is j . When there exists no such a closed walk, we say $g_{ij}(G, \sigma) = \infty$. By the definition of homomorphism of signed graphs, we have the following no-homomorphism lemma.

Lemma 1 [13]. *If $(G, \sigma) \rightarrow (H, \pi)$, then $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for each $ij \in \mathbb{Z}_2^2$.*

1.1 Circular Coloring of Signed Graphs

The notion of circular coloring of signed graphs defined in [14] is a common extension of circular coloring of graphs and $2k$ -coloring of signed graphs.

Given a real number r , a *circular r -coloring* of a signed graph (G, σ) is a mapping $\varphi : V(G) \rightarrow C^r$ such that for each positive edge uv of (G, σ) , $\varphi(u)$ and $\varphi(v)$ are at distance at least 1, and for each negative edge uv of (G, σ) , $\varphi(u)$ and the antipodal of $\varphi(v)$ are at distance at least 1. The *circular chromatic number* of a signed graph (G, σ) is defined as

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

For integers $p \geq 2q > 0$ such that p is even, the *signed circular clique* $K_{p;q}^s$ has the vertex set $[p] = \{0, 1, \dots, p-1\}$, in which ij is a positive edge if and only if $q \leq |i - j| \leq p - q$ and ij is a negative edge if and only if either $|i - j| \leq \frac{p}{2} - q$ or $|i - j| \geq \frac{p}{2} + q$. Moreover, let $\hat{K}_{p;q}^s$ be the signed subgraph of $K_{p;q}^s$ induced by vertices $\{0, 1, \dots, \frac{p}{2} - 1\}$. The following statements are equivalent:

1. (G, σ) has a circular $\frac{p}{q}$ -coloring;
2. (G, σ) admits an edge-sign preserving homomorphism to $K_{p;q}^s$;
3. (G, σ) admits a homomorphism to $\hat{K}_{p;q}^s$.

The next lemma is a straightforward consequence of the transitivity of the homomorphism relation.

Lemma 2. *If $(G, \sigma) \rightarrow (H, \pi)$, then $\chi_c(G, \sigma) \leq \chi_c(H, \pi)$.*

1.2 Homomorphism of Signed Bipartite Graphs

Given a signed graph (G, σ) , we define $T_l(G, \sigma)$ to be the signed graph obtained from (G, σ) by replacing each edge with a path of length l with the sign $-\sigma(uv)$. For a non-zero integer ℓ , we denote by C_ℓ the cycle of length $|\ell|$ whose sign agrees with the sign of ℓ . The *negative-girth* of a signed graph is defined to be the shortest length of a negative closed walk of it. In the following lemma, the k -coloring problem of graphs is captured by C_{-k} -coloring problem of signed graphs.

Lemma 3 [10]. *A graph G is k -colorable if and only if $T_{k-2}(G, +)$ is C_{-k} -colorable.*

In particular, the $2k$ -coloring problem of graphs is captured by the C_{-2k} -coloring problem of signed bipartite graphs. Thus we could restate the Four-Color Theorem as follows:

Theorem 1. *For any planar graph G , the signed bipartite planar graph $T_2(G, +)$ admits a homomorphism to C_{-4} .*

Moreover, the problem of mapping signed bipartite graphs to negative even cycles is equivalent to the question of bounding the circular chromatic number of signed bipartite graphs.

Proposition 1. *A signed bipartite graph (G, σ) admits a homomorphism to C_{-2k} if and only if $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$.*

Proof. Observe that the signed graph $\hat{K}_{4k;2k-1}^s$ is obtained from C_{-2k} by adding a negative loop at each vertex and $\chi_c(C_{-2k}) = \frac{4k}{2k-1}$. It suffices to prove that if $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$, then $(G, \sigma) \rightarrow C_{-2k}$. Let $\{x_1, \dots, x_{4k}\}$ be the vertex set of $K_{4k;2k-1}^s$ and let φ be an edge-sign preserving homomorphism of (G, σ) to $K_{4k;2k-1}^s$. For the rest of the proof, addition in indices of vertices are considered mod $4k$.

Recall that in $K_{4k;2k-1}^s$, each x_i is adjacent with positive edges to three vertices furthest from it, namely x_{i+2k-1} , x_{i+2k} and x_{i+2k+1} , and it is adjacent with negative edges to three vertices closest to it, namely x_{i-1} , x_i and x_{i+1} . Let $K_{4k;2k-1}^{s'}$ be the signed graph obtained from $K_{4k;2k-1}^s$ by removing negative loops and positive edges $x_i x_{i+2k}$ for each i . We claim that $(G, \sigma) \rightarrow K_{4k;2k-1}^{s'}$. Let (A, B) be a bipartition of vertices of (G, σ) and let (X, Y) be the bipartition of $K_{4k;2k-1}^{s'}$ where $X = \{x_1, x_3, \dots, x_{4k-1}\}$ and $Y = \{x_2, x_4, \dots, x_{4k}\}$. For any $u \in V(G)$ with $\varphi(u) = x_i$, we define $\phi(u)$ as follows: If either $u \in A$ and i is even or $u \in B$ and i is odd, then $\phi(u) = x_{i+1}$; Otherwise, $\phi(u) = x_i$. It's easy to verify that ϕ is an edge-sign preserving homomorphism of (G, σ) to $K_{4k;2k-1}^{s'}$. Since $K_{4k;2k-1}^{s'} \rightarrow C_{-2k}$, it completes the proof.

Restricted to signed planar graphs, there is a bipartite analogue question of Jaeger-Zhang conjecture proposed in [12]: Given an integer k , what is the smallest value $f(k)$ such that every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k} ?

2 Circular Coloring of Signed Bipartite Planar Graphs

For a class \mathcal{C} of signed graphs, we define $\chi_c(\mathcal{C}) = \sup\{\chi_c(G, \sigma) : (G, \sigma) \in \mathcal{C}\}$. In the sequel, we denote the class of signed bipartite planar graph of negative-girth at least $2k$ by \mathcal{SBP}_{2k} . For this special class of signed graphs, some bounds have been already studied. It has been proved in [14] that $\chi_c(\mathcal{SBP}_4) = 4$.

2.1 Signed Bipartite Planar Graphs of Negative-Girth at Least 6

In this section, we will show that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$ in which the negative edges form a matching. Its connection with circular coloring of signed bipartite planar graphs is presented in the following lemma.

Lemma 4. *A signed bipartite graph (G, σ) admits a homomorphism to $(K_{3,3}, M)$ if and only if $\chi_c(G, \sigma) \leq 3$.*

Proof. As $\chi_c(K_{3,3}, M) = 3$, it remains to show that if $(G, \sigma) \xrightarrow{s.p.} K_{6;2}^s$, then $(G, \sigma) \rightarrow (K_{3,3}, M)$. Let φ be an edge-sign preserving homomorphism of (G, σ) to $K_{6;2}^s$ and let (A, B) be the bipartition of (G, σ) . Let $\{x_1, x_2, \dots, x_6\}$ be the vertex set of $K_{6;2}^s$. Let $K_{6;2}^{s'}$ be the signed graph obtained from $K_{6;2}^s$ by deleting all the negative loops and positive edges $x_i x_{i+2}$ for all the i (the index addition is taken (mod 6)). First of all, $K_{6;2}^{s'}$ is bipartite and let (X, Y) be its bipartition where $X = \{x_1, x_3, x_5\}$ and $Y = \{x_2, x_4, x_6\}$. Secondly, $K_{6;2}^{s'}$ is switching equivalent to $(K_{3,3}, M)$.

For any vertex u with $\varphi(u) = x_i$, we define ϕ to be as follows: if $u \in A$ and i is even or $u \in B$ and i is odd, then $\phi(u) = x_i$; otherwise, we switch at u and $\phi(u) = x_{i+3}$. It's easy to verify that ϕ is a homomorphism of $(G, \sigma) \rightarrow K_{6;2}^{s'}$. Hence $(G, \sigma) \rightarrow (K_{3,3}, M)$.

A *Signed Projective Cube* of dimension k , denoted $SPC(k)$, is a projective cube $PC(k)$ (the graph with vertex set \mathbb{Z}_2^k where vertices u and v are adjacent if $u - v \in \{e_1, e_2, \dots, e_d\} \cup \{J\}$) together with an assignment such that all the edges uv satisfying that $u - v = J$ are negative. The following result is implied from an edge-coloring result of [4] and a result of [11]:

Theorem 2. *If (G, σ) is a signed planar graph satisfying that $g_{ij}(G, \sigma) \geq g_{ij}(SPC(5))$, then $(G, \sigma) \rightarrow SPC(5)$.*

As $g_{10}(SPC(5)) = 6$ and $g_{01}(SPC(5)) = g_{11}(SPC(5)) = \infty$, it means that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $SPC(5)$.

Theorem 3 [11]. *Let (G, σ) be a signed graph. We have that $(G, \sigma) \rightarrow SPC(k)$ if and only if there exists a partition of the edges of G , say E_1, E_2, \dots, E_{k+1} , such that for each $i \in \{1, 2, \dots, k + 1\}$, the signature σ_i which assigns $-$ to the edges in E_i is switching equivalent to σ .*

We are now ready to prove our main theorem:

Theorem 4. *Every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$. In other words, $\chi_c(\mathcal{SBP}_6) \leq 3$.*

Proof. Let (G, σ) be a signed bipartite planar graph of negative-girth at least 6 with a bipartition (A, B) . By Theorem 2, $(G, \sigma) \rightarrow SPC(5)$. Thus, by Theorem 3, there exists a partition of the edges of G , say E_1, E_2, \dots, E_6 , such that for each $i \in [6]$, there is a signature σ_i equivalent to σ satisfying that E_i is the set of all negative edges in (G, σ_i) .

We consider the signed graph (G, σ_1) where the set of negative edges is E_1 . Contracting all the negative edges in E_1 , we obtain a signed graph with only positive edges, denoted by G' . Observe that G' might contain parallel edges. It is also easily observed that a cycle C' of G' is odd if and only if it is obtained from a negative cycle C of (G, σ) by contraction. Since C must contain at least one edge from each of E_2, E_3, \dots, E_6 , C' is of length at least 5. In other words, G' is a planar graph with no loops and no triangle.

We conclude that G' is a triangle-free planar graph. Thus, by the Grötzsch theorem, G' admits a 3-coloring, say $\varphi : V(G') \rightarrow \{1, 2, 3\}$. Let (X, Y) be a bipartition of $(K_{3,3}, M)$ where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ such that $\{x_1y_1, x_2y_2, x_3y_3\}$ is the set of negative edges. We can now define a mapping ψ of (G, σ_1) to $(K_{3,3}, M)$ as follows:

$$\psi(u) = \begin{cases} x_i, & \text{if } u \in A \text{ and } \varphi(u) = i \\ y_i, & \text{if } u \in B \text{ and } \varphi(u) = i. \end{cases}$$

It's easy to verify that ψ is an edge-sign preserving homomorphism of (G, σ_1) to $(K_{3,3}, M)$. It completes the proof.

Examples of signed bipartite graphs of negative-girth 4 which do not map to $(K_{3,3}, M)$ are given in [9]. So the negative-girth 6 is the best possible girth condition for signed bipartite graph (G, σ) to satisfy $\chi_c(G, \sigma) \leq 3$.

2.2 Signed Bipartite Planar Graphs of Negative-Girth at Least 8

In this section, we include the result that every signed bipartite planar graph of negative-girth at least 8 admits a homomorphism to C_{-4} . Generalizing the notion of H -critical graph defined by Catlin [2], we say a signed graph (G, σ) is (H, π) -critical if $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$, it does not admit a homomorphism to (H, π) but any proper subgraph of it does. The girth condition implies, in particular, that every C_{-4} -critical signed graph is bipartite.

Theorem 5 [10]. *If (G, σ) is a C_{-4} -critical signed graph, then $|E(G)| \geq \frac{4|V(G)|-1}{3}$.*

Corollary 1 [10]. *Every signed bipartite planar graph of negative-girth at least 8 maps to C_{-4} .*

Here the negative-girth condition 8 is the best possible because there exists a signed bipartite planar graph of negative-girth 6 which does not admit a homomorphism to C_{-4} as shown in [10]. Furthermore, considering the signed bipartite planar graphs Γ_n introduced in [14], we have $\lim_{n \rightarrow \infty} T_2(\Gamma_n) = \frac{8}{3}$. Therefore, $\chi_c(\mathcal{SBP}_s) = \frac{8}{3}$.

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Results on the Graceful Game and Range-Relaxed Graceful Game

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Abstract. The range-relaxed graceful game is a maker-breaker game played in a simple graph G where two players, Alice and Bob, alternately assign an unused label $f(v) \in \{0, \dots, k\}$, $k \geq |E(G)|$, to an unlabeled vertex $v \in V(G)$. If both ends of an edge $vw \in E(G)$ are already labeled, then the label of the edge is defined as $|f(v) - f(w)|$. Alice's goal is to end up with a vertex labelling of G where all edges of G have distinct labels, and Bob's goal is to prevent this from happening. When it is required that $k = |E(G)|$, the game is called graceful game. The range-relaxed graceful game and the graceful game were proposed by Tuza in 2017. The author also posed a question about the least number of consecutive non-negative integer labels necessary for Alice to win the game on an arbitrary simple graph G and also asked if Alice can win the range-relaxed graceful game on G with the set of labels $\{0, \dots, k+1\}$ once it is known that she can win with the set $\{0, \dots, k\}$. In this work, we investigate the graceful game in Cartesian and corona products of graphs, and determine that Bob has a winning strategy in all investigated families independently of who starts the game. Additionally, we partially answer Tuza's questions presenting the first results in the range-relaxed graceful game and proving that Alice wins on any simple graph G with order n , size m and maximum degree Δ , for any set of labels $\{0, \dots, k\}$ with $k \geq (n-1) + 2\Delta(m-\Delta) + \frac{\Delta(\Delta-1)}{2}$.

Keywords: Graceful labeling · Graph labeling game · Maker-breaker game

1 Introduction

In the last decades, many optimization problems have been proposed where it is required to label the vertices or the edges of a given graph with numbers. Most of these problems [7–10] emerged naturally from modeling of optimization problems on networks and one of the oldest and most investigated is the problem of determining the gracefulfulness of a graph, proposed by Golomb [7] in 1972.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001, CAPES-PrInt project number 88881.310248/2018-01, CNPq project 423833/2018-9 and FAPERJ.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 214–220, 2021.
https://doi.org/10.1007/978-3-030-83823-2_34

Formally, given a simple graph $G = (V(G), E(G))$, where $|V(G)| = n$ and $|E(G)| = m$ and a set $\mathcal{L} \subset \mathbb{Z}$, a *labeling* of G is a vertex labeling $f: V(G) \rightarrow \mathcal{L}$ that induces an edge labeling $g: E(G) \rightarrow \mathbb{Z}$ in the following way: $g(uv)$ is a function of $f(u)$ and $f(v)$, for all $uv \in E(G)$, and g respects some specified restrictions. Given the set of consecutive integer labels $\mathcal{L} = \{0, \dots, k\}, k \geq |E(G)|$, a labeling $f: V(G) \rightarrow \mathcal{L}$ is *graceful* if: (i) $k = |E(G)|$; (ii) f is injective; and (iii) if each edge $uv \in E(G)$ is assigned the (*induced*) label $g(uv) = |f(u) - f(v)|$, then all induced edge labels are distinct. When condition (i) in the above definition is relaxed so as to allow $k \geq |E(G)|$, f is said to be a *range-relaxed graceful labeling* (RRG labeling). The least k needed for G to have a labeling f satisfying conditions (ii) and (iii) in the above definition is called the *gracefulness* of G and is denoted by $grac(G)$. Figure 1 exhibits a graceful labeling of complete graph K_4 and a range-relaxed graceful labeling of cycle C_5 .

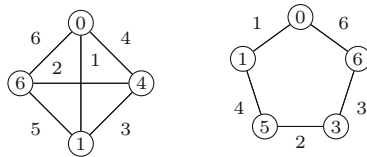


Fig. 1. A graceful labeling of K_4 and an RRG labeling of C_5 .

According to Golomb [7], range-relaxed graceful labelings arise in the following practical context: thinking of a graph G as a communication network with n terminals and m interconnections between terminals, we wish to assign a distinct non-negative integer to each terminal so that each interconnection is uniquely identified by the absolute value of the difference between the numbers assigned to its two end terminals. The objective is to minimize the largest number assigned to any terminal.

Graceful labelings were introduced by Rosa [14] in 1966 and were so named by Golomb [7], who also introduced the range-relaxed graceful variation later investigated by other authors [1, 2, 15]. Although it is known that the parameter $grac(G)$ is defined for every simple graph G [7], $grac(G)$ is not yet determined even for classic families of graphs such as complete graphs [13].

Graph labeling is an area of graph theory whose main concern consists in determining the feasibility of assigning labels to vertices or edges of a graph satisfying certain conditions. From the literature of graph labeling [6], it is notorious that labeling problems are usually studied from the perspective of determining whether a given graph has a required labeling. An alternative perspective is to analyze labeling problems from the point of view of combinatorial games. In most combinatorial games, two players—traditionally called Alice and Bob—alternately select and label vertices or edges (typically one vertex or edge in each step) in a graph G which is completely known for both players. One of the first graph combinatorial games is the coloring game [17] conceived by Brams, firstly published in 1981 by Gardner.

In 2017, Tuza [16] surveyed the area of labeling games and proposed new labeling games such as the *edge-sum distinguishing* and the *edge-difference distinguishing game (range-relaxed graceful game)*. The author also posed the following questions regarding the range-relaxed graceful (RRG) game:

Question 1. Given a simple graph G and a set of consecutive non-negative integer labels $\mathcal{L} = \{0, \dots, k\}$, for which values of k can Alice win the range-relaxed graceful game?

Question 2. If Alice can win the range-relaxed graceful game on a graph G with the set of labels $\mathcal{L} = \{0, \dots, k\}$, can she also win with $\mathcal{L} = \{0, \dots, k + 1\}$?

The graceful game was later studied by Fricke et al. [4], who investigated winning strategies (sequence of moves that leads to the victory one of the players) for Alice and Bob in some classic families of graphs such as paths, complete graphs, cycles, complete bipartite graphs, caterpillars, trees, gear graphs, helms, web graphs, prisms, hypercubes, and 2-powers of paths.

In this work, we examine the graceful game and study winning strategies for Alice and Bob in classes of corona products such as the corona product of cycles and complete graphs, $C_r \odot K_q$, and the corona product of a connected graph G with at least two vertices and an empty graph I_p with $p \geq 1$ vertices, $G \odot I_p$. We also study winning strategies for Alice and Bob in products of graphs, such as grids ($P_r \square P_q$), generalized book graphs ($P_q \square S_r$), stacked prisms ($P_q \square C_r$), the Cartesian product of paths and complete graphs, and toroidal grids ($C_r \square C_q$). We also investigate the classes of crown graphs (the bipartite graph with vertex set $X \cup Y$, where $X = \{x_0, \dots, x_{n-1}\}$ and $Y = \{y_0, \dots, y_{n-1}\}$, and edge set $\{x_i y_j : 0 \leq i, j \leq n-1, i \neq j\}$), and circular snake graphs (a connected simple graph with $k \geq 2$ blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to a cycle on s vertices). These results show structural properties, implied by the graceful labeling constraints, that aim to contribute to the study of the graceful labeling of graphs in which the gracefulness was not yet determined. Moreover, we present the first results concerning the range-relaxed graceful game and show an upper bound on the number of consecutive non-negative integer labels necessary for Alice to win the game on an arbitrary simple graph, which gives a partial answer to Questions 1 and 2 posed by Tuza [16]. The paper is organized as follows: Sect. 2 presents auxiliary results and definitions; Sects. 3 and 4 show our results on the graceful game and range-relaxed graceful game, respectively. Finally, we summarize our results in Sect. 5.

2 Basic Notation and Auxiliary Lemmas

Next, we present definitions and results used throughout the text. All graphs considered in this paper are finite, undirected, and simple. Let $G = (V(G), E(G))$ be a graph. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; in such a case, edge $e = uv$ and vertices u and v are called *incident*, and vertices u and v are also called *neighbors*. The set of neighbors of a vertex $v \in V(G)$ is usually denoted

by $N(v)$, this is called the *open neighborhood* of v . The *closed neighborhood* of v is defined as $N[v] = \{v\} \cup N(v)$. The *degree* of a vertex $v \in V(G)$ is the number of edges incident to v and is denoted by $d_G(v)$. The *maximum degree* of G is the number $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. The distance $d(u, v)$ between $u, v \in V(G)$ is the number of edges in a shortest path connecting u and v .

The *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 is the simple graph with $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ such that $(u_1, v_1)(u_2, v_2) \in E(G_1 \square G_2)$ if and only if either (i) $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$, or (ii) $v_1 v_2 \in E(G_2)$ and $u_1 = u_2$. Another product of graphs is the corona product of two graphs [5]. Formally, given a graph G with p vertices and a graph H , the *corona product* of G and H , $G \odot H$, is the graph obtained from G and p copies of H by joining each vertex of G to every vertex of its respective copy of H .

In 2017, Tuza [16] proposed the following maker-breaker game inspired on RRG labelings: given a graph G , two players, Alice and Bob, alternately assign an unused label $f(v) \in \mathcal{L} = \{0, \dots, k\}$, $k \in \mathbb{N}$, to an unlabeled vertex $v \in V(G)$. If both ends of an edge $vw \in E(G)$ are already labeled, then the (*induced*) *label* of the edge is defined as $g(vw) = |f(v) - f(w)|$. We call a vertex of G *unlabeled* if no label has been assigned to it. We say that a label assignment (move) is *legal* if, after it, all edge labels are distinct. Only legal moves are allowed during the game and both players play optimally. The game ends if there is no legal move possible or an RRG labeling is created. Note that after any move, the next player can only label unlabeled vertices. Alice *wins* if an RRG labeling of G is created, otherwise Bob *wins*. Tuza [16] called such a game *edge-difference distinguishing*. However, we call it a *range-relaxed graceful game* in order to match with the range-relaxed graceful labeling nomenclature previously established in the literature [1, 2, 15]. Note that, for the case where $k = |E(G)|$, Alice's goal is to end up with a graceful labeling of G and the game is called *graceful game*. We observe that not every graph is graceful and thus, for non-graceful graphs, we establish that Bob is the winner of the graceful game.

We present below an auxiliary result on the graceful game established by Frickes et al. [4] and used in our proofs.

Lemma 1 (Frickes et al. [4]). *Let G be a simple graph with m edges. In any step of the graceful game, Alice can only assign the label 0 (resp. m) to a vertex $v \in V(G)$ if v is adjacent to every remaining vertex not yet labeled or v is adjacent to a vertex already labeled by Bob with m (resp. 0).*

Next, we present new structural lemmas on the graceful game that are used in our proofs.

Lemma 2. *If a graph G has two vertices u and v of degree 1 such that $d(u, v) \geq 4$, then Bob wins the graceful game on G no matter who starts.*

Lemma 3. *Let G be a simple graph and $w \in V(G)$ be the center of an induced star subgraph, i.e., $G[N[w]] \cong K_{1,y}$, $2 \leq y \leq \Delta(G)$. Consider that Alice and Bob start playing the graceful game on G and that, after the third step of the game, $u, v, w \in N[w]$ are the only labeled vertices, such that: Alice assigns label*

$\ell \in \{1, \dots, m - 1\}$ to v on the first move and assigns m to u on the third move; and Bob assigns 0 to w on the second move. If there exists a vertex u' , with $u' \notin N(w)$, such that $d(u', u) \geq 2$, then Bob wins the graceful game.

Lemma 4. *Let G be a simple graph with m edges and $n \geq 9$ vertices. Let $w \in V(G)$ be a vertex with $N(w) = \{u, u', z, z'\}$ such that: u and u' are adjacent and $N(w) - \{u\}$ and $N(w) - \{u'\}$ are independent sets. Let $v \in V(G)$ such that $d(v, w) \geq 3$ and $x \in N(w)$ such that $x \in \{u, z\}$. Consider that Alice and Bob start playing the graceful game on G and that, after the third move, v, w and x are the only labeled vertices, such that: Alice assigned label $\ell \in \{1, \dots, m - 1\}$ to v on the first move and assigned m to x on the third move; and Bob assigned 0 to w on the second move. If there exists a vertex y in G such that $d(y, w) \geq 3$, $y \neq v$ and y not adjacent to v , then Bob wins the game.*

3 Graceful Game on Graph Products

A grid, $P_r \square P_q$, is a simple graph obtained from the Cartesian product of two paths P_r and P_q , with $r, q \in \mathbb{N}$ and $r, q \geq 2$. Jungreis and Reid [11] proved that grid graphs are graceful. Theorem 1 characterizes the graceful game for all grids.

Theorem 1. *Bob has a winning strategy for the graceful game on every grid $P_r \square P_q$, for $r, q \geq 2$.*

The *generalized book*, $B_{q,r}$, is the graph obtained from the Cartesian product $P_q \square S_r$ of a path P_q and a star S_r , where q is the number of vertices of the path and r is the number of edges of the star.

In 1980, Maheo [12] proved that generalized book graphs of the form $B_{2,2k}$ are graceful, and Delorme [3] proved that $B_{2,4k+1}$ are also graceful. Generalized book graphs $B_{2,4k+3}$ are not graceful since they do not satisfy Rosa’s parity condition for Eulerian graphs [14], which says that, if an Eulerian graph G with m edges is graceful, then $m \equiv 0, 3 \pmod{4}$. Corollary 1 characterizes the graceful game for all generalized book graphs and its proof follows from Theorem 1 since, for $1 \leq r \leq 2$, G is a grid and, for $r \geq 3$, each page of the book (that is, each of its r induced subgraphs isomorphic to $P_q \square S_1$) is a grid.

Corollary 1. *Bob has a winning strategy for the graceful game on all generalized books $P_q \square S_r$, for $q \geq 2$ and $r \geq 1$.*

Besides grid graphs and generalized books, we also characterized the graceful game for other families of graphs.

Theorem 2. *Bob has a winning strategy for: (1) the Cartesian products: stacked prisms, toroidal grids and cartesian product of paths and complete graphs; (2) the corona products: $C_r \odot K_q$, with $r \geq 4$ and $q \geq 1$, and the product $G \odot I_p$ for G a connected graph with at least two vertices and I_p an empty graph with $p \geq 1$ vertices; and (3) for crowns and k - C_n snake graphs.*

4 Results on the Range-Relaxed Graceful Game

Given a simple graph G , its *game graceful number*, denoted by $grac_g(G)$, is the minimum non-negative integer k such that Alice has a winning strategy playing the range-relaxed graceful game on G with a set of labels $\mathcal{L} = \{0, \dots, k\}$. Theorem 3 partially answers Question 1 posed by Tuza [16] presenting an upper bound for the value of k , for any simple graph G .

Theorem 3. *If G is a simple connected graph on n vertices, m edges and maximum degree Δ , then $grac_g(G) \leq (n - 1) + 2\Delta(m - \Delta) + \frac{\Delta(\Delta-1)}{2}$.*

The idea of the proof of Theorem 3 is that, at the beginning of the game, every vertex $v \in V(G)$ has a list of available labels $L(v) = \{0, \dots, k\}$. After each round, the sets $L(v)$ of unlabeled vertices $v \in V(G)$ are updated so that, in the last move, we guarantee that there exists at least one available label in the set $L(v)$ that can be assigned for the last vertex v in order to obtain a range relaxed graceful labeling of the graph. From the proof of Theorem 3, it is also possible to obtain the following result.

Corollary 2. *Let G be a simple connected graph on n vertices, m edges and maximum degree Δ . If Alice can win the range-relaxed graceful game on G with the set of labels $\mathcal{L} = \{0, \dots, k\}$, then she also wins with $\mathcal{L}' = \{0, \dots, k + 1\}$ for any integer $k \geq (n - 1) + 2\Delta(m - \Delta) + \frac{\Delta(\Delta-1)}{2}$.*

5 Concluding Remarks

In this work, we show that Bob has a winning strategy for the graceful game on several families defined by Cartesian and corona products of graphs. We observe that there exist cases where Alice has a winning strategy for graphs G_1 and G_2 , but she loses on the product of G_1 and G_2 . For example, Alice wins the graceful game on C_3 and K_1 [4], but she loses on the product $C_3 \odot K_1$. Another example is $P_q \square K_p$, where Alice wins on P_2 and K_3 [4], but she loses on $P_2 \square K_3$. We also study the RRG game and present the first upper bound for the parameter $grac_g(G)$ for an arbitrary connected simple graph G .

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New Bounds on the Modularity of Johnson Graphs and Random Subgraphs of Johnson Graphs

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Abstract. In this paper we study the modularity of the so called Johnson graphs, also known as $G(n, r, s)$ graphs. We obtain significant improvements for this value in case s and $r \geq cs^2$ are fixed and n tends to infinity. We also obtain results on the modularity of random subgraphs of $G(n, r, s)$ in Erdős–Rényi model.

Keywords: Modularity · Johnson graphs · Clusterization · Distance graphs

1 Introduction

In this paper we obtain new bounds on the modularity, a characteristic of graphs that shows if the vertices of the graph can be partitioned into dense clusters with connection between the clusters being quite weak. This characteristic was first introduced by Newman and Girvan in [23]. Since that time a lot of clustering algorithms relying on this value were designed (see [1, 4, 6, 7, 12–14, 18, 19, 21, 22, 24, 25]).

This paper considers the family of graphs known as $G(n, r, s)$ graphs, or Johnson graphs. These graphs play a huge role in Ramsey theory (see [5, 20, 28, 29, 33, 36]), combinatorial geometry (see [2, 3, 5, 15, 16, 26, 27, 30–35]) and coding theory (see [20]). We will also consider random subgraphs of Johnson graphs in Erdős–Rényi model and provide some results for them.

Let us now give rigorous definitions of the objects we study.

Definition 1. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a partition of the vertices of a graph G . The edge contribution is then defined as $\sum_{i=1}^k \frac{e(A_i)}{e(G)}$, where $e(A) = |\{(v_1, v_2) \in E(G) | v_1, v_2 \in A\}|$.

Definition 2. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a partition of the vertices of a graph G . The value

$$\sum_{i=1}^k \frac{(\sum_{v \in A_k} \text{deg}(v))^2}{4e^2(G)}$$

is called the degree tax.

Remark 1. Note that for d -regular graphs the degree tax can be rewritten as

$$\sum_{i=1}^k \frac{(|A_k|d)^2}{(d|G|)^2} = \sum_{i=1}^k \frac{|A_k|^2}{|G|^2}.$$

Definition 3. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a partition of the vertices of a graph G . The modularity of partition \mathcal{A} is defined as the difference between the edge contribution and the degree tax of this partition:

$$q(\mathcal{A}) = \sum_{A \in \mathcal{A}} \frac{e(A)}{e(G)} - \sum_{A \in \mathcal{A}} \frac{(\sum_{v \in A} \text{deg}(v))^2}{4e^2(G)}.$$

Definition 4. The modularity of graph G is defined as a maximum of modularities over all partitions of the vertices of G :

$$q^*(G) = \max_{\mathcal{A}} \{q(\mathcal{A})\}.$$

Let us also define the family of Johnson graphs.

Definition 5. Let $0 \leq s < r < n$ be integers. The $G(n, r, s)$ graph is a graph with set of vertices being equal to $\binom{[n]}{r}$ and vertices u and v joined by an edge if and only if $|u \cap v| = s$.

Remark 2. It is easy to see that $G(n, r, s)$ is a $\binom{r}{s} \binom{n-r}{r-s}$ -regular graph with $\binom{n}{r}$ vertices and $\frac{1}{2} \binom{r}{s} \binom{n-r}{r-s} \binom{n}{r}$ edges.

Finally we define random subgraphs of Johnson graphs in the following way.

Definition 6. Let $0 \leq s < r < n$. $G_p(n, r, s)$ is a random element such that for any graph H that is a spanning subgraph of $G(n, r, s)$ the following equality holds.

$$\mathbf{P}(G_p(n, r, s) = H) = p^{e(H)}(1 - p)^{e(G(n, r, s)) - e(H)}.$$

Modularity of Johnson graphs was studied in [9–11, 17]. The following theorems were proved.

Theorem 1 ([10]). Let $r \geq 2$ and $1 \leq s \leq \lceil \frac{r}{2} \rceil$. Then

$$\limsup_{n \rightarrow \infty} q^*(G(n, r, s)) \leq 1 - \frac{\binom{\lceil \frac{r}{2} \rceil}{s}}{2\binom{r}{s}}.$$

Theorem 2 ([9]).

$$q^*(G(n, 2, 1)) = \frac{1}{3} + \frac{2w(w-1)(w-2)}{3n(n-1)(n-2)} - \frac{w^2(w-1)^2}{n^2(n-1)^2} - \frac{4n-2}{3n(n-1)} + \frac{w(w-1)(4w-2)}{3n^2(n-1)^2} \quad (1)$$

for all $n \geq 5$, where $w = \lceil \frac{n}{2} \rceil + 1$. Its limit with $n \rightarrow \infty$ is $\frac{17}{48}$.

Theorem 3 ([17]). Let $r > s \geq 1$. Then

$$\liminf_{n \rightarrow \infty} q^*(G(n, r, s)) \geq \frac{s}{2r-s} \left(1 + \left(\frac{r-s}{r} \right)^{\frac{2r}{s}} \right).$$

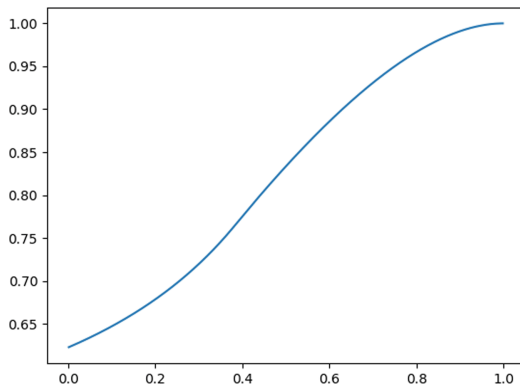
We have improved the result of Theorem 1 significantly.

Theorem 4. Let $\varepsilon \in (0, 1)$, $s \geq 1$, $r \geq -\frac{1}{\ln(1-\varepsilon)}s^2 + 2s - 1$. Then

$$\limsup_{n \rightarrow \infty} q^*(G(n, r, s)) \leq f(\varepsilon),$$

where $f(\varepsilon) = \max_{x \in [0,1]} \left(\frac{1+x-x^2}{2-x} - \max \left(\frac{x^2-\varepsilon x}{1-\varepsilon}, 0 \right) \right)$.

The graph below shows how $f(\varepsilon)$ behaves.



We also provide a table with numerical values of $f(\varepsilon)$ and bounds on r .

ε	$r \geq$	$f(\varepsilon)$
0.01	$99.4992s^2 + 2s - 1$	0.6246
0.1	$9.4912s^2 + 2s - 1$	0.6469
0.2	$4.4814s^2 + 2s - 1$	0.6783
0.3	$2.8037s^2 + 2s - 1$	0.7195
0.4	$1.9576s^2 + 2s - 1$	0.7750
0.5	$1.4427s^2 + 2s - 1$	0.8333
0.6	$1.0914s^2 + 2s - 1$	0.8857
0.7	$0.8306s^2 + 2s - 1$	0.9308
0.8	$0.6213s^2 + 2s - 1$	0.9667
0.9	$0.4343s^2 + 2s - 1$	0.9909
0.99	$0.2171s^2 + 2s - 1$	0.9999

We will finally mention one particular case to show that the new bounds are way better than the previous ones. Consider the graphs $G(n, r^2, r)$ for big enough r and $n \rightarrow \infty$. In this case the upper bound on the modularity from Theorem 4 will be approximately 0.9011. At the same time the upper bound from Theorem 1 is only $1 - \frac{\binom{r^2/2}{r}}{2\binom{r^2}{r}} > 1 - 2^{-r-1}$.

The main difficulty in the proof of Theorem 4 is to bound $e(U)$ in terms of $|U|$. The bound we provide is given in the following theorem.

Theorem 5. *Let*

$$\alpha, \beta \in (0, 1), \alpha < \beta^2, s \geq 1, r \geq -\frac{1}{\ln\left(\frac{1-\beta}{1-\alpha}\right)}s^2 + 2s - 1,$$

and also $U \subseteq V_G, |U| < \alpha \binom{n}{r}$. Then $e(U) \leq \frac{1+\beta-\beta^2}{2(2-\beta)} \binom{r}{s} \binom{n-s}{r-s} |U|$.

It is also worth mentioning that the result of Theorem 5 is not asymptotic and thus it can be used to obtain bounds on the modularity of $G(n, r, s)$ for fixed n as well.

The idea of the proof of Theorem 5 is to divide the edges between the vertices in $|U|$ into sets depending on the intersection of vertices these edges connect. We then prove that most of these sets are small and arrive to the bound mentioned above.

We also succeeded in proving the probabilistic version of Theorems 3 and 4.

Theorem 6. *Let $r \geq 2s, p = p(n) = \omega\left(n^{-\frac{r-s-1}{2}}\right)$. Then*

$$\limsup_{n \rightarrow \infty} q^*(G_p(n, r, s)) \leq 1 - \frac{\binom{\lfloor \frac{r}{2} \rfloor}{s}}{2\binom{r}{s}} \text{ a.s.}$$

Theorem 7. *Suppose $r, s, (r, s) \neq (2, 1)$ suite the conditions of Theorem 5, $p = p(n) = \omega\left(n^{-\frac{r-s-1}{2}}\right)$. Then*

$$\limsup_{n \rightarrow \infty} q^*(G_p(n, r, s)) \leq f(\varepsilon) \text{ a.s.,}$$

where $f(\varepsilon) = \max_{x \in [0,1]} \left(\frac{1+x-x^2}{2-x} - \max\left(\frac{x^2-\varepsilon x}{1-\varepsilon}, 0\right) \right)$.

One can notice that an extra requirement $(r, s) \neq (2, 1)$ appeared in Theorem 7 in comparison to those of Theorem 4. This requirement is not so restricting though, as Theorem 6 provides a better bound anyway.

Probabilistic version of Theorem 1 is proved as well, though some extra assumptions have to be made.

Theorem 8. *Let $r \geq \max(2s, s + 2), s \geq 1, p = p(n) = \omega\left(n^{-\frac{r-s-1}{2}}\right)$, Then*

$$\liminf_{n \rightarrow \infty} q^*(G_p(n, r, s)) \geq \frac{s}{2r-s} \left(1 + \left(\frac{r-s}{r}\right)^{\frac{2r}{s}} \right) \text{ a.s.}$$

The proof of all three theorems is similar. The main tool here is the Hoeffding inequality (see [8]).

Lemma 1 (Hoeffding). *Let X_1, \dots, X_n be independent random variables. Assume that for any i there exists a pair of numbers a_i, b_i , such that $\mathbf{P}(X_i \in [a_i, b_i]) = 1$. Then for $S_n = X_1 + \dots + X_n$ the following inequality holds.*

$$\mathbf{P}(|S_n - ES_n| \geq \varepsilon n) < 2 \exp \left(- \frac{2\varepsilon^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2} \right).$$

Using this theorem we one can prove that both $e(U)$ and $\sum_{v \in U} \deg(v)$ are very unlikely to deviate from their average even for one big enough U . The only thing remaining is to handle small parts of the partition, which is not too hard.

Acknowledgments. The work is done under the financial support of the grant of the president of the Russian Federation no. NSh-2540.2020.1. The second author is also supported by the grant of Theoretical Physics and Mathematics Advancement Foundation “BASIS”.

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On Bipartite Sum Basic Equilibria

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Abstract. A connected and undirected graph G of size $n \geq 1$ is said to be a sum basic equilibrium iff for every edge uv from G and any node v' from G , when performing the swap of the edge uv for the edge uv' the sum of the distances from u to all the other nodes is not strictly reduced. This concept comes from the so called *Network Creation Games*, a wide subject inside Algorithmic Game Theory that tries to better understand how Internet-like networks behave. It has been shown that the diameter of sum basic equilibria is $2^{O(\sqrt{\log n})}$ in general and at most 2 for trees. In this paper we see that the upper bound of 2 not only holds for trees but for bipartite graphs, too. Specifically, we show that the only bipartite sum basic equilibrium networks are the complete bipartite graphs $K_{r,s}$ with $r, s \geq 1$.

Keywords: Network creation game · Sum basic · Diameter · Nash equilibrium

1 Introduction

Definition of the model and context. In the *sum basic network creation game*, introduced by Alon et al. in 2010 [2], it is assumed that $n \geq 1$ players are the nodes of an undirected graph of size n . If G is connected and for every edge uv and every node v' , player u does not strictly reduce the sum of distances to all the other nodes by performing any single swap of the edge uv for the edge uv' , then such network is said to be a *sum basic equilibrium graph*.

Given an undirected graph G and a node u from G , a *deviation* in u is any swap of an edge uv from G for any other edge uv' with $v' \neq v$, v any other node from G . The *deviated graph* associated to any such deviation is the resulting graph obtained after applying the swap. Furthermore, the *cost difference* associated to any deviation in u is just the difference between the sum of distances from u to all the other nodes in the original graph minus the sum of the distances from u to all the other nodes in the deviated graph. Therefore, a connected and

This work has been partially supported by funds from the Spanish Ministry for Economy and Competitiveness (MINECO) and the European Union (FEDER funds) under grant GRAMM (TIN2017-86727-C2-1-R) and from the Catalan Agency for Management of University and Research Grants (AGAUR, Generalitat de Catalunya) under project ALBCOM 2017-SGR-786.

undirected graph G is a sum basic equilibrium iff for every node u in G the cost difference associated to every possible deviation in u is non-negative.

This network creation game is inspired by the *sum classical network creation game* introduced by Fabrikant et al. in 2003 [4] which is a relatively simple yet tractable model to better understand Internet-like networks. The main contributions of the several authors investigating the sum classical network creation game consist in showing improved bounds for the price of anarchy for this model, a measure that quantifies the loss of efficiency of the system due to the selfish behaviour of its agents. It turns to be that the price of anarchy in the sum classical network creation game is related to the diameter of equilibrium networks in the same model [3]. For this reason, one of the main interests in the sum basic network creation game is the study of the diameter of equilibrium networks, too.

One of the most important contributions from [2] is an upper bound on the diameter of any sum basic equilibrium of $2^{O(\sqrt{\log n})}$. However, this bound can be dramatically reduced if we restrict to the tree topology, in which case the diameter is shown to be at most 2. Furthermore, in [2] the authors establish a connection between sum basic equilibria of diameter larger than $2 \log n$ and *distance-uniform graphs*. The authors then conjecture that distance-uniform graphs have logarithmic diameter which would imply, using this connection, that sum basic equilibria have poly-logarithmic diameter. Unfortunately the conjecture is later refuted in [5]. After some years, in [6], Nikolettseas et al. use the probability principle to establish structural properties of sum basic equilibria. As a consequence of some of these properties, in some extremal situations, like when the maximum degree of equilibrium network is at least $n/\log^l n$ with $l > 0$, it is shown that the diameter is polylogarithmic.

Our Contribution. In this paper we focus our attention to sum basic equilibria restricted to bipartite graphs topology. We show that such networks are the complete bipartite graphs $K_{r,s}$ with $r, s \geq 1$ thus dramatically reducing the diameter to 2 when restricting to this particular case. Our approach consists in considering any 2-edge-connected component H of a non-tree sum basic equilibrium G . In Sect. 2 we consider all the collection of individual swaps uv for uv' for each $u, v, v' \in V(H)$ and $uv, vv' \in E(H)$. We show that if $\text{diam}(H) > 2$, then the sum of the cost differences off all these swaps will be < 0 , thus contradicting the fact that G is a sum basic equilibrium. In Sect. 3, we study further properties of any 2-edge-connected component of any non-tree sum basic equilibrium that work in general and which allow us to reach the main conclusion.

Notation. In this work we consider mainly undirected graphs G for which we denote by $V(G), E(G)$ its corresponding sets of vertices and edges, respectively.

Given an undirected graph G and any pair of nodes u, v from G we denote by $d_G(u, v)$ the distance between u, v . In this way, $D(u)$ is the sum of distances from u to all the other nodes, that is, $D(u) = \sum_{v \neq u} d_G(u, v)$ if G is connected or ∞ otherwise. Given a subgraph H from G , noted as $H \subseteq G$, the i -th distance layer in H with respect u is denoted as $\Gamma_{i,H}(u) = \{v \in V(H) \mid d_G(u, v) = i\}$. In particular, the neighbourhood of u in H , the set of nodes from $V(H)$ at distance one with respect u , is $\Gamma_{1,H}(u)$.

Given an undirected graph G and a property P , we say that H is a maximal subgraph of G satisfying P when for any other subgraph H' of G , if H' satisfies P then $H \not\subseteq H'$. An edge $e \in E(G)$ is said to be a *bridge* if its removal increases the number of connected components from G . In this way, a 2-edge-connected component H from G is any maximal connected subgraph of G not containing any bridge. Moreover, for a given 2-edge-connected component H from G and a vertex $u \in V(H)$, $W(u)$ is the connected component containing u in the subgraph induced by the vertices $(V(G) \setminus V(H)) \cup \{u\}$.

Finally, remind that a bipartite graph is any graph for which all cycles, that is, closed paths, have even length.

2 Local Swap Deviations

Given a non-tree bipartite sum basic equilibrium graph G , let H be any of its 2-edge-connected components. In this section we show that $diam(H) = 2$.

Given $u \in V(H)$ and $w \in V(G)$, we define $\delta_w^-(u)$ the subset of nodes v from $\Gamma_{1,H}(u)$ such that $d_G(w, v) = d_G(w, u) - 1$ and $\delta_w^+(u)$ the subset of nodes v from $\Gamma_{1,H}(u)$ such that $d_G(w, v) = d_G(w, u) + 1$. Since G is bipartite, for any $u \in V(H)$ and $w \in V(G)$, $\delta_w^-(u) \cup \delta_w^+(u) = \Gamma_{1,H}(u)$.

Moreover, given $u \in V(H)$ and $w \in V(G)$ such that $|\delta_w^-(u)| = 1$, we define $u_w^- \in \delta_w^-(u)$ to be the neighbour of u in H closer from w than u . Recall that, for any $u \in V(H)$ and $w \in V(G)$, if $|\delta_w^-(u)| = 1$ then clearly $\delta_w^+(u) \neq \emptyset$ because H is 2-edge-connected.

Now, let u, v be nodes with $u \in V(H)$ and $v \in \Gamma_{1,H}(u)$. We define $S(u, v)$ to be the sum of the cost differences associated to the swaps of the edge uv by the edges uv' with $v' \in \Gamma_{1,H}(v) \setminus \{u\}$ divided over $deg_H(v) - 1$. Then we define $S = \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} S(u, v)$.

Let u, v, w be nodes with $u \in V(H)$, $v \in \Gamma_{1,H}(u)$ and $w \in V(G)$ and define $\Delta_w(u, v)$ to be the sum of the distance changes from u to w due to the swaps of the edge uv by the edges uv' with $v' \in \Gamma_{1,H}(v) \setminus \{u\}$ divided over $deg_H(v) - 1$. Furthermore, let $\Delta_w = \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} \Delta_w(u, v)$.

In this way we have that $S = \sum_{w \in V(G)} \Delta_w$.

Before going to the main result of this section we first find a formula to compute the value of $\Delta_w(u, v)$ allowing us to obtain an expression for Δ_w .

Lemma 1. *For any nodes $u, v \in V(H)$ and $w \in V(G)$ such that $v \in \Gamma_{1,H}(u)$:*

$$\Delta_w(u, v) = \begin{cases} \frac{-|\delta_w^-(u_w^-)| + |\delta_w^+(u_w^-)| - 1}{deg_H(v) - 1} & \text{If } |\delta_w^-(u)| = 1 \text{ and } v = u_w^- \\ \frac{-|\delta_w^-(v)|}{deg_H(v) - 1} & \text{If } |\delta_w^-(u)| > 1 \text{ and } v \in \delta_w^-(u) \\ 0 & \text{otherwise} \end{cases}$$

Proof. If v is further from w than u , then clearly $\Delta_w(u, v) = 0$. Therefore, since G is bipartite the remaining case is that v is closer from w than u . We can see clearly that we need to distinguish the cases $|\delta_w^-(u)| = 1$ with $v = u_w^-$ and the case $|\delta_w^-(u)| > 1$ with $v \in \delta_w^-(u)$. In the first case the corresponding sum of

distance changes from u to w could get positive when the set of nodes $\delta_w^+(u_w^-)$ has size at least two. In contrast, in the second case the sum of distance changes is always no greater than zero because having at least another node distinct than v in the subset $\delta_w^-(u)$ guarantees that when making the corresponding deviation the distance from u to w does not increase.

Now we are ready to show the main result of this section.

Theorem 1. $diam(H) \leq 2$.

Proof. First, we claim that for every $w \in V(G)$, $\Delta_w \leq 0$. Applying Lemma 1:

$$\begin{aligned} \Delta_w &= \sum_{u \in V(H)} \sum_{v \in \Gamma_{1,H}(u)} \Delta_w(u, v) = \\ &= \left(\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{-|\delta_w^-(u_w^-)| + |\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} + \sum_{\{u \in V(H) \wedge |\delta_w^-(u)|>1\}} \sum_{v \in \delta_w^-(u)} \frac{-|\delta_w^-(v)|}{deg_H(v) - 1} \right) = \\ &= \left(\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} + \sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{-|\delta_w^-(v)|}{deg_H(v) - 1} \right) \end{aligned}$$

On the one hand:

$$\sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in V(H)} \sum_{u \in \delta_w^+(v)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in V(H)} \frac{|\delta_w^-(v)||\delta_w^+(v)|}{deg_H(v) - 1}$$

Now, let Z_w be the subset of nodes z from $V(H)$ such that $\delta_w^-(z) \neq \emptyset$ and $\delta_w^+(z) \neq \emptyset$. If $z \in Z_w$ then clearly $|\delta_w^-(z)||\delta_w^+(z)| \geq deg_H(z) - 1$. One possible way to see this is the following. Since H is bipartite, then $|\delta_w^-(z)|$ and $|\delta_w^+(z)|$ are positive integers that add up to $deg_H(z)$. Furthermore, any concave function defined on a closed interval attains its minimum in one of its extremes. Therefore, the conclusion follows when combining these two facts to the function $f(x) = x(deg_H(z) - x)$ defined in $[1, deg_H(z) - 1]$. In this way:

$$\sum_{u \in V(H)} \sum_{v \in \delta_w^-(u)} \frac{|\delta_w^-(v)|}{deg_H(v) - 1} = \sum_{v \in Z_w} \frac{|\delta_w^-(v)||\delta_w^+(v)|}{deg_H(v) - 1} \geq \sum_{v \in Z_w} 1 = |Z_w| \quad (1)$$

On the other hand for any u such that $|\delta_w^-(u)| = 1$:

$$\frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} \leq 1 \quad (2)$$

Notice that the equality in (2) holds exactly when $\delta_w^-(u_w^-) = \emptyset$. For any $w \in V(G)$ there exists exactly one node $t_w \in V(H)$ verifying $\delta_w^-(t_w) = \emptyset$ which is the unique node from $V(H)$ such that $w \in W(t_w)$. Therefore, the equality in (2) holds exactly for the nodes from $\Gamma_{1,H}(t_w)$.

In this way:

$$\sum_{\{u \in V(H) \wedge |\delta_w^-(u)|=1\}} \frac{|\delta_w^+(u_w^-)| - 1}{deg_H(u_w^-) - 1} \leq |\{u \in V(H) \wedge |\delta_w^-(u)| = 1\}| \quad (3)$$

Notice that since H is bipartite, $\Gamma_{1,H}(t_w) \subseteq \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$. Therefore, the equality in (3) holds only when $\Gamma_{1,H}(t_w) = \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$, otherwise, the inequality in (3) is strict.

Now, recall that $\{u \in V(H) \wedge |\delta_w^-(u)| = 1\} \subseteq Z_w$ because H is 2-edge-connected. Therefore, combining (1) with (3):

$$\Delta_w \leq -|Z_w| + |\{u \in V(H) \mid |\delta_w^-(u)| = 1\}| \leq 0$$

As we wanted to prove.

Now, suppose that $diam(H) > 2$ and take any path $\pi = x_1 - x_2 - x_3 - \dots$ of length $diam(H)$ inside H . Then, pick $x \in W(x_1)$ any node inside $W(x_1)$. Setting $w = x$ we have that $x_1 = t_w$ and $x_3 \in Z_w$ but $x_3 \notin \Gamma_{1,H}(t_w)$. If $x_3 \notin \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$ then the inclusion $\{u \in V(H) \mid |\delta_w^-(u)| = 1\} \subseteq Z_w$ is strict and then $\Delta_w < 0$. Otherwise, $x_3 \in \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$ but $x_3 \notin \Gamma_{1,H}(t_w)$ so that the inclusion $\Gamma_{1,H}(t_w) \subseteq \{u \in V(H) \mid |\delta_w^-(u)| = 1\}$ is strict and then $\Delta_w < 0$, too. Therefore, $S = \sum_{w \in V(G)} \Delta_w < 0$ and this contradicts the fact that G is an equilibrium graph.

3 2-Edge-Connectivity in the Sum Basic Equilibria

In this section, we investigate further topological properties of any 2-edge-connected component H from any sum basic equilibrium G . These properties help us to derive the main result of this paper.

Lemma 2. *If $wv \in E(G)$ is a bridge, then $deg(w) = 1$ or $deg(v) = 1$.*

Proof. Let $u_1u_2 \in E(G)$ be a bridge between two connected components G_1, G_2 in such a way that $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$. Furthermore, assume wlog that $|V(G_1)| \leq |V(G_2)|$. If we suppose the contrary, then we can find a node $v \in V(G_1)$ such that $vu_1 \in E(G_1)$. Then, let ΔC be the cost difference associated to the deviation in v that consists in swapping the edge vu_1 for the edge vu_2 . Clearly, we are getting one unit closer to every node from $V(G_2)$ and getting one unit distance further from at most all nodes in $V(G_1)$ except for the node v itself. Therefore, using the assumption $|V(G_1)| \leq |V(G_2)|$, we deduce:

$$\Delta C \leq |V(G_1)| - 1 - |V(G_2)| \leq -1 < 0$$

Lemma 3. *If H is any 2-edge-connected component of G then there exists at most one node $u \in V(H)$ such that $W(u) \neq \{u\}$.*

Proof. Suppose the contrary and we reach a contradiction. Let u_1, u_2 be two distinct nodes such that $W(u_1) \neq \{u_1\}$ and $W(u_2) \neq \{u_2\}$. Let $v_1 \neq u_1$ and $v_2 \neq u_2$ be two nodes from $W(u_1)$ and $W(u_2)$ respectively. By Lemma 2, $W(u_1)$ and $W(u_2)$ are stars. Assume wlog that $D(u_1) \leq D(u_2)$. When swapping the link v_2u_2 for the link v_2u_1 we can reach the nodes from $V(G) \setminus \{v_1\}$ at the distances seen by v_1 and, also, we are reducing in at least one unit distance the distance from v_2 to v_1 . Therefore, if ΔC is the cost difference associated to such swap, then: $\Delta C \leq D(u_1) - D(u_2) - 1 < 0$.

Therefore, combining these two lemmas with Theorem 1, we deduce that every non-tree bipartite sum basic equilibrium is the complete bipartite $K_{r,s}$ with some star S_k (the star with a central node and k edges) attached to exactly one of the nodes from $K_{r,s}$, let it be $x_0 \in V(K_{r,s})$. Then, if we consider any path $x_0 - x_1 - x_2$ in H of length 2, x_2 has an incentive to swap the link x_2x_1 for the link x_2x_0 unless $k = 0$, that is, unless $G = K_{r,s}$.

From here we reach the conclusion of this paper:

Corollary 1. *The set of bipartite sum basic equilibria is the set of complete bipartite graphs $K_{r,s}$, with $r, s \geq 1$ and therefore the diameter of every bipartite sum basic equilibrium graph is at most 2.*

4 Conclusion

Therefore, the diameter of sum basic equilibria is at most 2 not only when we consider trees, also when we consider bipartite graphs. Furthermore, notice that the crucial results in this paper have been obtained considering a sum of the cost differences associated to a family of deviations. This is nothing more than a disguised application of the probability principle, a technique used also in [6] for the same model. These results show that maybe this technique can be further explored in order to reach new insights for this model or for related ones.

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Characterization and Enumeration of Preimages Under the Queuesort Algorithm

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Abstract. Following the footprints of what have been done with the algorithm **Stacksort**, we investigate the preimages of the map associated with a slightly less well known algorithm, called **Queuesort**. Our results include a recursive description of the set of all preimages of a given permutation, a characterization of allowed cardinalities for preimages, the exact enumeration of permutations having 0, 1 and 2 preimages, respectively, and, finally, a closed formula for the number of preimages of those permutations whose left-to-right maxima are concentrated at the beginning and at the end.

Keywords: Sorting algorithm · Permutation · Queuesort · Preimage · Catalan numbers · Ballot numbers · Derangement numbers

1 Introduction

Stacksort is a classical and well-studied algorithm that attempts to sort an input permutation by (suitably) using a stack. It has been introduced and first investigated by Knuth [9] and West [13], and it is one of the main responsible for the great success of the notion of *pattern* for permutations. Among the many research topics connected with **Stacksort**, a very interesting one concerns the characterization and enumeration of preimages of the associated map, which is usually denoted with s (so that $s(\pi)$ is the permutation which is obtained after performing **Stacksort** on π). More specifically, given a permutation π , what is $s^{-1}(\pi)$? How many permutations does it contain? These questions have been investigated first by Bousquet-Melou [2], and more recently by Defant [4] and Defant, Engen and Miller [7].

In the present paper we address the same kind of problems for a similar sorting algorithm. Suppose to replace the stack with a queue in **Stacksort**. What is obtained is a not so useful algorithm, whose associated map is the identity (and so, in particular, the only permutations that it sorts are the identity permutations). However, if we allow one more operation, namely the bypass of the queue, the resulting algorithm (which we call **Queuesort**) is much more

interesting. This is not a new algorithm, and some properties of it can be found scattered in the literature [1, 8, 10, 12]. However, to the best of our knowledge, the problem of studying preimages under the map associated with `Queuesort` (similarly to what have been done for `Stacksort`) has never been considered. Our aim is thus to begin the investigation of this kind of matters, with a particular emphasis on enumeration questions.

2 A Recursive Characterization of Preimages

Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, the algorithm `Queuesort` attempts to sort π by using a queue in the following way: scan π from left to right and, called π_i the current element,

- if the queue is empty or π_i is larger than the last element of the queue, then π_i is inserted to the back of the queue;
- otherwise, compare π_i with the first element of the queue, then output the smaller one.

When all the elements of π have been processed, pour the content of the queue into the output. In the following, we will denote with q the map associated with `Queuesort`.

Our first important tool is an effective description of the behavior of the algorithm `Queuesort` directly on the input permutation, which is based on the notion of left-to-right maximum. An element π_i of a permutation π is called a *left-to-right maximum* (briefly, *LTR maximum*) when it is larger than every element to its left (that is, $\pi_i > \pi_j$, for all $j < i$).

Let π be a permutation of length n , and denote with m_1, m_2, \dots, m_k its LTR maxima, listed from left to right. Thus, in particular, $m_k = n$. Then $q(\pi)$ is obtained from π by moving its LTR maxima to the right according to the following instructions:

- for i running from k down to 1, repeatedly swap m_i with the element on its right, until such an element is larger than m_i .

For instance, if $\pi = 21543$, then there are two LTR maxima, namely 2 and 5; according to the above instructions, π is thus modified along the following steps: $21543 \rightsquigarrow 21435 \rightsquigarrow 12435$, and so $q(21543) = 12435$.

Notice that, as an immediate consequence of the above alternative description of `Queuesort`, the set of preimages of a permutation $\pi \in S_n$ is nonempty if and only if the last element of π is n .

The key ingredient to state our recursive characterization of preimages is a suitable decomposition of permutations, based on the notion of LTR maximum. Given a permutation π , we decompose it as $\pi = M_1P_1M_2P_2 \cdots M_{k-1}P_{k-1}M_k$, where the M_i 's are all the maximal sequences of contiguous LTR maxima of π (and the P_i 's collect all the remaining elements). This decomposition will be called the *LTR-max decomposition* of π . In particular, all the P_i 's are nonempty, and M_i is nonempty for all $i \neq k$. Moreover, $m_i = |M_i|$ denotes the length of

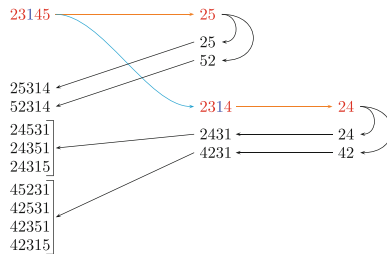
M_i , and analogously $p_i = |P_i|$ denotes the length of P_i , for all i . Sometimes we will also need to refer to the last element of M_i , which will be denoted μ_i . The sequence M_i with μ_i removed will be denoted M'_i . Also, in some cases we will use N and R in place of M and P , respectively. In order to avoid repeating the same things several times, the above notations for the LTR-max decomposition of a permutation will remain fixed throughout all the paper.

The next theorem contains the announced characterization of the set of preimages under **Queuesort** of a given permutation. We warn the reader that, with a little abuse of notation, we will assume that every finite sequence of distinct integers is a permutation, by simply rescaling its elements, that is by replacing the i -th smallest element with i , for all i .

Theorem 1. *Let $\pi = M_1P_1 \cdots M_{k-1}P_{k-1}M_k \in S_n$, with $M_k \neq \emptyset$, and suppose that π is different from the identity permutation. A permutation $\sigma \in S_n$ is a preimage of π if and only if exactly one of the following holds:*

- $\sigma = \tau\mu_{k-1}P_{k-1}M'_k$, where $\tau \in q^{-1}(M_1P_1 \cdots M_{k-2}P_{k-2}M'_{k-1}n)$ denotes a preimage of $M_1P_1 \cdots M_{k-2}P_{k-2}M'_{k-1}n$;
- if π' is defined by removing n from π and $\sigma' = N_1R_1 \cdots N_{s-1}R_{s-1}N_s$ is a preimage of π' , then σ is obtained by inserting n in one of the positions to the right of N_{s-1} .

To illustrate the above theorem, we now compute all the preimages of the permutation 23145. The various steps of the recursive procedure are depicted in the figure below:



3 Enumerative Results

Our first achievement of the present section is an important feature of the **Queuesort** algorithm: the number of preimages of a permutation depends only on the positions of its LTR maxima, and not on their values. We first need one more definition.

Given $\pi \in S_n$, denote with $LTR(\pi)$ the set of the positions of the LTR maxima of π , i.e. $LTR(\pi) = \{i \leq n \mid \pi_i \text{ is a LTR maximum of } \pi\}$.

The proof of the next theorem involves rather technical details, but the strategy is reasonably easy to describe and, in our opinion, quite interesting, so we have included a few words to illustrate it.

Theorem 2. *Let $\rho, \pi \in S_n$. If $LTR(\rho) \subseteq LTR(\pi)$, then $|q^{-1}(\rho)| \leq |q^{-1}(\pi)|$.*

Proof. (Sketch). Define a map $f : q^{-1}(\rho) \rightarrow q^{-1}(\pi)$ as follows. Given $\tau \in q^{-1}(\rho)$, suppose that, performing Queuesort on τ , the LTR maxima of τ in positions $i'_1 < \dots < i'_h$ are moved to the right to positions $i_1 < \dots < i_h$, respectively. Define $f(\tau) = \sigma$ as the permutation obtained from π by moving the elements in positions i_1, \dots, i_h to the left to positions i'_1, \dots, i'_h , respectively. First of all we need to prove that f is well defined, i.e. that indeed $\sigma \in q^{-1}(\pi)$. This can be done by showing that, performing Queuesort on σ , the only elements that are moved are precisely those in positions $i'_1 < \dots < i'_h$ and that such elements are moved precisely to positions $i_1 < \dots < i_h$. The fact that f is injective is easy.

Corollary 1. *If two permutations π and ρ have their LTR maxima in the same positions, then they have the same number of preimages.*

We now provide some results concerning permutations with a given number of preimages. It is easy to see that $\pi \in S_n$ has no preimages if and only if its last element is different from n . Therefore, setting $Q_n^{(k)} = \{\pi \in S_n \mid |q^{-1}(\pi)| = k\}$ and $q_n^{(k)} = |Q_n^{(k)}|$, we have that $Q_n^{(0)} = \{\pi \in S_n \mid \pi_n \neq 0\}$ and $q_n^{(0)} = (n-1)! \cdot (n-1)$. The next propositions deal with $Q_n^{(1)}$ and $Q_n^{(2)}$.

Proposition 1. *For all n , we have $Q_n^{(1)} = \{\pi \in S_n \mid \pi_n = n \text{ and } \pi \text{ does not have two adjacent LTR maxima}\}$. As a consequence,*

$$q_n^{(1)} = (n-1)! \cdot \sum_{i=0}^{n-1} \frac{(-1)^i}{i!},$$

that is the $(n-1)$ -th derangement number (sequence A000166 in [11]).

Proposition 2. *For all n , we have $Q_n^{(2)} = \{\pi \in S_n \mid \pi_n = n \text{ and } \pi \text{ does not have two adjacent LTR maxima except for the first two elements}\}$. As a consequence, $q_n^{(2)}$ satisfies the recurrence relation*

$$\begin{aligned} q_{n+1}^{(2)} &= (n-1)q_n^{(2)} + (n-1)q_{n-1}^{(2)}, & n \geq 3, \\ q_0^{(2)} = q_1^{(2)} = q_3^{(2)} &= 0, & q_2^{(2)} = 1. \end{aligned}$$

Sequence $q_n^{(2)}$ starts 0, 0, 1, 0, 2, 6, 32, 190... and is essentially A055596 in [11]. We thus deduce, for $n \geq 2$, the closed formula $q_n^{(2)} = (n-1)! - 2q_n^{(1)}$ (recall that $q_n^{(1)}$ equals the $(n-1)$ -th derangement number), as well as the exponential generating function

$$\sum_{n \geq 0} q_n^{(2)} \frac{x^n}{n!} = \frac{x(2-x-2e^{-x})}{1-x}.$$

We have thus seen that there exist permutations having 0, 1 or 2 preimages. We now show (Propositions 3 and 4) that there exist permutations having any number of preimages, except for 3.

Proposition 3. *Given $n \geq 2$, let $\pi = n(n-1)(n-2) \cdots 21(n+2)(n+3)(n+1)(n+4) \in S_{n+4}$. Then $|q^{-1}(\pi)| = n+2$.*

The previous proposition can be easily extended to the case $n = 0$, since it is immediate to check that $|q^{-1}(2314)| = 2$. However, it does not hold when $n = 1$, as $|q^{-1}(13425)| = 5$. The next proposition shows that this is no accident.

Proposition 4. *There exists no permutation π such that $|q^{-1}(\pi)| = 3$.*

To conclude our paper, we find an expression for the number of preimages of a generic permutation π of the form $\pi = M_1P_1M_2$. To this aim, we need to consider a statistic on 321-avoiding permutations.

For every $n, i \geq 1$, define $b_{n,i} = |\{\pi \in S_n(321) \mid \pi_i = n\}| = |\{\pi \in S_n(321) \mid \pi_n = i\}|$.

It can be shown that the $b_{n,i}$'s are essentially the well-known sequence of ballot numbers. To be more precise, $b_{n,i}$ equals the term of indices $(n-1, i-1)$ of sequence A009766 in [11].

Theorem 3. *Let $\pi = M_1P_1M_2 \in S_n$, with $M_2 \neq \emptyset$. Then*

$$|q^{-1}(\pi)| = \sum_{i=1}^{m_2} \sum_{j=0}^{i-1} \binom{i-1}{j} b_{m_1+j+1, m_1} \cdot b_{m_2+p_1-j, m_2-i+1}. \tag{1}$$

Another way to express formula (1) comes from expanding the ballot numbers of the previous corollary in terms of Catalan numbers.

Corollary 2. *For $\pi = M_1P_1M_2 \in S_n$, the quantity $|q^{-1}(\pi)|$ can be expressed as a linear combination of Catalan numbers. More precisely, for any fixed $m_2 = |M_2|$, we have that $|q^{-1}(\pi)|$ is a linear combination of the Catalan numbers $C_{m_1}, C_{m_1+1}, \dots, C_{m_1+m_2-1}$ with polynomial coefficients in p_1 , i.e.:*

$$|q^{-1}(\pi)| = \sum_{t=0}^{m_2-1} \omega_{m_2,t}(p_1)C_{m_1+t},$$

where $\omega_{m_2,t}(p_1)$ is a polynomial in p_1 of degree $m_2 - t - 1$, for all t .

Effective enumerative results can be obtained for small values of the parameter m_2 in formula (1). In particular, when $m_2 = 1, 2, 3$, we are able to get simple closed formulas.

Corollary 3. *For $\pi = M_1P_1M_2 \in S_n$, we get:*

- $|q^{-1}(\pi)| = C_{m_1}$, when $|M_2| = 1$;
- $|q^{-1}(\pi)| = C_{m_1+1} + (p_1 + 1)C_{m_1}$, when $|M_2| = 2$;
- $|q^{-1}(\pi)| = C_{m_1+2} + (p_1 + 1)C_{m_1+1} + \frac{1}{2}(p_1 + 1)(p_1 + 4)C_{m_1}$, when $|M_2| = 3$.

The above corollary, together with some further calculations, seems to suggest that $\omega_{m_2,t}(p_1) = \omega_{m_2+1,t+1}(p_1)$, for all m_2, t . This could clearly simplify the computations needed to determine $|q^{-1}(\pi)|$ when m_2 increases.

4 Conclusions and Further Work

In the spirit of previous work on `Stacksort`, we have investigated the preimages of the map associated with the algorithm `Queuesort`, obtaining a recursive description of all preimages of a given permutation and some enumerative results concerning the number of preimages. Our approach seems to suggest that, in some sense, the structure of the map associated with `Queuesort` is a little bit easier than that of the map associated with `Stacksort`, which allows us to obtain nicer results. For instance, we have been able to find a neat result concerning the possible cardinalities for the set of preimages of a given permutation; the same thing turns out to be much more troublesome for `Stacksort` [5].

Our paper can be seen as a first step towards a better understanding of the algorithm `Queuesort`, which appears to be much less studied than its more noble relative `Stacksort`. Since the structure of `Queuesort` appears to be slightly simpler, it is conceivable that one can achieve better and more explicit results. In this sense, there are many (classical and nonclassical) problems concerning `Stacksort` which could be fruitfully addressed also for `Queuesort`.

For instance, it could be very interesting to investigate properties of the iterates of the map associated with `Queuesort`. For any natural number n , define the rooted tree whose nodes are the permutations of length n , having id_n as its root and such that, given two distinct permutations $\tau, \sigma \in S_n$, σ is a son of τ whenever $q(\sigma) = \tau$. Studying properties of this tree could give some insight on the behavior of iterates of q . For instance, what can be said about the average depth of such a tree? Can we find enumerative results concerning some interesting statistic in the set of permutations having a fixed depth?

Another interesting issue could be the investigation of properties of the sets $Q_n^{(k)}$. For instance, how many permutations in $Q_n^{(k)}$ avoid a given pattern π ?

Moreover, following [3], we could consider devices consisting of two queues (both with bypass) in series, where the content of the first queue is constraint to avoid some pattern. What can we say about permutations that are sortable by such devices?

Finally, Defant discovered some surprising connections between `Stacksort` and free probability theory [6]. Can we find anything similar for `Queuesort`?

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On Off-Diagonal Ordered Ramsey Numbers of Nested Matchings

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Abstract. For two ordered graphs $G^<$ and $H^<$, the *ordered Ramsey number* $r_<(G^<, H^<)$ is the minimum N such that every red-blue coloring of the edges of the ordered complete graph $K_N^<$ contains a red copy of $G^<$ or a blue copy of $H^<$.

For $n \in \mathbb{N}$, a *nested matching* $NM_n^<$ is the ordered graph on $2n$ vertices with edges $\{i, 2n-i+1\}$ for every $i = 1, \dots, n$. We improve bounds on the numbers $r_<(NM_n^<, K_3^<)$ obtained by Rohatgi, we disprove his conjecture about these numbers, and we determine them exactly for $n = 4, 5$. This gives a stronger lower bound on the maximum chromatic number of k -queue graphs for every $k \geq 3$.

We expand the classical notion of Ramsey goodness to the ordered case and we attempt to characterize all connected ordered graphs that are n -good for every $n \in \mathbb{N}$. In particular, we discover a new class of such ordered trees, extending all previously known examples.

Keywords: Ordered Ramsey number · Ramsey goodness · Nested matching

1 Introduction

Ramsey theory is devoted to the study of the minimum size of a system that guarantees the existence of a highly organized subsystem. Given graphs G and H , their *Ramsey number* $r(G, H)$ is the smallest $N \in \mathbb{N}$ such that any two-coloring of the edges of K_N contains either G as a red subgraph or H as a blue subgraph of K_N . The case $G = H$ is called the *diagonal* case and in this case we use the abbreviation $r(G) = r(G, G)$.

The growth rate of Ramsey numbers has been of interest to many researchers. In general, it is notoriously difficult to find tight estimates on Ramsey numbers. For example, despite many efforts, the best known bounds on $r(K_n)$ are essentially

$$2^{n/2} \leq r(K_n) \leq 2^{2n} \tag{1}$$

obtained by Erdős and Szekeres [10], although some smaller term improvements are known. For a more comprehensive survey we can refer the reader to [8].

In this paper, we study ordered graphs. An *ordered graph* $G^<$ on n vertices is a graph whose vertex set is $[n] := \{1, \dots, n\}$ and it is ordered by the standard ordering $<$ of integers. For an ordered graph $G^<$, we use G to denote its unordered counterpart. An ordered graph $G^<$ on $[n]$ is an *ordered subgraph* of another ordered graph $H^<$ on $[N]$ if there exists a mapping $\phi : [n] \rightarrow [N]$ such that $\phi(i) < \phi(j)$ for $1 \leq i < j \leq n$ and also $\{\phi(i), \phi(j)\}$ is an edge of $H^<$ whenever $\{i, j\}$ is an edge of $G^<$. Definitions that are often stated for unordered graphs, such as vertex degrees, colorings, and so on, have their natural analogues for ordered graphs. Note that, for every $n \in \mathbb{N}$, there is a unique complete ordered graph $K_n^<$.

Motivated by connections to classical results such as the Erdős–Szekeres theorem on monotone subsequences [10], various researchers [2, 7] recently initiated the study of Ramsey numbers of ordered graphs. Given two ordered graphs $G^<$ and $H^<$, the *ordered Ramsey number* $r_<(G^<, H^<)$ is defined as the smallest N such that any two-coloring of the edges of $K_N^<$ contains either $G^<$ as a red ordered subgraph or $H^<$ as a blue ordered subgraph.

Observe that for any two ordered graphs $G_1^<$ and $G_2^<$ on n_1 and n_2 vertices, respectively, we have $r_<(G_1, G_2) \leq r_<(G_1^<, G_2^<) \leq r(K_{n_1}, K_{n_2})$. Thus, by (1), the number $r_<(G_1^<, G_2^<)$ is finite and, in particular, $r_<(G^<)$ is at most exponential in the number of vertices for every ordered graph $G^<$.

It is known that for dense graphs, there is not a huge difference in the growth rate of their ordered and unordered Ramsey numbers [2, 7]. On the other hand, ordered Ramsey numbers of sparse ordered graphs behave very differently from their unordered counterparts. For example, Ramsey numbers of *matchings* (that is, graphs with maximum degree 1) are clearly linear in the number of their vertices. However, it was proved independently in [2, 7] that there exist ordered matchings such that their diagonal ordered Ramsey numbers grow superpolynomially.

Theorem 1. [2, 7]. *There are arbitrarily large ordered matchings $M^<$ on n vertices that satisfy $r_<(M^<) \geq n^{\Omega(\frac{\log n}{\log \log n})}$.*

The bound from Theorem 1 is quite close to the truth as Conlon, Fox, Lee and Sudakov [7] proved that $r_<(G^<, K_n^<) \leq 2^{O(d \log^2(2n/d))}$ for every ordered graph $G^<$ on n vertices with degeneracy d . In particular, we have $r_<(G^<) \leq r_<(G^<, K_n^<) \leq n^{O(\log n)}$ if $G^<$ has its maximum degree bounded by a constant.

There has also been a keen interest in studying the off-diagonal ordered Ramsey numbers. Conlon, Fox, Lee and Sudakov [7] proved that there exist ordered matchings $M^<$ such that $r_<(M^<, K_3^<) = \Omega\left((n/\log n)^{4/3}\right)$. On the other hand, the best known upper bound on $r_<(M^<, K_3^<)$ is

$$r_<(M^<, K_3^<) \leq r_<(K_n^<, K_3^<) = r(K_n, K_3) = O\left(\frac{n^2}{\log n}\right),$$

which follows from the well-known result $r(K_n, K_3) = O\left(\frac{n^2}{\log n}\right)$ [1], which is tight [12]. Note that the first inequality only uses the fact that $M^<$ is an ordered

subgraph of $K_n^<$ and does not utilize any special properties of ordered matchings such as its sparseness. Conlon, Fox, Lee and Sudakov [7] expect that the upper bound is far from optimal and posed the following problem.

Problem 1. [7]. Does there exist an $\varepsilon > 0$ such that any ordered matching $M^<$ on $n \in \mathbb{N}$ vertices satisfies $r_<(M^<, K_3^<) = O(n^{2-\varepsilon})$?

Problem 1 remains open, but there was a recent progress obtained by Rohatgi [15], who resolved some special cases of this problem. In particular, he proved that if the edges of an ordered matching $M^<$ do not cross, then the ordered Ramsey number $r_<(M^<, K_3^<)$ is almost linear. The basic building block of the proof of this result is formed by so-called nested matchings. For $n \in \mathbb{N}$, a *nested matching* (or a *rainbow*) $NM_n^<$ is the ordered matching on $2n$ vertices with edges $\{i, 2n - i + 1\}$ for every $i \in [n]$. Rohatgi [15] determined the off-diagonal ordered Ramsey numbers of nested matchings up to a constant factor.

Proposition 1. [15]. *For every $n \in \mathbb{N}$, we have $4n - 1 \leq r_<(NM_n^<, K_3^<) \leq 6n$.*

He believed that the upper bound is far from optimal and posed the following conjecture, which he verified for $n \in \{1, 2, 3\}$.

Conjecture 1. [15]. For every $n \in \mathbb{N}$, we have $r_<(NM_n^<, K_3^<) = 4n - 1$.

The ordered graphs that do not contain $NM_m^<$ as an ordered subgraph for some $m \in \mathbb{N}$ are known to be equivalent to so-called $(m - 1)$ -queue graphs [11] and, in particular, 1-queue graphs correspond to *arched-leveled-planar graphs* [11]. As we will see, estimating the ordered Ramsey numbers $r_<(NM_m^<, K_n^<)$ is connected to extremal questions about $(m - 1)$ -queue graphs. In particular, there is a close connection to the problem of Dujmović and Wood [9] about determining the chromatic number of such graphs.

Problem 2. [9]. What is the maximum chromatic number χ_k of a k -queue graph?

Dujmović and Wood [9] note that $\chi_k \in \{2k + 1, \dots, 4k\}$ and they prove that the lower bound is attainable for $k = 1$.

2 Our Results

In this paper, we also focus on off-diagonal ordered Ramsey numbers. In particular, we improve and generalize the bounds on $r_<(NM_n^<, K_3^<)$ and we disprove Conjecture 1. We also consider ordered Ramsey numbers $r_<(G^<, K_n)$ for general connected ordered graphs $G^<$ and we introduce the concept of Ramsey goodness for ordered graphs.

2.1 Nested Matchings Versus Complete Graphs

First, we improve the leading constant in the upper bound from Proposition 1 and thus show that the bound by Rohatgi is indeed not tight. However, we believe that our estimate can be improved as well.

Theorem 2. *For every $n \in \mathbb{N}$, we have $r_{<}(NM_n^<, K_3^<) \leq (3 + \sqrt{5})n < 5.3n$.*

Next, we disprove Conjecture 1 by showing $r_{<}(NM_n^<, K_3^<) > 4n - 1$ for every $n \geq 4$. For $n \in \{4, 5\}$, we determine $r_{<}(NM_n^<, K_3^<)$ exactly.

Theorem 3. *For every $n \geq 6$, we have $r_{<}(NM_n^<, K_3^<) \geq 4n + 1$. Moreover, $r_{<}(NM_4^<, K_3^<) = 16$ and $r_{<}(NM_5^<, K_3^<) = 20$.*

We prove the lower bound $r_{<}(NM_n^<, K_3^<) \geq 4n + 1$ by constructing a specific red-blue coloring of the edges of $K_{4n}^<$ that avoids a red copy of $NM_n^<$ and a blue copy of $K_3^<$. To determine $r_{<}(NM_4^<, K_3^<)$ and $r_{<}(NM_5^<, K_3^<)$ exactly, we use a computer-assisted proof based on SAT solvers. For more details about the use of SAT solvers for finding avoiding colorings computationally, we refer the reader to the bachelor’s thesis of the second author [14]. The utility we developed for computing ordered Ramsey numbers $r_{<}(G^<, H^<)$ for small ordered graphs $G^<$ and $H^<$ is publicly available [13].

By performing the exhaustive computer search, we know that there are only 326 red-blue colorings of the edges of $K_{15}^<$ without a red copy of $NM_4^<$ and a blue copy of $K_3^<$. They all share the same structure except for 6 red edges that can be switched to blue while not introducing a blue triangle. Using the same computer search, we were able to find many red-blue colorings of the edges of $K_{19}^<$ without a red copy of $NM_5^<$ and a blue copy of $K_3^<$, some of which even had certain symmetry properties. There were no such symmetric colorings on 15 vertices, which suggests that the lower bound on $r_{<}(NM_n^<, K_3^<)$ might be further improved for larger values of n .

Using the lower bounds from Theorem 3, we can address Problem 2 about the maximum chromatic number χ_k of k -queue graphs. In particular, we can improve the lower bound $\chi_k \geq 2k + 1$ by 1 for any $k \geq 3$.

Corollary 1. *For every $k \geq 3$, the maximum chromatic number of k -queue graphs is at least $2k + 2$.*

We recall that the maximum chromatic number χ_1 of 1-queue graphs is 3 [9]. We use this result to prove the exact formula for the off-diagonal ordered Ramsey numbers $r_{<}(NM_2^<, K_n^<)$ of nested matchings with two edges.

Theorem 4. *For every $n \in \mathbb{N}$, we have $r_{<}(NM_2^<, K_n^<) = 3n - 2$.*

For general nested matchings versus arbitrarily large complete graphs, we can determine the asymptotic growth rate of their ordered Ramsey numbers, generalizing the linear bounds from Proposition 1 and Theorem 2.

Theorem 5. *For every $m, n \in \mathbb{N}$, we have $r_{<}(NM_m^<, K_{n+1}^<) = \Theta(mn)$.*

2.2 Ramsey Goodness for Ordered Graphs

To obtain the lower bound $r(G, K_n) \geq (m-1)(n-1)+1$ for a connected graph G on m vertices, one might consider a simple construction that is usually attributed to Chvátal and Harary [6]. Take $n-1$ red cliques, each with $m-1$ vertices, and connect vertices in different red cliques by blue edges. For some graphs G , this lower bound is the best possible and such graphs are called (*Ramsey*) n -good. That is, a connected graph G on m vertices is n -good if $r(G, K_n) = (m-1)(n-1)+1$. We call a graph *good* if it is n -good for all $n \in \mathbb{N}$. A famous result by Chvátal [5] states that all trees are good.

Studying n -good graphs is a well-established area in extremal combinatorics. Despite this, to the best of our knowledge, Ramsey goodness has not been considered for ordered graphs. Motivated by our results from Subsection 2.1, we thus extend the definition of good graphs to ordered graphs and we attempt to characterize all good connected ordered graphs. A connected ordered graph $G^<$ on m vertices is n -good if $r_<(G^<, K_n^<) = (m-1)(n-1)+1$. A connected ordered graph is *good* if it is n -good for all $n \in \mathbb{N}$.

A generalization of the well-known Erdős–Szekeres theorem on monotone subsequences states that $r_<(P_m^<, K_n^<) = (m-1)(n-1)+1$ for every $n \in \mathbb{N}$ and every *monotone path* $P_m^<$ [4], which is an ordered path on m vertices where edges connect consecutive vertices in $<$. In other words, any monotone path is good, which gives a first example of good ordered graphs. Note, however, that not all ordered paths are good, which follows immediately from Theorem 1.

First, we state some basic properties of good ordered graphs, some of them resembling their unordered counterparts. It can be shown as in the unordered case that if a connected ordered graph $G^<$ is $(n+1)$ -good, then it is n -good.

Let $G^<$ be an ordered graph containing an ordered cycle $C^<$ as an ordered subgraph. It is known that, for every cycle C_l on $l \geq 3$ vertices and for n going to infinity, the Ramsey number $r(C, K_n)$ grows superlinearly with n [3, 12]. Since $r_<(G^<, K_n^<) \geq r_<(C^<, K_n^<) \geq r(C, K_n)$, the number $r_<(G^<, K_n^<)$ is also superlinear in n and thus the ordered graph $G^<$ cannot be good. We thus obtain the following result that limits good ordered graphs to ordered trees.

Proposition 2. *Every good ordered graph is an ordered tree.*

In our attempt to characterize good ordered trees, we discovered a class of good ordered trees, which significantly extends the example with monotone paths. In order to describe this new class, we need to introduce some notation.

An *ordered star graph* $S_{l,r}^<$ is an ordered graph on $r+l-1$ vertices such that the l th vertex in the vertex ordering is adjacent to all other vertices and there are no other edges. We call an ordered star $S_{l,r}^<$ *one-sided* if $l=1$ or $r=1$.

For any two ordered graphs $G^<$ and $H^<$ on m and n vertices, respectively, the *join* $G^< + H^<$ is an ordered graph on $m+n-1$ vertices constructed by identifying the leftmost vertex of $H^<$ with the rightmost vertex of $G^<$. The join operation is associative and if $G^<$ and $H^<$ are both connected, then $G^< + H^<$ is connected as well. The following result gives a construction of good ordered graphs based on the join operation.

Theorem 6. *For all $n, r, l \in \mathbb{N}$, if a connected ordered graph $G^<$ is n -good, then the ordered graphs $G^< + S_{1,r}^<$, $G^< + S_{l,1}^<$, $S_{l,1}^< + G^<$, and $S_{1,r}^< + G^<$ are also n -good.*

Theorem 6 immediately implies that every ordered star graph is good. More generally, it follows that all ordered trees from the following class are good. An ordered graph $G^<$ is a *monotone caterpillar graph* if there exist positive integers $n, l_1, \dots, l_n, r_1, \dots, r_n$ such that $l_i = 1$ or $r_i = 1$ for each $i \in [n]$ and $G^< = S_{l_1, r_1}^< + \dots + S_{l_n, r_n}^<$. In other words, if $G^<$ can be obtained by performing joins on one-sided ordered star graphs. Note that monotone paths and ordered stars are all monotone caterpillar graphs.

Corollary 2. *All monotone caterpillar graphs are good.*

Computer experiments based on our SAT solver based utility [13] showed that all good ordered graphs up to 6 vertices are monotone caterpillar graphs. We believe that there are no other good ordered graphs.

To get a better understanding of good ordered graphs, we prove an alternative characterization of monotone caterpillar graphs stated in terms of the following four forbidden ordered subgraphs: $NM_2^<$, $K_3^<$, and two ordered graphs on the vertex set $\{1, 2, 3, 4\}$ with the edge sets $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 2\}, \{1, 4\}, \{3, 4\}\}$. We note that if we assume that $G^<$ is an ordered tree, then we can leave out $K_3^<$ and the characterization still holds.

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Bounds on Half Graph Orders in Powers of Sparse Graphs

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Abstract. Half graphs and their variants, such as ladders, semi-ladders and co-matchings, are combinatorial objects that encode total orders in graphs. Works by Adler and Adler (Eur. J. Comb.; 2014) and Fabiański et al. (STACS; 2019) prove that in the powers of sparse graphs, one cannot find arbitrarily large objects of this kind. We provide nearly tight asymptotic lower and upper bounds on the maximum order of half graphs, parameterized on the power, in the following classes of sparse graphs: planar graphs, graphs with bounded maximum degree, graphs with bounded pathwidth or treewidth, and graphs excluding a fixed clique as a minor. As an essential part of this work, we prove a fully polynomial bound on the neighborhood complexity in planar graphs.

1 Introduction

It is widely known that there is a huge array of algorithmic problems deemed to be computationally hard. One of the ways of circumventing this issue is limiting the set of possible instances of a problem by assuming a more manageable structure. For example, restricting our attention to graph problems, we can exploit the planarity of the graph instances through techniques such as the planar separator theorem [3]; analogously, for the graphs with bounded treewidth, we can solve multiple hard problems by means of dynamic programming on tree decompositions. These examples present some of the algorithmic techniques which allow us to utilize the structural sparsity of graph instances.

Nešetřil and Ossona de Mendez have proposed two abstract notions of sparsity in graphs: bounded expansion [4] and nowhere denseness [5]. Intuitively, a class \mathcal{C} of graphs has bounded expansion if one cannot obtain arbitrarily dense graphs by picking a graph $G \in \mathcal{C}$ and contracting pairwise disjoint connected subgraphs of G with fixed radius to single vertices. More generally, \mathcal{C} is nowhere dense if one cannot produce arbitrarily large cliques as graphs as a result of



The full version of this work is available on arXiv [9].

This work is a part of project TOTAL that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 677651).

the same process. For formal definitions of these notions, and for a comprehensive introduction to these classes of graphs, we refer to the book by Nešetřil and Ossona de Mendez [6] and to the lecture notes from the University of Warsaw [7].

Research in Sparsity provides a plethora of technical tools for structural analysis of sparse graphs, many of which are in the form of various graph parameters. In this work, we consider one kind of structural measures that behave nicely in sparse graphs, which are related to the concepts of half graphs, ladders, semi-ladders, and co-matchings.

Definition 1. In an undirected graph $G = (V, E)$, for an integer $\ell \geq 1$, 2ℓ different vertices $a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_\ell$ form:

- a **half graph** (or a **ladder**) of order ℓ if for each pair of indices i, j such that $i, j \in [1, \ell]$, we have $(b_i, a_j) \in E$ if and only if $i < j$ (Fig. 1(a));
- a **semi-ladder** of order ℓ if we have $(b_i, a_j) \in E$ for each pair of indices i, j such that $1 \leq i < j \leq \ell$, and $(b_i, a_i) \notin E$ for each $i \in [1, \ell]$ (Fig. 1(b));
- a **co-matching** of order ℓ if for each pair of indices i, j such that $i, j \in [1, \ell]$, we have $(b_i, a_j) \in E$ if and only if $i \neq j$ (Fig. 1(c)).

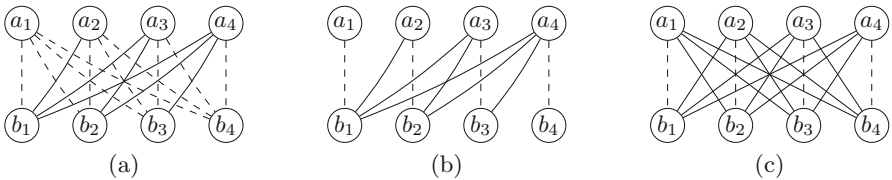


Fig. 1. The objects in Definition 1. Solid lines indicate pairs of vertices connected by an edge, and dashed edges indicate pairs of vertices not connected by an edge. Note that there are no restrictions on the existence of edges between the pairs of vertices on the same side of the objects (i.e., between a_i and a_j , or between b_i and b_j).

Naturally, each half graph, ladder, and co-matching is also a semi-ladder. Hence, if for a class \mathcal{C} of graphs, the orders of semi-ladders occurring in any graph in \mathcal{C} are bounded from above by some constant M , then M is also the corresponding upper bound for orders of half graphs, ladders and co-matchings. Conversely, uniform upper bounds on the orders of both half graphs and co-matchings in \mathcal{C} imply a uniform upper bound on the orders of semi-ladders [2].

In this work, we consider \mathcal{C} to be powers of nowhere dense classes of graphs. Formally, for an undirected graph G and an integer d , we define the graph G^d as an undirected graph with the same set of vertices as G , but in which two vertices u, v are adjacent if and only if $\text{dist}_G(u, v) \leq d$. Then, the d -th power of a class \mathcal{C} is defined as $\mathcal{C}^d = \{G^d \mid G \in \mathcal{C}\}$.

Even though \mathcal{C}^d may potentially contain dense graphs, it turns out that the objects from Definition 1 still behave nicely in \mathcal{C}^d :

Theorem 1 ([1, 2]). For every nowhere dense class \mathcal{C} of graphs, there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $d \in \mathbb{N}$, the orders of half graphs and semi-ladders in graphs from \mathcal{C}^d are uniformly bounded from above by $f(d)$.

It turns out that Theorem 1 has algorithmic implications. For instance, the DISTANCE- d DOMINATING SET problem admits a parameterized algorithm whose time complexity depends hugely on the asymptotic growth of the bound f [2]. However, the proofs of Theorem 1 are either non-constructive, or rely on the description of nowhere denseness through uniform quasi-wideness, yielding suboptimal bounds on the maximum order of half graphs. It is natural to pose a question: can we find good estimates on the maximum orders of half graphs occurring in the powers of well-studied classes of sparse graphs?

2 Our Results

We state and prove asymptotic lower and upper bounds on the orders of half graphs in the d -th powers of the following classes of sparse graphs: graphs with maximum degree bounded by Δ , planar graphs, graphs with pathwidth bounded by p , graphs with treewidth bounded by t , and graphs excluding the complete graph K_t as a minor (Fig. 2). Our results are asymptotically almost tight—they allow us to understand how quickly the maximum orders of semi-ladders grow, as functions of d , in these classes of graphs.

Class of graphs	Lower bound	Upper bound
DEGREE $\leq \Delta$	$\Delta^{\Omega(d)}$	$\Delta^d + 1$
PLANAR	$2^{\lceil \frac{d}{2} \rceil}$	$d^{O(d)}$
PATHWIDTH $\leq p$	$d^{p-O(1)}$	$(dp)^{O(p)}$
TREewidth $\leq t$	$2d^{\Omega(t)}$	$d^{O(d^{t+1})}$
K_t -MINOR-FREE	$2d^{\Omega(t)}$	$d^{O(d^{t-1})}$

Fig. 2. Bounds on the maximum orders of half graphs proved in this work.

The most important part of our work is the upper bound on the orders of half graphs in powers of planar graphs (Theorem 2). Here, we employ techniques of structural graph theory to analyze semi-ladders in planar graphs through the notion of cages, which expose a topological structure in semi-ladders. Moreover, as a vital part of the proof, we derive a fully polynomial bound on the neighborhood complexity of planar graphs (Theorem 3).

2.1 Upper Bounds on Half Graphs in Planar Graphs

We begin by stating the $d^{O(d)}$ upper bound on half graphs in the d -th powers of planar graphs. The result is actually slightly stronger as it provides an upper bound on the order of semi-ladders (cf. Definition 1). As each half graph is a semi-ladder, Theorem 2 also yields an upper bound on the order of the half graphs.

Theorem 2. *There exists a polynomial p of degree 22 such that every semi-ladder in the d -th power of any planar graph has order at most $d \cdot p(d)^d$.*

In the proof, we start with a huge semi-ladder in the d -th power of a chosen planar graph G . Using this semi-ladder, we find in G objects with more and more structure, which we call: quasi-cages, cages, ordered cages, identity ordered cages, neighbor cages, and separating cages, in this order. Each extraction step requires us to forfeit a fraction of the object, but we give good estimates on the maximum size loss incurred in the process. After all extractions, we produce a separating cage of large order; however, it can be proved that no large separating cages can exist in G . Retracing all the steps, we can easily figure out the upper bound on the maximum order of a distance- d semi-ladder in the class of planar graphs, which is of the form $d \cdot p(d)^d$ for some polynomial p .

2.2 Fully Polynomial Upper Bound on Neighborhood Complexity

As a crucial part of the proof of Theorem 2, we prove the following fact, implying a fully polynomial bound on the neighborhood complexity of planar graphs.

Theorem 3. *For a planar graph G , a vertex v , integer d , and a set of vertices A , we define the distance- d profile of v on A : $\pi_d[v, A] : A \rightarrow \{0, 1, \dots, d, +\infty\}$ as the function mapping the vertices of A into their distance to v , or $+\infty$ if this distance exceeds d . Then, the set $\{\pi_d[v, A] \mid v \in V(G)\}$ of all distinct distance- d profiles on A has at most $128|A|^3(d+2)^7$ elements.*

Sketch of the Proof. Fix G . We begin by stating the restricted variant of the theorem where the set A is replaced by a *noose*: a closed curve in the plane passing through vertices of G that does not intersect itself or the interiors of any edges.

Lemma 1. (Noose Profile Lemma, a variant). *If A is a non-empty set of vertices lying on some noose \mathcal{L} in G , then the set $\{\pi_d[v, A] \mid v \in V(G)\}$ contains at most $2|A|^3(d+2)^4$ elements.*

The proof of Lemma 1 is quite technical, so we omit it in this abstract.

We will now lift the lemma to Theorem 3. We apply an induction on the size of A , with $|A| = 1$ being trivial. If A can be partitioned into two non-empty subsets X and Y so that every two vertices from the different subsets are at distance at least $2d + 1$ from each other, then each distance- d profile on A is identically equal to $+\infty$ either on X or on Y . By the induction hypothesis applied to sets X and Y , the statement of the theorem follows.

Otherwise, let \mathcal{T} be the Steiner tree of A in G , i.e., the tree of the smallest possible size which is a subgraph of G and which contains all vertices of A . It can be argued that \mathcal{T} contains at most $(|A| - 1)(2d + 1)$ edges. We transform the graph G into G' by “cutting the plane open” along \mathcal{T} (Fig. 3). In G' , the tree \mathcal{T} becomes a noose \mathcal{L} containing at most $2(|A| - 1)(2d + 1)$ vertices. Lemma 1 now

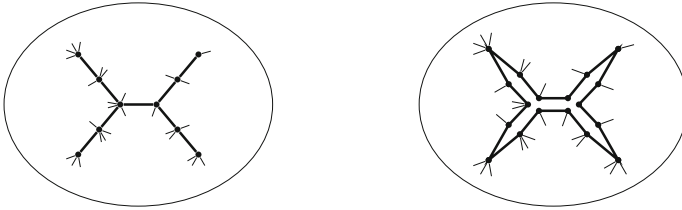


Fig. 3. Cutting the plane open along the Steiner tree \mathcal{T} (bold).

applies, yielding the upper bound of at most $128|A|^3(d + 2)^7$ different distance- d profiles on the vertices of \mathcal{L} in G' . Since each distance- d profile on \mathcal{L} in G' uniquely implies the distance- d profile on A in G , our theorem follows.

We remark that it has been known that the neighborhood complexity of planar graphs (and, more generally, of every class of graphs with bounded expansion) is bounded by a function linear in $|A|$, but exponential in d [8]. However, the polynomial dependence on d in Theorem 3 is crucial in the proof of Theorem 2. On a side note, it is an open problem whether there exists a bound on the neighborhood complexity of planar graphs that is both polynomial in d (as in Theorem 3) and linear in $|A|$ (as in the work of Reidl et al. [8]).

2.3 Upper Bounds on Half Graphs in Other Classes of Graphs

We move on to the upper bounds for the remaining considered sparse graph classes, which are proved in the full version of the paper. Again, each of the results below actually provides an upper bound on the orders of semi-ladders.

Theorem 4. *For each $\Delta \geq 2$, $d \geq 1$, every semi-ladder in the d -th power of a graph with maximum degree bounded by Δ has order at most $\Delta^d + 1$.*

Theorem 5. *For $p \geq 1$, $d \geq 1$, every semi-ladder in the d -th power of a graph with pathwidth at most p has order at most $(2d + 3)(p + 1)![(2d + 3)(d + 2)]^{p+1}$.*

Theorem 6. *For some polynomial p and for all $t \geq 4$, $d \geq 2$, every semi-ladder in the d -th power of a K_t -minor-free graph has order at most $d^{p(t) \cdot (2d+1)^{t-1}}$.*

In the proof of Theorem 6, we utilize the upper bound on the weak coloring numbers in K_t -minor-free graphs proved by van den Heuvel et al. [10] and the explicit upper bound on the semi-ladder orders in nowhere dense classes of graphs (Theorem 1) using the notion of uniform quasi-wideness.

Since graphs with treewidth at most t are K_{t+2} -minor-free, Theorem 6 yields a similar upper bound for the class of graphs with bounded treewidth.

2.4 Lower Bounds on Half Graphs

The lower bounds presented in Fig. 2 are achieved by three constructions. Each of them constructs a family of graphs whose d -th powers contain large half graphs.

Theorem 7. *For every odd $d \geq 1$ and even $\Delta \geq 4$, there exists a graph with maximum degree bounded by Δ whose d -th power contains a half graph of order $(\frac{\Delta}{2})^{\lceil d/2 \rceil}$. For $\Delta = 4$, this graph is planar.*

Theorem 8. *For each $d \geq 1$ and $p \geq 0$, there exists a graph $P_{p,d}$ with pathwidth at most $p + 2$ whose $(4d - 1)$ -st power contains a half graph of order $(2d + 1)^p$.*

Theorem 9. *For every even $d \geq 2$ and odd $t \geq 3$, there exists a graph $T_{t,d}$ with treewidth at most t whose d -th power contains a half graph of order $2^{\ell_{t,d}}$ where $\ell_{t,d} = \binom{\frac{1}{2}(d+t-5)}{\frac{1}{2}(t-3)}$.*

In the proofs of Theorems 8 and 9, recursive constructions of the postulated graphs are presented. In Theorem 8, we construct $P_{p,d}$ by creating $2d + 1$ copies of $P_{p-1,d}$, and linking them together so that the half graphs within each copy of $P_{p-1,d}$ are merged into a large half graph within $P_{p,d}$. The construction in Theorem 9 is a bit more involved: $T_{t,d}$ is created from $T_{t,d-2}$ and multiple copies of $T_{t-2,d}$. If done correctly, this yields a graph $T_{t,d}$ containing a half graph of order $2^{\ell_{d,t}} = 2^{\ell_{t,d-2}} \cdot 2^{\ell_{t-2,d}}$.

Again, as graphs with treewidth at most t are K_{t+2} -minor-free, Theorem 9 yields an analogous lower bound for the class of K_t -minor-free graphs.

Acknowledgements. We would like to thank: Michał Pilipczuk for pointing us to the problem and the proofreading of the full version of this work; Piotr Micek for reviewing the full version of the work; and an anonymous reviewer for their suggestions and comments.

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The Homotopy Type of the Independence Complex of Graphs with No Induced Cycles of Length Divisible by 3

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Abstract. We prove Engström's conjecture that the independence complex of graphs with no induced cycle of length divisible by 3 is either contractible or homotopy equivalent to a sphere. Our result strengthens a result by Zhang and Wu, verifying a conjecture of Kalai and Meshulam which states that the total Betti number of the independence complex of such a graph is at most 1. A weaker conjecture was proved earlier by Chudnovsky, Scott, Seymour, and Spirkl, who showed that in such a graph, the number of independent sets of even size minus the number of independent sets of odd size has values 0, 1, or -1 .

Keywords: Independence complexes · Homotopy type · Ternary graphs

1 Introduction

We assume all graphs are finite and contain no loops and no multiple edges. A subgraph of a graph G is an *induced subgraph* if it can be obtained from G by deleting vertices and all edges incident with those vertices. An *induced cycle* is an induced subgraph that is a cycle. An *independent set* is a set of pairwise non-adjacent vertices. The *independence complex* of a graph G is the abstract simplicial complex $I(G)$ on the vertex set $V(G)$ whose faces are the independent sets of G . A graph is *ternary* if it contains no induced cycle of length divisible by 3.

Here is our main theorem.

Theorem 1. *A graph is ternary if and only if every induced subgraph has the independence complex that is contractible or homotopy equivalent to a sphere.*

We can easily deduce the converse of Theorem 1 as follows. Kozlov [7, Proposition 5.2] showed that the independence complex of a cycle is not homotopy equivalent to a sphere if and only if the cycle has length divisible by 3. More precisely, if C_ℓ

J. Kim—This work was supported by the Institute for Basic Science (IBS-R029-C1).

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 255–261, 2021.
https://doi.org/10.1007/978-3-030-83823-2_40

is a cycle of length $\ell \geq 3$, then the homotopy type of the independence complex is given by

$$I(C_\ell) \simeq \begin{cases} S^k \vee S^k & \text{if } \ell = 3k + 3, \\ S^k & \text{if } \ell = 3k + 2 \text{ or } 3k + 4. \end{cases} \tag{1}$$

Therefore, if every induced subgraph of a graph G has the independence complex that is contractible or homotopy equivalent to a sphere, then G does not contain an induced cycle of length divisible by 3.

Our main result is motivated from a conjecture by Kalai and Meshulam. For a simplicial complex K , let $\tilde{H}_i(K)$ be the i -th reduced homology group of K over \mathbb{Z} and $\tilde{\beta}_i(K)$ the i -th reduced Betti number of K , which is the rank of $\tilde{H}_i(K)$. Let $\beta(K)$ be the total Betti number, that is, $\beta(K) = \sum_{i \geq 0} \tilde{\beta}_i(K)$. Decades ago, Kalai and Meshulam [5] conjectured that the independence complex of every ternary graph has total Betti number at most 1. This conjecture was recently proved by Zhang and Wu [8].

In general, for a graph G , $\beta(I(G)) \leq 1$ does not imply that $I(G)$ is contractible or homotopy equivalent to a sphere. To see why, observe that the barycentric subdivision of any cell complex can be expressed as the independence complex of some graph. For example, considering the real projective plane $\mathbb{R}P^2$, one can find a graph whose independence complex is homotopy equivalent to $\mathbb{R}P^2$, which has the total Betti number 0, but is neither contractible nor homotopy equivalent to a sphere.

Our result, as well as the result by Zhang and Wu, is a generalization of a result about the reduced Euler characteristic of the independence complex of ternary graphs. Given a simplicial complex K , the reduced Euler characteristic of K is defined as $\chi(K) = \sum_{i \geq 0} (-1)^i \tilde{\beta}_i(K)$. It is a well-known fact in algebraic topology that $\chi(K) = \sum_{A \in K} (-1)^{|A|}$ (see [4]). Therefore, for a graph G , $|\chi(I(G))|$ is the difference between the number of independent sets of G of even size and the number of those of odd size. Kalai and Meshulam [5] also posed a weaker conjecture that a graph G is ternary if and only if $|\chi(I(H))| \leq 1$ for every induced subgraph H , and this conjecture was proved by Chudnovsky, Scott, Seymour and Spirkl [1].

Earlier, Gauthier [3] proved a special case of the conjecture: if a graph G contains no (not necessarily induced) cycles of length divisible by 3, then $|\chi(I(G))| \leq 1$. By extending the work of Gauthier, Engström [2] showed that the independence complex of such a graph is either contractible or homotopy equivalent to a sphere. Engström conjectured that his result can be extended to ternary graphs. Theorem 1 confirms Engström’s conjecture.

Remark 1. This manuscript is an extended abstract. Full proofs and details can be found in the full paper [6].

2 Preliminaries

In this section, we introduce some topological background (for details, see [4]) and useful lemmas to determine the homotopy type of independence complexes.

For a graph G and $v \in V(G)$, let $N(v) := \{u \in V : uv \in E\}$ and $N[v] := N(v) \cup \{v\}$. For $W \subset V(G)$, let $N[W] := \cup_{w \in W} N[w]$. When $V(G) = \emptyset$, we call G the *null graph*.

2.1 Mayer-Vietoris Sequences

Let G be a graph on V . For each $v \in V$, observe that every independent set of G containing v is contained in $G - N(v)$. This implies

$$I(G) = I(G - v) \cup I(G - N(v)).$$

Note that $I(G - N(v))$ is a cone with apex v , thus it is contractible. Finally, we observe

$$I(G - v) \cap I(G - N(v)) = I(G - N[v]).$$

Then, by applying the Mayer-Vietoris sequence, we obtain the following long exact sequence:

$$\dots \rightarrow \tilde{H}_i(I(G - N[v])) \rightarrow \tilde{H}_i(I(G - v)) \rightarrow \tilde{H}_i(I(G)) \rightarrow \tilde{H}_{i-1}(I(G - N[v])) \rightarrow \dots \tag{2}$$

Let $G(X|Y)$ be the subgraph of G induced by $V - N[X] - Y$ if X is independent and $X \cap Y = \emptyset$, and the null graph otherwise. If X and Y are vertex subsets where X is independent and $X \cap Y = \emptyset$, and if $v \notin N[X] \cup Y$, then we obtain the following exact sequence by replacing G with $G - N[X] - Y$ in (2):

$$\begin{aligned} \dots \rightarrow \tilde{H}_i(I(G(X \cup \{v}|Y))) &\rightarrow \tilde{H}_i(I(G(X|Y \cup \{v}))) \rightarrow \tilde{H}_i(I(G(X|Y))) \\ \rightarrow \tilde{H}_{i-1}(I(G(X \cup \{v}|Y))) &\rightarrow \tilde{H}_{i-1}(I(G(X|Y \cup \{v}))) \rightarrow \tilde{H}_{i-1}(I(G(X|Y))) \rightarrow \dots \end{aligned} \tag{3}$$

Recalling that $\tilde{H}_i(S^k) = 0$ if $i \neq k$ and $\tilde{H}_k(S^k) \simeq \mathbb{Z}$, we can prove the following lemma.

Lemma 1. *Let A, B and C be simplicial complexes such that the following sequence is exact:*

$$\dots \rightarrow \tilde{H}_i(A) \rightarrow \tilde{H}_i(B) \rightarrow \tilde{H}_i(C) \rightarrow \tilde{H}_{i-1}(A) \rightarrow \dots$$

Suppose $A \simeq S^k$ and $B \simeq S^\ell$ for some non-negative integers k and ℓ . Then the following hold.

- (i) *If $k > \ell$, then $\tilde{\beta}_{k+1}(C) = \tilde{\beta}_\ell(C) = 1$.*
- (ii) *If $k = \ell$, then either $\tilde{\beta}_i(C) = 0$ for all non-negative integer i or both $\tilde{H}_{k+1}(C)$ and $\tilde{H}_k(C)$ are non-vanishing.*

2.2 Homotopy Type Theory

Let A and B be two topological spaces.

- A and B are *homotopy equivalent* if there are continuous maps $f : A \rightarrow B$ and $g : B \rightarrow A$ such that $g \circ f \simeq \text{id}_A$ and $f \circ g \simeq \text{id}_B$, where id_X is the identity map on X . We write $A \simeq B$ if A and B are homotopy equivalent. In particular, if A is contractible, we write $A \simeq *$.
- The *wedge sum* of A and B is the space obtained by taking the disjoint union of A and B and identifying a point of A and a point of B . We denote the wedge sum of A and B by $A \vee B$.
- Let \sim be an equivalence relation on A . Then we denote the quotient space of A under \sim by A/\sim . Let $B \subset A$. Then we define A/B as the quotient space A/\sim where for all $a \neq b$ in A , $a \sim b$ if and only if $a, b \in B$.
- The *suspension* of A is the quotient space $\Sigma A := A \times [0, 1]/\sim$ where for all $(a, s) \neq (b, t)$ in $A \times [0, 1]$, $(a, s) \sim (b, t)$ if and only if either $s = t = 0$ or $s = t = 1$.

Note that if S^n is the n -dimensional sphere, then $\Sigma S^n \simeq S^{n+1}$.

Now let K, K_1 and K_2 be simplicial complexes where $K_1 \cap K_2 \neq \emptyset$, and let $L \neq \emptyset$ be a subcomplex of K . Then,

- (A) If $K = K_1 \cup K_2$, then $K/K_2 \simeq K_1/(K_1 \cap K_2)$.
- (B) Suppose the inclusion map $L \hookrightarrow K$ is homotopic to a constant map $c : L \rightarrow K$, that is, L is contractible in K . Then $K/L \simeq K \vee \Sigma L$. In particular, $K/L \simeq \Sigma L$ when K is contractible, and $K/L \simeq K$ when L is contractible.

By applying (B), we can deduce the following well-known statement:

Lemma 2. *Let X be a simplicial complex, and Y be a subcomplex of X . If $X \simeq S^k$ and $Y \simeq S^\ell$ for some non-negative integers k and ℓ with $\ell < k$, then X/Y is homotopic to $S^k \vee S^{\ell+1}$.*

By a similar argument as in Sect. 2.1 and by applying (A), we obtain the following lemma about homotopy equivalence of independence complexes.

Lemma 3. *Let G be a graph and v a vertex of G . If X and Y are disjoint subsets of $V(G)$ such that X is independent, $v \notin N[X] \cup Y$, and $N[X] \cup N[v] \cup Y \neq V(G)$, then $I(G(X|Y)) \simeq I(G(X|Y \cup \{v\}))/I(G(X \cup \{v\}|Y))$.*

3 Proof of Theorem 1

In this section, we prove the main result. By (1), it is sufficient to show the following.

Theorem 2. *Let G be a ternary graph. Then $I(G)$ is either contractible or homotopy equivalent to a sphere.*

To prove Theorem 2 by contradiction, take a counter-example G on V which is minimal in the following sense: $I(G)$ is neither contractible nor homotopy equivalent to a sphere, but $I(H)$ is either contractible or homotopy equivalent to a sphere for every proper induced subgraph H of G .

Let X and Y be vertex subsets of G such that $X \cup Y \neq \emptyset$. We define $d(X|Y)$ as the following:

$$d(X|Y) = \begin{cases} d & \text{if } I(G(X|Y)) \simeq S^d, \\ * & \text{if either } I(G(X|Y)) \simeq * \text{ or } G(X|Y) \text{ is the null graph.} \end{cases}$$

We first describe all possible types of the triples of the form $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ under certain conditions.]

Lemma 4. *Let X and Y be vertex subsets of G such that $X \cup Y \neq \emptyset$. For every vertex $v \notin X \cup Y$, the triple $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\}))$ equals to one of the following:*

$$(*, *, *), (k, *, k), (*, k, k), (k + 1, k, *), (0, *, *)$$

for some non-negative integer k . In particular, if $(d(X|Y), d(X \cup \{v\}|Y), d(X|Y \cup \{v\})) = (0, *, *)$, then $N[X] \cup N[v] \cup Y = V$.

Note that Lemma 4 is a natural analogue of [8, Lemma 3.1], which is about the dimension that the Betti number of the independence complex is non-vanishing.

Lemma 5. *There is a non-negative integer k such that $d(\emptyset|v) = d(v|\emptyset) = k$ for all $v \in V$.*

Proof. Since G is a ternary graph, if $N[v] = V$ for some $v \in V$, we have $I(G) \simeq S^0$, which is a contradiction to the assumption on G . Thus, we may assume $N[v] \neq V$ for all $v \in V$. Then by Lemma 3, for any vertex $v \in V$, we have

$$I(G) \simeq I(G(\emptyset|v))/I(G(v|\emptyset)).$$

Note that each of $I(G(\emptyset|v))$ and $I(G(v|\emptyset))$ is either contractible or homotopy equivalent to a sphere. By (2.2),

- if $I(G(v|\emptyset)) \simeq *$, then $I(G) \simeq I(G(\emptyset|v))$, and
- if $I(G(\emptyset|v)) \simeq *$, then $I(G) \simeq \Sigma I(G(v|\emptyset))$.

In both cases, it is clear that $I(G)$ is either contractible or homotopy equivalent to a sphere. Thus we may assume both $I(G(\emptyset|v))$ and $I(G(v|\emptyset))$ are homotopy equivalent to spheres, that is, $d(v|\emptyset) = \ell$ and $d(\emptyset|v) = k$ for some non-negative integers k and ℓ . We claim $k = \ell$.

If $k > \ell$, then $I(G) \simeq S^k \vee S^{\ell+1}$ by Lemma 2. If $k < \ell$, then by Lemma 1, we have $\tilde{\beta}_k(I(G)) = \tilde{\beta}_{\ell+1}(I(G)) = 1$. In both cases, we have $\beta(I(G)) \geq 2$, which is a contradiction to a result by Zhang and Wu [8]: if H is a ternary graph, then $\beta(I(H)) \leq 1$.

Now suppose that there exist $u, v \in V$ such that $d(u|\emptyset) = d(\emptyset|u) = p$ and $d(v|\emptyset) = d(\emptyset|v) = q$ for two non-negative integers p, q with $p < q$. Since $q > 0$, by Lemma 4, $(d(v|\emptyset), d(u, v|\emptyset), d(v|u))$ should be either $(q, *, q)$ or $(q, q - 1, *)$. If $d(u, v|\emptyset) = q - 1$, then $(d(u|\emptyset), d(u, v|\emptyset), d(u|v)) = (p, q - 1, d(u|v))$ which is possible only when $p = q$ by Lemma 4. Thus we obtain

$$(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (q, *, q).$$

On the other hand, Lemma 4 implies

$$(d(u|\emptyset), d(u, v|\emptyset), d(u|v)) = (p, *, p).$$

Therefore, we have

$$(d(\emptyset|u), d(v|u), d(\emptyset|u, v)) = (p, q, d(\emptyset|u, v)).$$

However, Lemma 4 implies $q = p - 1$, which is a contradiction because we have $p < q$. □

Now suppose $k \geq 1$. We claim $d(u, v|\emptyset) = k - 1$ for any two distinct vertices u, v of G . By Lemma 4, $(d(v|\emptyset), d(u, v|\emptyset), d(v|u))$ is either $(k, *, k)$ or $(k, k - 1, *)$ and $(d(\emptyset|u), d(v|u), d(\emptyset|u, v))$ is either $(k, *, k)$ or $(k, k - 1, *)$. Then $d(v|u)$ must be $*$, and hence we obtain

$$(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (k, k - 1, *).$$

Since $d(u, v|\emptyset) = k - 1 \geq 0$, $G(u, v|\emptyset)$ should not be the null graph. Thus $\{u, v\}$ is an independent set of G . Since this holds for every pair of two distinct vertices u and v , we conclude that the whole vertex set V is an independent set of G . In this case, $I(G)$ is contractible, which is a contradiction to the assumption on G . Thus we may assume $k = 0$.

We first claim that $N[u] \cup N[v] = V$ for any $u, v \in V$ with $u \neq v$. By Lemma 4, $(d(v|\emptyset), d(u, v|\emptyset), d(v|u))$ is either $(0, *, *)$ or $(0, *, 0)$ and $(d(\emptyset|u), d(v|u), d(\emptyset|u, v))$ is either $(0, *, *)$ or $(0, *, 0)$. Then it must be

$$(d(v|\emptyset), d(u, v|\emptyset), d(v|u)) = (0, *, *).$$

Therefore, by Lemma 4, we have $N[u] \cup N[v] = V$. Now take $x \in V$. Since $I(G(x|\emptyset)) \simeq S^0$, $I(G(x|\emptyset))$ has exactly two connected components, say C_1 and C_2 . For any choice of $u \in V(C_1)$ and $v \in V(C_2)$, we know $N[u] \cup N[v] = V$. However, this cannot be true because $V(C_1) \cup V(C_2) \subset V \setminus N[x]$, that is, x is not adjacent to any of u and v .

In any case, we reach a contradiction, implying that our initial assumption of G being a counter-example cannot hold. Therefore, if G is ternary, then $I(G)$ is either contractible or homotopy equivalent to a sphere.

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The Maximum Number of Paths of Length Three in a Planar Graph

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Abstract. Let $f(n, H)$ denote the maximum number of copies of H possible in an n -vertex planar graph. The function $f(n, H)$ has been determined when H is a cycle of length 3 or 4 by Hakimi and Schmeichel and when H is a complete bipartite graph with smaller part of size 1 or 2 by Alon and Caro. We determine $f(n, H)$ exactly in the case when H is a path of length 3.

Keywords: Planar graph · Maximal planar graph · Apollonian networks

1 Introduction

In recent times, generalized versions of the extremal function $\text{ex}(n, H)$ have received considerable attention. For graphs G and H , let $\mathcal{N}(H, G)$ denote the number of subgraphs of G isomorphic to H . Let \mathcal{F} be a family of graphs, then a graph G is said to be \mathcal{F} -free if it contains no graph from \mathcal{F} as a subgraph. Alon and Shikhelman [3] introduced the following generalized extremal function (stated in higher generality in [4]),

$$\text{ex}(n, H, \mathcal{F}) = \max\{\mathcal{N}(H, G) : G \text{ is an } \mathcal{F}\text{-free graph on } n \text{ vertices}\}.$$

If $\mathcal{F} = \{F\}$, we simply write $\text{ex}(n, H, F)$. The earliest result of this type is due to Zykov [25] (and also independently by Erdős [6]), who determined $\text{ex}(n, K_s, K_t)$ exactly for all s and t . Erdős conjectured that asymptotically $\text{ex}(n, C_5, C_3) = (\frac{n}{5})^5$ (where the lower bound comes from considering a blown up C_5). This conjecture was finally verified half of a century later by Hatami, Hladký, Král, Norine and Razborov [19] and independently by Grzesik [16]. Recently, the asymptotic value of $\text{ex}(n, C_k, C_{k-2})$ was determined for every odd

k by Grzesik and Kielak [17]. In the opposite direction, the extremal function $\text{ex}(n, C_3, C_5)$ was considered by Bollobás and Győri [5]. Their results were subsequently improved in the papers [3, 7] and [8], but the problem of determining the correct asymptotic remains open. The problem of maximizing P_ℓ copies in a P_k -free graph was investigated in [13].

It is interesting that although maximizing copies of a graph H in the class of F -free graphs has been investigated heavily, maximizing H -copies in other natural graph classes has received less attention. In the setting of planar graphs such a study was initiated by Hakimi and Schmeichel [18]. Let $f(n, H)$ denote the maximum number of copies of H possible in an n -vertex planar graph. Observe that $f(n, H)$ is equal to $\text{ex}(n, H, \mathcal{F})$ where \mathcal{F} is the family of $K_{3,3}$ or K_5 subdivisions [21]. The case when H is a clique and \mathcal{F} is a family of clique minors has also been investigated (see, for example, [9, 20, 23]).

Hakimi and Schmeichel determined the function $f(n, H)$ when H is a triangle or cycle of length four. Moreover, they classified the extremal graphs attaining this bound (a small correction to their result was given in [1]).

Theorem 1 (Hakimi and Schmeichel [18]). *Let G be a maximal planar graph with $n \geq 6$ vertices, then $C_3(G) \leq 3n - 8$, where $C_3(G)$ is the number of cycles of length 3 in G . This bound is attained if and only if G is a graph is obtained from K_3 by recursively placing a vertex inside a face and joining the vertex to the three vertices of that face (graphs constructed in this way are referred to as Apollonian networks).*

Theorem 2 (Hakimi and Schmeichel [18], Alameddine [1]). *Let G be a maximal planar graph with $n \geq 5$ vertices, then $C_4(G) \leq \frac{1}{2}(n^2 + 3n - 22)$, where $C_4(G)$ is the number of cycles of length 4 in G . For $n \neq 7, 8$, the bound is attained if and only if G is the graph shown in Fig. 1(A). For $n = 7$, the bound is attained if and only if G is the graph in Fig. 1(A) or (B). For $n = 8$, the bound is attained if and only if G is the graph in Fig. 1(A) or (C).*

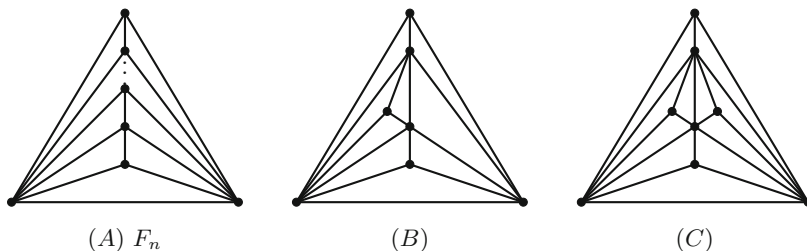


Fig. 1. Planar graphs maximizing the number of cycles of length 4.

Thus, we have $f(n, C_3) = 3n - 8$ when $n \geq 6$ and $f(n, C_4) = \frac{1}{2}(n^2 + 3n - 22)$ for $n \geq 5$. In [14], we extended the results of Hakimi and Schmeichel by determining $f(n, C_5)$ for all n .

In the case when H is a complete bipartite graph, Alon and Caro [2] determined the value of $f(n, H)$ exactly. They obtained the following results.

Theorem 3 (Alon and Caro [2]). *For all $k \geq 2$ and $n \geq 4$,*

$$f(n, K_{1,k}) = 2 \binom{n-1}{k} + 2 \binom{3}{k} + (n-4) \binom{4}{k}.$$

Theorem 4 (Alon and Caro [2]).

For all $k \geq 2$ and $n \geq 4$,

$$f(n, K_{2,k}) = \begin{cases} \binom{n-2}{k}, & \text{if } k \geq 5 \text{ or } k = 4 \text{ and } n \neq 6; \\ 3, & \text{if } (k, n) = (4, 6); \\ \binom{n-2}{3}, & \text{if } k = 3, n \neq 6; \\ 12, & \text{if } (k, n) = (3, 6); \\ \binom{n-2}{2} + 4n - 14, & \text{if } k = 2. \end{cases}$$

Other results in this direction include a linear bound on the maximum number of copies of a 3-connected planar graph by Wormald [24] and independently Eppstein [10]. The exact bound on the maximum number of copies of K_4 was given by Wood [22]. Let P_k denote the path of k vertices. It is well-known that $f(n, P_2) = 3n - 6$, and it follows from Theorem 3 that $f(n, P_3) = n^2 + 3n - 16$ for $n \geq 4$. The order of magnitude of $f(n, H)$ when H is a fixed tree was determined in [12]. In particular, for a path on k vertices, we have $f(n, P_k) = \Theta(n^{\lfloor \frac{k-1}{2} \rfloor + 1})$.

In this paper we determine $f(n, P_4)$, the maximum number of copies of a path of length three possible in n -vertex planar graph. Our main result is the following.

Theorem 5. *We have,*

$$f(n, P_4) = \begin{cases} 12, & \text{if } n = 4; \\ 147, & \text{if } n = 7; \\ 222, & \text{if } n = 8; \\ 7n^2 - 32n + 27, & \text{if } n = 5, 6 \text{ and } n \geq 9. \end{cases}$$

For integers $n \in \{4, 5, 6\}$ and $n \geq 9$, the only n -vertex planar graph attaining the value $f(n, P_3)$ is the graph F_n . For $n = 7$ and $n = 8$ the graphs pictured in Fig. 1(B) and 1(C), respectively are the only graphs attaining the value $f(n, P_4)$.

A few words about the proof:

It is known that the minimum degree of a maximal planar graph is at least 3 and at most 5. Proof of the theorem is a very involved mathematical induction of 14 pages distinguishing these three cases of the minimum degree. The base cases of the induction have surprisingly complicated proofs too. Details of the proof can be found in arxiv [15].

The asymptotic value of $f(n, P_5)$ was determined, too, it is n^3 (see [11]). However the exact upper bound is not known yet.

Acknowledgements. The research of the first, third and fifth authors is partially supported by the National Research, Development and Innovation Office – NKFIH, grant K 116769, K 132696 and SNN 117879. The research of the third author is partially supported by the Shota Rustaveli National Science Foundation of Georgia SRNSFG, grant number DI-18-118. The research of the fourth author is supported by the Institute for Basic Science (IBS-R029-C1).

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Strongly Pfaffian Graphs

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Abstract. An orientation of a graph G is *Pfaffian* if every even cycle C such that $G - V(C)$ has a perfect matching has an odd number of edges oriented in either direction of traversal. Graphs that admit a Pfaffian orientation permit counting the number of their perfect matchings in polynomial time.

We consider a strengthening of Pfaffian orientations. An orientation of G is *strongly Pfaffian* if every even cycle has an odd number of edges directed in either direction of the cycle. We show that there exist two graphs S_1 and S_2 such that a graph G admits a strongly Pfaffian orientation iff it does not contain a graph H as a subgraph which can be obtained from S_1 or S_2 by subdividing every edge an even number of times. Combining our main results with a result of Kawarabayashi et al. we show that given any graph the tasks of recognising whether the graph admits a strongly Pfaffian orientation and constructing such an orientation provided it exists can be solved in polynomial time.

Keywords: Graph theory · Perfect matching · Pfaffian orientation

1 Introduction

All graphs considered in this article are finite and do not contain loops. We also exclude parallel edges, unless we explicitly address the graphs as multi-graphs. Let G be a graph and $F \subseteq E(G)$ be a set of edges. F is called a *matching* if no two edges in F share an endpoint, a matching is *perfect* if every vertex of G is contained in some edge of F . A subgraph H of G is *conformal* if $G - V(H)$ has a perfect matching, and finally an orientation of G is a digraph \mathbf{G} such that (u, v) or (v, u) is an edge of \mathbf{G} iff $uv \in E(G)$, and \mathbf{G} does not contain (u, v) and (v, u) at the same time. An even cycle C of G is *oddly oriented* by an orientation \mathbf{G} if it has an odd number of edges directed in either direction around C . An orientation \mathbf{G} of G is *Pfaffian* if G has a perfect matching and every even conformal cycle C is oddly oriented by \mathbf{G} . A graph G is called *Pfaffian* if it admits a Pfaffian orientation.

Pfaffian orientations are significant as, given that a graph G admits a Pfaffian orientation, the number of perfect matchings of G can be computed in polynomial time. In general, the problem of counting the number of perfect matchings in a

graph is polynomial-time equivalent to computing the permanent of a matrix, which is known to be $\sharp\text{P}$ -hard [9]. By utilising a deep theorem of Lovász [5], Vazirani and Yannakakis [10] proved that, in terms of complexity, recognising a Pfaffian graph and finding a Pfaffian orientation can be seen as the same problem.

Theorem 1 ([10]). *The decision problems ‘Is a given orientation of a graph Pfaffian?’ and ‘Is a given graph Pfaffian?’ are polynomial-time equivalent.*

Let H be a graph and $e \in E(H)$ be some edge of H . We say that a graph H' is obtained from H by *subdividing*, or respectively *bisubdividing*, the edge e , if H' can be obtained from H by replacing e with a path of positive length, or respectively a path of odd length (possibly length one), whose endpoints coincide with the endpoints of e and whose internal vertices do not belong to H . A graph H'' is a *subdivision*, or respectively a *bisubdivision*, of H if it can be obtained by subdividing, or respectively bisubdividing, all edges of H . Notably, bisubdivision preserves path- and cycle-parities.

There exists a precise characterisation of Pfaffian bipartite graphs in terms of forbidden bisubdivisions.

Theorem 2 ([4]). *A bipartite graph G is Pfaffian iff it does not contain a conformal bisubdivision of $K_{3,3}$.*

While Theorem 2 characterises all Pfaffian bipartite graphs it does not immediately yield a polynomial time recognition algorithm. Such an algorithm was later found by McCuaig [6] and, independently, by Robertson et al. [8].

The case of non-bipartite Pfaffian graphs appears to be more illusive, and it remains an important open problem in graph theory whether Pfaffian graphs can be recognised in polynomial time. For several graph classes including non-bipartite graphs characterisations of Pfaffian graphs in these classes are known (see [1, 2]), but whether they can be recognised in polynomial time appears to be open. Moreover, there is no hope that all non-bipartite Pfaffian graphs can be described by excluding a finite number of minimal obstructions in a fashion similar to Theorem 2 [7]. For more Information on Pfaffian orientations and related problems we refer the reader to [6].

Our Contribution. Inspired by the inherent complexity of Pfaffian orientations, especially in non-bipartite graphs, we investigate a stronger, and therefore more restrictive version of Pfaffian orientations.

Definition 1. *Let G be a graph. An orientation \mathbf{G} of G is strongly Pfaffian if every even cycle C of G is oddly oriented by \mathbf{G} . A graph G that admits a strongly Pfaffian orientation is called strongly Pfaffian.*

For the class of strongly Pfaffian graphs it suffices to exclude bisubdivisions of S_1 and S_2 (see Fig. 1), as subgraphs to characterise the entire class.

Theorem 3. *A graph G is strongly Pfaffian iff it does not contain a bisubdivision of S_1 or S_2 as a subgraph.*

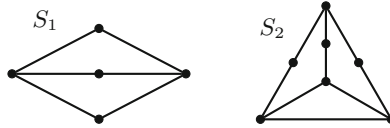


Fig. 1. The two obstructions for strongly Pfaffian graphs.

And furthermore, thanks to the results in [3], Theorem 3 allows us to deduce that the task of recognising strongly Pfaffian graphs can be performed in polynomial time, since we can simply check for the existence of the bisubdivided versions of S_1 and S_2 .

Theorem 4 ([3]). *Let H be a fixed graph, and for each edge $e \in E(H)$ let a value $p(e) \in \{0, 1\}$ be fixed. There exists a polynomial time algorithm for testing if a given graph G contains a subdivision of H in which for every $e \in E(H)$ the length of the subdivision-path representing e is congruent to $p(e)$ modulo 2.*

Corollary 1. *Strongly Pfaffian graphs can be recognised in polynomial time.*

Theorem 5. *Given a strongly Pfaffian graph, a strongly Pfaffian orientation can be constructed in polynomial time.*

Consequently given a graph G , we can test whether it is strongly Pfaffian and construct a strongly Pfaffian orientation in polynomial time.

2 A Structural Characterisation of Strongly Pfaffian Graphs

Let \mathcal{S} be the class of graphs which do not contain bisubdivisions of S_1 or S_2 as subgraphs. In order to show Theorem 3, we need to show that (1) every strongly Pfaffian graph is contained in \mathcal{S} and (2) every graph in \mathcal{S} admits a strongly Pfaffian orientation.

An important tool when working with strongly Pfaffian orientations is the *switching operation*. For this let \mathbf{G} be an orientation of a graph G and let $v \in V(G)$. Then *switching at v* in \mathbf{G} means reversing all arcs of \mathbf{G} incident with v to obtain a modified orientation of G . We say that two orientations of a graph are *switching-equivalent* if one can be obtained from the other by a sequence of switchings. Observe that if \mathbf{G}_1 and \mathbf{G}_2 are switching-equivalent orientations of the same graph, then \mathbf{G}_1 is strongly Pfaffian if and only the same holds for \mathbf{G}_2 . Using this fact we can give a short proof of (1).

Lemma 1. *If G is a strongly Pfaffian graph, then $G \in \mathcal{S}$.*

Proof. Since every subgraph of a strongly Pfaffian graph is strongly Pfaffian, it suffices to show that no bisubdivision of S_1 or S_2 is strongly Pfaffian.

First let G be a bisubdivision of S_1 and suppose towards a contradiction that a strongly Pfaffian orientation \mathbf{G} of G exists. Let u, v be the unique vertices of

degree three in G and let P_1, P_2, P_3 be the three disjoint u - v -paths in G . Since G is a bisubdivision of S_1 , all three paths P_1, P_2, P_3 have even length. Pause to note that by applying vertex-switchings, we may find a strongly Pfaffian orientation \mathbf{G}' of G that is switching-equivalent to \mathbf{G} such that $P_1, P_2 - v, P_3 - v$ each form directed paths in \mathbf{G}' starting at u . Since the cycles $P_1 \cup P_2$ and $P_1 \cup P_3$ must be oddly oriented by \mathbf{G}' , the last edges of both P_2 and P_3 must start in v . However, this means that the cycle $P_2 \cup P_3$ has an even number of edges oriented in either direction in \mathbf{G}' , a contradiction showing that G is not strongly Pfaffian.

Next suppose that G is a bisubdivision of S_2 . Let v_1, v_2, v_3, v_4 be the vertices of degree three in G and for $i, j \in \{1, 2, 3, 4\}$, let $P_{i,j}$ be the subdivision-path of G connecting v_i and v_j . Possibly after relabelling, we may assume that $P_{j,4}$ is even for $j \in \{1, 2, 3\}$ and $P_{1,2}, P_{2,3}, P_{1,3}$ are odd. Suppose towards a contradiction that there exists a strongly Pfaffian orientation \mathbf{G} of G . Possibly by performing switchings, we may assume w.l.o.g. that $P_{j,4}$ is directed from v_j to v_4 for $j \in \{2, 3, 4\}$. For $j \in \{1, 2, 3\}$, let p_j be the number of forward-edges on $P_{j,j+1}$ when traversing it from v_j to v_{j+1} (where $3 + 1 := 1$). By the pigeon-hole principle, we have $p_j \equiv p_{j+1} \pmod{2}$ for some $j \in \{1, 2, 3\}$, w.l.o.g. $j = 1$. Now $P_{1,2} \cup P_{2,3} \cup P_{3,4} \cup P_{4,1}$ is an even cycle in G with an even number of edges oriented in either direction. This contradiction shows that G is not strongly Pfaffian. \square

The main work for Theorem 3 needs to be done when proving (2). Here our approach is as follows: We give an explicit description of all graphs in the class \mathcal{S} and then later construct strongly Pfaffian orientations for all of these graphs.

We start by observing that it is sufficient to prove (2) for 2-vertex-connected graphs: As is easily noted, a graph is contained in \mathcal{S} iff the same is true for each of its blocks (maximal 2-connected subgraphs). Similarly, given a strongly Pfaffian orientation of each block of a graph, the union of these orientations forms a strongly Pfaffian orientation of the whole graph.

As a next step we reduce the proof of (2) to the case of subdivisions of 3-vertex-connected multi-graphs. In order to do so, we consider an operation that decomposes 2-connected graphs along 2-vertex-separations while preserving important information concerning parities of path lengths.

Definition 2. *Let G be a 2-vertex-connected graph, and let $u, v \in V(G)$ be distinct vertices such that $\{u, v\}$ forms a separator of G . Let H_1, H_2 be connected subgraphs of G such that $V(H_1) \cup V(H_2) = V(G)$, $V(H_1) \cap V(H_2) = \{u, v\}$. Finally, for $i \in \{1, 2\}$ let G_i be a supergraph of H_i obtained as follows: If there exists an odd u, v -path in H_{3-i} , then add an edge uv to H_i (unless it already exists). If there exists an even u, v -path in H_{3-i} , then add a new vertex $w \notin V(H_i)$ to H_i and connect it to u and v . Perform both operations simultaneously if both an even and an odd path from u to v in H_{3-i} exists. With these definitions, we call G a parity 2-sum of the two graphs G_1 and G_2 .*

The following result describes how parity sums interact with the class \mathcal{S} and strongly Pfaffian graphs. Due to the space restrictions, we omit its proof.

Lemma 2. *Let G be the parity 2-sum of two graphs G_1 and G_2 . Then $G \in \mathcal{S}$ iff $G_1, G_2 \in \mathcal{S}$, and G is strongly Pfaffian iff G_1 and G_2 are strongly Pfaffian.*

Using Lemma 2, it suffices to prove (2) for 2-vertex-connected graphs that cannot be written as the parity 2-sum of two smaller graphs. It is not hard to see that such graphs either have at most 2 vertices of degree larger than two (in this case, the proof of (2) becomes quite trivial); or they are subdivisions of 3-vertex-connected multi-graphs such that every subdivision-path has length at most two, and two parallel subdivision-paths only exist in the form of a direct edge and a 2-edge-path between the same two branch vertices. Since neither S_1 nor S_2 use two parallel subdivision-paths of this type, it turns out that a graph G as described above is contained in \mathcal{S} iff the same is true for every subdivision of a 3-connected simple graph contained in G (obtained by ignoring one of the two subdivision-paths of each parallel pair). Hence, in order to describe all such graphs that are contained in \mathcal{S} , we can further reduce to subdivisions of 3-connected simple graphs. To handle this case, we use the following key lemma.

Lemma 3. *Let $G \in \mathcal{S}$ be a subdivision of a 3-connected simple graph H . Then (A) $V(H)$ is the disjoint union of two induced cycles whose subdivisions in G are odd, or (B) for every odd cycle C in G the graph $G - V(C)$ is a forest.*

Proof (Sketch). We rely on two central observations. Let G be as in Lemma 3. First, we observe that if C_1 and C_2 are respectively even and odd cycles within G , then $V(C_1) \cap V(C_2) \neq \emptyset$. If this was not true consider three internally disjoint paths connecting C_1 and C_2 in G , which are guaranteed by the 3-connectivity of H . Without much effort, it can be deduced that no matter which parities these three paths respectively possess, we are guaranteed to find a bisubdivision of S_1 in G , contradicting $G \in \mathcal{S}$.

The second observation is slightly trickier. Let C_1 and C_2 be two disjoint odd cycles in G , then we claim that C_1 and C_2 contain all vertices of degree three or higher, and the underlying cycles C'_1 and C'_2 of C_1 and C_2 are induced in H . Consider a vertex $v \notin V(C_1) \cup V(C_2)$ for which two paths P and Q with $V(P) \cap V(Q) = \{v\}$ exist such that both P and Q end on C_i and do not intersect C_{3-i} , for $i \in \{1, 2\}$. Now clearly $C_i \cup P \cup Q$ contains an even cycle disjoint from C_{3-i} , contradicting our first observation. Thus we already know that the underlying cycles C'_1 and C'_2 of C_1 and C_2 are induced in H , as we can otherwise find such a vertex v on a subdivided chord of C_1 or C_2 . (Note in particular that H is a simple graph and thus no parallel edges will disturb our cycles.) Finally, if there exists a vertex u of degree three or higher outside $V(C_1) \cup V(C_2)$, then we can again use the 3-connectedness of H to find three internally disjoint paths between u and $V(C_1) \cup V(C_2)$. By the pigeonhole principle, two of these paths must now violate the condition we have just established for the vertices outside of $V(C_1) \cup V(C_2)$. From these two observations it is easy to derive the characterisation presented in the statement of the lemma. \square

Using the two cases suggested by Lemma 3 we can derive a partition of the graphs in \mathcal{S} which are subdivisions of 3-connected graphs into four classes and a handful of sporadic examples. This process is quite arduous and we omit the explicit definitions of the graphs from this article and opt instead to provide examples for each of the classes.

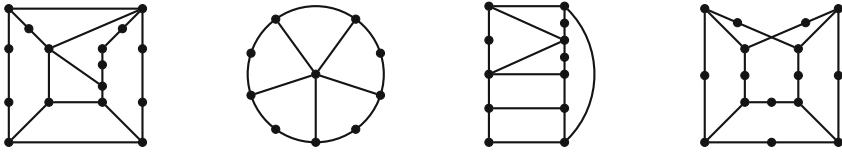


Fig. 2. Representatives for the four classes. From left to right: (a) prismoid, (b) wheeloid, (c) C_4 -cockade with handle, (d) möbioid.

The prismoids (see Fig. 2a), and five of the sporadic examples, all closely related to the prismoids, correspond to property A mentioned in Lemma 3. The wheeloids (see Fig. 2b), C_4 -cockades with handles (see Fig. 2c), möbioids (see Fig. 2d) and a specific subdivision of K_5 all correspond to property B in Lemma 3. The subdivision of K_5 and the möbioids are notably the only non-planar graphs in \mathcal{S} .

Using the structure of each of the classes, we can give concrete proofs for the existence of strongly Pfaffian orientations for all their members. This can then easily be extended to subdivisions of 3-connected multi-graphs as described above, all in all showing that every graph in \mathcal{S} admits a Pfaffian orientation. Using this approach we verify (2) and hence conclude the proof of Theorem 3.

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Strong Modeling Limits of Graphs with Bounded Tree-Width

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Abstract. The notion of first order convergence of graphs unifies the notions of convergence for sparse and dense graphs. Nešetřil and Ossona de Mendez [J. Symbolic Logic 84 (2019), 452–472] proved that every first order convergent sequence of graphs from a nowhere-dense class of graphs has a modeling limit and conjectured the existence of such modeling limits with an additional property, the strong finitary mass transport principle. The existence of modeling limits satisfying the strong finitary mass transport principle was proved for first order convergent sequences of trees by Nešetřil and Ossona de Mendez [Electron. J. Combin. 23 (2016), P2.52] and for first order sequences of graphs with bounded path-width by Gajarský et al. [Random Structures Algorithms 50 (2017), 612–635]. We establish the existence of modeling limits satisfying the strong finitary mass transport principle for first order convergent sequences of graphs with bounded tree-width.

Keywords: Combinatorial limit · Graph limit · First order convergence · Modeling limit

1 Introduction

The theory of combinatorial limits is an evolving area of combinatorics. The most developed is the theory of graph limits, which is covered in detail in a recent monograph by Lovász [21]. Further results concerning many other combinatorial structures exist, e.g. for permutations [12, 15, 16, 20] or for partial orders [14, 18]. In the case of graphs, limits of dense graphs [5–7, 22, 23], also see [8, 9] for a general theory of limits of dense combinatorial structures, and limits of sparse graphs [1–4, 10, 13] evolved to a large extent independently.

A notion of first order convergence was introduced by Nešetřil and Ossona de Mendez [31, 33] as an attempt to unify convergence notions in the dense and sparse regimes. This general notion can be applied in the setting of any

relational structures, see e.g. [17] for results on limits of mappings. Informally speaking, a sequence of relational structures is *first order convergent* if for any first order property, the density of ℓ -tuples of the elements having this property converges; a formal definition is given in Sect. 2. Every first order convergent sequence of dense graphs is convergent in the sense of dense graph convergence from [6, 7], and every first order convergent sequence of graphs with bounded degree is convergent in the sense of Benjamini-Schramm convergence from [2].

A first order convergent sequence of graphs can be associated with an analytic limit object, which is referred to as a *modeling limit* (see Sect. 2 for a formal definition). However, not every first order convergent sequence of graphs has a modeling limit [33] and establishing the existence of a modeling limit for first order convergent sequences of graphs is an important problem in relation to first order convergence of graphs: a modeling limit of a first order convergent sequence of dense graphs yields a graphon, the standard limit object for convergent sequences of dense graphs, and a modeling limit of a first order convergent sequence of sparse graphs that satisfies the strong finitary mass transport principle (see Sect. 2 for the definition of the principle) yields a graphing, the standard limit object for convergent sequences of sparse graphs.

Nešetřil and Ossona de Mendez [33] conjectured that every first order convergent sequence of graphs from a nowhere-dense class of graphs has a modeling limit. Nowhere-dense classes of graphs include many sparse classes of graphs, in particular, classes of graphs with bounded degree and minor closed classes of graphs; see [24–27, 29] for further details and many applications. The existence of modeling limits for convergent sequences of graphs from a nowhere-dense class of graphs was proven in [32].

Theorem 1 (Nešetřil and Ossona de Mendez [32]). *Let \mathcal{C} be a class of graphs. Every first order convergent sequence of graphs from \mathcal{C} has a modeling limit if and only if \mathcal{C} is nowhere-dense.*

Theorem 1 gives little control on the measure of vertex subsets in a modeling limit, which naturally have the same size in finite graphs, e.g., those joined by a perfect matching. The strong finitary mass transport principle, vaguely spoken, translates natural constraints on sizes of vertex subsets to measures of corresponding vertex subsets in a modeling limit. We refer to Sect. 2 for further details.

Nešetřil and Ossona de Mendez [32] conjectured that Theorem 1 can be strengthened by adding a condition that modeling limits satisfy the strong finitary mass transport principle.

Conjecture 1 (Nešetřil and Ossona de Mendez [32, Conjecture 6.1]). *Let \mathcal{C} be a nowhere-dense class of graphs. Every first order convergent sequence of graphs from \mathcal{C} has a modeling limit that satisfies the strong finitary mass transport principle.*

The existence of modeling limits satisfying the strong finitary mass transport principle is known for first order convergent sequences of trees of bounded depth

and more generally sequences of graphs with bounded tree-depth [33], sequences of trees [31] and sequences of graphs with bounded path-width [11], which can be interpreted by plane trees. Our main result (Theorem 2) establishes the existence of modeling limits satisfying the strong finitary mass transport principle for sequences of graphs with bounded tree-width.

Theorem 2. *Let k be a positive integer. Every first-order convergent sequence of graphs with tree-width at most k has a modeling limit satisfying the strong finitary mass transport principle.*

While it may seem at the first sight that a proof of Theorem 2 can be an easy combination of a proof of the existence of modeling limits satisfying the strong finitary mass transport principle for trees from [31] and for graphs with bounded path-width [11], this is actually not the case. In fact, the argument in [11] is based on interpretation of modeling limits of so-called plane trees, i.e., the results in both [11] and [31] on the existence of modeling limits satisfying the strong finitary mass transport principle do not go significantly beyond the class of trees.

We have not been able to find a first order interpretation of graphs with bounded tree-width by trees, and we believe that this is related to a possibly complex structure of vertex cuts in such graphs, which need to be addressed using a more general approach. Specifically, the proof of Theorem 2 is based on constructing modeling limits of rooted k -trees, which essentially encode the universal weak coloring orders studied in relation to sparse classes of graphs [29], so the proof may be amenable to an extension to graph classes with bounded expansion in principle.

The proof of Theorem 2, similarly to the proof for the existence of modeling limits of plane trees in [11], has two steps: the decomposition step, focused on distilling first order properties of graphs in the sequence, and the composition step, focused on constructing a modeling limit consistent with the identified first order properties. These two steps also appear implicitly in [31, 33], in particular, the decomposition step is strongly related to the comb structure results presented in [31, 33]. The arguments of the decomposition step of the proof of Theorem 2 are analogous to those used in [11]. The composition step however requires a conceptual extension of techniques used for modeling limits of trees as we had to deal with vertex separations of sizes larger than one. This was achieved by a careful analysis of different types of paths arising in an analogue of a weak coloring order. This allows defining the edge set of a modeling limit in a measurable and consistent way for vertex separations of sizes larger than one.

We remark that our arguments can be easily adapted to show the existence of modeling limits of first order convergent sequences of graphs with bounded tree-width that are residual, which then can be combined with the framework described in [31, Theorem 1] to an alternative proof of Theorem 2.

2 Statement of the Main Result

In order to formally state our results, we need to define the notion of first order convergence. This notion can be used for all relational structure and beyond, e.g.,

matroids [19], however, for simplicity, we limit our exposition to graphs, which may (but need not) be directed and edge-colored. If ψ is a first order formula with ℓ free variables and G is a (finite) graph, then the *Stone pairing* $\langle \psi, G \rangle$ is the probability that a uniformly chosen ℓ -tuple of vertices of G satisfies ψ . A sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is *first order convergent* if the limit $\lim_{n \rightarrow \infty} \langle \psi, G_n \rangle$ exists for every first order formula ψ . It follows from a straightforward argument that every sequence of graphs has a first order convergent subsequence, see e.g. [28, 30, 33].

A *modeling* M is a (finite or infinite) graph with a standard Borel space on its vertex set equipped with a probability measure such that the set of all ℓ -tuples of vertices of M satisfying a formula ψ is measurable in the product measure for every first order formula ψ with ℓ free variables. In the analogy to the graph case, the *Stone pairing* $\langle \psi, M \rangle$ is the probability that a randomly chosen ℓ -tuple of vertices satisfies ψ . If a finite graph is viewed as a modeling with a uniform discrete probability measure on its vertex set, then the Stone pairings for the graph and the modeling obtained in this way evidently coincide. A modeling M is a *modeling limit* of a first order convergent sequence $(G_n)_{n \in \mathbb{N}}$ if

$$\lim_{n \rightarrow \infty} \langle \psi, G_n \rangle = \langle \psi, M \rangle$$

for every first order formula ψ .

Every modeling limit M of a first order convergent sequence of graphs satisfies the *finitary mass transport principle*. This means that for any two given first order formulas ψ and ψ' , each with one free variable, such that every vertex v satisfying $\psi(v)$ has at least a neighbors satisfying ψ' and every vertex v satisfying $\psi'(v)$ has at most b neighbors satisfying ψ , it holds that

$$a \langle \psi, M \rangle \leq b \langle \psi', M \rangle.$$

For further details, we refer the reader to [31].

A stronger variant of this principle, known as the *strong finitary mass transport principle*, requires that the following holds for any measurable subsets A and B of the vertices of M : if each vertex of A has at least a neighbors in B and each vertex of B has at most b neighbors in A , then

$$a\mu(A) \leq b\mu(B)$$

where μ is the probability measure of M . Note that the assertion of the finitary mass transport principle requires this inequality to hold only for first order definable subsets of vertices. The strong finitary mass transport principle is satisfied by any finite graph when viewed as a modeling but it need not hold for modelings in general. In particular, the existence of a modeling limit of a first order convergent sequence of graphs does not a priori imply the existence of a modeling limit satisfying the strong finitary mass transport principle. The importance of the strong finitary mass transport principle comes from its relation to graphings, which are limit representations of Benjamini-Schramm convergent sequences of bounded degree graphs: a modeling limit of a first order convergent sequence

of bounded degree graphs is a limit graphing of the sequence if and only if M satisfies the strong finitary mass transport principle.

The proof of Theorem 2 follows from the following result concerning *rooted k -trees*, which are defined in a recursive way as follows. Any transitive tournament with at most k vertices is a rooted k -tree, and if G is a rooted k -tree and vertices v_1, \dots, v_k form a tournament, then the graph obtained from G by adding a new vertex v and adding an edge directed from v to v_i for every $i \in [k]$ is also a rooted k -tree. Observe that every rooted k -tree is an acyclic orientation of a k -tree (the converse need not be true). We will consider rooted k -trees with edges colored with two colors, which we will refer to as *2-edge-colored rooted k -trees*.

Theorem 3. *Fix a positive integer k . Every first order convergent sequence of 2-edge-colored rooted k -trees has a modeling limit satisfying the strong finitary mass transport principle.*

Since the tree-width of a graph G is the minimum k such that an orientation of G is a subgraph of a rooted k -tree, G can be considered as a 2-edge-colored rooted k -tree, where the coloring interprets the existence of edges in G . Hence, Theorem 2 follows immediately from Theorem 3.

Acknowledgements. This work received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 648509). The second and third authors were also supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University. This publication reflects only its authors’ view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.

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Loose Cores and Cycles in Random Hypergraphs

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Abstract. The *loose core* in hypergraphs is a structure inspired by loose cycles which mirrors the close relationship between 2-cores and cycles in graphs. We prove that the order of the loose core undergoes a phase transition at a certain critical threshold in the r -uniform binomial random hypergraph $H^r(n, p)$ for every $r \geq 3$. We also determine the asymptotic number of vertices and edges in the loose core of $H^r(n, p)$. Furthermore we obtain an improved upper bound on the length of the longest loose cycle in $H^r(n, p)$.

Keywords: Random hypergraphs · Loose cores · Loose cycles · Factor graphs · Peeling processes

1 Motivation and Main Results

The k -core of a graph G , defined as the maximal subgraph of minimum degree at least k , has been studied extensively in the literature (e.g. [3, 6, 8, 9]). In the binomial random graph $G(n, p)$, whp¹ the k -core is equal to the largest k -connected subgraph for each $k \geq 3$, and therefore it may be seen as a natural generalisation of the largest component. Among many other applications, cores can be used to study cycles, since any cycle must lie within the 2-core. In fact, the best known upper bounds on the length of the longest cycle in a random graph derive from a careful analysis of the 2-core (e.g. [4]).

There are many different ways of generalising the concept of a k -core to hypergraphs; some results for these cores can be found e.g. in papers by Molloy [7] and Kim [5]. However, in the case $k = 2$, all k -cores which have been studied so far do not fully capture the nice connection between the 2-core and cycles in graphs.

One of the most natural concepts of cycles in hypergraphs is *loose cycles*. Inspired by the substantial body of research on loose cycles, in this paper we introduce the *loose core*, a structure which does indeed capture the connection with loose cycles.

¹ With probability tending to one as $n \rightarrow \infty$.

Definition 1 (Loose cycle). A loose cycle of length ℓ in an r -uniform hypergraph is a sequence of vertices $v_1, \dots, v_{\ell(r-1)+1}$, all of which are distinct except that $v_{\ell(r-1)+1} = v_1$, and a sequence of edges e_1, \dots, e_ℓ , where $e_i = \{v_{(i-1)(r-1)+1}, \dots, v_{(i-1)(r-1)+r}\}$ for $i \in [\ell] := \{1, \dots, \ell\}$.

Definition 2 (Loose core). The loose core of an r -uniform hypergraph H is the unique maximal subhypergraph H' of H such that H' contains no isolated vertices and such that every $e \in E(H')$ contains at least two vertices which have degree at least two in H' .

Observe that a loose cycle must be contained in the loose core.

In this extended abstract we determine the asymptotic number of vertices and edges in the loose core (see Theorem 1) and derive an improved upper bound on the length of the longest loose cycle (see Theorem 2) in an r -uniform binomial random hypergraph. The previous best known upper bounds come from considering either the first moment (see [1]) or isolated vertices (see [2]).

Throughout this extended abstract we fix $d > 0$ and $r \geq 3$ and also fix the following further parameters. Let $p = p(r, n) := \frac{d}{\binom{n-1}{r-1}}$. In addition we define a function $F : [0, \infty) \rightarrow \mathbb{R}$ by setting $F(x) = F_{r,d}(x) := \exp(-d(1-x^{r-1}))$ and let $\rho_* = \rho_*(r, d)$ be the largest solution of the fixed-point equation $1 - \rho = F(1 - \rho)$. Define $\hat{\rho}_* := 1 - (1 - \rho_*)^{r-1}$ and let

$$\alpha = \alpha(r, d) := \rho_* (1 - d(r-1)(1 - \rho_*)^{r-1}),$$

$$\beta = \beta(r, d) := \frac{d}{r} (1 - (1 - \rho_*)^r - r\rho_*(1 - \rho_*)^{r-1}),$$

and

$$\gamma = \gamma(r, d) := 1 - \exp(-d\hat{\rho}_*) - d\hat{\rho}_* \exp(-d\hat{\rho}_*).$$

Let $H^r(n, p)$ denote the r -uniform binomial random hypergraph on vertex set $[n]$ in which each set of r distinct vertices forms an edge with probability p independently. Let $v(C_H)$ and $e(C_H)$ denote the number of vertices and edges in the loose core C_H of $H = H^r(n, p)$. Our first result concerns the asymptotic behaviour of $v(C_H)$ and $e(C_H)$.

Theorem 1. Let $H = H^r(n, p)$. Then whp

$$v(C_H) = (\alpha + o(1))n \quad \text{and} \quad e(C_H) = (\beta + o(1))n.$$

We can show that $d^* := 1/(r-1)$ is a threshold at which the solution set of $1 - \rho = F(1 - \rho)$ changes its behaviour from only containing 0 to containing a unique positive solution and therefore it is also a threshold for the existence of a loose core of linear size.

Our second main result is an improved upper bound on the length L_C of the longest loose cycle in $H^r(n, p)$.

Theorem 2. Let $H = H^r(n, p)$. Then whp $L_C \leq (\min\{\beta, \gamma\} + o(1))n$. In particular, if $\varepsilon > 0$ is a constant and $p = \frac{1+\varepsilon}{(r-1)\binom{n-1}{r-1}}$, then whp

$$\left(\frac{\varepsilon^2}{4(r-1)^2} + O(\varepsilon^3)\right)n \leq L_C \leq \left(\frac{2\varepsilon^2}{(r-1)^2} + O(\varepsilon^3)\right)n.$$

2 Sketch Proofs of Theorems 1 and 2

Instead of analysing the hypergraph $H^r(n, p)$ directly, it is more convenient to study its natural representation as a factor graph. Given a hypergraph H , the *factor graph* $G = G(H)$ of H is a bipartite graph on vertex classes $\mathcal{V} := V(H)$ and $\mathcal{F} := E(H)$, where the vertices of G are the vertices and edges of H (which we will call *variable* and *factor nodes*, respectively), and the edges of G represent incidences. Let

$$G^r(n, p) := G(H^r(n, p)),$$

i.e. the factor graph of the r -uniform binomial random hypergraph $H^r(n, p)$.

Given the loose core C_H of a hypergraph H , the vertices of degree one may be seen as passengers, not playing an active role in helping to fulfil the conditions of Definition 2. Therefore it is useful to study the (non-uniform) hypergraph obtained by deleting all vertices of degree 1 in C_H , but not the edges in which they are contained; we call this the *reduced core* of H . This structure has an easy description in the factor graph setting: The *reduced core* $R = R_G$ of a factor graph G is defined as the maximal subgraph of G with no nodes of degree 1. Note that the reduced core is very similar to the 2-core of G —the only difference is that we do not delete isolated nodes, so all original nodes are still present, which will be convenient when describing the degree distribution in R_G .

Critically, it is easy to reconstruct the loose core of $H^r(n, p)$ from the reduced core R of the factor graph $G^r(n, p)$ by moving to the corresponding (non-uniform multi-)hypergraph, deleting any isolated vertices and empty edges, then adding distinct vertices into each remaining edge until the hypergraph is r -uniform. Therefore our main results are implied by the following theorem about the reduced core R_G of the factor graph $G = G^r(n, p)$.

For a non-negative real number λ , let us denote by $\widetilde{\text{Po}}$ the distribution that is identical to the Po distribution except that values of 1 are replaced by 0. We define the $\widetilde{\text{Bi}}$ distribution analogously.

For each $j \in \mathbb{N}$, let ξ_j and $\hat{\xi}_j$ be the proportion of variable nodes and factor nodes of $G = G^r(n, p)$ respectively which have degree j in the reduced core R_G .

Theorem 3. *There exists a function $\varepsilon = \varepsilon(n) = o(1)$ such that whp for any constant $j \in \mathbb{N}$ we have*

$$\xi_j = \mathbb{P}(\widetilde{\text{Po}}(d\hat{\rho}_*) = j) \pm \varepsilon \quad \text{and} \quad \hat{\xi}_j = \mathbb{P}(\widetilde{\text{Bi}}(r, \rho_*) = j) \pm \varepsilon.$$

We prove Theorem 3 in Sect. 3.

Proof (Sketch proof of Theorem 1). By Theorem 3 we know the number of variable nodes in the reduced core R_G of $G = G^r(n, p)$. Any variable node of degree $j \geq 2$ in R_G also has degree j in the loose core of H . However, when moving from R_G to C_H certain isolated vertices receive degree one. Since we know the degree distribution of factor nodes in R_G it is easy to calculate how often this occurs, which proves the statement on the number of vertices in the loose core of H . The second statement is proven similarly.

Proof (Sketch proof of Theorem 2). The length of the longest loose cycle is bounded both by the number of variable nodes and the number of factor nodes which are not isolated in R_G . The upper bound then follows from Theorem 1. The lower bound follows from a result of [1] on loose *paths* together with a sprinkling argument.

3 Reduced Core: Proof of Theorem 3

In order to prove Theorem 3 we consider the obvious adaptation of the standard peeling process which gives the reduced core rather than the 2-core. In every round we check whether the factor graph has any nodes of degree one and delete edges incident to such nodes. We say that we *disable* a node if we delete its incident edges. For $\ell \geq 0$, let G_ℓ be the graph obtained from $G = G^r(n, p)$ after ℓ rounds of this process.

We recall the definition of ξ_j and $\hat{\xi}_j$ before Theorem 3 and observe that

$$\xi_j := \lim_{\ell \rightarrow \infty} \xi_j^{(\ell)} \quad \text{and} \quad \hat{\xi}_j := \lim_{\ell \rightarrow \infty} \hat{\xi}_j^{(\ell)},$$

where $\xi_j^{(\ell)}, \hat{\xi}_j^{(\ell)}$ are the proportions of variable nodes and factor nodes respectively which have degree j in G_ℓ for $\ell \in \mathbb{N}$. Theorem 3 follows immediately from the following two lemmas. The first describes the asymptotic distribution of $\xi_j^{(\ell)}$ and $\hat{\xi}_j^{(\ell)}$ for large ℓ .

Lemma 1. *There exist an integer $\ell = \ell(n) \in \mathbb{N}$ and a real number $\varepsilon_1 = \varepsilon_1(n) = o(1)$ such that whp, for any constant $j \in \mathbb{N}$*

$$\xi_j^{(\ell)} = \mathbb{P}(\widetilde{\text{Po}}(d\hat{\rho}_*) = j) \pm \varepsilon_1 \quad \text{and} \quad \hat{\xi}_j^{(\ell)} = \mathbb{P}(\widetilde{\text{Bi}}(r, \rho_*) = j) \pm \varepsilon_1.$$

The second lemma states that $\xi_j^{(\ell)}$ and $\hat{\xi}_j^{(\ell)}$ approximate ξ_j and $\hat{\xi}_j$, respectively.

Lemma 2. *Let ℓ, ε_1 be as in Lemma 1 and set $\varepsilon_2 := \sqrt{\varepsilon_1}$. Then, whp, for any constant $j \in \mathbb{N}$ we have $\xi_j = \xi_j^{(\ell)} \pm \varepsilon_2$ and $\hat{\xi}_j = \hat{\xi}_j^{(\ell)} \pm \frac{2\varepsilon_2 r}{d}$.*

4 CoreConstruct: Proofs of Lemmas 1 and 2

To prove Lemma 1, we introduce a procedure called **CoreConstruct**, which is related to the peeling process. We first need some more notation.

Definition 3. *Let G be a factor graph with variable node set \mathcal{V} and factor node set \mathcal{F} . We denote by $d_G(u, v)$ the distance between two nodes $u, v \in \mathcal{V} \cup \mathcal{F}$, i.e. the number of edges in a shortest path between them. For each $\ell \in \mathbb{N}$ and each $w \in \mathcal{V} \cup \mathcal{F}$, we define*

$$D_\ell(w) := \{u \in \mathcal{V} \cup \mathcal{F} : d_G(w, u) = \ell\} \quad \text{and} \quad d_\ell(w) := |D_\ell(w)|.$$

Let $D_{\leq \ell}(w) = \bigcup_{i=0}^\ell D_i(w)$ and $N_{\leq \ell}(w) := G[D_{\leq \ell}(w)]$, i.e. the subgraph of G induced on $D_{\leq \ell}(w)$.

Given a factor graph G on node set $\mathcal{V} \cup \mathcal{F}$ and a node $w \in \mathcal{V} \cup \mathcal{F}$, we consider the factor graph as being rooted at w . In particular, neighbours of a node v which are at distance $d_G(v, w) + 1$ from w are called *children* of v . Starting at distance $\ell \in \mathbb{N}$ and moving up towards the root w , we recursively delete any node with no (remaining) children; Algorithm 1 gives a formal description of this procedure. We will denote by $D_{\ell-i}^*(w)$ the set of nodes in $D_{\ell-i}(w)$ which survive round i and let $d_i^*(w) := |D_i^*(w)|$.

Algorithm 1: CoreConstruct

Input: Integer $\ell \in \mathbb{N}$, node $w \in \mathcal{V} \cup \mathcal{F}$, factor graph $N_{\leq \ell+1}(w)$

Output: $d_1^*(w)$

- 1 $D_{\ell+1}^*(w) = D_{\ell+1}(w)$
 - 2 **for** $1 \leq i \leq \ell$ **do**
 - 3 $D_{\ell-i+1}^*(w) \leftarrow D_{\ell-i+1}(w) \setminus \left\{ v : N(v) \cap D_{\ell-i+2}^*(w) = \emptyset \right\}$
 - 4 $d_{\ell-i+1}^*(w) \leftarrow |D_{\ell-i+1}^*(w)|$
-

CoreConstruct is intended to model the effect of the peeling process on the degree of w after ℓ steps. Although it does not mirror the peeling process precisely, we obtain the following important relation.

Lemma 3. *Let $\ell \geq 1$ and $w \in \mathcal{V} \cup \mathcal{F}$. If there are no cycles in $N_{\leq \ell+1}(w)$, then the output $d_1^*(w)$ of **CoreConstruct** with input ℓ, w and $N_{\leq \ell+1}(w)$ satisfies $d_{G_\ell}(w) = d_1^*(w)$ if $d_1^*(w) \neq 1$ and $d_{G_\ell}(w) \leq d_1^*(w)$ if $d_1^*(w) = 1$.*

We next describe the survival probabilities of internal (i.e. non-root) variable and factor nodes in each round of **CoreConstruct**. Recall that for any $i \in [\ell]$ the set $D_{\ell+1-i}^*(w)$ consists of nodes within $D_{\ell+1-i}(w)$ which survive the i -th round of **CoreConstruct**. We define the recursions $\rho_0 = 1$, $\hat{\rho}_t = \mathbb{P}(\text{Bi}(r-1, \rho_{t-1}) \geq 1)$ and $\rho_t = \mathbb{P}(\text{Po}(d\hat{\rho}_t) \geq 1)$.

Lemma 4. *Let $w \in \mathcal{V} \cup \mathcal{F}$ and ℓ be odd if $w \in \mathcal{V}$ or even if $w \in \mathcal{F}$. Let $t \in \mathbb{N}$ with $0 \leq t \leq \frac{\ell+1}{2}$ be given. If $N_{\leq \ell+1}(w)$ has no cycles, then for each $u \in D_{\ell+1-2t}(w)$ independently of each other and for each $a \in D_{\ell-2t}(w)$ independently of each other*

$$\mathbb{P}[u \in D_{\ell+1-2t}^*(w)] = \rho_t + o(1) \quad \text{and} \quad \mathbb{P}[a \in D_{\ell-2t}^*(w)] = \hat{\rho}_{t+1} + o(1).$$

A consequence of Lemma 4 is that, if ℓ is large, the distribution of the number of children of the root w which survive **CoreConstruct** is almost identical to one of the claimed distributions in Theorem 3 (depending on whether w is a variable or factor node). Thus we can asymptotically determine the expected degree distribution in G_ℓ .

Corollary 1. *There exist $\varepsilon = o(1)$ and $\ell = \ell(\varepsilon)$ such that for all $j \in \mathbb{N}$,*

$$\mathbb{E} \left(\xi_j^{(\ell)} \right) = \mathbb{P} \left(\widetilde{\text{Po}}(d\hat{\rho}_*) = j \right) \pm \varepsilon \quad \text{and} \quad \mathbb{E} \left(\hat{\xi}_j^{(\ell)} \right) = \mathbb{P} \left(\widetilde{\text{Bi}}(r, \rho_*) = j \right) \pm \varepsilon.$$

We achieve concentration of $\xi_j^{(\ell)}$ and $\hat{\xi}_j^{(\ell)}$ around their expectations using an Azuma-Hoeffding inequality applied to a variant of the vertex-exposure martingale. To ensure a sufficiently strong Lipschitz property, we apply this to the probability space conditioned on having maximum degree $\log n$, proving Lemma 1.

In order to prove Lemma 2, we also need to show that after some large number ℓ rounds of the peeling process on $G = G^r(n, p)$ have been completed, whp very few nodes will be disabled in subsequent rounds (at most $\varepsilon_2 n$), thus proving Lemma 2.

Definition 4 (Change process). *We will track the changes that the peeling process makes after reaching round ℓ by revealing information a little at a time as follows. Reveal the degrees of all nodes. While there are still nodes of degree one, pick one such node x_0 . Reveal its neighbour x_1 , delete the edge $x_0 x_1$ and update the degrees of x_0, x_1 . If x_1 now has degree one, continue from x_1 ; otherwise find a new x_0 (if there is one).*

By Lemma 1, whp at most $\varepsilon_1^{3/4} n$ vertices will change in round $\ell + 1$ of the peeling process. Each stage of the change process (from a new vertex x_0) stops if we reach a vertex of degree at least 3. Each time we reveal a neighbour, the probability that it has degree at least 3 is bounded away from 0 by Lemma 1. This can be modelled by an abstract subcritical branching process which provides an upper coupling on a stage of the change process. In particular, since whp each of the at most $\varepsilon_1^{3/4} n$ stages of the change process will die out quickly, whp the total number of vertices which change after round ℓ is at most $\sqrt{\varepsilon_1} n = \varepsilon_2 n$. Some elementary calculations complete the proof of Lemma 2.

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Building a Larger Class of Graphs for Efficient Reconfiguration of Vertex Colouring

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Abstract. We introduce a class of graphs called OAT graphs that generalizes P_4 -sparse, chordal bipartite, and compact graphs. We prove that if G is a k -colourable OAT graph then G is $(k+1)$ -mixing and the $(k+1)$ -recolouring diameter of G is $O(n^2)$, unifying and extending several results in the literature. We also identify a mistake in the literature and leave as an open problem whether a k -colourable P_5 -free graph is $(k+1)$ -mixing.

Keywords: Reconfiguration · Vertex colouring

1 Introduction

All graphs considered are finite and simple. We use n to denote the number of vertices of a graph. See [4] for standard graph-theoretic notation.

For a graph G , the *reconfiguration graph* of the k -colourings, $\mathcal{R}_k(G)$, is the graph whose vertices are the k -colourings of G and two colourings are joined by an edge in $\mathcal{R}_k(G)$ if they differ in colour on exactly one vertex. A graph G is *k -mixing* if $\mathcal{R}_k(G)$ is connected. In this case the *k -recolouring diameter* of G is defined to be the diameter of $\mathcal{R}_k(G)$. We say that a graph G is *quadratically k -mixing* if G is k -mixing and the diameter of $\mathcal{R}_k(G)$ is $O(n^2)$.

Bonamy et al. [2] asked whether a k -colourable perfect graph G is quadratically $(k+1)$ -mixing. One cannot hope for a smaller diameter since the 3-recolouring diameter of P_n , the path on n vertices, is $\Omega(n^2)$ [2]. Bonamy and Bousquet [1] answered this question in the negative, using an example of Cereceda, van den Heuvel, and Johnson [3], who gave an infinite family of bipartite graphs that are not k -mixing. Feghali and Fiala [5] found an infinite family of weakly chordal graphs that are k -colourable but not $(k+1)$ -mixing. It is known that a k -colourable graph G is quadratically $(k+1)$ -mixing if G is chordal, chordal bipartite [2], or P_4 -free [1]. Feghali and Fiala introduced a subclass of weakly chordal graphs called *compact* graphs that generalizes the class of co-chordal graphs, and proved that a k -colourable compact graph G is quadratically $(k+1)$ -mixing [5].

We introduce a new class of graphs which we call *OAT graphs* that generalizes the classes of chordal bipartite, P_4 -free, and compact graphs. Interestingly, not

all OAT graphs are perfect, see Fig. 1. We prove that a k -colourable OAT graph G is quadratically $(k + 1)$ -mixing, thus unifying and extending several results in the literature. We note that our proof leads to a polynomial time algorithm to find a path of length $O(n^2)$ between two nodes in $\mathcal{R}_{k+1}(G)$.

The third author’s Master’s thesis [7] includes a polynomial time algorithm to recognize OAT graphs and other details not in this paper.

Definition 1. *A graph G is an OAT graph if it can be constructed from single vertex graphs with a finite sequence of the following four operations. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex-disjoint OAT graphs.*

1. *Take the disjoint union of G_1 and G_2 , defined as $(V_1 \cup V_2, E_1 \cup E_2)$.*
2. *Take the join of G_1 and G_2 , defined as $(V_1 \cup V_2, E_1 \cup E_2 \cup \{xy \mid x \in V_1, y \in V_2\})$.*
3. *Add a vertex $u \notin V_1$ comparable to vertex $v \in V_1$, defined as $(V_1 \cup \{u\}, E_1 \cup \{ux \mid x \in X\})$, where $X \subseteq N(v)$.*
4. *Attach a complete graph $Q = (V_Q, E_Q)$ to a vertex v of G_1 , defined as $(V_1 \cup V_Q, E_1 \cup E_Q \cup \{qv \mid q \in V_Q\})$.*

The following is the main result of this paper.

Theorem 1. *Let G be an OAT graph and let $k \geq \chi(G)$. Then G is $(k+1)$ -mixing and the $(k + 1)$ -recolouring diameter of G is at most $4n^2$.*

Feghali and Fiala asked whether a k -colourable $(P_5, \text{co-}P_5, C_5)$ -free graph G is quadratically $(k + 1)$ -mixing. We give a positive answer for the subclass of P_4 -sparse graphs, which are exactly the $(P_5, \text{co-}P_5, C_5, P, \text{co-}P, \text{fork}, \text{co-fork})$ -free graphs [6]: every P_4 -sparse graph is an OAT graph [7].

Next we remark on the connectivity and diameter of $\mathcal{R}_{k+1}(G)$ for a k -colourable P_t -free graph G for $t \geq 5$. In the case $t \geq 6$, the bipartite graph B_k given by Cereceda, van den Heuvel, and Johnson [3] is not k -mixing, but is P_6 -free for every $k \geq 3$.

In the case of P_5 -free graphs, Bonamy and Bousquet [1] thought they had a 4-colourable P_5 -free graph G with an isolated vertex in $\mathcal{R}_5(G)$. This graph is in fact not P_5 -free as illustrated in Fig. 2. In addition, Bonamy and Bousquet thought they had a family of P_5 -free graphs $\{G_k \mid k \geq 3\}$ where G_k is $(k + 1)$ -colourable and where $\mathcal{R}_{2k}(G)$ has an isolated vertex. The graph G_k also contains an induced P_5 for every $k \geq 3$. We leave as an open problem whether a k -colourable P_5 -free graph is $(k + 1)$ -mixing.

2 Recolouring OAT Graphs

In this section, we prove Theorem 1. Our strategy uses a canonical $\chi(G)$ -colouring as a central vertex in the reconfiguration graph $\mathcal{R}_{k+1}(G)$. For any two colourings α and β in $\mathcal{R}_{k+1}(G)$, we show how to transform both into the canonical $\chi(G)$ -colouring γ by recolouring each vertex at most $2n$ times, so using at most $2n^2$ recolouring steps. This is asymptotically optimal since the path P_n is a 2-colourable OAT graph for which the 3-recolouring diameter is $\Omega(n^2)$ [2].

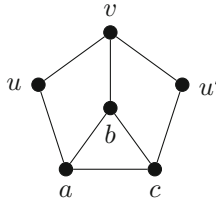


Fig. 1. A graph that is not perfect, but is an OAT graph. (Start with b , add v and $\{a, c\}$ via attaching complete graphs, then add u, u' comparable to b .)

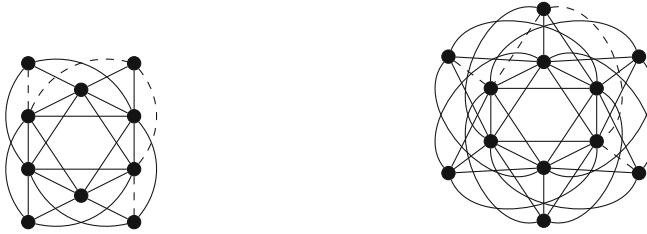


Fig. 2. Graphs mistaken to be P_5 -free [1]. An induced P_5 is marked with dashed edges.

Let \mathcal{S} be a set of k colours and let $\alpha : V(G) \rightarrow \mathcal{S}$ be a k -colouring of G . The set \mathcal{S} is called the *set of permissible colours* for α and we denote \mathcal{S} by $\mathcal{S}(\alpha)$ when α is not clear from the context. We also call α an \mathcal{S} -colouring when we want to emphasize its set of permissible colours.

Let $\mathcal{R}_{\mathcal{S}}(G)$ be the graph whose vertices are the \mathcal{S} -colourings of G such that two vertices of $\mathcal{R}_{\mathcal{S}}(G)$ are adjacent if and only if they differ by colour on exactly one vertex. If $|\mathcal{S}| = k$ then $\mathcal{R}_{\mathcal{S}}(G)$ is isomorphic to $\mathcal{R}_k(G)$.

Let $\mathcal{C}(\alpha)$ be the set of colours c that *appear* in α i.e., $\alpha(v) = c$ for some vertex $v \in V(G)$. Thus $\mathcal{C}(\alpha) \subseteq \mathcal{S}(\alpha)$ but they need not be equal. We say that a colouring α of G can be *transformed* into a colouring β of G in $\mathcal{R}_{\mathcal{S}}(G)$ if there is a path from α to β in $\mathcal{R}_{\mathcal{S}}(G)$. Let H be a subgraph of G . Let n_H denote the number of vertices of H . The *projection* of α onto H is the colouring $\alpha_H : V(H) \rightarrow \mathcal{S}(\alpha_H)$ where $\alpha_H(v) = \alpha(v)$ for all $v \in V(H)$.

A *build-sequence* of an OAT graph G is a finite sequence of the four defined operations that constructs G ; we assume in the following that one such build-sequence σ is fixed. We use σ to define a canonical $\chi(G)$ -colouring, which is unique if we assume that vertices are enumerated in some fixed arbitrary order.

Definition 2. Let G be an OAT graph and let \mathcal{C} be an ordered set of $\chi(G)$ colours. The canonical χ -colouring of G with respect to \mathcal{C} is the $\chi(G)$ -colouring of G constructed recursively as follows.

1. If G is a single vertex v , then v is coloured with the first colour of \mathcal{C} .

2. If G is the disjoint union of L and R , then take a canonical χ -colouring of L with respect to the first $\chi(L)$ colours of \mathcal{C} and a canonical χ -colouring of R with respect to the first $\chi(R)$ colours of \mathcal{C} .
3. If G is the join of L and R , then take a canonical χ -colouring of L with respect to the first $\chi(L)$ colours in \mathcal{C} and take a canonical χ -colouring of R with respect to the next $\chi(R)$ colours in \mathcal{C} ($|\mathcal{C}| = \chi(G) = \chi(L) + \chi(R)$).
4. If G is constructed by adding a comparable vertex u to a vertex v of a graph H , then take a canonical χ -colouring of H with respect to \mathcal{C} and colour u the same colour as v .
5. If G is constructed by attaching a complete graph Q to a vertex v of a graph H , then take a canonical χ -colouring of H with respect to the first $\chi(H)$ colours of \mathcal{C} . Let the induced order of vertices of Q be $\{q_1, q_2, \dots\}$. Colour the vertices q_1, q_2, \dots of Q in order with the first $|Q|$ colours of $\mathcal{C} \setminus c$ where c is the colour given to v in the canonical χ -colouring of H .

Our proofs use induction to recolour the subgraphs that build up the OAT graph in a fixed construction. There are generally two steps to these proofs. The first step is to recolour the vertices so that the partition of vertices into colour classes is the same as the canonical χ -colouring. The second step is to rename these colours so that the correct colour appears on the correct colour class. For this, we rely on the Renaming Lemma. The Renaming Lemma is an adaptation of an idea that is used in token swapping and was also rediscovered by Bonamy and Bousquet [1] who rephrased the lemma in terms of recolouring complete graphs. Our statement is expressed more generally.

Lemma 1 (Renaming Lemma [1]). *If α and β are two k -colourings of G that induce the same partition of vertices into colour classes, and if \mathcal{S} is a set of at least $k + 1$ colours such that the permissible colours $\mathcal{S}(\alpha)$ and $\mathcal{S}(\beta)$ are each a subset of \mathcal{S} , then α can be transformed into β in $\mathcal{R}_{\mathcal{S}}(G)$ by recolouring each vertex at most 2 times.*

The following lemma is used to prove Theorem 1.

Lemma 2. *Let G be an OAT graph. Let \mathcal{S} be a set of $k + 1$ colours where $k \geq \chi(G)$ and let \mathcal{C} be an ordered set of $\chi(G)$ colours such that $\mathcal{C} \subseteq \mathcal{S}$. Then any colouring α in $\mathcal{R}_{\mathcal{S}}(G)$ can be transformed into the canonical χ -colouring γ of G with respect to \mathcal{C} by recolouring each vertex at most $2n$ times.*

Proof. The proof is by induction on the number of vertices n of G . Clearly α can be recoloured into γ using at most 1 recolouring if $n = 1$, so assume G was constructed with one of the four operations defining OAT graphs.

Case 1. Suppose G is constructed as the disjoint union of the graphs L and R . Note that L and R can be recoloured independently since there are no edges between L and R . Let α_L be the projection of α onto L and define $\mathcal{S}(\alpha_L) = \mathcal{S}$ to be its set of permissible colours. Clearly $\chi(G) = \max\{\chi(L), \chi(R)\}$, so α_L is an \mathcal{S} -colouring of L with $|\mathcal{S}| \geq \chi(L) + 1$. By the induction hypothesis, we can transform α_L within $\mathcal{R}_{\mathcal{S}}(L)$ into the canonical χ -colouring of L with respect to

the first $\chi(L)$ colours of \mathcal{C} by recolouring each vertex of L at most $2n_L < 2n$ times. This reconfiguration sequence appears in $\mathcal{R}_S(G)$ since there are no edges between L and R . Similarly reconfigure the projection α_R of α onto R into the canonical χ -colouring of R by recolouring each vertex of R at most $2n_R < 2n$ times. Taking these two reconfiguration sequences consecutively gives the desired reconfiguration sequence.

Case 2. Suppose G is constructed as the join of the graphs L and R . Let α_L and α_R denote the projections of α onto L and R , respectively. Note that $\mathcal{C}(\alpha_L)$ is disjoint from $\mathcal{C}(\alpha_R)$ since there are all possible edges between L and R . We obtain four subcases by combining: ((1) $|\mathcal{C}(\alpha_L)| = \chi(L)$ OR (2) $|\mathcal{C}(\alpha_L)| > \chi(L)$) AND ((A) $|\mathcal{C}(\alpha_R)| = \chi(R)$ OR (B) $|\mathcal{C}(\alpha_R)| > \chi(R)$). Assume for now that (1B) does not happen. Recolour L first, using permissible colours $\mathcal{S}(\alpha_L)$ chosen as follows. If $|\mathcal{C}(\alpha_L)| > \chi(L)$, then set $\mathcal{S}(\alpha_L) = \mathcal{C}(\alpha_L)$. Otherwise (since (1B) does not happen) $|\mathcal{S}| > \chi(G) = \chi(L) + \chi(R) = |\mathcal{C}(\alpha_L)| + |\mathcal{C}(\alpha_R)|$. So some colour $c \in \mathcal{S}$ does not appear in α ; define $\mathcal{S}(\alpha_L) = \mathcal{C}(\alpha_L) \cup \{c\}$. Either way, α_L is an $\mathcal{S}(\alpha_L)$ -colouring of L and $|\mathcal{S}(\alpha_L)| > \chi(L)$. By the induction hypothesis, α_L can be transformed within $\mathcal{R}_{\mathcal{S}(\alpha_L)}(L)$ into the canonical χ -colouring of L with respect to the first $\chi(L)$ colours of $\mathcal{C}(\alpha_L)$ by recolouring each vertex at most $2n_L$ times. Furthermore, by our choice of $\mathcal{S}(\alpha_L)$, none of the intermediate colourings of L uses a colour from $\mathcal{C}(\alpha_R)$ so the same reconfiguration sequence appears within $\mathcal{R}_S(G)$.

Since the canonical χ -colouring of L uses $\chi(L)$ colours, some colour $c' \in \mathcal{S}(\alpha_L)$ does not appear in the current colouring of G . Define $\mathcal{S}(\alpha_R) = \mathcal{C}(\alpha_R) \cup \{c'\}$ as the set of permissible colours for α_R . Then α_R is an $\mathcal{S}(\alpha_R)$ -colouring of R and $|\mathcal{S}(\alpha_R)| > \chi(R)$. By the induction hypothesis, α_R can be transformed within $\mathcal{R}_{\mathcal{S}(\alpha_R)}(R)$ into the canonical χ -colouring of R with respect the first $\chi(R)$ colours of $\mathcal{C}(\alpha_R)$ by recolouring each vertex at most $2n_R$ times. The same reconfiguration sequence appears within $\mathcal{R}_S(G)$ since $\mathcal{S}(\alpha_R)$ is disjoint from the colours that appear on the vertices of L .

So we now have a colouring α' of G such that α'_L is a canonical $\chi(L)$ -colouring of L and α'_R is a canonical $\chi(R)$ -colouring of R . (If Case (1B) happens, then we obtain α' in a symmetric fashion by recolouring R before recolouring L .) Colouring α' and the canonical $\chi(G)$ -colouring γ of G must partition the vertices of G into the same colour classes. By Lemma 1, we can transform α' into γ by recolouring each vertex at most twice. Therefore we can transform α into γ by recolouring each vertex of G at most $2 \max\{n_L, n_R\} + 2 \leq 2n$ times.

Case 3. Suppose G is constructed by adding a vertex u comparable to a vertex v of the OAT graph $H = G \setminus \{u\}$. First recolour u the same colour as v . This is possible since u and v are non-adjacent and $N(u) \subseteq N(v)$. Let α_H be the projection of α onto H and define $\mathcal{S}(\alpha_H) = \mathcal{S}$ to be its set of permissible colours. Clearly $\chi(H) = \chi(G)$ and so α_H is an \mathcal{S} -colouring with $|\mathcal{S}| > \chi(H)$. By the induction hypothesis, α_H can be transformed within $\mathcal{R}_S(H)$ into the canonical $\chi(H)$ -colouring with respect to \mathcal{C} by recolouring each vertex of H at most $2n_H$ times. To extend this reconfiguration sequence to $\mathcal{R}_S(G)$, whenever v

is recoloured, recolour u the same colour. By definition, this colouring of G is the canonical χ -colouring of G with respect to \mathcal{C} . Each vertex of H was recoloured at most $2n_H < 2n$ times and u was recoloured at most $2n_H + 1 < 2n$ times.

Case 4. Suppose G is constructed by attaching a complete graph Q to some vertex v of an OAT graph H . Let α_H be the projection of α onto H and define $\mathcal{S}(\alpha_H) = \mathcal{S}$ to be its set of permissible colours. Clearly $\chi(H) \leq \chi(G)$ and so α_H is an \mathcal{S} -colouring of H with $|\mathcal{S}| > \chi(H)$. By the induction hypothesis, α_H can be transformed within $\mathcal{R}_{\mathcal{S}}(H)$ into the canonical χ -colouring γ_H of H with respect to the first $\chi(H)$ colours of \mathcal{C} . To extend this reconfiguration sequence to $\mathcal{R}_{\mathcal{S}}(G)$, whenever v is recoloured to some colour c , we may need to first recolour at most one vertex q of Q that is coloured c . Since $\chi(G) = \max\{\chi(H), n_Q + 1\}$ and $|\mathcal{S}| \geq \chi(G) + 1 \geq n_Q + 2$, and each vertex of Q has degree n_Q , there exists some colour c' that does not appear on the neighbourhood of q and is not the colour c . Recolour q with the colour c' and then continue by recolouring v with colour c . Now H is coloured with the canonical $\chi(H)$ -colouring γ_H .

Let $c^* = \gamma_H(v)$ and let α'_Q be the current colouring of Q and define $\mathcal{S}(\alpha'_Q) = \mathcal{S} \setminus \{c^*\}$ to be its set of permissible colours. Recall that the vertices of Q are ordered $\{q_1, q_2, \dots\}$. The canonical χ -colouring of Q with respect to \mathcal{C} is the colouring γ_Q such that q_i is coloured the i th colour of $\mathcal{C} \setminus \{c^*\}$. Since $|\mathcal{S}| \geq n_Q + 2$, then $|\mathcal{S} \setminus \{c^*\}| \geq n_Q + 1$. By the Renaming Lemma (Lemma 1), α'_Q can be transformed within $\mathcal{R}_{\mathcal{S}(\alpha'_Q)}(Q)$ into γ_Q by recolouring each vertex of Q at most twice. Since each vertex of Q is only adjacent to v in H and c^* was never used in this recolouring of Q , this reconfiguration sequence can extend to $\mathcal{R}_{\mathcal{S}}(G)$. Now by definition, the current colouring of G is the canonical χ -colouring of G with respect to \mathcal{C} . Each vertex of H was recoloured at most $2n_H$ times and each vertex of Q was recoloured at most $2n_H + 2 \leq 2n$ times.

We are now ready to prove Theorem 1.

Proof (Proof of Theorem 1). Fix $\mathcal{S} = \{1, 2, \dots, k+1\}$ to be the set of permissible colours used in the colourings of $\mathcal{R}_{k+1}(G)$ and let \mathcal{C} be an ordered set of $\chi(G)$ colours such that $\mathcal{C} \subseteq \mathcal{S}$. Let $\alpha, \beta : V(G) \rightarrow \mathcal{S}$ be two $(k+1)$ -colourings of G . Then by Lemma 2, we can transform both α and β into the canonical χ -colouring γ of G with respect to \mathcal{C} in $\mathcal{R}_{\mathcal{S}}(G)$ by recolouring each vertex at most $2n$ times. Then to transform α to β , follow the sequence from α to γ and then follow the sequence from β to γ in reverse. Therefore $\mathcal{R}_{k+1}(G)$ is connected with diameter at most $4n^2$.

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A Refinement of Cauchy-Schwarz Complexity, with Applications

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Abstract. Let $\Phi := (\phi_i)_{i \in I}$ be a finite collection of linear forms $\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p$. We introduce a 2-parameter refinement of Cauchy-Schwarz (CS) complexity, called *sequential Cauchy-Schwarz complexity*. We prove that if Φ has sequential Cauchy-Schwarz complexity at most (k, ℓ) , then $|\mathbb{E}_{x_1, \dots, x_d \in \mathbb{F}_p^n} \prod_{i \in I} f_i(\phi_i(x_1, \dots, x_d))| \leq \min_{i \in I} \|f_i\|_{U^{k+1}}^{2^{1-\ell}}$ for any 1-bounded functions $f_i : \mathbb{F}_p^n \rightarrow \mathbb{C}$, $i \in I$. For $\ell = 1$, this reduces to CS complexity, but for larger ℓ the two notions differ. For example, let $S_{k,M} := \{z \in [0, p-1]^M : z_1 + \dots + z_M < k\}$, and consider $\Phi_{k,M} := \{\phi_z(x, t_1, \dots, t_M) := x + z_1 t_1 + \dots + z_M t_M \mid z \in S_{k,M}\}$, a multivariable generalization of arithmetic progressions. We show that $\Phi_{k,M}$ has sequential CS complexity at most $(\min(k, M(p-1) + 1) - 2, \ell)$ for some finite ℓ , yet can have CS complexity strictly larger than $\min(k, M(p-1) + 1) - 2$. Moreover, we show that $\Phi_{k,M}$ has True complexity $\min(k, M(p-1) + 1) - 2$.

In [2], we use these results in a new proof of the inverse theorem for \mathbb{F}_p^n .

Keywords: Cauchy-Schwarz complexity · True complexity · Generalized Von Neumann

1 Introduction

This paper is a short version of [1].

Let G be a finite abelian group. Given $A \subset G$, a central problem in additive combinatorics consists in counting the number of solutions of a given system of linear equations inside A . For example, we may be interested in counting the number of (non-trivial) arithmetic progressions of length k inside A . To do so, since the work of Gowers on Szemerédi's theorem [3] it has become a standard technique to consider the functional $A_k(A) := \mathbb{E}_{x,r \in G} 1_A(x) 1_A(x+r) \cdots 1_A(x+$

$(k-1)r$), which gives us the (normalized) count of k -term arithmetic progressions inside A^1 , and to analyze this quantity using the Gowers uniformity norms.

In this paper we focus on the case $G = \mathbb{F}_p^n$ for some prime p . For technical reasons, one often considers more generally the averages $\Lambda((f_i)_{i \in I}) := \mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{i \in [r]} f_i(\phi_i(t_1, \dots, t_d))$ where $(f_i : \mathbb{F}_p^n \rightarrow \mathbb{C})_{i \in I}$ are 1-bounded functions and $(\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i \in I}$ are distinct linear forms. We are then interested in bounding $\Lambda((f_i)_{i \in I})$ in terms of $\min_{i \in [r]} (\|f_i\|_{U^{k+1}})$ where $\|\cdot\|_{U^{k+1}}$ is the Gowers uniformity norm of order $k+1$.² To state the first result in this direction let us begin with the following definition (introduced by Green and Tao in [8])³:

Definition 1 (Cauchy-Schwarz complexity). *Let $(\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i=1}^r$ be a collection of r distinct linear forms in d variables. Fix some $i \in [r]$ and let $k \geq 0$ be an integer. We say that this system has Cauchy-Schwarz complexity (or CS complexity for short) at most k at i if $\{\phi_1, \dots, \phi_e\} \setminus \{\phi_i\}$ can be partitioned into at most $k+1$ classes such that ϕ_i is not contained in the \mathbb{F}_p -linear span of any of these classes. If this holds for all $i \in [r]$, we say that the system has CS complexity at most k .*

It is not difficult to extract from (the arguments used in) [8] the following estimate:

Proposition 1. *Let $(\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i=1}^r$ be a collection of r distinct linear forms in d variables⁴. Suppose that it has CS complexity at most k . Then for every collection of 1-bounded functions $f_i : \mathbb{F}_p^n \rightarrow \mathbb{C}$, $i \in [r]$ we have*

$$|\mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{i \in [r]} f_i(\phi_i(t_1, \dots, t_d))| \leq \min_{i \in [r]} (\|f_i\|_{U^{k+1}}).$$

This result tells us (roughly) that if we control sufficiently well the U^k norms of the functions f_i then we can estimate the densities of various types of linear configurations. Currently we have a much better understanding of the U^k norms for small k . If $k = 1$ this (semi)norm collapses to the absolute value of the mean of the function. For $k = 2$ it can be proved that $\|f\|_{U^2(G)} = \|\widehat{f}\|_{l^4(G)}$ where \widehat{f} is the Fourier transform of f (see [12, (11.3)]). Similarly, we have a better understanding of the U^3 norm than of the U^k norm for $k > 3$, etc., see [4, 7, 9].

Thus, it is of interest to know which degree of Gowers norm is sufficiently large to control a given linear configuration. This question was raised by Gowers

¹ These arguments appear in [3] for the group $\mathbb{Z}/p\mathbb{Z}$, but they easily extend to any group G such that $|G|$ is coprime with $(k-1)!$.

² For a more detailed account on these norms and how to use them we refer the reader to [12, Chapter 11] or [8, Appendix B]. However, this paper includes all necessary results to understand the proof of the main theorem.

³ This notion appeared originally in [8] but we use the name of *Cauchy-Schwarz complexity* that comes from [5, Definition 1.1].

⁴ Note that we can regard the functions ϕ_i as linear functionals from \mathbb{F}_p^{dn} to \mathbb{F}_p^n in the obvious manner for any $n \geq 1$. This assumption will be made throughout the whole paper.

and Wolf in [5], where they introduced the following concept (we state here the adapted version for the group \mathbb{F}_p^n):

Definition 2 (True complexity). Let $\Phi := (\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i=1}^r$ be a collection of r distinct linear forms in d variables. The True complexity of Φ is the smallest k with the following property. For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n \geq 1$ and any system of 1-bounded functions $(f_i : \mathbb{F}_p^n \rightarrow \mathbb{C})_{i \in [r]}$ with $\min_{i \in [r]} (\|f_i\|_{U^{k+1}}) < \delta$ we have

$$|\mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{i \in [r]} f_i(\phi_i(t_1, \dots, t_d))| < \epsilon.$$

It is trivial that the CS complexity of a system of linear equations is at least its True complexity. Surprisingly, it turns out that there are systems of linear forms with True complexity strictly smaller than CS complexity. In [5] Gowers and Wolf conjectured a simple algebraic condition to compute the True complexity of a given linear system, and they proved their conjecture in the case of True complexity 1 and CS complexity 2. We omit the detailed statement of this result, as it is not needed for this paper. Let us briefly mention that for the groups that interest us, i.e. the groups \mathbb{F}_p^n , the full conjecture was eventually proved by Hatami, Hatami and Lovett in [10]. These works relied on the inverse conjecture for the Gowers norms, and thus they did not give effective bounds on ϵ in terms of δ . It was conjectured in [6, Problem 7.8] that the dependence between ϵ and δ cannot be too good (in particular, not polynomial) in general.

The only known result (apart from the ones mentioned) that gives good bounds for this dependence, for certain systems of linear equations, is due to Manners, who proved in [11, Theorem 1.5] that for 6 linear forms in 3 variables, if a system of linear equations has CS complexity 2 and True complexity 1, it is possible to obtain a polynomial dependence between ϵ and δ as in Definition 2. That is, there exists a constant $C > 0$ that does not depend on n such that $\epsilon = \delta^C$. The proof of this uses only the CS inequality together with clever changes of variables.

2 New Results

We begin with the following definition:

Definition 3 (Sequential CS complexity). Let $\Phi := (\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i=1}^r$ be a collection of r distinct linear forms in d variables. Fix some $i \in [r]$. We say that Φ has sequential CS complexity at most (k, ℓ) at i , for some integers $k \geq 0$ and $\ell \geq 1$, if there exists a sequence $j_1, \dots, j_\ell \in [r]$ of indices such that $j_\ell = i$ and for every $e \in \{1, \dots, \ell\}$, the set $\{\phi_1, \dots, \phi_r\} \setminus \{\phi_{j_1}, \dots, \phi_{j_e}\}$ can be partitioned into $k + 1$ classes such that $\{\phi_{j_1}, \dots, \phi_{j_e}\}$ is included in the complement of the linear span of each of these classes. If this happens for every $i \in [r]$ we say that Φ has sequential CS complexity at most (k, ℓ) .

Note that for $\ell = 1$ this definition reduces to Definition 1.

We can now state the main result of this paper:

Theorem 1. *Let $\Phi := (\phi_i : \mathbb{F}_p^d \rightarrow \mathbb{F}_p)_{i=1}^r$ be a collection of r distinct linear forms in d variables. Suppose that this system has sequential CS complexity at most (k, ℓ) for some integers $k \geq 0$ and $\ell \geq 1$. Then for every collection of 1-bounded functions $(f_i : \mathbb{F}_p^n \rightarrow \mathbb{C})_{i \in [r]}$ we have*

$$|\mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{i \in [r]} f_i(\phi_i(t_1, \dots, t_d))| \leq \min_{i \in [r]} (\|f_i\|_{U^{k+1}}^{2^{1-\ell}}).$$

As an application, we prove that this improves Proposition 1 in the sense that there are systems of linear forms such that their CS complexity is k' and their sequential CS complexity is at most (k, ℓ) with $k < k'$. Let $S_{k,M} := \{z \in [0, p-1]^M : z_1 + \dots + z_M < k\}$ (where the sum is in \mathbb{Z}). We consider the following system of linear forms $\Phi_{k,M} := \{\phi_z(x, t_1, \dots, t_M) := x + z_1 t_1 + \dots + z_M t_M \mid z = (z_1, \dots, z_M) \in S_{k,M}\}$, where now these forms are linear functions from \mathbb{F}_p^{M+1} to \mathbb{F}_p . This can be regarded as a multidimensional arithmetic progression of length k over \mathbb{F}_p . Note that for $M = 1$ this is a regular arithmetic progression of length k . In this case we can use Proposition 1 to prove that the True complexity of this system is at most $\min(k, p-1) - 2$. Indeed it can be proved that this is the True complexity of the system, and more generally for all M we have the following (for a proof see [1]):

Proposition 2. *Let k, M be positive integers and p a prime number. Then the system $\Phi_{k,M}$ has True complexity at least $\min(k, M(p-1) + 1) - 2$.*

Using Theorem 1 we can deduce the following (for a proof see [1]):

Theorem 2. *Let k, M be positive integers and p a prime number. Then the system $\Phi_{k,M}$ has True complexity equal to $k^* := \min(k, M(p-1) + 1) - 2$ and there exists $c = c_{k,M,p} \in (0, 1]$ such that for every collection of 1-bounded functions $(f_z : \mathbb{F}_p^n \rightarrow \mathbb{C})_{z \in S_{k,M}}$ we have*

$$|\mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{z \in S_{k,M}} f_z(\phi_z(t_1, \dots, t_d))| \leq \min_{z \in S_{k,M}} \|f_z\|_{U^{k^*+1}}^c. \tag{1}$$

We can take $c = 1$ for $k \leq p$ or $k \geq M(p-1) + 1$ and $c = 2^{1-|S_{k,M}|}$ otherwise.

Note that in the so-called *high-characteristic case* ($k \leq p$) Theorem 2 follows from Proposition 1, and similarly if $k \geq M(p-1) + 1$.

Finally, let us mention the following result, which proves that in many cases the CS complexity of the system $\Phi_{k,M}$ is larger than $k - 2$ (for a proof see [1]).

Proposition 3. *Let $p \geq 5$ and $M \geq 2$. Then the system $\Phi_{p+1,M}$ has CS complexity at least p .*

3 Proof of the Main Result

Let \mathbb{F}_p denote the field $\mathbb{Z}/p\mathbb{Z}$ for a prime p . A linear form in \mathbb{F}_p^d is a linear map $\phi : \mathbb{F}_p^d \rightarrow \mathbb{F}_p$, $(x_1, \dots, x_d) \mapsto a_1x_1 + \dots + a_dx_d$ for some $a_1, \dots, a_d \in \mathbb{F}_p$. More generally, we can consider ϕ as a linear form defined on $(\mathbb{F}_p^n)^d$ for any $n \geq 1$ via the formula $(x_1 \dots, x_d) \in (\mathbb{F}_p^n)^d \mapsto a_1x_1 + \dots + a_dx_d$ where now $x_i \in \mathbb{F}_p^n$ for all $i = 1, \dots, d$. We will use the following fact of linear algebra:

Proposition 4. *Let U_1, U_2 be subspaces of \mathbb{F}_p^M such that $U_1 \cap U_2 = \langle v \rangle$ for some $v \neq 0$. For $i = 1, 2$ suppose that there is a subspace $D_i \subset U_i$ with $D_i \cap \langle v \rangle = \{0\}$. Then $(D_1 + D_2) \cap U_i = D_i$ for $i = 1, 2$.*

We leave the proof of this fact to the reader (it can be found also in [1]).

Proof (of Theorem 1). The proof is by induction on ℓ , the case $\ell = 1$ being given by Proposition 1.

First, assume that the sequence that appears in Definition 3 is precisely ϕ_1, \dots, ϕ_ℓ . Recall that we are interested in bounding the average $\Lambda((f_i)_{i \in [r]}) := \mathbb{E}_{t_1, \dots, t_d \in \mathbb{F}_p^n} \prod_{i \in [r]} f_i(\phi_i(t_1, \dots, t_d))$. We claim that there exists an invertible matrix $T \in M_{d \times d}(\mathbb{F}_p)$ such that $\phi_1 T = (1, 0^{d-1})$ (where ϕ_1 is seen as a horizontal vector). Consider the matrix $T^* \in M_{nd \times nd}(\mathbb{F}_p)$ that is obtained by replacing each entry $T_{i,j}$ of T by $T_{i,j} \cdot \text{Id}_{n \times n}$ (where $\text{Id}_{n \times n}$ is the identity matrix of dimension n). It can be proved that $\det(T^*) = (\det(T))^n$ so this is indeed a valid change of variables.

Thus, without loss of generality we can assume that $\phi_1 = (1, 0^{d-1})$ and write $\Lambda((f_i)_{i \in [r]}) = \mathbb{E}_{t_1 \in \mathbb{F}_p^n} f_1(t_1) \mathbb{E}_{t_2, \dots, t_d \in \mathbb{F}_p^n} \prod_{i=2}^r f_i(\phi_i(t_1, \dots, t_d))$. We apply now the CS inequality to the t_1 variable and we obtain that $|\Lambda((f_i)_{i \in [r]})|$ is bounded by the square root of

$$\mathbb{E}_{t_1, t_2, \dots, t_d, t'_2, \dots, t'_d \in \mathbb{F}_p^n} \prod_{i \in [2, r]} f_i(\phi_i(t_1, t_2, \dots, t_d)) \prod_{i \in [2, r]} \overline{f_i(\phi_i(t_1, t'_2, \dots, t'_d))}. \quad (2)$$

Now we shall write this as an average over $2d - 1$ variables and check that the corresponding system of linear forms has sequential CS complexity at most $(k, \ell - 1)$. If we denote by $\phi_{i,j}$ the j -th entry of the linear form ϕ_i we define $\phi'_i := (\phi_{i+1,1}, \dots, \phi_{i+1,d}, 0^{d-1}) \in \mathbb{F}_p^{2d-1}$ if $i \in [r - 1]$ and $\phi'_i := (\phi_{i-r+2,1}, 0^{d-1}, \phi_{i-r+2,2}, \dots, \phi_{i-r+2,d}) \in \mathbb{F}_p^{2d-1}$ if $i \in [r, 2r - 2]$. And also, $g_i := f_{i+1}$ if $i \in [r - 1]$ and $g_i := \overline{f_{i-r+2}}$ if $i \in [r, 2r - 2]$. Thus, (2) equals

$$\mathbb{E}_{t_1, \dots, t_{2d-1} \in \mathbb{F}_p^n} \prod_{i \in [2r-2]} g_i(\phi'_i(t_1, \dots, t_{2d-1})). \quad (3)$$

And we just have to show that the system $(\phi'_i)_{i=1}^{2r-2}$ has sequential CS complexity at most $(k, \ell - 1)$ at $\ell - 1$ (recall that we have assumed that this is the index we are aiming for). Fix some $e \in [r - 1]$. We have to prove that we can partition

$\{\phi'_1, \dots, \phi'_{2r-2}\} \setminus \{\phi'_1, \dots, \phi'_e\}$ into $k + 1$ classes such that their \mathbb{F}_p -span do not meet $\{\phi'_1, \dots, \phi'_e\}$.

Note that there are clearly two types of linear forms, the ones for the indices $i \in [r - 1]$ all of which have their last $d - 1$ coordinates equal to 0 (let us denote this space by U_1) and the remaining forms for $i \in [r, 2r - 2]$, all of which have their second to the d -th coordinate equal to 0 (let us denote this space by U_2). Now, consider the trivial isomorphism between U_1 and \mathbb{F}_p^d . It is clear that the forms $\phi'_1, \dots, \phi'_{r-1}$ correspond to the points ϕ_2, \dots, ϕ_r . As $\{\phi_1, \dots, \phi_r\}$ satisfies Definition 3 we can partition $\{\phi_{e+1}, \dots, \phi_r\}$ into $k + 1$ classes such that their linear span do not meet $\{\phi_1, \dots, \phi_e\}$. Let us denote these linear spaces by $L_1, \dots, L_{k+1} \subset \mathbb{F}_p^d$ and consider the corresponding linear spaces seen inside U_1 (denote them by L'_1, \dots, L'_{k+1}). Now we argue similarly with U_2 , but this time we use the linear cover of $\{\phi_1, \dots, \phi_r\}$ that excludes just ϕ_1 . Thus, in U_2 we have linear subspaces D'_1, \dots, D'_r that cover every form ϕ'_i for $i \in [r, 2r - 2]$.

The key point now is that as we have eliminated ϕ_1 with the CS inequality, we can apply Proposition 4 to the pairs $L'_j \subset U_1, D'_j \subset U_2$ for $j \in [k + 1]$. Therefore the subspaces $L'_j + D'_j$ for $j \in [k + 1]$ form a covering of $\{\phi'_{e+1}, \dots, \phi'_{2r-2}\}$ that does not meet $\{\phi'_1, \dots, \phi'_e\}$ (as in Definition 3). In other words, this system has sequential CS complexity at most $(k, l - 1)$ at $l - 1$ and the result follows by induction.

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The Ising Antiferromagnet in the Replica Symmetric Phase

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Abstract. Partition functions are an important research object in combinatorics and mathematical physics [Barvinok, 2016]. In this work, we consider the partition function of the Ising antiferromagnet on random regular graphs and characterize its limiting distribution in the replica symmetric phase up to the Kesten-Stigum bound. Our proof relies on a careful execution of the method of moments, spatial mixing arguments and small subgraph conditioning.

Keywords: Ising antiferromagnet · Small subgraph conditioning

1 Introduction

1.1 Motivation

The Ising model, invented by Lenz in 1920 to explain magnetism, is a cornerstone in statistical physics. Consider any graph G with vertex set V and edge set E . Each vertex carries one of two possible spins ± 1 and the interactions between vertices are represented by E . For a spin configuration $\sigma \in \{\pm 1\}^V$ on G , we can consider the Hamiltonian \mathcal{H}_G

$$\mathcal{H}_G(\sigma) = \sum_{(v,w) \in E} \frac{1 + \sigma_v \sigma_w}{2}.$$

Together with a real parameter $\beta > 0$ the Hamiltonian gives rise to a distribution on spin configurations $\sigma \in \{\pm 1\}^V$ defined by

$$\mu_{G,\beta}(\sigma) = \frac{\exp(-\beta \mathcal{H}_G(\sigma))}{Z_{G,\beta}} \quad \text{where} \quad Z_{G,\beta} = \sum_{\tau \in \{\pm 1\}^V} \exp(-\beta \mathcal{H}_G(\tau)). \quad (1)$$

The probability measure $\mu_{G,\beta}$ is known as the Boltzmann distribution with the normalizing term $Z_{G,\beta}$ being the partition function. $\mu_{G,\beta}$ favors configurations

The authors thank Amin Coja-Oghlan for helpful discussions and insights. Philipp Loick is supported by DFG CO 646/3. The full version of this extended abstract can be found on [arXiv:2103.09775](https://arxiv.org/abs/2103.09775).

with few edges between vertices of the same spin which is known as the anti-ferromagnetic Ising model. There is a corresponding formulation of (1) where edges between vertices of the same spin are preferred - the ferromagnetic Ising model. Both models are of great interest in combinatorics and physics and the literature on each is vast [7].

In this paper, we study the Ising antiferromagnet on the random d -regular graph $\mathbb{G} = \mathbb{G}(n, d)$. One might be tempted to think that the regularities of this graph model provide a more amenable study object than its well-known Erdős-Rényi counterpart with fluctuating vertex degrees. However, for the Ising model the reverse seems to be true. Indeed, the independence of edges in the Erdős-Rényi-model greatly facilitates deriving the distribution of short cycles in the planted model and simplifies the calculation of both the first and second moment.

Clearly, $\mu_{\mathbb{G},\beta}$ gives rise to correlations between spins of nearby vertices. The degree of such correlations is governed by the choice of β . A question which is of keen interest in combinatorics and statistical physics is whether such correlations persist for two uniformly sampled (and thus likely distant) vertices. According to physics predictions, for small values of β we should observe a rapid decay of correlation [10] and thus no *long-range correlations*. This regime is known as the *replica symmetric phase*. It is suggested that there exists a specific β which marks the onset of long-range correlations in \mathbb{G} . This value is conjectured to be at the combinatorially meaningful Kesten-Stigum bound [3]

$$\beta_{\text{KS}} = \log \left(\frac{\sqrt{d-1} + 1}{\sqrt{d-1} - 1} \right).$$

The question of long-range correlations is tightly related to the partition function $Z_{\mathbb{G},\beta}$ from which also various combinatorially meaningful observables can be derived. The MAX CUT on random d -regular graphs is a case in point due to the well-known relation

$$\text{MAXCUT}(G) = \frac{dn}{2} + \lim_{\beta \rightarrow \infty} \frac{\partial}{\partial \beta} \log Z_{G,\beta}.$$

for any graph G . Thus, it is of key interest to understand the behavior of $Z_{\mathbb{G},\beta}$.

1.2 Result

In recent work, [3] were able to pinpoint the replica symmetry breaking phase transition at the Kesten-Stigum bound, thus charting the replica symmetric phase for the Ising antiferromagnet on random d -regular graphs. The key feature of the replica-symmetric phase is that w.h.p. two independent samples σ_1, σ_2 from the Boltzmann distribution $\mu_{\mathbb{G},\beta}$ exhibit an almost flat overlap in the sense that $|\sigma_1 \cdot \sigma_2| = o(n)$. To be precise, [3] determined $Z_{\mathbb{G},\beta}$ up to an error term $\exp(o(n))$ for $\beta < \beta_{\text{KS}}$. In this paper, we move beyond this crude approximation. By deriving the limiting distribution in the replica-symmetric phase, we

show that $Z_{\mathbb{G},\beta}$ is tightly concentrated with bounded fluctuations which we can quantify and attribute to short cycles in \mathbb{G} .

Theorem 1. *Assume that $0 < \beta < \beta_{KS}$ and $d \geq 3$. Let $(A_i)_i$ be a sequence of independent Poisson variables with $\mathbb{E}[A_i] = \lambda_i$ where $\lambda_i = \frac{(d-1)^i}{2^i}$. Further, let $\delta_i = \left(\frac{e^{-\beta}-1}{e^{-\beta}+1}\right)^i$. Then as $n \rightarrow \infty$ we have*

$$\log(Z_{\mathbb{G}(n,d),\beta}) - \frac{1}{2} \log\left(\frac{1+e^\beta}{2+de^\beta-d}\right) - n\left(\left(1-\frac{d}{2}\right)\log(2) + \frac{d}{2}\log(1+e^{-\beta})\right) \xrightarrow{d} \log(W) \quad \text{where} \quad W := \exp(-\lambda_1\delta_1 - \lambda_2\delta_2) \prod_{i=3}^{\infty} (1+\delta_i)^{A_i} \exp(-\lambda_i\delta_i).$$

The infinite product defining W converges a.s. and in L^2 .

Taking the expectation of this distribution readily recovers the first part of the result by [3]. The proof of Theorem 1 relies on the combination of the method of moments and small subgraph conditioning enriched in our case by spatial mixing arguments to make the calculation of the second moment tractable.

2 Techniques

2.1 Notation

Let $\mathbb{G} = \mathbb{G}(n, d)$ denote a random d -regular graph on n vertices. We consider sparse graphs with constant d as $n \rightarrow \infty$. Throughout the paper, we will employ standard Landau notation with the usual symbols $o(\cdot), O(\cdot), \Theta(\cdot), \omega(\cdot)$, and $\Omega(\cdot)$ to refer to the limit $n \rightarrow \infty$. We say that a sequence of events $(\mathcal{E}_n)_n$ holds *with high probability* (w.h.p.) if $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}_n) = 1$. When the context is clear we might drop the index of the expectation. Moreover, we will use the proportional \propto to hide necessary normalizations.

2.2 Outline

To get a handle on the distribution of $Z_{\mathbb{G},\beta}$ in the replica symmetric phase, we need to identify the sources of fluctuations of $Z_{\mathbb{G},\beta}$. One obvious source is the number of short cycles. Since \mathbb{G} is sparse and random, standard arguments reveal that \mathbb{G} contains only few short cycles. In the following, let $C_i(G)$ denote the number of short cycles of length i in a graph G and \mathcal{F}_ℓ the σ -algebra generated by the random variables $C_i(\mathbb{G})$ for $i \leq \ell$. A key quantity to consider is the variance of $Z_{\mathbb{G},\beta}$. By standard decomposition, we have

$$\text{Var}[Z_{\mathbb{G},\beta}] = \mathbb{E}\left[\mathbb{E}[Z_{\mathbb{G},\beta} | \mathcal{F}_\ell]^2 - \mathbb{E}[Z_{\mathbb{G},\beta}]^2\right] + \mathbb{E}\left[\mathbb{E}[Z_{\mathbb{G},\beta}^2 | \mathcal{F}_\ell] - \mathbb{E}[Z_{\mathbb{G},\beta} | \mathcal{F}_\ell]^2\right]$$

for any $\ell \geq 1$. Note that the first term of the r.h.s. describes the contribution to the variance by the fluctuations in the number of short cycles, while the second term accounts for the conditional variance given the number of short cycles. It turns out that as $\ell \rightarrow \infty$ after taking $n \rightarrow \infty$, the second summand vanishes. In other words, the entire variance of $Z_{\mathbb{G},\beta}$ is due to fluctuations in the number of short cycles.

To show this property formally, we leverage a result by [8] that stipulates conditions under which one is able to describe the limiting distribution of $Z_{\mathbb{G},\beta}$ (see Theorem 1 in [8]). One ingredient is the distribution of short cycles in \mathbb{G} and a planted model \mathbb{G}^* . In \mathbb{G}^* , we first select a spin configuration σ uniformly at random and subsequently sample a graph G with probability proportional to $\exp(-\beta\mathcal{H}_G(\sigma))$. While the distribution of short cycles in \mathbb{G} is well established, the distribution of short cycles in the planted model \mathbb{G}^* is a key contribution of this paper. The second ingredient is a careful application of the method of moments. Unfortunately, standard results on the first and second moment on random regular graphs (see i.e. [3]), do not suffice in our case and we have to sharpen our pencils to yield an error term of order $O(\exp(1/n))$. While the need for this lower error term prolongs calculations, it also poses some challenges that we resolve by a careful application of the Laplace’s method as suggested by [5] and spatial mixing arguments.

2.3 Short Cycles

To get started, let us write

$$\delta_i = \left(\frac{e^{-\beta} - 1}{e^{-\beta} + 1} \right)^i \quad \text{and} \quad \lambda_i = \frac{(d-1)^i}{2i}. \tag{2}$$

The first item on the agenda is to derive the distribution of short cycles in \mathbb{G} . This is a well-established result.

Fact 2 (Theorem 9.5 in [9]) *Let $\Lambda_i \sim \text{Po}(\lambda_i)$ be a sequence of independent Poisson random variables for $i \geq 3$. Then jointly for all i we have $C_i(\mathbb{G}) \xrightarrow{d} \Lambda_i$ as $n \rightarrow \infty$.*

Deriving the distribution of short cycles in the planted model \mathbb{G}^* informally introduced above requires some more work. Let us start with the definitions. Given $\sigma \in \{\pm 1\}^V$ and for any $\beta > 0$, let us define the distribution of $\mathbb{G}^*(\sigma)$ for any event \mathcal{A} as

$$\mathbb{P}[\mathbb{G}^*(\sigma) \in \mathcal{A}] \propto \mathbb{E}[\exp(-\beta\mathcal{H}_{\mathbb{G}}(\sigma)) \mathbf{1}\{\mathbb{G} \in \mathcal{A}\}]. \tag{3}$$

This definition gives rise to the following experiment. First, draw a spin configuration σ^* uniformly at random among all configurations $\{\pm 1\}^V$. In the next step, draw $\mathbb{G}^* = \mathbb{G}^*(\sigma^*)$ according to (3). Hereafter, \mathbb{G}^* will be denoted the planted model.

Proposition 1. *Let $\Xi_i \sim \text{Po}(\lambda_i(1 + \delta_i))$ be a sequence of independent Poisson random variables for $i \geq 3$. Then jointly for all i we have $C_i(\mathbb{G}^*) \xrightarrow{d} \Xi_i$ as $n \rightarrow \infty$.*

Establishing the distribution of short cycles in \mathbb{G}^* is one of the main contributions of this paper. To this end, we start off with similar arguments as used in [11], but need to diligently account for the subtle dependencies introduced by the regularities in \mathbb{G}^* .

Applying Fact 2 and Proposition 1 to Theorem 1 in [8] requires a slight detour via the Nishimori property. To this end, note that the random graph \mathbb{G} induces a reweighted graph distribution $\hat{\mathbb{G}}$ which for any event \mathcal{A} is defined by

$$\mathbb{P}[\hat{\mathbb{G}} \in \mathcal{A}] \propto \mathbb{E}[Z_{\mathbb{G},\beta} \mathbf{1}\{\mathbb{G} \in \mathcal{A}\}]. \tag{4}$$

Moreover, consider the distribution $\hat{\sigma}$ on spin configurations defined by

$$\mathbb{P}[\hat{\sigma} = \sigma] \propto \mathbb{E}[\exp(-\beta \mathcal{H}_{\mathbb{G}}(\sigma))] \tag{5}$$

for any $\beta > 0$. $\hat{\mathbb{G}}, \mathbb{G}^*, \hat{\sigma}, \sigma^*$, and the Boltzmann distribution from (1) are connected via the well-known Nishimori property.

Fact 3 (Proposition 3.2 in [4]) *For any graph G and spin configuration $\sigma \in \{\pm 1\}^V$ we have*

$$\mathbb{P}[\hat{\mathbb{G}} = G] \mu_G(\sigma) = \mathbb{P}(\hat{\sigma} = \sigma) \mathbb{P}(\mathbb{G}^* = G \mid \sigma^* = \sigma).$$

2.4 The First and Second Moment

The second key ingredient towards the proof of Theorem 1 is the method of moments. As standard random regular graph results are too crude, we need a more precise calculation. Fortunately, with some patience and equipped with the Laplace method as stated in [5], the first moment is not too hard to find.

Proposition 2. *Assume that $0 < \beta < \beta_{KS}$ and $d \geq 3$. Then we have*

$$\begin{aligned} \mathbb{E}[Z_{\mathbb{G},\beta}] &= \exp(-\lambda_1 \delta_1 - \lambda_2 \delta_2 + O(1/n)) \sqrt{(1 + e^\beta) / (2 + de^\beta - d)} \\ &\quad \cdot \exp(n((1 - d/2) \log(2) + d \log(1 + e^{-\beta})/2)) \end{aligned}$$

The second moment is not as amenable. The key challenge for applying the Laplace method is to exhibit that the obvious choice of the optimum is indeed a global maximum. We resolve this issue by resorting to results on the broadcasting process on an infinite d -regular tree and the disassortative stochastic block model. This spatial mixing argument allows us to focus our attention on an area close to the anticipated optimum. To this end, let us exhibit an event \mathcal{O} that is concerned with the location of two typical samples $\sigma_{\mathbb{G}}, \sigma'_{\mathbb{G}}$ from the Boltzmann distribution $\mu_{\mathbb{G},\beta}$, i.e.

$$\mathcal{O} = \{\mathbb{E}[\sigma_{\mathbb{G}} \cdot \sigma'_{\mathbb{G}} \mid \mathbb{G}] < \varepsilon_n n\} \tag{6}$$

for a sequence of $\varepsilon_n = o(1)$. Then we can leverage the following result from [3].

Lemma 1. *For the event \mathcal{O} defined in (6) we have for $d \geq 3, 0 < \beta < \beta_{KS}$*

$$\mathbb{E}[Z_{\mathbb{G},\beta} \mathbf{1}\{\mathcal{O}\}] = (1 - o(1))\mathbb{E}[Z_{\mathbb{G},\beta}]$$

Proof. The statement follows from Lemmas 2.1, 2.3, 2.5, 4.3 and Corollary 4.15 in [3].

Conditioning on \mathcal{O} greatly facilitates the calculation of the second moment.

Proposition 3. *For $0 < \beta < \beta_{KS}$ and $d \geq 3$ we have*

$$\begin{aligned} \mathbb{E}[Z_{\mathbb{G},\beta}^2 \mathbf{1}\{\mathcal{O}\}] &= \exp\left(\lambda_1 + \lambda_2 - \frac{4\lambda_1}{(1 + e^\beta)^2} - \frac{4\lambda_2(1 + e^{2\beta})^2}{(1 + e^\beta)^4} + O\left(\frac{1}{n}\right)\right) \\ &\cdot \frac{(1 + e^\beta)^2 \exp(n((2 - d)\log(2) + d\log(1 + e^{-\beta})))}{(de^\beta - d + 2)\sqrt{2e^{2\beta} + 2de^\beta - de^{2\beta} - d + 2}}. \end{aligned}$$

2.5 Proof of Theorem 1

We apply Theorem 1 in [8] to the random variable $Z_{\mathbb{G},\beta} \mathbf{1}\{\mathcal{O}\}$. Condition (1) readily follows from Fact 2. For Condition (2) let us write

$$\mathcal{C}(G) = \{C_1(G) = c_1, \dots, C_\ell(G) = c_\ell\}$$

for any graph G . Using standard reformulations and the definition of $\hat{\mathbb{G}}$ from (4) we find

$$\frac{\mathbb{E}[Z_{\mathbb{G},\beta} \mid \mathcal{C}(G)]}{\mathbb{E}[Z_{\mathbb{G},\beta}]} = \frac{\mathbb{E}[Z_{\mathbb{G},\beta} \mathbf{1}\{\mathcal{C}(G)\}]}{\mathbb{P}[\mathcal{C}(G)]\mathbb{E}[Z_{\mathbb{G},\beta}]} = \frac{\mathbb{P}[\mathcal{C}(\hat{\mathbb{G}})]}{\mathbb{P}[\mathcal{C}(G)]} = \frac{\mathbb{E}_{\hat{\sigma}}[\mathbb{P}[\mathcal{C}(\hat{\mathbb{G}}) \mid \hat{\sigma}]]}{\mathbb{P}[\mathcal{C}(G)]}.$$

Since a typical sample σ from $\hat{\sigma}$ has the property that $|\sigma \cdot \mathbf{1}| = O(n^{2/3})$, i.e. is relatively balanced, the Nishimori property (Fact 3) implies

$$\mathbb{E}_{\hat{\sigma}}[\mathbb{P}[\mathcal{C}(\hat{\mathbb{G}}) \mid \hat{\sigma}]] \sim \mathbb{P}[\mathcal{C}(G^*)].$$

Condition (2) now follows from Fact 2 and Proposition 1. For Condition (3) consider any $\beta = \beta_{KS} - \varepsilon$ for some small $\varepsilon > 0$. Letting $\eta = \eta(\varepsilon) > 0$ a simple calculation reveals

$$\sum_{i \geq 1} \lambda_i \delta_i^2 \leq \sum_{i \geq 1} \lambda_i \left(\frac{e^{-\beta_{KS} + \varepsilon} - 1}{e^{-\beta_{KS} + \varepsilon} + 1}\right)^{2i} = \sum_{i \geq 1} \frac{(1 - \eta)^i}{2i} < \infty.$$

Finally, by Lemma 1, Propositions 2 and 3 and the fact that for any $0 < x < 1$ $\log(1 - x) = -\sum_{i \geq 1} x^i/i$ we find for $0 < \beta < \beta_{KS}$ and $d \geq 3$

$$\frac{\mathbb{E} \left[Z_{\mathbb{G},\beta}^2 \mathbf{1} \{ \mathcal{O} \} \right]}{\mathbb{E} [Z_{\mathbb{G},\beta} \mathbf{1} \{ \mathcal{O} \}]^2} = (1 + o(1)) \frac{\mathbb{E} \left[Z_{\mathbb{G},\beta}^2 \mathbf{1} \{ \mathcal{O} \} \right]}{\mathbb{E} [Z_{\mathbb{G},\beta}]^2} = (1 + o(1)) \exp \left(\sum_{i \geq 3} \lambda_i \delta_i^2 \right)$$

establishing Condition (4) and thus the distribution of $Z_{\mathbb{G},\beta} \mathbf{1} \{ \mathcal{O} \}$. Since by Lemma 1 $\mathbb{E} [Z_{\mathbb{G},\beta} (1 - \mathbf{1} \{ \mathcal{O} \})] = o(\mathbb{E} [Z_{\mathbb{G},\beta}])$, Theorem 1 follows from Markov’s inequality.

3 Discussion

Studying partition functions has a long tradition in combinatorics and mathematical physics. k -SAT, q -coloring or the stochastic block model are just some noteworthy examples where the partition function reveals fundamental and novel combinatorial insights. Due to its connection to the MAX CUT problem and the disassortative stochastic block model, the Ising antiferromagnet fits nicely into this list. For random d -regular graphs, Coja-Oghlan et al. [3] pinpointed its replica symmetry breaking phase transition at the Kesten-Stigum bound. Using the method of moments and spatial mixing arguments, they determine $Z_{\mathbb{G},\beta}$ up to $\exp(o(n))$. In this paper, we move beyond this approximation and derive the limiting distribution of $Z_{\mathbb{G},\beta}$ in the replica symmetric regime. We note that the distribution of $Z_{\mathbb{G},\beta}$ above the Kesten-Stigum bound is fundamentally different. A similar analysis for the Erdős-Rényi-model was carried out in [11].

Using the combination of the method of moments and small subgraph conditioning underlying our proof was initially pioneered by Robinson & Wormald [12] to prove that cubic graphs are w.h.p. Hamiltonian. Janson [8] subsequently showed that small subgraph conditioning can be used to obtain limiting distributions. This strategy was successfully applied, among others, to the stochastic block model [11] and the Viana-Bray model [6]. For other problems, the second moment appears to be too crude for the entire replica symmetric phase and enhanced techniques are needed [2]. In this work, we enrich the classical strategy of the method of moments and small subgraph conditioning by spatial mixing arguments to cover the entire replica symmetric phase.

An interesting remaining question is to throw a bridge between the properties of the partition function $Z_{\mathbb{G},\beta}$ and long-range correlations in \mathbb{G} . While it should be a small step from Theorem 1 to vindicate the absence of long-range correlations in the replica symmetric phase, proving the presence of long-range correlations above the Kesten-Stigum bound is a more challenging, yet important endeavour.

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A Simple $(2 + \epsilon)$ -Approximation Algorithm for Split Vertex Deletion

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Abstract. A *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set. Given a graph G and weight function $w : V(G) \rightarrow \mathbb{Q}_{\geq 0}$, the SPLIT VERTEX DELETION (SVD) problem asks to find a minimum weight set of vertices X such that $G - X$ is a split graph. It is easy to show that a graph is a split graph if and only if it does not contain a 4-cycle, 5-cycle, or a two edge matching as an induced subgraph. Therefore, SVD admits an easy 5-approximation algorithm. On the other hand, for every $\delta > 0$, SVD does not admit a $(2 - \delta)$ -approximation algorithm, unless $P = NP$ or the Unique Games Conjecture fails.

For every $\epsilon > 0$, Lokshtanov, Misra, Panolan, Philip, and Saurabh [9] recently gave a *randomized* $(2 + \epsilon)$ -approximation algorithm for SVD. In this work we give an extremely simple deterministic $(2 + \epsilon)$ -approximation algorithm for SVD.

Keywords: Graph theory · Approximation algorithms · Induced subgraphs

A graph G is a *split graph* if $V(G)$ can be partitioned into two sets K and S such that K is a clique and S is a stable set. Split graphs are an important subclass of perfect graphs which feature prominently in the proof of the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, and Thomas [3].

Given a graph G and weight function $w : V(G) \rightarrow \mathbb{Q}_{\geq 0}$, the SPLIT VERTEX DELETION (SVD) problem asks to find a set of vertices X such that $G - X$ is a split graph and $w(X) := \sum_{x \in X} w(x)$ is minimum. A subset $X \subseteq V(G)$ such that $G - X$ is a split graph is called a *hitting set*. We denote by $\text{OPT}(G, w)$ the minimum weight of a hitting set.

It is easy to show that G is a split graph if and only if G does not contain C_4, C_5 or $2K_2$ as an induced subgraph, where C_ℓ denotes a cycle of length ℓ and $2K_2$ is a matching with two edges. Therefore, the following is an easy 5-approximation algorithm¹ for SVD in the unweighted case (the general case

¹ An α -approximation algorithm for SVD is a (deterministic) polynomial-time algorithm computing a hitting set X with $w(X) \leq \alpha \cdot \text{OPT}(G, w)$.

follows from the *local ratio method* [6]). If G is a split graph, then \emptyset is a hitting set, and we are done. Otherwise, we find an induced subgraph H of G such that $H \in \{C_4, C_5, 2K_2\}$. We put $V(H)$ into the hitting set, replace G by $G - V(H)$, and recurse.

On the other hand, there is a simple approximation preserving reduction from VERTEX COVER to SVD (see [9]). Therefore, for every $\delta > 0$, SVD does not admit a $(2 - \delta)$ -approximation algorithm, unless $P=NP$ or the Unique Games Conjecture fails [7].

For every $\epsilon > 0$, Lokshtanov, Misra, Panolan, Philip, and Saurabh [9] recently gave a *randomized* $(2 + \epsilon)$ -approximation algorithm for SVD. Their approach is based on the randomized 2-approximation algorithm for feedback vertex set in tournaments [8], but is more complicated and requires several new ideas and insights.

Our main result is a much simpler deterministic $(2 + \epsilon)$ -approximation algorithm for SVD.

Theorem 1. *For every $\epsilon > 0$, there is a (deterministic) $(2 + \epsilon)$ -approximation algorithm for SVD.*

As a quick comparison, the full version of [9] is 27 pages, while our entire proof fits into this extended abstract. Moreover, as far as we can tell, the easy 5-approximation described above was the previously best (deterministic) approximation algorithm for SVD. Before describing our algorithm and proving its correctness, we need a few definitions.

Let G be a graph and \mathcal{H} be a family of graphs. We say that G is \mathcal{H} -free if G does not contain H as an induced subgraph for all $H \in \mathcal{H}$. We let \overline{G} be the complement of G . A *cut* in a graph G is a pair (A, B) such that $A \cup B = V(G)$ and $A \cap B = \emptyset$. The cut (A, B) is said to *separate* a pair (K, S) where K is a clique, and S is a stable set if $K \subseteq A$ and $S \subseteq B$. A family of cuts \mathcal{F} is called a *clique-stable set separator* if for all pairs (K, S) where K is a clique and S is a stable set disjoint from K , there exists a cut (A, B) in \mathcal{F} such that (A, B) separates (K, S) . For each $k \in \mathbb{N}$, let P_k be the path on k vertices.

The main technical ingredient we require is the following theorem of Bousquet, Lagoutte and Thomassé [1].

Theorem 2. *For every $k \in \mathbb{N}$, there exists $c(k) \in \mathbb{N}$ such that every n -vertex, $\{P_k, \overline{P}_k\}$ -free graph has a clique-stable set separator of size at most $n^{c(k)}$. Moreover, such a clique-stable set separator can be found in polynomial time.*

We remark that [1] do not state that the clique-stable set separator can be found in polynomial time, but this is easy to check, where the relevant lemmas appear in [2, Theorem 4], [5, Theorem 1.1], and [4, Lemma 1.5]. Note that the abstract of [1] states that $c(k)$ is a tower function. However, the bound for $c(k)$ can be significantly improved by using [5, Theorem 1.1] instead of a lemma of Rödl [10] (which was used in an older version of [2]). The proof of [5, Theorem 1.1] does not use the Szemerédi Regularity Lemma [11], and provides much better quantitative estimates.

We are now ready to state and prove the correctness of our algorithm.

Proof. Let G be an n -vertex graph, $w : V(G) \rightarrow \mathbb{Q}_{\geq 0}$, and $\epsilon > 0$. We may assume that $w(v) > 0$ for all $v \in V(G)$, since we may delete vertices of weight 0 for free. Choose k sufficiently large so that $\frac{2k}{k-4} \leq 2 + \epsilon$. Let $\mathbf{1}$ be the weight function on $V(P_k)$ which is identically 1. Since the largest clique of P_k has size 2 and every vertex cover of P_k has size at least $\lfloor k/2 \rfloor$, every hitting set of P_k has size at least $\frac{k-4}{2}$. Therefore, $|V(P_k)|/\text{OPT}(P_k, \mathbf{1}) \leq 2 + \epsilon$, and so by the local ratio method [6], we may assume that G is P_k -free. Note that G is a split graph if and only if \overline{G} is a split graph. Thus, we may also assume that G is $\overline{P_k}$ -free. Now, by Theorem 2, there exists a constant $c(k)$ such that G has a clique-stable set separator \mathcal{F} such that $|\mathcal{F}| \leq n^{c(k)}$.

For each $(A, B) \in \mathcal{F}$, let ρ_A and ρ_B be the weights of the minimum vertex covers of $(\overline{G}[A], w)$ and $(G[B], w)$. Since there is a 2-approximation algorithm for vertex cover, for each $(A, B) \in \mathcal{F}$, we can find vertex covers X_A and X_B of $(\overline{G}[A], w)$ and $(G[B], w)$ such that $w(X_A) \leq 2\rho_A$ and $w(X_B) \leq 2\rho_B$. Note that $X_A \cup X_B$ is a hitting set. Let X^* be a minimum weight hitting set for (G, w) , and suppose that $V(G - X^*)$ is partitioned into a clique K^* and a stable set S^* . Since \mathcal{F} is a clique-stable set separator, there must be some $(A^*, B^*) \in \mathcal{F}$ such that $K^* \subseteq A^*$ and $S^* \subseteq B^*$. Therefore, if we choose $(A, B) \in \mathcal{F}$ such that $w(X_A) + w(X_B)$ is minimum, then $X_A \cup X_B$ is a hitting set such that $w(X_A \cup X_B) \leq 2w(X^*)$. Finally, since $|\mathcal{F}| \leq n^{c(k)}$, our algorithm clearly runs in polynomial time.

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On Graphs Without an Induced Path on 5 Vertices and Without an Induced *Dart* or *Kite*

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Abstract. The *kite* and the *dart* are the two graphs obtained from a $K_4 - e$ by adding a vertex u and making u adjacent to one vertex of degree 2 or 3 in $K_4 - e$, respectively. The optimal χ -binding function $f_{F_1, F_2, \dots, F_n}^* : \mathbb{N} \rightarrow \mathbb{N}$ of the class of (F_1, F_2, \dots, F_n) -free graphs is defined by

$$x \mapsto \max\{\chi(G) : \omega(G) = x, G \text{ is } (F_1, F_2, \dots, F_n)\text{-free}\}.$$

In this work, we prove $f_{P_5, kite}^*(x) \leq 2x - 2$ for every $x \geq 3$ and $f_{P_5, dart}^* = f_{3K_1}^*$. In other words, we show that every $(P_5, kite)$ -free graph, say, G_{kite} with clique number at least 3 is $(2\omega(G_{kite}) - 2)$ -colourable and every $(P_5, dart)$ -free graph, say, G_{dart} is $f_{3K_1}^*(\omega(G_{dart}))$ -colourable.

Keywords: P_5 -free graphs · *Dart*-free graphs · *Kite*-free graphs · Chromatic number · χ -binding function

1 Introduction

We use standard notation and terminology, and denote the chromatic number and the clique number of a graph G by $\chi(G)$ and $\omega(G)$, respectively.

The study of χ -binding functions for graph classes is nowadays one central problem in chromatic graph theory. Gyárfás [9] introduced this concept as follows: For a graph class \mathcal{G} , a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is a χ -binding function for \mathcal{G} if $\chi(G) \leq f(\omega(G))$ for every graph G that is an induced subgraph of a graph in \mathcal{G} .

Natural classes of graphs for which we study χ -binding functions are hereditary ones, that is, graph classes defined by a set of forbidden induced subgraphs. As Gyárfás [9] proved, χ -binding functions of graph classes defined by finite sets of forbidden induced subgraphs exist only if one of the forbidden induced subgraphs is a forest, and graph classes defined by a forbidden induced path are bounded by an exponential one.

On one hand, P_4 -free graphs are perfect by a result of Seinsche [12]. On the other hand we still do not know if there is a subexponential χ -binding function for the class of P_5 -free graphs. In view of this massive difference, one central and important problem posed by Gyárfás [9] more than 30 years ago asks for the order of magnitude of an optimal χ -binding function for the class of P_5 -free

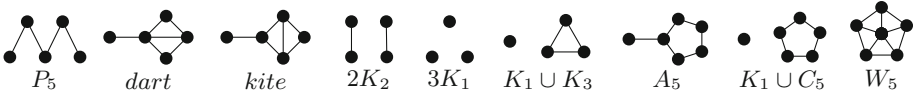


Fig. 1. Most frequently used forbidden induced subgraphs

graphs. In this context, the *optimal* χ -binding function $f_{F_1, F_2, \dots, F_n}^* : \mathbb{N} \rightarrow \mathbb{N}$ for the class of (F_1, F_2, \dots, F_n) -free graphs with finitely many graphs F_1, F_2, \dots, F_n is defined by

$$x \mapsto \max\{\chi(G) : \omega(G) = x, G \text{ is } (F_1, F_2, \dots, F_n)\text{-free}\}.$$

As Gyarfas problem seems tough and challenging, subclasses of P_5 -free graphs were investigated. However, even for some of such classes, e.g. (P_5, C_5) -free graphs as studied by Chudnovsky and Sivaraman [7], the order of magnitude of all known χ -binding functions is exponential. In contrast, linear χ -binding functions have been found, e.g., for (P_5, gem) -free graphs by Chudnovsky et al. [5], and quadratic ones have been found, e.g., for $(P_5, bull)$ -free graphs by Chudnovsky and Sivaraman [7] and for $(P_5, banner)$ -free graphs in [2].

It is known that $f_{P_5}^*$ is non-linear but to the best of our knowledge it could be possible that $f_{P_5}^*$ is quadratic. This fact motivates to determine subclasses of P_5 -free graphs that have linear and (sub)quadratic optimal χ -binding functions. In this work, we contribute two of these classes. In particular, we show two χ -binding functions for subclasses of P_5 -free graphs one of which is optimal and (sub)quadratic while the other one is linear and thus equals that of an optimal one in its order of magnitude.

Theorem 1. *Let G be a P_5 -free graph.*

1. *If G is kite-free and $\omega(G) \geq 3$, then $\chi(G) \leq 2\omega(G) - 2$.*
2. *If G is dart-free and $\omega(G) \geq 1$, then $\chi(G) \leq f_{3K_1}^*(\omega(G))$.*

Our result implies $f_{P_5, kite}^*(x) \in \Theta(x)$. Moreover, as every $3K_1$ -free graph is $(P_5, dart)$ -free, we find that $f_{P_5, dart}^* = f_{3K_1}^*$, and so we observe that the inequality in Property 2 of Theorem 1 is tight. By a result of Kim [10], the maximum order of a $3K_1$ -free graph with clique number at most x is $\Theta(x^2 / \log(x))$, and so $f_{P_5, dart}^*(x) \in \Theta(x^2 / \log(x))$. We note that there are just a few of these results determining a superlinear optimal χ -binding function of a graph class, e.g. [2, 3].

2 Sketch of the Proof

As some parts of our proof for Theorem 1 are very long, we omit technical details and focus on describing the framework and main ideas.

We first need to introduce some further notation and terminology. If S is a set, then we denote by 2^S its power set. Let G be a connected graph and $q : V(G) \rightarrow \mathbb{N}$ be a vertex weight function.

The q -clique number $\omega_q(G)$ of G is the largest weight k of a clique. The q -chromatic number $\chi_q(G)$ of G is the smallest integer k for which there is some colouring function $L: V(G) \rightarrow 2^{\{1,2,\dots,k\}}$ such that $|L(u)| = q(u)$ and $L(u) \cap L(v) = \emptyset$ for every vertex $u \in V(G)$ and every neighbour v of u . As we find that $\omega_q(G) = \omega(G)$ and $\chi_q(G) = \chi(G)$ if $q(u) = 1$ for every vertex $u \in V(G)$, the terminologies of q -clique number and q -chromatic number are generalisations of the clique number and the chromatic number of a graph, respectively.

The graph G is *candled* if its vertex set can be partitioned into three sets $X, Y,$ and Z such that $|X| = |Y| \geq 1, X$ is an independent set, Y is a clique, the set of edges between X and Y is a matching of size $|X|,$ no vertex of X has a neighbour in $Z,$ and every vertex of Y is adjacent to every vertex of $Z.$ Moreover, G is *matched co-bipartite* if its vertex set can be partitioned into two cliques X and Y with $|X| \leq |Y| \leq |X| + 1$ such that the set of edges between X and Y is a matching of size at least $|Y| - 1.$

If X is a set of vertices of $G, G - X$ is disconnected, X can be partitioned into modules X_1, X_2, \dots, X_n for some integer $n \geq 1,$ and every vertex of X_i is adjacent to every vertex of X_j for every two integers i and $j,$ then X is a *clique separator of modules.* If $|X_i| = 1$ for every integer $i,$ then X is a *clique separator.*

The set \mathcal{G}^* consists of all connected graphs G of independence number at least 3 such that, taken an arbitrary cycle C of length 5 in $G,$ we have that the set of vertices which are non-adjacent to a vertex of C is an independent set, and there are two non-adjacent vertices on C that have no neighbour in $V(G) \setminus V(C)$ while the other three vertices of C have the same neighbours in $V(G) \setminus V(C).$ We further let $G_1, G_2, G_3,$ and G_4 be the four graphs depicted in Fig. 2.

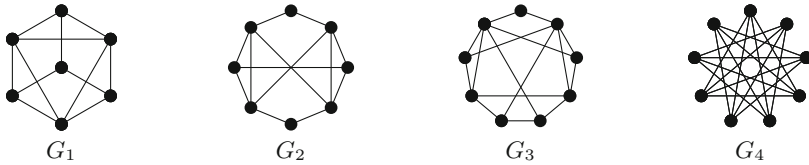


Fig. 2. Graphs $G_1, G_2, G_3,$ and G_4

2.1 (P_5, Kite) -Free Graphs

We first prove a structure theorem. Let G be a prime (P_5, kite) -free graph that has no clique separator. If G is disconnected, then, as G is prime, it consists of two non-adjacent vertices only, and so G is $(K_1 \cup K_3, 2K_2)$ -free. Assume that G is connected. By a result of Brandstädt and Mosca [1], G is $2K_2$ -free or a matched co-bipartite graph. In the case where G is $2K_2$ -free, we note that \bar{G} is a prime (fork, C_4) -free graph without universal vertices. Hence, a result of Chudnovsky et al. [4] implies that \bar{G} is *claw*-free or one of $\{G, \bar{G}\}$ is *candled.* As G has no clique separator, we find that neither G nor \bar{G} is *candled.* Therefore, G is $(K_1 \cup K_3, 2K_2)$ -free and we obtain the following structural result.

Lemma 1. *If H is a prime (P_5, kite) -free graph, then H contains a clique separator or H is a matched co-bipartite graph, or H is $(K_1 \cup K_3, 2K_2)$ -free.*

We are now in a position to prove Property 1 of Theorem 1, and first show

$$f_{P_5, \text{kite}}^*(x) \leq f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(x) \tag{1}$$

for every $x \geq 1$. We prove this inequality by a minimal counterexample approach and collect some properties of such a graph G . Clearly, G is connected. By the following lemma, G has no clique separator of modules.

Lemma 2 ([2]). *If H, H_1, H_2 are three graphs such that $H = H_1 \cup H_2$ and $V(H_1) \cap V(H_2)$ is a clique separator of modules in H , then*

$$\chi(H) = \max\{\chi(H_1), \chi(H_2)\} \quad \text{and} \quad \omega(H) = \max\{\omega(H_1), \omega(H_2)\}.$$

Additionally, if G has a module M in G whose vertices are adjacent to all vertices of $G - M$, then $\omega(G) = \omega(G[M]) + \omega(G - M)$ and $\chi(G) = \chi(G[M]) + \chi(G - M)$. As

$$f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(x) + f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(y) \leq f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(x + y)$$

for every $x, y \geq 1$, see [2], we have a contradiction to the fact that G is a minimal counterexample. Thus, we find that there is a vertex u in $G - M$ that has no neighbour in M for every module M . As our structural characterisation in Lemma 1 are for prime graphs only, we need to study the modules of G . For this purpose, we apply a result of [2] that reads for kite-free graphs as follows.

Lemma 3 ([2]). *If H is a kite-free graph, then, for each module M in H , $H[M]$ is K_2 -free or $N_H(M)$ is a clique separator of modules, or every vertex of $H - M$ is adjacent to every vertex in M .*

As a homogeneous set, say, M of G is independent by Lemma 3 and as two vertices, say, u_1, u_2 of M have the same neighbours in G , we find that $\chi(G) = \chi(G - u_1)$. By this contradiction to the fact that G is a minimal counterexample, we find that G is prime. By Lemma 1 and as G contains no clique separator (of modules), G is a matched co-bipartite graph or $(K_1 \cup K_3, 2K_2)$ -free. In the first case, it is easily seen that $\chi(G) = \omega(G)$, and so we find that G is $(K_1 \cup K_3, 2K_2)$ -free as $\chi(G) > \omega(G)$. Note that $G - N(u)$ is bipartite or contains an induced cycle on 5 vertices for every vertex $u \in V(G)$. We prove the following, which completes our proof for (1).

Proposition 1. *If H is a prime $(K_1 \cup K_3, 2K_2)$ -free graph without clique separators of modules, then $\chi(H) \leq f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(\omega(H))$.*

It remains to show $\chi(G') \leq 2\omega(G') - 2$ for every $(K_1 \cup K_3, K_1 \cup C_5, 2K_2)$ -free graph G' with $\omega(G') \geq 3$.

We prove this fact by induction. As $f_{2K_2}^*(3) = 4$ by a result of Gaspers and Huang [8], we assume $\omega(G') \geq 4$. Let W be a clique of size $\omega(G')$ in G' , w be

a vertex of W , and $S = V(G) \setminus N(w)$. Note that $\omega(G' - S) = \omega(G') - 1$ as $W \setminus \{w\}$ is a clique of size $\omega(G') - 1$ and every vertex of $G' - S$ is adjacent to w . Moreover, as S induces a bipartite graph and as $\omega(G' - S) \geq 3$, it follows

$$\chi(G') \leq \chi(G' - S) + 2 \leq 2\omega(G' - S) = 2\omega(G') - 2.$$

It follows

$$f_{P_5, kite}^*(x) \leq f_{K_1 \cup K_3, K_1 \cup C_5, 2K_2}^*(x) \leq 2x - 2$$

for every $x \geq 3$, and our proof of Property 1 of Theorem 1 is complete.

2.2 $(P_5, Dart)$ -Free Graphs

We are heading to a structural result that is similar to that of Lemma 1. In particular, we show the following.

Lemma 4. *If H is a prime $(P_5, dart)$ -free graph, then H is perfect or H is $3K_1$ -free or $H \in \mathcal{G}^*$, or H is isomorphic to an induced subgraph of G_1, G_2, G_3, G_4 .*

As $3K_1$ -free graphs are $(P_5, dart)$ -free, we analyse the structure of $(P_5, dart)$ -free graphs with independence number at least 3. We find that such graphs exist as each of the graphs in $\{G_1, G_2, G_3, G_4\}$ is $(P_5, dart)$ -free, connected, prime and has no clique separator but all these graphs have independence number at least 3. In view of the desired result, let G be a prime $(P_5, dart)$ -free graph that has three vertices which are pairwise non-adjacent. If G is disconnected, then G consists of two non-adjacent vertices only, and so we find that G is connected. Moreover, assume that none of $\{G_1, G_2, G_3, G_4\}$ contains an induced subgraph isomorphic to G . As G is not isomorphic to G_1 , we prove and make use of the following result.

Proposition 2. *If H is a prime $(P_5, dart)$ -free graph of independence number at least 3, then either H is W_5 -free and \bar{H} is A_5 -free, or H is isomorphic to G_1 .*

By Proposition 2, we obtain that G is W_5 -free and \bar{G} is A_5 -free. Natural candidates for graphs that are W_5 -free and whose complementary graphs are A_5 -free, are C_5 -free graphs. Thus, let us first consider the case where G is C_5 -free. As G is P_5 -free, each induced cycle of G has length at most four. Moreover, by proving the following result, we find that every induced cycle of odd length in \bar{G} is a triangle, and so G is perfect by the Strong Perfect Graph Theorem [6].

Proposition 3. *If H is a prime $dart$ -free graph of independence number at least 3, then \bar{H} is (C_7, C_9, \dots) -free or H contains an induced cycle and an induced path each of which is of order 5.*

In view of the desired result of Lemma 4, we next consider the case where G has an induced cycle of order 5. Again by Proposition 3, we obtain that \bar{G} is (C_7, C_9, \dots) -free. We complete our proof of Lemma 4 by showing the following result as our assumption states that none of $\{G_1, G_2, G_3, G_4\}$ contains an induced subgraph isomorphic to G .

Proposition 4. *If H is a prime (P_5, W_5, dart) -free graph that has an induced cycle of order 5 and for which \bar{H} is (A_5, C_7, C_9, \dots) -free, then $H \in \mathcal{G}^*$ or H is isomorphic to an induced subgraph of G_2, G_3 , or G_4 .*

We proceed by colouring (P_5, dart) -free graphs, and so we prove Property 2 of Theorem 1. We make use of a result in [2] that applied for our purpose reads as follows.

Lemma 5 ([2]). *Let H be a dart-free graph. If $\chi_q(H) \leq f_{3K_1}^*(\omega_q(H))$ for each vertex-weight function $q: V(H) \rightarrow \mathbb{N}$ for which $H[\{u : q(u) > 0\}]$ is prime, then*

$$\chi(H) \leq f_{3K_1}^*(\omega(H)).$$

By Lemma 5, we need to study the q -chromatic number of prime (P_5, dart) -free graphs. Let G be such a graph and $q: V(G) \rightarrow \mathbb{N}$ be an arbitrary vertex weight function. Let G_q be a graph that is obtained from G by replacing each vertex, say, v by a clique of size $q(v)$ and adding all edges between two of these cliques if and only if the corresponding vertices are adjacent. Note that $\chi_q(G) = \chi(G_q)$, $\omega_q(G) = \omega(G_q)$, and G_q is $3K_1$ -free if G is $3K_1$ -free. Moreover, G_q is perfect if and only if G is perfect as Lovász [11] showed in his famous proof for the Weak Perfect Graph Theorem. Consequently, $\chi(G_q) \leq f_{3K_1}^*(\omega_q(G))$ if G is $3K_1$ -free or if G is perfect. In view of the desired result and by Lemma 4, it remains to consider the cases where $G \in \mathcal{G}^*$ or where G is isomorphic to an induced subgraph of G_1, G_2, G_3 , or G_4 . We prove the following result.

Proposition 5. *If $H \in \mathcal{G}^* \cup \{G_1, G_2, G_3, G_4\}$ and $q: V(H) \rightarrow \mathbb{N}$, then*

$$\chi_q(H) \leq \left\lceil \frac{5\omega_q(H) - 1}{4} \right\rceil.$$

It remains to show $\lceil (5x-1)/4 \rceil \leq f_{3K_1}^*(x)$ for every $x \geq 2$. Let $G: c_1c_2c_3c_4c_5c_1$ be a cycle of length 5 and $q: V(G) \rightarrow \mathbb{N}$ be such that $q(c_1) = q(c_3) = q(c_5) = \lfloor x/2 \rfloor$ and $q(c_2) = q(c_4) = \lceil x/2 \rceil$. We find $f_{3K_1}^*(x) \geq \chi(G_q) \geq \lceil (5x-1)/4 \rceil$, which completes the proof of Property 2 of Theorem 1.

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Constant Congestion Brambles in Directed Graphs

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Abstract. The Directed Grid Theorem, stating that there is a function f such that a directed graph of directed treewidth at least $f(k)$ contains a directed grid of size at least k as a butterfly minor, after being a conjecture for nearly 20 years, has been proven in 2015 by Kawarabayashi and Kreutzer. However, the function f in the proof is very fast growing. Here, we show that if we relax *directed grid* to *bramble of constant congestion*, we obtain a polynomial bound. More precisely, we show that for every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of congestion at most 8 and size at least k .

Keywords: Directed graphs · Directed treewidth · Graph theory

1 Introduction

The Grid Minor Theorem, proven by Robertson and Seymour [14], is one of the most important structural characterizations of treewidth. Informally, it asserts that a *grid minor* is a canonical obstacle to small treewidth: a graph of large treewidth necessarily contains a big grid as a minor. The relation of “large” and “big” in this statement, being non-elementary in the original proof, after a series of improvements has been proven to be a polynomial of relatively small degree: *For every $k \geq 1$ there exists $t = \mathcal{O}(k^9 \text{polylog} k)$ such that every graph of treewidth at least t contains a $k \times k$ grid as a minor* [2].

This research is part of projects that have received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme Grant Agreement 714704. T.M. was partially supported by a postdoctoral fellowship at the Simon Fraser University through NSERC grants R611450 and R611368. M.S. was partially supported by Alexander von Humboldt Foundation. The full version of the paper can be found on arXiv [11].

In the mid-90s, Johnson, Robertson, Seymour, and Thomas [6] proposed an analog of treewidth for directed graphs, called *directed treewidth*, and conjectured an analogous statement (with the appropriate notion of a directed grid). After nearly 20 years, the Directed Grid Theorem was proven by Kawarabayashi and Kreutzer [8]. However, the proof yields a very high dependency between the required directed treewidth bound and the promised size of the directed grid.

While searching for better and better bounds for (undirected) Grid Minor Theorem, researchers investigated relaxed notions of a grid. In some sense, the “most relaxed” notion of a grid is a *bramble*: a family \mathcal{B} of connected subgraphs of a given graph such that every two $B_1, B_2 \in \mathcal{B}$ either share a vertex or there exists an edge with one endpoint in B_1 and one endpoint in B_2 . Brambles can be large; the notion of complexity of a bramble is its *order*: the minimum size of a vertex set that intersects every element of a bramble. We also refer to the *size* of a bramble as the number of its elements and the *congestion* of a bramble as a maximum number of elements that contain a single vertex; note that the size of the bramble is bounded by the product of its order and congestion.

What links brambles and grids is that a $k \times k$ grid contains a bramble of size k , order $\lceil k/2 \rceil$, and congestion 2 whose elements are subgraphs consisting of the i -th row and i -th column of the grid for every $1 \leq i \leq k$. In the other direction, brambles of small congestion can replace grids if one wants use the grid as an object that allows arbitrary interconnections of small congestion between different pairs of vertices on its boundary. This usage appears e.g. in arguments for the Disjoint Paths problem [1].

Surprisingly, as proven by Seymour and Thomas [15], brambles form a dual object tightly linked to treewidth: the maximum order of a bramble in a graph is *exactly* the treewidth of the graph plus one. However, as shown by Grohe and Marx [3] and sharpened by Hatzel et al. [5], brambles of high order may need to have exponential size: while a graph of treewidth k necessarily contains a bramble of order $\tilde{\Omega}(\sqrt{k})$ of congestion 2 (and thus of size linear in their order), there are classes of graphs (e.g., constant-degree expanders) where for every $0 < \delta < 1/2$ any bramble of order $\tilde{\Omega}(k^{0.5+\delta})$ requires size exponential in roughly $k^{2\delta}$.¹

In directed graphs, the notion of *bramble* generalizes to a family of strongly connected subgraphs such that every two subgraphs either intersect in a vertex, or the graph contains an arc with a tail in the first subgraph and a head in the second and an arc with a tail in the second subgraph and a head in the first. The order of a directed bramble is defined in the same way as in undirected graphs. We no longer have a tight relation between directed treewidth and maximum order of a directed bramble, but these two graph parameters are within a constant factor of each other [12]. However, the lower bound of Grohe and Marx [3] also applies to directed graphs: there are digraph families where a graph of directed treewidth k contains only brambles of order $k^{0.5+\delta}$ of exponential size, for any $0 < \delta < 0.5$.

Hence, it is natural to ask what order of a bramble of constant congestion we can expect in a directed graph of directed treewidth t . The lower bound of

¹ The notation $\tilde{\Omega}$ and \tilde{O} omits polylogarithmic factors.

Grohe and Marx shows that we cannot hope for a better answer than $\tilde{\mathcal{O}}(\sqrt{t})$. Since a directed grid contains a bramble of congestion 2 and order linear in the size of the grid, the Directed Grid Theorem (resp. a half-integral version) implies that for every $k \geq 1$ there exists $t = t(k)$ such that directed treewidth at least t guarantees an existence of a bramble of order k and congestion 2 (resp. congestion 4). However, the function $t = t(k)$ in the known proofs [7, 8] is very fast-growing.

In this work, we show that this dependency can be made polynomial, if we are satisfied with slightly larger congestion.

Theorem 1. *For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of congestion at most 8 and size at least k .*

So far, similar bounds were known only for planar graphs, where Hatzel, Kawarabayashi, and Kreutzer showed a polynomial bound for the Directed Grid Theorem [4]. Decreasing the congestion in Theorem 1, ideally to 2, while keeping the polynomial dependency of t on k , remains an open problem. Optimizing the parameters in the other direction would also be interesting: for all we know, obtaining the dependency $t = \tilde{\mathcal{O}}(k^2)$ for constant congestion may be possible.

On the technical level, the proof of Theorem 1 borrows a number of tools from previous works. From Reed and Wood [13], we borrow the idea of using Kostochka degeneracy bounds for graphs excluding a minor [9] to ensure the existence of a large clique minor in an intersection graph of a family of strongly connected subgraphs, if it turns out to be dense (which immediately gives a desired bramble). We also use their Lovász Local Lemma-based argument to find a large independent set in a multipartite graph of low degeneracy. Similarly as in the proof of Directed Grid Theorem [4, 8], we start from the notion of a path system and its existence (with appropriate parameters) in graphs of high directed treewidth. Finally, from our recent proof of half- and quarter-integral directed Erdős-Pósa property [10], we reuse their partitioning lemma, allowing us to find a large number of closed walks with small congestion. On top of the above, compared to [4] and [10], the proof of Theorem 1 offers a much more elaborate analysis of the studied path system, allowing us to find the desired bramble.

Preliminaries. For integers $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, 2, \dots, n\}$. An undirected graph is d -degenerate if its every subgraph has a vertex of degree at most d . For a family \mathcal{S} of sets, its *intersection graph* has vertex set \mathcal{S} , and two distinct sets $S_1, S_2 \in \mathcal{S}$ are adjacent if $S_1 \cap S_2 \neq \emptyset$.

For $A, B \subseteq V(G)$, such that $|A| = |B|$, a *linkage* from A to B in G is a set of $|A|$ pairwise vertex-disjoint paths in G , each with a starting vertex in A and ending vertex in B . A set $X \subseteq V(G)$ is *well-linked* if for every $A, B \subseteq X$, s.t. $|A| = |B|$ there are $|A|$ vertex-disjoint A - B -paths in $G - (X \setminus (A \cup B))$.

A *threaded linkage* is a pair (W, \mathcal{L}) , where $\mathcal{L} = \{L_1, \dots, L_\ell\}$ is a linkage and W is a walk such that there exist $\ell - 1$ paths $Q_1, \dots, Q_{\ell-1}$ such that W is the concatenation of $L_1, Q_1, L_2, Q_2, \dots, Q_{\ell-1}, L_\ell$ in that order. A threaded linkage

(W, \mathcal{L}) is *untangled* if for every i , the path Q_i may only intersect the rest of W in L_i or L_{i+1} . The *size* of an (untangled) threaded linkage (W, \mathcal{L}) is the size of linkage \mathcal{L} . The *overlap* of (W, \mathcal{L}) is $\max_{v \in V(G)} \text{oc}(v, W)$, where $\text{oc}(v, W)$ is the function counting the number of occurrences of vertex $v \in V(G)$ on W .

An (a, b) -*path system* $(P_i, A_i, B_i)_{i=1}^a$ consists of vertex-disjoint paths P_1, \dots, P_a , and, for every $i \in [a]$, sets $A_i, B_i \subseteq V(P_i)$, each of size b , where every vertex of B_i appears on P_i after all vertices of A_i and $\bigcup_{i=1}^a (A_i \cup B_i)$ is well-linked in G .

2 Sketch of the Proof of Theorem 1

Let us show some ideas used in the proof of Theorem 1. Let $k \in \mathbb{N}$ with $k > 1$ and let G be a graph of directed treewidth t . We show that if $t \geq c_t \cdot k^{48} \log^{13} k$ for appropriately chosen constant c_t , then G contains a bramble of congestion at most 8 and size k . We start by fixing the following parameters:

$$a = \tilde{\Theta}(k^2), \quad d_2 = \tilde{\Theta}(k^5), \quad d_1 = \Theta(a^2 d_2) = \tilde{\Theta}(k^9), \quad b = \Theta(a^2 d_1^2) = \tilde{\Theta}(k^{22}).$$

First, we aim to use the following result of Kawarabayashi and Kreutzer [8].

Lemma 1. *There is a constant c_{KK} such that for every $a, b \geq 1$ every directed graph G of directed treewidth at least $c_{KK} \cdot a^2 b^2$ contains an (a, b) -path system.*

By choosing c_t appropriately, we ensure that $t \geq c_{KK} a^2 b^2$, and thus, by Lemma 1, there is an (a, b) -path system $(P_i, A_i, B_i)_{i=1}^a$ in G .

Let $V = \{(i, j) \mid i, j \in [a] \wedge i \neq j\}$.

Claim 2.1. *For all $i, j \in [a]$, there exists a linkage $\mathcal{L}_{i,j}$ from B_i to A_j and a threaded linkage $(W_{i,j}, \mathcal{L}_{i,j})$ of size b and overlap at most 3.*

Proof. For every $i, j \in [a]$, we fix a linkage $\mathcal{L}_{i,j}$ from B_i to A_j and a linkage $\overleftarrow{\mathcal{L}}_{i,j}$ from A_j to B_i ; these linkages exist by well-linkedness of $\bigcup_{i=1}^a A_i \cup B_i$.

For every $P \in \mathcal{L}_{i,j}$ let $\rho_{i,j}(P)$ be the path of $\overleftarrow{\mathcal{L}}_{i,j}$ that starts at the ending point of P and let $\pi_{i,j}(P)$ be the path of $\mathcal{L}_{i,j}$ that starts at the ending point of $\rho_{i,j}(P)$. Note that $\pi_{i,j}$ is a permutation of $\mathcal{L}_{i,j}$. Let $\mathcal{C}_{i,j}$ be the family of cycles of the permutation $\pi_{i,j}$. Observe that every such a cycle corresponds to a closed walk composed of the paths in $\mathcal{L}_{i,j}$ and $\overleftarrow{\mathcal{L}}_{i,j}$.

From every cycle $C \in \mathcal{C}_{i,j}$ we arbitrarily select one path; we call it the *representative* of C . Let C_1, C_2, \dots, C_r be the elements of $\mathcal{C}_{i,j}$ in the order of the appearance of the starting points of their representatives along P_i . Define the walk $W_{i,j}$ as follows: follow P_i and for every $\ell \in [r]$, when we encounter the starting point of the representative of C_ℓ , follow the respective closed walk corresponding to C_ℓ , returning back to the starting point of the representative of C_ℓ , and then continue going along P_i . Finally, trim $W_{i,j}$ so that it starts and ends with a path of $\mathcal{L}_{i,j}$, as required by the definition of a threaded linkage.

Recall that the size of $(W_{i,j}, \mathcal{L}_{i,j})$ is the size of the linkage $\mathcal{L}_{i,j}$, i.e., b . One can verify that the overlap of $(W_{i,j}, \mathcal{L}_{i,j})$ is at most 3. ◁

Let us fix some $(i, j) \in V$ and denote $(W, \mathcal{L}) := (W_{i,j}, \mathcal{L}_{i,j})$. Let z be the length of W . For $1 \leq p \leq q \leq z$, by $W[p]$ we will denote the p -th vertex of W and by $W[p, q]$ we denote the subwalk $W[p], W[p + 1], \dots, W[q]$. A *useful walk* of W is a subwalk $W[p, q]$ of W , such that $W[p] = W[q]$ and $W[p, q]$ contains at least one path of \mathcal{L} as a subwalk. The pair (p, q) is called a *useful intersection*.

We greedily construct a sequence I_1, I_2, \dots, I_ℓ of useful walks as follows: $I_1 = W[p_1, q_1]$ is a useful walk of W such that q_1 is the smallest possible, and subsequently $I_{\xi+1} = W[p_{\xi+1}, q_{\xi+1}]$ is a useful walk of W such that $p_{\xi+1} > q_\xi$ and $q_{\xi+1}$ is the smallest possible. The greedy construction stops when there are no useful walks starting after q_ℓ . Let $Z \subseteq V$ be the set of those $(i, j) \in V$, for which the above procedure performs at least d_1 steps, i.e., $\ell \geq d_1$.

Claim 2.2. *For each $(i, j) \in V \setminus Z$ there exists an untangled threaded linkage $(W'_{i,j}, \mathcal{L}'_{i,j})$ of size $\Omega(a^2 d_1)$ and overlap at most 3, such that $W'_{i,j}$ is a subwalk of $W_{i,j}$ and $\mathcal{L}'_{i,j} \subseteq \mathcal{L}_{i,j}$.*

Proof. Let $(i, j) \in V \setminus Z$ and for brevity denote $(W, \mathcal{L}) := (W_{i,j}, \mathcal{L}_{i,j})$. Let ℓ and d be defined as above; recall that $\ell < d_1$. We select $\ell + 1$ subwalks $I'_1, I'_2, \dots, I'_{\ell+1}$ in W as follows. The subwalk I'_1 is defined as $W[1, q_1 - 1]$. Then, for $2 \leq \xi \leq \ell$, we define I'_ξ as $W[q_{\xi-1} + 1, q_\xi - 1]$. Finally, we define $I'_{\ell+1} := W[q_\ell + 1, z]$.

By the construction of the walks I_ξ , no I'_ξ contains a useful walk. Furthermore, the union of all walks I'_ξ covers W , except for q_1, \dots, q_ℓ . Hence, for at least $|\mathcal{L}| - \ell$ paths $P \in \mathcal{L}$ there is $\xi \in [\ell - 1]$ such that P is fully contained in I'_ξ . So there is some $\xi \in [\ell + 1]$, such that I'_ξ contains at least $\frac{|\mathcal{L}| - \ell}{\ell + 1} \geq \frac{b - (d_1 - 1)}{d_1} = \Omega(a^2 d_1)$ paths of \mathcal{L} . Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of paths contained in I'_ξ and let $W'_{i,j}$ be the walk I'_ξ , trimmed so that it starts and ends with a path of $\mathcal{L}'_{i,j}$. Note that $(W'_{i,j}, \mathcal{L}'_{i,j})$ is an untangled threaded linkage as I'_ξ contains no useful intersection. Moreover, the overlap of $(W'_{i,j}, \mathcal{L}'_{i,j})$ is at most 3. ◀

For both $\ell = 1, 2$, let $E_\ell \subseteq \binom{V}{2}$ be the set of those pairs $\{(i, j), (i', j')\} \in \binom{V}{2}$, for which the intersection graph of $\mathcal{L}'_{i,j}$ and $\mathcal{L}'_{i',j'}$ is not d_ℓ -degenerate. Define an undirected graph $H_\ell = (V, E_\ell)$. Since $d_1 \geq d_2$, we have $E_1 \subseteq E_2$. Let M_1 be a maximum matching in $H_1 - Z$. Let M_2 be a maximum matching in the graph $(V, E(H_2) \setminus \binom{V(M_1) \cup Z}{2})$, that is, in the graph that results from H_2 by removing all edges with both endpoints in $V(M_1) \cup Z$. We now observe that we can distinguish three subsets of V such that one of them is large enough for us.

Claim 2.3. *At least one of the following cases occurs:*

- Case 1. $|V \setminus (V(M_1) \cup Z)| \geq 0.6|V|$;
- Case 2. $|V(M_1) \cup V(M_2) \cup Z| \geq 0.6|V|$;
- Case 3. $|V \setminus V(M_2)| \geq 0.6|V|$.

It remains to show that every outcome of Claim 2.3 leads to a large bramble of congestion at most 8. To give some flavor of our approach, we sketch the proof of Case 1. The remaining two cases use a similar basic approach but need more tools to extract brambles from sets of pairs of linkages and make crucial use of the low overlap in Claims 2.1 and 2.2. See the full version for the details [11].

Let $\mathcal{I} := V \setminus (V(M_1) \cup Z)$. By the assumption of Case 1 we have $|\mathcal{I}| \geq 0.6a(a - 1)$. Observe that for every pair $(i, j), (i', j') \in \mathcal{I}$ the intersection graph of $\mathcal{L}'_{i,j}$ and $\mathcal{L}'_{i',j'}$ is d_1 -degenerate, and each of these linkages is relatively large. This allows us to apply the Lovász Local Lemma-based argument by Reed and Wood [13], in order to obtain a single path $P_{i,j} \in \mathcal{L}_{i,j}$ for each $(i, j) \in \mathcal{I}$, such that the paths in $\{P_{i,j}\}_{(i,j) \in \mathcal{I}}$ are pairwise disjoint.

Consider an auxiliary graph H with vertex set $[a]$ and $ij \in E(H)$ if both $(i, j) \in \mathcal{I}$ and $(j, i) \in \mathcal{I}$. Since $|\mathcal{I}| \geq 0.6 \cdot a(a - 1)$, we have $|E(H)| \geq 0.1 \cdot \binom{a}{2}$. By the celebrated result by Kostochka [9], H contains a clique minor of size $p = \tilde{\Omega}(a)$. Without loss of generality, we may assume that $p = \binom{q}{2}$ for some integer $q = \Omega(a^{1/2} / \log^{1/4} a)$. By appropriate choice of a we ensure that $q \geq k$. Let $(B_{x,y})_{\{x,y\} \in \binom{[q]}{2}}$ be the family of branch sets of the clique minor of size p in H . Observe that for every $x \in [q]$, the subgraph of H induced by $\bigcup_{y \in [q] \setminus \{x\}} B_{x,y}$ is connected; let T_x be its spanning tree. Note that for every two distinct $x, y \in [q]$, the trees T_x and T_y intersect in $B_{x,y}$. On the other hand, every vertex and every edge of H is contained in at most two trees T_x .

Now we transfer the set of trees above to a bramble in G . For every $i \in [a]$, let e_i be the last arc of P_i , whose tail is in A_i . Note that e_i is well-defined, as the set B_i follows A_i on P_i . For every edge $e = ij \in E(H)$, let W_e be a closed walk in G obtained as follows: Start with $P_{i,j}$, and then follow P_j until arriving at the starting vertex of $P_{j,i}$. Then follow $P_{j,i}$, and then P_i until the walk is closed. Note that W_e contains both e_i and e_j . For every $x \in [q]$, define a subgraph G_x of G as the union of all walks W_e for all $e \in E(T_x)$. Since for every $e = ij \in E(H)$, the walk W_e contains e_i and e_j , and T_x is connected, the graph G_x is strongly connected and contains all edges e_i for $i \in V(T_x)$. Thus, since every two trees T_x and T_y intersect in $B_{x,y}$, the family $(G_x)_{x \in [q]}$ is a bramble of size $q \geq k$. One can verify that the congestion of the bramble constructed in this case is at most 4.

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Component Behaviour of Random Bipartite Graphs

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Abstract. We study the component behaviour of the binomial random bipartite graph $G(n, n, p)$ near the critical point. We show that, as is the case in the binomial random graph $G(n, p)$, for an appropriate range of p there is a unique ‘giant’ component of order at least $n^{\frac{2}{3}}$ and determine asymptotically its order and excess. Our proofs rely on good enumerative estimates for the number of bipartite graphs of a fixed order, as well as probabilistic techniques such as the sprinkling method.

Keywords: Component behaviour · Random bipartite graphs · Critical point

1 Introduction

1.1 Background and Motivation

It was shown by Erdős and Rényi [3] that a ‘phase transition’ occurs in the uniform random graph model $G(n, m)$ when m is about $\frac{n}{2}$. This can also be cast in terms of the binomial random graph $G(n, p)$ at around $p = \frac{1}{n}$. More precisely, they showed that when $p = \frac{1-\epsilon}{n}$ for some fixed $\epsilon > 0$, with high probability¹ (whp for short) every component of $G(n, p)$ has order at most $O(\log n)$; when $p = \frac{1}{n}$ whp the order of largest component is $\Theta\left(n^{\frac{2}{3}}\right)$; and when $p = \frac{1+\epsilon}{n}$ whp $G(n, p)$ will contain a unique ‘giant component’ $L_1(G(n, p))$ of order $\Omega(n)$.

Whilst it may seem at first that the component behaviour of the model $G(n, p)$ exhibits quite a sharp ‘jump’ at around this point, subsequent investigations, notably by Bollobás [1] and Łuczak [6], showed that in fact if one chooses the correct parameterisation for p , this change can be seen to happen quite smoothly. In particular, Łuczak’s work implies the following. Throughout the paper let $L_i(G)$ denote the i th largest component of a graph G for $i \in \mathbb{N}$.

T. A. Do, J. Erde and M. Kang—Supported by Austrian Science Fund (FWF): I3747, W1230.

¹ With probability tending to one as $n \rightarrow \infty$.

Theorem 1. *Let $\epsilon = \epsilon(n) > 0$ be such that $\epsilon^3 n \rightarrow \infty$ and $\epsilon = o(1)$, and let $p = \frac{1+\epsilon}{n}$. Then whp $|L_1(G(n, p))| = (1 + o(1))2\epsilon n$ and $|L_2(G(n, p))| \leq n^{\frac{2}{3}}$. Furthermore, the excess² of $L_1(G(n, p))$ is $(1 + o(1))\frac{2}{3}\epsilon^3 n$.*

Theorem 2. *Let $\epsilon = \epsilon(n)$ be such that $|\epsilon^3|n \rightarrow \infty$ and $\epsilon = o(1)$ and let $p = \frac{1+\epsilon}{n}$ and $\delta = \epsilon - \log(1 + \epsilon)$. Let $\alpha = \alpha(n) > 0$ be an arbitrary function. Then the following hold in $G(n, p)$.*

- (1) *With probability at least $1 - e^{-\Omega(\alpha)}$ there are no tree components of order larger than $\frac{1}{\delta} (\log(|\epsilon|^3 n) - \frac{5}{2} \log \log(|\epsilon|^3 n) + \alpha)$.*
- (2) *With probability at least $1 - e^{-\Omega(\alpha)}$ there are no unicyclic components of order larger than $\frac{\alpha}{\delta}$.*
- (3) *If $\epsilon < 0$ then whp there are no complex components.*
- (4) *If moreover $\epsilon^3 n \rightarrow \infty$ then with probability at least $1 - O(\frac{1}{\epsilon^3 n})$ there are no complex components of order smaller than $n^{\frac{2}{3}}$.*

The aim of this article is to investigate the component behaviour of the binomial random bipartite graph $G(n, n, p)$, which is the random graph given by taking two partition classes N_1 and N_2 of order n , and including each edge between N_1 and N_2 independently and with probability p , which has been the object of recent study [2, 4, 5].

1.2 Main Results

Our first main result determines asymptotically the existence, order and excess of the ‘giant’ component in $G(n, n, p)$ near the critical point.

Theorem 3. *Let $\epsilon = \epsilon(n) > 0$ be such that $\epsilon^4 n \rightarrow \infty$ and $\epsilon = o(1)$, $p = \frac{1+\epsilon}{n}$ and let $L_i = L_i(G(n, n, p))$ for $i = 1, 2$. Then whp $|L_1 \cap N_1| = (1 + o(1))2\epsilon n$ and $|L_1 \cap N_2| = (1 + o(1))2\epsilon n$. Furthermore, whp $|L_2| \leq n^{\frac{2}{3}}$ and the excess of L_1 is $(1 + o(1))\frac{4}{3}\epsilon^3 n$.*

Our next theorem gives a finer picture of the component structure of $G(n, n, p)$ near the critical point.

Theorem 4. *Let $\epsilon = \epsilon(n)$ be such that $|\epsilon^3|n \rightarrow \infty$ and $\epsilon = o(1)$ and let $p = \frac{1+\epsilon}{n}$ and*

$$\delta = \epsilon - \log(1 + \epsilon). \tag{1}$$

Let $\alpha = \alpha(n) > 0$ be an arbitrary function. Then the following hold in $G(n, n, p)$.

- (1) *With probability at least $1 - e^{-\Omega(\alpha)}$ there are no tree components of order larger than $\frac{1}{\delta} (\log(|\epsilon|^3 n) - \frac{5}{2} \log \log(|\epsilon|^3 n) + \alpha)$.*
- (2) *With probability at least $1 - e^{-\Omega(\alpha)}$ there are no unicyclic components of order larger than $\frac{\alpha}{\delta}$.*
- (3) *If $\epsilon < 0$, then whp there are no complex components.*

² The excess of a connected graph G is $|E(G)| - |V(G)|$.

(4) If moreover $\epsilon^4 n \rightarrow \infty$, then with probability at least $1 - O\left(\frac{1}{\epsilon^4 n}\right)$ there are no complex components of order smaller than $n^{\frac{2}{3}}$.

The proof of Theorem 4 uses enumerative estimates on the number of bipartite graphs of a fixed order and excess to bound the probability of certain types of components existing. Using these estimates and Theorem 4 we can bound quite precisely the number of vertices contained in *large* components in $G(n, n, p)$, those of order at least $n^{\frac{2}{3}}$. Then, using a sprinkling argument, we show that whp there is a unique large component L_1 . Given the order of L_1 we can use these enumerative estimates again to give a weak bound on its excess, which we can then bootstrap to an asymptotically tight bound via a multi-round exposure argument.

2 Enumerative Estimates

Throughout this section, unless stated otherwise we let $\epsilon = \epsilon(n)$ be such that $|\epsilon^3|n \rightarrow \infty$ and $\epsilon = o(1)$ and let $p = \frac{1+\epsilon}{n}$. Given $i, j \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ let $X(i, j, \ell)$ denote the number of components in $G(n, n, p)$ with i vertices in N_1 , j vertices in N_2 , and excess ℓ and let $C(i, j, \ell)$ denote the number of connected bipartite graphs with i vertices in one partition class, j in the second, and $i + j + \ell$ many edges. Letting $i + j = k$ we have

$$\mathbb{E}(X(i, j, \ell)) = \binom{n}{i} \binom{n}{j} C(i, j, \ell) p^{k+\ell} (1-p)^{kn-ij-k-\ell}.$$

We can use this formula to estimate the expected number of tree, unicyclic and complex components, where a tree component has excess $\ell = -1$, a *unicyclic* component has excess $\ell = 0$ (and hence contains a unique cycle), and a *complex* component has excess $\ell \geq 1$. Hence, in order to have a good estimate for $\mathbb{E}(X(i, j, \ell))$ it will be useful to have good estimates for $C(i, j, \ell)$.

A classic result of Scoins [8] determines precisely $C(i, j, -1)$.

Theorem 5 [8]. *Let $i, j \in \mathbb{N}$, then $C(i, j, -1) = i^{j-1} j^{i-1}$.*

We give an exact formula for $C(i, j, 0)$.

Theorem 6. *For any $i, j \in \mathbb{N}$ we have*

$$C(i, j, 0) = \frac{1}{2} \sum_r \binom{i}{r} \binom{j}{r} (i^{j-r-1} j^{i-r} + j^{i-r-1} i^{j-r})$$

and so in particular if $i, j \rightarrow \infty$ and $\frac{1}{2} \leq \frac{i}{j} \leq 2$, then

$$C(i, j, 0) \sim \sqrt{\frac{\pi}{8}} \sqrt{i+j} i^{j-\frac{1}{2}} j^{i-\frac{1}{2}}.$$

Since a unicyclic graph is the union of a cycle and a forest, we are able to deduce Theorem 6 from a formula for the number of bipartite forests, which we derive using Prüfer codes. We also give an upper bound for $C(i, j, \ell)$.

Theorem 7. *There is a constant c such that for $i, j, \ell \in \mathbb{N}$ with $\ell \leq ij - i - j$ and $\frac{1}{2} \leq \frac{i}{j} \leq 2$,*

$$C(i, j, \ell) \leq i^{j+\frac{1}{2}} j^{i+\frac{1}{2}} (i+j)^{\frac{3\ell+1}{2}} \left(\frac{i}{j}\right)^{\frac{i-j}{2}} \left(\frac{c}{\ell}\right)^{\frac{\ell}{2}}.$$

Furthermore, if $\ell \geq i + j$, then $C(i, j, \ell) \leq i^{j-\frac{1}{2}} j^{i-\frac{1}{2}} (i+j)^{\frac{3\ell+1}{2}} \ell^{-\frac{\ell}{2}}$.

For $\ell \geq i + j$ Theorem 7 follows from the naive bound $C(i, j, \ell) \leq \binom{ij}{i+j+\ell}$, whereas for $\ell \leq i + j$ we follow an ingenious probabilistic argument of Łuczak [7] to bound $C(i, j, \ell)$. We note that for small enough ℓ , for example $\ell = O(1)$, the naive bound that follows from

$$C(i, j, \ell) \leq C(i, j, 0)(ij)^\ell \leq \sqrt{i+j} i^{j+\ell-\frac{1}{2}} j^{i+\ell-\frac{1}{2}}$$

will be more effective than the first part of Theorem 7.

As a consequence, we can derive good estimates for $\mathbb{E}(X(i, j, \ell))$. To ease notation we set

$$S(n, \epsilon, i, j) = \exp\left(-\frac{(i-j)^2}{2n} - \frac{i^3+j^3}{6n^2} + \frac{\epsilon ij}{n} + O\left(\frac{ij}{n^2} + \frac{i^4+j^4}{n^3}\right)\right).$$

Firstly, we can give an asymptotic formula for the expected number of tree and unicyclic components in $G(n, n, p)$.

Theorem 8. *For any $i, j \in \mathbb{N}$, δ is as in (1), and letting $k = i + j$, we have*

$$\mathbb{E}(X(i, j, -1)) \sim \frac{ne^{-\delta k}}{2\pi(ij)^{\frac{3}{2}}} \left(\frac{i}{j}\right)^{j-i} S(n, \epsilon, i, j).$$

Theorem 9. *For any $i, j \in \mathbb{N}$ with $\frac{1}{2} \leq \frac{i}{j} \leq 2$, δ is as in (1), and letting $k = i + j$, we have*

$$\mathbb{E}(X(i, j, 0)) \sim \frac{\sqrt{k}e^{-\delta k}}{4\sqrt{2\pi}ij} \left(\frac{i}{j}\right)^{j-i} S(n, \epsilon, i, j).$$

Also we can bound from above the expected number of complex components.

Theorem 10. *For any $i, j \in \mathbb{N}$ with $\frac{1}{2} \leq \frac{i}{j} \leq 2$, $\ell \leq ij - i - j$, and letting $k = i + j$, we have*

$$\begin{aligned} \mathbb{E}(X(i, j, \ell)) &\leq \sqrt{k}e^{-\delta k + \frac{\epsilon k^2}{4n}} \left(\frac{i}{j}\right)^{\frac{i-i}{2}} \\ &\times \exp\left(-\frac{(i-j)^2}{2n} + O\left(\frac{ij}{n^2}\right)\right) \left(\frac{ck^3}{\ell n^2}\right)^{\frac{\ell}{2}} \exp\left(\ell \log(1+\epsilon) + \frac{\ell(1+\epsilon)}{n}\right), \end{aligned}$$

and for $\ell = O(1)$, and δ is as in (1),

$$\mathbb{E}(X(i, j, \ell)) = O\left(\frac{\sqrt{k}}{n^\ell} e^{-\delta k + \frac{\epsilon k^2}{4n}} (ij)^{\ell-1} \left(\frac{i}{j}\right)^{i-j} \exp\left(-\frac{(i-j)^2}{2n} + O\left(\frac{ij}{n^2}\right)\right)\right).$$

3 A Finer Look at Component Structure of $G(n, n, p)$

Using the estimates from Sect. 2 we can describe more precisely the component structure of $G(n, n, p)$.

Below, particularly in Theorems 11–13, we let $\epsilon = \epsilon(n)$ be such that $|\epsilon^3|n \rightarrow \infty$ and $\epsilon = o(1)$ and let $p = \frac{1+\epsilon}{n}$. Let us first consider the number of tree components. As indicated in Theorem 4 (1), we will show that whp there are no tree components of order significantly larger than $\frac{1}{\delta}(\log(|\epsilon|^3n) - \frac{5}{2} \log \log(|\epsilon|^3n))$, and moreover that the number of components of order around this order will tend to a Poisson distribution.

Theorem 11.

- (1) Given $r_1, r_2 \in \mathbb{R}^+$ with $r_1 < r_2$ let Y_{r_1, r_2} denote the number of tree components in $G(n, n, p)$ of orders between

$$\frac{1}{\delta} \left(\log(|\epsilon|^3n) - \frac{5}{2} \log \log(|\epsilon|^3n) + r_1 \right)$$

and

$$\frac{1}{\delta} \left(\log(|\epsilon|^3n) - \frac{5}{2} \log \log(|\epsilon|^3n) + r_2 \right),$$

where δ is as in (1), and let $\lambda := \frac{1}{\sqrt{\pi}}(e^{-r_1} - e^{-r_2})$. Then Y_{r_1, r_2} converges in distribution to $Po(\lambda)$.

- (2) With probability at least $1 - e^{-\Omega(\alpha)}$, $G(n, n, p)$ contains no tree components of order larger than $\frac{1}{\delta}(\log(|\epsilon|^3n) - \frac{5}{2} \log \log(|\epsilon|^3n) + \alpha)$ for any function $\alpha = \alpha(n) > 0$.

Similarly, as indicated in Theorem 4 (2), we will show that whp there are no tree components of order significantly larger than $\frac{1}{\delta}$, and moreover that the number of components of order around this order will tend to a Poisson distribution.

Theorem 12.

- (1) Given $u_1, u_2 \in \mathbb{R}^+$ with $u_1 < u_2$ let Z_{u_1, u_2} denote the number of unicyclic components in $G(n, n, p)$ of orders between $\frac{u_1}{\delta}$ and $\frac{u_2}{\delta}$, where δ is as in (1), and let $\nu := \frac{1}{2} \int_{u_1}^{u_2} \frac{\exp(-t)}{t} dt$. Then Z_{u_1, u_2} converges in distribution to $Po(\nu)$.
- (2) With probability at least $1 - e^{-\Omega(\alpha)}$, $G(n, n, p)$ contains no unicyclic components of order larger than $\frac{\alpha}{\delta}$ for any function $\alpha = \alpha(n) > 0$.

Finally, as indicated in Theorem 4 (3) & (4), we will show that whp there are no large complex components, and in fact no complex components at all in the subcritical regime.

Theorem 13.

- (i) If $\epsilon < 0$ then with probability at least $1 - O(\frac{1}{\epsilon^3n})$, $G(n, n, p)$ contains no complex components.

(ii) If in addition $\epsilon^4 n \rightarrow 0$ then with probability at least $1 - O\left(\frac{1}{\epsilon^4 n}\right)$, $G(n, n, p)$ contains no complex components of order smaller than $n^{\frac{2}{3}}$.

Theorem 4 follows as a direct consequence of Theorems 11–13.

We now consider the largest and second largest components in $G(n, n, p)$ in the supercritical regime. Let ϵ' be defined as the unique positive solution of $(1 - \epsilon')e^{\epsilon'} = (1 + \epsilon)e^{-\epsilon}$.

Lemma 1. *Let $\epsilon = \epsilon(n) > 0$ be such that $\epsilon^4 n \rightarrow \infty$ and $\epsilon = o(1)$, $p = \frac{1+\epsilon}{n}$ and let $L_i := L_i(G(n, n, p))$ for $i = 1, 2$. With probability at least $1 - O\left((\epsilon^4 n)^{-\frac{1}{6}}\right)$ we have*

$$\left|L_1 - \frac{2(\epsilon + \epsilon')}{1 + \epsilon}n\right| < \frac{n^{\frac{2}{3}}}{50} \quad \text{and} \quad |L_2| \leq n^{\frac{2}{3}}.$$

Furthermore with probability at least $1 - O\left((\epsilon^3 n)^{-\frac{1}{6}}\right)$ we have that $|L_1 \cap N_1| = (1 \pm 2\sqrt{\epsilon})|L_1 \cap N_2|$.

The first part of Theorem 3 follows directly from Lemma 1 and for the second part we argue via a multi-round exposure argument, starting with some supercritical p' which is significantly smaller than p . Using the bound on $|L_1|$ we can show that the excess of the giant in $G(n, n, p')$ is $o(\epsilon^3 n)$, and we can also estimate quite precisely the change in the excess of the giant between each stage of the multi-round exposure, leading to the asymptotically tight bound on the excess of L_1 .

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Maximising Line Subgraphs of Diameter at Most t

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Abstract. We wish to bring attention to a natural but slightly hidden problem, posed by Erdős and Nešetřil in the late 1980s, an edge version of the degree–diameter problem. Our main result is that, for any graph of maximum degree Δ with more than $1.5\Delta^t$ edges, its line graph must have diameter larger than t . In the case where the graph contains no cycle of length $2t + 1$, we can improve the bound on the number of edges to one that is exact for $t \in \{1, 2, 3, 4, 6\}$. In the case $\Delta = 3$ and $t = 3$, we obtain an exact bound. Our results also have implications for the related problem of bounding the distance- t chromatic index, $t > 2$; in particular, for this we obtain an upper bound of $1.941\Delta^t$ for graphs of large enough maximum degree Δ , markedly improving upon earlier bounds for this parameter.

Keywords: Degree-diameter problem · Strong cliques · Distance edge-colouring

1 Introduction

Erdős in [9] wrote about a problem he proposed with Nešetřil:

“One could perhaps try to determine the smallest integer $h_t(\Delta)$ so that every G of $h_t(\Delta)$ edges each vertex of which has degree $\leq \Delta$ contains two edges so that the shortest path joining these edges has length $\geq t$... *This problem seems to be interesting only if there is a nice expression for $h_t(\Delta)$* ”.

Equivalently, $h_t(\Delta) - 1$ is the largest number of edges inducing a graph of maximum degree Δ whose line graph has diameter at most t . Alternatively, one could consider this an edge version of the (old, well-studied, and exceptionally difficult) degree–diameter problem, cf. [3].

It is easy to see that $h_t(\Delta)$ is at most $2\Delta^t$ always, but one might imagine it to be smaller. For instance, the $t = 1$ case is easy and $h_1(\Delta) = \Delta + 1$. For $t = 2$,

it was independently proposed by Erdős and Nešetřil [9] and Bermond, Bond, Paoli and Peyrat [2] that $h_2(\Delta) \leq 5\Delta^2/4 + 1$, there being equality for even Δ . This was confirmed by Chung, Gyárfás, Tuza and Trotter [7].

For the case $t = 3$, we suggest the following as a “nice expression”.

Conjecture 1. $h_3(\Delta) \leq \Delta^3 - \Delta^2 + \Delta + 2$, with equality if Δ is one more than a prime power.

As to the hypothetical sharpness of this conjecture, first consider the point–line incidence graphs of finite projective planes of prime power order q . Writing $\Delta = q + 1$, such graphs are bipartite, Δ -regular, and of girth 6; their line graphs have diameter 3; and they have $\Delta^3 - \Delta^2 + \Delta$ edges. At the expense of bipartiteness and Δ -regularity, one can improve on the number of edges in this construction by *one* by subdividing one edge, which yields the expression in Conjecture 1. We remark that for multigraphs instead of simple graphs, one can further increase the number of edges by $\lfloor \frac{\Delta}{2} \rfloor - 1$, by deleting some arbitrary vertex v and replacing it with a multiedge of multiplicity $\lfloor \frac{\Delta}{2} \rfloor$, whose endvertices are connected with $\lfloor \frac{\Delta}{2} \rfloor$ and $\lceil \frac{\Delta}{2} \rceil$ of the original Δ neighbours of v . This last remark contrasts to what we know for multigraphs in the case $t = 2$, cf. [4, 8].

For larger fixed t , although we are slightly less confident as to what a “nice expression” for $h_t(\Delta)$ might be, we believe that $h_t(\Delta) = (1 + o(1))\Delta^t$ holds for infinitely many Δ .

We contend that this naturally divides into two distinct challenges, the former of which appears to be more difficult than the latter.

Conjecture 2. For any $\varepsilon > 0$, $h_t(\Delta) \geq (1 - \varepsilon)\Delta^t$ for infinitely many Δ .

Conjecture 3. For $t \neq 2$ and any $\varepsilon > 0$, $h_t(\Delta) \leq (1 + \varepsilon)\Delta^t$ for all large enough Δ .

With respect to Conjecture 2, we mentioned earlier how it is known to hold for $t \in \{1, 2, 3\}$. For $t \in \{4, 6\}$, it holds due to the point–line incidence graphs of, respectively, a symplectic quadrangle with parameters $(\Delta - 1, \Delta - 1)$ and a split Cayley hexagon with parameters $(\Delta - 1, \Delta - 1)$ when $\Delta - 1 = q$ is a prime power. For all other values of t the conjecture remains open. Conjecture 2 may be viewed as the direct edge analogue of an old conjecture of Bollobás [3]. That conjecture asserts, for any positive integer t and any $\varepsilon > 0$, that there is a graph of maximum degree Δ with at least $(1 - \varepsilon)\Delta^t$ vertices of diameter at most t for infinitely many Δ . The current status of Conjecture 2 is essentially the same as for Bollobás’s conjecture: it is unknown if there is an absolute constant $c > 0$ such that $h_t(\Delta) \geq c\Delta^t$ for all t and infinitely many Δ . For large t the best constructions we are aware of are (ones derived from) the best constructions for Bollobás’s conjecture.

Proposition 1. *There is t_0 such that $h_t(\Delta) \geq 0.629^t \Delta^t$ for $t \geq t_0$ and infinitely many Δ .*

Proof. Canale and Gómez [6] proved the existence of graphs of maximum degree Δ , of diameter t' , and with more than $(0.6291\Delta)^{t'}$ vertices, for t' large enough and infinitely many Δ . Consider this construction for $t' = t - 1$ and each valid Δ . Now in an iterative process arbitrarily add edges between vertices of degree less than Δ . Note that as long as there are at least $\Delta + 1$ such vertices, then for every one there is at least one other to which it is not adjacent. Thus by the end of this process, at most Δ vertices have degree smaller than Δ , and so the resulting graph has at least $\frac{1}{2}((0.6291\Delta)^t \Delta - \Delta^2)$ edges, which is greater than $(0.629\Delta)^t$ for t sufficiently large. Furthermore since the graph has diameter at most $t - 1$, its line graph has diameter at most t . \square

By the above argument (which was noted in [17]), the truth of Bollobás’s conjecture would imply a slightly weaker form of Conjecture 2, that is, with a leading asymptotic factor of $1/2$. As far as we are aware, a reverse implication, i.e. from Conjecture 2 to some form of Bollobás’s conjecture is not known.

1.1 Notation

For a graph $G = (V, E)$, we denote the i^{th} neighbourhood of a vertex v by $N_i(v)$, that is, $N_i(v) = \{u \in V \mid d(u, v) = i\}$, where $d(u, v)$ denotes the distance between u and v in G . Similarly, we define $N_i(e)$ as the set of vertices at distance i from an endpoint of e .

Let $T_{k,\Delta}$ denote a tree rooted at v of height k (i.e. the leafs are exactly $N_k(v)$) such that all non-leaf vertices have degree Δ . Let $T_{k,\Delta}^1$ be one of the Δ subtrees starting at v , i.e. a subtree rooted at v of height k such that v has degree 1, such that $N_k(v)$ only contains leaves and all non-leaf vertices have degree Δ .

2 Results

Our main result is partial progress towards Conjecture 3 (and thus Conjecture 1). In order to discuss one consequence of our work, we can reframe the problem of estimating $h_t(\Delta)$ in stronger terms. Let us write $L(G)$ for the line graph of G and H^t for the t -th power of H (where we join pairs of distinct vertices at distance at most t in H). Then the problem of Erdős and Nešetřil framed at the beginning of the paper is equivalent to seeking optimal bounds on $|L(G)|$ subject to G having maximum degree Δ and $L(G)^t$ inducing a clique. Letting $\omega(H)$ denote the clique number of H , our main results are proven in terms of bounds on the *distance- t edge-clique number* $\omega(L(G)^t)$ for graphs G of maximum degree Δ .

We have settled Conjecture 3 in the special case of graphs containing no cycle C_{2t+1} of length $2t + 1$ as a subgraph in this more general framework.

Theorem 1. *Let $t \geq 2$ be an integer. Let G be a C_{2t+1} -free graph with maximum degree Δ . Then $\omega(L(G)^t) \leq |E(T_{t,\Delta})|$, in particular the line graph of C_{2t+1} -free graph of maximum degree Δ with at least $|E(T_{t,\Delta})| + 1$ edges has diameter greater than t .*

For $t \in \{1, 2, 3, 4, 6\}$, the statement is sharp due to the point–line incidence graphs of generalised polygons. The cases $t \in \{3, 4, 6\}$ are perhaps most enticing in Conjecture 3, and that is why we highlighted the case $t = 3$ first in Conjecture 1.

The bound in Theorem 1 is a corollary of the following proposition.

Proposition 2. *Let $H \subseteq G$ be a graph with maximum degree Δ_H which is a subgraph of a C_{2t+1} -free graph G with maximum degree Δ . Let v be a vertex with degree $d_H(v) = \Delta_H = j$ and let u_1, u_2, \dots, u_j be its neighbours. Suppose that in $L^t(G)$, every edge of H is adjacent to vu_i for every $1 \leq i \leq j$. Then $|E(H)| \leq |E(T_{t,\Delta})|$.*

The sketch of the proof is as follows. For fixed Δ , let H and G be graphs satisfying all conditions, such that $|E(H)|$ is maximised. Note that such a choice does exist since $|E(H)|$ is bounded, by e.g. Δ^{t+1} . With respect to the graph G , we write $N_i = N_i(v)$ for $0 \leq i \leq t + 1$. We start proving a claim that makes work easier afterwards.

Claim. For any $1 \leq i \leq t$, H does not contain any edge between two vertices of N_i .

Proof. Suppose it is not true for some $i \leq t - 1$. Take an edge $yz \in E(H)$ with $y, z \in N_i$. Construct the graph H' with $V(H') = V(H) \cup \{y', z'\}$ and $E(H') = E(H) \setminus yz \cup \{yy', zz'\}$, where y' and z' are new vertices, and let G' be the corresponding modification of G . Then $H' \subseteq G'$ also satisfies all conditions in Proposition 2 and $|E(H')| = |E(H)| + 1$, contradicting the choice of H .

Next, suppose there is an edge $yz \in E(H)$ with $y, z \in N_t$. Take a shortest path from u_1 to yz , which is without loss of generality a path P_y from u_1 to y . Note that a shortest path P_z from v to z will intersect P_y since G is C_{2t+1} -free. Let w be the vertex in $V(P_y) \cap V(P_z)$ that minimises $d_G(w, z)$.

If $d_G(w, u_i) = m - 1$ for every $1 \leq i \leq j$, we can remove yz again and add two edges yy' and zz' to get a graph H' satisfying all conditions, leading to a contradiction again. This is the blue scenario in Fig. 1.

In the other case there is some $1 < s \leq j$ such that $d_G(w, u_s) > m - 1$. Since $d_G(u_s, yz) = t - 1$, without loss of generality $d_G(u_s, z) = t - 1$, there is a shortest path from u_s to z which is disjoint from the previously selected shortest path P_y between u_1 and y . Hence together with the edges u_1v, vu_s and yz , this forms a C_{2t+1} in G , which again is a contradiction. This is sketched as the red scenario in Fig. 1. ◊

Now it is handy to track the number of edges incident with certain vertices. For every $1 \leq m \leq t + 1$, let A_m be the set of all vertices x in N_m such that $d_G(v, x) = d_G(u_i, x) + 1 = m$ for at least one index $1 \leq i \leq j$ and let $R_m = N_m \setminus A_m$. Also let $A_0 = \{v\}$. Let $A = \cup_{i=0}^{t+1} A_i$ and $R = \cup_{i=0}^{t+1} R_i$. By observing that no edge of H is incident to any vertex in R_{t+1} , one can find that the number of edges of H which contain at least one vertex of R can be upper bounded by $|R_1| \cdot \left(|E(T_{t,\Delta}^{(1)})| - 1 \right)$, which equals

$$(\deg(v) - \Delta_H) \left(\frac{1}{\Delta} |E(T_{t,\Delta})| - 1 \right). \tag{1}$$

To count the number of edges adjacent to a vertex in A , we define a weight function w that helps. Define w on the vertices x in A in such a way that for every $x \in A_m$ where $1 \leq m \leq t$ the value $w(x)$ equals the number of paths (in G) of length $m - 1$ between x and A_1 . Then for $x \in A_t$, by separately analysing the cases $w(x) < j$ and $w(x) \geq j$, we conclude that the number of edges in H adjacent to x will be bounded by $w(x)$. By doing so we find the desired bound on $|E(H[A])|$ to conclude.

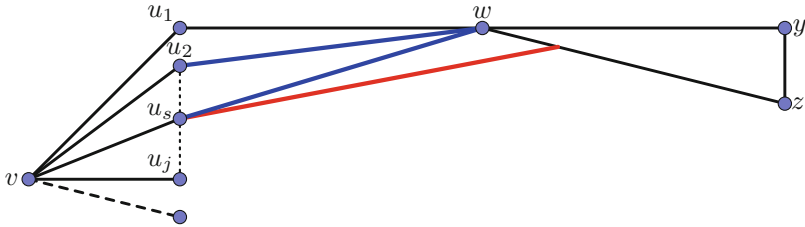


Fig. 1. Sketch of some scenarios in Claim 2

For the general case of Conjecture 3, we obtain the following weaker bound.

Theorem 2. $h_t(\Delta) \leq \frac{3}{2}\Delta^t + 1$.

Theorem 2 is a result/proof valid for all $t \geq 1$, but as we already mentioned there are better, sharp determinations for $t \in \{1, 2\}$.

We prove Theorem 2 by doing so for the following stronger form.

Theorem 3. For any graph G of maximum degree Δ , it holds that $\omega(L(G)^t) \leq \frac{3}{2}\Delta^t$.

The proof here is an extension of the proof sketched for Theorem 1. So one can prove the following analog of Proposition 2.

Proposition 3. Let $H \subseteq G$ be a graph with maximum degree Δ_H which is a subgraph of a graph G with maximum degree Δ . Let v be a vertex with degree $d_H(v) = \Delta_H = j$ and let u_1, u_2, \dots, u_j be its neighbours. Suppose that in $L^t(G)$, every edge of H is adjacent to vu_i for every $1 \leq i \leq j$. Then

$$|E(H)| \leq \sum_{m=1}^{t-1} \Delta(\Delta - 1)^{m-1} + \frac{3}{2}\Delta(\Delta - 1)^{t-1}.$$

Claim 2 is still true in this more general setting, except for $i = t$. So there are plausibly some edges between vertices in A_t . This will imply that we can bound the number of edges adjacent to vertices in A_t only by $\frac{3}{2} \sum_{a \in A_t} w(a)$, resulting in the weaker bound. Furthermore Proposition 3 itself is sharp, implying that one needs a more global perspective on the problem to improve on Theorem 3. For

example when $t = 2$, the following example (Fig. 2) shows that the blow-up of a C_5 is not extremal anymore when only taking into account the weaker conditions from Proposition 3. We should remark that Dębski and Śleszyńska-Nowak [17] announced a bound of roughly $\frac{7}{4}\Delta^t$. Note that the bound in Theorem 3 can be improved in the cases $t \in \{1, 2\}$: $\omega(L(G)) \leq \Delta + 1$ is trivially true, while $\omega(L(G)^2) \leq \frac{4}{3}\Delta^2$ is a recent result of Faron and Postle [10].

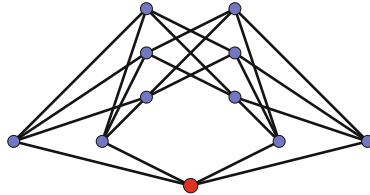


Fig. 2. An extremal graph for Proposition 3.

A special motivation for us is a further strengthened form of the problem. In particular, there has been considerable interest in $\chi(L(G)^t)$ (where $\chi(H)$ denotes the chromatic number of H), especially for G of bounded maximum degree. For $t = 1$, this is the usual chromatic index of G ; for $t = 2$, it is known as the *strong chromatic index* of G , and is associated with a more famous problem of Erdős and Nešetřil [9]; for $t > 2$, the parameter is referred to as the *distance- t chromatic index*, with the study of bounded degree graphs initiated in [13]. We note that the output of Theorem 3 may be directly used as input to a recent result [11] related to Reed’s conjecture [15] to bound $\chi(L(G)^t)$. This yields the following.

Corollary 1. *There is some Δ_0 such that, for any graph G of maximum degree $\Delta \geq \Delta_0$, it holds that $\chi(L(G)^t) < 1.941\Delta^t$.*

Proof. By Theorem 3 and [11, Theorem 1.6],

$$\begin{aligned} \chi(L(G)^t) &\leq \lceil 0.881(\Delta(L(G)^t) + 1) + 0.119\omega(L(G)^t) \rceil \\ &\leq \lceil 0.881(2\Delta^t + 1) + 0.119 \cdot 1.5\Delta^t \rceil < 1.941\Delta^t \end{aligned}$$

provided Δ is taken large enough. □

For $t = 1$, Vizing’s theorem states that $\chi(L(G)) \leq \Delta + 1$. For $t = 2$, the current best bound on the strong chromatic index [11] is $\chi(L(G)^2) \leq 1.772\Delta^2$ for all sufficiently large Δ . For $t > 2$, note for comparison with Corollary 1 that the local edge density estimates for $L(G)^t$ proved in [12] combined with the most up-to-date colouring bounds for graphs of bounded local edge density [11] yields only a bound of $1.999\Delta^t$ for all large enough Δ . We must say though that, for the best upper bounds on $\chi(L(G)^t)$, $t > 2$, rather than bounding $\omega(L(G)^t)$ it looks more promising to pursue optimal bounds for the local edge density of $L(G)^t$, particularly for $t \in \{3, 4, 6\}$. We have left this to future study.

Last, we mention that through a brief case analysis we also have confirmed Conjecture 1 in the case $\Delta = 3$.

Theorem 4. *The line graph of any (multi)graph of maximum degree 3 with at least 23 edges has diameter greater than 3. That is, $h_3(3) = 23$.*

Here we estimate upper bounds on $|E|$ by performing a breadth-first search rooted at some specified edge e up to distance 3. This idea implies that a counterexample G is a simple, triangle-free and 3-regular graph. A more thorough analysis shows that G contains no cycles C_4 , C_5 and finally that there does not exist a counterexample G .

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Unavoidable Hypergraphs

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Abstract. The following very natural problem was raised by Chung and Erdős in the early 80's. What is the minimum of the Turán number $\text{ex}(n, \mathcal{H})$ among all r -graphs \mathcal{H} with a fixed number of edges? Their actual focus was on an equivalent and perhaps even more natural question which asks what is the largest size of an r -graph that can not be avoided in any r -graph on n vertices and e edges?

In the original paper they resolve this question asymptotically for graphs, for most of the range of e . In a follow-up work Chung and Erdős resolve the 3-uniform case and raise the 4-uniform case as the natural next step. In this paper we make first progress on this problem in over 40 years by asymptotically resolving the 4-uniform case which gives us some indication on how the answer should behave in general.

Keywords: Turán numbers · Hypergraphs · Unavoidable graphs

1 Introduction

The Turán number $\text{ex}(n, \mathcal{H})$ of an r -graph \mathcal{H} is the maximum number of edges in an r -graph on n vertices which does not contain a copy of \mathcal{F} as a subhypergraph. For ordinary graphs (the case $r = 2$), a rich theory has been developed (see [28]), initiated by the classical Turán's theorem [38] dating back to 1941. The problem of finding the numbers $\text{ex}(n, \mathcal{H})$ when $r > 2$ is notoriously difficult, and exact results are very rare (see surveys [27, 29, 36, 37] and references therein).

The following very natural extremal question was raised by Chung and Erdős [6] almost 40 years ago. What is the minimum possible value of $\text{ex}(n, \mathcal{H})$ among r -graphs \mathcal{H} with a fixed number of edges? The focus of Chung and Erdős was on the equivalent inverse question which is perhaps even more natural. Namely, what is the largest size of an r -graph that we can not avoid in any r -graph on n vertices and e edges? This question was repeated multiple times over the years: it featured in a survey on Turán-type problems [27], in an Erdős open problem collection [5] and more recently in an open problem collection from AIM Workshop on Hypergraph Turán problems [32].

Following Chung and Erdős we call an r -graph \mathcal{H} as above (n, e) -*unavoidable*, so if every r -graph on n vertices and e edges contains a copy of \mathcal{H} . Their question now becomes to determine the maximum possible number of edges in an (n, e) -unavoidable r -graph. Let us denote the answer by $\text{un}_r(n, e)$. In the graph case, Chung and Erdős determined $\text{un}_2(n, e)$ up to a multiplicative factor for essentially the whole range. In a follow-up paper from 1987, Chung and Erdős [7] studied the 3-uniform case and identified the order of magnitude of $\text{un}_3(n, e)$ for essentially the whole range of e . In the same paper Chung and Erdős raise the 4-uniform case as the natural next step since the 3-uniform result fails to give a clear indication on how the answer should behave in general. In the present paper we resolve this question by determining $\text{un}_4(n, e)$ up to a multiplicative factor for essentially the whole range of e . Before stating the main result, let us give some notation.

If a set of vertices S in an r -graph G is contained in at least k edges, we say that S is k -*expanding*. For non-negative functions f and g we write either $f \lesssim g$ or $f = O(g)$ to mean there is a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n ; we write $f \gtrsim g$ or $f = \Omega(g)$ to mean there is a constant $c > 0$ such that $f(n) \geq cg(n)$ for all n ; we write $f \approx g$ to mean that $f \lesssim g$ and $f \gtrsim g$. To simplify the presentation we write $f \gg g$ or $g \ll f$ to mean that $f \geq Cg$ for a sufficiently large constant¹ C , which can be computed by analysing the argument.

Theorem 1. *The following statements hold.*

- (i) For $1 \leq e \leq n^2$, we have $\text{un}_4(n, e) \approx 1$.
- (ii) For $n^2 \leq e \leq n^3$, we have $\text{un}_4(n, e) \approx \min \left\{ (e/n^2)^{3/4}, (e/n)^{1/3} \right\}$.
- (iii) For $n^3 < e \ll \binom{n}{4}$, we have $\text{un}_4(n, e) \approx \min \left\{ e^{4/3}/n^{10/3}, \frac{e^{1/4} \log n}{\log \left(\binom{n}{4}/e \right)} \right\}$.

The optimal unavoidable hypergraphs, or in other words hypergraphs which minimise the Turán number, turn out to be certain combinations of sunflowers of different types. For this reason, it is essential for our proof of Theorem 1 to have a good understanding of the Turán numbers of sunflowers for a wide range of parameters. This turns out to be a well-studied problem in its own right.

1.1 Sunflowers

A family A_1, \dots, A_k of distinct sets is said to be a *sunflower* if there exists a *kernel* C contained in each of the A_i such that the *petals* $A_i \setminus C$ are disjoint. The original term for this concept was “ Δ -system”. The more recent term “sunflower” coined by Deza and Frankl [9] has recently become more prevalent. For $r, k \geq 1$, let $f_r(k)$ denote the smallest natural number with the property that any family of $f_r(k)$ sets of size r contains an (r -uniform) sunflower with k petals. The

¹ Note here that we are defining \gg , in a way which is more common in fields outside of combinatorics, namely $f \gg g$ does not mean $g = o(f)$ but is more similar to $g = O(f)$ with the exception that we are allowed to choose the constant in the big O as small as we like, as long as it remains fixed.

celebrated Erdős-Rado theorem [15] from 1960 asserts that $f_r(k)$ is finite; in fact Erdős and Rado gave the following bounds:

$$(k - 1)^r \leq f_r(k) \leq (k - 1)^r r! + 1. \tag{1}$$

They conjectured that for a fixed k the upper bound can be improved to $f_r(k) \leq O(k)^r$. Despite significant efforts, a solution to this conjecture remains elusive. The current record is $f_r(k) \leq O(k \log(kr))^r$, established in 2019 by Rao [34], building upon a breakthrough of Alweiss, Lovett, Wu and Zhang [1].

Some 43 years ago, Duke and Erdős [10] initiated the systematic investigation of a closely related problem. Denote by $Sf_r(t, k)$ the r -uniform sunflower with k petals, and kernel of size t . Duke and Erdős asked for the Turán number of $Sf_r(t, k)$. Over the years this problem has been reiterated several times [5, 27] including in a recent collaborative “polymath” project [33]. The case $k = 2$ of the problem has received considerable attention [20, 23, 25, 30, 31], partly due to its huge impact in discrete geometry [24], communication complexity [35] and quantum computing [3]. Another case that has a rich history [11, 13, 14, 17–19, 22] is $t = 0$ (a matching of size k is forbidden); the optimal construction in this case is predicted by the Erdős Matching Conjecture.

For fixed r, t and k with $1 \leq t \leq r - 1$ and $k \geq 3$ Frankl and Füredi [21, Conjecture 2.6] give a conjecture for the correct value of $\text{ex}(n, Sf_r(t, k))$ up to lower order terms, based on two natural candidates for near-optimal $Sf_r(t, k)$ -free r -graphs. They verify their conjecture for $r \geq 2t + 3$, but otherwise, with the exception of a few particular small cases, it remains open in general. If we are only interested in asymptotic results the answer of $\text{ex}(n, Sf_r(t, k)) \approx n^{\max\{r-t-1, t\}}$ was determined by Frankl and Füredi [20] and Füredi [26].

Another natural question is what happens if we want to find large sunflowers, in other words if we only fix the uniformity r and “type” of the sunflower, determined by its kernel size t , while allowing k to grow with n . Further motivation for this question is that it is easy to imagine that it could be very useful to know how big a sunflower of a fixed type we are guaranteed to be able to find in an r graph with n vertices and e -edges. In particular, it is precisely the type of statement we require when studying the unavoidability problem of Chung and Erdős. In the graph case $r = 2$ the question simply asks for the Turán number of a (big) star and the answer is easily seen to be $\text{ex}(n, Sf_2(1, k)) \approx nk$. In contrast, the 3-uniform case is already non-trivial: Duke and Erdős [10] and Frankl [16] showed $\text{ex}(n, Sf_3(1, k)) \approx nk^2$ while $\text{ex}(n, Sf_3(2, k)) \approx n^2k$. Chung [4] even managed to determine the answer in the 3-uniform case up to lower order terms, while Chung and Frankl [8] determined $\text{ex}(n, Sf_3(1, k))$ precisely for large enough n . Chung and Erdős [7] wrote in their paper that results for such large sunflowers with uniformity higher than 3 are far from satisfactory. Here we make first progress in this direction, by solving asymptotically the 4-uniform case.

Theorem 2. *For $2 \leq k \leq n$ we have*

- (i) $\text{ex}(n, Sf_4(1, k)) \approx k^2n^2$,
- (ii) $\text{ex}(n, Sf_4(2, k)) \approx k^2n^2$ and
- (iii) $\text{ex}(n, Sf_4(3, k)) \approx kn^3$.

1.2 General Proof Strategy

Our proof strategy for determining $f_r(n, e)$ for most of the range is as follows. In order to show an upper bound $f_r(n, e) \leq D$ we need to show there is no r -graph with more than D edges which is contained in every r graph with n vertices and e edges. With this in mind we consider a number of, usually very structured, n -vertex r -graphs on e or more edges, and argue they can not have a common subhypergraph with more than D edges. The hypergraphs we use are often based on Steiner systems or modifications thereof. A major benefit of this approach is that our collection of hypergraphs often imposes major structural restrictions on possible common graphs which have close to D edges as well and tells us where to look for our optimal examples of unavoidable hypergraphs which we need in order to show matching lower bounds, by upper bounding their Turán numbers.

2 Turán Numbers of 3-Uniform Sunflowers

For our proof of Theorem 2, we will need the following results about 3-uniform sunflowers, which were already established by Duke and Erdős [10] and Frankl [16]. We include our somewhat simpler proofs to illustrate the ideas we will use in the 4-uniform case. There are only two different types of 3-uniform sunflowers, namely $Sf_3(1, k)$ and $Sf_3(2, k)$; we consider them in the next two lemmas.

Lemma 1. *For $2 \leq k \ll n$ we have $ex(n, Sf_3(1, k)) \approx k^2n$.*

Proof. For the lower bound, we split $[n]$ into disjoint sets: A of size $n - k \geq n/2$, and B of size k . Let our 3-graph consist of all edges with one vertex in A and two vertices in B . This 3-graph has $\Omega(k^2n)$ edges and is $Sf_3(1, k)$ -free. Indeed, if we can find a copy of $Sf_3(1, k)$ each of its edges contains two vertices in B , one of which is not the common vertex, so it uses at least $k + 1$ vertices of B , which has size k , a contradiction. This shows $ex(n, Sf_3(1, k)) = \Omega(k^2n)$.

For the upper bound, we show that every 3-graph G with $4k^2n$ edges contains a copy of $Sf_3(1, k)$. Let G be such a 3-graph and suppose towards a contradiction that it does not contain an $Sf_3(1, k)$. For each $v \in V(G)$ let D_v denote the (2)-graph on V whose edges are the $2k$ -expanding pairs Y such that $v \cup Y \in E(G)$. D_v does not contain matchings and stars of size k ; if D_v contained a k -matching then v and this matching would make an $Sf_3(1, k)$ in G ; if D_v contained a star of size k then we can greedily extend each edge of the star by a new vertex to obtain an $Sf_3(1, k)$, since the edges are $2k$ -expanding. This implies that D_v can have at most $2k^2$ edges. The number of edges of G containing a $2k$ -expanding pair is upper bounded by $\sum_v |D_v| \leq 2k^2n$, so if we delete all such edges we are left with a 3-graph G' with at least $2k^2n$ edges with no $2k$ -expanding pairs of vertices. Now take a vertex v with degree at least $3|E(G')|/n \geq 4k^2$ in G' ; it cannot have a star of size $2k$ in its link graph, as then the pair v and centre of the star would be $2k$ -expanding. Hence there must be a k -matching in its link graph, which together with v forms an $Sf_3(1, k)$ in G , so we are done.

Lemma 2. *For $2 \leq k \ll n$ we have $ex(n, Sf_3(2, k)) \approx kn^2$.*

Proof. To prove the lower bound, we consider the linear 3-graph which is a fixed Steiner triple system $S(2, 3, n)$ on $[n]$ with $\Omega(n^2)$ edges. Let G be a union of $k - 1$ random copies of $S(2, 3, n)$, where each copy is obtained by randomly permuting the vertices of $S(2, 3, n)$. Since each pair of vertices lies in at most one edge from each copy of $S(2, 3, n)$, G does not contain a copy of $Sf_3(2, k)$. A fixed triple is chosen with probability $\Omega(1/n)$ in a random copy of $S(2, 3, n)$, independently between our $k - 1$ copies. Thus the probability that a given triple is chosen in one of our $k - 1$ copies is at least $\Omega(k/n)$ so the expected number of chosen triples is $\Omega(kn^2)$, giving $\text{ex}(n, Sf_3(2, k)) = \Omega(kn^2)$. We now turn to the upper bound. Let G be a 3-graph with kn^2 edges. By averaging, there must exist a pair of vertices belonging to at least k edges, which make a copy of $Sf_3(2, k)$ in G .

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2-Distance $(\Delta + 1)$ -Coloring of Sparse Graphs Using the Potential Method

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Abstract. A 2-distance k -coloring of a graph is a proper k -coloring of the vertices where vertices at distance at most 2 cannot share the same color. We prove the existence of a 2-distance $(\Delta + 1)$ -coloring for graphs with maximum average degree less than $\frac{18}{7}$ and maximum degree $\Delta \geq 7$. As a corollary, every planar graph with girth at least 9 and $\Delta \geq 7$ admits a 2-distance $(\Delta + 1)$ -coloring. The proof uses the potential method to reduce new configurations compared to classic approaches on 2-distance coloring.

Keywords: Sparse graphs · 2-distance coloring · Discharging method · Potential method

A k -coloring of the vertices of a graph $G = (V, E)$ is a map $\phi : V \rightarrow \{1, 2, \dots, k\}$. A k -coloring ϕ is a *proper coloring*, if and only if, for all edge $xy \in E$, $\phi(x) \neq \phi(y)$. In other words, no two adjacent vertices share the same color. The *chromatic number* of G , denoted by $\chi(G)$, is the smallest integer k such that G has a proper k -coloring. A generalization of k -coloring is k -list-coloring. A graph G is *L -list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring ϕ of G such that for all $v \in V(G)$, $\phi(v) \in L(v)$. If G is L -list colorable for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be *k -choosable* or *k -list-colorable*. The *list chromatic number* of a graph G is the smallest integer k such that G is k -choosable. List coloring can be very different from usual coloring as there exist graphs with a small chromatic number and an arbitrarily large list chromatic number.

In 1969, Kramer and Kramer introduced the notion of 2-distance coloring [20, 21]. This notion generalizes the “proper” constraint (that does not allow two adjacent vertices to have the same color) in the following way: a *2-distance k -coloring* is such that no pair of vertices at distance at most 2 have the same color (similarly to proper k -list-coloring, one can also define *2-distance k -list-coloring*). The *2-distance chromatic number* of G , denoted by $\chi^2(G)$, is the smallest integer k so that G has a 2-distance k -coloring.

For all $v \in V$, we denote $d_G(v)$ the degree of v in G and by $\Delta(G) = \max_{v \in V} d_G(v)$ the maximum degree of a graph G . For brevity, when it is clear from the context, we will use Δ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$). One

can observe that, for any graph G , $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex v with degree Δ , and v itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta^2 + 1$. Moreover, this bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ , are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. See Fig. 1.

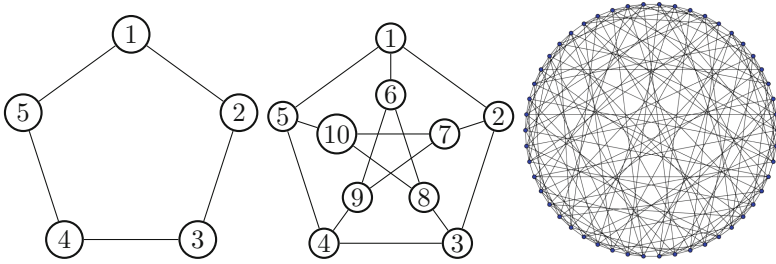


Fig. 1. Examples of Moore graphs for which $\chi^2 = \Delta^2 + 1$: **a.** The Moore graph of type $(2, 2)$: the odd cycle C_5 ; **b.** The Moore graph of type $(3, 2)$: the Petersen graph; **c.** The Moore graph of type $(7, 2)$: the Hoffman-Singleton graph.

By nature, 2-distance colorings and the 2-distance chromatic number of a graph depend a lot on the number of vertices in the neighborhood of every vertex. More precisely, the “sparser” a graph is, the lower its 2-distance chromatic number will be. One way to quantify the sparsity of a graph is through its maximum average degree. The *average degree* ad of a graph $G = (V, E)$ is defined by $ad(G) = \frac{2|E|}{|V|}$. The *maximum average degree* $mad(G)$ is the maximum, over all subgraphs H of G , of $ad(H)$. Another way to measure the sparsity is through the girth, i.e. the length of a shortest cycle. We denote $g(G)$ the girth of G . Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of planar graphs.

A graph is *planar* if one can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When G is a planar graph, Wegner conjectured in 1977 that $\chi^2(G)$ becomes linear in $\Delta(G)$:

Conjecture 1 (Wegner [25]). Let G be a planar graph with maximum degree Δ . Then,

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

The upper bound for the case where $\Delta \geq 8$ is tight (see Fig. 2a). Recently, the case $\Delta \leq 3$ was proved by Thomassen [24], and by Hartke *et al.* [17] independently. For $\Delta \geq 8$, Havet *et al.* [18] proved that the bound is $\frac{3}{2}\Delta(1+o(1))$, where

$o(1)$ is as $\Delta \rightarrow \infty$ (this bound holds for 2-distance list-colorings). Conjecture 1 is known to be true for some subfamilies of planar graphs, for example K_4 -minor free graphs [23].

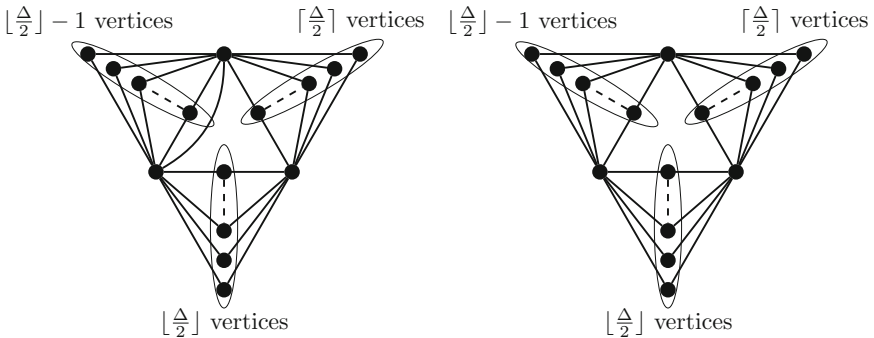


Fig. 2. Graphs with $\chi^2 \approx \frac{3}{2}\Delta$: **a.** A graph with girth 3 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$; **b.** A graph with girth 4 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$.

Wegner’s conjecture motivated extensive researches on 2-distance chromatic number of sparse graphs, either of planar graphs with high girth or of graphs with upper bounded maximum average degree which are directly linked due to Proposition 1.

Proposition 1 (Folklore).

For every planar graph G , $(\text{mad}(G) - 2)(g(G) - 2) < 4$.

As a consequence, any theorem with an upper bound on $\text{mad}(G)$ can be translated to a theorem with a lower bound on $g(G)$ under the condition that G is planar. Many results have taken the following form: every graph G of $\text{mad}(G) \leq m_0$ and $\Delta(G) \geq \Delta_0$ satisfies $\chi^2(G) \leq \Delta(G) + c(m_0, \Delta_0)$ where $c(m_0, \Delta_0)$ is a small constant depending only on m_0 and Δ_0 . Due to Proposition 1, as a corollary, we have the same results on planar graphs of girth $g \geq g_0(m_0)$ where g_0 depends on m_0 . Table 1 shows all known such results, up to our knowledge, on the 2-distance chromatic number of planar graphs with fixed girth, either proven directly for planar graphs with high girth or came as a corollary of a result on graphs with bounded maximum average degree.

For example, the result from line “7” and column “ $\Delta + 1$ ” from Table 1 reads as follows: “every planar graph G of girth at least 7 and of Δ at least 16 satisfies $\chi^2(G) \leq \Delta + 1$ ”. The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs G with $\chi^2(G) = \Delta + 2$ for arbitrarily large Δ [4, 16]. The lack of results for $g = 4$ is due to the fact that the graph in Fig. 2b has girth 4, and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$ for all Δ .

We are interested in the case $\chi^2(G) = \Delta + 1$ as $\Delta + 1$ is a trivial lower bound for $\chi^2(G)$. In particular, we were looking for the smallest integer Δ_0 such

Table 1. The latest results with a coefficient 1 before Δ in the upper bound of χ^2

g_0	$\chi^2(G)$							
	$\Delta + 1$	$\Delta + 2$	$\Delta + 3$	$\Delta + 4$	$\Delta + 5$	$\Delta + 6$	$\Delta + 7$	$\Delta + 8$
3	–			$\Delta = 3$ [17, 24]				
4	–							
5	–	$\Delta \geq 10^7$ [1]	$\Delta \geq 339$ [15]	$\Delta \geq 312$ [14]	$\Delta \geq 15$ [9]	$\Delta \geq 12$ [8]	$\Delta \neq 7, 8$ [14]	All Δ [13]
6	–	$\Delta \geq 17$ [3]	$\Delta \geq 9$ [8]		All Δ [10]			
7	$\Delta \geq 16$ [19]			$\Delta = 4$ [11]				
8	$\Delta \geq 9$ [22]		$\Delta = 5$ [7]					
9	$\Delta \geq 8$ [2]	$\Delta = 5$ [7]	$\Delta = 3$ [12]					
	$\Delta \geq 7$ (Corollary 1)							
10	$\Delta \geq 6$ [19]							
11		$\Delta = 4$ [11]						
12	$\Delta = 5$ [19]	$\Delta = 3$ [5]						
13								
14	$\Delta \geq 4$ [2]							
...								
22	$\Delta = 3$ [19]							

that every graph with maximum degree $\Delta \geq \Delta_0$ and $\text{mad} \leq \frac{18}{7}$ (which contains planar graphs with $\Delta \geq \Delta_0$ and girth at least 9) can be 2-distance colored with $\Delta + 1$ colors. Borodin et al. [6] showed that planar graphs of girth at least 9 and $\Delta \geq 10$ are 2-distance $(\Delta + 1)$ -colorable in 2008. In 2011, Ivanova [19] improved on the result with a simpler proof that planar graphs of girth at least 8 and $\Delta \geq 10$ are 2-distance $(\Delta + 1)$ -colorable. Later on, in 2014, Bonamy et al. [2] improved on this result once again by proving that graphs with $\text{mad} < \frac{18}{7}$ and $\Delta \geq 8$ are 2-distance $\Delta + 1$ -colorable. In this paper, we will improve this result to graphs with $\text{mad} < \frac{18}{7}$ and $\Delta \geq 7$ in Theorem 1. But most importantly, that breakthrough is obtained by using a new approach based on the potential method.

All of these results and most of the results in Table 1 are proven using the discharging method. Due to the extensive amount of work done on this subject, the classic discharging method is reaching its limit. The *discharging method* assigns a certain charge to each object (often vertices and faces when the graph is planar) of a minimal counter-example G to the result we want to prove. Then, using either Euler’s formula or the upper bound on the maximum average degree, we can prove that the total amount of charges is negative. However, by redistributing these charges via *discharging rules* that do not modify the total sum, we can prove that we have a nonnegative amount of charges under the reducibility of some configurations, which results in a contradiction. Since the initial total amount of charges is fixed, the improvements on these type of results rely on the reduction of new configurations and reducing a configuration relies on extending a precoloring of a subgraph of G . Until now, we have always assumed the worst case scenario for the precoloring. However, these assumptions can only get us so far when we can find unextendable precolorations. In order to avoid the worst case scenario, we need to add some vertices and edges to our subgraph but we might run into the risk of increasing our maximum average degree.

Then came the *potential method*, which introduces a potential function that can, more precisely, quantify the local maximum average degree in our subgraph, thus allowing us to add edges and vertices while staying in the same class of graphs. This breakthrough allowed for new configurations to become reducible and thus, improving on the limit of what the classic discharging method was able to reach.

Our main result is the following:

Theorem 1. *If G is a graph with $\text{mad}(G) < \frac{18}{7}$, then G is 2-distance $(\Delta(G)+1)$ -colorable for $\Delta(G) \geq 7$.*

For planar graphs, we obtain the following corollary:

Corollary 1. *If G is a planar graph with $g(G) \geq 9$, then G is 2-distance $(\Delta(G) + 1)$ -colorable for $\Delta(G) \geq 7$.*

Since Bonamy, Lévêque, and Pinlou has already proven in [2] that:

Theorem 2 (Bonamy, Lévêque, Pinlou [2]).

If G is a graph with $\text{mad}(G) < \frac{18}{7}$, then G is list 2-distance $(\Delta(G) + 1)$ -colorable for $\Delta(G) \geq 8$.

We will prove the following, which is a stronger version with $\text{mad}(G) \leq \frac{18}{7}$ instead of $\text{mad}(G) < \frac{18}{7}$:

Theorem 3. *If G is a graph with $\text{mad}(G) \leq \frac{18}{7}$, then G is 2-distance $(\Delta(G)+1)$ -colorable for $\Delta(G) = 7$.*

To prove Theorem 3, let us define the potential function, which is the key to the potential method.

Let $A \subseteq V(G)$, we define $\rho_G(A) = 9|A| - 7|E(G[A])|$. Note that $\rho_G(A) \geq 0$ for all $A \subseteq V(G)$ if and only if $\text{mad}(G) \leq \frac{18}{7}$. We define the *potential function* $\rho_G^*(A) = \min\{\rho_G(S) \mid A \subseteq S \subseteq V(G)\}$ for all $A \subseteq V(G)$. Since $\rho_G(A) \geq 0$ for all $A \subseteq V(G)$, the same holds for $\rho_G^*(A)$.

Thus, we will prove the following equivalent version of Theorem 3.

Theorem 4. *Let G be a graph such that $\rho_G^*(A) \geq 0$ for all $A \subseteq V(G)$, then G is 2-distance $(\Delta(G) + 1)$ -colorable for $\Delta(G) = 7$.*

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Graphs Where Search Methods Are Indistinguishable

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Abstract. Graph searching is one of the simplest and most widely used tools in graph algorithms. Every graph search method is defined using some particular selection rule, and the analysis of the corresponding vertex orderings can aid greatly in devising algorithms, writing proofs of correctness, or recognition of various graph families.

We study graphs where the sets of vertex orderings produced by two different search methods coincide. We characterise such graph families for ten pairs from the best-known set of graph searches: Breadth First Search (BFS), Depth First Search (DFS), Lexicographic Breadth First Search (LexBFS) and Lexicographic Depth First Search (LexDFS), and Maximal Neighborhood Search (MNS).

Keywords: Graph search methods · Breadth first search · Depth first search

1 Introduction

Graph search methods (for instance, Depth First Search and Breadth First Search) are among essential concepts classically taught at the undergraduate level of computer science faculties worldwide. Various types of graph searches have been studied since the 19th century, and used to solve diverse problems, from solving mazes, to linear-time recognition of interval graphs, finding minimal path-cover of co-comparability graphs, finding perfect elimination order, or optimal coloring of a chordal graph, and many others [1, 2, 5, 6, 9, 10, 13, 14].

In its most general form, a *graph search* (also *generic search* [7]) is a method of traversing vertices of a given graph such that every prefix of the obtained vertex ordering induces a connected graph. This general definition of a graph search leaves much freedom for a selection rule determining which node is chosen next. By defining some specific rule that restricts this choice, various different graph search methods are defined. Other search methods that we focus on in this paper are Breadth First Search, Depth First Search, Lexicographic Breadth First Search, Lexicographic Depth First Search, and Maximal Neighborhood Search.

This paper is structured as follows. In Sect. 2 we briefly present the studied graph search methods, and then state the obtained results in Sect. 3. In Sect. 4 we provide a short proof of Theorem 1, as it is the easiest to deal with. Due to lack of space we omit the proofs of Theorem 2 and 3, and provide some directions for further work in Sect. 5.

2 Preliminaries

We now briefly describe the above-mentioned graph search methods, and give the formal definitions. Note that the definitions below are not given in a historically standard form, but rather as so-called *three-point conditions*, due to Corneil and Kruger [7] and also Brändstadt et al. [4].

Breadth First Search (BFS), first introduced in 1959 by Moore [12], is a restriction of a generic search which puts unvisited vertices in a queue and visits a first vertex from the queue in the next iteration. After visiting a particular vertex, all its unvisited neighbors are put at the end of the queue, in an arbitrary order.

Definition 1. *An ordering σ of V is a BFS-ordering if and only if the following holds: if $a <_{\sigma} b <_{\sigma} c$ and $ac \in E$ and $ab \notin E$, then there exists a vertex d such that $d <_{\sigma} a$ and $db \in E$.*

Any BFS ordering of a graph G starting in a vertex v results in a rooted tree (with root v), which contains the shortest paths from v to any other vertex in G (see [8]). We use this property implicitly throughout the paper.

Depth First Search (DFS), in contrast with the BFS, traverses the graph as deeply as possible, visiting a neighbor of the last visited vertex whenever it is possible, and backtracking only when all the neighbors of the last visited vertex are already visited. In DFS, the unvisited vertices are put on top of a stack, so visiting a first vertex in a stack means that we always visit a neighbor of the most recently visited vertex.

Definition 2. *An ordering σ of V is a DFS-ordering if and only if the following holds: if $a <_{\sigma} b <_{\sigma} c$ and $ac \in E$ and $ab \notin E$, then there exists a vertex d such that $a <_{\sigma} d <_{\sigma} b$ and $db \in E$.*

The algorithm for DFS has been known since the nineteenth century as a technique for threading mazes, known under the name *Trémaux's algorithm* (see [11]).

Lexicographic Breadth First Search (LexBFS) was introduced in the 1970s by Rose, Tarjan and Lueker [13] as a part of an algorithm for recognizing chordal graphs in linear time. Since then, it has been used in many graph algorithms mainly for the recognition of various graph classes.

Definition 3. *An ordering σ of V is a LexBFS ordering if and only if the following holds: if $a <_{\sigma} b <_{\sigma} c$ and $ac \in E$ and $ab \notin E$, then there exists a vertex d such that $d <_{\sigma} a$ and $db \in E$ and $dc \notin E$.*

LexBFS is a restricted version of Breadth First Search, where the usual queue of vertices is replaced by a queue of unordered subsets of the vertices which is sometimes refined, but never reordered.

Lexicographic Depth First Search (LexDFS) was introduced in 2008 by Corneil and Krueger [7] and represents a special instance of a Depth First Search.

Definition 4. *An ordering σ of V is a LexDFS ordering if and only if the following holds: if $a <_{\sigma} b <_{\sigma} c$ and $ac \in E$ and $ab \notin E$, then there exists a vertex d such that $a <_{\sigma} d <_{\sigma} b$ and $db \in E$ and $dc \notin E$.*

Maximal Neighborhood Search (MNS), introduced in 2008 by Corneil and Krueger [7], is a common generalization of LexBFS, LexDFS, and MCS, and also of Maximal Label Search (see [3] for definition).

Definition 5. *An ordering σ of V is an MNS ordering if and only if the following statement holds: If $a <_{\sigma} b <_{\sigma} c$ and $ac \in E$ and $ab \notin E$, then there exists a vertex d with $d <_{\sigma} b$ and $db \in E$ and $dc \notin E$.*

The MNS algorithm uses the set of integers as the label, and at every step of iteration chooses the vertex with maximal label under set inclusion.

Corneil [7] exposed an interesting structural aspect of graph searches: the particular search methods can be seen as restrictions, or special instances of some more general search methods. For six well-known graph search methods they present a depiction, similar to the one in Fig. 1, showing how those methods are related under the set inclusion. For example, every LexBFS ordering is at the same time an instance of BFS and MNS ordering of the same graph. Similarly, every LexDFS ordering is at the same time also an instance of MNS, and of DFS (see Fig. 1). The reverse, however, is not true, and there exist orderings that are BFS and MNS, but not LexBFS, or that are DFS and MNS but not LexDFS.

3 Problem Description and Results

Since the connections in Fig. 1 represent relations of inclusion, it is natural to ask under which conditions on a graph G the particular inclusion holds also in the converse direction. More formally, we say that two search methods are *equivalent on a graph G* if the sets of vertex orderings produced by both of them are the same. We say that two graph search methods are *equivalent on a graph class \mathcal{G}* if they are equivalent on every member $G \in \mathcal{G}$. Perhaps surprisingly, three different graph families suffice to describe graph classes equivalent for the ten pairs of graph search methods that we consider. Those are described in Theorems 1 to 3 below, but first we need a few more definitions.

All the graphs considered in the paper are finite and connected. A k -pan is a graph consisting of a k -cycle, with a pendant vertex added to it. We say that a graph is *pan-free* if it does not contain a pan of any size as an induced subgraph. A 3-pan is also known as a *paw graph*.

Theorem 1. *Let G be a connected graph. Then the following is equivalent:*

- A1. *Graph G is $\{P_4, C_4, paw, diamond\}$ -free.*
- A2. *Every graph search of G is a DFS ordering of G .*
- A3. *Every graph search of G is a BFS ordering of G .*
- A4. *Any vertex-order of G is a BFS, if and only if it is a DFS.*

Theorem 2. *Let G be a connected graph. Then the following is equivalent:*

- B1. *Graph G is $\{pan, diamond\}$ -free.*
- B2. *Every DFS ordering of G is a LexDFS ordering of G .*
- B3. *Every BFS ordering of G is a LexBFS ordering of G .*
- B4. *Every graph search of G is an MNS ordering of G .*

Theorem 3. *Let G be a connected graph. Then the following is equivalent:*

- C1. *Graph G is $\{P_4, C_4\}$ -free.*
- C2. *Every MNS ordering of G is a LexDFS ordering of G .*
- C3. *Every MNS ordering of G is a LexBFS ordering of G .*

From Theorems 1 and 2 we can immediately derive similar results for two additional pairs of graph search methods.

Corollary 1. *Let G be a connected graph. Then the following is equivalent:*

- A1. *Graph G is $\{P_4, C_4, paw, diamond\}$ -free.*
- A5. *Every graph search of G is a LDFS ordering of G .*
- A6. *Every graph search of G is a LBFS ordering of G .*

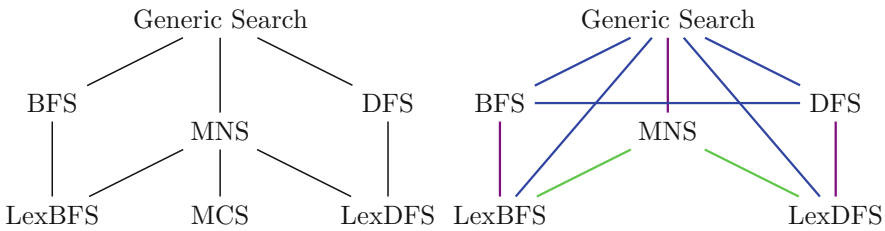


Fig. 1. On the left: Hasse diagram showing how graph searches are refinements of one another. On the right is a summary of our results: green pairs are equivalent on $\{P_4, C_4\}$ -free graphs. Violet pairs are equivalent on $\{pan, diamond\}$ -free graphs. Blue pairs are equivalent on $\{paw, diamond, P_4, C_4\}$ -free graphs. (Color figure online)

4 Proof of Theorem 1

The following lemma investigates the case when an input graph contains an induced subgraph from $\{P_4, C_4, \text{paw}, \text{diamond}\}$.

Lemma 1. *Suppose either of the following is true:*

1. every graph search of G is also a BFS, or
2. every graph search of G is also a DFS, or
3. a vertex-order of G is a BFS, if and only if it is a DFS.

Then G is a $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free graph (Fig. 2).

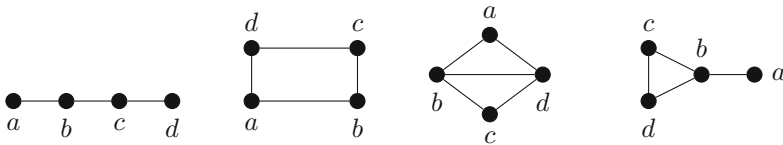


Fig. 2. In the examples above, ordering (c, b, a, d) is not BFS, while ordering (b, c, a, d) is not DFS. In the two rightmost examples above, ordering (c, b, a, d) is not MNS.

Proof. Suppose that G contains an induced copy of a graph from $\{P_4, C_4, \text{paw}, \text{diamond}\}$. In other words, G admits a subgraph H , where $V(H) = \{a, b, c, d\}$ and $\{ab, bc, cd\} \subseteq E(G)$ and $ac \notin E(G)$. We derive the negations for the three items from this claim.

1. Consider any generic search order of G starting with (c, b, a, \dots) . Observe that such a vertex-order violates the BFS search paradigm (see Definition 1) with the triplet (c, a, d) .
2. Now consider any generic search order of G starting with (b, c, a, \dots) . In this case observe that the prefix (b, c, a) of any such vertex-ordering violates Definition 2.
3. It is enough to find a vertex-ordering which is exactly of one among types $\{\text{BFS}, \text{DFS}\}$. To this end consider again any search order of G starting with (c, b, a) , and continuing so that DFS search order is respected. Similarly as in the item (1) notice that this search again violates the BFS search paradigm (see Definition 1), with the triplet (c, a, d) .

We proceed with the proof of the main claim of this section.

Theorem 1. *Let G be a connected graph. Then the following is equivalent:*

- A1. Graph G is $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free.
- A2. Every graph search of G is a DFS ordering of G .
- A3. Every graph search of G is a BFS ordering of G .

A4. Any vertex-order of G is a BFS, if and only if it is a DFS.

Proof. By Lemma 1 it is clear that Item A1 follows independently from either Item A2, A3, or A4

We now establish that G is $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free, if and only if it is a star, or a clique. The converse direction is trivial, as every star, as well as K_3 , are $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free. For the forward direction assume that G is a $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free connected graph. We distinguish two cases:

1. Graph G is triangle-free. Since it is also $\{P_4, C_4\}$ -free, G must be a tree of diameter at most two, which exactly corresponds with the family of stars.
2. Maximal clique C in G is of size at least three. If G itself is a clique we are done, so suppose that there exists an additional vertex $a \notin C$, such that $N(a) \cap C \neq \emptyset$. Let $b \in N(a) \cap C$ and let $c \in C$ be such that $ac \notin E(G)$ (such a vertex c exists by the maximality of C). Finally, since the C is of size at least three, let $d \in C \setminus \{b, c\}$ be an arbitrary remaining vertex of C . It remains to observe that (a, b, c, d) induce a paw, or a diamond.

To conclude the proof, it remains to show that every generic graph search in a clique or a star is also (both) a BFS as well as DFS search. Since in the clique all vertex-orderings are isomorphic, we only consider the case of stars. However, observe that stars only admit two non-isomorphic generic vertex orderings, namely the one starting in the center, and the one starting in a leaf. Since both of those vertex-orderings are at the same time also BFS and DFS orders, this concludes the proof of the claim.

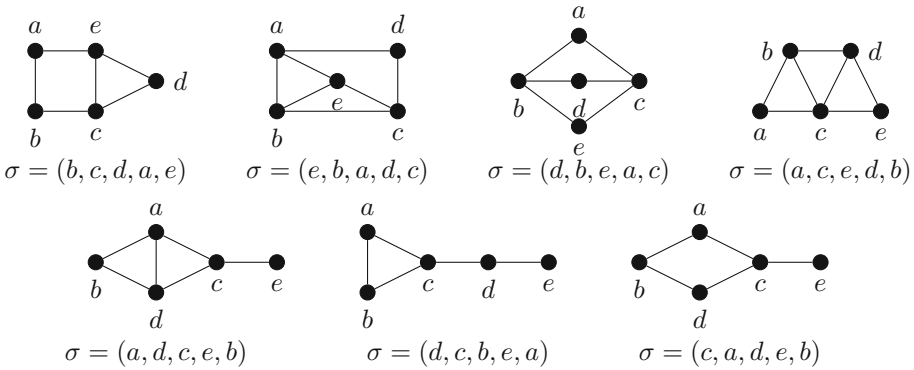


Fig. 3. Graphs and corresponding orderings that are MNS and not MCS orderings.

5 Conclusion and Further Work

In this paper we consider the major graph search methods and study the graphs in which vertex-orders of one type coincide with vertex-orders of some other

type. Interestingly, three different graph families suffice to describe graph classes equivalent for the ten pairs of graph search methods that we consider, which provides an additional aspect of similarities between the studied search methods.

Among the natural graph search methods not yet considered in this setting would be the *Maximum Cardinality Search* (MCS), introduced in 1984 (for definition see Tarjan and Yannakakis [15]). As shown on Fig. 1, every MCS is a special case of an MNS vertex-order. While it is easy to verify that $\{P_4, C_4, \text{paw}, \text{diamond}\}$ -free graphs do not distinguish between MNS and MCS vertex orders, Fig. 3 provides examples of graphs which admit MNS, but not MCS vertex orders. Characterising graphs equivalent for MNS and MCS remains an open question.

Acknowledgements. The authors would like to thank prof. Martin Milanič for the initial suggestion of the problem, and to Ekki Köhler and his research group, for introducing the diverse world of graph searches to us.

This work was partially supported by the Slovenian Research Agency (research programs P1-0285, P1-0383; research projects N1-0102, J1-9110, J1-1692, N1-0160, J1-9187; and a Young Researchers Grant). The first author acknowledges European Commission for funding the InnoRenew CoE project (Grant Agreement #739574) under the Horizon 2020 Widespread-Teaming program and the Republic of Slovenia.

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Discrete Helly-Type Theorems for Pseudohalfplanes

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Abstract. We prove discrete Helly-type theorems for pseudohalfplanes, which extend recent results of Jensen, Joshi and Ray about halfplanes. Among others we show that given a family of pseudohalfplanes \mathcal{H} and a set of points P , if every triple of pseudohalfplanes has a common point in P then there exists a set of at most two points that hits every pseudohalfplane of \mathcal{H} . We also prove that if every triple of points of P is contained in a pseudohalfplane of \mathcal{H} then there are two pseudohalfplanes of \mathcal{H} that cover all points of P .

To prove our results we regard pseudohalfplane hypergraphs, define their extremal vertices and show that these behave in many ways as points on the boundary of the convex hull of a set of points. Our methods are purely combinatorial.

Keywords: Pseudohalfplane · Geometric hypergraph · Helly

1 Introduction

Given a (finite) point set P and a family of regions \mathcal{R} (e.g., the family of all halfplanes) in the plane (or in higher dimensions), let \mathcal{H} be the hypergraph with vertex set P and for each region of \mathcal{R} having a hyperedge containing exactly the same points of P as this region. There are many interesting problems that can be phrased as a problem about hypergraphs defined this way, which are usually referred to as *geometric hypergraphs*. This topic has a wide literature, researchers considered problems where \mathcal{R} is a family of halfplanes, axis-parallel rectangles, translates or homothets of disks, squares, convex polygons, pseudo-disks and so on. There are many results and open problems about the maximum number of hyperedges of such a hypergraph, coloring questions and other properties. For a survey of some of the most recent results see the introduction of [2] and of [3], for an up-to-date database of such results with references see the webpage [1].

One of the most basic families is the family of halfplanes, about which already many problems are non-trivial. Among others one such problem was considered

in [8] where they prove that the vertices of every hypergraph defined by halfplanes on a set of points (i.e., P is a finite set of points and \mathcal{R} is the family of all halfplanes) can be k colored such that every hyperedge of size at least $2k + 1$ contains all colors. In [6] they considered generalizing this result by replacing halfplanes with the family of translates of an unbounded convex region (e.g., an upwards parabola). It turned out that this is true even when halfplanes are replaced by *pseudohalfplanes*. The main tool of proving this was an equivalent combinatorial definition of so called *pseudohalfplane graphs*, hypergraphs that can be defined on points with respect to pseudohalfplanes.¹ This formulation had the promise that many other statements about halfplane hypergraphs can be generalized to pseudohalfplane hypergraphs in the future. While this combinatorial formulation has the disadvantage of being less visual and thus somehow less intuitive than the geometric setting, it has many advantages, among others covering a much wider range of hypergraphs, also, being purely combinatorial might have algorithmic applications as well. One recent application is a similar polychromatic coloring result about disks all containing the origin [3] where after observing that in every quadrant of the plane the disks form a family of pseudohalfplanes they can apply the results from [6].

In [6] the equivalent of the convex hull vertices in the plane (more precisely, the points on the boundary of the convex hull) was defined for pseudohalfplane hypergraphs and called *unskippable vertices* and this made it possible to generalize the proof idea of [8] from halfplanes to pseudohalfplane hypergraphs. To ease intuition, we call unskippable vertices as *extremal vertices* from here on. Exact definitions of these notions are omitted from this extended abstract.

Recently Jensen, Joshi and Ray [5] proved discrete Helly-type theorems which can be formulated in terms of halfplane hypergraphs, their results are detailed in Sect. 1.2. In this paper we generalize their results to pseudohalfplane hypergraphs, in addition we also prove one missing variant for which even the halfplane counterpart was not considered yet. Again we make use of extremal vertices defined in [6], but we need to prove many new properties of extremal vertices which show that extremal vertices behave in many ways as convex hull vertices in the plane (more precisely, as the points on the boundary of the convex hull). We believe that these properties will be useful also for future research on pseudohalfplane hypergraphs. We also consider these problems for pseudohemisphere hypergraphs, a natural hypergraph family containing the family of pseudohalfplane hypergraphs.

We consider the following two types of problems: in a *primal discrete Helly theorem of type $k \rightarrow l$* let P be a set of n points (resp. vertex set) and \mathcal{F} be a family of regions (resp. hypergraph). If every k -tuple of regions (resp. hyperedges) in \mathcal{F} intersects at a point (resp. vertex) in P , then there exists a set of l points (resp. vertices) in P that intersects each $F \in \mathcal{F}$. In a *dual discrete Helly theorem of type $k \rightarrow l$* let P be a finite set of n points (resp. vertices) and \mathcal{F} be a family of regions (resp. hypergraph). If every subset of k points in P belongs to some

¹ The exact definition of pseudohalfplanes and pseudohalfplane hypergraphs are postponed to Sect. 1.1.

region (resp. hyperedge) $F \in \mathcal{F}$ then there exist l regions (resp. hyperedges) in \mathcal{F} whose union covers P .

In Table 1 we summarize our results. For all our results we show that they are optimal except for the ones about pseudohemispheres.

Proofs are omitted from this extended abstract due to space constraints but can be found in the full version of this paper [7].

Table 1. Summary of the considered Helly-type results

Halfplane		ABA-free	Pseudohalfplane		Pseudohemisphere
Primal	Dual	Primal/dual	Primal	Dual	Primal/dual
$3 \rightarrow 2$ [5] (Theorem 6)	$3 \rightarrow 2$ [5] (Theorem 4)	$2 \rightarrow 2$ (Theorem 8, Corollary 9)	$3 \rightarrow 2$ (Theorem 11)	$3 \rightarrow 2$ (Theorem 12)	$4 \rightarrow 2$ (in full version, [7])
$2 \rightarrow 3$ (Theorem 10)	$2 \rightarrow 3$ [5] (Theorem 5)		$2 \rightarrow 3$ (Theorem 10)	$2 \rightarrow 3$ (Theorem 13)	

1.1 Pseudohalfplanes and Pseudohalfplane Hypergraphs

Pseudohalfplane Hypergraphs. The definition of *pseudohalfplane hypergraphs* introduced in [6] is based on the definition of ABA-free hypergraphs and is as follows.

Definition 1 A hypergraph \mathcal{H} with an ordered vertex set is called ABA-free if \mathcal{H} does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.²

Definition 2 A hypergraph \mathcal{H} on an ordered set of vertices V is called a pseudohalfplane hypergraph if there exists an ABA-free hypergraph \mathcal{F} on V such that $\mathcal{H} \subset \mathcal{F} \cup \bar{\mathcal{F}}$.³

Pseudolines. A *pseudoline arrangement* is a finite collection of simple curves in the plane such that each curve cuts the plane into two components (i.e., both endpoints of each curve are at infinity) and any two of the curves are either disjoint or intersect once, and in the intersection point they cross. It is usually also required and so we require as well that they intersect exactly once.⁴ However, in our case we do not need this restriction and so an arrangement where not all pairs of pseudolines intersect we call a *loose* pseudoline arrangement. An

² We imagine the vertices on a horizontal line, and thus if $x < y$ then we may say that x is to the left from y and so on.

³ $\bar{\mathcal{F}}$ denotes the family of the complements of the hyperedges of \mathcal{F} . It was shown in [6] that $\bar{\mathcal{F}}$ is also ABA-free if \mathcal{F} is ABA-free.

⁴ They are usually defined in the projective plane, as a collection of simple closed curves whose removal does not disconnect the projective plane and for which every pair of the curves meets no more than once (hence exactly once and crossing).

arrangement of pseudolines is *simple* if no three pseudolines meet at a point. Wlog. we can assume that the pseudolines are x -monotone bi-infinite curves (see, e.g. [6]), such arrangements are sometimes called Euclidean or *graphic* pseudoline arrangements. For an introduction into pseudoline arrangements see Chap. 5 of [4] by Felsner and Goodman.

Pseudohalfplanes. Given a pseudoline arrangement, a *pseudohalfplane family* is the subfamily of the above defined components (one on each side of each pseudoline). A pseudohalfplane family is simple (resp. loose) if the boundaries form a simple (resp. loose) pseudoline arrangement. A pseudohalfplane family is upwards if we just take components that are above the respective pseudoline (here we use that the pseudolines are assumed to be x -monotone).

In [6] it is shown that given a family \mathcal{F} of pseudohalfplanes in the plane and a set of points P then the hypergraph whose hyperedges are the subsets that we get by intersecting regions of \mathcal{F} with P is a pseudohalfplane hypergraph, and that all pseudohalfplane hypergraphs can be realized this way.⁵ If \mathcal{F} is a family of upwards pseudohalfplanes then we get the ABA-free hypergraphs. Thus, all our results about pseudohalfplane hypergraphs implies the respective result about (loose and not loose) families of pseudohalfplanes where we replace vertices with points and hyperedges with pseudohalfplanes.

1.2 Helly-Type Theorems for Halfplanes

Helly's classic theorem in the plane is as follows:

Theorem 3 (Helly for convex sets). *Let P be a set of n points and \mathcal{C} be a family of convex sets in the plane. If every subset of 3 points in P belongs to some convex set $C \in \mathcal{C}$ then there exists a point (not necessarily in P) which is in every convex set of \mathcal{C} .*

Jensen, Joshi and Ray [5] regarded discrete versions of Helly's theorem, where they require that the point one finds also comes from the set P . First, their following simple construction shows that we cannot require this for convex sets, even if we replace 3 by some larger value k and we want to find only some bounded number of vertices that hit all sets: take a set P of n points in convex position, then every subset of points in P can be separated from the rest of the points in P by a convex set. Now for some fixed k let \mathcal{C} be the family of such separating convex sets for the subsets of points in P of size more than $n - n/k$. Then every subfamily of size k of \mathcal{C} has a common point in P , on the other hand no subset of points in P of size less than n/k hits every set in \mathcal{C} .

They show that replacing convex sets with halfplanes yields interesting problems and prove the following results:

⁵ In fact they prove that we can realize them with loose simple pseudoline arrangements but their argument can be easily modified to have a realization with a simple and not loose pseudoline arrangement as well.

Theorem 4 (Dual Discrete Helly for halfplanes, $3 \rightarrow 2$) [5]. *Let P be a set of n points and \mathcal{H} be a family of halfplanes. If every subset of 3 points in P belongs to some halfplane $H \in \mathcal{H}$ then there exist two halfplanes in \mathcal{H} whose union covers P .*

They give an example that this is tight, that is, 3 cannot be replaced by 2. They also show the following:

Theorem 5 (Dual Discrete Helly for halfplanes, $2 \rightarrow 3$) [5]. *Let P be a set of n points and \mathcal{H} be a family of halfplanes. If every pair of points in P belongs to some halfplane $H \in \mathcal{H}$ then there exists 3 halfplanes in \mathcal{H} whose union covers P .*

Theorem 6 (Primal Discrete Helly for halfplanes, $3 \rightarrow 2$) [5]. *Let P be a set of n points and \mathcal{H} be a family of halfplanes. If every triple of halfplanes in \mathcal{H} intersects at a point in P , then there exists a set of two points in P which intersects each $H \in \mathcal{H}$.*

The above two results are implied by their following result about convex pseudodisks:

Theorem 7 (Primal Discrete Helly for convex pseudodisks, $3 \rightarrow 2$) [5]. *Let P be a set of n points and \mathcal{D} be a family of convex pseudodisks. If every triple of pseudodisks in \mathcal{D} intersects at a point in P , then there exists a set of two points in P which intersects each $D \in \mathcal{D}$.*

1.3 Helly-Type Theorems for Pseudohalfplanes

We aim to prove results about pseudohalfplanes similar to the ones about halfplanes from the previous section. First we show discrete Helly-type results for ABA-free hypergraphs:

Theorem 8 (Primal Discrete Helly for ABA-free hypergraphs, $2 \rightarrow 2$). *Given an ABA-free \mathcal{H} such that every pair of hyperedges has a common vertex, there exists a set of at most two vertices that hits every hyperedge of \mathcal{H} .*

As the dual of an ABA-free hypergraph is also an ABA-free hypergraph, this implies (and is in fact equivalent to):

Corollary 9 (Dual Discrete Helly for ABA-free hypergraphs, $2 \rightarrow 2$). *Given an ABA-free \mathcal{H} on vertex set V of size $n \geq 2$ such that for every pair of vertices there is a hyperedge of \mathcal{H} containing both of them, there exists at most two hyperedges of \mathcal{H} whose union covers V .*

Applying this twice to the two ABA-free parts of a pseudohalfplane hypergraph implies easily that $2 \rightarrow 4$ is true for pseudohalfplanes but we can prove a better bound which is optimal (we note that this was not known earlier even in the special case of halfplanes):

Theorem 10 (Primal Discrete Helly for pseudohalfplanes, $2 \rightarrow 3$). *Given a pseudohalfplane hypergraph \mathcal{H} such that every pair of hyperedges has a common vertex, there exists a set of at most 3 vertices that hits every hyperedge of \mathcal{H} .*

We can also prove the following:

Theorem 11 (Primal Discrete Helly for pseudohalfplanes, $3 \rightarrow 2$). *Given a pseudohalfplane hypergraph \mathcal{H} such that every triple of hyperedges has a common vertex, there exists a set of at most 2 vertices that hits every hyperedge of \mathcal{H} .*

In the dual setting we have the following results about pseudohalfplanes:

Theorem 12 (Dual Discrete Helly for pseudohalfplanes, $3 \rightarrow 2$). *Let V be an ordered set of $n \geq 3$ vertices and \mathcal{H} a family of pseudohalfplanes. If every subset of 3 vertices in V belongs to some halfplane $H \in \mathcal{H}$ then there exist at most two pseudohalfplanes in \mathcal{H} whose union covers V .*

Theorem 13 (Dual Discrete Helly for pseudohalfplanes, $2 \rightarrow 3$). *Let V be an ordered set of $n \geq 2$ vertices and \mathcal{H} a family of pseudohalfplanes. If every pair of vertices in V belongs to some halfplane $H \in \mathcal{H}$ then there exist at most 3 pseudohalfplanes in \mathcal{H} whose union covers V .*

Constructions showing that these results are best possible and further results about pseudohemispheres (see Table 1) can be found in the full version of the paper [7].

Acknowledgement. Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under the grant LP2017-19/2017 and by the National Research, Development and Innovation Office – NKFIH under the grant K 132696. The author is grateful to D. Pálvölgyi for the many discussions about these results.

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Cycles of Many Lengths in Hamiltonian Graphs

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Abstract. In 1999, Jacobson and Lehel conjectured that for $k \geq 3$, every k -regular Hamiltonian graph has cycles of at least linearly many different lengths. This was further strengthened by Verstraëte, who asked whether the regularity can be replaced with the weaker condition that the minimum degree is at least 3. Despite attention from various researchers, until now the best partial result towards both of these conjectures was a \sqrt{n} lower bound on the number of cycle lengths. We resolve these conjectures asymptotically, by showing that the number of cycle lengths is at least $n^{1-o(1)}$.

Keywords: Cycle spectrum · Cycle lengths · Hamiltonian graph

1 Introduction

The study of cycles in graphs goes back to the early days of graph theory and has been fundamental ever since. Of particular interest are Hamilton cycles, i.e. cycles passing through all the vertices of a graph. Starting with the cornerstone theorem of Dirac [7], there are many results giving sufficient conditions for a graph to be Hamiltonian, for some other classical examples see [4–6, 11, 19]. In 1973, Bondy [3] made the “meta-conjecture” that any non-trivial condition which guarantees the existence of a Hamilton cycle, should also guarantee that the given graph is pancyclic, i.e. contains cycles of all possible lengths, with possibly a simple family of exceptions. This assertion turned out to be influential, and by now there are numerous appealing results of this type. For example, Bondy himself [2] proved that Ore’s sufficient condition for Hamiltonicity (that the sum of degrees of any pair of non-adjacent vertices is at least n), implies that the graph is either pancyclic or isomorphic to the complete bipartite graph $K_{n/2, n/2}$. Bauer and Schmeichel [1], relying on previous results of Schmeichel and Hakimi [20], have shown that the sufficient conditions for Hamiltonicity of Bondy [4], Chvátal [5] and Fan [11] all imply pancyclicity, barring a small family of exceptions. Jackson and Ordaz [15] conjectured that any graph G whose connectivity $\kappa(G)$ is strictly larger than its independence number $\alpha(G)$ must be pancyclic. This conjecture is motivated by the classical theorem of Chvátal and

Research supported in part by SNSF grant 200021_196965.

Erdős [6] that a graph with $\kappa(G) \geq \alpha(G)$ must be Hamiltonian. An approximate form of the conjecture was proven by Keevash and Sudakov [16], who showed that $\kappa(G) \geq 600\alpha(G)$ is already sufficient for pancyclicity.

Pancyclicity is just an instance of a wider class of problems, which study the properties of the set of cycle lengths of a graph with connection to other graph parameters. The set of cycle lengths of G is called its *cycle spectrum*, and denoted $\mathcal{C}(G)$. There are by now numerous results relating properties of $\mathcal{C}(G)$ to various graph parameters. For example, Erdős [9] conjectured that a graph G with girth g and average degree d must satisfy $|\mathcal{C}(G)| \geq \Omega(d^{\lfloor \frac{g-1}{2} \rfloor})$. The case $g = 5$ was settled by Erdős, Faudree, Rousseau, and Schelp [10]. Later, Sudakov and Verstraëte [21] proved the full conjecture in a strong form. Another example is a result of Gould, Haxell and Scott [14] that a graph with minimum degree cn must have a cycle of any even length between 4 and $ec(G) - K$, where $ec(G)$ is the length of a longest even cycle in G and K is a constant depending only on c . We should also mention the recent work of Gao, Huo, Liu and Ma [12], who proved several conjectures relating properties of $\mathcal{C}(G)$ to the minimum degree, connectivity or chromatic number of G .

Bondy's meta-conjecture is about conditions for Hamiltonicity which imply pancyclicity. A natural question in the opposite direction is as follows: Let us *assume* that a graph G is Hamiltonian; under which assumptions can we also guarantee that G is pancyclic? Since pancyclicity is sometimes too strong of a requirement, we can relax it and ask to find many cycle lengths. Questions of this type were first introduced by Jacobson and Lehel at the 1999 conference "Paul Erdős and His Mathematics", where they asked for the minimum size of the cycle spectrum of a k -regular Hamiltonian graph G on n vertices? The aforementioned result of Bondy [2] implies that if $k = \lceil n/2 \rceil$, then G is pancyclic unless $G = K_{n/2, n/2}$. At the other extreme, if $k = 2$ then G clearly has just one cycle. Jacobson and Lehel conjectured that already for $k \geq 3$, the number of cycle lengths should be linear in n . This is best possible, since they also observed that one cannot expect to have pancyclicity. Indeed, assuming $2k$ divides n , take $\frac{n}{2k}$ disjoint copies of $K_{k,k}$, ordered in a cycle, remove an edge from each of them, and add an edge between any two consecutive copies such that the resulting graph is k -regular. It is not hard to see that in this construction, the possible cycle lengths are precisely all the even integers between 4 and $2k$ and between $\frac{2n}{k}$ and n . This gives in total $\frac{n}{2} \cdot \frac{k-2}{k} + k$ different lengths.

Soon after the above question was first circulated, Gould, Jacobson and Pfender proved that $|\mathcal{C}(G)| \geq \Omega(\sqrt{n})$ for every k -regular n -vertex Hamiltonian graph G (with $k \geq 3$). This bound was subsequently obtained by several other authors. Yet, prior to our work, the \sqrt{n} bound was the best known result. In particular, Girão, Kittipassorn and Narayanan [13] remarked that improving this estimate would be of considerable interest. Furthermore, the following strengthening of the above conjecture of Jacobson and Lehel, which replaces the k -regularity condition with the assumption that the minimum degree is at least 3, was proposed by Verstraëte [25].

Conjecture 1. An n -vertex Hamiltonian graph G with $\delta(G) \geq 3$ has $\Omega(n)$ different cycle lengths.

While the special case of this conjecture for regular graphs already seems quite challenging, it is natural to expect that the full Conjecture 1 is even harder. The reason for this is that often problems become more difficult when the regularity requirement is replaced by a minimum degree assumption. One well-known example is a conjecture of Thomassen [22], that a graph with a large enough minimum degree contains a subgraph of large minimum degree and large girth. This conjecture is open even for girth 7. However, this statement becomes easy if the given graph is regular, see e.g. [18]. Such situations arise also for questions related to the one studied here: A classical result of Smith (see [24] and also [23]) states that every Hamiltonian 3-regular graph G contains a second Hamilton cycle. As was shown by Entringer and Swart [8], this is no longer true if instead of 3-regularity we assume that $\delta(G) \geq 3$ (even if all degrees are equal to 3 or 4). Girão, Kittipassorn and Narayanan [13] required an involved proof to even show that a Hamiltonian G with $\delta(G) \geq 3$ contains a second cycle of length at least $n - o(n)$. In contrast, for regular G this proof can be simplified considerably and gives a better bound.

It is worth noting that if one replaces the minimum degree requirement $\delta(G) \geq 3$, with the requirement that the average degree is at least 3, then the aforementioned lower bound of $\Omega(\sqrt{n})$ is tight. More generally, Milans, Pfender, Rautenbach, Regen and West [17] have shown that a graph G with n vertices and m edges satisfies $|\mathcal{C}(G)| \geq (1 - o(1))\sqrt{m - n}$, and this is tight. In this paper we prove the following theorem, which resolves Conjecture 1 asymptotically:

Theorem 1. *An n -vertex Hamiltonian graph G with $\delta(G) \geq 3$ contains cycles of $n^{1-o(1)}$ different lengths.*

2 A Sketch and Main Ideas

The most general overarching idea that we employ is to split the Hamilton cycle into pieces (usually paths or pairs of paths) and then find paths with lengths on a different “scale” in different parts. To illustrate what we mean, let us consider the following situation. Suppose that we managed to split our Hamilton cycle into two paths P_1, P_2 , such that there are still many chords inside the vertex-set of each P_i (or, more precisely, that inside each P_i there is a linear number of vertices touching a chord whose other endpoint is also on P_i). Suppose that we found $k = \Omega(\sqrt{n})$ paths Q_1, \dots, Q_k between the endpoints of P_1 (which only use the vertices of P_1), such that $|Q_1|, \dots, |Q_k|$ are all different and all belong to an interval of width \sqrt{n} . Suppose further that we found $\ell = \Omega(\sqrt{n})$ paths R_1, \dots, R_ℓ between the endpoints of P_2 (which only use the vertices of P_2), such that the lengths of any two of these paths are at least \sqrt{n} apart, namely, $||R_i| - |R_j|| > \sqrt{n}$ for all $i \neq j$. In this situation, we can combine any one of the Q_i ’s with any one the R_j ’s, joining them into a cycle of length $|Q_i| + |R_j|$. The crucial point is that the $k\ell$ numbers $|Q_i| + |R_j|$ are all different. In other words,

we use the “condensed” lengths Q_1, \dots, Q_k to “fill in the gaps” between the “spread-out” lengths R_1, \dots, R_ℓ . In total, this would give us $k\ell = \Omega(n)$ different cycle lengths. Hence, achieving both above goals would establish Conjecture 1.

We believe that both above statements should be true, namely, that one can find both $\Omega(\sqrt{n})$ distinct path lengths all contained in an interval of width \sqrt{n} and $\Omega(\sqrt{n})$ path lengths which are \sqrt{n} apart. Observe that both of these statements are essentially implied by Conjecture 1, and that our main result shows that both hold asymptotically (i.e., with \sqrt{n} replaced by $n^{1/2-o(1)}$). On the other hand, these statements shift the difficulty from finding many lengths (note that there have been a number of proofs that there are at least \sqrt{n} different lengths over the years) to controlling what kind of lengths we find.

Our actual strategy for tackling Conjecture 1 is a bit more involved. Instead of splitting our cycle into just two parts, we split it into a larger number k of parts (with k to be chosen as roughly $\sqrt{\log n}$). Here each part will be a pair of cycle sections (subpaths of the cycle) with at least $n^{1-o(1)}$ chords between them, with different section-pairs situated “on top of” each other (see Fig. 1). Now, with the goal of finding $n^{1-\varepsilon}$ different lengths (where ε is an appropriately chosen vanishing function of n), we shall proceed as follows. Inside the first of the k parts, we shall find $\Omega(n^\varepsilon)$ path lengths all belonging to an interval of width n^ε . Then, inside the second part, we shall find about $\Omega(n^\varepsilon)$ lengths $\ell_1 < \dots < \ell_t$ such that any two consecutive lengths are $\Theta(n^\varepsilon)$ apart, namely $\ell_{i+1} - \ell_i = \Theta(n^\varepsilon)$ for all i . Now, by combining the paths we found in these two parts, we will get $\Omega(n^{2\varepsilon})$ different path lengths, all belonging to an interval of width $O(n^{2\varepsilon})$, and only using vertices from the first two parts of the partition. Continuing in this manner, we will find inside the third part $\Omega(n^\varepsilon)$ lengths which are $\Theta(n^{2\varepsilon})$ apart, inside the fourth part $\Omega(n^\varepsilon)$ lengths which are $\Theta(n^{3\varepsilon})$ apart, and so on. This will always allow us to combine the new lengths we find with the lengths found so far to get $\Omega(n^{i\varepsilon})$ different path lengths, all belonging to an interval of width $O(n^{i\varepsilon})$, only using vertices from the first i parts. In each iteration we will actually lose a polylogarithmic factor in the number of paths we find, which will result in the optimal number of iterations being $\sqrt{\frac{\log n}{\log \log n}}$ (this corresponds to having $\varepsilon = \sqrt{\frac{\log \log n}{\log n}}$). After this number of iterations, we will find $n^{1-o(1)}$ different lengths.

Let us now focus on a single iteration and sketch the main ideas involved. For simplicity, suppose that this is the third iteration, namely, that our goal is to find (inside the third of the k parts of the partition) $\Omega(n^\varepsilon)$ path lengths $\ell_1 < \dots < \ell_t$ with $\ell_{i+1} - \ell_i = \Theta(n^{2\varepsilon})$ for all i . Up to this step, we have already found $\Theta(n^{2\varepsilon})$ path lengths in an interval of width $O(n^{2\varepsilon})$ inside the first two parts. Now, we consider a maximum collection e_1, \dots, e_m of chords inside the third part, such that for all $i \neq j$, the lengths of e_i and e_j differ by at least $n^{2\varepsilon}$. Each chord e_i gives rise to a path inside the third part (namely, the path that consists of the chord and pieces of the cycle), and the lengths of any two of these m paths differ by at least $n^{2\varepsilon}$. Now, observe that if $m \geq n^{1-3\varepsilon}$, then by combining these paths with the $\Theta(n^{2\varepsilon})$ path lengths we found in the first

two parts of the partition, we obtain altogether $m \cdot \Omega(n^{2\varepsilon}) = \Omega(n^{1-\varepsilon})$ different cycle lengths, and thus achieve our goal already at this stage. So we may assume that $m \leq n^{1-3\varepsilon}$. Since e_1, \dots, e_m is a maximal family, the length of any other chord must be at distance at most $n^{2\varepsilon}$ to that of one of the e_i 's. By averaging (and as each part of the partition contains $n^{1-o(1)}$ chords), we see that there is a family E of at least $n^{1-o(1)}/m \geq n^{3\varepsilon-o(1)}$ different chords, whose lengths all belong to an interval of width $n^{2\varepsilon}$. The reason such a family E is useful is as follows: Suppose we partition the left section of the third part into subpaths X_1, X_2, \dots of length $n^{2\varepsilon}$. Then, for any two such subpaths X_i, X_j which are not consecutive (and hence are at distance larger than $n^{2\varepsilon}$ on the path), any chord touching X_i must interlace (i.e. cross) any chord touching X_j . For if not, then the difference of the lengths of these two chords is larger than $n^{2\varepsilon}$, contradicting the fact that both lengths belong to an interval of width $n^{2\varepsilon}$. Letting Y_i be the neighbourhood of X_i on the right side, we see that E decomposes into pairwise-interlacing pieces X_i, Y_i , see Fig. 2 for an illustration. This structure, together with some additional arguments, then allows us to find the $\Omega(n^\varepsilon)$ desired path lengths $\ell_1 < \dots < \ell_t$. We remark that while it is not hard to find such lengths with $\ell_{i+1} - \ell_i = \Omega(n^{2\varepsilon})$, which already allows us to find $\Omega(n^{3\varepsilon})$ lengths, it is essential for the next iteration that these lengths are not too far apart, in other words ensuring in addition that $\ell_{i+1} - \ell_i \leq O(n^{2\varepsilon})$ is crucial in order to be able to continue our argument.

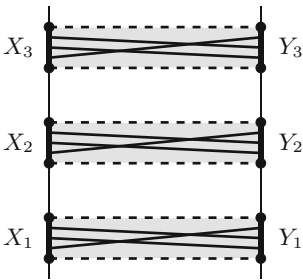


Fig. 1. Parallel subsection pairs

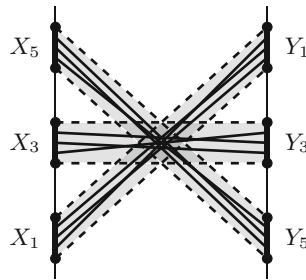


Fig. 2. Interlacing subsection pairs

3 Concluding Remarks

Our main result is that an n -vertex Hamiltonian graph of minimum degree 3 has cycles of $n^{1-o(1)}$ different lengths, which shows that Conjecture 1 holds asymptotically. Moreover, for the original question of Jacobson and Lehel (dealing with graphs of bounded degree), we can use our ideas to get a better quantitative bound of $\frac{n}{\text{polylog}(n)}$. Still, it would be very interesting to prove a linear bound on the number of cycles, even in the 3-regular case. Towards this we propose the following natural intermediate steps:

Conjecture 2. Every n -vertex Hamiltonian graph with minimum degree 3 has:

1. $\Omega(\sqrt{n})$ cycle lengths all belonging to an interval of width $O(\sqrt{n})$.
2. $\Omega(\sqrt{n})$ cycle lengths any two of which are at least $\Omega(\sqrt{n})$ apart.

Observe first that Conjecture 1 immediately implies Conjecture 2. On the other hand, we can show that a slight strengthening of Conjecture 2 already implies Conjecture 1.

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Maximal k -Wise ℓ -Divisible Set Families Are Atomic

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Abstract. Let $\mathcal{F} \subset 2^{[n]}$ such that the intersection of any two members of \mathcal{F} has size divisible by ℓ . By the famous Eventown theorem, if $\ell = 2$ then $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$, and this bound can be achieved by ‘atomic’ construction, i.e. splitting the ground set into disjoint pairs and taking their arbitrary unions. Similarly, splitting the ground set into disjoint sets of size ℓ gives a family with pairwise intersections divisible by ℓ and size $2^{\lfloor n/\ell \rfloor}$. Yet, for infinitely many ℓ , Frankl and Odlyzko constructed families \mathcal{F} as above of much bigger size $2^{\Omega(n \log \ell/\ell)}$. On the other hand, in 1983 they conjectured that for every ℓ there exists some k such that if any k distinct members of \mathcal{F} have an intersection of size divisible by ℓ , then $|\mathcal{F}| \leq 2^{(1+o(1))n/\ell}$. We completely resolve this old conjecture in a strong form, showing that $|\mathcal{F}| \leq 2^{\lfloor n/\ell \rfloor} + O(1)$ holds if k is chosen appropriately.

Keywords: Extremal combinatorics · Set systems · Intersections

1 Introduction

An *eventown* is a family $\mathcal{F} \subset 2^{[n]}$ such that $|A \cap B|$ is even for any $A, B \in \mathcal{F}$. The famous Eventown theorem of Berkelamp [2], also proved independently by Graver [6], states that if $\mathcal{F} \subset 2^{[n]}$ is an eventown, then $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$. This bound is also the best possible, and a simple construction showing this is given as follows. Say that a family $\mathcal{F} \subset 2^{[n]}$ is *atomic*, if there exist disjoint sets $A_1, \dots, A_d \subset [n]$ such that \mathcal{F} is the family of all sets A satisfying that either $A_i \subset A$, or $A_i \cap A = \emptyset$ for every $i \in [d]$, and A contains no element not covered by the sets A_i . Also, let $S(n, \ell)$ be the atomic family for which $d = \lfloor n/\ell \rfloor$ and all $A_i, i \in [d]$ have size exactly ℓ . Note that $|S(n, \ell)| = 2^{\lfloor n/\ell \rfloor}$, and the size of the intersection of any number of sets in $S(n, \ell)$ is divisible by ℓ . Therefore, the family $S(n, 2)$ is an eventown of size $2^{\lfloor n/2 \rfloor}$. This construction is not unique. Moreover, any eventown family can be completed to a maximal one of size $2^{\lfloor n/2 \rfloor}$, see e.g. the book of Babai and Frankl [1], which is also a general reference on intersection problems.

In general, one might be tempted to conjecture that the maximal families $\mathcal{F} \subset 2^{[n]}$, whose all pairwise intersections are divisible by ℓ , have size close to $2^{(1+o(1))n/\ell}$. However, this turns out to be far from the truth. Frankl and Odlyzko

[3] proved that if there exists a Hadamard matrix of order 4ℓ , then there exists such a family of size $2^{\Omega(n \log \ell/\ell)}$, and this bound is also the best possible up to the constant factor. On the other hand, Frankl and Tokushige [4] proved that if we consider uniform families, that is, $\mathcal{F} \subset [n]^{(r)}$, then $|\mathcal{F}| \leq \binom{\lfloor n/\ell \rfloor}{r/\ell}$ if n is sufficiently large given r and $\ell \mid r$. This bound is also the best possible as witnessed by the family $\mathcal{F} = [n]^{(r)} \cap S(n, \ell)$. Let us emphasize that the condition that n must be large compared to r is necessary, otherwise this would contradict the aforementioned construction of Frankl and Odlyzko.

Despite all the above, if we require that the intersection of *any number* of sets in $\mathcal{F} \subset 2^{[n]}$ must have size divisible by ℓ , then it is not difficult to show that $|\mathcal{F}| \leq 2^{\lfloor n/\ell \rfloor}$ for any n and ℓ . Moreover, in this case, \mathcal{F} is contained in some isomorphic copy of $S(n, \ell)$ (we say that two families in $2^{[n]}$ are isomorphic if they are equal up to a permutation of $[n]$). In 1983, Frankl and Odlyzko [3] asked whether a similar conclusion holds if we only require that the intersection of any k distinct sets in \mathcal{F} has size divisible by ℓ , where k is some constant only depending on ℓ . More precisely, they conjectured that for some k , we must have $|\mathcal{F}| \leq 2^{(1+o(1))n/\ell}$ for such a family \mathcal{F} . Until recently, it was not even known if the bound $2^{O(n \log \ell/\ell)}$ can be improved for any constant k . Indeed, while there are many tools to handle pairwise intersections as they correspond to the scalar product of characteristic vectors, k -wise intersections are usually harder to analyse, see, e.g., [5, 7–10] for related results. Also, it was shown in [8] that if the conjecture is true, k must depend on ℓ . In particular, if ℓ is a power of 2, there exist families $\mathcal{F} \subset 2^{[n]}$ such that the intersection of any k sets in \mathcal{F} has size divisible by ℓ , and $|\mathcal{F}| \geq 2^{c_k n \log \ell/\ell}$, where $c_k > 0$ is a constant only depending on k . In this paper we resolve the conjecture of Frankl and Odlyzko in the following strong form.

Theorem 1. *Let ℓ be a positive integer, then there exists $k = k(\ell)$ such that for every positive integer n the following holds. Let $\mathcal{F} \subset 2^{[n]}$ such that the intersection of any k distinct elements of \mathcal{F} is divisible by ℓ . Then $|\mathcal{F}| \leq 2^{\lfloor n/\ell \rfloor} + c$, where $c = c(\ell, k)$ is a constant, and $c = 0$ if $\ell \mid n$ and n is sufficiently large.*

Note that the error term c is needed if ℓ does not divide n . Indeed, in this case $S(n, \ell)$ is not extremal, one can add a constant number of sets contained in the nonempty set not covered by members of $S(n, \ell)$ while retaining the property that the intersection of every k distinct sets has size divisible by ℓ .

Our proof of Theorem 1 will proceed via a stability type argument, which might be of independent interest. We show that if the dimensions of the subspaces (over any field \mathbb{F}) generated by the characteristic vectors of the elements of \mathcal{F} and $\mathcal{F} \cdot \mathcal{F} = \{A \cap B : A, B \in \mathcal{F}\}$ do not differ by much, then \mathcal{F} must be close to an atomic family in a certain sense. Let us present some of the key ideas needed for the proof of Theorem 1.

2 Dimensionality

In this section, we present a theorem which implies Theorem 1 after a small amount of work. Before we state this theorem, let us introduce some notation.

For a vector v , we use $v(i)$ to denote the i th coordinate of v . As usual, \mathbb{Z}_ℓ denotes the ring of integers modulo ℓ , and if p is a prime, we write \mathbb{F}_p instead of \mathbb{Z}_p to emphasize that it is also a field. If $\mathcal{F} \subset \mathbb{Z}_\ell^n$, then $\langle \mathcal{F} \rangle_\ell \subset \mathbb{Z}_\ell^n$ (or simply $\langle \mathcal{F} \rangle$ if ℓ is clear from the context) is the set of all linear combinations of the elements of \mathcal{F} . If $S \subset [n]$ and $v \in \mathbb{Z}_\ell^n$, then $v|_S \in \mathbb{Z}_\ell^S$ is the restriction of v to the coordinates in S , and $\mathcal{F}|_S = \{v|_S : v \in \mathcal{F}\}$. Say that \mathcal{F} is *non-reducible* if \mathcal{F} does not vanish on any of the coordinates (namely, if there is no i such that $v(i) = 0$ for all $v \in \mathcal{F}$). Finally, let $\|v\| = \sum_{i=1}^n v(i)$.

Given $v, w \in \mathbb{Z}_\ell^n$, let $v \cdot w \in \mathbb{Z}_\ell^n$ be defined as $(v \cdot w)(i) = v(i)w(i)$ for $i \in [n]$, and let v^k be defined as $v^k(i) = v(i)^k$. Also, if $\mathcal{F}, \mathcal{F}' \subset \mathbb{Z}_\ell^n$, let $\mathcal{F} \cdot \mathcal{F}' = \{v \cdot w : v \in \mathcal{F}, w \in \mathcal{F}'\}$, and let $\mathcal{F}^k = \mathcal{F} \cdot \dots \cdot \mathcal{F}$, where the product contains k terms. Say that a set $\mathcal{F} \subset \mathbb{Z}_\ell^n$ is *k -closed* if $\|v\| = 0$ for every $1 \leq i \leq k$ and $v \in \mathcal{F}^i$. Note that if v and w are characteristic vectors of sets A and B , then $v \cdot w$ is the characteristic vector of $A \cap B$. We use the following simple, but important observation repeatedly.

Claim. If $\mathcal{F} \subset \mathbb{Z}_\ell^n$ is k -closed, then $\langle \mathcal{F} \rangle$ is also k -closed. Also, if $\mathcal{F}, \mathcal{F}' \subset \mathbb{Z}_\ell^n$, then $\langle \mathcal{F} \cdot \mathcal{F}' \rangle = \langle \mathcal{F} \rangle \cdot \langle \mathcal{F}' \rangle$.

Let p be a prime, and let $\mathcal{F} \subset \mathbb{F}_p^n$. Given $i, j \in [n]$, say that i and j are *siblings* in \mathcal{F} if there exists $\lambda \in \mathbb{F}_p, \lambda \neq 0$ such that $v(i) = \lambda v(j)$ for all $v \in \mathcal{F}$ and $v(i) \neq 0$ for at least one $v \in \mathcal{F}$. Also, if $\mathcal{F} \subset \mathbb{Z}_\ell^n$, say that i and j are *twins* if $v(i) = v(j)$ for every $v \in \mathcal{F}$, and $v(i) \neq 0$ for at least one $v \in \mathcal{F}$. If $S \subset [n]$, then say that S is a *set of siblings (or twins)* if any pair of elements in S are siblings (or twins), or $|S| = 1$. Also, say that S is a maximal set of siblings (twins) if it is maximal with respect to containment.

Let us collect some simple but important properties of siblings and twins.

Claim. Let $\mathcal{F} \subset \{0, 1\}^n$.

1. Whether i and j are twins for \mathcal{F} over \mathbb{Z}_ℓ is independent of ℓ .
2. If \mathcal{F} is non-reducible, then the maximal sets of twins for \mathcal{F} form a partition of $[n]$.
3. For every $\ell > 1$ and $k \geq 1$, the families $\langle \mathcal{F} \rangle_\ell$ and $\bigcup_{i=1}^k \mathcal{F}^i$ have the same sets of twins (over \mathbb{Z}_ℓ) as \mathcal{F} .
4. If i and j are siblings for \mathcal{F} over \mathbb{F}_p , then i and j are twins.

The main result of this section is the following variant of Theorem 1. We show that if $\mathcal{F} \subset 2^{[n]}$ is such that the intersection of any k not necessarily distinct elements of \mathcal{F} has size divisible by ℓ , then $|\mathcal{F}| \leq 2^{\lfloor n/\ell \rfloor}$, given k is sufficiently large with respect to ℓ . We also show that if \mathcal{F} is close to being extremal, then \mathcal{F} must be a subfamily of some isomorphic copy of $S(n, \ell)$.

Theorem 2. *Let ℓ be a positive integer, then there exists k such that the following holds. Let $\mathcal{F} \subset \{0, 1\}^n$ such that \mathcal{F} is k -closed over \mathbb{Z}_ℓ . Then $|\mathcal{F}| \leq 2^{\lfloor n/\ell \rfloor}$. Also, if $|\mathcal{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, then $[n]$ can be partitioned into sets A_1, \dots, A_d, A' such that A_i is a maximal set of twins for \mathcal{F} for $i \in [d]$, $|A_i| = \ell$, $|A'| \leq \ell - 1$, and \mathcal{F} vanishes on A' .*

Say that a subspace $V < \mathbb{F}_p^n$ of dimension d is *atomic* if $[n]$ can be partitioned into d sets of siblings for V . In order to prove Theorem 2, we prove a stability type result which tells us that if $V < \mathbb{F}_p^n$ and the dimension of $\langle V \cup (V \cdot V) \rangle$ is not much larger than the dimension of V , then V must be close to an atomic subspace.

Lemma 1. *Let p be a prime, $V < \mathbb{F}_p^n$, $d = \dim(V)$ and $\dim(\langle V \cup (V \cdot V) \rangle) = d + h$. Then $[n]$ can be partitioned into $d + 1$ sets A_1, \dots, A_d, B such that A_i is a maximal set of siblings for V for each $i \in [d]$, and $\dim(V|_B) \leq 2h$.*

Next, we show that if $\ell = p^\alpha$ is a prime power, and $\mathcal{F} \subset \{0, 1\}^n$ is k -closed over \mathbb{Z}_ℓ for some large constant k , then most sets of maximal twins for \mathcal{F} must have size divisible by ℓ , provided that the dimension of $\langle \mathcal{F} \rangle_p$ is large.

Lemma 2. *Let p be a prime and $\alpha \in \mathbb{Z}^+$. Let $\mathcal{F} \subset \{0, 1\}^n$ be $2(p + \alpha)$ -closed over \mathbb{Z}_{p^α} , let $\dim(\langle \mathcal{F} \rangle_p) = d$, and let A_1, \dots, A_d, B be a partition of $[n]$ such that A_i is a set of twins, and $\dim(\langle \mathcal{F}|_B \rangle_p) \leq h$. Then at least $d - 2\alpha h$ of the numbers $|A_1|, \dots, |A_d|$ are divisible by p^α .*

From this, we deduce the following.

Lemma 3. *Let p be a prime, and $\alpha, t \in \mathbb{Z}^+$. Let $\mathcal{F} \subset \{0, 1\}^n$ such that \mathcal{F} is non-reducible and $2^{t+1}(p + \alpha)$ -closed over \mathbb{Z}_{p^α} . Let A_1, \dots, A_d be the unique partition of $[n]$ into maximal sets of twins, and let*

$$B = \bigcup_{\substack{i \in [d] \\ |A_i| \not\equiv 0 \pmod{p^\alpha}}} A_i.$$

Then $\dim(\langle \mathcal{F}|_B \rangle_p) \leq \frac{6n\alpha}{t}$.

In order to prove Theorem 2, we apply Lemma 3 for the different prime powers dividing ℓ . We conclude that if \mathcal{F} is large, one of the maximal sets of twins for \mathcal{F} must have size divisible by ℓ . From this point, one can easily finish the proof by induction on n .

In order to prove Theorem 1, we just show that if $\mathcal{F} \subset 2^{[n]}$ is such that the intersection of any k distinct members of \mathcal{F} has size divisible by ℓ , then we can find $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{F}'$ is small, and the intersection of any k , not necessarily distinct, members of \mathcal{F}' has size divisible by ℓ . But then $|\mathcal{F}'| \leq 2^{\lfloor n/\ell \rfloor}$ by Theorem 2.

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A Trivariate Dichromate Polynomial for Digraphs

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Abstract. We define a trivariate polynomial combining the NL-coflow and the NL-flow polynomial, which build a dual pair counting acyclic colorings of directed graphs, in the more general setting of regular oriented matroids.

Keywords: Flow polynomial · Chromatic polynomial · Dichromatic number · Face lattice · Oriented matroids

1 Introduction

In 1954 Tutte introduced a bivariate polynomial of an undirected graph G and called it the *dichromate* of G [9]. Nowadays better known as the *Tutte polynomial* it features not only a variety of properties and applications, but also specializes to many graph-theoretic polynomials. Two of them, the chromatic and the flow polynomial, counting proper colorings and nowhere-zero flows, build a pair of dual polynomials in the sense that one polynomial becomes the other one by taking the dual graph.

Regarding directed graphs, or digraphs for short, acyclic colorings are a natural generalization of proper colorings. A digraph is *acyclically colorable* if no color class contains a directed cycle. This concept is due to Neumann-Lara [8].

In [5] a flow theory for digraphs transferring Tutte's theory of nowhere-zero flows to directed graphs has been developed and amplified in [1] and [6], where the authors introduce a pair of dual polynomials, counting acyclic colorings of a digraph and the dual equivalent called NL-flows.

In order to combine these two polynomials we will leave the setting of digraphs and enter the world of oriented matroids. This more general scenery provides a plethora of useful techniques as well as a common foundation upon which our new polynomial is built. This foundation is due to a construction of Brylawski and Ziegler [4] representing a dual pair of oriented matroids as complementary minors.

Our notation is fairly standard and follows the book of Björner et al. [2] if not explicitly defined. Due to space restrictions almost all proofs are omitted.

1.1 Notation and Previous Results

In [6] we found the following representation of the NL-coflow polynomial counting acyclic colorings in a digraph $D = (V, A)$.

Definition 1. Let μ_Q be the Möbius function of (Q, \subseteq) with $Q := \{B \subseteq A : D[B] \text{ is a totally cyclic subdigraph of } D\}$. Then

$$\psi_{NL}^D(x) = \sum_{B \in Q} \mu_Q(\emptyset, B)x^{rk(A/B)}$$

is called the NL-coflow polynomial of D , where, for $Y \subseteq A$, $rk(Y)$ is the rank of the incidence matrix of $D[Y]$, which equals $|V(Y)| - c(Y)$, i.e. the number of vertices minus the number of connected components of $D[Y]$.

Recall that in our definition of contraction (see [3]) no additional arcs (elements) are removed, i.e. parallel arcs and loops can occur. This holds for both the graphic and the matroid contraction.

We will now define this polynomial in the more general setting of (regular) oriented matroids. Note, that all of our results also work in the non-regular case. Since we are not aware of a meaningful interpretation in this case, all our matroids will be regular, if not explicitly pointed out.

Let M be an oriented matroid on the (finite) groundset E . The covectors of M , i.e. compositions of (signed) cocircuits, together with the partial order $0 \leq +$ and $0 \leq -$ form the *face lattice* \mathcal{L} of M with minimal element \emptyset . Since the NL-flow polynomial (see [1]) only considers directed cuts, we are only interested in the nonnegative part of \mathcal{L} , which we denote by $\mathcal{L}_+ := \mathcal{L} \cap \{0, +\}^E$. By \mathcal{L}^* we denote the face lattice of the dual M^* . Again, we are only interested in the nonnegative part \mathcal{L}_+^* which in the graphic case corresponds to the set of totally cyclic subdigraphs partially ordered by inclusion. Let μ and μ^* denote the Möbius function of \mathcal{L}_+ and \mathcal{L}_+^* , respectively. By rk and rk^* we denote the rank and corank of the respective matroid (minor) and by \underline{X} we denote the support of the covector X .

Now we can define the *NL-coflow polynomial* of an oriented matroid M as

$$\psi_{NL}^M(x) := \sum_{X \in \mathcal{L}_+^*} \mu^*(\emptyset, X)x^{rk(M/\underline{X})}.$$

Dually we define the *NL-flow polynomial* of M as

$$\phi_{NL}^M(x) := \sum_{X \in \mathcal{L}_+} \mu(\emptyset, X)x^{rk^*(M \setminus \underline{X})}.$$

It is easy to see that both coflow polynomials coincide in the graphic case. Our new definition of the NL-flow polynomial also coincides with the graphic one in [1] since $rk^*(Y) = |Y| - rk(Y)$ holds for any minor, in particular for $Y := M \setminus B, B \in \mathcal{C}^1$.

¹ In [1] the NL-flow polynomial of a digraph $D = (V, A)$ is defined on the poset (\mathcal{C}, \supseteq) with $\mathcal{C} := \{A \setminus C : \exists C_1, \dots, C_r \text{ directed cuts s.t. } C = \bigcup_{i=1..r} C_i\}$.

2 Setting

Since our polynomials are defined on different face lattices we have to find a common lattice including both. In [4] Brylawski and Ziegler give the following beautiful construction which provides the desired lattice.

Let M be an oriented matroid on the groundset $E = \{1, \dots, n\}$ with rank r and M^* its dual. Suppose that $\mathcal{B} := \{1, \dots, r\}$ is a basis of M and $\{r+1, \dots, n\}$ is the corresponding basis of M^* . Furthermore set $E_1 := \mathcal{B}, E_2 := E \setminus \mathcal{B}$ and

$$\hat{E} := E_1 \cup E_2 \cup A \cup B = E \cup A \cup B$$

with $A := \{n+1, \dots, n+r\}$ and $B := \{n+r+1, \dots, 2n\}$ and let M_1 be the oriented matroid on \hat{E} , that is obtained by extending M by elements $n+i$ that are parallel to the elements i for $1 \leq i \leq r$ and that are loops for $r+1 \leq i \leq n$. Similarly, let M_2 be the oriented matroid on \hat{E} that is obtained by extending M^* by elements $n+i$ that are loops for $1 \leq i \leq r$ and that are parallel to the elements i for $r+1 \leq i \leq n$. Then M_1 has rank r and M_2 has rank $n-r$. Their union (see Chap. 7.6 in [2])

$$\hat{M} := M_1 \cup M_2$$

has rank n . Note that the construction of the oriented matroid union highly depends on the choice of the basis \mathcal{B} . Due to Theorem 2 in [4] we have

$$\hat{M} \setminus A / B = M \text{ and } \hat{M} / A \setminus B = M^*.$$

In the case where M is realizable, \hat{M} is also realizable. Namely, if M can be represented by $(I_r \ C)$, where I_r denotes the identity matrix of rank r , then M_1 and M_2 are represented by $(I_r \ C \ I_r \ 0)$ and $(-C^T \ I_{n-r} \ 0 \ I_{n-r})$, respectively. Now let $(-C^T \ I_{n-r} \ 0 \ I_{n-r})^\epsilon$ be the matrix obtained by multiplying the i -th column by ϵ^{2n-i} for all $i \in \{1, \dots, 2n\}$ and $\epsilon > 0$ sufficiently small. Then the combined matrix

$$\begin{pmatrix} I_r & C & I_r & 0 \\ (-C^T & I_{n-r} & 0 & I_{n-r})^\epsilon \end{pmatrix}$$

represents \hat{M} (see Proposition 8.2.7 of [2] and [4]). Note that even if M and M^* are regular, this might not be true for \hat{M} . However, the face lattice of \hat{M} , which we will denote by $\hat{\mathcal{L}}$, will serve our purpose.

In the following subsections we will find a characterization of the covectors of M and its dual in this supermatroid \hat{M} .

2.1 Cocircuits and Covectors

Given a cocircuit D in M or in M^* we find a corresponding cocircuit \hat{D} in \hat{M} such that $\underline{D} \subseteq \underline{\hat{D}}$. Furthermore we will find that, given $D^- = \emptyset$, then also $\hat{D}^- = \emptyset$ holds. Due to the construction of \hat{M} we will first extend D to a cocircuit in M_1 , which then is already a cocircuit in \hat{M} . For the proof we will first look at the underlying unoriented matroid and then compute the signatures in a second step. We write $x||y$ iff x and y are parallel elements as constructed above.

Lemma 1. *Let D be a cocircuit in \underline{M} and set $D_1 := \{a \in A : a||e, e \in D \cap E_1\}$. Then $\hat{D} := D \cup D_1$ is a cocircuit in \underline{M}_1 . If $D = (D^+, D^-)$ is a signed cocircuit in M with $D^- = \emptyset$, then $\hat{D} := (D^+ \cup D_1, \emptyset)$ is a signed cocircuit in M_1 .*

We are left to prove that \hat{D} is also a cocircuit in \hat{M} . Again, we will first take a look at the underlying unoriented case, where the oriented matroid union becomes the usual matroid union. Let \mathcal{I}_1 and \mathcal{I}_2 be the independent sets in \underline{M}_1 and \underline{M}_2 respectively. Then $\hat{M} = (\hat{E}, \mathcal{I})$, where $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1 \text{ and } I_2 \in \mathcal{I}_2\}$ are the independent sets of \hat{M} . As an immediate result every basis b of \hat{M} can be written as $b = b_1 \cup b_2$, where b_1 is a basis of \underline{M}_1 and b_2 is a basis of \underline{M}_2 .

Lemma 2. *Let D be a cocircuit in \underline{M}_1 . Then D is also a cocircuit in \hat{M} .*

If $D = (D^+, D^-)$ is a signed cocircuit in M_1 , then D is a signed cocircuit in \hat{M} .

Analogously, one can define $D_2 := \{b \in B : b||e, e \in D \cap E_2\}$ and prove that if $D = (D^+, \emptyset)$ is a signed cocircuit in M^* , then $\hat{D} := (D^+ \cup D_2, \emptyset)$ is a signed cocircuit in \hat{M} . Since covectors are compositions of cocircuits, the results above readily yield:

Proposition 1.

- (i) *Let X be a covector in M and $\hat{A} := \{a \in A : a||e, e \in \underline{X} \cap E_1\}$. Then $\hat{X} := (X^+ \cup \hat{A}, \emptyset)$ is a covector in \hat{M} .*
- (ii) *Let X be a covector in M^* and $\hat{B} := \{b \in B : b||e, e \in \underline{X} \cap E_2\}$. Then $\hat{X} := (X^+ \cup \hat{B}, \emptyset)$ is a covector in \hat{M} .*

2.2 The Face Lattice of \hat{M}

We have already seen that both the covectors of M and of M^* can be found in the face lattice of \hat{M} . In the following we will show the converse: Having a covector of \hat{M} of that specific shape we determined in the previous section, its restriction to E corresponds to a covector of M or of M^* , respectively. Furthermore we will see that the corresponding Möbius functions coincide. The following lemma will be crucial for both. Here, $(\hat{X} \cap A)||(\hat{X} \cap E_1)$ means, that for all $x \in A, y \in E_1$ we have $x, y \in \hat{X}$ if and only if $x||y$.

Lemma 3. *Let $\hat{X} = (\hat{X}^+, \emptyset)$ be a signed cocircuit of \hat{M} with $\hat{X} \cap B = \emptyset$ ($\hat{X} \cap A = \emptyset$). Then $(\hat{X} \cap A)||(\hat{X} \cap E_1)$ (resp. $(\hat{X} \cap B)||(\hat{X} \cap E_2)$).*

Lemma 4. *Let $\hat{X} = (\hat{X}^+, \emptyset)$ be a signed cocircuit of \hat{M} .*

- (i) *If $\hat{X} \cap B = \emptyset$, then $X = \hat{X} \cap E := (\hat{X}^+ \cap E, \hat{X}^- \cap E)$ is a signed cocircuit of M and $X^- = \emptyset$.*
- (ii) *If $\hat{X} \cap A = \emptyset$, then $X = \hat{X} \cap E := (\hat{X}^+ \cap E, \hat{X}^- \cap E)$ is a signed cocircuit of M^* and $X^- = \emptyset$.*

Again, the previous lemma generalizes naturally to covectors. Let us now take a look at the corresponding rank functions. By $rk_{\mathcal{L}}, rk_{\mathcal{L}^*}$ and $rk_{\hat{\mathcal{L}}}$ we denote the rank functions of the respective face lattices of M, M^* and \hat{M} .

Lemma 5. *Let $X = (X^+, \emptyset)$ be a covector of M (of M^*) and let \hat{X} be the corresponding covector in \hat{M} . Then $rk_{\mathcal{L}}(X) = rk_{\hat{\mathcal{L}}}(\hat{X})$ ($rk_{\mathcal{L}^*}(X) = rk_{\hat{\mathcal{L}}}(\hat{X})$).*

As an immediate result, also the corresponding Möbius functions coincide. Aside from this we will find a common expression of the exponents of the NL-flow and the NL-coflow polynomial in terms of the rank in the face lattice of \hat{M} . In order to do so we will use Corollary 4.1.15 (i) in [2]:

$$rk_{\mathcal{L}}(X) = rk(M) - rk(M \setminus \underline{X}) \quad \forall X \in \mathcal{L}. \tag{1}$$

Lemma 6. *Let $\hat{X} \in \hat{\mathcal{L}}_+$ and $X := \hat{X} \cap E$.*

- (i) *If $\hat{X} \cap A = \emptyset$, then $rk(M/\underline{X}) = rk_{\hat{\mathcal{L}}}(\hat{X}) + |E \setminus \underline{X}| - (n - r)$.*
- (ii) *If $\hat{X} \cap B = \emptyset$, then $rk^*(M \setminus \underline{X}) = rk_{\hat{\mathcal{L}}}(\hat{X}) + |E \setminus \underline{X}| - r$.*

Proof. By Lemma 4, $X \in \mathcal{L}^*$. Dualizing, (1) and Lemma 5 yield

$$\begin{aligned} rk(M/\underline{X}) &= rk^*(M^* \setminus \underline{X}) = |E \setminus \underline{X}| - rk(M^* \setminus \underline{X}) \\ &= |E \setminus \underline{X}| + rk_{\mathcal{L}^*}(X) - rk(M^*) = |E \setminus \underline{X}| + rk_{\hat{\mathcal{L}}}(\hat{X}) - (n - r). \end{aligned}$$

3 A New Polynomial

Finally we are able to define a new polynomial in three variables which somehow generalizes both, the NL-flow and the NL-coflow polynomial. In order to switch between the NL-flow and the NL-coflow polynomial we use two of the three variables as some kind of toggle. Whenever the support of a covector of \hat{M} is non-empty in A (or in B resp.), this covector cannot correspond to one of M^* (or M resp.) and will be rejected. Due to Lemma 4, covectors that correspond neither to a covector of M nor to one of M^* will also be rejected, since they have non-empty support in A as well as in B . This is why we can define our polynomial on the whole face lattice $\hat{\mathcal{L}}_+$.

Definition 2. *Let M be a regular, oriented matroid on a finite groundset E , \mathcal{B} the basis of M chosen to construct \hat{M} and $\hat{\mu}$ the Möbius function of the face lattice of \hat{M} . Then we define*

$$\Omega_{NL}^{M, \mathcal{B}}(x, y, z) := \sum_{X \in \hat{\mathcal{L}}_+} \hat{\mu}(\emptyset, X) x^{rk_{\hat{\mathcal{L}}}(X) + |E \setminus (\underline{X} \cap E)|} y^{|\underline{X} \cap A|} z^{|\underline{X} \cap B|},$$

which we call the dichromate of a digraph representing M in the graphic case.

Theorem 1. *Let M be a regular, oriented matroid on E with $|E| = n$ and let r be its rank. Then*

$$\Omega_{NL}^{M, \mathcal{B}}(x, 0, 1) = x^{n-r} \cdot \psi_{NL}^M(x)$$

for any basis \mathcal{B} of M .

Proof. By Definition 2 it immediately follows that

$$\Omega_{NL}^{M,\mathcal{B}}(x, 0, 1) = \sum_{\substack{X \in \mathcal{L}_+ \\ X \cap A = \emptyset}} \hat{\mu}(\emptyset, X) x^{rk_{\mathcal{L}}(X) + |E \setminus (X \cap E)|}$$

for any basis \mathcal{B} of M . Lemma 4 (ii) yields that the sum only considers $X \cap E \in \mathcal{L}_+$, since it is a covector of M^* with positive entries only. The respective Möbius functions coincide due to Lemma 5. Lemma 6 (i) completes the proof.

Using Lemmas 4 (i), 5 and 6 (ii), the next can be proven completely analogously.

Theorem 2. *Let M be a regular, oriented matroid on E with $|E| = n$ and let r be its rank. Then*

$$\Omega_{NL}^{M,\mathcal{B}}(x, 1, 0) = x^r \cdot \phi_{NL}^M(x)$$

for any basis \mathcal{B} of M .

4 Outlook

We are not aware of any meaningful interpretation in the non-regular case. Nevertheless the polynomial exists in this case and since the union does not need to preserve regularity we have in any event already crossed this line.

Since the contraction of arcs might generate new directed cycles and loops it is clear that our polynomials do not satisfy the (classical) deletion-contraction formula. Presumably the most agreed concept of digraph minors in the context of acyclic colorings are *butterfly minors* (see [7]). Unfortunately digraphs that are not butterfly contractible can be arbitrarily complicated.

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Schur's Theorem for Randomly Perturbed Sets

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Abstract. A set of integers A is said to be *Schur* if any two-colouring of A results in monochromatic x, y and z with $x + y = z$. We study the following problem: How many random integers from $[n]$ need to be added to some $A \subset [n]$ to ensure that the resulting set is Schur with high probability? Hu showed in 1980 that when $|A| > \lceil \frac{4n}{5} \rceil$, no random integers are needed as A is already guaranteed to be Schur. Recently, Aigner-Horev and Person showed that for any dense set of integers $A \subseteq [n]$, adding $\omega(n^{1/3})$ random integers suffices, noting that this is optimal for sets A with $|A| \leq \lfloor \frac{n}{2} \rfloor$. Here we complete the picture by closing the gap between these two results. We show that if $A \subset [n]$, with $|A| = \lfloor \frac{n}{2} \rfloor + t < \lceil \frac{4n}{5} \rceil$ then adding $\omega(\min\{n^{1/3}, nt^{-1}\})$ random integers will result in a set that is Schur with high probability. Our result is optimal for all t , and we further provide a stability result showing that one needs far fewer random integers when A is not close in structure to the extremal example.

Keywords: Randomly perturbed sets · Ramsey theory · Combinatorial number theory · Schur triples

1 Introduction

A *Schur triple* in a set $A \subset \mathbb{N}$ is a triple $(x, y, z) \in A^3$ such that $x + y = z$. We say a set $A \subset \mathbb{N}$ is *r-Schur* if any r -colouring of the elements in A results in a monochromatic Schur triple. Note that the property of A being 1-Schur is just the property of containing a Schur triple. We call sets that are not 1-Schur *sum-free*. This terminology stems from a classic theorem of Schur [17] which asserts that for every r , there is some $n_0 = n_0(r)$ such that $[n]$ is r -Schur for all $n \geq n_0$.

Given this, it is natural to ask which subsets of $[n]$ are also r -Schur. From an extremal perspective, this leads to the question of establishing the maximum

S. Das—Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project 415310276.

P. Morris—Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 384–389, 2021.
https://doi.org/10.1007/978-3-030-83823-2_60

size a subset $A \subset [n]$ which is *not* r -Schur. It is a simple exercise to show that if $|A| > \lceil \frac{n}{2} \rceil$, A must be 1-Schur, and taking $A \subset [n]$ to be the set of all odd integers shows that this is best possible. For 2-colourings, one can take A to be all integers in $[n]$ which are *not* divisible by 5, colouring those which are congruent to 1 or 4 (mod 5) red and those congruent to 2 or 3 (mod 5) blue. This colouring gives no monochromatic Schur triples and hence there exists sets of size $\lceil \frac{4n}{5} \rceil$ which are not 2-Schur. Hu [13] showed with an elegant argument that one can not do better than this.

Theorem 1. *For any $n \in \mathbb{N}$ and $A \subset [n]$ with $|A| > \lceil \frac{4n}{5} \rceil$, A is 2-Schur.*

For $r \geq 3$, it is not known what density forces a subset to be r -Schur. Abbott and Wang [1] posed this question in 1977 and provided constructions which they conjecture to be best possible, while some upper bounds have been provided in [1, 12].

Deviating from the problem of determining the size of extremal sets, one can also study the behaviour of almost all subsets of $[n]$ by adopting a probabilistic perspective. For this, we fix some probability $p = p(n) \in [0, 1]$ and randomly sparsify the set $[n]$, defining $[n]_p$ to be the set obtained by taking each integer of $[n]$ into $[n]_p$ with probability p , independently of the other choices. The goal is then to understand for what p we can expect the resulting set to be r -Schur. Here, and throughout, we say an event holds with high probability (whp, for short) if the probability that it holds tends to 1 as n tends to infinity. Again, establishing the appearance of Schur triples is an easy task and standard tools (the first and second moment methods) give that if $p = o(n^{-2/3})$, then $[n]_p$ is sum-free whp whilst if $p = \omega(n^{-2/3})$ then $[n]_p$ will be 1-Schur whp. For more colours, the behaviour was determined by Graham, Rödl and Ruciński [9] for $r = 2$ and by Rödl and Ruciński [16] for $r \geq 3$.

Theorem 2. *For any $2 \leq r \in \mathbb{N}$ we have that if $p = o(n^{-1/2})$ then whp $[n]_p$ is not r -Schur whilst if $p = \omega(n^{-1/2})$ then whp $[n]_p$ is r -Schur.*

For the rest of the paper we restrict to the case $r = 2$ and say that a set $A \subseteq [n]$ is *Schur* if it is 2-Schur.

1.1 Randomly Perturbed Sets of Integers

As a combination of the extremal and probabilistic thresholds, Bohman, Frieze and Martin [4] initiated the study of combinatorial properties in random perturbed graphs by looking at how many random edges need to be added to an arbitrary dense graph to make it Hamiltonian. This has inspired several subsequent results on graphs and hypergraphs, with their Ramsey properties having been extensively studied (see e.g. [2, 6, 7, 14, 15]).

Aigner-Horev and Person were the first ones to transfer this to the setting of additive combinatorics. From our discussion above, if $|A| \leq \frac{4n}{5}$, one can ask how much we need to randomly perturb A in order to obtain a set that is Schur. For dense sets of integers A , Aigner-Horev and Person [3] showed the following.

Theorem 3. *Let $\varepsilon > 0$. If $A \subseteq [n]$, $|A| \geq \varepsilon n$, and $p = \omega(n^{-2/3})$, then whp $A \cup [n]_p$ is Schur.*

This can be interpreted as saying that any dense set is close to being Schur as a small random perturbation is enough to force the set to be Schur. From a probabilistic point of view, one can also see that, in comparison to Theorem 2, one can save a great deal of randomness by starting with an arbitrary set of positive density. Note that Theorem 3 is easily seen to be tight for $|A| \leq \lceil \frac{n}{2} \rceil$; taking A to be a sum-free set, we can colour A red and $[n]_p \setminus A$ blue. Then the only possible monochromatic Schur triples can come from $[n]_p$, and the threshold for their appearance is $p = n^{-2/3}$ as discussed above.

Our main result precisely describes the amount of randomness needed when the size of the dense set grows beyond $n/2$.

Theorem 4. *Let n and t be positive integers such that $\lceil \frac{n}{2} \rceil + t \leq \lceil \frac{4n}{5} \rceil$, and define $p(n, t) = \min \{n^{-2/3}, t^{-1}\}$. Then the following statements hold.*

- (0) *There exists a set $A \subseteq [n]$ with $|A| = \lceil \frac{n}{2} \rceil + t$ such that for $p = o(p(n, t))$, whp $A \cup [n]_p$ is not Schur.*
- (1) *For all $A \subseteq [n]$ with $|A| = \lceil \frac{n}{2} \rceil + t$ and $p = \omega(p(n, t))$, whp $A \cup [n]_p$ is Schur.*

In particular, if $|A| \geq \frac{n}{2} + \Omega(n)$ then adding a super-constant number of random integers already suffices to force the resulting set to be Schur. Along with Theorems 1 and 3, this completes our understanding of the behaviour of perturbed sets of integers when the starting set is dense. This follows a recent trend in the perturbed setting to look at these kind of transitions in more detail (see e.g. [5, 11]).

Furthermore, we can show stability for Theorem 4, demonstrating that any set that requires many random integers to be made 2-Schur must be close to the extremal example from the 0-statement.

Theorem 5. *If $A \subseteq [n]$ with $|A| = \lceil \frac{n}{2} \rceil + t$, and $q = \omega((nt)^{-1/2})$ is such that whp $A \cup [n]_q$ is not 2-Schur, then $|\lceil \frac{n}{2} \rceil, n \setminus A| = O(q^{-1})$.*

For $t = o(n)$ such that $t = \omega(n^{1/3})$ we have $(nt)^{-1/2} = o(\min \{n^{-2/3}, t^{-1}\})$, and so this shows that we can make significant savings in the amount of randomness required when the dense set A is far from the extremal construction.

2 Proof Sketch

Tools. Our proof relies on some powerful previously developed theory. We start with the following arithmetic removal lemma of Green [10].

Theorem 6. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $A \subseteq [n]$ is a set containing at most δn^2 Schur triples, then there is a sum-free $A' \subseteq A$ with $|A \setminus A'| \leq \varepsilon n$.*

The next powerful result we will use is a stability statement for large sum-free sets due to Deshouillers, Freiman, Temkin and Sós [8].

Theorem 7. *If $A \subseteq [n]$ is sum-free and $|A| > \frac{2}{3}n$, then either*

- (i) *A only consists of odd numbers, or*
- (ii) *$\min A > |A|$.*

We will also often need to find many arithmetic progressions, and the following result of Varnavides [18] will be repeatedly applied. A 4-AP in a set S is a sequence $a, a + d, a + 2d, a + 3d \in S$ and d is said to be the *difference* of the AP.

Theorem 8. *For every $\delta > 0$ there is a $\xi = \xi(\delta) > 0$ such that if $S \subset [m]$ is a set with $|S| \geq \delta m$, then S contains at least ξm^2 4-APs. In particular, there are at least ξm distinct differences of 4-APs in S .*

Finally, we shall require the following variations of a fact used by Aigner-Horev and Person [3].

Proposition 1. *Each of the following sets are 2-Schur:*

- (i) $L_1(a, x, d) = \{d, x, x + d\} \cup \{a + id, a + x + id : i = 0, 1, 2, 3\}$, and
- (ii) $L_2(a, x, d) = \{d, x - d, x\} \cup \{a + id, x - a - id : i = 0, 1, 2, 3\}$.

Given an element $x \in A$, we define $S_A^+(x) = \{y \in A : x + y \in A\}$, $S_A^-(x) = \{y \in A : x - y \in A\}$, and $S_A(x) = S_A^+(x) \cup S_A^-(x)$. The following result shows that it will suffice to find some structure in these links of Schur triples in the set A .

Lemma 1. *Suppose we have a set $X \subseteq A$ of size λ , and that, for each $x \in X$, there is a set D_x of size κ such that for every $d \in D_x$, either $S_A^+(x)$ or $S_A^-(x)$ contains a 4-AP with common difference d . Then, if $p = \omega(\max((\lambda\kappa)^{-1/2}, \kappa^{-1}))$, $A \cup [n]_p$ is 2-Schur with high probability.*

Let us sketch the proof of this lemma. Take $x \in X$ and $d \in D_x$ with d the difference of a 4-AP in $S_A^+(x)$. Letting a be the first term of the AP we have that $\{a + id : i = 0, 1, 2, 3\} \subseteq S_A^+(x) \subseteq A$, and thus, by definition of $S_A^+(x)$, we also have $\{a + x + id : i = 0, 1, 2, 3\} \subseteq A$. Note that $P(x, d) := \{d, x + d\} \subseteq [n]$ and $L_1(a, x, d) \setminus A \subseteq P(x, d)$. By Proposition 1 it follows that $A \cup [n]_p$ will be 2-Schur if $P(x, d) \subseteq [n]_p$. In a similar fashion if d is the difference of a 4-AP in $S_A^-(x)$, we obtain a pair of integers $P(x, d) := \{x, x - d\}$ whose appearance in $[n]_p$ gives a copy of $L_2(a, x, d)$ in $A \cup [n]_p$. One can verify that the map $(x, d) \mapsto P(x, d)$ is at most three-to-one and so by the hypothesis of the lemma, we have $\Omega(\lambda\kappa)$ pairs of integers $P(x, d)$ whose appearance in $[n]_p$ gives a Schur set. The expected number of these pairs that appear is $\omega(1)$ and an application of the second moment method (for which we need $p = \omega(\kappa^{-1})$) gives that one such pair appears whp, settling the lemma.

Proof of the 1-Statement of Theorem 4. We first split into two cases, depending on the number of Schur triples in A . Fix some small $\varepsilon > 0$ and let $\delta > 0$ be the resulting value from Theorem 6.

Case I: A has at least δn^2 Schur triples. In this case, we easily find many $x \in A$ for which $|S_A^+(x)| \geq \frac{1}{2}\delta n$ and so $S_A^+(x)$ contains many 4-APs by Theorem 8. This allows us to apply Lemma 1.

Case II: A has fewer than δn^2 Schur triples. By Theorem 6, we can remove at most εn elements from A to obtain a sum-free subset $A' \subseteq [n]$. It follows that $|A'| \geq (\frac{1}{2} - \varepsilon)n$, and hence we can apply Theorem 7 to obtain structural information about A' —it either consists entirely of odd integers, or of large integers.

Case II.1: A' is contained in the odd integers. In this case we have that A must contain t even integers x and each of these either has a linearly large set $S_A^+(x)$ or $S_A^-(x)$. In either case, we can apply Theorem 8 to find many 4-APs in the link of x and conclude again by using Lemma 1.

Case II.2: $\min A' > |A'|$. This case is similar but more technically involved. We omit the details, simply noting that we take X to be the smallest integers in A , allowing us to guarantee that the sets $S_A^+(x)$ for $x \in X$ are sufficiently large. \square

Finally, we remark that Theorem 5 follows from the proof outlined above, because if A fell under Cases I or II.1, then $p = \omega((nt)^{-1/2})$ would be sufficient to ensure that $A \cup [n]_p$ is whp Schur.

Proof of the 0-Statement of Theorem 4. In order to show the lower bound we give an explicit construction. When $t = \Omega(n)$, and $|A| = \lceil \frac{n}{2} \rceil + t \leq \lceil \frac{4n}{5} \rceil$, we can simply take A to be any set which is not 2-Schur (for example, the construction that removes the integers divisible by 5 discussed in the introduction). Then for $p = o(n^{-1})$, an application of Markov’s inequality gives that whp $[n]_p$ is empty and $A \cup [n]_p$ remains 2-colourable without monochromatic Schur triples.

For $1 \leq t = o(n)$, let $A = [\lceil \frac{n+1}{2} \rceil - t, n]$ and take some $p = o(\min\{n^{-2/3}, t^{-1}\})$. Write $B = [\lceil \frac{n+1}{2} \rceil - t, n - 2t]$, $C = [n - 2t + 1, n]$, and $R = [n]_p \setminus A = [\lceil \frac{n}{2} \rceil - t]_p$, noting that $A \cup [n]_p = B \cup C \cup R$. We colour B blue and $C \cup R$ red. A visualisation can be found in Fig. 1. Note that B is sum-free, and therefore we have no monochromatic Schur triples in blue. We also have that C is sum-free, and since $\min C > 2 \max R$, the only possible monochromatic red Schur triples are of the form $x + y = z$ with $x, y, z \in R$ or with $x \in R$ and $y, z \in C$. The former amounts to the random set containing a Schur triple, which we know whp does not happen for $p = o(n^{-2/3})$. For the latter, we require the element x to belong to the difference set $C - C$. Since C is an interval of

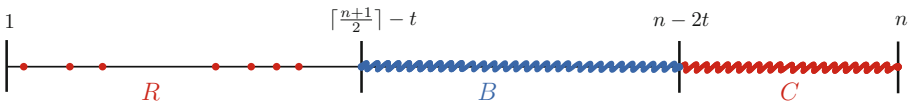


Fig. 1. Visualisation of the lower bound construction

length $2t$, there are $2t - 1$ possible differences. As $p = o(t^{-1})$, whp none of these elements x appear in R . Thus this colouring has no monochromatic Schur triples whp, thereby demonstrating that $A \cup [n]_p$ is not 2-Schur.

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Unit Disk Visibility Graphs

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Abstract. We study unit disk visibility graphs, where the visibility relation between a pair of geometric entities is defined by not only obstacles, but also the distance between them. This particular graph class models real world scenarios more accurately compared to the conventional visibility graphs. We first define and classify the unit disk visibility graphs, and then show that the 3-coloring problem is NP-complete when unit disk visibility model is used for a set of line segments (which applies to a set of points) and for a polygon with holes.

Keywords: Unit disk graphs · Visibility graphs · 3-coloring problem · NP-hardness

1 Introduction

A visibility graph is a simple graph $G = (V, E)$ defined over a set $\mathcal{P} = \{p_1, \dots, p_n\}$ of n geometric entities where a vertex $u \in V$ represents a geometric entity $p_u \in \mathcal{P}$, and the edge $uv \in E$ exists if and only if p_u and p_v are mutually visible (or see each other). In the literature, visibility graphs were studied considering various geometric sets such as a simple polygon [14], a polygon with holes [16], a set of points [2], a set of line segments [6], along with different visibility models such as line-of-sight visibility [8] and α -visibility [9].

Visibility graphs are used to describe real-world scenarios majority of which concern the mobile robots and path planning [1]. However, the physical limitations of the real world are usually overlooked or ignored while using visibility graphs. Since no camera, sensor, or guard (the objects represented by vertices of a visibility graph) has infinite range, two objects might not sense each other even though there are no obstacles in-between. Based on such a limitation, we assume that if a pair of objects are too far from each other, then they do not see each other. To model this notion, we adapt the unit disk graph model.

G is called a *unit disk visibility graph* of \mathcal{P} if the existence of an edge $uv \in E$ means that the straight line between p_u and p_v does not intersect any obstacles (e.g., some $p_w \notin \{p_u, p_v\}$), and the Euclidean distance between them is at most 1 unit. Unit disk point visibility graphs are well-defined by this definition whereas for unit disk segment graphs and polygon visibility graphs, the additional constraints are the following: *i*) the edges of unit disk segment visibility

This work has been supported by the Czech Science Foundation, project no. 20-04567S.

graphs cannot intersect any segment, *ii*) the edges of unit disk polygon visibility graphs must lie entirely inside the polygon, and *iii*) the segment lengths and the length of boundary edges of a polygon are at most 1 unit (Fig. 1).

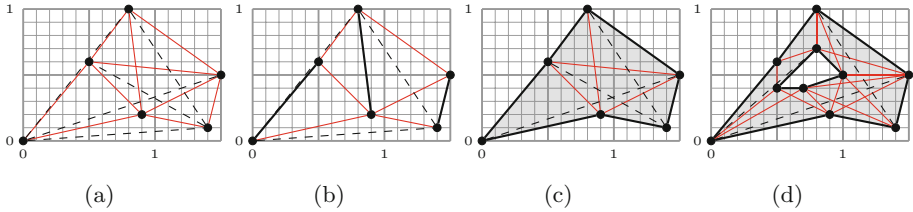


Fig. 1. Unit disk visibility relations of (a) a set of points, (b) a set of line segments, (c) a simple polygon, and (d) a polygon with a hole.

The *3-coloring* problem [7] is a famous NP-complete problem which asks if a graph has a (proper) 3-coloring, i.e. all vertices receive one of the three pre-given colors so that no two adjacent vertices receive the same color. In this paper, we tackle the 3-coloring problem, and show that it is also NP-complete on unit disk visibility graphs of a set of line segments, and a polygon with holes.

2 Preliminary Results

In this section, we prove that unit disk visibility graphs are not included in the (hierarchical) intersection of unit disk graphs and visibility graphs. We first show that visibility graphs are a proper subclass of unit disk visibility graphs. We assume that the given geometric set is a set of points, since every visibility graph considered in this paper can be embedded in the Euclidean plane; points, endpoints of a set of line segments, and the vertices of a polygon.

Lemma 1. *Consider a set $P = \{p_1, \dots, p_n\}$ of points, and the visibility graph $G(P)$ of P . There exists an embedding of P , such that the Euclidean distance between every pair $p, q \in P$ is at most one unit, preserving the visibility relations.*

By Lemma 1, a given set P of points can be scaled down to obtain P' so that every point in P' is inside a unit circle, and the visibility graph $G(P)$ of P is exactly the same as the unit disk visibility graph of P' . We get the following.

Lemma 2. *If a problem Ω is NP-hard for point visibility graphs, then Ω is also NP-hard for unit disk point visibility graphs.*

Remark 1. *By Lemma 2, the minimum vertex cover, the maximum independent set and the minimum dominating set problems which have been shown to be NP-hard for visibility graphs by [11, 12] are NP-hard for unit disk visibility graphs.*

We now obtain the following classification.

Lemma 3. *Unit disk visibility graphs are not a subclass of unit disk graphs.*

The idea used to prove Lemma 3 is that unit disk point, segment and polygon visibility graphs can contain an induced $K_{1,6}$ which is a forbidden induced subgraph for unit disk graphs $K_{1,6}$ [13].

Lemma 4. *Unit disk graphs are a proper subclass of unit disk point visibility graphs, and not a subclass of unit disk segment and polygon visibility graphs.*

Proof Sketch. Given a representation of a unit disk graph, we can simply perturb the disk centers slightly to obtain a set of points in general position in which no three points are collinear [4], which together with Lemma 3 shows that unit disk graphs are a proper subclass of unit disk point visibility graphs. However, unit disk segment visibility graphs require even number of vertices, unit disk visibility graphs for simple polygons require Hamiltonian cycles, and unit disk visibility graphs for polygons with holes require induced chordless cycles. Since these structures need not appear in unit disk graphs, unit disk segment and polygon visibility graphs are not a subclass of unit disk visibility graphs. \square

3 Main Results

In this section, we mention our NP-hardness reductions. A polynomial-time (NP-hardness) reduction from a (NP-hard) problem \mathfrak{Q} to another problem \mathfrak{P} is to map any instance Φ of \mathfrak{Q} to some instance Ψ of \mathfrak{P} such that Φ is a YES-instance of \mathfrak{Q} if and only if Ψ is a YES-instance of \mathfrak{P} , in polynomial time and polynomial space. We first show that the 3-coloring problem for unit disk segment visibility graphs is NP-hard, using a reduction from the *Monotone NAE3SAT* problem which is a 3SAT variant [15] with no negated variables, and to satisfy the circuit, at least one true variable and one false variable must appear in each clause.

Theorem 1. *There is a polynomial-time reduction from the Monotone NAE3SAT problem to the 3-coloring problem for unit disk segment visibility graphs.*

Proof sketch. Three main components of our reduction are as follows.

- (1) *A long edge* shown in Fig. 2c is used to transfer a color from one end to the other (similar to transferring the truth assignment of a variable). This configuration, no matter how long, has a unique 3-coloring (up to permutation).
- (2) *A clause gadget* shown in Fig. 2d is modeled by three line segments xx' , yy' , and zz' , not allowing three variables to have the same truth assignment. If all x, y, z have the same color, then this clause gadget cannot be 3-colored.
- (3) *An edge crossing gadget* shown in Fig. 2e describes a certificate for an edge crossing in the circuit so that it can be realized as a set of non-intersecting line segments. It has exactly two distinct 3-colorings (up to permutation).

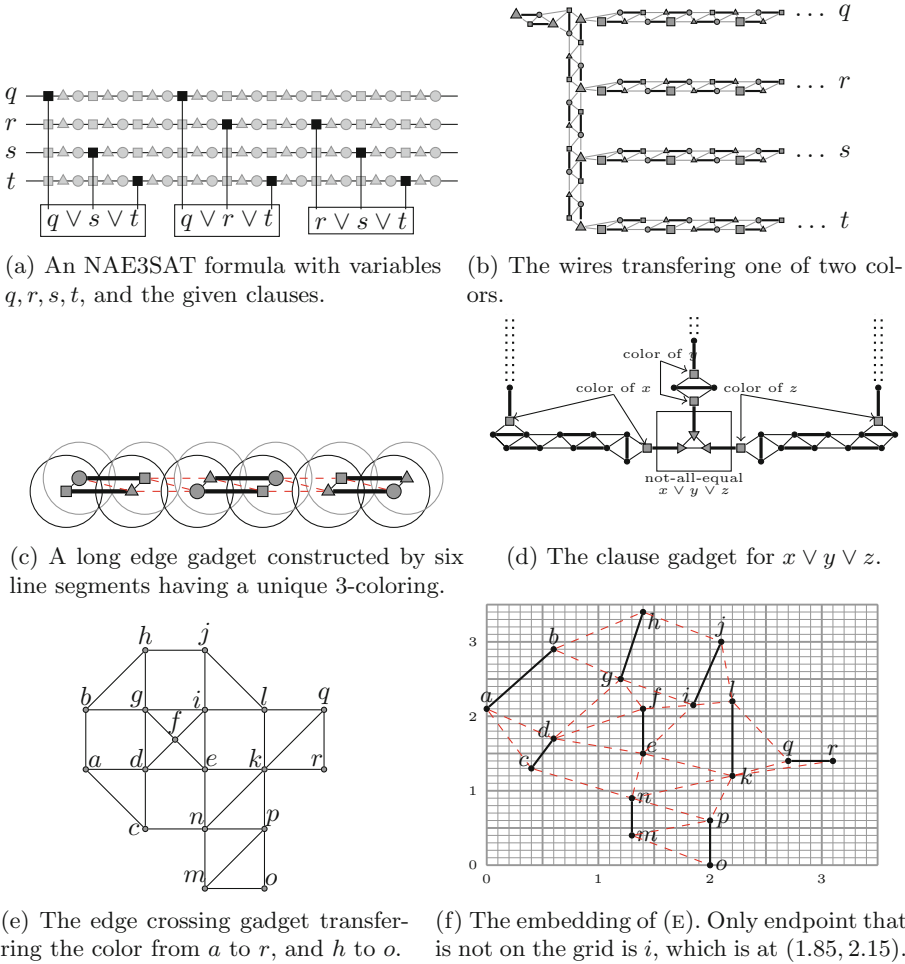


Fig. 2. The gadgets used in the proof of Theorem 1.

Given a Monotone NAE3SAT formula with m clauses C_1, \dots, C_m and n variables q_1, \dots, q_n , we construct the corresponding unit disk segment visibility graph G as follows:

- For each variable q_i , add a vertex v_i to G together with a long horizontal edge H_i transferring its color.
- For each clause C_i and each variable q_j in C_i , add a triangle T_i to G together with a long vertical edge V_j transferring the color of v_j .
- For each V_j crossing a H_i , add an edge crossing gadget (certificate) to G replacing the vertices in the intersection $V_j \cup H_i$.

Since this polynomial-time reduction works correctly, and the Monotone NAE3SAT problem is NP-complete [15], the 3-coloring problem for unit disk segment visibility graphs is also NP-complete. \square

Remark 2. *The 3-coloring problem for unit disk graphs is NP-complete [10], and by Lemma 4, it is NP-complete for unit disk point visibility graphs. For an alternative reduction to [10], our gadgets can be utilized with small modifications.*

We now show that the 3-coloring problem for unit disk visibility graphs of polygons with holes is NP-hard by giving a reduction from the 3-coloring problem for 4-regular planar graphs [5].

Theorem 2. *There is a polynomial-time reduction from the 3-coloring problem for 4-regular planar graphs to the 3-coloring problem for unit disk visibility graphs of polygons with holes.*

Proof sketch. Two main components of our reduction are as follows.

- (1) A *corridor* shown in Fig. 3a replaces the edges. This gadget makes sure that the two ends of an edge receive different colors.
- (2) A *chamber* shown in Fig. 3b replaces the vertices. It is an induced subgraph with 12 vertices. One of them acts as *the central vertex*, and the boundary vertices act as *the openings* to the corridors which connect it to other chambers.

Given a 4-regular planar graph H on n vertices v_1, \dots, v_n , we construct the corresponding polygon P with holes as follows:

- For each vertex v_i , add a chamber to P whose central vertex is vertex u_i .
- For each pair of adjacent vertices (v_i, v_j) , add a corridor to P between the chambers with central vertices u_i and u_j .

Since this polynomial-time reduction works correctly, and the 3-coloring problem for 4-regular planar graphs is NP-complete [5], this problem is also NP-complete for unit disk visibility graphs of polygons with holes. \square

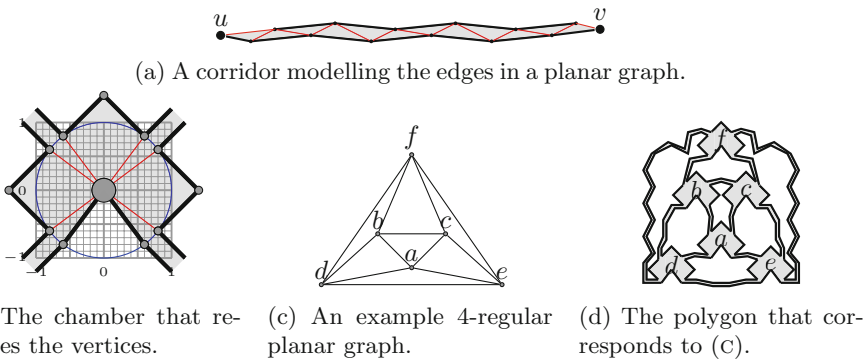


Fig. 3. The gadgets used in the proof of Theorem 2.

4 Conclusion

We have introduced the unit disk visibility graphs, and proved the following:

- Visibility graphs are a proper subclass of unit disk visibility graphs.
- Unit disk graphs are a proper subclass of unit disk point visibility graphs while they are neither a subclass nor a superclass of unit disk visibility graphs of a set of line segments, simple polygons or polygons with holes.
- The 3-coloring problem for unit disk segment visibility graphs and for unit disk visibility graphs of polygons with holes is NP-complete.

In the gadget used to prove NP-completeness of 3-coloring of unit disk segment visibility graphs, all line segments can be exactly one unit long except the edge crossings, and the rest of the gadget contains line segments either horizontal or vertical. Thus, we pose two interesting questions for reader's consideration:

Open Problem 1. *Is the 3-colorability of unit disk visibility graphs of line segments NP-hard when all the segments are exactly 1 unit long?*

Open Problem 2. *Is the 3-colorability of unit disk visibility graphs of line segments NP-hard when all the segments are either vertical or horizontal?*

In [3], it was proven that for visibility graphs of simple polygons, the 4-coloring problem can be solved in polynomial time, and the 5-coloring problem is NP-complete. Therefore, we would like to study the chromatic number problem on unit disk visibility graphs of simple polygons.

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Waiter-Client Games on Randomly Perturbed Graphs

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Abstract. Waiter-Client games are played on a hypergraph (X, \mathcal{F}) , where $\mathcal{F} \subseteq 2^X$ denotes the family of winning sets. During each round, Waiter offers a predefined amount (called bias) of elements from the board X , from which Client takes one for himself while the rest go to Waiter. Waiter wins the game if she can force Client to occupy any winning set $F \in \mathcal{F}$. In this paper we consider Waiter-Client games played on randomly perturbed graphs. These graphs consist of the union of a deterministic graph G_α on n vertices with minimum degree at least αn and the binomial random graph $G_{n,p}$. Depending on the bias we determine the order of the threshold probability for winning the Hamiltonicity game and the k -connectivity game on $G_\alpha \cup G_{n,p}$.

Keywords: Waiter-Client games · Randomly perturbed graphs · Connectivity · Hamilton cycles

1 Introduction

In general, a positional game is a perfect information game played by two players on a hypergraph (X, \mathcal{F}) . Throughout the game both players occupy elements of the *board* X according to some predefined rule, and the winner is determined through the *family of winning sets* \mathcal{F} . Research over the last decades has generated many interesting results in the area of positional games (see e.g. the monographs [1, 11]).

In this paper we are interested in Waiter-Client games (see e.g. [3, 9, 13]) where X is the edge set of a randomly perturbed graph G , and where \mathcal{F} is the family of all Hamilton cycles of G , or all k -vertex-connected spanning subgraphs of G .

O. Parczyk—Supported by the Deutsche Forschungsgemeinschaft (DFG, Grant PA 3513/1-1).

1.1 Randomly Perturbed Graphs

A randomly perturbed graph is the union of a dense graph and a binomial random graph. More precisely, let $\alpha > 0$ be fixed and let G_α be a sequence of n -vertex graphs with minimum degree at least αn . We consider the model $G_\alpha \cup G_{n,p}$ which was first introduced by Bohman, Frieze, and Martin [5] and recently attracted a lot of attention (see e.g. [7, 10]). We slightly abuse notation by writing $G_\alpha \cup G_{n,p}$ for $G_\alpha \cup G$ with $G \sim G_{n,p}$.

Bohman, Frieze, and Martin [5] showed that for any $\alpha > 0$ there exists a large enough constant C such that it is sufficient to take $p \geq \frac{C}{n}$ to ensure that a.a.s.¹ $G_\alpha \cup G_{n,p}$ is Hamiltonian. In order to see that taking any $p = o(\frac{1}{n})$ is not sufficient, one can consider $G_\alpha = K_{\alpha n, (1-\alpha)n}$. In that case, we can use at most $2\alpha n$ edges of G_α for a Hamilton cycle and a.a.s. $G_{n,p}$ only adds $o(n)$ edges. Regarding k -vertex-connectivity, it was shown by Bohman, Frieze, Krivelevich, and Martin [4] that for any fixed positive integer k the randomly perturbed graph $G_\alpha \cup G_{n,p}$ is k -vertex-connected when $p = \omega(\frac{1}{n^2})$.

1.2 Waiter-Client Games

A $(1 : b)$ Waiter-Client game on (X, \mathcal{F}) is played as follows: In each round, Waiter offers $(b + 1)$ elements of X . Client chooses one of them for himself, the remaining b go to Waiter. Waiter wins the game if she can force Client to claim all elements of a winning set $F \in \mathcal{F}$. Otherwise, Client wins. In [3], Bednarska-Bzdęga, Hefetz, Krivelevich, and Łuczak showed that for large enough n there exists a constant c such that Waiter wins the $(1 : b)$ Hamiltonicity game on K_n if $b \leq cn$ and otherwise Client wins. Further they showed that the same holds for the k -vertex connectivity game.

For Waiter-Client games played on a random graph $G_{n,p}$, Hefetz, Krivelevich, and Tan showed in [13], that given a constant $b \in \mathbb{N}$ Waiter a.a.s. wins the $(1 : b)$ Hamiltonicity game, if $p \geq (1 + o(1)) \frac{\ln(n)}{n}$. Moreover, it is easy to check with a variant of Beck’s criterion for Client-Waiter games (see [2] and [3, Theorem 2.2]) that given constants $k, b \in \mathbb{N}$ there exists a C such that Waiter a.a.s. wins the $(1 : b)$ Waiter-Client k -connectivity game, if $p \geq C \frac{\ln(n)}{n}$.

1.3 Our Results

The model of randomly perturbed graphs interpolates between the purely deterministic graph G_α and the random graph model $G_{n,p}$. In these models the $\ln n$ term and $\alpha \geq \frac{1}{2}$ are necessary conditions for the graphs to be connected. As discussed above in $G_\alpha \cup G_{n,p}$ this is no longer the case and our goal is to strengthen the results from [4, 5] to the setting of biased Waiter-Client games. We prove the following theorem for the Hamiltonicity game:

¹ Asymptotically almost surely (a.a.s.) is with probability tending to one as n tends to infinity.

Theorem 1. *For every real $\alpha > 0$ there exist constants $c, C > 0$ such that the following holds for large enough integers n . Let G_α be a graph on n vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq cn$ be an integer, and let $p \geq \frac{Cb}{n}$. Then a.a.s. the following holds: playing a $(1 : b)$ biased Waiter-Client game on the edges of $G_\alpha \cup G_{n,p}$, Waiter has a strategy to force Client to occupy a Hamilton cycle.*

Note that the bound on b is optimal up to the constant factor (as shown in [3]). The bound on p is optimal as well. To see this, consider G_α to be a complete bipartite graph $A \cup B$ with $|A| = \alpha n$ and $|B| = (1 - \alpha)n$. Now, every Hamilton cycle in $G_\alpha \cup G_{n,p}$ needs to contain at least $(1 - 2\alpha)n$ edges within B and there need to be $(1 - 2\alpha)(b + 1)n$ edges such that Waiter can force Client to take $(1 - 2\alpha)n$. However, a.a.s. $G_{n,p}$ contains fewer edges when $p = o(\frac{b}{n})$.

Note that this result also strengthens the result of Bohman, Frieze, and Martin [5] on the containment of Hamilton cycles. When $p \geq \frac{C}{n}$ for some large enough constant C , the graph $G_\alpha \cup G_{n,p}$ a.a.s. does not only contain a Hamilton cycle; instead it is so rich of this structure that Waiter even wins the $(1 : 1)$ Waiter-Client Hamiltonicity game on it.

Further, we prove the following theorem for the k -vertex-connectivity game:

Theorem 2. *For every real $\alpha > 0$ and every integer k there exist constants $C, c > 0$ such that the following holds for large enough integers n . Let G_α be a graph on n vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq cn$ be an integer, and let $p \geq \frac{Cb}{n^2}$. Then with probability at least $1 - \exp(-cnp^2)$ the following holds: playing a $(1 : b)$ Waiter-Client game on the edges of $G_\alpha \cup G_{n,p}$, Waiter has a strategy to force Client to claim a spanning k -vertex-connected graph.*

Again, the bound on b is optimal up to the constant factor [3]. Regarding the optimality of p , one can look at the graph consisting of (roughly) $\frac{1}{\alpha}$ disjoint cliques of size (roughly) αn . For $p = o(\frac{b}{n^2})$ a.a.s. there are less than b edges in $G_{n,p}$, and thus, Waiter is not able to connect Client’s graph. Note again that this theorem strengthens the result on the k -vertex-connectivity of $G_\alpha \cup G_{n,p}$ given in [4].

2 Hamiltonicity Game

In this section, we will sketch the proof of Theorem 1. A more detailed version of the proof can be found in [8]. We will use different arguments, depending on the bias b . The first argument works for any $b = o(\sqrt{n})$, while the second can be applied for all $b = \Omega(\ln n)$.

2.1 Sketch of the Proof for $b \leq n^{0.49}$

This part of the proof relies on the following sufficient condition for a graph to contain a Hamilton cycle.

Theorem 3 (Theorem 2.5 in [12]). *Let $12 \leq d \leq \sqrt{n}$ and let G be a graph on n vertices such that the following properties hold:*

- (i) $|N_G(S)| \geq d|S|$ for every $S \subset V(G)$ of size $|S| \leq \frac{n \ln d}{d \ln n}$,
- (ii) $e_G(A, B) > 0$ for every pair of disjoint sets $A, B \subset V(G)$ of size $|A|, |B| \geq \frac{n \ln d}{1035 \ln n}$.

Then, provided that n is large enough, G contains a Hamilton cycle.

We will apply Theorem 3 with $d = n^\delta$ for some appropriately chosen δ and state a strategy for Waiter which ensures that Client’s graph \mathcal{C} will fulfil the following properties at the end of the game:

- (1) $|N_{\mathcal{C}}(S)| \geq n^\delta |S|$ for every $S \subset V(G)$ of size $|S| \leq \delta n^{1-\delta}$,
- (2) $e_{\mathcal{C}}(A, B) > 0$ for every pair of disjoint sets $A, B \subset V(G)$ of size $|A|, |B| \geq \beta n$ for some appropriately chosen β .

By following that strategy, Waiter wins the Hamiltonicity game according to Theorem 3.

Let p and G_α be defined as in Theorem 1, and reveal $G_2 \sim G_{n,p}$. Let $G_1 = G_\alpha \setminus G_2$. Note that a.a.s. $\delta(G_1) \geq \frac{\alpha}{2}n$. Waiter first plays on $E(G_1)$ in such a way that Client claims a subgraph with property (1). Afterwards, Waiter plays on $E(G_2)$ in such a way that Client claims a subgraph with property (2). We look at the games on $E(G_1)$ and $E(G_2)$ separately.

Game on $E(G_1)$: To describe Waiter’s strategy, we first fix a partition $V(G_1) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that for each $i \in [4]$ we have $|V_i| = \frac{n}{4} \pm 1$ and for each $v \in V(G_1)$ we have

$$d_{G_1}(v, V_j) > \frac{\alpha}{5}|V_j| > \frac{\alpha}{25}n.$$

The existence of such a partition can be guaranteed by looking at a random partition. We then split G_1 into two edge-disjoint subgraphs

$$G_{1,1} = G_1[V_1, V_2] \cup G_1[V_2, V_3] \cup G_1[V_3, V_4] \cup G_1[V_4, V_1] \text{ and } G_{1,2} = G_1[V_1, V_3] \cup G_1[V_2, V_4].$$

On $G_{1,1}$, Waiter ensures that every vertex will have degree at least $\frac{\alpha n}{200b}$ in Client’s graph. She can do this by playing as follows: as long as there is some $i \in [4]$ and some vertex $v \in V_i$ with $d_{\mathcal{C}}(v) < \frac{\alpha n}{200b}$, she offers $(b+1)$ edges between v and V_{i+1} (with $V_5 := V_1$). On $G_{1,2}$, Waiter ensures that Client claims an edge in each set contained in

$$\mathcal{F}_1 = \left\{ E_{G_1}(A, B) : \begin{array}{l} \exists i, j \in [4] \text{ with } |i - j| = 2, \\ A \subset V_i, B \subset V_j, |A| = n^{0.5} \text{ and } |B| = (1 - \frac{\alpha}{10})|V_j| \end{array} \right\}.$$

One can verify that she is indeed able to do so by applying a variant of Beck’s criterion for Client-Waiter games (see [2,3]). With the achievements on $G_{1,1}$ and $G_{1,2}$, it follows that Client’s graph also satisfies property (1).

Game on $E(G_2)$: Waiter wins the game on $E(G_2)$ if she is able to force client to claim an element from each of the sets contained in

$$\mathcal{F}_2 = \{E_{G_2}(A, B) : A, B \subset V(G_2) \text{ disjoint and } |A| = |B| = \beta n\}.$$

Since a.a.s. for any $F \in \mathcal{F}$ it holds that $|F| \geq \frac{1}{2}p(\beta n)^2$, we can again apply the variant of Beck’s Criterion for Client-Waiter games (see [2,3]) to show that Waiter can indeed reach her goal on $E(G_2)$.

2.2 Sketch of the Proof for $b \geq \ln(n)$

In this case we use some ideas from [3, 14]. The strategy involves forcing Client to build a connected $\frac{n}{5}$ -expander in his graph, which is defined as follows:

Definition 1. *For an integer R , a graph G is called an R -expander if $|N_G(A)| \geq 2|A|$ for every $A \subset V(G)$ with $|A| \leq R$.*

Having built this expander, Waiter can force Client to finish the Hamilton cycle by only offering boosters in the remaining rounds. A booster for a graph G is any non-edge $e \notin E(G)$ such that $G + e$ is either Hamiltonian or $G + e$ contains a longer path than G .

Lemma 1 (see e.g. Lemma 8.5 in [6]). *If G is a connected non-Hamiltonian R -expander, then the set of boosters for G has size at least $\frac{R^2}{2}$.*

Let p and G_α be defined as in Theorem 1, and reveal $G_2 \sim G_{n,p}$. Let $G_1 = G_\alpha \setminus G_2$. Note that a.a.s. $\delta(G_1) \geq \frac{\alpha}{2}n$. Before we describe Waiter’s strategy, let us fix a partition $V = V_1^{(1)} \cup V_1^{(2)} \cup V_2^{(1)} \cup V_2^{(2)} \cup V_3^{(1)} \cup V_3^{(2)}$ with $|V_i^{(j)}| = \frac{n}{6} \pm 1$, for every $i \in [3]$ and $j \in [2]$, such that in G_1 every vertex has degree at least $\frac{\alpha n}{25}$ into each part $V_i^{(j)}$. Waiter’s strategy consists of the following three stages.

Stage I: In this stage, Waiter plays on G_1 for at most $\frac{n}{\epsilon^2}$ rounds. Here, she ensures that Client occupies a graph with the following property:

(P) for every $i \in [3]$ and every $A \subset V_i^{(1)} \cup V_i^{(2)}$ of size at most ϵn , there are at least $9|A|$ neighbours in $V_{i+1}^{(1)} \cup V_{i+1}^{(2)}$ (where we set $V_4^{(j)} := V_1^{(j)}$ for $j = 1, 2$).

Waiter can do this by first claiming an edge between any two appropriately chosen large subsets of $V_i^{(1)} \cup V_i^{(2)}$ and $V_{i+1}^{(1)}$ as in Sect. 2.1 using the variant of Beck’s Criterion for Client-Waiter games. Afterwards, there cannot be too many vertices in sets which contradict property (P). By ensuring that those vertices have pairwise disjoint neighbourhoods of size 9 in $V_{i+1}^{(2)}$, we can repair the contradicting sets and thus achieve property (P).

Stage II: Waiter plays on G_2 for at most $\frac{10n}{\epsilon^2}$ further rounds. Now she ensures that Client occupies a graph which has an edge between any two disjoint sets of size ϵn . As we have seen in Sect. 2.1, Waiter succeeds in this stage. By the achievement of the first two stages, one can see that Client’s graph is already a connected $\frac{n}{5}$ -expander.

Stage III: By offering only boosters, Waiter turns this expander into a Hamiltonian graph within less than n rounds. It remains to be seen that we can actually offer enough boosters. Following from Lemma 1, Client’s graph contains at least $\frac{n^2}{50}$ boosters. For any possible expander graph that Client could have claimed by now, the probability that less than $\frac{n^2 p}{100}$ of these boosters are edges of G_2 is at most $\exp\left(-\frac{n^2 p}{400}\right)$, by an application of Chernoff’s inequality. Taking a union bound over all potential expanders, we a.a.s. get that there are enough boosters in G_2 independent of the expander Client had claimed by the end of Stage II.

3 k -connectivity Game

In this section, we will sketch the proof of Theorem 2. Again, a more detailed version can be found in [8]. Before we describe Waiter’s strategy, we split the board into suitable subboards. As a first step, we fix a partition $V(G_\alpha) = U_1 \cup U_2 \cup \dots \cup U_k$ such that

- (1) $|U_i| = \frac{n}{k} \pm 1$ for all $i \in [k]$,
- (2) $e_{G_\alpha}(v, U_i) \geq \frac{\alpha}{2}|U_i|$ for all $v \in V(G)$ and $i \in [k]$.

Next, we additionally split each of the sets U_i , $i \in [k]$ to obtain a partition $U_i = U_{i,1} \cup U_{i,2} \cup \dots \cup U_{i,s_i}$ such that

- (a) $|U_{i,j}| \geq \frac{\alpha}{20k}n$, and
- (b) $G_\alpha[U_{i,j}]$ is $\frac{\alpha^2}{80k}n$ -vertex-connected

for every $j \in [s_i]$. This partitions and s_i can be found using [4, Lemma 1]. For every $i \in [k]$ and $j \in [s_i]$, set $G_{i,j} := G_\alpha[U_{i,j}]$. We further can find a partition $G_{i,j} = G_{i,j}^1 \cup G_{i,j}^2$ such that both parts are γn -vertex-connected graphs on $U_{i,j}$, for some appropriately chosen constant γ .

With the described partition at hand, we reveal the edges $G' \sim G_{n,p}$ and observe that with probability at least $1 - \exp(-cn^2p)$ the following holds:

- (c) $e_{G'}(U_{i,j_1}, U_{i,j_2}) \geq \frac{\alpha^2}{800k^2}n^2p$ for every i and $j_1 \neq j_2$.

We are now ready to describe Waiter’s strategy to force Client’s graph to contain a k -vertex-connected spanning subgraph of $G = G_\alpha \cup G'$. We will split Waiter’s strategy into the following four stages.

Stage I: In this stage, Waiter plays on $G_I := \bigcup_{i,j} E(G_{i,j}^1)$ to ensure that Client creates an ϵn -expander on $V(G)$ as defined in Definition 1. Since $G_{i,j}$ is γn -connected for every i, j , we have $\delta(G_I) \geq \gamma n$, and thus can use the same strategy as in Sect. 2.2. After Waiter succeeds, Client’s graph consists of at most $\frac{1}{\epsilon}$ components, each of size at least ϵn , where each of those components is a subset of some $U_{i,j}$ with $i \in [k]$ and $j \in [s_i]$.

Stage II: Waiter plays on the board $G_{II} := \bigcup_{i,j} E(G_{i,j}^2)$ in such a way that Client’s graph becomes connected on each of the sets $U_{i,j}$. Since $G_{i,j}$ is γn -connected for every i, j , and there are at most $\frac{1}{\epsilon}$ components within $U_{i,j}$ she will always find two components with $(b + 1)$ unclaimed edges between them, which she then offers.

Stage III: In this stage, Waiter forces Client to make U_i a connected component in his graph for every $i \in [k]$ by playing on the board $G_{III} := \bigcup_{i,j_1,j_2} E_{G'}(U_{i,j_1}, U_{i,j_2})$. By property (c) and an appropriate choice of C , there are more than $(b + 1)$ edges between U_{i,j_1} and U_{i,j_2} for each $i \in [k]$ and $j_1 \neq j_2 \in [s_i]$. Thus, Waiter can easily reach her goal.

Stage IV: Waiter considers the board $G_{IV} := \bigcup_{i_1,i_2} E_{G_\alpha}(U_{i_1}, U_{i_2})$. She ensures, that by the end of this stage Client’s graph \mathcal{C} satisfies the following: $e_{\mathcal{C}}(v, U_{i_2}) > 0$

for every $i_1 \neq i_2$ and $v \in U_{i_1}$. Using property (2), Waiter has enough edges to choose from to succeed in this stage.

It remains to argue that Client's graph is indeed k -vertex-connected after Waiter succeeds in every stage. This can be seen as follows. Let $K \subset V(G)$ be any set of size at most $k-1$. Then there exists some $i \in [k]$ such that $U_i \cap K = \emptyset$. By Stage III, U_i is a connected component in Client's graph; lastly by Stage IV, every other vertex in $V \setminus (K \cup U_i)$ has a neighbour in U_i ; i.e. Client's graph is still connected after the removal of K .

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Vector Choosability in Bipartite Graphs

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Abstract. The *vector choice number* of a graph G over a field \mathbb{F} , introduced by Haynes et al. (Electron. J. Comb., 2010), is the smallest integer k such that for every assignment of k -dimensional subspaces of a vector space over \mathbb{F} to the vertices, it is possible to choose nonzero vectors for the vertices from their subspaces so that the vectors received by adjacent vertices are orthogonal over \mathbb{F} . This work is concerned with the vector choice number of bipartite graphs over various fields. We first observe that the vector choice number of bipartite graphs can be arbitrarily large over any field. We then consider the problem of estimating, for a given integer k , the smallest integer m for which the vector choice number of the complete bipartite graph $K_{k,m}$ over \mathbb{F} exceeds k . We prove upper and lower bounds on this quantity, implying a substantial difference between the behavior of the (color) choice number and the vector choice number on bipartite graphs. For the computational aspect, we show a hardness result for deciding whether the vector choice number of a given bipartite graph over \mathbb{F} is at most k , provided that $k \geq 3$ and that \mathbb{F} is either the real field or any finite field.

Keywords: Graph coloring · Graph choosability · Orthogonality dimension

1 Introduction

Graph coloring is the problem of minimizing the number of colors in a vertex coloring of a graph G where adjacent vertices receive distinct colors. This minimum is known as the *chromatic number* of G and is denoted by $\chi(G)$. Being one of the most popular topics in graph theory, the graph coloring problem was extended and generalized over the years in various ways. One classical variant, initiated independently by Vizing in 1976 [9] and by Erdős, Rubin, and Taylor in 1979 [2], is that of *graph choosability*, also known as *list coloring*, which deals with vertex colorings with some restrictions on the colors available to each vertex. A graph $G = (V, E)$ is said to be *k -choosable* if for every assignment of sets S_v of k colors to the vertices $v \in V$, there exists a choice of a color $c_v \in S_v$ for each $v \in V$, resulting in a proper coloring of G (that is, $c_v \neq c_{v'}$ whenever v and v' are adjacent in G). The *choice number* of a graph G , denoted $\text{ch}(G)$, is the smallest integer k for which G is k -choosable. It is well known that the choice number

Research supported by the Israel Science Foundation (grant No. 1218/20).

$\text{ch}(G)$ behaves quite differently from the standard chromatic number $\chi(G)$. In particular, it can be arbitrarily large even for bipartite graphs (see, e.g., [2]). The choice number of graphs enjoys an intensive study in graph theory involving combinatorial, algebraic, and probabilistic tools (see, e.g., [1]). The computational decision problem associated with the choice number is unlikely to be tractable, as it is known to be complete for the complexity class Π_2 of the second level of the polynomial-time hierarchy even for bipartite planar graphs [2–4].

Another interesting variant of graph coloring, introduced by Lovász [6] in the study of Shannon capacity of graphs, is that of *orthogonal representations*, where the vertices of the graph do not receive colors but vectors from some given vector space. A t -dimensional orthogonal representation of a graph $G = (V, E)$ over \mathbb{R} is an assignment of a nonzero vector $x_v \in \mathbb{R}^t$ to every vertex $v \in V$, such that $\langle x_v, x_{v'} \rangle = 0$ whenever v and v' are adjacent in G .¹ The *orthogonality dimension* of a graph G over \mathbb{R} is the smallest integer t for which there exists a t -dimensional orthogonal representation of G over \mathbb{R} . The orthogonality dimension parameter is closely related to several other well-studied graph parameters, and in particular, for every graph G it is bounded from above by the chromatic number $\chi(G)$. The orthogonality dimension of graphs and its extensions to fields other than the reals have found a variety of applications in combinatorics, information theory, and theoretical computer science (see, e.g., [7, Chap. 10]). As for the computational aspect, the decision problem associated with the orthogonality dimension of graphs is known to be NP-hard [8].

In 2010, Haynes, Park, Schaeffer, Webster, and Mitchell [5] introduced another variant of the chromatic number of graphs that captures both the choice number and the orthogonality dimension. In this setting, which we refer to as *vector choosability*, each vertex of a graph G is assigned a k -dimensional subspace of some finite-dimensional vector space, and the goal is to choose for each vertex a nonzero vector from its subspace so that adjacent vertices receive orthogonal vectors. The smallest integer k for which such a choice is guaranteed to exist for all possible subspace assignments is called the *vector choice number* of the graph G , formally defined as follows.

Definition 1. For a graph $G = (V, E)$ and a function $f : V \rightarrow \mathbb{N}$, G is f -vector choosable over a field \mathbb{F} if for every integer t and for every assignment of subspaces $W_v \subseteq \mathbb{F}^t$ with $\dim(W_v) = f(v)$ to the vertices $v \in V$ (which we refer to as an f -subspace assignment), there exists a choice of a nonzero vector $x_v \in W_v$ for each vertex $v \in V$, such that $\langle x_v, x_{v'} \rangle = 0$ whenever v and v' are adjacent in G . For an integer k , the graph G is k -vector choosable over \mathbb{F} if it is f -vector choosable over \mathbb{F} for the constant function defined by $f(v) = k$. The vector choice number of G over \mathbb{F} , denoted $\text{ch-v}(G, \mathbb{F})$, is the smallest k for which G is k -vector choosable over \mathbb{F} .

¹ Orthogonal representations are sometimes defined in the literature with the orthogonality constraint on pairs of *non-adjacent* vertices.

Here and throughout the paper, we associate with the real field \mathbb{R} and with every finite field \mathbb{F} the inner product defined by $\langle x, y \rangle = \sum x_i y_i$, whereas for the complex field \mathbb{C} we consider, as usual, the one defined by $\langle x, y \rangle = \sum x_i \overline{y_i}$.

The work [5] has initiated the study of the vector choice number of graphs over the real and complex fields. Among other things, it is shown there that a graph is 2-vector choosable over \mathbb{R} if and only if it contains no cycles. We note that this is in contrast to the characterization given in [2] of the (chromatic) 2-choosable graphs, which include additional graphs such as even cycles. This implies that the choice number and the vector choice number do not coincide even on the 4-cycle graph. Over the complex field \mathbb{C} , however, it was shown in [5] that a graph is 2-vector choosable if and only if it either contains no cycles or contains only one cycle and that cycle is even. This demonstrates the possible effect of the field on the vector choice number.

1.1 Our Contribution

The current work studies the vector choice number of graphs over general fields. We put our focus on bipartite graphs, whose vector choice number gives rise to plenty of natural and interesting questions from combinatorial, algebraic, and computational perspectives.

We start with the observation that the vector choice number of a bipartite graph can be arbitrarily large over every field. As mentioned earlier, such a result was shown for the coloring setting by Erdős et al. [2], who proved that the choice number of the complete bipartite graph $K_{m,m}$ exceeds k for $m = \binom{2k-1}{k}$. Here we provide the following analogue result for vector choosability, holding simultaneously for all fields.

Proposition 1. *For every integer k and for every field \mathbb{F} , $\text{ch-v}(K_{m,m}, \mathbb{F}) > k$ for $m = \binom{2k-1}{k}$.*

Another question on color choosability studied in [2] concerns, for a given integer k , the choice number of the complete bipartite graphs $K_{k,m}$. It was observed there that the graph $K_{k,m}$ is k -choosable for every $m < k^k$ whereas $\text{ch}(K_{k,m}) = k + 1$ for every $m \geq k^k$. Considering the vector choice number of these graphs, it can be seen that for every integer m and for every field \mathbb{F} , it holds that $\text{ch-v}(K_{k,m}, \mathbb{F}) \leq k + 1$. Indeed, given an assignment of $(k + 1)$ -subspaces to the vertices of $K_{k,m}$, every choice of nonzero vectors for the vertices of the left side can be extended to a proper choice of vectors for the others. We consider here the problem of identifying the values of m for which this $k + 1$ upper bound is tight, and as explained below, we discover that the situation is quite different from the color choosability setting.

For an integer k and a field \mathbb{F} , let $m(k, \mathbb{F})$ denote the smallest integer m for which it holds that $\text{ch-v}(K_{k,m}, \mathbb{F}) = k + 1$. We first prove the following lower bound.

Theorem 1. *For every integer k and for every field \mathbb{F} , $m(k, \mathbb{F}) > \sum_{i=1}^{k-1} \left\lfloor \frac{k-1}{i} \right\rfloor$. In particular, for every field \mathbb{F} it holds that $m(k, \mathbb{F}) = \Omega(k \cdot \log k)$.*

We next provide a general approach for proving upper bounds on $m(k, \mathbb{F})$. The following theorem reduces this challenge to constructing families of vectors with certain linear independence constraints.

Theorem 2. *If there exists a collection of $m = k \cdot t + 1$ nonzero vectors in \mathbb{F}^k satisfying that every $t + 1$ of them span the entire space \mathbb{F}^k , then $m(k, \mathbb{F}) \leq m$.*

The above theorem allows us to derive upper bounds on $m(k, \mathbb{F})$ for various fields.

Corollary 1. *Let k be an integer and let \mathbb{F} be a field.*

1. *If $|\mathbb{F}| \geq k^2 - k + 1$ then $m(k, \mathbb{F}) \leq k^2 - k + 1$.*
2. *If \mathbb{F} is a finite field of size $q \geq k$ then $m(k, \mathbb{F}) \leq k \cdot \frac{q^{k-1} - 1}{q - 1} + 1$.*

We remark that Item 1 of Corollary 1 is obtained by applying Theorem 2 with collections of vectors defined as the columns of Vandermonde matrices. It implies that $m(k, \mathbb{F}) = O(k^2)$ whenever the field \mathbb{F} is infinite or sufficiently large as a function of k , leaving us with a nearly quadratic gap from the lower bound given in Theorem 1. Interestingly, our results yield a substantial difference between the behavior of the choice number and of the vector choice number on bipartite graphs. Indeed, while the results of [2] imply that $\text{ch}(K_{n,n}) = (1 + o(1)) \cdot \log_2 n$, Corollary 1 shows that $\text{ch-v}(K_{n,n}, \mathbb{F}) = \Omega(\sqrt{n})$ whenever \mathbb{F} is sufficiently large.

We finally consider the computational aspect of the vector choice number and prove the following hardness result.

Theorem 3. *Let $k \geq 3$ be an integer and let \mathbb{F} be either \mathbb{R} or some finite field. Then, the problem of deciding whether a given bipartite graph G satisfies $\text{ch-v}(G, \mathbb{F}) \leq k$ is NP-hard.*

The proof of Theorem 3 is inspired by the approach taken in a proof due to Rubinfeld [2] for the Π_2 -hardness of a decision problem associated with the (color) choice number. His proof involves a delicate construction of several gadget graphs used to efficiently map an instance of the Π_2 -variant of the satisfiability problem to an instance of the color choosability problem. These gadgets, however, do not fit the setting of vector choosability. In fact, the characterization of 2-vector choosable graphs over the reals, given in [5], implies that the instances produced by the reduction of [2] are never vector choosable over this field. To overcome this difficulty, we construct and analyze a different gadget graph that allows us, combined with ideas of Gutner and Tarsi [3, 4], to obtain the NP-hardness result stated in Theorem 3. Our analysis involves a characterization, stated below, of the 2-vector choosable graphs over finite fields, extending the characterizations given in [5] for the real and complex fields.

Proposition 2. *For every finite field \mathbb{F} , a graph is 2-vector choosable over \mathbb{F} if and only if it contains no cycles.*

While Theorem 3 indicates the hardness of efficiently determining the vector choice number of bipartite graphs, it would be natural to expect the stronger

notion of Π_2 -hardness to hold for this problem. This challenge is left here for future research.

Due to space limitation, most of the proofs have been deferred to the full version of this paper. We do include here, though, a proof of Theorem 2 and a description of the main component used in the proof of Theorem 3.

2 Proof of Theorem 2

Suppose that there exists a collection of $m = k \cdot t + 1$ nonzero vectors b_1, \dots, b_m in \mathbb{F}^k satisfying that every $t + 1$ of them span the space \mathbb{F}^k . To prove that $m(k, \mathbb{F}) \leq m$, we have to show that it is possible to assign a k -subspace over \mathbb{F} to each vertex of the graph $K_{k,m}$ so that no choice of a nonzero vector from each subspace satisfies that the vectors of the left vertices are orthogonal to the vectors of the right vertices.

Let u_1, \dots, u_k be the vertices of the left side, and let v_1, \dots, v_m be the vertices of the right side. For every $i \in [k]$, we assign to the vertex u_i the subspace U_i spanned by the k vectors $e_i \otimes e_1, \dots, e_i \otimes e_k$ of \mathbb{F}^{k^2} . Here, e_i stands for the vector in \mathbb{F}^k with 1 on the i th entry and 0 everywhere else, and \otimes stands for the tensor product operation of vectors. Viewing the vectors of \mathbb{F}^{k^2} as a concatenation of k parts of length k , this means that U_i is the k -subspace of all vectors that have zeros in all parts but the i th one. Then, for every $j \in [m]$, we assign to the vertex v_j the subspace V_j spanned by the k vectors $e_1 \otimes b_j, \dots, e_k \otimes b_j$ of \mathbb{F}^{k^2} . Namely, V_j is the k -subspace of all vectors in \mathbb{F}^{k^2} consisting of k parts, each of which is equal to the vector b_j multiplied by some element of \mathbb{F} .

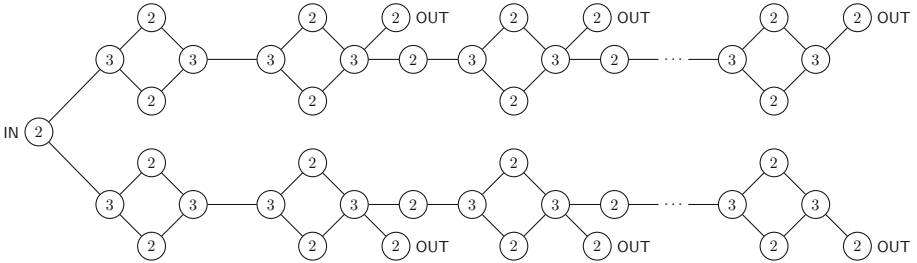
Assume for the sake of contradiction that there exist nonzero vectors $x_i \in U_i$ ($i \in [k]$) and $y_j \in V_j$ ($j \in [m]$) such that $\langle x_i, y_j \rangle = 0$ for all i and j . For any $i \in [k]$, let $\tilde{x}_i \in \mathbb{F}^k$ be the (nonzero) restriction of the vector x_i to the i th part. For any $j \in [m]$, write $y_j = \sum_{i \in [k]} \alpha_{i,j} \cdot e_i \otimes b_j$ for some coefficients $\alpha_{i,j} \in \mathbb{F}$. Since all the vectors y_j are nonzero, it clearly follows that at least m of the coefficients $\alpha_{i,j}$ are nonzero. Now, observe that for all $i \in [k]$ and $j \in [m]$, $\langle x_i, y_j \rangle = 0$ implies that $\langle \tilde{x}_i, \alpha_{i,j} \cdot b_j \rangle = 0$. However, combining the facts that \tilde{x}_i is nonzero and that every $t + 1$ vectors among b_1, \dots, b_m span \mathbb{F}^k , it follows that for every $i \in [k]$, at most t of the coefficients $\alpha_{i,j}$ with $j \in [m]$ are nonzero. This yields that the total number of nonzero coefficients $\alpha_{i,j}$ is at most $t \cdot k < m$, providing the desired contradiction. \square

3 Hardness Result

The main component of our hardness proof is the \exists -graph defined below, whose properties are given in Lemma 1 and used in the proofs of Theorems 4 and 3.

Definition 2 (\exists -graph). For any integers n_1, n_2 , define the \exists -graph $H = H_{n_1, n_2}$ and the function $f_H : V(H) \rightarrow \{2, 3\}$ as follows. The graph consists of a vertex labelled IN with degree 2, whose two neighbors serve as the starting points

of two subgraphs to which we will refer as the top and bottom branches. Each branch is composed of a sequence of 4-cycles connected by edges, as described in the following figure. In each branch, the vertex of largest distance from IN in every 4-cycle but the first has a neighbor labelled OUT and another neighbor separating it from the next 4-cycle (except for the last 4-cycle). The numbers of OUT vertices in the top and bottom branches are n_1 and n_2 respectively. The function f_H is defined on the vertices of H as indicated in the figure.



Lemma 1. *The \exists -graph H and the function f_H given in Definition 2 satisfy the following.*

1. *The graph H is bipartite, and every bipartition of H puts all OUT vertices in the same part.*
2. *For every f_H -subspace assignment of H over any field \mathbb{F} , any choice of a nonzero vector for IN can be extended to all vertices of each of the branches.*
3. *For every f_H -subspace assignment of H over any field \mathbb{F} and for each of the branches of H , there exists a choice of a nonzero vector for IN which is compatible with any choice of vectors for the OUT vertices of that branch.*
4. *Let \mathbb{F} be either \mathbb{R} or any finite field, and let $t \geq 8$ and $j \in [t]$ be some integers. Then, there exists an f_H -subspace assignment of H in \mathbb{F}^t such that for every valid choice of vectors for H there exists a branch all of whose OUT vertices are assigned vectors proportional to e_j .*

Theorem 4. *Let \mathbb{F} be either \mathbb{R} or any finite field. It is NP-hard to decide given a bipartite graph $G = (V, E)$ and a function $f : V \rightarrow \{2, 3\}$ whether G is f -vector choosable over \mathbb{F} .*

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Tight Bounds on the Expected Number of Holes in Random Point Sets

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Abstract. For integers $d \geq 2$ and $k \geq d + 1$, a k -hole in a set S of points in general position in \mathbb{R}^d is a k -tuple of points from S in convex position such that the interior of their convex hull does not contain any point from S . For a convex body $K \subseteq \mathbb{R}^d$ of unit volume, we study the expected number $EH_{d,k}^K(n)$ of k -holes in a set of n points drawn uniformly and independently at random from K .

We prove an asymptotically tight lower bound on $EH_{d,k}^K(n) \geq \Omega(n^d)$ for all fixed $d \geq 2$ and $k \geq d + 1$. For small holes, we even determine the leading constant $\lim_{n \rightarrow \infty} n^{-d} EH_{d,k}^K(n)$ exactly. We improve the best known lower bound on $\lim_{n \rightarrow \infty} n^{-d} EH_{d,d+1}^K(n)$ and we show that our bound is tight for $d \leq 3$. We show that $\lim_{n \rightarrow \infty} n^{-2} EH_{2,k}^K(n)$ is independent of K for every fixed $k \geq 3$ and we compute it exactly for $k = 4$, improving several earlier estimates.

Keywords: Stochastic geometry · Random point set · Convex position · Holes

M. Balko—was supported by the grant no. 21-32817S of the Czech Science Foundation (GAČR), by the Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004), and by the PRIMUS/17/SCI/3 project of Charles University. This article is part of a project that has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 810115).

M. Scheucher—was partially supported by DFG Grant FE 340/12-1 and by the internal research funding “Post-Doc-Funding” from Technische Universität Berlin. We also gratefully acknowledge support from the internal research program IFFP 2016–2020 by the FernUniversität in Hagen.

P. Valtr—was supported by the grant no. 21-32817S of the Czech Science Foundation (GAČR) and by the PRIMUS/17/SCI/3 project of Charles University.

1 Introduction

For a positive integer d , let S be a set of points from \mathbb{R}^d in *general position*. That is, for every $k \in \{1, \dots, d - 1\}$, no $k + 2$ points from S lie on a k -dimensional affine subspace of \mathbb{R}^d . Throughout the whole paper we only consider point sets in \mathbb{R}^d that are finite and in general position.

A point set P is in *convex position* if no point from P is contained in the convex hull of the remaining points from P . For an integer $k \geq d + 1$, a *k-hole* H in S is a set of k points from S in convex position such that the convex hull $\text{conv}(H)$ of H does not contain any point of S in its interior.

The study of k -holes in point sets was initiated by Erdős [7], who asked whether, for each $k \in \mathbb{N}$, every sufficiently large point set in the plane contains a k -hole. This was known to be true for $k \leq 5$, but, in the 1980s, Horton [11] constructed arbitrarily large point sets without 7-holes. The question about the existence of 6-holes was a longstanding open problem until 2007, when Gerken [10] and Nicolas [15] showed that every sufficiently large set of points in the plane contains a 6-hole.

The existence of k -holes was considered also in higher dimensions. Valtr [20] showed that, for $k \leq 2d + 1$, every sufficiently large set of points in \mathbb{R}^d contains a k -hole. He also constructed arbitrarily large sets of points in \mathbb{R}^d that do not contain any k -hole with $k > 2^{d-1}(P(d-1) + 1)$, where $P(d-1)$ denotes the product of the first $d - 1$ prime numbers. Very recently Bukh, Chao, and Holzman [6] improved this construction.

Estimating the number of k -holes in point sets in \mathbb{R}^d attracted a lot of attention; see [1]. In particular, it is well-known that the minimum number of $(d + 1)$ -holes (also called *empty simplices*) in sets of n points in \mathbb{R}^d is of order $O(n^d)$. This is tight, as every set of n points in \mathbb{R}^d contains at least $\binom{n-1}{d}$ $(d + 1)$ -holes [3, 12].

The tight upper bound $O(n^d)$ can be obtained by considering random point sets drawn from a convex body. More formally, a *convex body* in \mathbb{R}^d is a compact convex subset of \mathbb{R}^d with a nonempty interior. We use λ_d to denote the d -dimensional Lebesgue measure on \mathbb{R}^d and \mathcal{K}_d to denote the set of all convex bodies in \mathbb{R}^d of volume $\lambda_d(K) = 1$. For an integer $k \geq d + 1$ and a convex body $K \in \mathcal{K}_d$, let $EH_{d,k}^K(n)$ be the expected number of k -holes in a set S of n points chosen uniformly and independently at random from K . Note that S is in general position with probability 1.

Bárány and Füredi [3] proved the upper bound $EH_{d,d+1}^K(n) \leq (2d)^{2d^2} \cdot \binom{n}{d}$ for every $K \in \mathcal{K}_d$. Valtr [21] improved this bound in the plane by showing $EH_{2,3}^K(n) \leq 4\binom{n}{2}$ for any $K \in \mathcal{K}_2$. Very recently, Reitzner and Temesvari [16, Theorem 1.4] showed that this bound on $EH_{2,3}^K(n)$ is asymptotically tight for every $K \in \mathcal{K}_2$. This follows from their more general bounds

$$\lim_{n \rightarrow \infty} n^{-2} EH_{2,3}^K(n) = 2$$

and

$$\frac{2}{d!} \leq \lim_{n \rightarrow \infty} n^{-d} EH_{d,d+1}^K(n) \leq \frac{d}{(d + 1)} \frac{\kappa_{d-1}^{d+1} \kappa_{d^2}}{\kappa_d^{d-1} \kappa_{(d-1)(d+1)}} \tag{1}$$

for $d \geq 2$, where $\kappa_d = \pi^{\frac{d}{2}} \Gamma(\frac{d}{2} + 1)^{-1}$ is the volume of the d -dimensional Euclidean unit ball. Moreover, the upper bound in (1) holds with equality in the case $d = 2$, and if K is a d -dimensional ellipsoid with $d \geq 3$. Note that, by (1), there are absolute positive constants c_1, c_2 such that

$$d^{-c_1 d} \leq \lim_{n \rightarrow \infty} n^{-d} EH_{d,d+1}^K(n) \leq d^{-c_2 d}$$

for every $d \geq 2$ and $K \in \mathcal{K}_d$.

Considering general k -holes in random point sets in \mathbb{R}^d , the authors [2] recently proved that $EH_{d,k}^K(n) \leq O(n^d)$ for all fixed integers $d \geq 2$ and $k \geq d+1$ and every $K \in \mathcal{K}_d$. More precisely, we showed

$$EH_{d,k}^K(n) \leq 2^{d-1} \left(2d^{2d-1} \binom{k}{\lfloor d/2 \rfloor} \right)^{k-d-1} \frac{n(n-1) \cdots (n-k+2)}{(k-d-1)! \cdot (n-k+1)^{k-d-1}}. \quad (2)$$

In this paper, we also study the expected number $EH_{d,k}^K(n)$ of k -holes in random sets of n points in K . In particular, we derive a lower bound that asymptotically matches the upper bound (2) for all fixed values of k . Moreover, for some small holes, we even determine the leading constants $\lim_{n \rightarrow \infty} n^{-d} EH_{d,k}^K(n)$.

2 Our Results

First, we show that for all fixed integers $d \geq 2$ and $k \geq d+1$ the number $EH_{d,k}^K(n)$ is in $\Omega(n^d)$, which matches the upper bound (2) by the authors [2] up to the leading constant.

Theorem 1. *For all integers $d \geq 2$ and $k \geq d+1$, there are constants $C = C(d, k) > 0$ and $n_0 = n_0(d, k)$ such that, for every integer $n \geq n_0$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have $EH_{d,k}^K(n) \geq C \cdot n^d$.*

In particular, we see that random point sets typically contain many k -holes no matter how large k is, as long as it is fixed. This contrasts with the fact that, for every $d \geq 2$, there is a number $t = t(d)$ and arbitrarily large sets of points in \mathbb{R}^d without any t -holes [11, 20].

Theorem 1 together with (2) shows that $EH_{d,k}^K(n) = \Theta(n^d)$ for all fixed integers d and k and every $K \in \mathcal{K}^d$, determining the growth rate of $EH_{d,k}^K(n)$. We thus focus on determining the leading constants $\lim_{n \rightarrow \infty} n^{-d} EH_{d,k}^K(n)$, at least for small holes.

For a convex body $K \subseteq \mathbb{R}^d$ (of a not necessarily unit volume), we use p_d^K to denote the probability that the convex hull of $d+2$ points chosen uniformly and independently at random from K is a d -simplex. That is, the probability that one of the $d+2$ points falls in the convex hull of the remaining $d+1$ points. The problem of computing p_d^K is known as the d -dimensional *Sylvester's convex hull problem* for K and it has been studied extensively. Let $p_d = \max_K p_d^K$, where the maximum is taken over all convex bodies $K \subseteq \mathbb{R}^d$. We note that the maximum is

achieved, since it is well-known that every affine-invariant continuous functional on the space of convex bodies attains a maximum.

First, we prove the following lower bound on the expected number $EH_{d,d+1}^K(n)$ of empty simplices in random sets of n points in K , which improves the lower bound from (1) by Reitzner and Temesvari [16] by a factor of d/p_{d-1} .

Theorem 2. *For every integer $d \geq 2$ and every convex body $K \subseteq \mathbb{R}^d$ of unit volume, we have*

$$\lim_{n \rightarrow \infty} n^{-d} EH_{d,d+1}^K(n) \geq \frac{2}{(d-1)!p_{d-1}}.$$

Using the trivial fact $p_1 = 1$ with the inequality $EH_{2,3}^K(n) \leq 2(1 + o(1))n^2$ proved by Valtr [21], we see that the leading constant in our estimate is asymptotically tight in the planar case. An old result of Blaschke [4,5] implies that Theorem 2 is also asymptotically tight for simplices in \mathbb{R}^3 .

Corollary 1. *For every convex body $K \subseteq \mathbb{R}^3$ of unit volume, we have*

$$3 \leq \lim_{n \rightarrow \infty} n^{-3} EH_{3,4}^K(n) \leq \frac{12\pi^2}{35} \approx 3.38.$$

Moreover, the left inequality is tight if K is a tetrahedron and the right inequality is tight if K is an ellipsoid.

Note that, in contrast to the planar case, the leading constant in $EH_{3,4}^K(n)$ depends on the body K .

By Theorem 2, better upper bounds on p_{d-1} imply stronger lower bounds on $EH_{d,d+1}^K(n)$. The problem of estimating p_d is equivalent to the problem of estimating the expected d -dimensional volume EV_d^K of the convex hull of $d+1$ points drawn from a convex body $K \subseteq \mathbb{R}^d$ uniformly and independently at random, since $p_d^K = \frac{(d+2)EV_d^K}{\lambda_d(K)}$; see [14,18]. In the plane, Blaschke [4,5] showed that EV_2^K is maximized if K is a triangle, which we use to derive the lower bound in Corollary 1. For $d \geq 3$, it is one of the major problems in convex geometry to decide whether EV_d^K is maximized if K is a simplex [19].

We do not have a general upper bound on the probability p_d , but we can determine the growth rate of p_d^K for convex bodies K with small diameter.

Proposition 1. *Let $\varepsilon > 0$ and let $d \geq 1$ be an integer. Let $K \subseteq \mathbb{R}^d$ be a convex body. If there is a volume-preserving affine transformation $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $f(K)$ has diameter at most $d^{1-\varepsilon}\lambda_d(K)^{1/d}$, then $p_d^K \leq \frac{(d+2)d^{(1-\varepsilon)d}}{d!}$.*

We note that there are convex bodies that do not satisfy the assumption from Proposition 1, for example the regular d -dimensional simplex.

Besides empty simplices, we also consider larger k -holes. The expected number $EH_{2,4}^K(n)$ of 4-holes in random planar sets of n points was considered by Fabila-Monroy, Huemer, and Mitsche [9], who showed $EH_{2,4}^K(n) \leq 18\pi D^2 n^2 + o(n^2)$ for any $K \in \mathcal{K}_2$, where $D = D(K)$ is the diameter of K .

Since we have $D \geq 2/\sqrt{\pi}$, by the Isodiametric inequality [8], the leading constant in their bound is at least 72 for any $K \in \mathcal{K}_2$. This result was strengthened by the authors [2] to $EH_{2,4}^K(n) \leq 12n^2 + o(n^2)$ for every $K \in \mathcal{K}_2$. Here we determine the leading constant in $EH_{2,4}^K(n)$ exactly.

Theorem 3. *For every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have*

$$\lim_{n \rightarrow \infty} n^{-2}EH_{2,4}^K(n) = 10 - \frac{2\pi^2}{3} \approx 3.420.$$

Our computer experiments support this result. We sampled random sets of n points from a square and from a disk and the average number of 4-holes was around $3.42n^2$ for $n = 25000$ in our experiments. The source code of our program is available on the supplemental website [17].

For larger k -holes in the plane, we do not determine $\lim_{n \rightarrow \infty} n^{-2}EH_{2,k}^K(n)$ exactly, but we can show that it exists and does not depend on the convex body K . We recall that this is not true in larger dimensions already for empty simplices.

Theorem 4. *For every integer $k \geq 3$, there is a constant $C = C(k)$ such that, for every convex body $K \subseteq \mathbb{R}^2$ of unit area, we have $\lim_{n \rightarrow \infty} n^{-2}EH_{2,k}^K(n) = C$.*

Open Problems. We determined the leading constants $\lim_{n \rightarrow \infty} n^{-d}EH_{d,k}^K(n)$ exactly for small holes and, in particular, we showed that these limits exist in such cases. However, we do not have any argument that would yield the existence of these limits for all values of d and k . It is thus an interesting open problem to determine whether $\lim_{n \rightarrow \infty} n^{-d}EH_{d,k}^K(n)$ exists for all positive integers d and k with $k \geq d + 1$. It follows from a result by Reitzner and Temesvari [16, Theorem 1.4] and from Theorem 4 that this limit exists if $k = d + 1$ or if $k \geq 3$ and $d = 2$, respectively.

As we remarked earlier, any nontrivial upper bound on the probability p_{d-1} translates into a stronger lower bound on $\lim_{n \rightarrow \infty} n^{-d}EH_{d,d+1}^K(n)$. However, we are not aware of any such estimate on p_{d-1} . Kingman [13] found the exact formula for $p_d^{B^d}$, which is of order $d^{-\Theta(d)}$. We conjecture that the upper bound on p_d^K is of this order for any convex body.

Conjecture 1. There is a constant $c > 0$ such that, for every integer $d \geq 2$, we have $p_d \leq d^{-cd}$.

We also believe that our lower bound from Theorem 2 is tight for simplices in arbitrarily large dimension d , not only for $d \leq 3$.

Conjecture 2. For every $d \geq 2$, if K is a d -dimensional simplex of unit volume, then $\lim_{n \rightarrow \infty} n^{-d}EH_{d,d+1}^K(n) = \frac{2}{(d-1)!p_{d-1}}$.

As remarked earlier, it is widely believed that p_d^K is maximized if K is a simplex. If this is true, then it follows from the proof of Theorem 2 that Conjecture 2 is true as well.

It might also be interesting to determine $\lim_{n \rightarrow \infty} n^{-2}EH_{2,k}^K(n)$ exactly for as many values $k > 4$ as possible. Recall that, by Theorem 4, the number $\lim_{n \rightarrow \infty} n^{-2}EH_{2,k}^K(n)$ is the same for all convex bodies $K \in \mathcal{K}_2$.

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The Game of Toucher and Isolator

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Abstract. We introduce a new positional game called ‘Toucher-Isolator’, which is a quantitative version of a Maker-Breaker type game. The playing board is the set of edges of a given graph G , and the two players, Toucher and Isolator, claim edges alternately. The aim of Toucher is to ‘touch’ as many vertices as possible (i.e. to maximise the number of vertices that are incident to at least one of her chosen edges), and the aim of Isolator is to minimise the number of vertices that are so touched.

We analyse the number of untouched vertices $u(G)$ at the end of the game when both Toucher and Isolator play optimally, obtaining results both for general graphs and for particularly interesting classes of graphs, such as cycles, paths, trees, and k -regular graphs.

Keywords: Positional games · Maker-Breaker · Graphs

1 Introduction

One of the most fundamental and enjoyable mathematical activities is to play and analyse games, ranging from simple examples, such as snakes and ladders or noughts and crosses, to much more complex games like chess and bridge.

Many of the most natural and interesting games to play involve pure skill, perfect information, and a sequential order of play. These are known formally as ‘combinatorial’ games, see e.g. [4], and popular examples include Connect Four, Hex, noughts and crosses, draughts, chess, and go.

Often, a combinatorial game might consist of two players alternately ‘claiming’ elements of the playing board (e.g. noughts and crosses, but not chess) with the intention of forming specific winning sets, and such games are called ‘positional’ combinatorial games (for a comprehensive study, see [3] or [9]). In particular, much recent research has involved positional games in which the

The full version of this extended abstract is published as *The Toucher-Isolator game* in the Electronic Journal of Combinatorics 26(4) (2019), #P4.6

M. Kang—Supported by Austrian Science Fund (FWF): W1230.

M. Mikalački and M. Stojaković—Partly supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant nr. 174019.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 417–422, 2021.
https://doi.org/10.1007/978-3-030-83823-2_65

board is the set of edges of a graph, and where the aim is to claim edges in order to form subgraphs with particular properties.

A pioneering paper in this area was that of Chvátal and Erdős [5], in which the primary target was to form a spanning tree. Subsequent work has then also involved other standard graph structures and properties, such as cliques [2, 7], perfect matchings [11, 14], Hamilton cycles [11, 13], planarity [10], and given minimum degree [8]. Part of the appeal of these games is that there are several different versions. Sometimes, in the so-called strong games, both players aim to be the first to form a winning set (c.f. three-in-a-row in a game of noughts and crosses). In others, only Player 1 tries to obtain such a set, and Player 2 simply seeks to prevent her from doing so.

This latter class of games are known as ‘Maker-Breaker’ positional games. A notable result here is the Erdős-Selfridge Theorem [6], which establishes a simple but general condition for the existence of a winning strategy for Breaker in a wide class of such problems. A quantitative generalisation of this format then involves games in which Player 1 aims to form as many winning sets as possible, and Player 2 tries to prevent this (i.e. Player 2 seeks to minimise the number of winning sets formed by Player 1).

In this paper, we introduce a new quantitative version of a Maker-Breaker style positional game, which we call the ‘Toucher-Isolator’ game. Here, the playing board is the set of edges of a given graph, the two players claim edges alternately, the aim of Player 1 (Toucher) is to ‘touch’ as many vertices as possible (i.e. to maximise the number of vertices that are incident to at least one of her edges), and the aim of Player 2 (Isolator) is to minimise the number of vertices that are touched by Toucher (i.e. to claim all edges incident to a vertex, and do so for as many vertices as possible).

This problem is thus simple to formulate and seems very natural, with connections to other interesting games, such as claiming spanning subgraphs, matchings, etc. In particular, we note that it is related to the well-studied Maker-Breaker vertex isolation game (introduced by Chvátal and Erdős [5]), where Maker’s goal is to claim all edges incident to a vertex, and it is hence also related to the positive min-degree game (see [1, 9, 12]), where Maker’s goal is to claim at least one edge of every vertex.

Our Toucher-Isolator game can be thought of as a quantitative version of these games, where Toucher now wants to claim at least one edge on *as many* vertices as possible, while Isolator aims to isolate *as many* vertices as possible. However, the game has never previously been investigated, and so there is a vast amount of unexplored territory here, with many exciting questions. What are the best strategies for Toucher and Isolator? How do the results differ depending on the type of graph chosen? Which graphs provide the most interesting examples?

2 General Graphs

Given a graph $G = (V(G), E(G))$, we use $u(G)$ to denote the number of untouched vertices at the end of the game when both Toucher and Isolator play

optimally. We obtain both upper and lower bounds on $u(G)$, some of which are applicable to all graphs and some of which are specific to particular classes of graphs (e.g. cycles or trees).

Clearly, one of the key parameters in our game will be the degrees of the vertices (although, as we shall observe later, the degree sequence alone does not fully determine the value of $u(G)$). In our bounds for general G , perhaps the most significant is the upper bound of Theorem 1. Here, we find that it suffices just to consider the vertices with degree at most three (we again re-iterate that all our bounds are tight).

Theorem 1. *For any graph G , we have*

$$d_0 + \frac{1}{2}d_1 - 1 \leq u(G) \leq d_0 + \frac{3}{4}d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3,$$

where d_i denotes the number of vertices with degree exactly i .

Proof (Sketch). **Upper bound:** Toucher uses a pairing strategy to touch enough vertices for the statement to hold. We define a collection of disjoint pairs of edges, and Toucher’s strategy will be to wait (and play arbitrarily) until Isolator claims an edge within a pair, and then immediately respond by claiming the other edge (unless she happens to have already claimed it with one of her previous arbitrary moves, in which case she can again play arbitrarily). This way, Toucher will certainly claim at least one edge in every pair. To create a pairing, we add an auxiliary vertex and connect it to all odd degree vertices of G . The graph created in this way is even, and each of its components has an Eulerian tour. For each of these Eulerian tours, we then arbitrarily choose one of two orientations. Removing the auxiliary vertex leaves an orientation of G . Now, for each vertex that has at least 2 incoming edges, we take two such arbitrary edges and pair them. Some degree 3 vertices and all vertices of degree at least 4 will be covered by such pairing, and now we should consider the vertices of degree 1 and 2 and the remaining vertices of degree 3. We deal with this in the following way: we collect them and pair them arbitrarily. If their number is odd, Toucher takes one of these edges in the very first move (before Isolator claimed anything).

Lower Bound: Lower bound is obtained by the fact that Isolator can claim at least half of all edges whose at least one endpoint has degree 1, including at least half of the edges whose both endpoints have degree 1. □

For certain degree sequences, the bounds given in Theorem 1 can be improved by our next result.

Theorem 2. *For any graph G , we have*

$$\sum_{v \in V(G)} 2^{-d(v)} - \frac{|E(G)| + 7}{8} \leq u(G) \leq \sum_{v \in V(G)} 2^{-d(v)},$$

where $d(v)$ denotes the degree of vertex v .

Equivalently, we have

$$\sum_{i \geq 0} 2^{-i} d_i - \frac{|E(G)| + 7}{8} \leq u(G) \leq \sum_{i \geq 0} 2^{-i} d_i,$$

where d_i again denotes the number of vertices with degree exactly i .

Note also that $|E(G)|$ will be small if the degrees are small, and so Theorem 2 then provides a fairly narrow interval for the value of $u(G)$ (observe that Theorem 1 already provides a narrow interval if the degrees are large).

Proof (Sketch). The proof relies on the approach of Erdős and Selfridge [6] and their “danger” function, defined as follows:

A vertex touched by Toucher has the danger value 0, while a vertex untouched by Toucher incident with k free edges (edges unclaimed by anyone) has danger value 2^{-k} .

The total danger of the graph is the sum of the danger values for all vertices. When the game is over, the total danger of the graph is precisely the number of untouched vertices.

When Isolator claims an edge, the total danger increases by the sum of the dangers of the endpoints of that edge. On the other hand, when Toucher claims an edge, the total danger decreases by the sum of the dangers of the endpoints of that edge.

Upper Bound: The upper bound is obtained by adding the strategy of Toucher to all the aforementioned. Toucher will always choose the edge that maximises the sum of danger values of the two vertices that are touched. Therefore, after two consequent moves of Toucher and Isolator, the total danger never increases throughout the game. The given upper bound follows.

Lower Bound: For the lower bound one has to carefully track change in total danger value after one round in the game, i.e. the consecutive moves of Isolator and of Toucher, given that Isolator plays in such a way to maximise the sum of danger of the endpoints of the claimed edge. The total danger value decreases by at most $\frac{1}{4}$ after one round, and also, after the first move of Toucher, the total danger decreases by at most one. Noting that there are $\lfloor \frac{|E(G)|-1}{2} \rfloor$ rounds after first Toucher’s move, the given lower bound follows. □

Remark 1. Note that the upper bound of Theorem 2 will be better than the upper bound of Theorem 1 if

$$\sum_{i \geq 4} 2^{3-i} d_i < 2d_1 + 2d_2 + d_3.$$

Remark 2. Note that the lower bound of Theorem 2 will be better than the lower bound of Theorem 1 if

$$|E(G)| < 1 + \sum_{i \geq 2} 2^{3-i} d_i.$$

As $2|E(G)| = \sum_{i \geq 1} id_i$, this will occur if d_2 is sufficiently large (e.g. consider a path or a cycle, in which case the lower bound of Theorem 1 is ineffective).

3 Specific Graphs

Moving on from these general bounds, it is already interesting to play on relatively small graphs (such as cycles, paths and 2-regular graphs), and to try to determine the optimal strategies and the proportion of untouched vertices. We again obtain tight upper and lower bounds, both for C_n and for the closely related game on P_n (the path on n vertices).

Theorem 3. *For all n , we have*

$$\frac{3}{16}(n - 3) \leq u(C_n) \leq \frac{n}{4}.$$

Theorem 4. *For all n , we have*

$$\frac{3}{16}(n - 2) \leq u(P_n) \leq \frac{n + 1}{4}.$$

We also extend the game to general 2-regular graphs (i.e. unions of disjoint cycles). Our main achievement here is to obtain a *tight* lower bound of $u(G) \geq \frac{n-3}{6}$, which (by a comparison with the lower bound of Theorem 3) also demonstrates that $u(G)$ is not solely determined by the degree sequence.

Theorem 5. *For any 2-regular graph G with n vertices, we have*

$$\frac{n - 3}{6} \leq u(G) \leq \frac{n}{4}.$$

An interesting and natural extension of the game on paths is obtained by considering general trees, although this additional freedom in the structure can make the problem significantly more challenging. Here, we derive the following tight bounds, and provide the examples of graphs satisfying these bounds exactly.

Theorem 6. *For any tree T with $n > 2$ vertices, we have*

$$\frac{n + 2}{8} \leq u(T) \leq \frac{n - 1}{2}.$$

It follows from Theorem 1 that there will be no untouched vertices in k -regular graphs if $k \geq 4$, and it is natural to consider what happens in the 3-regular case.

A direct consequence of Theorem 2 is the following.

Corollary 1. *For any 3-regular graph G with n vertices, we have*

$$u(G) \leq \frac{n}{8}.$$

We observe that there are 3-regular graphs for which $u(G) = 0$, and one might expect that this could be true for all such graphs. However, we in fact manage to construct a class of examples for which a constant proportion of vertices remain out of Toucher's reach.

Theorem 7. *For all even $n \geq 4$, there exists a 3-regular graph G with n vertices satisfying*

$$u(G) \geq \left\lfloor \frac{n}{24} \right\rfloor.$$

4 Concluding Remarks

We cannot hope to obtain exact results just by looking at the degree sequence of the graph. Hence, we are curious to know if any other properties or parameters of the graph can be utilised to give more precise bounds.

Finally, what is the largest possible proportion of untouched vertices for a 3-regular graph? By Theorem 7 and Corollary 1, we know that this is between $\frac{1}{24}$ and $\frac{1}{8}$.

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Hermitian Rank-Metric Codes

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Abstract. In the space $H_n(q^2)$ of Hermitian matrices over \mathbb{F}_{q^2} of order n we can define a d -code as subset C of $H_n(q^2)$ such that $\text{rk}(A - B) \geq d$ for every $A, B \in C$ with $A \neq B$. In $H_n(q^2)$ we give two possible equivalence definitions:

- 1) the ones coming from the maps that preserve the rank in $\mathbb{F}_{q^2}^{n \times n}$;
- 2) the ones that come from restricting to those maps preserving both the rank and the space $H_n(q^2)$.

As pointed out by Zhou in [15], there are examples of Hermitian d -codes C_1 and C_2 for which there not exist maps of form 2) sending C_1 in C_2 , but there exists a map ψ of the form 1) such that $\psi(C_1) = C_2$. We prove that when $q > 2$, $d < n$ and the codes considered are maximum additive d -codes and $(n - d)$ -designs, these two equivalence relations coincide. As a consequence, we get that the idealisers of such codes are not distinguishers, as it usually happens for rank-metric codes. The results rely on the paper [13].

Keywords: Hermitian matrices · Rank-metric code · Association scheme

1 Preliminaries on Hermitian Rank-Metric Codes

Let us consider $\mathbb{F}_q^{n \times n}$, the set of the square matrices of order n defined over \mathbb{F}_q , with q a prime power. It is well-known that $\mathbb{F}_q^{n \times n}$ equipped with

$$d(A, B) = \text{rk}(A - B),$$

where $A, B \in \mathbb{F}_q^{n \times n}$, is a metric space. If C is a subset of $\mathbb{F}_q^{n \times n}$ with the property that for each $A, B \in C$ then $d(A, B) \geq d$ with $1 \leq d \leq n$, then we say that C is a d -code. Furthermore, we say that C is *additive* if C is an additive subgroup of $(\mathbb{F}_q^{n \times n}, +)$, and C is \mathbb{F}_q -*linear* if C is an \mathbb{F}_q -subspace of $(\mathbb{F}_q^{n \times n}, +, \cdot)$, where $+$ is the classical matrix addition and \cdot is the scalar multiplication by an element of \mathbb{F}_q .

Consider $\bar{\cdot} : x \in \mathbb{F}_{q^2} \mapsto x^q \in \mathbb{F}_{q^2}$ the conjugation map over \mathbb{F}_{q^2} . Let $A \in \mathbb{F}_{q^2}^{n \times n}$ and denote by A^* the matrix obtained from A by conjugation of each entry and transposition. A matrix $A \in \mathbb{F}_{q^2}^{n \times n}$ is said *Hermitian* if $A^* = A$. Denote by $H_n(q^2)$ the set of all Hermitian matrices of order n over \mathbb{F}_{q^2} . In [11, Theorem 1], Schmidt proved that if C is an additive d -code contained in $H_n(q^2)$, then $|C| \leq q^{n(n-d+1)}$. When the parameters of C satisfy the equality in this bound, we say that C is a *maximum* (additive) Hermitian d -code.

2 The Association Scheme of Hermitian Matrices

By [1, Section 9.5] we have that $H_n(q^2)$ gives rise to an *association scheme* whose classes are $(A, B) \in R_i \Leftrightarrow \text{rk}(A - B) = i$. Let $\chi : \mathbb{F}_q \rightarrow \mathbb{C}$ be a nontrivial character of $(\mathbb{F}_q, +)$ and let $\langle A, B \rangle = \chi(\text{tr}(A^*B))$, with $A, B \in H_n(q^2)$ and tr denotes the matrix trace. Denoting by H_i the subset of $H_n(q^2)$ of matrices having rank equal to i , the *eigenvalues* of such association scheme are $Q_k(i) = \sum_{A \in H_k} \langle A, B \rangle$, for $B \in H_i$, with $i, k \in \{0, 1, \dots, n\}$, see [2, 11, 12].

Let $C \subseteq H_n(q^2)$. The *inner distribution* of C is (A_0, A_1, \dots, A_n) of rational numbers given by $A_i = \frac{|(C \times C) \cap R_i|}{|C|}$. Therefore, C is a d -code if and only if $A_1 = \dots = A_{d-1} = 0$. The *dual inner distribution* of C is $(A'_0, A'_1, \dots, A'_n)$ where

$$A'_k = \sum_{i=0}^n Q_k(i) A_i.$$

Also, we have that $A'_0 = |C|$, $A'_k \geq 0$ for each $k \in \{0, 1, \dots, n\}$ and if C is additive then $|C|$ divides A'_i for each $i \in \{0, \dots, n\}$.

If $A'_1 = \dots = A'_t = 0$, we say that C is a t -*design*. Of course, if C is additive the A_i 's count the number of matrices in C of rank i with $i \in \{0, 1, \dots, n\}$.

Moreover, in such a case we can associate with C its dual in $H_n(q^2)$; i.e.,

$$C^\perp = \{X \in H_n(q^2) : \langle X, Y \rangle = 1 \text{ for each } Y \in C\},$$

and it is possible to show that the coefficients $\frac{A'_k}{|C|}$ count exactly the number of matrices in C^\perp of rank i with $i \in \{0, 1, \dots, n\}$.

Also in [11] the author proved the following results on combinatorial properties of maximum additive Hermitian d -codes when d is odd.

Theorem 1. [11, Theorem 1] *If $C \subseteq H_n(q^2)$ is a Hermitian additive d -code with odd d , then it is maximum if and only if C is an $(n - d + 1)$ -design.*

Consider m and ℓ two non-negative integers, *negative q -binomial coefficient* is defined as

$$\begin{bmatrix} m \\ \ell \end{bmatrix} = \prod_{i=1}^{\ell} \frac{(-q)^{m-i+1} - 1}{(-q)^i - 1}.$$

We will need the following property for negative q -binomial coefficients. Let k and i be two non-negative integers, then

$$\sum_{j=i}^k (-1)^{j-i} (-q)^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} = \delta_{k,i}, \tag{1}$$

where $\delta_{k,i}$ is the Kronecker delta function, see [11, Equation (6)] and [7, Equation (10)].

If C is a Hermitian additive d -code and a $(n - d)$ -design, then its inner distribution has been determined.

Theorem 2. [11, Theorem 3] *If C is a Hermitian additive d -code and a $(n - d)$ -design, then*

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} (-q)^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{|C|}{q^{nj}} (-1)^{(n+1)j} - 1 \right),$$

for each $i \in \{0, 1, \dots, n - 1\}$.

3 The Equivalence Issue

For given $a \in \mathbb{F}_q^*$, $\rho \in \text{Aut}(\mathbb{F}_{q^2})$, $A \in \text{GL}(n, q^2)$ and $B \in H_n(q^2)$, the map

$$\Theta : C \in H_n(q^2) \mapsto aAC^\rho A^* + B \in H_n(q^2), \tag{2}$$

where C^ρ is the matrix obtained from C by applying ρ to each of its entry, preserves the rank distance and conversely, see [14]. For two subset C_1 and C_2 of $H_n(q^2)$, if there exists Θ as in (2) such that $C_2 = \{\Theta(C) : C \in C_1\}$ we say that C_1 and C_2 are *equivalent in $H_n(q^2)$* . Nevertheless, we may consider the maps of $\mathbb{F}_{q^2}^{n \times n}$ preserving the rank distance, which by [14] are all of the following kind $\Psi : C \in \mathbb{F}_{q^2}^{n \times n} \mapsto AC^\sigma B + R \in \mathbb{F}_{q^2}^{n \times n}$ or $\Psi : C \in \mathbb{F}_{q^2}^{n \times n} \mapsto A(C^\sigma)^T B + R \in \mathbb{F}_{q^2}^{n \times n}$, where $A, B \in \text{GL}(n, q^2)$, $\sigma \in \text{Aut}(\mathbb{F}_{q^2})$, $R \in \mathbb{F}_{q^2}^{n \times n}$ and C^T denotes the *transpose* of C . For two subset C_1 and C_2 of $H_n(q^2)$, if there exists Ψ as above such that $C_2 = \{\Psi(C) : C \in C_1\}$ we say that C_1 and C_2 are said *extended equivalent*. Clearly, if C_1 and C_2 of $H_n(q^2)$ are equivalent in $H_n(q^2)$, they are also extended equivalent. However, when maximum d -codes are considered, the converse statement is not true. In fact, from what Yue Zhou points out in [15], it follows that constructions of commutative *semifields* exhibited in [3] and in [16] provide examples of maximum n -codes in $H_n(q^2)$ say C , with the property that there exist $A, B \in \text{GL}(n, q^2)$ such that $ACB \subseteq H_n(q^2)$, where $A \neq aB^*$ for each $a \in \mathbb{F}_q$.

Along the lines of what has been done by Zhou in [15], in next section we will investigate on the conditions that guarantee the identification of the aforementioned types of equivalence for maximum Hermitian d -codes.

3.1 A Partial Solution for Maximum Additive Hermitian d -codes

In this section we will show that, under some assumptions, the equivalence of two maximum additive hermitian d -codes in $H_n(q^2)$ coincides with extended equivalence in $\mathbb{F}_{q^2}^{n \times n}$.

First recall the following incidence structures introduced in [9]

$$S(\infty) = \{(\mathbf{0}, \mathbf{y}) : \mathbf{y} \in \mathbb{F}_{q^2}^n\},$$

$$S(X) = \{(\mathbf{x}, \mathbf{x}X) : \mathbf{x} \in \mathbb{F}_{q^2}^n\}, \text{ for } X \in C.$$

The *kernel* $K(C)$ of C is defined as the set of all the endomorphism μ of the group $(\mathbb{F}_{q^2}^{2n}, +)$ such that $S(X)^\mu \subseteq S(X)$ for every $X \in C \cup \{\infty\}$.

When considering a maximum additive d -code in $H_n(q^2)$ the following result can be proved.

Theorem 3. [13, Theorem 3.3] *Let d be a positive integer and let C be a maximum additive d -code in $H_n(q^2)$. If there exist $a \in \mathbb{F}_q^*$ and $P \in GL(n, q^2)$ such that $I_n \in aP^*XP$, then $K(C)$ is isomorphic to a finite field containing \mathbb{F}_q . In particular, if $d < n$ then $K(C)$ is isomorphic to \mathbb{F}_q .*

Lemma 1. *If C is a Hermitian maximum additive d -code and an $(n - d)$ -design with $d < n$. Then there is at least one invertible matrix in C .*

Proof. If $d = 1$, then $C = H_n(q^2)$ and the assertion holds. So assume that $1 < d < n$: our aim is to prove that $A_n \neq 0$. By Theorem 2, we have that

$$A_{n-i} = \sum_{j=i}^{n-d} (-1)^{j-i} (-q)^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{|C|}{q^{nj}} (-1)^{(n+1)j} - 1 \right),$$

for each $i \in \{0, 1, \dots, n - 1\}$. For $i = 0$, we get

$$A_n = \sum_{j=0}^{n-d} (-1)^j (-q)^{\binom{j}{2}} \begin{bmatrix} j \\ 0 \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{|C|}{q^{nj}} (-1)^{(n+1)j} - 1 \right). \tag{3}$$

Recalling that $|C| = q^{n(n-d+1)}$, the above formula can be written as follows

$$A_n = \sum_{j=0}^{n-d} (-1)^j (-q)^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \left(q^{n(n-d-j+1)} - 1 \right) \equiv - \sum_{j=0}^n (-1)^j (-q)^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \pmod{q^{n-d}}.$$

Therefore, by Equation (1) we have $A_n \equiv -1 \pmod{q^{n-d}}$, so that $A_n \neq 0$.

We are ready to prove the main result of this section.

Theorem 4. *If C_1 and C_2 are two maximum additive Hermitian d -codes and $(n - d)$ -designs with $d < n$. Then they are equivalent in $H_n(q^2)$ if and only if they are extended equivalent.*

Proof. Clearly, if C_1 and C_2 are equivalent in $H_n(q^2)$ then they are also extended equivalent. Now assume that C_1 and C_2 are extended equivalent, i.e. there exist two invertible matrices $A, B \in GL(n, q^2)$, $\rho \in \text{Aut}(\mathbb{F}_{q^2})$ and $R \in \mathbb{F}_{q^2}^{n \times n}$ such that $C_1 = AC_2^\rho B + R$. Since C_1 and C_2 are additive, we may assume that $R = O$, i.e. $C_1 = AC_2^\rho B$. We are going to prove that $A = zB^*$ for some $z \in \mathbb{F}_q^*$. So,

$$C_2 = AC_1^\sigma B = (A(B^*)^{-1})B^*C_1^\sigma B = MC_3,$$

where $M = A(B^*)^{-1}$ and $C_3 = B^*C_1^\sigma B \subseteq H_n(q^2)$. As a consequence, we have that $MX \in H_n(q^2)$ for each $X \in C_3$, i.e. $MX = (MX)^* = XM^*$, for all $X \in C_3$. Hence the matrix $\begin{pmatrix} M & O \\ O & M^* \end{pmatrix} \in K(C_3)$. By Lemma 1, there exists in C_3 an invertible matrix, which implies the existence of $a \in \mathbb{F}_q$ and $D \in GL(n, q)$ such that $I_n \in aD^*C_3D$. Now, by Theorem 3 we have that $K(C_3) = \mathbb{F}_q$ and hence $M = zI_n$ for some $z \in \mathbb{F}_q^*$, i.e. $A = zB^*$.

As a consequence of Theorem 1 we get the following.

Corollary 1. *If C_1 and C_2 are two Hermitian maximum additive d -codes with d odd, $d < n$. Then they are equivalent in $H_n(q^2)$ if and only if they are extended equivalent.*

4 Idealisers Are Not Distinguishers in $H_n(q^2)$

In the classical rank-metric context, to establish whether or not two codes are equivalent could be quite difficult. One of the strongest tool for such issue is given by the automorphism groups of such codes, which is usually very hard to determine. In some cases it is enough to study some subgroups of the automorphism group which are invariant under the equivalence and which are easier to calculate, such as the *idealisers* introduced in [8] and deeply investigated in [9].

Let C be an additive rank-metric code in $\mathbb{F}_q^{n \times n}$, its *left idealiser* $I_\ell(C)$ is defined as

$$I_\ell(C) = \{Z \in \mathbb{F}_q^{n \times n} : ZX \in C \text{ for all } X \in C\}$$

and its *right idealiser* $I_r(C)$ is defined as

$$I_r(C) = \{Z \in \mathbb{F}_q^{n \times n} : XZ \in C \text{ for all } X \in C\}.$$

Idealisers have been used to *distinguish* examples of MRD-codes, see e.g. [4–6, 10]. In the next we prove that for maximum additive Hermitian d -codes left and right idealisers are isomorphic to \mathbb{F}_{q^2} , i.e. they cannot be used as distinguishers in the Hermitian setting.

Theorem 5. *Let C be a maximum Hermitian additive d -code and a $(n - d)$ -design with $d < n$. Then $I_\ell(C)$ and $I_r(C)$ are both isomorphic to \mathbb{F}_q .*

Proof. Let us consider the left idealiser case and let $M \in I_\ell(\mathbb{C})$. We have that $MX \in H_n(q^2)$ for each $X \in \mathbb{C}$, i.e. $MX = (MX)^* = XM^*$ for all $X \in \mathbb{C}$. Hence the matrix $\begin{pmatrix} M & O \\ O & M^* \end{pmatrix} \in K(\mathbb{C})$, and as in the proof of Theorem 4, we get that $M = aI_n$ for some $a \in \mathbb{F}_q$. Similar arguments can be performed to obtain the same result for the right idealiser.

As a consequence of Theorem 1 we get that when considering a maximum Hermitian additive d -code \mathbb{C} with d odd, $d < n$, then $I_\ell(\mathbb{C})$ and $I_r(\mathbb{C})$ are both isomorphic to \mathbb{F}_q .

We conclude this abstract with the following question.

Problem 1. Is it possible to adapt these considerations to the case of skew-symmetric matrices setting?

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The Flat Wall Theorem for Bipartite Graphs with Perfect Matchings

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Abstract. Matching minors are a specialised version of minors fit for the study of graphs with perfect matchings. The first major appearance of matching minors was in a result by Little who showed that a bipartite graph is Pfaffian if and only if it does not contain $K_{3,3}$ as a matching minor. Later it was shown, that $K_{3,3}$ -matching minor free bipartite graphs are essentially, that is after some clean-up and with a single exception, bipartite planar graphs glued together at 4-cycles.

We generalise these ideas by giving an approximate description of bipartite graphs excluding $K_{t,t}$ as a matching minor in the spirit of the famous Flat Wall Theorem of Robertson and Seymour. In essence, we show that every bipartite $K_{t,t}$ -matching minor free graph is locally $K_{3,3}$ -matching minor free after removing an apex set of bounded size.

Keywords: Graph theory · Perfect matchings · Bipartite graphs · Flat wall

1 Introduction

The aim of Matching Theory is to study the structural properties of graphs with perfect matchings and, within its context, a plethora of results revealing rich structural properties of graphs with perfect matchings have appeared. For an in-depth exhibition of Matching Theory, we refer to [8]. In particular, bipartite graphs are further well-studied from the scope of Matching Theory [9, 10, 14].

All graphs considered in this article are finite and do not contain parallel edges or loops. Let G be a graph and $F \subseteq E(G)$ be a set of edges. F is called a *matching* if no two edges in F share an endpoint, a matching is *perfect* if every vertex of G is contained in some edge of F . A set $X \subseteq V(G)$ is *conformal* if $G - X$ has a perfect matching of subgraph H of G is *conformal* if $V(H)$ is conformal. If M is a perfect matching of G and contains a perfect matching of H we say that H is M -conformal. A *bicontraction* is the operation of contracting both edges incident with a vertex of degree two at the same time. Finally, a *matching minor* is a graph H that can be obtained by a series of bicontractions from a conformal subgraph of G .

S. Wiederrecht—Supported by the ANR project ESIGMA (ANR-17-CE23-0010).

The significance of matching minors was first¹ discovered by Little who characterised the bipartite Pfaffian graphs as exactly those which exclude $K_{3,3}$ as a matching minor [6]. However, this did not yield a polynomial time recognition algorithm. Pfaffian graphs themselves are of importance as they are a large canonical class on which the number of perfect matchings can be counted efficiently, a problem which is $\sharp\text{P}$ -hard in general [17].

A graph G is called *matching covered* if it is connected and each of its edges is contained in a perfect matching. Let G be a graph and $X \subseteq V(G)$. We denote by $\partial(X)$ the *edge cut* around X , we call the sets X and $V(G) \setminus X$ the *shores* of $\partial(X)$. Let us now assume G to be matching covered. An edge cut $\partial(X)$ is *tight* if $|\partial(X) \cap M| = 1$ for every perfect matching of G and it is *trivial* if $|X| = 1$ or $|V(G) \setminus X| = 1$. A graph G' obtained from a graph G by identifying a shore of a non-trivial tight cut into a single vertex and removing all resulting loops and parallel edges is called a *tight cut contraction* of G . A matching covered bipartite graph without a non-trivial tight cut is called a *brace*. Please note that any bipartite matching covered graph G can be decomposed, by repeated tight cut contractions, into a list of braces, a classic theorem of Lovász shows that this list of braces is uniquely determined by G [7].

One can observe that for $t \geq 3$, $K_{t,t}$ is a matching minor of a bipartite matching covered graph G if and only if it is a matching minor of a brace of G [1], so it suffices to recognise $K_{3,3}$ -matching minor free braces. This was finally achieved by [10,14] through a structural characterisation. Let G_1 and G_2 be braces, each containing a 4-cycle C_i , $i \in \{1,2\}$, respectively. We say that the graph G obtained by identifying the vertices of C_1 and C_2 and possibly deleting some of the edges of the resulting cycle is obtained from G_1 and G_2 by a C_4 -sum. If G is a brace obtained by a sequence of C_4 -sums from planar braces G_i , $1 \leq i \leq h \in \mathbb{N}$, then the G_i are called the *summands* of G .

Theorem 1 ([10,14]). *A brace G is $K_{3,3}$ -matching minor free if and only if it is isomorphic to the Heawood graph (see Fig. 1) or it can be obtained from planar braces by means of C_4 -sums.*

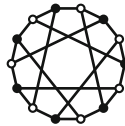


Fig. 1. The Heawood graph H_{14} .

¹ Please note that the term is ‘matching minor’ is much younger.

2 A Matching Theoretic Version of Flatness

The Flat Wall Theorem of Robertson and Seymour [5,13] describes K_t -minor free graphs G in one of two ways. Either the treewidth of G is bounded by a function in t and a parameter $r \in \mathbb{N}$, or G contains a set $A \subseteq V(G)$ of size depending solely on t , and a wall W of height r such that the subgraph of $G - A$ which attaches to the inside of W can be made planar by means of clique sums of order at most 3^2 . This property of W in $G - A$ is called ‘flatness’. In order to obtain a similar result for bipartite graphs with perfect matchings we need two ingredients: a) A notion of treewidth fit for the study of matching minors, and b) a matching theoretic version of flatness. Introduced by Norine [11] as a possible approach to solve the non-bipartite Pfaffian recognition problem which is still open, the parameter *perfect matching width* can be seen as the appropriate version of treewidth. Indeed, it was shown in [3] that, in bipartite graphs, large perfect matching width forces the existence of a large grid as a matching minor.

A *cross* over a cycle C is a pair of disjoint paths P_1 and P_2 with endpoints $a_1, b_1, a_2,$ and b_2 respectively such that $a_1, a_2, b_1,$ and b_2 appear on C in the order listed and P_1 and P_2 are internally disjoint from C . The famous Two-Paths Theorem [4,12,15,16] states that a cycle C has no cross in a graph G if and only if G can be constructed by using clique sums of order at most three from a family of graphs H_1, \dots, H_h such that $C \subseteq H_1, H_1$ is planar, and C bounds a face in H . For a matching theoretic version let G be a brace and C be a four-cycle in G . A cross P_1, P_2 over C is *conformal* if $C + P_1 + P_2$ is a conformal subgraph of G .

Theorem 2. *Let G be a brace and C be a four-cycle in G . Then there exists a conformal cross over C in G if and only if G contains $K_{3,3}$ as a matching minor.*

Please note that Theorem 2 means that, in case there is no conformal cross over C , G has a summand H which is planar and contains C , thereby, in some sense, replicating the Two-Paths Theorem³. Please note that for our application it suffices to consider 4-cycles, since they can be added as a gadget to allow checking for the required two disjoint paths.

We can now approach a matching theoretic definition of flatness. In case G is bipartite, let us denote by V_1 and V_2 the two colour classes of G and let us always assume that a bipartite graph G is given with a two-colouring. Let G and H be bipartite graphs with a perfect matching such that H has a single brace J that is not isomorphic to C_4 . We say that H is a *J-expansion*. A brace B of G is said to be a *host of H* if B contains a conformal subgraph H' that is a *J-expansion* and can be obtained from H by repeated applications of tight cut contractions. The graph H' is called the *remnant of H* .

² See [5] for the exact statements and definitions.

³ Further inspection reveals that C bounds a face of H .

Definition 1. Let G and H be bipartite graphs with perfect matchings and assume H to be matching covered, planar, and a J -expansion for some planar brace J . Moreover, let P be a collection of pairwise vertex disjoint faces of H such that P is a conformal subgraph of H . At last, let $A \subseteq V(G)$ be a conformal set. Then H is P -flat in G with respect to A if there exists a tuple (X_1, Y, Z, X_2) such that

1. $X_1 \cup X_2 \cup Y \cup Z = V(G - A)$, $X_1 \cap X_2 = \emptyset$, $X_i \cap (Y \cup Z) = \emptyset$ for both $i \in \{1, 2\}$, and $Y \cap Z \subseteq V(P)$,
2. no edge of $G - A$ has an endpoint in $X_i \cap V_{3-i}$ and the other in $X_{3-i} \cup Y \cup Z$ for both $i \in \{1, 2\}$, moreover, no edge of $\partial X_1 \cup \partial X_2$ belongs to a perfect matching of $G - A$,
3. $G[Y \cup Z]$ is matching covered and conformal in $G - A$, H is a conformal subgraph of $G[Z]$, and no edge in $G - A$ has one endpoint in $Y \setminus Z$ and the other in $Z \setminus Y$, and
4. $G[Z]$ has a brace B that has no $K_{3,3}$ -matching minor, is a host of H , and B has a planar summand B' containing a remnant H' of H such that every remnant of a face from P in H' bounds a face of B' .

Note that the tuple (Y, Z) works, essentially, as the separation in the original definition of flatness from [5, 13], while the sets X_1 and X_2 take care of those portions of the graph which are, in a matching theoretic sense, already disconnected by deleting the apex set A . The conditions on the edges from the $\partial(X_i)$ are consequences of the so called *Dulmage-Mendelsohn* decomposition [2].

Let G and B be bipartite graphs with perfect matchings such that B is a conformal subgraph of G . We say that B grasps a brace J as a matching minor of G if there exists a conformal J -expansion H in G such that every vertex of degree at least three in H belongs to B and $B \cup H$ is conformal in G .

3 The Matching Flat Wall Theorem for Bipartite Graphs

What is left is a definition of the walls themselves. The *elementary matching k -wall* W is defined as the graph with vertex set $[k] \times [4k]$ and edge set $\{(i, j), (i', j')\} : i = i' \text{ and } j' = j + 1 \text{ or } i' = i - 1, j' = j - 1 \text{ and } j \bmod 4 = 0 \text{ or } i' = i + 1, j' = j - 1 \text{ and } j \bmod 4 = 2\}$ and its canonical (perfect) matching M is the set $\{(i, j), (i, j + 1)\} : i \in [k] \text{ and } j \bmod 2 = 1\}$. The *matching k -wall* W' is a subdivision of the elementary k -wall W where each edge is subdivided an even number of times and a perfect matching M' of W' is *canonical* if any path P_e corresponding to a subdivided edge e of the underlying elementary k -wall is M' -conformal if $e \in M$ and internally M -conformal, otherwise. The *perimeter* of a matching k -wall W' , $\text{Per}(W')$ is the union of the outermost and innermost M' -conformal cycles. See Fig. 2 for an example of the elementary matching 3-wall, its canonical matching and its perimeter.

With this, we are ready to state our matching theoretic version of the Flat Wall Theorem.

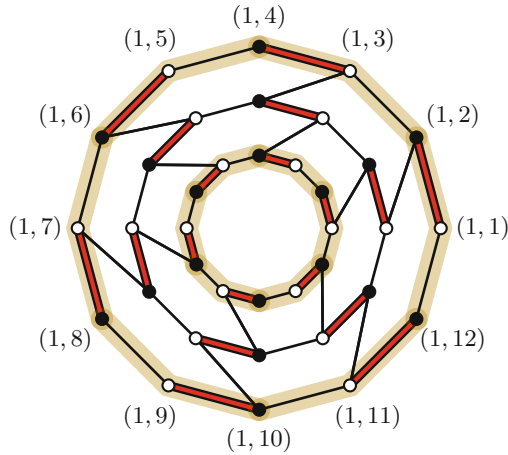


Fig. 2. The elementary 3-wall with its canonical matching (red edges) and its perimeter (marked cycles).

Theorem 3. *Let $r, t \in \mathbb{N}$ be positive integers. There exist functions $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ and $\rho: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every bipartite graph G with a perfect matching M the following is true: If W is an M -conformal matching $\rho(t, r)$ -wall in G such that $M \cap E(W)$ is the canonical matching of W , then either*

1. G has a $K_{t,t}$ -matching minor grasped by W , or
2. there exist an M -conformal set $A \subseteq V(G)$ with $|A| \leq \alpha(t)$ and an M -conformal matching r -wall $W' \subseteq W - A$ such that W' is $\text{Per}(W')$ -flat in B with respect to A .

A Weak Structure Theorem. With Theorem 3 at hand, we can give an approximate characterisation of all bipartite graphs with perfect matchings that exclude $K_{t,t}$ as a matching minor for some $t \in \mathbb{N}$. This weak structure theorem is similar to [5] and in some sense can be seen as a generalisation of Theorem 1 in conjunction with results from [6, 10].

Theorem 4. *Let $r, t \in \mathbb{N}$ be positive integers, α and ρ be the two functions from Theorem 3, and G be a bipartite graph with a perfect matching.*

- *If G has no $K_{t,t}$ -matching minor, then for every conformal matching $\rho(t, r)$ -wall W in G and every perfect matching M of G such that $M \cap E(W)$ is the canonical matching of W , there exist an M -conformal set $A \subseteq V(G)$ with $|A| \leq \alpha(t)$ and an M -conformal matching r -wall $W' \subseteq W - A$ such that W' is $\text{Per}(W')$ -flat in G with respect to A .*
- *Conversely, if $t \geq 2$ and $r \geq \sqrt{2\alpha(t)}$, and for every conformal matching $\rho(r, t)$ -wall W in G and every perfect matching M of G such that $M \cap E(W)$ is the canonical perfect matching of W , there exist an M -conformal set $A \subseteq V(G)$ with $|A| \leq \alpha(t)$ and an M -conformal matching r -wall $W' \subseteq W - A$*

such that W' is $\text{Per}(W')$ -flat in G with respect to A , then G has no matching minor isomorphic to $K_{t',t'}$, where $t' = 16\rho(t, r)^2$.

Proof. The first part of the theorem follows immediately from Theorem 3, since in case B does not have $K_{t,t}$ as a matching minor, the first part of Theorem 3 can never be true and thus every matching $\rho(t, r)$ -wall must be flat in B .

For the reverse, note that an elementary matching $\rho(t, r)$ -wall has exactly $16\rho(t, r)^2$ vertices. Now suppose B has a matching minor model $\mu: K_{t',t'} \rightarrow B^4$. Then there exists a perfect matching M such that μ is M -conformal. Indeed, $K_{t',t'}$ contains an $M|_{K_{t',t'}}$ -conformal elementary matching $\rho(t, r)$ -wall, and thus $\mu(K_{t',t'})$ contains an M -conformal matching $\rho(t, r)$ -wall W . Indeed, for every vertex w of degree three in W there exists a unique vertex $u_w \in V(K_{t',t'})$ such that $w \in V(\mu(u_w))$, and in case $w \neq w'$ are both vertices of degree three in W , then $u_w \text{Nequ}_{w'}$. Moreover, if P is a path in W whose endpoints w and w' have degree three in W and all internal vertices are vertices of degree two in W , then $V(P) \subseteq V(\mu(u_w)) \cup V(\mu(u_{w'}))$. By assumption there exist an M -conformal set $A \subseteq V(B)$ and an M -conformal matching r -wall $W' \subseteq W$ such that W' is $\text{Per}(W')$ -flat in B with respect to A . Now W' has $16r^2$ many vertices of degree three in W' , $16r$ of which lie on $\text{Per}(W')$. Since $r \geq \sqrt{2\alpha(t)}$, we have at least $32\alpha(t)$ many such degree three vertices. Thus, with $|A| \leq \alpha(t)$ and $t \geq 2$, there exist $w_1, \dots, w_6 \in V(W' - \text{Per}(W'))$ such that $V(\mu(u_{w_i})) \cap A = \emptyset$ for all $i \in [1, 6]$. This, however, means that for every conformal and matching covered subgraph $H \subseteq G - A$ that contains W' as a conformal subgraph, every brace J of H that is a host of W' must contain $K_{3,3}$ as a matching minor. Hence W' cannot be $\text{Per}(W')$ -flat in B with respect to A and we have reached a contradiction. \square

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Big Ramsey Degrees and Forbidden Cycles

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Abstract. Using the Carlson–Simpson theorem, we give a new general condition for a structure in a finite binary relational language to have finite big Ramsey degrees.

Keywords: Structural ramsey theory · Big ramsey degrees · Dual ramsey theorem

1 Introduction

We consider standard model-theoretic (relational) structures in finite binary languages formally introduced below. Such structures may be equivalently seen as edge-labelled digraphs with finitely many labels, however the notion of structures is more standard in the area. Structures may be finite or countably infinite. Given structures \mathbf{A} and \mathbf{B} , we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings from \mathbf{A} to \mathbf{B} . We write $\mathbf{C} \longrightarrow (\mathbf{B})_{k,l}^{\mathbf{A}}$ to denote the following statement: for every colouring χ of $\binom{\mathbf{C}}{\mathbf{A}}$ with k colours, there exists an embedding $f: \mathbf{B} \rightarrow \mathbf{C}$ such that χ does not take more than l values on $\binom{f(\mathbf{B})}{\mathbf{A}}$. For a countably infinite structure \mathbf{B} and its finite substructure \mathbf{A} , the *big Ramsey degree* of \mathbf{A} in \mathbf{B} is the least number $l' \in \mathbb{N} \cup \{\infty\}$ such that $\mathbf{B} \longrightarrow (\mathbf{B})_{k,l'}^{\mathbf{A}}$ for every $k \in \mathbb{N}$. We say that the *big Ramsey degrees of \mathbf{B} are finite* if for every finite substructure \mathbf{A} of \mathbf{B} the big Ramsey degree of \mathbf{A} in \mathbf{B} is finite.

We focus on structures in binary languages L and adopt some graph-theoretic terminology. Given a structure \mathbf{A} and distinct vertices u and v , we say that u and v are *adjacent* if there exists $R \in L$ such that either $(u, v) \in R_{\mathbf{A}}$ or $(v, u) \in R_{\mathbf{A}}$. A structure \mathbf{A} is *irreducible* if any two distinct vertices are adjacent. A sequence $v_0, v_1, \dots, v_{\ell-1}$, $\ell \geq 3$, of distinct vertices of a structure \mathbf{A} is called a *cycle* if v_i is adjacent to v_{i+1} for every $i \in \{0, \dots, \ell - 2\}$ as well as v_0 adjacent to v_{ℓ} . A cycle is *induced* if none of the other remaining pairs of vertices in the sequence is adjacent.

Following [8, Section 2], we call a homomorphism $f: \mathbf{A} \rightarrow \mathbf{B}$ (see Sect. 2) a *homomorphism-embedding* if f restricted to any irreducible substructure of \mathbf{A} is an embedding. The homomorphism-embedding f is called a (*strong*) *completion of \mathbf{A} to \mathbf{B}* provided that \mathbf{B} is irreducible and f is injective.

Our main result, which applies techniques developed by the third author in [7], gives the following condition for a given structure to have finite big Ramsey degrees.

Theorem 1. *Let L be a finite language consisting of unary and binary symbols, and let \mathbf{K} be a countably-infinite irreducible structure. Assume that every countable structure \mathbf{A} has a completion to \mathbf{K} provided that every induced cycle in \mathbf{A} (seen as a substructure) has a completion to \mathbf{K} and every irreducible substructure of \mathbf{A} of size at most 2 embeds into \mathbf{K} . Then \mathbf{A} has finite big Ramsey degrees.*

This can be seen as a first step towards a structural condition implying bounds on big Ramsey degrees, giving a strengthening of results by Hubička and Nešetřil [8] to countable structures.

The study of big Ramsey degrees originates in the work of Laver who in 1969 showed that the big Ramsey degrees of the ordered set of rational numbers are finite [11, Chapter 6]. The whole area has been revitalized recently; see [6, 7] for references. Our result can be used to identify many new examples of structures with finite big Ramsey degrees. Theorem 1 is particularly fitting to examples involving metric spaces. In particular, the following corollary may be of special interest.

Corollary 1. *The following structures have finite big Ramsey degrees:*

- (i) *Free amalgamation structures described by forbidden triangles,*
- (ii) *S -Urysohn space for finite distance sets S for which S -Urysohn space exists,*
- (iii) *Λ -ultrametric spaces for a finite distributive lattice Λ [3],*
- (iv) *metric spaces associated to metrically homogeneous graphs of a finite diameter from Cherlin's list with no Henson constraints [5].*

Vertex partition properties of Urysohn spaces were extensively studied in connection to oscillation stability [10] and determining their big Ramsey degrees presented a long standing open problem: Corollary 1 (i) is a special case of the main result of [12], (ii) is a strengthening of [7, Corollary 6.5 (3)], (iii) strengthens [9] and (iv) is a strengthening of [1] to infinite structures.

To see these connections, observe that a metric space can be also represented as an irreducible structure in a binary language having one relation for each possible distance. Possible obstacles to completing a structure in this language to a metric space are irreducible substructures with at most 2 vertices and induced *non-metric cycles*. These are cycles with the longest edge of a length exceeding the sum of the lengths of all the remaining edges; see [1].

Note that all these proofs may be modified to yield Ramsey classes of finite structures. Thus, for example, (ii) generalizes [8, Section 4.3.2].

Our methods yield the following common strengthening of the main results from [12] and Theorem 1. To obtain this result, which is going to appear in [2], we found a new strengthening of the dual Ramsey theorem.

Theorem 2. *Let L be a finite language consisting of unary and binary symbols, and let \mathbf{K} be a countably-infinite irreducible structure. Assume that there exists $c > 0$ such that every countable structure \mathbf{A} has a completion to \mathbf{K} provided that every induced cycle in \mathbf{A} has a completion to \mathbf{K} and every irreducible substructure of \mathbf{A} of size at most c embeds into \mathbf{K} . Then \mathbf{A} has finite big Ramsey degrees.*

2 Preliminaries

A relational language L is a collection of (relational) symbols $R \in L$, each having its *arity*. An L -structure \mathbf{A} on A is a structure with the *vertex set* A and with relations $R_{\mathbf{A}} \subseteq A^r$ for every symbol $R \in L$ of arity r . If the set A is finite, then we call \mathbf{A} a *finite structure*. A *homomorphism* $f: \mathbf{A} \rightarrow \mathbf{B}$ is a mapping $f: A \rightarrow B$ such that for every $R \in L$ of arity r we have $(x_1, x_2, \dots, x_r) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \dots, f(x_r)) \in R_{\mathbf{B}}$. A homomorphism f is an *embedding* if f is injective and the implication above is an equivalence. If the identity is an embedding $\mathbf{A} \rightarrow \mathbf{B}$, then we call \mathbf{A} a *substructure* of \mathbf{B} . In particular, our substructures are always induced.

Hubička [7] connected big Ramsey degrees to an infinitary dual Ramsey theorem for parameters spaces. We now review the main notions used. Given a finite alphabet Σ and $k \in \omega \cup \{\omega\}$, a k -parameter word is a (possibly infinite) string W in the alphabet $\Sigma \cup \{\lambda_i : 0 \leq i < k\}$ containing all symbols $\lambda_i : 0 \leq i < k$ such that, for every $1 \leq j < k$, the first occurrence of λ_j appears after the first occurrence of λ_{j-1} . The symbols λ_i are called *parameters*. Given a parameter word W , we denote its *length* by $|W|$. The letter (or parameter) on index j with $0 \leq j < |W|$ is denoted by W_j . Note that the first letter of W has index 0. A 0-parameter word is simply a *word*. Let W be an n -parameter word and let U be a parameter word of length $k \leq n$, where $k, n \in \omega \cup \{\omega\}$. Then $W(U)$ is the parameter word created by *substituting* U to W . More precisely, $W(U)$ is created from W by replacing each occurrence of $\lambda_i, 0 \leq i < k$, by U_i and truncating it just before the first occurrence of λ_k in W . Given an n -parameter word W and a set S of parameter words of length at most n , we define $W(S) := \{W(U) : U \in S\}$.

We let $[\Sigma] \binom{n}{k}$ be the set of all k -parameter words of length n , where $k \leq n \in \omega \cup \{\omega\}$. If k is finite, then we also define $[\Sigma]^* \binom{\omega}{k} := \bigcup_{k \leq i < \omega} [\Sigma] \binom{i}{k}$. For brevity, we put $\Sigma^* := [\Sigma]^* \binom{\omega}{0}$.

Our main tool is the following infinitary dual Ramsey theorem, which is a special case of the Carlson–Simpson theorem [4, 11].

Theorem 3. *Let $k \geq 0$ be a finite integer. If $[\emptyset]^* \binom{\omega}{k}$ is coloured by finitely many colours, then there exists $W \in [\emptyset] \binom{\omega}{\omega}$ such that $W([\emptyset]^* \binom{\omega}{k})$ is monochromatic.*

Definition 1 ([7]). *Given a finite alphabet Σ , a finite set $S \subseteq \Sigma^*$ and $d > 0$, we call $W \in [\emptyset]^* \binom{\omega}{d}$ a d -parametric envelope of S if there exists a set $S' \subseteq \Sigma^*$*

satisfying $W(S') = S$. In such case the set S' is called the embedding type of S in W and is denoted by $\tau_W(S)$. If d is the minimal integer for which a d -parameter envelope W of S exists, then we call W a minimal envelope.

Proposition 1 ([7]). *Let Σ be a finite alphabet and let $k \geq 0$ be a finite integer. Then there exists a finite $T = T(|\Sigma|, k)$ such that every set $S \subseteq \Sigma^*$, $|S| = k$, has a d -parameter envelope with $d \leq T$. Consequently, there are only finitely many embedding types of sets of size k within their corresponding minimal envelopes. Finally, for any two minimal envelopes W, W' of S , we have $\tau_W(S) = \tau_{W'}(S)$.*

We will thus also use $\tau(S)$ to denote the type $\tau_W(S)$ for some minimal W .

3 Proof of Theorem 1

The proof is condensed due to the space limitations, but we believe it gives an idea of fine interplay of all building blocks. Throughout this section we assume that \mathbf{K} and L are fixed and satisfy the assumptions of Theorem 1. Following ideas from [7, Section 4.1], we construct a special L -structure \mathbf{G} with finite big Ramsey degrees and then use \mathbf{G} to prove finiteness of big Ramsey degrees for \mathbf{K} .

Lemma 1. *Let $h: \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism-embedding. If \mathbf{B} has a completion $c: \mathbf{B} \rightarrow \mathbf{K}$, then there exists a completion $d: \mathbf{A} \rightarrow \mathbf{K}$.*

Proof. It is clearly enough to consider the case where c is the identity and h is surjective and almost identity, that is, there is a unique vertex $v \in A$ such that $h(v) \neq v$. Let \mathbf{B}' be the structure induced by \mathbf{K} on B . We create a structure \mathbf{B}'' from \mathbf{B}' by duplicating the vertex $h(v)$ to v' and leaving $h(v)$ not adjacent to v' . Since \mathbf{B}' is irreducible, it is easy to observe that all induced cycles in \mathbf{B}'' are already present in \mathbf{B}' . By the assumption on \mathbf{K} , there is a completion $c': \mathbf{B}'' \rightarrow \mathbf{K}$. Now, the completion $d: \mathbf{A} \rightarrow \mathbf{K}$ can be constructed by setting $d(v) = c'(v')$ and $d(u) = c'(u)$ for every $u \in A \setminus \{v\}$.

We put $\Sigma = \{\mathbf{A} : A = \{0, 1\}\}$ and there exists an embedding $\mathbf{A} \rightarrow \mathbf{K}$. For $U \in \Sigma^*$, we will use bold characters to refer to the letters (e.g. \mathbf{U}_0 is the structure corresponding to the first letter of U) to emphasize that Σ consists of structures.

Given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \Sigma$, there is at most one structure \mathbf{D} with the vertex set $\{u, v, w\}$ satisfying the following three conditions: (i) mapping $0 \mapsto u, 1 \mapsto v$ is an embedding $\mathbf{A} \rightarrow \mathbf{D}$, (ii) the mapping $0 \mapsto v, 1 \mapsto w$ is an embedding $\mathbf{B} \rightarrow \mathbf{D}$, and (iii) the mapping $0 \mapsto u, 1 \mapsto w$ is an embedding $\mathbf{C} \rightarrow \mathbf{D}$. If such a structure \mathbf{D} exists, we denote it by $\Delta(\mathbf{A}, \mathbf{B}, \mathbf{C})$ (since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle). Otherwise we leave $\Delta(\mathbf{A}, \mathbf{B}, \mathbf{C})$ undefined.

Definition 2. *Let \mathbf{G} be the following structure.*

1. *The vertex set G consists of all finite words W of length at least 1 in the alphabet Σ that satisfy the following condition.*
 - (A1) *For all i and j with $0 \leq i < j < |W|$, the structure induced by \mathbf{W}_i on $\{1\}$ is isomorphic to the structure induced by \mathbf{W}_j on $\{1\}$.*

2. Let U, V be vertices of \mathbf{G} with $|U| < |V|$ that satisfy the following condition.
 (A2) The structure $\Delta(\mathbf{U}_i, \mathbf{V}_{|U|}, \mathbf{V}_i)$ is defined for every i with $0 \leq i < |U|$ and it has an embedding to \mathbf{K} .
 Then the mapping $0 \mapsto U, 1 \mapsto V$ is an embedding of type $\mathbf{V}_{|U|} \rightarrow \mathbf{G}$.
3. There are no tuples in the relations $R_{\mathbf{G}}, R \in L$, other than the ones given by 2.

Lemma 2. Every induced cycle in \mathbf{G} has a completion to \mathbf{K} . Since every irreducible substructure of size at most 3 embeds into \mathbf{K} there is a completion $\mathbf{G} \rightarrow \mathbf{K}$.

Proof. Suppose for contradiction that there exists ℓ and a sequence $U^0, U^1, \dots, U^{\ell-1}$ forming an induced cycle \mathbf{C} in \mathbf{G} such that \mathbf{C} has no completion to \mathbf{K} . Without loss of generality, we assume that $|U^0| \leq |U^k|$ for every $1 \leq k < \ell$. We create a structure \mathbf{D} from \mathbf{C} by adding precisely those tuples to the relations of \mathbf{D} such that the mapping $0 \mapsto \mathbf{U}^0, 1 \mapsto \mathbf{U}^k$ is an embedding from $\mathbf{U}_{|U^0|}^k$ to \mathbf{D} for every k satisfying $2 \leq k < \ell$ and $|U^0| < |U^k|$.

For simplicity, consider first the case that we have $|U^0| < |U^k|$ for every $1 \leq k \leq \ell - 1$. In this case, we produced a triangulation of \mathbf{D} : all induced cycles are triangles containing the vertex U^0 . It follows from the construction of \mathbf{G} that, for every $2 \leq k \leq \ell$, the triangle induced by \mathbf{D} on U^0, U^k and U^{k+1} is isomorphic either to $\Delta(\mathbf{U}_{|U^0|}^k, \mathbf{U}_{|U^k|}^{k+1}, \mathbf{U}_{|U^0|}^{k+1})$ (if $|U^k| < |U^{k+1}|$) or to $\Delta(\mathbf{U}_{|U^0|}^{k+1}, \mathbf{U}_{|U^{k+1}|}^k, \mathbf{U}_{|U^0|}^k)$. By (A2) the triangle has an embedding to \mathbf{K} , hence all induced cycles in \mathbf{D} have a completion to \mathbf{K} , which implies that \mathbf{D} has a completion $c: \mathbf{D} \rightarrow \mathbf{K}$. We get completion $c: \mathbf{C} \rightarrow \mathbf{K}$, a contradiction.

It remains to consider the case that there are multiple vertices of \mathbf{D} of length $|U^0|$. We then set $M := \{U^k : |U^k| = |U^0|\}$. By the construction of \mathbf{G} , the vertices in M are never neighbours. Moreover, for every $U, V \in M$, the structure induced on $\{U\}$ by \mathbf{C} is isomorphic to structure induced on $\{V\}$ by \mathbf{C} , which, by (A2), is isomorphic to the structure induced on $\{0\}$ by $\mathbf{W}_{|U^0|}$ for every $W \in C \setminus M$. Consequently, it is possible to construct a structure \mathbf{E} from \mathbf{D} by identifying all vertices in M and to obtain a homomorphism-embedding $f: \mathbf{D} \rightarrow \mathbf{E}$. Observe that the structure \mathbf{E} is triangulated and every triangle is known to have a completion to \mathbf{K} . By Lemma 1, \mathbf{D} also has a completion to \mathbf{K} .

The following result follows directly from the definition of substitution.

Observation 4. For every $W \in [\emptyset] \binom{\omega}{\omega}$ and all $U, V \in G$, the structure induced by \mathbf{G} on $\{U, V\}$ is isomorphic to the structure induced by \mathbf{G} on $\{W(U), W(V)\}$.

Without loss of generality we assume that $K = \omega \setminus \{0\}$. Let \mathbf{K}' be the structure \mathbf{K} extended by the vertex 0 such that there exists an embedding $\mathbf{K}' \rightarrow \mathbf{K}$. Such a structure \mathbf{K}' exists, because duplicating the vertex 1 does not introduce new induced cycles. We define the mapping $\varphi: \omega \setminus \{0\} \rightarrow G$ by setting $\varphi(i) = U$, where U is a word of length i defined by setting, for every $0 \leq j < i$, \mathbf{U}_j as the unique structure in Σ such that $0 \mapsto j, 1 \mapsto i$ is an embedding $\mathbf{U}_j \rightarrow \mathbf{K}'$. It is easy to check that φ is an embedding $\varphi: \mathbf{K} \rightarrow \mathbf{G}$. We prove Theorem 1 in the following form.

Theorem 5. *For every finite $k \geq 1$ and every finite colouring of subsets of G with k elements, there exists $f \in \binom{G}{G}$ such that the colour of every k -element subset S of $f(G)$ depends only on $\tau(S) = \tau(f^{-1}[S])$.*

By Proposition 1, we obtain the desired finite upper bound on the number of colours. By the completion $c: \mathbf{G} \rightarrow \mathbf{K}$ given by Lemma 2, the colouring of substructures of \mathbf{K} yields a colouring of irreducible substructures of \mathbf{G} . Embedding $f \in \binom{G}{G}$ can be restricted $f' \in \binom{G}{K}$ and gives $c \circ f' \in \binom{K}{K}$ and thus Theorem 5 indeed implies Theorem 1.

Proof (Sketch). Fix k and a finite colouring χ of the subsets of G of size k . Proposition 1 bounds number of embedding types of subsets of G of size k . Apply Theorem 3 for each embedding type. By Observation 4, we obtain the desired embedding; see [7, proof of Theorem 4.4] for details.

Acknowledgement. D. Ch., J. H., M. K. and J. N. are supported by the project 21-10775S of the Czech Science Foundation (GAČR). This is part of a project that has received funding from the European Research Council (ERC) under the EU Horizon 2020 research and innovation programme (grant agreement No 810115). M. B. and J. H. were supported by the Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004). M. K. was supported by the Charles University Grant Agency (GA UK), project 378119.

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The Ramsey Number for 4-Uniform Tight Cycles

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Abstract. A 4-uniform tight cycle is a 4-uniform hypergraph with a cyclic ordering of its vertices such that its edges are precisely the sets of 4 consecutive vertices in that ordering. We prove that the Ramsey number for the 4-uniform tight cycle on $4n$ vertices is $(5 + o(1))n$. This is asymptotically tight.

Keywords: Ramsey number · Hypergraph · Tight cycle

1 Introduction

For k -graphs H_1, \dots, H_m , the *Ramsey number* $r(H_1, \dots, H_m)$ is the smallest integer N such that any m -edge-colouring of the complete k -graph $K_N^{(k)}$ contains a monochromatic copy of H_i in the i -th colour for some $1 \leq i \leq m$. If H_1, \dots, H_m are all isomorphic to H then we let $r_m(H) = r(H_1, \dots, H_m)$ and call it the *m -colour Ramsey number for H* . We also write $r(H)$ for $r_2(H)$ and simply call it the *Ramsey number for H* .

The Ramsey number for cycles in graphs has been determined in [2, 3] and [15]. In particular, for $n \geq 5$, we have

$$r(C_n) = \begin{cases} \frac{3}{2}n - 1, & \text{if } n \text{ is even,} \\ 2n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Note that there is a dependence on the parity of the length of the cycle. For the m -colour Ramsey number, Jenssen and Skokan [8] proved that for $m \geq 2$ and any large enough odd integer n we have $r_m(C_n) = 2^{m-1}(n - 1) + 1$.

Some Ramsey numbers for k -graphs have also been studied. A k -uniform tight cycle $C_n^{(k)}$ is a k -graph on n vertices with a cyclic ordering of its vertices such that its edges are the sets of k consecutive vertices. The Ramsey number of the 3-uniform tight cycle on n vertices $C_n^{(3)}$ was determined asymptotically by Haxell, Łuczak, Peng, Rödl, Ruciński and Skokan, see [6] and [7]. They showed that $r(C_{3n}^{(3)}) = (1 + o(1))4n$ and $r(C_{3n+i}^{(3)}) = (1 + o(1))6n$ for $i \in \{1, 2\}$.

We define the *k -uniform tight path on n vertices* $P_n^{(k)}$ to be the k -graph obtained from $C_{n+1}^{(k)}$ by deleting a vertex. Using the bound on the Turán number

for tight paths that was recently shown by Füredi, Jiang, Kostochka, Mubayi and Verstraëte [4], we deduce that $r(P_n^{(k)}) \leq k(n - k + 1)$ for any even $k \geq 2$.

The Ramsey number for loose cycles have also been studied. We denote by $LC_n^{(k)}$, where $n = \ell(k - 1)$, the k -uniform loose cycle on n vertices, that is the k -graph with vertex set $\{v_1, \dots, v_n\}$ and edges $e_i = \{v_{1+i(k-1)}, \dots, v_{k+i(k-1)}\}$ for $0 \leq i \leq \ell - 1$, where indices are taken modulo n . Gyárfás, Sárközy and Szemerédi [5] showed that

$$r(LC_n^{(k)}) = (1 + o(1)) \frac{2k - 1}{2k - 2} n.$$

Recently, the exact values of Ramsey numbers for loose cycles have been determined for various cases, see [16] for more details.

Another problem of interest in this area is determining the Ramsey number of a complete graph and a cycle. For graphs, Keevash, Long and Skokan [9] showed that there exists an absolute constant $C \geq 1$ such that

$$r(C_\ell, K_n) = (\ell - 1)(n - 1) + 1 \text{ provided } \ell \geq C \frac{\log n}{\log \log n}.$$

Analogous problems for hypergraphs have also been considered. See [12, 13] and [14] for the analogous problem with loose, tight and Berge cycles, respectively.

We will consider the Ramsey number for tight cycles. We determine the Ramsey number for the 4-uniform tight cycle on n vertices $C_n^{(4)}$ asymptotically in the case where n is divisible by 4.

Theorem 1. *Let $\varepsilon > 0$. For n large enough we have $r(C_{4n}^{(4)}) \leq (5 + \varepsilon)n$.*

It is easy to see that this is asymptotically tight.

Proposition 1. *For $n, k \geq 2$, we have that $r(C_{kn}^{(k)}) \geq (k + 1)n - 1$.*

Proof. Let $N = (k + 1)n - 2$. We show that there exists a red-blue edge-colouring of $K_N^{(k)}$ that does not contain a monochromatic copy of $C_{kn}^{(k)}$. We partition the vertex set of $K_N^{(k)}$ into two sets X and Y of sizes $n - 1$ and $kn - 1$, respectively. We colour every edge that intersects the set X red and any other edge blue. It is easy to see that this red-blue edge-colouring of $K_N^{(k)}$ does not even contain a monochromatic matching of size n and thus also cannot contain a monochromatic copy of $C_{kn}^{(k)}$. There is no red matching of size n since every red edge must intersect X and $|X| = n - 1$. Moreover, there is no blue matching of size n since all blue edges are entirely contained in Y and $|Y| = kn - 1$. \square

Theorem 1 also implies the following corollary about the Ramsey number for the 4-uniform tight path.

Corollary 1. *We have $r(P_n^{(4)}) = (5/4 + o(1))n$.*

1.1 Sketch of the Proof of Theorem 1

We now sketch the proof of Theorem 1. We use a hypergraph version of the connected matching method of Łuczak [11] as follows. We consider a red-blue edge-colouring of $K_N^{(4)}$ for $N = (5/4 + \varepsilon)n$. We begin by applying the Hypergraph Regularity Lemma. More precisely, we use the Regular Slice Lemma of Allen, Böttcher, Cooley and Mycroft [1]. This gives us a reduced graph \mathcal{R} , which is a red-blue edge-coloured almost complete 4-graph on $(5/4 + \varepsilon)n'$ vertices. To prove Theorem 1, it now suffices to find a *monochromatic tightly connected matching* of size $n'/4$ in \mathcal{R} . A monochromatic tightly connected matching is a monochromatic matching M such that for any two edges $f, f' \in M$, there exists a tight walk¹ in \mathcal{R} of the same colour as M connecting f and f' .

Let γ be a constant such that $0 < \gamma \ll \varepsilon$ and let M be a maximal monochromatic tightly connected matching in \mathcal{R} . Suppose that M has size less than $n'/4$ and is red. We show that we can find a monochromatic tightly connected matching of size at least $\gamma n'$ greater than M . By iterating this we get our desired result. (We actually find a *fractional matching* instead. Note that by taking a blow-up of \mathcal{R} we can then convert it back to an integral matching.) For simplicity, let us further assume that \mathcal{R} has only one red and one blue tight component². Then any monochromatic matching is tightly connected. Consider an edge $f \in M$ and a vertex w not covered by M . Observe that if all the edges in $\mathcal{R}[f \cup \{w\}]$ are red, then we get a larger red fractional matching. Thus for almost all the edges $f \in M$ there is a blue edge f' such that $|f \cap f'| = 3$. This gives us a blue matching M' of almost the same size as M . Note that the set of leftover vertices $W = V(\mathcal{R}) \setminus V(M \cup M')$ has size at least $\varepsilon n'$. By the maximality of M , any edge in $\mathcal{R}[W]$ must be blue. So we can extend M' by adding a matching in W to get the desired matching.

However, \mathcal{R} may contain many monochromatic tight components (instead of just two). Hence we need to choose monochromatic tight components carefully. To do this we use a novel auxiliary graph called the blueprint which the authors introduced in [10]. The blueprint is a graph with the key property that monochromatic connected components in it correspond to monochromatic tight components in the 4-graph we are considering. Since the blueprint is red-blue edge-coloured and almost complete, it contains an almost-spanning monochromatic tree. Using the key property of blueprints this shows that \mathcal{R} contains a large monochromatic tight component.

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¹ A tight walk in a k -graph is a sequence of edges e_1, \dots, e_t such that $|e_i \cap e_{i+1}| = k - 1$ for all $1 \leq i \leq t - 1$.

² A *red (blue) tight component* is a maximal set F of red (blue) edges such that for any two edges $f, f' \in F$ there is a tight walk in F connecting f and f' .

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Chip-Firing on the Complete Split Graph: Motzkin Words and Tiered Parking Functions

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Abstract. We highlight some results from studying chip-firing on the the complete split graph [5]. In this work it is shown that recurrent states can be characterised in terms of Motzkin words and can also be characterised in terms of a new type of parking function that we call a *tiered parking function*. These new parking functions arise by assigning a tier (or colour) to each of the cars, and specifying how many cars of a lower-tier one wishes to have parked before them.

Keywords: Chip-firing · Recurrent states · Complete split graph · Motzkin words · Parking functions

1 Introduction

The complete split graph is a bipartite graph consisting of two distinct parts, a clique part in which all distinct pairs of vertices are connected by a single edge, and an independent part in which no two vertices are connected to an edge. There is precisely one edge between every vertex in the clique part and every vertex in the independent part. We denote the complete split graph which has vertices $\{v_1, \dots, v_m\}$ in the clique part and vertices $\{w_1, \dots, w_n\}$ in the independent part by $S_{m,n}$. The graph $S_{5,4}$ is illustrated in Fig. 1. The graph $S_{m,n}$ contains the complete graph K_m as a subgraph, but is also a bipartite graph in its own right, and it is this dual feature that we find interesting to examine in terms of chip-firing.

Chip-firing has been studied on several classes of graphs, and rich connections to other combinatorial structures have been established in each of the cases. Cori and Rossin [3] showed that the set of recurrent states of chip-firing on the complete graph are in one-to-one correspondence with parking functions of order n . The author in collaboration with others [1, 6–8] showed that the set of recurrent states of chip-firing on the complete bipartite graph admits a characterization in terms of planar animals called *parallelogram polyominoes*. Cori and Poulalhon in [2] showed that the recurrent states of chip-firing on the complete tri-partite graph $K_{1,p,q}$ admits a description in terms of a parking function for cars of two different colours.

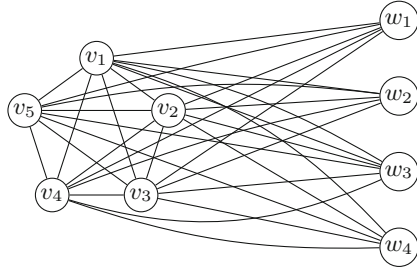


Fig. 1. The complete split graph $S_{5,4}$

We will highlight some of the results from our paper [5] that characterizes recurrent states of chip-firing on the complete split graph. The characterization is in terms of Motzkin words and we give a second characterisation of the recurrent states in terms of a new type of parking function that we call a *tiered parking function*. These parking functions are characterised by assigning a tier (or colour) to each of the cars, and specifying how many cars of a lower-tier one wishes to have parked before them in a one-way street.

2 Recurrent States for Chip-Firing on a Graph

The chip-firing game, also known as the Abelian sandpile model (ASM) in the literature, may be defined on any undirected graph G with a designated vertex s called the *sink*. A *configuration* on G is an assignment

$$c : \text{Vertices}(G) \setminus \{s\} \mapsto \mathbb{N} = \{0, 1, 2, \dots\}.$$

The number $c(v)$ is sometimes referred to as the number of *grains* at vertex v . Given a configuration c , a vertex v is said to be *stable* if the number of grains at v is strictly smaller than the threshold of that vertex, which is the degree of v , denoted $\text{deg}(v)$. Otherwise v is *unstable*. A configuration is called *stable* if all non-sink vertices are stable.

If a vertex is unstable then it may *topple*, which means the vertex donates one grain to each of its neighbors. The sink vertex has no height associated with it and it only absorbs grains, thereby modelling grains exiting the system. Given this, it is possible to show that starting from any configuration c and toppling unstable vertices, one eventually reaches a stable configuration c' . Moreover, c' does not depend on the order in which vertices are toppled in this sequence. We call c' the *stabilisation* of c . We use the notation $\text{ASM}(G, s)$ to indicate that we are considering the Abelian sandpile model as described here on the graph G with sink s .

Starting from the empty configuration, one may indefinitely add any number of grains to any vertices in G and topple vertices should they become unstable. Certain stable configurations will appear again and again, that is, they *recur*, while other stable configurations will never appear again. These *recurrent configurations* are the ones that appear in the long term limit of the system. In [4, Sec. 6], Dhar describes the so-called *burning algorithm*, which establishes whether a given stable configuration is recurrent:

Proposition 1 ([4, Sect. 6.1]. *Let G be a graph with sink s . A stable configuration c on G is recurrent if and only if there exists an ordering $v_0 = s, v_1, \dots, v_n$ of the vertices of G such that, starting from c , for any $i \geq 1$, toppling the vertices v_0, \dots, v_{i-1} causes the vertex v_i to become unstable.*

A configuration on $ASM(S_{m,n}, s)$ is a vector $c = (c_1, \dots, c_{i-1}, -, c_{i+1}, \dots, c_{m+n-1})$ whereby c_i represents the number of chips/grains at the i th vertex in the sequence $(v_1, \dots, v_m, w_1, \dots, w_n)$. The dash indicates the location corresponding to the sink. We will use a semi-colon in the configurations to distinguish between the clique and independent parts, e.g. $c = (a_1, \dots, a_{m-1}, -; b_1, \dots, b_n)$. Let $Rec(ASM(S_{m,n}, s))$ be the set of recurrent states of the model. We will restrict our analysis to those configurations that are weakly decreasing with respect to vertex labels. This restriction will allow us to focus our analysis to consider characterizing all those ‘different’ configurations, and from which we can generate all configurations through permutations.

3 Motzkin Words and Recurrent States for a Clique-Sink

We will use Motzkin words for the characterisation of recurrent states. A *Motzkin path* P of length p is a lattice path in the plane from $(0, 0)$ to $(p, 0)$ which never passes below the x -axis and whose permitted steps are the up step $U = (1, 1)$, the down step $D = (1, -1)$, and the horizontal step $H = (1, 0)$. An example of a Motzkin path is given in Fig. 2a. The Motzkin word of a path is a listing of the p steps of the path in the order they appear from left to right, e.g. the Motzkin word of Fig. 2a is $\alpha = HUHHDHUDD$.

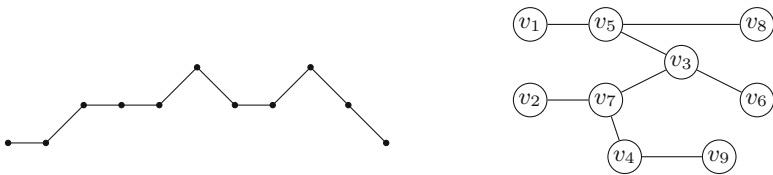


Fig. 2. (a) Example of a Motzkin path of length 10 (b) A spanning tree of the complete split graph $S_{5,4}$

Definition 1. A Motzkin word is a word α consisting of the letters U, D, H with the properties (i) in counting α from left to right the U count is always

greater than or equal to the D count, and (ii) the total U count is equal to the D count. Let $\text{Motzkin}_{m,n}$ be the set of Motzkin words with m occurrences of U and n occurrences of H .

The deletion of the H letters in a Motzkin word gives a Dyck word and, conversely, a Motzkin word with m occurrences of U and n occurrences of H is obtained by taking a Dyck word with m occurrences of U and inserting n occurrences of H in any positions in this word. This decomposition provides an enumeration of such words: $|\text{Motzkin}_{m,n}| = \frac{1}{m+1} \binom{2m}{m} \binom{2m+n}{n}$.

Definition 2. To any Motzkin word $\alpha \in \text{Motzkin}_{m-1,n}$ we associate a configuration $f(\alpha) = (a_1, \dots, a_{m-1}, -; b_1, b_2, \dots, b_n)$ on $S_{m,n}$ as follows:

- a_i is such that $a_i + 1$ is equal to the number of occurrences of the letter D plus the number of occurrences of the letter H appearing in α after the i -th occurrence of the letter U .
- b_i is equal to the number of occurrences of the letter D appearing after the i -th occurrence of the letter H in α .

Observe that, by construction, the sequences (a_1, \dots, a_{m-1}) and (b_1, \dots, b_n) are weakly decreasing. For the Motzkin word example above, the associated configuration on $\text{ASM}(S_{4,4}, v_4)$ is $f(\alpha) = (5, 3, 1, -; 3, 3, 3, 2)$.

Theorem 1. A weakly decreasing stable configuration $c = (a_1, \dots, a_{m-1}, -; b_1, \dots, b_n)$ on $\text{ASM}(S_{m,n}, v_m)$ is recurrent iff it corresponds, via the construction f , to a unique Motzkin word α in $\text{Motzkin}_{m-1,n}$.

Since the number of such decreasing recurrent configurations is equal to the number $|\text{Motzkin}_{m-1,n}|$, we have the number of weakly decreasing recurrent configurations on $S_{m,n}$ with sink v_m is $\frac{1}{m} \binom{2m-2}{m-1} \binom{2m-2+n}{n}$.

3.1 Prüfer Code Decomposition for Spanning Trees of $S_{m,n}$

It is a well established fact the the number of recurrent states of the Abelian sandpile model on a graph is equal to the number of spanning trees of that graph. We will enumerate the set of recurrent states by enumerating the spanning trees of the complete split graph. We will do this by presenting a bijective proof that uses the Prüfer code of the spanning trees.

To do this we use a different labelling of the vertices of the complete split graph. Let $S_{m,n}$ have clique vertices $\{v_1, v_2, \dots, v_m\}$ and independent vertices $\{v_{m+1}, \dots, v_{m+n}\}$. Suppose these vertex labels are totally ordered:

$$v_1 < v_2 < \dots < v_m < v_{m+1} < \dots < v_{m+n}.$$

Any spanning tree T of $S_{m,n}$ is a tree with vertices labelled v_1, \dots, v_{m+n} such that for any edge (v_i, v_j) of T , one has $i \leq m$ or $j \leq m$.

The Prüfer code of this tree is obtained by successively deleting the leaves of T with minimal label and recording the vertex to which they were attached,

until T has only one edge. The unique edge that is obtained at the end of the procedure is (v_{m+n}, v_i) where $i \leq m$. Two words \mathbf{f}, \mathbf{g} may be built using the alphabet consisting of the labels of the vertices. Initially both \mathbf{f} and \mathbf{g} are empty, and at each step of the procedure a letter is added either to \mathbf{f} or to \mathbf{g} . It is added to \mathbf{f} if the leaf deleted is a clique vertex and added to \mathbf{g} if it is an independent vertex. The letter added is the label of the vertex neighbour of the deleted leaf.

For the tree given in Fig. 2b, the vertices are deleted in the following order

$$v_1, v_2, v_6, v_8, v_5, v_3, v_7.$$

The Prüfer code of the tree is $v_5, v_7, v_3, v_5, v_3, v_7, v_4$ and the construction of the two words \mathbf{f} and \mathbf{g} gives

$$\mathbf{f} = v_5, v_7, v_3, v_7 \qquad \mathbf{g} = v_3, v_5, v_4.$$

Proposition 2. *There is a bijection between spanning trees of $S_{m,n}$ and pairs of words (\mathbf{f}, \mathbf{g}) such that \mathbf{f} has length $m - 1$ and has letters in the alphabet $\{v_1, \dots, v_{m+n}\}$, while \mathbf{g} has $n - 1$ letters in the alphabet $\{v_1, \dots, v_m\}$.*

Corollary 1. *The number of spanning trees of $S_{m,n}$ is $(m + n)^{m-1}m^{n-1}$ and, consequently, this is the number of recurrent configurations of the Abelian sandpile model on the complete split graph $S_{m,n}$.*

4 Tiered Parking Functions

We may also offer the following definition of recurrent configurations as a new type of parking function that we will call a *tiered parking function*. Parking functions were mentioned earlier in the paper in relation to the recurrent states of the sandpile model on the complete graph. Moreover, the G -parking functions of Postnikov and Shapiro [9] provide a useful language in which an alternative description of recurrent states of the sandpile model on a general graph may be given.

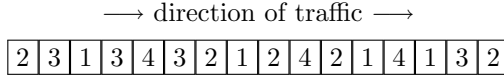
The application of G -parking functions to the complete split graph is different to what we present in this section. Our aim is to provide a new ‘type’ of parking function, and provide a setting in which recurrence is quite easily established in this new context. We refer the interested reader to Yan [11] for a discussion of G -parking functions and their relation to the ASM. Our definition is inspired by Cori and Poulalhon’s [2] concept of (p, q) -parking functions.

Definition 3 (k -tiered parking function). *Let m_1, \dots, m_k be a sequence of positive integers with $m_1 + \dots + m_k = M$. Suppose that there are m_i cars of colour/tier i and there are M parking spaces. We will call a sequence $P = (m_1; P_2, \dots, P_k)$ of sequences $P_i = (p_1^{(i)}, \dots, p_{m_i}^{(i)})$ a k -tiered parking function of order (m_1, \dots, m_k) if there exists a parking configuration of the M cars that satisfies the following preferences for all drivers:*

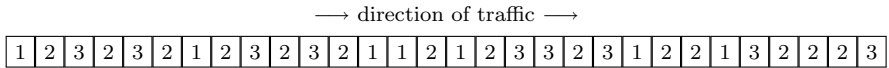
*the driver of the j th car having colour $i > 1$ asks that there be **at least** $p_j^{(i)}$ cars of colours $\{1, \dots, i - 1\}$ parked before him.*

Drivers of cars having colour 1 have no preferences with regard to other coloured cars, which is why we only list their number m_1 in P .

Example 1. The sequence $P = (4; (2, 1, 0, 4, 2), (8, 2, 1, 2), (4, 10, 8))$ is a 4-tiered parking function of order $(4, 5, 4, 3)$. This is realised by the following parking configuration where the leftmost entry represents the first parking spot, and the colour of the parked car is indicated.



Example 2. The sequence $P = (9; (8, 2, 9, 4, 6, 9, 7, 8, 2, 7, 9, 4), (18, 2, 14, 6, 21, 13, 7, 13, 3))$ is a 3-tiered parking function of order $(9, 12, 9)$. This is realised by the following parking configuration:



We now connect these tired parking functions to the recurrent states of the sandpile model for the complete split graph.

Theorem 2. *A configuration $c = (a_1, \dots, a_{m-1}, -; b_1, \dots, b_n)$ on $ASM(S_{m,n}, v_m)$ is recurrent iff there exists a 3-tiered parking function of order $(m - 1, n, m - 1)$ with $P = ((b_1, \dots, b_n), (a_1 + 1, \dots, a_{m-1} + 1))$.*

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Long Shortest Cycle Covers in Cubic Graphs

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Abstract. A well known conjecture of Alon and Tarsi (1985) states that every bridgeless graph admits a cycle cover of length not exceeding $\frac{7}{5} \cdot m$, where m is the number of edges. Although there exist infinitely many cubic graphs with covering ratio $7/5$, there is an extensive evidence that most cyclically 4-edge-connected cubic graphs have covering ratio close to the natural lower bound of $4/3$. In line with this observation, Brinkmann et al. (2013) proposed a conjecture that every cyclically 4-edge-connected cubic graph has a cycle cover of length at most $\frac{4}{3}m + o(m)$. In this paper we disprove the conjecture.

Keywords: Cycle cover · Cubic graph · Perfect matching · Snark

1 Introduction

A *cycle cover* of a graph G is a collection of cycles (even subgraphs) such that each edge of G belongs to at least one member of the collection. The problem is to find, for a given graph G , a cycle cover with minimum total number of edges. In 1985, Alon and Tarsi [2] (and independently Jaeger) proposed the following conjecture, which quickly took its place among the most important problems in graph theory.

Conjecture 1. (Shortest cycle cover conjecture) Every bridgeless graph G has a cycle cover of length at most $\frac{7}{5} \cdot |E(G)|$.

The conjecture has the form of an optimisation problem, nonetheless, its validity would have very strong structural implications. For example, the conjecture, if true, would imply the assertion of the celebrated cycle double cover conjecture [11]. Several other problems, such as the Chinese postman problem or the Petersen colouring conjecture, are also closely related to the $7/5$ conjecture [9, 10].

In 1983, Bermond et al. [3] and in 1985 Alon and Tarsi [2] independently proved that every bridgeless graph G has a cycle cover of length at most $\frac{5}{3} \cdot |E(G)|$. In spite of a great effort (see for example [7, 12, 13]) this result remains the best general approximation of the shortest cycle cover conjecture to date.

As with other problems related to cycles and flows in graphs, cubic graphs are crucial for the shortest cycle cover problem. In particular, the largest known values of covering ratio occur among cubic graphs. Besides the exceptional Petersen graph whose covering ratio equals $7/5$ there exist infinite families of 2-connected and 3-connected cubic graphs reaching the same value (see, for example, [3]). However, all such examples can be obtained from the Petersen graph in a straightforward manner.

For cyclically 4-edge-connected cubic graphs the situation is quite different. Note that the covering ratio of any cubic graph must be at least $4/3$, because at least one third of the edges must be covered more than once. It is known [8, 16] that the value $4/3$ is met by cubic graphs whose edges can be covered with at most four perfect matchings. This set includes all 3-edge-colourable cubic graphs as well as a significant portion of snarks (2-connected non-3-edge-colourable cubic graphs). Cyclically 4-edge-connected cubic graphs whose covering ratio is strictly greater than $4/3$ are therefore very difficult to find. Among the millions of cyclically 4-edge-connected cubic graphs of girth at least 5 on up to 36 vertices, generated in [4], there are only two graphs whose covering ratio is greater than $4/3$, the Petersen graph and a graph on 34 vertices (see [4, Figure 3]). In both cases the minimum length of a cycle cover equals $\frac{4}{3}m + 1$, where m denotes the number of edges. The first infinite family of cyclically 4-edge-connected cubic graphs whose shortest cycle cover has length at least $\frac{4}{3}m + 1$ was constructed by Esperet and Mazzuoccolo [6] in 2014. The same authors also constructed a cyclically 4-edge-connected cubic graph on 106 vertices whose shortest cycle cover has length $\frac{4}{3}m + 2$, see [6, Figure 8].

The difficulties to find cyclically 4-edge-connected cubic graphs with covering ratio significantly greater than $4/3$ seem to suggest that the value by which the length of their shortest cycle cover exceeds $\frac{4}{3}m$ should be small with respect to m . This is exactly what is formally expressed by the following conjecture, proposed Brinkmann et al. in [4] and is additionally supported by results of [8] and [16, Section 2.C].

Conjecture 2. The length of a shortest cycle cover of any cyclically 4-edge-connected cubic graph with m edges is bounded above by $\frac{4}{3} \cdot m + o(m)$.

The aim of the present paper is to disprove Conjecture 2 by establishing the following result.

Theorem 1. *For every integer $k \geq 2$ there exists a cyclically 4-edge-connected cubic graph G_k on $46k$ vertices whose shortest cycle cover has length at least*

$$\left(\frac{4}{3} + \frac{1}{69}\right) \cdot |E(G_k)|.$$

The graphs G_k are in fact nontrivial snarks (which means that they have cyclic connectivity at least 4 and girth at least 5); the smallest graph of the family is shown in Fig. 1. Since the length of a shortest cycle cover of each G_k is greater than $\frac{4}{3} \cdot |E(G_k)|$, the results of [8, Theorem 5.4] and [16, Theorem 3.1]

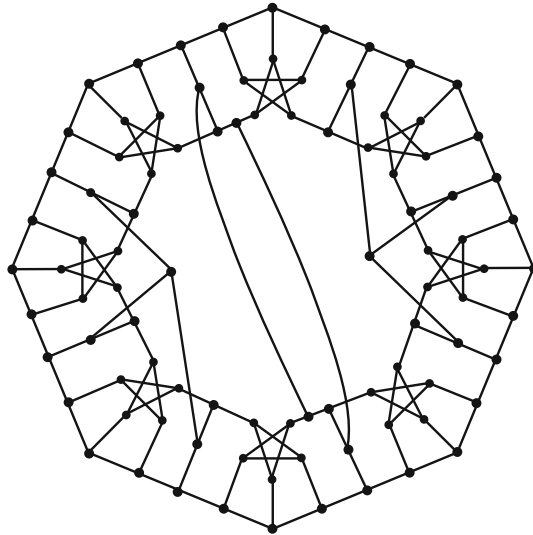


Fig. 1. The graph G_2 from Theorem 1

imply that each G_k requires at least five perfect matchings to cover all its edges. Theorem 1 thus provides a new family of snarks whose perfect matching index is greater than 4. Other such families can be found in [1, 5, 6].

2 Results

The family $(G_k)_{k \geq 2}$ will be assembled from smaller building blocks, called *multipoles*. Multipoles are similar to graphs, except that they are permitted to contain edges having one end-vertex and one free end. Such edges are called *dangling edges*. Dangling edges of a multipole can be grouped into pairwise disjoint sets, called *connectors*. A multipole with exactly two connectors of equal size is called a *dipole*. One of its connectors is chosen to be the *input connector*, the other connector is its *output connector*. A dipole where the size of each connector is m is called an (m, m) -pole.

Cycle covers of our graphs will be usually composed from cycle covers of the constituting multipoles. Cycle covers of multipoles are defined similarly as for graphs, with a slightly different definition of a circuit. A *circuit* of a multipole M is a connected 2-regular subgraph of M . Observe that a circuit of a multipole is either a circuit in the usual sense or a path starting and ending with a dangling edge. A *cycle* of M is an edge-disjoint union of circuits. The *length* of a cycle is the number of its edges. The *length* of a cycle cover \mathcal{C} , denoted by $\ell(\mathcal{C})$, is the sum of lengths of the cycles from \mathcal{C} .

Consider a cubic graph G with a fixed cycle cover \mathcal{C} ; clearly, G is bridgeless. The *weight* of an edge e with respect to \mathcal{C} , denoted by $w(e)$, is the number of cycles of \mathcal{C} that contain e . The *weight* of a vertex v , denoted by $w(v)$, is the sum

of weights of all edges incident with v . Observe that with respect to any cycle cover the weight of a vertex must be a positive even integer.

Let $\text{scc}(G)$ denote the minimum length of a cycle cover of G . As previously noted, $\text{scc}(G) \geq \frac{4}{3} \cdot |E(G)|$ for every bridgeless cubic graph G . Cubic graphs where the equality is attained are exactly those for which there exists a cycle cover with every vertex of weight 4; such a cycle cover will be called *light*.

In this paper we focus on cubic graphs that do not admit light cycle covers. Take such a graph G and form a $(2, 2)$ -pole X by removing from G a pair of adjacent vertices and by putting the edges formerly incident with the same vertex into the same connector. We say that the dipole X obtained in this way is *robust*. The basic robust dipole arises from the Petersen graph; it will be denoted by Q .

In general, robust dipoles may admit light cycle covers. An example of such a dipole is the dipole Q obtained from the Petersen graph. However, as we shall see, robust dipoles may be combined into structures that fail to have light cycle covers. One such example is the dipole $Z(X_1, X_2, X_3, X_4)$ represented in Fig. 2, where $X_1, X_2, X_3,$ and X_4 denote arbitrary robust dipoles (with input edges drawn bold). The smallest such dipole is obtained by taking the dipole Q for each X_i . The resulting dipole $Z = Z(Q, Q, Q, Q)$ has 46 vertices.

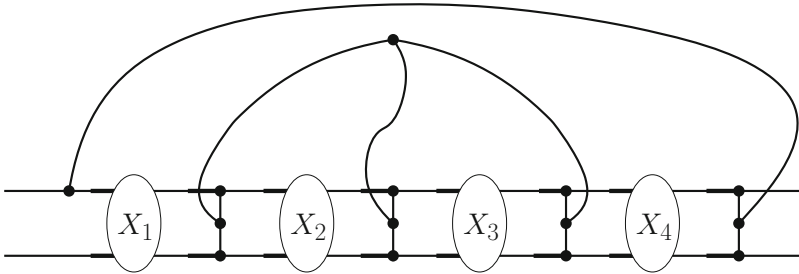


Fig. 2. The basic building block for $(G_k)_{k \geq 2}$

Proposition 1. *If $X_1, X_2, X_3,$ and X_4 are robust $(2, 2)$ -poles, then the $(2, 2)$ -pole $Z(X_1, X_2, X_3, X_4)$ has no light cycle cover.*

The proof of this result is by contradiction and requires a careful analysis of how a hypothetical light cycle cover of $Z(X_1, X_2, X_3, X_4)$ could traverse individual parts of $Z(X_1, X_2, X_3, X_4)$. The following lemma is the main tool of the analysis.

Lemma 1. *If \mathcal{C} is a light cycle cover of a cubic graph G , then every vertex v of G is traversed by exactly two cycles D_1 and D_2 of \mathcal{C} in such a way that one edge incident with v is contained only in D_1 , another edge incident with v is contained only in D_2 , and the third edge at v is contained in both D_1 and D_2 . In particular, $w(e) \leq 2$ for each edge e of G .*

We defer a detailed proof of Proposition 1 to our paper [15]. A significantly easier task than to prove Proposition 1 is to show that the $Z(X_1, X_2, X_3, X_4)$

cannot be covered with four perfect matchings. As mentioned earlier, the latter property is a necessary condition for a graph (or a multipole) not to have a light cycle cover (see [8, Theorem 5.4] and [16, Theorem 3.1]). The fact that $Z(X_1, X_2, X_3, X_4)$ cannot be covered with four perfect matchings can be established with the help of the theory of tetrahedral flows developed by the present authors in [14]. Theorem 8.6 and Corollary 5.5 from [14] are particularly useful for this purpose.

We proceed to the construction of the family $(G_k)_{k \geq 2}$.

Construction. Let B_k be an arbitrary 4-edge-connected 4-regular graph with $k \geq 2$ vertices; parallel edges are not excluded. Replace each vertex v of B_k with a copy Z_v of $Z(Q, Q, Q, Q)$, where Q is the robust dipole on eight vertices constructed from the Petersen graph, and identify the four dangling edges of Z_v with the four edges of B_v incident with v arbitrarily. Finally, for each edge $e = uv$ of B_k glue the dangling edge of Z_u corresponding to e with that of Z_v . Since each Z_v has 46 vertices, the resulting graph $G_k = G(B_k)$ has order $46k$. The simplest choice for the base graph B_k is the graph $C_k^{(2)}$ obtained from the k -cycle C_k by doubling each edge. Clearly, $C_k^{(2)}$ is 4-edge-connected.

Now we are ready for the proof of our main result.

Proof of Theorem 1. Let us consider the graph G_k for an arbitrary fixed integer $k \geq 2$. Obviously, G_k contains k vertex-disjoint copies of the dipole $Z = Z(Q, Q, Q, Q)$.

First of all, it is not difficult to check that each G_k is cyclically 4-edge-connected. Indeed, the edge-cut that separates any copy of $Z(Q, Q, Q, Q)$ from the rest of G_k is cycle-separating (that is, both components resulting from the removal of the cut contain cycles) and has size 4. On the other hand, from the construction of G_k it is clear that G_k has no bridges, 2-edge-cuts, and nontrivial 3-edge-cuts.

Next we show that every cycle cover of G_k has length at least $(\frac{4}{3} + \frac{1}{69}) \cdot |E(G_k)|$. To this end, recall that Z does not admit a light cycle cover, by Proposition 1. Let \mathcal{C} be an arbitrary cycle cover of G_k . Every copy of Z in G_k contains at least one vertex of weight at least 6 with respect to \mathcal{C} . Thus there will be at least k pairwise distinct vertices of weight at least 6 in G_k , one in each copy of Z . Since each vertex of G_k has weight at least 4, we have $\ell(\mathcal{C}) \geq (4 \cdot 46k + 2k)/2 = 93k$. As G_k has $\frac{3}{2} \cdot 46k = 69k$ edges, we obtain

$$\ell(\mathcal{C}) \geq \frac{93}{69} \cdot |E(G_k)| = \left(\frac{4}{3} + \frac{1}{69}\right) \cdot |E(G_k)|,$$

as required. □

3 Concluding Remarks

The constant $\frac{4}{3} + \frac{1}{69}$ occurring in the statement of Theorem 1 is very unlikely to be best possible. We therefore propose the following problem.

Problem 1. What is the supremum of covering ratios of all cyclically 4-edge-connected cubic graphs different from the Petersen graph?

Another natural question arises if connectivity 4 is raised to 5.

Problem 2. Does there exist a cyclically 5-edge-connected cubic graph different from the Petersen graph whose covering ratio is strictly greater than $4/3$?

Acknowledgements. The authors acknowledge partial support from the grants VEGA 1/0813/18 and APVV-19-0308.

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Powers of Hamilton Cycles of High Discrepancy Are Unavoidable

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Abstract. The Pósa-Seymour conjecture asserts that every graph on n vertices with minimum degree at least $(1 - 1/(r + 1))n$ contains the r^{th} power of a Hamilton cycle. Komlós, Sárközy and Szemerédi famously proved the conjecture for large n . The notion of discrepancy appears in many areas of mathematics, including graph theory. In this setting, a graph G is given along with a 2-coloring of its edges. One is then asked to find in G a copy of a given subgraph with a large discrepancy, i.e., with many more edges in one of the colors. For $r \geq 2$, we determine the minimum degree threshold needed to find the r^{th} power of a Hamilton cycle of large discrepancy, answering a question posed by Balogh, Csaba, Pluhár and Treglown. Notably, for $r \geq 3$, this threshold approximately matches the minimum degree requirement of the Pósa-Seymour conjecture.

Keywords: Graph theory · Discrepancy · Hamilton cycles

1 Introduction

Classical discrepancy theory studies problems of the following kind: given a family of subsets of a universal set \mathcal{U} , is it possible to partition the elements of \mathcal{U} into two parts such that each set in the family has roughly the same number of elements from each part? One of the first significant results in this area is a criterion for a sequence to be uniformly distributed in the unit interval proved by Hermann Weyl. Since then, discrepancy theory has had wide applicability in many fields such as ergodic theory, number theory, statistics, geometry, computer science, etc. For a comprehensive overview of the field, see the books by Beck and Chen [3] and by Chazelle [5].

This paper studies a problem in the discrepancy theory of graphs. To discuss the topic, we start with a definition.

Definition 1. Let $G = (V, E)$ be a graph and $f: E \rightarrow \{-1, 1\}$ a labelling of its edges. Given a subgraph H of G , we define its discrepancy $f(H)$ as

$$f(H) = \sum_{e \in E(H)} f(e).$$

Furthermore, we refer to the value $|f(H)|$ as the absolute discrepancy of H .

One of the central questions in graph discrepancy theory is the following. Suppose we are given a graph G and a spanning subgraph H . Does G , for every edge labelling $f: E(G) \rightarrow \{-1, 1\}$, contain an isomorphic copy of H of high absolute discrepancy with respect to f ? Erdős, Füredi, Loeb and Sós [6] proved the first result of this kind. They show that, given a tree on n vertices T_n with maximum degree Δ and a $\{-1, 1\}$ -coloring of the edges of the complete graph K_n , one can find a copy of T_n with absolute discrepancy at least $c(n - 1 - \Delta)$, for some absolute constant $c > 0$.

A commonly studied topic in extremal combinatorics are *Dirac-type problems* where one is given a graph G on n vertices with minimum degree at least αn and wants to prove that G contains a copy of a specific spanning subgraph H . In the discrepancy setting it is natural to ask whether we can also find a copy of H with large absolute discrepancy. Balogh, Csaba, Jing and Pluhár studied this problem for spanning trees, paths and Hamilton cycles. Among other results, they determine the minimum degree threshold needed to force a Hamilton cycle of high discrepancy.

Theorem 1 (Balogh, Csaba, Jing and Pluhár[1]). *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n -vertex graph with*

$$\delta(G) \geq (3/4 + c)n$$

and $f: E(G) \rightarrow \{-1, 1\}$, then G contains a Hamilton cycle with absolute discrepancy at least $cn/32$ with respect to f . Moreover, if 4 divides n , there is an n -vertex graph with $\delta(G) = 3n/4$ and an edge labelling $f: E(G) \rightarrow \{-1, 1\}$ such that any Hamilton cycle in G has discrepancy 0 with respect to f .

Very recently, Freschi, Hyde, Lada and Treglown [7], and independently, Gishboliner, Krivelevich and Michaeli [8] generalized this result to edge-colorings with more than two colors.

A fundamental result in extremal graph theory is the Hajnal-Szemerédi theorem. It states that if r divides n and G is a graph on n vertices with $\delta(G) \geq (1 - 1/r)n$, then G contains a perfect K_r -tiling, i.e. its vertex set can be partitioned into disjoint cliques of size r . Balogh, Csaba, Pluhár and Treglown [2] proved a discrepancy version of this theorem.

Theorem 2 (Balogh, Csaba, Pluhár and Treglown[2]). *Suppose $r \geq 3$ is an integer and let $\eta > 0$. Then there exists $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices where r divides n and where*

$$\delta(G) \geq \left(1 - \frac{1}{r+1} + \eta\right)n.$$

Given any function $f: E(G) \rightarrow \{-1, 1\}$ there exists a perfect K_r -tiling \mathcal{T} in G so that

$$\left| \sum_{e \in E(\mathcal{T})} f(e) \right| \geq \gamma n.$$

The r^{th} power of a graph G is the graph on the same vertex set in which two vertices are joined by an edge if and only if their distance in G is at most r . The Pósa-Seymour conjecture asserts that any graph on n vertices with minimum degree at least $(1 - 1/(r + 1))n$ contains the r^{th} power of a Hamilton cycle. Komlós, Sárközy and Szemerédi [12] proved the conjecture for large n . In [2] the authors posed the question of determining the minimum degree needed to force the r^{th} power of a Hamilton cycle with absolute discrepancy linear in n . Because the r^{th} power of a Hamilton cycle contains a (almost) perfect $(r + 1)$ -tiling, they suggested the minimum degree required should be $(1 - 1/(r + 2) + \eta)n$, based on their result for K_r -tilings. We prove this value is correct for $r = 2$. However, we show that for $r \geq 3$, a minimum degree of $(1 - 1/(r + 1) + \eta)n$, for arbitrarily small $\eta > 0$, is sufficient, approximately matching the minimum degree required for finding any r^{th} power of a Hamilton cycle. As far as the author knows, this is the first Dirac-type discrepancy result in which the threshold for finding a spanning subgraph of large discrepancy is the same, up to an arbitrarily small linear term, as the minimum degree required for finding any copy of the subgraph.

Theorem 3. *For any integer $r \geq 3$ and $\eta > 0$, there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. Suppose a graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (1 - 1/(r + 1) + \eta)n$ and an edge coloring $f: E(G) \rightarrow \{-1, 1\}$ are given. Then in G there exists the r^{th} power of a Hamilton cycle H^r satisfying*

$$\left| \sum_{e \in E(H^r)} f(e) \right| \geq \gamma n.$$

Interestingly, the minimum degree needed for finding the r^{th} power of a Hamilton cycle of large discrepancy is the same for $r \in \{1, 2, 3\}$ and equals $(\frac{3}{4} + \eta)n$. The cases $r = 1, 3$ being resolved in [1] and by the previous theorem, respectively, we also prove this for $r = 2$.

Theorem 4. *For any $\eta > 0$, there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. Suppose a graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (3/4 + \eta)n$ and an edge coloring $f: E(G) \rightarrow \{-1, 1\}$ are given. Then in G there exists the square of a Hamilton cycle H^2 satisfying*

$$\left| \sum_{e \in E(H^2)} f(e) \right| \geq \gamma n.$$

These results are tight in the following sense. If we weaken the minimum degree requirement by replacing the term ηn with a sublinear term, then there are examples in which any r^{th} power of a Hamilton cycle has absolute discrepancy $o(n)$.

2 Lower Bound Examples

We present simple lower bound constructions showing our results are best possible in a certain sense. For $\eta = \eta(n) = o(1)$, the condition $\delta(G) \geq (1 - \frac{1}{r+1} + \eta)n$

when $r \geq 3$ or $\delta(G) \geq (3/4 + \eta)n$ when $r = 2$, is not enough to guarantee an r^{th} power of a Hamilton cycle with absolute discrepancy linear in n . Moreover, for $\eta = 0$, there exists a graph in which every r^{th} power of a Hamilton cycle has discrepancy 0.

First consider $r \geq 3$. We construct a graph G as follows. Let t be even and V_1, \dots, V_{r+1} disjoint clusters of size t . Additionally, let V_0 be a cluster of size m . We construct a graph on the vertex set $V = \bigcup_{i=0}^{r+1} V_i$. We put an edge between any two vertices from different clusters and we put all edges connecting two vertices in V_0 . Let $n = |V_0| = (r + 1)t + m$ and note that $\delta(G) = rt + m = \left(1 - \frac{1}{r+1} + \frac{m}{(r+1)((r+1)t+m)}\right)n$.

Next we describe the coloring f of the edges. We color the edges incident to vertices in V_0 arbitrarily. For each $V_i, i \geq 1$ we denote half of its vertices as *positive* and the other half as *negative*. For a vertex $v \in V_i$ and any vertex $u \in V_j$ where $1 \leq j < i$ we set $f(v, u) = 1$ if v is positive and $f(v, u) = -1$ if v is negative.

Let H^r be an r^{th} power of a Hamilton cycle in G viewed as a $2r$ -regular subgraph of G . Call a vertex $v \in V \setminus V_0$ a *bad* vertex if at least one of its neighbours in H^r is from the cluster V_0 , otherwise call it *good*. If a vertex $v \in V_i$ is good then in H^r it has precisely two neighbours from each of the clusters $V_j, 1 \leq j \leq r + 1, j \neq i$. Note that for $i \geq 1$ at most 2 vertices from V_i can be adjacent to a vertex $v \in V_0$, so there are at most $2m$ bad vertices in V_i . Now consider only positive good vertices and their edges towards vertices from clusters with a smaller index. Thus, the number of edges labelled 1 in H^r is at least

$$\sum_{i=1}^{r+1} 2(i - 1)(t/2 - 2m) = r(r + 1)(t/2 - 2m).$$

Hence, we have

$$f(H^r) \geq -nr + 2r(r + 1)(t/2 - 2m) \geq -5r(r + 1)m.$$

Completely analogously, $f(H^r) \leq 5r(r + 1)m$. Therefore, if $m = 0$, we have $f(H^r) = 0$ and if $m = o(n)$, we get $|f(H^r)| = o(n)$.

For $r = 2$, the following construction was given in [1], where the case $r = 1$ was considered. Let G be the 4-partite Turán graph on $n = 4k$ vertices, so $\delta(G) = \frac{3}{4}n$. Color all edges incident to one of the parts with -1 and the rest with 1. Any square of a Hamilton cycle contains $4k$ edges labelled -1 , exactly 4 edges for each vertex in the special class. As it has a total of $8k$ edges, its discrepancy is 0. Similarly as above, we can add $m = o(n)$ vertices connected to every other vertex and still any square of a Hamilton cycle has absolute discrepancy $o(n)$.

3 Outline of the Proofs

Our proof follows a very similar structure to that of Balogh, Csaba, Pluhár and Treglown [2] for perfect K_r -tilings.

We start by applying the regularity lemma on G to obtain the reduced graph R and the corresponding edge labelling f_R .

Before proving the conjecture for large n , Komlós, Sárközy and Szemerédi [11] proved an approximate version, namely they proved it for n -vertex graphs with minimum degree at least $(1 - 1/(r + 1) + \varepsilon)n$. We make slight modifications to their proof to establish two important claims.

We prove that a perfect K_{r+1} -tiling of R with linear discrepancy with respect to f_R can be used to construct an r^{th} power of a Hamilton cycle in G with linear discrepancy with respect to f . Combined with Theorem 2, this is enough to deduce the case $r = 2$ (Theorem 4).

To discuss the second claim, we need a definition.

Definition 2 (C^r -template). *Let F be a graph. A C^r -template of F is a collection $\mathcal{F} = \{C_1, C_2, \dots, C_s\}$ of not necessarily distinct cycles whose r^{th} powers are subgraphs of F . In a C^r -template each vertex appears the same number of times. Moreover, we require that each cycle C_i has length between $r + 1$ and $10r^2$. The discrepancy of a C^r -template is given as $f(\mathcal{F}) = \sum_{i=1}^s f(C_i^r)$, where for a cycle $C = (v_1, \dots, v_k)$, we define*

$$f(C^r) = \sum_{i=1}^k \sum_{j=1}^r f(v_i, v_{i+j}),$$

where we identify $v_{k+j} = v_j$, $1 \leq j \leq r$.

In our context, a C^r -template can be viewed as a generalization of a K_{r+1} -tiling. We prove a very useful property of R : suppose that F is a small subgraph of R and there are two C^r -templates of F covering the vertices the same number of times, but having different discrepancies with respect to f_R . Then in G there exists the r^{th} power of a Hamilton cycle with linear discrepancy with respect to f .

From this point on, we only ‘work’ on the reduced graph R . To use the last claim, we need a subgraph F on which we can find two different C^r -templates, so the simplest subgraph we can study is an $(r + 2)$ -clique. We prove that every $(r + 2)$ -clique in R is either monochromatic or such that one of the colors induces a copy of $K_{1,r+1}$ and the other a copy of K_{r+1} . As R has large minimum degree, every clique of size $k \leq r + 2$ can be extended to a clique of size $r + 2$. This shows that every clique of size $k \leq r + 2$ is of the same form: either it is monochromatic, or one of the colors induces a copy of $K_{1,k-1}$ and the other a copy of K_{k-1} .

We assume G has no r^{th} power of a Hamilton cycle with large absolute discrepancy. By the Hajnal-Szemerédi theorem, we can find a perfect K_{r+1} -tiling \mathcal{T} of R . The previous arguments show that only four types of cliques appear in \mathcal{T} and \mathcal{T} has a small discrepancy with respect to f_R . This tells us that the numbers of each of the four types of cliques in \mathcal{T} are balanced in some way.

We consider two cliques in \mathcal{T} of different types. For several relevant cases when there are many edges between the two cliques, we construct two C^r -templates of different discrepancies, which contradicts our assumption by the

claim about C^r -templates. Finally, this restricts the number of edges between different cliques which leads to a contradiction with the minimum degree assumption on R .

Acknowledgment. The author would like to thank Miloš Trujić for introducing him to the problem, helpful discussions and many useful comments. Additionally, the author is grateful to the anonymous referees whose feedback helped improve the clarity of this extended abstract.

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Low Diameter Algebraic Graphs

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Abstract. Let D be a division ring, n a positive integer, and $\text{GL}_n(D)$ the set of invertible square matrices of size n and values in D , called the general linear group. We address the intersection graph of subgroups of $\text{GL}_n(D)$ and prove that it has diameter at most 3. Two particular cases of its induced subgraphs are then investigated: by cyclic subgroups, and by almost subnormal subgroups. We prove that the latter case results in a connected graph whose diameter is sharply bounded by 2. In the former case, we completely characterise the connectivity of the induced graph with respect to D , where, in case of connectivity, we prove that it has diameter at most 7 in general, and at most 5 if D is a locally finite field of characteristic not 2 different from \mathbb{F}_3 and \mathbb{F}_9 .

Keywords: Division ring · Intersection graph · Graph connectivity

1 Introduction

Geographically, human activities can naturally be embedded into an Euclidean space \mathbb{R}^n with $n = 3$ or $n = 2$, by extending infinitely the region on Earth under study and by projecting latitude, longitude and altitude coordinates of that region on the space. We formalise graphs arising from Euclidean spaces as in the sense of intersection graphs [18]: given a set of geometric objects in the space, the corresponding intersection graph has these objects as vertices, and the vertices are adjacent if and only if the objects have a non-trivial non-empty intersection. When the space is a plane, that is, $n = 2$, intersection graph theory sows the way for important research results: social network simulations can stem from the assumption where adjacency likeliness increases in function of the geographic distance of disks in the space [21], planar graph characterisation and coloring can rely on intersection of segments in the space [6, 20], and RNA

Supported by Courtanet – Sorbonne Université convention C19.0665 and ANRT grant 2019.0485.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 465–471, 2021.
https://doi.org/10.1007/978-3-030-83823-2_73

structures can benefit from approximations using the proximity of intervals (and pairs of intervals) on a line [7, 12].

More generally, geometric graphs are well captured in the (d, e) -subspace intersection model proposed in Laison-Qing's classification [17]. Following these termes, d denotes the space dimension and e the dimension of the geometric objects: Penrose's random geometric networks belong to $(2, 2)$ -subspace intersection graphs, Scheinerman planar graph characterisation to $(2, 1)$ -subspace intersection graphs, and RNA approximations to $(1, 1)$ -subspace intersection graphs.

Generalising further, an important step has been taken by Yaraneri's studies on submodules [22]. Here, the real field \mathbb{R} is left behind in favor of ring structures. Rings are important because cryptosystems can rely on hardness results of their isomorphisms [16] in order to define new cybersecurity schemes [4, 13]. Historically, studying intersection graphs of elements of a ring has been initiated by Bosák [3] and further developed with Csákány-Pollák's theorems [8]. Following these works, many research efforts have been put in studying the structure of intersection graphs stemming from group structures [5, 11]. However, to the best of our knowledge, very little results are known for the most general case of a division ring D . This is precisely the shortage where Yaraneri's results help rectifying when studying the intersection graph of submodules of a (left/right) module over a division ring D . For a simpler introduction, let us ignore the case of infinite dimension, and define D_R^n as a right D -module of dimension n . Then, in a nut shell, column vectors of module D_R^n over ring D are natural extensions of vectors of space \mathbb{R}^n over the field \mathbb{R} of real numbers. This way, one can retrieve most of the nice topology results from the segments of a plane in the submodules of a module [22]. However, Yaraneri left unstudied the extension of column vectors to matrices over D . In particular, the graph structure of ring morphisms is not well understood. In this paper, we help paving the road toward this research direction with invertible square matrices.

More precisely, we lift the intersection model of geometric graphs to the algebraic graph defined by non-trivial intersections of proper subgroups of the general linear group $\text{GL}_n(D)$, that we denote by graph $\Gamma(\text{GL}_n(D))$. Here, the general linear group is the set of invertible square matrices of size n . Its subgroups are sets of invertible square matrices closed under multiplication and inversion. When $n \geq 2$ and $D = F$ is a field with at least 3 elements, Bien and Viet recently proved that $\Gamma(\text{GL}_n(F))$ has diameter either 2 or 3 [2].

In Sect. 2, we extend Bien-Viet's theorem from fields to arbitrary division rings, and prove furthermore for $n = 1$ that $\Gamma(\text{GL}_1(D))$ with D infinite has diameter at most 3, and exactly 2 when D is so-called weakly locally finite. Proving these results requires a careful analysis of namely the non-central subgroups of $\text{GL}_n(D)$. Thus, we push further in Sect. 3 our analysis on prominent ring structures with a non-trivial center. One of our major results is that they need not be complex: the non-central almost subnormal subgroups, when exist, always induce a clique, while the almost subnormal subgroups induce a subgraph of diameter bounded by a surprisingly sharp 2. We then investigate the subgraph $\Gamma_c(\text{GL}_n(D))$ induced by cyclic subgroups of $\text{GL}_n(D)$, and found a very

close relationship with the proper power graph $P^*(\text{GL}_n(D))$. This helps characterising the graph's connectivity, in which case we show that it has diameter at most 7. Examining particular cases, when $n = 2$ and D is either Galois fields \mathbb{F}_3 or \mathbb{F}_9 , we prove that $\Gamma_c(\text{GL}_n(D))$ is not connected. Surprisingly, when $n \geq 3$ and D is either above Galois fields, we result in that $\Gamma_c(\text{GL}_n(D))$ is connected, therefore, has diameter at most 7. Except for the previous cases, when $n \geq 2$ and $D = F$ is a locally finite field of characteristic not 2, we prove that the diameter of $\Gamma_c(\text{GL}_n(D))$ is at most 5. In Sect. 4, we close the paper with concluding remarks and perspectives for research on algebraic graphs. By space restriction, properties marked with (\star) are given without a proof.

2 General Linear Groups

For a loopless simple undirected (not necessarily finite) graph Γ , we note $V(\Gamma)$ its vertex set, $E(\Gamma) \subseteq \binom{V(\Gamma)}{2}$ its edge set, and $u \sim v$ every edge $\{u, v\} \in E(\Gamma)$. A path joining u to v of length k is a sequence of $k + 1$ distinct vertices $u = v_0 \sim v_1 \sim v_2 \sim \dots \sim v_k = v$. The geodesic distance $d(u, v)$ between u and v is $+\infty$ if no such path exists, and the minimum length of such a path otherwise. Graph Γ is connected if there exists a path joining any pair of its vertices. When Γ is connected, if moreover the set $\{d(u, v) \mid u, v \in V, u \neq v\}$ is bounded, then we define the diameter of Γ as $\text{diam}(\Gamma) = \max\{d(u, v) \mid u, v \in V, u \neq v\}$.

The intersection graph $\Gamma(G)$ of a group G has as vertex set the non-trivial proper subgroups of G . Two vertices of $\Gamma(G)$ are adjacent if and only if their intersection is not reduced to $\{1_G\}$. A division ring D is a ring where every non-zero element is invertible. In particular, D can as well be non-commutative or commutative, where it is called a field. The general linear group $\text{GL}_n(D)$ is the set of invertible square matrices of size $n \geq 1$ and values in D . The center of D is its subset containing every element $c \in D$ with $cx = xc$ for every $x \in D$. When $n = 1$, the general linear group coincides with the multiplicative group, and is noted $D^* = \text{GL}_1(D)$. The Galois field with q elements is noted \mathbb{F}_q .

Theorem 1 (\star) . *Let D be a division ring whose center F contains at least 3 elements, and n an integer. When $n = 1$, assume additionally that D is infinite. Then, for every $n \geq 1$, the intersection graph $\Gamma(\text{GL}_n(D))$ of subgroups of the general linear group $\text{GL}_n(D)$ has diameter at most 3. Moreover, if $n \geq 2$, then $\Gamma(\text{GL}_n(D))$ has diameter at least 2. When $n = 1$, the requirement for D to be infinite cannot be relieved: $\Gamma(\mathbb{F}_7^* = \text{GL}_1(\mathbb{F}_7))$ is not a connected graph.*

In the sequel, we compare Theorem 1 with known results when $D = F$ is a field [2]. A division ring D with center F is weakly locally finite if for every finite subset S of D the division subring $F(S)$ generated by S over F is a finite dimensional vector space over its center [9]. Theorem 2 below generalizes [2, Theorem 4.2], which holds for $n \geq 2$ and $D = F$ a field. Both theorems contrast sharply with the finite case of a field $D = F$ with $D^* = \text{GL}_1(D)$ of prime order, where the diameter is exactly 3 instead of 2 [2, Proposition 3.5]. We stress that every locally finite division ring is weakly locally finite [14, Theorem 4], however, the converse is not necessarily true, for infinitely many instances [10].

Theorem 2 (\star). *Let D be an infinite weakly locally finite division ring. For every $n \geq 1$, $\Gamma(\text{GL}_n(D))$ has diameter exactly 2.*

3 Almost Subnormal Subgroups and Cyclic Subgroups

A subgroup N of a group G is *almost subnormal* if there exists a sequence of subgroups $N = N_r < N_{r-1} < N_{r-2} < \dots < N_1 < N_0 = G$ such that for every $r \geq i > 0$, either N_i is a normal subgroup of N_{i-1} , noted $N_i \triangleleft N_{i-1}$, or the index $[N_{i-1} : N_i]$ is finite, as in the sense of [15]. Following a previous work [19], we conjecture that the almost subnormal subgroups of $\text{GL}_n(D)$ induce a clique in $\Gamma(\text{GL}_n(D))$. When $n \geq 2$, [19, Theorem 3.3] implies that the almost subnormal subgroups of $\text{GL}_n(D)$ are also normal subgroups, and the conjecture follows. Unfortunately, for $n = 1$ the almost subnormal subgroups are not necessarily normal subgroups, leaving the conjecture whether they form a clique in $D^* = \text{GL}_1(D)$ unanswered. Our main contribution in this topic is a tailored proof for below Theorem 3, bridging the gap for $n = 1$ with a positive answer.

Theorem 3 (\star). *For any division ring D , the family of non-central almost subnormal subgroups of $D^* = \text{GL}_1(D)$ is closed under intersection. It induces a clique in $\Gamma(D^*)$ when not empty. The family of almost subnormal subgroups of D^* induces a connected subgraph of diameter at most 2. The bound is sharp.*

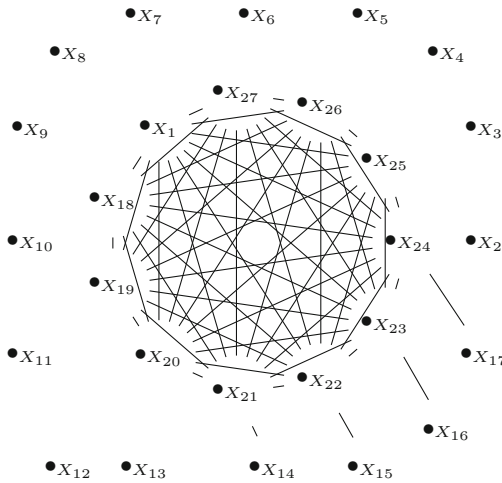


Fig. 1. Graph $\Gamma_c(\text{GL}_2(\mathbb{F}_3))$.

The *cyclic subgroup intersection graph* $\Gamma_c(G)$ of a group G is the subgraph of $\Gamma(G)$ induced by all cyclic subgroups of G . A field is said to be *locally finite*

if every finitely generated subfield over its prime subfield is finite. Theorem 4 below goes over our results on cyclic subgroups. In particular, for every $n \geq 3$ we found that $\Gamma_c(\text{GL}_n(\mathbb{F}_3))$ has diameter at most 7. However, we result surprisingly in that $\Gamma_c(\text{GL}_2(\mathbb{F}_3))$ is disconnected, cf. Fig. 1.

Theorem 4 (\star). *Let $n \geq 2$ be an integer and D a division ring of characteristic not 2. If $\Gamma_c(\text{GL}_n(D))$ is connected, then D is a locally finite field. Conversely, if F is a locally finite field whose characteristic is not 2, then the following statements hold.*

1. Both $\Gamma_c(\text{GL}_2(\mathbb{F}_3))$ and $\Gamma_c(\text{GL}_2(\mathbb{F}_9))$ are disconnected.
2. If $n \geq 3$, then both $\Gamma_c(\text{GL}_n(\mathbb{F}_3))$ and $\Gamma_c(\text{GL}_n(\mathbb{F}_9))$ have diameter at most 7.
3. In all other cases, $\Gamma_c(\text{GL}_n(F))$ has diameter at most 5.

The characteristic requirement cannot be omitted: $\Gamma_c(\text{GL}_2(\mathbb{F}_2))$ is disconnected.

Theorem 4 also helps answering positively to below Question 1 when the group is the general linear group. The *power graph* $P(G)$ of a group G has G as vertex set, and two distinct vertices are adjacent if and only if one of them is a positive power of the other. The *proper power graph* $P^*(G)$ of G is the subgraph of $P(G)$ induced by all non-identity elements of G . Note for a periodic group G that $P(G)$ has diameter at most 2 since every vertex is connected to 1_G . However, $P^*(G)$ is not necessarily connected, for instance when $G = S_3$ is the symmetric group of degree 3.

Question 1. [1, Question 39] Which groups do have the property that the proper power graph is connected?

Theorem 5 (\star). *Given an integer $n \geq 2$ and let D be a division ring of characteristic not 2. Then, the following conditions are equivalent:*

1. The graph $P^*(\text{GL}_n(D))$ is connected.
2. The graph $\Gamma_c(\text{GL}_n(D))$ is connected.
3. $D = F$ is a locally finite field, and $\text{GL}_n(F) \neq \text{GL}_2(\mathbb{F}_3); \text{GL}_2(\mathbb{F}_9)$.

4 Conclusion and Perspectives

We show that most intersection graphs of subgroups of the general linear group over a division ring have a low diameter. We hope this can help understanding ring morphisms and leave open the question whether cyclic subgroups induce clique-like components, like those of $\text{GL}_2(\mathbb{F}_3)$ presented in Fig. 1.

Acknowledgements. We are grateful to the anonymous reviewers for their helpful insights and pointers.

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Complexity of $3 + 1/m$ -coloring P_t -free Graphs

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Abstract. The 4-coloring problem is NP-complete for P_7 -free graphs whereas the 3-coloring problem can be solved in quasi-polynomial time on P_t -free graphs for any fixed t . We consider circular coloring to locate precisely the complexity gap between 3 and 4 colors: for every fixed integer $m \geq 2$, the $3 + 1/m$ -coloring problem is NP-complete on P_{30} -free graphs.

Keywords: Graph homomorphism · NP-completeness

1 Introduction

We consider the complexity of coloring problems restricted to hereditary graph classes.

Theorem 1. [6] *Let H be a (fixed) graph, and let $k \geq 3$. If the k -coloring problem can be solved in polynomial time when restricted to the class of H -free graphs, then every connected component of H is a path.*

Thus if H is connected, then the question of determining the complexity of k -coloring H -free graph is reduced to studying the complexity of k -coloring P_t -free graphs for certain t . Let us review the main results obtained on this topic.

Theorem 2. [2,5] *For all $k \geq 5$, the k -coloring problem can be solved in polynomial time for the class of P_5 -free graphs and is NP-complete for the class of P_6 -free graphs.*

Theorem 3. [3,5] *The 4-coloring problem can be solved in polynomial time for the class of P_6 -free graphs and is NP-complete for the class of P_7 -free graphs.*

Theorem 4. [1,7] *The 3-coloring problem can be solved in polynomial time for the class of P_7 -free graphs. For every fixed t , the 3-coloring problem can be solved for the class of P_t -free graphs in $n^{O(\log^2 n)}$.*

P. Ochem—This work is supported by the ANR project HOSIGRA (ANR-17-CE40-0022).

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 472–476, 2021.
https://doi.org/10.1007/978-3-030-83823-2_75

Thus, for every integer $t \geq 7$, the k -coloring problem on P_t -free graphs is NP-complete for $k \geq 4$ and seems easier for $k = 3$. We narrow this complexity gap between $k = 3$ and $k = 4$ using the framework of circular coloring.

Recall that if $q = \frac{n}{d}$ is a rational number, then the circular clique K_q is the graph with vertex set $\{x_0, x_1, \dots, x_{n-1}\}$ such that x_u and x_v are adjacent if and only if $d \leq |u - v| \leq n - d$. The circular chromatic number $\chi_c(G)$ is the smallest q such that G admits a homomorphism to K_q .

Theorem 5. *For every fixed integer $m \geq 2$, deciding whether a P_{30} -free graph maps to $K_{3+\frac{1}{m}}$ is NP-complete.*

2 Proof

We reduce from 3-SAT. Let x_0, \dots, x_{3m} be the vertices/colors of the target graph $K_{3+\frac{1}{m}}$ in cyclic order. The *pseudo distance* between two vertices x_i and x_j is the distance in the chordless cycle x_0, \dots, x_{3m} . Two vertices of $K_{3+\frac{1}{m}}$ are *consecutive* if their pseudo distance is 1. A k -*vertex* is a vertex of degree k . We split the proof into the cases $m \geq 3$ and $m = 2$. In each case, we construct a graph G from an instance I of 3-SAT.

The case $m \geq 3$.

- (1) For every variable v of I , the variable gadget consists of two non-adjacent vertices v and \bar{v} called *variable-literals* as they correspond to the literals of the variable v .
- (2) For every clause c of I , the clause gadget is a copy of a 9-cycle with vertices $c_0, m_1, m_2, c_3, m_4, m_5, c_6, m_7, m_8$. The vertices c_i are called *clause-literals* and correspond to the three literals of the clause c . The vertices m_i are called *middle* vertices.
- (3) Every middle vertex is adjacent to every variable-literal.
- (4) Every clause-literal is joined to the corresponding variable gadget by a *literal-gadget* depicted in Fig. 1. The literal-gadget consists of a copy of $K_{3+\frac{1}{m}}$ represented by a dotted circle and two vertices of degree 1 and 2. The two-headed arrows indicate the pseudo distance between the considered vertices of $K_{3+\frac{1}{m}}$. The 1-vertex labelled ℓ is identified with the clause-literal. If $\ell = v$, then the other vertex labelled ℓ in $K_{3+\frac{1}{m}}$ is identified with the variable-literal v and the vertex labelled $\bar{\ell}$ is identified with the variable-literal \bar{v} . And vice-versa if $\ell = \bar{v}$.
- (5) Finally, G contains a copy of $K_{3+\frac{1}{m}}$ called the *synchronizer*. The vertices x_0 and x_{3m} of the synchronizer are adjacent to every middle vertex. For every clause-literal c_i , we add a new 2-vertex adjacent to c_i and to the vertex x_{m+1} of the synchronizer.

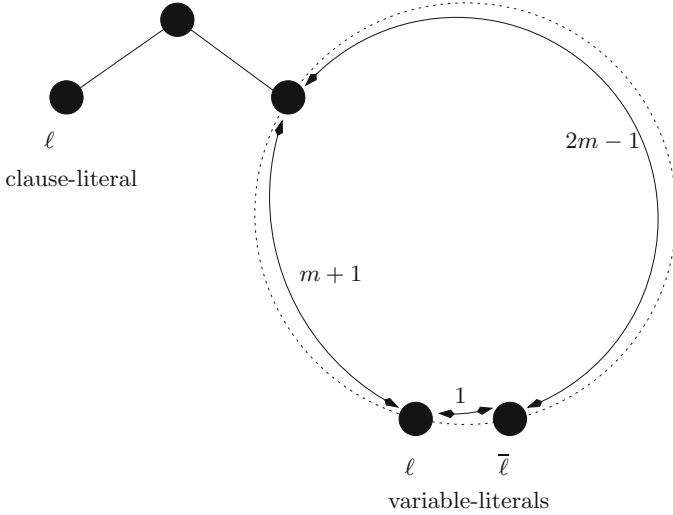


Fig. 1. The literal-gadget for $m \geq 3$.

Let h be a homomorphism from G to $K_{3+\frac{1}{m}}$. Without loss of generality we can assume that every vertex x_i of the synchronizer maps to the vertex x_i of $K_{3+\frac{1}{m}}$. The graph induced by the middle vertices is a matching. By (5), every edge of this matching maps to the edge $x_m x_{2m}$. For every variable v , the variable-literals v and \bar{v} satisfy $\{h(v), h(\bar{v})\} \subset \{x_0, x_{3m}\}$ by (3). Since v or \bar{v} occurs in at least one clause, v and \bar{v} are consecutive in at least one literal-gadget. Thus $\{h(v), h(\bar{v})\} = \{x_0, x_{3m}\}$. Assigning true (resp. false) to the variable v corresponds to setting $h(v) = x_0$ and $h(\bar{v}) = x_{3m}$ (resp. $h(v) = x_{3m}$ and $h(\bar{v}) = x_0$). By (5), every clause-literal c_i is connected to x_{m+1} via a walk of length two, so that $h(c_i) \in \{x_0, x_1, \dots, x_{2m+1}, x_{2m+2}\}$. If a literal is assigned true, then the clause-literal forces the same coloring constraints as the synchronizer on c_i , that is, $h(c_i) \in \{x_0, \dots, x_{2m+2}\}$. If a literal is assigned false, then the coloring constraints due to the clause-literal on c_i are symmetrical to the constraints due to the synchronizer with respect to the involution which maps x_i to x_{3m-i} . In this case, $h(c_i) \in \{x_0, \dots, x_{2m+2}\} \cap \{x_{m-2}, \dots, x_{3m}\} = \{x_{m-2}, \dots, x_{2m+2}\}$.

Now we argue that I is satisfiable if and only if G maps to $K_{3+\frac{1}{m}}$. It is sufficient to check that a clause is satisfied if and only if the clause gadget maps to $K_{3+\frac{1}{m}}$. If a clause is satisfied, then at least one literal, say the one corresponding to c_0 , is true. So we can set $h(c_0) = x_0$, $h(m_1) = h(c_3) = h(m_5) = h(m_7) = x_m$, and $h(m_2) = h(m_4) = h(c_6) = h(m_8) = x_{2m}$. If a clause is not satisfied, then in particular the 9-cycle of the clause gadget has to map to the subgraph of $K_{3+\frac{1}{m}}$ induced by $\{x_{m-2}, \dots, x_{2m+2}\}$. If $m \geq 5$, the contradiction is straightforward since this subgraph is bipartite. In the general case $m \geq 3$, recall that the three

edges of the clause gadget whose extremities are middle vertices must map to the edge $x_m x_{2m}$. So at least one of the clause-literals, say c_0 , is adjacent to a middle vertex colored x_m and a middle vertex colored x_{2m} . This implies that $h(c_0) \in \{x_0, x_{3m}\}$, which contradicts that $h(c_i) \in \{x_{m-2}, \dots, x_{2m+2}\}$.

The case $m = 2$. It corresponds to the homomorphism to $K_{\frac{7}{2}}$, which is known to be NP-complete for planar graphs [4]. The previous construction does not work with $m = 2$ because the constraint $h(c_i) \in \{x_{m-2}, \dots, x_{2m+2}\}$ for a false literal becomes $h(c_i) \in \{x_0, \dots, x_{3m}\}$, which gives no constraint.

We consider the graph N depicted in Fig. 2. Every homomorphism h of N to $K_{\frac{7}{2}}$ satisfies $h(x) \neq h(y)$. Conversely, for every distinct vertices a and b in $K_{\frac{7}{2}}$, N admits a homomorphism h such that $h(x) = a$ and $h(y) = b$. We say that N links two vertices of G if one vertex is identified with x and the other vertex is identified with y .

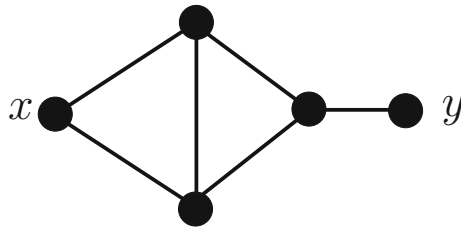


Fig. 2. The graph N .

We use the previous construction of G with $m = 2$, except for the following modifications.

- (a) The variable gadget consists of a copy of $K_{\frac{7}{2}}$ with two specified consecutive vertices as the variable-literals v and \bar{v} .
- (b) The *literal-gadget* consists of a copy of N linking the clause-literal corresponding to ℓ and the variable-literal corresponding to $\bar{\ell}$.
- (c) For every clause-literal c_i , we add a copy of N linking c_i and the vertex x_6 of the synchronizer.

Again, true corresponds to x_0 and false corresponds to x_6 . If c_i is assigned to true, then $h(c_i) \in \{x_0, \dots, x_5\}$ and $h(c_i) \in \{x_1, \dots, x_5\}$ otherwise. Every edge whose extremities are middle vertices must map to the edge $x_2 x_4$. If a clause is satisfied, then at least one literal, say c_0 , is true. So we can set $h(c_0) = x_0$, $h(m_1) = h(c_3) = h(m_5) = h(m_7) = x_2$, and $h(m_2) = h(m_4) = h(c_6) = h(m_8) = x_4$. If a clause is not satisfied, then at least one of the clause-literals, say c_0 , is adjacent to a middle vertex colored x_2 and a middle vertex colored x_4 . This implies that $h(c_0) \in \{x_0, x_6\}$, which contradicts that $h(c_i) \in \{x_1, \dots, x_5\}$.

P_t-freeness. In our construction of G , we have not tried to minimize the length of the longest induced path. Also, we do not try to determine exactly the smallest t such that G is P_t -free, which should depend on m anyway. We only sketch the arguments that G is P_{30} -free for every m .

We say that a vertex of G is *gentle* if it is a variable-literal or x_0 or x_{3m} of the synchronizer. If H is a copy of $K_{3+\frac{1}{m}}$ in G , we say that a vertex v of H is a *gate* if v has a neighbor outside of H .

It is not hard to check that $K_{3+\frac{1}{m}}$ is $3K_2$ -free. In particular, it is P_8 -free. Also, every copy H of $K_{3+\frac{1}{m}}$ in G has at most 3 gates, two of them being gentle.

If an induced path of G contains $a \geq 1$ middle vertices and $b \geq 1$ gentle vertices, then $a + b \leq 3$, since otherwise the path would contain $K_{1,3}$ or C_4 as a subgraph. So G cannot contain a very long path, since it would have to go through many clause gadgets and variable gadgets.

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Integer Covering Problems and Max-Norm Ramsey Theorems

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Abstract. We prove a conjecture of Schmidt–Tuller on the optimal covering of \mathbb{Z} by translated copies of a three-point set. We relate linear coverings to some problems in high-dimensional geometric Ramsey theory. More concretely, we obtain several essentially tight bounds on the chromatic number of \mathbb{R}^n with the max-norm for a large class of forbidden monochromatic configurations.

Keywords: Coverings · Ramsey theory · Chromatic number

1 Integer Covering Problems

Covering problems are among the most important elements that forms the view of modern combinatorial geometry. The main question here is the following: given $S, T \subset \mathbb{R}^d$, what is the ‘economical’ way to cover T by the translates of S ? This question has been extensively studied in case when $T = \mathbb{R}^d$ and S is a bounded convex body by Rogers [20], Erdős and Rogers [6], Tóth [25], Kuperberg [12] and many others.

In the present paper we deal with another important case of this general problem, namely when T is \mathbb{Z}^d and S is a finite nonempty subset of T (see, e.g., Newman [16, 17], Schmidt and Tuller [22, 23], and Bollobás, Janson, and Riordan [3]). In this setting, a set $A \subset \mathbb{Z}^d$ is called S -covering if $S + A = \mathbb{Z}^d$, where $S + A = \{s + a : s \in S, a \in A\}$. Similarly, a set $A \subset \mathbb{Z}^d$ is called $(S, -S)$ -covering if both sets $S + A$ and $-S + A$ coincide with \mathbb{Z}^d . We define the values $d_c(\mathbb{Z}^d; S)$ and $d_c(\mathbb{Z}^d; S, -S)$ as the minimum possible lower density of S -covering and $(S, -S)$ -covering sets respectively. Note that since every $(S, -S)$ -covering set is clearly also S -covering by definition, it is easy to see that

$$d_c(\mathbb{Z}^d; S) \leq d_c(\mathbb{Z}^d; S, -S)$$

for all $d \in \mathbb{N}$ and for all finite nonempty $S \subset \mathbb{Z}^d$.

In case $d = 1$ Newman [17] showed that for all finite nonempty $S \subset \mathbb{Z}$ the value $d_c(\mathbb{Z}; S)$ is achieved on some periodic S -covering subset which period is bounded in terms of S . In particular, the value $d_c(\mathbb{Z}; S)$ is always rational. The

same argument goes for $d_c(\mathbb{Z}; S, -S)$ as well. Recently Bhattacharya [2] proved the similar statement in case $d = 2$. In case $d \geq 3$ there are partial results due to Szegedy [24] and Greenfeld and Tao [9]. However, the conjecture that for all $d \in \mathbb{N}$ and for all finite nonempty $S \subset \mathbb{Z}^d$, the value $d_c(\mathbb{Z}^d; S)$ is rational remains widely open.

A simple averaging argument implies that $d_c(\mathbb{Z}^d; S) \geq \frac{1}{|S|}$. Moreover, this bound is sometimes tight. Indeed, in case $S = [k] = \{1, 2, \dots, k\}$, $A = k\mathbb{Z}$ we have $S + A = \mathbb{Z}$ and $-S + A = \mathbb{Z}$. Hence, $d_c(\mathbb{Z}; [k]) = d_c(\mathbb{Z}; [k], -[k]) = \frac{1}{k}$. As for the upper bounds, Newman [16] showed that $d_c(\mathbb{Z}; S) \leq \frac{2}{5}$ for all three-point $S \subset \mathbb{Z}$ and provided an example $S = \{0, 1, 3\}$ showing that this result is tight. Newman also conjectured that the tight upper bound for four-point $S \subset \mathbb{Z}$ is $\frac{1}{3}$ (see [26]). This conjecture has been recently proven by Axenovich et al. [1]. In case $|S| = 5$ or 6 Bollobás et al. [3] conjectured that the tight upper bounds are $\frac{3}{11}$ and $\frac{1}{4}$ respectively, but their conjecture remains open. For large $|S|$ Newman [16] showed that $d_c(\mathbb{Z}^d; S) \leq \frac{\ln |S| + O(1)}{|S|}$. The proof was simplified by Bollobás et al. [3]. We slightly modify their ideas to show that the same bound holds for $d_c(\mathbb{Z}^d; S, -S)$ as well. Moreover, we show that the ‘worst case’ is when S is one-dimensional.

Proposition 1. *Given $d \in \mathbb{N}$ and a finite subset $S' \subset \mathbb{Z}^d$, there is $S \subset \mathbb{Z}$ of the same cardinality such that $d_c(\mathbb{Z}^d; S') \leq d_c(\mathbb{Z}; S)$ and $d_c(\mathbb{Z}^d; S', -S') \leq d_c(\mathbb{Z}; S, -S)$.*

Proposition 2. *There are positive constants C_1, C_2 such that the following two statements hold. First, for all $k \in \mathbb{N}$ and for all $S \subset \mathbb{Z}$ such that $|S| = k$, one has $d_c(\mathbb{Z}; S, -S) \leq \frac{\ln k + C_2}{k}$. Second, for all $k \in \mathbb{N}$, there is $S \subset \mathbb{Z}$ such that $|S| = k$ and $d_c(\mathbb{Z}; S, -S) \geq \frac{\ln k + C_1}{k}$.*

The problem of finding the explicit expression for $d_c(\mathbb{Z}; S)$ and $d_c(\mathbb{Z}; S, -S)$ is probably very hard in general. In case $|S| = 2$ it is almost obvious that $d_c(\mathbb{Z}; S) = d_c(\mathbb{Z}; S, -S) = \frac{1}{2}$. Schmidt and Tuller [22] stated a conjecture concerning the value of this function for $|S| = 3$ (depending on S). We confirm their conjecture.

Theorem 1. *Let λ'_1, λ'_2 be two integers, μ be their greatest common divider, and $S = \{0, \lambda'_1, \lambda'_1 + \lambda'_2\}$ be a three-point set of a general form. Set $\lambda_1 = \frac{\lambda'_1}{\mu}$, $\lambda_2 = \frac{\lambda'_2}{\mu}$. Then*

$$d_c(\mathbb{Z}; S) = d_c(\mathbb{Z}; S, -S) = \min \left(\frac{\lceil \frac{1}{3}(\lambda_1 + 2\lambda_2) \rceil}{\lambda_1 + 2\lambda_2}, \frac{\lceil \frac{1}{3}(2\lambda_1 + \lambda_2) \rceil}{2\lambda_1 + \lambda_2} \right).$$

In the next section, we discuss the relation between $d_c(\mathbb{Z}; S, -S)$ and a certain class of geometric Ramsey-type questions.

2 Max-Norm Ramsey Theorems

The well-known problem of Nelson about finding the chromatic number $\chi(\mathbb{R}^n)$ of the n -dimensional Euclidean space can be generalized in many ways. One

of them is the following: given a subset $S \subset \mathbb{R}^d$ (with induced metric), find the minimum number of colors $\chi(\mathbb{R}^n; S)$ needed to color all points of the Euclidean space \mathbb{R}^n with no monochromatic isometric copy $S' \subset \mathbb{R}^n$ of S . A set S is called *Ramsey* if $\chi(\mathbb{R}^n; S) \rightarrow \infty$ as $n \rightarrow \infty$.

A systematic study of such questions on the interface of geometry and Ramsey theory, called *Euclidean Ramsey theory*, begins with the paper [4] of Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus. They showed that each Ramsey set must be finite and *spherical* and conjectured that this criterion is sufficient. Recently Leader, Russell and Walters [14] proposed a ‘rival’ conjecture that S is Ramsey if and only if it is *subtransitive*, but the problem remains widely open.

Relatively few sets are known to be Ramsey. Frankl and Rödl [8] proved that the vertex sets of simplices and hyperrectangles are Ramsey. Kříž [10] showed that each ‘fairly symmetric’ set is Ramsey, e.g., the set of vertices of each regular polytope in any dimension.

The best known lower and upper bounds on $\chi(\mathbb{R}^n; S)$ for these S are relatively far from each other. For example, in the simplest two cases, when S is a pair of points (clearly, in this case $\chi(\mathbb{R}^n; S) = \chi(\mathbb{R}^n)$) or S is a set of vertices of an equilateral triangle Δ it is only known that

$$(1.239\dots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n,$$

$$(1.01446\dots + o(1))^n \leq \chi(\mathbb{R}^n; \Delta) \leq (2.732\dots + o(1))^n,$$

as $n \rightarrow \infty$. The lower and upper bounds on $\chi(\mathbb{R}^n)$ are due to Raigorodskii [19] and Larman and Rogers [13] respectively. The lower and upper bounds on $\chi(\mathbb{R}^n; \Delta)$ are due to Nashund [15] (see also the paper [21] of the third author) and Prosanov [18] respectively.

The last two authors [11] recently showed that such questions become much simpler if we replace the Euclidean n -dimensional space \mathbb{R}^n by the n -dimensional space \mathbb{R}_∞^n with the max-norm, defined for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$\|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$

Theorem 2. ([11]). *Any finite metric space S is exponentially Ramsey in the max-norm, i.e., there is a constant $\chi_S > 1$ such that the minimum number of colors $\chi(\mathbb{R}_\infty^n; S)$ needed to color all points of \mathbb{R}_∞^n with no monochromatic isometric copy of S satisfies $\chi(\mathbb{R}_\infty^n; S) > (\chi_S + o(1))^n$ as $n \rightarrow \infty$.*

In the upcoming paper of all three authors [7] we prove several following explicit results that extend Theorem 2. First of all, we find the optimal value χ_S for all ‘one-dimensional’ metric spaces in the notation of the first section.

Theorem 3. *Let $k \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{N}$ be a sequence of positive integers. Set $S = \{0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \dots + \lambda_k\}$. Then*

$$\chi(\mathbb{R}_\infty^n; S) = \left(\frac{1}{1 - d_c(\mathbb{Z}; S, -S)} + o(1) \right)^n$$

as $n \rightarrow \infty$.

We find the base of the exponent of $\chi(\mathbb{R}_\infty^n; S)$ for ‘non-integer’ one-dimensional finite sets too, but the exact statement is rather cumbersome, so we present only two of its corollaries here.

Corollary 1. *Let $k \in \mathbb{N}$ be an integer, and $\lambda_1, \dots, \lambda_k$ be a set of linearly independent over \mathbb{Z} real numbers. Set $S = \{0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \dots + \lambda_k\}$. Then*

$$\chi(\mathbb{R}_\infty^n; S) = \left(1 + \frac{1}{k} + o(1)\right)^n$$

as $n \rightarrow \infty$.

Corollary 2. *There is a constant $C > 0$ such that for all $S \subset \mathbb{R}$ of cardinality k , one has*

$$\left(1 + \frac{1}{k} + o(1)\right)^n \leq \chi(\mathbb{R}_\infty^n; S) \leq \left(1 + \frac{\ln k + C}{k} + o(1)\right)^n$$

as $n \rightarrow \infty$. So, the base of the exponent of $\chi(\mathbb{R}_\infty^n; S)$ tends to 1 with ‘almost linear’ speed as $|S|$ tends to infinity.

We also determine the base of the exponent of $\chi(\mathbb{R}_\infty^n; S)$ for several families of higher-dimensional sets S . The hyperrectangles are among the most interesting of them.

Theorem 4. *Let $S \subset \mathbb{R}_\infty^d$ be a set of vertices of a hyperrectangle. Then*

$$\chi(\mathbb{R}_\infty^n; S) = (2 + o(1))^n$$

as $n \rightarrow \infty$.

However, there is a special class of problems that become more difficult in the max-norm setting in comparison with the Euclidean one. It is known (see [5]) that for all $n \in \mathbb{N}$ and for all infinite $S \subset \mathbb{R}^n$ we have $\chi(\mathbb{R}^n; S) = 2$ in the Euclidean case. We prove the following analogue of this statement for the max-norm, but the general picture is not yet clear.

Theorem 5. *For all $n \in \mathbb{N}$ and for all infinite $S \subset \mathbb{R}_\infty^n$, we have $\chi(\mathbb{R}_\infty^n; S) \leq n + 1$. Moreover, this bound is tight, i.e., for all $n \in \mathbb{N}$, there is an infinite $S \subset \mathbb{R}_\infty^n$ such that $\chi(\mathbb{R}_\infty^n; S) = n + 1$.*

Acknowledgments. The research of the third author was supported in part by the Moebius Contest Foundation for Young Scientists. The third author is a Young Russian Mathematics award winner and would like to thank its sponsors and jury. Part of this research was done while the first author was a Visiting Fellow at the London School of Economics, supported by an LMS Early Career Fellowship.

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Unit Disks Hypergraphs Are Three-Colorable

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Abstract. We prove that any finite point set \mathcal{P} in the plane can be three-colored so that every unit disk intersecting \mathcal{P} in at least 1025 points contains points of at least two different colors.

Keywords: Discrete geometry · Geometric hypergraph coloring · Decomposition of multiple coverings

1 Introduction

Coloring problems for hypergraphs defined by geometric range spaces have been studied extensively in different settings [1–21]. A pair $(\mathcal{P}, \mathcal{S})$, where \mathcal{P} is a set of points in the plane and \mathcal{S} is a family of subsets of the plane (the *range space*), defines a (primal) hypergraph $\mathcal{H}(\mathcal{P}, \mathcal{S})$ whose vertex set is \mathcal{P} , and edge set is $\{S \cap \mathcal{P} \mid S \in \mathcal{S}\}$. Given any hypergraph \mathcal{G} , a planar realization of \mathcal{G} is defined as a pair $(\mathcal{P}, \mathcal{S})$ for which $\mathcal{H}(\mathcal{P}, \mathcal{S})$ is isomorphic to \mathcal{G} . If \mathcal{G} can be realized with some pair $(\mathcal{P}, \mathcal{S})$, where \mathcal{S} is from some family \mathcal{F} , then we say that \mathcal{G} is realizable with \mathcal{F} .

It is an easy consequence of the properties of Delaunay-triangulations and the Four Color Theorem that the vertices of any hypergraph realizable with disks can be four-colored such that every edge that contains at least two vertices contains two differently colored vertices. But are less colors sufficient if all edges are required to contain at least m vertices for some large enough constant m ? The authors settled this question recently [6], showing that three colors are not enough for any m , i.e., for any m , there exists an m -uniform hypergraph that is not three-colorable and that permits a planar realization with disks.

For unit disks in arbitrary position, Pach and Pálvölgyi [16] showed that for any m , there exists an m -uniform hypergraph that is not two-colorable and that permits a planar realization with unit disks. Our main result is showing that for large enough m three colors are sufficient for unit disks.

Theorem 1. *Any finite point set \mathcal{P} can be three-colored such that any unit disk that contains at least 1025 points from \mathcal{P} contains two points colored differently.*

2 Hypergraph Colorings

It is important to distinguish between two types of hypergraph colorings that we will use, the *proper* coloring and the *polychromatic* coloring.

Definition 1. A hypergraph is properly k -colorable if its vertices can be colored with k colors such that each edge contains points from at least two color classes. Such a coloring is called a proper coloring.

Definition 2. A hypergraph is polychromatic k -colorable if its vertices can be colored with k colors such that each edge contains points from each color class. Such a coloring is called a polychromatic coloring.

Polychromatic colorability was studied for many geometric families. For hypergraphs determined by pseudohalfplanes (defined as the regions on one side of each pseudoline in some pseudoline arrangement) the following is known.

Theorem 2. (Keszegh-Pálvölgyi [12]). Given a finite collection of points and pseudohalfplanes, the points can be k -colored such that every pseudohalfplane that contains at least $2k - 1$ points contains all k colors.

Polychromatic colorability is a much stronger property than proper colorability. Any polychromatic k -colorable hypergraph is proper 2-colorable. We generalize this trivial observation to the following statement about unions of polychromatic k -colorable hypergraphs.

Theorem 3. Let $\mathcal{H}_1 = (V, E_1), \dots, \mathcal{H}_{k-1} = (V, E_{k-1})$ be hypergraphs on a common vertex set V . If $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ are polychromatic k -colorable, then the hypergraph $\bigcup_{i=1}^{k-1} \mathcal{H}_i = (V, \bigcup_{i=1}^{k-1} E_i)$ is proper k -colorable.

Proof. Let $c_i : V \rightarrow \{1, \dots, k\}$ be a polychromatic k -coloring of \mathcal{H}_i . Choose $c(v) \in \{1, \dots, k\}$ such that it differs from each $c_i(v)$. We claim that c is a proper k -coloring of $\bigcup_{i=1}^{k-1} \mathcal{H}_i$. To prove this, it is enough to show that for every edge $H \in \mathcal{H}_i$ and for every color $j \in \{1, \dots, k - 1\}$, there is a $v \in H$ such that $c(v) \neq j$. We can pick $v \in H$ for which $c_i(v) = j$. This finishes the proof.

Theorem 3 is sharp in the sense that for every k there are $k - 1$ polychromatic k -colorable hypergraphs such that their union is not properly $(k - 1)$ -colorable.

3 Proof of Theorem 1

Let \mathcal{P} denote the points and let \mathcal{D} denote the unit (radius) disks that contain at least 1025 points from \mathcal{P} .

The first step of the proof is a classic divide and conquer idea [15]. Divide the plane into a grid of squares of side length $\frac{1}{\sqrt{10}} \approx 0.31$ such that no point of \mathcal{P} falls on the boundary of a grid square. Since a square of side length two intersects

at most eight rows and eight columns of the grid, each unit disk intersects at most 64^1 squares. Let $D \in \mathcal{D}$ be one of the unit disks. Since D contains at least $1025 = 64 \cdot 16 + 1$ points from \mathcal{P} , by the pigeonhole principle there is a square S such that $S \cap D$ contains at least 17 points from \mathcal{P} .

Hence it is enough to show the following theorem. Applying it separately for the points in each square of the grid provides a proper three-coloring of the whole point set.

Lemma 1. *Suppose \mathcal{P} is a finite point set inside a square of side length $\frac{1}{\sqrt{10}}$. Then we can color the points of \mathcal{P} by three colors such that any unit disk, that contains at least 17 points from \mathcal{P} , will contain points from all three colors.*

Proof. Since $2 \cdot (\frac{1}{\sqrt{10}})^2 < 1$, if the center of a unit disk lies in the square, then the disk contains the whole square. As we will use more than one color to color the points in the square, such disks cannot be monochromatic. The sidelines of the square divide the plane into nine regions. Denote the unbounded closed quadrant regions by Q_1, Q_2, Q_3, Q_4 and the unbounded open half-strip regions by S_1, S_2, S_3, S_4 , numbered in a clockwise order, according to Fig. 1. We need to assure that no matter in which of these eight regions the center of a unit disk lies, it is not monochromatic.

Let ℓ_x be a horizontal halving line for \mathcal{P} , that is, a horizontal line such that both (closed) halfplanes bounded by ℓ_x contain at least $\frac{|\mathcal{P}|}{2}$ points. Similarly, let ℓ_y be a vertical halving line for \mathcal{P} and let O denote the intersection of ℓ_x and ℓ_y . These lines divide the square into four (closed) rectangular regions R_1, R_2, R_3, R_4 , indexed according to Fig. 1. The usefulness of this further subdivision comes from the following observation.

Observation 1. *If the center of a unit disk lies in Q_i and the disk contains O , then the disk contains the whole region R_i .*

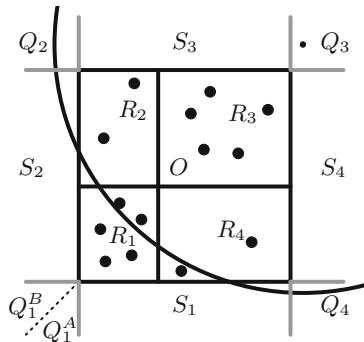


Fig. 1. Regions around a grid square.

¹ In fact less, but we are not trying to optimize our constants.

We will color the four regions R_1, \dots, R_4 separately, but Observation 1 reduces the number of disks that have to be considered for each region.

Let $\mathcal{D}_i \subset \mathcal{D}$ denote the disks that contain at least 5 point from $R_i \cap \mathcal{P}$. We will color the points of R_i with three colors such that for each $D \in \mathcal{D}_i$ the following holds: either $D \cap R_i \cap \mathcal{P}$ is not monochromatic or D contains the whole region R_{i+2} (indexed modulo 4).

By symmetry it is enough to consider R_1 . If $|R_1 \cap \mathcal{P}| \leq 4$, then \mathcal{D}_i is empty and we are done. Otherwise we divide the disks in \mathcal{D}_i into three groups. The line of the diagonal from the bottom-left corner of the square to its top-right corner splits Q_1 into two parts, as marked with a dashed line on Fig. 1. Denote by Q_1^A the bottom-right part of Q_1 and by Q_1^B its upper-left part. Let $\mathcal{A} \subset \mathcal{D}_1$ be the disks whose center lies in $Q_1^A \cup S_1 \cup Q_4 \cup S_4$. Let $\mathcal{B} \subset \mathcal{D}_1$ be the disks whose center lies in $Q_1^B \cup S_2 \cup Q_2 \cup S_3$. Let $\mathcal{C} \subset \mathcal{D}_1$ be the disks whose center lies in the closed quadrant Q_3 .

If a disk is in \mathcal{C} , then it contains O , thus by Observation 1 it also contains the whole region R_3 , and the coloring of the points $\mathcal{P} \cap R_3$ will ensure that it cannot be monochromatic. Hence, it is enough to properly three-color the hypergraph $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A} \cup \mathcal{B})$. First we show that both $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A})$ and $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{B})$ are realizable with pseudohalfplanes. We use the following geometric lemma.

Lemma 2. *If we take two disks from \mathcal{A} , or two disks from \mathcal{B} , their boundaries intersect at most once inside R_1 .*

Proof. Let $R = \cup_{i=1}^4 R_i$ denote the square and define two trapezoidal regions around R as follows. Denote by X^* the reflection of any region X to the bottom-right corner of the square R . One trapezoid is $(Q_1^A \cup S_1) \cap S_4^*$, and the other is $(Q_1^A \cup S_1)^* \cap S_4$, see the shaded regions on Fig. 2. The trapezoids have $45^\circ, 90^\circ$ and 135° degree angles and the ratio of their sides is $1 : 1 : 2 : \sqrt{2}$.

Let D_1 and D_2 be two disks from \mathcal{A} and let o_1, o_2 denote their centers. If the boundaries of D_1 and D_2 intersect twice inside R , then the midpoint of $o_1 o_2$ falls into R . It is easy to see that this implies that o_1 and o_2 must be located in the shaded regions shown in Fig. 2. If we place o_1 outside of the shaded region, then the possible locations for o_2 fall outside of $Q_1^A \cup S_1 \cup Q_4 \cup S_4$. On the other hand if o_1, o_2 are in the shaded region, then D_1 and D_2 contains R as $(\frac{3}{\sqrt{10}})^2 + (\frac{1}{\sqrt{10}})^2 = 1$. This contradicts that their boundaries intersect inside R .

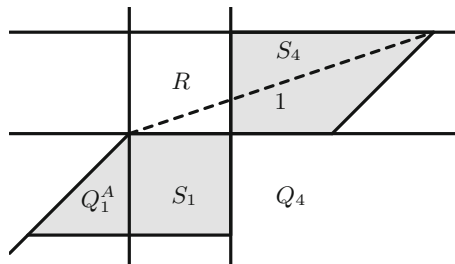


Fig. 2. Locations for two points in $Q_1^A \cup S_1 \cup Q_4 \cup S_4$ whose midpoint lies in R .

A similar argument holds for \mathcal{B} , finishing the proof of Lemma 2.

We remark that Lemma 2 holds also for squares of side length $\frac{1}{\sqrt{5}}$ with a more careful argument.

Therefore, $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A})$ and $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{B})$ are hypergraphs that can be realized by pseudohalfplanes. By definition each edge in these hypergraphs contains at least 5 vertices. Thus by Theorem 2 they are polychromatic three-colorable, and by Theorem 3, $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A} \cup \mathcal{B})$ is proper three-colorable.

We apply the previous argument for each R_i . To see that the resulting coloring is good, take any disk $D \in \mathcal{D}$. Since D contains at least $17 = 4 \cdot 4 + 1$ points from \mathcal{P} , there is a region R_i such that D contains at least 5 points from $R_i \cap \mathcal{P}$, that is $D \in \mathcal{D}_i$. Therefore either D contains two points of different colors in R_i , or D contains the whole region R_{i+2} . Since ℓ_x and ℓ_y are halving lines $|\mathcal{P} \cap R_i| = |\mathcal{P} \cap R_{i+2}|$ (indexed modulo 4). Hence region R_{i+2} contains at least 5 point from \mathcal{P} . The points inside R_{i+2} are not monochromatic, hence D is not monochromatic in either case.

4 Concluding Remarks

Let the m -fat edges of a hypergraph be those edges whose cardinality is at least m . We can restate Theorem 1 the following way. If \mathcal{P} is a set of point in the plane and \mathcal{S} is a set of unit disks, then the m -fat edges of the hypergraph $\mathcal{H}(\mathcal{P}, \mathcal{S})$ form a hypergraph that is properly three-colorable.

One can consider other geometric families for \mathcal{S} . For example, let C be a convex compact set whose boundary is smooth and let \mathcal{S} be a family of translates of C . A small refinement of the argument above shows that there is an $m = m(C)$ such that the m -fat edges of $\mathcal{H}(\mathcal{P}, \mathcal{S})$ form a three-colorable hypergraph. It was shown in [16] that for every smooth compact set C and for every m there is a non-two-colorable hypergraph that can be realized by C . We can show that this result extends to several sets whose boundary is only partly smooth, such as a halfdisk, answering an open problem from [16]. The construction is essentially the same as in [13, 16, 19], using the arrangement shown in Fig. 3 for the recursive step.

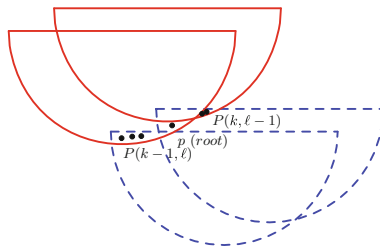


Fig. 3. Recursive step for the non-two-colorable half disk construction.

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Characterization of FS-Double Squares Ending with Two Distinct Squares

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Abstract. A square is a concatenation of two identical words. A long-standing open conjecture is that the number of distinct squares in a word is bounded by its length. When two squares start at a location for the last time in a word, the longer square is called FS-double square. One way to increase the number of distinct squares in a word is to add as many FS-double squares as possible. For any FS-double square, the first letter always adds two distinct squares. However, the last letter can be in at most two distinct squares. We give a structure of an FS-double square where removing any of the terminal letters removes two distinct squares. We show that the maximum number of such FS-double squares that are adjacent to each other in a word w is less than $\frac{|w|}{11}$. We also show that the distinct squares introduced by the terminal letters of an FS-double square are conjugates.

Keywords: Distinct squares · FS-double squares · Bordered FS squares

1 Introduction

Repetitions in words are used to find the properties of words and is a well-researched topic in word combinatorics. There are different definitions of repetitions, and each is explored in detail [2–4]. A square is a repetition of the form xx where x is a non-empty word. Frankel et al. [4] conjectured that the number of distinct squares in a word is always less than its length. The existing bounds for the conjecture are based on the result that a location in a word can start with at most two rightmost squares. When a location starts with two rightmost squares, the longer square is named FS-double square [2]. A word of length n can accommodate a maximum of $\frac{5n}{6}$ FS-double squares and, therefore, the number of distinct squares in a word is less than $\frac{11n}{6}$ [2].

In this work, we explore properties of FS-double squares that can possibly be used to get a better bound for the square conjecture. To do so, we find the

structure of words where both the first and the last letters are part of two distinct squares. We then count the maximum number of such letters in a prefix and in a suffix of a word. Similar to our study, the structure of words that begins with two squares has been explored in [1, 3, 5] to find the occurrences of neighbouring squares. A case study with 14 partially solved cases is done in [5] and the results are used for efficient computation of the repetitions in a word. The results of this paper can be further extended to solve some of the unsolved cases specified in [5]. The main contributions of our work are listed below.

- (a) We identify the structure of an FS-double square in which terminal letters are part of two distinct squares,
- (b) We compute the length of the longest sequence of consecutive FS-double squares where the square follows the structure mentioned in (a), and
- (c) We describe the types of distinct squares introduced by terminal letters of an FS-double square.

The rest of the paper is organized as follows. The next section gives definitions and the existing results that are used later in the paper. In Sect. 3, we find the structure of an FS-double square where the terminal letters are part of two distinct squares. In Sect. 4, we explore the properties of words with adjacent FS-double squares. Using these properties, we count the maximum number of consecutive FS-double squares that have the structure identified in Sect. 3.

2 Background

We use Σ to denote an alphabet. A word is a concatenation of letters drawn from Σ . The length of a word w is the number of letters in it and is denoted by $|w|$. The length of an empty word is zero. For a non-empty word $w = xyz$, the words x and z are called prefix and suffix, respectively. Here, x, z are terminal letters when $|x| = |z| = 1$. We use $LCP(x, y)$ to denote the longest prefix that is common in words x and y . If a word u is both a prefix and a suffix of word w , we refer to it as a border of w . The symbol $\mathbb{B}(w)$ represents a set of borders of word w . We refer conjugate of a word $w = w_1w_2$ by a word w_2w_1 where $w_1, w_2 \in \Sigma^+$.

A repetition is the word of the form x^k where $x \in \Sigma^+$ and the value of integer k is greater than one. A square is a word obtained with $k = 2$. We write the square xx as x^2 and refer to the word x as a base of the square. The set $DS(w)$ contains all the distinct squares that are present in a non-empty word w . A word w is a primitive word if $w = x^k$ implies $k = 1$. Every repetition can be represented using a primitive word. In such a representation, the primitive word is called primitive root. If the base of a square is primitive, we call it a primitive square. Otherwise, the square is a non-primitive square.

The square conjecture predicts that the number of distinct squares in a word is always less than its length [4]. The lower bound for the square conjecture is obtained by exploring the properties of primitive words [6] while the upper bound is given by mapping the rightmost occurrences of distinct squares with the locations of words [2, 4]. Given a location that starts with two such squares,

Deza et al. [2] referred to the longer square as FS-double square and its structure is given in Lemma 1.

Lemma 1 (FS-double square [2]). *Let SQ^2 be an FS-double square that begins with another shorter square sq^2 such that $|sq| < |SQ| < 2|sq|$. Then, $SQ = (xy)^{p_1}(x)(xy)^{p_2}$, where x, y are non-empty words and p_1, p_2 are integers such that $p_1 \geq p_2 \geq 1$. Here, xy is a primitive word and $sq^2 = ((xy)^{p_1}(x))^2$ is a unique square in SQ^2 .*

In the rest of the paper, we refer to the structure given in Lemma 1 for an FS-double square SQ^2 unless mentioned otherwise. In the next section, we explore the structure of an FS-double square that ends with two distinct squares.

3 FS-Double Squares Ending with Two Distinct Squares

An FS-double square, SQ^2 , begins with two distinct squares and these squares end after the first instance of SQ . In this section, we identify the structure of FS-double squares, where each of the terminal letters belongs to two distinct squares. Moreover, the lengths of these squares are greater than $|SQ|$. For example, consider an FS-double square $SQ^2 = awb$ where $a, b \in \Sigma, w \in \Sigma^+$. The difference between the number of distinct squares in SQ^2 and the same in aw (or wb) must be two. We use some of the existing results of FS-double squares and properties of primitive words to obtain such words. The following lemma describes the types of squares that start at the beginning of an FS-double square.

Lemma 2 ([1]). *The following statements hold for an FS-double square SQ^2 . (a) SQ^2 is a primitive square, and (b) sq^2 is a primitive square for $p_2 > 1$.*

Lemma 3. *Every conjugate of a primitive word is distinct.*

Fan et al. [3] gave “The new periodicity lemma” that classifies the squares in an FS-double square based on their structures and locations. This lemma is revisited in [1] and is given as follows.

Lemma 4 ([1]). *Let u^2 be a square in an FS-double square SQ^2 . Then, one of the statements holds: (a) $|u| = |SQ|$, (b) $|u| < |sq|$, (c) If $|SQ| > |u| \geq |sq|$, then the primitive root of u is a conjugate of xy .*

We find the structure of FS-double squares that ends with two distinct squares in Theorem 1. The results shown in Lemmas 5 to 7 are used to prove the theorem.

Lemma 5. *Let SQ^2 be an FS-double square that ends with a square, v^2 , such that $|SQ| < 2|v|$ and $p_1 = p_2 = 1$. Then, $|v| = |sq|$ and $x \in \mathbb{B}(xy)$.*

Proof. We omit the proof of this lemma due to space restrictions.

Lemma 6. *Let v^2 be a suffix of an FS-double square, SQ^2 , where $2|v| > |SQ|$. If $p_1 = p_2$, then $x \in \mathbb{B}(xy)$ and $|v| = |sq|$.*

Proof. The given statement holds for $p_1 = p_2 = 1$ (refer Lemma 5). There are five possibilities for v^2 to start in SQ^2 as shown in Fig. 1 for $p_1 = p_2 = p > 1$. The number in the figure indicates the beginning of v^2 in SQ^2 and the respective case number. As v^2 is a suffix of SQ^2 and $|v| > p|xy|$, the second occurrence of v in all of the five cases ends with $(xy)^p$. The first occurrence of v ends with either of the suffixes: (a) xy , (b) y_2xy_1 where $y = y_1y_2$ or, (c) x_2yx_1 where $x = x_1x_2$. According to Lemma 4, the primitive root of v is a conjugate of xy . Thus, the relations obtained in cases (b) and (c) imply that two conjugates of xy are equal. This contradicts Lemma 3, so we discard the words with these two cases. The possible structures of v^2 for case (a) are verified in Table 1 where $|v| = |sq|$ is the only valid case. The structure of v^2 in this case implies $x \in \mathbb{B}(xy)$.



Fig. 1. Starting location of v^2 in SQ^2 for $p_1 = p_2 = p$

Table 1. First occurrences of v ending with xy in SQ^2

Case No.	Possible structure of base v	Condition	Remark
1	$v = (xy)^q(x)(xy)^p(xy)^s = (xy)^s(xy)^q(x)(xy)^p$	$p > q > s > 0$	$xy = yx$
2	$v = x(xy)^p(xy)^q = (xy)^qx(xy)^p$	$p > q > 0$	$xy = yx$
3	$v = x_2(xy)^p(xy)^q = (xy)^{p-q}(x_1x_2)(xy)^p$	$p, q, p - q > 0$	$xy = yx$
4	$v = (xy)^p(xy)^s = (xy)^{p-s}x(xy)^p$	$p, s, p - s > 0$	$xy = yx$
5	$v = y_2(xy)^p = x(xy)^p$	$p > 0$	$x \in \mathbb{B}(xy)$

Lemma 7. *Given an FS-double square SQ^2 with $p_1 > p_2 > 1$ that ends with v^2 where $|SQ| < 2|v|$. Then, $|v| = |SQ|$.*

Proof. Assume $p_2 = p$ and $p_1 = p + q$ for some positive integers p, q . In Fig. 2, we marked all the possible starting locations of v^2 in SQ^2 . Here, we show that every case leads to the relation $xy = yx$ and the relation contradicts Lemma 3. Similar to Lemma 6, the first occurrence of v ends with either x_2yx_1 , y_2xy_1 or xy assuming $x = x_1x_2, y = y_1y_2$. We discard the first two types of squares since occurrences of v 's violate Lemma 3. The first occurrence of v that starts at one of the marked locations 1, 3, 5, 7 or 8 never ends with xy . If v^2 begins at location 2, 4 or 6, then equating the structures of two v 's always gives $xy = yx$.

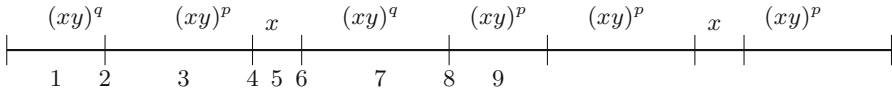


Fig. 2. Beginning of v^2 in SQ^2 where $p_1 > p_2$

Theorem 1 (Bordered FS Square). *Let $SQ^2 = s.a$ be an FS-double square, where $s \in \Sigma^+$ and $a \in \Sigma$. Then, $|DS(SQ^2)| - |DS(s)| = 2$ iff SQ^2 ends with a conjugate of sq^2 .*

Proof. **(If)** The statement follows from Lemma 6 and 7.

(Only if) The last letter of every FS-double square is a part of the FS-square itself. We assume SQ^2 ends with a conjugate of sq^2 , say v^2 . So, xy in Eq. (1) ends with x . Thus, we can write $xy = y'x$ for $y' \in \Sigma^+$ and $|y'| = |y|$. Now, the reverse of SQ^2 is a word that starts with two distinct squares, and these two squares satisfy the premise of Lemma 1. Thus, v^2 is a unique square and removing the last letter of SQ^2 removes two distinct squares.

$$SQ^2 = (xy)^p(x)(xy)^{p-1} \underbrace{xy.(xy)^p(x)(xy)^p}_{y'v^2} \tag{1}$$

4 Consecutive Bordered FS Squares

A primitive square can be extended with its prefix to get consecutive equal length squares [4,6]. With the same approach, it is possible to add letters to an FS-double square and get a sequence of consecutive FS-double squares. The next lemmata explain the characteristics of consecutive squares of equal lengths.

Lemma 8. *If u^2 and v^2 be two equal length squares that start at adjacent locations, then u and v are conjugates and u^2 is appended by its prefix.*

Proof. Let a letter ‘ a ’ be the prefix of u^2 that ends before v^2 such that $u = au'$. Similarly, $v = v'b$ assuming v ends with a letter ‘ b ’. So, the first occurrences of u, v give $ub = av$. This shows that u is followed by ‘ b ’ in u^2 implying $u = bu'$. Hence, $a = b$. Also, we get $au'a = av'a$ implying $u = au', v = u'a$.

Lemma 9. *Let w begins with two equal length consecutive FS-double squares, SQ^2 followed by $\overline{SQ^2}$. Then, the respective shorter squares, sq^2 and $\overline{sq^2}$, are conjugates.*

Proof. We know, $SQ = (xy)^{p_1}(x)(xy)^{p_2}$ and $sq = (xy)^{p_1}(x)$. Assume SQ starts with a letter ‘ a ’ such that $x = ax'$. We get $\overline{SQ} = (x'ya)^{p_1}(x'a)(x'ya)^{p_2}$ (see Lemmas 6 and 8). The structure of \overline{SQ} implies $\overline{sq} = (x'ya)^{p_1}(x'a)$.

Lemma 10 (Consecutive FS-double square). *Let SQ^2 be an FS-double square with $p_1 = p_2 = p$. Then, the maximum number of consecutive FS-double squares of length $2|SQ|$ is $\min(|LCP(xy, yx)|, |x|)$.*

Proof. The shorter squares of consecutive FS-double squares are conjugates (refer Lemma 9). Further, Lemma 8 and the highlighted part (after sq^2) in Eq. (2) show that $k = |LCP(xy, yx)|$. However, sq^2 reappears if SQ^2 is extended with more than $|x|$ letters violating Lemma 1. Thus, a word can have $\min(|LCP(xy, yx)|, |x|)$ consecutive FS-double squares.

$$SQ^2 = (xy)^p(x)(xy)^p x(\mathbf{y}x)^p(xy)^p \tag{2}$$

Lemma 11. *Let w begins with k equal length consecutive bordered FS squares. If the first bordered FS square is $SQ^2 = (xyxxy)^2$, then $k = |LCP(x, y)| + 1$ and $|y| > |x|$.*

Proof. Given $SQ^2 = (xyxxy)^2$ and Theorem 1 shows that $x \in \mathbb{B}(xy)$. So, either x is a suffix of y or $|y| < |x|$. Let SQ^2 and \overline{SQ}^2 be two consecutive bordered FS squares. Assume x begins with a letter ‘ a ’ such that $x = ax'$. So, $\overline{SQ} = (\mathbf{x}'\mathbf{y}a)(x'a)(x'ya)$ and $x'a \in \mathbb{B}(x'ya)$. The latter condition holds provided y begins with ‘ a ’ (refer the prefix in bold in the structure of \overline{SQ}) and ya ends with $x'a$. Thus, the value of $|LCP(x, y)|$ must be at-least one to get two consecutive bordered FS squares. Similarly, the conjugate of \overline{SQ}^2 adjacent to it is bordered FS square if $|LCP(x, y)| = 2$. Thus, k consecutive bordered FS squares are possible when $|LCP(x, y)| = k - 1$. In case of $|y| < |x|$, the two bases SQ and \overline{SQ} are non-primitive. This contradicts Lemma 2. So, $|y| > |x|$.

We compute the maximum number of equal length consecutive bordered FS squares in Theorem 2. The proof of the theorem is based on the properties of bordered FS squares that are explained in Lemma 11.

Theorem 2. *Let w contains k equal length consecutive bordered FS squares. Then, $11k < |w|$.*

Proof. The value of $\frac{k}{|w|}$ is maximum if k consecutive bordered FS squares are at the beginning of w and k^{th} bordered FS square is a suffix of w . So, assume w begins with an FS-double square SQ^2 followed by $k - 1$ consecutive bordered FS squares of size $2|SQ|$. From Eq. (3), the value of $\frac{k}{|w|}$ is maximum for $p = 1$. Lemma 10 shows that SQ^2 can be extended with at-most $|x| - 1$ letters to obtain consecutive FS-double squares. Here, $k = |x|$ and $SQ = (xy'x)(x)(xy'x)$ where y' is some non-empty word (refer Lemma 11). The ratio is computed below.

$$\frac{k}{|w|} = \frac{|x|}{2((p + 1)|x| + p|y|) + |x| - 1} < \frac{|x|}{2(5|x| + 2|y'|) + |x| - 1} < \frac{1}{11} \tag{3}$$

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On Asymmetric Hypergraphs

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Abstract. In this paper, we prove that for any $k \geq 3$, there exist infinitely many minimal asymmetric k -uniform hypergraphs. This is in a striking contrast to $k = 2$, where it has been proved recently that there are exactly 18 minimal asymmetric graphs.

We also determine, for every $k \geq 1$, the minimum size of an asymmetric k -uniform hypergraph.

Keywords: Asymmetric hypergraphs · k -uniform hypergraphs · Automorphism

1 Introduction

Let us start with graphs: A graph G is called *asymmetric* if it does not have a non-identical automorphism. Any non-asymmetric graph is also called *symmetric* graph. A graph G is called *minimal asymmetric* if G is asymmetric and every non-trivial induced subgraph of G is symmetric. (Here G' is a *non-trivial subgraph* of G if G' is a subgraph of G and $1 < |V(G')| < |V(G)|$.) In this paper all graphs are finite.

It is a folkloristic result that most graphs are asymmetric. In fact, as shown by Erdős and Rényi [2] most graphs on large sets are asymmetric in a very strong sense. The paper [2] contains many extremal results (and problems), which motivated further research on extremal properties of asymmetric graphs, see e.g. [5, 10]. This has been also studied in the context of the reconstruction conjecture [4, 6].

The second author bravely conjectured long time ago that there are only finitely many minimal asymmetric graphs, see e.g. [1]. Partial results were given in [7, 8, 11]. Recently this conjecture has been confirmed by Pascal Schweitzer



This paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115 – DYNASNET).

and Patrick Schweitzer [9] (the list of 18 minimal asymmetric graphs has been isolated in [7]):

Theorem 1 [9]. *There are exactly 18 minimal asymmetric undirected graphs up to isomorphism.*

An *involution* of a graph G is any non-identical automorphism ϕ for which $\phi \circ \phi$ is an identity. It has been proved in [9] that all minimal asymmetric graphs are in fact minimal involution-free graphs.

In this paper, we consider analogous questions for k -graphs (or k -uniform hypergraphs), i.e. pairs (X, \mathcal{M}) where $\mathcal{M} \subseteq \binom{X}{k} = \{A \subseteq X; |A| = k\}$. Induced subhypergraphs, asymmetric hypergraphs and minimal asymmetric hypergraphs are defined analogously as for graphs. Instead of hypergraphs we often speak just about k -graphs.

We prove two results related to minimal asymmetric k -graphs.

Denote by $n(k)$ the minimum number of vertices of an asymmetric k -graph.

Theorem 2. $n(2) = 6, n(3) = k + 3, n(k) = k + 2$ for $k \geq 4$.

Theorem 2 implies the existence of small minimal asymmetric k -graphs. Our second result disproves analogous minimality conjecture (i.e. a result analogous to Theorem 1) for k -graphs.

Theorem 3. *For every integer $k \geq 3$, there exist infinitely many k -graphs that are minimal asymmetric.*

In fact we prove the following stronger statement.

Theorem 4. *For every integer $k \geq 3$, there exist infinitely many k -graphs (X, \mathcal{M}) such that*

1. (X, \mathcal{M}) is asymmetric.
2. If (X', \mathcal{M}') is a k -subgraph of (X, \mathcal{M}) with at least two vertices, then (X', \mathcal{M}') is symmetric.

Such k -graphs we call *strongly minimal asymmetric*. So strongly minimal asymmetric k -graphs do not contain any non-trivial (not necessarily induced) asymmetric k -subgraph. Note that some of the minimal asymmetric graphs fail to be strongly minimal.

Theorem 4 is proved by constructing a sequence of strongly minimal asymmetric k -graphs. We have two different constructions of increasing strength: In Sect. 3 we give a construction with all vertex degrees bounded by 3. A stronger construction which yields minimal asymmetric k -graphs ($k \geq 6$) with respect to involutions is not presented due to space limitations (see the arXiv version [3]).

2 The Proof of Theorem 2

Lemma 1. For $k \geq 3$, we have $n(k) \geq k + 2$.

Proof. Assume that there exists an asymmetric k -graph (X, \mathcal{M}) with $|X| = k + 1$. If for each vertex $u \in X$, there is a hyperedge $M \in \mathcal{M}$ such that $u \notin M$, then $\mathcal{M} = \binom{X}{k}$, which is symmetric. Otherwise there exists $u, v \in X$ such that $\{u, v\} \subset M$ for every edge $M \in \mathcal{M}$, or there exist $u', v' \in X$ and $M_1, M_2 \in \mathcal{M}$ such that $u' \notin M_1$ and $v' \notin M_2$. In the former case, there is an automorphism ϕ of (X, \mathcal{M}) such that $\phi(u) = v$ and $\phi(v) = u$. In the later case there is an automorphism ϕ of (X, \mathcal{M}) such that $\phi(u') = v'$ and $\phi(v') = u'$. In both cases we have a contradiction. \square

For a k -graph $G = (X, \mathcal{M})$, the *set-complement* of G is defined as a $(|X| - k)$ -graph $\bar{G} = (X, \bar{\mathcal{M}}) = (X, \{X - M \mid M \in \mathcal{M}\})$. Denote by $Aut(G)$ the set of all the automorphisms of G and thus we have $Aut(G) = Aut(\bar{G})$. We define the degree of a vertex v in a k -graph G as $d_G(v) = |\{M \in \mathcal{M}; v \in M\}|$.

Lemma 2. For $k \geq 4$, we have $n(k) = k + 2$.

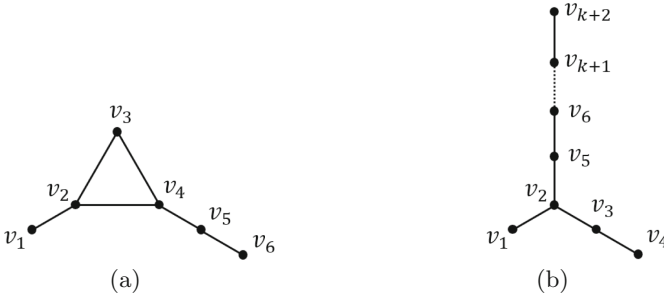


Fig. 1. The asymmetric graphs (a) X_1 , (b) T_{k+2}

Proof. First, we construct an asymmetric k -graph (X, \mathcal{M}) with $|X| = k + 2$ for each $k \geq 4$. Examples of such graphs X_1 and T_{k+2} are depicted in Fig. 1.

For $k = 4$, take the set-complement of X_1 . For every $k \geq 5$, take the set-complement of T_{k+2} . It is known that X_1 and T_{k+2} ($k \geq 5$) are asymmetric. Thus set-complements \bar{X}_1 and \bar{T}_{k+2} ($k \geq 5$) are also asymmetric k -graphs.

Let H be any of these set-complements. H is minimum as any non-trivial induced sub- k -graph $H' \subset H$ has no more than $k + 1$ elements. Thus we can use Lemma 1. \square

The case $k = 3$ (i.e. proof of $n(3) = 6$) is a (rather lengthy) case analysis which has to be omitted here.

3 Proofs of Theorem 4

In this section, we outline two different proofs of Theorem 4.

Firstly, we give a proof with bounded degrees.

For $k \geq 3, t \geq k - 2$, we define the following k -graphs.

$$G_{k,t} = (X_{k,t}, \mathcal{E}_{k,t}),$$

$$X_{k,t} = \{v_i; i \in [tk]\} \cup \{u_i; i \in [tk]\} \cup \{v_i^j; i \in [tk], j \in [k - 3]\},$$

$\mathcal{E}_{k,t} = \{E_i; i \in [tk]\} \cup \{E_{i,j}; j \in [k - 3], i = j + sk, s \in \{0, 1, 2, \dots, t - 1\}\}$, where $E_i = \{v_i, u_i, v_i^1, v_i^2, \dots, v_i^{k-3}, v_{i+1}\}$, $E_{i,j} = \{v_i^j, v_{i+1}^j, \dots, v_{i+k-1}^j\}$ and $y_{tk+t} = y_t$ for every $y \in \{v, u, v^1, v^2, \dots, v^{k-3}\}$.

$$G_{k,t}^\circ = \{X_{k,t} \cup \{x\}, \mathcal{E}_{k,t} \cup \{E\}\}, \text{ where } E = \{v_1, u_1, v_1^1, v_1^2, \dots, v_1^{k-3}, x\}.$$

The k -graphs $G_{k,t}$ and $G_{k,t}^\circ$ are schematically depicted in Fig. 2.

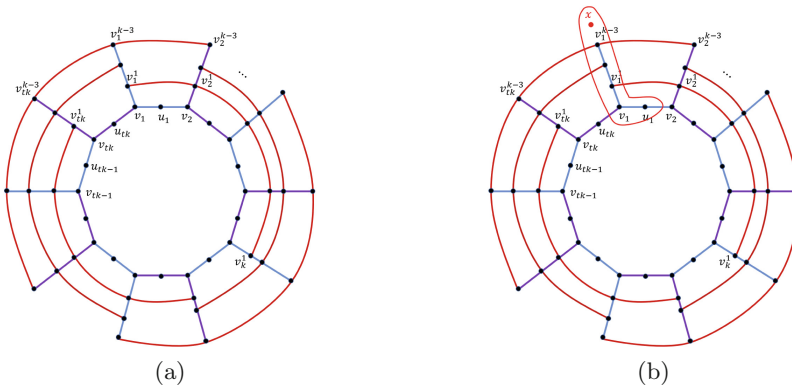


Fig. 2. The k -graphs (a) $G_{k,t}$, (b) $G_{k,t}^\circ$

The proof of Theorem 4 follows from the following two lemmas. The proofs are omitted due to space limitations:

- Lemma 3.** 1) The k -graph $G_{k,t}$ is symmetric and every non-identical automorphism ϕ of G_k satisfies that there exists an integer c (which does not divide tk) such that for every $i \in [tk], j = i + c - \lfloor \frac{i+c}{tk} \rfloor, \phi(E_i) = E_j$ (i.e. for each vertex $v \in E_i, \phi(v) \in E_j$).
- 2) The only automorphism of $G_{k,t}$ which leaves the set E_1 invariant (i.e. for each vertex $v \in E_1, \phi(v) \in E_1$) is the identity.
- 3) Every non-trivial k -subgraph of $G_{k,t}$ containing the vertices in E_1 has a non-identical automorphism ϕ which leaves the set E_1 invariant.

- Lemma 4.** 1) The k -graph $G_{k,t}^\circ$ is asymmetric.
- 2) Every non-trivial k -subgraph of $G_{k,t}^\circ$ has a non-identical automorphism.

It is easy to observe that the k -graphs $G_{k,t}^\circ$ have vertex degree at most 3. However note that in this construction, some of the strongly minimal asymmetric k -graphs $G_{k,t}^\circ$ are not minimal involution-free. In fact, when $k \geq 3$, $t \geq k - 2$ is odd, the k -subgraph $G_{k,t}^\circ - x$ of $G_{k,t}^\circ$ is involution-free. However the most interesting form of Theorem 4 relates to minimal asymmetric graphs for involutions. This can be stated as follows:

Theorem 5. *For every $k \geq 6$, there exist infinitely many k -graphs (X, \mathcal{M}) such that*

1. (X, \mathcal{M}) is asymmetric.
2. If (X', \mathcal{M}') is a k -subgraph of (X, \mathcal{M}) with at least two vertices, then (X', \mathcal{M}') has an involution.

This is more involved and the proof is omitted due to space limitations.

4 Concluding Remarks

Of course one can define the notion of asymmetric graph also for directed graphs, binary relations and k -nary relations $R \subseteq X^k$.

One has then the following analogy of Theorem 1: There are exactly 19 minimal asymmetric binary relations. (These are symmetric orientations of 18 minimal asymmetric (undirected) graphs and the single arc graph $(\{0, 1\}, \{(0, 1)\})$.)

Here is a companion problem about extremal asymmetric oriented graphs and one of the original motivation of the problem [1]:

Let $G = (V, E)$ be an asymmetric graph with at least two vertices. We say that G is *critical asymmetric* if for every $x \in V$ the graph $G - x = (V \setminus \{x\}, \{e \in E; x \notin e\})$ fails to be asymmetric. An oriented graph is a relation not containing two opposite arcs.

Conjecture 1. Let G be an oriented asymmetric graph. Then it fails to be critical asymmetric. Explicitly: For every oriented asymmetric graph G , there exists $x \in V(G)$ such that $G - x$ is asymmetric.

Wójcik [11] proved that a critical oriented asymmetric graph has to contain a directed cycle.

This research indicates a particular role of binary structures with respect to automorphism and asymmetry. While for higher arities there are infinitely many minimal asymmetric graphs, for binary structures this may be always finite. We formulate this in graph language as follows:

Let L be a finite set of colours. An L -graph is a finite graph where each edge gets one of the colours from L . Automorphisms are defined as colour preserving automorphisms. The following is a problem which generalizes the problem (and its solution [7]) which motivated the present note:

Conjecture 2. For every finite L , there are only finitely many minimal asymmetric L -graphs.

This is open for any $|L| > 1$. By results of this paper we know that for non-binary languages the analogous problem has negative solution.

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On Multicolour Ramsey Numbers and Subset-Colouring of Hypergraphs

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Abstract. For $n \geq s > r \geq 1$ and $k \geq 2$, write $n \rightarrow (s)_k^r$ if every k -colouring of all r -subsets of an n -element set has a monochromatic subset of size s . Improving upon previous results by Axenovich et al. (*Discrete Mathematics*, 2014) and Erdős et al. (*Combinatorial set theory*, 1984) we show that

$$\text{if } r \geq 3 \text{ and } n \not\rightarrow (s)_k^r \text{ then } 2^n \not\rightarrow (s+1)_{k+3}^{r+1}.$$

This yields an improvement for some of the known lower bounds on multicolour hypergraph Ramsey numbers.

Given a hypergraph $H = (V, E)$, we consider the Ramsey-like problem of colouring all r -subsets of V such that no hyperedge of size $\geq r + 1$ is monochromatic. We give upper and lower bounds on the number of colours necessary in terms of the chromatic number $\chi(H)$. We show that this number is $O(\log^{(r-1)}(r\chi(H)) + r)$.

Keywords: Ramsey numbers · Hypergraph colouring · Stepping-up lemma

1 Introduction

Even though Ramsey theory has attracted much attention from its inception almost a century ago, many questions remain elusive.

An extended preprint is available at <https://arxiv.org/abs/2103.12627>.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 503–508, 2021.
https://doi.org/10.1007/978-3-030-83823-2_81

Notations. For any natural number $n \in \mathbb{N}$, put $\llbracket n \rrbracket = \{0, \dots, n - 1\}$. For any set S and $k \in \mathbb{N}$, the set $\llbracket S \rrbracket^k$ is $\{T \subset S : |T| = k\}$, i.e., the set of all k -subsets of S . We write $\mathcal{P}(S)$ for the powerset of S . If $f : A \rightarrow B$ is a function and $X \subset A$, we write $f \upharpoonright X = \{f(x) : x \in X\}$ instead of the more usual (outside set theory) but ambiguous $f(X)$. Throughout the paper, \log is the *binary* logarithm (although the choice of base is inconsequential in most places).

Definition 1 (Rado’s arrow notation). *Given $k \geq 2, n > r \geq 1$, an r -subset k -colouring of $\llbracket n \rrbracket$ is a function $f : \llbracket \llbracket n \rrbracket \rrbracket^r \rightarrow \llbracket k \rrbracket$. A set $X \subset \llbracket n \rrbracket$ is monochromatic (under f , in the colour $i \in \llbracket k \rrbracket$) if $\llbracket X \rrbracket^r \subset f^{-1}(i)$, or equivalently $f \upharpoonright \llbracket X \rrbracket^r = \{i\}$. For k integers $(s_i)_{i \in \llbracket k \rrbracket}$ all satisfying $n \geq s_i > r$, we write*

$$n \rightarrow (s_0, \dots, s_{k-1})^r,$$

or more concisely $n \rightarrow (s_i)_{i \in \llbracket k \rrbracket}^r$, to mean that for every $f : \llbracket \llbracket n \rrbracket \rrbracket^r \rightarrow \llbracket k \rrbracket$ there is a colour $i \in \llbracket k \rrbracket$ in which a subset of $\llbracket n \rrbracket$ of size s_i is monochromatic. When $s_0 = s_1 = \dots = s_{k-1} = s$ (the diagonal case) this is further abbreviated to $n \rightarrow (s)_k^r$.

The logical negation of any arrow relation is written similarly, replacing \rightarrow with \nrightarrow .

Let $n \geq s > r \geq 1$ and $k \geq 2$. The (*multicolour, hypergraph, diagonal*) Ramsey number¹ $r_k(s; r)$ is the smallest n for which $n \rightarrow (s)_k^r$. The fact that these numbers exist is Ramsey’s 1930 theorem [11].

1.1 Previous Results

The values of $r_k(s; r)$ remain unknown except for several simple cases; in fact, for $r \geq 3$ only $r_2(4; 3) = 13$ is known [9]. See the book by Graham, Rothschild and Spencer [8] for background on finite Ramsey theory and the survey by Radziszowski [10] for recent bounds.

Some lower bounds on Ramsey numbers are obtained through *stepping-up lemmata* such as the following:

$$\text{if } r \geq 3 \text{ and } n \nrightarrow (s)_k^r \text{ then } 2^n \nrightarrow (s + 1)_{k'}^{r+1}, \tag{1}$$

for any $k' \geq 2k + 2r - 4$ [1] or $k' \geq k + 2^r + 2^{r-1} - 4$ [7, Lemma 24.1].

2 A Stepping-Up Lemma

Our first and main result is that (1) still holds with $k' \geq k + \eta(r)$, with the number of additional colours

$$\eta(r) = \begin{cases} 1 & \text{if } r = 3, \\ 2 & \text{if } r > 3 \text{ is even,} \\ 3 & \text{if } r > 3 \text{ is odd.} \end{cases} \tag{2}$$

¹ Different authors use different notations: cf. $r_k(K_s^r)$ [1], $r_r(s; k)$ [3], $R_k(s; r)$ [10].

(This improvement also extends to transfinite cardinals, strengthening results of Erdős et al. [6, Chap. 24] in certain ranges of parameters.)

Our main result is as follows.

Theorem 1. Fix integers $k \geq 2$ and $n > r \geq 3$ and k integers $(s_i)_{i \in \llbracket k \rrbracket}$ all satisfying $n \geq s_i \geq r + 1$.

$$\text{If } n \not\rightarrow (s_i)_{i \in \llbracket k \rrbracket}^r \text{ then } 2^n \not\rightarrow (s_0 + 1, \dots, s_{k-1} + 1, \underbrace{r + 2, \dots, r + 2}_{\substack{\eta(r) \text{ terms,} \\ \text{each } r + 2}})^{r+1},$$

with the integer $\eta(r) \leq 3$ defined above in (2).

To prove Theorem 1 let $k, n, r, (s_i)_{i \in \llbracket k \rrbracket}$ be as in the hypotheses. In particular, there exists $f_r : \llbracket n \rrbracket^r \rightarrow \llbracket k \rrbracket$ under which, for every $i \in \llbracket k \rrbracket$, no set of size s_i is monochromatic in colour i . We use f_r to construct $f_{r+1} : \llbracket 2^n \rrbracket^{r+1} \rightarrow \llbracket k + \eta(r) \rrbracket$ under which, for every $i \in \llbracket k \rrbracket$, no set of size $s_i + 1$ is monochromatic in colour i , and no set of size $r + 2$ is monochromatic in either of the additional colours $k, k + 1, \dots, k + \eta(r) - 1$.

2.1 Description of the Colouring

Splitting Indices. For every natural number n , let $d(n) \subset \mathbf{N}$ be the unique finite set of integers such that $n = \sum_{i \in d(n)} 2^i$. In other words, $d(n)$ is the set of non-zero indices in the binary representation of n . Given a finite set S of natural numbers, $|S| \geq 2$, its *first splitting index* is $s(S) = \max\{i \in \mathbf{N} : \exists x, y \in S : i \in d(x) \setminus d(y)\}$. That is, $s(S)$ is the index of the most significant digit where two elements in S differ in their binary representation.

The first splitting index partitions S into two disjoint, non-empty subsets $S_0 = \{x \in S : s(S) \notin d(x)\}$ and $S_1 = \{x \in S : s(S) \in d(x)\}$ with $\max S_0 < \min S_1$. This partition is unique and exists as soon as $|S| \geq 2$; we denote it by $S = (S_0 \mid S_1)$.

Caterpillars and Types. Consider the following process: if $S = (L \mid R)$ and $|L| = 1$ recurse on R , or if $|R| = 1$ recurse on L . This process either ends at a singleton, and we say that S is a *caterpillar*, or at $(L \mid R)$ with $|L|, |R| \geq 2$, and we say that the *type* $t(S) = (|L|, |R|)$. Thus each finite set of natural numbers that is not a caterpillar has a type in $\mathbf{N} \times \mathbf{N}$. For example, $t(\{0, 1, 2, 3, 4, 8\}) = t(\{0, 1, 2, 3\}) = (2, 2)$. Let \mathcal{C} denote the set of all caterpillars.

To each caterpillar $S \in \mathcal{C}$ we associate $\delta(S) = \{s(\{x, y\}) : x, y \in S, x \neq y\}$. For example, the reader may check that $\delta(\{1, 3, 6, 31\}) = \{1, 2, 4\}$.

Description. We can now define our colouring $f_{r+1} : \llbracket 2^n \rrbracket^{r+1} \rightarrow \llbracket k + 3 \rrbracket$. Let $S \in \llbracket 2^n \rrbracket^{r+1}$.

If $S \in \mathcal{C}$ then $\delta(S) \in \llbracket [n] \rrbracket^r$, and we let $f_{r+1}(S) = f_r(\delta(S)) \in \llbracket [k] \rrbracket$. Otherwise $S \notin \mathcal{C}$ has a type $t(S) = (p, q)$ with $p, q \geq 2$ and $p + q \leq r + 1$, and we let

$$f_{r+1}(S) = \begin{cases} k & \text{if } p + q = r + 1, p \text{ even,} \\ k + 1 & \text{if } p + q < r + 1, p + q \text{ even,} \\ 0 & \text{if } p + q = r + 1, p \text{ odd and } r \text{ odd,} \\ k + 2 & \text{if } p + q = r + 1, p \text{ odd and } r \text{ even,} \\ 1 & \text{if } p + q < r + 1, p + q \text{ odd.} \end{cases}$$

Claim. For every $i \in \llbracket [k] \rrbracket$, no set of size s_i is monochromatic in colour i under f_{r+1} .

3 Lower Bounds on Multicolour Hypergraph Ramsey Numbers

Erdős, Hajnal and Rado obtained the first bounds on $r_k(s; r)$ [5,6]. Here the *tower functions* are defined by $\text{twr}_1(x) = x$ and $\text{twr}_{r+1}(x) = 2^{\text{twr}_r(x)}$.

Theorem 2 (Erdős, Hajnal and Rado). *Let $r \geq 2$. There exists $s_0(r)$ such that:*

- For any $s > s_0$, we have $r_k(s; r) \geq \text{twr}_r(c'k)$,
- For any $s > r$, we have $r_k(s; r) \leq \text{twr}_r(ck \log k)$,

where $c' = c'(s, r)$ and $c = c(s, r) \leq 3(s - r)$.

Unlike the upper bound, the lower bound holds only for sufficiently large values of s . Duffus, Leffman and Rödl [4] gave a tower-function lower bound for all $s \geq r + 1$, but said bound, $\text{twr}_{r-1}(c''k)$, is significantly weaker than the bound of Theorem 2. Conlon, Fox, and Sudakov [3] proved that the lower bound of Theorem 2 holds whenever $s \geq 3r$. Axenovich et al. [1] matched the lower bound of Theorem 2 for all $s > r$, but only for sufficiently large k .

Theorem 3 (Axenovich et al.). *For any $s > r \geq 2$ and any $k > r2^r$, we have*

$$r_k(s; r) > \text{twr}_r\left(\frac{k}{2^r}\right).$$

In particular the results of [3] give $r_k(s; 3) \geq 2^{2^{c'k}}$ for some constant c' whenever $s \geq 9$, and the results of [1] give $r_k(4; 3) > 2^{2^{k/8}}$ for all $k > 24$.

As a consequence of Theorem 1, we prove that the lower bound of Theorem 2 holds for values of k much closer to r than in Theorem 3, and also improve the constant inside the tower function.

Corollary 1. *There are absolute constants $\alpha \simeq 1.678$ and β such that for $r = 3$ and any $k \geq 4$ or for $r \geq 4$ and $k \geq \lfloor 5r/2 \rfloor - 5$, we have:*

$$r_k(r + 1; r) > \text{twr}_r \left(\frac{\alpha}{2} \cdot \left(k - \frac{5r}{2} \right) + \beta \right), \tag{3}$$

and for $r = 3$ and any $k \geq 2$ or for $r \geq 4$ and $k \geq \lfloor 5r/2 \rfloor - 7$,

$$r_k(r + 2; r) > \text{twr}_r \left(\alpha \cdot \left(k - \frac{5r}{2} \right) + \beta \right). \tag{4}$$

To the best of our knowledge, the current best lower bound for $r_k(5; 3)$ (for large k) is $\text{twr}_3(k + O(1))$ [2]. Compare with our $r_k(5; 3) > \text{twr}_3(1.678k + O(1))$.

4 Subset Colouring in Hypergraphs

Our second result addresses a hypergraph colouring problem. Given a hypergraph H and $r \in \mathbf{N}$, we are interested in the smallest number $k = k(H; r)$ for which there exists a k -colouring of all r -subsets of vertices without any monochromatic hyperedge of size $\geq r + 1$. Note that if all the hyperedges in H are of size at least 2, then $k(H; 1) = \chi(H)$, the standard *vertex chromatic number* of H (i.e. the least number of colours in a colouring of V in which no hyperedge is monochromatic).

In his work on simplicial complexes, Sarkaria related this quantity (as the *weak r -th chromatic number*) to embeddability properties [12–14].

We show that for any H , the number of colours $k(H; r)$ is not much larger than the corresponding number of colours for the complete $(r+1)$ -uniform hypergraph with the same vertex chromatic number, that is, $k(K_{r\chi(H)}^{(r+1)}; r)$. Hence finding $k(n, r) = \max\{k(H; r) : \chi(H) = n\}$ for any n, r , is essentially equivalent to the problem of finding the Ramsey number $r_{k'}(r + 1; r)$ for an appropriate value of k' .

Theorem 4. *For any positive integers n and r ,*

$$k(K_{rn}^{(r+1)}; r) \leq k(n, r) \leq k(K_{rn}^{(r+1)}; r) + 5. \tag{5}$$

Together with our lower bound on multicolour Ramsey numbers and the previously known upper bound, we obtain:

$$\Omega \left(\frac{\log^{(r-1)}(rn)}{\log^{(r)}(rn)} \right) < k(n, r) < O \left(\log^{(r-1)}(rn) + r \right).$$

Theorem 4 mostly follows from Theorem 5 below.

Theorem 5. *Let $r, k \geq 2$ and let n be such that $n \not\rightarrow (r+1)_k^r$. If the hypergraph (V, E) admits a vertex-colouring $V \rightarrow \llbracket n \rrbracket$ under which no hyperedge of size $\geq r+1$ is monochromatic, then there is an r -subset colouring $[V]^r \rightarrow \llbracket k + f(r) \rrbracket$ such that no hyperedge of size $\geq r+1$ is monochromatic. Here*

$$f(r) = \begin{cases} 1 & \text{if } r = 2, \\ 3 & \text{if } r = 3, \\ 4 & \text{if } r \geq 4 \text{ and } r + 1 \text{ is prime,} \\ 5 & \text{otherwise.} \end{cases}$$

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Max Cuts in Triangle-Free Graphs

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Abstract. A well-known conjecture by Erdős states that every triangle-free graph on n vertices can be made bipartite by removing at most $n^2/25$ edges. This conjecture was known for graphs with edge density at least 0.4 and edge density at most 0.172. Here, we will extend the edge density for which this conjecture is true; we prove the conjecture for graphs with edge density at most 0.2486 and for graphs with edge density at least 0.3197. Further, we prove that every triangle-free graph can be made bipartite by removing at most $n^2/23.5$ edges improving the previously best bound of $n^2/18$.

Keywords: Extremal combinatorics · Graph theory · Triangle-free graphs

1 Introduction

How many edges need to be removed from a triangle-free graph on n vertices to make it bipartite? Erdős [2] asked this question and conjectured that $n^2/25$ edges would always be sufficient. This would be sharp as the balanced blow-up of C_5 with class sizes $n/5$ needs at least $n^2/25$ edges removed to be made bipartite. For a graph G , denote $D_2(G)$ the minimum number of edges which have to be removed to make G bipartite.

Conjecture 1. (Erdős [2]) For every triangle-free graph G on n vertices

$$D_2(G) \leq \frac{n^2}{25}. \quad (1)$$

An elementary probabilistic argument (see e.g. [5]) resolves Conjecture 1 for graphs G with at most $2/25n^2$ edges: Take a random bipartition where each vertex, independently from each other, is placed with probability $1/2$ in one of the two classes. The expected number of edges inside both of the classes is $|E(G)|/2$. Thus, there exists a bipartition with at most $|E(G)|/2$ edges inside the classes. Note that this argument does not use that G is triangle-free. Erdős, Faudree, Pach and Spencer [4] slightly improved this random cut argument utilizing triangle-freeness.

Theorem 1 (Erdős, Faudree, Pach, Spencer [4]). *For every triangle-free graph with n vertices and m edges*

$$D_2(G) \leq \min \left\{ \frac{m}{2} - \frac{2m(2m^2 - n^3)}{n^2(n^2 - 2m)}, m - \frac{4m^2}{n^2} \right\} \leq \frac{n^2}{18}. \tag{2}$$

This confirmed Conjecture 1 for graphs with roughly at most $0.086n^2$ edges and graphs with at least $n^2/5$ edges. It also gives the current best bound on the Erdős problem; one can remove at most $n^2/18$ edges to make a triangle-free graph bipartite. We improve this result and extend the range for which Erdős’ conjecture is true.

Theorem 2. *Let G be a triangle-free graph on n vertices. Then, for n large enough,*

- (a) $D_2(G) \leq \frac{n^2}{23.5}$,
- (b) $D_2(G) \leq \frac{n^2}{25}$ when $|E(G)| \geq 0.3197 \binom{n}{2}$,
- (c) $D_2(G) \leq \frac{n^2}{25}$ when $|E(G)| \leq 0.2486 \binom{n}{2}$.

Sudakov studied a related question; he [12] determined the maximum number $D_2(G)$ for K_4 -free graph G . Recently, Hu, Lidický, Martins, Norin and Volec [7] announced a proof for determining the maximum number $D_2(G)$ for n -vertex K_6 -free graphs G . They use the method of flag algebras, developed by Razborov [11], to describe local cuts which leads to the solution. We use a similar idea of encoding local cuts.

Our proof of Theorem 2 also extends on the ideas from Erdős, Faudree, Pach, Spencer [4]. While their proof uses two different ways of finding bipartitions, our proof uses many ways. In order to handle a large amount of bipartitions, we use the method of flag algebras. It relies on formulating a problem as a semidefinite program and then using a computer to solve it.

We will handle graphs with edge density close to $2/5$ (the density of the conjectured extremal example) separately. In this range we use standard techniques from extremal combinatorics, such as a minimum degree removing algorithm. Additionally, we will make use of the following result by Erdős, Győri and Simonovits [5].

A C_5 -blow-up H is a graph with vertex set $V(H) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ and edges $xy \in E(H)$ iff $x \in A_i$ and $y \in A_{i+1}$ for some $i \in [5]$, where $A_6 := A_1$.

Theorem 3 (Erdős, Győri and Simonovits [5]). *Let G be a K_3 -free graph on n vertices with at least $n^2/5$ edges. Then there exists an unbalanced blow-up of C_5 H such that*

$$D_2(G) \leq D_2(H). \tag{3}$$

Note that this result recently was extended to cliques by Korándi, Roberts and Scott [8] confirming a conjecture from [1].

There is a local version of Conjecture 1.

Conjecture 2. (Erdős [2]) Every triangle-free graph on n vertices contains a vertex set of size $\lfloor n/2 \rfloor$ that spans at most $n^2/50$ edges.

Erdős [3] offered \$250 for the first solution of this conjecture. As pointed out by Krivelevich [9], for regular graphs Conjecture 2 would imply Conjecture 1. We are wondering if similar methods we are using could be used to make progress towards proving Conjecture 2.

This extended abstract is organized as follows. In Sect. 2.1 we present our setup for flag algebras to give a sketch of the proof of the main part of Theorem 2. In Sect. 2.2 we sketch the proof of Conjecture 1 in the edge range slightly below edge density $2/5$.

2 Proof Sketch of Theorem 2

2.1 Setup for Flag Algebras

Towards contradiction assume that there is a triangle-free graph G on n vertices with $D_2(G) \geq n^2/25$. This means that whenever we create a bipartition of $V(G)$, then it has at least $n^2/25$ edges inside the two parts. Using flag algebras, one can define bipartitions and count edges inside of the two parts.

For example, in a graph G one could fix a vertex v and define the bipartition of G as $V(G) = N(v) \cup \overline{N(v)}$. If one uses this bipartition, all edges in $\overline{N(v)}$ need to be removed while $N(v)$ is independent since G is triangle-free. This can be written in flag algebras in the following way

$$\begin{array}{c}
 \bullet \text{---} \bullet \\
 \square \\
 v
 \end{array}
 \geq \frac{2}{25},
 \tag{4}$$

where the depicted graph represents its expected induced density when unordered pair of black vertices is picked uniformly at random while the yellow vertex is fixed. In proving Theorem 1, Erdős, Faudree, Pach, Spencer [4] used this cut and the following cut. Let uv be two adjacent vertices. Let $\overline{N(u)}$ be one part and $N(v)$ be the other part. The remaining vertices in $\overline{N(u)} \cup N(v)$ are partitioned uniformly at random with probability $1/2$ to either of the two parts. Since G is K_3 -free, one obtains the following equation for flag algebras

$$\frac{1}{2}
 \begin{array}{c}
 \bullet \text{---} \bullet \\
 \square \text{---} \square \\
 u \quad v
 \end{array}
 + \frac{1}{2}
 \begin{array}{c}
 \bullet \text{---} \bullet \\
 \square \text{---} \square \\
 u \quad v
 \end{array}
 + \frac{1}{2}
 \begin{array}{c}
 \bullet \text{---} \bullet \\
 \square \text{---} \square \\
 u \quad v
 \end{array}
 \geq \frac{2}{25}.
 \tag{5}$$

This idea of defining cuts can be generalized by rooting on more vertices. Pick a copy of a labeled graph H on k vertices in G . This will partition the rest of $V(G)$ into classes X_1, \dots, X_{2^k} based on the adjacencies to the fixed k vertices. Now we construct a bipartition of $V(G)$ into sets A and B . For each class X_i fix $p_i \in [0, 1]$ and for each vertex in X_i we put it to A with probability p_i and to put it to B otherwise, i.e., with probability $(1 - p_i)$.

This creates a bipartition and it is possible to count the edges that need to be removed using flag algebras. We can include all cuts rooted on at most 4 vertices and C_5 .

1. $|V(H)| \leq 2$ and $p_i \in \{0, 0.5, 1\}$, gives 10 cuts,
2. $|V(H)| \leq 3$ and $p_i \in \{0, 0.5, 1\}$, gives 108 cuts,
3. $|V(H)| = 4$ and $p_i \in \{0, 1\}$, gives 953 cuts,
4. $H = C_5$, and $p_i \in \{0, 1\}$, gives 125 cuts.

However, for $k \geq 6$, there are more possible inequalities than computers can reasonably handle. Therefore we have to decide on which we want to use. We will present two particular important ones here.

Norin and Ru Sun [10] observed that the Clebsch graph, see Fig. 1, is particularly unfriendly when applying local cuts. We add cuts that are specially designed to cut the Clebsch graph. The root is a 4-cycle $v_0v_1v_2v_3v_0$ and two additional vertices v_4 and v_5 with edges v_4v_0 and v_1v_5 . Although this is a bipartite graph, we create a bipartition as if v_1, v_2, v_5 and v_1, v_3, v_4 were in the same parts respectively.

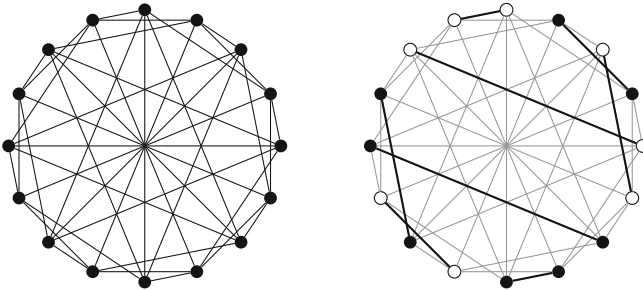


Fig. 1. Clebsch graph and its cutting

Another inequality that made a big difference is an extension of (5). While (5) partitions neighbors of the chosen two vertices very well, the non-neighbors can be partitioned better. In particular, we pick another K_2 in the non-neighborhood and do the same partition once more. This results in rooting on $K_2 \cup K_2 \cup K_2$.

Our flag algebra proof cannot deal with the density range close to $2/5$, i.e. close to the conjectured extremal example. In the following section we explain how this density range can be handled.

2.2 High Density Range

In this section we provide a sketch of the proof of Erdős’ conjecture for graphs with edge density slightly below $2/5$.

Theorem 4. *There exists n_0 such that for all $n \geq n_0$ the following holds. Let G be an n -vertex triangle-free graph with $|E(G)| \geq (0.2 - \varepsilon)n^2$ edges, where $\varepsilon = 10^{-8}$. Then $D_2(G) \leq n^2/25$.*

Let $G_n := G$ be a triangle-free graph on n vertices with $|E(G)| \geq (0.2 - \varepsilon)n^2$ edges. Assume, towards contradiction, $D_2(G) > n^2/25$. We iteratively remove a vertex of minimum degree from G . This means $G_i = G_{i+1} - x$, where $\deg(x) = \delta(G_{i+1})$. We stop this algorithm if $\delta(G_i) > \frac{3}{8}i$ or after $\lfloor 5 \cdot 10^{-7}n \rfloor$ rounds. Let m be the stage in which the algorithm stops.

Lemma 1. *We have*

$$D_2(G) \leq \frac{3}{32}(n^2 - m^2 + n - m) + D_2(G_m). \tag{6}$$

This Lemma can be verified by taking a smallest cut of G_m and adding the remaining vertices to the set where they have smaller neighborhood in.

Depending on when the algorithm stops we perform a different analysis. If the algorithm stops “late”, then G_m has edge density of slightly more than $2/5$. By Lemma 1, we can assume that

$$D_2(G_m) \geq \frac{n^2}{25} - \frac{3}{32}(n^2 - m^2 + n - m). \tag{7}$$

By Theorem 3 we can find a C_5 -blow-up H on m vertices with classes A_1, A_2, A_3, A_4, A_5 satisfying $|E(H)| \geq |E(G_m)|$ and $D_2(H) \geq D_2(G_m)$. In fact, it can also be assumed that the class sizes of H are symmetric, that is $|A_2| = |A_5| + o(n)$ and $|A_3| = |A_4| + o(n)$. A straight-forward optimization of the number of edges in H gives a contradiction with $|E(H)| \geq |E(G_m)|$.

If the algorithm stops early, we make use of a result by Häggkvist [6] who proved that every triangle-free graph on n vertices with minimum degree more than $3n/8$ is a subgraph of a C_5 -blow-up. Having this particular structure, it can be calculated that $D_2(G) \leq n^2/25$, we omit the detailed computations.

2.3 Concluding Remarks

Note that Theorem 2 only holds for $n \geq n_0$ for some n_0 large enough. However, this is not an actual restriction towards proving Conjecture 1. Assuming Conjecture 1 were to hold for all $n \geq n_0$, then it actually holds for all n by the following argument. Let G be a triangle-free graph on $n < n_0$ vertices and assume, towards contradiction, that $D_2(G) > n^2/25$. Consider the blow-up G' of G , where each vertex is replaced by an independent set of size $\lceil \frac{n_0}{n} \rceil$ and two vertices in different sets are made adjacent iff the corresponding vertices in G were adjacent. This new graph G' is still triangle-free and has at least n_0 vertices. A result by Erdős, Györi and Simonovits [5, Theorem 7] gives

$$\frac{D_2(G')}{(\lceil \frac{n_0}{n} \rceil n)^2} \geq \frac{D_2(G)}{n^2} > \frac{1}{25}, \tag{8}$$

contradicting that we assumed Conjecture 1 holds for all $n \geq n_0$ and therefore in particular for G' .

We believe Theorem 2 can be improved by adding more cuts to the calculation and possibly lead to the proof of Conjecture 1. Adding more cuts lead to marginal improvements so far. We are looking at other cuts as well but the time needed to perform the calculations grows quickly and it may take a while until a significant improvement is obtained.

Acknowledgement. We thank Humberto Naves, Florian Pfender, and Jan Volec for fruitful discussions in early stages of this project. The first author is supported by NSF grants DMS-1764123, DMS-1937241, the Langan Scholar Fund (UIUC), and the Simons Fellowship. The first and second authors are supported by the Arnold O. Beckman Research Award (UIUC RB 18132). The last author is supported by NSF grant DMS-1855653.

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On the Cycle Rank Conjecture About Metric Dimension and Zero Forcing Number in Graphs

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Abstract. The *metric dimension* $\dim(G)$ of a graph G is the minimum cardinality of a subset S of vertices of G such that each vertex of G is uniquely determined by its distances to S . The *zero forcing number* $Z(G)$ of G is the minimum cardinality of a subset S of *black* vertices of G such that all the vertices will be turned black after applying finitely many times the following rule: *a non black vertex is turned black if it is the only non black neighbor of a black vertex*.

Eroh, Kang and Yi conjectured in 2017 that, for every graph G , $\dim(G) \leq Z(G) + c(G)$ where $c(G)$ is the cycle-rank of G . We prove a weaker version of the conjecture: $\dim(G) \leq Z(G) + 6c(G)$ holds for any graph. We also prove that the conjecture is true for cactus graphs.

Keywords: Graph theory · Metric dimension · Zero-forcing number

1 Introduction

A *zero forcing set* is a subset of vertices colored in black which colors the whole vertex set in black when we iteratively apply the following rule: *A vertex is colored black if it is the unique non black neighbor of a black vertex* (see Fig. 1 for an illustration). The *zero forcing number* of a graph is the minimal size of a zero forcing set, denoted by $Z(G)$. The zero forcing number has been introduced to bound the rank of some families of adjacency matrices in [3]. Deciding if the zero forcing number of a graph is at most k is NP-complete [7].

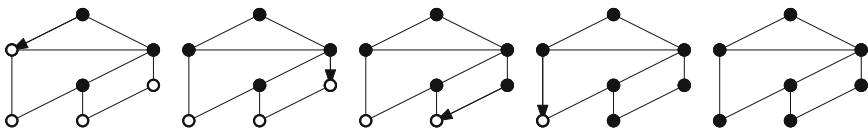


Fig. 1. Iterations of the change color rule. On the graph on the left, the three black vertices form a zero forcing set.

A *resolving set* of a graph G is a subset S of vertices of G such that any vertex of G is identified by its distances to the vertices of S . In Fig. 2, the set $\{A, B\}$ is a resolving set of the graph because all the vertices have a different *distance vector* to $\{A, B\}$, so the knowledge of the distance to A and B identifies uniquely a vertex. The *metric dimension* of G , denoted by $\dim(G)$, is the minimum size of a resolving set. A vertex w *resolves* a pair of vertices (u, v) if $d(w, u) \neq d(w, v)$. A set of vertices S resolves a set W if S resolves all the pairs of W . A set S is a resolving set if S resolves V . This notion has been introduced by Slater [2] for trees and by Harary and Melter [4] for graphs. Since its introduction the notion became an important notion in graph theory and has been widely studied. Resolving sets and related notions have many applications e.g. the navigation of a robot in an Euclidean space [5]. Determining the minimum size of a resolving set is NP-complete [6] even restricted to planar graphs [8].

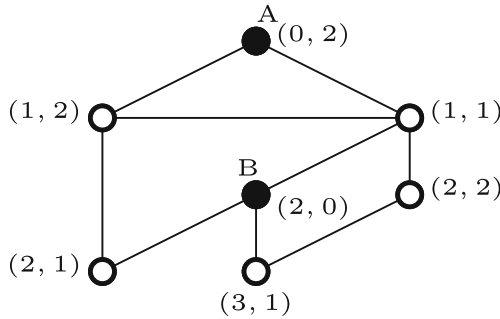


Fig. 2. The black vertices form a resolving set. For each vertex x the vector next to x , $(d(A, x), d(B, x))$ is unique.

In general, the gap between metric dimension and zero-forcing number can be arbitrarily large. But for some restricted sparse graph classes like paths or cycles, both the parameters and optimal sets are the same. Eroh, Kang and Yi have started a systematic comparison between them in [1]. They proved that $\dim(G) \leq Z(G)$ when G is a tree and that $\dim(G) \leq Z(G)+1$ when G is *unicyclic* (G is a tree plus an edge). On the other hand, $\dim(G)$ can be arbitrarily larger than the zero forcing number when the number of cycles increases. They made the following conjecture:

Conjecture 1 (Cycle-rank conjecture [1]). Every connected graph G satisfies $\dim(G) \leq Z(G) + c(G)$ where $c(G)$ is the minimal number of edges that have to be removed from G to obtain a tree.

Conjecture 1 is tight for infinitely many graphs. Consider the graph G_c composed of a path of 3 vertices plus c cycles of size 4 intersecting in the central vertex of the path (see Fig. 3). Then $\dim(G_c) = 2c + 1$ and $Z(G_c) = c + 1$.

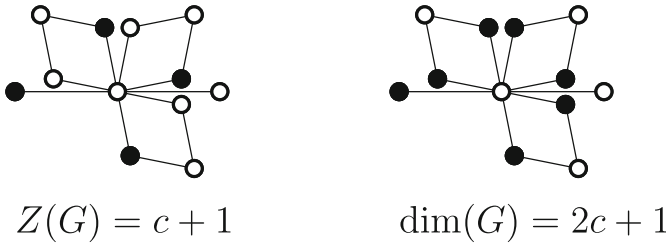


Fig. 3. Tightness of Conjecture 1

Towards this conjecture, Eroh et al. proved in [1] that $\dim(G) \leq Z(G) + 2c(G)$ when G has no even cycles.

In this paper, we prove Conjecture 1 in several particular cases and prove a weaker version of the conjecture in general. We first give a proof of Conjecture 1 for unicyclic graphs which is much shorter and simpler than the one of [1]. We also show that $\dim(G) \leq Z(G)$ when the unique cycle of G has odd length. We then prove Conjecture 1 for cactus graphs (graphs with edge-disjoint cycles) which generalizes the result for unicyclic graphs (and in particular prove the conjecture for graphs without even cycles, improving the result of [1]).

We finally prove a weaker version of Conjecture 1 which is our main result:

Theorem 1. *For every graph G of cycle-rank at least one, we have*

$$\dim(G) \leq Z(G) + 6c(G) - 5.$$

As far as we know, it is the first upper bound of $\dim(G)$ of the form $Z(G) + f(c(G))$. One can wonder if the dependency on $c(G)$ can be removed and if $\dim(G)$ can be upper bounded by a function of $Z(G)$ only. The answer is negative: for G_n a path of length n connected to a universal vertex, $Z(G_n) = 2$ for any $n \geq 2$ but $\dim(G_n)$ is a linear function in n .

In our proof of Theorem 1, feedback vertex sets (subsets of vertices whose deletion leave an acyclic graph) are playing an important role. Actually, our result is even stronger since we prove that

$$\dim(G) \leq Z(G) + \min\{5c(G) + \tau(G); 3c(G) + 5\tau(G)\} - 5$$

where $\tau(G)$ is the size of a minimum feedback vertex set in G .

We raise the following question, which is a weakening of Conjecture 1:

Conjecture 2. There exists a function f such that, for any connected graph G , $\dim(G) \leq Z(G) + c(G) + f(\tau(G))$ where $c(G)$ is the minimal number of edges that have to be removed from G to obtain a tree and $\tau(G)$ is a minimum size of a feedback vertex set of G .

2 Cactus Graphs

For acyclic graphs, the following holds:

Lemma 1. [1] *For every tree T , $\dim(T) \leq Z(T)$.*

Conjecture 1 is proved in [1] for unicyclic graphs. We give a shorter proof of this result. The main ingredient of our proof is the following lemma.

Lemma 2. *Let $G = (V, E)$ be a graph which is not a tree and $C \subseteq V$ a cycle in this graph. Then, there exists an edge $e \in E(C)$ such that $Z(G - e) \leq Z(G)$.*

Proof. Let $Z \subseteq V$ be a minimum zero forcing set of G . For $e = uv \in E$ we say e is a *forcing edge* if at some step u is black and v is the only white neighbor of u . Let $F \subseteq E$ be the *forcing edges* in a sequence starting from Z .

We claim that at least one edge of C is not in F . Indeed, if u forces v then u is turned black before v . So the first vertex w of the cycle that is turned black cannot be turned black because of an edge of C . Let w_1, w_2 be the two neighbors of w on C . The vertex w can force at most one of its two neighbors. So without loss of generality, w_2 is not forced by w and is turned black after w . So removing the edge $e = ww_2$ leaves a forcing set of $G - e$ where the same sequence of applications of the color change rule turns G into black. Therefore $Z(G - e) \leq Z(G)$.

Lemma 1 together with the following lemma will allow us to prove Conjecture 1 for unicyclic graphs.

Lemma 3. [1] *Let $G = (V, E)$ be a graph and C be a cycle of G and let $V(C) = \{v_0, v_1, \dots, v_k\}$ be the vertices of C . Denote by $G_i = (V_i, E_i)$ the connected component of the vertex u_i in $G \setminus E(C)$. If, for every $i \neq j$, $V_i \cap V_j = \emptyset$ then for any $e \in E(C)$, $\dim(G) \leq \dim(G - e) + 1$.*

Corollary 1. *Let G be a unicyclic graph. Then, $\dim(G) \leq Z(G) + 1$.*

Proof. A unicyclic graph contains exactly one cycle C . Let e be an edge of C such that $Z(G - e) \leq Z(G)$. Such an edge exists by Lemma 2. By Lemma 3, $\dim(G) \leq \dim(G - e) + 1$. Moreover, by Lemma 1, $\dim(G - e) \leq Z(G - e)$ since $G - e$ is a tree. The combination of these three inequalities gives $\dim(G) \leq Z(G) + 1$.

We also give a more precise result in the case where the cycle has odd length.

Theorem 2. *Let G be a unicyclic graph, if its cycle has odd length, then $\dim(G) \leq Z(G)$.*

This result is tight and cannot be extended to unicyclic graphs with an even cycle as shown in Fig. 4.

We prove Conjecture 1 is true for cactus graph. We prove this result by induction on the cycle-rank using similar arguments than for the unicyclic case.

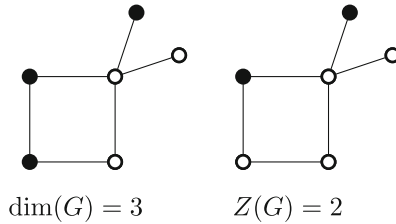


Fig. 4. Black vertices form respectively a metric basis and a minimal zero forcing set.

Theorem 3. *Let $G = (V, E)$ be a cactus graph. Then, $\dim(G) \leq c(G) + Z(G)$.*

It improves a result of [1] that ensures that $\dim(G) \leq Z(G) + 2c(G)$ for G with no even cycles. Indeed graphs without even cycles are cactus graphs since if a graph contains two odd cycles that share one edge, it also contains an even cycle.

3 General Bound

We sketch the proof of the following theorem that implies Theorem 1 since $\tau(G) \leq c(G)$.

Theorem 4. *Let $G = (V, E)$ be a graph of cycle rank at least one. Then,*

$$\dim(G) \leq Z(G) + \min(5c(G) + \tau(G) - 5, 3c(G) + 5\tau(G) - 5).$$

The main idea to prove Theorem 4 is to remove a small number of vertices M of G in order to guarantee that $G \setminus M$ is a forest where each tree is connected to at most two vertices of M . To construct M , we start with a minimal feedback vertex set X of G and add few vertices to X in the following way. For each connected component G_i of $G \setminus X$, let N_i be the subset of vertices of G_i adjacent to X in G . Let T_i be the minimal subtree of G_i containing N_i . Let M_i be the set of vertices of T_i of degree at least three and let $M = \bigcup_i (M_i) \cup X$. Then, any component of $G \setminus M$ is connected to M by at most two edges.

Then, to construct a resolving set of G , the idea is to take M and resolving sets in all the trees of $G \setminus M$. However, this is not enough to identify in which connected component of $G \setminus M$ a vertex belongs to. For that, we need a set of vertices P constructed as follows. Let H be a connected component of $G \setminus M$ with exactly two edges between H and $G \setminus H$. Let x and y be the two endpoints of the edges between H and M . Let ρ_H be a vertex on the middle of the path between x and y in H . In other words, ρ_H must satisfy $|d_H(x, \rho_H) - d_H(y, \rho_H)| \leq 1$. Let P be the union of the vertices ρ_H .

We can now find a resolving set in G . For each connected component G_i of $G \setminus X$, let S_i be a metric basis of G_i .

Lemma 4. *The set $S = M \cup P \cup (\bigcup_i S_i)$ is a resolving set of G . In particular, $\dim(G) \leq |M| + \sum_i |S_i| + |P|$.*

The size of P and M are bounded by functions of $c(G)$ and $\tau(G)$:

Lemma 5. *We have $|P| \leq c(G) + |M| - 1$ and $|M| \leq \min\{2c(G) - 2; c(G) + 2\tau(G) - 2\}$.*

Proof. We give the proof for the first inequality. The second one is proved in a similar way. Let K be the multigraph (with loops) with vertex set M and an edge between two vertices x and y if and only if there exists in $G \setminus M$ a connected component H adjacent to x and y . One can easily notice that in K there is an edge xy with multiplicity k if and only if in $G \setminus M$ there are k connected components attached to x and y . This give an isomorphism between P and the edges of K . Since K is a minor of G , $c(K) \leq c(G)$. As K contains $|M|$ vertices it contains at most $c(G) + |M| - 1$ edges. So we have $|P| \leq c(G) + |M| - 1$.

Finally, we prove that a zero-forcing set of G is almost a zero-forcing set in each tree of $G \setminus M$. Using Lemma 1, it leads to the following bound:

Lemma 6. $|\bigcup_i S_i| - \tau(G) \leq Z(G)$.

Proof. Let Z be a minimal zero forcing set for G . Let \mathcal{F} be a sequence of forcings that turns G into black starting from Z . Let Z' be the set of vertices turned black in \mathcal{F} because of a vertex of X . Note that $|Z'| \leq \tau(G)$. Let G_i be a connected component of $G[V \setminus X]$. We claim that $(Z \cup Z') \cap V(G_i)$ is a zero-forcing set of G_i . So, by Lemma 1, we have $|S_{G_i}| \leq |Z' \cap G_i|$. When we consider the union over all the components, it gives $|Z \cup Z'| \geq \bigcup_i |S_{G_i}|$. Since Z' has size at most $\tau(G)$, we have $Z(G) \geq |\bigcup_i S_{G_i}| - \tau(G)$.

Theorem 4 is a direct consequence of Lemmas 4, 5 and 6.

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Tight Bounds for Powers of Hamilton Cycles in Tournaments

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Abstract. A basic result in graph theory says that any n -vertex tournament with in- and out-degrees larger than $\frac{n-2}{4}$ contains a Hamilton cycle, and this is tight. In 1990, Bollobás and Häggkvist significantly extended this by showing that for any fixed k and $\epsilon > 0$, and sufficiently large n , all tournaments with degrees at least $\frac{n}{4} + \epsilon n$ contain the k -th power of a Hamilton cycle. Given this, it is natural to ask for a more accurate error term in the degree condition.

We show that if the degrees are at least $\frac{n}{4} + cn^{1-1/\lceil k/2 \rceil}$ for some constant $c = c(k)$, then the tournament contains the k -th power of a Hamilton cycle. We also present a construction which, modulo a well-known conjecture on Turán numbers for complete bipartite graphs, shows that the error term must be of order at least $n^{1-1/\lceil (k-1)/2 \rceil}$, which matches our upper bound for all even k . For $k = 3$, we improve the lower bound by constructing tournaments with degrees $\frac{n}{4} + \Omega(n^{1/5})$ and no cube of a Hamilton cycle.

1 Introduction

Hamiltonicity is one of the most central notions in graph theory, and it has been extensively studied by numerous researchers. The problem of deciding Hamiltonicity of a graph is NP-complete, but there are many important results which derive sufficient conditions for this property. One of them is the classical Dirac's theorem [4], which states that every graph with minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle, and that this is tight. Another natural and more general property is to contain the k -th power of a Hamilton cycle. Extending Dirac's theorem and confirming a conjecture of Seymour [13], Komlós, Sárközy and Szemerédi [10] determined the minimum degree condition for a graph to contain the k -th power of a Hamilton cycle. They proved that for large n , a minimum degree of $\frac{kn}{k+1}$ is enough.

Clearly, one can ask similar questions for directed graphs (see [12]), which tend to be more difficult. In 1979, Thomassen [14] asked the question of determining the *minimum semidegree* $\delta^0(G)$ (that is, the minimum of all in- and out-degrees) which implies the existence of a Hamilton cycle in an oriented graph G . This was only answered thirty years later by Keevash, Kühn and Osthus [8], who showed that $\delta^0(G) \geq \frac{3n-4}{8}$ forces a Hamilton cycle, which is tight by

a construction of Häggkvist [7]. Already the problem for squares of Hamilton cycles is not well understood. Treglown [15] showed that $\delta^0(G) \geq \frac{5n}{12}$ is necessary, which was subsequently improved by DeBiasio (personal communication). He showed that $\delta^0(G) \geq \frac{3n}{7} - 1$ is needed, using a slightly unbalanced blowup of the Paley tournament on seven vertices. It would be interesting to determine, even asymptotically, the optimal value of $\delta^0(G)$ which implies the existence of the square of a Hamilton cycle. For clarity, by the k -th power of the directed path $P_l = v_0 \dots v_l$ we mean the directed graph P_l^k on the same vertex set with an edge $v_i v_j$ if and only if $i < j \leq i + k$. The k -th power of a directed cycle is similarly defined.

Due to the difficulty of these problems in general, it is natural to ask what happens in tournaments. It is a very basic result that every tournament with minimum semidegree $\frac{n-2}{4}$ has a Hamilton cycle and that this is best possible. By how much do we need to increase the degrees in order to guarantee a k -th power? The remarkable result by Bollobás and Häggkvist [2] given below says that a little bit is already enough.

Theorem 1. *For every $\epsilon > 0$ and k , there exists a $n_0 = n_0(\epsilon, k)$ such that every tournament T on $n \geq n_0$ vertices with $\delta^0(T) \geq \frac{n}{4} + \epsilon n$ contains the k -th power of a Hamilton cycle.*

This theorem suggests two questions. For fixed ϵ , how large should n_0 be as a function of k and what is the correct order of magnitude of the additive error term in the degree condition? The proof of Bollobás and Häggkvist needs n_0 to grow faster than $t(\lceil \log_2(1/\epsilon) \rceil + 2)$, where t is a tower-type function defined by letting $t(0) = 2^k$ and $t(i+1) = \frac{1}{2}(2\epsilon)^{-t(i)}$. It also does not give any additional information about the error term, apart from showing that it is $o(n)$.

In this paper we address both these questions, resolving the first one and obtaining nearly tight bounds for the second one. We start with the additive error in the degree condition.

Theorem 2. *There exists a constant $c = c(k) > 0$ such that any tournament T on n vertices with $\delta^0(T) \geq \frac{n}{4} + cn^{1-1/\lceil k/2 \rceil}$ contains the k -th power of a Hamilton cycle.*

In particular, we show that a constant error term is enough for the tournament to contain the square of a Hamilton cycle.

It appears that this theorem is nearly tight. This follows from a somewhat surprising connection between our question and the Turán problem for complete bipartite graphs. Indeed, suppose that v is a vertex in some tournament T and is such that the bipartite graph with parts $N^-(v)$ and $N^+(v)$ (respectively, the in and out-neighbourhoods of v) generated by the edges in T which are directed from $N^-(v)$ to $N^+(v)$ is $K_{r,r}$ -free. Then, it is easy to see that T cannot contain the $2r$ -th power of a Hamilton cycle. Using this observation, the next result gives a construction of tournaments with large minimum semidegree which do not contain the k -th power of a Hamilton cycle. As usual, for a fixed graph H we let $\text{ex}(n, H)$ denote the maximal number of edges in a n -vertex graph which does not contain H as a subgraph.

Theorem 3. *Let $k \geq 2$ and $r = \lceil \frac{k-1}{2} \rceil$. For all sufficiently large $n = 3 \pmod{4}$, there exists a n -vertex tournament T with $\delta^0(T) \geq \frac{n+1}{4} + \Omega\left(\frac{\text{ex}(n, K_{r,r})}{n}\right)$ which does not contain the k -th power of a Hamilton cycle.*

Modulo a well-known conjecture on Turán numbers for complete bipartite graphs, this result implies that in addition to $n/4$, the semidegree bound must have an additive term of order at least $n^{1-1/\lceil(k-1)/2\rceil}$, which matches the bound in Theorem 2 for all even k . Indeed, the celebrated result of Kővári, Sós and Turán [11], says that $\text{ex}(n, K_{r,r}) = O(n^{2-1/r})$ and this estimate is widely believed to be tight. Moreover, for unbalanced complete bipartite graphs, it was proven by Alon, Kollár, Rónyai and Szabó [1, 9] that $\text{ex}(n, K_{r,s}) = \Omega(n^{2-1/r})$ when $s > (r - 1)!$. It is also known that $\text{ex}(n, K_{r,r}) = \Omega(n^{2-1/r})$ for $r = 2, 3$ [3, 6], which corresponds to $k = 4, 6$ in our problem.

For odd values of k there is still a small gap between the results in Theorems 2 and 3, which would be interesting to bridge. We make a step in this direction, showing that for $k = 3$, the constant error term in Theorem 3 can be improved to a power of n .

Theorem 4. *For infinitely many values of n , there exists a tournament on n vertices with minimum semidegree $\frac{n}{4} + \Omega(n^{1/5})$ and no cube of a Hamilton cycle.*

In the next section, we will give a sketch of the proof of the main theorem and in the final section, we will make some concluding remarks.

2 Proof Outline of Theorem 2

The main idea is based on a dichotomy that occurs in the structure of tournaments. We say that a tournament T is δ -cut-dense if any balanced partition (X, Y) of $V(T)$ is such that $\vec{e}(X, Y) \geq \delta|X||Y|$. Note in particular, that every tournament with minimum semidegree at least $\frac{n}{4} + \frac{\delta n}{2}$ is δ -cut-dense.

We will first consider tournaments which are cut-dense and show that they contain the k -th power of a Hamilton cycle even if the minimum semidegree is slightly below $\frac{n}{4}$. After this, we consider a tournament which has a balanced cut that is sparse in one direction. An overview of what we do for each case is given below.

Cut-Dense Tournaments

The following theorem deals with the case of cut-dense tournaments and also provides an answer to the first question raised in the introduction. It shows that the Bollobás-Häggkvist theorem holds already when n is exponential in k .

Theorem 5. *Let $k \geq 2$, $\delta > 0$ and $n \geq \left(\frac{3}{\delta}\right)^{1000k}$. Then, any δ -cut-dense tournament T such that $\delta^0(T) \geq \frac{n}{4} - \frac{\delta n}{200}$ has the k -th power of a Hamilton cycle.*

As noted above, tournaments with minimum semidegree at least $\frac{n}{4} + \epsilon n$ are 2ϵ -cut-dense. Thus, this result implies that we can take $n_0 = \epsilon^{-O(k)}$ in Theorem 1. On the other hand, using the observation mentioned before Theorem 3, it is not difficult to see that this behaviour of n_0 is optimal.

The first idea in the proof of the above theorem is to partition the tournament into so-called chains C , which are ordered structures with the following properties:

- *Robustness.* C is such that even if we delete some of its vertices which are somewhat sparsely distributed in C , we get a structure which contains the k -th power of a path.
- *Large neighborhoods.* The first k vertices in C have a large common in-neighborhood, and the last k have a large common out-neighborhood.

In order to find this partition into chains we use the recent result in [5], which shows that one can always find the k -th power of a long path in a tournament. We apply this iteratively, until a certain constant number of vertices is left. By using the semidegree condition, we also absorb these vertices into other chains. Call the obtained (disjoint) chains $\mathcal{C} = \{C_1, \dots, C_t\}$. To finish the proof, we “link” the chains, by always connecting the last k vertices of C_i to the first k vertices of C_{i+1} (and C_t to C_1) with k -th powers of paths. In order to create the links between the chains, we are free to use the internal vertices of the other chains in \mathcal{C} , but in such a way that the robustness property ensures that after deleting the used vertices from the chains in \mathcal{C} , we still have k -th powers of paths. This gives the desired k -th power of a Hamilton cycle.

The rough idea behind how two chains are linked is the following. Suppose we want to link C_i to C_{i+1} . We consider the set of vertices A which are in some sense *reachable* by k -th powers of paths starting at the set of last k -vertices in C_i ; similarly, we consider the set B of vertices which can reach the first k vertices of C_{i+1} . Because of the minimum semidegree condition, A and B will be of sizes close to $n/2$. Then we consider two cases. Either the intersection $S = A \cap B$ is large, and thus we can find a connection between C_i and C_{i+1} which passes through S ; or S is small, and then we can use the δ -cut-dense property of T , to find a connection between A and B , and consequently establish a connection between C_i and C_{i+1} .

Tournaments with a Sparse Cut

In the second part of the proof, we consider the case of T having a balanced cut which is sparse in one of the directions, that is, the number of edges in this direction is $o(n^2)$. The first thing to do is to convert this cut into sets A, B, R which partition the vertex set and are such that $|R| = o(n)$ and both tournaments $T[A], T[B]$ are almost regular, of size $\frac{n}{2} - o(n)$ and such that $\vec{e}(A, B) = o(n^2)$.

Let’s first consider the case when $R = \emptyset$ and $|A| = |B| = \frac{n}{2}$ in order to give a rough outline of some ideas. As a preliminary, note that since the average out-degree in $T[A]$ is at most $\frac{n}{4}$, we have that $\vec{e}(A, B) \geq c|A|n^{1-1/\lceil k/2 \rceil} =$

$\Omega(n^{2-1/\lceil k/2 \rceil})$. Naturally, the first step is to find a way to cross from A to B and from B to A . Specifically, we will want to find transitive subtournaments $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ of size k such that (A_1, B_1) forms a k -th power of a path of size $2k$ starting at A_1 and ending at B_1 and (B_2, A_2) forms a k -th power starting at B_2 and ending at A_2 . Now, since the density from B to A is $1 - o(1)$, finding A_2 and B_2 is not difficult. The bottleneck of the problem is in finding A_1 and B_1 , which is heavily dependent on the number of edges going from A to B .

Indeed, assume that such A_1 and B_1 exist and define A'_1 to be the set of last $\lceil k/2 \rceil$ vertices of A_1 and B'_1 the set of first $\lceil k/2 \rceil$ vertices of B_1 . Then, we have that every vertex in A'_1 dominates every vertex in B'_1 , which creates a $K_{\lceil k/2 \rceil, \lceil k/2 \rceil}$ in the graph formed by the edges going from A to B . Therefore, in general, to find such sets A_1 and B_1 we would need the number of edges from A to B to be at least the Turán number of $K_{\lceil k/2 \rceil, \lceil k/2 \rceil}$, which is believed to be $\Theta(n^{2-1/\lceil k/2 \rceil})$. In fact, we are able to show that this number of edges is also sufficient to find A_1, B_1 as above.

After constructing the sets A_1, A_2, B_1, B_2 , it is simple to finish. Recall that $T[A]$ and $T[B]$ are almost regular. Therefore, $T[A \setminus (A_1 \cup A_2)]$ and $T[B \setminus (B_1 \cup B_2)]$ are $\frac{1}{4}$ -cut-dense tournaments and so, by Theorem 5 they contain spanning chains C_A and C_B . We can link the end of the chain C_A to A_1 , then link B_1 to the start of C_B , the end of C_B to B_2 and finally link A_2 to the start of C_A . This produces the k -th power of a Hamilton cycle.

The general case builds on the above approach. The main goal will be to cover the set R with a collection \mathcal{B} of $o(n)$ many vertex-disjoint structures, which we will call bridges. Informally, a bridge is the k -th power of a path which intersects $A \cup B$ in at most $4k$ vertices, such that the sets of first k vertices and last k are contained entirely in A or B . We will say that a bridge goes from B to A , for example, if the first k vertices are in B and the last k are in A . We also want these first k to have large common in-neighborhood in B (so that it can be linked later to the rest of B) and the last k to have large common out-neighborhood in A . Finally, we will crucially need the collection \mathcal{B} to satisfy the following property. The number of bridges going from A to B is positive and equal to the number of them going from B to A . Indeed, note that if this is the case, we can then construct the k -th power of a Hamilton cycle by using these bridges to cover R and then using the almost regularity of both $T[A]$ and $T[B]$ to link them to the rest of the vertices in A and B , like we did in the case $R = \emptyset$. Since the number of bridges going from A to B is positive, we are able to cross from A to B at least once, and further, since we have the same number of bridges going from B to A , we are able to cross the same number of times from B to A . The construction of bridges is delicate and requires several ideas.

3 Concluding Remarks

We resolved, for all even k , the question of determining the minimum semidegree condition which ensures that a tournament contains the k -th power of a Hamilton

cycle. For odd k , although we made a very significant improvement on what was previously known, there is still a small gap between our bounds and it would be very interesting to close it. Here, the first open case is $k = 3$. In this case, our Theorem 4 gives a lower bound of order $n^{1/5}$ on the additive error term. On the other hand, the upper bound in this case, coming from Theorem 2, has order \sqrt{n} and it is not clear what the truth should be.

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On Sufficient Conditions for Hamiltonicity in Dense Graphs

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Abstract. We study structural conditions in dense graphs that guarantee the existence of vertex-spanning substructures such as Hamilton cycles. Recall that every Hamiltonian graph is connected, has an almost perfect matching and, excluding the bipartite case, contains an odd cycle. Our main result states that any large enough graph that robustly satisfies these properties must already be Hamiltonian. Moreover, the same holds for powers of cycles and the bandwidth setting subject to natural generalizations of connectivity, matchings and odd cycles.

This solves the embedding problem that underlies multiple lines of research on sufficient conditions for Hamiltonicity. As an application, we recover several old and new results, and prove versions of the Bandwidth Theorem under Ore-type degree conditions, Pósa-type degree conditions, deficiency-type conditions and for balanced partite graphs.

Keywords: Hamilton cycles · Bandwidth · Degree conditions

1 Introduction

An old question in discrete mathematics is to understand the presence of certain vertex-spanning substructures in graphs, such as Hamilton cycles. Since the corresponding decision problems are often computationally intractable, we do not expect to find ‘simple’ characterisations of the graphs that contain a particular spanning structure. The extremal approach to these questions has therefore focused on easily-verifiable sufficient conditions. A classic example in this direction is Dirac’s theorem (1952), which states that every graph on n vertices and minimum degree at least $n/2$ contains a Hamilton cycle. Since its inception, Dirac’s theorem has been extended in numerous ways [7, 15]. Here we propose a general framework that covers many of these extensions and also leads to new ones.

It is natural to ask whether the assumptions of Dirac’s theorem can be weakened. A simple example shows that a minimum degree of $n/2$ is best-possible. However, as it turns out, not all of the vertices need to have this degree to ensure that the graph is Hamiltonian. A well-known theorem of Ore (1960) states that a graph on n vertices contains a Hamilton cycle if $\deg(u) + \deg(v) \geq n$ for all non-adjacent vertices u and v . Pósa (1962) extended this by showing that a

graph on n vertices contains a Hamilton cycle provided that its degree sequence $d_1 \leq \dots \leq d_n$ satisfies $d_i \geq i + 1$ for all $i \leq n/2$.¹ Later on, Chvátal (1972) gave a complete characterisation of the degree sequences that guarantee Hamiltonicity. More recently, conditions such as local density together with large minimum degree or inseparability [5, 18] and deficiency [16] have been investigated.

Another way to generalize Dirac's theorem is to strengthen its outcome by embedding more complex substructures, such as clique factors and powers of cycles. A k -clique factor in a graph G consists of pairwise disjoint k -cliques (complete graphs on k vertices) that cover all vertices of G (the case $k = 2$ corresponds to perfect matchings). It was conjectured by Erdős and proved by Hajnal and Szemerédi [8] that graphs on n vertices with n divisible by k and minimum degree at least $(k - 1)n/k$ have a k -clique factor. Similarly, the notion of cycles can be generalised in terms of their powers. The k th power (or square when $k = 2$) of a graph G is obtained from G by joining any two vertices of distance at most k . Pósa (for $k = 3$) and Seymour (for $k \geq 3$) conjectured that any graph on n vertices with minimum degree at least $(k - 1)n/k$ contains the $(k - 1)$ th power of a Hamilton cycle. This was confirmed by Komlós, Sárközy and Szemerédi [12–14] for sufficiently large n . While powers of cycles might appear to be a somewhat particular class of graphs, their embedding turned out to be an important milestone with regards to embedding the much richer class of k -colourable graphs with bounded degree and sublinear bandwidth (as defined below). This last result is known as the Bandwidth Theorem and was proved by Böttcher, Schacht and Taraz [3].

In the recent years there has been a surge of activity in combining the above detailed weaker assumptions with stronger outcomes, such as obtaining powers of cycles under Pósa-type degree conditions. (We will survey some of these results later.) Many of these results are proved using similar embedding techniques, but differ in their structural analysis. One might therefore wonder whether there is a common structural base that ‘sits between’ all of these assumptions and (variations of) Hamiltonicity. The purpose of this article is to propose such a structural base, which we call a *Hamilton framework*. Our main result states that graphs that have a robust Hamilton framework are (in a strong sense) Hamiltonian. As an application we can easily recover many of the above mentioned contributions and also prove several new results.

2 Results

To provide an overview of our outcomes, we introduce some further notation. A graph H admits an ordering with *bandwidth at most b* if the vertices of H can be labelled with $\{1, \dots, n\}$ such that $|i - j| \leq b$ for all edges ij . A β -block of $\{1, \dots, n\}$ is an interval of the type $\{(i - 1)\lceil \beta n \rceil + 1, \dots, i\lceil \beta n \rceil\}$ for some $1 \leq i \leq \beta^{-1}$. A $(k + 1)$ -colouring $\chi: \{1, \dots, n\} \rightarrow \{0, 1, \dots, k\}$ is said to be (z, β) -zero-free if, among every z consecutive β -blocks, at most one of them uses

¹ Indeed, a quick exercise shows that every graph satisfying the conditions of Ore's theorem also satisfies the conditions of Pósa's theorem.

the colour 0. Intuitively, in those $(k + 1)$ -colourings there is one colour that is used only in few vertices. A graph G on n vertices is (z, β, Δ, k) -Hamiltonian if G contains every graph H on n vertices with $\Delta(H) \leq \Delta$ which admit an ordering with bandwidth at most βn and a (z, β) -zero-free $(k + 1)$ -colouring. Note that (z, β, Δ, k) -Hamiltonicity yields the existence of $(k - 1)$ th powers of Hamilton cycles for $\Delta = 2k, z = 1, \text{ any } \beta > 0, \text{ and } n \text{ large enough.}$

2.1 Ore-Type Conditions

Kierstead and Kostochka [9] gave optimal Ore-type conditions which ensure the existence of k -clique factors. For sufficiently large graphs, Ch au [4] proved a generalisation of Ore’s theorem for squares of Hamilton cycles.

Ch au [4] also conjectured generalizations of this for all $k \geq 4$. The following result proves this conjecture in a strong sense. The result is tight up to the μn term, as witnessed by k -colourable Tur an graphs.

Theorem 1 (Bandwidth theorem for Ore-type conditions). *For $k, \Delta \in \mathbb{N}$ and $\mu > 0$ there are $z, \beta > 0$ and $n_0 \in \mathbb{N}$ with the following property. Let G be a graph on $n \geq n_0$ vertices with $\deg(x) + \deg(y) \geq 2\frac{k-1}{k}n + \mu n$ for all $xy \notin E(G)$. Then G is (z, β, Δ, k) -Hamiltonian.*

2.2 P osa-Type Conditions

Balogh, Kostochka and Treglown [1, 2] raised the question of which type of degree conditions would guarantee the existence of clique factors and powers of Hamilton cycles. Treglown [19] proved a P osa-type theorem for clique factors under optimal conditions. Balogh, Kostochka and Treglown [1] asked whether one could improve the degree sequences which ensure the existence of $(k - 1)$ th power of a Hamilton cycle, by allowing a non-negligible number of vertices to have degree less than $(k - 1)n/k$. Staden and Treglown [17] answered this question for $k = 3$, showing that the same P osa-type conditions that guarantee the existence of a triangle factor also imply the existence of a squared Hamilton cycle. They also conjectured that this can be extended to all $k \geq 4$ [17].

We confirm these conjectures in a strong sense by showing them in the bandwidth setting. Here, Knox and Treglown [11] had previously proved a bandwidth theorem for degree sequences in the case of $k = 2$. Staden and Treglown [17] conjectured such a result could be true for $k = 3$, and Treglown [20] extended the conjecture to all $k \geq 3$, so our results confirm these conjectures as well.

Theorem 2 (Bandwidth theorem for P osa-type conditions). *For $k, \Delta \in \mathbb{N}$ and $\mu > 0$ there are $z, \beta > 0$ and $n_0 \in \mathbb{N}$ with the following property. Let G be a graph with degree sequence d_1, \dots, d_n such that $d_i \geq \frac{k-2}{k}n + i + \mu n$ for every $i \leq n/k$. Then G is (z, β, Δ, k) -Hamiltonian.*

We remark that the degree conditions are essentially tight: by adapting examples of Balogh, Kostochka and Treglown [1], we can show Theorem 2 becomes false if μn is replaced by $o(n^{1/2})$. Finally, let us mention that (unlike the case for Hamilton cycles), the conditions of Theorem 2 and of Theorem 1 do not imply each other, as constructions show.

2.3 Other Results

We also show bandwidth theorems under conditions of local density and inseparability. This recovers the work of Ebsen, Maesaka, Reiher, Schacht and Schülke [5], which itself extended work of Staden and Treglown [18]. We also consider so-called deficiency conditions introduced by Nenadov, Sudakov and Wagner [16]. We show that, up to a lower order term, the conditions ensuring k -clique factors yield already $(k - 1)$ th powers of Hamilton cycles, and the corresponding bandwidth versions. This extends results of Freschi, Hyde and Treglown [6] for $k = 2$. Finally, we generalise the results of Keevash and Mycroft [10] on degree conditions which yield k -clique factors in balanced partite graphs, again showing that the same conditions yield already the corresponding powers of cycles and bandwidth versions.

3 Hamilton Frameworks

To motivate the following definitions in a simpler setting, consider a non-bipartite Hamiltonian graph G . Then G must be connected and have a perfect fractional matching. Since G is not bipartite, it must also contain an odd cycle. We call a graph that satisfies these properties a *Hamilton framework*. Is this property equivalent to Hamiltonicity? The answer is no, as witnessed by two vertex-disjoint odd cycles joined by a single edge. On the other hand, the properties of this graph are somewhat fragile. Deleting few vertices or few edges (at every vertex) quickly leads to a disconnected graph, or a graph without an almost perfect matching. To exclude examples like this, we could restrict our attention to Hamilton frameworks that are robust against such operations and ask again whether this already guarantees the existence of a Hamilton cycle. As it turns out, this is the case. In fact, the same conditions allow us to even embed 2-colourable graphs H of bounded bandwidth and maximum degree into robust Hamilton frameworks G . If G has in addition a triangle, this can be extended to graphs H with suitable zero-free 3-colourings. (Note that this condition is necessary at least in some sense, since such graphs might contain triangles.) This discussion corresponds to the case $k = 2$. In the following we introduce the terminology to formalize these ideas for $k \geq 3$.

The general definition of Hamilton frameworks takes place in the hypergraph setting. A k -graph is a hypergraph where every edge has exactly k vertices. A *tight (Hamilton) cycle* $C \subseteq G$ is a (spanning) subgraph whose vertices can be cyclically ordered such that its edges consist precisely of all k consecutive vertices under this ordering. For an (ordinary) graph G , we write $K_k(G)$ to mean the k -clique k -graph of G , which is the k -graph with vertex set $V(G)$ and a k -edge X whenever X is a k -clique in G . Observe that G contains the $(k - 1)$ th power of a Hamilton cycle if and only if $K_k(H)$ contains a tight Hamilton cycle.

A *(closed) tight walk* in a k -graph G is a (cyclically) ordered multi-set of vertices such that every interval of k consecutive vertices forms an edge of G . Note that vertices and edges are allowed to be visited more than once in a closed tight walk. The *length* of a tight walk is its number of vertices. Finally,

a subgraph C of a k -graph G is *tightly connected*, if there is a closed tight walk that contains all edges of C .

A *matching* M in a k -graph G is a subgraph of vertex disjoint edges. We can define a linear programming relaxation of this as follows. A *perfect fractional matching* is an edge weighting $\mathbf{w}: E(G) \rightarrow \{0, 1\}$ such that $\sum_{e \ni v} \mathbf{w}(e) = 1$ for every vertex $v \in V(H)$.

Definition 1 (Hamilton framework). A pair (G, H) , where G is a graph and $H \subseteq K_k(G)$, is a zero-free Hamilton framework, if

- (F1) H is contained in a tightly connected subgraph of $K_k(G)$, (connected)
- (F2) H admits a perfect fractional matching, (matchable)
- (F3) H contains a $(k + 1)$ -clique. (zero-freeness)

Next, we formalize the notion of robustness. For two k -graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we write $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

Definition 2 (Robustness). A Hamilton framework (G, H) on n vertices is μ -robust, if the following holds. For every subgraph $G' \subseteq G$ with

- (i) $\deg_{G'}(v) \geq \deg_G(v) - \mu n$ for all $v \in V(G')$ and
- (ii) $|V(G')| \geq (1 - \mu)n$,

the pair $(G', H \cap K_k(G'))$ is a zero-free Hamilton framework.

Finally, we call an edge e in a k -graph a *second neighbour* of a vertex $v \notin e$, if there is an edge f with $v \in f$ and $|e \cap f| = k - 1$. Now we can state our main result from which the outcomes in Sect. 2 can be derived.

Theorem 3. For $k, \Delta \in \mathbb{N}$ and $\mu > 0$ there are $\beta, z > 0$ and $n_0 \in \mathbb{N}$ with the following property. Let G be a graph on $n \geq n_0$ vertices. Suppose there exists a k -uniform μ -robust zero-free Hamilton framework (G, H) , and suppose that every vertex of G has at least μn^k second neighbours in H . Then G is (z, β, Δ, k) -Hamiltonian.

We also obtain results under weaker assumptions on G , e.g. if $K_k(G)$ contains walks of length coprime to k (but not $(k + 1)$ -cliques) or whenever G is balanced k -partite. In those cases, we can still embed suitably-defined families of graphs of low bandwidth (including $(k - 1)$ th powers of Hamilton cycles).

To illustrate the method, we sketch a proof of Theorem 2 using Theorem 3. We are given a graph G which satisfies the Pósa-type degree conditions with an added μn in the degree. Set $H = K_k(G)$. For appropriate $\nu \ll \mu$, the Graph Removal Lemma shows that there are at least νn^k second neighbours in H for every vertex in $V(G)$. To apply Theorem 3, it remains to show that $(G, K_k(G))$ is a ν -robust zero-free Hamilton framework. Now consider an arbitrary $G' \subseteq G$ as in Definition 2, with ν instead of μ , and let $H' = K_k(G')$. To show that (G', H') is a zero-free Hamilton framework, we have to check connectivity, matchability and zero-freeness from Definition 1. The key point is that such graphs G' will

inherit, up to a small error term, the Pósa-like degree conditions of G . So, for instance, the fractional matching can be deduced quickly from the results of Treglown [19], and the $(k+1)$ -clique is similarly easy. The only missing ingredient is connectivity, which we obtain from an elementary (but non-trivial) argument using induction on k .

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On the Maximum Number of Weakly-Rainbow-Free Edge-Colorings

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Abstract. A pattern P of a graph F is a partition of its edge set. Given a family \mathcal{P} whose elements are pairs (F, P) , where F is graph and P is a pattern of F , we consider n -vertex graphs with the largest number of r -edge-colorings that avoid copies of F colored according to the pattern P , for any $(F, P) \in \mathcal{P}$. In particular, if $k \geq 3$ and \mathcal{P}_k is a family of patterns of K_k that contains the rainbow pattern, there is $r_0(\mathcal{P}_k)$ such that the Turán graph $T_{k-1}(n)$ admits the largest number of colorings for $r \geq r_0$ and large n . We find bounds on $r_0(\mathcal{P}_k)$ for several families \mathcal{P}_k .

Keywords: Edge-coloring · Erdős-Rothschild problem · Turán problem

1 Introduction

There has been a lot of progress in the Erdős-Rothschild problem and its variations in the last years. The original version was first stated in a paper by Erdős [6]. Given an integer $r \geq 2$ and a fixed graph F , Erdős and Rothschild considered the maximum number of r -edge-colorings¹ of an n -vertex graph G that avoid monochromatic copies of F . Among other things, they asked whether, for all or almost all choices of graph F and any $\varepsilon > 0$, every large n -vertex graph G admits at most $r^{(1+\varepsilon)\text{ex}(n,F)}$ distinct r -colorings with no monochromatic copy of F . As usual, $\text{ex}(n, F)$ denotes the maximum number of edges in an F -free n -vertex graph, that is, in an n -vertex graph G with no copy of F as a subgraph.

This question may be restated more generally in terms of graph patterns. For a fixed graph F , a *pattern* P of F is a partition of its edge set. Let $\gamma(P)$ denote the number of classes in this partition. An r -coloring of a graph G is said to be (F, P) -free if G does not contain a copy of F in which the partition of the edge

¹ For simplicity, we will henceforth refer to them as r -colorings.

This work has been performed as part of a CAPES-DAAD Probral Program (CAPES Proc. 88881.143993/2017-01, DAAD 57518130). C. Hoppen was partially supported by CNPq (Proj. 308054/2018-0) and FAPERGS (19/2551-0001727-8).

set induced by the coloring is isomorphic to P . In particular, the original Erdős-Rothschild question is concerned with the pattern P_F^M of F given by $\{E(F)\}$, so that $\gamma(P_F^M) = 1$. Another pattern that has attracted considerable attention is the *rainbow* pattern $P_F^R = \{\{e\} : e \in E(F)\}$, which satisfies $\gamma(P_F^R) = |E(F)|$.

Here, we consider a yet more general problem dealing with colorings that avoid patterns in a pattern family \mathcal{P} , namely a set of pairs (F, P) for which F is a graph and P is a pattern of F . An r -colored graph \widehat{G} is \mathcal{P} -free if \widehat{G} is (F, P) -free for all pairs $(F, P) \in \mathcal{P}$. Given such a pattern family \mathcal{P} , an integer $r \geq 2$, and a graph G , let $\mathcal{C}_{r, \mathcal{P}}(G)$ be the set of all \mathcal{P} -free r -colorings of a graph G . We write $c_{r, \mathcal{P}}(G) = |\mathcal{C}_{r, \mathcal{P}}(G)|$ and

$$c_{r, \mathcal{P}}(n) = \max \{c_{r, \mathcal{P}}(G) : |V(G)| = n\}. \tag{1}$$

An n -vertex graph G is (r, \mathcal{P}) -*extremal* if $c_{r, \mathcal{P}}(n) = c_{r, \mathcal{P}}(G)$. Solving this version of the Erdős-Rothschild problem consists of determining $c_{r, \mathcal{P}}(n)$ and the corresponding (r, \mathcal{P}) -extremal graphs. The question of Erdős and Rothschild mentioned above could be asked in the following more general terms. For which values of r , graphs F and families \mathcal{P}_F of patterns of F , does the following hold for all $\varepsilon > 0$ and sufficiently large n :

$$c_{r, \mathcal{P}}(n) \leq r^{(1+\varepsilon)\text{ex}(n, F)} ? \tag{2}$$

In addition to generalizing the original Erdős-Rothschild problem, the framework of (1) clearly generalizes the classical Turán problem: for a graph F , consider the pattern family \mathcal{P}_F that contains all possible patterns of F . Clearly, a graph G admits a \mathcal{P}_F -free coloring if and only if it is F -free, in which case $c_{r, \mathcal{P}_F}(G) = r^{|E(G)|}$. Thus G is (r, \mathcal{P}_F) -extremal if and only if it is F -extremal for the Turán problem.

To refer to previous results, we go back to pattern families with a single pattern. Several results have been obtained for the complete graph $F = K_k$ on k -vertices with pattern $P_k^M = P_F^M$. As usual, given $n \geq \ell \geq 2$, the *Turán graph* $T_\ell(n)$ is the balanced, complete, ℓ -partite graph on n vertices. Yuster [11] showed that $T_2(n)$ is the only $(2, (K_3, P_3^M))$ -extremal graph on n vertices for all $n \geq 6$. This result has been extended by Alon, Balogh, Keevash, and Sudakov [1], who proved that $T_{k-1}(n)$ is the only n -vertex $(r, (K_k, P_k^M))$ -extremal graph for large n and $r \in \{2, 3\}$. Note that these results give graphs F and values of r for which the upper bound in (2) holds in the case (F, P_F^M) . On the other hand, the results in [1] also imply that this upper bound does not hold for (F, P_F^M) whenever $r \geq 4$ and F is not bipartite. For more information about $(r, (K_k, P_k^M))$ -extremal graphs when $r \geq 4$, see [5] and [10], and their references.

With regard to rainbow patterns, Odermann and two of the current authors [9] proved that, for $k \geq 4$, $T_{k-1}(n)$ is the only n -vertex $(r, (K_k, P_k^R))$ -extremal graph for $r \geq r_0 = \binom{k}{2}^{8k-4}$ and large n . They also showed that the version of this result for $k = 3$ holds for all $r \geq 5$. This has been extended to $r = 4$ by Balogh and Li [3], which is best possible. The authors of [9] have shown that some of their results could be extended to other forbidden graphs

and patterns. In a different direction, they proved that there exists r_1 such that for all $r \geq r_1$, there is $\varepsilon_0 > 0$ with the following property. There exist n -vertex graphs of arbitrarily large order that admit more than $r^{(1+\varepsilon_0)\text{ex}(n, K_k)}$ distinct r -colorings such that every copy of K_k is assigned the rainbow pattern.

2 Main Results

Regarding general pattern families, some results follow immediately from the definition of (r, \mathcal{P}) -extremality. Given a pattern family \mathcal{P} , let $\gamma_{\min}(\mathcal{P})$ and $\chi_{\min}(\mathcal{P})$ be the minima of $\gamma(P)$ and $\chi(F)$ over all $(F, P) \in \mathcal{P}$, where $\chi(F)$ denotes the (vertex) chromatic number of F .

Proposition 1. *Let \mathcal{P}_1 and \mathcal{P}_2 be pattern families, and let $n, r \geq 2$ be integers. If $r < \gamma_{\min}(\mathcal{P}_1)$, then $c_{r, \mathcal{P}_1}(n) = r^{\binom{n}{2}}$ and K_n is the unique (r, \mathcal{P}_1) -extremal graph. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $c_{r, \mathcal{P}_1}(n) \geq c_{r, \mathcal{P}_2}(n)$. Moreover, if $s = \chi_{\min}(\mathcal{P}_2) - 1$ and the Turán graph $T_s(n)$ is (r, \mathcal{P}_1) -extremal, then $T_s(n)$ is also (r, \mathcal{P}_2) -extremal.*

This paper is particularly concerned with pattern families \mathcal{P}_k whose elements are patterns of K_k for a fixed integer $k \geq 3$. The results in [9] mentioned in the introduction and Proposition 1 immediately imply the following.

Proposition 2. *Let $k \geq 3$ and let \mathcal{P}_k be a pattern family whose elements are patterns of K_k . There exists r_0 such that $T_{k-1}(n)$ is (r, \mathcal{P}_k) -extremal for all $r \geq r_0$ and sufficiently large n if and only if $P_k^R \in \mathcal{P}_k$.*

For each family \mathcal{P}_k with $P_k^R \in \mathcal{P}_k$, let $r_0(\mathcal{P}_k)$ be the least r_0 satisfying the property in Proposition 2. It is clear that $r_0(\mathcal{P}_k) \leq r_0(\{P_k^R\})$.

In this paper, we give a general result about (r, \mathcal{P}_k) -extremal graphs when $r < r_0$ or $P_k^R \notin \mathcal{P}_k$, and we find upper bounds on $r_0(\mathcal{P}_k)$ for a class of pattern families that contain P_k^R . The first result, which generalizes [4, Theorem 1.1], shows that we may always find a complete multipartite (r, \mathcal{P}) -extremal n -vertex graph $G = (V, E)$, i.e., one with a partition $V = V_1 \cup \dots \cup V_\ell$ such that $E = \{\{u, v\} : u \in V_i, v \in V_j, i \neq j\}$.

Theorem 1. *Let \mathcal{P} be a pattern family whose elements are patterns of complete graphs and let $r \geq 2$ be an integer. For any positive integer n , there exists an n -vertex complete multipartite graph G^* that is (r, \mathcal{P}) -extremal.*

Our proof of Theorem 1 [8] has the following useful consequence.

Corollary 1. *Let \mathcal{P} be a pattern family of complete graphs and $n, r \geq 2$ be integers. If there exists an (r, \mathcal{P}) -extremal graph that is not complete multipartite, then there exist at least two non-isomorphic (r, \mathcal{P}) -extremal complete multipartite graphs on n vertices.*

Our second result improves the upper bound on $r_0(\{P_k^R\})$ given in [9], and identifies *weakly-rainbow* pattern families \mathcal{P}_k of K_k for which much better bounds, or even the actual value of $r_0(\mathcal{P}_k)$ may be computed. Let $k \geq 3, r \geq 2$

and $s \leq \binom{k}{2}$ be positive integers, and let $\mathcal{P}_k^{\geq s}$ be the pattern family that contains all patterns of K_k with s or more classes. The statement splits the interval $2 \leq s \leq \binom{k}{2}$ into three intervals for which bounds of different orders of magnitude are obtained. The third interval includes $s = \binom{k}{2}$, for which $\mathcal{P}_k^{\geq s} = \{(K_k, P_k^R)\}$.

In order to specify $r_0(\mathcal{P}_k^{\geq s}) = r_0(k, s)$, it is convenient to use the following quantity. For $j \in \{2, \dots, k - 1\}$, let $A(k, j) = \binom{k}{2} - \text{ex}(k, K_{j+1})$ which is the minimum number of edges that must be deleted from a complete graph K_k to make it j -partite. Let

$$s_0(k) = A(k, 2) + 2 = \binom{k}{2} - \left\lfloor \frac{k}{2} \right\rfloor \cdot \left\lceil \frac{k}{2} \right\rceil + 2, \text{ and } s_1(k) = \binom{k}{2} - \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

For $s \leq s_0(k)$, let i^* be the least integer i such that $A(k, k - i) \geq s - 2$, which implies $i^* \leq \min\{s - 2, k - 2\}$. Let $r_0(k, s)$ be the least integer greater than

$$(s - 1)^{\frac{k-1}{k-2}} \prod_{i=2}^{i^*} (s - A(k, k - i + 1) - 1)^{\frac{1}{(k-i-1)(k-i)}}. \tag{3}$$

For $3 \leq s \leq s_0(k)$, we have $r_0(k, s) \leq (s - 1)^2$. For $s > s_0(k)$, we set

$$r_0(k, s) = s^7 \text{ if } s \leq s_1(k) \text{ and } r_0(k, s) = s^{\frac{4(k-1)}{\binom{k}{2}-s+2}+2} \text{ if } s > s_1(k). \tag{4}$$

This shows that $r_0(k, s)$ is bounded above by a polynomial in s whenever $s \leq s_1$. Moreover, $r_0(k, \binom{k}{2}) \leq k^{4k}/2^{2k}$, which improves on the bound in [9].

Theorem 2. *Let $k \geq 4$ and $2 \leq s \leq \binom{k}{2}$ be integers. Fix $r \geq r_0(k, s)$, defined above. There is $n_0 = n_0(r, k, s)$ for which the following holds. Every graph $G = (V, E)$ on $n > n_0$ vertices satisfies $|\mathcal{C}_{r, (K_k, \mathcal{P}_k^{\geq s})}(G)| \leq r^{\text{ex}(n, K_k)}$. Moreover, equality holds if and only if G is isomorphic to $T_{k-1}(n)$.*

Our proof of Theorem 2 is based on the stability method. Precisely, it relies on the following colored stability result.

Theorem 3. *Let $k \geq 3$ and $2 \leq s \leq \binom{k}{2}$ be integers. Fix $r \geq r_0(k, s)$, defined above. For any $\delta > 0$, there is $n_0 = n_0(\delta, r, k, s)$ such that the following holds. If $G = (V, E)$ is a graph on $n > n_0$ vertices such that $|\mathcal{C}_{r, (K_k, \mathcal{P}_k^{\geq s})}(G)| \geq r^{\text{ex}(n, K_k)}$, then there is a partition $V = W_1 \cup \dots \cup W_{k-1}$ such that at most δn^2 edges have both endpoints in a same class W_i .*

The proof of Theorem 3 is involved and uses the regularity lemma combined with a linear programming approach. The values of $s_0(k), s_1(k), r_0(k, s)$ are based on the solutions to linear programs that appear in our proof of Theorem 3. Improvements to the values $r_0(k, s)$ would typically lead to improvements in the statement of Theorem 2.

As it turns out, the values of $r_0(k, s)$ in (4) can be improved using more involved formulas [7]. Table 1 gives the best values of $r_0(k, s)$ for which the

statement of Theorem 2 is currently known to hold, for a few values of k and s . The symbols $*$ and \clubsuit are used to indicate the first value of s such that $s > s_0$ and such that $s > s_1$. For comparison, it is easy to see that, if $r \leq r_1(k, s) = \lceil (s - 1)^{(k-1)/(k-2)} - 1 \rceil$, then $|\mathcal{C}_{r, (K_k, \geq s)}(K_n)| > |\mathcal{C}_{r, (K_k, \geq s)}(T_{k-1}(n))|$ for large values of n , so that Theorem 2 cannot possibly be extended to such values of r (see Table 2). Note that, for any fixed s , we have $\lim_{k \rightarrow \infty} r_1(k, s) = s - 1$ and, by (3), $\lim_{k \rightarrow \infty} r_0(k, s) = s$. This raises the natural question of whether the behavior observed for $k = 3$, where K_n is $(r, (K_3, P_3^R))$ -extremal for $r \leq 3$ and $T_2(n)$ is $(r, (K_3, P_3^R))$ -extremal for $r \geq 4$, also holds for larger values of r and s , for all sufficiently large n .

Table 1. $r_0(k, s)$ for some small values of k and s .

$k \setminus s$	2	3	4	5	6	7	8	9	10	12	13	15
4	2	3	8	222*	5434							
5	2	3	5	11	19	457*	3270	55507	218896			
6	2	3	5	7	15	24	35	606*	3528	309393	933907	$1.44 \cdot 10^{12} \clubsuit$

Table 2. $r_1(k, s)$ for some values of k and s .

$k \setminus s$	3	4	5	6	7	8	9	10	12	13	15
4	2	5	7	11							
5	2	4	6	8	10	13	15	18			
6	2	3	5	7	9	11	13	15	20	22	27

To conclude the paper, we sketch the proof of Theorem 2 (see [7]).

Proof. Let $k \geq 4$ and $2 \leq s \leq \binom{k}{2}$ be integers. Fix $r \geq r_0(k, s)$, with r_0 defined in (3) and (4) for $s \leq s_0(k)$, $s_0(k) < s \leq s_1(k)$ and $s > s_1(k)$, respectively. Fix a sufficiently small positive constant δ . Fix $n_0 = n_0(\delta, r, k, s)$ as in Theorem 3.

To reach a contradiction, suppose that there is an n -vertex graph $G = (V, E)$ that is $(r, K_k, \mathcal{P}_k^{\geq s})$ -extremal, but $G \neq T_{k-1}(n)$. By Corollary 1, we may assume that G is a complete multipartite graph. Let $V = V'_1 \cup \dots \cup V'_p$ be the multipartition of G , where $p \geq k$.

Let $V = V_1 \cup \dots \cup V_{k-1}$ be a partition of the vertex set of G such that $\sum_{i=1}^{k-1} e(V_i)$ is minimized, so that $\sum_{i=1}^{k-1} e(V_i) \leq \delta n^2$ by Theorem 3, which implies $|V_i - n/(k-1)| \leq \sqrt{2\delta}n$. The minimality of this partition ensures that, if $v \in V_i$, then $|V_j \cap N(v)| \geq |V_i \cap N(v)|$, for every $j \in [k-1]$, where $N(v)$ denotes the set of neighbors of v . Given that $p > k-1$, there must be an edge $\{x, y\} \in E$ whose endpoints are contained in the same class of the partition, say $x, y \in V_{k-1}$. Since

x and y are in different classes of $V'_1 \cup \dots \cup V'_p$, any $z \in V_{k-1} - \{x, y\}$ must be adjacent to x or y . We conclude that x or y are adjacent to the majority of vertices in their own class. This restricts the number of $(K_k, \mathcal{P}_k^{\geq s})$ -free r -colorings of G . For $2 \leq s \leq \binom{k-1}{2} + 1$, counting arguments show that the number of suitable r -colorings of G is less than $r^{\text{ex}(n, K_k)}$, which contradicts our choice of G hence $p = k - 1$. For $s > \binom{k-1}{2} + 1$, we suppose that G has $r^{\text{ex}(n, K_{k-1})+m}$ distinct $(K_k, \mathcal{P}_k^{\geq s})$ -free colorings, where $m \geq 0$. We then prove that the graph $G - x$ must have at least $r^{\text{ex}(n-1, K_k)+m+1}$ such colorings. This conclusion will lead to the desired contradiction, as we could apply this argument iteratively until we obtain a graph G' on n_0 vertices with at least $r^{\text{ex}(n_0, K_k)+m+n-n_0} > r^{n_0^2} \geq r^{|E(G')|}$ many $(K_k, \mathcal{P}_k^{\geq s})$ -free colorings. \square

3 Concluding Remarks

In this paper we considered pattern families \mathcal{P}_k of the complete graph K_k containing the rainbow pattern P_k^R . Our focus was on the value of $r_0(\mathcal{P}_k)$, which is the least value of r_0 such that (2) holds for all $r \geq r_0$.

The version of the Erdős-Rothschild problem considered here raises several other natural questions. For instance, if \mathcal{P}_k is a pattern family of K_k such that $P_k^R \notin \mathcal{P}_k$, there may be $r \geq 2$ such that (2) holds. This is true for $r \in \{2, 3\}$ if $P_k^M \in \mathcal{P}_k$. The results in [2] imply that (2) holds for $r = 2$ and any \mathcal{P}_k that contains a pattern with two classes. In these situations, determining the largest value of r for which (2) holds is an interesting question.

The same questions would be interesting for fixed non-complete graphs F with $\chi(F) \geq 3$. In particular, we can ask whether, for any family \mathcal{P} of patterns of F containing F^R , there exists r_0 such that, if $r \geq r_0$, then (2) holds for any $\varepsilon > 0$ and large n . More generally, for any family \mathcal{P} of patterns of F , can one ensure the existence of complete multipartite (r, \mathcal{P}) -extremal graphs for any fixed $r \geq 2$ and n sufficiently large?

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Degree Conditions for Tight Hamilton Cycles

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Abstract. We develop a framework to study minimum d -degree conditions in k -uniform hypergraphs, which guarantee the existence of a tight Hamilton cycle. Our main theoretical result deals with the typical absorption, path-cover and connecting arguments for all k and d at once, and thus sheds light on the underlying structural problems. Building on this, we show that one can study minimum d -degree conditions of k -uniform tight Hamilton cycles by focusing on the inner structure of the neighbourhoods. This reduces the matter to an Erdős–Gallai-type question for $(k - d)$ -uniform hypergraphs.

As an application, we derive two new bounds. First, we generalize a classic result of Rödl, Ruciński and Szemerédi for $d = k - 1$, and determine asymptotically best possible degree conditions for $d = k - 2$ and all $k \geq 3$. This was proved independently by Polcyn, Reiher, Rödl and Schülke.

Secondly, we also provide a general upper bound of $1 - 1/(2(k - d))$ for the tight Hamilton cycle d -degree threshold in k -graphs, narrowing the gap to the lower bound of $1 - 1/\sqrt{k - d}$ due to Han and Zhao.

Keywords: Hamilton cycles · Degree conditions · Hypergraphs

1 Introduction

A widely researched question in modern graph theory is whether a given graph contains certain vertex-spanning substructures such as a perfect matching or a Hamilton cycle. Since the corresponding decision problems are usually computationally intractable, we do not expect to find a ‘simple’ characterisation of the (hyper)graphs that contain a particular spanning structure. The extremal approach to these questions has therefore focused on easily verifiable sufficient conditions. A classic example of such a result is Dirac’s theorem [5], which states that a graph, whose minimum degree is at least as large as half the size of the vertex set, contains a Hamilton cycle. Since its inception, Dirac’s theorem has been generalised in numerous ways [9, 12]. Here we study its analogues for hypergraphs.

To formulate Dirac-type problems in hypergraphs, let us introduce the corresponding degree conditions. In this paper, we consider k -uniform hypergraphs

(or shorter k -graphs), in which every edge consists of exactly k vertices. The *degree*, written $\deg(S)$, of a subset of vertices S in a k -graph H is the number of edges which contain S . We often find convenient to state results and problems in terms of the *relative degree* $\overline{\deg}(S) = \deg(S) / \binom{n-d}{k-d}$, where n is the number of vertices of H and d is the size of S . Note that $0 \leq \overline{\deg}(S) \leq 1$ for all S . The *minimum relative d -degree* of a k -graph H , written $\overline{\delta}_d(H)$, is the minimum of $\overline{\deg}(S)$ over all sets S of d vertices. The case of $d = k - 1$ is also known as the *minimum codegree*. Observe that minimum degrees exhibit a monotone behaviour:

$$\overline{\delta}_{k-1}(H) \leq \dots \leq \overline{\delta}_1(H).$$

It is therefore not surprising that minimum degree conditions for the existence of Hamilton cycles were first studied for the structurally richer settings when d is close to k . Before we come to this, let us specify the notion of cycles that we are interested in.

Cycles in hypergraphs have been considered in several ways. The oldest variant are *Berge cycles* [2], whose minimum degree conditions were studied by Bermond, Germa, Heydemann and Sotteau [3]. In the last two decades, however, research has increasingly focused on a stricter notion of cycles introduced by Kierstead and Katona [11]. A *tight cycle* in k -graph is a cyclically ordered set of vertices such that every interval of k subsequent vertices forms an edge.

Asymptotic Dirac-type results for hypergraphs can be stated compactly using the notion of thresholds.

Definition 1 (Threshold for tight Hamilton cycles). *For $k \geq 2$ and $1 \leq d \leq k - 1$, the minimum d -degree threshold for tight Hamilton cycles, denoted by $hc_d(k)$, is the smallest number $\delta > 0$ with the following property:*

For every $\mu > 0$ there is an $n_0 \in \mathbb{N}$ such that every k -graph H on $n \geq n_0$ vertices with $\overline{\delta}_d(H) \geq \delta + \mu$ contains a tight Hamilton cycle.

For instance, we have $hc_1(2) = 1/2$ by Dirac’s theorem. Codegree thresholds for tight Hamilton cycles of larger uniformity were first investigated by Katona and Kierstead [11], who showed that $1/2 \leq hc_{k-1}(k) \leq 1 - 1/(2k)$ and conjectured the threshold to be $hc_{k-1}(k) = 1/2$. This was confirmed in a seminal contribution by Rödl, Ruciński and Szemerédi [20, 21]. For $k = 3$ and large enough hypergraphs, the same authors also obtained an exact result [22]. For more background on these problems and their history, we refer the reader to the surveys of Zhao [24] and Simonovits and Szemerédi [23].

2 Results

Here, we investigate the thresholds $hc_d(k)$ when $1 \leq d \leq k - 2$. As noted by Kühn and Osthus [12] and Zhao [24], it appears that the problem gets significantly harder in this setting. After preliminary results of Glebov, Person and Weps [8], Rödl and Ruciński [18], Rödl, Ruciński, Schacht and Szemerédi [19] and Cooley and Mycroft [4], it was shown by Reiher, Rödl, Ruciński, Schacht

and Szemerédi [16] that $hc_1(3) = 5/9$, which resolves the case of $d = k - 2$ when $k = 3$. With regards to general bounds, Rödl and Ruciński [17] conjectured that the threshold $hc_d(k)$ coincides with the analogous threshold guaranteeing perfect matchings. However, this conjecture was disproved by Han and Zhao [10]. Their constructions imply (amongst other things) that $hc_d(k) \geq 1 - 1/\sqrt{k-d}$. As a consequence, we have $hc_d(k) \rightarrow 1$ when the difference $k - d$ goes to infinity. This behaviour differs from the Dirac-type matching thresholds, which are known to be bounded away from 1, independent of k and d (see Ferber and Jain [6]).

Before our work, the best general upper bound for $h_d(k)$ was due to Glebov, Person and Weps [8], who proved that $hc_d(k) \leq hc_1(k) \leq 1 - 1/(Ck^3)^{k-1}$ for some $C > 1$ independent of d and k . Note that this is a function of k only. Given the known lower bounds, it is natural to ask whether $hc_d(k)$ can be bounded by a function of $k - d$ instead. Here we answer this question in the affirmative.

Theorem 1. *For all $k \geq 2$ and $1 \leq d \leq k - 1$, we have $hc_d(k) \leq 2^{-\frac{1}{k-d}} \leq 1 - 1/(2(k-d))$.*

We remark that Theorem 1 gives the currently best-known results for $hc_d(k)$ for all $k \geq 4$ and all $d \leq k - 3$. For instance, in the smallest unsolved case $k = 4$ and $d = 1$, we now know that $hc_1(4) \leq 2^{-1/3} \approx 0.7937$. The best-known lower bounds [10] give $hc_1(4) \geq 5/8$.

In the particular case of $k \geq 3$ and $d = k - 2$, the construction of Han and Zhao shows that $hc_{k-2}(k) \geq 5/9$. Very recently, Polcyn, Reiher, Rödl, Ruciński, Schacht and Schülke [13] proved that $hc_2(4) = 5/9$. They also conjectured that $hc_{k-2}(k) = 5/9$ for all $k \geq 5$ [15]. Here we resolve this problem.

Theorem 2. *For all $k \geq 3$, we have $hc_{k-2}(k) = 5/9$.*

We remark that the last result has also been obtained independently by Polcyn, Reiher, Rödl and Schülke [14].

2.1 Overview of Our Methods

The above presented bounds on the tight Hamilton cycle thresholds follow from a method that is suitable to approach these issues in a general manner. The argument has three parts, which we sketch now.

First, it is shown that thresholds of tight Hamilton cycles can be studied in a structurally cleaner setting related to what we call *Hamilton frameworks*. In broad terms, a Hamilton framework is a k -graph that exhibits some of the key properties of k -uniform Hamilton cycles (such as tight connectivity and a perfect fractional matching) and in addition to this has a sufficiently large vertex degree. To obtain Hamilton cycles from a Hamilton framework, we deploy a (rather involved) combination of hypergraph regularity and the absorption method.

Secondly, we reduce the task of finding k -uniform Hamilton frameworks under minimum d -degree conditions to an Erdős–Gallai-type question for $(k-d)$ -graphs (by which we mean finding a large tight cycle in subgraphs, given density conditions), which is of independent interest. To show this, we use techniques from

hypergraph Turán-type problems together with ideas of Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1] on the Erdős Matching Conjecture.

Finally, we use the previous reductions to derive Theorem 1 and 2 by means of two short solutions to the respective Erdős–Gallai-type problems. Here, we rely on results and methods from extremal set theory such as the Kruskal–Katona theorem and results by Frankl [7] on hypergraph matchings.

3 Open Problems

In light of our work, the most basic question is how the minimum d -degree threshold of k -uniform tight Hamilton cycles behaves for $k - d \geq 3$. The construction of Han and Zhao [10] together with Theorem 1 shows that there are constants $c, C > 0$ such that $1 - c(k - d)^{-1/2} \leq hc_d(k) \leq 1 - C(k - d)^{-1}$ for all $k > d \geq 1$. In our view, the left side of these two inequalities is more likely to reflect reality. In order to understand the asymptotic behaviour of the tight Hamilton cycle threshold it would be interesting to investigate whether $hc_d(k) \leq 1 - C'(k - d)^{-1/2}$ for some $C' > 0$.

A more challenging task consists in determining exact bounds for $hc_d(k)$. In context of this, let us review the construction of Han and Zhao [10].

Construction 1. For $1 \leq d \leq k - 2$, let $\ell = k - d$. Choose $0 \leq j \leq k$ such that $(j - 1)/k < \lceil \ell/2 \rceil / (\ell + 1) < (j + 1)/k$. Let H be a k -graph on n vertices and a subset of vertices X with $|X| = \lceil \ell/2 \rceil n / (\ell + 1)$, such that H contains precisely the edges S for which $|S \cap X| \neq j$.

Consider a k -graph H as in the Construction 1. By design, there is no tight walk between the edges S which satisfy $|S \cap X| > j$ and the edges S' which satisfy $|S' \cap X| < j$. On the other hand, a simple averaging argument shows that a tight Hamilton cycle must contain an edge of each (intersectional) type. Hence H does not admit a tight Hamilton cycle. Computing the minimum degree of H , we obtain for instance that $hc_d(k) \geq 5/9, 5/8, 408/625$ for $k - d = 2, 3, 4$, respectively. (It can be shown that the value of j and the size of X in the construction maximises these bounds.) We believe that the lower bounds obtained by Han and Zhao are best possible.

Conjecture 1. The minimum d -degree threshold for k -uniform tight Hamilton cycles coincides with the lower bounds given by Construction 1.

As mentioned earlier, our results also allow us to reduce the problem of finding tight Hamilton cycles to an Erdős–Gallai-type question. This is formalized as follows. A *tight walk* in a k -graph G is a sequence of vertices such that every interval of k consecutive vertices forms an edge. We say that G is *tightly connected* if any two edges are on a common tight walk.¹

¹ It is not hard to see that this is equivalent to the line graph $L(G)$ being connected, where $L(G)$ has vertex set $E(G)$ and an edge ef whenever $|e \cap f| = k - 1$.

Definition 2. For $\ell \in \mathbb{N}$, let $\text{eg}(\ell)$ be the smallest number $\delta > 0$ such that, for every $\mu > 0$, there are $\gamma > 0$ and $n_0 \in \mathbb{N}$ with the following property.

Suppose that G is an ℓ -graph on $n \geq n_0$ vertices with at least $(\delta + \mu) \binom{n}{\ell}$ edges. Then there is a subgraph $C \subseteq G$ which is

- (i) tightly connected, (connectivity)
- (ii) has a fractional matching of density $1/(\ell + 1) + \gamma$ and (space)
- (iii) has edge density at least $1/2 + \gamma$. (density)

It turns out that these properties (applied to auxiliary $(k - d)$ -graphs) are sufficient to construct Hamilton frameworks in k -graphs H with $\bar{\delta}_d(H) \geq \text{eg}(k - d)$. Thus we obtain the following results as a corollary of our previous reductions.

Corollary 1. For $1 \leq d \leq k - 1$, we have $\text{hc}_d(k) \leq \text{eg}(k - d)$.

We believe that this is tight when $k - d$ is odd.

Conjecture 2. If $k - d$ is odd, then $\text{hc}_d(k) = \text{eg}(k - d)$.

A first step towards this conjecture would be to show that $\text{eg}(3) = 5/8$, where the lower bound follows from (the link graphs of) Construction 1. Finally, let us remark that a similar (but notationally more involved) conjecture can be formulated for even $k - d$.

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An Approximate Structure Theorem for Small Sumsets

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Abstract. Let A and B be randomly chosen s -subsets of the first n integers such that their sumset $A + B$ has size at most Ks . We show that asymptotically almost surely A and B are almost fully contained in arithmetic progressions P_A and P_B with the same common difference and cardinalities approximately $Ks/2$. The result holds for $s = \omega(\log^3 n)$ and $2 \leq K = o(s/\log^3 n)$. Our main tool is an asymmetric version of the method of hypergraph containers which was recently used by Campos to prove the result in the special case $A = B$.

Keywords: Additive combinatorics · Sumsets · Hypergraph containers

1 Introduction and Main Result

The general framework of problems in additive combinatorics is to ask for the structure of a set A subject to some additive constraint in an additive group. The celebrated theorem of Freiman [5] provides such a structural result in terms of arithmetic progressions when the sumset $A + A$ is small. Classical results like the Kneser theorem in abelian groups or the Brunn–Minkowski inequality in Euclidian spaces naturally address a similar problem for the addition of distinct sets A and B . The proof by Ruzsa of the theorem of Freiman does provide the same structural result for distinct sets A, B with the same cardinality when their sumset is small. When the sumset $A + A$ is very small, then another theorem of Freiman shows that the set is dense in one arithmetic progression, and this result

M. Campos—is partially supported by CNPq. M.Co. is supported by the Spanish Ministerio de Economía y Competitividad through the project MTM2017-82166-P and the FPI-scholarship PRE2018-083621. O.S. is partially supported by the Spanish AEI under project MTM2017-82166-P. M.W. acknowledges financial support from the Fondo Social Europeo and the Agencia Estatal de Investigación through the FPI grant number MDM-2014-0445-16-2 and the Spanish Ministry of Economy and Competitiveness, through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445), as well as through the project MTM2017-82166-P.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 546–552, 2021.
https://doi.org/10.1007/978-3-030-83823-2_88

has been also extended to distinct sets A and B by Lev and Smeliansky [10] showing that both sets are dense in arithmetic progressions with the same common difference. Discrete versions of the Brunn–Minkowski inequality have also been addressed for distinct sets by Ruzsa [12] and Gardner and Gronchi [6].

Motivated by the Cameron–Erdős conjecture on the number of sum-free sets in the first N integers, there has been a quest to analyze the typical structure of sets satisfying some additive constraint. One of the most efficient techniques to address this problem is the method of hypergraph containers first introduced explicitly by Balogh, Morris and Samotij [2] and independently by Saxton and Thomason [13], which has been successfully applied to a number of problems of this flavour.

A conjecture by Alon, Balogh, Morris and Samotij [1] on the number of sets A of size $s \geq C \log n$ contained in the first N integers which have sumset $|A+A| \leq K|A|$, $K \leq s/C$ was proved by Green and Morris [7] for K constant and recently extended by Campos [3] to $K = o(s/(\log n)^3)$. These counting results are naturally connected to the typical structure of these sets, showing that they are almost contained in an arithmetic progression of length $(1 + o(1))Ks/2$. We build on the later work by Campos to adapt the result to distinct sets. Our main result is the following.

Theorem 1. *Let s and n be integers and $K \geq 2$ such that*

$$s = \omega((\log n)^3) \text{ and } K = o(s/(\log n)^3).$$

Let $A, B \subset [n]$ be a uniformly chosen pair of sets satisfying

$$|A| = |B| = s \text{ and } |A + B| \leq Ks.$$

Then asymptotically almost surely, there exist arithmetic progressions P and Q with the same common difference of size at most

$$|P|, |Q| \leq \left(\frac{1}{2} + o(1)\right) Ks$$

such that $|A \cap P| \geq (1 - o(1))s$, and similarly, $|B \cap Q| \geq (1 - o(1))s$.

An example discussed in [3] that can easily be adapted to the asymmetric case shows that the range of K for which a closely related counting statement holds cannot be improved to $K \geq s/\log n$. It is not clear whether the same holds true for the structure theorem, and an interesting question would be to investigate whether Theorem 1 might be true for any $K = o(s)$. On the other hand, one should keep in mind the strength of the current statement. If $s = n^\alpha$ for some $0 < \alpha \leq 1/2$, then $K = n^{\alpha(1-\epsilon)} = s^{1-\epsilon}$ is a valid choice for any fixed $\epsilon > 0$, and hence Theorem 1 states that for almost every pair of sets A, B of size $s = n^\alpha$ such that $|A + B| \leq s^{2-\epsilon}$, both A and B are (essentially up to scaling and translating) almost contained in an interval of size $O(n^{\alpha(2-\epsilon)}) = o(n)$.

2 An Overview of the Proof

One of the key techniques used in [3] is based on an asymmetric version of the container lemma introduced by Morris, Samotij and Saxton [11] which allows for applications to forbidden structures with some sort of asymmetry. This asymmetry can be interpreted as considering bipartite hypergraphs. The first key component to prove Theorem 1 is to further extend this bipartite version to a multipartite one as follows.

Let r be a positive integer. For an r -vector $x = (x_1, \dots, x_r)$ we call an r -partite hypergraph \mathcal{H} with vertex set $V(\mathcal{H}) = V_1 \cup \dots \cup V_r$ x -bounded if $|E \cap V_i| \leq x_i$ for every hyperedge $E \in E(\mathcal{H})$ and every $1 \leq i \leq r$. Denote by \mathcal{I} the family of independent sets of \mathcal{H} , and for any $m \in \mathbb{N}$, define

$$\mathcal{I}_m(\mathcal{H}) := \{I : I \in \mathcal{I} \text{ and } |I \cap V_r| \geq |V_r| - m\}.$$

For a subset of vertices $L \subset V(\mathcal{H})$, the codegree is defined as

$$d_{\mathcal{H}}(L) = |\{E \in E(\mathcal{H}) : L \subset E\}|.$$

Also, given a vector $v = (v_1, v_2, \dots, v_r) \in \mathbb{Z}^r$, denote

$$\Delta_v(\mathcal{H}) := \max\{d(L) : L \subset V(\mathcal{H}), |L \cap V_i| = v_i, 1 \leq i \leq r\}.$$

Finally, for any vector y , $|y|$ will denote its 1-norm $\sum |y_i|$.

Theorem 2. *For all non-negative integers r, r_0 and each $R > 0$ the following holds. Suppose that \mathcal{H} is a non-empty r -partite $(1, \dots, 1, r_0)$ -bounded hypergraph with $V(\mathcal{H}) = V_1 \cup V_2 \cup \dots \cup V_r$, $m \in \mathbb{N}$, $w = (|V_1|, |V_2|, \dots, |V_{r-1}|, m)$ and b, q are integers with $b \leq \min_i w_i$ and $q \leq m$, satisfying*

$$\Delta_v(\mathcal{H}) \leq R \left(\prod_{i=1}^r w_i^{v_i} \right)^{-1} b^{|v|-1} e(\mathcal{H}) \left(\frac{m}{q} \right)^{\mathbb{1}_{\{v_r > 0\}}}$$

for every vector $v = (v_1, v_2, \dots, v_r) \in \left(\prod_{i=1}^{r-1} \{0, 1\} \right) \times \{0, 1, \dots, r_0\}$. Then there exists a family $\mathcal{S} \subset \prod_{i=1}^r \binom{V_i}{\leq b}$ and functions $f: \mathcal{S} \rightarrow \prod_{i=1}^r 2^{V_i}$ and $g: \mathcal{I}_m(\mathcal{H}) \rightarrow \mathcal{S}$, such that, letting $\delta = 2^{-(r_0+r-1)(2r_0+r)} R^{-1}$, the following three things are true.

- (i) If $f(g(I)) = (A_1, A_2, \dots, A_r)$ with $A_i \subset V_i$, then $I \cap V_i \subset A_i$ for all $1 \leq i \leq r$.
- (ii) For every $(A_1, A_2, \dots, A_r) \in f(\mathcal{S})$, either $|A_i| \leq (1 - \delta)|V_i|$ for some $1 \leq i \leq r - 1$, or $|A_r| \leq |V_r| - \delta q$.
- (iii) If $g(I) = (S_1, S_2, \dots, S_r)$ and $f(g(I)) = (A_1, A_2, \dots, A_r)$, then $S_i \subset I \cap V_i$ for all $1 \leq i \leq r$. Furthermore, $|S_i| > 0$ only if $|A_j| \leq |V_j| - \delta w_j$ for some $j \geq i$.

The iterated application of Theorem 2 to the hypergraph described below provides the appropriate setting to prove the main result. It is formulated in the context of general groups (not necessarily abelian) and iterated sumsets (not only two summands) as our main project is to address the problem in this more general setting.

For a group G and finite subsets $F_1, \dots, F_r \subset G$, define the r -partite and $(1, \dots, 1)$ -bounded hypergraph $\mathcal{H}(F_1, \dots, F_r)$ as follows. It has vertex set $\bigsqcup_{i \in [r]} F_i$ and for any collection of $r - 1$ elements f_1, \dots, f_{r-1} with $f_i \in F_i$, if their product $f_r = f_1 f_2 \cdots f_{r-1}$ is contained in F_r , then $\{f_1, \dots, f_r\}$ is an edge. Note that the sets F_i need not actually be disjoint.

Theorem 3. *Let G be a group, $h \geq 2$ an integer and $\epsilon > 0$. Suppose n, m, s_1, \dots, s_h are integers such that $\log n \leq \max s_i \leq m \leq \log n (\min s_i)^h$, and let F_1, \dots, F_h be subsets of G of cardinality $|F_i| = n$ with product set $F = F_1 F_2 \cdots F_h$. Then there exists a family $\mathcal{A} \subset \prod_{i \in [h]} 2^{F_i} \times 2^F$ of $(h + 1)$ -tuples (A_1, \dots, A_h, B) of size*

$$|\mathcal{A}| \leq \exp\left(2^{(h+1)(h+5)} \epsilon^{-h} m^{1/h} (\log n)^{(2h-1)/h}\right)$$

such that the following two things are true:

1. For all $X_i \subset F_i, Y \subset F$ with $|X_i| = s_i, X_1 X_2 \cdots X_h \subseteq Y$ and $|Y| \leq m$, there exists a tuple $(A_1, \dots, A_h, B) \in \mathcal{A}$ such that $B \subset Y$ and $X_i \subset A_i$ for all $i \in [h]$.
2. For every $(A_1, \dots, A_h, B) \in \mathcal{A}$ it holds that $|B| \leq m$ and either $\max_i |A_i| < m / \log n$ or there are at most $\epsilon^h \prod |A_i|$ tuples $(a_1, \dots, a_h) \in \prod A_i$ such that $a_1 a_2 \cdots a_h \notin B$.

One can now analyze the structure of the containers using supersaturation and stability results. For the former, we utilize the following statement that essentially was already present in Campos' original result.

Proposition 1. *Let $A_1, A_2, B \in \mathbb{Z}$ be finite and non-empty sets and $0 < \epsilon < 1/2$. If $|A_1| + |A_2| \geq (1 + 2\epsilon)(|B| + 1)$, then there are at least $\epsilon^2 |A_1| |A_2|$ pairs $(a_1, a_2) \in A_1 \times A_2$ such that $a_1 + a_2 \notin B$.*

For the stability statement, results in the literature were mainly concerned with handling the case of sets having the same cardinality. The problem here is that while the pairs of sets that are counted in Theorem 1 have the same size, the containers obtained via Theorem 3 might differ slightly. We modify the recently obtained robust version of Freiman's $3k - 4$ theorem by Shao and Xu [15], which itself built on earlier work by Lev [9] to handle this, and obtain the following result.

Proposition 2. *Let $\epsilon > 0$ and $U, V \subset \mathbb{Z}$ finite sets with $|V| \leq |U|$. If there exists a set $\Gamma \subset U \times V$ with $|\Gamma| \geq (1 - \epsilon)|U||V|$ and*

$$|U + V| < (1 - 13\epsilon^{1/2})|U| + \frac{3}{2}|V|,$$

then there are arithmetic progressions P and Q of the same common difference and length $|P| \leq (1 + 5\epsilon^{1/2})|U|$, $|Q| \leq |V| + 5\epsilon^{1/2}|U|$ such that $|P \cap U| \geq (1 - \epsilon^{1/2})|U|$ and $|Q \cap V| \geq (1 - \epsilon^{1/2})|V|$.

The containers we need to apply this to will have size roughly $(1 + o(1))Ks/2$, which results in the following statement.

Corollary 1. *Let $0 < \epsilon \leq 2^{-9}$. If $A_1, A_2, B \subset \mathbb{Z}$, such that*

$$(1 - \epsilon)|B| \leq |A_1| + |A_2| \text{ and } \max\{|A_1|, |A_2|\} \leq (1 + 2\sqrt{\epsilon})|B|/2.$$

Then one of the following holds:

1. *There are at least $\epsilon^2|A_1||A_2|$ pairs $(a_1, a_2) \in A_1 \times A_2$ such that $a_1 + a_2 \notin B$.*
2. *There are arithmetic progressions P_1, P_2 of length at most $|B|/2 + 4\sqrt{\epsilon}|B|$ with the same common difference such that P_i contains all but at most $\epsilon|A_i|$ points of A_i .*

The further direction is now clear. The second property of Theorem 3 tells us that essentially, case one of Corollary 1 cannot happen, and hence the containers either have a structure as outlined in the second case, or one of the requirements of the corollary must not be satisfied, that is, either the sum of cardinalities of the containers is very small, or they must be somewhat unbalanced.

Both of the latter two situations are handled by counting *all* pairs of s -subsets of such containers, irrespective of whether they correspond to sets with small sumset. This is done via standard inequalities for binomial coefficients and products thereof, with the result being that there are $o(1)\binom{Ks/2}{s}^2$ of such pairs.

What remains is the case that the containers themselves are close to being contained in arithmetic progressions. But if this happens, we argue that there cannot be many pairs of s -subsets not satisfying a slightly weaker containment, since any of the *bad* points of a set in such a pair would need to be chosen from the exceptional set of its container, which we know is small. The final conclusion is that in this case, the number of pairs of s -subsets not satisfying a containment as stated in Theorem 1 will also be $o(1)\binom{Ks/2}{s}^2$.

On the other hand, there are clearly at least $\binom{Ks/2}{s}^2$ good pairs, since one can fix a single pair of disjoint arithmetic progressions of size $Ks/2$ and take any pair of s -subsets of them.

3 Conclusions

The set up for the proof of the main result is devised to address a number of additional applications. On one hand the setting allows to handle multiple set addition: if A_1, \dots, A_k is a family of independently chosen random sets of integers with cardinality s among those satisfying a constraint on their sumset, say $|A_1 + \dots + A_k| \leq Ks$ then with high probability each of the sets is almost contained in an arithmetic progression of size Ks/k having the same common

difference. The quantitative aspects of the statement depend on the strength of supersaturation and stability results analogous to Propositions 1 and 2, which can likely be derived from the generalized Kneser theorem of DeVos, Goddyn, and Mohar [4] and the structural results for multiple set addition of Lev [8].

A second direction is to consider the problem in general abelian groups, not only in the integers. In the symmetric case $A = B$, Campos addresses this case providing a counting result. More precisely, he shows that if X is a set of size n in an abelian group G , then number of sets $A \subset X$ of size s having doubling at most K is at most $2^{o(s)} \binom{Ks+\beta}{s}$, where β is a parameter that measures the largest subgroup of G still smaller than Ks . A similar result should hold in our setting as well. Getting a structural result akin to Theorem 1 seems to be more challenging, although the recent robust version of the Balog-Szemerédi-Gowers theorem obtained by Shao [14] might allow one to prove it for specific groups other than the integers. A less explored direction would be to address the problem in general groups, not necessarily abelian.

Finally, there are specific substructures that might be interesting to investigate. Note that Campos' version of Theorem 1 handling the case $A = B$ does not directly follow from ours, since the theorem only compares the number of pairs (A, B) satisfying some structural statement to those that do not, but it is not clear a priori that it will hold in a relative sense when only looking at pairs (A, A) . The same is true for other instances of specific pairs (A, B) . One interesting problem is the case of $B = \lambda * A$, the *dilation* of A by some factor λ . It is known that $A + \lambda * A$ has size at least $(\lambda + 1)|A|$, and in cases where λ is prime the structure of extremal cases is known. Can one prove an approximate structure result similar in scope to Theorem 1 for this specific situation?

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Classification of Local Problems on Paths from the Perspective of Descriptive Combinatorics

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Abstract. We classify which local problems with inputs on oriented paths have so-called Borel solution and show that this class of problems remains the same if we instead require a measurable solution, a factor of iid solution, or a solution with the property of Baire.

Together with the work from the field of distributed computing [Balliu et al. PODC 2019], the work from the field of descriptive combinatorics [Gao et al. arXiv:1803.03872, Bernshteyn arXiv:2004.04905] and the work from the field of random processes [Holroyd et al. Annals of Prob. 2017, Grebík, Rozhoň arXiv:2103.08394], this finishes the classification of local problems with inputs on oriented paths using complexity classes from these three fields.

A simple picture emerges: there are four classes of local problems and most classes have natural definitions in all three fields. Moreover, we now know that randomness does *not* help with solving local problems on oriented paths.

Keywords: Descriptive combinatorics · Factors of iid processes · Distributed algorithms

1 Introduction

The full version of this paper is available online [15].

Locally checkable problems (LCLs) is a class of graph problems where the correctness of a solution can be checked locally. This class includes problems like vertex or edge coloring, perfect matching or a maximal independent set, as well as problems with inputs such as list colorings etc. In this paper, we study LCLs on oriented paths from three different perspectives: the perspective of *descriptive combinatorics*, *distributed algorithms*, and *random processes*. Each perspective offers different procedures and computational power to solve LCLs, as well as different scales to measure complexity of LCLs. In particular, the notion of easy/local vs hard/global problems varies in different settings.

In this work we completely describe the connections between these complexity classes and show that the problem of deciding to what class a given LCL belongs is decidable. This extends the known classification of LCLs without inputs on oriented paths (see the full version) and is in contrast with the fact that on \mathbb{Z}^d , where $d > 1$, this problem is not decidable.

Moreover, we hope that our work helps to clarify the relationships between the three different perspectives and, since the situation is both non-trivial and well understood, may serve as a basis of a common theory. We note that for different class of graphs, i.e., grids, regular trees, etc., describing possible complexity classes of LCLs and the relationship between them is not known, is already much harder and contains many exciting questions, see [16].

We will now briefly explain all three fields and then the particular setup of LCLs on paths that we are interested in. Then we state Theorem 1, our main theorem that, together with previous work, gives a classification of complexity classes of LCL problems on oriented paths that relates the three different perspectives to each other. To make the presentation more precise, we introduce the formal definition of LCLs.

Definition 1 (LCLs on oriented paths). *A locally checkable problem (LCL) on an oriented path is a quadruple $\Pi = (\Sigma_{in}, \Sigma_{out}, r, \mathcal{P})$, where Σ_{in} and Σ_{out} are finite sets, r is a positive integer, and \mathcal{P} is a finite collection of finite rooted Σ_{in} - Σ_{out} -labeled paths of diameter (at most) r .*

A correct solution of an LCL problem Π in a Σ_{in} -labeled oriented path G (finite cycle or infinite path) is a map $f : V(G) \rightarrow \Sigma_{out}$ such that the Σ_{in} - Σ_{out} -labeled rooted r -neighborhood of every node $v \in V(G)$ is in \mathcal{P} . Every such f is called a Π -coloring.

Example of an LCL problem is k -coloring: there, $\Sigma_{in} = \emptyset$, $\Sigma_{out} = \{1, 2, \dots, k\}$, $r = 1$, and \mathcal{P} contains all pairs $(\sigma_1, \sigma_2) \in \Sigma_{out}^2$ such that $\sigma_1 \neq \sigma_2$. Other examples of problems without inputs are: maximal independent set, edge coloring, or perfect matching. Problems with inputs include for example list coloring (we formally require the set of all colors in the lists to be finite).

The fact that we allow for input labelling is important. Without it, the problem is substantially simpler. The reason we care about inputs is twofold. First, problems with inputs contain more general setups such as the setup in the circle squaring problem that we discuss next. Second, understanding problems on lines with inputs serves as an intermediate step towards understanding problems on trees, at least in the distributed computing area [1, 8].

Distributed Algorithms. The study of the LOCAL model of distributed algorithm [20] is motivated by understanding distributed algorithms in huge networks. The LOCAL model formalizes this setup: there is a graph such that each of its nodes at the beginning knows only its size n , and perhaps some other parameter like the maximum degree Δ . Moreover, each node starts with a unique identifier from a range polynomial in the size of the graph n . In one round, each node can exchange any message with its neighbors and can perform an arbitrary

computation. We want to find a solution to a problem in as few communication rounds as possible.

Importantly, there is an equivalent view of t -round LOCAL algorithms: such an algorithm is simply a function that maps all possible t -hop neighbourhoods to the final output. An algorithm is correct if and only if applying this function to the t -hop neighborhood of each node solves the problem.

The simplest possible setting to consider in the LOCAL model is if one restricts their attention to the simplest graph: a sufficiently long consistently oriented cycle. This graph is the simplest model, since all local neighborhoods of it look like an oriented path. This case is very well understood: It is known that any LCL problem can have only one of three complexities on oriented paths and the randomized complexity is always the same as the deterministic one. First, there are problems solvable in $O(1)$ local computation rounds – think of the problems like “color all nodes with red color” or “how many different input colors are there in my 5-hop neighborhood?”. Second, there is a class of basic symmetry breaking problems solvable in $\Theta(\log^* n)$ rounds – this includes problems like 3-coloring, list-coloring with lists of size at least 3, or maximal independent set. Finally, there is a class of “global” problems that cannot be solved in $o(n)$ rounds – these include e.g. 2-coloring or perfect matching.

Descriptive Combinatorics. In 1990s Laczkovich resolved the famous Circle Squaring Problem of Tarski[19]: A square of unit area can be decomposed into finitely many pieces that can be translated to form a disk of unit area. This result was improved in recent years to make the pieces measurable in various sense [14, 22, 23]. This theorem and its subsequent strengthenings are the highlights of a field nowadays called descriptive combinatorics [18, 24] that has close connections to distributed computing as was shown in an insightful paper by Bernshteyn [2].

The simplest non-trivial setup of descriptive combinatorics is the following [24]. Consider the unit cycle \mathbb{S}^1 in \mathbb{R}^2 , i.e., the set of pairs (x, y) such that $x^2 + y^2 = 1$. Imagine rotating this cycle by a rotation $\alpha > 0$ that is irrational with respect to the full rotation, i.e., $\alpha/(2\pi) \notin \mathbb{Q}$. This rotation naturally induces a directed graph $G_\alpha = (\mathbb{S}^1, E)$ on the cycle \mathbb{S}^1 where a directed edge in E points from (x, y) to (x', y') if one gets (x', y') by rotating (x, y) by α . This graph has uncountably many connected components, each of which is a directed path going to infinity in both directions.

We could now ask questions like: What is the chromatic number of G ? If there were no other restrictions, the answer is 2, i.e., color each path separately. However, for this type of argument we implicitly use the axiom of choice. The main goal of descriptive combinatorics is to understand what happens if some additional definability/regularity requirements are posed on the colorings. That is, what if we require each color class to be an interval, open, Borel, or Lebesgue measurable set? With any of those requirements, the chromatic number of G now increases to 3.

For a general LCL without input labelings the situation is completely understood. It turns out that problems solvable on G (with the additional definability requirements) are exactly those that have local complexity $O(\log^* n)$.

Our main theorem generalizes to the setup with input labels. In the example with \mathbb{S}^1 this means that adversary first partitions \mathbb{S}^1 into sets that are indexed by input labels, i.e., $\mathbb{S}^1 = \bigsqcup_{\sigma \in \Sigma_{in}} A_\sigma$. Again, we require each input color class to satisfy some definability properties, e.g., union of intervals, open, Borel, or Lebesgue measurable set. We may view any such partition as labeling of nodes simply by thinking that a node $x \in \mathbb{S}^1$ is labeled by σ if $x \in A_\sigma$. This induces a labeling of each oriented doubly infinite path of the graph G . Now we ask for a definable coloring with colors from Σ_{out} that would solve Π . As we demonstrate more LCL problems can be solved in this setup than with local algorithms. However, the picture is still very clean and in our view nicely explains the power of descriptive combinatorics when compared with distributed algorithms.

Factors of iid Processes. The third field that offers a yet different view on local problems is the area of randomized processes where one studies whether a solution can be constructed as a so-called factor of iid process. We describe this area only in the full version [15], due to space constraints.

1.1 Our Contribution

We extend the classification of LCLs on oriented paths from the perspective of distributed algorithms [1] to the perspective of descriptive combinatorics and random processes. A simple picture emerges. We find that with inputs the latter setups offer more complexity classes than distributed algorithms.

We now state our main theorem and its corollary which is a classification of LCL problems from three different perspectives. It contains several classes, some of which we introduce only in the full version. One should think of the class BOREL as in the example where the adversary partitions the circle to Borel measurable pieces and our final solution needs to be Borel measurable. The classes MEASURE, BAIRE are traditionally studied in descriptive combinatorics and the class fiid is studied in the area of random processes. These classes offer more computational power. Intuitively, the additional power of MEASURE and fiid when compared with BOREL is the same as the additional power of a randomized algorithm when compared with a deterministic one.

Theorem 1. *For the classes of LCL problems on infinite oriented lines we have that*

$$\text{BOREL} = \text{MEASURE} = \text{fiid} = \text{BAIRE}.$$

Moreover, deciding whether $\Pi \in \text{BOREL}$ is a PSPACE-complete problem.

As a corollary, this finishes the classification of complexity classes coming from the three analysed areas. The big picture containing all relevant classes is given in Fig. 1. It follows from the work in distributed algorithms [1, 5, 6, 8], descriptive combinatorics [2, 3, 12], and finitary factors of iid processes [4, 16, 17].

This classification is complete in the sense that we are not aware of other used classes of problems. It describes an unified picture of locality for one particular class of graphs and it helps to clarify the computational powers and limits of different approaches.

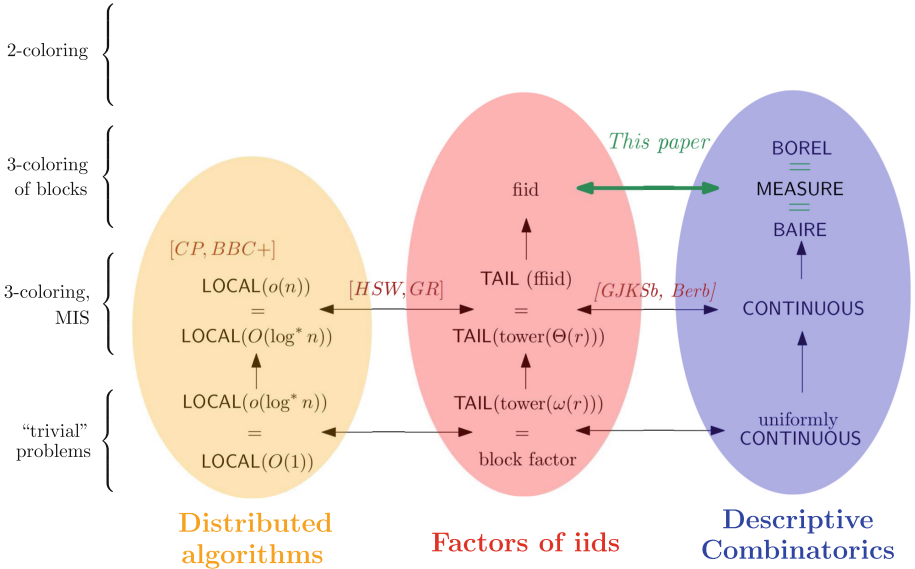


Fig. 1. The figure shows a classification of LCL problems on paths from all three considered perspectives. A clear picture emerges. There are four classes of problems. First, there is class that contains trivial problems such as “how many different input colors are there in my 5-hop neighborhood”. Then, there is a class of basic symmetry breaking problems such as 3-coloring or MIS. Then, there is a class of problems that we can solve with basic symmetry breaking tools, but we cannot do it locally. An example problem is the 3-coloring-of-blocks problem described in the full version. Finally, there is a class of “global” problems that contain e.g. 2-coloring. We do not discuss all classes present in the picture. They are discussed in [16] for the more general setup of grids.

Derandomization Perspective. The relation of classes MEASURE and fiid to BOREL is analogous to the relation of randomized algorithms to deterministic algorithms. This means that Theorem 1 can be seen as a *derandomization* result. The topic of derandomization is of a big interest in distributed algorithms [6, 9, 10, 13, 25] and in complexity theory in general. We are not aware of similar derandomization results in the area of descriptive combinatorics, except the case of LCLs without inputs on paths (see full version). For concrete problems, like the famous circle squaring problem, the derandomization of the construction (that is, replacing measurable pieces by borel measurable pieces) was done in the work of [23] that improved the previous “measurable version” [14].

In general, it is known that randomness helps if we do not bound the expansion of the graph class under consideration. As an example we recall that proper vertex 3-coloring or perfect matching is not in the class BOREL for infinite 3-regular trees [21]. This implies no nontrivial deterministic local algorithm. On the other hand, the two problems are in the class MEASURE and BAIRE [7] and the 3-coloring problem, in fact, admits a nontrivial randomized local algorithm [11]. What if the graph family is of subexponential growth? The popular conjecture of Chang and Pettie that the deterministic complexity of the Lovász Local Lemma (LLL) problem is $O(\log n)$ would imply that if the graph class is of subexponential growth, the class of “LLL-type problems”, the only local class where randomness helps essentially, is not present [6, 8]. We conjecture that it is a general phenomenon that randomness does not help in graphs of subexponential growth. In particular, we conjecture that our result that BOREL = MEASURE = fiid on paths holds for all graphs of subexponential growth. We note that this is not known even for 2-dimensional grids, see [16].

The classification from Fig. 1 is decidable, though, in fact, PSPACE-hard (this follows from Theorem 1 and [1]). We think it is an exciting complexity-theoretic problem to understand whether it is in fact PSPACE-complete [1]. The fact that the classification from Fig. 1 is decidable corresponds to the fact that there is a reasonable combinatorial classification of problems in each class in Fig. 1.

Acknowledgment. The first author was supported by Leverhulme Research Project Grant RPG-2018-424. This project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No. 853109).

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On the Enumeration of Plane Bipolar Posets and Transversal Structures

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Abstract. We show that plane bipolar posets (i.e., plane bipolar orientations with no transitive edge) and transversal structures can be set in correspondence to certain (weighted) models of quadrant walks, via suitable specializations of a bijection due to Kenyon, Miller, Sheffield and Wilson. We then derive exact and asymptotic counting results, and in particular we prove that the number t_n of transversal structures on $n + 2$ vertices satisfies (for some $c > 0$) $t_n \sim c(27/2)^n n^{-1-\pi/\arccos(7/8)}$, which also ensures that the associated generating function is not D-finite.

Keywords: Bijections · Oriented planar maps · Quadrant walks

1 Introduction

The combinatorics of planar maps (i.e., planar multigraphs endowed with an embedding on the sphere) has been a very active research topic ever since the early works of W.T. Tutte. In the last few years, after tremendous progress on the enumerative and probabilistic theory of maps, the focus has started to shift to planar maps endowed with *constrained orientations*. Indeed constrained orientations capture a rich variety of models [7] with connections to (among other) graph drawing, pattern-avoiding permutations, Liouville quantum gravity, or theoretical physics. From an enumerative perspective, these new families of maps are expected to depart (e.g. [6]) from the usual *algebraic generating function* pattern followed by many families of planar maps with local constraints [11]. From a probabilistic point of view, they lead to new models of random graphs and surfaces, as opposed to the universal Brownian map limit capturing earlier models. Both phenomena are first witnessed by the appearance of new critical exponents $\alpha \neq 5/2$ in the generic $\gamma^n n^{-\alpha}$ asymptotic formulas for the number of maps of size n .

A fruitful approach to oriented planar maps is through bijections (e.g. [1]) with walks with a specific step-set in the quadrant, or in a cone, up to shear transformations. We rely here on a recent such bijection [10] that encodes plane bipolar orientations by certain quadrant walks (so-called tandem walks): we show in Sect. 2 that it can be furthermore adapted to other models by introducing properly chosen weights. Building on these specializations, in Sect. 3 we obtain

exact enumeration results for plane bipolar posets and transversal structures. In particular we show that the number b_n of plane bipolar posets on $n+2$ vertices is equal to the number of plane permutations of size n recently studied in [4], and that a reduction to small-steps quadrant walks models (which makes coefficient computation faster) can be performed for the number e_n of plane bipolar posets with n edges and the number t_n of transversal structures on $n+2$ vertices. In Sect. 4 we obtain asymptotic formulas for the coefficients b_n, e_n, t_n all of the form $c\gamma^n n^{-\alpha}$ with $c > 0$ and with $\gamma, \alpha \neq 5/2$ explicit, and by the approach of [3] we deduce from these estimates that the generating functions for e_n and t_n are not D-finite.

Note: An extended version on these results is available at [arXiv:2105.06955](https://arxiv.org/abs/2105.06955).

2 Oriented Planar Maps and Quadrant Tandem Walks

A *plane bipolar orientation* B is a planar map endowed with an acyclic orientation having a single source S and a single sink N , which both lie in the outer face, see Fig. 1(a). It is known that the contour of each face f of B (including the outer one) splits into a left lateral path L_f and a right lateral path R_f (which share the same origin and end); the *type* of f is the pair (i, j) where $i+1$ (resp. $j+1$) is the length of L_f (resp. R_f). The *outer type* of B is the type of the outer face. The *pole-type* of B is the pair (p, q) such that $p+1$ is the degree of S and $q+1$ is the degree of N .

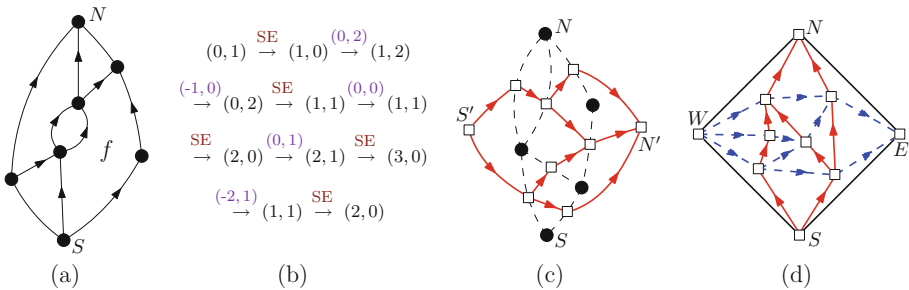


Fig. 1. (a) A plane bipolar orientation of outer type $(1, 2)$ (the marked inner face f has type $(2, 1)$). (b) A quadrant tandem walk from $(0, 1)$ to $(2, 0)$ (actually the one associated to (a) by the KMSW bijection). (c) From a plane bipolar orientation (round vertices) with n edges and $f+2$ vertices to one of the associated plane bipolar posets (square vertices) with $n+2$ vertices and f inner faces. (d) A 4-triangulation endowed with a transversal structure (blue edges are dashed).

On the other hand, a *tandem walk* (see Fig. 1(b)) is defined as a walk on \mathbb{Z}^2 with steps in $SE \cup \{(-i, j), i, j \geq 0\}$; it is a *quadrant walk* if it stays in \mathbb{N}^2 all along. Every step $(-i, j)$ in such a walk is called a *face-step*, and the pair (i, j) is called its *type*. We will crucially rely on the following bijective result:

Theorem 1 (KMSW bijection [10]). *Plane bipolar orientations of outer type (a, b) with $n + 1$ edges are in bijection with quadrant tandem walks of length n from $(0, a)$ to $(b, 0)$. Every non-pole vertex corresponds to a SE-step, and every inner face corresponds to a face-step, of the same type.*

An edge $e = (u, v) \in B$ is called *transitive* if there is a path from u to v avoiding e . If B has no transitive edge it is called a *plane bipolar poset*.

Remark 1. Let B be a plane bipolar orientation. Then B is a plane bipolar poset iff it has no inner face whose type has a zero entry. Hence the KMSW bijection specializes into a bijection (with same parameter-correspondence) between plane bipolar posets of outer type (a, b) and quadrant tandem walks from $(0, a)$ to $(b, 0)$ such that the type of every face-step has no zero-entry.

In Remark 1 the primary parameter of the poset (the one corresponding to the walk length) is the number of edges (minus 1). We will see below another way to relate plane bipolar posets to (weighted) quadrant tandem walks, this time with the number of vertices as the primary parameter. Other oriented maps to be related below to weighted quadrant tandem walks are transversal structures [8]. A *4-triangulation* is a map whose outer face contour is a (simple) 4-cycle and whose inner faces are triangles; the outer vertices are denoted W, N, E, S in clockwise order, and V denotes the set of inner vertices. A *transversal structure* on such a map (see Fig. 1(d)) is an orientation and bicoloration of its inner edges (in blue or red) so that red (resp. blue) edges form a bipolar poset with V as the set of non-pole vertices and (S, N) (resp. (W, E)) as the pair (source,sink), and moreover any intersection of a blue path with a red path is a crossing where the blue path arrives from the left side of the red path.

For w a function from \mathbb{N}^2 to \mathbb{N} , a *w-weighted* plane bipolar orientation is a bipolar orientation where every inner face f carries an integer $\iota(f)$ in $[1..w(i, j)]$ with (i, j) the type of f . A *w-weighted* tandem walk is a tandem walk where every face-step s carries an integer $\iota(s)$ in $[1..w(i, j)]$ with (i, j) the type of s .

Proposition 1. *For $w : (i, j) \rightarrow \binom{i+j}{i}$, plane bipolar posets of pole-type (p, q) , with $n + 2$ vertices and f inner faces, are in bijection with w -weighted plane bipolar orientations of outer type (p, q) , with n edges and $f + 2$ vertices. These correspond (via KMSW) to w -weighted quadrant tandem walks of length $n - 1$ from $(0, p)$ to $(q, 0)$ with f SE-steps.*

For $w : (i, j) \rightarrow \binom{i+j-2}{i-1}$ (with $w(i, j) = 0$ if $i = 0$ or $j = 0$), transversal structures having n inner vertices and m blue edges are in bijection with w -weighted plane bipolar posets of outer type $(1, 1)$ having $n + 4$ vertices and $m + 4$ edges. These correspond (via KMSW) to w -weighted quadrant tandem walks from $(0, 1)$ to $(1, 0)$ of length $m + 3$ with $n + 2$ SE-steps.

Proof. The first correspondence (see Fig. 1(c)) is adapted from [9]. Starting from a plane bipolar orientation B , insert a square vertex in the middle of each edge (these are to be the non-pole vertices of the bipolar poset). Then in each inner face f , with (i, j) its type, insert $i + j + 1$ non-crossing edges from the square

vertices on $L(f)$ to the square vertices on $R(f)$; there are precisely $w(i, j) = \binom{i+j}{i}$ ways to do so (so the chosen way can be encoded by an integer $\iota(f) \in [1..w(i, j)]$). Finally create a square vertex S' (resp. N') in the left (resp. right) outer face and connect it to all square vertices on the left (resp. right) lateral path of B . Then the bipolar poset is obtained by erasing the vertices and edges of B in the obtained figure.

The second correspondence relies on the fact that a transversal structure is completely encoded by its red bipolar poset (augmented by the 4 outer edges oriented from S to N) and the knowledge of how each inner face is transversally triangulated by blue edges: if the face has type (i, j) then there are precisely $\binom{i+j-2}{i-1}$ ways to do so.

3 Exact Counting Results

Let $P_a^w(x, y)$ denote the generating series of w -weighted quadrant tandem walks starting in position $(0, a)$, with respect to the number of steps (variable t), end positions (variables x and y) and number of SE steps (variable u). A last step decomposition immediately yields the following *master equation* in the ring $\mathbb{Q}((\bar{x}))[[y, t]]$ of formal power series in t and y with coefficients that are Laurent series in $\bar{x} = 1/x$, where $W_k(\bar{x}, y) = \sum_{i \geq k, j \geq 0} w(i, j) \frac{y^j}{\bar{x}^i}$:

$$P_a^w(x, y) = y^a + tu \frac{x}{y} (P_a^w(x, y) - P_a^w(x, 0)) + tW_0(\bar{x}, y)P_a^w(x, y) - t \sum_{k \geq 0} W_{k+1}(\bar{x}, y)x^k [x^k]P_a^w(x, y).$$

In the case of plane bipolar posets enumerated by vertices, we have (cf. Proposition 1) $w(i, j) = \binom{i+j}{i}$ for $i, j \geq 0$, so that $W_k(\bar{x}, y) = \frac{1}{1-(\bar{x}+y)} \frac{\bar{x}^k}{(1-y)^k}$ in $\mathbb{Q}[[y, \bar{x}]]$. For $B(x, y) \equiv P_0^w(x, y)$ the master equation then rewrites

$$B(x, y) = 1 + t \frac{x}{y} (B(x, y) - B(x, 0)) + \frac{t}{1-y} \frac{1}{x - \frac{1}{1-y}} \left(xB(x, y) - \frac{1}{1-y} B\left(\frac{1}{1-y}, y\right) \right).$$

Let b_n denote the number of plane bipolar posets with $n+2$ vertices. It is also, by adding a new sink of degree 1 (connected to the former sink), the number of plane bipolar posets of pole-type $(0, b)$ with $n+3$ vertices and arbitrary $b \geq 0$, so that $b_n = [t^n]B(1, 0)$. Then we prove¹ that b_n is also the number of plane permutations of size n which are studied in [4]: to see this let $S(u, v) := x(B(x, y) - 1)$ under the change of variable relation $\{y = 1 - \bar{u}, x = v\}$ (note that $B(1, 0) = 1 + S(1, 1)$), and observe that the equation for S derived from the above equation for B is exactly [4, Eq. (2)] (they use (x, y, z) for our (t, u, v)). Furthermore $B(1, 0) = 1 + S(1, 1)$ is D-finite [4, Proposition 13], and there are single sum expressions for b_n [4, Theorem 14].

¹ Our proof relies on generating function manipulations, but a similar bijective approach as in [2] also applies, as detailed in the extended version.

The case of bipolar posets counted by edges corresponds to having $w(i, j) = \mathbf{1}_{i \neq 0, j \neq 0}$ (cf. Remark 1). By some manipulations on the functional equation in that case, we can show that the number e_n of plane bipolar orientations with n edges coincides with the number of quadrant excursions of length $n - 1$ with steps in $\{0, E, S, NW, SE\}$. While the series $\sum_n e_n t^n$ is non D-finite (as discussed in the next section) the reduction to a quadrant walk model with small steps allows to compute the sequence e_1, \dots, e_n with time complexity $O(n^4)$ using $O(n^3)$ bit space. The sequence starts as 1, 1, 1, 2, 5, 12, 32, 93, 279, 872, 2830, ...

The case of transversal structures corresponds to having $w(i, j) = \binom{i+j-2}{i-1}$ for $i, j \geq 1$, 0 otherwise. The corresponding weighted quadrant walks can be turned into unweighted quadrant walks with small steps (see the extended version for details), ensuring that the number t_n of transversal structures on $n + 2$ vertices is equal to the coefficient $d_{3n-2}(1, 0)$, where $d_n(i, j)$ and $u_n(i, j)$ are coefficients specified by the recurrence

$$\begin{cases} d_n(i, j) = d_{n-1}(i-1, j+1) + u_{n-1}(i-1, j+1), \\ u_n(i, j) = d_{n-2}(i+1, j-1) + u_{n-2}(i+1, j-1) + u_{n-1}(i+1, j) + u_{n-1}(i, j-1), \end{cases}$$

with boundary conditions $d_n(i, j) = u_n(i, j) = 0$ for any (n, i, j) with $n \leq 0$ or $i < 0$ or $j < 0$, with the exception (initial condition) of $d_0(0, 1) = 1$. The recurrence allows us again to compute the sequence t_3, \dots, t_n with $O(n^4)$ bit operations using $O(n^3)$ bit space, giving an alternative to the recurrence in [12] (again the series of t_n is non D-finite). The sequence starts as 1, 2, 6, 24, 116, 642, 3938, ...

4 Asymptotic Counting Results

We adopt here the method by Bostan, Raschel and Salvy [3] (itself relying on results by Denisov and Wachtel [5]) to obtain asymptotic estimates for the counting coefficients of plane bipolar posets (by vertices and by edges) and transversal structures (by vertices). Let $\mathcal{S} = SE \cup \{(-i, j), i, j \geq 0\}$ be the tandem step-set. Let $w : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ satisfying the symmetry property $w(i, j) = w(j, i)$. The induced weight-assignment on \mathcal{S} is $w(s) = 1$ for $s = SE$ and $w(s) = w(i, j)$ for $s = (-i, j)$. Let $a_n^{(w)}$ be the weighted number (i.e., each walk σ is counted with weight $\prod_{s \in \sigma} w(s)$) of quadrant tandem walks of length n , for some fixed starting and ending points. Let $S(z; x, y) := \frac{x}{y} z^{-2} + \sum_{i, j \geq 0} w(i, j) \frac{y^j}{x^i} z^{i+j}$, let $S(z) := S(z; 1, 1)$, and let ρ be the radius of convergence (assumed here to be strictly positive) of $S(z) - z^{-2}$. Let $\tilde{w}(s) := \frac{1}{\gamma} w(s) z_0^{y(s)-x(s)}$ be the modified weight-distribution where $\gamma, z_0 > 0$ are adjusted so that $\tilde{w}(s)$ is a probability distribution (i.e. $\sum_{s \in \mathcal{S}} \tilde{w}(s) = 1$) and the drift is zero, which is here equivalent to having $z = z_0 \in (0, \rho)$ solution of $S'(z) = 0$ (one solves first for z_0 and then takes $\gamma = S(z_0)$). Then according to [3] we have, for some $c > 0$,

$$a_n^{(w)} \sim c \gamma^n n^{-\alpha}, \text{ where } \alpha = 1 + \pi/\arccos(\xi), \text{ with } \xi = -\frac{\partial_x \partial_y S(z_0; 1, 1)}{\partial_x \partial_x S(z_0; 1, 1)}.$$

Plane bipolar posets counted by vertices correspond to $w(i, j) = \binom{i+j}{i}$, giving $S(z; x, y) = \frac{x}{y}z^{-2} + \frac{1}{1-z/x-zy}$, $z_0 = \frac{3-\sqrt{5}}{2} \approx 0.38$, $\gamma = \frac{1}{2}(11 + 5\sqrt{5}) \approx 11.09$, $\xi = \frac{1}{4}(1 + \sqrt{5}) \approx 0.81$, and $\alpha = 6$. We recover, as expected in view of the previous section, the asymptotic constants γ and α for plane permutations, which were obtained in [4] (where c was also explicitly computed).

Plane bipolar posets counted by edges correspond to taking $w(i, j) = \mathbf{1}_{i \neq 0, j \neq 0}$, which gives $S(z; x, y) = \frac{x}{y}z^{-2} + \frac{z/x}{1-z/x} \frac{zy}{1-zy}$. We find that $z_0 \approx 0.54$ is the unique positive root of $z^4 + z^3 - 3z^2 + 3z - 1$, $\gamma = 5z_0^3 + 7z_0^2 - 13z_0 + 9 \approx 4.80$, $\xi = 1 - z_0/2 \approx 0.73$, and $\alpha \approx 5.14$. With the method in [3] one can also check that α is irrational (this amounts to checking that the minimal polynomial $P(X)$ of ξ is such that no prime factor of $P(\frac{1}{2}(X + 1/X))$ is cyclotomic) so the generating function of plane bipolar posets by edges is not D-finite.

Finally for **transversal structures** we take $w(i, j) = \binom{i+j-2}{i-1}$ but to count by vertices we aggregate the steps into groups formed by a SE step followed by a (possibly empty) sequence of non-SE steps. The series for one (aggregated) step is $S(z; x, y) = \frac{xy^{-1}z^{-2}}{1-yx^{-1}z^2/(1-zx^{-1}-zy)}$, which gives $z_0 = 1/3$, $\gamma = 27/2$, $\xi = 7/8$, and $\alpha \approx 7.21$. Again the method of [3] ensures that the associated series is not D-finite. Another consequence of our estimate is that the coding procedure in [12] can be made asymptotically optimal, as it yields the bound $\gamma \leq 27/2$.

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The Characterization of Graphs Whose Sandpile Group has Fixed Number of Generators

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Abstract. Let \mathcal{K}_k be the family of connected graphs G whose sandpile groups have minimal number of generators equal to $|G| - k - 1$, where $|G|$ is the number of vertices of G . We survey previous result on the characterization of \mathcal{K}_1 and \mathcal{K}_2 , including complete characterizations for both cases. Furthermore, we shed some light on the characterization of \mathcal{K}_3 . Particularly, we give a minimal list of graphs that are forbidden for the regular graphs in \mathcal{K}_3 and we use them to give a characterization of the regular graphs in \mathcal{K}_3 .

Keywords: Characteristic ideal · Sandpile group · Forbidden induced subgraph

1 Introduction

The dynamics of the *Abelian sandpile model* was firstly studied by Bak, Tang and Wiesenfeld in [4]. The sandpile group has been studied under different names, for example: *chip-firing game*, *critical group*, *group of components*, *Jacobian group*, *Laplacian unimodular equivalence*, or *Picard group*. We recommend the reader the book [6] which is an excellent reference on the theory of chip-firing game and its relations with other combinatorial objects like rotor-routing, hyperplane arrangements, parking functions and dominoes.

Recall the Laplacian matrix $L(G)$ of a graph G is given such that the (u, v) -entry of $L(G)$ is defined as

$$L(G)_{u,v} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m(u, v) & \text{otherwise,} \end{cases}$$

where $m(uv)$ is the number of edges between u and v . We will introduce the Smith normal form (SNF) of a matrix, since in our context the SNF is relevant to compute the structure of the sandpile group, which is isomorphic to the torsion part of the cokernel of the Laplacian matrix of G , see [6, Chapter 4]. In the following, let $K(G)$ denote the sandpile group of G .

Two matrices M and N are said to be *equivalent* if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$, and denoted by $N \sim M$. Given a square integer matrix M , the SNF of M is the unique equivalent diagonal matrix $\text{diag}(d_1, d_2, \dots, d_n)$ whose non-zero entries are non-negative and satisfy d_i divides d_{i+1} whenever $d_i > 0$. The diagonal elements of the SNF are known as *invariant factors* or *elementary divisors*. This is because if $N \sim M$, then $\text{coker}(M) = \mathbb{Z}^n / \text{Im}M \cong \mathbb{Z}^n / \text{Im}N = \text{coker}(N)$. In particular, the fundamental theorem of finitely generated Abelian groups states $\text{coker}(M) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \dots \oplus \mathbb{Z}_{d_r} \oplus \mathbb{Z}^{n-r}$, where r is the rank of M . Therefore, the structure of $K(G)$ is obtained by the invariant factors of $L(G)$. The minimal number of generators of the torsion part of the cokernel of M equals the number of invariant factors of $\text{SNF}(M)$ greater than 1. One of the interesting features of the sandpile group of connected graphs is that the order $|K(G)|$ is equal to the number $\tau(G)$ of spanning trees of the graph G . Let $f_1(G)$ and $\phi(G)$ denote the number of invariant factors of $L(G)$ equal to 1 and the minimal number of generators of $K(G)$, respectively. If G is a graph with n vertices and c connected components, then $n - c = f_1(G) + \phi(G)$.

The characterization of the n -vertex graphs having sandpile group with $n - 3$ and $n - 4$ minimal number of generators has been of great interest. Note this is the same that the characterization of the family \mathcal{K}_k of simple connected graphs such that the SNFs of their Laplacian matrices have k invariant factors equal to 1, where k is either 2 or 3, respectively. Probably, it was initially posed by R. Cori¹. However, the first result appeared in 1991 when D. Lorenzini [8] and A. Vince [11] noticed, independently, that the graphs in \mathcal{K}_1 consist only of complete graphs. After, C. Merino in [9] posed interest on the characterization of \mathcal{K}_2 and \mathcal{K}_3 . In this extended abstract, we survey previous results on the topic, and introduce our recent contribution [1] on this problem: the characterization of the regular graphs in \mathcal{K}_3 .

2 Previous Sandpile Groups Characterizations

In the following, we will consider only connected graphs. Given two graphs G and H , the *disjoint union* will be denoted by $G + H$ and the *join* by $G \vee H$, that is the graph obtained from $G + H$ when each vertex in G is adjacent with each vertex in H .

Trees and complete graphs are extremes on the whole spectrum of sandpile groups possibilities. It is an standard exercise to verify that the complete graph K_n with n vertices has $K(K_n) \cong \bigoplus_{i=1}^{n-2} \mathbb{Z}_n$ and $\phi(K_n) = n - 2$. On the other hand, the sandpile group of any tree T consists of only one element. This inspired

¹ Personal communication with C. Merino.

two questions. Which graphs have cyclic sandpile group? And which n -vertex graphs have sandpile group with $n - 3$ and $n - 4$ minimal number of generators?

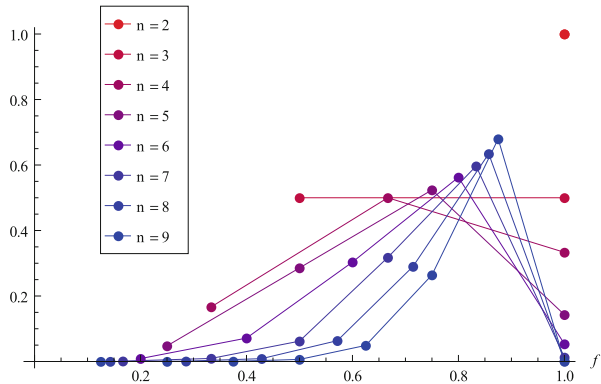


Fig. 1. Scaled number of connected graphs with n vertices and f_1 invariant factors equal to 1

In [7] and [12] D. Lorenzini and D. Wagner, based on numerical data, suggested we could expect to find a substantial proportion of graphs having a cyclic sandpile group. The reader might appreciate in Fig. 1 how the number of n -vertex connected graphs with $n - 2$ invariant factors equal to 1 grows as n increase. Based on this, D. Wagner conjectured [12] that almost every connected simple graph has a cyclic sandpile group. A recent study [13] concluded that the probability that the sandpile group of a random graph is cyclic is asymptotically at most 0.7935212, differing from Wagner’s conjecture.

In the following we will focus on the second question. This can be rephrased as which graphs are in \mathcal{K}_2 and \mathcal{K}_3 ? Our first reaction might be to ask for the graphs belonging to \mathcal{K}_1 . This was answered independently by D. Lorenzini and A. Vince.

Proposition 1 [8,11]. *The family \mathcal{K}_1 consists of complete graphs.*

Now, let us focus on results on the family \mathcal{K}_2 . In the following, we will denote by $d_i(G)$, the i -th invariant factor of the SNF of the Laplacian matrix of graph G .

Let G and H be two graphs. The *one point union* $G \bullet H$ of G and H is the graph obtained from the union of G and H and identifying a vertex of G with a vertex of H . In [5], it was noticed that the graphs in \mathcal{K}_2 with a cut vertex must be isomorphic to $K_n \bullet K_m$ such that $\gcd(n, m) > 1$. Then we should focus on the 2-connected simple graphs.

We already noticed that complete graphs with $n \geq 3$ vertices have $d_3 = n$. In [10], it was characterized the graphs in \mathcal{K}_2 whose third invariant factor is equal to $n, n - 1, n - 2$, or $n - 3$.

Proposition 2 [10]. *Let G be a simple connected graph with $n \geq 5$ vertices such that $G \neq K_n$. Let d_3 be the third invariant factor of $\text{SNF}(L(G))$. Then*

- (a) $d_3 = n$ if and only if $G = K_n - e$, where e is an edge of K_n ,
- (b) $d_3 = n - 1$ if and only if G is K_{n-1} with an additional vertex adjacent with a vertex in K_{n-1} ,
- (c) $d_3 = n - 2$ if and only if G is either $K_5 - 2e$ or $K_5 - C_4$,
- (d) $d_3 = n - 3$ if and only if G is one of the following graphs: $K_{2,3}$, $K_5 - C_3$, $K_6 - C_3$, $K_7 - 2C_3$, $K_{3,3}$ or $K_7 - K_{3,3}$.

One disadvantage of the family \mathcal{K}_k , that make difficult its characterization, is that it is not closed under induced subgraphs. For instance, consider the cone $c(S_3)$ of the star with 3 leaves, we have $K(c(S_3)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$ and $c(S_3) \in \mathcal{K}_2$, but S_3 belongs to \mathcal{K}_3 . Similarly, $K_6 \setminus \{2P_2\}$ belongs to \mathcal{K}_3 meanwhile $K_5 \setminus \{2P_2\}$ belongs to \mathcal{K}_2 . Moreover, if H is an induced subgraph of G it is not always true that $K(H) \trianglelefteq K(G)$. However, in [5] it was proved that the graphs shown in Fig. 2 are induced forbidden subgraphs for graphs in \mathcal{K}_2 . Then, Hou, Shiu and Chan noticed that graphs in \mathcal{K}_2 must have diameter at most 2, and characterized the complete multipartite graphs in \mathcal{K}_2 .

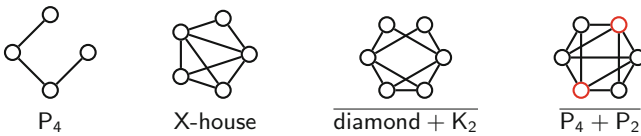


Fig. 2. Some forbidden graphs for \mathcal{K}_2

Proposition 3. *Let G be a complete multipartite graph on $n > 3$ vertices. Then, $G \in \mathcal{K}_2$ if and only if G is one of the following graphs:*

- (a) $K_{2,n-2}$,
- (b) K_{n_1,n_2} with $2 < n_1 \leq n_2$ and $\text{gcd}(n_1, n_2) > 1$,
- (c) K_{n_1,n_2,n_3} with $2 \leq n_1 \leq n_2 \leq n_3$ and $n \equiv n_i \pmod 2$ for $i = 1, 2, 3$,
- (d) K_{1,n_1,n_2} with $\text{gcd}(n_1 + 1, n_2 + 1) > 1$,
- (e) $K_{n-2} \vee \overline{K_2}$,
- (f) $K_2 \vee \overline{K_{n-2}}$,
- (g) $K_{n-l} \vee \overline{K_l}$ with $l \leq 3$, $n \leq 6$ and $\text{gcd}(n, l) > 1$.

Note the list of forbidden graphs in Fig. 2 can be refined, for instance by removing red nodes from $P_4 + P_2$, we obtain P_4 . In [2] there were found that the minimal forbidden induced graphs for $\mathcal{K}_{\leq 2}$ (the family of simple connected graphs such that the SNFs of their Laplacian matrices have at most 2 invariant factors equal to 1) are P_4 , X-house, $K_6 \setminus M_2$, cricket and dart, see Fig. 3.

By using the minimal forbidden graphs for $\mathcal{K}_{\leq 2}$, a complete characterization of \mathcal{K}_2 was obtained in [2].

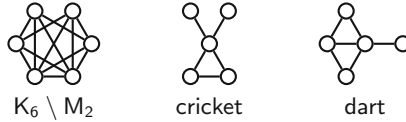


Fig. 3. More forbidden graphs for \mathcal{K}_2

Proposition 4. *Let G be a simple graph. Then $G \in \mathcal{K}_2$ if and only if G is one of the following:*

- (a) K_{n_1, n_1, n_3} with $n_1 \geq n_2 \geq n_3$ satisfying the following conditions:
 - (i) $n_1 = 2$ and $n_2 = 1$,
 - (ii) $n_1 \geq 2 = n_2$ and $n_3 = 0$,
 - (iii) $n_1, n_2 \geq 2, n_3 = 0$ and $\gcd(n_1, n_2) \neq 1$,
 - (iv) $n_1 \geq 3$ and $n_2 = n_3 = 1$,
 - (v) $n_1, n_2 \geq 3, n_3 = 1$ and $\gcd(n_1 + 1, n_2 + 1) \neq 1$, or
 - (vi) $n_1, n_2, n_3 \geq 2$ with the same parity.
- (b) $(nK_1) \vee (K_{m_1} + K_{m_2})$ with $m_1 \geq m_2$ and n satisfying the following conditions:
 - (i) $n \geq 2 = m_1$ and $m_2 = 0$,
 - (ii) $n = 2, m_1 \geq 3$ and $m_2 = 0$,
 - (iii) $n, m_1 \geq 3, m_2 = 0$ and $\gcd(n, m_1) \neq 1$,
 - (iv) $n \geq 1$ and $m_1 = m_2 = 1$,
 - (v) $m_1 \geq 1$ and $n = m_2 = 1$,
 - (vi) $n, m_1 \geq 2, m_2 = 1$ and $\gcd(n - 1, m_1 + 1) \neq 1$,
 - (vii) $n = 1, m_1, m_2 \geq 2$ and $\gcd(m_1 + 1, m_2 + 1) \neq 1$, or
 - (viii) $n, m_1, m_2 \geq 2$ with the same parity.

In [3], there were found 49 forbidden graphs for \mathcal{K}_3 . However, a complete characterization of \mathcal{K}_3 seems to be a hard problem.

3 Regular Graphs in \mathcal{K}_3

Recently, in [1] we approach the characterization of the regular graphs in \mathcal{K}_3 . For this, we first obtained a set of minimal forbidden induced graphs for the regular graphs in $\mathcal{K}_{\leq 3}$ (the family of simple connected graphs such that the SNFs of their Laplacian matrices have at most 3 invariant factors equal to 1), see Fig. 4.

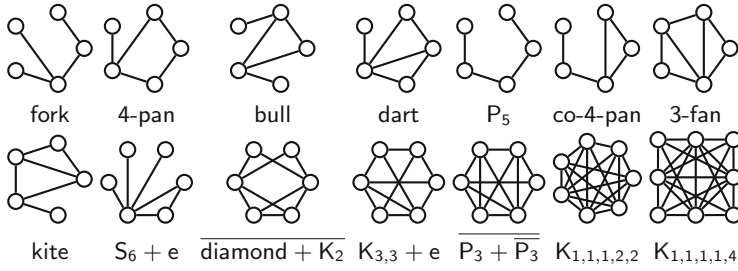


Fig. 4. The family of graphs \mathcal{F} .

Then, we obtained a characterization of the regular graphs in $\mathcal{K}_{\leq 3}$. The proof can be found in the full version [1] of this extended abstract.

Theorem 1 [1]. *Let G be a connected simple regular graph. Then $G \in \mathcal{K}_{\leq 3}$ if and only if G is one of the following:*

- (a) C_5 ,
- (b) $K_3 \square K_2$,
- (c) a complete graph K_r ,
- (d) a regular complete bipartite graph $K_{r,r}$,
- (e) a regular complete tripartite graph $K_{r,r,r}$,
- (f) a regular complete 4-partite graph $K_{r,r,r,r}$,
- (g) $C_4^{(-r,-r,-r,-r)}$, for any $r \in \mathbb{N}$.

where $C_4^{(-r,-r,-r,-r)}$ is the graph obtained by replacing the vertices in C_4 for cliques of size r and preserving adjacency.

Acknowledgements. We are grateful to the Graduate Research Workshop in Combinatorics (GRWC) 2018 for allowing the initial framework for this collaboration.

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Kissing Number in Non-Euclidean Spaces of Constant Sectional Curvature

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Abstract. This paper provides upper and lower bounds on the kissing number of congruent radius $r > 0$ spheres in hyperbolic \mathbb{H}^n and spherical \mathbb{S}^n spaces, for $n \geq 2$. For that purpose, the kissing number is replaced by the kissing function $\kappa_H(n, r)$, resp. $\kappa_S(n, r)$, which depends on the dimension n and the radius r .

After we obtain some theoretical upper and lower bounds for $\kappa_H(n, r)$, we study their asymptotic behaviour and show, in particular, that $\kappa_H(n, r) \sim (n-1) \cdot d_{n-1} \cdot B(\frac{n-1}{2}, \frac{1}{2}) \cdot e^{(n-1)r}$, where d_n is the sphere packing density in \mathbb{R}^n , and B is the beta-function. Then we produce numeric upper bounds by solving a suitable semidefinite program, as well as lower bounds coming from concrete spherical codes. A similar approach allows us to locate the values of $\kappa_S(n, r)$, for $n = 3, 4$, over subintervals in $[0, \pi]$ with relatively high accuracy.

Keywords: Sphere packing · Kissing number · Semidefinite programming

1 Preliminaries

The kissing number $\kappa(n)$ is the maximal number of unit spheres that can simultaneously touch a central unit sphere in n -dimensional Euclidean space \mathbb{R}^n without pairwise overlapping. The research on the kissing number leads back to 1694, when Isaac Newton and David Gregory had a discussion whether $\kappa(3)$ is equal to 12 or 13 [6].

The exact value of $\kappa(n)$ is only known for $n = 1, 2, 3, 4, 8, 24$, whereas for $n = 1, 2$ the problem is trivial. In 1953, Schütte and van der Waerden proved that $\kappa(3) = 12$ [24]. Furthermore, Delsarte, Goethals, and Seidel [9] developed a linear programming (LP) bound, which was used by Odlyzko and Sloane [22] and independently by Levenshtein [18] to prove $\kappa(8) = 240$ and $\kappa(24) = 196560$. Later, Musin [21] showed that $\kappa(4) = 24$ by using a stronger version of the LP bound.

The study of the Euclidean kissing number and the contact graphs of kissing configurations in \mathbb{R}^n is still an active area in geometry and optimisation, with a few recent developments [2, 3, 10, 16].

In this work we consider an analogous problem in n -dimensional hyperbolic space \mathbb{H}^n , as well as in n -dimensional spherical space \mathbb{S}^n . In these cases we are able to reduce our consideration to the Euclidean picture, and it turns out that the kissing number for all of these spaces equals the maximal cardinality of a spherical code with certain minimal angular distance. Then the classical spherical cap method [13, § X.50] can be applied, as well as the recent approaches using semidefinite programs (SDP) as exemplified in the works of Bachoc and Vallentin [1], Mittelman and Vallentin [20], and Machado and Oliveira [19].

Some other non-Euclidean geometries, notably the ones on Thurston’s list of the eight 3-dimensional model spaces, have been studied in [25, 26].

By a sphere of radius r in the hyperbolic space \mathbb{H}^n we mean the set of points at a given geodesic distance $r > 0$ from a specific point, which is called the centre of the sphere.

In a kissing configuration each sphere in \mathbb{H}^n has the same radius. Unlike in Euclidean spaces, the kissing number in \mathbb{H}^n depends on the radius r , and we denote it by $\kappa_H(n, r)$.

The kissing number in \mathbb{S}^n is denoted by $\kappa_S(n, r)$, and also depends on the radius r of the spheres in kissing configuration, although here r belongs to the bounded interval $(0, \pi]$. Since \mathbb{S}^n is a compact metric space, $\kappa_S(n, r)$ is a decreasing function of r , while $\kappa_H(n, r)$ increases with r exponentially fast.

The proofs of all theorems are contained in the expanded version of the paper available on the arXiv [11]. The accompanying computer code [12] accompanying the paper allows to reproduce our numerical results obtained via semidefinite programming.

2 Kissing Number in Hyperbolic Space

In this section we present some theoretical upper and lower bounds for the kissing function $\kappa_H(n, r)$, so that we can analyse its asymptotic behaviour in the dimension n , and in the radius r , for large values of the respective parameters. We refer the reader to [23, § 2.1] and [23, § 4.5] for the necessary facts about hyperbolic and spherical geometry.

2.1 Upper and Lower Bounds

The principal tool in the proof of the upper bound is passing to the Euclidean setting and then using the classical “spherical cap” method, cf. [13, § X.50].

Theorem 1. *For any integer $n \geq 2$ and a non-negative number $r \geq 0$, we have that*

$$\kappa_H(n, r) \leq \frac{2 B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\frac{\operatorname{sech}^2 r}{4}, \frac{n-1}{2}, \frac{1}{2}\right)},$$

where $B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt$, for all $x \in [0, 1]$ and $y, z > 0$, is the incomplete beta-function, and $B(y, z) = B(1; y, z)$.

By using a purely Euclidean picture of the arrangement of $k \leq \kappa_H(n, r)$ spheres of radius r in \mathbb{H}^n seen in the Poincaré ball model, we obtain that the kissing number in \mathbb{H}^n coincides with the maximal number of spheres of radius $\frac{1}{2} (\tanh \frac{3r}{2} - \tanh \frac{r}{2})$ that can simultaneously touch a central sphere of radius $\tanh \frac{r}{2}$ in \mathbb{R}^n without pairwise intersecting. This is the main observation that allows us to obtain the upper bound on $\kappa_H(n, r)$. An analogous lower bound for $\kappa_H(n, r)$ also holds.

Theorem 2. *For any integer $n \geq 2$ and a non-negative number $r \geq 0$, we have that*

$$\kappa_H(n, r) \geq \frac{2 B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\operatorname{sech}^2 r - \frac{\operatorname{sech}^4 r}{4}; \frac{n-1}{2}, \frac{1}{2}\right)},$$

where $B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt$, for all $x \in [0, 1]$ and $y, z > 0$, is the incomplete beta-function, and $B(y, z) = B(1; y, z)$.

2.2 Asymptotic Behaviour

One of the main corollaries of the above bounds is that they are asymptotically sharp and produce the following corollary.

Theorem 3. *As $r \rightarrow \infty$, the following asymptotic formula holds:*

$$\kappa_H(n, r) \sim (n-1) d_{n-1} B\left(\frac{n-1}{2}, \frac{1}{2}\right) e^{(n-1)r},$$

where d_n be the best packing density of \mathbb{R}^n by unit spheres.

Note, that the exponential asymptotic behaviour of the kissing number $\kappa_H(2, r)$, as $r \rightarrow \infty$, follows readily from the work by Bowen [5].

3 Kissing Number in Spherical Space

As the standard sphere \mathbb{S}^n can be considered as a submanifold of \mathbb{R}^{n+1} with the induced metric of constant sectional curvature $+1$, one can also use the corresponding Euclidean picture in order to obtain the following upper and lower bounds for the spherical kissing number $\kappa_S(n, r)$.

Theorem 4. *For any integer $n \geq 2$ and a non-negative number $r \leq \frac{\pi}{3}$, we have that*

$$\kappa_S(n, r) \leq \frac{2 B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\frac{\sec^2 r}{4}; \frac{n-1}{2}, \frac{1}{2}\right)},$$

where $B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt$, for all $x \in [0, 1]$, and $y, z > 0$, is the incomplete beta-function, and $B(y, z) = B(1; y, z)$.

Theorem 5. For any integer $n \geq 2$ and a non-negative number r , we have that

$$\kappa_S(n, r) \geq \begin{cases} \frac{2B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{B\left(\sec^2 r - \frac{\sec^4 r}{4}; \frac{n-1}{2}, \frac{1}{2}\right)}, & \text{if } 0 \leq r \leq \frac{\pi}{4}, \\ \frac{2B\left(\frac{n-1}{2}, \frac{1}{2}\right)}{2B\left(\frac{n-1}{2}, \frac{1}{2}\right) - B\left(\sec^2 r - \frac{\sec^4 r}{4}; \frac{n-1}{2}, \frac{1}{2}\right)}, & \text{if } \frac{\pi}{4} \leq r \leq \frac{\pi}{3}. \end{cases}$$

where $B(x; y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt$, for all $x \in [0, 1]$, and $y, z > 0$, is the incomplete beta-function, and $B(y, z) = B(1; y, z)$.

4 Semidefinite Programming Bounds

In order to obtain much more precise upper bounds for $\kappa_H(n, r)$, we adapt the SDP by Bachoc and Vallentin [1]. On the other hand, concrete kissing configurations provide lower bounds. A great deal of the latter is taken from the spherical codes produced and collected by Hardin, Smith, and Sloane [17], which often turn out to be optimal.

For $n \geq 3$, let $P_k^n(u)$ denote the Jacobi polynomial of degree k and parameters $((n-3)/2, (n-3)/2)$, normalized by $P_k^n(1) = 1$. If $n = 2$, then $P_k^n(u)$ denotes the Chebyshev polynomial of the first kind of degree k . For a fixed integer $d > 0$, we define Y_k^n to be a $(d-k+1) \times (d-k+1)$ matrix whose entries are polynomials on the variables u, v, t defined by $(Y_k^n)_{i,j}(u, v, t) = P_i^{n+2k}(u)P_j^{n+2k}(v)Q_i^{n-1}(u, v, t)$, for $0 \leq i, j \leq d-k$, where

$$Q_k^{n-1}(u, v, t) = ((1-u^2)(1-v^2))^{k/2} P_k^{n-1}\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

The symmetric group on three elements \mathcal{S}_3 acts on a triple (u, v, t) by permuting its components. This induces the action $\sigma p(u, v, t) = p(\sigma^{-1}(u, v, t))$ on $\mathbb{R}[u, v, t]$, where $\sigma \in \mathcal{S}_3$. By taking the group average of Y_k^n , we obtain the matrix $S_k^n(u, v, t) = \frac{1}{6} \sum_{\sigma \in \mathcal{S}_3} \sigma Y_k^n(u, v, t)$, whose entries are invariant under the action of \mathcal{S}_3 .

Let $A(n, \theta)$ be the maximal number of points on the unit sphere with minimal angular distance θ . In [1], Bachoc and Vallentin proved the following theorem, where J denotes the ‘‘all 1’s’’ matrix.

Theorem 6. Any feasible solution of the following optimisation program gives an upper bound on $A(n, \theta)$:

$$\begin{aligned} \min \quad & 1 + \sum_{k=1}^d a_k + b_{11} + \langle J, F_0 \rangle, \\ & a_k \geq 0 \text{ for } k = 1, \dots, d, \\ & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0, \\ & F_k \in \mathbb{R}^{(d-k+1) \times (d-k+1)} \text{ and } F_k \succeq 0 \text{ for } k = 0, \dots, d, \end{aligned}$$

$$\sum_{k=1}^d a_k P_k^n(u) + 2b_{12} + b_{22} + 3 \sum_{k=0}^d \langle S_k^n(u, u, 1), F_k \rangle \leq -1 \text{ for } (u, u, 1) \in \Delta_0,$$

$$b_{22} + \sum_{k=0}^d \langle S_k^n(u, v, t) F_k \rangle \leq 0 \text{ for } (u, v, t) \in \Delta,$$

where $\Delta_0 \subset \mathbb{R}^3$ and $\Delta \subset \mathbb{R}^3$ are certain domains determined by the geometry of the problem.

We show that the following modifications of the SDP in Theorem 6 give the upper bounds on the non-Euclidean kissing numbers $\kappa_H(n, r)$ and $\kappa_S(n, r)$.

Theorem 7. *Any feasible solution of the optimisation program in Theorem 6 with $\Delta = \left\{ (u, v, t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq 1 - \frac{1}{1+\cosh(2r)} \text{ and } 1 + 2uvt - u^2 - v^2 - t^2 \geq 0 \right\}$ and $\Delta_0 = \left\{ (u, u, 1) : -1 \leq u \leq 1 - \frac{1}{1+\cosh(2r)} \right\}$ provides an upper bound on $\kappa_H(n, r)$.*

For certain dimensions and radii, we compute the lower bounds by using the results of Sect. 2, and the upper bounds due to Levenshtein [18] and Coxeter [4, 7, 15]. Then, we compare them with the lower bounds given by concrete kissing configurations and with the SDP upper bounds, respectively.

Theorem 8. *Any feasible solution of the optimisation program in Theorem 6 with $\Delta = \left\{ (u, v, t) \in \mathbb{R}^3 : -1 \leq u \leq v \leq t \leq 1 - \frac{1}{1+\cos(2r)} \text{ and } 1 + 2uvt - u^2 - v^2 - t^2 \geq 0 \right\}$ and $\Delta_0 = \left\{ (u, u, 1) : -1 \leq u \leq 1 - \frac{1}{1+\cos(2r)} \right\}$ provides an upper bound on $\kappa_S(n, r)$.*

The “jumps” for the kissing number $k_S(3, r)$ can be computed by using the solutions of Tammes’ problem [8, 14, 24]. In order to compute the approximate shape of $\kappa_S(4, r)$, we need the upper bounds from SDPs, and lower bounds from concrete configurations [17]. For each radius r of these exact spherical codes, we compute the lower bound given by Theorem 5, as well as the upper bound by the SDP, together with the bounds due to Levenshtein [18] and Coxeter [4, 7, 15]. The tables with the data are voluminous and can be accessed in the expanded version of the paper available on the arXiv [11]. The computer code used to produce the results (written in SageMath and Julia) is available on GitHub [12].

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Decomposing Cubic Graphs with Cyclic Connectivity 5

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Abstract. Let G be a cyclically 5-connected cubic graph with a cycle separating 5-edge-cut separating G into two components G_1 and G_2 . We prove that each component G_i can be completed to a cyclically 5-connected cubic graph by adding three vertices, unless G_i is a cycle of length five. Our work extends similar results proved by Andersen et al. for cyclic connectivity 4 in 1988.

Keywords: Cubic graphs · Decomposition · Cyclic connectivity · Girth

1 Introducton

An *edge-cut* of a connected graph G , or a *cut* for short, is any set S of edges of G such that $G - S$ is disconnected. An edge-cut is *cycle-separating* if at least two components of $G - S$ contain a cycle.

We say that a connected graph G is *cyclically k -edge-connected* if it contains no cycle-separating edge-cut consisting of fewer than k edges. The *cyclic edge-connectivity* of G , denoted by $\zeta(G)$, is the largest number $k \leq \beta(G)$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the cycle rank of G , for which G is cyclically k -edge-connected (cf. [9, 10]). Note that the graphs K_4 and $K_{3,3}$ are cyclically k -edge-connected for every positive integer k because they contain no cycle separating cut, although $\zeta(K_4) = 3$ and $\zeta(K_{3,3}) = 4$.

The cyclic connectivity of every cubic graph G is bounded above by the *girth* of G , denoted by $g(G)$, which is the length of a shortest cycle in G [9, 10]. One can easily check that for a cubic graph G with $\zeta(G) \leq 3$, the value $\zeta(G)$ is equal to the usual vertex-connectivity and edge-connectivity of G . Furthermore, the cyclic edge-connectivity and the cyclic vertex-connectivity, which is defined in a similar manner, of every cubic graph coincide. Therefore, we shall use terms *cyclically k -connected* and *cyclic connectivity* instead of cyclically k -edge-connected and cyclic edge-connectivity. Note that for a cubic graph G a cycle-separating cut S of minimum size consists of independent edges and that $G - S$ has exactly two components called *cyclic parts* or *fragments*.

Cubic graphs offer a convenient approach to several widely-open conjectures such as the Tutte's 5-flow conjecture, the cycle double cover conjecture, or the

Berge-Fulkerson conjecture. It is known that minimal counterexamples to these conjectures are cubic and also not 3-edge-colourable. Such graphs are commonly called *snarks*. Moreover, it has been proven that the minimal counterexample to the 5-flow conjecture is cyclically 6-connected [6]; it is cyclically 4-connected for the cycle double cover conjecture [12], and cyclically 5-connected for the Berge-Fulkerson conjecture [7]. On the other hand, Jaeger and Swart conjectured that there are no cyclically 7-connected snarks [2]. Cyclic connectivity almost inevitably emerges in the study of many other problems concerning cubic graphs. For instance, it is fundamental in an approach to a conjecture that every Cayley graph (of order more than two) has a Hamilton cycle [3], and to Tutte's 3-edge-colouring conjecture [11]. Hence, cyclic connectivity plays a crucial role in the study of aforementioned conjectures.

Small edge-cuts in cubic graphs enable us to use inductive arguments. If G is a graph from some class \mathcal{C} with a small cycle-separating cut, it is useful, if possible, to decompose G along the cut into two smaller graphs contained in \mathcal{C} . Andersen et al. [1] established such results for the class of cyclically 4-connected cubic graphs. They showed that each cyclic part of a cyclically 4-connected cubic graph can be extended to a cyclically 4-connected cubic graph by adding a pair of adjacent vertices and restoring 3-regularity. Moreover, they characterised graphs where it is sufficient to add only two additional edges. Using this result they proved a lower bound on the number of removable edges in a cyclically 4-connected cubic graph [1]. Later, Goedgebeur et al. constructed and classified all snarks with cyclic connectivity 4 and oddness 4 up to order 44 [4, 5].

In this paper, we examine how a cyclic part H of a cubic graph with cyclic connectivity 5 can be completed to a cyclically 5-connected cubic graph. For more detailed proofs, we refer the reader to [8].

2 Preliminaries

All graphs considered here are simple and cubic; subcubic graphs occur as subgraphs of cubic graphs. A subgraph of a graph G induced by a set of vertices X is denoted by $G[X]$. The set of edges of the graph G that have one end in X and the other in $V(G) - X$ is denoted by $\delta_G(G[X])$, or $\delta_G(X)$. In this notation, we omit the graph G whenever it is clear from the context. Also, we will write only $\deg_X(v)$ instead of $\deg_{G[X]}(v)$ to denote the degree of vertex v in the induced subgraph $G[X]$.

The following proposition [9, Proposition 4] of Nedela and Škoviera implies that each cyclic part of a cyclically 5-connected graph is 2-connected.

Proposition 1. *Let G be a connected cubic graph. Then each cyclic part of G is connected. Moreover, if $\zeta(G) > 3$, then each cyclic part is 2-connected.*

If H is a non-empty induced subgraph of a cyclically 5-connected cubic graph G , then it is either cyclic, and thus $|\delta_G(H)| \geq 5$, or H is acyclic. In the latter case the relation between the number $|\delta_G(H)|$ and the number of vertices of H is determined by following well-known lemma. Since H is non-empty, we get bound on $|\delta_G(H)|$.

Lemma 1. *Let M be a connected acyclic induced subgraph of a cubic graph G . Then $|\delta_G(M)| = |V(M)| + 2$.*

Corollary 1. *If M is a non-empty induced subgraph of a cyclically 5-connected cubic graph G , then $|\delta_G(M)| \geq 3$.*

In general, a cyclic induced subgraph H of a cyclically 5-connected cubic graph with $|\delta_G(H)| = 6$ needs not to be 2-connected, since H may contain a bridge. However, then H contains only one bridge which is additionally in a special position.

Lemma 2. *Let H be a connected induced subgraph of a cyclically 5-connected cubic graph G such that $|\delta_G(H)| = 6$. Then exactly one of the following conditions holds:*

- (i) H is acyclic;
- (ii) H contains exactly one bridge whose one end x is incident with two edges from $\delta_G(H)$ and $H - x$ is a 2-connected cyclic part of G ;
- (iii) all the edges from $\delta_G(H)$ are independent and H is 2-connected.

Finally, we formalise the process of completing a cyclic part to a cubic graph by adding three new vertices lying on a path of length two.

Definition 1. *Let H be a cyclic part of a cubic graph G with $\zeta(G) = 5$ and let $a_1, a_2, a_3, a_4,$ and a_5 be the vertices of H of degree 2. We add to H three vertices x, y and z and edges $xy, yz, xa_1, xa_2, ya_3, za_4,$ and za_5 . We denote the graph obtained in this way by $H(a_1, a_2, a_3, a_4, a_5)$. Throughout this paper, the three newly added vertices will be consistently denoted by x, y and z .*

3 Extensions Without Small Cycles

In this section we show that each cyclic part $H \not\cong C_5$ of a cubic graph with $\zeta(G) = 5$ can be extended to a cubic graph $\bar{H} = H(a_1, a_2, a_3, a_4, a_5)$ which has girth at least 5.

Lemma 3. *Let H be a cyclic part of a cubic graph with $\zeta(G) = 5$ which is not a 5-cycle and let A be the set of vertices of H of degree 2. Then each vertex from A has at most one neighbour in A .*

Proof. Suppose that a_1 is a vertex from A with two neighbours a_2 and a_3 in A . Then the induced subgraph $G[V(H) - \{a_1, a_2, a_3\}]$ has only four outgoing edges, hence it is acyclic and contains only two vertices due to Lemma 1. Therefore, H is a 5-cycle; a contradiction.

Lemma 4. *Let H be a cyclic part of a cubic graph G with $\zeta(G) = 5$ that is not a 5-cycle. Then there exists a permutation $a_1a_2a_3a_4a_5$ of vertices of H of degree 2 such that $H(a_1, a_2, a_3, a_4, a_5)$ has girth at least five.*

Proof. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of the five degree 2 vertices of H . Furthermore, let D be the graph with vertex set A where $a_i a_j \in E(D)$ if $\text{dist}_H(a_i, a_j) = 2$ for each $a_i, a_j \in A$. Note that if $a_i a_j \in E(D)$, then there exist exactly one path $a_i v a_j$ in H . Observe that the graph $H(a_1, a_2, a_3, a_4, a_5)$ has girth at least five if and only if

$$\text{deg}_A(a_3) = 0 \quad \text{and} \quad a_1 a_2, a_4 a_5 \notin E(D) \cup E(H[A]).$$

By Lemma 3, there are at most two edges between the vertices from A . We divide the proof into three cases according to the number of edges in $H[A]$.

Case (i). Assume that the induced subgraph $H[A]$ contains two edges, say $a_1 a_5$ and $a_2 a_4$. We show that all the vertices a_1, a_2, a_4 , and a_5 have at most one neighbour in D among the vertices $\{a_1, a_2, a_4, a_5\}$. Hence, there exists a permutation $b_2 b_4$ of $\{a_2, a_4\}$ such that $a_1 b_2, a_5 b_4 \notin E(D)$ and thus $g(H(a_1, b_2, a_3, b_4, a_5)) \geq 5$.

Case (ii). Let $H[A]$ contain only one edge, which we denote by $a_1 a_5$ in such a way that $\text{deg}_D(a_5) \leq \text{deg}_D(a_1)$. We show that $\text{deg}_D(a_1) \leq 2$. It follows that one of the edges $a_5 a_4$ and $a_5 a_3$, say it is $a_5 a_4$, is not in $E(D)$ and then the graph $H(a_1, a_2, a_3, a_4, a_5)$ has girth at least five.

Case (iii). Finally, assume that $H[A]$ contains no edges. We show that one can choose four distinct vertices $b_1, b_2, b_4, b_5 \in V(D)$ such that $b_1 b_2, b_4 b_5 \notin E(D)$. It is a simple matter to verify that if this is not possible, then D contains K_4 as a subgraph, or it contains two vertices of degree 4. One can easily show that both alternatives lead to a contradiction.

The graph $H(a_1, b_2, a_3, b_4, a_5)$ from the previous lemma need not to be cyclically 5-connected. However, if it is not, due to its girth, we are able to specify the position of a small cut. Other positions of a small cut lead to a subgraph with at most four outgoing edges, necessarily acyclic, which produces together with the vertex x or z a small cycle.

Lemma 5. *Let H be a cyclic part of a cubic graph G with $\zeta(G) = 5$ and let a_1, a_2, a_3, a_4 , and a_5 be the vertices of degree 2 in H . Assume that the graph $\bar{H} = H(a_1, a_2, a_3, a_4, a_5)$ has girth 5 and that \bar{H} contains a minimum cycle-separating cut S of size smaller than 5. Then $|S| = 4$ and the cut S separates $\{a_1, a_2, x\}$ from $\{a_4, a_5, z\}$.*

4 Main Result

Theorem 1. *Let H be a cyclic part of a cubic graph G with $\zeta(G) = 5$. If H is not a cycle of length five, then H can be extended to a cyclically 5-connected cubic graph by adding three new vertices on a path of length two and by restoring 3-regularity.*

Proof (sketch). By Lemma 4, the graph $H_1 = H(a_1, a_2, a_3, a_4, a_5)$ has girth 5 for some permutation $a_1 a_2 a_3 a_4 a_5$ of the degree 2 vertices of H . If $\zeta(H_1) \geq 5$, we are done, so we assume that H_1 contains a cycle-separating cut S_1 whose removal leaves components C'_1 and C'_2 . According to Lemma 5 we know that $|S_1| = 4$, and without loss of generality $a_1, a_2 \in C'_1$ and $a_3, a_4, a_5 \in C'_2$. Put $C_1 = C'_1 - \{x, y, z\}$ and $C_2 = C'_2 - \{x, y, z\}$.

Since H_1 has girth at least 5, the subgraphs C_1 and C_2 have to be cyclic for otherwise their vertices together with x or z would produce a short cycle. According to Lemma 2, the component C_2 is 2-connected or contains a bridge incident with some a_i and $C_2 - a_i$ is 2-connected.

Suppose that C_2 is 2-connected. We choose $\{i, j, k\} = \{3, 4, 5\}$ in such a way that $a_i a_j, a_i a_k \notin E(H)$, which is possible due to Lemma 3. It follows that the graph $H_2 = H(a_1, a_j, a_i, a_2, a_k)$ has girth at least 5 (see Fig. 1a).

We show that the graph H_2 is cyclically 5-connected. Suppose that S_2 is a smallest cut of H_2 of size at most 4. From Lemma 5 we know that S_2 separates $\{x, a_1, a_j\}$ from $\{z, a_2, a_k\}$. Therefore S_2 contains one of the edges xy and yz . Moreover, since a_1 and a_2 lie in the component C_1 , which is 2-connected, S_2 contains at least two edges from C_1 . Similarly, S_2 contains at least two edges from the 2-connected component C_2 . Therefore $|S_2| \geq 5$, which is a contradiction.

If the subgraph C_2 contains a bridge connecting a_2 to the 2-connected subgraph $D_2 = C_2 - a_i$, then we put $H_b = H(a_1, a_j, a_i, a_k, a_2)$ (see Fig. 1b) and proceed analogously.

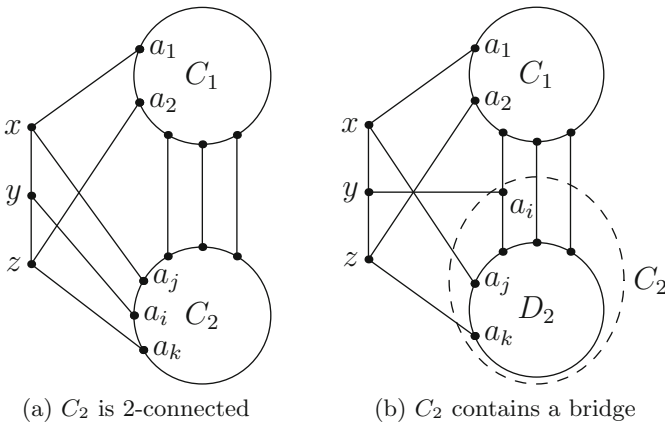


Fig. 1. Completions of the cyclic part H to a cyclically 5-connected cubic graph

5 Concluding Remarks

There is only one way how a cyclic part H of a cubic graph with cyclic connectivity 5 can be completed to a cubic graph by adding fewer than three vertices.

Namely, we can add one new vertex and connect it to three 2-valent vertices of H , and add one new edge between the remaining two 2-valent vertices. Assume that H contains three vertices a_1 , a_2 and a_3 of degree 2 such that all of them have some common neighbour v , or there is a 6-cycle $a_1v_1a_2v_2a_3v_3$ in H (cf. [1, Lemma 9]). Then, in every case, two of the vertices a_1 , a_2 and a_3 are connected by an edge or are connected to the newly added vertex which yields a 3-cycle or a 4-cycle, respectively. Thus, the cyclic part H cannot be completed to a cyclically 5-connected cubic graph by adding only one vertex. Clearly, there are infinitely many cyclic parts satisfying one of the two aforementioned conditions. This stands in contrast to the only exception (the 5-cycle) for completing H by adding a path of length two. It remains an open problem to characterise all such cyclic parts that cannot be completed by adding only one new vertex.

Acknowledgements. The authors acknowledge partial support from the grants VEGA 1/0813/18 and APVV-19-0308.

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Dirac-Type Conditions for Spanning Bounded-Degree Hypertrees

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Abstract. We prove that for fixed k , every k -uniform hypergraph on n vertices and of minimum codegree at least $n/2 + o(n)$ contains every spanning tight k -tree of bounded vertex degree as a subgraph. This generalises a well-known result of Komlós, Sárközy and Szemerédi for graphs. Our result is asymptotically sharp.

Keywords: Hypergraphs · Trees

1 Introduction

Forcing spanning substructures with minimum degree conditions is a central topic in extremal graph theory. For instance, a classic result of Dirac [5] from 1952 states that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. In the same spirit, Bollobás [2] conjectured in the 1970s that all graphs on n vertices with minimum degree at least $n/2 + o(n)$ would contain every n -vertex tree of bounded maximum degree as a subgraph. Komlós, Sárközy and Szemerédi [6] proved this conjecture in 1995, introducing a prototype version of what is now known as the Blow-Up Lemma.

In recent years, many efforts have been made to extend Dirac's theorem to k -uniform hypergraphs, also called k -graphs, using various notions of degrees or cycles. Notably, Rödl, Ruciński and Szemerédi [9] proved that k -graphs with *minimum codegree* at least $n/2 + o(n)$ contain a *tight Hamiltonian cycle*. For more results, we refer the reader to [10] and the references therein.

Our main result is an extension of the Komlós–Sárközy–Szemerédi theorem to k -graphs. There is more than one definition of ‘hypergraphs trees’ in the literature, and we will focus on the one most appropriate to our context, the *tight k -trees*, which we call here simply k -trees. Their definition is a simple generalisation of defining (non-trivial) trees in graphs: a k -tree is a k -graph whose edges can be ordered in such a way that every edge e , except the first one, contains a vertex v which is not in any previous edge, and furthermore, $e \setminus \{v\}$ is contained in some previous edge.

To state our main result, we need two more definitions. For a k -graph H , the *maximum 1-degree* $\Delta_1(H)$ is the maximum number m such that some vertex

of H is contained in m edges. The *minimum codegree* $\delta_{k-1}(H)$ is the largest m such that every set of $k - 1$ vertices is contained in at least m edges of H .

Theorem 1. *For all $k, \Delta \geq 2$ and $\gamma > 0$ there is n_0 such that every k -graph H on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ contains every k -tree T on n vertices with $\Delta_1(T) \leq \Delta$.*

Observe that, for $k = 2$, this yields the aforementioned result of Komlós, Sárközy and Szemerédi [6] about spanning bounded-degree trees in graphs.

The full proof of Theorem 1 can be found in [8]. It relies on the absorption method, combined with structural results about bounded-degree k -trees which allow us to decompose a large k -tree into smaller k -trees of controlled size. A sketch of our proof is given in Sects. 3 and 4.

We remark that the condition on $\delta_{k-1}(H)$ in Theorem 1 is best possible, up to the term γn and a term depending on k only. The constructions exhibiting these lower bounds will be presented in the next section.

Proposition 1. *For every $k \geq 2$ and for every k -tree T on $n \geq k$ vertices, there exists $f(T) \leq 2^k + k - 1$ and a k -graph H on n vertices not containing T , with $\delta_{k-1}(H) \geq \lfloor n/2 \rfloor - f(T)$. Moreover, there are k -trees T with $f(T) = k - 1$.*

2 Lower Bounds

To show Proposition 1, we need a basic fact about k -trees. We say that a k -graph H is *k -partite* if there is a partition $\{V_1, \dots, V_k\}$ of $V(H)$ such that $|e \cap V_i| = 1$ for each $e \in E(H)$ and $i \in [k]$. Using induction on the number of vertices, it is easy to show that every k -tree is k -partite with a unique possible k -partition.

We will use the following family of k -graphs.

Definition 1. *For disjoint sets A, B , and $0 \leq i \leq k$, let $H_i = \{e \subseteq A \cup B : |e| = k, |e \cap A| = i\}$, and $I = \{i \in \{0, \dots, k\} : i \not\equiv \lfloor k/2 \rfloor \pmod{2}\}$. Define $H(A, B) := \bigcup_{i \in I} H_i$.*

Assuming that $|A \cup B| \geq k$, note that $\delta_{k-1}(H(A, B)) \geq \min\{|A|, |B|\} - k + 1$. There are not many ways to embed a k -tree into $H(A, B)$. Essentially, the k -partition of a k -tree must respect the partition $\{A, B\}$ of $H(A, B)$ in any embedding, as shown in the following lemma.

Lemma 1. *Let $k, n \in \mathbb{N}$, let $H(A, B)$ be as in Definition 1, with $|A \cup B| = n \geq k$. Let T be a k -tree, with a unique k -partition $V_1 \cup \dots \cup V_k$, and an embedding $\phi : V(T) \rightarrow V(H(A, B))$. Then, for each $1 \leq i \leq k$ either $\phi(V_i) \subseteq A$ or $\phi(V_i) \subseteq B$.*

Now we are ready for the proof of Proposition 1.

Proof (of Proposition 1). Given T , let $\{V_1, \dots, V_k\}$ be the unique partition of $V(T)$ which makes it k -partite. Let $a(T)$ be the largest integer such that $a(T) \leq n/2$ and $a(T) \neq |\bigcup_{j \in J} V_j|$ for all $J \subseteq [k]$. The definition clearly implies $a(T) \geq \lfloor n/2 \rfloor - 2^k$. Set $f(T) = \lfloor n/2 \rfloor - a(T) + k - 1$.

Let A, B be disjoint sets such that $|A| = a(T)$ and $|A \cup B| = n$, and consider the k -graph $H(A, B)$ as in Definition 1. Then $\delta_{k-1}(H(A, B)) \geq a(T) - k + 1 = \lfloor n/2 \rfloor - f(T)$ (by the observation after Definition 1), and T does not embed in $H(A, B)$ because of Lemma 1.

To finish, note that $a(T) = \lfloor n/2 \rfloor$ for some trees. An example are star-like k -trees with $n \geq 2k$, consisting of a $(k - 1)$ -set which is contained in $n - k + 1$ many edges.

3 The Structure of Tight Hypertrees

We will make a quick overview of how k -trees can be decomposed into smaller hypertrees of controlled size. To do that, we will formalise the inductive definition of k -trees presented in the introduction, and introduce new definitions.

By definition, every k -tree T on n vertices has orderings e_1, \dots, e_{n-k+1} of its edges, and v_1, \dots, v_n of its vertices such that $e_1 = \{v_1, \dots, v_k\}$, and for all $i \in \{k + 1, \dots, n\}$,

- (a) $\{v_i\} = e_{i-k+1} \setminus \bigcup_{1 \leq j < i-k+1} e_j$, and
- (b) there exists $j \in [i - 1]$ such that $e_{i-k+1} \setminus \{v_i\} \subseteq e_j$.

Clearly, an ordering of the edges implies an ordering of the vertices (and vice versa). Any ordering of $E(T)$ or $V(T)$ with properties (a) and (b) will be called a *valid ordering*. Note that a k -tree on n vertices has exactly $n - k + 1 \geq 1$ edges. If $j \in [i - 1]$ is the smallest index such that (b) holds for e_i and v_i then we call e_j the *parent* of e_{i-k+1} and e_{i-k+1} a *child* of e_j .

A *k -subtree* of T is a k -tree T' such that $T' \subseteq T$. For instance, given $1 \leq r \leq n - k + 1$, the first r edges in a valid ordering of T induce a k -subtree. In particular, the tree $T - v_n$ obtained by removing v_n and e_{n-k+1} from T is a k -subtree of T .

Our objective is to partition a large k -tree into a constant number of smaller k -subtrees, in a similar way as has been done for trees in graphs [1] to tackle embedding problems. More precisely, we will show that for any $\beta > 0$, one can partition the edges of any k -tree with n edges into at most β^{-1} parts, so that each of these parts spans a k -tree of size $O(\beta n)$. Under a suitable definition of ‘rooted k -tree’, we will show that the parts can be ordered and each of them can be rooted so that the first ℓ parts, for any ℓ , form a connected k -subtree, which contains the root of part $\ell + 1$.

The key object in our proof will be the *layering* of a k -tree T . This is a vertex-partition of T , corresponding to a fixed initial $(k - 1)$ -subset r of some edge of T . More precisely, r is an ordered set, so $r = (r_1, \dots, r_{k-1})$. A layering for this choice of (T, r) is a partition $\{V_1, \dots, V_\ell\}$ of $V(T)$ such that (among other properties), $r_i \in V_i$ for all $1 \leq i < k$, and every edge of T intersects k

consecutive sets V_i . Also, in the layering, no edge e of T can lie strictly ‘above’ the edges of the unique path connecting e and r . With this concept, we can now sketch the construction of the desired partition into bounded-size trees. We will transverse a tree according to the ordering given by a layering (T, r) . At each step we find a k -subtree of appropriate size, ‘rooted’ at some $(k - 1)$ -set which is ‘far away from r ’ in the layering order. We remove the small k -subtree and iterate. For details, see [8].

4 Sketch of the Proof of Theorem 1

Given a k -graph H with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ and a k -tree T with $\Delta_1(T) \leq \Delta$ as in Theorem 1, the embedding of T into H will be performed in three steps. In the first step, we embed a small k -subtree T' of T into vertices of H that have some special properties. More precisely, T' will contain our desired ‘absorber’. In the second step we embed the bulk of $T - T'$, by using the decomposition in small trees sketched in the previous section. In the third step, there only remains a few vertices of T to be embedded. At this point, we use the absorber included in T' to embed the few missing vertices.

4.1 Step One: The Absorber

We first select a k -subtree $T' \subseteq T$ on $\Omega(n)$ vertices and embed it carefully in H to use absorption. The design of our absorbers is inspired by constructions appearing in [3, 4].

Given a k -graph H and a vertex v , the *link graph* $H(v)$ of v in H is the $(k - 1)$ -graph consisting of the $(k - 1)$ -sets f such that $f \cup \{v\} \in H$. The key observation is that in a k -tree all its link graphs are $(k - 1)$ -trees. Since T' has bounded vertex-degree we will be able to find a $(k - 1)$ -tree X such that, for $\Omega(n)$ vertices $v \in V(T')$, the link graph of v in T' is isomorphic to X . A pair of such $(v, H(v))$ will be called an *X -tuple* in T' .

We will also find *X -tuples in H* , meaning vertices whose link-graphs in H contain copies of X as well. An *X -tuple in H* is a pair (v, X_v) , where $v \in V(H)$ and X_v is a fixed copy of X in $H(v)$. Each k -tuple of vertices in H has an associated set of *absorbing X -tuples* (more about the absorbing mechanism will be said in the third step). Using a probabilistic argument, we find a family \mathcal{A} of vertex-disjoint X -tuples in H , having the property that *every* k -tuple in H has many absorbing X -tuples in \mathcal{A} . Using the codegree conditions of H , we are able to embed T' in H , mapping the X -tuples in T' to the X -tuples in \mathcal{A} .

4.2 Step Two: Embedding the Bulk of the Tree

To embed the bulk of $T - T'$, we start by decomposing $T - T'$ into a constant number of smaller k -subtrees, as described in the previous section. Recall that the k -subtrees can be ordered, and each of them can be rooted, so that the first ℓ parts, for any ℓ , form a connected k -subtree which contains the root of part

$\ell + 1$. The idea is to proceed by embedding these small k -trees T_1, \dots, T_t one by one, following the given ordering. We will also make sure the embedding connects correctly with the previously embedded k -subtree T' , which is achieved by fixing the embedding of the root of $T - T'$. Thus in each step, the part of the k -tree which is currently embedded forms a connected k -subtree of T which includes T' .

Our plan to embed the small parts is to locate in H a matching \mathcal{M} consisting of ‘regular tuples’ of k disjoint sets of vertices, which together cover most of H . We apply the Weak Regularity Lemma for hypergraphs, and then find the desired matching by using the codegree condition on H . At this point we remark that, in contrast with other approaches to hypergraph embedding, we do not need the strength of the more advanced versions of hypergraph regularity. In fact, if the host hypergraph H satisfies some mild conditions of ‘quasirandomness’ we do not need regularity at all, and an arbitrary partition will suffice.

We then embed the parts T_1, \dots, T_t successively. At each step, we embed one part T_i (except its root, which is already embedded). For each T_i we find a suitable edge $\{V_{i_1}, \dots, V_{i_k}\} \in \mathcal{M}$ with sufficient free space and convenient density. We embed a constant number of layers ‘between’ the already embedded root of T_i and the edge $\{V_{i_1}, \dots, V_{i_k}\}$ (see the next paragraph). We then embed the remainder of T_i (which is actually most of T_i) into $V_{i_1} \cup \dots \cup V_{i_k}$. Because of the way we chose this edge, this can be done using a simple greedy argument.

In order to make the connections between the already embedded root of T_i and the target edge, we use a part of H that we have separated earlier only for this purpose (making all such connections). This is the *reservoir*, a very small set $R \subseteq V(H)$ having (among others) the property that every $(k - 1)$ -set has several neighbours in R . The reservoir is found using a probabilistic argument. A Connecting Lemma will then allow us to find many short walks between arbitrary pairs of ordered $(k - 1)$ -sets, whose internal vertices are all inside the reservoir. An enhanced version of this lemma allows us to embed not only walks or paths but instead bounded-size k -trees of bounded degree into the reservoir, joining given pairs of $(k - 1)$ -edges. This is what we need to finish the connection step described in the previous paragraph, and thus the embedding of T_i .

4.3 Step Three: Finalising the Embedding

Finally, after embedding almost all of T , we use an Absorbing Lemma. We can assume that the now-embedded k -subtree can be completed to an embedding of T by adding leaves, one at a time (as in the definition of k -trees). Suppose we want to add a leaf v_{i+1} such that $\{u_1, \dots, u_{k-1}\} \in \partial T$ is already embedded and $\{v_{i+1}, u_1, \dots, u_{k-1}\} \in T$. Let us denote by φ_i the current embedding. Given an unused vertex $x_{i+1} \in V(H)$, we find an absorbing X -tuple $(u^*, X_{u^*}) \in \mathcal{A}$ for $\{x_{i+1}, \varphi_i(u_1), \dots, \varphi_i(u_{k-1})\}$ which was used in φ_i for an X -tuple of T' . The absorbing X -tuple (u^*, X_{u^*}) satisfies the following two properties:

- $X_{u^*} \subseteq H(u^*) \cap H(x_{i+1})$, and
- $\{u^*, \varphi_i(u_1), \dots, \varphi_i(u_{k-1})\} \in H$.

Now we can exchange x_{i+1} with u^* in order to embed v_{i+1} . More precisely, we can set $\varphi_{i+1}(\varphi_i^{-1}(u^*)) = x_{i+1}$ and $\varphi_{i+1}(v_{i+1}) = u^*$, and $\varphi_{i+1}(v) = \varphi_i(v)$ for all other vertices v . We thus obtain a new embedding φ_{i+1} in which an extra vertex of T is now incorporated. Iterating this argument, adding a vertex at a time, we can complete the embedding of T .

5 Open Questions

It remains open to consider embedding of trees of unbounded degree. Komlós, Sarközy and Szemerédi [7] showed the following strengthening of the main result of [6], which ensures the existence of an even larger family of spanning trees under the same conditions in the host graph.

Theorem 2 (Komlós, Sarközy and Szemerédi [7]). *For all $\gamma > 0$ there is n_0 and $c > 0$ such that every graph H on $n \geq n_0$ vertices with $\delta(H) \geq (1/2 + \gamma)n$ contains every tree T on n vertices with $\Delta_1(T) \leq cn/\log n$.*

The value $cn/\log n$ is best possible, as witnessed by the case where H is a dense random graph and T is as follows: it is an n vertex tree consisting of a root r with $\Theta(\log n)$ children, and each of the remaining vertices is distributed as evenly as possible as children of the children of r . An adaptation of this example to k -graphs [8, Sect. 12.2] shows that there are k -trees with $\Delta_1(T) = \Omega(n/\log n)$ which are not present in all k -graphs H satisfying $\delta_{k-1}(H) \geq (1/2 + o(1))n$. It would be interesting to determine the largest class of hypertrees which are present as spanning subgraphs in the setting of Theorem 1.

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Outer 1-String Graphs of Girth at Least Five are 3-Colorable

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Abstract. An outer 1-string graph is the intersection graph of curves contained in a disk with one endpoint in its boundary, such that two curves intersect at most once. In an attempt to tackle a problem by Kostochka and Nešetřil (European J. of Comb. 19, 1998), we study coloring of outer 1-string graphs with girth $g \geq 5$. In the process, we generalize the results of Ageev (Discrete Math. 195, 1999), and Esperet and Ochem (Discrete Math. 309, 2009) on circle graphs.

Keywords: String graphs · Chromatic number · Girth

1 Overview

The main purpose of this work¹ is to attack a problem by Kostochka and Nešetřil [8] on coloring *1-string graphs* of girth five. In this regard, we show that *outer 1-string graphs* of girth g at least five and minimum degree at least two have $(g - 4)$ vertices of degree two, which induce a path. This leads to the first step towards our main goal. Furthermore, our result seems to be a natural milestone as it generalizes similar results on *circle graphs*, by Ageev [1], and by Esperet and Ochem [5]. We begin with some definitions.

The *intersection graph* of a family of sets \mathcal{S} is the graph with vertex set \mathcal{S} and edge set $\{XY \mid X, Y \in \mathcal{S}, X \neq Y, X \cap Y \neq \emptyset\}$. Here \mathcal{S} is the *intersection representation* of the intersection graph. A *curve* or *string*² is a homeomorphic image of the interval $[0, 1]$ in the plane. A *1-string graph* is the intersection graph of a finite collection of strings such that two curves intersect at most once. We assume that whenever two strings intersect, they *cross* each other. When the strings in a 1-string representation are contained in a disk with one endpoint in the boundary of the disk, the resulting intersection graphs are known as *outer 1-string graphs*. *Circle graphs* are intersection graphs of chords of a circle. We

¹ The result of this paper was presented in EuroCG '21 for outerstring graphs [4].

Here we present the result for outer 1-string graphs, which has simpler definitions and is easier to verify; although, the proof strategy is the same for both classes.

² We can safely assume that the strings are *simple* (non self-intersecting).

denote the class of 1-string graphs by \mathcal{S}_1 , and the class of outer 1-string graphs by \mathcal{OS}_1 .

Many interesting problems on the *vertex chromatic number* (denoted χ) of intersection graphs of geometric objects have been studied. One such class of problems explores its dependence on *girth* [1, 2, 5, 7–9].

Given a class of intersection graphs \mathcal{G} and a positive integer k , with $k \geq 4$, find or bound $\chi(\mathcal{G}, k)$, where

$$\chi(\mathcal{G}, k) := \max_{G \in \mathcal{G}} \{\chi(G) \mid \text{girth}(G) \geq k\}.$$

One of the popular problems in this regard was posed by Erdős (see [6, Problem 1.9]) in the 1970s³. His question can be translated to the following: Is $\chi(\mathcal{I}, 4) < \infty$ (Problem 1 in [8]), where \mathcal{I} is the class of intersection graphs of line segments in the plane. Further Kratochvíl and Nešetřil asked a similar problem (see [7]): Is $\chi(\mathcal{S}_1, 4) < \infty$? (Problem 2 in [8].)

However, these questions were recently resolved in the negative in a breakthrough paper by Pawlick et al. [9]. They proved that $\chi(\mathcal{I}, 4)$ can be arbitrarily large, by constructing triangle-free segment intersection graphs with an arbitrarily high chromatic number.

Earlier, motivated by the above problems, Kostochka and Nešetřil [8] studied 1-string graphs with girth at least five. In particular, they proved that $\chi(\mathcal{S}_1, 5) \leq 6$. They also posed if $\chi(\mathcal{S}_1, 5) > 3$. Hence, the best known bounds are $3 \leq \chi(\mathcal{S}_1, 5) \leq 6$.

Our main objective is to improve the bounds of $\chi(\mathcal{S}_1, 5)$. A standard approach in handling such problems is via proving degeneracy of graphs in such classes. A graph is *k-degenerate* if every subgraph has a vertex with degree at most k . A greedy coloring scheme implies that a k -degenerate graph is $(k + 1)$ -colorable. In our context, we begin by studying outer 1-string graphs of girth at least five.

Theorem 1. *Every outer 1-string graph with girth at least five is 2-degenerate.*

Studying degeneracy in outer 1-string graphs is a natural approach in improving the upper bound of $\chi(\mathcal{S}_1, 5)$ due to the following reason. Given a 1-string representation (with girth $g \geq 5$), we can treat its *outer envelope* as the boundary of the disk containing the outer 1-string representation. The graph induced by the other strings intersecting this *boundary* is an outer 1-string graph. As we shall see, the target string (corresponding to the vertex with degree at most two) in an outer 1-string representation is in some sense *closest* to the boundary. This would result in finding a degree three vertex in the 1-string graph (because of the girth restriction), proving them to be 4-colorable. There are some hidden details. We shall address this in future work.

The next corollary follows from Theorem 1, as all odd cycles are outer 1-string graphs.

³ An approximate date was confirmed in a personal communication with András Gyárfás and Janós Pach to the authors of [9] (Pawlick et al.). See footnote 2 in [9].

Corollary 1. $\chi(\mathcal{OS}_1, 5) = 3$.

In this article, we further strengthen Theorem 1. Our main result on outer 1-string graphs generalizes similar results on circle graphs (intersection graphs of chords of a circle), first by Ageev [1], and then by Esperet and Ochem [5].

Ageev [1] proved the following result on degeneracy in circle graphs.

Theorem 2 (Ageev [1]). *Every circle graph with girth at least five is 2-degenerate.*

Esperet and Ochem [5] generalized Theorem 2 by proving the following.

Theorem 3 (Esperet and Ochem [5]). *Every circle graph with girth $g \geq 5$ and minimum degree at least two contains a chain of $(g - 4)$ vertices of degree two.*

We first prove the following extension of Theorem 3 to outer 1-string graphs. Outer 1-string graphs generalize circle graphs. See Fig. 1 for a separating example between circle graphs and outer 1-string graphs with girth five and minimum degree two. See [3] for a proof.



Fig. 1. (a) Separating example and (b) its Outer 1-string representation.

Theorem 4. *Every outer 1-string graph with girth $g \geq 5$ and minimum degree at least two contains a chain of $(g - 4)$ vertices of degree two.*

We shall only prove Theorem 4, as Theorem 1 directly follows from it. Also, it suffices to consider only connected outer 1-string graphs.

2 Definition and Notations

Basic Definitions. Henceforth, we use the following equivalent⁴ definition of outer 1-string graphs. An outer 1-string graph is the intersection graph of curves that pairwise intersect at most once and are contained in a halfplane with one

⁴ Define a homeomorphism from a closed half-plane to a closed disk minus one boundary point.

endpoint in the boundary of the halfplane. We call this boundary the *stab line* of the representation. For an outer 1-string graph OS_1 , we denote its outer 1-string representation as \mathbb{OS}_1 .

The outer 1-string representation induced by a subset of strings in \mathbb{OS}_1 is called its *sub-representation*. We assume the stab line in \mathbb{OS}_1 to be the Y-axis and the strings lie in the halfplane $x \leq 0$. For each string s_i in \mathbb{OS}_1 , its endpoint on the stab line is its *fixed end* and the other endpoint is its *free end*. We can safely assume that all the free ends of the strings are intersection points. We relax the *crossing* assumption at these points. Since the stab line is the Y-axis, we can order all the fixed ends of strings; so the terms *above*, *below*, *topmost* and *bottommost* are well defined.

A region $\mathcal{A} \subset \mathbb{R}^2$ is *arc-connected* if for any two points in \mathcal{A} there is a curve lying completely in \mathcal{A} that connects them. A *face* in \mathbb{OS}_1 is a maximal arc-connected component of $\mathbb{R}^2 \setminus \bigcup s_i$. (Here the ambient space is \mathbb{R}^2 , and not only the halfplane $x \leq 0$.) An *arrangement* induced by a set of curves in \mathbb{R}^2 is the embedding of these curves in \mathbb{R}^2 .

A *n-chain* in OS_1 is a path of length $n + 1$ in OS_1 , whose n internal vertices have degree two in OS_1 . By abuse of notation, we shall also call the corresponding representation in \mathbb{OS}_1 as a *n-chain*.

Left Envelope. In \mathbb{OS}_1 , let t_0 (respectively, b_0) be the topmost fixed end (respectively, bottommost fixed end) of \mathbb{OS}_1 . Let t_0b_0 be the line segment (on the stab line) from t_0 to b_0 . Consider the arrangement induced by the strings in \mathbb{OS}_1 and t_0b_0 in \mathbb{R}^2 . Consider the unbounded face of this arrangement. The *left envelope* of \mathbb{OS}_1 is the part of the boundary of this unbounded face after removing t_0b_0 except its end points. We say a string s *belongs* to the left envelope \mathcal{E} if $s \cap \mathcal{E} \neq \emptyset$. Similarly, we can define the left envelope of a sub-representation of \mathbb{OS}_1 .

Zone. This region is about a collection of faces in the arrangement induced by the strings of \mathbb{OS}_1 and the stab line, specific to any two intersecting strings. In an outer 1-string representation, given two intersecting strings s_t and s_b , the *zone* of s_t and s_b in \mathbb{OS}_1 is the bounded closed face in $\mathbb{R}^2 \setminus \{s_t, s_b, tb\}$ that contains tb in its boundary. The strings s_t and s_b are called as the *defining strings* of this zone: with s_t as its *top defining string* and s_b as its *bottom defining string*. Thus, every pair of intersecting strings corresponds to a zone.

A zone \mathcal{Z} of s_t and s_b is *filled* if there exists a string s in \mathbb{OS}_1 whose fixed end is in \mathcal{Z} , and s does not intersect with s_t and s_b . Thus, $s \cap \mathcal{Z} = s$ and $s \cap s_t = s \cap s_b = \emptyset$: we say that \mathcal{Z} *supports* s .

k-Face. In an outer 1-sting graph OS_1 with girth $g \geq 5$, let \mathcal{F} be a bounded face in \mathbb{OS}_1 . Form the cyclic sequence of strings encountered while traversing the boundary of \mathcal{F} with the following restriction. Ignore the strings that contribute just a point in the boundary of \mathcal{F} . Soon we shall prove that the strings in this sequence do not repeat (see proof of Observation 5). We call these strings as *bounding strings*. We say \mathcal{F} is a *k-face* if it has k bounding strings. One can

check that $k \geq g$. Next, we show that the cycle contributed by the k -face \mathcal{F} is an induced cycle in OS_1 . See [3] for a proof.

Observation 5. *The vertices corresponding to the bounding strings of a k -face \mathcal{F} in \mathbb{OS}_1 of an outer 1-string graph OS_1 (with girth $g \geq 5$) form an induced cycle in OS_1 .*

Extended Face. In an outer 1-string graph OS_1 with girth $g \geq 5$, consider an induced cycle C on k vertices. As the strings corresponding to the adjacent vertices intersect, we can find a closed curve in \mathbb{OS}_1 using the parts of strings corresponding to the vertices of C . Since C is an induced cycle, this closed curve is a Jordan curve. Furthermore, this Jordan curve is unique as a pair of strings intersect at most once. We call the closed inside region of this Jordan curve as the *extended face* of C .

3 Outline of Proof of Theorem 4

The proof of Theorem 4 consists of three parts. See [3] for a proof. Here we give a detailed outline. The main proof is by strong induction. To this end, first, we need to study the outer 1-string representations of cycles (for the base step in induction). Then we study some basic properties of outer 1-string representations, followed by the outline of the main proof.

Cycle Representations: Consider an outer 1-string representation \mathbb{C} of a cycle C on $n \geq 5$ vertices. We first prove that if \mathbb{C} has a filled zone \mathcal{Z} , then every string s in \mathbb{C} , other than the defining strings of \mathcal{Z} , has an intersection point in $Int(\mathcal{Z})$. Then we prove the following.

Claim (3). There is at least one filled zone in \mathbb{C} . Every filled zone in \mathbb{C} contains a $(n - 4)$ -chain.

We call such strings that are *supported* by the filled zone as *intermediate strings*. These intermediate strings induce a $(n - 4)$ -chain. While proving Claim 3, we found that \mathbb{C} can have either one or three filled zones depending on the number of intersection points in its left envelope. See [3] for details.

Some Basic Properties of Outer 1-string Representations: Consider an outer 1-string graph OS_1 with girth $g \geq 5$ and minimum degree $\delta \geq 2$. Let \mathcal{Z} be a filled zone in \mathbb{OS}_1 . Such a filled zone always exists (see Claim 3). We prove the following.

Subclaim 6. *For every filled zone \mathcal{Z} in \mathbb{OS}_1 , there exists a k -face in \mathcal{Z} .*

A key argument used in the proof of Subclaim 6, as well as in the main proof, is called the *branching argument*. If a filled zone is not allowed to have a k -face in it and $\delta \geq 2$, then consider the forest/tree induced by the strings supported by the zone. The leaves (except one) of this tree have degree one in OS_1 , else a k -face is formed in the filled zone. This contradicts our minimum degree restriction. Hence Subclaim 6 follows.

Finally, we prove the following stronger version of Theorem 4.

Claim (7). Given an outer 1-string graph OS_1 with girth $g \geq 5$ and $\delta \geq 2$, any filled zone \mathcal{Z} in \mathbb{OS}_1 completely contains a $(g - 4)$ -chain, that is, the strings in this chain are supported by \mathcal{Z} .

3.1 Outline of Proof of Claim 7

We proceed by strong induction on the number of vertices of OS_1 . The base case is easy to verify, as it is a cycle on g vertices (see Claim 3). Now assume the induction hypothesis: in any outer 1-string graph on less than l vertices with girth $g \geq 5$ and $\delta \geq 2$, every filled zone contains a $(g - 4)$ -chain in its outer 1-string representation.

Let OS_1 be an outer 1-string graph on l vertices with girth $g \geq 5$ and $\delta \geq 2$, with outer 1-string representation \mathbb{OS}_1 . Consider a filled zone \mathcal{Z} in \mathbb{OS}_1 : \mathcal{Z} exists as OS_1 has a cycle in it (see Claim 3). Let s_t and s_b be the top and bottom defining strings of \mathcal{Z} , respectively.

Subclaim 7. *We can safely assume that every string in \mathbb{OS}_1 has an intersection point in $\text{Int}(\mathcal{Z})$.*

Next, we can also assume that OS_1 is not isomorphic to the cycle on l vertices. Indeed, we have proved Claim 7 for cycles (see Claim 3). Hence there are at least two (induced) cycles in OS_1 . Next, we prove the following.

Subclaim 8. *\mathbb{OS}_1 has at least two filled zones.*

Using this, we show that two filled zones are restricted to attain one of two possible configurations. In each of these configurations, we exhaustively study all cases and exhibit a filled zone and a string that does not have an intersection point in the interior of the filled zone, in each of these cases, thereby contradicting Subclaim 7. This completes the proof of Claim 7. \square

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The Rainbow Turán Number of Even Cycles

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Abstract. The rainbow Turán number $\text{ex}^*(n, H)$ of a graph H is the maximum possible number of edges in a properly edge-coloured n -vertex graph with no rainbow subgraph isomorphic to H . We prove that for any integer $k \geq 2$, $\text{ex}^*(n, C_{2k}) = O(n^{1+1/k})$. This is tight and establishes a conjecture of Keevash, Mubayi, Sudakov and Verstraëte. We use the same method to prove several other conjectures in various topics.

For example, we answer a question of Jiang and Newman by showing that there exists a constant $c = c(r)$ such that any n -vertex graph with more than $cn^{2-1/r}(\log n)^{7/r}$ edges contains the r -blowup of an even cycle. We also prove that the r -blowup of C_{2k} has Turán number $O(n^{2-\frac{1}{r} + \frac{1}{k+r-1} + o(1)})$, which can be used to disprove an old conjecture of Erdős and Simonovits.

Keywords: Turán number · Even cycles · Proper edge-colouring

1 Introduction

In this paper we develop a method that allows us to find cycles with suitable extra properties in graphs with sufficiently many edges. We give applications in three different areas, which are introduced in the next three subsections. For the full version of this paper, see [10].

1.1 Rainbow Turán Numbers

For a family of graphs \mathcal{H} , the Turán number (or extremal number) $\text{ex}(n, \mathcal{H})$ is the maximum number of edges in an n -vertex graph which does not contain any $H \in \mathcal{H}$ as a subgraph. When $\mathcal{H} = \{H\}$, we write $\text{ex}(n, H)$ for the same function. This function is determined asymptotically by the Erdős–Stone–Simonovits [6, 7] theorem when H has chromatic number at least 3. However, for bipartite graphs H , even the order of magnitude of $\text{ex}(n, H)$ is unknown in general. For example, a result of Bondy and Simonovits [2] states that $\text{ex}(n, C_{2k}) = O(n^{1+1/k})$, but a matching lower bound is only known when $k \in \{2, 3, 5\}$.

A variant of this function was introduced by Keevash, Mubayi, Sudakov and Verstraëte in [14]. In an edge-coloured graph, we say that a subgraph is rainbow

if all its edges are of different colour. The rainbow Turán number of the graph H is then defined to be the maximum number of edges in a properly edge-coloured n -vertex graph that does not contain a rainbow H as a subgraph. This number is denoted by $\text{ex}^*(n, H)$. Clearly, $\text{ex}^*(n, H) \geq \text{ex}(n, H)$ for every n and H . Keevash et al. [14] proved, among other things, that for any non-bipartite graph H , we have $\text{ex}^*(n, H) = (1 + o(1))\text{ex}(n, H)$. Hence, the most challenging case again seems to be when H is bipartite. Keevash et al. showed that $\text{ex}^*(n, K_{s,t}) = O(n^{2-1/s})$, which is tight when $t > (s - 1)!$ [1, 15]. The function has also been studied for trees (see [8, 12, 13]). About even cycles, Keevash et al. proved the following lower bound.

Theorem 1 (Keevash–Mubayi–Sudakov–Verstraëte [14]). *For any integer $k \geq 2$,*

$$\text{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k}).$$

They conjectured that this is tight.

Conjecture 1 (Keevash–Mubayi–Sudakov–Verstraëte [14]). For any integer $k \geq 2$,

$$\text{ex}^*(n, C_{2k}) = \Theta(n^{1+1/k}).$$

They have verified their conjecture for $k \in \{2, 3\}$. For general k , Das, Lee and Sudakov proved the following upper bound.

Theorem 2 (Das–Lee–Sudakov [4]). *For every fixed integer $k \geq 2$,*

$$\text{ex}^*(n, C_{2k}) = O\left(n^{1+\frac{(1+\varepsilon_k)\ln k}{k}}\right),$$

where $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

In this paper we prove Conjecture 1 by establishing the following result.

Theorem 3. *For any integer $k \geq 2$, we have*

$$\text{ex}^*(n, C_{2k}) = O(n^{1+1/k}).$$

The theta graph $\theta_{k,t}$ is the union of t paths of length k which share the same endpoints but are pairwise internally vertex-disjoint. We remark that our proof can be easily modified to show that $\text{ex}^*(n, \theta_{k,t}) = O(n^{1+1/k})$ for any fixed k and t .

Keevash et al. also asked how many edges a properly edge-coloured n -vertex graph can have if it does not contain any rainbow cycle. They constructed such graphs with $\Omega(n \log n)$ edges. Note that this is quite different from the uncoloured case, since any n -vertex acyclic graph has at most $n - 1$ edges. Das, Lee and Sudakov proved that if $\eta > 0$ and n is sufficiently large, then any properly edge-coloured n -vertex graph with at least $n \exp\left((\log n)^{\frac{1}{2}+\eta}\right)$ edges contains a rainbow cycle. We prove the following improvement.

Theorem 4. *There exists an absolute constant C such that if n is sufficiently large and G is a properly edge-coloured graph on n vertices with at least $Cn(\log n)^4$ edges, then G contains a rainbow cycle of even length.*

1.2 Colour-Isomorphic Even Cycles in Proper Colourings

Conlon and Tyomkyn [3] have initiated the study of the following problem. We say that two subgraphs of an edge-coloured graph are colour-isomorphic if there is an isomorphism between them preserving the colours. For an integer $r \geq 2$ and a graph H , they write $f_r(n, H)$ for the smallest number C so that there is a proper edge-colouring of K_n with C colours containing no r vertex-disjoint colour-isomorphic copies of H . They proved various results about this function, such as the bound $f_2(n, C_6) = \Omega(n^{4/3})$.

One of the several open problems they posed is the following question.

Question 1 (Conlon–Tyomkyn [3]). Is it true that for every $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that, for all $k \geq k_0$, $f_2(n, C_{2k}) = \Omega(n^{2-\varepsilon})$?

Later, Xu, Zhang, Jing and Ge made a more precise conjecture.

Conjecture 2 (Xu–Zhang–Jing–Ge [16]). For any $k \geq 3$, $f_2(n, C_{2k}) = \Omega(n^{2-\frac{2}{k}})$.

We prove this conjecture in a more general form.

Theorem 5. *Let $k, r \geq 2$ be fixed integers. Then $f_r(n, C_{2k}) = \Omega\left(n^{\frac{r}{r-1} \cdot \frac{k-1}{k}}\right)$.*

1.3 Turán Number of Blow-Ups of Cycles

For a graph F , the r -blowup of F is the graph obtained by replacing each vertex of F with an independent set of size r and each edge of F by a $K_{r,r}$. We write $F[r]$ for this graph. The systematic study of the Turán number of blow-ups was initiated by Grzesik, Janzer and Nagy [9]. They proved that for any tree T we have $\text{ex}(n, T[r]) = O(n^{2-1/r})$. They have also made the following general conjecture.

Conjecture 3 (Grzesik–Janzer–Nagy [9]). Let r be a positive integer and let F be a graph such that $\text{ex}(n, F) = O(n^{2-\alpha})$ for some $0 \leq \alpha \leq 1$ constant. Then

$$\text{ex}(n, F[r]) = O(n^{2-\frac{\alpha}{r}}).$$

Their result mentioned above proves this conjecture when F is a tree. It is easy to see that the conjecture holds also when $F = K_{s,t}$ and $\alpha = 1/s$.

In the case of forbidding all r -blowups of cycles, an earlier question was formulated by Jiang and Newman [11]. To state this question, we write $\mathcal{C}[r] = \{C_{2k}[r] : k \geq 2\}$.

Question 2 (Jiang–Newman [11]). Is it true that for any positive integer r and any $\varepsilon > 0$, $\text{ex}(n, \mathcal{C}[r]) = O(n^{2-\frac{1}{r}+\varepsilon})$?

We answer this question affirmatively in a stronger form.

Theorem 6. *For any positive integer r ,*

$$\text{ex}(n, \mathcal{C}[r]) = O(n^{2-1/r}(\log n)^{7/r}).$$

Random graphs show that $\text{ex}(n, \mathcal{C}[r]) = \Omega(n^{2-1/r})$. It would be interesting to decide whether the logarithmic factor in Theorem 6 can be removed.

Finally, we establish an upper bound for the Turán number when only one blowup cycle is forbidden.

Theorem 7. *For any integers $r \geq 1$ and $k \geq 2$, we have*

$$\text{ex}(n, C_{2k}[r]) = O\left(n^{2-\frac{1}{r}+\frac{1}{k+r-1}} (\log n)^{\frac{4k}{r(k+r-1)}}\right).$$

This is still quite a long way from the conjectured $\text{ex}(n, C_{2k}[r]) = O(n^{2-\frac{1}{r}+\frac{1}{kr}})$. However, it can be used to disprove the following conjecture of Erdős and Simonovits.

Conjecture 4 (Erdős–Simonovits [5]). Let H be a bipartite graph with minimum degree s . Then there exists $\varepsilon > 0$ such that $\text{ex}(n, H) = \Omega(n^{2-\frac{1}{s-1}+\varepsilon})$.

To see that this is false, note that the graph $C_{2k}[r]$ has minimum degree $2r$, but, by Theorem 7, for any $\delta > 0$, we have $\text{ex}(n, C_{2k}[r]) = O(n^{2-\frac{1}{r}+\delta})$ for sufficiently large k . This means that there exists, for any even $s \geq 4$ and any $\delta > 0$, a bipartite graph H with minimum degree s which has $\text{ex}(n, H) = O(n^{2-\frac{2}{s}+\delta})$, disproving Conjecture 4 for all even $s \geq 4$. On the other hand, a simple application of the probabilistic method shows that if H is a bipartite graph with minimum degree $s \geq 2$, then there exists $\varepsilon > 0$ such that $\text{ex}(n, H) = \Omega(n^{2-\frac{2}{s}+\varepsilon})$.

In the next section we present the key lemma which is used in the proof of Theorem 3 and Theorem 4.

2 Bounding Non-rainbow Homomorphic Cycles

In what follows, for graphs H and G we write $\text{hom}(H, G)$ for the number of graph homomorphisms $V(H) \rightarrow V(G)$. P_k will denote the path with k edges and we use the convention $C_2 = P_1$. For vertices $x, y \in V(G)$, $\text{hom}_{x,y}(P_\ell, G)$ denotes the number of walks of length ℓ in G between x and y . We write $\Delta(G)$ for the maximum degree of G .

With a slight abuse of terminology, we call a homomorphism $H \rightarrow G$ a *homomorphic copy of H in G* . That is, a homomorphic copy of $C_{2\ell}$ is a tuple $(x_1, \dots, x_{2\ell}) \in V(G)^{2\ell}$ such that $x_1x_2, x_2x_3, \dots, x_{2\ell}x_1 \in E(G)$. A rainbow homomorphic copy of H is one in which the images of distinct edges of H have different colour. Our key lemma is an upper bound on the number of those (homomorphic) 2ℓ -cycles which are not rainbow.

Lemma 1. *Let $\ell \geq 2$ be a positive integer and let G be a properly edge-coloured graph. Then the number of homomorphic copies of $C_{2\ell}$ which are not rainbow is at most*

$$16\ell (\ell\Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}.$$

Proof. Let $c(e)$ be the colour of the edge $e \in E(G)$. We want to prove that the number of $(x_1, x_2, \dots, x_{2\ell}) \in V(G)^{2\ell}$ with $x_1x_2, \dots, x_{2\ell}x_1 \in E(G)$ such that $c(x_1x_2), \dots, c(x_{2\ell}x_1)$ are not all distinct is at most $16\ell(\ell\Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}$. By symmetry, it suffices to prove that the number of $(x_1, x_2, \dots, x_{2\ell}) \in V(G)^{2\ell}$ with $x_1x_2, \dots, x_{2\ell}x_1 \in E(G)$ for which there exists $2 \leq i \leq \ell + 1$ such that $c(x_1x_2) = c(x_i x_{i+1})$ is at most $8(\ell\Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}$.

For a positive integer s , let α_s be the number of walks of length $\ell - 1$ in G whose endpoints y and z have $2^{s-1} \leq \text{hom}_{y,z}(P_{\ell-1}, G) < 2^s$ and let β_s be the number of walks of length ℓ in G whose endpoints y and z have $2^{s-1} \leq \text{hom}_{y,z}(P_\ell, G) < 2^s$. Clearly,

$$\sum_{s \geq 1} \alpha_s 2^{s-1} \leq \text{hom}(C_{2\ell-2}, G) \tag{1}$$

and

$$\sum_{s \geq 1} \beta_s 2^{s-1} \leq \text{hom}(C_{2\ell}, G). \tag{2}$$

For positive integers s and t , write $\gamma_{s,t}$ for the number of homomorphic copies $x_1x_2 \dots x_{2\ell}x_1$ of $C_{2\ell}$ for which there exists $2 \leq i \leq \ell + 1$ such that $c(x_1x_2) = c(x_i x_{i+1})$, $2^{s-1} \leq \text{hom}_{x_1, x_{\ell+2}}(P_{\ell-1}, G) < 2^s$ and $2^{t-1} \leq \text{hom}_{x_2, x_{\ell+2}}(P_\ell, G) < 2^t$. Observe that

$$\gamma_{s,t} \leq \alpha_s \cdot \Delta(G) \cdot 2^t. \tag{3}$$

Indeed, if $x_1x_2 \dots x_{2\ell}x_1$ is a homomorphic $C_{2\ell}$ with $2^{s-1} \leq \text{hom}_{x_1, x_{\ell+2}}(P_{\ell-1}, G) < 2^s$ and $2^{t-1} \leq \text{hom}_{x_2, x_{\ell+2}}(P_\ell, G) < 2^t$, then there are at most α_s ways to choose $(x_{\ell+2}, x_{\ell+3}, \dots, x_{2\ell}, x_1)$, given such a choice there are at most $\Delta(G)$ choices for x_2 , and given these there are at most 2^t choices for $(x_3, \dots, x_{\ell+1})$. On the other hand,

$$\gamma_{s,t} \leq \beta_t \cdot \ell \cdot 2^s. \tag{4}$$

Indeed, there are at most β_t ways to choose $(x_2, \dots, x_{\ell+2})$. Given such a choice, there are at most ℓ possibilities for x_1 , since $c(x_1x_2) = c(x_i x_{i+1})$ for some $2 \leq i \leq \ell + 1$, the edges $x_2x_3, \dots, x_{\ell+1}x_{\ell+2}$ are already fixed and c is a proper colouring. Finally, there are at most 2^s ways to complete this to a suitable homomorphic copy of $C_{2\ell}$.

Clearly, the total number of homomorphic copies $x_1x_2 \dots x_{2\ell}x_1$ of $C_{2\ell}$ with $c(x_1x_2) = c(x_i x_{i+1})$ for some $2 \leq i \leq \ell + 1$ is $\sum_{s,t \geq 1} \gamma_{s,t}$. We give an upper bound for this sum as follows. Let q be the integer for which $(\frac{\ell \text{hom}(C_{2\ell}, G)}{\Delta(G) \text{hom}(C_{2\ell-2}, G)})^{1/2} \leq 2^q < 2(\frac{\ell \text{hom}(C_{2\ell}, G)}{\Delta(G) \text{hom}(C_{2\ell-2}, G)})^{1/2}$. Now, using Eqs. (4) and (2),

$$\begin{aligned} \sum_{s,t: s \leq t-q} \gamma_{s,t} &\leq \ell \sum_{s,t: s \leq t-q} 2^s \beta_t \leq \ell \cdot \sum_{t \geq 1} 2^{t-q+1} \beta_t \leq \ell \cdot 2^{-q+2} \text{hom}(C_{2\ell}, G) \\ &\leq 4(\ell\Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}. \end{aligned}$$

Also, using Eqs. (3) and (1),

$$\begin{aligned} \sum_{s,t:s>t-q} \gamma_{s,t} &\leq \Delta(G) \sum_{s,t:s>t-q} 2^t \alpha_s \leq \Delta(G) \sum_{s \geq 1} 2^{s+q} \alpha_s \leq \Delta(G) 2^{q+1} \text{hom}(C_{2\ell-2}, G) \\ &\leq 4(\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}. \end{aligned}$$

Thus,

$$\sum_{s,t \geq 1} \gamma_{s,t} \leq 8(\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}.$$

This completes the proof.

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Trees with Few Leaves in Tournaments

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Abstract. We prove that there exists $C > 0$ such that any $(n + Ck)$ -vertex tournament contains a copy of every n -vertex oriented tree with k leaves, improving the previously best known bound of $n + O(k^2)$ vertices to give a result tight up to the value of C . Furthermore, we show that, for each k , there exists n_0 , such that, whenever $n \geq n_0$, any $(n + k - 2)$ -vertex tournament contains a copy of every n -vertex oriented tree with at most k leaves, confirming a conjecture of Dross and Havet.

Keywords: Tournament theory · Median orders · Sumner's conjecture

1 Introduction

The study of trees in tournaments has been motivated largely by Sumner's universal tournament conjecture from 1971, which states that every $(2n - 2)$ -vertex tournament should contain a copy of every n -vertex oriented tree (see, e.g., [12]). In 1991, Häggkvist and Thomason [5] gave the first proof that $O(n)$ vertices in a tournament are sufficient to find any n -vertex oriented tree. Following several subsequent improvements to the implicit constant [4, 7, 9], Dross and Havet [3] recently showed that $\lceil \frac{21}{8}n - \frac{47}{16} \rceil$ vertices are in fact sufficient, giving the best known bound which holds for all n . On the other hand, Sumner's conjecture is known to be true for sufficiently large n , as shown in 2010 by Kühn, Mycroft and Osthus [11], using regularity methods.

If true, Sumner's conjecture would be tight for each n , as demonstrated by the n -vertex star with every edge oriented out from the root vertex. The appearance of many trees can, however, be ensured with far fewer than $2n - 2$ vertices in the tournament. Indeed, confirming a conjecture of Rosenfeld [14], Thomason [15] showed in 1986 that there is some n_0 such that, whenever $n \geq n_0$, any n -vertex tournament contains a copy of every n -vertex oriented path. In 2000, Havet and Thomassé [10] showed that the optimal value of n_0 is 8, a result recently given a shorter proof by Hanna [6].

Answering the natural question arising from the different behaviour here between stars and paths, Häggkvist and Thomason [5] showed in 1991 that the number of additional vertices required in the tournament can be bounded by the number of leaves in the tree. That is, for each k , there is some smallest $g(k)$ such that every $(n + g(k))$ -vertex tournament contains a copy of every n -vertex tree with k leaves. The upper bound shown by Häggkvist and Thomason on $g(k)$ is

exponential in k^3 , but was recently improved to $144k^2 - 280k + 124$ by Dross and Havet [3]. Havet and Thomassé [8] conjectured in 2000 that $g(k) \leq k - 1$ for each $k \geq 2$. That is, generalising Sumner’s conjecture, they conjectured that every $(n + k - 1)$ -vertex tournament contains a copy of every n -vertex oriented tree with k leaves.

In this paper, we give the first linear bound on $g(k)$, as follows.

Theorem 1. *There is some $C > 0$ such that every $(n + Ck)$ -vertex tournament contains a copy of every n -vertex oriented tree with k leaves.*

If true, Havet and Thomassé’s conjecture would be tight whenever $k = n - 1$ (i.e., whenever it is covered by Sumner’s conjecture), but for general n and k , we only have examples showing that the tournament may need to have at least $n + k - 2$ vertices (as described below). From the result of Havet and Thomassé [10] on oriented paths we know that $n + k - 2$ is best possible if $k = 2$ and $n \geq 8$, while Ceroi and Havet [2] proved that $n + k - 2$ is also best possible if $k = 3$ and $n \geq 5$. Dross and Havet [3] conjectured that, for each k , if n is sufficiently large then $n + k - 2$ is best possible.

In this paper, we confirm this conjecture, as follows.

Theorem 2. *For each k , there is some n_0 such that, for each $n \geq n_0$, every $(n + k - 2)$ -vertex tournament contains a copy of every n -vertex oriented tree with k leaves.*

The following well-known example shows that this is best possible. Form a tree $T_{n,k}$ by taking a directed path P with $n - k + 1$ vertices and attaching $k - 1$ out-leaves to the last vertex of P . The resulting oriented tree $T_{n,k}$ has n vertices and k leaves. Construct the following $(n + k - 3)$ -vertex tournament G . Let $V(G) = A \cup B$, where $|A| = n - k$, and $|B| = 2k - 3$. Orient the edges of G so that $G[B]$ is a regular tournament, $G[A]$ is an arbitrary tournament, and all edges are directed from A to B . As $d_G^+(v) = k - 2$ for each $v \in B$, if G contains a copy of $T_{n,k}$ then the last vertex of P must be copied to A . Then, as every edge between A and B is oriented into B , every vertex of P must be copied into A , a contradiction as $|A| = n - k$. Thus the n -vertex tree $T_{n,k}$ with k leaves does not appear in the $(n + k - 3)$ -vertex tournament G .

The proofs for Theorems 1 and 2 are sketched in Sects. 3 and 4, while the full proofs can be found in [1]. Both proofs make use of median orders, a tool discussed in Sect. 2. We have not optimised the value of C reachable with our methods as this will not reach a plausibly optimal bound, but we show that Theorem 1 holds for some $C < 500$.

2 Median Orders

Median orders were first used to embed trees in tournaments by Havet and Thomassé [9]. Given a tournament G , an ordering $\sigma = v_1, \dots, v_n$ of $V(G)$ is a *median order* if it maximises the number of pairs $i < j$ with $v_i v_j \in E(G)$. The following lemma gives two simple fundamental properties of median orders (see, e.g., [3, Lemma 9]).

Lemma 1. *Let G be a tournament and v_1, \dots, v_n a median order of G . Then, for any two indices i, j with $1 \leq i < j \leq n$, the following properties hold.*

- i) v_i, v_{i+1}, \dots, v_j is a median order of the subtournament induced by G on $\{v_i, \dots, v_j\}$.*
- ii) v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_j$, and v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \dots, v_{j-1}$. In particular, each vertex v_i , $1 \leq i < n$, dominates its successor v_{i+1} .*

In both proofs, we will take a median order of a tournament, and carefully partition this order into intervals before embedding different parts of the tree into each interval. The parts embedded into each interval will then need connecting paths attached to them, to recover a copy of the tree.

For connecting paths that are directed, we embed along the path into a consistent order under σ . From Lemma 1, it can be deduced that there is a directed path of length at most 2 from any vertex to any vertex appearing later in a median order. The following lemma shows that directed paths are still possible across intervals of a median order containing some forbidden vertices.

Lemma 2. *Suppose G is an n -vertex tournament with a median order $\sigma = v_1, \dots, v_n$. Then, for any set $A \subset V(G) \setminus \{v_1, v_n\}$ with $|A| \leq (n - 8)/6$, there is a directed v_1, v_n -path in $G - A$ with length 3.*

In order to embed connecting paths that are not directed, we may instead use a suitable application of the following result of Thomason [15].

Theorem 3. ([15, Theorem 5]). *Let P be a oriented path of order $n \geq 5$, which changes direction at its first internal vertex, and also at its last internal vertex. Let G be a tournament of order $n+2$ and X and Y be two disjoint subsets of $V(G)$ of order at least 2. Then there is a copy of P in G with first vertex in X and last vertex in Y .*

Median orders have been used particularly effectively to embed arborescences in tournaments. An *out-arborescence* (respectively, *in-arborescence*) is an oriented tree T with a root vertex $t \in V(T)$ such that, for every $v \in V(T)$, the path between t and v in T is directed from t to v (respectively, from v to t). Dross and Havet [3] used median orders to prove that any $(n + k - 1)$ -vertex tournament contains a copy of any n -vertex arborescence with k leaves. We will use their result in the following slightly stronger form (see [3, Theorem 12]).

Theorem 4. *Let A be an n -vertex out-arborescence with $k \geq 1$ out-leaves and root r . Let G be a tournament on $n + k - 1$ vertices and let $\sigma = v_1, \dots, v_{n+k-1}$ be a median order of G . Then, there is an embedding ϕ of A in G such that $\phi(r) = v_1$.*

We will also need some linear bound on the number of vertices required in a tournament, which we then apply for small trees. Any linear bound would suffice, but here we quote a result of El Sahili [4].

Theorem 5. *For each $n \geq 2$, every $(3n - 3)$ -vertex tournament contains a copy of every n -vertex oriented tree.*

3 Sketch Proof of Theorem 1

To prove Theorem 1, we first show that it is enough to prove the case where all bare paths of T are directed. That is, we reduce the proof to showing the following result.

Theorem 6. *There is some $C > 0$ such that each $(n + Ck)$ -vertex tournament contains a copy of every n -vertex oriented tree with k leaves in which every bare path is a directed path.*

Sketch Proof of Theorem 1 Using Theorem 6. Let T be an n -vertex tree with k leaves. For each maximal bare path of T with several changes of direction, remove most of the middle section, and duplicate each new leaf created by this removal. Calling the resulting forest T' , if we have an embedding of T' then the duplication of a leaf gives us two options to embed the original vertex from T . Therefore, by Theorem 3, the removed paths may be reattached to T' , giving a copy of T .

Not every maximal bare path in T' will be directed, but each such path will have only a few changes of direction. Adding a dummy leaf at any vertex where the path direction changes will give a forest T'' containing T' whose maximal bare paths are all directed, allowing us to apply Theorem 6 to each component. Importantly, T' , and hence T'' , will still have $O(k)$ leaves. Therefore, by Theorem 6, any $(n + O(k))$ -vertex tournament contains a copy of T'' , and hence a copy of T .

Sketch Proof of Theorem 6. Let T be an n -vertex tree with k leaves, in which every bare path is a directed path. Remove long directed paths to leave a forest T' with $|T'| = O(k)$. Take consecutive intervals $V_1, U_1, V_2, U_2, \dots, V_{s-1}, U_{s-1}, V_s$ of a median order, with sizes chosen carefully. We then embed components of T' into the intervals V_1, \dots, V_s , such that, if some removed path was directed from component $T_1 \subseteq T$ to component $T_2 \subseteq T$, then there is some $i \in [s - 1]$ for which T_1 is embedded into V_i and T_2 is embedded into V_{i+1} . Because $|T'| = O(k)$, Theorem 5 shows this is possible, provided $\sum_{i=1}^s |V_i| = C'k$ for some absolute constant C' .

All that remains is to reattach the long directed paths removed to obtain T' , by disjointly copying their internal vertices to intervals U_1, \dots, U_{s-1} as appropriate. Most vertices of these paths can be embedded efficiently in parallel along a median order, by modifying an algorithm of Dross and Havet [3]. When we near the end of each path, it is necessary to connect the path to its desired endpoint, which can be handled using Lemma 2. The number of forbidden vertices when applying Lemma 2 can be shown to be $O(k)$ (essentially, the only vertices we must avoid are those in the intervals V_1, \dots, V_s , which is $O(k)$, plus a constant multiple of the number of paths being embedded in parallel, which is $O(k)$ in total). Therefore, the number of extra vertices required in the tournament for the embedding procedure to be successful is also $O(k)$, completing the proof.

4 Sketch Proof of Theorem 2

As an illustrative case, let us sketch Theorem 2 for trees consisting of a directed path between two arborescences. Suppose we have a directed path P , an in-arborescence S_1 with root the first vertex of P , and an out-arborescence S_2 with root the last vertex of P , and suppose that $S_1 \cup P \cup S_2$ is an oriented tree with n vertices. Say S_1 has k_1 in-leaves and S_2 has k_2 out-leaves, and the tournament G has $m := n + k_1 + k_2 - 2$ vertices and a median order v_1, \dots, v_m . Using Lemma 1 i) and Theorem 4 (via directional duality), we can embed S_1 into $G[\{v_1, \dots, v_{|S_1|+k_1-1}\}]$ with the root vertex embedded to $v_{|S_1|+k_1-1}$. Similarly, we can embed S_2 into $G[\{v_{m-|S_2|-k_2+2}, \dots, v_m\}]$ with the root vertex of S_2 embedded to $v_{m-|S_2|-k_2+2}$. Finally, by Lemma 1 ii), we have $v_{|S_1|+k_1-1} \rightarrow v_{|S_1|+k_1} \rightarrow \dots \rightarrow v_{m-|S_2|-k_2+2}$, so we can use this path to embed the $n - |S_1| - |S_2| + 2 = m - |S_1| - |S_2| - k_1 - k_2 + 4$ vertices of P and complete an embedding of T into G .

Essentially, all our embeddings will look like this, where P will be a very long path, but with some additional subtrees and paths found within the interval we use to embed P . For example, suppose now the tree T also has a subtree F which shares one vertex, t say, with S_1 , where t only has out-neighbours in F . If P is a long path (compared to $|F|, |S_1|, |S_2|$) then we can embed $T = F \cup S_1 \cup P \cup S_2$ into a tournament G with $m := |T| + k_1 + k_2 - 2$ vertices as follows. Carry out the above embedding of S_1 and S_2 into the start and end respectively of a median order v_1, \dots, v_m of G and note that the path $Q := v_{|S_1|+k_1-1} \rightarrow v_{|S_1|+k_1} \rightarrow \dots \rightarrow v_{m-|S_2|-k_2+2}$ has $|F| - 1 + |P|$ vertices. If s is the embedding of $t \in V(S_1)$, then by Lemma 1 ii) and as $|Q| \geq |P| - 1 \gg |F|, |S_1|$, s will have many out-neighbours in this path, enough that we can easily embed $F - t$ among the out-neighbours of s in Q (using, in particular, Theorem 5). However, we wish to do this so that there is a directed path between $v_{|S_1|+k_1-1}$ and $v_{m-|S_2|-k_2+2}$ covering exactly the $|Q| - (|F| - 1) = |P|$ vertices of $V(Q)$ which are not used to embed $F - t$.

To do this, before embedding F , we first use a random procedure to find a short directed $v_{|S_1|+k_1-1}, v_{m-|S_2|-k_2+2}$ -path R with vertices in $V(Q)$ so that every vertex in $V(Q)$ has at least one out-neighbour on R occurring after some in-neighbour on R . The path R will be short enough that we can embed $F - t$ in the out-neighbours of s in $V(Q)$ while avoiding $V(R)$. Once $F - t$ has been embedded, we slot the remaining vertices in $V(Q)$ into R one by one. Note that, in the language of absorption (as codified by Rödl, Ruciński and Szemerédi [13]), R is a path which can absorb any set of vertices from the interval of the median order between its first and last vertex.

More generally, we can embed small trees attached with an out-edge from $S_1 \cup P \cup S_2$, as long as the attachment point is not too late in P , or in S_2 , by embedding such small trees within the interval for the path P . Similarly, we can embed small trees attached with an in-edge from $S_1 \cup P \cup S_2$, as long as the attachment point is not too early in P , or in S_1 . We can also use Lemma 2 to add short paths between vertices in the interval from P that are not too close

together. This allows us to embed a class of digraphs which can be constructed in this way, called *good decompositions*.

The method to prove Theorem 2 is then as follows. Given an n -vertex tree T with k leaves, where $1/n \ll 1/k$, we will be able to assume, using Theorem 3, that T mostly consists of directed bare paths. When this is the case, then, with the possible addition of some dummy edges, we will be able to construct a good decomposition D containing T , where the arborescences S_1, S_2 are subtrees of T . From the discussion above, a copy of D may be found in any $(n + k_1 + k_2 - 2)$ -vertex tournament, where k_1 is the number of in-leaves of S_1 , and k_2 is the number of out-leaves of S_2 . Because the total number of leaves of T is at least $k_1 + k_2$, and $T \subseteq D$, we may conclude that any $(n + k - 2)$ -vertex tournament contains a copy of T .

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Uncommon Systems of Equations

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Abstract. A system of linear equations L over \mathbb{F}_q is *common* if the number of monochromatic solutions to L in any two-colouring of \mathbb{F}_q^n is asymptotically at least the number of monochromatic solutions in a random two-colouring of \mathbb{F}_q^n . The line of research on common systems of linear equations was recently initiated by Saad and Wolf. They were motivated by existing results for specific systems (such as Schur triples and arithmetic progressions), as well as extensive research on common and Sidorenko graphs. Building on earlier work, Fox, Pham and Zhao characterised common linear equations. For systems of two or more equations, only sporadic results were known.

We prove that any system containing an arithmetic progression of length four is uncommon, confirming a conjecture of Saad and Wolf. This follows from a stronger result which allows us to deduce the uncommonness of a general system from considering certain one- or two-equation subsystems.

Keywords: Ramsey theory · Linear systems · Fourier analysis

1 Introduction

A classical theorem of Goodman states that over all 2-edge-colourings of the complete graph K_n , the number of monochromatic triangles is asymptotically minimised by a random 2-colouring. Erdős conjectured that in Goodman's result, the triangle can be replaced by any fixed clique K_s , and Burr and Rosta extended the conjecture to any fixed graph. Erdős' conjecture was disproved by Thomason, motivating numerous results on *common and Sidorenko graphs*, including the famous Sidorenko conjecture. In the arithmetic setting, Graham, Rödl and Ruciński asked about the minimal number of Schur triples (triples satisfying $x + y - z = 0$) in 2-colourings of $[n] = \{1, 2, \dots, n\}$. Questions of this type for linear systems of equations were studied more systematically in [1, 4, 7], and we continue this line of research.

Research supported by ARC Discovery Project DP180103684.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 611–616, 2021.
https://doi.org/10.1007/978-3-030-83823-2_98

Following Fox, Pham and Zhao [4], we work in the finite field model – we fix a finite field \mathbb{F}_q , where q is a prime power, and consider a linear homogeneous system L on k variables with coefficients in \mathbb{F}_q . We say that the system L is *common* if the number of monochromatic solutions in any two-colouring of \mathbb{F}_q^n is asymptotically at least the number of monochromatic solutions in a random two-colouring of \mathbb{F}_q^n . Formal definitions will be given later. Let us briefly discuss systems consisting of a single equation $a_1x_1 + \dots + a_kx_k = 0$ with coefficients $a_i \in \mathbb{F}_q \setminus \{0\}$, which are now completely characterised. Cameron, Cilleruelo and Serra [1] showed that in fact, any such linear equation with an odd number of variables k is common. For even k , Saad and Wolf [7] proved that the equation is common whenever a_1, \dots, a_k can be partitioned into pairs, each pair summing to zero. They conjectured that when k is even, this sufficient condition is also necessary, which was confirmed by Fox, Pham and Zhao [4].

Much less is known when L consists of more than one equation. Saad and Wolf [7] showed that arithmetic progressions of length four (4-APs) over \mathbb{F}_5 are uncommon, and conjectured that any system *containing* a 4-AP is uncommon. Their conjecture can be seen as an analogue of the famous result of Jagger, Štoviček and Thomason [6], showing that any graph containing a K_4 is *uncommon*. Fox, Pham and Zhao [4] asked for a characterisation of common systems of equations, hoping that it might lead to a better understanding of the analogous properties for graphs and hypergraphs, but noted that they do not have a guess for such a characterisation.

Confirming the conjecture of Saad and Wolf, we show that any system L containing a 4-AP is uncommon. This result follows from a more general theorem which provides a sufficient condition for a system to be uncommon, based on certain one- or two-equation *subsystems* of L . Using this theorem, we display two large classes of uncommon systems. The reduction to one- or two-equation systems opens up avenues for using discrete Fourier analysis in studying systems with two or more equations.

We also give examples of common systems based on intricate relations between the condensed equations, indicating that a characterisation of common systems might be rather elusive.

2 Results

Before stating our results, let us introduce some notation. In a slight abuse of notation, we identify a system L with an $m \times k$ matrix L , so that the solution set of L in \mathbb{F}_q^n is

$$\text{sol}(L) = \{\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{F}_q^n)^k : L\mathbf{x}^T = 0\}.$$

We state the definitions and results in terms of functions $f : \mathbb{F}_q^n \rightarrow \mathbb{R}$, rather than subsets of \mathbb{F}_q^n . This is standard in arithmetic combinatorics, since a function can be used to sample a *random* subset of \mathbb{F}_q^n , and thus the commonness property for sets is equivalent to its functional version. This correspondence

between functions and sets is explained in more detail in [4]. The *density* of solutions to a system L in f is

$$\Lambda_L(f) = \frac{1}{|\text{sol}(L)|} \sum_{\mathbf{x} \in \text{sol}(L)} f(x_1)f(x_2)\dots f(x_k).$$

We refer to a system L with k variables and m equations as an $m \times k$ system or a k -variable system, and we have $m \leq k$ throughout. An $m \times k$ system is non-degenerate if its rank is m and there are no variables x_i and x_j such that the equation $x_i = x_j$ can be derived from the system. A non-degenerate $m \times k$ system is *common* if for every $f : \mathbb{F}_q^n \rightarrow [0, 1]$

$$\Lambda_L(f) + \Lambda_L(1 - f) \geq 2^{1-k}.$$

Note that the right-hand side is the expected density of monochromatic solutions in a random two-colouring of \mathbb{F}_q^n , and if f is the indicator function of a set A , then $\Lambda_L(f)$ is the density of solutions in A . Hence this definition corresponds to the intuitive definition given above. Any degenerate system can be easily reduced to the corresponding non-degenerate system, so we restrict our attention to non-degenerate systems throughout the note.

Consider a k -variable system L and a 4-variable system M (such as a 4-AP). We say that L *contains* M if there are coordinates $a, b, c, d \in [k]$, such that whenever $(x_1, \dots, x_k) \in (\mathbb{F}_q^n)^k$ is a solution to L , (x_a, x_b, x_c, x_d) is a solution to M . This is equivalent to saying that the equations for M (with relabelled variables) can be derived from L using elementary row operations. We can now state our first result, which confirms a conjecture of Saad and Wolf [7], when the system M is taken to be a 4-AP.

Theorem 1. *Let M be a non-degenerate 2×4 system. Any non-degenerate system containing M is uncommon.*

Even the fact that a four-variable system M itself is uncommon is a new result, and finding a function ψ which certifies that (for any M) is not straightforward. Indeed, previously it was only known that 4-APs are uncommon over \mathbb{F}_5 and \mathbb{Z}_N [5, 7], and the functions used there rely on the geometric structure of 4-APs. For a system L containing a 4-variable system M , we start with the above-mentioned function ψ , and turn it into a ‘uniform’ function using a trick due to Gowers [5], which in some sense isolates the contribution of the system M .

For our second result, we will introduce the notion of condensed equations of a system L , which turn out to be the crucial equations ‘forcing’ the uncommonness of L . A specific example can be found in Sect. 2.1. In reducing the properties of L to its condensed equations, we build on the key idea from [4], where a random function f is specified by sampling its Fourier coefficients.

Let L be an $m \times k$ matrix, which corresponds to an $m \times k$ system of m equations on k variables. We call a set $B \subseteq [k]$ *generic* if the matrix obtained from L by removing the columns corresponding to B has rank m . We define $s(L)$ to be the minimal order of a non-generic set. For example, when L is a 4-AP,

we have $s(L) = 3$ as all column sets of order two are generic. Our next theorem deals with the systems L with $s(L)$ even, for which we define a collection of *critical* sets

$$\mathcal{C}(L) = \{B \subseteq [k] : |B| = s(L) \text{ and } B \text{ is not generic}\}.$$

Note that for $B \in \mathcal{C}(L)$, the rank of the matrix obtained from L after removing the columns corresponding to B is $m - 1$. Hence there is a unique equation L_B (up to rescaling) derived from L by eliminating the variables x_i for $i \notin B$. This equation is called the *condensed equation* for B , denoted L_B . The following theorem describes a rather general class of uncommon systems L with even $s(L)$.

Theorem 2. *Let L be a system with $s(L)$ even. Suppose that for every set $B \in \mathcal{C}(L)$, the condensed equation L_B is uncommon. Then the system L is uncommon.*

Recall that a single equation L_B of even length is only common if its coefficients can be partitioned into pairs, each summing to zero. Thus in some sense, a ‘typical’ equation is uncommon, so we may say that a ‘typical’ system with $s(L)$ even is uncommon. The hypothesis that $s(L)$ is even is more than an artefact of our proofs, and is implicitly present in the results of [1, 4, 5].

2.1 A General Theorem and an Example

We will now describe our main theorem whose consequences are Theorem 1 and Theorem 2. For this purpose, we need to generalise our notion of condensed systems. Recall that $s(L)$ is the minimal order of a non-generic set B . We define a collection of sets

$$\mathcal{C}(L) = \begin{cases} \{B \subseteq [k] : |B| = s(L) \text{ and } B \text{ is not generic}\}, & \text{if } s(L) \text{ is even,} \\ \{B \subseteq [k] : |B| = s(L) + 1 \text{ and } B \text{ is not generic}\}, & \text{if } s(L) \text{ is odd} \end{cases}.$$

Each set $B \in \mathcal{C}(L)$ corresponds to a condensed system L_B consisting of one or two equations. We do not define L_B here, but it has the key property that any solution $(x_i : i \in B)$ to L_B extends to a solution to L .

Recall the definition of $\Lambda_L(f)$. Our main theorem reduces the uncommonness of an $m \times k$ system L to the ‘cumulative’ uncommonness of its condensed systems.

Theorem 3. *Let L be a non-degenerate $m \times k$ system over \mathbb{F}_q . L is uncommon whenever there is a positive integer n and a function $f : \mathbb{F}_q^n \rightarrow [-\frac{1}{2}, \frac{1}{2}]$ with $\mathbb{E}f = 0$ and*

$$\sum_{B \in \mathcal{C}(L)} \Lambda_{L_B}(f) < 0.$$

We finish with examples of common systems which will hopefully motivate further research and unveil some subtle phenomena. For instance, unlike in the single-equation case [4], the multiplicative structure of the field plays an important role in commonness.

Example 1. We consider a class $\mathcal{L}(q)$ consisting of 2×5 systems L over \mathbb{F}_q with $s(L) = 4$. (Note that $s(L) = 4$ is equivalent to the property that all 2×2 determinants of L are non-zero). In this case, we can also deduce the commonness of L from considering its condensed equations. Systems in $\mathcal{L}(q)$ have five critical sets $\mathcal{C}(L) = \{B \subset [5] : |B| = 4\}$ and five corresponding condensed equations.

1. Let M be the system whose matrix is

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 & -2 \end{pmatrix}.$$

The remaining condensed equations are

$$\begin{pmatrix} 0 & -1 & -2 & 1 & 2 \\ 2 & -3 & 0 & -1 & 2 \\ -1 & 0 & -3 & 2 & 2. \end{pmatrix}.$$

If $q \in \{5, 7\}$ the system is common as 3 can be written as -2 or -2^2 respectively, so the coefficients ‘align’ in a peculiar way for an application of Cauchy’s inequality. For $q > 7$, the system is uncommon.

2. The system L generated by the equations $2x_1 + x_2 + 3x_4 - 6x_5 = 0$ and $x_1 + 2x_2 + 3x_3 - 6x_5 = 0$ is common over all fields \mathbb{F}_q with $q \geq 5$. We suspect that there are no ‘similar’ systems L
3. If all five condensed equations of $L \in \mathcal{L}(q)$ are uncommon, the system is uncommon by Theorem 2. There is also an abundance of uncommon systems with one common condensed equation. One example is $x_1 + 3x_2 - x_3 - 3x_4 = 0$, $x_1 - 2x_2 - 3x_3 + 4x_5 = 0$.

3 Remarks and Open Problems

There are numerous avenues for further exploration. We select several of the problems which we find most interesting, and state only the simplest open case.

Systems with Many Uncommon Condensed Equations. Theorem 2 states that if $s(L)$ is even and all the condensed equations are uncommon, then L is uncommon. Our computational tests confirm the intuition that the conclusion holds even if the ‘majority’ of the condensed equations are uncommon. In the following conjecture, we propose such a class of two-equation systems.

Conjecture 1. For odd $k \geq 20$, any $2 \times k$ system L with $s(L) = k - 1$ is uncommon.

Partition Regularity for Linear Systems. For simplicity, we discuss systems with integer coefficients. A system L is *2-partition-regular* if any 2-colouring of \mathbb{Z} contains a monochromatic solution to L . The famous theorem of Rado characterises partition-regular systems for many colours, but 2-partition regularity seems to be much less understood. Two classes of known 2-partition regular systems are (i) single equations with at least three variables and (ii) translation-invariant systems [2].

Question 1. Is there a system L with $s(L) \geq 3$ which is not 2-partition-regular?

Commonness and Translation-Invariance. Translation-invariance is a sufficient condition for 2-partition regularity, but certainly not necessary (e.g. all one-equation systems of odd length are common). Still, it seems difficult to construct a larger common systems which is not translation-invariant.

Question 2. Is there a system with at least two equations which is common, but not translation-invariant?

How Uncommon Can An Equation Be? Even if a system is uncommon, it is still natural to enquire about the minimum density of monochromatic solutions. For single equations, this minimum density can be expressed as an apparently simple optimisation problem in terms of Fourier coefficients (see, e.g., Eq. (3) in [4]). This leads to the following question.

Question 3. Let \mathcal{L}_k be the collection of equations of length k over \mathbb{F}_q . What is the asymptotically minimal density of monochromatic solutions to L , over all colourings of \mathbb{F}_q^n and all $L \in \mathcal{L}_{2k}$?

Note that for odd k and all $L \in \mathcal{L}_k$, the density of monochromatic solutions depends only on the size of the colour classes. The analogous question for graphs has also been investigated [3].

Finally, many of the previous results have been generalised to the setting of arbitrary abelian groups [1, 8]. We have not attempted to extend our results in this direction.

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Exchange Properties of Finite Set-Systems

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Abstract. In a recent breakthrough, Adiprasito, Avvakumov, and Karasev constructed a triangulation of the n -dimensional real projective space with a subexponential number of vertices. They reduced the problem to finding a small downward closed set-system \mathcal{F} covering an n -element ground set which satisfies the following condition: For any two disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$, or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$. Denoting by $f(n)$ the smallest cardinality of such a family \mathcal{F} , they proved that $f(n) < 2^{O(\sqrt{n} \log n)}$, and they asked for a nontrivial lower bound. It turns out that the construction of Adiprasito *et al.* is not far from optimal; we show that $2^{(1.42+o(1))\sqrt{n}} \leq f(n) \leq 2^{(1+o(1))\sqrt{2n \log n}}$.

We also study a variant of the above problem, where the condition is strengthened by also requiring that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| > |B|$, there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$. In this case, we prove that the size of the smallest \mathcal{F} satisfying this stronger condition lies between $2^{\Omega(\sqrt{n} \log n)}$ and $2^{O(n \log \log n / \log n)}$.

Keywords: Extremal combinatorics · Set-systems · Exchange property

1 Introduction

It is an old problem to find a triangulation of the n -dimensional real projective space with as few vertices as possible. Recently, Adiprasito, Avvakumov, and Karasev [1] broke the exponential barrier by finding a construction of size $2^{O(\sqrt{n} \log n)}$. For the proof, they considered the following problem in extremal set theory.

What is the *minimum* cardinality of a system \mathcal{F} of subsets of $[n] = \{1, 2, \dots, n\}$, which satisfies three conditions:

1. \mathcal{F} is *atomic*, that is, $\emptyset \in \mathcal{F}$ and $\{a\} \in \mathcal{F}$ for every $a \in [n]$;
2. \mathcal{F} is *downward closed*, that is, if $A \in \mathcal{F}$, then $A' \in \mathcal{F}$ for every $A' \subset A$;
3. for any two disjoint members $A, B \in \mathcal{F} \setminus \{\emptyset\}$, there exist $a \in A$ and $b \in B$ such that

either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$,
or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$.

Letting $f(n)$ denote the minimum size of a set-system \mathcal{F} with the above three properties, Adiprasito *et al.* proved

$$f(n) \leq 2^{(1/2+o(1))\sqrt{n} \log n}, \tag{1}$$

where \log always denotes the base 2 logarithm. They used the following construction. Let $s, t > 0$ be integers, $n = st$. Fix a partition $[n] = X_1 \cup \dots \cup X_t$ of the ground set into t parts of equal size, $|X_1| = \dots = |X_t| = s$. Let

$$\mathcal{F} = \cup_{i=1}^t \mathcal{F}_i, \quad \text{where } \mathcal{F}_i = \{F \subseteq [n] : |F \cap X_j| \leq 1 \text{ for every } j \neq i\}, \tag{2}$$

for $1 \leq i \leq t$. (In the definition of \mathcal{F}_i , there is no restriction on the size of $F \cap X_i$.) It is easy to verify that \mathcal{F} meets the requirements. We have

$$|\mathcal{F}| = (t2^s - (s + 1)(t - 1))(s + 1)^{t-1} < 2^{s+t \log(s+1)+\log t}.$$

Substituting $s = (1/\sqrt{2} + o(1))\sqrt{n \log n}$ and $t = (\sqrt{2} + o(1))\sqrt{n/\log n}$, we obtain that

$$f(n) \leq 2^{(1/\sqrt{2}+o(1))\sqrt{n \log n}+(\sqrt{2}+o(1))\sqrt{n/\log n} \cdot \log \sqrt{n}} = 2^{(1+o(1))\sqrt{2n \log n}}. \tag{3}$$

This is slightly better than (1). (The authors of [1] remarked that their bound can be improved by a ‘‘subpolynomial factor.’’) Any further improvement on the upper bound would result in a smaller triangulation of the projective space.

Our first theorem implies that (3) is not far from optimal.

The *rank* of a set-system \mathcal{F} , denoted by $\text{rk}(\mathcal{F})$, is the size of the largest set $F \in \mathcal{F}$; see, e.g., [2].

We denote by $\lfloor x \rfloor$ the integer closest to x .

Theorem 1. *Let \mathcal{F} be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$, either there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$, or there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Then we have*

- (i) $|\mathcal{F}| \geq e^{(2e^{-1/\sqrt{2}}+o(1))\sqrt{n}} \geq 2^{(1.42+o(1))\sqrt{n}}$;
- (ii) $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$, and this bound is best possible.

It follows from part (i) that $f(n) \geq 2^{(1.42+o(1))\sqrt{n}}$.

We remark that the assumptions of Theorem 1 are weaker than those made by Adiprasito *et al.*, in two different ways: we do not require that \mathcal{F} is downward closed (which is their condition 2), and the exchange condition between two disjoint sets is also less restrictive than condition 3. Nevertheless, we know no significantly smaller set-systems satisfying these weaker conditions than the ones described in (2), for which $|\mathcal{F}| = 2^{(1+o(1))\sqrt{2n \log n}}$. Note that if we assume that \mathcal{F} is downward closed, then $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$ immediately implies that $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$. However, this is slightly weaker than the lower bound stated in part (i).

While part (ii) of Theorem 1 is tight, we suspect that part (i) and the lower bound $f(n) \geq 2^{\Omega(\sqrt{n})}$ can be improved. As a first step, we slightly strengthen the assumptions of Theorem 1, in order to obtain a better lower bound on $|\mathcal{F}|$.

Theorem 2. *Let \mathcal{F} be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$, either there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$, or there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Moreover, suppose that if $|A| < |B|$, then the second option is true. Then we have $|\mathcal{F}| \geq 2^{(1/2+o(1))\sqrt{n} \log n}$.*

This lower bound exceeds the upper bound in (3). Therefore, construction (2) cannot satisfy the stronger assumptions in Theorem 2. For example, set

$$A = \{a_1\} \cup \{a_2, a'_2\} \cup \{\emptyset\} \cup \dots \cup \{\emptyset\} \in \mathcal{F}_2 \subset \mathcal{F},$$

$$B = (X_1 \setminus \{a_1\}) \cup \emptyset \cup \{\emptyset\} \cup \dots \cup \{\emptyset\} \in \mathcal{F}_1 \subset \mathcal{F},$$

where $a_1 \in X_1$ and $a_2, a'_2 \in X_2$. If $s > 4$, then $|A| < |B|$, but there is no element of B that can be added to A such that the resulting set also belongs to \mathcal{F} . If $s \leq 4$, then the conditions of Theorem 2 are satisfied, but the construction is uninteresting, as $|\mathcal{F}| = 2^{\Theta(n)}$ and $\text{rk}(\mathcal{F}) = \Theta(n)$. A nontrivial construction is given here.

Theorem 3. *There exists an atomic downward closed set-system $\mathcal{F} \subset 2^{[n]}$ with the property that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| \leq |B|$, there is $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$, and*

- (i) $|\mathcal{F}| \leq 2^{(2+o(1))n \log \log n / \log n}$,
- (ii) $\text{rk}(\mathcal{F}) \leq (2 + o(1))n / \log n$.

The proofs of Theorems 1, 2, and 3 are presented in Sects. 2, 3, and 4, respectively.

2 Proof of Theorem 1

We start with a statement which immediately implies the inequality in part (ii).

Lemma 2.1. *Let $k \geq 1$ be an integer, $n > \binom{k}{2}$, and let \mathcal{F} be an atomic family of subsets of $[n]$ satisfying the condition in Theorem 1.*

Then we have $\text{rk}(\mathcal{F}) \geq k$. This bound cannot be improved.

Proof. By induction on k . For $k = 1$, the claim is trivial. Suppose that $k > 1$ and that the lemma has already been proved for $k - 1$.

Let $\mathcal{F} \subset 2^{[n]}$ be an atomic family, where $n > \binom{k+1}{2}$. By the induction hypothesis, there is a member $A \in \mathcal{F}$ of size at least k . If $|A| \geq k + 1$, we are done. Suppose that $|A| = k$, and consider the family $\mathcal{F}' = \{F \in \mathcal{F} : F \cap A = \emptyset\}$. Obviously, \mathcal{F}' is an atomic family on the ground set $[n] \setminus A$, and we have $|[n] \setminus A| > \binom{k+1}{2} - k = \binom{k}{2}$. Hence, we can apply the induction hypothesis to \mathcal{F}' to find a set $B \in \mathcal{F}'$ of size at least k which is disjoint from A . Using the exchange property in Theorem 1 for the sets A and B , at least one of them can

be enlarged to obtain a member of \mathcal{F} with at least $k + 1$ elements. Therefore, we have $\text{rk}(\mathcal{F}) \geq k + 1$, as required.

To show the tightness, let X_1, \dots, X_{k-1} be pairwise disjoint sets with $|X_i| = i$, for every i . Then $V = X_1 \cup \dots \cup X_{k-1}$ is a set of $\binom{k}{2}$ elements. For $i = 1, \dots, k - 1$, define

$$\mathcal{F}_i = \{F \subseteq V : |F \cap X_j| = 0 \text{ for every } j < i \text{ and } |F \cap X_j| \leq 1 \text{ for every } j > i\}. \tag{4}$$

In the definition of \mathcal{F}_i , there is no restriction on the size of $F \cap X_i$. Let $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{k-1}$. Obviously, every member of \mathcal{F}_i has at most $|X_i| + k - 1 - i = k - 1$ elements, which yields that $\text{rk}(\mathcal{F}) = \max_{i=1}^k \text{rk}(\mathcal{F}_i) = k - 1$. Furthermore, \mathcal{F} is atomic and any two disjoint members of \mathcal{F} satisfy the exchange condition in Theorem 1. Hence, the lemma is tight. \square

We remark that the maximal sets in the above \mathcal{F} form the same hypergraph as the one defined in Example 3 of [4] for $v = 1$.

To prove the inequality $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$ in part (ii) of Theorem 1, we have to find the largest k for which we can apply Lemma 2.1. It is easy to verify by direct computation that

$$\max\{k : \binom{k}{2} < n\} = \lfloor \sqrt{2n} \rfloor.$$

If $n = \binom{k}{2}$ for some $k \geq 1$, then the tightness of part (ii) of Theorem 1 follows from the tightness of Lemma 2.1. Suppose next that $\binom{k}{2} < n < \binom{k+1}{2}$. Let X_1, \dots, X_k be pairwise disjoint sets with $|X_i| = i$ for every $i < k$ and let $|X_k| = n - \binom{k}{2}$. Set $V = X_1 \cup \dots \cup X_k$. For $i = 1, \dots, k$, define \mathcal{F}_i as in (4), and let $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$. Then \mathcal{F} has the exchange property and $\text{rk}(\mathcal{F}) = k = \lfloor \sqrt{2n} \rfloor$. This proves part (ii) of Theorem 1.

It remains to establish part (i). Let \mathcal{F} be a family of subsets of $[n]$ satisfying the conditions. To each $F \in \mathcal{F}$ with $|F| \geq k$, assign a k -element subset $F' \subseteq F$. Let \mathcal{F}' denote the k -uniform hypergraph (i.e., family of k -element sets) consisting of all sets F' .

The independence number $\alpha(\mathcal{H})$ of a hypergraph \mathcal{H} is the maximum cardinality of a subset of its ground set which contains no element (hyperedge) of \mathcal{H} . It follows from Lemma 2.1 that any subset $S \subseteq [n]$ of size $|S| = \binom{k}{2} + 1$ contains at least one element of \mathcal{F} whose size is at least k . Therefore, any such set contains at least one element of \mathcal{F}' , which means that $\alpha(\mathcal{F}') \leq \binom{k}{2}$.

We need a result of Katona, Nemetz, and Simonovits [3] which is a generalization of Turán’s theorem to k -uniform hypergraphs.

Lemma 2.2 [3]. *Let \mathcal{H} be a k -uniform hypergraph on an n -element ground set. If the independence number of \mathcal{H} is at most α , then we have*

$$|\mathcal{H}| \geq \binom{n}{k} / \binom{\alpha}{k}.$$

Applying Lemma 2.2 to the hypergraph $\mathcal{H} = \mathcal{F}'$ with $k = (\sqrt{2}e^{-1/\sqrt{2}} + o(1))\sqrt{n}$ and $\alpha = \binom{k}{2}$, we obtain

$$|\mathcal{F}| \geq |\mathcal{F}'| \geq e^{(2e^{-1/\sqrt{2}} + o(1))\sqrt{n}} \geq 2^{(1.42 + o(1))\sqrt{n}},$$

completing the proof of part (i). This bound is slightly better than the inequality $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$, which immediately follows from part (ii), under the stronger assumption that \mathcal{F} is downward closed.

3 Proof of Theorem 2

Let \mathcal{F} be an atomic set-system on an n -element ground set X , where n is large, and let s and t be two positive integers to be specified later. We describe a procedure to identify $\sum_{i=0}^t s^i$ distinct members of \mathcal{F} . To explain this procedure, we fix an s -ary tree T of depth t . At the end, each of the s^t root-to-leaf paths in T will correspond to a unique member of \mathcal{F} .

Each *non-leaf vertex* v will be associated with an s -element subset $X(v) \subset X$ such that along every root-to-leaf path $p = v_0v_1 \dots v_t$, the sets $X(v_0), X(v_1), \dots, X(v_{t-1})$, associated with the root and with the internal vertices of p , will be pairwise disjoint. See Fig. 1 for an example.

Each *edge* $e = vu$ of T will be labelled with an element $x(e) \in X(v)$, in such a way that every edge from v to one of its s children gets a different label. Thus,

$$\{x(vu) : u \text{ is a child of } v\} = X(v).$$

Denoting the root by v_0 , we choose $X(v_0)$ to be an arbitrary s -element subset of the ground set X , and set $F(v_0) = \emptyset \in \mathcal{F}$. For any non-root vertex v , let

$$F(v) = \{x(e) : e \text{ is an edge along the root-to-}v \text{ path}\}.$$

We will choose $X(v)$ such that $F(v) \in \mathcal{F}$ for every v . All of the sets $F(v)$ will be distinct, as any two different paths starting from the root diverge somewhere, unless one contains the other.

Suppose that we have already determined the set $X(u)$ for all ancestors of some non-leaf vertex v at level $\ell < t$ of T . At this point, we already know the set $F(v) \in \mathcal{F}$, where $|F(v)| = \ell$, and we want to determine $X(v)$. The next lemma guarantees that there is a good choice for $X(v)$.

Lemma 3.1. *There is an s -element subset $X(v) \subset X$ such that for every $x \in X(v)$, we have $F(v) \cup \{x\} \in \mathcal{F}$.*

Proof. Let $Z = \cup\{X(u) : u \text{ lies on the root-to-}v \text{ path, } u \neq v\}$. If v is at level $\ell < t$, we have $|Z| = \ell s \leq (t - 1)s$. Let

$$Y = \{y \in X \setminus Z : F(v) \cup \{y\} \notin \mathcal{F}\}.$$

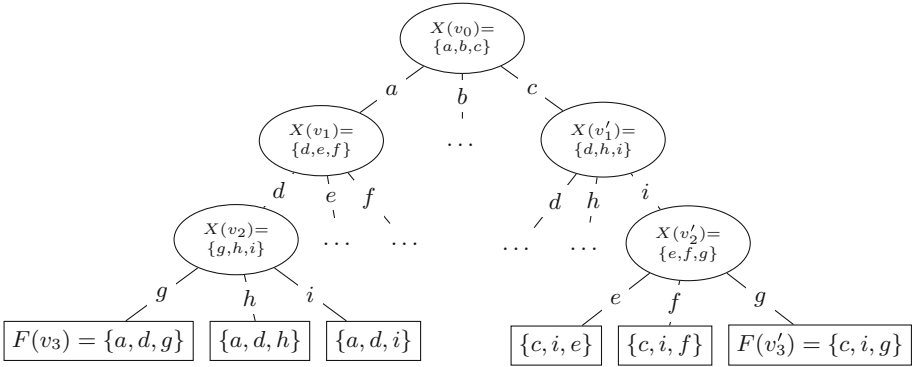


Fig. 1. Construction of the auxiliary tree T for the proof of Theorem 2.

In other words, Y consists of all elements of $X \setminus Z$ that cannot be added to $F(v)$ to obtain a set in \mathcal{F} .

Consider the atomic family $\mathcal{F}' = \{F \in \mathcal{F} : F \subseteq Y\}$. If $|Y| > \binom{t}{2}$, then Lemma 2.1 implies that $\text{rk}(\mathcal{F}') \geq t$. Thus, there is a set $B \in \mathcal{F}'$ with $|B| \geq t > |F(v)|$. In this case, we can apply the exchange condition in Theorem 2 to the sets $F(v)$ and B , to conclude that there exists $b \in B$ for which $F(v) \cup \{b\} \in \mathcal{F}$. However, this contradicts the fact that $b \in Y$.

Thus, we can assume that $|Y| \leq \binom{t}{2}$. Now we have

$$|(X \setminus Z) \setminus Y| \geq n - (t - 1)s - \binom{t}{2}.$$

If the right-hand side of this inequality is at least s , there is a proper choice for the set $X(v)$. For this, it is enough if $n \geq \frac{t^2}{2} + ts$, or, equivalently, $2 \geq (\frac{t}{\sqrt{n}})^2 + \frac{t}{\sqrt{n}} \frac{s}{\sqrt{n}}$. To achieve this, let n be large, $s = \lfloor \sqrt{n} / \log^2 n \rfloor$, and $t = \lfloor (1 - 1/\log n) \sqrt{2n} \rfloor$. \square

By the above procedure, we can recursively assign a different set $F(v) \in \mathcal{F}$ to each vertex v of T . This gives the desired

$$|\mathcal{F}| \geq \sum_{i=0}^t s^i \geq (n^{1/2} / \log^2 n)^{(1-o(1))\sqrt{2n}} = n^{\sqrt{(1/2+o(1))n}}.$$

4 Proof of Theorem 3

Assume for simplicity that n is a multiple of k , and fix a partition $[n] = X_1 \cup \dots \cup X_{n/k}$ into n/k parts, each of size k . That is, let $|X_1| = \dots = |X_{n/k}| = k$, where k is the largest number for which $2^{k-2} \leq n/k$; this gives $k = (1 + o(1)) \log n$. We will also assume $n, k \geq 3$.

For any $A \subseteq [n]$ and $0 \leq i \leq k$, let $p_A(i)$ and $s_A(i)$ denote the number of parts X_t which intersect A in *precisely* i elements and in *at least* i elements, respectively. Thus, we have $s_A(i) = \sum_{j=i}^k p_A(j)$ and $|A| = \sum_{i=1}^k ip_A(i) = \sum_{i=1}^k s_A(i)$. Define the *profile vector* of A , as

$$p_A = (p_A(k), p_A(k-1), \dots, p_A(0)),$$

and let

$$s_A = (s_A(k), s_A(k-1), \dots, s_A(0)).$$

That is, $p_A(0)$ is the number of parts that are disjoint from A , while $s_A(0)$ is always equal to n/k . We claim that the set-system

$$\mathcal{F} = \{A \subseteq [n] : s_A(k) \leq 1 \text{ and } s_A(i) \leq 2^{k-1-i} \text{ for every } 2 \leq i \leq k-1\}$$

meets the requirements of the theorem. Notice that for $i = 0$ and $i = 1$, there is no restriction on $s_A(i)$ other than the trivial bounds $0 \leq s_A(i) \leq n/k$. In particular, if A is an element of \mathcal{F} with maximum cardinality, we have

$$\begin{aligned} s_A &= (1, 1, 2^1, 2^2, 2^3, \dots, 2^{k-3}, n/k, n/k), \text{ and} \\ p_A &= (1, 0, 1, 2^1, 2^2, \dots, 2^{k-4}, n/k - 2^{k-3}, 0). \end{aligned}$$

Here, we used that $2^{k-3} \leq n/k$, by assumption. Thus, the size of such a largest set is

$$\text{rk}(\mathcal{F}) = |A| = \sum_{i=1}^k s_A(i) = n/k + \sum_{i=1}^{k-1} 2^{k-i-1} + 1 = n/k + 2^{k-2} \leq 2n/k = (2 + o(1))n/\log n.$$

This also gives the following simple bound on the size of the family.

$$|\mathcal{F}| \leq \sum_{i=0}^{\text{rk}(\mathcal{F})} \binom{n}{i} \leq \sum_{i=0}^{2n/k} \binom{n}{i} \leq \left(\frac{en}{2n/k}\right)^{2n/k} \leq \left(\frac{e \log n}{2}\right)^{(2+o(1))n/\log n} \leq 2^{(2+o(1))n \log \log n / \log n}.$$

The proof that \mathcal{F} meets the requirements of the theorem can be found in the full version of this paper.

5 Concluding Remarks

If we strengthen the condition of our results by requiring that for any two non-empty disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \in \mathcal{F}$ both hold, then the problem becomes trivial. Any atomic set-system $\mathcal{F} \subseteq 2^{[n]}$ with this property must contain all subsets of $[n]$. Indeed, every set $F = \{x_1, \dots, x_k\}$ can be built up, sequentially applying the condition to the sets $\{x_1, \dots, x_i\}$ and $\{x_{i+1}\}$, for $i = 1, \dots, k-1$.

In Theorems 1 and 2, we only assume that \mathcal{F} is atomic. However, our best constructions have the stronger property that \mathcal{F} is downward closed. Could we substantially strengthen these results under the stronger assumption? The proof

of the bound $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$, which is only slightly weaker than Theorem 1 (i), becomes much easier if we assume that \mathcal{F} is downward closed, and the proof of Theorem 2 can also be simplified if \mathcal{F} is downward closed.

The property of the set-system described in Theorems 2 and 3 is reminiscent of the *independent set exchange property* of matroids; see [5]. A common generalization of these two properties would be to require that for any two members $A, B \in \mathcal{F}$, if either $|A| = |B|$ and $A \cap B = \emptyset$, or $|A| < |B|$ (but they are not necessarily disjoint), then there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. A downward closed set-system \mathcal{F} has this property if and only if \mathcal{F} is the family of independent sets in a matroid in which no subspace has two disjoint generators A and B , i.e., $A \cap B = \emptyset$ and $\text{rk}(A) = \text{rk}(B) = \text{rk}(A \cup B)$ is forbidden. We do not know whether this question has been studied before.

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On a Question of Vera T. Sós About Size Forcing of Graphons

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Abstract. The k -sample $\mathbb{G}(k, W)$ from a graphon $W : [0, 1]^2 \rightarrow [0, 1]$ is the random graph on $\{1, \dots, k\}$, where we sample $x_1, \dots, x_k \in [0, 1]$ uniformly at random and make each pair $\{i, j\} \subseteq \{1, \dots, k\}$ an edge with probability $W(x_i, x_j)$, with all these choices being mutually independent. Let the random variable $X_k(W)$ be the number of edges in $\mathbb{G}(k, W)$.

Vera T. Sós asked in 2012 whether two graphons U, W are necessarily weakly isomorphic provided the random variables $X_k(U)$ and $X_k(W)$ have the same distribution for every integer $k \geq 2$. This question when one of the graphons W is a constant function was answered positively by Endre Csóka and independently by Jacob Fox, Tomasz Łuczak and Vera T. Sós. Here we investigate the question when W is a 2-step graphon and prove that the answer is positive for a 3-dimensional family of such graphons.

We also present some related results.

Keywords: Graphons · Weak isomorphism · Sample

1 Introduction

The k -sample $\mathbb{G}(k, W)$ from a graphon W (i.e. a measurable symmetric function $[0, 1]^2 \rightarrow [0, 1]$) is the random graph on $[k] := \{1, \dots, k\}$ obtained by sampling $x_1, \dots, x_k \in [0, 1]$ uniformly at random and making each pair $\{i, j\} \subseteq [k]$ an edge with probability $W(x_i, x_j)$, with all these choices being mutually independent. The (*homomorphism*) *density* $t(F, W)$ of a graph F on $[k]$ in W is the probability that $E(F) \subseteq E(\mathbb{G}(k, W))$, that is, every adjacent pair in F is also adjacent in $\mathbb{G}(k, W)$. Equivalently, $t(F, W) := \int_{[0, 1]^k} \prod_{\{i, j\} \in E(F)} W(x_i, x_j) dx_1 \dots dx_k$. Let us call two graphons U and W *weakly isomorphic* if the random graphs

Oliver Cooley and Mihyun Kang were supported by Austrian Science Fund (FWF) Grant I3747.

Oleg Pikhurko was supported by Leverhulme Research Project Grant RPG-2018-424.

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J. Nešetřil et al. (Eds.): *Extended Abstracts EuroComb 2021*, TM 14, pp. 625–630, 2021.
https://doi.org/10.1007/978-3-030-83823-2_100

$\mathbb{G}(k, U)$ and $\mathbb{G}(k, W)$ have the same distribution for every $k \in \mathbb{N}$. This is equivalent to $t(H, U) = t(H, W)$ for every connected graph H . Borgs, Chayes and Lovász [1] showed that all graphons in the weak isomorphism class of W can, roughly speaking, be obtained from W by applying measure-preserving transformations of the variables.

A *graphon parameter* f is a function that assigns to each graphon W a real number or a real vector $f(W)$ such that $f(W) = f(U)$ whenever U and W are weakly isomorphic. We say that a family $(f_i)_{i \in I}$ of graphon parameters *forces* a graphon W if every graphon U with $f_i(U) = f_i(W)$ for every $i \in I$ is weakly isomorphic to W . For example, the famous result of Chung, Graham and Wilson [2] on p -quasirandom graphs states, in this language, that the constant- p graphon is forced by $t(K_2, \cdot)$ and $t(C_4, \cdot)$, i.e. by the edge and 4-cycle densities.

Call a family $(f_i)_{i \in I}$ of graphon parameters *forcing* if it forces every graphon W . For example, the densities $t(F, \cdot)$, where F ranges over all connected graphs, form a forcing family. The authors are not aware of any results where a substantially smaller set of parameters than the densities of all connected graphs is shown to be forcing. Vera T. Sós [9] posed some questions in this direction, and in particular considered the following problem. For a graphon W and an integer $k \in \mathbb{N}$, let $X_k(W) := |E(\mathbb{G}(k, W))|$ be the size of, i.e. number of edges in, the k -sample $\mathbb{G}(k, W)$ from W . We identify the random variable $X_k(W)$ with the vector of probabilities $\mathbb{P}(X_k(W) = i)$ for $0 \leq i \leq \binom{k}{2}$, viewing it as a graphon parameter. Let \mathcal{W}_S be the family of graphons W that are forced by the sequence $(X_k(W))_{k \in \mathbb{N}}$, i.e. by the distributions of sizes of samples from W .

Question 1 (Size Forcing Question (Sós [9])). *Is every graphon in \mathcal{W}_S ?*

Alon (unpublished, see [4]) and independently Sliacan [8] proved that the constant- $\frac{1}{2}$ graphon is in the family \mathcal{W}_S . Then Csóka [4] and independently Fox, Łuczak and Sós [5] proved that constant- p graphon is in the family \mathcal{W}_S for any $p \in (0, 1)$. A natural next step would be to try to determine whether $W \in \mathcal{W}_S$ when W is a *2-step graphon*, i.e. we have a partition of $[0, 1]$ into measurable sets A and B such that W is constant on each of the sets A^2, B^2 and $(A \times B) \cup (B \times A)$. We need four parameters to describe a 2-step graphon: the measure of A as well as the three possible values of W . We can prove that $W \in \mathcal{W}_S$ for the following 3-dimensional subset of 2-step graphons.

Theorem 1. *Let W be the 2-step graphon with parts $A := [0, a]$ and $B := [a, 1]$ such that its values on $A^2, (A \times B) \cup (B \times A)$ and B^2 are respectively $0, p \in (0, 1]$ and $q \in (0, 1]$. If $(1 - a)q \leq (1 - 2a)p$, then $W \in \mathcal{W}_S$.*

We can also answer Question 1 for some other families of 2-step graphons. We present two further examples (Theorems 2 and 3) where a *finite* set of some natural real-valued parameters suffices. The first is motivated by the result of Csóka [4] who in fact proved that the constant- p graphon is forced by X_4 alone.

Theorem 2. *Let $p \in [0, 1]$ and let W be the graphon which is 0 on $[0, 1/2]^2 \cup [1/2, 1]^2$ and p everywhere else. Then W is forced by X_5 alone.*

Let the *independence ratio* $\alpha(W)$ of a graphon W be the supremum of the measure of $A \subseteq [0, 1]$ such that $W(x, y) = 0$ for a.e. $(x, y) \in A^2$. As was observed by Hladký, Hu and Piguet [6, Lemma 2.4], the supremum is in fact a maximum (that is, it is attained by some A). Also, the *clique ratio* $\omega(W) := \alpha(1 - W)$ is the maximum measure of $A \subseteq [0, 1]$ with W being 1 a.e. on A^2 .

Theorem 3. *Given $a, p \in [0, 1]$, set $A := [0, a)$ and $B := [a, 1]$, and let W be the graphon which is 0 on A^2 , 1 on B^2 , and p everywhere else. Then W is forced by (α, ω, X_4) .*

By using a basic version of the container method, we show that the value of α (and thus of ω) is determined by any infinite subsequence of $(X_k)_{k \in \mathbb{N}}$. More precisely, the following holds.

Theorem 4. $\alpha(W) = \lim_{k \rightarrow \infty} (\mathbb{P}(X_k(W) = 0))^{1/k}$ for every graphon W .

We note that Theorem 4, by relating $\alpha(W)$ to graph densities, fills one missing entry in [7, Table 1].

By combining Theorems 3 and 4, we directly obtain the following result.

Corollary 1. *Let W be a graphon as in Theorem 3 (that is, W is 0 on $[0, a)^2$, 1 on $[a, 1]^2$, and p everywhere else). Then $W \in \mathcal{W}_S$. \square*

Call a family \mathcal{F} of graphs *forcing* if the corresponding family of parameters $(t(F, \cdot))_{F \in \mathcal{F}}$ is forcing. Sós [9] also asked if one can find substantially smaller forcing families than taking all connected graphs. We show that two natural examples, namely the family of all cycles and the family of all complete bipartite graphs, do not suffice.

Proposition 1. (i) *The family of all connected graphs with at most one cycle is not forcing. In particular, the family of all cycles is not forcing.*
(ii) *For every integer d , the family of all graphs of diameter at most d is not forcing. In particular, the family of all complete bipartite graphs is not forcing.*

Full proofs of all the results stated in this extended abstract can be found in [3].

2 Some Auxiliary Results

We first present some known or easy auxiliary results that we need for the proofs of the main results. For graphs H_1, H_2 , let $H_1 \sqcup H_2$ denote their *disjoint union*. Let $\mathcal{G}_{k,m}$ consist of isomorphism classes of all graphs with at most k vertices and exactly m edges that do not contain any isolated vertices. For example, $\mathcal{G}_{5,3} = \{K_3, P_4, P_3 \sqcup K_2, K_{1,3}\}$.

Lemma 1. (i) *For any graphon W and for any graphs H_1 and H_2 , we have $t(H_1 \sqcup H_2, W) = t(H_1, W)t(H_2, W)$.*
(ii) *If W is a p -regular graphon and F' is obtained from a graph F by attaching a pendant edge then $t(F', W) = pt(F, W)$.*

The following result implicitly appears in Csóka [4] (see also [3, Lemma 12]). Let $(r)_k := r(r - 1) \dots (r - k + 1)$ denote the falling factorial.

Lemma 2. *Let integers k and m satisfy $1 \leq m \leq \binom{k}{2}$. Then for every graphon W we have*

$$\mathbb{E}((X_k(W))_m) = \sum_{F \in \mathcal{G}_{k,m}} c_{k,F} t(F, W),$$

where $c_{k,F} > 0$ is the number of graphs on $[k]$ that, after discarding isolated vertices, are isomorphic to F .

A useful consequence, which we will apply frequently, is that if two graphons U, W have k -samples $X_k(U), X_k(W)$ with identical distributions, and if $t(F, U) = t(F, W)$ for all $F \in \mathcal{G}_{k,m}$ except for some F_0 , then the F_0 -densities are also equal.

We will also need the following bipartite analogue of the Chung-Graham-Wilson Theorem, which can be proved either by passing to finite graphs converging to U (and adapting the original proof of Chung, Graham and Wilson [2]) or, using analytic methods, by dealing directly with graphons (see [3, Lemma 14]).

Lemma 3. *Let A and B be sets of measure a and b respectively that partition $[0, 1]$. (Thus $a + b = 1$.) Let $p \in [0, 1]$. Let U be a graphon taking value 0 on $A^2 \cup B^2$ such that $t(K_2, U) = 2abp$ and $t(C_4, U) = 2a^2b^2p^4$. Then $U(x, y) = p$ for a.e. $(x, y) \in (A \times B) \cup (B \times A)$.*

The following result can be proved using the container method (see [3, Theorem 15]). Let $\mathcal{I}(G)$ denote the family of all independent sets in a graph G and let $\mathcal{I}_k(G) := \{I \in \mathcal{I}(G) : |I| = k\}$ consist of all independent sets of size k .

Theorem 5. *For every $\delta > 0$ there exists $\varepsilon > 0$ such that for any $k \geq 1/\varepsilon$ there exists n_0 such that for every graph G on $n \geq n_0$ vertices and every real α , if $|\mathcal{I}_k(G)| \geq (\alpha - \varepsilon)^k \binom{n}{k}$, then there exists $A \subseteq V(G)$ with $|A| \geq (\alpha - \delta)n$ and $e(G[A]) \leq \delta n^2$.*

3 Proof Outlines of Main Results

Proof of Theorem 2. Let U be an arbitrary graphon such that the distribution of $X_5(U)$ is the same as the distribution of $X_5(W)$. Let us denote this common distribution by X_5 . The aim is to successively prove the following properties of U ; each step is individually relatively easy to prove given the previous properties.

- U is $(p/2)$ -regular, i.e. $\deg^U(x) := \int_0^1 U(x, y) dy = p/2$ for a.e. $x \in [0, 1]$.
- If $t(H, U) = t(H, W)$ for some graph H , then $t(H', U) = t(H', W)$ for any graph H' that is obtained from H by adding a pendant edge.
- $t(H, U) = t(H, W)$, where H is any one of $K_{1,3}, P_4, P_2 \sqcup K_2, K_3, C_4, C_4$ with a pendant edge, $C_5, K_{2,3}$.
- Let the random variable Z be $\text{codeg}^U(x, y) := \int_0^1 U(x, z)U(z, y) dz$, the density of copies of P_2 which have x, y as endpoints, where x and y are chosen uniformly and independently from $[0, 1]$. Then $\mathbb{P}(Z = 0) = \mathbb{P}(Z = p^2/2) = \frac{1}{2}$.
- Let C consist of those $(x, y) \in [0, 1]^2$ for which $\text{codeg}^U(x, y) = p^2/4$ and let $\deg_C(x)$ denote the measure of $N_C(x) := \{y : (x, y) \in C\}$, for $x \in [0, 1]$. Then $\deg_C(x) = 1/2$ for a.e. $x \in [0, 1]$.
- For a.e. $x \in [0, 1]$, the set $N_C(x)$ is independent in U .
- U is weakly isomorphic to W .

Proof of Theorem 4. For $k \in \mathbb{N}$, let $\alpha_k(W) := \mathbb{P}(X_k(W) = 0)$. It is easy to see that the limit $\alpha_\infty(W) := \lim_{k \rightarrow \infty} (\alpha_k(W))^{1/k}$ exists. Clearly, $\alpha_\infty(W)$ remains the same if we replace W by any weakly isomorphic graphon.

The inequality $\alpha_\infty(W) \geq \alpha(W)$ is easy to prove by picking an independent set $A \subseteq [0, 1]$ in W of measure $\lambda(A) = \alpha(W)$ (which exists by [6, Lemma 2.4]) and observing that $\Pr(X_k(W) = 0) \geq \lambda(A)^k$.

To show the converse inequality, we pick sufficiently large $k \ll n$ (so in particular $\alpha_k(W) \approx \alpha_\infty(W)$) and let $G \sim \mathbb{G}(n, W)$ be the n -sample from W . We consider the step graphon W_G encoding the adjacency relation in G . In an appropriate sense, a typical outcome W_G is “close” to W , and therefore it suffices to show that G contains an almost independent set of size close to $\alpha_k(W) \cdot n$, which will then transfer to an independent set in W of size close to $\alpha_k(W)$. The existence of this almost independent set can be proved by applying Theorem 5.

Proof of Theorem 1. Let U be an arbitrary graphon such that for every $k \in \mathbb{N}$ the distributions of $X_k(U)$ and $X_k(W)$ are the same; let us denote this random variable by X_k .

By Theorem 4, U contains an independent set of measure a , which we may assume is $A = [0, a)$.

We next claim that for almost every $x \in B$, we have $\text{deg}_A^U(x) \geq ap$. This can be proved by contradiction: if there is a set $B' \subset B$ of measure at least ε such that each point in B' has A -degree at most $ap - \varepsilon$, then some careful calculation shows that $\alpha_k(U) \geq \alpha_k(W)$ for sufficiently large k , which is a contradiction.

Let U' be the graphon obtained from U by averaging it over $(A \times B) \cup (B \times A)$ and over B^2 . A further averaging argument considering the density of P_3 and applying the Cauchy-Schwarz inequality shows that in fact $U' = W$.

Next, we show that for almost every $(x, y) \in B^2$, we have that $\text{codeg}_A^U(x, y) = ap^2$ where we denote $\text{codeg}_A^U(x, y) := \int_A U(x, z)U(y, z) dz$. If this were not true, then some careful calculation shows that $\mathbb{P}(X_k(U) \leq 1) > \mathbb{P}(X_k(W) \leq 1)$ for sufficiently large k , which is a contradiction.

We next deduce that the triangle densities in U, W are identical even if we specify which vertices are in A and B , and that the same is true for triangles with a pendant edge. It follows from Lemma 2 that U and W have the same density of 4-cycles. Since due to codegree considerations U and W have the same densities of $ABAB$ -cycles, Lemma 3 implies that U is constant p on $A \times B$. Finally, since we know the densities of all types of 4-cycles except those lying inside B , we also know the density of these 4-cycles, and the Chung-Graham-Wilson Theorem implies that U is also constant on B , so $U = W$.

Proof of Theorem 3. There are subsets $C, D \subseteq [0, 1]$ of measures a and $1 - a$ respectively such that U is 0 on C^2 a.e. and U is 1 on D^2 a.e., and we may assume that $C = A$ and $D = B$.

We first show that $\text{deg}_B^U(x) = (1 - a)p$ for a.e. $x \in A$ by showing that, viewed as a random variable when x is chosen uniformly at random from A , the first and third moments are $(1 - a)p$ and $((1 - a)p)^3$ respectively, which is only possible if

the random variable is constant $(1 - a)p$ a.e. on A . Similarly we can show that $\text{deg}_A^U(x) = ap$ for a.e. $x \in B$.

Let K_4^- be the 4-clique minus an edge, the unique graph on 4 vertices with 5 edges. A 4-sample from U can only form a K_4^- if it has either two or three vertices in B . Since we know the density of the second type, and the densities of K_4^- in U and W are identical, we can also deduce the density of the second type, which exactly matches the density of $ABAB$ -cycles in both U and W . It follows that U is constant a.e. on $A \times B$, and therefore weakly isomorphic to W .

Proof of Proposition 1. To prove (i), take the unit vectors

$$\mathbf{x}_1 := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3 := \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix},$$

let $\varepsilon := 1/4$ and set

$$A := \mathbf{x}_1 \mathbf{x}_1^T + \varepsilon \mathbf{x}_2 \mathbf{x}_2^T \quad \text{and} \quad A' := \mathbf{x}_1 \mathbf{x}_1^T + \varepsilon \mathbf{x}_3 \mathbf{x}_3^T.$$

Let W and W' be the 3-step graphons, with steps of measure $1/3$, whose values are given by the symmetric matrices $A, A' \in [0, 1]^{3 \times 3}$. It is simple to calculate that W, W' have the same densities of k -cycles for every k , and indeed the same densities of all unicyclic graphs, but are not weakly isomorphic since their limiting density of K_k as $k \rightarrow \infty$ is different.

To prove (ii), let $G := P_{d+2} \sqcup P_{d+2}$ and $G' := P_{d+3} \sqcup P_{d+1}$, and let W, W' be the step graphons with $2d + 4$ steps of equal measure encoding their adjacency relations. They are not weakly isomorphic because the induced density of P_{d+3} is zero in W but not in W' , but their densities of any graph of diameter at most d are identical, so this family is not forcing.

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On Deeply Critical Oriented Cliques

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Abstract. In this work we consider arc criticality in colourings of oriented graphs. We study deeply critical oriented graphs, those graphs for which the removal of any arc results in a decrease of the oriented chromatic number by 2. We prove the existence of deeply critical oriented cliques of every odd order $n \geq 9$, closing an open question posed by Borodin et al. [*Journal of Combinatorial Theory, Series B*, 81(1):150–155, 2001]. Additionally, we prove the non-existence of deeply critical oriented cliques among the family of circulant oriented cliques of even order.

Keywords: Oriented graph · Oriented chromatic number · Deeply critical graphs · Circulant graphs

1 Introduction and the Main Results

In 1994, Courcelle [3] defined oriented colouring as part of his seminal work on the monadic second order logic of graphs in which he established the illustrious Courcelle’s Theorem [4]. In the years following, oriented colouring and the oriented chromatic number gained popularity and developed into an independent field of research. We refer the reader to Sopena’s updated survey [9] for a broad overview of the state of the art.

An *oriented graph* \vec{G} is a directed graph without any directed cycle of length one or two. That is, it is a directed graph that is irreflexive and anti-symmetric. We denote the set of vertices and arcs of an oriented graph \vec{G} by $V(\vec{G})$ and $A(\vec{G})$, respectively. Its underlying simple graph is denoted by G .

By generalizing to oriented graphs the interpretation of graph colouring as homomorphism to a complete graph, one arrives at the following definition of oriented graph colouring.

An *oriented k -colouring* of an oriented graph \vec{G} is a function $\phi : V(\vec{G}) \rightarrow \{1, 2, \dots, k\}$ so that

S. Sen—Research partially supported by IFCAM MA/IFCAM/18/39 and SRG/2020/001575.

- (i) $\phi(x) \neq \phi(y)$ for all $xy \in A(\vec{G})$; and
- (ii) for all $xy, uv \in A(\vec{G})$, if $\phi(x) = \phi(v)$, then $\phi(y) \neq \phi(u)$.

The vertices of the target of homomorphism correspond to the colours. Condition (i) ensures that the target of the homomorphism is irreflexive. Condition (ii) ensures that the target of the homomorphism is anti-symmetric.

When $y = u$, condition (ii) implies vertices connected by a directed path of length 2 (i.e., a 2-dipath) must receive different colours.

The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is the minimum k for which \vec{G} admits an oriented k -colouring.

A major theme in oriented colourings research is the study of analogous versions of graph colouring concepts for oriented graphs. In 2004 Klostermeyer and MacGillivray [7] generalized the notion of clique to oriented colouring, studying those oriented graphs for which $\chi_o(\vec{G}) = |V(\vec{G})|$. This work was continued by Nandi, Sen, and Sopena [8]. Such oriented graphs are called *absolute oriented cliques* and admit the following classification.

Theorem 1 (Klostermeyer and MacGillivray). *An oriented graph is an absolute oriented clique if and only if every pair of non-adjacent vertices are connected by a 2-dipath.*

In 2001 Borodin et al. [1], extended the notion of arc criticality for graph colouring to oriented colourings. Notably they gave examples of oriented graphs for which the removal of any arc decreases the oriented chromatic number by 2, the maximum possible. Formally, a *deeply critical oriented graph* is an oriented graph \vec{G} for which

$$\chi_o(\vec{G} - xy) = \chi_o(\vec{G}) - 2$$

for each arc $xy \in A(\vec{G})$.

Borodin et al. [1] gave an infinite family of deeply critical oriented graphs that were also absolute oriented cliques. For convenience, we refer to such an oriented graph as a *deeply critical oriented clique*. By way of example, we invite the reader to verify that the directed cycle on 5 vertices is a deeply critical oriented clique.

Theorem 2 (Borodin et al. [1]). *There exists a deeply critical oriented clique of order n for every $n = 2 \cdot 3^m - 1$, where $m \geq 1$.*

In their work Borodin et al. [1] speculated the existence of deeply critical oriented cliques of odd order n for all $n \geq 33$ and left it open. We close this long-standing open problem by proving the following result.

Theorem 3. *Let n be a positive odd integer. There exists a deeply critical oriented clique of order n , if and only if $n \geq 5$, and $n \neq 7$.*

We outline the proof of this theorem in Sect. 2.

A curious aspect of the study of deeply critical oriented cliques is a lack of examples of such oriented graphs of even order, despite intensive computer search. We conjecture such deeply critical oriented cliques not to exist.

Conjecture 4. *There exists a deeply critical oriented clique of order n , if and only if n is odd, $n \geq 5$, and $n \neq 7$.*

Let n be an integer and let $S \subseteq \mathbb{Z}_n$ so that for all $k \in \mathbb{Z}_n$ if $k \in S$, then $-k \notin S$. Recall that the *oriented circulant graph* $\vec{C}(n, S)$ is the oriented graph with vertex set \mathbb{Z}_n so that $ij \in A(\vec{C}(n, S))$ when $j - i$ is congruent modulo n to an element of S .

We provide further evidence towards Conjecture 4 by proving no deeply critical oriented clique appears among the family of oriented circulant graphs of even order.

Theorem 5. *There does not exist any deeply critical oriented circulant clique of even order.*

Our work proceeds as follows. We provide outlines of the proofs of Theorems 3 and 5 in Sects. 2 and 3, respectively. In the former section we provide a method to construct a deeply critical oriented clique for any odd integer $n \geq 5$, exclusive of $n = 7$. In the latter section we give a full classification of deeply critical oriented circulant cliques. We provide concluding remarks and suggestions for future work in Sect. 4. We refer the reader to [2] for definitions of standard graph theoretic terminology and notation not defined herein. For full proofs of the results in Sect. 2 and 3 see [5].

2 Proof of Theorem 3

We begin by defining the following terms and notations. Let \vec{G} be an oriented graph. An *extending partition* of \vec{G} is a partition of its set of vertices $V(\vec{G}) = X_1 \sqcup X_2 \sqcup X_3$ so that

- (i) there is no arc from a vertex of X_{i+1} to a vertex of X_i , for all $i \in \{1, 2, 3\}$;
- (ii) for each $u \in X_i$, there exists a vertex $v \in X_{i+1}$ such that $N^-(v) \cap X_i = \{u\}$, for all $i \in \{1, 2, 3\}$; and
- (iii) for each $v \in X_{i+1}$, there exists a vertex $u \in X_i$ such that $N^+(u) \cap X_{i+1} = \{v\}$, for all $i \in \{1, 2, 3\}$,

where addition in indices is taken modulo 3.

We say an oriented graph is *extendable* when it admits an extending partition. For such graphs we define the following supergraphs. Let \vec{G} be an oriented graph with extending partition $V(\vec{G}) = X_1 \sqcup X_2 \sqcup X_3$. The *6-extension* of \vec{G} is the graph \vec{G}_6 constructed from \vec{G} as follows (see Fig. 1):

- Include six new vertices $x_1^-, x_1^+, x_2^-, x_2^+, x_3^-, x_3^+$ to the graph \vec{G} , and add the arcs $x_1^-x_2^+, x_1^+x_2^-, x_2^-x_3^+, x_2^+x_3^-, x_3^-x_1^+,$ and $x_3^+x_1^-$.
- Add all the arcs of the form x_1^-x and xx_1^+ for all $x \in X_1$.
- Add all the arcs of the form x_2^-x and xx_2^+ for all $x \in X_2$.
- Add all the arcs of the form x_3^-x and xx_3^+ for all $x \in X_3$.

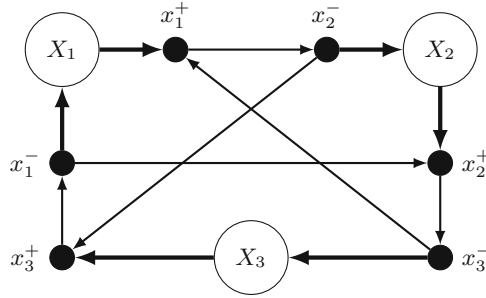


Fig. 1. Construction of \vec{G}_6 . Thickened arcs indicate the existence of all possible arcs between the vertex and the set X_i . Arcs between sets of vertices are not shown.

Following the definition of 6-extension, we define two further extensions of \vec{G} , which arise as induced subgraphs of \vec{G}_6 . The 4-extension of \vec{G} is the graph obtained by deleting the vertices x_1^+ and x_2^- from \vec{G}_6 . The 2-extension of \vec{G} is the graph obtained by deleting the vertices x_1^-, x_2^+, x_3^- and x_3^+ from \vec{G}_6 .

Lemma 1. *Let \vec{G} be an extendable deeply critical oriented clique. The 2-extension, 4-extension, and the 6-extension of \vec{G} are deeply critical oriented cliques.*

Lemma 1 implies that given an oriented deeply critical oriented clique on n vertices, one may construct deeply critical oriented cliques on $n + 2$, $n + 4$ and $n + 6$ vertices. We note, however that computer search yields many examples of deeply critical oriented cliques that do not arise as an extension of a smaller deeply critical oriented clique. Figure 2 gives such an example. Curiously, though generated by computer search, the oriented graph in Fig. 2 does arise as a 6-extension of a directed three cycle.

Given \vec{G} , a deeply critical oriented clique with extending partition $V(\vec{G}) = X_1 \sqcup X_2 \sqcup X_3$, one may verify $V(\vec{G}_6) = X'_1 \sqcup X'_2 \sqcup X'_3$ where $X'_i = X_i \cup \{x_i^-, x_i^+\}$ for all $i \in \{1, 2, 3\}$ is an extending partition of \vec{G}_6 .

Lemma 2. *The 6-extension of an extendable deeply critical oriented clique is extendable.*

With these two lemmas in place, we provide a proof of Theorem 3.

Proof (Theorem 3). The directed cycle on 5 vertices is a deeply critical oriented clique. By computer search there is no deeply critical oriented clique on 7 vertices.

The oriented graph given in Fig. 2 is a deeply critical oriented clique with 9 vertices. This oriented graph admits the following extending partition: $X_1 = \{6, 2, 7\}$, $X_2 = \{1, 5, 0\}$, $X_3 = \{4, 8, 3\}$. The result now follows inductively from the following observation: by Lemmas 1 and 2, if \vec{H} is an extendable deeply critical oriented clique with n vertices, then there exists deeply critical oriented clique on $n + 2$ and $n + 4$ vertices, and an extendable deeply critical oriented clique on $n + 6$ vertices. □

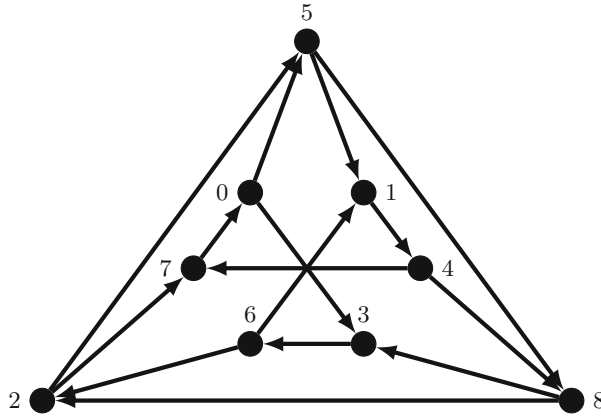


Fig. 2. A deeply critical oriented clique on 9 vertices.

3 Proof of Theorem 5

We provide a proof of Theorem 5 by first giving a full classification of deeply critical oriented circulant cliques.

Lemma 3. *The circulant graph $\vec{C}(n, S)$ is a deeply critical oriented clique if and only if for every $k \in \mathbb{Z}_n$*

- (a) *there exists $x, y \in S \cup \{0\}$ so that $k \equiv x + y \pmod{n}$ or $k \equiv -(x + y) \pmod{n}$; and*
- (b) *if k is even and $\frac{k}{2} \in S$ then the only way to express k as in (a) is by taking $x = y = \frac{k}{2}$, and writing $k \equiv (x + y) \pmod{n}$.*

Part (a) of the lemma ensures that $\vec{C}(n, S)$ is an absolute oriented clique. As circulant oriented graphs are vertex transitive we need only verify that (a) is equivalent to vertex 0 being either adjacent or connected by a 2-dipath to every vertex in $\vec{C}(n, S)$. When $k \equiv x + y \pmod{n}$ or $k \equiv -(x + y) \pmod{n}$ and $x, y \neq 0$, there is a 2-dipath, in some direction, between 0 and k . Otherwise if $x = 0$ or $y = 0$, then 0 and k are adjacent.

Part (b) of the lemma ensures $\vec{C}(n, S)$ is deeply critical. We appeal to vertex transitivity and use the stated condition to verify that for all vertices k not adjacent to 0 there is exactly one 2-dipath between 0 and k .

Proof (Proof of Theorem 5). Let $\vec{C}(n, S)$ be an oriented circulant graph so that $\vec{C}(n, S)$ is an absolute clique and n is even. As n is even we have $\frac{n}{2} \equiv -\frac{n}{2} \pmod{n}$. Therefore $\frac{n}{2} \notin S$. Subsequently vertices 0 and $\frac{n}{2}$ are not adjacent in $\vec{C}(n, S)$.

Since $\vec{C}(n, S)$ is an absolute clique, by Theorem 1 there is a 2-dipath connecting vertices 0 and $\frac{n}{2}$. Thus, there exists $x, y \in S$ satisfying $(x + y) \equiv \frac{n}{2}$

$(\text{mod } n)$ or $-(x + y) \equiv \frac{n}{2} \pmod{n}$. Therefore, $2x \equiv x + x \equiv -(y + y) \pmod{n}$ or $-2x \equiv -x - x \equiv y + y \pmod{n}$, violating part (b) of Lemma 3. Therefore, $\vec{C}(n, S)$ is not a deeply critical oriented clique. \square

4 Conclusions and Outlook

Work in [1] and herein provide examples of infinite families of deeply critical oriented cliques. These constructions and extensive computer search have yielded no examples of deeply critical oriented cliques of even order. These observations together with the result of Theorem 5 lend support the statement of Conjecture 4.

Our computer search has yielded surprising insight into the density of deeply critical oriented cliques among the family of absolute oriented cliques. We identified 9917 examples of previously unknown sporadic deeply critical oriented cliques on up to 17 vertices. In addition to these examples, our search of oriented circulants found 28 examples of previously unknown deeply critical circulant oriented cliques on up to 49 vertices. A classification of odd orders for which there exists a deeply critical circulant oriented clique remains open.

A result of Erdős [6] implies that asymptotically almost surely, every oriented graph is an absolute oriented clique. Though attempts to extend this result to deeply critical oriented cliques of odd order have so far been unsuccessful, it is possible that an analogous statement is true for deeply critical oriented cliques.

Remark: The proofs of Lemmas 1, 2, and 3 are omitted due to space constraint.

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Big Ramsey Degrees of the Generic Partial Order

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Abstract. As a result of 33 intercontinental Zoom calls, we characterise big Ramsey degrees of the generic partial order in a similar way as Devlin characterised big Ramsey degrees of the generic linear order (the order of rationals).

Keywords: Structural Ramsey theory · Big Ramsey degrees · Devlin type

1 Introduction

Given partial orders \mathbf{A} and \mathbf{B} , we denote by $(\mathbf{B}^{\mathbf{A}})$ the set of all embeddings from \mathbf{A} to \mathbf{B} . We write $\mathbf{C} \longrightarrow (\mathbf{B}^{\mathbf{A}})_{k,l}$ to denote the following statement: for every colouring χ of $(\mathbf{C}^{\mathbf{A}})$ with k colours, there exists an embedding $f: \mathbf{B} \rightarrow \mathbf{C}$ such that χ does not take more than l values on $(f^{\mathbf{B}^{\mathbf{A}}})$. For a countably infinite partial order \mathbf{B} and its finite suborder \mathbf{A} , the *big Ramsey degree* of \mathbf{A} in \mathbf{B} is the least number $l \in \mathbb{N} \cup \{\infty\}$ such that $\mathbf{B} \longrightarrow (\mathbf{B}^{\mathbf{A}})_{k,l}$ for every $k \in \mathbb{N}$.

A partial order is *homogeneous* if every isomorphism between two of its finite suborders extends to an automorphism. It is well known that up to isomorphism there is a unique homogeneous partial order $\mathbf{P} = (P, \leq_{\mathbf{P}})$ such that every countable partial order has an embedding to \mathbf{P} . The order \mathbf{P} is called the *generic partial order*. We refine the following recent result.

Theorem 1 (Hubička [4]). *The big Ramsey degree of every finite partial order in the generic order \mathbf{P} is finite.*

We characterise the big Ramsey degrees of \mathbf{P} using special sets of finite words. Our characterisation also leads to a big Ramsey structure with applications in topological dynamics [7].

Currently, there are only relatively few classes of structures where big Ramsey degrees are fully understood. The Ramsey theorem implies that the big Ramsey degree of every finite linear order in the order of ω is 1. In 1979, Devlin refined upper bounds by Laver and characterised big Ramsey degrees of the order of rationals [2, 6]. Laflamme, Sauer and Vuksanović characterised big Ramsey degrees of the Rado (or random) graph and related random structures in binary languages [5]. Recently, a characterisation of big Ramsey degrees of the triangle-free Henson graph was obtained by Dobrinen [3] and independently by the remaining authors of this abstract. See also [1]. Here, we characterize the structures that give the big Ramsey degrees for the generic partial order.

Our construction makes use of the following partial order introduced in [4] (which is closely tied to the order of 1-types within a fixed enumeration of \mathbf{P}). Let $\Sigma = \{L, X, R\}$ be an *alphabet* ordered by $<_{\text{lex}}$ as $L <_{\text{lex}} X <_{\text{lex}} R$. We denote by Σ^* the set of all finite words in the alphabet Σ , by \leq_{lex} their lexicographic order, and by $|w|$ the length of the word w (whose characters are indexed by natural numbers starting at 0). For $w, w' \in \Sigma^*$, we set $w \prec w'$ if and only if there exists i such that $0 \leq i < \min(|w|, |w'|)$, $(w_i, w'_i) = (L, R)$, and $w_j \leq_{\text{lex}} w'_j$ for every $0 \leq j < i$. It is not difficult to check that (Σ^*, \prec) is a partial order and that $(\Sigma^*, \leq_{\text{lex}})$ is one of its linear extensions [4].

We write $w \perp w'$ if $w_i <_{\text{lex}} w'_i$ and $w'_j <_{\text{lex}} w_j$ for some $0 \leq i, j < \min(|w|, |w'|)$. Note that it is not necessarily true that $(w \not\prec w' \wedge w' \not\prec w) \iff w \perp w'$, as, for example, $LR \preceq RL, LR \perp RL, X \not\prec R, X \not\prec R$. However, we will construct subsets of Σ^* where this is satisfied. We call words w and w' *related* if one of expressions $w \preceq w', w' \preceq w$ or $w \perp w'$ holds, otherwise they are *unrelated*.

Given a word w and an integer $i \geq 0$, we denote by $w|_i$ the *initial segment* of w of length i . For $S \subseteq \Sigma^*$, we let \bar{S} be the set $\{w|_i : w \in S, 0 \leq i \leq |w|\}$. The set Σ^* can be seen as a rooted ternary tree and sets $S = \bar{S} \subseteq \Sigma^*$ as its subtrees. Given $i \geq 0$, we let $S_i = \{w \in S : |w| = i\}$ and call it the *level i of S* . A word $w \in S$ is called a *leaf* of S if there is no word $w' \in S$ extending w . Let $L(S)$ be the set of all leaves of S . Given a word w and a character $c \in \Sigma$, we denote by $w \frown c$ the word created from w by adding c to the end of w . We also set $S \frown c = \{w \frown c : w \in S\}$.

To characterise big Ramsey degrees of \mathbf{P} , we introduce the following technical definition, whose intuitive meaning is explained at the end of this section.

Definition 1. *A set $S \subseteq \Sigma^*$ is called a poset-type if $S = \bar{S}$ and precisely one of the following four conditions is satisfied for every i with $0 \leq i < \max_{w \in S} |w|$:*

1. **Leaf:** *There is $w \in S_i$ related to every $u \in S_i \setminus \{w\}$ and $S_{i+1} = (S_i \setminus \{w\}) \frown X$.*
2. **Branching:** *There is $w \in S_i$ such that*

$$S_{i+1} = \{z \in S_i : z <_{\text{lex}} w\} \frown X \cup \{w \frown X, w \frown R\} \cup \{z \in S_i : w <_{\text{lex}} z\} \frown R.$$

3. **New \perp :** There are unrelated words $v <_{\text{lex}} w \in S_i$ such that

$$\begin{aligned}
 S_{i+1} = & \{z \in S_i : z <_{\text{lex}} v\} \frown X \cup \{v \frown R\} \\
 & \cup \{z \in S_i : v <_{\text{lex}} z <_{\text{lex}} w \text{ and } z \perp v\} \frown X \\
 & \cup \{z \in S_i : v <_{\text{lex}} z <_{\text{lex}} w \text{ and } z \not\perp v\} \frown R \\
 & \cup \{w \frown X\} \cup \{z \in S_i : w <_{\text{lex}} z\} \frown R.
 \end{aligned}$$

Moreover, the following assumption is satisfied:

(A) For every $u \in S_i$, $v <_{\text{lex}} u <_{\text{lex}} w$ implies that at least one of $u \perp v$ or $u \perp w$ holds.

4. **New \preceq :** There are unrelated words $v <_{\text{lex}} w \in S_i$ such that

$$\begin{aligned}
 S_{i+1} = & \{z \in S_i : z <_{\text{lex}} v \text{ and } z \perp v\} \frown X \cup \{z \in S_i : z <_{\text{lex}} v \text{ and } z \not\perp v\} \frown L \\
 & \cup \{v \frown L\} \cup \{z \in S_i : v <_{\text{lex}} z <_{\text{lex}} w\} \frown X \cup \{w \frown R\} \\
 & \cup \{z \in S_i : w <_{\text{lex}} z \text{ and } w \perp z\} \frown X \\
 & \cup \{z \in S_i : w <_{\text{lex}} z \text{ and } w \not\perp z\} \frown R.
 \end{aligned}$$

Moreover, the following assumptions are satisfied:

- (B1) For every $u \in S_i$ such that $u <_{\text{lex}} v$, at least one of $u \preceq w$ or $u \perp v$ holds.
- (B2) For every $u \in S_i$ such that $w <_{\text{lex}} u$, at least one of $v \preceq u$ or $w \perp u$ holds.

Given a finite partial order \mathbf{Q} , we let $T(\mathbf{Q})$ be the set of all poset-types S such that $(L(S), \preceq)$ is isomorphic to \mathbf{Q} . As our main result, we determine the big Ramsey degrees of \mathbf{P} .

Theorem 2. For every finite partial order \mathbf{Q} , the big Ramsey degree of \mathbf{Q} in the generic partial order \mathbf{P} equals $|T(\mathbf{Q})| \cdot |\text{Aut}(\mathbf{Q})|$.

Here, we only outline the main constructions giving the lower bound on big Ramsey degrees of \mathbf{P} ; see Sect. 2. The upper bound follows from a refinement of [4] and will appear in the full version of this abstract.

Poset-types, which can be compared to *Devlin types* [6], have a relatively intuitive meaning that we now outline. Words $u \leq_{\text{lex}} v$ are *compatible* if there is no $i < \min(|u|, |v|)$ such that $(u_i, v_i) = (R, L)$, and if there exists $j < \min(|u|, |v|)$ such that $(u_j, v_j) = (L, R)$ then $u \prec v$ and $u \not\perp v$. Conditions (A), (B1) and (B2) from Definition 1 originate from the following properties of compatible words.

Proposition 1. Let $u \leq_{\text{lex}} v \leq_{\text{lex}} w \leq_{\text{lex}} z$ be mutually compatible words from Σ_i^* for some $i \in \omega$.

1. If $v \prec w$ then the following two conditions are satisfied: (a) at least one of $u \prec w$ and $u \perp v$ holds; (b) at least one of $v \prec z$ and $w \perp z$ holds.
2. If $u \perp z$ and $u <_{\text{lex}} v <_{\text{lex}} z$ then at least one of $u \perp v$ and $v \perp z$ is satisfied.

Proof. To prove 1(a), we choose j such that $(v_j, w_j) = (L, R)$. Now, if $u_j = L$, we have $u_j \prec w_j$ by compatibility. If $u_j \in \{X, R\}$, we have $u \perp v$, since $u <_{\text{lex}} v$. Part 1(b) follows from symmetry.

For part 2, we choose j such that $z_j <_{\text{lex}} u_j$. If u is unrelated to v or if $u \preceq v$, we have $z_j <_{\text{lex}} u_j \leq_{\text{lex}} v_j$ (by compatibility) and, since $v <_{\text{lex}} z$, it follows that $v \perp z$. The case when v is unrelated to z or $v \preceq z$ follows analogously. \square

One can view a poset-type S as a binary branching tree and each level S_i as a structure $\mathbf{S}_i = (S_i, \leq_{\text{lex}}, \preceq, \perp)$. It follows that all words in S are mutually compatible and comparing with Proposition 1 one can verify that if a level S_i is a leaf level, then \mathbf{S}_{i+1} is isomorphic to \mathbf{S}_i with one vertex removed. If a level S_i is a branching level, then \mathbf{S}_{i+1} is isomorphic to \mathbf{S}_i with one vertex $w \in S_i$ duplicated to $w \wedge X, w \wedge R \in S_{i+1}$. Observe also that $w \wedge X$ and $w \wedge R$ are unrelated. If a level S_i has new \preceq (or \perp), then \mathbf{S}_{i+1} is isomorphic to \mathbf{S}_i extended by one pair in the relation \preceq (or \perp).

2 The Lower Bound

Without loss of generality, we can assume that $P = \omega$ and thus we fix an (arbitrary) enumeration of \mathbf{P} . We define a function $\varphi: \omega \rightarrow \Sigma^*$ by mapping $j \in P$ to a word w of length $2j + 2$ defined by putting $(w_{2j}, w_{2j+1}) = (L, R)$ and, for every $i < j$, (w_{2i}, w_{2i+1}) to (L, L) if $j \leq_{\mathbf{P}} i$, (R, R) if $i \leq_{\mathbf{P}} j$ and (X, X) otherwise. We set $T = \overline{\varphi[P]}$. The following result is easy to prove by induction.

Proposition 2 (Proposition 4.7 of [4]). *The function φ is an embedding $\mathbf{P} \rightarrow (\Sigma^*, \preceq)$. Moreover, $\varphi(v)$ is a leaf of T for every $v \in P$, all words in T are mutually compatible, and if $v, w \in P$ are incomparable, we have $\varphi(v) \perp \varphi(w)$.*

We will need the following refinement of this embedding.

Theorem 3. *There exists an embedding $\psi: \mathbf{P} \rightarrow (\Sigma^*, \preceq)$ such that $Q = \overline{\psi[\omega]}$ is a poset-type and $\psi(i)$ is a leaf of Q for every $i \in P$.*

Proof. We proceed by induction on levels of T . For every ℓ , we define an integer N_ℓ and a function $\psi_\ell: T_\ell \rightarrow \Sigma_{N_\ell}^*$. We will maintain the following invariants:

1. The set $\overline{\psi_\ell[T_\ell]}$ satisfies the conditions of Definition 1 for all levels with the exception of $N_\ell - 1$.
2. If $\ell > 0$, then, for every $u \in T_\ell$, the word $\psi_\ell(u)$ extends $\psi_{\ell-1}(u|_{\ell-1})$.

We let $N_0 = 0$ and put ψ_0 to map the empty word to the empty word. Now, assume that $N_{\ell-1}$ and $\psi_{\ell-1}$ are already defined. We inductively define a sequence of functions $\psi_\ell^i: T_\ell \rightarrow \Sigma_{N_{\ell-1}+i}^*$. Put $\psi_\ell^0(u) = \psi_{\ell-1}(u|_{\ell-1})$. Now, we proceed in steps. At step j , apply the first of the following constructions that can be applied and terminate the procedure if none of them applies:

1. If there are distinct words w, w' from T_ℓ such that $\psi_\ell^{j-1}(w) = \psi_\ell^{j-1}(w')$ and $w'|_{j-1} = w|_{j-1}$, we construct ψ_ℓ^j by extending each word in $\psi_\ell^{j-1}[T_\ell]$ by an additional character so that the conditions on the branching of $\psi_\ell^{j-1}(w)$ in Definition 1 are satisfied.
2. If there are words w and w' with $w <_{\text{lex}} w'$ such that $w \perp w'$ and $\psi_\ell^{j-1}(w) \not\perp \psi_\ell^{j-1}(w')$ and condition (A) is satisfied for the value range of ψ_ℓ^{j-1} , we construct ψ_ℓ^j to satisfy the conditions on new \perp for $\psi_\ell^{j-1}(w)$ and $\psi_\ell^{j-1}(w')$ as given by Definition 1.

3. If there are words w and w' with $w <_{\text{lex}} w'$ such that $w \prec w'$ and $\psi_\ell^{j-1}(w) \not\prec \psi_\ell^{j-1}(w')$ and conditions (B1) and (B2) are satisfied for the value range of ψ_ℓ^{j-1} , we construct ψ_ℓ^j to satisfy the conditions on new \prec for $\psi_\ell^{j-1}(w)$ and $\psi_\ell^{j-1}(w')$ as given by Definition 1.

Let J be the last index j for which ψ_ℓ^j is defined. It is possible to prove that ψ_ℓ^J is actually an isomorphism $(T_\ell, \leq_{\text{lex}}, \preceq, \perp) \rightarrow (\psi_\ell^J[T_\ell], \leq_{\text{lex}}, \preceq, \perp)$. To do so, we need to verify that the procedure does not terminate early. Clearly, all the branching can be realized since there are no conditions on step 1. We also have $\psi_\ell^J(w) \perp \psi_\ell^J(w') \implies w \perp w'$ and $\psi_\ell^J(w) \preceq \psi_\ell^J(w') \implies w \preceq w'$ for $w, w' \in T_\ell$. To see that these implications are in fact equivalences, we define the *distance* of w and w' as $|\{u : u \in T_\ell, w <_{\text{lex}} u \leq_{\text{lex}} w'\}|$. By Proposition 1, one can add all pairs to the relation \perp in the order of increasing distances to ensure that condition (A) is satisfied and then it is possible to add all pairs of \preceq in the order of decreasing distances so that conditions (B1) and (B2) are satisfied.

Finally, we put $N_\ell = |\psi_\ell^J(w)|$ for some $w \in T_\ell$ and $\psi_\ell = \psi_\ell^J$. Once all functions ψ_ℓ are constructed, we can set $\psi(i) = \psi_i(i)$. □

Given $S \subseteq \Sigma^*$, we call a level \bar{S}_i *interesting* if the structure $\bar{S}_i = (\bar{S}_i, \leq_{\text{lex}}, \preceq, \perp)$ is not isomorphic to $\bar{S}_{i+1} = (\bar{S}_{i+1}, \leq_{\text{lex}}, \preceq, \perp)$ or there exist incompatible $u, v \in S_{i+1}$ such that $u|_i$ and $v|_i$ are compatible. Let $I(S)$ be the set of all interesting levels in \bar{S} . Let $\tau_S : \bar{S} \rightarrow \Sigma^*$ be a mapping assigning every $w \in S$ the word created from w by deleting all characters with indices not in $I(S)$. Define $\tau(S) = \tau_S[S]$ and call it the *embedding type* of S . The following observation, which is a direct consequence of Definition 1, establishes that a sub-type of a poset-type is also a poset-type.

Observation 4. *For a poset-type S and $S' \subseteq L(S)$, $\tau(\bar{S}') = \overline{\tau(S')}$ is a poset-type.* □

Given a finite partial order \mathbf{A} , we construct a function (colouring) $\chi_{\mathbf{A}} : \binom{\mathbf{P}}{\mathbf{A}} \rightarrow T(\mathbf{A})$ by setting $\chi_{\mathbf{A}}(f) = \tau(\psi[f[A]])$ for every $f \in \binom{\mathbf{P}}{\mathbf{A}}$. We show that $\chi_{\mathbf{A}}$ is an *unavoidable coloring* in the following sense, which then implies Theorem 2.

Theorem 5. *For every finite partial order \mathbf{A} and every $f \in \binom{\mathbf{P}}{\mathbf{P}}$, we have $\{\chi_{\mathbf{A}}[f \circ g] : g \in \binom{\mathbf{P}}{\mathbf{A}}\} = T(\mathbf{A})$.*

We outline the proof of Theorem 5 in the rest of this Section. We fix $f \in \binom{\mathbf{P}}{\mathbf{P}}$ and an arbitrary embedding $f' : (\Sigma^*, \preceq) \rightarrow f[\mathbf{P}]$ (which exists, as every countable partial order embeds to \mathbf{P}). We set $h = \psi \circ f'$ and observe that it is an embedding $(\Sigma^*, \preceq) \rightarrow (\Sigma^*, \preceq)$. By Theorem 3 and Observation 4, we know that $\tau(h(\Sigma^*))$ is a poset-type and images of Σ^* correspond to its leaves. We will show a way to embed an arbitrary poset-type to $\tau(h(\Sigma^*))$.

We adapt a proof by Laflamme, Sauer and Vuksanović [5]. A word $u \in \Sigma^*$ is a *successor* of $w \in \Sigma^*$ if $|u| \geq |w|$ and $u|_{|w|} = w$. A subset $A \subseteq \Sigma^*$ is *dense above* u if every successor u' of u has a successor in A . Given $u, v \in \Sigma^*$, we say that v is *u-large* if the set $\{u' : h(u')$ is a successor of $v\}$ is dense above u .

Observation 6. *Let $u, v \in G$ be vertices such that v is u -large. Then*

- (i) *v is u' -large for every successor u' of u , and*
- (ii) *for every $\ell > |v|$ there exists a successor v' of v of length ℓ and a successor u' of u such that v' is u' -large. □*

Proof (of Theorem 5, Sketch). We construct functions $\alpha: \Sigma^* \rightarrow \Sigma^*$, $\beta: \Sigma^* \rightarrow \Sigma^*$ and sequences of integers M_i and N_i , $i \in \omega$, satisfying:

- (I) For every $i \in \omega$, $u, v \in \Sigma_i^*$ it holds that $M_i < |\alpha(u)| \leq M_{i+1}$, $N_i < |\beta(u)| \leq |h(\alpha(u))| \leq N_{i+1}$ and if $u <_{\text{lex}} v$, then $|h(\alpha(u))| < |\beta(v)|$.
- (II) For every $u, v \in \Sigma^*$ and every $j < \min(|\alpha(u)|, |\alpha(v)|)$ such that $\alpha(u)_j \neq \alpha(v)_j$ there exists $i < \min(|u|, |v|)$ satisfying $M_i = j$, $\alpha(u)_j = u_{i-1}$ and $\alpha(v)_j = v_{i-1}$.
- (III) For every $u \in \Sigma^*$ it holds that $\beta(u)$ is $\alpha(u)$ -large and $h(\alpha(u))$ is a successor of $\beta(u)$.
- (IV) For every $u, u' \in \Sigma^*$ such that u is a successor of u' it holds that $\alpha(u)$ is a successor of $\alpha(u')$ and $\beta(u)$ is a successor of $\beta(u')$.

Functions α and β are built by an induction on levels of Σ^* by repeated applications of Observation 6. By (III) the partial maps α, β always extend. Moreover:

Claim. For every poset-type S and every $i > 0$, the structures $\mathbf{S}_i = (S_i, \leq_{\text{lex}}, \preceq, \perp)$, $\mathbf{S}'_i = (\alpha[S_i], \leq_{\text{lex}}, \preceq, \perp)$ and $\mathbf{S}''_i = (\beta[S_i], \leq_{\text{lex}}, \preceq, \perp)$ are mutually isomorphic.

Using the Claim, it is possible to verify that for every poset-type S it holds that $\tau(h[\alpha[L(S)]]) = S$. From this, Theorem 5 follows: For every type $S \in T(\mathbf{A})$ we have that $h(\alpha(L(S)))$ gives a copy of \mathbf{A} within $f(\mathbf{P})$ of the given type.

Acknowledgement. J. H. and M. K. are supported by the project 21-10775S of the Czech Science Foundation (GAČR). This article is part of a project that has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 810115). M. B. and J. H. were supported by the Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004). M. K. was supported by the Charles University Grant Agency (GA UK), project 378119. N. D. is supported by National Science Foundation grant DMS-1901753. A. Z. was supported by National Science Foundation grant DMS-1803489. L. V. is supported by Beatriu de Pinós BP2018, funded by the AGAUR (Government of Catalonia) and by the Horizon 2020 programme No 801370.

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The Square of a Hamilton Cycle in Randomly Perturbed Graphs

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Abstract. We investigate the appearance of the square of a Hamilton cycle in the model of randomly perturbed graphs, which is, for a given $\alpha \in (0, 1)$, the union of any n -vertex graph with minimum degree αn and the binomial random graph $G(n, p)$. This is known when $\alpha > 1/2$, and we determine the exact perturbed threshold probability in all the remaining cases, i.e., for each $\alpha \leq 1/2$. Our result has implications on the perturbed threshold for 2-universality, where we also fully address all open cases.

Keywords: Randomly perturbed graphs · Square of Hamilton cycle

1 Introduction

Our goal is to completely settle the question when the square of a Hamilton cycle appears in randomly perturbed graphs. Given a graph H and a non-negative integer $m \in \mathbb{N}$, the m -th power H^m of H is the graph on vertex set $V(H)$ in which two vertices are adjacent if and only if their distance in H is at most m . As randomly perturbed graphs interpolate between random graph theory and extremal graph theory, before stating our results, we recall what is already known in these two fields on the containment problem for C_n^m , the m -th power of a cycle on n vertices.

We start with the binomial random graph $G(n, p)$. Since the expected number of copies of C_n^m in $G(n, p)$ is $\frac{1}{2}(n-1)!p^{nm}$, the threshold for the appearance of a copy of C_n^m is at least $n^{-1/m}$. For $m = 1$, a famous result by Pósa [16] shows that the threshold for the containment of a Hamilton cycle is $n^{-1} \log n$. For $m \geq 3$ a more general result of Riordan [17], that is proved using the second moment method, gives that $n^{-1/m}$ is indeed the threshold. The case of the square is more subtle: applications of the second moment method for $p = \Theta(n^{-1/2})$ were not successful and variants of the absorption technique only gave the threshold within a polylog-term. It was only recently proved by Kahn, Narayanan, and Park [12] that also in this case the lower bound from above is the truth.

O. Parczyk was supported by DFG grant PA 3513/1-1.

Let us now turn to minimum degree conditions in dense graph. Given $0 \leq \alpha \leq 1$, let G_α be any n -vertex graph with minimum degree $\delta(G_\alpha) \geq \alpha n$. For fixed $m \in \mathbb{N}$, we are interested in conditions on α that guarantee the containment of C_n^m in any such graph G_α . The case $m = 1$ is Dirac's theorem [8]: $\alpha \geq 1/2$ suffices and is best possible. For larger values of m , it was conjectured by Pósa that $\alpha \geq 2/3$ suffices when $m = 2$, and this conjecture was further generalised by Seymour to all m with $\alpha \geq \frac{m}{m+1}$. The conjecture is tight and was solved for all m by Komlós, Sarközy, and Szemerédi [13] for large enough n (depending on m).

One question that recently obtained quite some attention is how many random edges need to be added to any graph G_α with $\alpha \geq \frac{m}{m+1}$ to guarantee the containment of an even larger power of a Hamilton cycle. We formalise this question and state related results for the model of randomly perturbed graphs, which was introduced by Bohman, Frieze, and Martin [5] and allows to investigate how containment properties change if random edges are added. A *randomly perturbed graph* $G_\alpha \cup G(n, p)$ is the graph obtained by adding to a deterministic graph G_α on n vertices with minimum degree at least αn a random graph $G(n, p)$ on the same vertex set.

Definition 1 (perturbed threshold). *Let $m > 0$ be an integer and let $\alpha \in (0, 1)$. The perturbed threshold for the containment of the m -th power of a Hamilton cycle is $\hat{p} = \hat{p}(n, \alpha, m)$ if there exist constants C and c (depending on m and α) such that the following holds. For any $p \geq C\hat{p}$ and for any sequence of graphs G_n with $\delta(G_n) \geq \alpha n$ we have $\lim_{n \rightarrow \infty} \mathbb{P}(C_n^m \subseteq G_n \cup G(n, p)) = 1$, and for any $p \leq c\hat{p}$ there exists a sequence of graphs G_n with $\delta(G_n) \geq \alpha n$ such that $\lim_{n \rightarrow \infty} \mathbb{P}(C_n^m \subseteq G_n \cup G(n, p)) = 0$.*

Bohman, Frieze, and Martin studied when Hamilton cycles appear in randomly perturbed graphs. They showed in [5] that for any $\alpha \in (0, 1/2)$, there is a constant C such that a.s. for any n -vertex graph G_α , the perturbed graph $G_\alpha \cup G(n, p)$ is Hamiltonian, provided $p \geq C/n$. Moreover, this condition on p is optimal as the graph $K_{\alpha n, (1-\alpha)n}$ has minimum degree αn and misses a linear number of edges to be Hamiltonian. Therefore, using the notation of Definition 1, they showed that $\hat{p}(n, \alpha, 1) = n^{-1}$ for any $\alpha \in (0, 1/2)$. For higher powers of Hamilton cycles, one of the first results was obtained in [6], which showed that for any $\alpha \in (0, 1)$ there exists $\eta > 0$ such that $\hat{p}(n, \alpha, m) \leq n^{-1/m-\eta}$, and asked for the optimal η .

In the range $\alpha \in (1/2, 2/3)$, Bennett, Dudek, and Frieze [4] showed that $\hat{p}(n, \alpha, 2) \leq n^{-2/3}(\log n)^{1/3}$. This was improved and generalised by Dudek, Reiher, Ruciński, and Schacht [9]. They showed that for $\alpha \in (\frac{m}{m+1}, \frac{m+1}{m+2})$, not only G_α contains the m -th power of a Hamilton cycle, but adding a linear number of random edges suffices to get the $(m + 1)$ -st power, that is, $\hat{p}(n, \alpha, m + 1) = n^{-1}$. Nenadov and Trujić [14] then proved that in fact, with α in the same range, this suffices for the $(2m + 1)$ -st power and thus $\hat{p}(n, \alpha, 2m + 1) = n^{-1}$. They also conjectured that $\hat{p}(n, \frac{m}{m+1}, 2m + 1) = n^{-1} \log n$ for $\alpha = \frac{m}{m+1}$. When $\alpha > 1/2$, even higher powers have been studied by Antoniuk, Dudek, Reiher, Ruciński, and Schacht [3], who proved that in many cases the threshold is guided by the largest clique required from $G(n, p)$.

Observe that the exact results obtained so far all deal with the range $\alpha \in (1/2, 1)$ and already [3] asked for similar exact results for the case $\alpha \in (0, 1/2]$ and, in particular, for $m = 2$. We give the first optimal results on the perturbed threshold of the square of a Hamilton cycles for $\alpha \in (0, 1/2]$, answering the questions from [3] in a strong form.

Theorem 1. *We have*

$$\hat{p}(n, \alpha, 2) = \begin{cases} n^{-1} & \text{for } \alpha \in (\frac{1}{3}, \frac{2}{3}), \\ n^{-1} \log n & \text{for } \alpha = \frac{1}{3}. \end{cases}$$

Note that our result allows $\alpha \in (1/2, 2/3)$, but this was already covered in [9]. Theorem 1 has immediate consequences for the 2-universality of randomly perturbed graphs, that is, the containment of all graphs of maximum degree two. Indeed, it is easy to see that the square of the Hamilton cycle on n vertices contains each n -vertex graph with maximum degree two as a subgraph. This significantly strengthens one of our results from [7] on the containment of a triangle factor under the same conditions, and is optimal (see the discussion after Theorems 1.3 and 2.2 in [7]). The threshold for 2-universality in randomly perturbed graphs was studied in [15], which showed that for $\alpha \in (0, 1/3)$ the perturbed threshold is $n^{-2/3}$. In $G(n, p)$ alone, Ferber, Kronenberg, and Luh [10] showed that the threshold is $n^{-2/3}(\log n)^{1/3}$. Moreover, Aigner and Brandt [1] showed that for $\alpha \geq 2/3$, the graph G_α is already 2-universal. Thus, our Theorem 1, together with these results, fully solves the question for 2-universality in randomly perturbed graphs.

When α gets smaller than $1/3$, the thresholds for the square of a Hamilton cycle and that for 2-universality behave differently, as in the former case we need to increase the probability to ensure that we can find many copies of the square of a short path (see also the beginning of Sect. 2). However, we are still able to determine precisely the perturbed threshold for the square of a Hamilton cycle for all remaining α .

Theorem 2. *For any integer $k \geq 2$ we have*

$$\hat{p}(n, \alpha, 2) = \begin{cases} n^{-(k-1)/(2k-3)} & \text{for } \alpha \in (\frac{1}{k+1}, \frac{1}{k}), \\ n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)} & \text{for } \alpha = \frac{1}{k+1}. \end{cases}$$

Observe that Theorem 1 is a direct consequence of Theorem 2 and [9]. In the next section we provide an overview of the proof of this result and also explain what is the intuition behind the probabilities appearing there. Our theorem shows that the perturbed threshold $\hat{p}(n, \alpha, 2)$, regarded as a function of α , exhibits countable many jumps at $\alpha = 2/3$ and $\alpha = 1/k$ for each integer $k \geq 2$. Moreover for α tending to zero (i.e. for k tending to infinity), $\hat{p}(n, \alpha, 2)$ tends to $n^{-1/2}$, which is precisely the threshold for the square of a Hamilton cycle in $G(n, p)$ alone as discussed above.

It would be interesting to investigate larger powers of Hamilton cycles for $\alpha \leq 1/2$. A natural candidate to start with is the third power of a Hamilton

cycle, for $\alpha \geq 1/4$ and $p \geq Cn^{-1/2}$. However, this seems to be a more challenging problem, as it requires working with the square of a Hamilton cycle in $G(n, p)$ at the threshold of appearance.

2 Proof Overview

In this section we will sketch the proof of Theorem 2. We start with some notation, discuss the idea of our embedding strategy and explain how this leads to the threshold probabilities given in Theorem 2. We then turn to the arguments for the lower bound on $\hat{p}(n, \alpha, 2)$, and afterwards split the upper bound into two theorems depending on the structure of the dense graph G_α .

Let F be the square of a path P_k^2 with vertices v_1, v_2, \dots, v_k and edges $v_i v_j$, $1 \leq |i - j| \leq 2$. We call (v_1, v_2) the *start-tuple* of F and (v_{k-1}, v_k) the *end-tuple* of F . We also refer to v_i as the i -th vertex of F . Given $k \geq 2$, $\alpha, p \in [0, 1]$, and any n -vertex graph G with minimum degree αn , we want to find the square of a Hamilton cycle C_n^2 in the graph $G \cup G(n, p)$. We now describe a *decomposition* of the edges of the square of a long path or a cycle into deterministic edges (to be embedded to G) and random edges (to be embedded to $G(n, p)$) that we will use in our proof(s). We would like vertex disjoint copies F_1, \dots, F_t of the square of a path on k vertices P_k^2 in the random graph $G(n, p)$ such that the following holds. For each $i = 1, \dots, t - 1$, if we denote by (x_i, y_i) and (u_i, w_i) the start-tuple and end-tuple of F_i , then $w_i x_{i+1}$ is also an edge in $G(n, p)$. Moreover, there exist $t - 1$ additional vertices v_1, \dots, v_{t-1} such that, for $i = 1, \dots, t - 1$, all four edges $v_i u_i, v_i w_i, v_i x_{i+1}, v_i y_{i+1}$ are edges in G . This gives the square of a path on $t(k + 1) - 1$ vertices with edges of $G \cup G(n, p)$. Note that by requiring the edge $w_i x_1$ from $G(n, p)$ and adding another vertex v_t joined to u_t, w_t, x_1, y_1 in G , we get the square of a cycle on $t(k + 1)$ vertices. In order to find C_n^2 and for some technical reasons, our proof(s) will allow some of F_1, \dots, F_t to be the squares of paths of different lengths.

This decomposition already justifies the probabilities that appear in Theorem 2. Indeed, $n^{-(k-1)/(2k-3)}$ is the threshold in $G(n, p)$ for a linear number of copies of P_k^2 (by a standard application of Janson’s inequality), while $n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$ is the threshold in $G(n, p)$ for the existence of a P_k^2 -factor (this follows from a general result of Johansson, Kahn, and Vu [11]).

2.1 Lower Bounds

For any $\alpha \in (0, 1/2)$, let H_α be the complete bipartite n -vertex graph with vertex classes A and B of size αn and $(1 - \alpha)n$, respectively. We start with a sketch for the lower bound on $\hat{p}(n, \alpha, 2)$ for $\alpha \in (\frac{1}{k+1}, \frac{1}{k})$. We want to argue that for some constant $c \in (0, 1)$ depending on α and $p \leq cn^{-(k-1)/(2k-3)}$ a.a.s. $H_\alpha \cup G(n, p)$ does not contain C_n^2 . In B there are a.a.s. at most $2cn$ copies of P_k^2 (by an upper tail bound on the distribution of small subgraphs [18]) and at most $o(n)$ copies of P_{k+1}^2 (by the first moment method). On the other hand, in any embedding of C_n^2 into $H_\alpha \cup G(n, p)$, an α -fraction of the vertices is mapped into A and, because of the bound on the number of P_{k+1}^2 in B , two such vertices can only

rarely be of distance more than $k + 1$. From this it follows that there are at least $\frac{1-\alpha k}{2}n$ copies of P_k^2 in B , which is a contradiction if $c < \frac{1-\alpha k}{4}$.

We argue similarly for the lower bound of $\hat{p}(n, \frac{1}{k+1}, 2)$. We show that with $p \leq \frac{1}{2}n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$ a.a.s. $H_{1/(k+1)} \cup G(n, p)$ does not contain C_n^2 . Indeed, in this regime with $c = \frac{1}{2k}$, a.a.s. (by the first moment method) at least n^{1-2c} vertices from B are not contained in any copy of P_k^2 within B and B contains at most n^{1-c} copies of P_{k+1}^2 . Therefore, in any embedding of C_n^2 , the distance between two vertices mapped into A can only n^{1-c} often be larger than $k + 1$, but exactly one in $k + 1$ vertices is mapped into A . This implies that all but n^{1-c} vertices from B are contained in a copy of P_k^2 within B , which gives a contradiction.

2.2 Stability

It turns out that the additional $(\log n)^{1/(2k-3)}$ -term in $\hat{p}(n, \frac{1}{k+1}, 2)$ is only necessary if the deterministic graph G is really *close* to $H_{1/(k+1)}$. The next definition formalises what we mean by *close*.

Definition 2. For $\alpha, \beta > 0$ we say that an n -vertex graph G is (α, β) -stable if there exists a partition of $V(G)$ into two sets A and B of size $|A| = (\alpha \pm \beta)n$ and $|B| = (1 - \alpha \pm \beta)n$ such that the minimum degree of the bipartite subgraph $G[A, B]$ of G induced by A and B is at least $\frac{1}{4}\alpha n$, all but at most βn vertices from A have degree at least $|B| - \beta n$ into B , all but at most βn vertices from B have degree at least $|A| - \beta n$ into A , and $G[B]$ contains at most βn^2 edges.

We can prove the following stability version for the upper bound on $\hat{p}(n, \frac{1}{k+1}, 2)$ in Theorem 2.

Theorem 3. For every $k \geq 2$ and every $0 < \beta < \frac{1}{6k}$, there exists $\gamma > 0$ and $C > 0$ such that the following holds. Let G be any n -vertex graph with minimum degree at least $(\frac{1}{k+1} - \gamma)n$ that is not $(\frac{1}{k+1}, \beta)$ -stable. Then a.a.s. $G \cup G(n, p)$ contains the square of a Hamilton cycle, provided that $p \geq Cn^{-(k-1)/(2k-3)}$.

Only when the graph G is stable we need the $(\log n)^{1/(2k-3)}$ -term. This case is treated by the following theorem.

Theorem 4. For every $k \geq 2$ there exists $\beta > 0$ and $C > 0$ such that the following holds. Let G be any n -vertex graph with minimum degree at least $\frac{1}{k+1}n$ that is $(\frac{1}{k+1}, \beta)$ -stable. Then a.a.s. $G \cup G(n, p)$ contains the square of a Hamilton cycle, provided that $p \geq Cn^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$.

We sketch the ideas for the proof of these two theorems in the following two subsections. Together with the lower bounds, Theorem 3 and 4 imply Theorem 2.

2.3 Extremal Case

We now sketch the proof of Theorem 4. Suppose that G is an n -vertex $(\frac{1}{k+1}, \beta)$ -stable graph, and let $p \geq C(\log n)^{1/(2k-3)}n^{-(k-1)/(2k-3)}$. From the stability we

get a partition $A \cup B$ of $V(G)$ as in Definition 2. We would like to embed copies F_i of P_k^2 into B and vertices v_i into A , as described in the decomposition above. However this is only possible if $|B| = k|A|$ and, therefore, we first embed squares of short paths of different lengths to ensure this is the case. Moreover, we cover similarly all vertices in A and B that do not have high degree to the other part. Then we cover the remaining vertices in B with copies of P_k^2 , which is possible by [11] with our p . We let \mathcal{F} be the set of the copies of squares of paths that we obtain during these steps and for each $F \in \mathcal{F}$, denote its start-tuple by (x_F, y_F) and its end-tuple by (u_F, w_F) .

To turn this into an embedding of the square of a Hamilton cycle, we now reveal additional edges of $G(n, p)$ and encode this in an auxiliary directed graph \mathcal{T} on vertex set \mathcal{F} as follows. There is a directed edge (F, F') if and only if the edge $w_F x_{F'}$ appears in $G(n, p)$. It is easy to see that all directed edges in \mathcal{T} are revealed with probability p independently of all the others and, therefore, with [2], we can find a directed Hamilton cycle \vec{C} in \mathcal{T} . We finally match to each edge (F, F') of \vec{C} a vertex $v \in A$ not yet covered by any $F \in \mathcal{F}$ such that $u_F, w_F, x_{F'}, y_{F'}$ are all neighbours of v in the graph G . Owing to the minimum degree conditions, this easily follows from Hall's matching theorem. Thus we get the square of a Hamilton cycle, as desired.

2.4 Non-extremal Case

We now turn to the proof of Theorem 3. Assume that G is not $(\frac{1}{1+k}, \beta)$ -stable and let $p \geq Cn^{-(k-1)/(2k-3)}$. After applying the regularity lemma to G , with the help of a variant of [7, Lemma 4.4] is not hard to show that the reduced graph R can be vertex-partitioned into copies of stars $K_{1,k}$ and matching edges $K_{1,1}$, such that there are not too many stars. We would like to cover each such star and matching edge with the square of a Hamilton path, and then connect these paths to get the square of a Hamilton cycle. However since we do not have an additional log-term in p , we need the centre cluster of each star to be larger than the other clusters. Moreover, to ensure that we can connect the Hamilton paths, we need to setup some connections between the stars and matching edges in advance.

Therefore, we first remove some vertices from the leaf cluster of each star to make it unbalanced and ensure that all pairs are super-regular. We then label the stars and matching edges arbitrarily as S_1, \dots, S_s and for $i = 1, \dots, s$ find the square of a short path, that we denote by Q_i , with start- and end-tuple in leaf clusters of S_i and S_{i+1} (where indices are modulo s). We let V_0 be the sets of vertices not any more contained in any of the stars or matching edges. We cover V_0 by appending its vertices to the paths Q_i . Here we use that any vertex $v \in V_0$ has degree at least $(\frac{1}{k+1} - \alpha)n$ in G and, as we do not have too many stars, v has also many neighbours in some clusters which are not centres of stars. This is crucial to ensure that in each star the centre cluster from each star remains large enough in comparison to the leaf clusters.

Then, for any star S_i , we connect the end-tuple of Q_{i-1} with the start-tuple of Q_i while covering all vertices in all clusters of S_i . We emphasise again that, since our p does not have log-terms, this is only possible since each centre cluster is larger than the leaf-clusters and so we do not need to cover all vertices in the leaf clusters with copies of P_k^2 . For any matching edge S_i , we split its clusters and obtain two stars $K_{1,k}$ that also allow us to connect Q_{i-1} to Q_i and covering all vertices of S_i , as before. This gives the square of a Hamilton cycle in $G \cup G(n, p)$ we wanted.

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On Extremal Problems Concerning the Traces of Sets

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Abstract. Given two non-negative integers n and s , define $m(n, s)$ to be the maximal number m such that every hypergraph \mathcal{H} on n vertices and with at most m edges has a vertex x such that $|\mathcal{H}_x| \geq |E(\mathcal{H})| - s$, where $\mathcal{H}_x = \{H \setminus \{x\} : H \in E(\mathcal{H})\}$. The problem of determining the limit $m(s) = \lim_{n \rightarrow \infty} \frac{m(n, s)}{n}$ was posed by Füredi and Pach and by Frankl and Tokushige. While the first results were only for specific small values of s , Frankl determined $m(2^{d-1} - 1)$ for all $d \in \mathbb{N}$. Here we prove that $m(2^{d-1} - c) = \frac{(2^d - c)}{d}$ for every $c, d \in \mathbb{N}$ with $d \geq 4c$ and give an example showing that this equality does not hold anymore for $d = c$.

The other line of research on this problem is to determine $m(s)$ for small values of s . In this line, our second result determines $m(2^{d-1} - c)$ for $c \in \{3, 4\}$. This solves more instances of the problem for small s and in particular solves a conjecture by Frankl and Watanabe.

Keywords: Extremal set theory · Traces of sets · Abstract simplicial complexes

1 Introduction

A hypergraph \mathcal{H} is a pair (V, \mathcal{F}) where V is the set of vertices and $\mathcal{F} \subseteq 2^V$ is the set of edges. In the literature, the problems we consider in this work are often presented in the context of families rather than hypergraphs. If not necessary, it is then not distinguished between the family $\mathcal{F} \subseteq 2^V$ and the hypergraph (V, \mathcal{F}) . We will follow this notational path.

Let V be an n -element set and let \mathcal{F} be a family of subsets of V . For a subset T of V , define the *trace* of \mathcal{F} on T by $\mathcal{F}_{|T} = \{F \cap T : F \in \mathcal{F}\}$. For integers n, m, a , and b , we write

$$(n, m) \rightarrow (a, b)$$

if for every family $\mathcal{F} \subseteq 2^V$ with $|\mathcal{F}| \geq m$ and $|V| = n$, there is an a -element set $T \subseteq V$ such that $|\mathcal{F}_{|T}| \geq b$ (we also say that (n, m) *arrows* (a, b)).

In this context, Füredi and Pach [5] and, more recently, Frankl and Tokushige [3] posed the following problem¹:

¹ There have been slightly different versions in use for the arrowing notation and for what we denote by $m(n, s)$. In this work, we follow the notation in [3].

Problem 1. Given non-negative integers n and s , what is the maximum value $m(n, s)$ such that for every $m \leq m(n, s)$, we have

$$(n, m) \rightarrow (n - 1, m - s).$$

As described in the short abstract, this problem can also be formulated as finding the maximal number $m(n, s)$ such that the following holds. In every hypergraph \mathcal{H} with some n -set V as vertex set and with at most $m(n, s)$ edges, there is a vertex x such that $|\mathcal{H}_x| \geq |\mathcal{H}| - s$, where $\mathcal{H}_x = \mathcal{H}_{|V \setminus \{x\}} = \{H \setminus \{x\} : H \in \mathcal{H}\}$.

A family \mathcal{F} is *hereditary* if for every $F' \subseteq F \in \mathcal{F}$, we have that $F' \in \mathcal{F}$. In [2], Frankl proves that among families with a fixed number of edges and vertices, the trace is minimised by hereditary families. Thus, the problems considered here, and in particular Problem 1, can be reduced to hereditary families. Note that in hereditary families, Problem 1 is asking for the maximum number of edges such that there is always a vertex of small degree (as usual, we define the degree of a vertex v as the number of edges that contain v).

The investigation of this problem started with Bondy [1] and Bollobás [7] determining $m(n, 0)$ and $m(n, 1)$, respectively. Later Frankl [2] and Frankl and Watanabe [4] proved the following identities

$$m(n, 2^{d-1} - 1) = \frac{n}{d}(2^d - 1) \quad \text{and} \quad m(n, 2^{d-1} - 2) = \frac{n}{2}(2^d - 2) \tag{1}$$

for $d, n \in \mathbb{N}$ and $d \mid n$.

Consider a family consisting of a set of size d and all possible subsets, and take n/d vertex disjoint copies of it. The resulting family has minimum degree 2^{d-1} and $\frac{n}{d}(2^d - 1) + 1$ edges. Thus, this family is an extremal construction for the first identity of (1). By taking out all sets of size d , we obtain an extremal construction for the second one.

More generally, for an integer $c \geq 1$, if we arbitrarily take out $(c - 1)$ sets in each of those d -sets, then the minimum degree is at least $2^{d-1} - c + 1$ and the number of edges is $\frac{n}{d}(2^d - c) + 1$. More precisely for an arbitrary family $\mathcal{A} \subseteq 2^{[d]}$ of size $(c - 1)$ we consider the family

$$\mathcal{F}_c(\mathcal{A}) = \left\{ F + (i - 1)d : F \in 2^{[d]} \setminus \mathcal{A} \text{ and } i \in \left[\frac{n}{d} \right] \right\} \subseteq 2^{[n]}.$$

These families show that $m(n, 2^{d-1} - c) \leq \frac{n}{d}(2^d - c)$. The following theorem says that in fact we have equality as long as $c \leq \frac{d}{4}$.

Theorem 1. (Main theorem). *Let $d, c, n \in \mathbb{N}$ with $d \geq 4c$ and $d \mid n$. Then*

$$m(n, 2^{d-1} - c) = \frac{n}{d}(2^d - c).$$

Remark 1. In fact, our proof yields that for $d \geq 4c$ and $m \leq \frac{n}{d}(2^d - c)$, we have $(n, m) \rightarrow (n - 1, m - (2^{d-1} - c))$ without any divisibility conditions on n . The assumption $d \mid n$ is only necessary for the extremal constructions showing the maximality of $\frac{n}{d}(2^d - c)$. Analogous remarks hold for the identities in (1) above and Theorem 2 below.

One might also try to solve Problem 1 for small values of s . Apart from the aforementioned results by Bondy and Bollobás, progress was made by Frankl [2], Watanabe [11, 12], and by Frankl and Watanabe [4]. In [4], they conjectured that $m(n, 12) = (28/5 + o(1))n$. Theorem 1 does not consider cases for which d is very small in terms of c . The following results extend the identities in (1) for $c = 3$ and $c = 4$ and every $d \geq 5$ (for smaller d the respective $m(n, s)$ is not defined or has been determined previously). In particular, it proves the conjecture of Frankl and Watanabe for $s = 12$ in a strong sense.

Theorem 2. *Let $d, n \in \mathbb{N}$ with $d \geq 5$ and $d|n$. Then*

1. $m(n, 2^{d-1} - 3) = \frac{n}{d}(2^d - 3)$ and
2. $m(n, 2^{d-1} - 4) = \frac{n}{d}(2^d - 4)$. In particular, $m(n, 12) = \frac{28}{5}n$.

Note that for larger d , this theorem is of course a special case of Theorem 1.

2 Idea of the Proof

Here we present a sketch of the proof. For the complete proof we refer the reader to [8].

We need to show that for every hereditary hypergraph \mathcal{F} on n vertices with minimum degree at least $2^{d-1} - c + 1$, we have that

$$|\mathcal{F}| \geq \frac{n}{d}(2^d - c) + 1.$$

In the proofs of the identities in (1) in [2, 4], they observe that by double counting we have $|\mathcal{F} \setminus \{\emptyset\}| = \sum_{v \in V} \sum_{H \in L_v} \frac{1}{|H|+1}$, where $L_v = \{A \subseteq V : A \cup \{v\} \in \mathcal{F}\}$ is the *link of the vertex v* . Subsequently, they used a generalised form of the Kruskal-Katona Theorem to obtain a lower bound for $\sum_{H \in L_v} \frac{1}{|H|+1}$ which is the same for every vertex v . Due to the aforementioned double counting this in turn yields the lower bound on the number of edges.

For $c \geq 3$, there are extremal families which show that a general bound on $\sum_{H \in L_v} \frac{1}{|H|+1}$ for every vertex v is not sufficient to provide the desired bound on the number of edges. To overcome this difficulty, first observe that the double counting argument can be generalised by interpreting $\sum_{H \in L_v} \frac{1}{|H|+1}$ as the weight $w_{\mathcal{F}}(v)$ of a vertex v . We will refer to this weight as *uniform weight* since it can be imagined as uniformly distributing the unit weight of an edge to each of its vertices. In contrast, to prove Theorems 1 and 2, we will use non-uniform weights. Moreover, instead of bounding the weight of single vertices we will bound the weight of sets of vertices.

To this end, take a maximal set \mathcal{L} of “light” vertices with neighbourhoods² of size at most $d - 1$ such that the neighbourhoods of all vertices in \mathcal{L} are pairwise disjoint. For all $v \in \mathcal{L}$, we call the set $V_v = N(v) \cup \{v\}$ *cluster*. Observe that if the size of the neighbourhood of a vertex is at most $d - 1$, then it has to intersect

² For $v \in V$, the neighbourhood of v is $N(v) = \{w \in V \setminus \{v\} : \exists A \in \mathcal{F} : \{v, w\} \subseteq A$.

one of the clusters. For vertices whose neighbourhood does not intersect any cluster (and which therefore have a neighbourhood of size at least d), we use the uniform weight. To bound these uniform weights, we introduce a “local” lemma which is a close relative to a general form of the Kruskal-Katona theorem. Given a vertex of fixed degree, it provides a lower bound on the uniform weight and furthermore the minimum weight surplus if its link deviates enough from the minimising link. Since the link of every vertex whose neighbourhood does not intersect any cluster indeed deviates enough from the minimising link (because their neighbourhood contains at least d vertices), the lemma then gives that these vertices will have a large uniform weight.

The next step is to bound the weight of vertices in the clusters. The difficulty is that the weights of different vertices in a cluster might vary. Here, the first key idea is used. Instead of bounding the weight of each single vertex, we bound the average weight of the vertices in a cluster. Even if the number of edges inside a cluster is not large enough, \mathcal{F} being hereditary and the minimum degree of \mathcal{F} still provide some lower bound for the number of edges in each cluster. Then a second local lemma yields that there are several vertices within that cluster whose degree with respect to the cluster is not the minimum degree in \mathcal{F} . Therefore, there exist several crossing edges, i.e., edges containing vertices from both the inside and the outside of the cluster. If we use the uniform weight, these crossing edges will contribute enough to the weight of the cluster, even more than needed.

At this point, we still need to bound the weight of vertices with neighbourhoods of size at most $d - 1$ lying outside of any cluster. As mentioned above, the neighbourhood of every such vertex intersects some cluster, meaning every such vertex is contained in a crossing edge. Recall that in fact, a uniform weight on crossing edges would contribute more weight than needed for the inside of a cluster. Now the second idea comes into play: the unit weight of these edges will be distributed non-uniformly among its vertices. Hence, when splitting the unit weight of such a crossing edge according to the aforementioned imbalance, both sides will get a share that is big enough.

We remark that this strategy is in some sense compatible with the extremal constructions in so far as that those are composed of disjoint copies of almost complete families on d vertices (corresponding to the clusters in the proof).

3 Further Remarks and Open Problems

As in the abstract, consider $m(s)$ to be the following limit

$$m(s) := \lim_{n \rightarrow \infty} \frac{m(n, s)}{n}.$$

It is not difficult to check that $m(s)$ is well-defined (see [4]). Rephrased by means of this definition, Theorem 1 implies that for $c \leq d/4$, we have that $m(2^{d-1} - c) = \frac{2^d - c}{d}$. Further, given $d \geq 1$, define $c_*(d)$ to be the maximum

integer such that for every $c \leq c_*(d)$,

$$m(2^{d-1} - c) = \frac{2^d - c}{d}. \tag{2}$$

In view of Theorem 1 we have that $c_*(d) \geq \lfloor \frac{d}{4} \rfloor$. The following construction shows that for $d \geq 5$, $c_*(d) < d$.

Construction 1. Let k be a positive integer and set $n = 2dk$. Take V to be a set of n vertices. Consider U_1, \dots, U_{2k} to be a partition of V into sets of size d , and for every set U_i , arbitrarily pick a vertex $x_i \in U_i$. Define

$$\begin{aligned} \mathcal{G} &= \{S \subseteq V : \exists i \in [2k] \text{ with } S \subseteq U_i \text{ and } |S| \leq d - 2\}, \\ \mathcal{H} &= \{U_i \setminus \{x_i\} : i \in [2k]\}, \text{ and} \\ \mathcal{I} &= \{\{x_i, x_{i+1}\} : i \in \{1, 3, 5, \dots, 2k - 1\}\}. \end{aligned}$$

One can check that the number of edges of the family $\mathcal{F} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{I}$ is given by

$$|\mathcal{G}| + |\mathcal{H}| + |\mathcal{I}| = \frac{2^d - d - 2}{d}n + 1 + \frac{n}{d} + \frac{n}{2d} = \frac{2^d - d - \frac{1}{2}}{d}n + 1.$$

Finally, since every vertex in V has degree $s = 2^{d-1} - d + 1$, we obtain

$$m(n, 2^{d-1} - d) \leq \frac{n}{d} \left(2^d - d - \frac{1}{2} \right) < \frac{n}{d} (2^d - d),$$

and so $c_*(d) < d$ follows.

It would be interesting to understand the behaviour of $m(2^{d-1} - c)$ for $c > d$. To this end, we suggest the following three problems.

Problem 2. Given $\varepsilon > 0$ sufficiently small, determine $m(2^{d-1} - c)$ for all $d \in \mathbb{N}$ and $c \in \mathbb{N}$ with $d < c \leq (1 + \varepsilon)d$.

Problem 3. Given $\varepsilon > 0$ sufficiently small, determine $m(2^{d-1} - c)$ for all $d \in \mathbb{N}$ and $c \in \mathbb{N}$ with $d < c \leq d^{1+\varepsilon}$.

The following problem seems very difficult, and even estimates might be interesting.

Problem 4. Given $\varepsilon > 0$ sufficiently small, determine $m(\lfloor (1 - \varepsilon)2^{d-1} \rfloor)$ for all $d \in \mathbb{N}$.

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The Chromatic Number of Signed Graphs with Bounded Maximum Average Degree

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Abstract. A signed graph is a simple graph with two types of edges: positive and negative. A homomorphism from a signed graph G to another signed graph H is a mapping $\varphi : V(G) \rightarrow V(H)$ that preserves vertex adjacencies and balance of closed walks (the balance is the parity of the number of negative edges). The chromatic number $\chi_s(G)$ of a signed graph G is the order of a smallest signed graph H such that there is a homomorphism from G to H .

The maximum average degree $\text{mad}(G)$ of a graph G is the maximum of the average degrees of all the subgraphs of G .

The girth $g(G)$ of a graph G is the length of a shortest cycle of G .

In this paper, we consider signed graphs with bounded maximum average degree and we prove that:

- If $\text{mad}(G) < \frac{20}{7}$ and $g(G) \leq 7$ then $\chi_s(G) \leq 5$.
- If $\text{mad}(G) < \frac{17}{5}$ then $\chi_s(G) \leq 10$.
- If $\text{mad}(G) < 4 - \frac{8}{q+3}$ then $\chi_s(G) \leq q + 1$ where q is a prime power congruent to 1 modulo 4.

The first result implies that the chromatic number of planar signed graphs of girth at least 7 is at most 5.

Keywords: Signed graph · Chromatic number · Homomorphism · Maximum average degree · Planar graph

1 Introduction

There exist several notions of colorings of signed graphs which are all natural extensions and generalizations of colorings of simple graphs. It is well-known that a (classical) k -coloring of a graph is no more than a homomorphism to the complete graph on k vertices. Using the notion of homomorphism of signed graphs introduced by Guenin [9] in 2005, a corresponding notion of coloring of signed graphs can be defined. This has attracted a lot of attention since then and the general question of knowing whether every signed graph of a given family admits a homomorphism to some H has been extensively studied. We can for example cite the papers by Naserasr et al. [11, 12] in which they develop many aspects of this notion.

Coloring planar graphs has become a famous problem in the middle of the 19th century thanks to the Four Color Theorem, that states that four colors are

enough to color any simple planar graph. Various branches of this topic then arose, one of which being devoted to the coloring of *sparse* planar graphs. A way to measure the sparseness of a planar graph is to consider its girth (i.e. the length of a shortest cycle): the higher the girth is, the sparser the graph is. Colorings of signed sparse planar graphs have already been considered in the last decade (see e.g. [1, 4, 10, 11, 13, 14]).

A way to get results on sparse planar graphs is to consider graphs (not necessarily planar) with bounded maximum average degree thanks to the well-known relation that links the maximum average degree and the girth of a planar graph: Every planar graph of girth at least g has maximum average degree less than $\frac{2g}{g-2}$.

In this paper, we consider homomorphisms of signed graphs with bounded maximum average degree.

Signed Graphs. A *signed graph* $G = (V, E, s)$ is a simple graph (V, E) (we do not allow parallel edges nor loops) with two kinds of edges: positive and negative edges. The signature $s : E(G) \rightarrow \{-1, +1\}$ assigns to each edge its sign. *Switching* a vertex v of a signed graph corresponds to reversing the signs of all the edges that are incident to v . Two signed graphs G and G' are *switching equivalent* if it is possible to turn G into G' after some number of switches. The *balance* of a closed walk of a signed graph is the parity of its number of negative edges; a closed walk is said to be *balanced* (resp. *unbalanced*) if it has an even (resp. odd) number of negative edges. We can note that a switch does not alter the balance of any closed walk since a switch reverses the sign of an even number of edges of a closed walk. Therefore, Zaslavsky [16] showed the following:

Theorem 1 (Zaslavsky [16]). *Two signed graphs are switching equivalent if and only if they have the same underlying graph and the same set of balanced cycles.*

Homomorphisms of Signed Graphs. Given two signed graphs G and H , the mapping $\varphi : V(G) \rightarrow V(H)$ is a *homomorphism* if φ preserves vertex adjacencies (i.e. $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$) and the balance of closed walks (i.e. the closed walk $\varphi(v_1)\varphi(v_2) \dots \varphi(v_k)$ in H has the same balance as the closed walk $v_1v_2 \dots v_k$ in G). In that case we write $G \rightarrow H$. This type of homomorphism was introduced by Guenin [9] in 2005 and arises naturally from the fact that the balance of closed walks is central in the field of signed graphs.

There exists an alternate way to define homomorphisms of signed graphs using the notion of *sign-preserving homomorphisms*. Given two signed graphs G and H , the mapping $\varphi : V(G) \rightarrow V(H)$ is a *sign-preserving homomorphism* (sp-homomorphism for short) if φ preserves vertex adjacencies and the signs of edges. In that case we write $G \xrightarrow{sp} H$. Naserasr et al. [12] showed that, given two signed graphs G and H , we have $G \rightarrow H$ if and only if there exists a signed graph G' switching equivalent to G such that $G' \xrightarrow{sp} H$.

The *chromatic number* $\chi_s(G)$ (resp. *sign-preserving chromatic number* $\chi_{sp}(G)$) of a signed graph G is the order of a smallest graph H such that $G \rightarrow H$

(resp. $G \xrightarrow{sp} H$). The (sign-preserving) chromatic number $\chi_{s/sp}(\mathcal{C})$ of a class of signed graphs \mathcal{C} is the maximum of the (sign-preserving) chromatic numbers of the graphs in the class. Clearly, an sp-homomorphism is a homomorphism and thus $\chi_s(G) \leq \chi_{sp}(G)$ for any signed graph G .

If G admits a (sp-)homomorphism φ to H , we say that G is $H(-sp)$ -colorable and that φ is an $H(-sp)$ -coloring of G .

Target Graphs. Let q be a prime power with $q \equiv 1 \pmod{4}$. Let \mathbb{F}_q be the finite field of order q . The signed Paley graph SP_q has vertex set $V(SP_q) = \mathbb{F}_q$. Two vertices u and $v \in V(SP_q)$, $u \neq v$, are connected with a positive edge if $u - v$ is a square in \mathbb{F}_q and with a negative edge otherwise. See Fig. 1 for a picture of the signed Paley graph on five vertices.

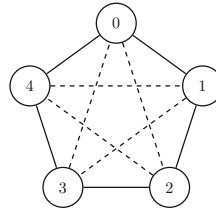


Fig. 1. SP_5 , the signed Paley graph on 5 vertices.

Notice that this definition is consistent since $q \equiv 1 \pmod{4}$ ensures that -1 is always a square in \mathbb{F}_q and if $u - v$ is a square then $v - u$ is also a square.

Given a signed graph SP_q , we denote by SP_q^- the graph obtained from SP_q by removing any vertex (SP_q is vertex-transitive) and by SP_q^+ the graph obtained from SP_q by adding a vertex that is connected with a positive edge to every other vertex.

Such graphs SP_q , SP_q^+ and SP_q^- have remarkable structural properties but due to lack of space, we will not list them (see [14] for more details). We use these target graphs to obtain our results.

2 State of the Art and Results

Let us denote by \mathcal{P}_g the class of planar signed graphs of girth at least g and by \mathcal{M}_d the class of signed graphs with maximum average degree less than d .

Table 1. Known results on the chromatic number of signed planar graphs with given girth and signed graphs with bounded maximum average degree.

Graph families	χ_s	Remarks	Refs
\mathcal{P}_3	$10 \leq \chi_s \leq 40$		[10, 13]
\mathcal{P}_4	$6 \leq \chi_s \leq 25$		[14]
$\mathcal{M}_{\frac{10}{3}}$	$\chi_s \leq 10$	$\mathcal{P}_5 \subset \mathcal{M}_{\frac{10}{3}}$	[10]
\mathcal{M}_3	$\chi_s \leq 6$	$\mathcal{P}_6 \subset \mathcal{M}_3$	[10]
$\mathcal{M}_{\frac{18}{7}}$	$\chi_s = 4$	$\mathcal{P}_9 \subset \mathcal{M}_{\frac{18}{7}}$	[6]

Note first that for planar graphs, the gap between the lower and upper bounds is huge ($10 \leq \chi_s(\mathcal{P}_3) \leq 40$) and in 2020, Bensmail et al. [2] conjectured that

$\chi_s(\mathcal{P}_3) = 10$. Recently, Bensmail et al. [1] proved that if this conjecture is true, then the target graph is necessarily SP_9^+ . This question remains widely open.

Finally, note that for maximum average degree less than $\frac{10}{3}$, 3, and $\frac{18}{7}$ (lines 4–6 of Table 1), this gives bounds for planar graphs of girth at least 5, 6, and 9. Note that since unbalanced even cycles have chromatic number 4 (see [8]), the bound for maximum average degree $\frac{18}{7}$ is tight.

In this paper, we prove the following theorem, improving several above-mentioned results:

Theorem 2. *Let G be a signed graph.*

- (1) *If $G \in \mathcal{M}_{4-\frac{8}{q+3}}$, then $G \rightarrow SP_q^+$. Thus $\chi_s(G) \leq q + 1$, with $q \equiv 1 \pmod{4}$ and q is a prime power.*
- (2) *If $G \in \mathcal{M}_{\frac{17}{5}}$, then $G \rightarrow SP_9^+$. Thus $\chi_s(G) \leq 10$.*
- (3) *If $G \in \mathcal{M}_{\frac{20}{7}}$ and $g(G) \geq 7$, then $G \rightarrow SP_5$. Thus $\chi_s(G) \leq 5$.*

It is not hard to see that signed cliques in which each edge is subdivided once have a maximum average degree that tends to 4 as the number of vertices grows. Such signed graphs have unbounded chromatic number and Theorem 2(1) gives an upper bound on the chromatic number of signed graphs of maximum average degree $4 - \varepsilon$ in function of ε . Theorem 2(2) improves the previous known result of Montejano et al. [10] saying that $\chi_s(\mathcal{M}_{\frac{10}{3}}) \leq 10$ by reaching the same upper bound for a superclass of graphs ($\mathcal{M}_{\frac{10}{3}} \subset \mathcal{M}_{\frac{17}{5}}$). Theorem 2(3) gives, as a corollary, that $\chi_s(\mathcal{P}_7) \leq 5$ since $\mathcal{P}_7 \subset \mathcal{M}_{\frac{20}{7}}$, which are new results that contribute to the above-mentioned collection of known results.

3 Proof Techniques

To prove our results, let us first introduce what we call antitwinned graphs. Given a signed graph G of signature s_G , we can create the signed graph $\rho(G)$ as follows: We take two copies G^{+1} , G^{-1} of G , hence $V(\rho(G)) = V(G^{+1}) \cup V(G^{-1})$; the edge set is defined as $E(\rho(G)) = \{u^i v^j : uv \in E(G), i, j \in \{-1, +1\}\}$ and the signature as $s_{\rho(G)}(u^i v^j) = i \times j \times s_G(u, v)$. A signed graph G is said to be antitwinned if there exists a signed graph H such that $G = \rho(H)$.

Antitwinned signed graphs play a central role for our proofs thanks to the following lemma:

Lemma 1 ([5]). *Given two signed graphs G and H , G admits an sp-homomorphism to $\rho(H)$ if and only if G admits a homomorphism to H .*

Therefore, Theorem 2 will be proved by showing that:

- (1) If $G \in \mathcal{M}_{4-\frac{8}{q+3}}$, then $G \xrightarrow{sp} \rho(SP_q^+)$.
- (2) If $G \in \mathcal{M}_{\frac{17}{5}}$, then $G \xrightarrow{sp} \rho(SP_9^+)$.
- (3) If $G \in \mathcal{M}_{\frac{20}{7}}$, then $G \xrightarrow{sp} \rho(SP_5)$.

We prove these results by contradiction, by assuming that they have counterexamples. Among all of these counterexamples, we take a graph G with the fewest number of vertices. Our goal is to prove that G satisfies structural properties incompatible with having a maximum average degree smaller than a certain value, hence the conclusion.

For each theorem, we start by introducing sets of so-called forbidden configurations, which by minimality G cannot contain. We then strive to reach a contradiction with the bounded maximum average degree. To do so, we use the discharging method. This means that we give some initial weight to vertices of G , we then redistribute those weights and obtain a contradiction by double counting the total weight. We present appropriate collections of discharging rules, and argue that every vertex of G ends up with non-negative weight while the total initial weight was negative.

The discharging method was introduced more than a century ago to study the Four-Color Conjecture [15], now a theorem. It is especially well-suited for studying sparse graphs, and leads to many results, as shown in two recent surveys [3, 7].

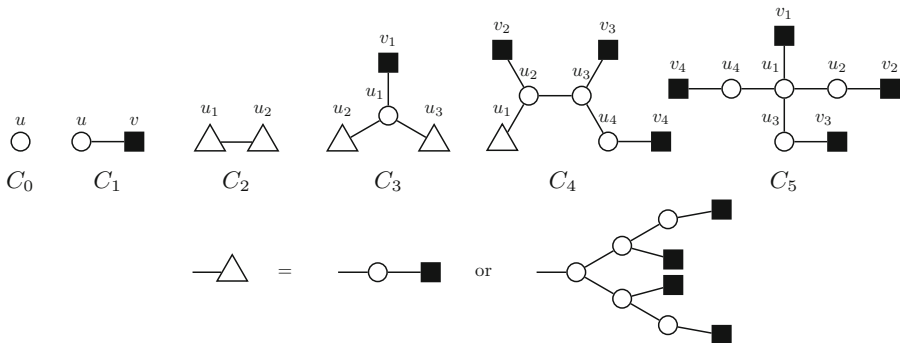


Fig. 2. Forbidden configurations. Every edge incident to round vertices is represented. Square vertices can be of any degree. Triangle vertices are replaced by one of the two represented structures.

Due to lack of space, let us just give the sketch of the proof of Theorem 2(3). To prove this theorem, we prove that every signed graph of maximum average degree less than $\frac{20}{7}$ and girth at least 7 admits a sp-homomorphism to $\rho(SP_5)$ which implies the theorem by Lemma 1.

Let G be a smallest signed graph with $\text{mad}(G) < \frac{20}{7}$ and girth at least 7 admitting no sp-homomorphism to $\rho(SP_5)$. We start by proving that the configurations depicted in Fig. 2 cannot appear in G .

We then define the weighting $\omega(v) = d(v) - \frac{20}{7}$ for each vertex v of degree $d(v)$. By construction, the sum of all the weights $\sum_{v \in V(G)} \omega(v)$ is negative since the maximum average degree of G (and therefore its average degree) is strictly

smaller than $\frac{20}{7}$. We say that a k -vertex (resp. k^+ -vertex) is a vertex of degree k (resp. at least k). A 3-vertex is said to be 3-worse if its adjacent to a 2-vertex, 3-bad if it is adjacent to two 3-worse vertices or 3-good otherwise. We then introduce the following discharging rules:

- (R_1) Every 3^+ -vertex gives $\frac{3}{7}$ to each of its 2-neighbors.
- (R_2) Every 3-good, 3-bad or 4^+ -vertex gives $\frac{1}{7}$ to each of its 3-worse-neighbors.
- (R_3) Every 3-good or 4^+ -vertex gives $\frac{1}{7}$ to each of its 3-bad-neighbors.

Finally, we show that every vertex has a positive final weight by using the fact that the configurations of Fig. 2 cannot appear in G , a contradiction.

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Maker–Breaker Games with Constraints

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Abstract. We analyse the unbiased WalkerMaker–WalkerBreaker game, a variant of the well-known Maker–Breaker positional game where both players Maker and Breaker are constrained to choose their edges according to a walk. Here, we consider two standard graph games - the Connectivity game and the Hamilton Cycle game played on the edge set of the complete graph, K_n , on n vertices, and show how fast WalkerMaker can build desired spanning structures in these games.

Keywords: Positional games · Maker–Breaker games ·
Spanning tree · Hamilton cycle

1 Introduction

We study WalkerMaker–WalkerBreaker games (or *WMaker–WBreaker* games, for brevity), a variant of well-known Maker–Breaker positional games. A positional game is described with the *board* of the game (a finite set X), the *family of winning sets* ($\mathcal{F} \subseteq 2^X$), and the *winning condition*. In the $(a : b)$ Maker–Breaker game on X , two players Maker and Breaker alternately claim unclaimed elements of the board until all elements are claimed. Maker claims a elements and Breaker claims b elements. Parameters a and b define the *bias* of the game. If both a and b are equal to 1, then the game is referred to as the *unbiased* game. In the Maker–Breaker game the players have opposite goals. Maker’s goal is to claim all elements of some $F \in \mathcal{F}$, while Breaker’s goal is to claim at least one element from every winning set in order to prevent Maker from winning. More about Maker–Breaker games and different aspects of the theory of positional games can be found in the book of Beck [1] and in the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [11].

It is very common to play Maker–Breaker games on the edges of a graph $G = (V, E)$ with $|V| = n$. In this case, the board of the game is E and the

A full version of this extended abstract is published in the journal of Discrete Applied Mathematics [9].

M. Mikalački—Partly supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant nr. 174019.

winning sets are all edge sets of subgraphs of G which possess some given graph property. For example, in the *Perfect Matching game* on G the winning sets are all sets containing $\lfloor |V|/2 \rfloor$ independent edges of G . In the *Connectivity game* the winning sets are all spanning trees of G . In the *Hamilton Cycle game* the winning sets are the edge sets of all Hamilton cycles of G . In the *k -vertex-connectivity game* the winning sets are all spanning k -vertex-connected subgraphs of G .

Most of the unbiased Maker–Breaker games played on $E(K_n)$ are an easy win for Maker. For example, in the Connectivity game Maker can win in $n - 1$ moves [13]. It is also known that other unbiased games are in a favor of Maker. So, for such games, the more interesting question to consider is *how fast Maker can win*. Studying fast winning strategies of Maker has received a lot of attention in recent years (see [3, 4, 7, 8, 10, 12]). For example, it is shown that Maker can win the unbiased Perfect Matching game in $n/2 + 1$ moves (for n even) [10], and the unbiased Hamilton Cycle game in $n + 1$ moves [12].

One way to compensate for Maker’s advantage are biased $(1 : b)$ Maker–Breaker games, where $b > 1$, the study of which was initiated by Chvátal and Erdős in [2]. Another approach is to reduce the number of winning sets by making the base graph sparser and to play on a random board, as proposed by Stojaković and Szabó in [14].

Recently, Espig, Frieze, Krivelevich, and Pegden in [6] introduced Walker–Breaker games. In these games Maker is restricted to claim her edges according to a *walk*. For her starting position, Walker (having the role of Maker) can choose any vertex. In every other round, she needs to claim an edge, not previously claimed by Breaker, incident with a vertex in which she has finished her previous move. On the other hand, Breaker has no restrictions on the way he moves. So, these games increase Breaker’s power and make up for Maker’s advantage in the unbiased Maker–Breaker games. The decrease of Maker’s power, as a walker, is evident in the Walker–Breaker Connectivity game, since Breaker is able to isolate a vertex from Walker’s graph simply by fixing a vertex after Walker’s first move and then claiming the edges between that fixed vertex and Walker’s current position in every other round. The maximum number of vertices that Walker can visit in the $(1 : 1)$ game on K_n is $n - 2$, as it is shown by Espig, Frieze, Krivelevich, and Pegden in [6].

Due to their recent appearance, little is known about Walker–Breaker games (see [5], [6]) and lots of questions are still open. The question of interest in this paper is the following:

Question 1. *What happens if Breaker is also a walker?*

Here we address this question and consider WMaker–WBreaker games in which each player has to claim her/his edges according to a walk. We focus on the Connectivity game and Hamilton Cycle game on $E(K_n)$ for large enough n and show that without wasting too many moves WMaker can win in these games.

1.1 Notation

Given a graph G , $V(G)$ and $E(G)$ denote its sets of vertices, respectively edges, and $v(G) = |V(G)|$ and $e(G) = |E(G)|$ their cardinalities. Given two vertices $x, y \in V(G)$ an edge in G is denoted by xy . Given a vertex $x \in V(G)$, we use $d_G(x)$ to denote the degree of vertex x in G . For a set $A \subseteq V(G)$ and $x \in V(G) \setminus A$, let $d_G(x, A)$ denote the degree of x towards A .

Assume that a WMaker–WBreaker game on the edge set of a given graph G is in progress. At any point of the game, let M and B denote the graphs spanned by the edges of WMaker and WBreaker, respectively, claimed so far.

For some vertex v we say that it is *visited* by a player if he/she has claimed *at least one* edge incident with v . A vertex is *isolated/unvisited* if no edge incident to it is claimed. Let U be the set of vertices that are still unvisited by WMaker, i.e. $U = V(G) \setminus V(M)$. The edges in $E(G) \setminus E(M \cup B)$ are called *free*. Unless otherwise stated, we assume that WBreaker starts the game, i.e. one *round* in the game consists of a move by WBreaker followed by a move of WMaker.

2 Results

As a walker, Maker is not able to build a spanning structure even when she plays against the Breaker’s bias $b = 1$. In this section we want to show that the situation changes when both Maker and Breaker are walkers, i.e. if both players are restricted in the same way, we show that Maker is able to win in the WMaker–WBreaker Connectivity game and Hamilton Cycle game. Also, we are interested to see how fast she can build a spanning structure.

As it is known, in the standard Maker–Breaker Connectivity game on $E(K_n)$ Maker can win in the optimal number of moves, $n - 1$, so it is natural to ask whether she, as a walker, can achieve such a quick winning now when Breaker is a walker too.

To win as soon as possible, WMaker will use the following strategy in the first stage of both the Connectivity game and the Hamilton Cycle game.

Strategy S. For her starting vertex, WMaker chooses the vertex v_1 , in which WBreaker has finished his first move, and claims an edge v_1u such that $d_B(u) = 0$ (ties broken arbitrarily). In every other round WMaker checks if there exists an edge $e \in E(B)$, $e = pq$, s.t. $p, q \in U$, and from her current position w claims wp , or wq , whichever is free. If both wp and wq are free she chooses wp if $d_B(p) > d_B(q)$, and wq , if $d_B(q) > d_B(p)$ (ties are broken arbitrarily). If no such edge exists, WMaker from her current position w claims a free edge wu such that $u \in U$ and $d_B(u) = \max\{d_B(v) : v \in U \text{ and } vw \text{ is free}\}$, ties broken arbitrarily, for as long as $|U| \geq 3$. If all free edges wu are such that $d_B(u) = 0$ for all $u \in U$, then WMaker claims an arbitrary free edge wu .

The following theorem shows that Maker, as walker, needs two additional moves than it is optimal.

Theorem 1 *In the (1 : 1) WMaker–WBreaker Connectivity game on $E(K_n)$, WMaker, as the second player, has a strategy to win in at most $n + 1$ moves.*

Sketch of the Proof. WMaker’s strategy is divided into two stages.

Stage 1. WMaker builds a path of length $n - 4$ in $n - 4$ rounds, by playing according to the strategy \mathcal{S} .

Stage 2. WMaker visits the three remaining untouched vertices in at most 5 additional moves. More details can be found in [9]. □

In the Hamilton Cycle game, it becomes more challenging for WMaker to win fast. The following theorem shows that she needs at most five more moves than is the case of the standard Maker–Breaker Hamilton Cycle game.

Theorem 2 *In the (1 : 1) WMaker–WBreaker Hamilton cycle game on $E(K_n)$, WMaker, as the second player, has a strategy to win in at most $n + 6$ moves.*

Sketch of the Proof. WMaker’s strategy is divided into three stages.

Stage 1. WMaker builds a path of length $n - 4$ in $n - 4$ rounds, by playing according to the strategy \mathcal{S} .

Stage 2. WMaker closes a cycle of length $n - 2$ in round $n - 2$ or cycle of length $n - 1$ at latest in round n .

Stage 3. WMaker completes a Hamilton cycle at latest in round $n + 6$. More details can be found in [9]. □

Next, we want to look at WBreaker’s possibilities to postpone WMaker’s win in the Connectivity game. The following theorem shows that WBreaker as the second player can force WMaker to play at least n moves in order to win the game.

Theorem 3 *In the (1 : 1) WMaker–WBreaker Connectivity game on $E(K_n)$, WBreaker, as the second player, has a strategy to postpone WMaker’s win by at least n moves.*

Sketch of the Proof. WBreaker’s strategy is as follows.

WBreaker plays arbitrarily until $|U| = 3$. To be able to visit $n - 3$ vertices, WMaker needs to play at least $n - 4$ moves. Let $u_1, u_2, u_3 \in U$ after round $r \geq n - 4$.

If in round $r + 1 \geq n - 3$ WMaker moves to some $u_i, i \in \{1, 2, 3\}$, WBreaker will move to $u_j, j \neq i$. WBreaker is able to move to u_j since $u_j \in U$ and there is no WMaker’s edge between WBreaker’s current position and vertex u_j . In the following round WBreaker moves to $u_k, k \neq i, j$. WMaker could claim $u_i u_j$ or $u_i u_k$ in round $r + 2 \geq n - 2$. Since $u_j u_k \in E(B)$, after the round $r + 3 \geq n - 1$, WMaker needs to make at least one more move in order to visit the remaining vertex from U . □

3 Concluding Remarks

We proved that WMaker is able to win in the $(1 : 1)$ WMaker–WBreaker Connectivity game and Hamilton Cycle game. From Theorems 1 and 3 it follows that WMaker needs t , $n \leq t \leq n + 1$ moves to make a spanning tree. To win in the Hamilton Cycle game she needs to play at most $n + 6$ moves, according to Theorem 2. As WMaker cannot make a spanning tree in less than n moves, it follows that WMaker needs at least $n + 1$ moves to create a Hamilton cycle.

The following natural question to consider could be what is the largest WBreaker’s bias b for which WMaker can win. However, for $b = 2$ WBreaker can isolate a vertex in WMaker’s graph. In each round, WBreaker can use one move to return to some fixed vertex along the previously claimed edge, and the other to claim the edge between this particular vertex and WMaker’s current position.

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Parallelisms of $PG(3, 5)$ with an Automorphism Group of Order 25

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Abstract. We construct all parallelisms of $PG(3, 5)$ that are invariant under an automorphism group of order 25. Up to isomorphism their number is 14873. Using them we obtain 12 transitive deficiency one parallelisms, two of which belong to an infinite family constructed by Johnson.

Keywords: Projective space · Parallelism · Automorphism · Transitive

1 Introduction

Let $PG(n, q)$ be the n -dimensional projective space over the finite field F_q . A set of lines, such that each point is in exactly one of these lines, is called a *spread*. Two spreads are isomorphic if an automorphism of $PG(n, q)$ maps one to the other. A *parallelism* is a partition of the set of all lines of the projective space to spreads. Two parallelisms are isomorphic if there is an automorphism of $PG(n, q)$ which maps the spreads of one parallelism to spreads of the other. A *deficiency one parallelism* is a partial parallelism with one spread less than the parallelism. Each deficiency one parallelism can be uniquely extended to a parallelism. A (partial) parallelism is called transitive if it has an automorphism group which is transitive on the spreads. More details on projective spaces, spreads and parallelisms can be found, for instance, in [14] or [19].

General constructions of parallelisms of $PG(n, 2)$ are presented in [1] and [25], of $PG(2^n - 1, q)$ in [4], and of $PG(3, q)$ in [7, 10, 13, 16]. Spreads and parallelisms are related to translation planes [19], network coding [9], error-correcting codes [12], design theory and cryptography [18]. All parallelisms of $PG(3, 2)$ and $PG(3, 3)$ are known [2, 14]. For larger projective spaces the classification problem is open. That is why computer-aided constructions of parallelisms with certain

S. Topalova—Partially supported by the Bulgarian National Science Fund under Contract No KP-06-Russia/33/17.12.2020.

S. Zhelezova—Partially supported by the National Scientific Program “Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)” , financed by the Ministry of Education and Science.

predefined automorphism groups (for $PG(3, 5)$ in [17, 21, 22, 24]) contribute significantly to the study of the properties and possible applications of parallelisms.

An infinite class of transitive deficiency one parallelisms of $PG(3, q)$ is described by Johnson [13] for $q = p^r$ if p is odd and further a group-theoretic characterization of the constructed parallelisms is presented by Johnson and Pomareda [15]. Properties of the automorphism groups and the spreads of transitive deficiency one parallelisms of $PG(3, q)$ are derived by Biliotti, Jha, and Johnson [5], and Diaz, Johnson, and Montinaro [8], who show that the deficiency spread must be Desarguesian, and the automorphism group should contain a normal subgroup of order q^2 (see also [14, chapter 38]). All transitive deficiency one parallelisms of $PG(3, 3)$ and $PG(3, 4)$ are known [2, 20]. The construction of such parallelisms in $PG(3, 5)$ is one of the aims of the present work.

We perform a computer-aided classification of all the parallelisms of $PG(3, 5)$ with an automorphism group of order 25. The construction method is described in Sect. 2, a comment on the results is given in Sect. 3, and concluding remarks can be found in Sect. 4.

2 Construction

2.1 The Automorphism Groups

The projective space $PG(3, 5)$ has 156 points and 806 lines. We denote by G its full automorphism group, where $G \cong P\Gamma L(4, 5)$ and $|G| = 2^9 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$. Each spread contains 26 lines which partition the point set and each parallelism has 31 spreads. The Sylow 5-subgroup of $P\Gamma L(4, 5)$ is of order 5^6 and all its elements are of order 5. The elements are partitioned to 4 conjugacy classes under G [26] presented here in Table 1. We denote by $G_{5_1}, G_{5_2}, G_{5_3}$ and G_{5_4} groups of order 5 that are generated by an element of each of these classes respectively. The number of elements in each class is given in the last column of the table. The normalizer of G_{5_i} in G is defined as $N_G(G_{5_i}) = \{\gamma \in G \mid \gamma G_{5_i} \gamma^{-1} = G_{5_i}\}$. We use GAP [11] to obtain all the necessary groups and their conjugacy classes.

Table 1. Conjugacy classes (under G) of elements of the Sylow 5-subgroup

Class	Group	$N_G(G_5)$	Number of elements
C_1	G_{5_1}	6000000	344
C_2	G_{5_2}	300000	880
C_3	G_{5_3}	10000	6400
C_4	G_{5_4}	500	8000

We establish that there are no parallelisms invariant under G_{5_1} or G_{5_3} , so we use the elements of C_2 and C_4 to obtain the possible automorphism groups of order 25. Such a group is generated by two elements α and β of order five. There

Table 2. Conjugacy classes of groups of order 25

Class	Generators	Group	$N_G(G_{25_i})$	Fixed points	Fixed lines	Line orbits of	
						Length 5	Length 25
A	$\alpha, \beta \in C_2$	G_{25_1}	4000	1	6	10	30
B	$\alpha, \beta \in C_2$	G_{25_2}	80000	6	11	34	25
C	$\alpha, \beta \in C_2$	G_{25_3}	180000	6	1	36	25
D	$\alpha, \beta \in C_2$	G_{25_4}	250000	6	6	35	25
E	$\alpha, \beta \in C_4$	G_{25_5}	500	1	1	6	31
F	$\alpha, \beta \in C_4$	G_{25_6}	2500	1	1	11	30
E	$\alpha \in C_2, \beta \in C_4$	G_{25_5}	500	1	1	6	31

are three cases to consider, namely $\alpha, \beta \in C_2$, $\alpha, \beta \in C_4$, and $\alpha \in C_2, \beta \in C_4$. The obtained groups of order 25 are in 6 conjugacy classes presented in Table 2, where groups of class *E* can be generated by $\alpha, \beta \in C_4$ as well as by $\alpha \in C_2, \beta \in C_4$.

Further considerations show that a parallelism admits only automorphism groups from classes *C* and *E*. Without loss of generality we construct parallelisms invariant under one group from class *C* and one from *E*, namely under G_{25_3} and G_{25_5} . Each of them is noncyclic and has 6 subgroups of order 5 that are conjugate to G_{5_2} or G_{5_4} .

2.2 Spread Orbits Under the Action of G_{25_3} or G_{25_5}

Under the action of each of these groups there are fixed spreads, spreads with orbits of length 5, and spreads with orbits of length 25. We consider them below.

A Fixed Spread. The short line orbits under each of the two groups cannot participate in a spread together because they share points. That is why a fixed spread consists of a fixed line and a line orbit of length 25.

A Spread with An Orbit of Length 5. Such a spread is fixed by one of the six subgroups of order 5 of G_{25_i} . It has 6 lines from different line orbits of length 5. We denote by b_i^j a line from the i -th short orbit $\{b_1^i, b_2^i, \dots, b_5^i\}$, $i = 1, 2, \dots, 6$. Besides these lines, the spread contains 4 orbits $\{c_1^i, c_2^i, \dots, c_5^i\}$, $i = 1, \dots, 4$ of length 5 under one of the subgroups of G_{25_i} . Their lines are from different line orbits under the remaining five subgroups. Each one of the 4 orbits mentioned above is a part of a full line orbit under G_{25_5} $\{c_1^i, c_2^i, \dots, c_{25}^i\}$, $i = 1, 2, \dots, 4$. This way a spread orbit of length 5 contains the following spreads:

b_1^1	b_2^1	b_3^1	b_4^1	b_5^1	b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4
b_2^1	b_3^1	b_4^1	b_5^1	b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4	
b_3^1	b_4^1	b_5^1	b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4		
b_4^1	b_5^1	b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4			
b_5^1	b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4				
b_6^1	c_1^1	c_2^1	c_3^1	c_4^1	c_5^1	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4					
b_1^2	b_2^2	b_3^2	b_4^2	b_5^2	b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4					
b_2^2	b_3^2	b_4^2	b_5^2	b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4						
b_3^2	b_4^2	b_5^2	b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4							
b_4^2	b_5^2	b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4								
b_5^2	b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4									
b_6^2	c_1^2	c_2^2	c_3^2	c_4^2	c_5^2	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4										
b_1^3	b_2^3	b_3^3	b_4^3	b_5^3	b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4										
b_2^3	b_3^3	b_4^3	b_5^3	b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4											
b_3^3	b_4^3	b_5^3	b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4												
b_4^3	b_5^3	b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4													
b_5^3	b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4														
b_6^3	c_1^3	c_2^3	c_3^3	c_4^3	c_5^3	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4															
b_1^4	b_2^4	b_3^4	b_4^4	b_5^4	b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4															
b_2^4	b_3^4	b_4^4	b_5^4	b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4																
b_3^4	b_4^4	b_5^4	b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4																	
b_4^4	b_5^4	b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4																		
b_5^4	b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4																			
b_6^4	c_1^4	c_2^4	c_3^4	c_4^4	c_5^4																				

A spread with An Orbit of Length 25. Such a spread contains lines from 26 different line orbits of length 25.

2.3 Computer Search for Parallelisms

We construct the parallelisms using software of both authors that is written in C++ and performs backtrack search with rejection of some of the equivalent partial solutions. The general approach is described in [23] and the details are similar to those in [3].

3 Properties of the Constructed Parallelisms

A *regulus* of $PG(3, q)$ is a set R of $q + 1$ mutually skew lines such that any line intersecting three elements of R intersects all elements of R . Such a line is called *transversal*. All the transversals of a regulus form its *opposite regulus*. A spread S of $PG(3, q)$ is *regular* if for every three distinct elements of S , the unique regulus determined by them is a subset of S . A spread is called *Hall spread* if it can be obtained from a regular spread by a replacement of one regulus by its opposite. A spread is called *conical flock spread* if it has q reguli which have exactly one common line. A spread is called *derived conical flock spread* if it can be obtained from a conical flock spread by a replacement of one regulus by its opposite.

There are 21 nonisomorphic spreads in $PG(3, 5)$ [6]. To distinguish them we use invariants based on their relation to the reguli of the projective space. All the spreads that take part in the parallelisms we construct, are distinguished by two numbers - the number of whole reguli in the spread and the number of reguli which share exactly 4 lines (out of all 6 lines) with the spread. For the regular spread of $PG(3, 5)$ these invariants are (130, 0), for the Hall spread (31, 105), for the conical flock spread (5, 200), and for the derived conical flock spread (1, 210). A parallelism is *uniform* if all its spreads are isomorphic to each other.

3.1 Parallelisms with G_{25_3}

The group fixes one line pointwise. Each fixed point is incident with the fixed line and with all the lines of 6 line orbits of length 5, and thus a spread can have a line of at most one of these 6 orbits. Therefore a parallelism invariant under G_{25_3} must have one fixed spread and 6 spread orbits of length 5. We obtain 14851 nonisomorphic parallelisms. Their properties are presented in Table 3. The fixed spread is regular for all of them. The other 30 spreads are either Hall spreads, or derived conical flock spreads. There are 4435 uniform deficiency one parallelisms made of Hall spreads. They admit rich automorphism groups. The 12 parallelisms with automorphism groups of order at least 600 yield transitive deficiency one parallelisms. Two of the parallelisms (with full automorphism groups of order 1200) are invariant under the full central collineation group of order $q^2(q^2 - 1) = 600$ (the group fixes the fixed by G_{25_3} line) and therefore their corresponding transitive deficiency one parallelisms belong to the infinite family

Table 3. Spreads and automorphisms of parallelisms with G_{25_3}

Fixed spread (130,0)	Orbits of length 5		Order of the full auomorphism group							
	(31,105)	(1,210)	25	50	100	200	400	600	1200	2400
1	5	15	576							
1	10	20	3840							
1	15	15	3936							
1	20	10	2064							
1	30	-	4124	120	80	82	17	4	6	2

constructed by Johnson [13, 15]. It is shown in [15, Corollary 26] that for $PG(3, 5)$ the number of Johnson’s parallelisms is 2 and our results comply with this. Up to our knowledge, the remaining 10 transitive deficiency one parallelisms have not been constructed before this work.

3.2 Parallelisms with G_{25_5}

The fixed point is incident with the fixed line, and with all the lines of one line orbit of length 5 and one of length 25. Since each point has to be in each spread, a parallelism consists of a fixed spread, a spread orbit of length 5 and a spread orbit of length 25. We construct 22 nonisomorphic parallelisms. The fixed spread of all of them is the conical flock spread (Table 4), and the spread orbit of length 5 is made either of Hall spreads, or of derived conical flock spreads.

Table 4. Spreads and automorphisms of parallelisms with G_{25_5}

Fixed Spread	Spread orbit of Length 5	Spread orbit of Length 25	Full auomorphism group of order 25
(5,200)	(31,105)	(4,78)	6
(5,200)	(31,105)	(1,82)	4
(5,200)	(31,105)	(0,104)	4
(5,200)	(31,105)	(1,138)	2
(5,200)	(1,210)	(0,72)	2
(5,200)	(1,210)	(4,78)	4

4 Concluding Remarks

All the constructed parallelisms are available online. They can be downloaded from <http://www.moi.math.bas.bg/moiuser/~stela>.

Our results comply with the known theoretical investigations on transitive deficiency one parallelisms [14, chapter 38]. The 12 transitive deficiency one

parallelisms that we construct have the spread structure of the parallelisms from Johnson's infinite family [13, 15], but only two of them belong to it. The present results might lead to a future generalization of that family.

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On Alternation, VC-dimension and k -fold Union of Sets

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Abstract. Alternation of Boolean functions is a measure of non-monotonicity of the function. In this paper, we asymptotically characterize the VC-dimension of family of Boolean functions parameterized by the maximum alternation of the Boolean functions in the family. Enroute to our main result, we show exact bounds for VC-dimension of functions which has alternation 1, which strictly contains monotone functions and hence generalizes the bounds in [3]. As an application, we show tightness of VC-dimension bounds for k -fold union, by explicitly constructing a family \mathcal{F} of subsets of $\{0, 1\}^n$ such that k -fold union of the family, $\mathcal{F}^k = \left\{ \bigcup_{i=1}^k F_i \mid F_i \in \mathcal{F} \right\}$ must have VC-dimension at least $\Omega(dk)$ and that this bound holds even when the union is over disjoint sets from \mathcal{F} . This provides a non-geometric set system achieving this bound.

Keywords: VC dimension · Alternation · Extremal sets · Boolean functions

1 Introduction

Vapnik-Chervonenkis Dimension (VC-dimension) is a combinatorial measure of a set system of subsets of a universe, developed by Vapnik and Chervonenkis in the 1960s and has found deep applications the area of statistical learning theory, discrete and computational geometry.

Let U be universe and \mathcal{F} be a set of subsets of U . The family \mathcal{F} is said to *shatter* a subset $S \subseteq U$ if for all subset $S' \subseteq S$, there is a $F \in \mathcal{F}$ such that $S \cap F = S'$. Notice that it is easy to shatter small sets, especially the empty set, and U is not shattered unless $\mathcal{F} = \mathcal{P}(U)$. The VC-dimension of \mathcal{F} is the largest d such that there is a set S of size d shattered by the family.

In the case when $U = \{0, 1\}^n$, each $F \in \mathcal{F}$ can be interpreted as the positive inputs of a Boolean function. That is, as the set $f^{-1}(1) = \{x \in \{0, 1\}^n \mid f(x) = 1\}$ of some Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$. Hence, any family \mathcal{F} can be equivalently interpreted as a family of Boolean functions. Under this interpretation, the VC-dimension of a class is directly related to the complexity of learning the Boolean functions in the class in the Probably Approximately Correct (PAC) model [13]: where it yields matching upper [2] and lower [4] bounds for the number of samples required in order to learn functions from \mathcal{F} .

Motivated by this, there has been several works exploring tight VC dimensions bounds various families of Boolean functions with n variables. Tight bounds are known for VC-dimension of various subfamilies of Boolean functions such as Boolean terms (conjunction of literals) - $O(n)$ [9], k -Decision Lists - n^k [10], Symmetric Functions - n [4].

An important class of functions for which a characterization is known is the class of monotone functions [3] where the VC-dimension was established to be exactly $\binom{n}{n/2}$. A subclass of this family - namely the monotone terms were also studied in [9] to establish a tight linear bound for VC-dimension. A natural question is how to generalize these bounds to non-monotone functions as well.

For $x, y \in \{0, 1\}^n$ we say $x \preceq y$ if $\forall i \in [n], x_i \leq y_i$ where x_i represents the i^{th} bit of x . Recall that a function f is said to be monotone, if $\forall x, y \in \{0, 1\}^n, x \preceq y$ then $f(x) \leq f(y)$. Consider a maximal chain of distinct inputs $x_0, x_1, x_2 \dots, x_n \in \{0, 1\}^n$ satisfying $x_0 \preceq x_1 \preceq x_2 \preceq \dots \preceq x_n$. The alternation of f (denoted by $\text{alt}(f)$) is defined as $\max \{ \text{alt}(f, \mathcal{C}) \mid \mathcal{C} \text{ is a maximal chain in } \mathcal{B}_n \}$ where $\text{alt}(f, \mathcal{C})$ is $|\{i \mid f(x_{i-1}) \neq f(x_i), x_i \in \mathcal{C}, i \in [n]\}|$. Indeed, for a monotone f , $\text{alt}(f) = 1$ and for any Boolean function f , $\text{alt}(f) \leq n$. Thus, it forms a measure of how much non-monotone the Boolean function is.

Our Results: In this paper, we initiate a study of the VC-dimension of Boolean function families parameterized by the *alternation* and show the following results:

Exact VC-dimension for family of functions with alternation 1: We show that family of functions with alternation 1 has VC-dimension exactly $\binom{n}{n/2} + 1$. Theorem 1. We also show that this family is shattering extremal as defined by [8] and hence has some potentially useful combinatorial properties.

Tight bounds for VC-dimension for family of functions with alternation k : We show (Theorem 2 and 3) that the family of functions \mathcal{F} with alternation k has VC-dimension satisfying: $\sum_{i=\frac{n-k}{2}}^{\frac{n+k}{2}} \binom{n}{i} \leq \text{VC}(\mathcal{F}_k) \leq O\left(k \times \binom{n}{n/2}\right)$. For $k \leq \sqrt{n}$ the upper and lower bounds is asymptotically of the same order and hence the bound is tight in general.

Application to VC-dimension of disjoint union of families: If a family \mathcal{F} is of VC-dimension $\leq d$, how large can the VC-dimension of the k -fold union family, defined as, $\mathcal{F}^k = \{\cup_{i=1}^k A_i \mid A_i \in \forall i, A_i \in \mathcal{F}\}$ be? Blumer *et al.* [1] and Haussler and Welzl [6] showed that the VC-dimension is at most $O(dk \log k)$. This bound was shown to be tight by Eisentat and Angluin [5] who shows existence of a geometric family with VC-dimension at most d and the k -fold union has VC-dimension at least $\Omega(dk \log k)$. The family constructed were point sets in a plane. Using our methods, we construct a family of Boolean functions such that k -fold union has VC-dimension at least $\Omega(dk)$ even when the unions in the k -fold union are restricted to k -fold disjoint union.

2 VC-dimension Bounds for Functions with Alternation 1

As a warm up towards later sections, in this section, we describe VC-dimension bounds and extremal properties of a family of functions related to alternation and

monotonicity. As mentioned in the introduction, [3] computes the VC-dimension of family of monotone functions (denoted by \mathcal{M}) as $\text{VC}(\mathcal{M}) = \binom{n}{\lfloor n/2 \rfloor}$

A function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is said to be k -slice function if $f(x) = 0$ for every $x \in \{0, 1\}^n$, $\sum_{i=1}^n x_i < k$ and 1 if $\sum_{i=1}^n x_i > k$. Define \mathcal{M}^* to be the family of all *slice*-functions.

Proposition 1. $\text{VC}(\mathcal{M}^*) = \binom{n}{\lfloor n/2 \rfloor}$

Observe that *slice*-functions are a subclass of monotone family. So the VC-dimension is upper bounded by $\binom{n}{\lfloor n/2 \rfloor}$ and *slice*-functions can shatter only antichain similar to monotone family. Thus the above bound.

Now consider the family of functions where each function has alternation at most 1. $\mathcal{F}_1 = \{f \mid \text{either } f \text{ or } \neg f \text{ is monotone}\}$. We compute the VC-dimension of this family exactly.

Theorem 1. $\text{VC}(\mathcal{F}_1) = \binom{n}{\lfloor n/2 \rfloor} + 1$

Proof. Lower Bound: We show the lower bound by shattering a set $S \subseteq U$ of cardinality $\binom{n}{\lfloor n/2 \rfloor} + 1$. $S = \{x \in \{0, 1\}^n \mid \sum_{i=1}^n x_i = \lfloor n/2 \rfloor\} \cup \{w\}$ where w is any arbitrary point in $\{0, 1\}^n$ such that $\sum_{i=1}^n x_i < n/2$. Let $S' \subseteq S$. We need to give an $F \in \mathcal{F}$ such that $F \cap S = S'$. We consider the following cases: Case (1): $S' = \{x \in \{0, 1\}^n \mid \sum_{i=1}^n x_i = n/2\}$. F such that $F \cap S = S'$ is given by the characteristic function $f = \bigvee_{z \in S'} \bigwedge_{z_i=1} x_i$. Case (2): $S' = X \cup \{w\}$ where $\forall x \in X, \sum_{i=1}^n x_i = n/2$. F such that $F \cap S = S'$ is given by the characteristic function $f = \bigwedge_{z \in S'} \bigvee_{z_i=1} \bar{x}_i$. Observe that negation of this function is monotone.

Upper Bound: Suppose \mathcal{F}_1 shatters a set S such that $|S| \geq \binom{n}{\lfloor n/2 \rfloor} + 2$. We first obtain certain properties that S cannot have through the following lemma.

Lemma 1. \mathcal{F}_1 cannot shatter a set S if it satisfies at least one of the following:

1. **Parallel Chain:** When there are elements $p, p', q, q' \in S$ such that all of them are distinct and $p \preceq p'$ and $q \preceq q'$.
2. **Triplet Chain:** If $p, q, r \in S$ such that $p \preceq q \preceq r$ then it is said to form a triplet chain.

Proof. For parallel chain, without loss of generality let us suppose that $p \preceq p'$ and $q \preceq q'$. Suppose $S' = \{p', q\}$. Now any monotonically increasing function will obtain the set $\{p', q, q'\}$ and any monotonically decreasing function will obtain the set $\{p, p', q\}$ but never $\{p', q\}$ alone. Hence it cannot be shattered.

For triplet chain, suppose $p, q, r \in S$ such that $p \preceq q \preceq r$. Consider the set $S' = \{p, r\}$. We observe that there does not exist a function $f \in \mathcal{F}_1$ such that it is true on p and r and evaluates to false on q . Hence S cannot be shattered.

Now we claim that set S which is shattered must be a disjoint unions of at most 2 maximal antichains. To see this, suppose $S = S_1 \uplus S_2 \uplus S_3$ such that the sets are maximal antichain. Without loss of generality, consider an element $p \in$

S_1 . Now there exists a point $q \in S_2$ such that p and q are comparable (otherwise they will be in the same set). Now if this is the case then neither p nor q can be related to any point in S_3 as it will either form a chain length of 3 i.e. $p < q < r$ (triplet chain) or parallel chains because of which the set S cannot be shattered (see Lemma 1). Now if p and q are not comparable then we can have a larger antichain by including either p or q in the set S_3 which contradicts the maximality of antichain set S_3 . Using similar argument for each, we conclude that, the set S can be disjoint union of at most 2 maximal antichains.

Hence we conclude that there must be $S = S_1 \uplus S_2$ where S_1, S_2 are maximal antichains i.e. no elements from S_1 can be put into S_2 and vice versa. Using Lemma 1 again, we have that S_1 and S_2 does not have either parallel chain or triplet chain. But that contradicts the maximality of S_1 and S_2 . Consider $p', q' \in S_2$. Then $\exists p \in S_1$ such that $p \preceq p'$ and $p \preceq q'$. But we can obtain a larger antichain by including p', q' into S_1 . Thus contradicting the maximality.

We now show that \mathcal{M} exhibit a special property (Proposition 2) which is also known as *s-extremal* or *shattering extremal* family. Let $Sh(\mathcal{F})$ denote set of shattered sets by a family \mathcal{F} .

Proposition 2. \mathcal{M} is shattering extremal i.e. $|Sh(\mathcal{M})| = |\mathcal{M}|$.

This can be proved using the fact that there is one-to-one correspondence between a monotone function and an antichain and \mathcal{M} shatters only antichains.

3 Bounds for VC-dimension in Terms of $alt(f)$

In this section, we derive VC-dimension bounds for families of Boolean functions, parameterized by the maximum alternation of functions in the family. We need the following known theorem.

Lemma 2 (Characterization of Alternation [1]). Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$. Then there exists $k = alt(f)$ monotone functions g_1, \dots, g_k each from $\{0, 1\}^n$ to $\{0, 1\}$ such that $f(x) = \bigoplus_{i=1}^k g_i$ if $f(0^n) = 0$ and $f(x) = \neg \bigoplus_{i=1}^k g_i$ if $f(0^n) = 1$.

We use this theorem to establish the following upper bound:

Theorem 2. Let $k > 1$. If \mathcal{F}_k is the family of Boolean functions f such that $alt(f) \leq k$. Then, $VC(\mathcal{F}_k) \leq O\left(k \binom{n}{n/2}\right)$

Proof. Using Lemma 2, we get $\mathcal{F}_k = \{(\neg \oplus \text{ or }) \bigoplus_{i=1}^k f_i \mid f_i \in \mathcal{M}\}$. We look at a family $\mathcal{G} = \{f \oplus g \mid f = \bigoplus_{i=1}^k f_i, f_i \in \mathcal{M}, g = const\}$ where $g(x) = 1$ if $f(0^n) = 1$ and $g(x) = 0$ if $f(0^n) = 0$. Observe that $\mathcal{F}_k \subseteq \mathcal{G}$ and hence $VC(\mathcal{F}_k) \leq VC(\mathcal{G})$. We now show an upper bound in general for such constructed families. Given k classes of n -bit Boolean functions $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$, and a fixed Boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$. We define: $\mathcal{F}(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k) = \{f(f_1(\cdot), \dots, f_k(\cdot)) \mid f_i \in \mathcal{F}_i, i \in [k]\}$. We have,

Lemma 3 ([2, 6, 11]). Let $d = \max_{i \in [k]}(VC(\mathcal{F}_i))$. $VC(\mathcal{F}(\mathcal{F}_1, \dots, \mathcal{F}_k)) \leq O(dk \log k)$

Applying this lemma we obtain for the above family $\text{VC}(\mathcal{F}_k) \leq O\left(k \binom{n}{n/2} \log k\right)$. We show below how to improve the bound. The idea is simple counting: we have $|\mathcal{G}| \leq |\mathcal{M}|^{k+1}$. We know that $\text{VC}(\mathcal{G}) \leq \log(|\mathcal{G}|)$, which gives us $\text{VC}(\mathcal{G}) \leq (k + 1) \log(\mathcal{M})$. This bound is the *Dedekind's number* and we use the following bound due to *Kleitman et al.* (refer [7]): $\log(\mathcal{M}) \leq \binom{n}{n/2} \left(1 + O\left(\frac{\log n}{n}\right)\right)$. This gives, $\text{VC}(\mathcal{F}_k) \leq (k + 1) \binom{n}{n/2} \left(1 + O\left(\frac{\log n}{n}\right)\right)$ Hence, we have $\text{VC}(\mathcal{F}_k) \leq O(k \binom{n}{n/2})$.

Now we turn to the lower bound. Using the fact that for any $k \geq 1$, the family \mathcal{F}_k also includes the set of monotone functions \mathcal{M} , the $\text{VC-dimension}(\mathcal{F}_k) \geq \text{VC-dimension}(\mathcal{M})$. Hence $\text{VC-dimension}(\mathcal{F}_k) \geq \binom{n}{n/2}$. We can improve this:

Theorem 3. *Let $k > 1$. If \mathcal{F}_k is the family of Boolean functions f such that $\text{alt}(f) \leq k$. Then, $\text{VC}(\mathcal{F}_k) \geq \sum_{i=n/2-k/2}^{n/2+k/2} \binom{n}{i}$*

Proof. We shatter the set $S = \{x \in \{0, 1\}^n \mid n/2 - k/2 \leq \sum_{i=1}^n x_i \leq n/2 + k\}$. To obtain any $S' \subseteq S$, we give $f \in \mathcal{F}_k$ as $f(x) = 1$ whenever $x \in S'$ and $f(x) = 0$ otherwise. It remains to show that $\text{alt}(f) \leq k$. Since the number of 1s in $x \in S'$ can only be in the range $[n/2 - k/2, n/2 + k/2]$, any chain in this part will have alternation at most k and in the remaining part 0. Hence we obtain all $S' \subseteq S$.

VC-dimension of Read-Once Functions: A Boolean function is said to be *read-once* if there is a Formula (a Boolean circuit where every gate has fanout at most 1) computing the function f such that every variable appears only once in the formula. Monotone read-once functions have no negations in the formula computing the function. The following lemma motivates the study of VC-dimension under composition between read-once functions and functions with alternation at most 1. We defer the proof to the full version.

Lemma 4. *Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ such that $\text{alt}(f) \leq k$. Then $f = g(f_1, f_2, \dots, f_k)$ where g is a monotone-Read Once formula and $f_i \in \mathcal{F}_1$ where \mathcal{F}_1 is the family of functions with alternation at most 1.*

The above lemma follows from Lemma 2 and the fact that $f_i \rightarrow f_{i+1}$. This motivates the study of VC-dimension of monotone read-once functions which could potentially be applied to improve the bound of $\text{alt-}k$ family. Let $\mathcal{R} = \{f : \{0, 1\}^n \rightarrow \{0, 1\} \mid f \text{ is monotone read-once}\}$.

Proposition 3. $n \leq \text{VC}(\mathcal{R}) \leq O(n \log n)$

Lower bound follows from the fact that Monomial family is a subclass of Read-once family. Upper Bound follows from counting all Read-once functions.

4 Application to VC-dimension of k -fold Union

In this section we show VC-dimension bound for a non-geometric family which is a k -union. We also remark in the end that even if we restrict our family to have only disjoint union of k functions, we obtain a VC-dimension bound of $\Omega(dk)$.

Lemma 5. *Let $\mathcal{F}_{2k} = \{f : \{0, 1\}^n \rightarrow \{0, 1\} \mid \text{alt}(f) \leq 2k\}$. Then this family is same¹ as $\mathcal{G} = \left\{ \bigcup_{i=1}^k g_i \mid g_i : \{0, 1\}^n \rightarrow \{0, 1\}, \text{alt}(g_i) \leq 2 \right\}$*

Proof. Let $f \in \mathcal{F}_{2k}$. Due to alternation characterization described in Lemma 2 we have, $f = \bigoplus_{i=1}^{2k} f_i = \bigoplus_{i=1}^k (\neg f_{2i-1} \wedge f_{2i}) \vee (f_{2i-1} \wedge \neg f_{2i})$. It can be observed from the construction of [1], that $f_i \rightarrow f_{i+1}$. Now using this fact we obtain $f = \bigvee_{i=1}^k (\neg f_{2i-1} \wedge f_{2i}) = \bigvee_{i=1}^k g_i$ such that $\text{alt}(g_i) \leq 2$. In fact, it can also be observed that f is the disjoint union of k sets (See full version for details).

For the reverse direction, we have a Boolean function $g : \{0, 1\}^n \rightarrow \{0, 1\}$, $g = \bigvee g_i$ where $g_i : \{0, 1\}^n \rightarrow \{0, 1\}$ and $\text{alt}(g_i) \leq 2$. We need to show that $\text{alt}(g) \leq 2k$. We use the property that $\text{alt}(g_1 \vee g_2) \leq \text{alt}(g_1) + \text{alt}(g_2)$ iteratively to conclude that $\text{alt}(g) \leq 2k$ (See full version for details.)

Theorem 4. *Let $\mathcal{F}_2 = \{f : \{0, 1\}^n \rightarrow \{0, 1\} \mid \text{alt}(f) \leq 2\}$. Consider the family $\mathcal{F}^{k\cup} = \left\{ \bigcup_{i=1}^k f_i \mid f_i \in \mathcal{F}_2 \right\}$. For $k \leq \Theta(\sqrt{n})$, we have $\text{VC}(\mathcal{F}^{k\cup}) = \Theta(k \binom{n}{n/2})$.*

Proof. From Lemma 5, we have that the family $\mathcal{F}^{k\cup}$ can alternately be represented as parity-composition of a family of monotone Boolean functions. So $\mathcal{F}^{k\cup} = \mathcal{F}_{2k}$. We conclude using Theorem 2 that $\text{VC}(\mathcal{F}^{k\cup}) \leq O(k \binom{n}{n/2})$. From Theorem 3 we have: $\text{VC}(\mathcal{F}^{k\cup}) \geq \sum_{i=n/2-k}^{n/2+k} \binom{n}{i}$. Using standard bounds (c.f. [12]), when

$$i \leq c\sqrt{n}, \text{ we obtain } \binom{n}{n/2+i} = c_1 \binom{n}{n/2}, c_1 > 0 \text{ which yields } \Omega(k \binom{n}{n/2}).$$

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¹ Under the interpretation of the sets in the system as $f^{-1}(1)$ for Boolean functions, \vee and \cup are used interchangeably.

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Weak Components of the Directed Configuration Model

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Abstract. We study the threshold for the existence of a linear order weakly connected component in the directed configuration model, confirming analytic but non-rigorous results recently obtained by Kryven [8]. We also establish convergence in probability of the fraction of vertices and edges that are contained in the largest component. As a consequence of our results, we obtain that the “separation” between the thresholds for the existence a giant weakly and strongly connected component is in some sense independent from the in-/out-degree correlation. We formalise this idea using bond percolation.

Keywords: Directed random graphs · Directed configuration model · Weak connected components · Multi-type branching processes

1 Introduction

The study of the component structure of random graphs with given degrees, and in particular of the configuration model, CM, was pioneered by the work of Molloy and Reed [9] who provided a criterion to determine if a degree sequence typically produces a linear order connected component (known as the *giant*) or if its largest component has sublinear order. Since then, it has become one of the central topics in random graph theory [1, 6, 7].

Directed models are much less understood. Newman, Strogatz and Watts [10] initiated the study of the directed configuration model, DCM, and located the threshold for the existence of a giant strongly connected component (SCC). Later, Cooper and Frieze [4] provided a rigorous proof for the existence of such threshold under certain conditions of the degree sequence. This problem has been recently revisited by Cai and the second author [3], extending the range of applicability of the result.

Weakly connected components (WCC) naturally arise in areas such as epidemiology, data mining or communication networks. In the physics community,

G. Perarnau—Supported by the Spanish Ministerio de Economía y Competitividad project MTM2017-82166-P and the MSCA-RISE-2020-101007705 - ‘RandNet’.

the study of WCC’s has been neglected under the assumption that it effectively behaves like the undirected case (see e.g. [10]). Kryven [8] observed that assumption is wrong and predicted an alternative threshold for the appearance of the giant WCC, supported with an analytical but non-rigorous approach based on generating functions for bounded bi-degree distributions. The aim of this paper is to provide a formal proof for the existence of the giant WCC threshold in the directed configuration model under the much weaker assumption of bounded second moments.

Let $[n] := \{1, \dots, n\}$ be a set of n vertices. Let $\mathbf{d}_n = ((d_1^-, d_1^+), \dots, (d_n^-, d_n^+))$ be a bi-degree sequence with $m_n := \sum_{i \in [n]} d_i^+ = \sum_{i \in [n]} d_i^-$. The directed configuration model, $\text{DCM} = \text{DCM}(\mathbf{d}_n)$, is the random directed multigraph on vertex set $[n]$ generated by assigning d_i^- in half-edges (*heads*) and d_i^+ out half-edges (*tails*) to vertex i , and then choosing a uniformly random matching between the set of heads and the set of tails. Let $n_{k,\ell} = \{i : (d_i^-, d_i^+) = (k, \ell)\}$. Let $D_n = (D_n^-, D_n^+)$ be the degree pair of a vertex chosen uniformly at random, that is $\mathbb{P}(D_n = (k, \ell)) = n_{k,\ell}/n$.

Let $(\mathbf{d}_n)_{n \geq 1}$ be a sequence of bi-degree sequences. We will consider sequences that satisfy the following.

Condition 1. *There exists a discrete probability distribution $D = (D^-, D^+)$ on $\mathbb{Z}_{\geq 0}^2$ with $\lambda_{k,\ell} := \mathbb{P}(D = (k, \ell))$ such that we have:*

- (i) *convergence in distribution, for $k, \ell \geq 0$, $\lim_{n \rightarrow \infty} \frac{n_{k,\ell}}{n} = \lambda_{k,\ell}$;*
- (ii) *convergence of expected values, $\lim_{n \rightarrow \infty} \mathbb{E}[D_n^\pm] = \mathbb{E}[D^\pm] =: \lambda \in (0, \infty)$;*
- (iii) *convergence of second moments,*
 $\lim_{n \rightarrow \infty} \mathbb{E}[D_n^- (D_n^- - 1)] = \mathbb{E}[D^- (D^- - 1)] =: \mu_{2,0} \in (0, \infty)$,
 $\lim_{n \rightarrow \infty} \mathbb{E}[D_n^- D_n^+] = \mathbb{E}[D^- D^+] =: \mu_{1,1} \in (0, \infty)$, and
 $\lim_{n \rightarrow \infty} \mathbb{E}[D_n^+ (D_n^+ - 1)] = \mathbb{E}[D^+ (D^+ - 1)] =: \mu_{0,2} \in (0, \infty)$.

Define the *in- and out-size biased* distributions of D by

$$\mathbb{P}(D_{\text{in}} = (k - 1, \ell)) = \frac{k\lambda_{k,\ell}}{\lambda}, \quad \mathbb{P}(D_{\text{out}} = (k, \ell - 1)) = \frac{\ell\lambda_{k,\ell}}{\lambda}. \tag{1}$$

Consider the random matrix and its mean matrix,

$$\Xi := \begin{pmatrix} D_{\text{out}}^- & D_{\text{out}}^+ \\ D_{\text{in}}^- & D_{\text{in}}^+ \end{pmatrix}, \quad M := \mathbb{E}[\Xi] = \begin{pmatrix} \mu_{1,1} & \mu_{0,2} \\ \mu_{2,0} & \mu_{1,1} \end{pmatrix}, \tag{2}$$

where D_{out} and D_{in} are independent and M has largest eigenvalue, $\rho := \mu_{1,1} + \sqrt{\mu_{2,0}\mu_{0,2}}$. Let $\mathbf{q} = (q_-, q_+)$ be the extinction probability vector of a 2-type branching process with offspring Ξ .

Let \mathcal{W}_n be the largest WCC in DCM. Let $v(\mathcal{W}_n)$ and $e(\mathcal{W}_n)$ be the number of vertices and edges in \mathcal{W}_n , respectively. Our main result is that the existence of a giant WCC undergoes a phase transition at $\rho = 1$:

Theorem 2. *Suppose that $(\mathbf{d}_n)_{n \geq 1}$ satisfies Condition 1. If $\rho > 1$, then*

$$\frac{v(\mathcal{W}_n)}{n} \rightarrow \eta \in (0, 1] \quad \text{and} \quad \frac{e(\mathcal{W}_n)}{n} \rightarrow \zeta \in (0, \lambda], \tag{3}$$

in probability, where $\eta := \sum_{k,\ell \geq 0} \lambda_{k,\ell}(1-q_-^k q_+^\ell)$ and $\zeta := \sum_{k,\ell \geq 0} k \lambda_{k,\ell}(1-q_-^k q_+^\ell)$. If $\rho < 1$, then, in probability,

$$\frac{v(\mathcal{W}_n)}{n} \rightarrow 0 \quad \text{and} \quad \frac{e(\mathcal{W}_n)}{n} \rightarrow 0. \tag{4}$$

Remark 3. The SCC’s of DCM have been studied in [3, 4] under Condition 1. Denote by \mathcal{S}_n the largest SCC. Then, if $\mu_{1,1} > 1$ or $\mu_{1,1} < 1$, analogues of (3) and (4) hold, respectively. So, the existence of a giant SCC undergoes a phase transition at $\mu_{1,1} = 1$.

The combination of the previous results allows us to quantify the “separation” between the WCC and the SCC thresholds through bond percolation. Suppose that \mathbf{d}_n satisfies $\mu_{1,1} > 1$. It is easy to show that for any $p \in [0, 1]$ the p -percolated random digraph $\text{DCM}_p(\mathbf{d}_n)$ is distributed in law as $\text{DCM}(\mathbf{d}_n^p)$, where \mathbf{d}_n^p is the p -thinned version of \mathbf{d}_n conditional on having equal number of heads and tails (for an undirected analogue of this result, see Lemma 3.2 in [5]).

Define $p_{\text{SCC}} = (\mu_{1,1})^{-1/2}$ and $p_{\text{WCC}} = (\mu_{1,1} + \sqrt{\mu_{2,0}\mu_{0,2}})^{-1/2}$, and note that $0 < p_{\text{WCC}} < p_{\text{SCC}} < 1$. We obtain a two-point threshold phenomenon for $\text{DCM}_p(\mathbf{d}_n)$:

- if $p \in [0, p_{\text{WCC}})$: whp no a giant WCC exists;
- if $p \in (p_{\text{WCC}}, p_{\text{SCC}})$: whp a giant WCC, but no giant SCC exists;
- if $p \in (p_{\text{SCC}}, 1]$: whp a giant SCC exists.

For an SCC-critical sequence \mathbf{d}_n (i.e. satisfying $\mu_{1,1} = 1$), $p_{\text{WCC}} \in (0, 1)$ does not depend on the in-/out-degree correlation, only on the marginals.

Remark 4. Theorem 2 fails for sequences with infinite second moment. For instance, let $\Delta = n^{2/3}$ and consider the sequence with one vertex with degrees $(\Delta, 0)$ and all other vertices with degrees $(1, 0)$ or $(0, 1)$. The largest eigenvalue ρ of the mean matrix M diverges but clearly the largest WCC has order $n^{2/3} + 1$.

2 Sketch of the Proof Of Theorem 2

2.1 Multitype Branching Processes

Fix $p \in \mathbb{N}$. We write $\mathbf{0}$ and $\mathbf{1}$ for the all zeros and all ones vectors of length p , respectively. For any $\omega \in \mathbb{R}$, we write $\boldsymbol{\omega} = \omega \mathbf{1}$. Let $\Xi = (\xi_{ij})$ be a random $p \times p$ matrix with entries in $\mathbb{Z}_{\geq 0}$ and let $(\Xi(m; t))_{m \geq 1, t \geq 0}$ be iid (independent and identically distributed) copies of Ξ . Let $\mathbf{z} = (z_1, \dots, z_p)$. For $i \in [p]$, define the generating function

$$h_i(\mathbf{z}) = \sum_{k_1, \dots, k_p \geq 0} \mathbb{P}(\cap_{j \in [p]} \{\xi_{ij} = k_j\}) \prod_{j \in [p]} z_j^{\xi_{ij}}$$

and $\mathbf{h}(\mathbf{z}) = (h_1(\mathbf{z}), \dots, h_p(\mathbf{z}))$. Denote by $m_{ij} = \mathbb{E}[\xi_{ij}]$ and by $M = (m_{i,j})$ the mean matrix. We say that M is finite if $m_{i,j} < \infty$ for all i, j . We say that M is irreducible if for every pair i, j there exists $t \in \mathbb{N}$ such that $(M^t)_{i,j} > 0$. Let ρ be the largest eigenvalue of M .

A p -type branching process with offspring distribution Ξ starting at $a \in [p]$ is a stochastic process $(\mathbf{X}^{(a)}(t) = (X_1^{(a)}(t), \dots, X_p^{(a)}(t)))_{t \geq 0}$ defined as follows:

$$X_j^{(a)}(t) = \begin{cases} \mathbb{1}_{a=j} & (t = 0) \\ \sum_{i \in [p]} \sum_{m=1}^{X_i^{(a)}(t-1)} \xi_{ij}(m; t-1) & (t \geq 1) \end{cases}$$

If $\mathbf{X}^{(a)}(t) \neq \mathbf{0}$ for all $t \in \mathbb{N}$, the branching process is said to *survive*; otherwise, it is said to *become extinct*. Let $s^{(a)} = \mathbb{P}(\cap_{t \geq 0} \{\mathbf{X}^{(a)}(t) \neq \mathbf{0}\})$ and $q^{(a)} = 1 - s^{(a)}$, be the survival and extinction probabilities, respectively. Write $\mathbf{s} = (s^{(1)}, \dots, s^{(p)})$ and $\mathbf{q} = (q^{(1)}, \dots, q^{(p)})$.

If we condition on the process becoming extinct, then $\mathbf{X}^{(a)}(t)$ becomes a subcritical branching process with offspring having generating function $\hat{\mathbf{h}}(\mathbf{z}) = \mathbf{q}^{-1} \mathbf{h}(\mathbf{qz})$, where $\mathbf{q}^{-1} = (1/q^{(1)}, \dots, 1/q^{(p)})$. Let $\hat{\rho} < 1$ be the largest eigenvalue of the mean matrix of the conditioned process.

We develop the following result about subcritical growth of supercritical branching processes conditioned on survival.

Theorem 5. *For $a \in [p]$, let $(\mathbf{X}^{(a)}(t))_{t \geq 0}$ be an irreducible p -type branching process with offspring distribution Ξ and mean matrix M with $\rho \in (1, \infty)$ and $\mathbf{q} > \mathbf{0}$. Then there exist constants $c, C > 0$ and a function $\tau(\ell) = (1 + o_\ell(1)) \log_\rho \ell$ such that*

$$c\hat{\rho}^t \leq \mathbb{P}\left(\cap_{r=1}^t \{\mathbf{0} \neq \mathbf{X}^{(a)}(r) < \boldsymbol{\omega}\}\right) \leq C\hat{\rho}^{t-\tau(\boldsymbol{\omega})}, \quad \text{for all } t \geq 1, \boldsymbol{\omega} \geq t. \quad (5)$$

This allows us to control the $\boldsymbol{\omega}$ -expansion time of the process.

Lemma 1. *For $a \in [p]$, let $(\mathbf{X}^{(a)}(t))_{t \geq 0}$ be an irreducible p -type branching process with offspring distribution Ξ and mean matrix M with $\rho \in (1, \infty)$. Let $T_\omega^{(a)} := \inf\{t : \mathbf{X}^{(a)}(t) \not\prec \boldsymbol{\omega}\}$. Then for all $\varepsilon > 0$ and as $\boldsymbol{\omega} \rightarrow \infty$,*

$$\mathbb{P}\left(T_\omega^{(a)} \leq (1 + \varepsilon) \log_\rho \boldsymbol{\omega}\right) \rightarrow 1 - q^{(a)}, \quad \mathbb{P}\left(T_\omega^{(a)} \in ((1 + \varepsilon) \log_\rho \boldsymbol{\omega}, \infty)\right) \rightarrow 0. \quad (6)$$

This can be easily generalised to multiple iid branching processes. We omit this result for brevity.

2.2 Exploration and Coupling of the Weak Components

We will use a Breadth First Search (BFS) digraph exploration process on the weak components of DCM starting at a vertex $v \in [n]$. This is equivalent to the usual BFS process on the graph obtained by removing the directions of the edges in DCM. Let $F_v(t)$ be the tree generated by the BFS process starting at v up to time t , where we only add edges if they reveal a vertex not exposed yet and assign them a mark depending on the direction we traverse them.

The *in- and out-size biased* distributions of D_n are defined as in (1). Then, by (i) of Condition 1, $(D_n)_{\text{in}} \rightarrow D_{\text{in}}$ and $(D_n)_{\text{out}} \rightarrow D_{\text{out}}$ in distribution. Similarly we define the sequence of random matrices $(\Xi_n)_{n \geq 0}$ with mean matrices M_n .

By Condition 1, Ξ_n converges in distribution and in \mathcal{L}^1 to Ξ and furthermore, M_n and M are finite and irreducible. Let ρ_n be the largest eigenvalue of Ξ_n . Let $\mathbf{q}_n = (q_n^-, q_n^+)$ be the extinction probability vector for the 2-type processes with offspring Ξ_n . Note that $\rho_n \rightarrow \rho$ and $\mathbf{q}_n \rightarrow \mathbf{q}$.

Let $\text{GW}_{\Xi}^{(d_1, d_2)}$ be $d_1 + d_2$ independent 2-type Galton-Watson trees with offspring distribution Ξ , the first d_1 ones starting with a particle of type 1, and the last d_2 , starting with a particle of type 2.

One can define two new sequences Ξ_n^\uparrow and Ξ_n^\downarrow that will stochastically dominate and be stochastically dominated by Ξ_n , respectively. This allows us to couple the digraph exploration process with the 2-type branching processes defined previously (see [2, Lemma 5.3] for a unidimensional version of it).

Lemma 2. *Let $v \in [n]$ with $d_v^- = d_1$ and $d_v^+ = d_2$. Let $\beta > 0$ be sufficiently small. For every rooted marked tree F with $\ell := |V(F)| \leq n^\beta$ we have*

$$(1 - o(1))\mathbb{P}\left(\text{GW}_{\Xi_n^\downarrow}^{(d_1, d_2)} \cong F\right) \leq \mathbb{P}(F_v(\ell) = F) \leq (1 + o(1))\mathbb{P}\left(\text{GW}_{\Xi_n^\uparrow}^{(d_1, d_2)} \cong F\right).$$

2.3 Expansion, Connection Probabilities and the Supercritical Case

We say that a half-edge f is at distance t from $v \in [n]$ if the shortest walk from v to the vertex incident to f has length t . Denote by $\mathcal{N}_t^\pm(v)$ (and $\mathcal{N}_{\leq t}^\pm(v)$) the set of head/tails at distance (at most) t from v .

Fix $\omega := \log^6 n$. Define the ω -expansion time of v as

$$t_\omega(v) := \inf \{t \geq 1 : \max\{|\mathcal{N}_t^-(v)|, |\mathcal{N}_t^+(v)|\} \geq \omega\}. \tag{7}$$

Combining Lemma 1 with $p = 2$ and Lemma 2, we obtain the following.

Lemma 3. *Suppose $\rho > 1$. Fix $\varepsilon \in (0, 1/2)$ and distinct $u, v \in [n]$. Define the event $A_1(v, \varepsilon) := \{t_\omega(v) \leq (1 + \varepsilon) \log_\rho \omega\}$. As $n \rightarrow \infty$,*

$$\mathbb{P}(A_1(u, \varepsilon)) = (1 + o(1))(1 - q_u^- q_u^+) \tag{8}$$

$$\mathbb{P}(A_1(u, \varepsilon) \cap A_1(v, \varepsilon)) = (1 + o(1))(1 - q_u^- q_u^+)(1 - q_v^- q_v^+). \tag{9}$$

Using Lemma 1 again, we prove that expansions are unlikely to happen late.

Lemma 4. *Assume that $\rho > 1$. Fix $\varepsilon \in (0, 1/2)$ and $v \in [n]$. As $n \rightarrow \infty$,*

$$\mathbb{P}(t_\omega(v) \in ((1 + \varepsilon) \log_\rho \omega, \infty)) = o(1). \tag{10}$$

Finally, we show that any pair of large sets are connected by a path.

Lemma 5. *For all $\varepsilon > 0$ and sets $\mathcal{X}, \mathcal{Y} \subseteq [n]$ with $|\mathcal{X}|, |\mathcal{Y}| \geq \omega$,*

$$\mathbb{P}(\text{dist}(\mathcal{X}, \mathcal{Y}) > (1 + \varepsilon) \log_\rho n) = o(n^{-2}). \tag{11}$$

By Lemma 5, with high probability, all but at most $o(n)$ vertices with finite ω -expansion time are in the same weakly connected component and by Lemma 4 there are at most $o(n)$ additional vertices in this component. Moreover, Lemma 3 allows us to find the expected order and size of this component, compute their second moment, and thus prove concentration around their expectations.

2.4 Subcritical Case

Define $\mathcal{C}(v)$ to be the component of DCM which contains v . Then, (4) follows by Markov's inequality from the next result.

Lemma 6. *If $\rho < 1$, then $\mathbb{E}_v[|\mathcal{C}(v)|] = o(n)$.*

We give a sketch of the proof. Fix $v \in [n]$ and $\beta > 0$ sufficiently small. For $h \in \mathbb{N}$, write $\mathcal{N}_h = \mathcal{N}_h^-(v) \cup \mathcal{N}_h^+(v)$. We shall call v *big* if any of the following hold and *small* otherwise:

- i) $d_v^- + d_v^+ \geq n^{\beta/3}$;
- ii) $|\mathcal{N}_h| \geq n^{\beta/2}$ for some $h \in \mathbb{N}$;
- iii) $|\mathcal{N}_{h_0}| \geq 1$ for $h_0 = \log_{1/\rho} n$.

All small vertices are contained in components of order at most $n^{\beta/2} \log_{1/\rho} n = o(n)$. We can use classical results on multitype branching processes and coupling, to bound the probability of each property i)–iii) in turn. Then, Markov's inequality implies that there are $o(n)$ big vertices. It follows that

$$\mathbb{E}_v[|\mathcal{C}(v)|] = \frac{1}{n} \mathbb{E} \left(\sum_{v \in [n]} |\mathcal{C}(v)| \right) = o(n).$$

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Decomposing and Colouring Locally Out-Transitive Oriented Graphs

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Abstract. We study the dichromatic number of a digraph, defined as the minimum number of parts in a partition of its vertex set into acyclic induced subdigraphs. We consider the class of oriented graphs such that the out-neighbourhood of any vertex induces a transitive tournament and prove for it a decomposition theorem. As a consequence, we obtain that oriented graphs in this class have dichromatic number at most 2, proving a conjecture of Naserasr and the first and third authors of this paper in *Extension of the Gyárfás-Sumner conjecture to digraphs* arXiv:2009.13319.

Keywords: Directed graphs · Graph colouring

1 Introduction

1.1 Notation

In this paper, directed graphs (*digraphs* in short) are simple, i.e. contain no loop and no multi-arc. If in addition it contains no *digon* (a cycle on two vertices), we say that it is an *oriented graph*. This paper is mostly concerned with oriented graphs.

An *acyclic colouring* of a digraph is an assignment of colours to the vertices such that each colour induces an *acyclic subdigraph*, that is a subdigraph containing no directed cycle. The *acyclic chromatic number*, or simply *dichromatic number*, of a digraph D , denoted $\overrightarrow{\chi}(D)$, is defined to be the smallest number of colours required for an acyclic colouring of D . This notion was first introduced in 1982 by Neumann-Lara [11] and has attracted a lot of attention in the past decade (see for example [4, 7, 10, 14]) as it seems to be the natural generalization for digraphs of the usual chromatic number.

Let D be a digraph. It is *strongly connected* if there is a directed path between each ordered pair of vertices. In this case we say that D is a *strong* digraph. For a vertex x of D , we denote by $x^+(D)$ (resp. $x^-(D)$) the set of its out-neighbours (resp. in-neighbours):

$$x^+(D) = \{y \in V(D), xy \in A(D)\}$$

$$x^-(D) = \{y \in V(D), yx \in A(D)\}$$

If there is no ambiguity on the digraph, we will simply use x^+ and x^- . If H is a subdigraph of D , we define the *contraction* D/H as the digraph obtained by removing all vertices of H , then adding a new vertex h such that xh (resp. hx) is an arc of D/H if $x^+ \cap H$ (resp. $x^- \cap H$) is non empty. Beware that this graph might contain digons even if D does not.

We denote by \overleftrightarrow{K}_2 the digraph on two vertices with an arc in both directions and by C_k the directed cycle on k vertices. The oriented graph on three vertices with a vertex of out-degree 2 (resp of in-degree 2) and two vertices of in-degree 1 (resp. of out-degree 1) is called S_2^+ (resp. S_2^-). A tournament is an orientation of a complete graph. We denote by TT_k the unique acyclic tournament on k vertices, called *transitive tournament*.

Given a set \mathcal{F} of digraphs we denote by $\text{Forb}_{ind}(\mathcal{F})$ the set of digraphs which have no member of \mathcal{F} as an induced subdigraph. If \mathcal{F} is explicitly given by a list of digraphs F_1, F_2, \dots, F_k we will write $\text{Forb}_{ind}(F_1, F_2, \dots, F_k)$ instead of $\text{Forb}_{ind}(\{F_1, F_2, \dots, F_k\})$. For example, $\text{Forb}_{ind}(\overleftrightarrow{K}_2)$ is the class of oriented graphs.

For a property P of digraphs (like tournament, acyclic), a digraph is *locally out- P* (resp. *locally in- P*) if for every vertex x , x^+ (resp. x^-) induces a digraph in P . We will make one exception for one of the main classes of this paper: for the oriented graphs for which the out-neighborhood of every vertex is a transitive tournament, we will use the term “out-transitive oriented graphs” instead of the heavier and possibly confusing “out-transitive tournament oriented graphs”.

1.2 Context and Presentation of the Main Results

Many theorems or conjectures about chromatic number revolve around the following question: what induced substructures are expected to be found inside a graph if we assume it has very large chromatic number? Or equivalently what are the minimal families \mathcal{F} such that graphs that do not contain any graph in \mathcal{F} as an induced subgraph has bounded chromatic number? In the case where \mathcal{F} is finite, Gyárfás and Sumner proposed the following tantalizing conjecture:

Conjecture 1 (Gyárfás-Sumner, [8, 16]). Given a finite set of graphs \mathcal{F} , graphs in $\text{Forb}_{ind}(\mathcal{F})$ have bounded chromatic number if and only if \mathcal{F} contains a complete graph and a forest.

In [2], Reza Naserasr and the first and third authors of this paper investigated such questions in the setting of digraphs. A *hero* is a tournament H such that every tournament not containing H have bounded dichromatic number. In [4], Berger et al. give a full description of heroes. Extending the notion of hero, a set \mathcal{F} of digraphs is said to be *heroic* if every digraph in $\text{Forb}_{ind}(\mathcal{F})$ has bounded dichromatic number. In [2], the following conjecture was given, along with a proof of the only if part. It can be seen as an analogue to the Gyárfás-Sumner Conjecture for oriented graphs.

Conjecture 2 ([2]). Let H be a hero and let F be an oriented forest. The set $\{\overleftrightarrow{K}_2, H, F\}$ is heroic if and only if:

- either F is the disjoint union of oriented stars,
- or H is a transitive tournament.

This conjecture is still widely open. The first case that was left in [2] is the case $F = S_2^+$, and $H = C_3$ and it was conjectured that digraphs in $\text{Forb}_{\text{ind}}(\overrightarrow{K_2}, C_3, S_2^+)$ have dichromatic number at most two. Here we prove this result, and in fact a stronger result. Before stating it, let us note that $\text{Forb}_{\text{ind}}\{\overrightarrow{K_2}, S_2^+\}$ is the class of oriented graphs such that the out-neighbourhood of any vertex induces a tournament. These objects, also called *locally out-tournaments*, are already well studied (see Chapter 6 in [3]), and often constitute the next step when trying to generalize results on tournaments.

Theorem 3 1. *Any oriented graph such that the out-neighbourhood of any vertex induces a transitive tournament has dichromatic number at most 2.*

In other words, locally out-transitive oriented graphs are 2-dicolorable. Note that it indeed is stronger than the conjecture mentioned above as it amounts to forbidding $\overrightarrow{K_2}$, S_2^+ and the hero on 4 vertices made of a directed triangle C_3 plus a vertex with an arc going to the three other vertices. As already said, forbidding S_2^+ implies that every out-neighbourhood is a tournament, and forbidding this hero implies that the tournament must be acyclic, hence transitive.

The proof of Theorem 3 relies on a structural decomposition theorem (see Theorem 4) for the class of out-transitive oriented graphs, that is a theorem whose statement is of the kind: either a graph in this class is “basic” (belongs to some simple subclass, that will be described in Sect. 2), or it can be decomposed in some prescribed ways. These kind of decomposition theorems proved to be a very powerful strategy in the world of undirected graphs (the most famous example being the celebrated proof of the perfect graph conjecture by Chudnovsky et al. in [5]), but there are, up to our knowledge, not many theorems of this kind in the world of digraphs, and there is no reason to believe that it could not be as effective in this setting.

Remark 1. A few days prior to the submission of this paper, R. Steiner published on arXiv a nice paper [15] containing another proof of Theorem 3 (as well as other results about Conjecture 2). Even though some of the ingredients are in common, his proof is longer and very different as it is an entirely inductive proof whereas ours relies on the structure theorem mentioned above.

2 Round and In-Round Oriented Graphs

A *linear order* on a digraph D is an order $O := (v_1, v_2, \dots, v_n)$ of its vertices. Two linear orders O_1 and O_2 are equivalent if there exists an integer k such that $O_1 = (v_1, v_2, \dots, v_n)$ and $O_2 = (v_k, v_{k+1}, \dots, v_n, v_1, v_2, \dots, v_{k-1})$. An equivalence class for this relation is called a *cyclic order* of D . Given a linear order $O = (v_1, v_2, \dots, v_n)$ we define the length $l(a)$ of an arc $a = v_i v_j$ to be equal to

$j - i$ if $j \geq i$ and $n - i + j$ if $j < i$. For two vertices v_i and v_j the *cyclic interval* $[v_i, v_j]$ is defined as follow:

$$[v_i, v_j] = \begin{cases} \{v_k, k \in [i, j]\} & \text{if } i < j \\ \{v_k, k \notin]j, i[\} & \text{if } i \geq j \end{cases}$$

Note that length of an arc and cyclic intervals only depend on the cyclic order and not on a linear order chosen as a representative. As usual, $]v_i, v_j[= [v_i, v_j] \setminus \{v_i, v_j\}$.

A *round oriented graph* $D = (V, A)$ is an oriented graph such that there is a cyclic order of its vertices satisfying:

$$\forall xy \in A, \forall z \in]x, y[, xz \in A \text{ and } zy \in A$$

In other words, for every vertex x , x^+ (resp. x^-) consists in a cyclic interval placed just after (resp. before) x in the cyclic order. Note that a round oriented graph is strongly connected if and only if every vertex has at least one in-neighbour. This is because the cyclic order given by the theorem above is an Hamiltonian cycle. By a similar observation, if a round oriented graph is not strong, then it is in fact acyclic.

Remember that an oriented graph is locally transitive if the in-neighborhood and the out-neighborhood of each vertex induces a transitive tournament. Huang [9] proved that round oriented graphs and locally transitive oriented graphs are actually the same. (He actually proved a more general version of this result that applied on digraphs in place of oriented graphs, but as we only need it for oriented graphs, we only state this one).

Theorem 1 (Huang [9]). *If $D = (V, A)$ is a connected oriented graph, then the two conditions below are equivalent*

1. *for every vertex x , both x^+ and x^- induce a transitive tournament.*
2. *there exists a cyclic order of the vertices of D such that*

$$\forall xy \in A, \forall z \in]x, y[, xz \in A \text{ and } zy \in A$$

Our first theorem is a generalization of the theorem above in the particular case of strong oriented graphs. Following the terminology of [3], any oriented graphs satisfying Condition 2. of the theorem below will be called *in-round*.

Theorem 2 *Let D be a strong oriented graph. Then conditions below are equivalent.*

1. *for every vertex x , x^+ induces a tournament and x^- induces an acyclic digraph*
2. *there exists a cyclic order of the vertices of D such that*

$$\forall xy \in A, \forall z \in]x, y[, zy \in A$$

Again 2. can be seen as the property that for every vertex x , x^- consists in a cyclic interval placed just before x in the order.

Proof (Proof of Theorem 2). The easy direction is 2. implies 1. Indeed, let x, y, z be such that $y, z \subset x^+$ and assume w.l.o.g that $z \in [x, y]$. Then by 2. we have that $zy \in A$, so that indeed x^+ is a tournament. If x^- contains a directed cycle C , let y denote the vertex of C such that $C \setminus y \subset [y, x]$ (the left most vertex of C in the representative of the cyclic order which ends in x). Let z be the predecessor of y on C . Now we have that $x \in [zy]$ and so by 2. there must be an arc xy , which contradicts the fact that $y \in x^-$ (there are no digon here).

Now assume 1.. For every vertex x , x^- induces a non empty acyclic oriented graph, and hence contains a vertex y such that $y^+ \cap x^- = \emptyset$ (take the last vertex in a topological ordering of x^-). For every x we arbitrarily choose one such vertex and denote it by $f(x)$. If z is an out-neighbour of $f(x)$, then since $f(x)^+$ induces a tournament z and x must be connected by an arc, and this cannot be zx by definition of $f(x)$ so there must be an arc xz . We have therefore $f(x)^+ \setminus \{x\} \subset x^+$ for all x .

Now let H be the graph induced by the arcs $f(x)x$. Each vertex of H has in-degree exactly 1, so H contains a cycle C . If C does not span all vertices of D , then since D is strong, there exists an arc xy in D where $x \in C$ and $y \notin C$. Because of the property above the whole cycle C must be contained in y^- , which contradicts 1.. So H consists in an Hamiltonian cycle and the property $f(x)^+ \subset x^+$ exactly translates into 2. for the cyclic order defined by this Hamiltonian cycle.

Remark 2. We point out that the proof above shows that Property 2. implies in fact that every out-neighbourhood in an in-round oriented graph induces a *transitive* tournament (even if it is not necessary to add it in Property 1. to get the equivalence). Thus in-round strong oriented graphs can also be seen as strong oriented graphs that are locally out-transitive and locally in-acyclic.

Using the cyclic structure given by the result above, it is not difficult to deduce that in-round oriented graphs have dichromatic number at most 2.

Proposition 1. *Every in-round oriented graph has dichromatic number at most 2. More precisely, for every vertex x , there exists a valid 2-dicolouring such that $\{x\} \cup x^+$ is monochromatic.*

Proof. We only need to prove it when the oriented graph is strong since a 2-colouring of each strong component yields a valid 2-colouring of the whole oriented graph. Consider the cyclic order given by the definition of in-round and pick any vertex x . Let y be the vertex such that xy is a longest arc, that is the arc such that the interval $[x, y]$ contains the maximum number of vertices. This implies that in the linear order given by the interval $]y, x[$ all arcs go forward since a back arc $x'y'$ would force the arc xy' contradicting the maximality of xy . So $]y, x[$ induces an acyclic oriented graph. Moreover $[x, y]$ induces an acyclic oriented graph since it is included in y^- and by definition of the in-round cyclic order. This concludes the proof.

3 Decomposing and Colouring Locally Out-Transitive Oriented Graphs

3.1 Statement of the Results and Outline of the Proofs

In this subsection we state our results on the class of locally out-transitive oriented graphs. Remember that an oriented graph D is out-transitive if for every vertex x of D , x^+ is a transitive tournament, and that strong in-round oriented graphs are exactly strong oriented graphs that are both locally out-transitive and locally in-acyclic (see Theorem 2). Hence, the theorem below generalises Proposition 1.

Theorem 3. *Every locally out-transitive oriented graph has dichromatic number at most 2.*

As mentioned in the introduction, the proof of the above theorem follows from the following structural result that describes strong locally out-transitive oriented graphs.

Theorem 4. *If D is a strong locally out-transitive oriented graph, then there exists a partition of its set of vertices into strong subdigraphs H_1, \dots, H_k such that the digraph obtained by contraction of every H_i is a strong in-round oriented graph.*

Due to lack of space, we only outline the proofs here. Like every other proofs missing in this paper, it can be found in the long version [1].

If D is an oriented graph, a subdigraph H of D is a *hub* if it satisfies the following

- H is strong,
- there exists $x \notin H$ such that $H \subseteq x^-$.

A hub is *non-trivial* if it has at least two vertices. Hubs are the key structure to understand locally out-transitive oriented graphs.

Let D be a locally out-transitive oriented graph. We first prove that the set of maximal (inclusion-wise) hubs forms a partition of the vertex set with the additional property that, if H_1 and H_2 are two maximal hubs, then we are in one of the three following situations:

- either there is no arc between H_1 and H_2 ,
- either there are all arcs from H_1 to a subset of H_2 inducing a transitive tournament, and no other arc,
- or there are all arcs from H_2 to a subset of H_1 inducing a transitive tournament, and no other arc.

We then prove that the digraph F obtained by contraction of every maximal hub is an oriented graph with no non-trivial hub, and then that every in-neighborhood is acyclic and every out-neighborhood is a tournament. This in turns implies, by Theorem 2, that the obtained graph is a strong oriented in-round graph, which concludes the proof of Theorem 4.

Using the 2-colouring of F given by Proposition 1, we are able to obtain a 2-colouring of D (we prove in fact as in Proposition 1 that for any vertex x there exists a 2-colouring such that the closed neighbourhood of x is monochromatic)

3.2 A Short Application of Theorem 4 to the Caccetta-Häggkvist Conjecture

A beautiful and famous conjecture due to Caccetta and Häggkvist states states the following.

Conjecture 3 (Caccetta-Häggkvist). Let $k \geq 2$ be an integer. Every digraph D on n vertices with no directed circuits of length at most k contains a vertex of out-degree less than n/k .

The case $k = 2$ is trivial but the case $k = 3$ is still widely open and has attracted a lot of attention. In [12] (see page 3), it is mentioned that for $k = 3$, while adding the hypothesis that the graph has no S_2^+ makes it very easy, the dual case of forbidding S_2^- was proven by Seymour but is "substantially more difficult". We can prove that this comes as an easy consequence of Theorem 4 and Theorem 2, for any value $k \geq 3$.

Theorem 5. *Let D be a locally in-tournament oriented graph on n vertices with no directed cycle of length at most k . Then D contains a vertex of out-degree less than n/k .*

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Ramsey Expansions of 3-Hypertournaments

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Abstract. We study Ramsey expansions of certain homogeneous 3-hypertournaments. We show that they exhibit an interesting behaviour and, in one case, they seem not to submit to current gold-standard methods for obtaining Ramsey expansions. This makes these examples very interesting from the point of view of structural Ramsey theory as there is a large demand for novel examples.

Keywords: Homogeneous hypertournaments · Ramsey property

Structural Ramsey theory studies which homogeneous structures have the so-called Ramsey property, or at least are not far from it (can be *expanded* by some relations to obtain a structure with the Ramsey property). Recently, the area has stabilised with general methods and conditions from which almost all known Ramsey structures follow. In particular, the homogeneous structures offered by the classification programme are well-understood in most cases. Hence, there is a demand for new structures with interesting properties.

In this abstract we investigate Ramsey expansions of four homogeneous 4-constrained 3-hypertournaments identified by the first author [3] and show that they exhibit an interesting range of behaviours. In particular, for one of them the current techniques and methods cannot be directly applied. There is a big demand for such examples in the area, in part because they show the limitations of present techniques, in part because they might lead to a negative answer to the question whether every structure homogeneous in a finite relational language has a Ramsey expansion in a finite relational language, one of the central questions of the area asked in 2011 by Bodirsky, Pinsker and Tsankov [2].

1 Preliminaries

We adopt the standard notions of languages (in this abstract they will be relational only), structures and embeddings. A structure is *homogeneous* if every

isomorphism between finite substructures extends to an automorphism. There is a correspondence between homogeneous structures and so-called (*strong*) *amalgamation classes* of finite structures, see e.g. [5]. A structure \mathbf{A} is *irreducible* if every pair of vertices is part of a tuple in some relation of \mathbf{A} .

In this abstract, an *n-hypertournament* is a structure \mathbf{A} in a language with a single *n*-ary relation R such that for every set $S \subseteq A$ with $|S| = n$ it holds that the automorphism group of the substructure induced on S by \mathbf{A} is precisely $\text{Alt}(S)$, the alternating group on S . This in particular means that exactly half of *n*-tuples of elements of S with no repeated occurrences are in $R^{\mathbf{A}}$. For $n = 2$ we get standard tournaments, for $n = 3$ this correspond to picking one of the two possible cyclic orientations on every triple of vertices. It should be noted however, that another widespread usage, going back at least to Assous [1], requires a unique instance of the relation to hold on each *n*-set. A *holey n-hypertournament* is a structure \mathbf{A} with a single *n*-ary relation R such that all irreducible substructures of \mathbf{A} are *n*-hypertournaments. A *hole* in \mathbf{A} is a set of 3 vertices on which there are no relations at all.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be structures. We write $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$ to denote the statement that for every 2-colouring of embeddings of \mathbf{A} to \mathbf{C} , there is an embedding of \mathbf{B} to \mathbf{C} on which all embeddings of \mathbf{A} have the same colour. A class \mathcal{C} of finite structures has the *Ramsey property (is Ramsey)* if for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$ and \mathcal{C}^+ is a *Ramsey expansion* of \mathcal{C} if it is Ramsey and can be obtained from \mathcal{C} by adding some relations. By an observation of Nešetřil [8], every Ramsey class is an amalgamation class under some mild assumptions.

1.1 Homogeneous 4-Constrained 3-Hypertournaments

Suppose that $\mathbf{T} = (T, R)$ is a 3-hypertournament and pick an arbitrary linear order \leq on T . One can define a 3-uniform hypergraph $\hat{\mathbf{T}}$ on the set T such that $\{a, b, c\}$ with $a \leq b \leq c$ is a hyperedge of $\hat{\mathbf{T}}$ if and only if $(a, b, c) \in R$. (Note that by the definition of a 3-hypertournament, it always holds that exactly one of (a, b, c) and (a, c, b) is in R .) This operation has an inverse and hence, after fixing a linear order, we can work with 3-uniform hypergraphs instead of 3-hypertournaments. There are three isomorphism types of 3-hypertournaments on 4 vertices:

- \mathbf{H}_4 The homogeneous 3-hypertournament on 4 vertices. For an arbitrary linear order \leq on H_4 , $\hat{\mathbf{H}}_4$ contains exactly two hyperedges. Moreover, they intersect in vertices $a < b$ such that there is exactly one $c \in H_4$ with $a < c < b$.
- \mathbf{O}_4 The odd 3-hypertournament on 4 vertices. For an arbitrary linear order \leq , $\hat{\mathbf{O}}_4$ will contain an odd number of hyperedges. Conversely, any ordered 3-uniform hypergraph on 4 vertices with an odd number of hyperedges will give rise to \mathbf{O}_4 .
- \mathbf{C}_4 The cyclic 3-hypertournament on 4 vertices. There is a linear order \leq on C_4 such that $\hat{\mathbf{C}}_4$ has all four hyperedges. In other linear orders, $\hat{\mathbf{C}}_4$ might have no hyperedges or exactly two which do not intersect as in \mathbf{H}_4 .

We say that a class \mathcal{C} of finite 3-hypertournaments is *4-constrained* if there is a non-empty subset $S \subseteq \{\mathbf{H}_4, \mathbf{O}_4, \mathbf{C}_4\}$ such that \mathcal{C} contains precisely those finite 3-hypertournaments whose every substructure on four distinct vertices is isomorphic to a member of S . There are four 4-constrained classes of finite 3-hypertournaments which form a strong amalgamation class [3]. They correspond to the following sets S :

$S = \{\mathbf{C}_4\}$ The *cyclic* ones. These can be obtained by taking a finite cyclic order and orienting all triples according to it. Equivalently, they admit a linear order such that the corresponding hypergraph is complete.

$S = \{\mathbf{C}_4, \mathbf{H}_4\}$ The *even* ones. The corresponding hypergraphs satisfy the property that on every four vertices there are an even number of hyperedges.

$S = \{\mathbf{C}_4, \mathbf{O}_4\}$ The *\mathbf{H}_4 -free* ones. Note that in some sense, this generalizes the class of finite linear orders: As $\text{Aut}(\mathbf{H}_4) = \text{Alt}(4)$, one can define \mathbf{H}_n to be the $(n - 1)$ -hypertournament on n points such that $\text{Aut}(\mathbf{H}_n) = \text{Alt}(n)$. For $n = 3$, we get that \mathbf{H}_3 is the oriented cycle on 3 vertices and the class of all finite linear orders contains precisely those tournaments which omit \mathbf{H}_3 .

$S = \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}$ The class of all finite 3-hypertournaments.

2 Positive Ramsey Results

In this section we give Ramsey expansions for all above classes with the exception of the \mathbf{H}_4 -free ones. Let \mathcal{C}_c be the class of all finite cyclic 3-hypertournaments. Let $\vec{\mathcal{C}}_c$ be a class of finite linearly ordered 3-hypertournaments such that $(A, R, \leq) \in \vec{\mathcal{C}}_c$ if and only if for every $x < y < z \in A$ we have $(x, y, z) \in R$. Notice that for every $(A, R) \in \mathcal{C}$ there are precisely $|A|$ orders \leq such that $(A, R, \leq) \in \vec{\mathcal{C}}_c$ (after fixing a smallest point, the rest of the order is determined by R), and conversely, for every $(A, R, \leq) \in \vec{\mathcal{C}}_c$ we have that $(A, R) \in \mathcal{C}_c$.

It is a well-known fact that every Ramsey class consists of linearly ordered structures [7]. We have seen that after adding linear orders freely, the class of all finite ordered even 3-hypertournaments corresponds to the class of all finite ordered 3-uniform hypergraphs which induce an even number of hyperedges on every quadruple of vertices. These structures are called *two-graphs* and they are one of the reducts of the random graph (one can obtain a two-graph from a graph by putting hyperedges on triples of vertices which induce an even number of edges). Ramsey expansions of two-graphs have been discussed in [4] and the same ideas can be applied here.

Let $\vec{\mathcal{C}}_e$ consist of all finite structures (A, \leq, E, R) such that (A, \leq) is a linear order, (A, E) is a graph, (A, R) is a 3-hypertournament and for every $a, b, c \in A$ with $a < b < c$ we have that $(a, b, c) \in R$ if and only if there are an even number of edges (relation E) on $\{a, b, c\}$. Otherwise $(a, c, b) \in R$.

Theorem 1. *The 4-constrained classes of finite 3-hypertournaments with $S \in \{\{\mathbf{C}_4\}, \{\mathbf{C}_4, \mathbf{H}_4\}, \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}\}$ all have a Ramsey expansion in a finite language. More concretely:*

1. $\vec{\mathcal{C}}_c$ is Ramsey.
2. $\vec{\mathcal{C}}_e$ is Ramsey.
3. The class of all finite linearly ordered 3-hypertournaments is Ramsey.

We remark that these expansions can be shown to have the so-called *expansion property* with respect to their base classes, which means that they are the optimal Ramsey expansions (see e.g. Definition 3.4 of [6]).

Proof. In $\vec{\mathcal{C}}_c$, R is definable from \leq and we can simply use Ramsey’s theorem. Similarly, in $\vec{\mathcal{C}}_e$, R is definable from \leq and E , hence part 2 follows from the Ramsey property of the class of all ordered graphs [9].

To prove part 3, fix a pair of finite ordered 3-hypertournaments \mathbf{A} and \mathbf{B} and use the Nešetřil–Rödl theorem [9] to obtain a finite ordered holey 3-hypertournament \mathbf{C}' such that $\mathbf{C}' \rightarrow (\mathbf{B})_2^{\mathbf{A}}$. The holes in \mathbf{C}' can then be filled in arbitrarily to obtain a linearly ordered 3-hypertournament \mathbf{C} such that $\mathbf{C} \rightarrow (\mathbf{B})_2^{\mathbf{A}}$.

3 The \mathbf{H}_4 -Free Case

Let $\mathbf{A} = (A, R)$ be a holey 3-hypertournament. We say that $\bar{\mathbf{A}} = (A, R')$ is a *completion* of \mathbf{A} if $R \subseteq R'$ and $\bar{\mathbf{A}}$ is an \mathbf{H}_4 -free 3-hypertournament. Most of the known Ramsey classes can be proved to be Ramsey by a result of Hubička and Nešetřil [6]. In order to apply the result for \mathbf{H}_4 -free 3-hypertournaments, one needs a finite bound c such that whenever a holey 3-hypertournament has no completion, then it contains a substructure on at most c vertices with no completion. (Completions defined in [6] do not directly correspond to completions defined here. However, the definitions are equivalent for structures considered in this paper.) We prove the following.

Theorem 2. *There are arbitrarily large holey 3-hypertournaments \mathbf{B} such that \mathbf{B} has no completion but every proper substructure of \mathbf{B} has a completion.*

This theorem implies that one cannot use [6] directly for \mathbf{H}_4 -free hypertournaments. However, a situation like in Theorem 2 is not that uncommon. There are two common culprits for this, either the class contains orders (for example, failures of transitivity can be arbitrarily large in a holey version of posets) or it contains equivalences (again, failures of transitivity can be arbitrarily large). In the first case, there is a condition in [6] which promises the existence of a linear extension, and thus resolves the issue. For equivalences, one has to introduce explicit representatives for equivalence classes (this is called *elimination of imaginaries*) and unbounded obstacles to completion again disappear.

For \mathbf{H}_4 -free hypertournaments neither of the two solutions seems to work. This means that something else is happening which needs to be understood in order to obtain a Ramsey expansion of \mathbf{H}_4 -free tournaments. Hopefully, this would lead to new, even stronger, general techniques.

In the rest of the abstract we sketch a proof of Theorem 2.

Lemma 1.

1. Let $\mathbf{G} = (G, R)$ be a holey 3-hypertournament with $G = \{1, 2, 3, 4\}$ such that $(1, 3, 4) \in R$, $(1, 4, 2) \in R$ and $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are holes. Let (G, R') be a completion of \mathbf{G} . If $(1, 2, 3) \in R'$, then $(2, 3, 4) \in R'$.
2. Let $\mathbf{G}^\neg = (G, R)$ be a holey 3-hypertournament with $G = \{1, 2, 3, 4\}$ such that $(2, 4, 3) \in R$, $(1, 4, 2) \in R$ and $\{1, 2, 3\}$ and $\{1, 3, 4\}$ are holes. Let (G, R') be a completion of \mathbf{G}^\neg . If $(1, 2, 3) \in R'$, then $(1, 3, 4) \notin R'$.

Proof. In the first case, suppose that $(1, 2, 3) \in R'$. If $(2, 4, 3) \in R'$, then (G, R') is isomorphic to \mathbf{H}_4 . Hence $(2, 3, 4) \in R'$. The second case is proved similarly.

Suppose that $\mathbf{A} = (A, R)$ is a holey 3-hypertournament. For $x, y, z, w \in A$, we will write $xyz \Rightarrow yzw$ if the map $(1, 2, 3, 4) \mapsto (x, y, z, w)$ is an embedding $\mathbf{G} \rightarrow \mathbf{A}$ and we will write $xyz \Rightarrow \neg xzw$ if the map $(1, 2, 3, 4) \mapsto (x, y, z, w)$ is an embedding $\mathbf{G}^\neg \rightarrow \mathbf{A}$. Using the complement of \mathbf{G} , we can define $\neg xyz \Rightarrow \neg yzw$, and using the complement of \mathbf{G}^\neg we can define $\neg xyz \Rightarrow xzw$. This notation can be chained as well, e.g. $xyz \Rightarrow yzw \Rightarrow zwu \Rightarrow \neg zuv$ means that all of $xyz \Rightarrow yzw$, $yzw \Rightarrow zwu$, $zwu \Rightarrow \neg zuv$ are satisfied.

Let $n \geq 6$. We denote by $\mathbf{O}_n = (O_n, R)$ the holey 3-hypertournament with vertex set $O_n = \{1, \dots, n\}$ such that

$$123 \Rightarrow 234 \Rightarrow 345 \Rightarrow \dots \Rightarrow (n-2)(n-1)n \Rightarrow \neg(n-2)n1 \Rightarrow \neg n12 \Rightarrow \neg 123.$$

All triples not covered by these conditions are holes.

Lemma 2.

1. There is a completion (O_n, R') of \mathbf{O}_n .
2. If (O_n, R') is a completion of \mathbf{O}_n , then $(1, 2, 3) \notin R'$.
3. For every $v \in O_n \setminus \{1, 2, 3\}$ there is a completion $(O_n \setminus \{v\}, R')$ of the structure induced by \mathbf{O}_n on $O_n \setminus \{v\}$ such that $(1, 2, 3) \in R'$.

Proof. For part 1, observe that every set of four vertices of \mathbf{O}_n with at least two different subsets of three vertices covered by a relation is isomorphic to \mathbf{G} , \mathbf{G}^\neg or the complement of \mathbf{G} . It follows that whenever $x, y, z \in O_n$ is a hole such that $x < y < z$, we can put (x, z, y) and its cyclic rotations in R' to get a completion. Part 2 follows by induction on the conditions.

For part 3, we put $(1, 2, 3), (2, 3, 4), \dots, (v-3, v-2, v-1) \in R'$, $(v+1, v+3, v+2), \dots, (n-2, n, n-1) \in R'$ and $(n-2, n, 1), (n, 1, 2) \in R'$. It can be verified that this does not create any copies of \mathbf{H}_4 . A completion of (O_n, R') exists as the class of all finite \mathbf{H}_4 -free tournaments has strong amalgamation.

Similarly, for $n \geq 6$, we define $\mathbf{O}_n^\neg = (O_n^\neg, R)$ the holey 3-hypertournament with vertex set $O_n^\neg = \{1, \dots, n\}$ such that

$$\neg 123 \Rightarrow \neg 234 \Rightarrow \neg 345 \Rightarrow \dots \Rightarrow \neg(n-2)(n-1)n \Rightarrow (n-2)n1 \Rightarrow n12 \Rightarrow 123$$

and there are no other relations in R . In any completion (O_n^\neg, R') of \mathbf{O}_n^\neg it holds that $(1, 2, 3) \in R'$, in fact, an analogue of Lemma 2 holds for \mathbf{O}_n^\neg .

Let \mathbf{B}_n be the holey 3-hypertournament obtained by gluing a copy of \mathbf{O}_n with a copy of \mathbf{O}_n^- , identifying vertices 1, 2 and 3. (This means that \mathbf{B}_n has $2n - 3$ vertices). We now use $\{\mathbf{B}_n : n \geq 6\}$ to prove Theorem 2.

Proof (of Theorem 2). Assume that (B_n, R') is a completion of \mathbf{B}_n . So in particular, it is a completion of the copies of \mathbf{O}_n and \mathbf{O}_n^- . By Lemma 2 and its analogue for \mathbf{O}_n^- , we have that $(1, 2, 3) \notin R'$ and $(1, 2, 3) \in R'$, a contradiction.

Pick $v \in B_n$ and consider the structure \mathbf{B}_n^v induced by \mathbf{B}_n on $B_n \setminus \{v\}$. We prove that \mathbf{B}_n^v has a completion. If $v \notin \{1, 2, 3\}$, one can use part 3 of Lemma 2 and its analogue for \mathbf{O}_n^- to complete the copy of \mathbf{O}_n and \mathbf{O}_n^- (one of them missing a vertex) so that they agree on $\{1, 2, 3\}$. Using strong amalgamation, we get a completion of \mathbf{B}_n^v . If $v \in \{1, 2, 3\}$, we pick an arbitrary completion of \mathbf{O}_n and \mathbf{O}_n^- , remove v from both of them, and let the completion of \mathbf{B}_n to be the strong amalgamation of the completions over $\{1, 2, 3\} \setminus v$.

The following question remains open.

Question 1. What is the optimal Ramsey expansion for the class of all finite \mathbf{H}_4 -free hypertournaments? Does it have a Ramsey expansion in a finite language?

Acknowledgement. This is part of a project that has received funding from the European Research Council (ERC) under the EU Horizon 2020 research and innovation programme (grant agreement No 810115). J. H., M. K. and J. N. are supported by the project 21-10775S of the Czech Science Foundation (GAČR). J. H. is supported by the Center for Foundations of Modern Computer Science (Charles University project UNCE/SCI/004). M. K. is supported by the Charles University Grant Agency (GA UK), project 378119.

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Path Decompositions of Random Directed Graphs

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Abstract. In this work we consider extensions of a conjecture due to Alspach, Mason, and Pullman from 1976. This conjecture concerns edge decompositions of tournaments into as few paths as possible. There is a natural lower bound for the number paths needed in an edge decomposition of a directed graph in terms of its degree sequence; the conjecture in question states that this bound is correct for tournaments of even order. The conjecture was recently resolved for large tournaments, and here we investigate to what extent the conjecture holds for directed graphs in general. In particular, we prove that the conjecture holds with high probability for the random directed graph $D_{n,p}$ for a large range of p .

Keywords: Path decomposition · Random directed graph

1 Introduction

There has been a great deal of recent activity in the study of decompositions of graphs and hypergraphs. The prototypical question in the area asks whether, for some given class \mathcal{C} of graphs, directed graphs, or hypergraphs, the edge set of each $H \in \mathcal{C}$ can be decomposed into parts satisfying some given property. Here, one often wishes to minimise the number of parts; e.g., in the case of edge colourings, determining the chromatic index amounts to partitioning the edges of a graph into as few matchings as possible. In this paper, we will be concerned with decomposing the edges of directed graphs into as few (directed) paths as possible.

Let D be a directed graph (or digraph for short) with vertex set $V(D)$ and edge set $E(D)$. A path decomposition of D is a collection of paths P_1, \dots, P_k of D whose edge sets $E(P_1), \dots, E(P_t)$ partition $E(D)$. Given any directed graph D , it is natural to ask what the minimum number of paths is in a path decomposition of D . This is called the path number of D and is denoted $\text{pn}(D)$. A natural lower bound on $\text{pn}(D)$ is obtained by examining the degree sequence of D . For each vertex $v \in V(D)$, write $d_D^+(v)$ (resp. $d_D^-(v)$) for the number of edges exiting (resp. entering) v . The excess at vertex v is defined to be $\text{ex}_D(v) := d_D^+(v) - d_D^-(v)$. We

note that, in any path decomposition of D , at least $|\text{ex}_D(v)|$ paths must start (resp. end) at v if $\text{ex}_D(v) \geq 0$ (resp. $\text{ex}_D(v) \leq 0$). Therefore, we have

$$\text{pn}(D) \geq \text{ex}(D) := \frac{1}{2} \sum_{v \in V(D)} |\text{ex}_D(v)|,$$

where $\text{ex}(D)$ is called the excess of D . Any digraph for which equality holds above is called *consistent*. Clearly not every digraph is consistent; in particular, any Eulerian digraph D has excess 0 and so cannot be consistent.

For the class of tournaments (that is, orientations of the complete graph), Alspach, Mason, and Pullman [1] conjectured that every tournament with an even number of vertices is consistent.

Conjecture 1. Every tournament T with an even number of vertices is consistent.

Many cases of this conjecture were resolved by the second author together with Lo, Skokan, and Talbot [9], and the conjecture has very recently been completely resolved (asymptotically) by Girão, Granet, Kühn, Lo, and Osthus [3]. Both results relied on the robust expanders technique, developed by Kühn and Osthus with several coauthors, which has been instrumental in resolving several conjectures about edge decompositions of graphs and directed graphs; see, e.g., [2, 7, 8].

The conjecture seems likely to hold for many digraphs other than tournaments: indeed, the conjecture was stated only for even tournaments probably because it considerably generalised the following conjecture of Kelly, which was wide open at the time. Kelly’s conjecture states that every regular tournament has a decomposition into Hamilton cycles. The asymptotic solution of Kelly’s conjecture was one of the first applications of the robust expanders technique [7].

A natural question then arises from Conjecture 1: which directed graphs are consistent? It is NP-complete to determine whether a digraph is consistent [11], and so we should not expect to characterise consistent digraphs. Nonetheless, here we begin to address this question by showing that the large majority of digraphs are consistent. We consider the random digraph $D_{n,p}$, which is constructed by taking n isolated vertices and inserting each of the $n(n - 1)$ possible directed edges independently with probability p . Our main result is the following theorem.

Theorem 1. *Let $\log^4 n/n^{1/3} \leq p \leq 1 - \log^3 n/n^{1/5}$. Then, with high probability¹ $D_{n,p}$ is consistent.*

Notice that some upper bound on p , as in the above theorem, is necessary because, when $p = 1$, we have that $\text{ex}(D_{n,p}) = 0$ (with probability 1) and so $D_{n,p}$ cannot be consistent. Moreover the property of being consistent is not a monotone property, that is, adding edges to a consistent digraph does not imply the resulting digraph is consistent. Therefore, unlike many other properties, we

¹ That is, with probability tending to 1 as n goes to infinity.

should not necessarily expect a threshold for the consistency of random digraphs. We believe that the theorem holds true for much smaller values of p , and perhaps even that no lower bound on p is necessary.

In fact, the proof of Theorem 1 does not use randomness in a very significant way. We give a set of sufficient conditions for a digraph to be consistent and show that the random digraph (for suitable p) satisfies these conditions with high probability. Broadly, our proof relies on the use of the so-called absorption technique, an idea due to Rödl, Ruciński, and Szemerédi [10] (with special forms appearing in earlier work, e.g., [6]). We adapt and refine the absorption ideas used in [9], which we explain in the next section. In contrast to the previous work on this question [3, 9], our proof does not make use of robust expanders. Preliminary ideas for this work came from de Vos [11].

2 Proof Sketch

Let $D = D_{n,p}$. We divide the vertices of D into sets A^+, A^-, A^0 depending on whether $\text{ex}_D(v) > t$, $\text{ex}_D(v) < -t$, or $-t \leq \text{ex}_D(v) \leq t$, respectively, for a suitable choice of t . One can show that, with high probability, A^+ and A^- have roughly the same size and A^0 is small. To simplify our exposition, we will assume $A^0 = \emptyset$.

It turns out that it is useful to have a collection \mathcal{A} of edges of D where every edge of \mathcal{A} is directed from A^+ to A^- , such that each vertex v of A^+ (resp. A^-) has exactly $|\text{ex}_D(v)| \geq t$ edges of \mathcal{A} exiting (resp. entering) v . This immediately implies that the total number of edges of \mathcal{A} is $\text{ex}(D)$. The set \mathcal{A} is our absorbing structure. In $D_{n,p}$ we can obtain (with high probability) a set of edges having properties close to the set \mathcal{A} described above, but for this sketch we assume we have obtained \mathcal{A} for simplicity. The digraph D' obtained from D by removing the edges in \mathcal{A} has in-degree equal to out-degree for every vertex and can therefore be decomposed into cycles. We should think of \mathcal{A} as a collection of $\text{ex}(D)$ single edge paths, and our goal is to slowly combine edges of \mathcal{A} with edges of D' to create longer paths in such a way that we maintain exactly $\text{ex}(D)$ paths at every stage. If we manage to combine all the edges of D' in this way, then we have decomposed D into $\text{ex}(D)$ paths, proving that D is consistent.

To begin the process of absorption, we apply a recent result of Knierim, Larcher, Martinsson and Noever [5] (improving on an earlier result of Huang, Ma, Shapira, Sudakov and Yuster [4]) to decompose the edges of D' into $O(n \log n)$ cycles. The core idea then is to combine edges from \mathcal{A} with each cycle C and to decompose the union into paths; we refer to this as *absorbing* the cycle. Crucially, in order to keep the number of paths invariant, we will combine each cycle C with a set \mathcal{A}_C of 2 edges from \mathcal{A} and decompose $C \cup \mathcal{A}_C$ into 2 paths (and thereafter, the edges in \mathcal{A}_C are no longer available for use in absorbing other cycles), as illustrated in Fig. 1.

We must then allocate absorbing edges to the cycles. The two main challenges here are (i) that the absorbing edges need to *fit* with the specific cycle, meaning they and the cycle can be decomposed into two paths, and (ii) that we only

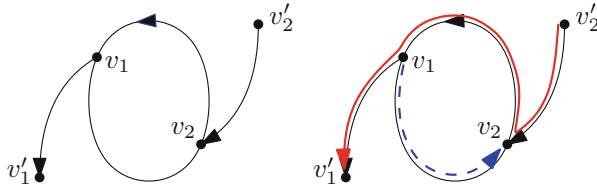


Fig. 1. Left: One example of absorbing a cycle using two absorbing edges. We have v_1, v_2 on our cycle C with $v_1 \in A^+, v_2 \in A^-$. We find the additional edges (v_1, v'_1) at v_1 and (v'_2, v_2) at v_2 with $v'_1 \in A^- \setminus V(C)$ and $v'_2 \in A^+ \setminus V(C)$. Right: The solid red and dashed blue lines show the two paths $P_1 := v'_2 v_2 C v_1 v'_1$ and $P_2 := v_1 C v_2$, which use all involved edges. Note that under certain circumstances, if v'_1, v'_2 lie on C , we can still decompose all involved edges into two paths.

have a limited number of absorbing edges available at each vertex. In order to be economical with absorbing edges, we employ different strategies to assign absorbing edges depending on the number of vertices that the cycle has. We divide the cycles into three sets and treat them separately.

For cycles that are long, we greedily choose two edges that fit the cycle. This is possible as each cycle contains a large number of vertices, so there are many choices for the possible absorbing edges, and we can always find two that fit appropriately with the cycle.

For cycles of medium length, we use a flow problem to assign vertices to cycles in such a way that each cycle is assigned a suitably large number of vertices dependent on its length, but such that no vertex is assigned to too many cycles. It turns out that this choice of assignment means we can find two assigned vertices per cycle and pick one edge each from each of those vertices that fit the cycle. This strategy is wasteful in the sense that we sometimes assign more than two vertices to a cycle and thereby reserve more absorbing edges than we use, but this approach allows us to find fitting absorbing edges.

For cycles that are short, it is easier to find fitting edges, as we are guaranteed to find absorbing edges that have their other end off the cycle as in the example in Fig. 1, but it is harder to ensure that we do not use too many edges per vertex. We also use a flow problem to assign vertices to cycles, but we take multiple rounds and only decompose certain ‘safe’ cycles in each round. In addition, we decompose certain closed walks in each round, so we need to apply the result by Knierim et al. between rounds in order to refactor the remaining edges into cycles, and this may generate new long or medium cycles. Absorbing the short cycles is the most complicated process of the three, but it is the process we apply first so that the long and medium cycles that are produced as a byproduct can be absorbed by the appropriate processes described above.

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Large Multipartite Subgraphs in H -free Graphs

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Abstract. In this work, we discuss a strengthening of a result of Füredi that every n -vertex K_{r+1} -free graph can be made r -partite by removing at most $T(n, r) - e(G)$ edges, where $T(n, r) = \frac{r-1}{2r} n^2$ denotes the number of edges of the n -vertex r -partite Turán graph. As a corollary, we answer a problem of Sudakov and prove that every K_6 -free graph can be made bipartite by removing at most $4n^2/25$ edges. The main tool we use is the flag algebra method applied to locally defined vertex-partitions.

Keywords: Max-Cut · Turán graph · Flag Algebras

1 Introduction

Let $G = (V, E)$ be an n -vertex graph and $r \geq 2$ an integer. Denote by $\text{del}_r(G)$ the minimum size of an edge-subset $X \subseteq E$ such that the graph $G - X$ is r -partite. Note that $\text{del}_2(G)$ is the dual problem to Max-Cut, i.e., finding the largest bipartite subgraph in G . For convenience, we also define $\text{del}_1(G) := e(G)$.

Our aim is to obtain upper bounds on $\text{del}_r(G)$ and $\text{del}_2(G)$, respectively, when G is a K_{r+1} -free graph, i.e., a graph with no complete subgraph on $r + 1$ vertices. A beautiful stability-type argument of Füredi [6] provides the following upper bound on $\text{del}_r(G)$.

Theorem 1. (Füredi [6]). *Fix an integer $r \geq 2$. If G is an n -vertex K_{r+1} -free graph, then $\text{del}_r(G) \leq \frac{r-1}{2r} \cdot n^2 - e(G)$.*

Note that the number of edges in every K_{r+1} -free graph on n vertices is bounded from above by the number of edges in the Turán graph $T(n, r)$, which is equal

to $\frac{r-1}{2r} \cdot n^2$. In other words, the result of Füredi can be stated as follows: if a K_{r+1} -free graph is missing t edges to being extremal, then removing at most t edges from it makes it r -partite.

When the number of edges of G is very close to the extremal value, Theorem 1 was sharpened in [2, 7]. Here we focus on a global improvement, and conjecture that Theorem 1 can be strengthened as follows.

Conjecture 1. Fix an integer $r \geq 2$. If G is an n -vertex K_{r+1} -free graph, then $\text{del}_r(G) \leq 0.8 \left(\frac{r-1}{2r} \cdot n^2 - e(G) \right)$.

If true, Conjecture 1 would be best possible, and we present tight constructions in Sect. 3. Note that for $r \geq 4$, the conjecture does not have a unique extremal example. To provide an evidence for Conjecture 1, we prove it for $r \in \{2, 3, 4\}$.

Theorem 2. Fix an integer $r \in \{2, 3, 4\}$. If G is an n -vertex K_{r+1} -free graph, then $\text{del}_r(G) \leq 0.8 \left(\frac{r-1}{2r} \cdot n^2 - e(G) \right)$.

We also establish the following general improvement on Theorem 1.

Theorem 3. For every $r \geq 5$ there exists $\varepsilon := \varepsilon(r) > 0$ such that the following holds. If G is an n -vertex K_{r+1} -free graph, then $\text{del}_r(G) \leq (1 - \varepsilon) \left(\frac{r-1}{2r} \cdot n^2 - e(G) \right)$.

The bound on $\varepsilon(r)$ we establish monotonically decreases to 0 as r tends to infinity, while Conjecture 1 claims that $\varepsilon(r) = 0.2$ for every r .

A closely related problem inspired by a well-known problem of Erdős on Max-Cuts in dense triangle-free graphs is the following conjecture of Sudakov [9].

Conjecture 2. Fix $r \geq 3$. For every K_{r+1} -free graph G , it holds that

$$\text{del}_2(G) \leq \begin{cases} \frac{(r-1)^2}{4r^2} \cdot n^2 & r \text{ odd, and} \\ \frac{r-2}{4r} \cdot n^2 & r \text{ even.} \end{cases}$$

Note that the conjectured value corresponds to the value of $\text{del}_2(T(n, r))$. Sudakov [9] proved the conjecture for $r = 3$.

Theorem 4. (Sudakov [9]). An n -vertex K_4 -free graph G can be made bipartite by removing $n^2/9$ edges, i.e., $\text{del}_2(G) \leq n^2/9$. Moreover, if $\text{del}_2(G) = n^2/9$, then G is the Turán graph $T(n, 3)$.

We prove the conjecture for $r = 5$.

Theorem 5. If G is an n -vertex K_6 -free graph, then $\text{del}_2(G) \leq 4n^2/25$. Moreover, if $\text{del}_2(G) = 4n^2/25$, then G is the Turán graph $T(n, 5)$.

As we have already mentioned, Erdős [4] made a conjecture on the size of the largest bipartite subgraph in triangle-free graphs. Specifically, he conjectured that $\text{del}_2(G) \leq n^2/25$ for every triangle-free n -vertex graph G . A result of Erdős, Faudree, Pach, and Spencer [5] states that $\text{del}_2(G) \leq n^2/18$. Using flag algebras in a manner analogous to the one we use here, an improvement on the last bound was recently announced by Balogh, Clemen, and Lidický [3].

Note that for all the theorems in this section, a straightforward application of the regularity lemma yields the corresponding asymptotic results for H -free graphs, where H is a fixed r -colorable graph.

In our work, we extensively use flag algebras, a versatile tool developed by Razborov [8], applied to K_{r+1} -free graph limits. We use as a convention that unlabeled vertices are depicted as black circles, labeled vertices as yellow squares, and edges as blue lines. Dashed lines indicate that both edge and non-edge are admissible. We write $\llbracket \cdot \rrbracket$ to denote the so-called unlabeled/averaging operator.

The rest of this extended abstract is organized as follows: In Sect. 2, we describe an alternative proof of Theorem 1 using flag algebras, which demonstrates the technique we use. In Sect. 3, we examine the set of possible extremal constructions for Conjecture 1, and give a sketch of the proof of Theorem 2 for the case $r = 2$. We conclude the extended abstract by Sect. 4, where we briefly discuss the case $r \geq 3$ as well as the ideas for the proof of Theorem 5.

2 Theorem 1 in Flag Algebras

As a warm-up to our flag algebra technique, we present a proof of Theorem 1. Suppose Theorem 1 is false, and let r be the smallest integer for which it fails. Let G be an n -vertex K_{r+1} -free graph G such that $\text{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$.

For a vertex $v \in V(G)$, consider an r -partition of $V(G)$ with $A_r := V \setminus N(v)$ being one part, and $(A_1, A_2, \dots, A_{r-1})$ being an $(r - 1)$ -partition of $N(v)$ given by Theorem 1 if $r \geq 3$, and $A_1 := N(v)$ in case $r = 2$. Note that if $r = 2$ then $N(v)$ induces no edges in G . It follows that the number of edges inside the parts is at most $e(G[A_r]) + \text{del}_{r-1}(G[N(v)])$, which is at most

$$e(G[A_r]) + \frac{r-2}{r-1} \cdot \frac{|N(v)|^2}{2} - e(G[N(v)]). \tag{1}$$

On the other hand, this is at least $\text{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$. This is in direct contradiction with the following simple flag algebra proposition, which shows that if we choose a vertex v uniformly at random, then the expectation of (1) is at most $\frac{r-1}{2r} \cdot n^2 - e(G)$.

Proposition 1. *Fix $r \geq 2$. If ϕ is a K_{r+1} -free graph limit, then*

$$\phi \left(\llbracket \begin{array}{c} \bullet \\ \diagdown \\ \square \end{array} + \frac{r-2}{r-1} \times \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \square \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \square \end{array} - \frac{r-1}{r} \times \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \square \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \square \end{array} \rrbracket \right) \leq 0.$$

Proof. We will show that the following identity holds for every $r \geq 2$.

$$\begin{aligned} (r-r^2) \cdot \llbracket \begin{array}{c} \bullet \\ \diagdown \\ \square \end{array} + \frac{r-2}{r-1} \times \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \square \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \square \end{array} - \frac{r-1}{r} \times \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \square \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \square \end{array} \rrbracket \\ = \llbracket \left((r-1) \times \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \\ \diagdown \\ \square \end{array} - \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \square \end{array} \right)^2 \rrbracket. \end{aligned}$$

Note that the identity immediately proves the statement since the right-hand side is non-negative while $r - r^2 < 0$. Firstly, observe that the left-hand side is equal to

$$\llbracket (1 - r^2) \left(\begin{array}{c} \bullet \\ \square \end{array} \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet \bullet \right) + (r - 1)^2 \left(\begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \square \end{array} \bullet \right) - (r - 1) \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet \rrbracket.$$

By the definition of $\llbracket \cdot \rrbracket$, the previous expression averages to the following:

$$(r - 1)^2 \times \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{(r - 1)(r - 3)}{3} \times \begin{array}{c} \bullet \\ \bullet \end{array} \bullet - \frac{2r - 3}{3} \times \begin{array}{c} \bullet \\ \square \end{array} \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet. \quad (2)$$

On the other hand, the right-hand side of the identity is equal to

$$(r - 1)^2 \times \left(\begin{array}{c} \bullet \\ \square \end{array} \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet \right) - (r - 1) \times \left(\begin{array}{c} \bullet \\ \square \end{array} \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet \right) + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet + \begin{array}{c} \bullet \\ \square \end{array} \bullet \bullet \bullet,$$

which again averages to (2). This finished the proof. □

Proposition 1 and the following lemma yield the statement of Theorem 1.

Lemma 1. *Fix positive integers r, b and ℓ . If G is a K_{r+1} -free graph then its b -blow-up $G[b]$ is K_{r+1} -free and $\text{del}_\ell(G[b]) = b^2 \cdot \text{del}_\ell(G)$.*

An inspection of the just presented proof yields that the bound in Theorem 1 is tight only if G is a Turán graph. Indeed when $G = T(n, r)$, Theorem 1 does not allow to remove any edge. However, this is rather a technical “obstacle” and Conjecture 1 can be seen as a way how to bypass it.

3 Tight Constructions for Conjecture 1

Clearly, Conjecture 1 is tight for Turán graphs since the bound $T(n, r) - e(G)$ does not allow deletion of any edges. When $r = 2$, the complete balanced bipartite graph and a balanced blow-up of C_5 attains the bound $0.8(n^2/4 - e(G))$. Therefore, blow-ups of C_5 behave similarly as a complete bipartite graph with respect to Conjecture 1, and this propagates to larger r .

Given $r \geq 2$, a tight construction for Conjecture 1 can be obtained as follows: Let H be a join of a copies of K_1 and b copies of C_5 , where $a + 2b = r$. Let G be a blow-up of H , such that all the vertices corresponding to K_1 s have the weight $1/r$ and all the vertices corresponding to C_5 s have the weight $2/(5r)$.

When $r \in \{2, 3, 4\}$, we prove the above description of the tight constructions for Theorem 2 is complete, see also Fig. 1.

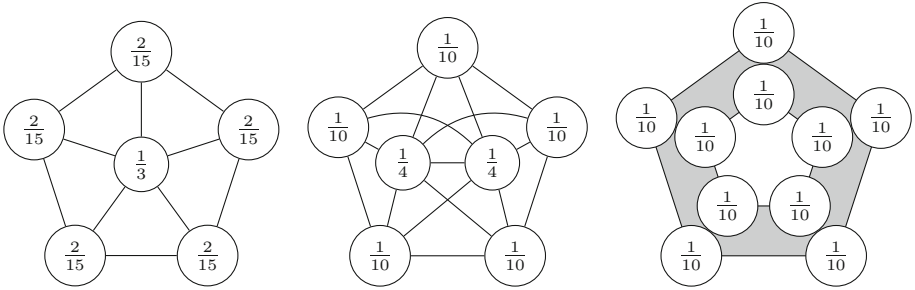


Fig. 1. Non-Turán tight constructions for Theorem 2 when $r = 3$ and $r = 4$.

3.1 Proof of Theorem 2 When $r = 2$

Let N be the non-edge type with labels u and w , and let C be the combination of N -flags that expresses the size of the cut (L, R) with $L := N(u) \cup N(v)$ and $R := V \setminus L$. Next, we define

$$O := \overline{K_3^N} \times (C - 0.8(1/2 - d(G))) = \overline{K_3^N} \times (C - 0.4(d(\overline{G}) - d(G))),$$

which can be expressed using flag algebras as follows:

Notice that $\frac{1}{2} - d(G)$ is the density of missing edges to the complete bipartite graph, and $0.8(\frac{1}{2} - d(G))$ is the normalized number of edges we are allowed to delete in Conjecture 1 when $r = 2$. In order to prove Conjecture 1, we need to show that the expression O is non-positive in triangle-free graphs.

Theorem 6. *If ϕ is a K_3 -free graph limit, then $\phi(\llbracket O \rrbracket) \leq 0$. Moreover, if $\phi(\llbracket O \rrbracket) = 0$, then $\phi^1 \left(\begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} \right) \in \{0.4, 0.5\}$ almost surely.*

Proof. First, let $F_1 := \left(\begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} - \begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} \right) \times \left(6 \times \begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} - 4 \times \begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} \right)$. Observe that if $\phi(\llbracket F_1^2 \rrbracket) = 0$ then $\phi^1 \left(\begin{smallmatrix} \bullet \\ \text{---} \\ \square \end{smallmatrix} \right) \in \{0.4, 0.5\}$ almost surely.

Next, consider the following two vectors X and Y of σ -flags, where σ is the one-vertex type and the co-cherry type, respectively, and the following 7 linear combinations of flags using X and Y :

$$X = \left(\begin{array}{c} \bullet \\ \square \end{array}, \begin{array}{c} \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \text{---} \\ \bullet \end{array} \right), \tag{3}$$

$$Y = \left(\begin{array}{c} \bullet \\ \square \end{array}, \begin{array}{c} \square \\ \square \\ \text{---} \\ \square \end{array}, \begin{array}{c} \bullet \\ \square \\ \text{---} \\ \square \end{array}, \begin{array}{c} \bullet \\ \square \\ \text{---} \\ \square \end{array}, \begin{array}{c} \bullet \\ \square \\ \text{---} \\ \square \end{array}, \begin{array}{c} \bullet \\ \square \\ \text{---} \\ \square \end{array} \right). \tag{4}$$

$$\begin{aligned} F_1 &= X \cdot (4, 4, -5, -5, 6), & F_4 &= Y \cdot (0, 1, -1, 1, -1), & F_7 &= Y \cdot (6, 1, 1, -4, -4), \\ F_2 &= X \cdot (6, -9, 0, 0, -6), & F_5 &= Y \cdot (0, 1, -1, 2, -2), & F_8 &= Y \cdot (2, -2, -2, 1, 1). \\ F_3 &= X \cdot (4, 0, -3, -4, 4), & F_6 &= Y \cdot (0, 2, -2, 1, -1), \end{aligned}$$

We express each term as a linear combination of 5-vertex unlabeled flags and establish the following estimate on $\llbracket O \rrbracket$ for some non-positive rationals w_1, w_2, \dots, w_8 :

$$\llbracket O \rrbracket \leq \sum_{i \in \{1, 2, \dots, 8\}} w_i \times \llbracket F_i^2 \rrbracket.$$

Hence, $\phi(\llbracket O \rrbracket) \leq 0$. Moreover, if the equality is attained for some limit ϕ , then $\phi(\llbracket F_i^2 \rrbracket) = 0$ for all $i \in [8]$ by complementary slackness. In particular, we have $\phi^1 \left(\begin{array}{c} \bullet \\ \text{---} \\ \square \end{array} \right) \in \{0.4, 0.5\}$ for almost every choice of the root. □

Lemma 1 readily translates Theorem 6 to the setting of finite graphs, and a result of Andrásfai, Erdős and Sós [1] yields that the only non-bipartite tight graph in Theorem 2 when $r = 2$ is a balanced blow-up of C_5 .

4 Concluding Remarks

An analogous approach to Conjecture 1 when $r = 2$ can be applied to the cases $r = 3$ and $r = 4$, although more locally defined partitions and more sum-of-squares are needed. The proof of Theorem 5 is also very similar, and in fact the simplest form we have found consists only of five sum-of-squares, a natural partition tuned to perform optimally on the corresponding Turán graphs, and an application of Theorem 6.

One of the main reasons why the complexity of the proof grows with r is the increasing number of tight constructions, and it is not obvious how to generalize this approach to all r . Nevertheless, bootstrapping from Theorem 6, we establish a much more modest improvement described in Theorem 3.

Acknowledgements. The first author is supported in part by National Natural Science Foundation of China grants 11801593 and 11931002. The second author is supported in part by NSF grant DMS-1855653. The fourth author is supported in part by an NSERC Discovery grant.

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Kempe Changes in Bounded Treewidth Graphs

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Abstract. We consider Kempe changes on the k -colourings of a graph on n vertices. If the graph is $(k - 1)$ -degenerate, then all its k -colourings are equivalent up to Kempe changes. However, the sequence between two k -colourings that arises from the corresponding proof may be exponential in the number of vertices. An intriguing open question is whether it can be turned polynomial. We prove this to be possible under the stronger assumption that the graph has treewidth at most $k - 1$. Namely, any two k -colourings are equivalent up to $O(kn^2)$ Kempe changes.

Keywords: Reconfiguration · Colouring · Treewidth · Graph theory

1 Introduction

In 1879, Kempe introduced an elementary operation on the k -colourings of a graph that became known as a Kempe change [1]. Given a coloured graph, a *Kempe chain* is a connected component in the subgraph induced by two given colours. A *Kempe change* consists in swapping the two colours in a Kempe chain. Two k -colourings of a graph are *Kempe equivalent* if one can be obtained from the other through a series of Kempe changes.

The study of Kempe changes has a vast history, see e.g. [2] for a comprehensive overview or [3] for a recent result on general graphs. We refer the curious reader to the relevant chapter of a 2013 survey by Cereceda [4]. Kempe equivalence falls within the wider setting of combinatorial reconfiguration, which [4] is also an excellent introduction to. Perhaps surprisingly, Kempe equivalence has direct applications in approximate counting and applications in statistical physics (see e.g. [5, 6] for nice overviews). Closer to graph theory, Kempe equivalence can be studied with a goal of obtaining a random colouring by applying random walks and rapidly mixing Markov chains, see e.g. [7].

Kempe changes were introduced as a mere tool, and are decisive in the proof of Vizing's edge colouring theorem [8]. However, the equivalence class they define on the set of k -colourings is itself highly interesting. In which cases is there a single equivalence class? In which cases does every equivalence class contain a colouring that uses the minimum number of colours? Vizing conjectured in

1965 [9] that the second scenario should be true in every line graph, no matter the choice of k .

A Kempe change is *trivial* if it involves a single vertex. Reconfiguration restricted to trivial Kempe changes is another well-studied topic, known as vertex recolouring. All the $(k + 2)$ -colourings of a k -degenerate graph are equivalent up to trivial Kempe changes, as proved in [10]. However, the sequence between two $(k + 2)$ -colourings that arises from the corresponding proof may be exponential in the number n of vertices. Cereceda conjectured that there exists one of length $O(n^2)$. In a breakthrough paper, Bousquet and Heinrich proved that there exists a sequence of length $O(n^{k+1})$ [11]. However, Cereceda’s conjecture remains open, even for $k = 2$. Bonamy and Bousquet [12] confirmed the conjecture for graphs of treewidth k . Note that a graph of treewidth k is k -degenerate, while there are 2-degenerate graphs with arbitrarily large treewidth.

When considering all Kempe changes – not only trivial ones – all the $(k + 1)$ -colourings of a k -degenerate graph are Kempe equivalent. However, the same haunting question remains. Is there always a short sequence between two $(k + 1)$ -colourings? The proof of Bousquet and Heinrich [11] does not extend to this setting, and even a polynomial upper-bound on the number of changes would be highly interesting. Here, we extend [12] to non-trivial Kempe changes with one fewer colour.

Theorem 1. *For every k and n , any two $(k + 1)$ -colourings of an n -vertex graph with treewidth at most k are equivalent up to $O(kn^2)$ Kempe changes.*

Additionally, the proof of Theorem 1 is constructive and yields an algorithm to compute such a sequence in time $f(k) \cdot \text{Poly}(n)$. Given a witness that the graph has treewidth at most k , the complexity drops to $k \cdot \text{Poly}(n)$.

The rest of the paper is organized as follows. In Sect. 2, we introduce various definitions, notation and observations. In Sect. 3, we present the algorithm behind Theorem 1, prove its correctness and analyse its complexity.

2 Basic Notions

Let $k, n \in \mathbb{N}$. Given a k -colouring α of a graph G and $c \in [k]$, we denote $K_{u,c}(\alpha, G)$ the k -colouring obtained from α by swapping c and $\alpha(u)$ in the connected component of $G[\alpha^{-1}(\{c, \alpha(u)\})]$ containing u . We may drop the parameter G when there is no ambiguity.

We denote $N(u)$ and $N[u]$ the open and closed neighbourhoods of a vertex u , respectively. Given an ordering $v_1 \prec \dots \prec v_n$ of the vertices of G , we denote $N^+(v_i) = \{v_j \in N(v_i) \mid j \geq i\}$ and $N^-(v_i) = \{v_j \in N(v_i) \mid j \leq i\}$.

A graph H is *chordal* if every induced cycle is a triangle. Equivalently, there is an ordering of the vertices such that for all vertex v , $N^+[v]$ is a clique. As a consequence, the chromatic number $\chi(H)$ of a chordal graph H is equal to the size $\omega(H)$ of a largest clique in H . We will use the following proposition.

Proposition 1 [13]. *Given an n -vertex chordal graph G and an integer p , any two p -colourings of G are equivalent up to at most n Kempe changes.*

The proof presented in [13] for Proposition 1 is constructive and the corresponding algorithm runs in linear time.

The *treewidth* of a graph measures how much a graph “looks like” a tree. Out of the many equivalent definitions of treewidth, we use the following: a graph G has treewidth at most k if there exists a chordal graph H such that G is a (not necessarily induced) subgraph of H , with $\omega(H) = \chi(H) \leq k + 1$. For example, if G is a tree, we note that G is chordal and 2-colourable, take $H = G$ and derive that G has treewidth at most 1.

A graph G is *k-degenerate* if there exists an ordering of the vertices such that for all $v \in V$, $N^+(v)$ is of size at most k . If G has treewidth at most k , then it is *k-degenerate*.

3 Main Proof

Let G be an n -vertex graph of treewidth k . Let H be a chordal graph such that G is a (not necessarily induced) subgraph of H , with $\omega(H) = \chi(H) \leq k + 1$ and $V(H) = V(G)$. Computing H is equivalent to computing a so-called tree decomposition of G , which can be done in time $f(k) \cdot n$ [14].

Since G and H are defined on the same vertex set, there may be confusion when discussing neighbourhoods and other notions. When useful, we write G or H in index to specify. There is an ordering $v_1 \prec \dots \prec v_n$ of the vertices of G such that $\forall v \in V(G), N_H^+[v]$ induces a clique in H . The ordering can be computed from H in $O(n)$ using Lex-BFS [15].

The core of the main proof lies in Proposition 2: any $(k + 1)$ -colouring of G is equivalent up to $O(k \cdot n^2)$ Kempe changes to a $(k + 1)$ -colouring of G that yields a $(k + 1)$ -colouring of H .

Proof (of Theorem 1 assuming Proposition 2). Let α and β be two $(k + 1)$ -colourings of G . By Proposition 2, there exists a $(k + 1)$ -colouring α' (resp. β') that is equivalent to α (resp. β) up to $O(kn^2)$ Kempe changes. Additionally, both α' and β' yield $(k + 1)$ -colourings of H . Since H is chordal, by Proposition 1, there exists a sequence of at most kn Kempe changes in H from α' to β' . Each of these Kempe changes in H can be simulated by at most n Kempe changes in G , which results in a sequence of length $O(kn^2)$ between α and β . □

Proposition 2. *Given any $(k + 1)$ -colouring α of G , there exists a $(k + 1)$ -colouring α' of G that is equivalent to α up to $O(kn^2)$ Kempe changes and such that $\alpha'(u) \neq \alpha'(v)$ for all $w \in E(H)$.*

To prove Proposition 2 and obtain a $(k + 1)$ -colouring of H , we gradually “add” to G the edges in $E(H) \setminus E(G)$. To add an edge, we first reach a $(k + 1)$ -colouring where the extremities have distinct colours, then propagate any later Kempe change involving one extremity to the other extremity. We formalize this process through Algorithm 1. Let $v_1w_1 \prec \dots \prec v_qw_q$ be the edges in $E(H) \setminus E(G)$ in the lexicographic order, where $v_i \prec w_i$ for every i .

Algorithm 1: Going from α to α'

Input : G , a $(k + 1)$ -colouring α of G and $v_1w_1 \prec \dots \prec v_qw_q$ such that $G + \{v_1w_1, \dots, v_qw_q\}$ is a $(k + 1)$ -colourable chordal graph H

Output: A $(k + 1)$ -colouring $\tilde{\alpha}$ of H , that is Kempe equivalent to α

Let $\tilde{G} \leftarrow G$ and $\tilde{\alpha} \leftarrow \alpha$;

1 **for** j from 1 to q **do**

2 **if** $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$ **then**

Let $c \in [k + 1] \setminus \tilde{\alpha}(N_H^+[v_j])$;

// Possible because $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$ and $|N_H^+[v_j]| \leq k$.

Let $U = N_{\tilde{G}}^-(v_j) \cap N_{\tilde{G}}^-(w_j) = \{u_1 \prec \dots \prec u_p\}$;

3 **for** $i = p$ down to 1 **do**

4 **if** $\tilde{\alpha}(u_i) = c$ **then**

Let $c_i \in [k + 1] \setminus \tilde{\alpha}(N_{\tilde{G}}^+[u_i])$;

// Possible because $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$ and $|N_{\tilde{G}}^+[u_i]| \leq k$.

5 $\tilde{\alpha} \leftarrow K_{u_i, c_i}(\tilde{\alpha}, \tilde{G})$;

6 **end**

7 **end**

// Now $c \notin \tilde{\alpha}(\{x \in N_{\tilde{G}}^-(v_j) \mid x \geq u_i\})$

8 **end**

// Now $c \notin \tilde{\alpha}(N_{\tilde{G}}^-(v_j))$

9 $\tilde{\alpha} \leftarrow K_{v_j, c}(\tilde{\alpha}, \tilde{G})$;

10 **end**

11 $\tilde{G} \leftarrow \tilde{G} \cup \{v_jw_j\}$;

12 **end**

We will prove the following three claims. Note that Proposition 2 follows from Claims 1 and 2, while Claim 3 simply guarantees that the proof of Theorem 1 is indeed constructive.

Claim (1). Algorithm 1 outputs a $(k + 1)$ -colouring α' of G that is Kempe equivalent to α and such that $\alpha'(u) \neq \alpha'(v)$ for all $uv \in E(H)$.

Claim (2). Algorithm 1 performs $O(kn^2)$ Kempe changes in G to obtain α' from α .

Claim (3). Algorithm 1 runs in $O(kn^4)$ time.

In Algorithm 1, the variable \tilde{G} keeps track of how close we are to a $(k + 1)$ -colouring of H . Before the computations start, $\tilde{G} = G$. When the algorithm terminates, $\tilde{G} = H$. At every step, G is a subgraph of \tilde{G} . To refer to \tilde{G} or $\tilde{\alpha}$ at some step of the algorithm, we may say the *current* graph or *current* colouring. The Kempe changes that we discuss are performed in \tilde{G} . Consequently, the corresponding set of vertices might be disconnected in G , and every Kempe change in \tilde{G} may correspond to between 1 and n Kempe changes in G .

Proof (of Claim 1). By construction, at every step $\tilde{\alpha}$ is Kempe equivalent to α . We prove the following loop invariant: at every step, $\tilde{\alpha}$ is a $(k + 1)$ -colouring

of \tilde{G} . Since $\tilde{G} = H$ at the end of the algorithm, proving the loop invariant will yield the desired conclusion.

The invariant holds at the beginning of the algorithm, when $\tilde{G} = G$.

Assume that at the beginning of the j -th iteration of the loop 1, $\tilde{\alpha}$ is a proper colouring of \tilde{G} . All the Kempe changes in the loop are performed in \tilde{G} , so we only need to prove that at the end of the iteration, $\tilde{\alpha}(v_j) \neq \tilde{\alpha}(w_j)$.

This follows from the validity of comments 4 and 5. The latter is a direct consequence of the former, so we focus on arguing that after the step i of the inner loop 2, we have $c \notin \tilde{\alpha}(\{x \in N_{\tilde{G}}(v_j) \mid x \geq u_i\})$. The key observation is that at the i -th step of the inner loop 2, the Kempe changes performed at line 3 involve only vertices smaller than u_i . We prove by induction the stronger statement the Kempe chain T involved in the Kempe change $K_{u_i, c_i}(\tilde{\alpha}, \tilde{G})$ is a tree rooted at u_i in which all the nodes are smaller than their father.

- $c_i \notin \tilde{\alpha}(N_{\tilde{G}}^+(u_i))$ so all the vertices at distance 1 in T from u_i are smaller than u_i .
- Let x at distance $d + 1$ from u_i in T . Let y be a neighbour of x at distance d from u_i . Assume by contradiction that $x \succ y$. By induction hypothesis, there is a unique neighbour z of y at distance $d - 1$ from u_i , with $y \prec z$. Both z and x are in $N_H^+(y)$ and since H is chordal, this implies $zx \in E(H)$. We have $z \prec u_i \prec v_j$ so $zx \prec v_j w_j$ and $zx \in E(\tilde{G})$. In particular, x is at distance d from u_i in T , which raises a contradiction and proves $x \prec y$.

Assume by contradiction that x is adjacent to two vertices y, z at distance d from u_i in T . Then y and z are identically coloured so $yz \notin E(\tilde{G})$. Moreover $y, z \in N_H^+(x)$ and H is chordal, hence yz is an edge of H . Since $y, z \prec v_j$, we have $yz \prec v_j w_j$. Thus, yz belongs to \tilde{G} , raising a contradiction.

As a result, the Kempe change of line 6 does not recolour any vertex larger than u_i with colour c , and the comment 5 is true. Therefore at the beginning of line 6, $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$ and $c \notin \alpha(N(v_j))$. At the end of line 6, v_j and w_j are coloured differently.

Proof (of Claim 2). We now prove that the number of Kempe changes in G performed by the algorithm is $O(kn^2)$.

We first prove that for each vertex x , there exists at most one step j of the loop 1 for which $v_j = x$ and we enter the conditional statement $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$. Indeed, the first time we enter the conditional statement, the vertex x is recoloured with a colour c not in $N_H^+(x)$ at line 6. Note that all the edges xy with $y \succ x$ are consecutive in the ordering of $E(H) \setminus E(G)$. Therefore, once the vertex x is recoloured, all the remaining edges xy are handled without Kempe change, as the conditional statement is not satisfied. This implies directly that line 6 is executed at most n times.

Now, we bound the number of times x plays the role of u_i in the Kempe change at line 3. For each step j of loop 1 for which it happens, we have $v_j, w_j \in N_H^+(x)$. Since $|N_H^+(x)| \leq k$ and each v_j is involved at most once by the above argument, we obtain that x plays this role at most k times.

Consequently the overall number of Kempe changes performed in \tilde{G} by the algorithm is $O(kn)$. Performing a Kempe change in \tilde{G} is equivalent to performing a Kempe change in all the connected component of G of the Kempe chain of \tilde{G} . Therefore, the number of Kempe changes performed in G by the algorithm is $O(kn^2)$.

Proof (of Claim 3). In total, the loop 2 is executed at most once for every pair of vertices in $N_H^+(u)$ for each $u \in V(G)$, that is $O(k^2n)$ times. However, we also take into account the number of Kempe changes that need to be performed. By Claim 2, only $O(kn^2)$ Kempe changes are performed in G . As a result, the total complexity of the algorithm is $O(k^2n + kn^4) = O(kn^4)$ (performing Kempe changes in a naive way in G).

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No Selection Lemma for Empty Triangles

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Abstract. In this paper we show that for any integer n and real number $0 \leq \alpha \leq 1$ there exists a point set of size n with $\Theta(n^{3-\alpha})$ empty triangles such that any point of the plane is in $O(n^{3-2\alpha})$ empty triangles.

Keywords: Combinatorial geometry · Empty triangles · Horton sets · Squared Horton sets · Selection Lemma

1 Introduction

Let S be a set of n points in general position¹ in the plane. A *triangle of S* is a triangle whose vertices are points of S . We say that a point p of the plane *stabs* a triangle Δ if it lies in the interior of Δ . Boros and Füredi [7] showed that for any point set S in general position in the plane, there exists a point in the plane which stabs a constant fraction ($\frac{n^3}{27} + O(n^2)$) of the triangles of S . Bárány [3] extended the result to \mathbb{R}^d ; he showed that there exists a constant $c_d > 0$, depending only on d , such that for any point set $S_d \subset \mathbb{R}^d$ in general position, there exists a point in \mathbb{R}^d which is in the interior of $c_d n^{d+1}$ d -dimensional simplices spanned by S_d . This result is known as *First Selection Lemma* [12].

Later, researchers considered the problem of the existence of a point in many triangles of a given family, \mathcal{F} , of triangles of S . Bárány, Füredi and Lovász [4] showed that for any point set S in the plane in general position and any family \mathcal{F} of $\Theta(n^3)$ triangles of S , there exists a point of the plane which stabs $\Theta(n^3)$ triangles from \mathcal{F} . This result, generalized to \mathbb{R}^d by Alon et al. [1], is now also known as *Second Selection Lemma* [12].

¹ A point set $S \subset \mathbb{R}^d$ is in general position if for every integer $1 < k \leq d+1$, no subset of k points of S is contained in a $(k-2)$ -dimensional flat.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 734922.

D.P. and B.V. were partially supported by the Austrian Science Fund within the collaborative DACH project *Arrangements and Drawings* as FWF project I 3340-N35.

Both results for the plane require families of $\Theta(n^3)$ triangles of an n -point set. It is natural to ask about families of smaller cardinality. For this question, Aronov et al. [2] showed that for every $0 \leq \alpha \leq 1$ and every family \mathcal{F} of $\Theta(n^{3-\alpha})$ triangles of an n -point set, there exists a point of the plane which stabs $\Omega(n^{3-3\alpha}/\log^5 n)$ triangles of \mathcal{F} . This lower bound was improved by Eppstein [8] to the maximum of $n^{3-\alpha}/(2n-5)$ and $\Omega(n^{3-3\alpha}/\log^2 n)$. A mistake in one of the proofs was later found and fixed by Nivasch and Sharir [13]. Furthermore, Eppstein [8] constructed point sets S and families of $n^{3-\alpha}$ triangles of S such that every point of the plane is in at most $n^{3-\alpha}/(2n-5)$ triangles for $\alpha \geq 1$ and in at most $n^{3-2\alpha}$ triangles for $0 \leq \alpha \leq 1$. Hence, for the number of triangles of a family \mathcal{F} that can be guaranteed to simultaneously contain some point of the plane, there is a continuous transition from a linear fraction for $|\mathcal{F}| = O(n^2)$ to a constant fraction for $|\mathcal{F}| = \Theta(n^3)$.

A triangle of S is said to be *empty* if it does not contain any points of S in its interior. Let $\tau(S)$ be the number of empty triangles of S . It is easily shown that $\tau(S)$ is $\Omega(n^2)$; Katchalski and Meir [11] showed that there exist n -point sets S with $\tau(S) = O(n^2)$. Note that for such point sets, an edge of S is on average part of a constant number of empty triangles of S . However, Erdős [9] conjectured that there is always an edge of S , which is part of a superconstant number of empty triangles of S . Bárány et al. [5] proved this conjecture for random n -point sets, showing that for such sets, $\Theta(n/\log n)$ empty triangles are expected to share an edge, where the expected total number of empty triangles is $\Theta(n^2)$; see [15].

Erdős' conjecture suggests that perhaps there is always a point of the plane stabbing many empty triangles of S . Naturally, the mentioned lower bounds for the number of triangles stabbed by a point of the plane also apply for the family of all empty triangles of S . In contrast, the upper bound constructions of Eppstein do not apply, since they contain non-empty triangles or do not contain all empty triangles of their underlying point sets. In this paper, we show that we cannot guarantee the existence of a point in more triangles than these upper bounds for general families of triangles; hence the title of our paper. Specifically, we prove the following.

Theorem 1. *For every integer n and every $0 \leq \alpha \leq 1$, there exist sets S of n points with $\tau(S) = \Theta(n^{3-\alpha})$ empty triangles where every point of the plane stabs $O(n^{3-2\alpha})$ empty triangles of S .*

To prove Theorem 1 for $\alpha = 1$, we utilize so called *Horton sets* and *squared Horton sets*. Horton [10] constructed a family of arbitrary large sets without large empty convex polygons. Valtr [14] generalized this construction and named the generalised sets Horton sets. Squared Horton sets were defined by Valtr [14] (as set A_k in Sect. 4). Bárány and Valtr [6] showed that squared Horton sets of size n span only $\Theta(n^2)$ empty triangles.

Outline. The remainder of this extended abstract is organized as follows: In Sect. 2, we define Horton sets and show several properties of them that will be

of use for later sections. Section 3 considers squared Horton sets and contains a proof of Theorem 1 for the case $\alpha = 1$ (Theorem 2). And in Sect. 4 we present a generalized construction based on squared Horton sets, which we analyze to prove Theorem 1. Due to space constraints, the full proofs of most statements in this extended abstract are deferred to the full version.

2 Horton Sets

Let X be a set of n points in the plane such that no two points have the same x -coordinate. Suppose that the points in X are labeled in increasing order by their x -coordinates, such that $X = \{p_0, p_1, \dots, p_{n-1}\}$. For any such labeled point set X , we denote with $X_0 = \{p_0, p_2, \dots\}$ the subset of points of X that have even labels and with $X_1 = \{p_1, p_3, \dots\}$ the subset of points of X with odd labels.

Now consider two point sets X and Y in the plane such that no two points of $X \cup Y$ have the same x -coordinate. We say that Y is *high above* X if every line passing through two points of Y is above every point of X . Similarly, X is *deep below* Y if every line through two points of X is below every point of Y .

A set H of n points in the plane, with no two points having the same x -coordinate, is called a *Horton set* if it satisfies the following properties:

- If $|H| = 1$, then H is a Horton set.
- If $|H| > 1$, then H_0 and H_1 are Horton sets, H_1 is high above H_0 , and H_0 is deep below H_1 .

The following two statements on empty triangles in Horton sets will turn out useful for proving our main theorem.

Lemma 1. *Let H be a Horton set of n points. Then every point of the plane stabs $O(n \log n)$ empty triangles of H .*

Lemma 2. *Let H be a Horton set of n points. Then every point of H is incident to $O(n \log n)$ empty triangles of H .*

3 Squared Horton Sets

For n being a squared integer, we denote with G an integer grid of size $\sqrt{n} \times \sqrt{n}$. Otherwise, G is a subset of an integer grid of size $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$, from which some consecutive points of the topmost row and possibly the leftmost column are removed to have n points remaining. An ε -*perturbation* of G is a perturbation of G where every point p of G is replaced by a point at distance at most ε to p .

A *squared Horton set* H of size n is a specific ε -perturbation of G such that triples of non-collinear points in G keep their orientations in H and such that points along each non-vertical line in G are perturbed to points forming a Horton set in H . Points along each vertical line are perturbed to points forming a rotated copy of a Horton set in H . See [6, 14] for more details. The following lemma is a direct consequence of this definition.

Lemma 3. *Let H be a squared Horton set obtained from an ε -perturbation of G and let ℓ and ℓ' be two parallel lines spanned by G . Then the subset of H formed by the ε -perturbations of $(G \cap \ell) \cup (G \cap \ell')$ is also a Horton set.*

Let Δ be a (possibly degenerate) triangle with vertices on G . Let e be an edge of Δ and let p be the vertex of Δ opposite to e . We say that the *height* of Δ w.r.t. e is zero if p is on the straight line spanned by e ; otherwise, it is one plus the number of lines between e and p , parallel to e , and containing points of G . We call the area bounded by two such neighboring lines a *strip* of G . The *height* of Δ is the minimum of the heights w.r.t. its edges and the edge defining the height of Δ is the *base edge*.

Lemma 4. *Any interior-empty triangle of G has height at most 2.*

For the proof of our next statement, we will use Euler’s totient function $\varphi(d)$, which is the number of integers k with $1 \leq k \leq d$ that are relative primes with d . Clearly, $\varphi(d) \leq d$. Note that for $d > 1$, $\varphi(d)$ is also the number of points (d, a) with $a < d$ on the integer grid such that the segment from the origin to the point (d, a) does not contain any grid point.

Theorem 2. *Let H be a squared Horton set of n points. Then every point of the plane stabs $O(n)$ empty triangles of H .*

Proof. Let H be a squared Horton set of n points. Let $q \in \mathbb{R}^2 \setminus H$. Obviously, no point of H stabs any empty triangle of H . By Lemma 4, every unperturbed triangle lies in at most two neighboring strips of G , parallel to the base edge of the triangle. We count each triangle that possibly contains q for this direction.

We start with the number of triangles of height zero. For each such triangle Δ , q has to be on the same perturbed ‘line’ as Δ . For each $1 \leq d \leq \lceil \sqrt{n} \rceil$, there are at most $4 \cdot \varphi(d)$ lines through q with at most \sqrt{n}/d points each. As for each line, the points form a Horton set in H , and by Lemma 1 the number of such empty triangles containing q is at most $c \sum_{d=1}^{\lceil \sqrt{n} \rceil} \varphi(d) (\lceil \sqrt{n} \rceil / d) \cdot \log_2(\lceil \sqrt{n} \rceil / d) \leq c \lceil \sqrt{n} \rceil \sum_{d=1}^{\lceil \sqrt{n} \rceil} \log_2(\lceil \sqrt{n} \rceil / d) = O(n)$.

We now consider the triangles of height one. For q to stab such a triangle, q has to lie on the boundary or in the interior of the strip defining the height of Δ . For each direction, there are at most two relevant strips for q , each containing a Horton set of size $2 \lceil \sqrt{n} \rceil / d$. Hence we obtain $c \sum_{d=1}^{\lceil \sqrt{n} \rceil} \varphi(d) (2 \lceil \sqrt{n} \rceil / d) \cdot \log_2(2 \lceil \sqrt{n} \rceil / d) = O(n)$ such empty triangles that could be stabbed by q .

Finally, consider the triangles of height 2. For each direction there are at most five grid lines that could contain the unperturbed base edge of an empty triangle containing q : at most three lines bounding the 1 – 2 strip(s) containing q , plus the neighboring lines above and below. Also, the third point p of this triangle stems from one of these lines. For each point $p \in H$ from these lines, the (possibly) empty triangles of height 2 with base edge on these lines are pairwise interior-disjoint. Therefore, for each point p there is at most one such empty triangle of height two that contains q . Hence, the total number of those triangles per direction is $O(\sqrt{n}/d)$. Summing over all $1 \leq d \leq \sqrt{n}$, we obtain a bound of $O(n)$ triangles of H of height 2 stabbed by q . □

The following result on the number of empty triangles incident to a fixed point of a squared Horton set can be proven in a similar way as Theorem 2.

Lemma 5. *Every point of a squared Horton set of n points is incident to $O(n)$ empty triangles.*

4 \sqsupset -Squared Horton Sets

We denote by \sqsupset a point set as depicted in Fig. 1. It is obtained by placing four points on the corners of a square and adding further points along four slightly concave arcs between adjacent corners, such that on each arc there is almost the same number of points.

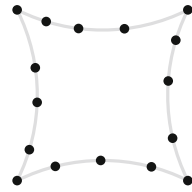


Fig. 1. A \sqsupset point set.

Let H be a squared Horton set with m points. A \sqsupset -squared Horton set H_{\sqsupset} is the set we obtain by replacing every point of H by a small \sqsupset with k points. Thus, H_{\sqsupset} consists of $n = km$ points. We denote the points of H by p_i and the corresponding \sqsupset set by \sqsupset_i . Further, the sets $\sqsupset_i, i \in \{1, \dots, m\}$, are sufficiently small such that for any $i \neq j \neq l \in \{1, \dots, m\}$, any point triple $q_i \in \sqsupset_i, q_j \in \sqsupset_j, q_l \in \sqsupset_l$ has the same orientation as p_i, p_j, p_l . This is possible since the underlying point set is in general position. Moreover, the arcs of each \sqsupset_i are such that for any \sqsupset_j with $i \neq j$, all points of one arc of \sqsupset_i and one arc of \sqsupset_j form a convex point set.

Lemma 6. *The number of empty triangles in H_{\sqsupset} is $\Theta(m^2k^3)$.*

Proof. We split the empty triangles of H_{\sqsupset} into three groups, depending on the number of different \sqsupset subsets of H_{\sqsupset} that contain vertices of a triangle.

Case 1. Triangles spanned by three points of $\sqsupset_i, i \in \{1, \dots, m\}$. Each \sqsupset_i spans $O(k^3)$ such empty triangles. Summing up over the m different subsets $\sqsupset_1, \dots, \sqsupset_m$ yields $O(mk^3)$ empty triangles of H_{\sqsupset} for this case.

Case 2. Triangles spanned by two points in \sqsupset_i and one point in \sqsupset_j , for some $i \neq j \in \{1, \dots, m\}$. There are $\Theta(m^2)$ pairs $(\sqsupset_i, \sqsupset_j)$. For each of \sqsupset_i and \sqsupset_j , there are at least $k/4$ and at most k choices for a vertex of an empty triangle. So we have $\Theta(m^2k^3)$ empty triangles of H_{\sqsupset} in this case.

Case 3. Triangles spanned by one point in each of $\sqsupset_i, \sqsupset_j, \sqsupset_l$, for some $i \neq j \neq l \in \{1, \dots, m\}$. Then p_i, p_j, p_l is an empty triangle of H . For each of p_i, p_j , and p_l , we have at least $k/4$ and at most k choices for a point of its corresponding \sqsupset such that the resulting triangle of H_{\sqsupset} is empty. As H has $\Theta(m^2)$ empty triangles, we obtain $\Theta(m^2k^3)$ empty triangles of H_{\sqsupset} for this case as well. \square

Lemma 7. *Every point of the plane stabs $O(mk^3)$ empty triangles of H_{\sqsupset} .*

Proof Sketch. First we fix a point s of the plane. We again we split the empty triangles into three groups: all three vertices are in the same \sqsupset , two vertices in one \sqsupset and the third one in a different \sqsupset , or all vertices in different \sqsupset s. The number

of empty triangles with s inside can be bounded relatively straightforward for triangles of the first two groups. Triangles of the third group again stem from empty triangles of the underlying squared Horton set H . For this group, we show we can find a point s' depending on s such that if s stabs an empty triangle Δ of H_{\square} , then either s' stabs the corresponding triangle Δ' of H or s' is on the boundary of Δ' . Using this in combination with our results for squared Horton sets (namely, Theorem 2 and Lemma 5) completes the proof. \square

With these lemmata we can finally show our main result.

Proof of Theorem 1. By Lemma 6 there are $\Theta(m^2k^3)$ empty triangles in S . With $k = n^{1-\alpha}$ there are $\Theta(n^{3-2\alpha})$ empty triangles. By Lemma 7 every point stabs $O(mk^3)$ empty triangles. Hence every point stabs $O(n^{3-2\alpha})$ empty triangles. \square

Acknowledgements. Research on this work has been initiated at a workshop of the H2020-MSCA-RISE project 73499 - CONNECT, held in Barcelona in June 2017. We thank all participants for the good atmosphere as well as for discussions on the topic.

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The Mod k Chromatic Index of Random Graphs

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Abstract. The *mod k chromatic index* of a graph G is the minimum number of colors needed to color the edges of G in a way that the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod k). Recently, we proved that the mod k chromatic index of every graph is at most $198k - 101$, improving, for large k , a result of Scott [*Discrete Math.* 175, 1–3 (1997), 289–291]. In this paper we study the mod k chromatic index of random graphs. More specifically, we prove that for every integer $k \geq 2$, there is $C_k > 0$ such that, if $p \geq C_k n^{-1} \log n$ and $n(1-p) \rightarrow \infty$, the following holds: if k is odd, then the mod k chromatic index of $G(n, p)$ is asymptotically almost surely equal to k ; and if k is even, then the mod k chromatic index of $G(2n, p)$ is asymptotically almost surely equal to k , while the mod k chromatic index of $G(2n+1, p)$ is asymptotically almost surely equal to $k+1$.

Keywords: Mod k coloring · Edge colorings · Random graphs

1 Introduction

Throughout this paper, all graphs considered are simple and $k \geq 2$ is an integer. A χ'_k -coloring of G is a coloring of the edges of G in which the subgraph spanned by the edges of each color has all degrees congruent to 1 (mod k). The *mod k chromatic index* of G , denoted $\chi'_k(G)$, is the minimum number of colors in a

This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil – CAPES – Finance Code 001. F. Botler is partially supported by CNPq (423395/2018-1) and FAPERJ (211.305/2019). L. Colucci is supported by FAPESP (2020/08252-2). Y. Kohayakawa is partially supported by CNPq (311412/2018-1, 423833/2018-9) and FAPESP (2018/04876-1, 2019/13364-7). The research that led to this paper started at WoPOCA 2019, which was financed by CNPq (425340/2016-3, 423833/2018-9) and FAPESP (2015/11937-9). FAPERJ is the Rio de Janeiro Research Foundation. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

χ'_k -coloring of G . In 1991, Pyber [4] proved that $\chi'_2(G) \leq 4$ for every graph G , and in 1997 Scott [5] proved that $\chi'_k(G) \leq 5k^2 \log k$ for every graph G . Recently, we proved [2] that $\chi'_k(G) \leq 198k - 101$ for any G , which improves Scott's bound for large k and is sharp up to the multiplicative constant. In this paper, we study the behavior of the mod k chromatic index of the random graph $G(n, p)$ for a wide range of $p = p(n)$. More specifically, we prove the following result.

Theorem 1. *Let $k \geq 2$ be an integer. There is a constant C_k such that if $p \geq C_k n^{-1} \log n$ and $n(1 - p) \rightarrow \infty$, then the following holds as $n \rightarrow \infty$.*

(i) *If k is even, then*

$$\begin{aligned} \mathbb{P}(\chi'_k(G(2n, p)) = k) &\rightarrow 1; \\ \mathbb{P}(\chi'_k(G(2n + 1, p)) = k + 1) &\rightarrow 1. \end{aligned}$$

(ii) *If k is odd, then*

$$\mathbb{P}(\chi'_k(G(n, p)) = k) \rightarrow 1.$$

Theorem 1 extends a theorem of the authors [1] that dealt with the case $k = 2$ and $C\sqrt{n^{-1} \log n} < p < 1 - 1/C$ for some C .

In Sect. 2, we present some technical lemmas, and in Sect. 3, we prove Theorem 1. While Theorem 1 tells us that the typical value of the mod k chromatic index is at most $k + 1$ for a wide range of edge densities, in Sect. 4, we give a sequence G_3, G_4, \dots of graphs for which $\chi'_k(G_k) \geq k + 2$ for every $k \geq 3$. Finally, in Sect. 5, we present some concluding remarks and discuss future work. Owing to space limitations, we omit the proofs of some results.

2 Technical Lemmas

In this section, we present some technical lemmas used in the proof of Theorem 1. We say that a bipartite graph with bipartition (A, B) is *balanced* if $|A| = |B|$. The next lemma is a generalization of Lemma 4.7 in [3].

Lemma 1. *For every positive constant c and integer $k \geq 2$, the following holds: if $p = \Theta(n^{-1} \log n)$, then a.a.s. every induced balanced bipartite subgraph of $G(n, p)$ with minimum degree at least $c \log n$ contains k disjoint perfect matchings.*

Given a graph G , for each $i \in \{1, \dots, k\}$, we denote by $V_i(G)$ the set of vertices of G with degree in G congruent to i modulo k . Let $n_i(G) = |V_i(G)|$ and denote by G_i the graph $G[V_i(G)]$. The sets V_1, \dots, V_k are the *degree classes* of G .

Lemma 2. *Let $k \geq 2$ be an integer and let $\varepsilon > 0$ be given. There is a constant $C = C_{\varepsilon, k} > 0$ such that the following holds: suppose that $p > Cn^{-1} \log n$ and $n(1 - p) \rightarrow \infty$, and let $G = G(n, p)$ be generated as the union of two random graphs $H_1 = G(n, p_1)$ and $H_2 = G(n, p_2)$ on the same vertex set (in particular, $(1 - p) = (1 - p_1)(1 - p_2)$), where $p_1 = \Theta(n^{-1} \log n)$. Then, as $n \rightarrow \infty$, for every $1 \leq i, j \leq k$, we have*

(i)

$$\mathbb{P} \left(\left| n_i(G) - \frac{n}{k} \right| < \varepsilon \frac{n}{k} \right) \rightarrow 1;$$

(ii)

$$\mathbb{P} \left(\left| |N_{H_1}(v) \cap V_j(G)| - \frac{np_1}{k} \right| < \varepsilon \frac{np_1}{k} \text{ for all } v \in V_i(G) \right) \rightarrow 1.$$

The following result is a straightforward consequence of Lemma 2.

Corollary 1. *Let $k \geq 3$ be an integer. There is a constant C_k such that if $p > C_k n^{-1} \log n$ and $n(1-p) \rightarrow \infty$, then $G = G(n, p)$ satisfies the following with probability $1 - o(1)$: there are $k - 2$ vertex-disjoint stars S_3, \dots, S_k in G such that, for each $i \in \{3, \dots, k\}$, the star S_i has $k + i - 1$ edges, is centered in G_i and its leaves belong to G_2 .*

We shall also make use of the following lemma.

Lemma 3. *For every integer $k \geq 2$, there is a constant $C_k > 0$ such that the following holds: if $p > C_k n^{-1} \log n$, then $G(n, p)_1$, the subgraph induced by the vertices of degree 1 modulo k in $G(n, p)$, is connected a.a.s.*

3 Main Theorem

We say that a graph G is a *mod k graph* if all its non-isolated vertices have degree congruent to 1 mod k . In what follows, we prove Theorem 1. The strategy of the proof is to use Corollary 1 to remove the edges of a set of vertex-disjoint stars of suitable sizes to fix the parity of the cardinality of some degree classes. Then we use Lemma 1 to remove perfect matchings inside each degree class, so that the remaining graph is a mod k graph. This is performed in a way that the removed edges form a graph that can be colored with $k - 1$ colors.

Proof (Proof of Theorem 1). First, we observe that a.a.s. G contains a vertex of nonzero degree congruent to 0 mod k , and hence $\mathbb{P}(\chi'_k(G(n, p)) \geq k) \rightarrow 1$ as $n \rightarrow \infty$ regardless of the parity of k .

The key idea of this proof when k and n are even or k is odd is (1) to use Corollary 1 to find a set of vertex-disjoint stars whose removal of their edges yields a graph G' such that, for each $i > 2$, the graph G'_i , obtained by removing the center of the star centered at G_i (if there is such a star), has an even number of vertices, and then (2) use Lemma 1 to remove, for each $i > 1$, an $(i - 1)$ -regular bipartite spanning graph inside each G'_i (the union of $i - 1$ disjoint perfect matchings). The removed graphs are vertex-disjoint and bipartite, and hence their union can be colored with $k - 1$ colors (see Fig. 1). Moreover, the obtained graph is a mod k graph, which can be colored with the k th color. The remaining case, namely when k is even and n is odd, then follows by removing a vertex v , using the first case, and coloring most of the edges incident to v with the $(k + 1)$ st color. In what follows, we divide the proof according to these cases.

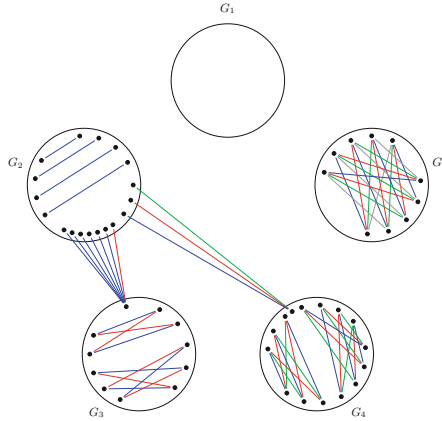


Fig. 1. Illustration of the subgraphs removed from G in the case $k = 5$: in this example, G_2 and G_5 have an even number of vertices, and G_3 and G_4 have an odd number of vertices. Each circle represents a degree class. A bipartite graph inside a degree class spans all of the vertices of that class, except for the vertices of the stars.

Case 1. k even. Let G_e and G_o denote the subgraphs of G induced by the vertices of even and odd degree, respectively. The proof of this case is divided depending on the parity of n .

Case 1.1 n even. We generate $G = G(n, p)$ as the union of $H_1 = G(n, p_1)$ and $H_2 = G(n, p_2)$, as in Lemma 2. By Corollary 1, there are vertex-disjoint stars S_3, \dots, S_k , such that, for each $i \in \{3, 4, \dots, k\}$, the star S_i has $i - 1$ edges, is centered in G_i and its leaves belong to G_2 . Let G' be the graph obtained from G by removing the edges of S_i for each $i \in \{3, 4, \dots, k\}$ if $|V(G_i)|$ is odd, and let G'_i be the subgraph of G' induced by the vertices with degree (in G') congruent to i modulo k . Note that G'_2 is precisely the graph obtained from G_2 by removing the leaves of the stars removed from G ; and, for $i \in \{3, 4, \dots, k\}$, G'_i is precisely the graph obtained from G_i by removing the their centers. Since n is even, both $|V(G_o)|$ and $|V(G_e)|$ are even, and hence, the number of $j \in \{2, 4, \dots, k\}$ for which $|V(G_j)|$ is odd is even. This implies that $|V(G'_i)|$ is even for every $i \geq 2$ and that the degree of each vertex that belongs to some star is 1 modulo k in G' . By Lemma 2 (ii), almost surely every G_i ($1 \leq i \leq k$) satisfies $\delta_{H_1}(G_i) \geq np_1/(2k) > c' \log n$ for some c' . One can prove, then, that every G'_i ($2 \leq i \leq k$) contains a spanning bipartite subgraph with equal sizes and minimum degree at least $(c'/3) \log n$ in H_1 , and hence, Lemma 1 implies that G'_i contains $i - 1$ disjoint perfect matchings. Note that the graph obtained from G' by removing these matchings is a mod k graph, and that the stars and matchings above form a bipartite graph with maximum degree at most $k - 1$, and hence can be colored with $k - 1$ colors, as desired. This finishes the proof of Case 1.1.

Case 1.2 n odd. Fix a vertex $u \in V(G)$. We generate $G(n, p)$ by first exposing the edges in $G' = G - u$ and then the edges incident to u . In this way, it is

clear that almost surely u has at least $k - 1$ (indeed, at least $(1 - o(1))npk^{-1}$) neighbors in G'_1 . By Case 1.1, almost surely G' can be colored with colors $1, \dots, k$. Moreover, by the previous case, the coloring can be done in a way that all edges incident to vertices with degree $1 \pmod k$ (in G') are colored with the same color, say 1. Now, we color the edges incident to u . Suppose that $d(u) \equiv d \pmod k$, for some $d \in \{1, 2, \dots, k\}$. If $d \neq 1$, then we assign each of the colors $2, \dots, d$ once to an edge joining u to vertices in G'_1 (this is possible since there are at least $k - 1$ such edges), leaving a number $1 \pmod k$ of uncolored edges incident to u . We assign these edges a new color. Therefore, G can be colored with $k + 1$ colors.

Now, suppose that G can be colored with k colors. Note that this implies that the edges incident to a fixed vertex in G_1 must get the same color. By Lemma 3, G_1 is a.a.s. connected, whence all the edges of G incident to vertices of G_1 must be colored with the same color, say 1. Moreover, by Lemma 2 (ii), $V(G_1)$ is a dominating set. This implies that the edges of color 1 span a spanning subgraph of G in which every degree is odd, which is a contradiction since $n = |V(G)|$ is odd.

Case 2. k odd. In this case, the stars given by Corollary 1, are taken either with $i - 1$ or with $k + i - 1$ edges. If there is at least one $i \geq 3$ with $|G_i|$ odd, it is clear that there is a choice of the size of the stars (for each odd $|V(G_i)|$, $i \geq 3$) for which $|V(G'_2)|$ is even (since $i - 1$ and $k + i - 1$ have opposite parity), regardless of the parity of n . Otherwise, if the only $i \geq 2$ for which $|V(G_i)|$ is odd is $i = 2$, we take a star with k edges, center in G_1 and leaves in G_2 , so $|V(G'_2)|$ is again even. The rest of the proof follows Case 1.1 *mutatis mutandis*.

4 A Lower Bound for $\max \chi'_k(G)$

In this section, we present a lower bound for the maximum mod k chromatic index of graphs. Namely, the next result states that $\chi'_k(K_{1,k,k}) \geq k + 2$, where $K_{1,k,k}$ is the complete 3-partite graph with vertex classes of size 1, k and k .

Proposition 1. *If $G = K_{1,k,k}$, then $\chi'_k(G) = k + 2$.*

Proof. Let G be the complete 3-partite graph $K_{1,k,k}$, and let $(\{u\}, A, B)$ be the vertex partition of G . Suppose, for a contradiction, that G can be colored with c colors, where $c \leq k + 1$. Note that some color, say 1, must be used to color precisely $k + 1$ edges incident to u , and hence, every other edge incident to u must be colored with a different color. In particular, this implies that $c \geq k$. On the other hand, given a vertex $v \neq u$, the only ways to color the edges incident to v is either (a) by coloring all the edges incident with the color used in uv ; or (b) by coloring each of its edges with a different color. We say call a vertex with such a coloring *monochromatic* or *rainbow*, respectively.

Since there are $k + 1$ edges incident to u with color 1, we may assume without loss of generality that there are two vertices x and y in A and a vertex z in B for which ux , uy and uz have color 1. We claim that all the vertices $v \in A \cup B$

are rainbow. First, suppose that uv has color 1. If $v \in A$ is monochromatic, then z has two edges with color 1, and hence z must be monochromatic. This implies that x and y have both two edges with color 1, and hence x and y must also be monochromatic, which implies that every vertex in B are monochromatic, and hence every edge of G is colored with color 1, which is a contradiction. Therefore, v is rainbow. Analogously, v is rainbow if $v \in B$ and uv has color 1. Thus, we may assume that uv has a color different from 1, say 2. Suppose $v \in A$ is a monochromatic vertex. In this case, since x is rainbow, there is an edge xw with color 2. Then xw and vw are two edges colored with 2. This implies that w must be a monochromatic vertex, and hence uw has color 2. Therefore u has two edges with color 2, a contradiction. Now, note that the subgraph of G spanned by the edges of color 2 is a spanning subgraph of G , but all its degrees are equal to 1, a contradiction to $|V(G)| = 2k + 1$ being odd.

5 Concluding Remarks and Future Work

In this paper we determined the mod k chromatic index of $G(n, p)$ for $p = p(n)$ such that $p \geq C_k n^{-1} \log n$ and $n(1 - p) \rightarrow \infty$. It is natural to investigate the remaining ranges of p . For instance, if G is a forest, it is not hard to prove that $\chi'_k(G) = \max \{r \in \{1, \dots, k\} : r \equiv d(v) \pmod{k} \text{ for some } v \in V(G)\}$. This observation settles the case $p \ll n^{-1}$, since in this range $G(n, p)$ is asymptotically almost surely a forest. The next step in this direction would be $p = cn^{-1}$ for some $c \in (0, 1)$, in which case the components of $G(n, p)$ are asymptotically almost surely trees and unicyclic graphs. Unfortunately, the formula above for $\chi'_k(G)$ does not extend to all unicyclic graphs: for instance, it is not hard to prove that if G is any graph that contains a cycle of length $\ell \geq 3$ in which $\ell - 1$ vertices have degree precisely $k + 1$, and one vertex has degree at most k , then $\chi'_k(G) \geq k + 1$. Quite possibly, the most challenging range would be $n^{-1} \leq p \leq cn^{-1} \log n$, where c is a smallish constant (Theorem 1 deals with the case in which $c \geq C_k$ for some constant C_k that depends only on k).

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Constructions of Betweenness-Uniform Graphs from Trees

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Abstract. Betweenness centrality is a measure of the importance of a vertex x inside a network based on the fraction of shortest paths passing through x . We study a blow-up construction that has been shown to produce graphs with uniform distribution of betweenness. We disprove the conjecture about this procedure's universality by showing that trees with a diameter at least three cannot be transformed into betweenness-uniform by the blow-up construction. It remains open to characterize graphs for which the blow-up construction can produce betweenness-uniform graphs.

Keywords: Graph theory · Betweenness centrality · Betweenness uniform

In our vibrant society, everything is moving. Goods are being transported between factories, warehouses and stores, ideas are communicated among people, data are passing through the Internet. Such transfers are often realized via shortest paths, and thus the structure of the underlying network plays an essential part in the workload of particular nodes. One way to measure such expected workload of a node in a network is betweenness centrality. This measure is based on the fraction of shortest path passing through a given vertex. More precisely, for a graph $G = (V, E)$ with $x \in V(G)$ we define *betweenness centrality* as

$$B(x) = \sum_{\{u,v\} \in \binom{V(G) \setminus \{x\}}{2}} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}},$$

where $\sigma_{u,v}$ denotes the number of shortest paths between u and v and $\sigma_{u,v}(x)$ is the number of shortest paths between u and v passing through x . [1] Indeed, it seems to be a highly useful notion of measuring the importance of nodes, as shown by its numerous applications in neuroscience [9], chemistry [12], sociology [4], and transportation [11]. From a more theoretical perspective,

betweenness centrality has also been shown to be a helpful measure in modelling random planar graphs [7].

Once we measure the tendency to put uneven workload to different vertices, we might want to optimize the underlying network such that the communication is spread more evenly. Apart from preventing overload and potential collapse, it might also be important from the strategic perspective, as we do not want to have a single point of failure. The extremal case is then the class of graphs with uniform distribution of betweenness. A graph is called *betweenness-uniform*, if the value of betweenness is the same for all vertices. It is easily observable that vertex-transitive graphs are betweenness-uniform [10]. However, for any fixed n , there are superpolynomially many betweenness-uniform graphs, which are not vertex-transitive [3]. Apart from the fact that the class of distance-regular graphs is betweenness-uniform [2,3], not much more is known about the characterization of betweenness-uniform graphs. An important property of betweenness-uniform graphs is that they are always 2-connected [3]. Actually, any betweenness-uniform graph is 3-connected, unless it is isomorphic to a cycle [5]. There exist other studies concerned with both the values of betweenness in betweenness-uniform graphs [6] and edge betweenness-uniform graphs [8].

Throughout this text we use a standard notation. A graph G has a vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = |\{u : uv \in E(G)\}| = |N_G(v)|$. Subscript is omitted whenever G is clear from the context. The *distance* $d(u, v)$ between $u, v \in V(G)$ is the length of the shortest path connecting these vertices. We say that a graph G has *diameter* k if $k = \max_{u, v \in V(G)} \{d(u, v)\}$. We denote by P_n the path on n vertices, by K_n the complete graph on n vertices and by I_n the edge-less graph on n vertices and by S_{n-1} the star with one central vertex and $n - 1$ leaves. We write $G \cong H$ whenever graph G is isomorphic to graph H . The set $\{1, \dots, n\}$ is denoted by $[n]$.

Let G be a graph with vertex set $\{v_1, \dots, v_n\}$ and H_1, \dots, H_n be a set of graphs. The graph $G[H_1, \dots, H_n]$ is defined on a vertex set obtained from $V(G)$ via substituting each $v_i \in V(G)$ by set of vertices $V(H_i)$ and has edge set $E(G[H_1, \dots, H_n]) = \bigcup_{i=1}^n E(H_i) \cup \{uv \mid u \in V(H_i), v \in V(H_j), v_i v_j \in E(G)\}$. We will call H_i as the *blow-up* of v_i .

Our primary motivation in this work is to study graphs potentially generated via the following conjecture.

Conjecture 1 (Coroničová Hurajová, Gago, Madaras, 2013 [3]). *For any graph G on a vertex set $\{v_1, \dots, v_n\}$ there exist graphs H_1, \dots, H_n such that $G[H_1, \dots, H_n]$ is betweenness-uniform.*

In the following, we deal with generating graphs from paths. We show the new blow-up constructions for P_3 and stars leading to betweenness-uniform graphs. Contrary to that, we show that no such construction exists for P_4 and further generalize this claim to trees with diameter at least three, disproving thus the conjecture.

By $v \in V(H_i)$ we mean that v is also in a subgraph of $G[H_1, \dots, H_n]$ defined on vertices blown-up from v_i . Whenever we consider a graph $G[H_1, \dots, H_n]$, we assume that the underlying graph G has at least two vertices and is connected.

From these assumptions it is not hard to realize that the distance of any two vertices inside the same blow-up graph H_i is at most two.

While considering betweenness of a vertex v in $G[H_1, \dots, H_n]$, we will distinguish the paths contributing to $B(v)$ according to their endpoints x, y . Either $x, y \in V(H_i)$ for some $i \in [n]$, or $x \in V(H_i), y \in V(H_j)$ for $i, j \in [n]$ such that $i \neq j$. We say that the first type of paths contributes to the *local* betweenness of v for given $H_i, B^{H_i}(v)$, and that the second type of paths contributes to the *global* betweenness of $v, B^G(v)$.

Observation 2. For $v \in V(H_i), B(v) = B^G(v) + B^{H_i}(v) + \sum_{j: v_j \in N_G(v_i)} B^{H_j}(v)$.

This observation enables us to ignore any paths longer than two between $x, y \in V(H_i)$ for any i , as such paths do not influence betweenness of any vertex. As a result, the only long paths influencing betweenness of any vertex are the paths between distinct H_i 's. Furthermore, the shortest paths between H_i and H_j for $j \neq i$ contain at most one vertex from each $H_k, k \in [n]$.

Our goal is to find a betweenness uniform graph using the blow-up construction for a path P_k . We iterate candidates via different values of k . Starting from the smallest paths, $P_2 = K_2$ is betweenness-uniform and thus $P_2[K_m, K_m]$ is betweenness-uniform for any integer m as has been shown in the article [3].

Even though the path P_3 and star S_k are not betweenness-uniform, but we still can find a blow-up resulting in a betweenness-uniform graph.

Observation 3. For P_3 , the graph $P_3[K_1, I_2, K_1]$ is isomorphic to C_4 and thus betweenness-uniform. Furthermore, $P_3[I_a, I_{a+b}, I_b]$ is betweenness-uniform for any a, b positive integers.

Observation 4. For any star S_k with vertices $\{v_1, \dots, v_k\}$ of degree one and $\deg(v_{k+1}) = k$ there exists a blow-up construction $S_k[I_{s_1}, \dots, I_{s_k}, I_{\sum_{j=1}^k s_j}]$ giving a betweenness-uniform graph.

Before moving to longer paths, we observe some properties of the blow-up graphs. From the fact that any betweenness-uniform graph is 2-connected [3], $|V(H_i)| > 1$ for any $v_i \in V(G)$ of degree at least two.

Let $u, v \in V(G[H_1, \dots, H_n])$ such that $u, v \in V(H_i)$ and $G' = G[H_1, \dots, H_n]$. We denote $\sigma_{u,v}^{H_i, G'}$ the number of uv -paths inside H_i , which have the same length as the shortest uv -path in G' . Specifically, $\sigma_{u,v}^{H_i, G'}$ is one if $uv \in E(H_i)$ and zero whenever $uv \notin E(H_i)$ and there is no uv -path of length two in H_i .

For simplicity, we denote $n_i = \sum_{j: v_j \in N_G(v_i)} |V(H_j)|$ the sum of sizes of the neighbouring blow-up graphs.

Observation 5. Let $G' = G[H_1, \dots, H_n], x \in V(H_i)$ for some $i \in [n]$ and $v_i v_j \in E(G)$. Then

$$B^{H_j}(x) = \sum_{\substack{u, v \in V(H_j) \\ uv \notin E(H_j)}} \frac{1}{\sigma_{u,v}^{H_j, G'} + \sum_{v_k \in N_G(v_j)}^k |V(H_k)|} = \sum_{\substack{u, v \in V(H_j) \\ uv \notin E(H_j)}} \frac{1}{\sigma_{u,v}^{H_j, G'} + n_j}$$

Now we focus on $G = P_4$ and denote $G' = P_4[H_1, H_2, H_3, H_4]$. We start by expressing betweenness centrality of a vertex $x \in V(H_1)$ and $y \in V(H_2)$ for $\deg_G(v_1) = 1$ and $v_1v_2 \in E(G)$.

Using Observation 2, $B(x) = B^G(x) + B^{H_1}(x) + B^{H_2}(x)$. From v_1 being an endpoint we have $B^G(v_1) = 0$. Furthermore, for $N_1(x) = N(x) \cap V(H_1)$,

$$B^{H_1}(x) = \sum_{\substack{x_1, x_2 \in N_1(x) \\ x_1x_2 \notin E(G')}} \frac{1}{\sigma_{x_1x_2}^{H_1, G'} + |V(H_2)|} \text{ and } B^{H_2}(x) = \sum_{\substack{y_1y_2 \in V(H_2) \\ y_1y_2 \notin E(G')}} \frac{1}{\sigma_{y_1y_2}^{H_2, G'} + n_2}.$$

There are no other paths contributing to $B(x)$ for $x \in V(H_1)$.

We can use Observations 2 and 5 to express the betweenness of the vertex y ,

$$B(y) = B^G(y) + B^{H_2}(y) + \sum_{j: v_j \in N_G(v_2)} \sum_{\substack{u, v \in V(H_j) \\ uv \notin E(H_j)}} \frac{1}{\sigma_{u,v}^{H_j, G'} + n_j}$$

The global betweenness of y is closely related to the betweenness of v_2 in G . All paths between $V(H_1)$ and $V(G') \setminus \{V(H_1) \cup V(H_2)\}$ contribute to each vertex of $V(H_2)$ by the same amount, so $B^G(y) = \frac{|V(H_1)| \cdot (|V(G')| - |V(H_1)| - |V(H_2)|)}{|V(H_2)|}$. An interesting conclusion is that the global betweenness of y grows heavily with the growing size of $|V(G')| - |V(H_2)|$ and gets small when $V(H_2)$ forms a large fraction of $V(G')$.

The local contributions to y behave similarly to the local contributions of x :

$$B^{H_2}(y) = \sum_{\substack{y_1y_2 \in N(y) \cap V(H_2) \\ y_1y_2 \notin E(G')}} \frac{1}{\sigma_{y_1y_2}^{H_2, G'} + n_2} \text{ and } B^{H_j}(y) = \sum_{\substack{y_1y_2 \in V(H_j) \\ y_1y_2 \notin E(G')}} \frac{1}{\sigma_{y_1y_2}^{H_j, G'} + n_j}$$

for each j such that $v_j \in N_G(v_2)$.

Note that the sizes of the local contributions shrink with the growing sizes of H_i 's in their neighbourhood, because the denominator grows.

Observation 6. For $\deg_G(v_1) = 1$ and $v_1v_2 \in E(G)$, $B^{H_1}(x) \leq B^{H_1}(y)$.

Using previous observation and the definition of B^{H_i} we infer

$$B_{y-x}^{H_1} = B^{H_1}(y) - B^{H_1}(x) = \sum_{\substack{y_1 \in V(H_1), y_2 \in V(H_1) \setminus N(x) \\ y_1y_2 \notin E(G')}} \frac{1}{\sigma_{y_1y_2}^{H_1, G'} + |V(H_2)|}. \tag{1}$$

Similarly, we obtain $B^{H_2}(x) \geq B^{H_2}(y)$ and thus

$$B_{x-y}^{H_2} = B^{H_2}(x) - B^{H_2}(y) = \sum_{\substack{y_1 \in V(H_2), y_2 \in V(H_2) \setminus N(y) \\ y_1y_2 \notin E(G')}} \frac{1}{\sigma_{y_1y_2}^{H_2, G'} + n_2}. \tag{2}$$

When considering $B(x) = B(y)$ and using Eqs. (1) and (2) we obtain

$$B_{x-y}^{H_2} = B^G(y) + B_{y-x}^{H_1} + \sum_{j:v_j \in N_G(v_2), j \neq 1} B^{H_j}(y) \tag{3}$$

We will show that even when trying to maximize $B(x)$, it will always be smaller than $B(y)$ for sufficiently large G' .

We denote

$$\Delta(x, y) = \frac{B_{x-y}^{H_2}}{B^G(y) + B_{y-x}^{H_1} + \sum_{j:v_j \in N_G(v_2), j \neq 1} B^{H_j}(y)}.$$

If $B(x) = B(y)$, $\Delta(x, y) = 1$. For $B(x) < B(y)$ we obtain $\Delta(x, y) < 1$ and vice versa.

Lemma 7. *Let $m = |H_2|$ be fixed. Then $\Delta(x, y)$ is maximized for $H_2 \cong I_m$.*

Lemma 8. *Let $m = |H_1|$ be fixed. Then $\Delta(x, y)$ is maximized for $H_1 \cong K_m$.*

In order to produce betweenness uniform graph using the above-described blow-up construction we need to maximize $\Delta(x, y)$, i.e. maximize corresponding $B(x)$ and minimize $B(y)$. From the lemmas above we obtain $H_1 \cong K_a, H_2 \cong I_b, H_3 \cong I_c, H_4 \cong K_d$ for some positive integers a, b, c and d . By this assumption, Eq. (3) simplifies. We transform it into an inequality, as by maximizing $\Delta(x, y)$ we must allow the case when $B(x) > B(y)$. Note that by adding edges into H_3 betweenness of vertices in both H_2 and H_4 decreases. By using the inequality for both $x \in V(H_1), y \in V(H_2)$ and $x' \in V(H_4), y' \in V(H_3)$ we obtain

$$\frac{\binom{b}{2}}{(a+c)} \geq \frac{a(c+d)}{b} + 0 + \frac{\binom{c}{2}}{(b+d)} \text{ and } \frac{\binom{c}{2}}{(b+d)} \geq \frac{d(a+b)}{c} + 0 + \frac{\binom{b}{2}}{(a+c)}$$

By substituting the second inequality to the first one we get

$$0 \geq \frac{a(c+d)}{b} + \frac{d(a+b)}{c} \text{ implying } 0 \geq ac(c+d) + bd(a+b)$$

which cannot be fulfilled by a, b, c, d positive integers and thus $B(x) < B(y)$ and G' is not betweenness-uniform.

By realizing that the global betweenness of y grows with growing number of vertices in the path while the betweenness of the endpoints of the path stays the same, we obtain a result summarizing betweenness-uniform blow-up constructions of paths.

Proposition 9. *For P_k with $k \geq 4$ there is no blow-up construction resulting in a betweenness-uniform graph.*

Theorem 10. *Let G be a tree with diameter d at least three and $|V(G)| = n$. Then there are no graphs H_1, \dots, H_n such that $G[H_1, \dots, H_n]$ is a betweenness-uniform graph.*

The idea of the proof is that P_d is contained in G as a subgraph such that its endpoints have degree one. It is not hard to see that by adding subtrees adjacent to the internal vertices of the path, the betweenness of end-vertices is not increasing and betweenness of the internal vertices is not decreasing.

Our results show that there are some non-betweenness-uniform graphs, such as P_3 and S_k , which can be transformed to a betweenness-uniform graph by the blow-up construction, but there are other non-betweenness-uniform graphs, which cannot be transformed into a betweenness-uniform graph by the blow-up construction for any H_1, \dots, H_n .

Problem 11. *Determine a class \mathcal{B} of graphs that can be transformed into betweenness uniform via blow-up construction with a suitable choice of H_1, \dots, H_n .*

Conjecture 12. *Let G be a graph of order n with diameter at least three having a vertex cut of size one. Then there are no graphs H_1, \dots, H_n such that $G[H_1, \dots, H_n]$ is betweenness-uniform.*

Acknowledgments. David Hartman and Aneta Pokorná were partially supported by ERC Synergy grant DYNASNET grant agreement no. 810115.

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On the Homomorphism Order of Oriented Paths and Trees

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Abstract. A partial order is universal if it contains every countable partial order as a suborder. In 2017, Fiala, Hubička, Long and Nešetřil showed that every interval in the homomorphism order of graphs is universal, with the only exception being the trivial gap $[K_1, K_2]$. We consider the homomorphism order restricted to the class of oriented paths and trees. We show that every interval between two oriented paths or oriented trees of height at least 4 is universal. The exceptional intervals coincide for oriented paths and trees and are contained in the class of oriented paths of height at most 3, which forms a chain.

Keywords: Graph homomorphism · Homomorphism order · Oriented path · Oriented tree · Universal poset · Fractal property

1 Introduction

Let G_1, G_2 be two finite directed graphs. A *homomorphism* $f : G_1 \rightarrow G_2$ is an arc preserving map $f : V(G_1) \rightarrow V(G_2)$. If such a map exists we write $G_1 \leq G_2$. The relation \leq defines a quasiorder on the class of directed graphs which, by considering homomorphic equivalence classes, becomes a partial order. A *core* of a digraph is its minimal-size homomorphic equivalent digraph. As the core of a digraph is unique up to isomorphism, see [5], it is natural to choose a core as the representative of each homomorphic equivalence class.

The first and second author were supported by the project 21-10775S of the Czech Science Foundation (GAČR). This is part of a project that has received funding from the European Research Council (ERC) under the EU Horizon 2020 research and innovation programme (grant agreement No. 810115). The fourth author was supported by the Spanish Research Agency under project MTM2017-82166-P.

Given a partial order (\mathcal{P}, \leq) and $a, b \in \mathcal{P}$ satisfying $a < b$, the interval $[a, b]$ is a *gap* if there is no $c \in \mathcal{P}$ such that $a < c < b$. We say that a partial order is *dense* if it contains no gaps. Finally, a partial order is *universal* if it contains every countable partial order as a suborder.

The homomorphism order of graphs has proven to have a very rich structure [5]. For instance, Welzl showed that undirected graphs, except for the gap $[K_1, K_2]$, are dense [8]. Later, Nešetřil and Zhu [7] proved a density theorem for the class of oriented paths of height at least 4. Recently, Fiala et al. [3] showed that every interval in the homomorphism order of undirected graphs, with the only exception of the gap $[K_1, K_2]$, is universal. The question whether this “fractal” property was also present in the homomorphism order of other types of graphs was formulated. In this context, the following result was shown when considering the homomorphism order of the class oriented trees.

Theorem 1 ([1]). *Let T_1, T_2 be oriented trees such that $T_1 < T_2$. If the core of T_2 is not a path, then the interval $[T_1, T_2]$ is universal.*

Theorem 1 was presented in the previous Eurocomb [1] but the result was not published as intervals of the form $[T_1, T_2]$ where the core of T_2 is a path remained to be characterized. In particular, the characterization of the universal intervals in the class of oriented paths was left as an open question. Here, we answer these questions and complete the characterization of the universal intervals in the class of oriented paths and trees by proving the following results (see Sect. 2 for definitions and notation).

Theorem 2. *Let P_1, P_2 be oriented paths such that $P_1 < P_2$. If the height of P_2 is greater or equal to 4 then the interval $[P_1, P_2]$ is universal. If the height of P_2 is less or equal to 3 then the interval $[P_1, P_2]$ forms a chain.*

Theorem 3. *Let T_1, T_2 be oriented trees such that $T_1 < T_2$. If the height of T_2 is greater or equal to 4 then the interval $[T_1, T_2]$ is universal. If the height of T_2 is less or equal to 3 then the interval $[T_1, T_2]$ forms a chain.*

It is known that the core of an oriented tree of height at most 3 is a path. Thus, oriented paths of height less or equal to 3 are the only exception when considering the presence of universal intervals in both the class of oriented paths and the class of oriented trees. Hence, the nature of its intervals in terms of density and universality is completely determined.

Let \mathbf{P}_n denote the directed path consisting on $n + 1$ vertices and n consecutive forward arcs. Let L_k denote the oriented path with vertex sequence $(a, b_0, c_0, b_1, c_1, \dots, b_k, c_k, d)$ and arcs $(ab_0, b_0c_0, b_1c_0, b_1c_1, \dots, b_kc_{k-1}, b_kc_k, c_kd)$, as shown in Fig. 1.

It is easy to check that $\mathbf{P}_0 < \mathbf{P}_1 < \mathbf{P}_2$ and there is no oriented path strictly between P_0 and P_1 nor P_1 and P_2 . Among oriented paths of height equal to three, the only cores are the paths L_k for all $k \geq 0$, and it is also easy to see that $L_k \leq L_m$ if and only if $k \geq m$. Combining these observations we get that oriented

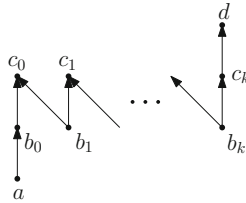


Fig. 1. The path L_k .

paths of height at most 3 are found at the “bottom” of the homomorphism order of paths and trees and form the following chain

$$\mathbf{P}_0 < \mathbf{P}_1 < \mathbf{P}_2 < \dots < L_{k+1} < L_k < L_{k-1} < \dots < L_2 < L_1 < L_0 = \mathbf{P}_3.$$

From the above results we can deduce the following corollary for both the homomorphism order of paths and the homomorphism order of trees.

Corollary 1. *An interval $[G_1, G_2]$ in the homomorphism order of oriented paths (resp. trees) is universal if and only if it contains two paths (resp. trees) which are incomparable in the homomorphism order.*

It might seem that combining Theorems 1 and 2 one would get a proof of Theorem 3. However, this is not true as we have to consider intervals of form $[T_1, P_2]$ where the core of T_1 is not a path and the height of P_2 is greater or equal to 4. For this reason, we need a new density theorem.

Theorem 4. *Let T_1 be a oriented tree and P_2 a oriented path such that $T_1 < P_2$. If the height of P_2 is greater or equal to 4 then there exists an oriented tree T satisfying $T_1 < T < P_2$.*

The proof of Theorem 4 is a more general version of the density theorem for paths proved by Nešetřil and Zhu [7]. Due to space limitations we omit the proof in this note.

It is now straightforward to prove Theorem 3. Given oriented trees T_1, T_2 with $T_1 < T_2$ and T_2 with height greater or equal to 4, we consider two cases: if the core of T_2 is not a path we are done by applying Theorem 1; else, the core of T_2 is a path so by Theorem 4 there exists a tree T satisfying $T_1 < T < T_2$. Now, if the core of T is a path we can apply Theorem 2 to the interval $[T, T_2]$, else the core of T is not a path and we can apply Theorem 1 to the interval $[T_1, T]$.

2 Notation

A (oriented) path $P = (V(P), A(P))$ is a sequence of vertices $V(P) = (p_0, \dots, p_n)$ together with a sequence of arcs $A(P) = (a_1, \dots, a_n)$ such that, for each $1 \leq i \leq n$, $a_i = p_{i-1}p_i$ or $p_i p_{i-1}$. We denote by $i(P) = p_0$ and $t(P) = p_n$ the initial and terminal vertex of P respectively. We denote by \mathcal{P} the set of all

oriented paths. Let \mathbf{P}_n denote the path with vertex set $V(\mathbf{P}_n) = (0, 1, \dots, n)$ and arc set $A(\mathbf{P}_n) = (01, 12, \dots, (n-1)n)$. The *height* of a path P , denoted $h(P)$, is the minimum $k \geq 0$ such that $P \rightarrow \mathbf{P}_k$. It is known that for any path P there is a unique homomorphism $l : P \rightarrow \mathbf{P}_{h(P)}$. Given such homomorphism $l : P \rightarrow \mathbf{P}_{h(P)}$, the *level* of a vertex $v \in V(P)$ is the integer $l(v)$. We define the *level* of an arc $a \in A(P)$ as the greatest level of its incident vertices and denote it $l(a)$.

We write P^{-1} for the *reverse* path of P , where $V(P^{-1}) = (p_n, p_{n-1}, \dots, p_0)$ and $pp' \in A(P^{-1})$ if and only if $p'p \in A(P)$. That is, P^{-1} is the path obtained from flipping the path P . Given paths P, P' , the *concatenation* PP' is the path with vertex set $V(PP') = (p_0, \dots, p_n = p'_0, p'_1, \dots, p'_n)$ and arc set $A(PP') = (a_1, \dots, a_n, a'_1, \dots, a'_n)$.

A *zig-zag of length n* consists on a path Z with $n + 1$ vertices whose arcs consecutively alternate from forward to backward. Note that all arcs in a zig-zag must have the same level. When considering a zig-zag Z as a subpath of some path P , we define the *level of Z* as the level of its arcs in P , and denote it $l(Z)$.

3 Proofs

The proof of Theorem 2 consists of the construction of an embedding from the homomorphism order of all oriented paths, which was proven to be universal [2], into the interval $[P_1, P_2]$. The construction is based on the standard indicator technique initiated by Hedrlín and Pultr [4], see also [5]. The method takes oriented paths Q and I and creates the path $\Phi_I(Q) := Q * I(a, b)$, obtained from Q by replacing each arc $qq' \in A(Q)$ with a copy of the path I by identifying the vertex q with $i(I)$ and the vertex q' with $t(I)$. If the path I is well chosen we can guarantee that Φ_I induces an embedding from (\mathcal{P}, \leq) into the interval $[P_1, P_2]$. For an arc $qq' \in Q$, we denote $I_{qq'} := \Phi_I(qq')$ to the copy of I replacing qq' .

Lemma 1. *Let I be a path. Let I_1, I_2 be copies of I and let $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. Suppose that, for every $Q \in \mathcal{P}$, the following conditions hold:*

- (i) $P_1 < \Phi_I(Q) < P_2$;
- (ii) Every homomorphism $f : I \rightarrow \Phi_I(Q)$ satisfies $f(I) \subseteq I_{qq'}$ for some $qq' \in A(Q)$;
- (iii) Every homomorphism $g : I_1^{\epsilon_1} I_2^{\epsilon_2} \rightarrow \Phi_I(Q)$ satisfies that for every $z, z' \in \{i, t\}$,

$$z(I_1) = z'(I_2) \text{ implies } z(I_{q_1q'_1}) = z'(I_{q_2q'_2})$$

where $g(I_1) \subseteq I_{q_1q'_1}$ and $g(I_2) \subseteq I_{q_2q'_2}$.

Then Φ_I is a poset embedding from the class \mathcal{P} of paths into the interval $[P_1, P_2]$.

The power of the method is that the construction of the embedding reduces to finding a suitable gadget I satisfying the above conditions.

To construct I , first consider a path P such that $P_1 < P < P_2$. We know such a path exist since paths of height greater or equal to 4 are dense [7]. Without loss

of generality let P be a core. Consider a surjective homomorphism $h : P \rightarrow P_2$. We can assume that such homomorphism always exists, since otherwise we will be able to find another path P' , with $P < P' < P_2$, which admits a surjective homomorphism into P_2 . Then, since h is surjective, there exist two different vertices $v_1, v_2 \in V(P)$ such that $h(v_1) = h(v_2)$. Without loss of generality suppose that v_1 appears before v_2 in the sequence $V(P)$. Note that $l(v_1) = l(v_2)$ as homomorphisms preserve level vertex difference. The path P is then naturally split into three subpaths: the subpath A from $i(P)$ to v_1 , the subpath B from v_1 to v_2 , and the subpath C from v_2 to $t(P)$. So $P = ABC$.

Let a_1 be the last arc of A and let a_2 be the first arc of C .

We first assume the case that neither $B = Z'_1$, $B = Z'_2$ nor $B = Z'_1Z'_2$ for some zig-zags Z'_1 and Z'_2 of level $l(a_1)$ and $l(a_2)$ in P respectively.

Next, we choose as a gadget the following path

$$I = Z_1A^{-1}ABCC^{-1}Z_2$$

where Z_1 and Z_2 are long enough zig-zags of even length and levels $l(a_1)$ and $l(a_2)$ in I respectively.

It only remains to check that I satisfies the conditions of Lemma 1.

Let $Q \in \mathcal{P}$. It follows by construction that $P_1 < \Phi_I(Q) < P_2$. To see this, first note that $P \subset I$ so it is clear that $P_1 < \Phi_I(Q)$. Now, observe that there is a homomorphism $\rho : I \rightarrow P$ which collapses the zig-zags Z_1 and Z_2 into the arcs a_1 and a_2 respectively, and maps A^{-1} into A and C^{-1} into C identifying the correspondent vertices. Then, the map $\rho' : \Phi_I(Q) \rightarrow P_2$ defined for each copy $I_{qq'}$ of I as $\rho'(I_{qq'}) := (h \circ \rho)(I_{qq'})$ is well defined since $h(v_1) = h(v_2)$. Finally, suppose that there is a homomorphism $f : P_2 \rightarrow \Phi_I(Q)$. Since the zig-zags are long enough we must have $f(P_2) \subset I_{qq'}$ for some $qq' \in A(Q)$, implying that $P_2 \rightarrow I \rightarrow P$, a contradiction. Thus, I satisfies condition (i) of Lemma 1.

We say that a digraph is *rigid* if its only automorphism is the identity map. To check conditions (ii) and (iii) of Lemma 1 we shall use several times the fact that the core of a path is rigid, see Lemma 2.1 in [7].

Another trivial property of homomorphisms is also that they can never increase the distance between two vertices when considering its images, see [5]. That is, every homomorphism $f : G_1 \rightarrow G_2$ satisfies $d(v_1, v_2) \geq d(f(v_1), f(v_2))$ for every $v_1, v_2 \in V(G_1)$.

Let $f : I \rightarrow \Phi_I(Q)$ be a homomorphism. Since the zig-zags Z_1 and Z_2 are long enough, we have, for $ABC \subset I$, that $f(ABC) \subset I_{qq'}$ for some $qq' \in A(Q)$. Because $I_{qq'} \rightarrow P$ and P is rigid, we must have that f maps the path ABC in I to the copy of ABC in $I_{qq'}$ via the identity map. It follows by a distance argument that $f(I) \subset I_{qq'}$, so condition (ii) holds.

Finally, let $g : I_1I_2 \rightarrow \Phi_I(Q)$ be a homomorphism. Note that $t(I_1) = i(I_2)$. Let $q_1q'_1, q_2q'_2 \in A(Q)$ such that $g(I_1) \subset I_{q_1q'_1}$ and $g(I_2) \subset I_{q_2q'_2}$. Let w_1 be the terminal vertex of the subpath ABC in I_1 and let w_2 be the initial vertex of the subpath ABC in I_2 . Recall from the paragraph above that any homomorphism $f : I \rightarrow \Phi_I(Q)$ maps the path ABC in I to the path ABC in $I_{qq'}$ fixing all vertices. Thus, $g(w_1)$ is the terminal vertex of the subpath ABC in $I_{q_1q'_1}$ and

$g(w_2)$ is the initial vertex of the subpath ABC in $I_{q_2q'_2}$. We can not have $q_1 = q'_2$ since in such case $d(g(w_1), g(w_2)) > d(w_1, w_2)$. Both cases $q_1 = q_2$ and $q'_1 = q'_2$ imply that there is a homomorphism $g' : I_1I_2 \rightarrow I$. Suppose that this is the case. Observe that w_1 is joined to w_2 by the path $C^{-1}Z_1Z_2A^{-1}$ in I_1I_2 . We would also have that $g'(w_2)$ is joined to $g'(w_1)$ by the path ABC in I . Thus, we must have $B = Z'_1, Z'_2$ or $Z'_1Z'_2$ for some zig-zags of level equal to Z_1 and Z_2 respectively, which contradicts our first assumption. The only remaining case is $q'_1 = q_2$.

The above argument is valid for the case $\epsilon_1 = \epsilon_2 = 1$ in condition (iii). However, the other cases are analogous if not even simpler. Hence, applying Lemma 1 to the path I completes the proof.

We have assumed that neither $B = Z'_1, B = Z'_2$ nor $B = Z'_1Z'_2$. If this is not the case what we do is take $l(a_1) \neq l(a_2)$ and consider the auxiliary path

$$P' = ABCC^{-1}Z_2CC^{-1}Z_1A^{-1}ABC$$

if $B \neq Z'_1$, or

$$P' = ABCC^{-1}Z_2A^{-1}AZ_1A^{-1}ABC$$

otherwise, where again Z_1 and Z_2 are long enough zig-zags of even length and level $l(a_1)$ and $l(a_2)$ in P' respectively. Then we show that P' satisfies the required properties for P above and we apply the arguments to P' .

4 Final Remarks


For directed graphs, and even more, for general relational structures, the gaps of the homomorphism order are characterized by Nešetřil and Tardif in full generality [6]. They are all related to trees. However the homomorphism order of these general relational structures does not enjoy the simplicity of the “bottom” of the homomorphism order of paths and trees. We think that Corollary 1 may shed light on the characterization of universal intervals in the homomorphism order of these more general structures.

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On Chromatic Number of (n, m) -graphs

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Abstract. An (n, m) -graph is a graph with n types of arcs and m types of edges. The chromatic number of an (n, m) -graph G is the minimum number of colors to color the vertices of G such that if we identify the vertices of the same color we get a simple (n, m) -graph by identifying the arcs (resp., edges) between two vertices which have the same direction and color. We study this chromatic number for the family of sparse graphs, partial 2-trees when $2n + m = 3$ and for graphs with bounded maximum degree and degeneracy.

Keywords: Colored mixed graphs · Graph homomorphisms · Chromatic number · Maximum average degree · Planar graphs

1 Introduction and Main Results

In 2000, Nešetřil and Raspaud [1] generalized the notion of graph homomorphisms by introducing the concept of colored homomorphisms of colored mixed graphs.

An (n, m) -colored mixed graph, or simply a (n, m) -graph is a graph G with a set of vertices $V(G)$, a set of arcs $A(G)$, and set of edges $E(G)$. Moreover, each arc is colored with one of the n colors from $\{1, 2, \dots, n\}$ and each edge is colored with one of the m colors from $\{n + 1, n + 2, \dots, n + m\}$. The underlying undirected graph of G is denoted by $U(G)$. In this article, we restrict ourselves to (n, m) -graph G where $U(G)$ is simple.

If u is adjacent to v via a forward arc (resp., edge) of color α , then v is an α -neighbor of u where $\alpha \in \{1, 2, \dots, n + m\}$. Moreover, if α is the color of a forward arc, then v is a $(-\alpha)$ -neighbor of u . Furthermore, the set of all α -neighbors of u is denoted by $N^\alpha(u)$ where $\alpha \in \{\pm 1, \pm 2, \dots, \pm n, n + 1, n + 2, \dots, n + m\}$.

Observe that, for $(n, m) = (0, 1)$, $(1, 0)$, and $(0, m)$, the (n, m) -graphs are the same as undirected graphs, oriented graphs [7, 11], and edge-colored graphs [3, 5], respectively. These types of graphs and their homomorphisms are well-studied. It is worth mentioning that, a variation of homomorphisms of $(0, 2)$ -graphs, called

A. Lahiri—Research supported by ISF grant 822/18.

S. Sen—Research partially supported by IFCAM MA/IFCAM/18/39 and SRG/2020/001575.

homomorphisms of signed graphs, have gained popularity in recent times due to its strong connection with classical graph theory (especially, coloring and graph minor theory) [8]. It is known that, homomorphisms of signed graphs are in one-to-one correspondence with a specific restriction of homomorphisms of $(0, 2)$ -graphs [12]¹. Thus, the notion of colored homomorphism truly manages to unify a lot of important theories related to graph homomorphisms.

A (colored) *homomorphism* of a (n, m) -graph G to another (n, m) -graph H is a function $f : V(G) \rightarrow V(H)$ such that if uv is an arc (resp., edge) of G , then $f(u)f(v)$ is also an arc (resp., edge) of H having the same color as uv . The notation $G \rightarrow H$ is used to denote that G admits a homomorphism to H .

Using the notion of homomorphism, one can define the chromatic number of colored mixed graphs that generalizes [13] the chromatic numbers defined for simple graphs, oriented graphs, edge-colored graphs, etc. The (n, m) -colored *mixed chromatic number* or, simply, (n, m) -*chromatic number* of a (n, m) -graph G is given by

$$\chi_{n,m}(G) = \min\{|V(H)| : G \rightarrow H\}.$$

For a family \mathcal{F} of simple graphs, the (n, m) -chromatic number is given by

$$\chi_{n,m}(\mathcal{F}) = \max\{\chi_{n,m}(G) : U(G) \in \mathcal{F}\}.$$

One of the major results proved for (n, m) -chromatic number of graphs is the one relating acyclic chromatic number to the parameter.

Theorem 1 [1]. *Let \mathcal{A}_k be the family of simple graphs having acyclic chromatic number at most k . Then*

$$\chi_{n,m}(\mathcal{A}_k) \leq k(2n + m)^{k-1}.$$

Using the above result and the fact that the acyclic chromatic number of planar graphs is at most 5 [2], an upper bound for the (n, m) -chromatic number of the family of planar graphs was established. Later Fabila-Monroy, Flores, Heumer, Montejano [4] proved lower bounds for the same, improving the previous lower bounds due to Nesětril and Raspaud [1]. We present these bounds in a consolidated manner below.

Let \mathcal{P}_g denote the family of planar graphs having *girth* (length of a shortest cycle) at least g .

Theorem 2 [1, 4]. *For all non-negative integers n and m where $(2n + m) \geq 2$ we have,*

$$(2n + m)^3 + 2(2n + m)^2 + (2n + m) + 1 \leq \chi_{n,m}(\mathcal{P}_3) \leq 5(2n + m)^4, \text{ for } m > 0 \text{ even}$$

$$(2n + m)^3 + (2n + m)^2 + (2n + m) + 1 \leq \chi_{n,m}(\mathcal{P}_3) \leq 5(2n + m)^4, \text{ otherwise.}$$

¹ In categorical terms, there is a bijective covariant functor from the category induced by homomorphisms of signed graphs to a subcategory of the category induced by homomorphisms of $(0, 2)$ -graphs.

In an attempt to reduce the gap between the above lower and upper bounds Montejano, Pinlou, Raspaud, Sopena [6] considered the parameter for sparse planar graphs. This led to a tight linear bound for the (n, m) -chromatic number of families of planar graphs having large girth.

Theorem 3 [6]. *For all non-negative integers n and m where $(2n + m) \geq 2$ and for $g \geq 10(2n + m) - 4$, we have*

$$\chi_{n,m}(\mathcal{P}_g) = 2(2n + m) + 1.$$

One of the main aims of this paper is to show that the (n, m) -chromatic number of planar graphs can be tight and linear even for lesser girths. To do so, we consider the family of graphs having bounded maximum average degree, given by

$$mad(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G \right\}.$$

Theorem 4. *For all non-negative integers n and m where $(2n + m) \geq 2$ and for $mad(G) < 2 + \frac{2}{4(2n+m)-1}$, we have*

$$\chi_{n,m}(G) \leq 2(2n + m) + 1.$$

Using a well known result, that a planar graph G having girth at least g has $mad(G) < \frac{2g}{g-2}$, we obtain the following result as a corollary of Theorem 4.

Theorem 5. *For all non-negative integers n and m where $(2n + m) \geq 2$ and for $g \geq 8(2n + m)$, we have*

$$\chi_{n,m}(\mathcal{P}_g) = 2(2n + m) + 1.$$

One more important result on (n, m) -chromatic number, due to Das, Nandi and Sen [9], relates maximum degree of the graph to the parameter. Let \mathcal{G}_Δ denote the family of connected graphs with maximum degree Δ .

Theorem 6 [9]. *For all non-negative integers n and m where $(2n + m) \geq 2$ and for all $\Delta \geq 5$, we have*

$$(2n + m)^{\Delta/2} \leq \chi_{n,m}(\mathcal{G}_\Delta) \leq 2(\Delta - 1)^{2n+m} \cdot (2n + m)^{(\Delta-1)} + 2.$$

We significantly improve the above bounds and generalize the result by restricting the degeneracy of the graph, for large values of Δ . The proof uses probabilistic method and generalises the method used by Aravind and Subramanian [10] in the context of oriented graphs. Let $\mathcal{G}_{\Delta,d}$ denote the family of graphs having maximum degree Δ and degeneracy d .

Theorem 7. *For all non-negative integers n and m where $(2n + m) \geq 2$, we have*

$$\chi_{n,m}(\mathcal{G}_{\Delta,d}) \leq 16\Delta d(2n + m)^d.$$

The last result in this paper is related to Theorem 2. The family of partial 2-trees is an important subclass of the family of planar graphs. The best known bounds for this family \mathcal{T}^2 of partial 2-trees are the following.

Theorem 8 [1,4]. *For all non-negative integers n and m where $(2n + m) \geq 2$, we have*

$$(2n + m)^2 + 2(2n + m) + 1 \leq \chi_{n,m}(\mathcal{T}^2) \leq 3(2n + m)^2, \text{ for } m > 0 \text{ even}$$

$$(2n + m)^2 + (2n + m) + 1 \leq \chi_{n,m}(\mathcal{T}^2) \leq 3(2n + m)^2, \text{ otherwise.}$$

Note that, for $(2n + m) = 2$ case, it is known that the lower bounds are tight. Our next result proves that this is not the case even for $(2n + m) = 3$ where $(n, m) = (1, 1)$ and $(n, m) = (0, 3)$. Further, we give closer bounds for this particular case.

Theorem 9. *For the family \mathcal{T}^2 of partial 2-trees we have*

- (i) $14 \leq \chi_{0,3}(\mathcal{T}^2) \leq 15.$
- (ii) $14 \leq \chi_{1,1}(\mathcal{T}^2) \leq 16.$

The sections with proof sketches are given below.

2 Proof of Theorem 4

Let us consider a Hamiltonian decomposition C_0, C_1, \dots, C_p of K_{2p+1} , where $p = 2n + m$. We convert this K_{2p+1} into a complete (n, m) -graph using the decomposition.

For each $\alpha \in \{1, 2, \dots, n\}$ convert the cycles $C_{2\alpha-1}$ and $C_{2\alpha}$ with directed cycles having arcs of color α . For each edge $\beta \in \{n + 1, n + 2, \dots, n + m\}$, convert the cycle $C_{2n+\beta}$ into a cycle having all edges of color β . Thus we obtain a complete (n, m) -mixed graph on $2p + 1$ vertices. As the graph is completely defined using the Hamiltonian decomposition, we can call it the *complete (n, m) -graph of the decomposition C_1, C_2, \dots, C_p* .

Lemma 1. *There exists a Hamiltonian decomposition of K_{2p+1} such that its complete (n, m) -graph has the following property: for every $S \subsetneq V(K_{2p+1})$ we have $|S| < |N^\alpha(S)|$ for all $\alpha \in \{-n, -(n - 1), \dots, -1, 1, 2, \dots, (n + m)\}$.*

Let T be a complete (n, m) -graph on $2p + 1$ vertices satisfying the condition of Lemma 1. We want to show that $G \rightarrow T$ whenever $mad(G) < 2 + \frac{2}{4p-1}$. That is, it is enough to prove the following lemma.

Lemma 2. *If $mad(G) < 2 + \frac{2}{4p-1}$, then $G \rightarrow T$.*

We prove the above lemma by contradiction. Hence we assume a minimal (with respect to number of vertices) (n, m) -graph M having $mad(M) < 2 + \frac{2}{4p-1}$ which does not admit a homomorphism to T .

Lemma 3. *The graph M does not contain a vertex having degree one.*

A path with all internal vertices of degree two is called a *thread*, and in particular a *k-thread* is a thread having k internal vertices. The end vertices (assume them to always have degree at least 3) of a (k) -thread are called (k) -thread adjacent.

Lemma 4. *The graph M does not contain k -thread adjacent vertices with $k \geq 2p - 1$.*

Let us describe the configuration C_l . Let v be a vertex, thread-adjacent to exactly l vertices v_1, v_2, \dots, v_l , each having degree at least three. Let the threads between v and v_i have k_i internal vertices. This is configuration C_l .

Lemma 5. *The graph M does not contain the configuration C_l as an induced subgraph if*

$$\sum_{i=1}^l k_i \geq (2p - 1)l - 2p$$

where $p = (2n + m)$.

Now we are ready to start the discharging procedure. First we define a charge function on the vertices of M .

$$ch(x) = \deg(x) - \left(2 + \frac{2}{4p - 1}\right), \text{ for all } x \in V(M).$$

Observe that, $\sum_{x \in V(M)} ch(x) < 0$ as $mad(M) < 2 + \frac{2}{4p - 1}$. After the completion of the discharging procedure, all updated charges become non-negative implying a contradiction. The discharging rule is the following:

(R1) : *Every vertex having degree three or more donates $\frac{1}{4p - 1}$ to the degree two vertices which are part of its incident threads.*

Let $ch^*(x)$ be the updated charge.

Lemma 6. *For any degree two vertex $x \in V(M)$, we have $ch^*(x) = 0$.*

Proof. As M does not have any degree one vertex due to Lemma 3, every degree two vertex x must be internal vertex of a thread. Thus, by rule (R1) the vertex x must receive $\frac{1}{4p - 1}$. Hence the updated charge is

$$ch^*(x) = ch(x) + \frac{2}{4p - 1} = \deg(x) - 2 - \frac{2}{4p - 1} + \frac{2}{4p - 1} = 0.$$

Thus we are done.

Lemma 7. *For any vertex x having degree three or more, we have $ch^*(x) \geq 0$.*

Proof. Let x be a degree d vertex of M . Thus by Lemma 5

$$\begin{aligned} ch^*(x) &\geq ch(x) - \frac{(2p-1)d-2p}{4p-1} = d-2 - \frac{2}{4p-1} - \frac{2pd-d-2p}{4p-1} \\ &= \frac{4pd-8p-d+2-2-2pd+d+2p}{4p-1} = \frac{2p(d-3)}{4p-1} \geq 0 \end{aligned}$$

for $d \geq 3$.

This implies $0 > \sum_{x \in V(M)} ch(x) = \sum_{x \in V(M)} ch^*(x) \geq 0$, a contradiction. Thus, the proof of Lemma 2 is completed, which implies Theorem 4.

3 Proof Sketch of Theorem 7

This result is a generalization of a theorem by Aravind and Subramanian [10] where they prove it for the case $(n, m) = (1, 0)$. The proof uses a probabilistic method.

The major modification in our proof is in its probabilistic model using which a complete (n, m) -graph with certain property is found. In our case, the probability of a particular type of adjacency between two vertices of the said complete graph is $\frac{1}{2n+m}$ instead of $\frac{1}{2}$. Moreover, the coveted property in the complete graph also changes according to the value of (n, m) .

4 Proof Sketch of Theorem 9

In this proof, first we show that all (n, m) - partial 2-trees admit a homomorphism to T if and only if $N^\alpha(u) \cap N^\beta(v) \neq \emptyset$ for all $\alpha, \beta \in \{-n, -(n-1), \dots, -1, 1, 2, \dots, (n+m)\}$, whenever u, v are adjacent in T . Moreover, we note that such a T must exist on $\chi_{n,m}(T^2)$ vertices. The lower bound follows by proving non-existence of such T on 13 vertices and the upper bound follows by showing existence of such T on 15 vertices.

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On a Problem of Füredi and Griggs

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Abstract. The Shadow Minimization Problem in the Boolean lattice asks for the minimum cardinality of the shadow of a family of k -sets of $[n]$ among families of the same cardinality. The well-known Kruskal–Katona theorem says that the initial segments $I_{n,k}(m)$ of length m in the colex order are solutions to the problem. Füredi and Griggs showed that, for some set of cardinalities m , the solution to this problem is unique (up to automorphisms of the Boolean lattice). They gave examples showing that this unicity may fail to hold for other cardinalities and raised the question of characterizing the extremal sets for this problem. We give a structural result for these extremal sets which shows in particular that, for every extremal family S of k -subsets of $[n]$ and every $t > c \log \log n$, the t iterated lower shadow of S is an initial segment in the colex order. Moreover, for an asymptotically dense set of cardinalities, initial segments in the colex order still are essentially the unique solution to this shadow minimization problem. These results illustrate the robustness of the colex order as a solution of this problem. A key property of the cardinalities for which solutions other than initial segments in the colex order exist is that the coefficients of their k -binomial decompositions decrease very fast, according to a family of numbers which extend a classical sequence of the so-called hypotenusal numbers. We also provide an algorithm linear in n and polynomial in k deciding, given a cardinality m and an integer t , if there is an extremal family S of k -subsets of $[n]$ such that $\Delta^t(S)$ is not an initial segment in the colex order and, if the answer is positive, provides a construction of such a set.

Keywords: Shadow minimization problem · Colex order · Kruskal–Katona theorem

1 Introduction

The well-known Kruskal–Katona Theorem [4, 5] on the minimum shadow of a family of k -subsets of $[n] = \{1, 2, \dots, n\}$ is a central result in Extremal Combinatorics with multiple applications, see e.g. [1]. The *shadow* of a family $S \subset \binom{[n]}{k}$ is the family $\Delta(S) \subset \binom{[n]}{k-1}$ of $(k-1)$ -subsets which are contained in some set in S . The Shadow Minimization Problem asks for the minimum cardinality of $\Delta(S)$ for families of k -sets S with a given cardinality $m = |S|$. The answer given by the

Kruskal–Katona theorem can be stated in terms of k -binomial decompositions. The k -binomial decomposition of a positive integer m is

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \cdots + \binom{a_t}{k-t}, \tag{1}$$

where the coefficients satisfy $a_0 > a_1 > \cdots > a_t \geq k-t \geq 1$, $t \in [0, k-1]$, are uniquely determined by m and k . We also refer to the t -tuple $(a_0 > a_1 > \cdots > a_t)$ as the k -binomial sequence of m .

Theorem 1 (Kruskal–Katona [3,5]). *Let $S \subset \binom{[n]}{k}$ be a non-empty family of k -subsets of $[n]$ and let*

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \cdots + \binom{a_t}{k-t}$$

be the k -binomial decomposition of $m = |S|$. Then

$$|\Delta S| \geq KK(m) := \binom{a_0}{k-1} + \binom{a_1}{k-2} + \cdots + \binom{a_t}{k-t-1}.$$

and, more generally

$$|\Delta^i S| \geq KK^i(m) := \binom{a_0}{k-i} + \binom{a_1}{k-1-i} + \cdots + \binom{a_t}{k-t-i}.$$

where $\Delta^i(S)$ is the i -th iterated shadow of S defined recursively by $\Delta^i(S) = \Delta(\Delta^{i-1}(S))$, $1 \leq i \leq k-1$.

In this context, we say that a family S is *extremal* if the cardinality of its lower shadow achieves the lower bound in Theorem 1, so $|\Delta S| = KK(|S|)$.

For every m , the initial segment of length m in the colex order is an extremal family. We recall that the colex order on the k -subsets of $[n]$ is defined by $x \leq_{\text{colex}} y$ if and only if $\max((x \setminus y) \cup (y \setminus x)) \in y$. We denote by $I_{n,k}(m)$ the initial segment of length m in the colex order in $\binom{[n]}{k}$. In what follows, we refer to initial segments up to automorphisms of the Boolean lattice, induced by any permutation of $[n]$.

Füredi and Griggs [2] (see also Mörs [7]) proved that, for cardinalities m for which the k -binomial decomposition has length $t+1 < k$, these initial segments $I_{n,k}(m)$ are in fact the unique extremal families (up to automorphisms of the Boolean lattice \mathcal{B}_n). They also displayed some non-trivial examples which show that this is not the case when $t+1 = k$. This prompted the authors to ask about the characterization of the extremal families. We note that, for fixed k , the set of integers with k -binomial decomposition of length k has upper asymptotic density one. Thus, the unicity of the extremal families can be ensured only on a thin set of cardinalities. The aim of this paper is to further study the relation between the binomial decompositions, the extremal families and their relation with the colex order, and provide an answer to these questions.

From the definition of $KK(m)$ it is clear that all initial segments $I_{n,k}(m)$ with m having a k -binomial decomposition sequence of the form $(a_0, a_1, \dots, a_{k-2}, i)$ for $1 \leq i < a_{k-1} - 1$ have the same lower shadow. Hence, if these decompositions define the integer interval $[m_0, m_1]$, then every family of the form $I_{n,k}(m_0) \cup J$ is extremal for every $J \subset I_{n,k}(m_1) \setminus I_{n,k}(m_0)$. This provides trivial examples of extremal families different from initial segments $I_{n,k}(m)$. On the other hand, the $(k - 1)$ -iterated shadow $\Delta^{k-1}(S)$ of a family S is always an initial segment in the colex order (a family of singletons). It is thus reasonable to measure how far is an extremal family from an initial segment by the smallest t such that $\Delta^t(S)$ is an initial segment of the colex order. Our first result shows that, from this perspective, extremal families can not be very far away from initial segments.

Theorem 2. *Let $S \subset \binom{[n]}{k}$ be an extremal family with cardinality m . Then, for every $t \geq c \log \log n$, where c is an absolute constant, we have*

$$\Delta^t(S) \cong I_{n,k-t}(KK^t(m)).$$

Another result which illustrates the robustness of the colex order as a solution of the shadow minimization problem extends the range of cardinalities for which unicity of solutions hold from sparse to dense.

Theorem 3. *Let $k < n$ be positive integers and $N = \binom{n}{k}$. Let $U \subset [N]$ be the set of integers for which all extremal families $S \subset \binom{[n]}{k}$ with cardinality $m \in U$ satisfy that $\Delta^3(S) \cong I_{n,k-3}(KK^3(m))$. Then*

$$\lim_{n \rightarrow \infty} \frac{|U|}{N} = 1.$$

Both results follow from a strengthened version of the following theorem, which shows that requiring $\Delta^t(S)$ being different from the colex order implies a fast decreasing of the coefficients in the k -binomial decomposition of $m = |S|$. This rate of decrease can be measured by a sequence of numbers introduced by Lucas [6, Page 496] which he called *hypotenusal* numbers. The n -th hypotenusal number h_n can be described by the following process. Suppose that we have a collection of bins indexed by the nonnegative integers and a moving wall placed at a nonnegative integer i separating bin i from bin $i + 1$. At each integer time k we denote by $w(k)$ the position of the wall, by $b(k, i)$ the number of balls in bin i and by $b(k)$ the total number of balls. At an initial time there is one ball in the bin 0 and the wall is located at position 0. At time k , we number the balls at time $k - 1$ from 1 to $b(k - 1)$ and for each ball j at bin i , we remove it and we add one ball to bins $i, i + 1, \dots, w(k - 1) + j - 1$ (it can be checked that the resulting number of balls at time k in bin i does not depend on the particular numbering of the balls at time $k - 1$). The wall is placed at position $w(k) = w(k - 1) + b(k - 1)$. The n -th hypotenusal number is $h_n = w(n) - w(n - 1)$. See Fig. 1 for an illustration.

The first hypotenusal numbers are

$$1, 1, 2, 6, 36, 876, 408696, 83762796636, \dots$$

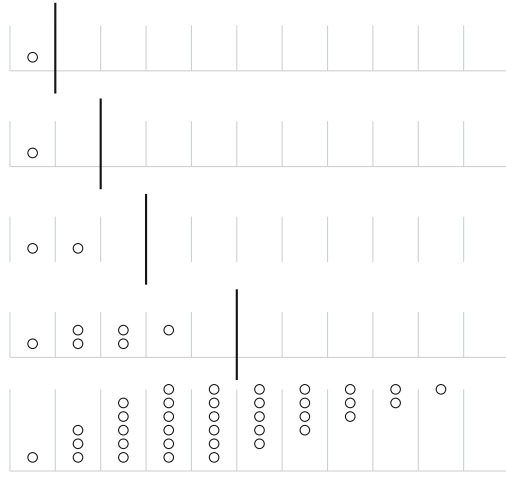


Fig. 1. The hypotenusal process

The resulting sequence is A001660 in the Encyclopedia of Integer Sequences [8]. It is related to the problem of finding extremal families by the following theorem.

Theorem 4 (Hypotenusal numbers). *Let $S \subset \binom{[n]}{k}$ be an extremal family of k -sets with $|\partial^{k-1}(S)| = n$. Let t be such that $\Delta^t(S)$ is not an initial segment of the colex order and let*

$$|S| = \binom{a_0}{k} + \dots + \binom{a_{k-1}}{1},$$

be the k -binomial decomposition of $|S|$. Then we have

$$a_{k-2-t+i} - a_{k-1-t+i} \geq h_i + 1, \quad 1 \leq i < t, \quad \text{and} \quad a_{k-2} - a_{k-1} \geq h_t, \quad (2)$$

where h_i is the i -th hypotenusal number.

Moreover, there are m, n and an extremal family $S \in \binom{[n]}{k}$ with cardinality m satisfying that $\Delta^t(S)$ is not an initial segment of length m in the colex order and the coefficients in the k -binomial decomposition of m satisfy equality in (2).

From the hypotenusal process described above it can be checked that the growth of the hypotenusal numbers is doubly exponential in n : there is a constant $h > 1$ such that $h_n = \Omega(h^{2^n})$. This is a lower bound on the growth of the last t coefficients of the binomial sequence for extremal sets S for which $\Delta^t(S)$ is not an initial segment in the colex order and provides a proof of Theorem 2.

Furthermore, extremal families S for which $\Delta^t(S)$ is not an initial segment in the colex order can be constructed in time $O(npoly(k))$ from a given k -binomial sequence when they exist. More precisely, we have the following algorithmic result.

Theorem 5 (Algorithmic construction). *Let $k, n, 2 \leq t < k - 1$ and m be given. The existence of an extremal family S with cardinality m with $|\Delta^{k-1}(S)| = n$ such that $\Delta^t(S)$ is not an initial segment of the colex order can be decided in time $O(npoly(k))$. If such a family exists, an instance of S can be constructed within the same time complexity.*

2 Hypergraph Representation

The constructive part of Theorem 5 reveals the structure of extremal families. The key idea in describing this structure consists in associating to a family S of k -sets the hypergraph of minimal elements in the complement of the simplicial complex

$$\mathcal{C}(S) = \cup_{i=0}^k \Delta^i(S)$$

generated by S , the family of all subsets y such that $y \subset x$ for some $x \in S$.

Definition 1 (Hypergraph of a family of k -sets). *Given a family of k -sets $S \subset \binom{[n]}{k}$ with support $\Delta^{k-1}(S) = [n]$, the hypergraph $H = H(S)$ of S has vertex set $V(H) = [n]$ and the edges of H are the minimal (with respect to containment) elements in $2^{V(H)} \setminus \mathcal{C}(S)$, the complement of the simplicial complex generated by S . In particular,*

$$S = \binom{[n]}{k} \setminus \bigcup_{e \in E(H)} \nabla^{k-|e|}\{e\},$$

and $H(S)$ uniquely determines S . Moreover, for each $0 \leq j \leq k - 1$,

$$\Delta^{k-j}(S) = \binom{[n]}{j} \setminus \bigcup_{e \in E, |e| \leq j} \nabla^{j-|e|}\{e\}.$$

For example, the hypergraph $H = H(S)$ of the initial segment in the colex order $S = I_{5,3}(5) = \{123, 124, 134, 234, 125\}$ has edge set $E(H) = \{35, 45\}$. It turns out that the structure of $E(H)$ is simpler to analyze than the family S itself. Moreover, the structure of $E(H)$ can be quantitatively linked to the cardinalities of the iterated lower shadows of S . If $E = \{e_1, \dots, e_h\}$, we may write $\bigcup_{e \in E, |e| \leq j} \nabla^{j-|e|}\{e\}$ as the disjoint union:

$$\bigcup_{e \in E, |e| \leq j} \nabla^{j-|e|}\{e\} = \cup_{i=1}^h \left(\nabla^{j-|e_i|}\{e_i\} \setminus \left[\cup_{t < i} \nabla^{j-|e_t|}\{e_t\} \right] \right)$$

The cardinality of each term $(\nabla^{j-|e_i|}\{e_i\} \setminus [\cup_{t=1}^{i-1} \nabla^{j-|e_t|}\{e_t\}])$ can be written as the sum of binomial coefficients of the type

$$\binom{n - |e_i| - j - p}{k - |e_i| - j - q}$$

for some appropriate nonnegative pairs of integers p and q ; the key point is that the sequence of p 's and q 's only depends on the previous edges e_1, \dots, e_{i-1}

but not on j . Then, by using binomial identities, we can find the binomial k -decomposition of $|\Delta^{k-j}(S)|$ and check the extremality of the family. Moreover, the same arguments can be reused to find $|\Delta^{k-j-1}(S)|$ giving a complete picture of the iterate lower shadows of S . The binomial identities lead to a strengthening of Theorem 4 in terms of so-called generalized hypotenusal numbers which can be described by the hypotenusal process with distinct initial configurations, and grow significantly larger than the hypotenusal numbers themselves, yet still being doubly exponential. We omit here the technical details.

Acknowledgements. The authors acknowledge financial support from the *Agencia Estatal de Investigación* of the Spanish Ministry of Science under grant MTM2017-82166-P. The second author was supported by the project 18-13685Y of the Czech Science Foundation (GAČR) and by the postdoctoral fellowship programme *Beatriu de Pinós* BP2018, funded by the AGAUR (Government of Catalonia) and by the Horizon 2020 programme of the European Union under the grant agreement No 801370.

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Gallai's Path Decomposition for Planar Graphs

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Abstract. In 1968, Gallai conjectured that the edges of any connected graph with n vertices can be partitioned into $\lceil \frac{n}{2} \rceil$ paths. Although this conjecture has been tackled and partially solved over the years, it is still open as of today. We prove that the conjecture is true for every planar graph. More precisely, we show that every connected planar graph except K_3 and K_5^- (K_5 minus one edge) can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths.

Keywords: Graph theory · Graph decomposition · Paths

1 Introduction

Given a graph G , a k -path decomposition of G is a partition of the edges of G into k paths. In 1968, Gallai stated this simple but surprising conjecture [8]: every finite undirected connected graph on n vertices admits a $\lceil \frac{n}{2} \rceil$ -path decomposition. Gallai's conjecture is still unsolved as of today, and has only been confirmed on very specific classes of graphs: graphs of degree 2 or 4 [6], graphs whose vertices of even degree induce a forest [9], series-parallel graphs [7], graphs with maximum degree at most 5 [1], or planar 3-trees [4]. Recently, Botler et al. proved that the conjecture is true in the case of triangle-free planar graphs [2]. Chu et al. confirmed the conjecture on graphs of maximum degree 6, under the condition that vertices of degree 6 form an independent set [5].

An *odd semi-clique* is obtained from a clique on $2k + 1$ vertices by deleting at most $k - 1$ edges. Bonamy and Perrett asked the following question [1, Question 1.1]: Does every connected graph G that is not an odd semi-clique admit a $\lfloor \frac{n}{2} \rfloor$ -path decomposition?

We answer this question positively for planar graphs. Only two odd semi-cliques are planar: the triangle K_3 and K_5 minus one edge, which we denote by K_5^- (see Fig. 1). We can therefore state the result as follows:

Theorem 11. *Every connected planar graph G on n vertices, except K_3 and K_5^- , can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths.*

To prove this result, we proceed with a standard approach for coloring problems, by considering a planar graph that is a counterexample to our theorem and

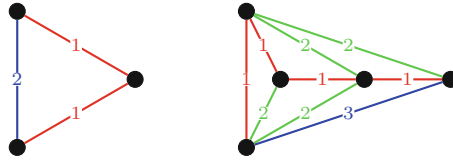


Fig. 1. A 2-path decomposition of K_3 and a 3-path decomposition of K_5^-

is vertex-minimum with respect to this property. We can prove that such a *minimum counterexample* (MCE) cannot contain a certain set of configurations, by providing for each of these configurations a reduction rule that takes advantage of the properties of the MCE and yields a contradiction. This technique is widely used in the literature on graph coloring, and especially on Gallai's conjecture [1–3, 5]. More precisely, these reducible configurations deal with vertices of small degree (at most 5), so after showing that our MCE cannot contain any of these configurations, we know that it has mostly vertices of degree at least 6. Finding these reducible configurations and their associated reduction rules makes up the bulk of the proof, and this is the part we develop in the next section. We finally use Euler's formula for planar graphs and structural arguments to prove that there is no such graph.

2 Reducible Configurations

We say that a path decomposition of a graph on n vertices is *good* if it contains at most $\lfloor \frac{n}{2} \rfloor$ paths. Given a planar graph G , a *2-family* is a set U of two vertices of G of degree at most 4. A *4-family* is a set of four vertices of degree 5. We say that a graph with a 4-family U is almost *4-connected* w.r.t. U if it is 3-connected and does not contain a set $A = \{a_1, a_2, a_3\}$ of vertices such that A separates two vertices $u_1, u_2 \in U$, or such that there is a vertex $u \in U \cap A$, and A separates two neighbors of u . The main lemma of the paper is the following.

Lemma 21. *Let G be a connected planar graph on n vertices, other than K_3 and K_5^- , and assume that G is a minimum counterexample to Theorem 11, i.e. that G does not admit a $\lfloor \frac{n}{2} \rfloor$ -path decomposition and is vertex-minimum for this property. Then G does not contain any of the following configurations:*

- (C_I) : a 2-family;
- (C_{II}) : an almost 4-connected component with respect to a 4-family.

The general idea is the following: we prove by contradiction that (C_I) or (C_{II}) configurations cannot occur in an MCE by considering such a graph G with a 2-family or 4-family U . We delete the vertices of U , and add or remove some edges. We call the resulting graph G' the *reduced graph*. G' is smaller than G , so it is made up of connected components that are either K_3 , K_5^- or have a good path decomposition. We are able to build a decomposition of G' into

paths and cycles, such that the total number of paths and cycles is $\lfloor \frac{|V(G')|}{2} \rfloor$. We describe a specific reduction rule for each case, such that the decomposition of the reduced graph associated with this case can be turned into a good path decomposition of G . Since G is a counterexample, it is a contradiction, hence after each case is dealt with we deduce that our MCE does not contain (C_I) nor (C_{II}) configurations.

2.1 Configurations (C_I)

The first set of configurations deals with all the cases of our MCE G containing 2 vertices of degree at most 4, which we call *special vertices*. We start by considering a shortest path S between u_1 and u_2 . Each special vertex has one neighbor belonging to S , and up to 3 other *remaining neighbors*. We distinguish between whether or not the special vertices have remaining neighbors in common.

If the special vertices u_1, u_2 have no common remaining neighbor, we deal with the case by considering a *composite rule* made up two *partial rules*. The reduced graph G' will be created by removing the vertices u_1, u_2 and the edges of S from G , as well as adding or removing some edges specified by each partial rule.



Fig. 2. A partial rule for 2 non-adjacent remaining neighbors

Figure 2 features an example of partial rule. The leftmost drawing represents the initial configuration in G , in this case the special vertex u_1 has degree 3, with a neighbor in the path S (in red) and two non-adjacent remaining neighbors v_1, v_2 . The middle drawing depicts what edges we add to or remove from the reduced graph G' , in this case we add the edge v_1v_2 . We consider an arbitrary path decomposition of G' , and assume the (green) path Q contains the edge v_1v_2 . The rightmost drawing represents a decomposition of G that we can build based on the decomposition of G' : all paths from the decomposition of G' remain unchanged in G , except Q which is deviated on the edges v_1u_1 and u_1v_2 . Since $|V(G)| - |V(G')| = 2$, we are allowed one more path, that we use to color the edges of S . Having one end of such a path on each special vertex is convenient, for example for the rule of Fig. 3, where it is extended to help handle edges within the neighborhood of u .

The composite rule is defined by application of the two partial rules, to define the reduced graph G' and draw a path decomposition of G from a path decomposition of G' . Since the special vertices do not have common remaining

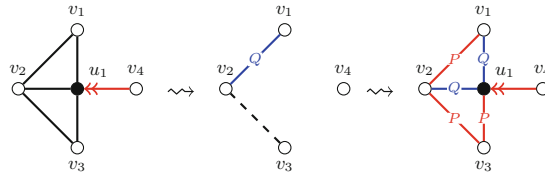


Fig. 3. A partial rule for 3 remaining neighbors

neighbors, the partial rules do not interfere with each other and can be applied independently.

When the two special vertices have common remaining neighbors, we treat both of them at the same time with a common rule, such as the one depicted in Fig. 4. The leftmost drawing depicts the configuration in G : two adjacent special vertices u_1, u_2 of degree 4 with a common neighbor v , and such that u_1 has two other neighbors v_1, v_3 , u_2 has two other neighbors v_2, v_4 , the vertices v_1, v_2, v_3, v_4 are distinct and v_1, v_2 are non-adjacent. The middle drawing depicts the reduced graph G' : after removing u_1, u_2 , we add the edge v_1v_2 , and this edge belongs to a path Q in a path decomposition of G' . The rightmost drawing depicts the path decomposition of G that we build from the one of G' : we deviate the path Q on the edges v_1u_1, u_1u_2, u_2v_2 , and we use an extra path for the edges v_3u_1, u_1v, vu_2 and u_2v_4 .

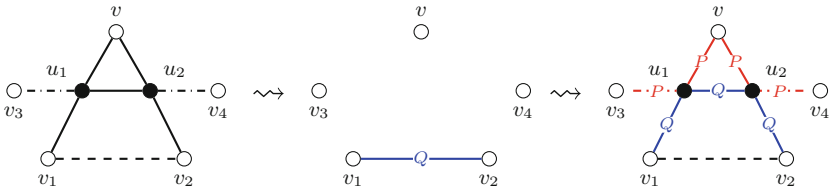


Fig. 4. A reduction rule for special vertices with 1 common neighbor

We find a set of about 30 rules that covers all possible cases of (C_I) configurations. We thus deduce a contradiction with the nature of G , which proves that such a minimum counterexample does not contain a configuration (C_I) .

2.2 Configurations (C_{II})

We proceed in the same way for the configurations (C_{II}) , by generalizing the concepts of the previous section for our set of 4 special vertices of degree 5. We again consider an MCE G and assume it contains a configuration (C_{II}) . Instead of considering a path S , we consider a subdivision of a certain graph, rooted on our 4 special vertices. The goal of this part is again to define valid reduction rules that cover each possible case. The rules operate like in the previous section: we

remove the special vertices and the edges of the subdivision, and add or remove some edges in the neighborhood of the special vertices, as specified in each rule. The resulting reduced graph can be decomposed into the right number of paths and cycles, and we are able to deduce from the rule a good path decomposition of our MCE G .

The subdivisions we consider are K_4 -subdivisions (Fig. 5a) or C_{4+} -subdivisions (Fig. 5b) rooted on our 4 special vertices (where C_{4+} is the graph made up of a cycle on 4 vertices with two parallel edges doubled).

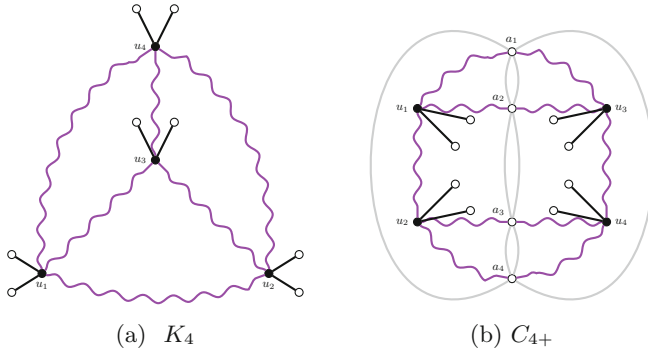


Fig. 5. A K_4 - and a C_{4+} -subdivision

We use a result by Yu [10] to show that in an almost 4-connected configuration, there exists indeed a K_4 - or C_{4+} -subdivision rooted on the four special vertices. These subdivisions can be decomposed into 2 paths, which corresponds to the number of extra paths we are allowed to use in our reductions. Additionally, these paths have their four ends on each of the four special vertices, which allows us once again to extend those paths if needed in order to cover all edges within the neighborhood of our special vertices. Each special vertex u has three incident edges that belong to the subdivision, and two other *remaining neighbors* v, v' . The edges uv, uv' are the ones we need to cover in the good path decomposition of G that we want to build, for each special vertex u .

We generalize our concept of partial rules for our 4 special vertices. Each partial rule treats the neighborhood of one or two special vertices at once, so we now consider composite rules made up of a subdivision and between 2 and 4 partial rules. If these partial rules affect disjoint areas of the graph and are disjoint from the subdivision, they can be applied independently, but it is not always the case.

In order to find a composite rule made up of compatible partial rules, we apply a series of alterations to the subdivision, to eliminate some unwanted configurations. When all modifications have been applied, we do a careful case analysis and find a composite rule for each case, such as the one in Fig. 6.

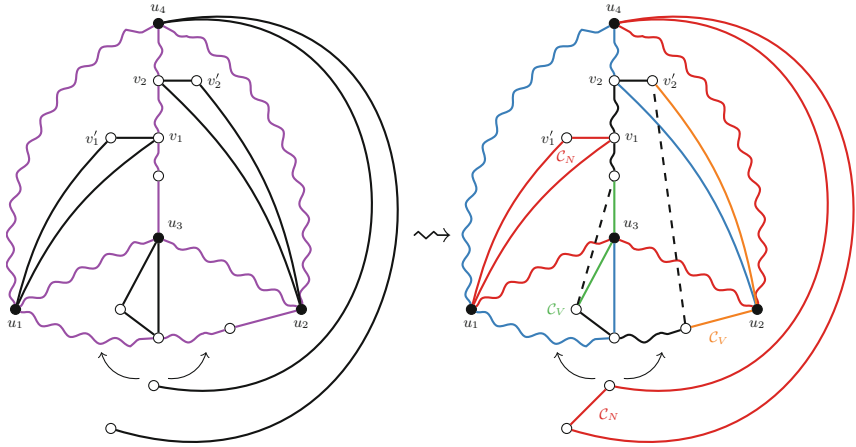


Fig. 6. Example of a composite rule on a K_4 -subdivision

The configuration depicted on the first drawing consists in a K_4 -subdivision S , such that some of the remaining neighbors of the special vertices belong to S , which prevents partial rules from being applied as-is. The reduction rule consists in finding an alternate subdivision S' (in this case the K_4 -subdivision is turned into a C_{4+} -subdivision), describing a 2-path decomposition of S' (in red and blue on the second drawing) and describing a set of partial rules (in this case C_V and C_N , which deals with two adjacent remaining neighbors) for each special vertex, while making sure these partial rules are compatible with one another and with the subdivision. We find a set of around 20 composite rules that covers all cases of (C_{II}) configurations.

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Tuza's Conjecture for Threshold Graphs

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Abstract. Tuza famously conjectured in 1981 that in a graph without $k + 1$ edge-disjoint triangles, it suffices to delete at most $2k$ edges to obtain a triangle-free graph. The conjecture holds for graphs with small treewidth or small maximum average degree, including planar graphs. However, for dense graphs that are neither cliques nor 4-colourable, only asymptotic results are known. Here, we confirm the conjecture for threshold graphs, i.e. graphs that are both split graphs and cographs.

Keywords: Tuza's conjecture · Packing · Covering · Threshold graphs

1 Introduction

If we can “pack” at most k disjoint objects of some type in a given graph, how many elements do we need to “cover” all appearances of such an object in the graph? Erdős and Pósa famously proved that if a graph contains at most k



This research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreements No. 714704 (Ł. Bożyk, T. Masařík, J. Novotná, and K. Okrasa), No. 648509 (A. Grzesik), and No. 648527 (M. Hatzel). T. Masařík was supported by a post-doctoral fellowship at SFU (NSERC grants R611450 and R611368). The full version of the paper can be found on arXiv [3] where we show that similar tools with more involved analysis can be used to verify Conjecture 1 also for specific co-chain graphs.

pairwise vertex-disjoint cycles, then there is a set of at most $f(k)$ vertices that intersects every cycle [7]. While the exact best value of function f is yet unknown, the asymptotic behaviour was recently determined to be $f(k) = \Theta(k \log k)$ [5].

In this paper, we focus on edge-disjoint triangles; we refer the interested reader to [12] for a dynamic survey on other objects. For a graph G , we call every family of pairwise edge-disjoint triangles a *triangle packing*, and every subset of edges intersecting all triangles in G a *triangle hitting*. We denote by $\mu(G)$ the maximum size of a triangle packing in G , and by $\tau(G)$ the minimum size of a triangle hitting in G . Trivially, there is a set of at most $3\mu(G)$ edges that intersect every triangle. We are concerned with improving that bound, following Tuza's conjecture from 1981.

Conjecture 1 (Tuza [13]). For any graph G it holds $\tau(G) \leq 2\mu(G)$.

Conjecture 1, if true, is tight for K_4 and K_5 . Gluing together copies of K_4 and K_5 along vertices, it is easy to build an infinite family of connected graphs for which Conjecture 1 is tight. However, for larger cliques, it is known that the ratio $\tau(K_p)/\mu(K_p)$ tends to $3/2$ as p increases [8]. In addition, Haxell and Rödl [10] proved that $\tau(G) \leq 2\mu(G) + o(|V(G)|^2)$ for any graph G , meaning Conjecture 1 is asymptotically true when $\tau(G)$ is quadratic with respect to $|V(G)|$. Those seem to indicate that Conjecture 1 should be easier for dense graphs than for sparse graphs. Conversely, it is asymptotically tight in some classes of dense graphs [2]. If we focus on *hereditary graph* classes (i.e. classes that contain every induced subgraph of a graph in the class), the conjecture has only been confirmed for a few graph classes. Those classes include most notably graphs of treewidth at most 6 [4], 4-colourable graphs [1], and graphs with maximum average degree less than 7 [11].

A good candidate for an interesting dense hereditary graph class is the class of *split graphs*, i.e. graphs whose vertex set can be partitioned into two sets: one that induces a clique, the other inducing an independent set. However, Conjecture 1 remains a real challenge even when restricted to split graphs. Another good candidate for an interesting dense hereditary graph class is the class of *cographs*, i.e. graphs with no induced path on four vertices. As an initial step, we focus on graphs that are both split graphs and cographs, i.e. *threshold* graphs. While this may seem like a small step, it is arguably the first dense hereditary superclass of cliques where the conjecture is confirmed.

Theorem 1. *If G is a threshold graph, then $\tau(G) \leq 2\mu(G)$.*

Finally, it is worth mentioning that Conjecture 1 is known to hold as soon as we consider *multi*-packing [6], and in particular it holds in its fractional relaxation. Another angle of attack consists of lowering the bound of 3 step by step for all graphs. The best, and in fact only, such bound is slightly under 2.87 [9].

1.1 Preliminaries

All graphs in this paper are undirected and simple. Let $G = (V, E)$ be a graph. For all $v \in V$ the set $N(v) := \{u \mid \{u, v\} \in E\}$ is called the *neighbourhood* of v

and $N[v] := N(v) \cup \{v\}$ is its *closed neighbourhood*. A vertex $v \in V$ is *complete* to $A \subseteq V, v \notin A$ if v is adjacent to all vertices in A . Disjoint sets $A, B \subseteq V$ are *complete* to each other if E contains all edges between A and B .

Lemma 1. *The edge set of a clique K with $|K| = k$ can be decomposed into k edge disjoint maximal matchings for k odd and $k - 1$ edge disjoint maximal matchings for k even.*

A graph $G = (V, E)$ is a *star* if $V = \{c, s_1, \dots, s_k\}$ and $E = \{\{c, s_i\} \mid 1 \leq i \leq k\}$, the vertex c is called the *center vertex* of the star. A graph G is a *complete split graph* if its vertex set can be partitioned into sets K and S , such that S is independent, K induces a clique, and K and S are complete to each other.

The following lemma describes how to pack triangles in complete split graphs.

Lemma 2 [8]. *Let K be a clique, S an independent set such that they are complete to each other and $|K| = |S| = k$. Then we can find an (optimal) triangle packing TP of size $\binom{k}{2}$ such that:*

1. *It uses all edges from K and each triangle in TP contains exactly one edge from K .*
2. *If k is odd, the remaining edges (not used in TP) create a matching between K and S , otherwise they create a star with its center vertex in S . Moreover, we can choose the unused matching and the center vertex of the unused star arbitrarily.*

Corollary 1. *Let K be a clique and S an independent set such that they are complete to each other.*

- (a) *If $|S| < |K|$, then we can find a triangle packing of size $|S| \cdot \lfloor |K|/2 \rfloor$.*
- (b) *If $|S| \geq |K|$, then we can find a triangle packing of size $\binom{|K|}{2}$.*

We say that we *pack edges of K with vertices of S* when we use triangle packings from Corollary 1.

2 Threshold Graphs

A graph $G = (V, E)$ is a *threshold graph* if its vertex set can be partitioned into two sets $K = \{c_1, \dots, c_k\}$ and $S = \{u_1, \dots, u_s\}$ such that $G[K]$ is a clique and S is an independent set in G , and $N[c_{i+1}] \subseteq N[c_i]$ for all $1 \leq i < k$ and $N(u_i) \subseteq N(u_{i+1})$ for all $1 \leq i < s$. We identify K with the clique $G[K]$ and say $G = (K \cup S, E)$ is a threshold graph with given *threshold representation* (K, S) .

The threshold representation of a threshold graph may not be unique. We prove that it can be chosen such that the clique contains a vertex which is not adjacent to any vertex of the independent set.

Lemma 3. *For every threshold graph $G = (V, E)$ there exists a threshold representation (K, S) such that there is a vertex $v \in K$ with $N(v) \cap S = \emptyset$.*

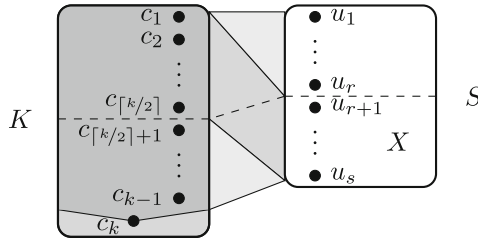


Fig. 1. The structure of threshold graph G .

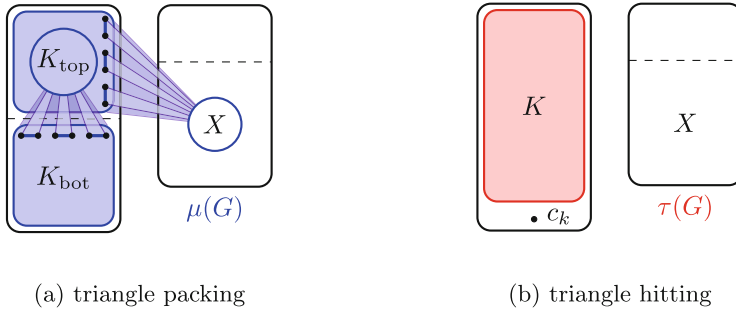


Fig. 2. The (a) triangle packing and (b) triangle hitting for $|X| \geq k/2$.

We can now prove that Conjecture 1 holds for all threshold graphs.

Proof (of Theorem 1). Let $G = (K \cup S, E)$ be a threshold graph with $K = \{c_1, \dots, c_k\}$ and $S = \{u_1, \dots, u_s\}$ such that $N(c_k) \cap S = \emptyset$. By Lemma 3, such a representation exists. Let $r \in \{1, \dots, s\}$ be chosen minimal such that $\{c_1, \dots, c_{\lceil k/2} \rceil\} \subseteq N(u_r)$ and let X be the subset $\{u_r, \dots, u_s\}$ of S (see Fig. 1). Note that X is complete to the set $\{c_1, \dots, c_{\lceil k/2} \rceil\}$. We distinguish two cases, based on the parity of k . First, we focus on the case that k is even. In this case we consider two cliques K_{top} and K_{bot} of equal size, induced by vertices $\{c_1, \dots, c_{k/2}\}$ and $\{c_{k/2+1}, \dots, c_k\}$, respectively.

We construct a triangle packing TP of G using Corollary 1 as follows: we pack the edges of K_{bot} with vertices in K_{top} , and the edges of K_{top} with vertices in X (see Fig. 2(a)).

If $|X| \geq \frac{k}{2}$, then TP is a triangle packing of size $2 \binom{k/2}{2}$. On the other hand, a triangle hitting of size $\binom{k-1}{2}$ can be obtained by taking all edges from K except those incident to c_k (see Fig. 2(b)). Thus, we obtain a lower bound on the triangle packing and an upper bound on the triangle hitting yielding:

$$\tau(G) \leq \binom{k-1}{2} = \frac{k-2}{2} \cdot (k-1) \leq \frac{k-2}{2} \cdot k = 4 \binom{k/2}{2} \leq 2\mu(G).$$

If $|X| < \frac{k}{2}$, then TP is of size at least $\binom{k/2}{2} + |X| \cdot \lfloor k/4 \rfloor \geq \binom{k/2}{2} + |X|(k/4 - 1/2)$. On the other hand, the edges inside K_{top} and inside K_{bot} together with all edges

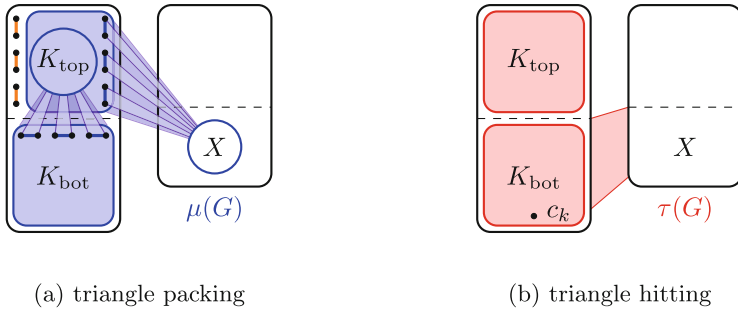


Fig. 3. The (a) triangle packing and (b) triangle hitting for $|X| < k/2$.

between S and K_{bot} build a triangle hitting of G (cf. Fig. 3(b)) of size at most $2\binom{k/2}{2} + |X|(k/2 - 1)$. Indeed, recall that c_k does not have any neighbours in S , therefore we have at most $|X|\left(\frac{k}{2} - 1\right)$ edges between X and K_{bot} , and by definition of X , there are no vertices in K_{bot} having neighbours in $S \setminus X$. Thus, we again obtain a lower bound on the triangle packing and an upper bound on the triangle hitting yielding:

$$\tau(G) \leq 2\binom{k/2}{2} + |X|\left(\frac{k}{2} - 1\right) = 2\binom{k/2}{2} + 2|X|\left(\frac{k}{4} - \frac{1}{2}\right) \leq 2\mu(G).$$

We are left with the case that k is odd. We consider the cliques K_{top} and K_{bot} induced by sets $\{c_1, \dots, c_{(k+1)/2}\}$ and $\{c_{(k+1)/2+1}, \dots, c_k\}$, respectively.

Again, we look at the size of X and in case it is large, we can derive a similar argument as in the previous case, using Corollary 1. More precisely, assume that $|X| \geq \frac{k+1}{2}$. Then we pack the edges of K_{bot} into $\binom{(k+1)/2}{2}$ triangles with vertices in K_{top} , and the edges of K_{top} into $\binom{(k-1)/2}{2}$ triangles with vertices in X . Together, this gives a triangle packing of size $\binom{(k+1)/2}{2} + \binom{(k-1)/2}{2} = (k-1)^2/4$. The triangle hitting again consists of all edges from K except those adjacent to c_k , therefore has size $\binom{k-1}{2}$ (recall Fig. 2). These two bounds together yield:

$$\tau(G) \leq \binom{k-1}{2} = \frac{k-1}{2} \cdot (k-2) \leq \frac{(k-1)^2}{2} \leq 2\mu(G).$$

It remains to consider the case $|X| < \frac{k+1}{2}$. In order to find a triangle packing, we define K'_{top} and K'_{bot} to be induced by $\{c_1, \dots, c_{(k-1)/2}\}$ and $\{c_{(k+1)/2}, \dots, c_k\}$, respectively (so $K'_{\text{top}} = K_{\text{top}} \setminus \{c_{(k+1)/2}\}$ is of size $\frac{k-1}{2}$ and $K'_{\text{bot}} = K_{\text{bot}} \cup \{c_{(k+1)/2}\}$ is of size $\frac{k+1}{2}$). We build a triangle packing analogously to before, using Corollary 1. The edges of K'_{bot} can be packed into $\lfloor \frac{(k+1)/2}{2} \rfloor \cdot \frac{k-1}{2} \geq \frac{k-1}{4} \cdot \frac{k-1}{2}$ triangles with vertices in K'_{top} . Moreover, $\min\left\{|X| \cdot \lfloor \frac{k-1}{4} \rfloor, \binom{(k-1)/2}{2}\right\} \geq |X| \frac{k-3}{4}$ edges of K'_{top} can be packed into triangles with vertices in X (see Fig. 4(a)). This gives a triangle packing of size at least $(k-1)/2 \cdot (k-1)/4 + |X|(k-3)/4$. To find a

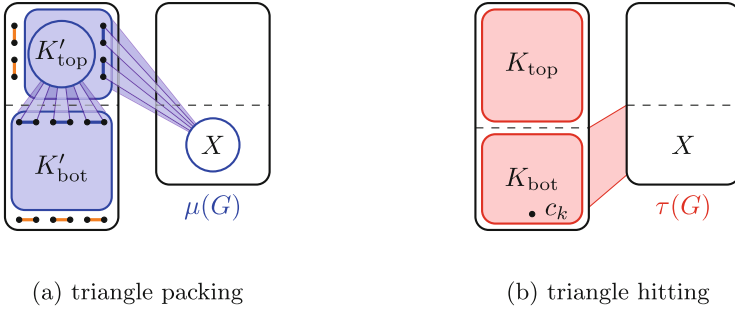


Fig. 4. The (a) triangle packing and (b) triangle hitting for $|K|$ odd and $|X| < (k+1)/2$.

triangle hitting, we again consider the partition of K into K_{top} and K_{bot} . We take all edges inside K_{top} and inside K_{bot} together with all edges between S and K_{bot} (see Fig. 4(b)). Again, recall that $c_k \in K_{\text{bot}}$ does not have any neighbours in S , and there are no vertices in K_{bot} having neighbours in $S \setminus X$. Thus, this yields a triangle hitting of size $\binom{(k+1)/2}{2} + \binom{(k-1)/2}{2} + |X| \binom{k-3}{2}$. Therefore, we obtain the following which concludes the proof:

$$\begin{aligned} \tau(G) &\leq \binom{\frac{k+1}{2}}{2} + \binom{\frac{k-1}{2}}{2} + |X| \frac{k-3}{2} \\ &= \frac{(k-1)^2}{4} + |X| \frac{k-3}{2} = 2 \cdot \frac{k-1}{2} \cdot \frac{k-1}{4} + 2|X| \frac{k-3}{4} \leq 2\mu(G). \end{aligned}$$

□

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Covering Three-Tori with Cubes

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Abstract. Let $\mu(\varepsilon)$ be the minimum number of cubes of side ε needed to cover the unit three-torus $[\mathbb{R}/\mathbb{Z}]^3$. We prove new lower and upper bounds for $\mu(\varepsilon)$ and find the exact value for all $\varepsilon \geq \frac{7}{15}$ and all $\varepsilon \in \left[\frac{1}{r+1/(r^2+r+1)}, \frac{r^2-1}{r^3-r-1} \right)$ for any integer $r \geq 3$.

Keywords: Coverings · Cubes · Tori

1 Introduction

Let d be a positive integer, $\varepsilon \in (0, 1)$. Consider the torus $T^d := [\mathbb{R}/\mathbb{Z}]^d$ and the set \mathcal{J}_ε of ‘sub-cubes’ of the form $\{(x_1, \dots, x_d) : x_i \in [x_i^0, x_i^0 + \varepsilon]\}$. The question is, what is the minimum number $\mu := \mu(d; \varepsilon)$ of sets A_1, \dots, A_μ from \mathcal{J}_ε needed to cover T^d (i.e., $T^d = A_1 \cup \dots \cup A_\mu$)?

In [1], it is proven that $\mu \geq \lceil 1/\varepsilon \rceil^{(d)}$, where $\lceil x \rceil^{(i)} = \lceil x \lceil x \rceil^{(i-1)} \rceil$ and $\lceil x \rceil^{(1)} = \lceil x \rceil$. Moreover, it is shown that, for $d = 2$, this lower bound is sharp, i.e. $\mu(2; \varepsilon) = \lceil 1/\varepsilon \lceil 1/\varepsilon \rceil \rceil$.

In our paper, we consider $d = 3$. Since $\mu(1; \varepsilon) = \lceil 1/\varepsilon \rceil$ and $\mu(2; \varepsilon) = \lceil 1/\varepsilon \lceil 1/\varepsilon \rceil \rceil$, we get that

$$\lceil 1/\varepsilon \lceil 1/\varepsilon \lceil 1/\varepsilon \rceil \rceil \leq \mu(3; \varepsilon) \leq \lceil 1/\varepsilon \lceil 1/\varepsilon \rceil \lceil 1/\varepsilon \rceil. \tag{1}$$

In [1], the authors also noticed that the lower bound for $d = 3$ is not sharp. For example, $\mu(3; 3/7) > \lceil 7/3 \lceil 7/3 \lceil 7/3 \rceil \rceil$. Unfortunately, nothing better than (1) is known.

In this paper, we have found the exact value of $\mu(3; \varepsilon)$ for $\varepsilon \geq 7/15$. We have also found exact values of $\mu(3; \varepsilon)$ for ε close to $1/r$, $r \in \mathbb{N}$. In Sect. 2, we state new results. In Sect. 3, we show that to solve the problem for $\varepsilon \in [1/r, 1/(r-1))$, $r \in \{2, 3, \dots\}$, it is sufficient to solve it for a finite set of rational numbers with denominator at most r^3 . In Sect. 4, we prove lemmas from Sect. 3. In Sect. 5, we give a complete proof of Theorem 2 and outline the proofs of other results.

2 New Results

Theorem 1. *If $\varepsilon \geq 2/3$, then the lower bound is the answer:*

$$\mu(3; \varepsilon) = 5, \varepsilon \in [2/3, 3/4), \quad \mu(3; \varepsilon) = 4, \varepsilon \in [3/4, 1).$$

If $1/2 \leq \varepsilon < 2/3$, then the upper bound is the answer, i.e. $\mu(3; \varepsilon) = 8$.

Notice that, in contrast to $d = 2$, for $\varepsilon \in [1/2, 1)$, $\mu(d = 3; \varepsilon)$ equals the lower bound in (1) if and only if $\varepsilon \in [1/2, 4/7) \cup [2/3, 1)$.

Since, for an integer $r \geq 2$ and $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1/r^2}\right)$, the lower bound and the upper bound in (1) are equal, the value of $\mu(3; \varepsilon)$ is straightforward and equals r^3 . We have also found left-neighborhoods of all $1/r$ where the lower bound is the correct answer (notice that for such ε the difference between the upper and the lower bounds is, conversely, large).

Theorem 2. *Let $r \in \mathbb{N}$. If $\varepsilon \in \left[\frac{1}{r+1/(r^2+r+1)}, \frac{1}{r}\right)$, then the lower bound is the answer, i.e. $\mu(3; \varepsilon) = r^3 + r^2 + r + 1$.*

Moreover, we have proved that the trivial right-neighborhoods of all $1/r$ where the upper bound is the correct answer can be extended in the following way.

Theorem 3. *Let $r \geq 2$ be an integer. If $\varepsilon \in \left[\frac{1}{r}, \frac{r^2-1}{r^3-r-1}\right)$, then the upper bound is the answer, i.e. $\mu(3; \varepsilon) = r^3$.*

We have also improved the lower bound from (1) in some special cases.

Theorem 4. *Let $r \geq 2$ be an integer, $\xi \in \{0, 1, \dots, r\}$ be such that*

$$\xi^2 \leq \xi + (r + 1) \left\lfloor \frac{\xi^2}{r + 1} \right\rfloor.$$

Let

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + 2\xi r + \left\lfloor \frac{\xi^2}{r+1} \right\rfloor$ *be coprime with s .*

Then $\mu\left(3; \frac{s}{t}\right) > t$, i.e. bigger than the lower bound.

The condition $\xi^2 \leq \xi + (r + 1) \left\lfloor \frac{\xi^2}{r+1} \right\rfloor$ implies that either $\xi \geq \sqrt{r + 1}$ or $\xi = 1$. In the interval $[1/3, 1/2)$, there are two such $\frac{k}{n} \in \left\{ \frac{7}{16}, \frac{8}{21} \right\}$.

Finally, we have improved the upper bound from (1) in some special cases.

Theorem 5. *Let $r \geq 2$ be an integer, $\xi \in \{2, 3, \dots, r\}$. Let*

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + \xi(r + 1)$.

Then $\mu(3; \frac{s}{t}) \leq t$, i.e. smaller than the upper bound.

Both Theorem 4 and Theorem 5 imply improvements of the bounds in (1) for ε from a certain right-interval of the respective s/t (see Lemma 1 from Sect. 3).

Notice that, for $\varepsilon \in [1/3, 1/2)$, Theorems 2, 3, 4, 5 and Lemma 1 from Sect. 3 imply that

- for $\varepsilon \in [\frac{1}{3}, \frac{8}{23})$, $\mu(3; \varepsilon) = 27$ (by Theorem 3 and Lemma 1);
- for $\varepsilon \in [\frac{8}{21}, \frac{5}{13})$, $\mu(3; \varepsilon) \in [22, 24]$ (by Theorem 4 and Lemma 1);
- for $\varepsilon \in [\frac{7}{16}, \frac{4}{9})$, $\mu(3; \varepsilon) \in [17, 21]$ (by Theorem 4 and Lemma 1);
- for $\varepsilon \in [\frac{4}{9}, \frac{7}{15})$, $\mu(3; \varepsilon) \in [16, 18]$ (by Theorem 5 and Lemma 1);
- for $\varepsilon \in [\frac{7}{15}, \frac{1}{2})$, $\mu(3; \varepsilon) = 15$ (by Theorem 2 and Lemma 1).

3 Integer Lattices

On the one hand, there are continuously many ε left for which the answer is not known. On the other hand, the below lemma implies that the problem reduces to a countable set.

Let $d \geq 2$ be an integer.

Lemma 1. *There exists an infinite sequence of rational numbers $1 > \frac{s_1}{t_1} > \frac{s_2}{t_2} > \dots > 0$ such that, for every $i \in \mathbb{N}$, $t_i \leq \mu(d; s_i/t_i)$ and $\mu(d; \varepsilon) = \mu(d; s_i/t_i)$ for all $\varepsilon \in [s_i/t_i, s_{i-1}/t_{i-1})$, where $s_0 = t_0 = 1$.*

For $d = 3$, due to (1), the denominator of the critical point $\frac{s_i}{t_i}$ is at most $\lceil \frac{t_i}{s_i} \rceil \lceil \frac{t_i}{s_i} \lceil \frac{t_i}{s_i} \rceil \rceil$. Therefore, for every integer $r \geq 2$, on $[\frac{1}{r}, \frac{1}{r-1})$ there are at most $\frac{r^2(r^3+1)}{2(r-1)} + r^3$ candidates on the role of a critical point.

We give the proof of Lemma 1 in Sect. 4. From this proof it immediately follows (see Remark 1 in Sect. 4) that the problem can be equivalently reformulated for integer lattices (as stated below in Lemma 2).

Let $s \leq t$ be positive integers. Consider the torus $[\mathbb{Z}/t\mathbb{Z}]^d$ and the set of its ‘sub-cubes’ with edges of size s : $\{(x_1, \dots, x_d) : x_i^0 \leq x_i \leq x_i^0 + s - 1 \pmod t, x_i^0 \in \mathbb{Z}/t\mathbb{Z}\}$. Let $\mu_0(d; s, t)$ be the minimum number of such ‘sub-cubes’ needed to cover the torus.

Lemma 2. *Let $r \geq 2$ be an integer, $\varepsilon \in [\frac{1}{r}, \frac{1}{r-1})$. Let $\frac{s}{t} \leq \varepsilon$ be the closest rational number to ε with $t \leq r^d$. Then $\mu(d; \varepsilon) = \mu_0(d; s, t)$.*

Since $\mu(d; s/t) = \mu_0(d; s, t)$, we get that $\mu_0(d; s, t)$ depends only on s/t .

4 Proofs of Lemmas

In this section, we prove Lemma 1 and note that this proof implies Lemma 2 as well.

Let $\mu \in \mathbb{N}$. Consider the minimum value of $\varepsilon \in (0, 1]$ such that there is a covering \mathcal{A} of the torus T^d by μ ‘sub-cubes’

$$A_j := \left\{ (x_1, \dots, x_d) : x_i \in [x_i^j, x_i^j + \varepsilon] \right\}, \quad j \in [\mu] := \{1, \dots, \mu\}.$$

Let $i \in [d]$. Consider the segments $[x_i^j, x_i^j + \varepsilon]$, $j \in [\mu]$. Let us draw a graph $G_i(\mathcal{A})$ with vertex set $[\mu]$. Let vertices $j_1, j_2 \in [\mu]$ be adjacent in G if and only if the sets $\{x_i^{j_1}, x_i^{j_1} + \varepsilon\}$, $\{x_i^{j_2}, x_i^{j_2} + \varepsilon\}$ are not disjoint (i.e., the respective segments have at least one common endpoint).

Let us show that there exists a covering $\tilde{\mathcal{A}}$ of T^d by μ ‘sub-cubes’ such that $G_i(\tilde{\mathcal{A}})$ is connected. Assume that, in $G_i(\mathcal{A})$, there are several connected components H_1, \dots, H_ℓ , $\ell \geq 2$. Let j_1, \dots, j_v be the vertices of H_1 . Denote by ρ the distance between the set of endpoints of segments labeled by j_1, \dots, j_v and the set of endpoints of the rest segments, i.e.

$$\rho = \min \left\{ |a-b|, a \in \{x_i^j, x_i^j + \varepsilon, j \in \{j_1, \dots, j_v\}\}, b \in \{x_i^j, x_i^j + \varepsilon, j \notin \{j_1, \dots, j_v\}\} \right\}$$

Let us shift all segments labeled by j_1, \dots, j_v in the direction to the closest segment that is not labeled by any of j_1, \dots, j_v at the distance ρ . Clearly, we get a covering \mathcal{A}_1 of T^d with $G_i(\mathcal{A}_1)$ consisting of at most $\ell - 1$ components. If $G_i(\mathcal{A}_1)$ is not connected, make the same procedure with \mathcal{A}_1 and obtain a covering \mathcal{A}_2 with $G_i(\mathcal{A}_2)$ having at most $\ell - 2$ components. Proceeding in this way, we will reach the desired covering $\tilde{\mathcal{A}}$.

Assume now that, for every $i \in [d]$, $G_i(\mathcal{A})$ is connected. Suppose that there is no integer $q \in \{1, \dots, \mu\}$ such that $q\varepsilon \in \mathbb{N}$ (otherwise, we get the statement of Lemma 1).

Fix $i \in [d]$ and consider the following relation $<$ on the set of segments $[x_i^j, x_i^j + \varepsilon]$:

$$\text{if } x_i^{j_1} + \varepsilon = x_i^{j_2}, \text{ then } [x_i^{j_1}, x_i^{j_1} + \varepsilon] < [x_i^{j_2}, x_i^{j_2} + \varepsilon].$$

Since $q\varepsilon \notin \mathbb{N}$ for any $q \in \{1, \dots, \mu\}$, we get that $<$ is linear order.

Let $[x_i^1, x_i^1 + \varepsilon]$ be a minimal segment respectively (there can be several equal segments). Without loss of generality, we may assume that $x_i^1 = 0$. Let $0 = \alpha_0 < \alpha_1 < \dots < \alpha_r < 1$ be all distinct endpoints of segments $[x_i^j, x_i^j + \varepsilon]$, $j \in [\mu]$. Let

$$\gamma = \min\{\alpha_{k+1} - \alpha_k, k \in \{0, 1, \dots, r\}\},$$

where $\alpha_{r+1} = 1$.

Let $\delta < \gamma/r$ be a positive number. For every $j \in [\mu]$, let r_j be the number of (distinct) segments less than $[x_i^j, x_i^j + \varepsilon]$. Let us replace A_j with the cube A'_j having the same (but the i th) projections, the i th projection is replaced with $[(x')^j_i := x_i^j - r_j\delta, (x')^j_i + \varepsilon - \delta]$. Let us prove that $\mathcal{A}' = \{A'_1, \dots, A'_\mu\}$ covers T^d as well.

Clearly, for every $y' \in [0, 1)$ there exists $y \in [0, 1)$ such that

$$y' \in [(x')^j_i, (x')^j_i + \varepsilon - \delta] \text{ if and only if } y \in [x_i^j, x_i^j + \varepsilon].$$

Therefore, $T^d|_{x_i=y'}$ is covered by the same $(d - 1)$ -dimensional cubes (hyperplanes of A'_j with $x_i = y'$) as $T^d|_{x_i=y}$. Since \mathcal{A} covers T^d we get that \mathcal{A}' covers T^d as well. Since we can reduce lengths of segments for every $i \in [d]$, we get a contradiction with the minimality of ε .

Remark 1. Clearly, this proof implies Lemma 2 as well. Indeed, if for a given $\varepsilon \in (0, 1)$, there are no positive integers $s, t \leq \mu(d; \varepsilon)$, such that $\varepsilon = s/t$, then we can reduce edges of ‘sub-cubes’ using the above construction. Length of new sides of cubes will be equal to the desired $\frac{s}{t}$, where $s, t \in \mathbb{N}, t \leq \mu(d; \varepsilon)$. Moreover, the vertices $\{x_1^0, \dots, x_d^0\}$ of these cubes will be exactly in the points of the lattice $\{(j_1 \frac{s}{t}, \dots, j_d \frac{s}{t}), j_1, \dots, j_d \in \mathbb{Z}_+\}$. Lemma 2 follows since $\mu(d; \varepsilon) = \mu(d; s/t)$.

5 Proof of Theorem 2

Due to (1), it is sufficient to prove that, for $\varepsilon = \frac{1}{r+1/(r^2+r)}$, there exists a covering by $r^3 + r^2 + r + 1$ cubes. By Lemma 2, we should construct a covering of $[\mathbb{Z}/t\mathbb{Z}]^3$ consisting of $t = r^3 + r^2 + r + 1$ ‘sub-cubes’ with edges of size $s = r^2 + r + 1$.

The main idea here is the same as in [1]: to take each next ‘sub-cube’ shifted relative to the previous one by a fixed integer vector v .

More formally, let us say that the ‘sub-cube’ $\{(x_1, x_2, x_3) : x_i^0 \leq x_i \leq x_i^0 + s - 1 \pmod t\}$ has the *base vertex* (x_1^0, x_2^0, x_3^0) . It remains to notice that the ‘sub-cubes’ with base vertices $(i, ri, (r^2+r+1)i)$ for $i \in \{0, 1, \dots, t-1\}$ cover the torus $[\mathbb{Z}/t\mathbb{Z}]^3$. The latter clearly follows (the points of the lattice with the same first coordinate are covered by s sequential cubes since each move of the first coordinate of the base vertex equals 1, and there are exactly $t - 1$ moves) from the fact that, in dimension 2, the squares with base vertices $(ri, (r^2 + r + 1)i)$ for $i \in \{0, 1, \dots, s - 1\}$ cover $[\mathbb{Z}/t\mathbb{Z}]^2$.

Theorem 5 can be proven in the same way by choosing the same shift vector, i.e. $(1, r, r^2 + r + 1)$.

Due to Lemma 1, Theorem 2 and Theorem 3, to prove Theorem 1, it is sufficient to find $\mu(3; 2/3)$ and $\mu(3; 3/5)$. The proof of $\mu(3; 2/3) = 5$ is constructive since it equals to the lower bound. We skip here the proof of $\mu(3; 3/5) = 8$ due to the space constraints.

Proofs of new lower bounds in Theorems 3 and 4 use completely different ideas. The main properties of possible coverings in smaller number of cubes are

1. absence of cubes with the base vertices having at least one same coordinate,
2. such coverings are ‘very close’ to each other (we may get one covering from another one by small shifts of sub-cubes).

Having this, it is not hard to prove that there are no such coverings.

Acknowledgements. The study was supported by the Russian Federation Government (Megagrant number 075-15-2019-1926).

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New Bounds on k -Geodetic Digraphs and Mixed Graphs

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Abstract. We study a generalisation of the degree/girth problem to the setting of directed and mixed graphs. We say that a mixed graph or digraph G is k -geodetic if there is no pair of vertices u, v such that G contains distinct non-backtracking walks of length $\leq k$ from u to v . The order of a k -geodetic mixed graph with minimum undirected degree r and minimum directed out-degree z in general exceeds the *mixed Moore bound* $M(r, z, k)$ by some small *excess* ϵ . Bannai and Ito proved that there are no non-trivial undirected graphs with excess one. In this paper we investigate the structure of digraphs with excess one and derive results on the permutation structure of the *outlier function* that rules out the existence of certain digraphs with excess one. We also present strong bounds on the excess of k -geodetic mixed graphs and show that there are no k -geodetic mixed graphs with excess one for $k \geq 3$.

Keywords: Excess · Geodeticity · Digraph · Mixed graph

1 The Degree/Geodeticity Problem

The well-known degree/girth problem asks for the smallest possible order of a d -regular graph G with given girth g . For odd girth $g = 2k + 1$, counting the vertices in a tree of depth k (called a *Moore tree*) rooted at an arbitrary vertex u of G shows that the order of G is bounded below by the *undirected Moore bound* $1 + d + d(d - 1) + \dots + d(d - 1)^{k-1}$.

A walk W in a graph is *non-backtracking* if it does not include a sub-walk of the form $u \sim v \sim u$, i.e. the same edge cannot appear in consecutive positions in W in opposite directions. We define a graph G to be k -geodetic if for any pair of vertices u, v of G there is at most one non-backtracking walk from u to v with length $\leq k$; this is equivalent to all of the vertices in any Moore tree of G with depth k being distinct, so G is k -geodetic if and only if it has girth $\geq 2k + 1$. It is easily seen that the Moore bound is attained by a graph G if and only if G is d -regular, has diameter k and is k -geodetic.

A graph that attains the Moore bound is a *Moore graph*. It is known that for odd g this bound can be met only if $d = 2$ (with the cycle C_{2k+1} being the extremal graph), $k = 1$ (complete graphs K_{d+1}) or $k = 2$ and $d \in \{3, 7, 57\}$. There exist Moore graphs with $(d, k) = (3, 2)$ (the Petersen graph) and $(d, k) =$

$(7, 2)$ (the Hoffman-Singleton graph), but the existence of a Moore graph with $(d, k) = (57, 2)$ is a famous unsolved problem. In general then the order of a k -geodetic graph with degree d will exceed the Moore bound by some (hopefully small) excess ϵ . Bannai and Ito showed in [1] using spectral theory that the only undirected graphs with excess one are the cycles. A survey of the degree/girth problem is given in [4].

In this paper we discuss one of several possible extensions of this problem to directed graphs and mixed graphs. A mixed graph is a graph containing both undirected links and directed arcs (allowing either the edge set or arc set to be empty, we can construe undirected and directed graphs as special cases of mixed graphs); we do not allow a mixed graph to contain loops or parallel edges and arcs. By a small abuse of notation, we define a *mixed path* of length ℓ in a mixed graph G to be a sequence $u_0, e_0, u_1, e_1, \dots, e_{\ell-1}, u_\ell$ such that for $0 \leq i \leq \ell$ each u_i is a vertex of G and for $0 \leq i \leq \ell - 1$ each e_i is either an edge $u_i \sim u_{i+1}$ or an arc $u_i \rightarrow u_{i+1}$, such that for no $0 \leq i \leq \ell - 2$ do we have $e_i = e_{i+1}$. The mixed graph G is *k -geodetic* if there do not exist vertices u, v of G with distinct mixed paths of length $\leq k$ from u to v ; this is equivalent to all vertices in a *mixed Moore tree* being distinct.

Recently in [7] Sillasen posed the following question, which we call the *degree/geodeticity problem*.

Question 1 (Degree/geodeticity problem). What is the smallest possible order of a k -geodetic digraph with minimum out-degree d ?

We call an extremal digraph for this problem a (d, k) -*geodetic-cage* and we denote the order of this digraph by $N(d, k)$. It is easily seen that the order of such a digraph is bounded below by the directed Moore bound $M(d, k) = 1 + d + d^2 + \dots + d^k$. However, it is known that there are no interesting digraphs that meet this lower bound [2]. We define a $(d, k; +\epsilon)$ -digraph to be a k -geodetic digraph with minimum out-degree d and order $M(d, k) + \epsilon$.

2 Results for Digraphs

Miller, Miret and Sillasen proved the following theorems for digraphs with excess $\epsilon = 1$. A digraph G is *diregular* if there exists d such that every vertex of G has both in- and out-degree d .

Theorem 1 [7]. *There are no diregular $(2, k; +1)$ -digraphs for $k \geq 2$.*

Theorem 2 [5]. *All $(d, k; +1)$ -digraphs are diregular. There are no $(d, k; +1)$ -digraphs for $k = 2$ and $d \geq 8$ or for $k = 3, 4$ and $d \geq 3$.*

The author extended these results in the papers [9–11] as follows.

Theorem 3. *For $k \geq 3$ there are no $(2, k; +2)$ -digraphs (diregular or otherwise) and no diregular $(2, k; +3)$ -digraphs. We have $N(2, 2) = 9$ and there are two $(2, 2)$ -geodetic-cages up to isomorphism, which are shown in Fig. 1.*

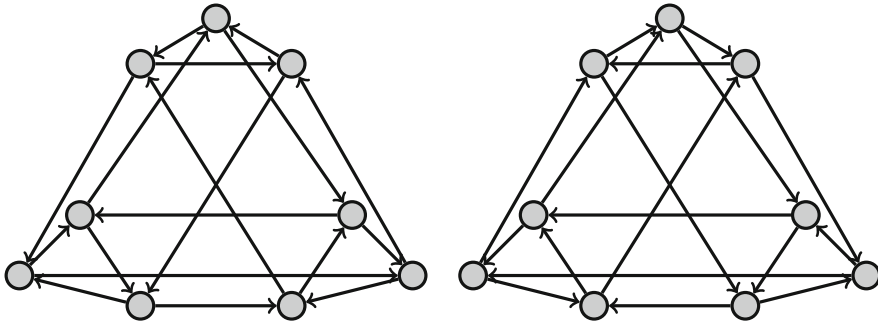


Fig. 1. The two (2,2)-geodetic-cages

A natural next step is to investigate whether there exist $(3, k; +1)$ -digraphs. Any such digraph must satisfy a strong structural lemma.

Lemma 1. *If u and v are any two distinct vertices of a $(3, k; +1)$ -digraph, then u and v have at most one common out-neighbour and at most one common in-neighbour.*

Using Lemma 1, it is elementary to show that there are no $(3, 2; +1)$ -digraphs. We can also deduce non-existence for other values of d and k by analysing the automorphism group of a $(d, k; +1)$ -digraph. In [8] in the context of the degree/diameter problem Sillarsen deduces information on the possible fixed sets of a non-identity automorphism of a digraph with order one less than the Moore bound; this analysis carries through for digraphs with excess one. Combined with the result that there are no $(2, k; +1)$ -digraphs [7], this restricts the possible sets of fixed points of an automorphism of a $(d, k; +1)$ -digraph to one of four possible forms.

Lemma 2. *If G is a $(d, k; +1)$ -digraph and ϕ is a non-identity automorphism of G , then the subdigraph $\text{Fix}(\phi)$ of G induced by the fixed points of ϕ is either the null digraph, a pair of isolated vertices, a directed $(k + 2)$ -cycle or a $(d', k; +1)$ -digraph, where $3 \leq d' \leq d - 1$.*

For every vertex u of a $(d, k; +1)$ -digraph there is a unique vertex $o(u)$ that lies at distance $> k$ from u ; this vertex is called the *outlier* of u and o is the *outlier function* of G . In fact the function o is also a digraph automorphism [7]; necessarily o is fixed-point-free. We make a definition that will help us to analyse the structure of the outlier automorphism.

Definition 1. *The order of a vertex u of a $(d, k; +1)$ -digraph G is the smallest value of $r \geq 1$ such that $o^r(u) = u$. The index $\omega(G)$ of G is the smallest vertex order in G . G is outlier-regular if its outlier function is a regular permutation.*

Applying Lemma 2 to a $(3, k; +1)$ -digraph G tells us about the subdigraph of G induced by the vertices with order equal to the index.

Corollary 1. *If G is a $(3, k; +1)$ -digraph, then one of the following holds:*

- G is outlier-regular,
- the outlier function o of G contains a unique transposition, or
- the vertices of G with order $\omega(G)$ form a directed $(k + 2)$ -cycle.

If the second case holds, then $k \not\equiv 3, 5 \pmod{6}$ and if $k \equiv 0, 2 \pmod{6}$, then G contains two vertices of order two, with all other vertices of G having order six. If the third case holds, then $k + 2$ divides $\frac{M(3,k)-k-1}{2}$. It follows that if $k \geq 2$ is such that i) $k \equiv 3$ or $5 \pmod{6}$, ii) $k + 2$ does not divide $\frac{M(3,k)-k-1}{2}$ and iii) $M(3, k) + 1$ is prime, then there is no $(3, k; +1)$ -digraph.

Corollary 2. *There are no $(3, 3; +1)$, $(3, 15; +1)$ - or $(3, 63; +1)$ -digraphs.*

Applying these ideas to $(d, 2; +1)$ -digraphs also allowed us to close the open cases for $k = 2$ in Theorem 2.

Theorem 4. *If $d, k \geq 2$, then any $(d, k; +1)$ -digraph has $d \geq 3$ and $k \geq 5$.*

A spectral argument also shows that there are no involutory digraphs with excess one, i.e. any $(d, k; +1)$ -digraph contains a vertex with order at least three. A computer search by the authors has also identified further geodetic cages for small d and k .

Theorem 5. *$N(2, 3) = 20$ and $N(3, 2) = 16$. Up to isomorphism there are two $(2, 3)$ -geodetic-cages and the $(3, 2)$ -geodetic-cage is unique.*

All known Moore graphs, as well as many cages, are vertex-transitive. It is therefore of interest to ask if there exist vertex-transitive digraphs with small excess. We derived the following divisibility condition for the existence of a vertex-transitive $(d, k; +1)$ -digraph.

Theorem 6. *Let G be a vertex-transitive $(d, k; +1)$ -digraph. Then $(k+1)$ divides $2d + d^2 + d^3 + \dots + d^{k+1}$ and $(k + t)$ divides $(M(d, k) + 1)(d^t - d^{t-1})$ for all $2 \leq t \leq k - 1$.*

Values of d and k that satisfy this condition are very rare. In particular, it follows from Theorem 4 that there are no vertex-transitive $(d, k; +1)$ -digraphs in the range $2 \leq d \leq 12$ and $2 \leq k \leq 10000$. Similar divisibility conditions hold for the broader range $1 \leq \epsilon < d$ if we assume the stronger property of arc-transitivity.

3 Bounds for Mixed Graphs

We can extend the degree/geodeticity problem to mixed graphs in a straightforward way. The Moore bound $M(r, z, k)$ for k -geodetic mixed graphs with minimum undirected degree r and minimum directed out-degree z was derived in [3]. We define an $(r, z, k; +\epsilon)$ -graph to be a k -geodetic mixed graph with undirected

degree r , directed out-degree z and order exceeding the Moore bound by excess ϵ . It is known that there are no mixed Moore graphs for $k \geq 3$ [6].

A mixed graph is *totally regular* if the subgraph induced by the edges of G is regular and the subdigraph induced by the arcs of G is diregular. The structure of a mixed graph is much easier to analyse if it is totally regular.

Theorem 7 [12]. *All $(r, z, 2; +1)$ -graphs are totally regular.*

Using counting arguments we prove a strong lower bound on the excess of totally regular mixed graphs.

Theorem 8. *For $k \geq 3$, the excess ϵ of a totally regular $(r, z, k; +\epsilon)$ -graph satisfies*

$$\epsilon \geq \frac{r}{\phi} \left[\frac{\lambda_1^{k-1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1} - 1}{\lambda_2 - 1} \right],$$

where $\phi = \sqrt{(r + z - 1)^2 + 4z}$, $\lambda_1 = \frac{1}{2}(r + z - 1 + \phi)$ and $\lambda_2 = \frac{1}{2}(r + z - 1 - \phi)$.

Trivially this bound shows that for $k \geq 4$ there are no mixed graphs with excess one. A closer analysis of the structure of $(r, z, 3; +1)$ -graphs yields the following result.

Corollary 3. *There are no totally regular $(r, z, k; +1)$ -graphs for $k \geq 3$.*

It is much more difficult to rule out the existence of mixed graphs with small excess that are not totally regular. However, we found recently that Theorem 8 can be generalised to all mixed graphs at the expense of reducing the bound by a factor $\frac{z}{2r+3z}$.

Theorem 9. *For $k \geq 3$, the excess of any $(r, z, k; +\epsilon)$ -graph satisfies*

$$\epsilon \geq \frac{rz}{(2r + 3z)\phi} \left[\frac{\lambda_1^{k-1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{k-1} - 1}{\lambda_2 - 1} \right].$$

This new bound allows us to expand Corollary 3 to rule out the existence of all k -geodetic mixed graphs with excess one for $k \geq 3$, thereby extending the result of Bannai and Ito [1].

Corollary 4. *For $r, z \geq 1$, if G is an $(r, z, k; +1)$ -graph, then $k = 2$, G is totally regular and either i) $r = 2$, ii) $4r + 1 = c^2$ for some $c \in \mathbb{N}$ and $c | (16z^2 - 24z + 25)$, or iii) $4r - 7 = c^2$ for some $c \in \mathbb{N}$ and $c | (16z^2 + 40z + 9)$.*

These bounds, combined with computer search, produce the new record mixed graphs in Table 1.

Table 1. Smallest $(r, z, k; +\epsilon)$ -graphs for given r, z and k (* = smallest known)

r	z	k	Moore bound	Order	Excess	Comment
1	1	3	11	16	5	Cages classified
1	1	4	19	30	11	Cages classified
1	1	5	32	54*	22*	No graphs of order less than 50
2	1	2	11	12	1	Cayley graph of D_{12}
2	1	3	28	48*	20*	No graphs of order less than 32
2	2	2	19	21	2	Cages not classified

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Counting C_k -free Orientations of $G(n, p)$

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Abstract. We determine the order of growth, up to polylogarithmic factors, of the number of orientations of the binomial random graph $G(n, p)$ containing no directed cycle of length k for every $k \geq 3$. This solves a conjecture of Kohayakawa, Morris and the last two authors.

Keywords: Orientations · Random graphs · Directed cycles

1 Introduction

An orientation \vec{H} of a graph H is an oriented graph obtained by assigning an orientation to each edge of H . The study of the number of \vec{H} -free orientations of a graph G , denoted by $D(G, \vec{H})$, was initiated by Erdős [8], who posed the problem of determining $D(n, \vec{H}) := \max \{D(G, \vec{H}) : |V(G)| = n\}$. For tournaments, this problem was solved by Alon and Yuster [2], who proved that $D(n, T_k) = 2^{\text{ex}(n, K_k)}$ holds for any tournament T_k and all sufficiently large $n \in \mathbb{N}$.

Let C_ℓ° denote the directed cycle of length ℓ . Bucić and Sudakov [6] determined $D(n, C_{2k+1}^\circ)$ for every $k \geq 1$ as long as n is sufficiently large, extending the proof in [2]. Another extension of the results in [2] was given by Araújo, Botler, and the last author [3] who determined $D(n, C_3^\circ)$ for every $n \in \mathbb{N}$ (see also [5]).

In the context of random graphs, Allen, Kohayakawa, Parente, and the last author [1] investigated the problem of determining the typical number of C_k° -free orientations of the Erdős–Rényi random graph $G(n, p)$. They proved that, for every $k \geq 3$, with high probability as $n \rightarrow \infty$ we have $\log_2 D(G(n, p), C_k^\circ) =$

M. Campos—Partially supported by CNPq.

G. O. Mota—Partially supported by CNPq (306620/2020-0, 428385/2018-4) and FAPESP (2018/04876-1, 2019/13364-7). FAPESP is the São Paulo Research Foundation. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brazil (CAPES), Finance Code 001.

$o(pn^2)$ for $p \gg n^{-1+1/(k-1)}$, and $\log_2 D(G(n, p), C_k^\circ) = (1 + o(1))p \binom{n}{2}$ for $n^{-2} \ll p \ll n^{-1+1/(k-1)}$. This result was improved in the case of triangles by Kohayakawa, Morris and the last two authors [7], who proved, among other things, the following result¹.

Theorem 1 [7, Theorem 1.2]. *If $p \gg n^{-1/2}$, then, with high probability as $n \rightarrow \infty$, $\log D(G(n, p), C_3^\circ) = \tilde{\Theta}(n/p)$.*

A first step towards determining $\log D(G(n, p), C_k^\circ)$ for $k \geq 4$ was also given in [7], where it was proved that $\log D(G(n, p), C_k^\circ) = \tilde{O}(n/p)$. Moreover, they proved that a natural generalization of the lower bound construction used in the proof of Theorem 1 gives

$$\log D(G(n, p), C_k^\circ) = \Omega \left(\frac{n}{p^{1/(k-2)}} \right) \tag{1}$$

with high probability when $p \gg n^{-1+1/(k-1)}$. They conjectured that this lower bound is sharp up to polylogarithmic factors, and we confirm this conjecture by proving the following result.

Theorem 2. *Let $k \geq 3$ and $p = p(n) \gg n^{-1+1/(k-1)}$. Then, with high probability as $n \rightarrow \infty$, $\log D(G(n, p), C_k^\circ) = \tilde{\Theta} \left(\frac{n}{p^{1/(k-2)}} \right)$.*

The proof of this result will be outlined in the next section, and it consists of new ideas together with a significant generalisation of the strategy from [7]. Roughly, we define a pseudorandomness condition on the number of directed r -paths between small sets for $1 \leq r \leq k - 2$, and split into two cases according to whether the condition holds. The graph container method (introduced by Kleitman and Winston [9], and Sapozhenko [10]) is used when the pseudorandomness condition is satisfied, that is the orientation contains many directed $(k - 2)$ -paths. However, most of the work is in the case where the pseudorandomness condition does not hold and the graph container lemma does *not* apply. In this situation, we will show a way to efficiently “encode” the number of orientations that do not extend many directed $(r - 1)$ -paths to directed r -paths, for $2 \leq r \leq k - 2$.

2 Outline of the Proof of Theorem 2

In this section we give an outline of the proof of Theorem 2. But first we present a quick sketch of the proof of Theorem 2 for C_3° as proved in [7].

The Proof in [7]. The idea is to obtain, by induction on the number of vertices, a general bound on the number of C_3° -free orientations of an n -vertex graph G which only depends on n and $\alpha(G)$. More precisely, one gets the bound

$$\binom{n}{\leq \alpha(G)}^{2n}. \tag{2}$$

¹ The $\tilde{\Theta}(\cdot)$ and $\tilde{O}(\cdot)$ notation are analogous to Θ and O notation but with polylogarithmic factors omitted. From now on \log will denote the natural logarithm for convenience.

In order to obtain such bound, let G be a graph on vertex set V and consider $v \in V$. Let $H = G[V \setminus \{v\}]$ and suppose that the number of C_3° -free orientations of H is

$$\binom{n-1}{\leq \alpha(H)}^{2n-2}. \tag{3}$$

Then, for each C_3° -free orientation \vec{G} of G , pick $\vec{T} \subset E(\vec{G}) \setminus E(\vec{H})$ minimal such that \vec{G} is the only C_3° -free orientation of G containing $\vec{T} \cup \vec{H}$. The key observation is that, by the minimality of T , the vertex sets $T^+ := N_{\vec{T}}^+(v)$ and $T^- := N_{\vec{T}}^-(v)$ are independent sets in H , so there are at most $\binom{n}{\leq \alpha(G)}^2$ choices for \vec{T} . This together with (3) and $\alpha(H) \leq \alpha(G)$ completes the proof. Since $\alpha(G(n, p)) \leq \frac{3 \log n}{p}$ holds with high probability, the bound (2) implies Theorem 2 for C_3° .

We will generalise the ideas depicted above in two ways, which will be described in the next two subsections. It will also be the case that directed paths of length $k - 2$ starting and ending in the neighbourhood of a vertex play a key role. For this reason, we fix $\ell = k - 2 \geq 1$ and avoid copies of $C_{\ell+2}^\circ$.

Pseudorandomness. We start by defining a ‘‘pseudorandom’’ oriented graph property, and we proceed to separately count the $C_{\ell+2}^\circ$ -free orientations depending on whether the already oriented subgraph \vec{H} is pseudorandom. Then we use the randomness of $G(n, p)$ in a stronger way, using a bit of the randomness in each step of the induction.

Let us define the pseudorandom property we mentioned in the previous paragraph. We write \vec{P}_r for the directed path with r edges and, given a oriented graph \vec{G} , we denote by \vec{G}^r the digraph such that (u, v) is an edge whenever there is a \vec{P}_r from u to v in \vec{G} . We say an oriented graph \vec{G} is r -locally dense (see Definition 1), if

$$e_{(\vec{G}[V \setminus X])^r}(A, B) \geq \frac{1}{2} p^{\frac{\ell-r+1}{\ell}} |A||B|,$$

for all disjoint sets $A, B, X \subset V(G)$ of size roughly $\frac{\log n}{p}$, where $e_{(\vec{G}[V \setminus X])^r}(A, B)$ denotes the number of edges between A and B in the digraph $(\vec{G}[V \setminus X])^r$.

Observe that being 1-locally dense does not depend on the orientation of the graph (and the set X plays no role in this case), i.e., being 1-locally dense is a pseudorandom property that depends only on the underlying graph G . In particular any orientation of $G(n, p)$ is 1-locally dense with high probability and one may think of this property as a strengthening of the fact that $\alpha(G(n, p)) \leq \frac{3 \log n}{p}$.

Sketch of the Proof. We will count separately the orientations \vec{G} which are ℓ -locally dense and the orientations which are not r -locally dense but are $(r - 1)$ -locally dense for some $2 \leq r \leq \ell$. In the former case (see Lemma 2 ii), we proceed similarly to the proof in [7]: let $v \in V$ and put $H = G[V \setminus \{v\}]$. Let $\vec{T} \subset E(\vec{G}) \setminus E(\vec{H})$ be minimal such that \vec{G} is the only orientation of G containing $\vec{T} \cup \vec{H}$. Note that $T^+ := N_{\vec{T}}^+(v)$ and $T^- := N_{\vec{T}}^-(v)$ are independent sets in \vec{H}^ℓ .

Since \vec{G} is ℓ -locally dense, the density of edges in \vec{H}^ℓ is of order $p^{1/\ell}$. This allows us to prove, using the graph container lemma, that the largest independent set in $N_{\vec{G}}(v)$ has size roughly $(\log n)/p^{1/\ell}$.

In the case where \vec{G} is not r -locally dense but is $(r - 1)$ -locally dense (see Lemma 2 (i)), we use the fact that there are disjoint sets $A, B \subset V$ and $A' \subset A$, with $|A'| \geq |A|/2$, such that for all $a \in A'$ it holds that $\vec{d}_{\vec{G}^r}(a, B) \leq p^{\frac{\ell-r+1}{\ell}}|B|$, where $\vec{d}(a, B)$ denotes $d^+(a, B) + d^-(a, B)$ (in this outline we assume $X = \emptyset$ for simplicity). Put $\vec{H} = \vec{G}[V \setminus A']$ and note that, since \vec{G} is $(r - 1)$ -locally dense, \vec{H}^{r-1} has many edges between any two “sufficiently large” sets². Now given $a \in A'$ we may choose $S^+ \subset V(H) \cap N_{\vec{G}}^+(v)$, the set of all $x \in N_{\vec{G}}^+(v)$ such that $d_{\vec{H}^{r-1}}^+(x, B) \geq d_{\vec{H}^{r-1}}^-(x, B)$ and $S^- \subset V(H) \cap N_{\vec{G}}^-(v)$, the set of all $x \in N_{\vec{G}}^-(v)$ such that $d_{\vec{H}^{r-1}}^-(x, B) > d_{\vec{H}^{r-1}}^+(x, B)$. We claim that $|S^+|, |S^-| \leq Cp^{-1/\ell}$. Indeed, for “almost” all $x \in S^+, S^-$ we have $\vec{d}_{\vec{H}^{r-1}}(x, B) \geq p^{\frac{\ell-r+2}{\ell}}|B|$, so if $|S^+|$ or $|S^-|$ is larger than $Cp^{-1/\ell}$ then $\vec{d}_{\vec{G}^r}(a, B)$ would be too large. Then, S^+ and S^- are small sets that fully determine the orientation of all edges of G between a and $V(H)$.

We remark that our proof depends on using the randomness of the edges between A' and $V(H)$ after choosing the orientation \vec{H} . The problem is that usually one should have to reveal all of the edges $G(n, p)$ before choosing the orientation. To circumvent this fact we will actually bound the expected number of 1-locally dense orientations of $G(n, p)$. When estimating this expectation we will be able to split the expectation in a way that makes possible to apply the induction and use part of the randomness after orienting part of the edges.

3 Forbidding Directed Cycles

In this section, our goal is to count orientations of $G(n, p)$ containing no copies of $C_{\ell+2}^\circ$. Recall that given an oriented graph \vec{G} , we denote by \vec{G}^r the digraph such that (u, v) is an edge whenever there is a \vec{P}_r from u to v in \vec{G} . In what follows we fix $\ell \geq 1$. We prove the following result, that implies Theorem 2.

Theorem 3. *With high probability, $G(n, p)$ admits at most $\exp\left(\frac{32\ell n(\log n)^2}{p^{1/\ell}}\right)$ $C_{\ell+2}^\circ$ -free orientations.*

We postpone the proof of Theorem 3 to the end of the section to make the required preparation. Throughout the rest of the paper, ℓ, n and p will be fixed, and all graphs will have vertex set contained in $[n]$. Furthermore, put $\alpha = 2^6(\log n)/p$. The following definition will be used to “encode” orientations of $C_{\ell+2}^\circ$ -free graphs.

² This is the point where we need X in the definition of r -locally dense graphs, because we want $\vec{H}^{r-1} = \vec{G}[V \setminus A']^{r-1}$ to have many edges between pairs of sets.

Definition 1. Given $1 \leq r \leq \ell$, an oriented graph \vec{G} is r -locally dense if for every pairwise disjoint sets A, B, X of $V(G)$ such that

$$|A| = \alpha, \quad r\alpha \leq |B| \leq \ell\alpha, \quad \text{and} \quad |X| \leq (\ell + 1 - r)\alpha, \quad (4)$$

we have

$$\vec{e}_{(\vec{G} \setminus X)_r}(A, B) \geq p^{\frac{\ell-r+1}{\ell}} |A||B|/2. \quad (5)$$

Otherwise, the orientation \vec{G} is called r -locally-sparse.

Even though the following lemma is a trivial application of Chernoff’s inequality (together with the fact that the definition of 1-locally-dense depends solely on the underlying undirected graph and not on the orientation of the edges), it will be crucial.

Lemma 1. *With high probability every orientation of $G(n, p)$ is 1-locally-dense.*

Given a graph G and an induced subgraph H of G , an orientation \vec{G} of G is an *extension* of an orientation \vec{H} of H if $E(\vec{H}) \subset E(\vec{G})$. Furthermore, we say that \vec{G} *extends* \vec{H} . Due to Lemma 1, we may restrict ourselves to counting 1-locally-dense, $C_{\ell+2}^\circ$ -free orientations in the rest of the paper.

Definition 2. Let $G = (V, E)$ be a graph and let \vec{H} be an orientation of an induced subgraph H of G . We denote by $\mathcal{D}_r(G, \vec{H})$ (resp. $\mathcal{S}_r(G, \vec{H})$) the set of all 1-locally-dense, $C_{\ell+2}^\circ$ -free orientations of G that extend \vec{H} and are r -locally-dense (resp. r -locally-sparse). For convenience, we also write $\mathcal{D}_r(G)$ for $\mathcal{D}_r(G, \emptyset)$ and $\mathcal{S}_r(G)$ for $\mathcal{S}_r(G, \emptyset)$.

In view of Lemma 1 and using the language of Definition 2, our goal is to estimate $|\mathcal{D}_1(G(n, p))|$.

If $\vec{G} \in \mathcal{S}_r(G)$, then let A, B and X be pairwise disjoint sets of $V(G)$ satisfying (4) such that (5) fails to hold. In this case, there exists $A' \subset A$ with $|A'| = |A|/2$ such that $\vec{d}_{(\vec{G} \setminus X)_r}(a, B) \leq p^{\frac{\ell-r+1}{\ell}} |B|$ for every $a \in A'$. This motivates the following definition. Recall that $\alpha = 2^6(\log n)/p$.

Definition 3. Let G be a graph and H be an induced subgraph of G , and let B, X be disjoint subsets of $V(H)$. The triple (H, B, X) is a G -root if

$$|V(G) \setminus V(H)| = \alpha/2, \quad r\alpha \leq |B| \leq \ell\alpha, \quad \text{and} \quad |X| \leq (\ell + 1 - r)\alpha.$$

Given a G -root (H, B, X) , an orientation \vec{G} of G is an (r, B, X) -sparse extension of an orientation \vec{H} of H if for every $a \in V(G) \setminus V(H)$, we have $\vec{d}_{(\vec{G} \setminus X)_r}(a, B) \leq p^{\frac{\ell-r+1}{\ell}} |B|$.

We let $\mathcal{S}_r(G, \vec{H}, B, X)$ denote the set of all orientations of \vec{G} that are (r, B, X) -sparse extensions of \vec{H} . Taking $\vec{H} = \vec{G} \setminus A'$ in the previous discussion, it follows that

$$\mathcal{S}_r(G) = \bigcup_{(H, B, X)} \bigcup_{\vec{H} \in \mathcal{D}_1(H)} \mathcal{S}_r(G, \vec{H}, B, X), \quad (6)$$

where the first union is over all G -roots.

Given a graph H and a set A disjoint from $V(H)$, we define $G(H, A, p)$ as the graph G with $V(G) = V(H) \cup A$ such that $E(G[V(H)]) = E(H)$ and we add each edge of $\binom{V(G)}{2} \setminus \binom{V(H)}{2}$ with probability p independently at random. We are now ready to state the main tool for the proof of Theorem 3, whose proof is omitted in this extended abstract.

Lemma 2. *Let H be a graph, A be a set disjoint from $V(H)$ with $|A| = \alpha/2$ and put $G := G(H, A, p)$. If (H, B, X) is a G -root, then for every orientation \vec{H} of H ,*

$$\begin{aligned} (i) \quad & \mathbb{E} \left[\left| \mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G, \vec{H}, B, X) \right| \right] \leq \exp \left(8A|p^{-\frac{1}{2}} \log n \right), \\ (ii) \quad & \mathbb{E} \left[\left| \mathcal{D}_\ell(G, \vec{H}) \right| \right] \leq \exp \left(4(\ell + 1)|A|p^{-\frac{1}{2}} (\log n)^2 \right). \end{aligned}$$

Using this result, we can prove Theorem 3.

Proof. Recall that n is fixed, and put $z := \exp \left(8(\ell + 1)p^{-\frac{1}{2}} (\log n)^2 \right)$. We will show by induction on $n' \leq n$ that

$$\mathbb{E} \left[\left| \mathcal{D}_1(G(n', p)) \right| \right] \leq z^{n'}, \tag{7}$$

which together with Lemma 1 and Markov’s inequality implies the desired result. To do so, we start with the following claim.

Claim. Let $n' \leq n$ and ℓ be positive integers. If $p \gg \log n/n$, then $\mathbb{E} \left[\left| \mathcal{D}_1(G(n', p)) \right| \right] \leq z^{\alpha/2} \cdot \mathbb{E} \left[\left| \mathcal{D}_1(G(n' - \alpha/2, p)) \right| \right]$.

Proof (of claim). Let $G = G(n', p)$. Notice that, since every 1-locally-dense orientation is either ℓ -locally dense or admits a minimal $2 \leq r \leq \ell$ for which it is r -locally sparse, we have

$$\mathcal{D}_1(G) = \mathcal{D}_\ell(G) \cup \bigcup_{r=2}^{\ell} (\mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G)), \tag{8}$$

so it suffices to bound the expected sizes of the sets in the right-hand side. Let A be a set of size $\alpha/2$, and let $H = G[V \setminus A]$ and $F = G \setminus \binom{V \setminus A}{2}$. Observe that F and H are independent. Therefore,

$$\mathbb{E} \left[\left| \mathcal{D}_\ell(G) \right| \right] = \mathbb{E}_H \left[\mathbb{E}_F \left[\sum_{\vec{H}} \left| \mathcal{D}_\ell(G, \vec{H}) \right| \mid H \right] \right] = \mathbb{E}_H \left[\sum_{\vec{H}} \mathbb{E}_F \left[\left| \mathcal{D}_\ell(G, \vec{H}) \right| \mid H \right] \right], \tag{9}$$

where the sums are over $\vec{H} \in \mathcal{D}_1(H)$. Conditioned on H , $F \cup H$ is distributed as $G(H, A, p)$, and so we have by Lemma 2(ii) that $\mathbb{E}_F \left[\left| \mathcal{D}_\ell(G, \vec{H}) \right| \mid H \right] \leq z^{|A|/2}$,

and hence, by (9), $\mathbb{E} \left[\left| \mathcal{D}_\ell(G) \right| \right] \leq \mathbb{E}_H \left[\sum_{\vec{H} \in \mathcal{D}_1(H)} z^{|A|/2} \right] = z^{\alpha/4} \cdot \mathbb{E} \left[\left| \mathcal{D}_1(G(n - \right.$

$\alpha/2, p)$], bounding one of the terms of the right-hand side of (8). To bound the expected sizes of the remaining sets, we proceed analogously, using (6) to bound $\mathbb{E}\left[|\mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G)|\right] \leq \mathbb{E}\left[\sum_{(H,B,X)} \sum_{\vec{H}} |\mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G, \vec{H}, B, X)|\right] = \mathbb{E}_H\left[\sum_{(H,B,X)} \sum_{\vec{H}} \mathbb{E}_F\left[|\mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G, \vec{H}, B, X)| \mid H\right]\right]$, where the sums are over all G -roots and all $\vec{H} \in \mathcal{D}_1(H)$, respectively. Applying Lemma 2(i) and the fact that there are at most $n^{3\ell\alpha/2} \leq z^{\alpha/3}$ G -roots, we have: $\mathbb{E}\left[|\mathcal{D}_{r-1}(G) \cap \mathcal{S}_r(G)|\right] \leq \mathbb{E}_H\left[\sum_{(H,B,X)} \sum_{\vec{H} \in \mathcal{D}_1(H)} z^{|A|/3}\right] = z^{\alpha/2} \cdot \mathbb{E}\left[|\mathcal{D}_1(G(n - \alpha/2), p)|\right]$, finishing the proof of the claim.

We can now prove (7). For the base cases, note that if $2 \leq n' \leq p^{-1-1/\ell}(\log n)^2$ then by an application of Chernoff: $\mathbb{E}\left[|\mathcal{D}_1(G(n', p))|\right] \leq \prod_{v \in V} \mathbb{E}\left[2^{|\mathcal{N}(v)|}\right] \leq \exp\left(8pn'^2\right) \leq z^{n'}$. Otherwise, $n' > p^{-1-1/\ell}(\log n)^2$ implies $p \gg \log n/n$, so from Claim 3 and the induction hypothesis we obtain $\mathbb{E}\left[|\mathcal{D}_1(G(n', p))|\right] \leq z^{\alpha/2}$. $\mathbb{E}\left[|\mathcal{D}_1(G(n' - \alpha/2, p))|\right] \leq z^{\alpha/2} \cdot z^{n' - \alpha/2} = z^{n'}$, proving (7) by induction. By Markov's inequality and (7), the number of 1-locally-dense orientations of $G(n, p)$ is $\exp\left(32\ell p^{-\frac{1}{\ell}} n(\log n)^2\right)$ with high probability. Moreover, by Lemma 1, with high probability every orientation of $G(n, p)$ is 1-locally-dense. This finishes the proof.

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Counting Circuit Double Covers

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Abstract. Several recent results and conjectures study counting versions of classical existence statements. We ask the same question for circuit double covers of cubic graphs. We prove an exponential bound for planar graphs: Every bridgeless cubic planar graph with n vertices has at least $(5/2)^{n/4-1/2}$ circuit double covers. The method we used to obtain this bound motivates a general framework for counting objects on graphs using linear algebra which might be of independent interest. We also conjecture that every bridgeless cubic graph has at least $2^{n/2-1}$ circuit double covers.

Keywords: Graph theory · Cycle double cover · Circuit double cover

Definition 1. Let G be a graph. A multiset¹ of circuits (resp. cycles) \mathcal{C} is a circuit (resp. cycle) double cover if every edge of G is contained in exactly two elements of \mathcal{C} . It is a k -cycle double cover if $|\mathcal{C}| \leq k$.

We want to study a way to effectively calculate the number of circuit double covers (CDCs for short) of graphs. All graphs considered in this paper are cubic, bridgeless and may contain parallel edges and loops unless noted otherwise. We count circuit double covers (circuit is a subgraph isomorphic to C_l for some $l \geq 3$) instead of cycle double covers (cycle is a subgraph with all degrees even, i.e., a union of edge-disjoint circuits) because counting the later allows to increase the number by just grouping the circuits into cycles in different ways.² The original motivation was examination of the flower snarks for which our following theorem does not apply because their embeddings have representativity at most two (as proved by Mohar and Vodopivec [2]).

¹ We must allow repetition, otherwise there would not exist any circuit (or cycle) double of C_k . For an introduction into the topic see e.g., [1].

² Given a k -cycle double cover, we can split the cycle with t circuits into two cycles and obtain 2^{t-1} different $(k+1)$ -cycle double covers.

This work was supported by grant 19-21082S of the Czech Science Foundation.

Theorem 2 (Hušek, Šámal [3]). *Every cyclically 4-edge-connected cubic graph with a closed cell embedding with representativity at least 4 and f faces³ has $2^{\Omega(\sqrt[3]{f})}$ circuit double covers.*

This study led to interesting results like an exponential lower-bound on the number of CDCs of bridgeless cubic planar graphs. We first present a general framework for computing graph parameters which might be of independent interest. Then we describe counting circuit double covers in this framework and apply linear programming to obtain the exponential lower bound.

1 The Framework

The basic idea is to construct graphs by joining gadgets and to extend the definition of a graph parameter so it can be effectively computed along this construction. It is important to note that all vertices and edges have identity – i.e., given a graph with two vertices and three parallel edges, we can tell which edge is which.

A *gadget* is a graph with some half-edges – half-edge is an edge which is incident to only one vertex in the gadget and its other end is “floating” so it can be connected to the other half-edge. The half-edges are ordered – we treat them as numbered by 1, 2, 3, . . . We identify graphs with the gadgets of size zero.

Definition 3. *A gadget g is a tuple (V, E, O, f, \prec) such that V and O are disjoint sets, E is a set, \prec is a total order on O , $(V \cup O, E, f)$ is a multigraph ($f : E \rightarrow \binom{V \cup O}{2}$ maps each edge to its endpoints), $|f(e) \cap O| \leq 1$ for all $e \in E$ and $|\{e \in E : o \in f(e)\}| = 1$ for all $o \in O$. Size of the gadget is $|O|$ and it is denoted $|g|$.*

We say k -gadget instead of gadget of size k . We denote the class of all graphs \mathcal{G} , the class of k -gadgets \mathcal{G}^k and the class of all gadgets \mathcal{G}^* . We create gadgets from other gadgets by the following three elementary types of operations defined for all $k, k' \in \mathbb{N}$:

1. *Disjoint union* $\uplus^{k,k'} : \mathcal{G}^k \times \mathcal{G}^{k'} \rightarrow \mathcal{G}^{k+k'}$.
2. *Join of half-edges*, denoted $\mathcal{J}_{i,j}^{k+2} : \mathcal{G}^{k+2} \rightarrow \mathcal{G}^k$ joining half-edges i and j .
3. *Permutation of half-edge labels*, denoted $\pi^k[\sigma] : \mathcal{G}^k \rightarrow \mathcal{G}^k$ for permutation σ .

We usually omit the k and k' as they can be inferred from context when needed.

In the rest of the paper we restrict ourselves to cubic graphs. We will still say “all graphs” and “all gadgets” but we will mean only cubic ones.

When we say just *join* (denoted $\mathcal{J} : \prod_{i \in [n]} \mathcal{G}^{k_i} \rightarrow \mathcal{G}^r$ where n is the number of input gadgets, k_i their sizes and r size of the resulting gadget), we mean function created by composing the elementary operations.

³ Such an embedding that the closure of every face is a disk and every non-contractible curve crosses at least 4 edges.

Definition 4. A function $\mathcal{P} : \mathcal{G} \rightarrow R$ such that $G_1 \cong G_2 \Rightarrow \mathcal{P}(G_1) = \mathcal{P}(G_2)$ is called a graph parameter. The set R is the range of this parameter.

We want to enrich the descriptions of graph parameters so that they capture how the parameters are computed along the gadget decomposition. We will model this using universal algebras: We first define the gadget algebra where operations are joins and then describe the parameters using homomorphisms from the gadget algebra.

The joins as defined above are defined only on gadgets of some size. This would lead to a partial algebra. To avoid this we extend the gadget algebra by a special object **None** and extend the joins to all gadgets but returning **None** on the new elements of their domains. To simplify the notation, we denote $\mathcal{G}' = \mathcal{G}^* \uplus \mathbf{None}$. More precisely we extend any join $\mathcal{J} : \prod_{i \in [n]} \mathcal{G}^{k_i} \rightarrow \mathcal{G}^r$ to a mapping $\mathcal{J} : \prod_{i \in [n]} \mathcal{G}' \rightarrow \mathcal{G}'$ such that $\mathcal{J}(x) = \mathbf{None}$ for all $x \notin \prod_{i \in [n]} \mathcal{G}^{k_i}$. This yields the same results as using the partial algebras with the strong homomorphisms as described in [4, Chap. 2].

Definition 5. The gadget algebra \mathbb{G} is an algebra whose objects are $\mathcal{G}' = \mathcal{G}^* \uplus \{\mathbf{None}\}$ and operations are the constant **None** and all the joins $\mathcal{J} : \mathcal{G}'^n \rightarrow \mathcal{G}'$.

Definition 6. The linear representation \mathcal{P} of the graph parameter \mathcal{P}' over the field \mathbb{Q} consists of

- the representation algebra \mathbb{P} ,
- an algebra homomorphism $h_{\mathcal{P}} : \mathbb{G} \rightarrow \mathbb{P}$, and
- a mapping $f_{\mathcal{P}} : \mathbb{P} \rightarrow \mathbb{Q}$.

The representation algebra \mathbb{P} is a vector space over \mathbb{Q} additionally equipped with operations corresponding to those of the gadget algebra \mathbb{G} . We let $\mathcal{J}_{\mathcal{P}}$ denote the operation corresponding to the join \mathcal{J} . We fix an orthonormal basis e_b of \mathbb{P} and index it by a set $B_{\mathcal{P}}$ so we can define support $\text{supp}(v) \subset B_{\mathcal{P}}$. We also treat \mathcal{P} as a graph parameter $\mathcal{G} \rightarrow \mathbb{Q}$ defined $\mathcal{P}(G) = f_{\mathcal{P}}(h_{\mathcal{P}}(G))$. We require that

1. $h_{\mathcal{P}}(\mathbf{None}) = 0$,
2. $\text{supp}(h_{\mathcal{P}}(g_1)) \cap \text{supp}(h_{\mathcal{P}}(g_2)) = \emptyset$ for all gadgets g_1, g_2 of different sizes,
3. all the functions $\mathcal{J}_{\mathcal{P}}$ are linear in all their arguments,
4. $f_{\mathcal{P}}$ is also a linear function and
5. $\mathcal{P}(G) = \mathcal{P}'(G)$ for all $G \in \mathcal{G}$.

We call elements of the set $B_{\mathcal{P}}^k = \bigcup_{g \in \mathcal{G}^k} \text{supp}(h_{\mathcal{P}}(g)) \subset B_{\mathcal{P}}$ the k -boundaries and the elements of $O_{\mathcal{P}}^k = \mathbb{Q}^{B_{\mathcal{P}}^k}$ (as a subspace of \mathbb{P}) k -multiplicity vectors. We omit the subscript \mathcal{P} whenever possible. We say that the representation is finite if all the sets $B_{\mathcal{P}}^k$ are finite.

Observation 7. Let \mathcal{P} be a linear representation and let S be some set of k -gadgets. Define matrix $A_{g_1, g_2}^k = \mathcal{P}(\mathcal{J}_g^k(g_1, g_2))$ where $\mathcal{J}_g^k : \mathcal{G}^k \times \mathcal{G}^k \rightarrow \mathcal{G}$ is gluing⁴ and $g_1, g_2 \in S$. Then $\text{rank}(A^k) \leq |B_{\mathcal{P}}^k|$ for every linear representation \mathcal{P}' of the same graph parameter.

⁴ It joins the half-edges with the same labels. The result is a gadget of the size zero – i.e., a graph.

On the other hand we do not expect this lower bound to be always tight but leave this as an open problem:

Problem 8. *Given arbitrary graph parameter with a finite linear representation over \mathbb{Q} . Is there such a representation \mathcal{P} of this parameter that $\text{rank}(A^k) = |B_{\mathcal{P}}^k|$ for $S = \mathcal{G}^k$ and all k ?*

Problem 9. *Characterize graph parameters which have a finite linear representation over \mathbb{Q} , \mathbb{R} or \mathbb{C} . What if we restrict the growth of $|B^k|$?*

2 The Number of CDCs as a Linear Representation

With the general framework in place, we can get back to the circuit double covers of cubic graphs. We will model the number of CDCs as a finite linear representation. We let $\nu(G)$ denote the number of circuit double covers of graph G , we also use ν to denote the linear representation we construct.

We describe CDCs by crossings on edges. We take some drawing of the graph in a plane – vertices are distinct points, edges are simple curves that do not go through vertices except at the ends but the edges can cross each other. We can define a walk along the edges in such a drawing and circuit double cover can be described by specifying whether its walks swap sides or not on each edge.

It is natural to extend this definition to gadgets. The CDC on a gadget is a (multi)set of circuits and walks which covers every edge of the gadget twice (including the half-edges). Both ends of each walk must be half-edges and no edge can appear twice in one walk. The crossings on regular edges are already determined but the crossings on half-edges will be determined when the half-edge is joined with another half-edge.

How do the gadget joins act on the CDCs? The disjoint union of the underlying gadgets is just a union of the CDCs and it always succeeds. Permuting half-edges does nothing, the only interesting operation is joining two half-edges together. When we are joining two half-edges, there are two walks on each of them and we must determine in which of the two possible ways we can join them.

What determines whether we can join two walks? We can always join a walk to itself, creating a circuit. If the walks are different, we only need them to not share an edge (otherwise we would create a self-touching walk which could not be completed into a circuit). Note that circuits do not participate in the joins in any way. We can formalize this and use the descriptions of the structure of the CDCs at the half edges as boundaries to construct a linear representation.

Observation 10. *The described representation has $2^{\Theta(k^2)}$ boundaries of size k .*

Theorem 11. *Any linear representation counting the number of circuit double covers must have at least $2^{\Omega(k \log k)}$ boundaries of size k .*

On the other hand for small k the asymptotic behaviour is not important and the values itself are more interesting – $|B^0|$ up to $|B^7|$ are 1, 0, 1, 1, 33, 744, 69920 and 13 710 912.

Observation 12. *Replacing a 3-vertex with a triangle doubles the number of circuit double covers.*

Corollary 13. *Any n -vertex graph created from three parallel edges by repeatedly expanding vertices to triangles has exactly $2^{n/2-1}$ circuit double covers.*

The results we obtained so far motivate our following conjecture:

Conjecture 14. *Every bridgeless cubic graphs with n vertices has at least $2^{n/2-1}$ circuit double covers.*

3 Reducing Cycles – The General Method

We want to show that graphs in some class \mathcal{C} have many CDCs and we know that there is a small set of gadgets \mathcal{S} such that every graph of \mathcal{C} has some gadget of \mathcal{S} as an induced subgraph.

We can for every gadget $s \in \mathcal{S}$ choose a set of smaller gadgets \mathcal{R}_s and try to prove that the number of CDCs of a graph G containing s can be lowerbounded by the number of CDCs of G with s replaced by elements of \mathcal{R}_s . If this is true, the modified graphs are all smaller and they also belong into \mathcal{C} then we obtain some lower bound on the number of CDCs for class \mathcal{C} .

Proving the lower bound for each s is where the linear programming comes into a play. We saw a special case of this approach before in Observation 12. But in that case there was only one boundary and one substitution gadget, so no linear program was needed.

Theorem 15. *If the objective value of the linear program P described below is at least one then the following holds for all gadgets g :*

$$\nu(\mathcal{J}(g, s)) \geq \min_{r \in \mathcal{R}_s} c^{n(s)-n(r)} \nu(\mathcal{J}(g, r))$$

where $n(g)$ is the number of vertices of gadget g and the linear program P is:

- The variables are $m \in \mathbb{R}^{B^{|s|}}$ which we interpret as coefficients of a multiplicity vector and they are non-negative.
- The objective is to minimize $\mathcal{J}_\nu(m, h_\nu(s))$ – the number of CDCs of the join of the worst “gadget” and s .
- For every $r \in \mathcal{R}_s$ there is a condition $c^{n(s)-n(r)} \mathcal{J}_\nu(m, h_\nu(r)) \geq 1$.

4 Reducing Cycles – Application to Planar Graphs

We are interested in bridgeless cubic planar graphs. We know that every such graph contains a cycle of size at most 5 because its dual is also a planar graph and so it contains a vertex of degree at most 5 (due to Euler’s formula). Moreover for $c \leq \sqrt{2}$ we may assume that the graph is cyclically 4-edge-connected. So we take all 4-edge-connected planar cubic graphs as the class \mathcal{C} .

We replace the 4-cycles with the 2 possible noncrossing matchings and the 5-cycles with a cubic vertex and free edge (again drawn in a noncrossing way) in all the 5 possible rotations. The following theorems show the results of this replacements:

Theorem 16. *Let G be a cyclically 4-edge-connected cubic graph with a 4-cycle. Let G_1 and G_2 be the two possible graphs obtained from G by deleting two opposite edges of the 4-cycle and suppressing vertices of degree 2. Then $\nu(G) \geq 4 \min\{\nu(G_1), \nu(G_2)\}$.*

Proof. Because the graph is cyclically 4-edge-connected and we are deleting non-adjacent edges, the resulting graph is still 2-edge-connected. We apply Theorem 15 with $c = \sqrt{2}$, 4-cycle as s and the two non-crossing matchings as \mathcal{R}_s . We obtain the following linear program:

$$\max \left\{ \sum_{i=1}^{33} o_i m_i : \sqrt{2}^4 \sum_{i=1}^{33} a_i m_i \geq 1 \wedge \sqrt{2}^4 \sum_{i=1}^{33} b_i m_i \geq 1 \right\}$$

where m_i are the variables and o_i , a_i and b_i are constants evaluated by a computer (although they might be computed by hand in this case). Each variable corresponds to a boundary and $|B^4| = 33$ hence there is 33 variables. Each inequality corresponds to an elements of \mathcal{R}_s . Plugging in the values, taking dual and removing conditions obviously implied by other conditions, we get:

$$\max \left\{ \frac{1}{4}x_0 + \frac{1}{4}x_1 : x_0 \leq 2 \wedge x_1 \leq 2 \right\}$$

The objective value of this linear program is 1. This satisfies the conditions of the theorem so we obtain:

$$\nu(G) = \nu(\mathcal{J}(g, s)) \geq \min_{r \in \mathcal{R}_s} \sqrt{2}^4 \nu(\mathcal{J}(g, r)) = 4 \min\{\nu(G_1), \nu(G_2)\}.$$

Theorem 17. *Let G be a cyclically 4-edge-connected cubic graph with a 5-cycle and no 4-cycle. Let G_1, G_2, \dots, G_5 be the 5 possible graphs obtained from G by replacing the 5-cycle by a cubic vertex and an edge in non-crossing way (assuming the 5-cycle is a face). Then $\nu(G) \geq 5/2 \min_i \nu(G_i)$. If we replace the 5-cycle by a cubic vertex and an edge in the all possible ways (i.e., breaking planarity) we get $\nu(G) \geq 3.75 \min_i \nu(G_i)$.*

The previous theorems together with the fact that the hypothetical minimal counterexample does not have 2-cut nor nontrivial 3-cut lead to the following corollary:

Corollary 18. *Every bridgeless planar graph with no vertex of degree two has at least $(5/2)^{n/4-1/2} \approx 2^{0.33n}$ circuit double covers.*

Corollary 19. *A minimal counterexample (the one with the smallest number of vertices) to Conjecture 14:*

1. *does not have 2-edge-cut,*
2. *does not have non-trivial 3-edge-cut,*
3. *does not contain triangle,*
4. *does not contain 4-cycle, and*
5. *has at least 22 vertices.*

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Oriented Graphs with Lower Orientation Ramsey Thresholds

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Abstract. For a graph G and an oriented graph \vec{H} , let $G \rightarrow \vec{H}$ denote the property that every orientation of G contains a copy of \vec{H} . We investigate the threshold $p_{\vec{H}} = p_{\vec{H}}(n)$ for $G(n, p) \rightarrow \vec{H}$, where $G(n, p)$ is the binomial random graph. Similarly to the classical (edge-colouring) Ramsey setting, $p_{\vec{H}} \leq n^{-1/m_2(\vec{H})}$, where $m_2(\vec{H})$ denotes the maximum 2-density of \vec{H} . While $n^{-1/m_2(\vec{H})}$ gives the correct order of magnitude for acyclic orientations of cycles and complete graphs with at least 4 vertices, this is known not always to be the case. We extend the examples in that category, describing a large family of oriented graphs \vec{H} such that $p_{\vec{H}} \ll n^{-1/m_2(\vec{H})}$.

Keywords: Ramsey theory · Random graphs · Oriented graphs

1 Introduction

Given a graph G and an oriented graph \vec{H} , we write $G \rightarrow \vec{H}$ to mean that every orientation of G contains a copy of \vec{H} . This property has been investigated by a range of authors (see, e.g., [5, 7–10, 12–16, 20, 21, 24, 25]), focusing on orientations of complete graphs.

We study this property in the context of the *binomial random graph* $G(n, p)$, which is the random graph formed from the complete graph K_n by deleting each edge independently with probability $1 - p$. More precisely, we are interested in determining, for any acyclically oriented graph \vec{H} , the threshold function

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil (CAPES), Finance Code 001. BPC was supported by FAPESP 2018/05557-7. YK was partially supported by CNPq (311412/2018-1, 423833/2018-9) and FAPESP (2018/04876-1, 2019/13364-7). GOM was partially supported by CNPq (306620/2020-0, 428385/2018-4) and FAPESP (2018/04876-1, 2019/13364-7). TN was supported by FAPESP (2019/04375-5, 2018/04876-1, 2019/13364-7). FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

$p_{\vec{H}} = p_{\vec{H}}(n)$ of $G(n, p) \rightarrow \vec{H}$. We call $p_{\vec{H}} = p_{\vec{H}}(n)$ a *threshold* for $G(n, p) \rightarrow \vec{H}$ if

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, p) \rightarrow \vec{H}] = \begin{cases} 0 & \text{if } p \ll p_{\vec{H}} \\ 1 & \text{if } p \gg p_{\vec{H}}, \end{cases}$$

where $a \ll b$ (or, equivalently, $b \gg a$) means $\lim_{n \rightarrow \infty} a_n/b_n \rightarrow 0$ (we speak of ‘the threshold $p_{\vec{H}}$ ’, since $p_{\vec{H}}$ is unique up to constant factors).

Thresholds for Ramsey-type properties are widely studied (see, e.g., [11, 18] and the many references therein). Note that since every graph admits an acyclic orientation, $G \not\rightarrow \vec{H}$ whenever \vec{H} contains a directed cycle. On the other hand, if \vec{H} is acyclic, then the property $G(n, p) \rightarrow \vec{H}$ is non-trivial and monotone, and hence (as proved by Bollobás and Thomason [3]) it has a threshold $p_{\vec{H}} = p_{\vec{H}}(n)$.

Let H be a graph. As usual, let $v(H)$ and $e(H)$ denote the number of vertices and edges in H . An important parameter for estimating thresholds of Ramsey-type properties involving H is the *maximum 2-density* $m_2(H)$ of H (for $v(H) \geq 3$), given by $\max_J (e(J) - 1)/(v(J) - 2)$, where the maximum is taken over all $J \subset H$ with $v(J) \geq 3$. We also consider the *maximum density* $m(H)$, defined as $\max_J e(J)/v(J)$, where the maximum is taken over all $J \subset H$ with $v(J) \geq 1$. Analogous definitions are used for every oriented graph \vec{H} : we denote by H the (undirected) graph we obtain from \vec{H} by ignoring the orientation of its arcs, and set $m_2(\vec{H}) := m_2(H)$ and $m(\vec{H}) := m(H)$.

For any acyclic oriented graph \vec{H} , we can obtain an upper bound for $p_{\vec{H}}$ using the regularity method (it suffices to combine ideas from [11, Sect. 8.5] and, say, [6]). For an alternative approach giving the same upper bound, based on the methods of [19], see [4].

Theorem 1 (Cavalar [4]). *For every acyclically oriented graph \vec{H} there exists a constant $C = C(\vec{H})$ such that if $p \geq Cn^{-1/m_2(\vec{H})}$ then a.a.s. $G(n, p) \rightarrow \vec{H}$.*

Barros, Cavalar, Kohayakawa and Naia [2] proved that $p_{\vec{H}} = n^{-1/m_2(\vec{H})}$ for every acyclic orientation of a complete graph or a cycle with at least 4 vertices. Furthermore, they showed that the threshold is lower for transitive triangles.

Theorem 2 (Barros, Cavalar, Kohayakawa and Naia [2]). *If \vec{H}_t is an acyclic orientation of a complete graph or a cycle with t vertices, then*

$$p_{\vec{H}_t} = \begin{cases} n^{-1/m(K_4)} & \text{if } t = 3 \\ n^{-1/m_2(\vec{H}_t)} & \text{if } t \geq 4. \end{cases}$$

We shall describe a large family of oriented graphs \vec{H} such that $p_{\vec{H}} \ll n^{-1/m_2(\vec{H})}$. In order to define this family, consider an oriented graph \vec{H} and let r be an arbitrary vertex of \vec{H} . Sometimes we say that \vec{H} is *rooted* at r , or (equivalently) that r is the *root* of \vec{H} . Given an oriented graph \vec{F} , we denote by $\vec{F} \circ \vec{H}$ the *rooted product* of \vec{F} and \vec{H} , defined as $\vec{F} \circ \vec{H} = (V, E)$ where

$V = V(\vec{F}) \times V(\vec{H})$, and

$$E = \{((f, r), (f', r)) : (f, f') \in E(\vec{F})\} \cup \bigcup_{x \in V(\vec{F})} \{((x, h), (x, h')) : (h, h') \in E(\vec{H})\}.$$

See Fig. 1 for an example of the rooted product between oriented graphs. Also, let $\overrightarrow{\text{TT}}_3$ be the transitive tournament on 3 vertices. We prove that, for the rooted product $\vec{F} \circ \vec{H}$ of some oriented graphs \vec{F} and \vec{H} , the threshold $p_{\vec{F} \circ \vec{H}}$ is asymptotically smaller than $n^{-1/m_2(\vec{F} \circ \vec{H})}$. Note that $m_2(F) < 2$ whenever F is a tree or cycle with at least four vertices.

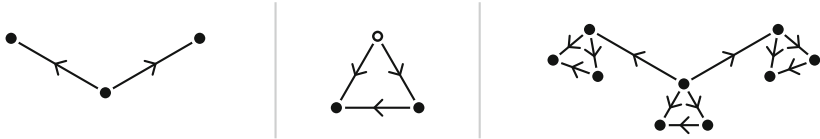


Fig. 1. An oriented tree \vec{F} , a rooted $\overrightarrow{\text{TT}}_3$ and their rooted product $\vec{F} \circ \overrightarrow{\text{TT}}_3$.

Theorem 3. *Let \vec{H} be a rooted $\overrightarrow{\text{TT}}_3$. If $m_2(\vec{F}) < 2$ and \vec{F} is acyclic, then $p_{\vec{F} \circ \vec{H}} \ll n^{-1/m_2(\vec{F} \circ \vec{H})}$.*

In order to obtain Theorem 3, we prove that sublinearly-sized subsets of vertices of $G(n, p)$ inherit the orientation Ramsey property. Our proof uses the method of hypergraph containers [1, 22] applied to thresholds of Ramsey properties, closely following ideas of Nenadov and Steger [19].

Theorem 4. *Let \vec{H} be an acyclically oriented graph. For every $0 < \gamma < 1$, there exists D such that the following holds. If $p \geq (D \ln n)n^{-(1-\gamma)/m_2(\vec{H})}$ then for $G = G(n, p)$ we a.a.s. have that $G[S] \rightarrow \vec{H}$ for all $S \in \binom{V(G)}{n^{1-\gamma}}$.*

In the next section we prove Theorem 3. In Sect. 3 we observe that Theorem 3 holds even for trees \vec{F} whose order is a small polynomial in n (see Theorem 5). Finally, in Sect. 4 we outline the proof of Theorem 4.

2 Proof of Theorem 3

We prove Theorem 3 using a restricted version of a general result (see, e.g. [11, 23]).

Lemma 1 (Schürger [23]). *If $\delta > 0$ and $p \geq n^{\delta-1/m(K_4)}$, then a.a.s. $G(n, p)$ contains $\Theta(n^{6\delta})$ vertex-disjoint copies of K_4 .*

Given an oriented graph \vec{H} and a positive integer k , we denote by $k\vec{H}$ the disjoint union of k copies of \vec{H} . For brevity, we omit some calculations.

Proof (Theorem 3; outline). We prove that there are constants γ, D and $\delta > 0$, such that $p = (D \log n)n^{-(1-\gamma)/m_2(\vec{F})} \geq n^{\delta-1/m_2(K_4)}$, $6\delta > 1 - \gamma$ and $p \ll n^{-1/m_2(\vec{F} \circ \vec{H})}$. Lemma 1 and the first inequality above imply that a.a.s. $G(n, p)$ contains $n^{6\delta}$ vertex-disjoint copies of K_4 . The second inequality, together with the fact that $K_4 \rightarrow \vec{H}$, implies that a.a.s. $G(n, p) \rightarrow n^{1-\gamma}\vec{H}$. Theorem 4 implies that a.a.s. in every orientation of $G(n, p)$ the root vertices in copies of \vec{H} induce a copy of \vec{F} . Hence $p_{\vec{F} \circ \vec{H}} \leq p$, and the third inequality completes the proof. \square

3 Larger Trees (proof of Theorem 5)

In this section we prove Theorem 3 for “large” trees.

Theorem 5. *Let \vec{F} be an oriented tree and \vec{H} be a rooted $\overrightarrow{\text{TT}}_3$. For every $\varepsilon > 0$ there is $\gamma > 0$ such that the following holds. If $v(\vec{F}) \leq n^{1/2-\varepsilon}$ and $p \geq n^{-1/2-\gamma}$, then a.a.s. $G(n, p) \rightarrow \vec{F} \circ \vec{H}$. In particular, $p_{\vec{F} \circ \vec{H}} \ll n^{-1/m_2(\vec{F} \circ \vec{H})}$.*

To prove Theorem 5 we need the following result.

Theorem 6 (Naia [17]). *If G is a graph and T is an oriented tree whose order is at most $\chi(G)/\log_2 v(G)$, then $G \rightarrow T$.*

We can now prove Theorem 5.

Proof (of Theorem 5). Fix $\varepsilon > 0$ and let $\gamma = \varepsilon/8$. Let $\delta := 1/6 - \gamma$ and $p := n^{\delta-1/m(K_4)} = n^{-1/2-\gamma}$. Note that $p \ll n^{-1/m_2(K_3)} = n^{-1/m_2(\vec{F} \circ \vec{H})}$. We will show that a.a.s. $G(n, p) \rightarrow \vec{F} \circ \vec{H}$.

By Lemma 1, a.a.s. $G = G(n, p)$ contains m vertex-disjoint copies H_1, \dots, H_m of K_4 , where $m \geq Cn^{6\delta}$ for some $C > 0$. Moreover, by an easy application of the first moment method, we have that a.a.s. $\alpha(G) < 3p^{-1} \log n$, so for all $S \subseteq V(G)$ with $|S| \geq Cn^{6\delta}$ we have $\chi(G[S]) \geq |S|/\alpha(G(n, p)) > Cn^{1/2-7\gamma}/(3 \log n)$.

Fix an orientation of G . Since $K_4 \rightarrow \overrightarrow{\text{TT}}_3$, for each $i \in [m]$, the orientation of H_i contains a copy of $\overrightarrow{\text{TT}}_3$ whose root corresponds to some $r_i \in V(H_i)$. By Theorem 6, the fixed orientation of $G' = G[\{r_1, \dots, r_m\}]$ contains every oriented tree with $\chi(G')/\log_2 m > n^{1/2-\varepsilon}$ vertices, which follows from the choice of γ . \square

4 Inheritance by Sublinear Sets (Proof of Theorem 4)

Let \vec{H} be an acyclically oriented graph. In this section we sketch a proof that, if $\gamma > 0$ and p lies slightly above $n^{-(1-\gamma)/m_2(\vec{H})}$, then a.a.s. the Ramsey property $G(n, p) \rightarrow \vec{H}$ is inherited (simultaneously) by all vertex subsets of size $n^{1-\gamma}$. An important ingredient in the proof of Theorem 4 is the following container theorem for digraphs, which we obtain by applying the hypergraph container lemma of Saxton and Thomason [22] and Balogh, Morris and Samotij [1]. We omit the

somewhat technical proof of Theorem 7 below for conciseness. For convenience, given numbers n, s and t , define

$$\mathcal{T}(n, s, t) := \left\{ (T_1, \dots, T_s) \in E(K_n)^s : \left| \bigcup_{i \in [s]} T_i \right| \leq t \right\}.$$

Theorem 7. *Let \vec{H} be an acyclic oriented graph. There exists $\alpha > 0$ and positive integers n_0, s and c such that the following holds for all $n \geq n_0$. For each graph G of order n such that $G \not\rightarrow \vec{H}$, there exists $T = (T_1, \dots, T_s) \in E(G)^s$ and a set $C = C(T) \subseteq E(K_n)$ depending only on T such that*

- (a) $\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C$,
- (b) $|E(K_n) \setminus C| \geq \alpha n^2$, and
- (c) $T \in \mathcal{T}(n, s, cn^{2-1/m_2(\vec{H})})$.

Let us outline the proof of Theorem 4. Our calculations follow those of Nenadov and Steger [19], but we require an extra union bound, over vertex sets of size $n^{1-\gamma}$. For each $S \subseteq V(G)$ denote by $\mathcal{E}(S)$ and $\mathcal{E}^c(S)$, respectively, the events $G[S] \rightarrow \vec{H}$ and $G[S] \not\rightarrow \vec{H}$. For each $S \in \binom{V(G)}{n^{1-\gamma}}$, if $G[S] \not\rightarrow \vec{H}$, then by Theorem 7 (applied to $G[S]$) there exists an s -tuple T with

$$T = (T_1, \dots, T_s) \in \mathcal{T}(n^{1-\gamma}, s, cn^{(2-1/m_2(\vec{H})) (1-\gamma)}) \tag{1}$$

and a set $C(T) \subseteq E(K_n)$ such that $\bigcup_{i \in [s]} T_i \subseteq E(G[S]) \subseteq C(T)$ and $|E(K_{n^{1-\gamma}}) \setminus C(T)| \geq \alpha n^{2(1-\gamma)}$. Crucially, $G[S]$ avoids $E(K_{n^{1-\gamma}}) \setminus C(T)$. We can bound $\mathbb{P}[\mathcal{E}^c(S)]$ by bounding the probability that there exists an s -tuple T from (1) such that $E_0(T) := E(K_{n^{1-\gamma}}) \setminus C(T)$ is edge-disjoint from $G[S]$. Thus, writing $t := cn^{(2-1/m_2(\vec{H})) (1-\gamma)}$, we have

$$\mathbb{P}[\mathcal{E}^c(S)] \leq \sum_{T \in \mathcal{T}(n^{1-\gamma}, s, t)} \mathbb{P} \left[\left(\bigcup_{i \in [s]} T_i \subseteq E(G[S]) \right) \wedge (E(G[S]) \cap E_0(T) = \emptyset) \right].$$

Note that the two events in the above probability are independent and hence

$$\mathbb{P}[\mathcal{E}^c(S)] \leq \sum_{T \in \mathcal{T}(n^{1-\gamma}, s, t)} p^{\left| \bigcup_{i \in [s]} T_i \right|} \cdot (1-p)^{\alpha n^{2(1-\gamma)}}.$$

The sum can be bounded by first deciding on the number of edges

$$j := \left| \bigcup_{i \in [s]} T_i \right| \leq cn^{(2-1/m_2(H))(1-\gamma)},$$

then choosing j edges, and finally deciding, for each edge, in which of the s -tuples T_1, \dots, T_s it appears. It is then possible to show that

$$\mathbb{P}[\mathcal{E}^c(S)] \ll \exp(-n^{(2-1/m_2(\vec{H})) (1-\gamma)} \ln n).$$

Applying the union bound over all choices of S , we deduce that $\mathbb{P}[\bigcup_S \mathcal{E}^c(S)] \leq \sum_{S \in \binom{V(G)}{n^{1-\gamma}}} \mathbb{P}[\mathcal{E}^c(S)] = o(1)$, since $m_2(\vec{H}) \geq 1$ and $\gamma < 1$, and Theorem 4 follows.

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Random 2-Cell Embeddings of Multistars

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Abstract. By using permutation representations of maps, one obtains a bijection between all maps whose underlying graph is isomorphic to a graph G and products of permutations of given cycle types. By using statistics on cycle distributions in products of permutations, one can derive information on the set of all 2-cell embeddings of G . In this paper, we study multistars—loopless multigraphs in which there is a vertex incident with all the edges. The well known genus distribution of the two-vertex multistar, also known as a dipole, can be used to determine the expected genus of the dipole. We then use a result of Stanley to show that, in general, the expected genus of every multistar with n nonleaf edges lies in an interval of length $2/(n+1)$ centered at the expected genus of an n -edge dipole. As an application, we show that the face distribution of the multistar is the same as the face distribution gained when adding a new vertex to a 2-cell embedded graph, and use this to obtain a general upper bound for the expected number of faces in random embeddings of graphs.

Keywords: Random maps · Genus distribution · Dipole · Multistar

1 Introduction

By an *embedding* of a graph G , we mean a 2-cell embedding of G in some oriented surface. Two embeddings of G are *equivalent* if there is an orientation-preserving

T. Masařík—was supported by a postdoctoral fellowship at the Simon Fraser University through NSERC grants R611450 and R611368.

B. Mohar—was supported in part by the NSERC Discovery Grant R611450 (Canada) and by the Research Project J1-8130 of ARRS (Slovenia).

R. Šámal—was partially supported by grant 19-21082S of the Czech Science Foundation. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 810115). This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 823748.

The full version of this extended abstract is available as [3].

homeomorphism of the surface mapping the graph in one embedding onto the graph in the other, and the restriction of the homeomorphism to the graph is the identity isomorphism. Equivalent embeddings are considered the same since they define the same *map*, where a map is considered as the incidence structure of vertices, edges and faces of the embedding. It is well known [4, 10] that equivalence classes of 2-cell embeddings of G (i.e., maps whose underlying graph is G) are in bijective correspondence with *local rotations*, where for each vertex $v \in V(G)$ we prescribe a cyclic permutation π_v of the half-edges, or *darts*, incident with v .

We consider the ensemble of all maps of G , endowed with the uniform probability distribution. The genus and the number of faces of a random map of G become random variables in this setting. This gives rise to the notion of the *average genus* of the graph and leads to *random topological graph theory* as termed by White [11]. It turns out that considering all embeddings of a graph is useful not only in graph theory and combinatorics, but also in applications in algebra and in theoretical physics. We refer to Lando and Zvonkin [2] for an overview of such applications.

Two special cases of random embeddings are well understood. The first one is when the graph is a bouquet of n loops (also called a *monopole*), which is the graph with a single vertex and n loops incident with the vertex. By duality, the maps of the monopole with n loops correspond to unicellular maps [1] with n edges. The second well studied case is the *n -dipole*, a two-vertex graph with n edges joining the two vertices.

The main object of study in this paper are *multi-stars*, or loopless multigraphs in which there is a vertex incident with all the edges. Formally, we have one center vertex v_0 incident to n edges; these edges lead to $k \geq 1$ other vertices, v_1, \dots, v_k , with n_i edges between v_0 and v_i ($1 \leq i \leq k$). As $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$, where $\sum_{i=1}^k n_i = n$, we see that multi-stars with n edges are in bijective correspondence with partitions of n . An expression for the genus polynomial of multi-stars was obtained by Stanley [9], and our main results use this formulation to derive precise bounds for the expected genus of these graphs (see Sect. 3).

Although most previous works in random topological graph theory concern the (average) genus, it turns out that the number of faces is a more natural statistic. Fixing a graph G , the Euler-Poincaré formula allows us to switch between the genus of a map of G and the number of faces in the map. If G has n vertices and e edges, then a map of G of genus g has $f = e - n + 2 - 2g$ faces. This provides an easy exchange between the average genus and the expected number of faces.

The paper is organized as follows. In Sect. 2, we show that the expected number of faces for a random embedding of a dipole with n edges is precisely $H_{n-1} + \lceil \frac{n}{2} \rceil^{-1}$, where $H_{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$ is the harmonic sum (see Corollary 1). Previously, Stahl [8] proved that the average number of faces is at most $H_{n-1} + 1$. It is worth noting that we are able to obtain our exact result with a relatively simple combinatorial proof.

In Sect. 3 we extend the dipole result to multistars, showing that they have the same expected number of faces as dipoles up to a difference of $\pm \frac{1}{n+1}$ (see Theorem 3). In Sect. 4 we note that the result for multistars can be used in a more general setting, where we consider a map to which we add a new vertex and consider the expected number of new faces obtained after doing so. In particular, our Theorem 4 shows that the expected number of new faces obtained when adding a new vertex of degree d is at most $\log(d) + 1$ (where we use $\log(\cdot)$ to denote the natural logarithm). We apply this result to obtain new upper bounds for the expected number of faces of several families of graphs on n vertices. A notable outcome is for d -regular graphs, where the conclusion is that the expected number of faces is at most $n \log(d)$. More generally, the same result works for d -degenerate graphs (see Theorem 5 and Corollary 6). This also improves an old result of Stahl [7] that the expected number of faces in a random embedding of an arbitrary graph of order n is at most $n \log(n)$.

2 The Dipole

We will start by studying random embeddings of the *dipole* D_n : the graph with two vertices and n parallel edges joining them. Each embedding of D_n is determined by the local rotations at both vertices. In this case, each local rotation is a full cycle in C_n . This means there is a bijection between embeddings of D_n and pairs $\{(\sigma, \tau) : \sigma, \tau \in C_n\}$. It is then fairly easy to see that the faces in an embedding given by (σ, τ) correspond to the cycles in the permutation product $\sigma\tau$.

Calculating the expected number of faces in an embedding of D_n is therefore equivalent to calculating the expected number of cycles in a product of two full cycles taken randomly from C_n . The labelling on the symbols in S_n is arbitrary, so we may fix one of the full cycles to be $\sigma = (1\ 2\ 3\ \dots\ n)$ and just consider the set $\{(\sigma, \tau) : \tau \in C_n\}$. Let F be the random variable for the number of cycles in $\sigma\tau$ when τ is chosen uniformly at random from C_n . Therefore the expected number of faces in a random embedding of D_n is equal to $\mathbb{E}[F]$. Using a result of Stanley [9], we get the following:

Corollary 1. *Let F be the number of faces in a random embedding of D_n , where $n \geq 2$. Then*

$$\mathbb{E}(F) = \begin{cases} H_{n-1} + \frac{2}{n}, & \text{if } n \text{ is even;} \\ H_{n-1} + \frac{2}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

3 Multi-stars

As mentioned in the introduction, multistars with n edges are in bijective correspondence with partitions of n . If $n = \lambda_1 + \dots + \lambda_k$, we denote the partition as $\lambda = (\lambda_1, \dots, \lambda_k)$ and write $\lambda \vdash n$. We denote by C_λ the set of all permutations of type λ . We consider the *multistar of type λ* : the multistar with k outer vertices and λ_i edges from the central vertex to the i^{th} outer vertex. Call this $K_\lambda(n)$. Let $r(\lambda)$ denote the number of parts of size 1 in λ , and note that a vertex of

degree 1 in a multistar has no effect on the number of faces. Stanley [9] gives the generating function we will use (the *shift operator* E is defined by the rule $E(f(q)) = f(q - 1)$).

Theorem 2 ([9]). *Let $f_\lambda(j)$ denote the number of permutations in C_λ , whose product with the full cycle $(1\ 2\ \dots\ n)$ is a permutation with j cycles. Then:*

$$\sum_{j=1}^n f_\lambda(j)q^j = \frac{|C_\lambda|}{(n + 1)!} \left(\prod_{i=1}^k (1 - E^{\lambda_i}) \right) (q + n)_{n+1}.$$

We use this result to derive our main result of this section.

Theorem 3. *Let $F_\lambda(n)$ be the random variable denoting the number of faces in a random embedding of $K_\lambda(n)$ and let $n' = n - r(\lambda)$. Then*

$$\mathbb{E}(F_\lambda(n)) \in \left(\Delta_{n'} - \frac{1}{n'+1}, \Delta_{n'} + \frac{1}{n'+1} \right),$$

where $\Delta_{n'} = H_{n'-1} + \left\lceil \frac{n'}{2} \right\rceil^{-1}$.

Let us observe that the value $\Delta_{n'} = H_{n'-1} + \left\lceil \frac{n'}{2} \right\rceil^{-1}$ in Theorem 3 is precisely the same as the expected number of faces for the dipole with n' edges in Corollary 1.

4 General Graphs

Next we use the results of the previous sections to get bounds on the expected number of faces for random embeddings of more general classes of graphs.

Now suppose we have a fixed embedding of some graph G , and want to add a new vertex to this graph. The new vertex v is connected to some vertices of G . If u is a neighbor of v , we fix one of the appearances of u on the facial walks of G . This is where the edge uv will emanate from u in the local clockwise order around u . Finally, choosing a local rotation at v randomly gives an embedding of $G' = G + v$.

If two edges incident with v end up in the same face of G we call them equivalent; this defines a partition $\lambda \vdash \deg(v) =: d$. We can relate the number of faces incident with v and the number of faces in an embedding of the multistar $K_\lambda(d)$, which we studied in the previous section. As a result we get the following.

Theorem 4. *Suppose a vertex v of degree d is added to an embedding of a graph G by giving v a random local rotation. Then the expected number of faces containing v is less than*

$$h(d) := H_{d-1} + \left\lceil \frac{d}{2} \right\rceil^{-1} + \frac{1}{d + 1}.$$

Theorem 4 implies general bounds on the expected number of faces in random embeddings of graphs. Given a graph G , let F denote the random variable for the number of faces in a random embedding of G . It immediately follows from Theorem 4 that, if G has maximum degree $d \geq 2$, then $\mathbb{E}[F] < n(\log(d) + \frac{5}{3})$.

In fact, we can obtain a stronger upper bound. The graph is said to be d -degenerate if there is an ordering of its vertices v_1, \dots, v_n such that $d_i \leq d$ for each $i \in [n]$, where d_i denotes the number of neighbors v_j of v_i with $j < i$. We call d_i the *back-degree*. We make a slight change to the definition above. Whenever $d_i = 0$ we redefine it to the value $d_i = 1$ instead.

Theorem 5. *Let G be a connected graph of order n . Given a linear order of vertices v_1, \dots, v_n with respective back-degrees d_i ($1 \leq i \leq n$), we have*

$$\mathbb{E}[F] \leq 1 + \sum_{i=3}^n \log d_i^*, \tag{1}$$

where $d_i^* := d_i$ if $d_i \neq 2$ and $d_i^* := e$ if $d_i = 2$. We also have

$$\mathbb{E}[F] \leq 1 + \sum_{i=3}^n H_{d_i-1}. \tag{2}$$

Corollary 6. *Let G be a connected d -degenerate graph. If $d = 2$, then $\mathbb{E}[F] \leq n - 1$. If $d \geq 3$, then $\mathbb{E}[F] \leq 1 + (n - 2) \log d$.*

Theorem 5 improves upon the previous best known general bound, proven by Stahl in [7]. A similar improvement has been made with (2), which should be compared with the bound $\mathbb{E}(F) \leq n + \sum_{i=1}^n H_{d'_i-1}$ from [7].

In a separate paper [6], Stahl described some infinite families of graphs for which $\mathbb{E}(F)$ is linear in the number of vertices. All of them arised by linking together n copies of a fixed graph H “in a consistent manner so as to form a chain”. The collection of graphs for which the expected number of faces is linear is much richer though, as the following proposition illustrates.

Proposition 7. *Let G be a graph and let \mathcal{C} be a family containing cycles (2 -regular connected subgraphs) of G . Suppose that each cycle in \mathcal{C} has length at most ℓ and all of its vertices have degree at most d . If $d \neq 2$, then the expected number of faces in a random embedding of G is at least $\frac{2|\mathcal{C}|}{(d-1)^\ell}$.*

Note that two cycles in \mathcal{C} are allowed to intersect. Combining Theorem 5 with Proposition 7, we see that any graph with the bounded maximum degree and linearly many short cycles has linearly many expected faces. Although Proposition 7 describes general classes of graphs with the linear expected number of faces, it is believed that this is rare. In fact, Stahl conjectured [5, Conjecture 4.3] that for almost all graphs with q edges, the expected number of faces in a random embedding is close to H_{2q} .

We were unable to find any graph family with unbounded degeneracy for which the bound in Theorem 5 is tight. Indeed, we believe that such graphs do not exist.

Conjecture 8. *The expected number of faces in a random embedding of a graph on n vertices with maximum edge multiplicity μ is $O(n \log(2\mu))$.*

Conjecture 8 would imply that the expected number of faces in a random embedding of any simple graph is at most linear in the number of vertices. Notice that the dipole, considered in Sect. 2, gives a family of graphs for which Conjecture 8 is tight. Moreover, a long path in which every second edge is replaced by a dipole with μ edges gives a tight family in which each of n and μ can independently tend to infinity.

Acknowledgements. The authors would like to thank Amarpreet Rattan for pointing out relevant literature, and Ladislav Stacho for helpful initial discussions on the topic.

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Cycle Saturation in Random Graphs

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Abstract. For a fixed graph F , the minimum number of edges in an edge-maximal F -free subgraph of G is called the F -saturation number. The asymptotics of the F -saturation number of the Erdős–Rényi random graph $G(n, p)$ for constant $p \in (0, 1)$ was established for any complete graph and any star graph. We obtain the asymptotics of the C_m -saturation number of $G(n, p)$ for $m \geq 5$. Also we prove non-trivial linear (in n) lower bounds and upper bounds for the C_4 -saturation number of $G(n, p)$ for some fixed values of p .

Keywords: Random graph · Saturation · Cycle · Extremal graph theory

One of the fundamental problems of the extremal graph theory was posed by Turán [1]. It is concerned with finding the maximum number of edges in a graph on n vertices without a copy of a given graph F as a subgraph. This value is denoted by $ex(n, F)$. Zykov [2] and later independently Erdős, Hajnal and Moon [3] raised a dual question of finding the minimum number of edges in an edge-maximal F -free graph on n vertices. Let us give a formal definition.

Let F and H be graphs. H is said to be F -saturated if it is a maximal F -free graph, i.e. H does not contain any copy of F as a subgraph, but adding any missing edge to H creates one. The saturation number $sat(n, F)$ is defined to be the minimum number of edges in an F -saturated graph on n vertices.

If F is an m -clique then $sat(n, F)$ is known. It was proven in [3] that when $n \geq m \geq 2$, then

$$sat(n, K_m) = (m - 2)n - \binom{m - 1}{2}. \tag{1}$$

Denote by $K_{1,m}$ the star graph on $m + 1$ vertices. Its saturation number is also known. It was proven by Kászonyi and Tuza [4] that

$$sat(n, K_{1,m}) = \begin{cases} \binom{m}{2} + \binom{n-m}{2}, & m + 1 \leq n \leq \frac{3m}{2}; \\ \lceil \frac{(m-1)n}{2} - \frac{m^2}{8} \rceil, & n \geq \frac{3m}{2}. \end{cases} \tag{2}$$

Finding $sat(n, C_m)$ is harder. The problem is solved only for $m = 4, 5$. It was determined in [5] by Ollman that for $n \geq 5$

$$sat(n, C_4) = \left\lfloor \frac{3n - 5}{2} \right\rfloor. \tag{3}$$

An upper bound of

$$\left\lceil \frac{10}{7}(n - 1) \right\rceil \tag{4}$$

for $\text{sat}(n, C_5)$ was given in [6] by Fisher, Fraughnaugh and Langley. In a very technical paper [7] by Chen it was shown that for $n \geq 21$ this upper bound is also a lower bound for $\text{sat}(n, C_5)$.

Luo, Shigeno and Zhang in [8] established that

$$\begin{aligned} \text{sat}(n, C_6) \leq \left\lfloor \frac{3n - 3}{2} \right\rfloor, \quad \text{for } n \geq 9, \\ \text{sat}(n, C_6) \geq \left\lfloor \frac{7n}{6} \right\rfloor - 2, \quad \text{for } n \geq 6. \end{aligned} \tag{5}$$

Finally, Füredi and Kim [9] showed that for all $m \geq 7$ and $n \geq 2m - 5$

$$\left(1 + \frac{1}{m + 2}\right)n - 1 < \text{sat}(n, C_m) < \left(1 + \frac{1}{m - 4}\right)n + \binom{m - 4}{2}. \tag{6}$$

Observe that there is a gap between upper and lower bounds though Füredi and Kim conjectured that the constructions yielding the upper bound are optimal.

More results concerning the saturation problem can be found in a survey [10] by J. Faudree, R. Faudree, Schmitt and in references therein.

Many classical extremal questions were extended to random settings. Korándi and Sudakov [11] initiated the study of the saturation problem for random graphs.

Recall that the *random graph* $G(n, p)$ is a random element of the set of all graphs G on $[n] := \{1, \dots, n\}$ with probability distribution $p^{|E(G)|}(1 - p)^{\binom{n}{2} - |E(G)|}$ (or, in other words, every pair of vertices is adjacent with probability $0 \leq p \leq 1$ independently). We say that a graph property Q holds with high probability (whp), if $\mathbb{P}(G(n, p) \in Q) \rightarrow 1$ as $n \rightarrow \infty$.

For fixed graphs F and G we say that a subgraph $H \subseteq G$ is *F-saturated* in G if H is a maximal F -free subgraph of G . The minimum number of edges in an F -saturated graph in G is denoted by $\text{sat}(G, F)$.

Korándi and Sudakov [11] asked a question of determining the saturation number of $G(n, p)$ when F is a complete graph K_m on m vertices. They proved that for every fixed $p \in (0, 1)$ and fixed integer $m \geq 3$ whp

$$\text{sat}(G(n, p), K_m) = (1 + o(1))n \log_{\frac{1}{1-p}} n. \tag{7}$$

The saturation number of $G(n, p)$ when F is a star graph was studied in a couple of papers. Note that by the definition $\text{sat}(G, K_{1,2})$ coincides with the minimum cardinality of a maximal matching in G . Zito [12] showed that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{n}{2} - \log_{\frac{1}{1-p}}(np) < \text{sat}(G(n, p), K_{1,2}) < \frac{n}{2} - \log_{\frac{1}{1-p}} \sqrt{n} \right) = 1. \tag{8}$$

Mohammadian and Tayfeh-Rezaie [13] proved that for every fixed $p \in (0, 1)$ and fixed integer $m \geq 3$ whp

$$sat(G(n, p), K_{1,m}) = \frac{(m - 1)n}{2} - (1 + o(1))(m - 1) \log_{\frac{1}{1-p}} n. \tag{9}$$

Observe that their upper bound is stronger than the one in (8) whereas the lower bound is weaker.

It is interesting to analyze the stability of the saturation number. When F is a complete graph the comparison of (1) and (7) shows that it gets roughly logarithm times bigger. When F is a star on $m + 1$ vertices the contrast of (2) and (8),(9) shows that there is an asymptotical stability.

It is natural to ask a question about the asymptotic behavior of a C_m -saturation number of $G(n, p)$.

The first result of the present paper establishes the asymptotic behavior of the C_m -saturation number of $G(n, p)$ when $m \geq 5$.

Theorem 1. *Let $p \in (0, 1)$ be fixed. For every $m \geq 5$ and $\varepsilon > 0$ whp*

$$n - 1 \leq sat(G(n, p), C_m) \leq n(1 + \varepsilon). \tag{10}$$

To prove this we show the existence of a certain structure in $G(n, p)$ whp. Let the vertex set of $G(n, p)$ be $\{1, \dots, n\}$. Let $k := \lceil m/2 \rceil$.

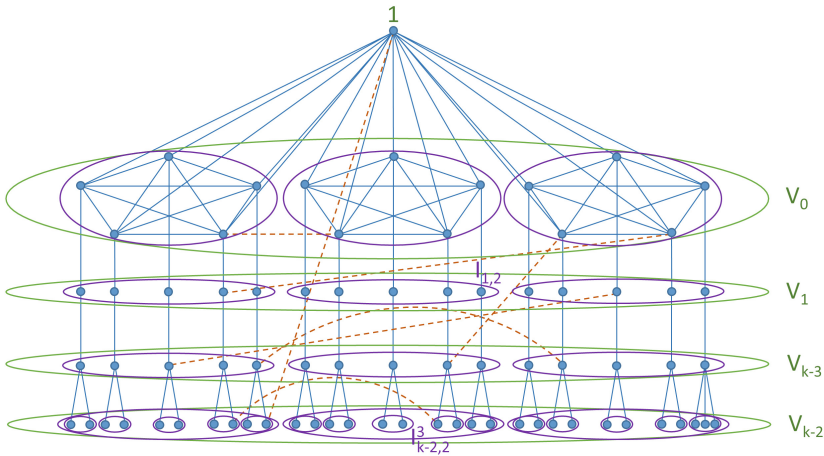


Fig. 1. Graphs $G(n, p)$ and A for odd m . Dashed edges may be present in $G(n, p)$ but not present in A

To settle the problem for odd m (see Fig. 1) we show that whp there exists a spanning subgraph A of $G(n, p)$ which consists of a vertex 1 adjacent only to a certain number of disjoint cliques of size k . There is a copy of the path P_{k-2}

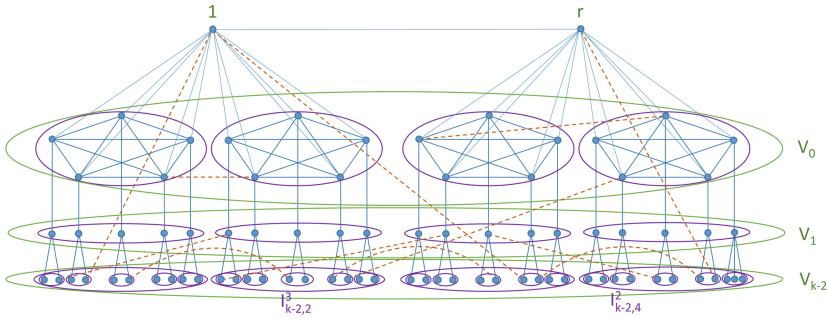


Fig. 2. Graphs $G(n, p)$ and A for even m . Dashed edges may be present in $G(n, p)$ but not present in A

emanating from each vertex of every clique. The last vertex of every path is an internal vertex of a copy of some star graph. There are no other edges in A , and it does not contain copies of C_m . We also show that whp there only may be present edges between vertices of paths or leaves of stars emanating from different cliques, and edges between 1 and leaves of any star graph in $G(n, p)$. Addition of any such edge to A creates a copy of C_m .

The construction for even m is similar (see Fig. 2). We show that whp there exists a spanning subgraph A of $G(n, p)$ which consists of two neighbors 1 and r , each of which is adjacent only to a unique set of the same certain number of vertex-disjoint cliques of size $m/2 + 1$. There is a copy of the path $P_{m/2-2}$ emanating from each vertex of every clique. The last vertex of every path is an internal vertex of a copy of some star graph. There are no other edges in A , and it does not contain copies of C_m . We also show that whp there only may be present edges between vertices of paths or leaves of stars emanating from different cliques, and edges between 1 or r and leaves of any star graph in $G(n, p)$. Addition of any such edge to A creates a copy of C_m .

The second result of the paper provides the upper bound for the C_4 -saturation number of $G(n, p)$.

Theorem 2. *Let $p \in (0, 1)$ be fixed. For every $\varepsilon > 0$ there exists a constant $C(p)$ such that whp*

$$\text{sat}(G(n, p), C_4) \leq C(p) \cdot n(1 + \varepsilon). \tag{11}$$

In particular, when $p = 1/2$ then for every $\varepsilon > 0$ whp

$$\text{sat}(G(n, p), C_4) \leq \frac{27}{14} \cdot n(1 + \varepsilon). \tag{12}$$

It is simpler to describe our construction for $p > 1 - 1/\sqrt[3]{7}$, so we only outline it for such p . Denote by $A[V]$ an induced graph on a vertex set V by the graph A . Denote by $N_A(v)$ the set of neighbors of the vertex v in A . Let $\delta > 0$, $r := \lceil (1/2 + \delta) \log_{\frac{1}{1-p}} n \rceil$. To prove the theorem we show that whp $G(n, p)$ contains a

spanning subgraph A and sets of its vertices $V^{r+1} \subset V^r \subset \dots \subset V^1 = [n]$ such that for every $i \in \{1, \dots, r\}$ the induced subgraph $A[V^i]$ is as in Fig. 3 and the graph $A[V^{r+1}]$ is C_4 -saturated with $o(n)$ edges. There may be present any other edges in $G(n, p)$. Addition of any such edge creates C_4 in A . The third result of the paper provides the lower bound for the C_4 -saturation number of $G(n, p)$.

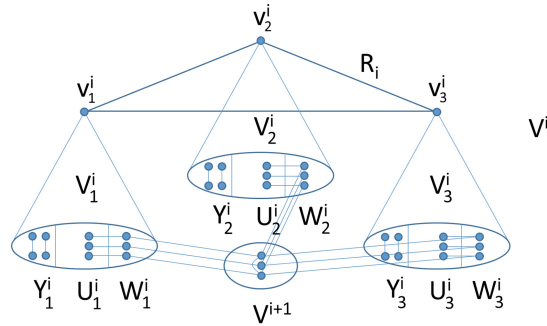


Fig. 3. Graph $A[V^i] : R_i = \{v_1^i, v_2^i, v_3^i\}$; for every $j \in \{1, 2, 3\} V_j^i = N_A(v_j^i), V_j^i = Y_j^i \sqcup U_j^i \sqcup W_j^i, E(A[Y_j^i])$ is a perfect matching, $E(A[U_j^i \sqcup W_j^i])$ is a perfect matching between U_j^i and W_j^i , $E(A[W_j^i \sqcup V^{i+1}]) \setminus E(A[V^{i+1}])$ is a perfect matching between W_j^i and V^{i+1} .

Theorem 3. Let $p \in (0, 1)$ be fixed. For every $\varepsilon > 0$ whp

$$sat(G(n, p), C_4) \geq \frac{3}{2}n(1 - \varepsilon). \tag{13}$$

Let A be a C_4 -saturated graph in $G(n, p)$. Find the set of all vertices of degree 2 in A and denote it by U . Recursively remove from A and $G(n, p)$ adjacent vertices of degree 2 included in U that have a common neighbor not in U . Clearly, these operations preserve C_4 -saturation. After such deletions there also remain bad configurations such as vertices of degree two in U that have two neighbors not in U , induced simple paths of length at least 5 with endpoints not in U , isolated vertices. We show that there are $o(n)$ of them and they do not affect the asymptotics of the number of edges. If all remaining vertices have degree at least 3 then the problem is solved. Otherwise the analysis of the degrees of neighbors of adjacent vertices with degree 2 in U that do not have a common neighbor allows to solve the problem.

Let us analyze the stability of the saturation number when F is a cycle C_m . The comparison of (4), (5), (6) and our result (10) implies that the order of growth is stable for $m > 4$, but there is no stability since the constants before n are different. Contrast of (3) and (11), (12), (13) shows that we can not make a conclusion about the stability of C_4 . Remarkably, our results also demonstrate that so far C_m -saturation number for $m > 4$ is a unique example when the saturation number drops.

Acknowledgements. This work was carried out with the financial support of the Ministry of Education and Science of the Russian Federation, in the framework of Megagrant no. 075-15-2019-1926. The work of the first author was also supported by the Simons Foundation.

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On the Maximum Cut in Sparse Random Hypergraphs

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Abstract. The paper deals with the max-cut problem for random hypergraphs. We consider a binomial model of a random k -uniform hypergraph $H(n, k, p)$ for some fixed $k \geq 3$, growing n and $p = p(n)$. For given natural number q , the max- q -cut for a hypergraph is the maximal possible number of edges that can be properly colored with q colors under the same coloring. Generalizing the known results for graphs we show that in the sparse case (when $p = cn/\binom{n}{k}$ for some fixed $c > 0$ not depending on n) there exists a limit constant $\gamma(c, k, q)$ such that

$$\frac{\text{max-q-cut}(H(n, k, p))}{n} \xrightarrow{P} \gamma(c, k, q)$$

as $n \rightarrow +\infty$. We also prove some estimates for $\gamma(c, k, q)$ of the form $A_{k,q} \cdot c + B_{k,q} \cdot \sqrt{c} + o(\sqrt{c})$.

Keywords: Random hypergraphs · Max-cut · Interpolation method

1 Introduction

The paper is devoted to the maximum cut problem for random hypergraphs. Let us recall some definitions.

1.1 Definitions

Let $H = (V, E)$ be a hypergraph and q be a natural number. A q -Cut (or a q -coloring) is a partition of the vertex set into q disjoint subsets $V = V_1 \sqcup \dots \sqcup V_q = V$. The size of the cut is the number of edges that are not entirely contained in some V_i , i.e. $|\{e \mid e \in E(H), \forall i : e \not\subseteq V_i\}|$. The max- q -cut(H) is the maximal value of a q -cut over all partitions. In terms of colorings, max- q -cut(H) is equal to the maximal possible number of edges which can be properly colored with q colors under the same coloring.

The problem of estimating max- q -cut will be considered in the setting of random hypergraphs. A classical binomial model of a random k -uniform hypergraph, denoted by $H(n, k, p)$, can be described as a Bernoulli scheme on k -subsets of an

n -element set of vertices: every k -subset is drawn independently with probability $p = p(n)$. It is easy to see that the probability of getting exactly a hypergraph $H' = (V, E')$ is equal to

$$P(H(n, k, p) = (V, E')) = p^{|E'|} \cdot (1 - p)^{\binom{n}{k} - |E'|}.$$

For $k = 2$, $H(n, 2, p)$ is a well-known Erdős–Renyi random graph $G(n, p)$.

1.2 Known Results

For fixed given $q \geq 2$, the problem of estimating the max- q -cut value of $G(n, p)$ is mostly interesting in the sparse case when $p = cn/\binom{n}{2}$ and $c > 0$ does not depend on n , because for np tending to infinity, it is easy to see that

$$\text{max-}q\text{-cut}(G(n, p)) \sim \left(1 - \frac{1}{q}\right) p \binom{n}{2}.$$

The fundamental result concerning the sparse case is the theorem of Bayati, Gamarnik and Tetali [1]. It states that for any $c > 0$, $q \geq 2$ there exists $\gamma(c, q)$ such that

$$\frac{\text{max-}q\text{-cut}(G(n, cn/\binom{n}{2}))}{n} \xrightarrow{\text{Pr}} \gamma(c, q) \text{ as } n \rightarrow +\infty.$$

The estimation of the value $\gamma(c, q)$ was intensively studied for decades. Most of the bounds deal with large c (in comparison with q) and have the following asymptotic representation:

$$\gamma(c, q) = (1 - q^{-1}) \cdot c + B_q \cdot \sqrt{c} + o(\sqrt{c}).$$

For the most challenging case, $q = 2$, Bertoni et al. [2] proved that $B_2 \leq \sqrt{\frac{\ln 2}{2}}$. Later, Coppersmith et al. [3] proved that $B_2 \in [0.37613, 0.58870]$. Gamarnik and Li [4] improved this and showed that $B_2 \in [0.47523, 0.55909]$. Finally, Dembo, Montanari and Sen [5] found the exact value of B_2 which can be numerically approximated by a solution of some partial differential equation.

The general case, $q > 2$, is not studied so well. Coja-Oghlan, Moore and Sanwalani [6] obtained the following estimates for B_q when q is large enough:

$$\frac{4}{3} \sqrt{\frac{\ln q}{q}} \left(1 - O\left(\frac{\ln \ln q}{\ln q}\right)\right) \leq B_q \leq \sqrt{\frac{2 \ln q}{q}} \sqrt{1 - \frac{1}{q}}.$$

The aim of our work was to generalize some of the above results to the hypergraph setting.

1.3 New Results

Our first result provides the existence of the limit constant in the max- q -cut problem for random hypergraphs in the sparse case, i.e. when $p = cn/\binom{n}{k}$ and $c > 0$ does not depend on n .

Theorem 1. For arbitrary fixed $k \geq 3, q \geq 2, c > 0$ and $p = cn/\binom{n}{k}$ there is a constant $\gamma(c, k, q)$ such that:

$$\frac{\text{max-q-cut}(H(n, k, cn/\binom{n}{k}))}{n} \xrightarrow{\text{Pr}} \gamma(c, k, q) \text{ as } n \rightarrow +\infty.$$

This is a natural generalization of the result from [1]. However, this means that the proof of Theorem 1 does not provide any bounds for the value $\gamma(c, k, q)$. The next theorem gives some estimates for $\gamma(c, k, q)$. Again, we assume that c is large enough in comparison with k and q .

Theorem 2. For any large enough $c > c_0(k, q)$,

$$\gamma(c, k, q) \leq c \cdot (1 - q^{1-k}) + \sqrt{c} \cdot A_{k,q} + o(\sqrt{c})$$

$$\gamma(c, k, q) \geq c \cdot (1 - q^{1-k}) + \sqrt{c} \cdot C_{k,q} + o(\sqrt{c})$$

where

$$A_{k,q} = \frac{1}{q^{k-1}} \cdot \sqrt{2 \ln q \cdot (q^{k-1} - 1)},$$

$$C_{k,q} = \frac{\sqrt{8 \ln q}}{k + 1} \cdot \sqrt{\frac{k}{q^{k-1}}} \cdot \left(1 - O\left(\frac{\ln \ln q}{\ln q}\right)\right).$$

Theorem 2 generalizes the results from [6] to the case of hypergraphs and shows that for large q , we have the gap of the order \sqrt{k} between the bounds. It would be interesting to improve the obtained estimates, by the way the advances in the graph case have been done by using the results concerning the Sherrington-Kirkpatrick model of spin glasses.

In the next section we will provide the sketches of the proofs.

2 Ideas of the Proofs

2.1 Proof of Theorem 1

The proof of Theorem 1 follows the argument of Bayati, Gamarnik and Tetali [1] and uses the interpolation method.

First of all, we establish that it is sufficient to show that there exists a limit for the expected value of max- q -cut:

$$\gamma(c, k, q) = \lim_{n \rightarrow +\infty} \frac{\mathbb{E} \text{max-q-cut}(H(n, k, cn/\binom{n}{k}))}{n}.$$

This follows from the fact that $\text{max-q-cut}(H(n, k, cn/\binom{n}{k}))$ is highly concentrated around its mean and can be established by application of the Talagrand inequality.

On the second step we want to show that the sequence

$$a(n) = \mathbb{E} \text{max-q-cut}(H(n, k, cn/\binom{n}{k}))$$

is superadditive, i.e.

$$a(n) \geq a(n_1) + a(n_2) - O(n^{1/2})$$

for any n, n_1, n_2 such that $n_1 + n_2 = n$. Standard calculus shows that this will imply the existence of the limit $a(n)/n$.

So, let us fix arbitrary n, n_1, n_2 such that $n_1 + n_2 = n$ and apply the interpolation method. Let V be a vertex set, $|V| = n$, and let us fix its partition into two parts V_1 and V_2 such that $|V_i| = n_i, i = 1, 2$. Consider the following sequence of random hypergraphs on the vertex set $V, H^{(t)}(n, k, m), t = 0, \dots, m$, where $m = \lfloor cn \rfloor$. The hypergraph $H^{(t)}(n, k, m)$ consists of t green edges and $m - t$ red edges, all the edges are chosen independently.

- A green edge is chosen uniformly from V^k (i.e. we choose k vertices with replacement).
- A red edge with probability n_i/n is chosen uniformly from $V_i^k, i = 1, 2$.

Note that $H^{(t)}(n, k, m)$ can contain nonproper (less than k different vertices) or coinciding edges. However $H^{(m)}(n, k, m)$ looks like the usual uniform model with m edges and $H^{(0)}(n, k, m)$ looks like the union of two small hypergraphs $H^{(m_i)}(n_i, k, m_i), m_i = \lfloor cn_i \rfloor, i = 1, 2$. The following lemma is the heart of the method.

Lemma 1. *For any $t = 0, \dots, m - 1$,*

$$\mathbb{E} \max\text{-}q\text{-cut}(H^{(t+1)}(n, k, m)) \geq \mathbb{E} \max\text{-}q\text{-cut}(H^{(t)}(n, k, m)).$$

Note that for the hypergraphs $H^{(t)}(n, k, m)$ we still want to make nonproper edges nonmonochromatic while counting the max- q -cut.

During the final step of the proof we establish via the coupling technique that

$$\left| \mathbb{E} \max\text{-}q\text{-cut}(H^{(m)}(n, k, m)) - \mathbb{E} \max\text{-}q\text{-cut}(H(n, k, cn / \binom{n}{k})) \right| = O(n^{1/2}).$$

2.2 Proof of the Upper Bound in Theorem 2

Here we follow the ideas from [2] and [6]. It will be convenient again to switch to another model of a random hypergraph. Consider the classical uniform model $H(n, k, m)$ where different $m = \lfloor cn \rfloor$ k -subsets of the vertex set are chosen at random. To prove some upper bound for the max- q -cut, say x , we need to estimate the number of k -uniform hypergraphs on n vertices with m edges and max- q -cut greater than x .

For a hypergraph $H = (V, E)$, consider a vertex q -partition $V = V_1 \sqcup \dots \sqcup V_q$. Let E_i denote the set of edges that are entirely contained in V_i and let E_0 denote the set of edges that are not contained in any V_i . Define $T = (V_1, \dots, V_q, E_0, \dots, E_q)$ and consider the number of all such collections over all k -uniform hypergraphs with vertex set V, m edges and size of E_0 at least x :

$$t(n, m, k, q, x) = |\{T : |E_0| \geq x\}|$$

$$= \sum_{\substack{n_1, \dots, n_q \\ m_1, \dots, m_q}} |\{T : |V_i| = n_i, |E_i| = m_i, |E_0| \geq x\}|$$

Now let $h(n, m, k, q, x)$ denote the maximal summand in the above sum. The following lemma gives the asymptotic behavior of this value.

Lemma 2. For $x > m/2$,

$$h(n, m, k, q, x) \sim \frac{n!}{((n/q)!)^q} \cdot \left(\frac{\binom{n/q}{k}}{\frac{m-x}{q}}\right)^q \cdot \binom{\binom{n}{k} - q \cdot \binom{n/q}{k}}{x}$$

The proof is based on the solution of some optimization problem. Lemma 2 helps to estimate the probability that a random hypergraph $H(n, k, m)$ has max- q -cut greater than x . Clearly, it is less than

$$\frac{h(n, m, k, q, x) \cdot n^{q-1} \cdot (n^k)^q}{\binom{\binom{n}{k}}{m}}$$

It remains to show that this value tends to 0 for

$$x = n \cdot (c \cdot (1 - q^{1-k}) + \sqrt{c} \cdot A_{k,q} + o(\sqrt{c})).$$

2.3 Proof of the Lower Bound in Theorem 2

Here we follow the approach from [3] and [6] and use the greedy procedure to get a large enough cut. Consider all the vertices one by one and color them as follows: assign a color i that will add the maximal number of edges to the current cut. For a step t , let $m(t)$ denote the number of edges that become monochromatic after the coloring of the vertex number $t + 1$ and let $z(t)$ denote the number of edges added to the cut. The following lemma allows us to estimate the expected value of $z(t)$.

Lemma 3

$$\begin{aligned} \mathbb{E}z(t) &\geq \frac{cn}{\binom{n}{k}} \cdot \binom{t}{k-1} \\ &- \left(\frac{c \cdot (t/n)^{k-1} \cdot k}{q^{k-1}} + r_q \cdot \sqrt{\frac{c \cdot (t/n)^{k-1} \cdot k}{q^{k-1}}} \right) \cdot \left(1 + O\left(\frac{1}{n}\right) \right) + o(\sqrt{c}), \end{aligned}$$

where $r_q = \mathbb{E} \min\{\xi_1, \dots, \xi_q\}$, ξ_i are independent $\mathcal{N}(0, 1)$ random variables.

In the proof we first estimate the expected value of $m(t)$. Since the worst case is the case of a balanced coloring on the step t , we have:

$$\mathbb{E}m(t) \leq \mathbb{E} \min\{u_1, \dots, u_q\},$$

where u_1, \dots, u_q are independent random variables with $\text{Bin}(\binom{t/q}{k-1}, p)$ distribution. Further, we use several limit theorems which allow us to get an approximation by the normal distribution.

During the final step of the proof we need to estimate the sum $\sum_{t=1}^{n-1} \mathbf{E}m(t)$ from below by using the expression from Lemma 3. The careful analysis leads us to the required lower bound.

Acknowledgments. The article was prepared within the framework of the HSE University Basic Research Program and funded by RFBR and INSF, project number 20-51-56017.

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Asymptotics for Connected Graphs and Irreducible Tournaments

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Abstract. We compute the whole asymptotic expansion of the probability that a large uniform labeled graph is connected, and of the probability that a large uniform labeled tournament is irreducible. In both cases, we provide a combinatorial interpretation of the involved coefficients.

1 Introduction

Let us consider the Erdős-Rényi model of random graphs $G(n, 1/2)$, where for each integer $n \geq 0$, we endow the set of undirected simple graphs on the set $\{1, \dots, n\}$ with the uniform probability: each graph appears with probability $1/2^{\binom{n}{2}}$. The probability p_n that such a random graph of size n is connected goes to 1 as n goes to ∞ . In 1959, Gilbert [2] provided a more accurate estimation and proved that

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

In 1970, Wright [8] computed the first four terms of the asymptotic expansion of this probability:

$$p_n = 1 - \binom{n}{1} \frac{1}{2^{n-1}} - 2 \binom{n}{3} \frac{1}{2^{3n-6}} - 24 \binom{n}{4} \frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

The method can be used to compute more terms, one after another. However, it does not allow to provide the structure of the whole asymptotic expansion, since no interpretation is given to the coefficients $1, 2, 24, \dots$

The first goal of this paper is to provide such a structure: the k th term of the asymptotic expansion of p_n is of the form

$$i_k 2^{k(k+1)/2} \binom{n}{k} \frac{1}{2^{kn}},$$

where i_k counts the number of irreducible labeled tournaments of size k . A tournament is said *irreducible* if for every partition $A \sqcup B$ of the set of vertices there exist an edge from A to B and an edge from B to A . Equivalently, a tournament is irreducible if, and only if, it is strongly connected [6, 7].

Theorem 1 (Connected graphs). *For any positive integer r , the probability p_n that a random graph of size n is connected satisfies*

$$p_n = 1 - \sum_{k=1}^{r-1} i_k \binom{n}{k} \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where i_k is the number of irreducible labeled tournaments of size k .

In particular, as there are no irreducible tournament of size 2, this explains why there is no term in $\binom{n}{2} \frac{1}{2^{2n}}$ in Wright’s formula. This result might look surprising as it relates asymptotics of undirected objects with directed ones.

A similar development happened for irreducible tournaments. For $n \geq 0$, we endow the set of tournaments on the set $\{1, \dots, n\}$ with the uniform probability: each tournament appears with probability $1/2^{\binom{n}{2}}$. In 1962, Moon and Moser [5] gave a first estimation of the probability q_n that a labeled tournament of size n is irreducible, which was improved in [4] into

$$q_n = 1 - \frac{n}{2^{n-2}} + O\left(\frac{n^2}{2^{2n}}\right).$$

In 1970, Wright [9] computed the first four terms of the asymptotic expansion of the probability that a labeled tournament is irreducible:

$$q_n = 1 - \binom{n}{1} 2^{2-n} + \binom{n}{2} 2^{4-2n} - \binom{n}{3} 2^{8-3n} - \binom{n}{4} 2^{15-4n} + O(n^5 2^{-5n}).$$

Here again, we provide the whole structure of the asymptotic expansion, together with a combinatorial interpretation of the coefficients (they are not all powers of two):

Theorem 2 (Irreducible tournaments). *For any positive integer r , the probability q_n that a random labeled tournament of size n is irreducible satisfies*

$$q_n = 1 - \sum_{k=1}^{r-1} (2i_k - i_k^{(2)}) \binom{n}{k} \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $i_k^{(2)}$ is the number of labeled tournaments of size k with two irreducible components.

We can notice that the coefficients cannot be interpreted as counting a single class of combinatorial objects, since the coefficient $2i_2 - i_2^{(2)} = 0 - 2$ is negative.

2 Notations, Strategy and Tools

Let us denote, for every integer n , g_n the number of labeled graphs of size n , c_n the number of connected labeled graphs of size n , t_n the number of labeled

tournaments of size n , and i_n the number of irreducible labeled tournaments of size n . We have $p_n = c_n/g_n$ and $q_n = i_n/t_n$.

Looking for a proof of Theorem 1, we see two issues: finding a formal relation between connected graphs and irreducible tournaments, and proving the convergence. A tool to settle the first issue is the symbolic method: we associate to each integer sequence its exponential generating function:

$$G(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}, \quad C(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \quad T(z) = \sum_{n=0}^{\infty} t_n \frac{z^n}{n!}, \quad I(z) = \sum_{n=0}^{\infty} i_n \frac{z^n}{n!}.$$

Since $g_n = t_n = 2^{\binom{n}{2}}$, we have

$$G(z) = T(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}. \tag{1}$$

Note that, while the number of labeled tournaments of size n is equal to the number of labeled graphs of size n , their associated species are not isomorphic: for $n = 2$, the two labeled tournaments are isomorphic (by swapping the vertices), while the two labeled graphs are not, so this equality is somewhat artificial.

Since every labeled graph can be uniquely decomposed as a disjoint union of connected labeled graphs, we have

$$G(z) = \exp(C(z)). \tag{2}$$

It remains to find a relation between tournaments and irreducible tournaments.

Lemma 1. *Any tournament can be uniquely decomposed into a sequence of irreducible tournaments.*

In terms of generating functions, Lemma 1 translates to

$$T(z) = \frac{1}{1 - I(z)}. \tag{3}$$

Hence, part of the work will be to let those expressions interplay.

Regarding asymptotics, we will rely on Bender's Theorem [1]:

Theorem 3 (Bender). *Consider a formal power series*

$$A(z) = \sum_{n=1}^{\infty} a_n z^n$$

and a function $F(x, y)$ which is analytic in some neighborhood of $(0, 0)$. Define

$$B(z) = \sum_{n=1}^{\infty} b_n z^n = F(z, A(z)) \quad \text{and} \quad D(z) = \sum_{n=1}^{\infty} d_n z^n = \frac{\partial F}{\partial y}(z, A(z)),$$

Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$, and that for some integer $r \geq 1$ we have

$$(i) \quad \frac{a_{n-1}}{a_n} \rightarrow 0 \text{ as } n \rightarrow \infty; \quad (ii) \quad \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r}) \text{ as } n \rightarrow \infty.$$

Then

$$b_n = \sum_{k=0}^{r-1} d_k a_{n-k} + O(a_{n-r}).$$

3 Proofs

Proof (Proof of Lemma 1). Let T be a tournament. It is either irreducible, and all is done, or it consists of two nonempty parts A and B such that all edges between A and B are directed from A to B . Applying the same argumentation recursively to A and B , we obtain a decomposition of T into a sequence of subtournaments T_1, \dots, T_k , such that each T_i is irreducible and for every pair $i < j$, all edges go from T_i to T_j (see Fig. 1). Since T_i are also the strongly connected components of T , the decomposition is unique.

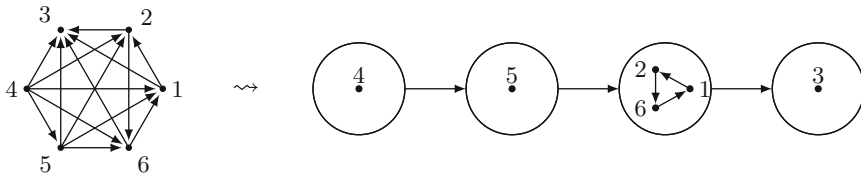


Fig. 1. Decomposition of a tournament as a sequence of irreducible components.

Proof (Proof of Theorem 1). Let us apply Bender’s theorem (Theorem 3) the following way: take

$$A(z) = G(z) - 1 \quad \text{and} \quad F(z, w) = \ln(1 + w).$$

Then, in accordance with formulas (1), (2) and (3),

$$B(z) = \ln(G(z)) = C(z) \quad \text{and} \quad D(z) = \frac{1}{G(z)} = \frac{1}{T(z)} = 1 - I(z).$$

Check the conditions of Theorem 3. In the case at hand, condition (i) has the form:

$$\frac{a_{n-1}}{a_n} = \frac{2^{\binom{n-1}{2}} n!}{(n-1)! 2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

To establish condition (ii), consider $x_k = n!a_k a_{n-k} = \binom{n}{k} 2^{\binom{k}{2} + \binom{n-k}{2}}$, where $r \leq k \leq n-r$. Then (x_k) decreases for $r \leq k \leq n/2$ and increases symmetrically for $n/2 \leq k \leq n-r$. Bounding each summand by the first term is not enough, but bounding each summand (except for the first and last) by the second term gives the following:

$$\begin{aligned} \sum_{k=r}^{n-r} a_k a_{n-k} &\leq \frac{1}{n!} \binom{n}{r} 2^{\binom{r}{2} + \binom{n-r}{2} + 1} + \frac{n-2r-1}{n!} \binom{n}{r+1} 2^{\binom{r+1}{2} + \binom{n-r-1}{2}} \\ &= O\left(\frac{2^{\binom{n^2-(2r+1)n}{2}}}{(n-r)!}\right) + O\left(\frac{2^{\binom{n^2-(2r+3)n}{2}}}{(n-r-2)!}\right) = O(a_{n-r}). \end{aligned}$$

Hence, Bender’s theorem implies

$$b_n = \frac{c_n}{n!} = \frac{2^{\binom{n}{2}}}{n!} - \sum_{k=1}^{r-1} \frac{i_k}{k!} \frac{2^{\binom{n-k}{2}}}{(n-k)!} + O\left(\frac{2^{\binom{n-r}{2}}}{(n-r)!}\right).$$

Dividing by $g_n/n! = 2^{\binom{n}{2}}/n!$, we get

$$p_n = \frac{c_n}{g_n} = 1 - \sum_{k=1}^{r-1} i_k \binom{n}{k} \frac{2^{\binom{n-k}{2}}}{2^{\binom{n}{2}}} + O\left(\frac{n^r}{2^{nr}}\right).$$

Proof (Proof of Theorem 2).

Let us apply Bender’s theorem (Theorem 3) for

$$A(z) = T(z) - 1 \quad \text{and} \quad F(z, w) = -\frac{1}{1+w}.$$

Then, in accordance with formula (3),

$$B(z) = -\frac{1}{T(z)} = -1 + I(z) \quad \text{and} \quad D(z) = \frac{1}{(T(z))^2} = (1 - I(z))^2.$$

Since $(I(z))^2$ is the generating function for the class of labeled tournaments which can be decomposed into a sequence of two irreducible tournaments, we can rewrite the latter identity in the form

$$D(z) = 1 - \sum_{n=1}^{\infty} (2i_k - i_k^{(2)}) \frac{z^n}{n!}.$$

In the case at hand, the conditions that are needed to apply Theorem 3 are the same as in the proof of Theorem 1, since the sequence (a_n) is the same. Hence,

$$b_n = \frac{i_n}{n!} = \frac{2^{\binom{n}{2}}}{n!} - \sum_{k=1}^{r-1} \frac{2i_k - i_k^{(2)}}{k!} \frac{2^{\binom{n-k}{2}}}{(n-k)!} + O\left(\frac{2^{\binom{n-r}{2}}}{(n-r)!}\right).$$

Dividing by $t_n/n! = 2^{\binom{n}{2}}/n!$, we get

$$q_n = \frac{i_n}{t_n} = 1 - \sum_{k=1}^{r-1} (2i_k - i_k^{(2)}) \binom{n}{k} \frac{2^{\binom{n-k}{2}}}{2^{\binom{n}{2}}} + O\left(\frac{n^r}{2^{nr}}\right).$$

4 Further Results

With a bit more work, we can compute the probability that a random graph of size n has exactly m connected components, and the probability that a random tournament of size n has exactly m irreducible components as n goes to ∞ .

In another direction, we can also generalize Theorem 1 to the Erdős-Rényi model $G(n, p)$, where the constant 2 in the formulas is replaced by $\rho = 1/(1-p)$ and the sequence (i_k) is replaced by a sequence of polynomials $(P_k(\rho)) = 1, \rho - 2, \rho^3 - 6\rho + 6, \rho^6 - 8\rho^3 - 6\rho^2 + 36\rho - 24, \dots$ with an explicit combinatorial interpretation.

The methods presented here can also be extended to some geometrical context where connectedness questions appear. In particular, we will provide asymptotics for combinatorial maps, square tiled surfaces, constellations, random tensor model [3]. In some of the models, the coefficients in the asymptotic expansions show connections with indecomposable tuples of permutations and perfect matchings.

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Interval Representation of Balanced Separators in Graphs Avoiding a Minor

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Abstract. We show that for any sufficiently large graph G avoiding K_k as a minor, we can map vertices $v \in V(G)$ to intervals $I(v) \subseteq [0, 1]$ so that (1) $I(u) \cap I(v) \neq \emptyset$ for each edge uv (2) the sum of the squares of the lengths of these intervals is $O(k^6 \log k)$, and (3) the average distance between the intervals is at least $1/25$. Balanced separators of G of sublinear size (with various additional properties) can be read off this representation.

Keywords: Graph theory · Small separators · Minor-closed

1 Interval Representation and Balanced Separators

For a fixed constant $c < 1$, we say that a set X of vertices of an n -vertex graph G is a *balanced separator* if each component of $G - X$ has at most cn vertices. It is customary to take $c = 2/3$, but any constant smaller than 1 gives qualitatively the same results; for the purposes of this paper, we take $c = 0.99$. If necessary, one can usually improve the balance by iteratively adding to X a balanced separator of the largest component of $G - X$.

Let $s(G)$ denote the minimum size of a balanced separator in G , and for a class \mathcal{G} of graphs, let $s_{\mathcal{G}} : \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$s_{\mathcal{G}}(n) = \max\{s(G) : G \in \mathcal{G}, |V(G)| \leq n\}.$$

Classes with sublinear separators (i.e., classes \mathcal{G} with $s_{\mathcal{G}}(n) = o(n)$) are of interest from the computational perspective, as they naturally admit divide-and-conquer style algorithms.

Most known examples of classes of sublinear separators arise from topological or geometric considerations (planar graphs [9], graphs on a fixed surface [7], intersection graphs of connected subsets of a surface with subquadratic number of edges [8], finite element meshes and overlap graphs [11], nearest-neighbor graphs [10], ...). Proper minor-closed classes were shown to have sublinear separators by Alon et al. [1], and provide an interesting example of such graph classes

Supported by the ERC-CZ project LL2005 (Algorithms and complexity within and beyond bounded expansion) of the Ministry of Education of Czech Republic.

that are not explicitly geometric (though they are related to graphs on surfaces via the structure theorem [12]).

We use a variation on the technique developed by Biswal, Lee, and Rao [2] to provide a geometric representation for graphs from proper minor-closed classes from which one can directly read off a balanced separator. For an interval J , let $|J|$ denote the length of J . For two closed intervals J_1 and J_2 , let $d(J_1, J_2)$ denote the distance between them, i.e., the minimum of $|x - y|$ for $x \in J_1$ and $y \in J_2$. For a graph G , an *interval representation of G in $[0, 1]$* is a function I assigning to each vertex of G a closed subinterval of $[0, 1]$ such that for every $uv \in E(G)$, we have $I(u) \cap I(v) \neq \emptyset$. For $c > 0$, the representation I is *c-thrifty* if $\sum_{v \in V(G)} |I(v)|^2 \leq c$. The representation I is *scattered* if $\sum_{u,v \in V(G)} d(I(u), I(v)) \geq |V(G)|^2/25$, i.e., if the average distance between the intervals representing the vertices of G is at least $1/25$.

Theorem 1. *For every positive integer k , there exists a positive real number $c(k) = O(k^6 \log k)$ such that every graph G with at least $10^7 c(k)$ vertices not containing K_k as a minor has a $c(k)$ -thrifty scattered interval representation in $[0, 1]$.*

It is easy to read off a balanced separator in G from such a representation.

Lemma 2. *Let c be a positive real number. If a graph G with n vertices has a c -thrifty scattered interval representation I in $[0, 1]$, then $s(G) \leq 49\sqrt{cn}$.*

Proof. Let $x \in [0, 1]$ be chosen uniformly at random. Let $X = \{v \in V(G) : x \in I(v)\}$.

$$E[|X|] = \sum_{v \in V(G)} \Pr[v \in X] = \sum_{v \in V(G)} |I(v)| \leq \sqrt{n \sum_{v \in V(G)} |I(v)|^2} \leq \sqrt{cn}.$$

Moreover, note that for $v_1, v_2 \in V(G)$, if x separates $I(v_1)$ from $I(v_2)$, then v_1 and v_2 belong to different components of $G - X$. Hence, letting A consist of vertices $v \in V(G)$ such that $\max I(v) < x$ and B consist of vertices $v \in V(G)$ such that $x < \min I(v)$, both A and B are unions of components of $G - X$. Note that

$$\begin{aligned} E[|A|^2 + |B|^2] &\leq n^2 - 2E[|A||B|] = n^2 - 2E[\{(u, v) : \max I(u) < x < \min I(v)\}] \\ &= n^2 - 2 \sum_{u,v \in V(G)} \Pr[\max I(u) < x < \min I(v)] \\ &= n^2 - \sum_{u,v \in V(G)} d(I(u), I(v)) \leq \frac{24}{25}n^2. \end{aligned}$$

Consequently,

$$E[|A|^2 + |B|^2 + \frac{n^{3/2}}{50\sqrt{c}}|X|] \leq \frac{49}{50}n^2,$$

and thus there exists a choice of $x \in [0, 1]$ for which $|A|^2 + |B|^2 + \frac{n^{3/2}}{50\sqrt{c}}|X| \leq \frac{49}{50}n^2$. It follows that $|A|, |B| \leq 0.99n$, i.e., X is a balanced separator, and $|X| \leq 49\sqrt{cn}$.

Let us remark that Biswal et al. [2] obtain a similar representation without the requirement that the intervals representing vertices are subintervals of $[0, 1]$ (and trading off a weaker scattering bound for better thriftiness). Consequently, their representation is less explicit about the separators, as it only implies the existence of a (not necessarily balanced) separator X splitting the graph into two parts A and B such that $|X| = O(n^{-1/2} \min(|A|, |B|))$; thus, in order to obtain a balanced sublinear separator, they have to iterate this process, further splitting the larger part.

Thrifty scattered representations in $[0, 1]$ also capture more general kinds of separators.

Lemma 3. *Let c be a positive real number and suppose a graph G with n vertices has a c -thrifty scattered interval representation I in $[0, 1]$. Then, for any assignment $w : V(G) \rightarrow \mathbb{R}_0^+$ of non-negative weights to vertices of G and any $\varepsilon > 0$, there exist sets $X, Y \subseteq V(G)$ such that $w(X) \leq \varepsilon w(V(G))$, $|Y| \leq 9604c/\varepsilon$, and $X \cup Y$ is a balanced separator in G .*

Let us remark that having the small set Y of potentially unbounded weight in such a balanced separator cannot be avoided; e.g., if G is the star $K_{1,n-1}$ with the center of the star having weight $n-1$ and each of the leaves having the weight 1, then for $\varepsilon < 1/200$, there is no balanced separator X with $w(X) \leq \varepsilon w(V(G))$.

2 Thrifty Scattered Metrics

Consider a graph and a function $r : V(G) \rightarrow \mathbb{R}_0^+$, which we view as assigning a diameter to each vertex of G . For an edge $uv \in E(G)$, we naturally define its length $\ell_r(uv) = \frac{1}{2}(r(u) + r(v))$, and for a path P , we set $\ell_r(P) = \sum_{e \in E(P)} \ell_r(e)$. This defines a pseudometric on $V(G)$, $\mu_r(u, v) = \min(\{\ell_r(P) : P \in \mathcal{P}_G(u, v)\})$, where $\mathcal{P}_G(u, v)$ is the set of all paths from u to v in G . We will also consider a capped version of this pseudometric, $\mu_{r/1}(u, v) = \min(1, \mu_r(u, v))$. For $c, s > 0$, we say that r is c -thrifty if $\sum_{v \in V(G)} r^2(v) \leq c$, and that $\mu_{r/1}$ is s -scattered if $\sum_{u, v \in V(G)} \mu_{r/1}(u, v) \geq s|V(G)|^2$.

Observation 4. *Let c be a positive real number and suppose a graph G has a c -thrifty scattered interval representation I in $[0, 1]$. Let $r(v) = |I(v)|$ for every $v \in V(G)$. Then r is c -thrifty and $\mu_{r/1}$ is $1/25$ -scattered.*

Proof. The fact that r is c -thrifty is trivial. To see that $\mu_{r/1}$ is $1/25$ -scattered, it suffices to observe that $d(I(u), I(v)) \leq \mu_{r/1}(u, v)$.

We now aim to show that a weak converse holds if G avoids a fixed graph as a minor. Let \mathcal{C} be a partition of $V(G)$, let μ be a metric on $V(G)$, and let $\beta > 0$ be a real number. The (μ, β) -center of \mathcal{C} is the set of vertices $v \in V(G)$ such that $\{u \in V(G) : \mu(u, v) \leq \beta\}$ is a subset of a single part of \mathcal{C} , i.e., such that all vertices at μ -distance at most β from v belong to the same part of \mathcal{C} as v . The partition \mathcal{C} is (μ, β) -padded if the (μ, β) -center of \mathcal{C} contains at least half of the vertices of G . For $\Delta > 0$, the partition \mathcal{C} is (μ, Δ) -bounded if for every $C \in \mathcal{C}$,

all vertices $u, v \in C$ satisfy $\mu(u, v) \leq \Delta$. We use the following key property of graphs avoiding a minor, which follows from Theorem 1 of Fakcharoenphol and Talwar [6].

Theorem 5. *There exists a function $\alpha(k) = O(k^2)$ such that for any positive integer k , any graph G avoiding K_k as a minor, any pseudometric μ in $V(G)$ induced by an assignment of non-negative lengths to edges of G , and any $\Delta > 0$, there exists a (μ, Δ) -bounded $(\mu, \Delta/\alpha(k))$ -padded partition of $V(G)$.*

Clearly, we can apply this result to $\mu_{r/1}$ and $\Delta \leq 1$, since $\mu_r(u, v)$ and $\mu_{r/1}(u, v)$ agree up to distance 1. By a argument similar to Theorem 4.4 of Biswal et al. [2], Theorem 5 implies the following claim.

Corollary 6. *Let α be the function from Theorem 5. Let c be a positive real number. Let k be a positive integer and let G be a graph with $n \geq 10^8 c \alpha^2(k)$ vertices avoiding K_k as a minor. Let r be a c -thrifty assignment of non-negative diameters to vertices of G . If $\mu_{r/1}$ is $1/2$ -scattered, then G has a $64\alpha^2(k)c$ -thrifty scattered representation by intervals in $[0, 1]$.*

Theorem 1 now follows from the existence of an assignment of diameters with these properties. Such an assignment can be obtained using a modification of the flow duality argument of Biswal et al. [2].

Lemma 7. *There exists a constant $\kappa > 0$ such that the following claim holds. Let k be a positive integer and let $c = 720\kappa^2 k^2 \log k$. If G is a graph with $n \geq 400c$ vertices and K_k is not a minor of G , then there exists a c -thrifty assignment $r : V(G) \rightarrow \mathbb{R}_0^+$ such that $\mu_{r/1}$ is $1/2$ -scattered.*

3 Generalizations and Open Problems

Similarly to Biswal et al. [2], we can perform the same argument for graphs from any class with polynomial expansion (which is equivalent to having strongly sublinear separators in all subgraphs [4]). We then obtain an $O(n^\beta)$ -thrifty scattered representation for some positive $\beta < 1$. This is good enough to ensure sublinear separators, but not very good in the setting of Lemma 3, where this forces us to have $\Omega(n^{\beta/\varepsilon})$ exceptional vertices in the set Y (while it is known that much fewer suffice, the dependence on n is sublogarithmic [5] and possibly might even be constant).

Since the aforementioned graphs do not have to have balanced separators of order $O(\sqrt{n})$ but rather $O(n^\gamma)$ for some $\gamma < 1$, we cannot hope to directly generalize Theorem 1 for them while keeping c constant. To overcome this issue, it is natural to consider requiring $\sum_{v \in V(G)} r^k(v)$ for some $k > 2$ to be bounded by a constant instead of the current c -thriftness condition. However, this fails even for quite simple graphs already at the stage where we seek a suitable scattered metric (analogue to Lemma 7). Let T_n be the cartesian product of a balanced rooted binary tree B with $n/\log n$ vertices with a path P with $\log n$ vertices. It can be shown that T_n does not admit an assignment of diameters r such that

both r has constant thriftiness and $\mu_{r/1}$ has constant scattering, and thus it also does not admit a scattered representation by intervals in $[0, 1]$ with constant thriftiness.

Let us note that in contrast, T_n satisfies the conclusions of Lemma 3: For any assignment w of weights to vertices of T_n and any $\varepsilon > 0$, there exist sets $X, Y \subseteq V(T_n)$ such that $X \cup Y$ is a balanced separator, $w(X) \geq \varepsilon w(V(T_n))$, and $|Y| \leq 3/\varepsilon$. Indeed, if $\varepsilon < 3/\log n$, then we can set $X = \emptyset$ and let Y be the vertex set of the copy of P corresponding to the root of B . If $\varepsilon \geq 3/\log n$, note that among the middle third of the vertices of P , there exists at least one such that the corresponding copy B' of B satisfies $w(V(B')) \leq w(V(G))/(|V(P)|/3) \leq \varepsilon w(V(G))$, and thus we can set $X = V(B')$ and $Y = \emptyset$.

Moreover, note that T_n is actually quite close to planar graphs, in the following sense: Every planar graph is known to be a subgraph of the strong product of a graph of treewidth 8 with a path [3]. Hence, it seems that Theorem 1 (in the strong form with c being independent on the number of vertices of G) cannot be extended much beyond proper minor-closed classes.

Motivated by the example of T_n , we can consider a weaker concept, allowing different representations for different values of ε . For a polynomial p , a *p-thrifty system of scattered interval representations in $[0, 1]$* is a system $\{I_\varepsilon : \varepsilon > 0\}$, where for each $\varepsilon > 0$, I_ε is a scattered interval representation of G in $[0, 1]$ such that $|I_\varepsilon(v)| \geq \varepsilon$ for at most $p(1/\varepsilon)$ vertices $v \in V(G)$. The existence of such a system implies an analogue of Lemma 3, and promisingly, conversely Lemma 8 below holds. For a polynomial q , we say a graph G has *low-weight separators with exception growth q* if for every assignment $w : V(G) \rightarrow \mathbf{R}_0^+$ and every $\varepsilon > 0$, there exists a balanced separator $X \cup Y$ in G such that $w(X) \leq \varepsilon w(V(G))$ and $|Y| \leq q(1/\varepsilon)$. By the linear programming duality we can prove the following.

Lemma 8. *Let q be a polynomial. If a graph G has low-weight separators with exception growth q , then for every $\varepsilon > 0$, there exists an assignment $r_\varepsilon : V(G) \rightarrow [0, 1]$ such that $r_\varepsilon(v) \geq \varepsilon$ for at most $q(1/\varepsilon)/\varepsilon$ vertices $v \in V(G)$ and $\mu_{r/1}$ is $\frac{49}{5000}$ -scattered.*

We run into further problems when we try to convert such a metric into an interval representation, since Theorem 5 is not true for all classes with low-weight separators with bounded exception growth. However, it is possible the following claim can be proven using some other argument.

Conjecture 9. *For every polynomial q , there exists a polynomial p such that every graph with low-weight separators with exception growth q has a p -thrifty system of scattered interval representations in $[0, 1]$.*

The example of the graph T_n shows that such a system must contain at least two different representations. We do not know whether a fixed number of representations always suffices, or whether the number of distinct representations in the system must grow with $|V(G)|$.

Finally, let us remark that a c -thrifty scattered interval representation in $[0, 1]$ only describes separators in the whole graph G , not in its subgraphs. For

example, the disjoint union of any two graphs with the same number of vertices has a 0-thrifty scattered interval representation, mapping the vertices one of the graphs to the interval $[0, 0]$ and the vertices of the other one to $[1, 1]$. For hereditary graph classes with sublinear separators (e.g., for proper minor-closed classes), it is natural to ask whether the graphs from these classes admit a geometric representation that would also certify sublinear separators in all their subgraphs. Presumably, one could expect to obtain our thrifty scattered interval representation in $[0, 1]$ from this hypothetical representation by a suitable projection.

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Lines in the Manhattan Plane

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Abstract. A well-known theorem in plane geometry states that any set of n non-collinear points in the plane determines at least n lines. Chen and Chvátal asked whether an analogous statement holds within the framework of finite metric spaces, with lines defined using the notion of *betweenness*.

In this paper, we prove that in the plane with the L_1 (also called Manhattan) metric, a non-collinear set induces at least $\lceil n/2 \rceil$ lines. This is an improvement of the previous lower bound of $n/37$, with substantially different proof.

Keywords: Metric spaces · Lines · de Bruijn–Erdős theorem · L_1 distance

1 Lines in Finite Metric Spaces

A well-known theorem in plane geometry states that n points in the plane are either collinear, or they induce at least n lines. Erdős noticed in [5] that this is a corollary of the Sylvester-Gallai theorem (which states that for any non-collinear set X of points in the plane, some line passes through exactly two points of X). Also, it is a special case of a theorem proved by de Bruijn and Erdős [4] about set systems satisfying certain properties.

The proof of de Bruijn and Erdős involves neither measurements of distances, nor measurements of angles. As such it is part of *ordered geometry*, which revolves around the ternary notion of *betweenness*: a point w is between u and v if it is an interior point of the line segment with endpoints u and v . We will write $[uvw]$ to indicate that w is between u and v . In terms of Euclidean distance ρ ,

$$w \text{ is between } u \text{ and } v \iff u, v, w \text{ are distinct points and} \\ d(u, v) = d(u, w) + d(w, v).$$

In an arbitrary metric space, this notion becomes *metric betweenness*, introduced in [7]. The concept of a line is also generalized quite naturally:

A line $\langle u, v \rangle$ consists of u and v and all points w such that one of u, v, w is between the other two.

Supported by grant 19-04113 of the Czech Science Fountain (GACR) and by Charles University project UNCE/SCI/004.

Having introduced this definition in 2006 [2], Chen and Chvátal asked which properties of lines in Euclidean space translate into the setting of arbitrary finite metric spaces. In particular, they posed the following question:

Is it true that every finite metric space (X, ρ) induces at least $|X|$ lines, or there is a line containing all of X ?

This question is still open, although a number of interesting results related to it have been proved; these are surveyed in [3]. Among them, let us mention for future reference a theorem of Aboulker, Chen, Huzhang, Kapadia, and Supko ([1], Theorem 3.1): In an arbitrary metric space, every non-collinear set of n points induces $\Omega(\sqrt{n})$ lines.

In this paper, we concentrate on the plane with the L_1 (also called Manhattan) metric, defined by $\rho((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. We encounter two very different kinds of lines in this case: those induced by two points that share a coordinate (horizontal or vertical pairs), and those induced by pairs that do not share a coordinate (so called increasing or decreasing pairs). The two types of lines (for increasing/decreasing pairs of points, and for horizontal/vertical pairs) can be seen in Fig. 1.

Balázs Patkós and this author proved in [6] that every non-collinear set of n points in this particular metric space induces at least $n/37$ lines. Moreover, if no two of the points share their x - or y -coordinate, then there are at least n lines, i.e., the Chen—Chvátal question has an affirmative answer. The proof relies on a lemma that was later extended and used by Aboulker, Chen, Huzhang, Kapadia, and Supko in deriving the weaker lower bound $\Omega(\sqrt{n})$ valid for all metric spaces.

In the present paper, we improve the lower bound $n/37$ by a completely different method.

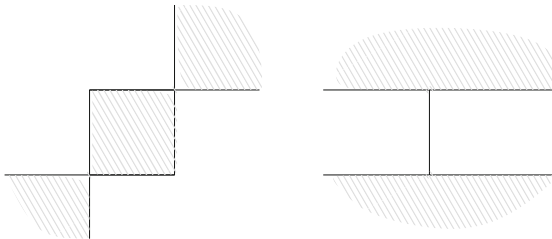


Fig. 1. Line induced by an increasing pair and by a vertical pair.

Theorem 1. *Let X be a set of n points in the plane with the L_1 metric. Either there is a line containing all of X , or X induces at least $\lceil n/2 \rceil$ lines.*

In the rest of this paper, X will be a set of n points in the plane with the L_1 metric and we will assume that there is no line containing all of X . We will only consider lines induced by increasing and decreasing pairs of points.

2 Increasing and Decreasing Pairs, Blue and Red Arrows

In order to prove Theorem 1, we will introduce an auxiliary graph G on the vertex set X as follows. Fix a line L and suppose that F is the family of all increasing pairs that induce L . We select a member of F that is in some sense minimal and put a (directed) edge between the two points. Repeat this for all lines. Then repeat the process for all lines and decreasing pairs as well. With a few exceptions, distinct edges represent distinct lines. Whenever two edges represent the same line, delete one of the edges. The resulting graph G' may have some isolated vertices (and indeed, the original graph G may have them as well). But we can show that for each isolated vertex there is another point nearby, which has degree at least 2. Counting the degrees, we find out that there are at least the required number of edges, and hence at least the required number of lines.

For $p \in X$ we define $x(p)$ to be the x -coordinate of p and $y(p)$ the y -coordinate. Let $\{p, q\}$ be a pair of points in the plane. We say that it is an *increasing pair* if $(x(p) - x(q)) \cdot (y(p) - y(q)) > 0$. A *decreasing pair* has the inequality reversed. A *horizontal pair* has $y(p) = y(q)$, while *vertical pair* satisfies $x(p) = x(q)$. The line defined by points u, v will be denoted by $\langle u, v \rangle$.

Our goal now is to introduce a partial order on the set of increasing pairs that induce a given line. In order to do that, we first introduce a partial order on the points themselves. If $c, d \in X$ with $x(c) \leq x(d)$ and $y(c) \leq y(d)$, we write $c \leq_I d$. This is a partial order on X . Let us consider all increasing pairs that induce a line L . For two such pairs, $\{p, q\}$ with $p \leq_I q$ and $\{a, b\}$ with $q \leq_I b$, we put $\{p, q\} \leq_I^* \{a, b\}$ if either $q <_I b$, or if the first pair is “nested” in the second pair, that is, $p \geq_I a$ and $q \leq_I b$. All pairs in $\{p, q, a, b\}$ are comparable in \leq_I , so \leq_I^* is a linear order on the increasing pairs inducing L . For each L , pick the least element $\{a, b\}$ and suppose that $a \leq_I b$. We say that the ordered pair (a, b) is a *blue arrow*.

We repeat the process for decreasing pairs: we write $c \leq_D d$ if $x(c) \leq x(d)$ and $y(c) \geq y(d)$. We define \leq_D^* and use it to select some decreasing pairs and call them *red arrows*, all in complete analogy with the increasing case. The graph G mentioned in the first paragraph will have red and blue arrows as its edges. (See left part of Fig. 2 for an example of configuration with an isolated point and some blue and red arrows).

Note that we may have situations where, e.g., $\langle u, v \rangle = \langle w, v \rangle$ for increasing pairs $\{u, v\}$, $\{w, v\}$, and $\{u, w\}$ is a horizontal pair. That is why we defined \leq_I and \leq_D in such a way that they accommodate horizontal and vertical pairs as well.

3 What Happens Around Isolated Points?

The aim of this section is to show that there are many arrows. In particular, if a is an isolated vertex in the graph G , there is some nearby vertex that is the endpoint of two arrows.

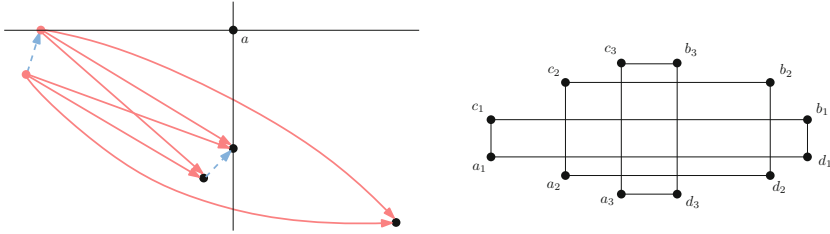


Fig. 2. Left: a configuration with an isolated point and some red (solid) and blue (dashed) arrows. Right: blue and red arrows generating the same lines.

The statements in this section are proved by a careful analysis of point configurations and are outside the scope of this abstract. Let us note that any point $a \in X$ defines four quadrants of the plane respective to this point. These are numbered anticlockwise in the usual manner and are considered to be open (i.e., they do not contain the boundary lines).

Lemma 1. *If a is an isolated vertex of the graph G , and X has some points in the union of the first, second, and fourth quadrants relative to a , then there are points r, r', s and red or blue arrows $(r, s), (r', s)$ as depicted in Fig. 3.*

Moreover, the line segments between r and a and between s and a are empty. Also, it is not hard to prove that without loss of generality, X contains no points with first, second, and fourth quadrants empty.

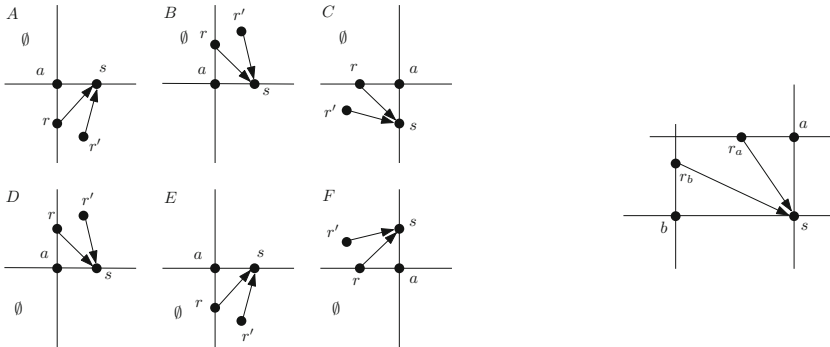


Fig. 3. Six situations for isolated vertices, and an example of conflict.

For each isolated vertex a , Lemma 1 provides a point s_a such that $\{a, s_a\}$ is a horizontal or vertical pair and there are two blue or two red arrows into s_a . Define $f(a) = (s_a, \text{blue})$ if the two arrows are blue, $f(a) = (s_a, \text{red})$ otherwise. If I is the set of isolated vertices, then this f maps I into $(X \setminus I) \times \{\text{blue, red}\}$.

In the following lemma, we note that the situation depicted in the right side of Fig. 3 never happens.

Lemma 2. *The mapping f is injective.*

4 The Lines Induced by Arrows are (Mostly) Distinct

Two red arrows by definition correspond to different lines. The same is true for two blue arrows. One can prove that if a blue and a red arrow induce the same line, then the four points are located at corners of a rectangle in the plane. Moreover, if we have more such pairs of arrows, the rectangles are arranged in a way described by Lemma 3, which can be seen on the right side of Fig. 2.

Lemma 3. *Let (a_i, b_i) for $i = 1, \dots, k$ be blue arrows and (c_i, d_i) be red arrows such that $\langle a_i, b_i \rangle = \langle c_i, d_i \rangle$ for all i . We may number the points in such a way that $a_1 <_D a_2 <_D \dots <_D a_k, b_1 >_D \dots >_D b_k, c_1 <_I \dots <_I c_k$ and $d_1 >_I \dots >_I d_k$.*

Lemma 4. *Let (a_i, b_i) for $i = 1, \dots, k$ be blue arrows and (c_i, d_i) be red arrows such that $\langle a_i, b_i \rangle = \langle c_i, d_i \rangle$ for all i . For each i there is an arrow starting in a_i , other than (a_i, b_i) . Also for each c_i , there is an arrow starting in c_i , other than (c_i, d_i) .*

We will not prove these lemmas in this extended abstract.

5 Counting Degrees

Proof (Proof of Theorem 1). Let us consider the directed graph G on the vertex set X , where the edges are the red and blue arrows. This graph might have some isolated vertices. In Sect. 3, we defined a mapping f that assigns a pair (s_a, red) or (s_a, blue) to each isolated a , where s_a is a point located close to a and the second entry denotes the color of the two arrows ending in s_a that are assigned to this a . We have also noted that this mapping is injective.

Let c be the number of isolated vertices. We have c pairs (a, s_a) , and with each pair we associate two arrows of the same color ending in s_a . We may have $s_a = s_b$ for distinct isolated vertices a and b , but by the injectivity of f , the pairs of arrows are pairwise disjoint. If I is the set of isolated vertices, the set $A := I \cup \{s_a; a \in I\}$ has at most $2c$ vertices, and

$$\sum_{v \in A} \text{deg}^+(v) \geq 2c.$$

As before, let (a_i, b_i) and (c_i, d_i) for $i \in \{1, \dots, k\}$ be the coinciding red and blue arrows. Let $B := \{a_1, \dots, a_k, c_1, \dots, c_k\}$. We have shown in Sect. 4 that each vertex in B has outdegree at least 2. We have $|A \cup B| \leq 2c + 2k$, so

$C := X \setminus (A \cup B)$ has size at least $n - 2c - 2k$. The vertices outside $A \cup B$ have at least one arrow starting or ending in them. We have

$$\begin{aligned} 2|E(G)| &= \sum_{v \in X} \deg^+(v) + \sum_{v \in X} \deg^-(v) \\ &\geq \sum_{v \in A} \deg^+(v) + \sum_{v \in B} \deg^-(v) + \sum_{v \in C} \deg^+(v) + \sum_{v \in C} \deg^-(v) \\ &\geq 2c + 2 \cdot 2k + (n - 2c - 2k) \\ &= n + 2k. \end{aligned}$$

Now delete the arrow (a_i, b_i) for each $i = 1, \dots, k$. This modified graph G' has $2|E(G')| \geq n$. It follows that G' has at least $n/2$ arrows, each corresponding to a distinct line of the original metric space.

6 L_∞ Metric

Let us briefly consider the L_∞ metric on \mathbb{R}^d , defined by

$$d((u_1, \dots, u_d), (v_1, \dots, v_d)) = \max_{1 \leq j \leq d} |u_j - v_j|.$$

Whenever we have a finite set in the plane with the L_∞ metric, rotating the plane by 45 degrees transforms the L_∞ -lines into L_1 -lines. The following theorem is an easy consequence of Theorem 1.

Theorem 2. *Let X be a set of n points in the plane with the L_∞ metric. Either there is a line containing all of X , or X induces at least $\lceil n/2 \rceil$ lines.*

This may be of interest because any finite metric space can be regarded as an L_∞ space in \mathbb{R}^d for some d . However, the required d may be very high, which makes handling it difficult.

Acknowledgment. I would like to thank Vašek Chvátal for his many insights into the topic of lines in metric spaces, as well as useful remarks concerning the full version of this paper.

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A Rainbow Connectivity Threshold for Random Graph Families

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Abstract. Given a family \mathcal{G} of graphs on a common vertex set X , we say that \mathcal{G} is *rainbow connected* if for every vertex pair $u, v \in X$, there exists a path from u to v that uses at most one edge from each graph of \mathcal{G} . We consider the case that \mathcal{G} contains s graphs, each sampled randomly from $G(n, p)$, with $n = |X|$ and $p = \frac{c \log n}{sn}$, where $c > 1$ is a constant. We show that there exists a threshold of at most three consecutive integer values such that when s is greater than or equal to this threshold, \mathcal{G} is a.a.s. rainbow connected, and when s is below this threshold, \mathcal{G} is a.a.s. not rainbow connected.

Keywords: Graph theory · Random graph · Rainbow path

1 Introduction

We consider random graphs using the *Erdős-Rényi model*, that are defined as follows. Let n be a positive integer and $0 \leq p \leq 1$. We construct a graph G with $V(G) = [n] := \{1, \dots, n\}$ by independently letting each edge $e \in \binom{[n]}{2}$ belong to $E(G)$ with probability p . We say that G is a *random graph* in $G(n, p)$. When a statement involving a value n holds with probability approaching 1 as n approaches infinity, we say that the statement holds *asymptotically almost surely*, or *a.a.s.* for short.

One particular property of random graphs that has been the focus of extensive research is the diameter. Recall that the *diameter* $\text{diam}(G)$ of a graph G is the maximum distance $\text{dist}(u, v)$ taken over all vertex pairs u, v in the graph. In a seminal paper on random graphs from 1959, Erdős and Rényi [8] showed that if G is a random graph in $G(n, p)$, where $p = \frac{c \log n}{n}$ and c is a constant, then G is a.a.s. connected when $c > 1$ and a.a.s. disconnected when $c < 1$. This result of Erdős and Rényi was essentially the first result on the diameter of random graphs, giving a probability threshold for when the diameter of a random graph is finite. Later, in 1974, Burtin [5] determined that when $p \gg n^{-\frac{d-1}{d}}$ for a positive integer d , a random graph in $G(n, p)$ has diameter at most d a.a.s. Klee and Larman [12] rediscovered this result in 1981. Bollobás [3] then showed in 1984 that when a graph G on n vertices has $\frac{c \log n}{n} \binom{n}{2}$ randomly placed edges (where $c > 1$ is a constant), its diameter is a.a.s. equal to one of at most four consecutive

integer values. Chung and Lu [6] later translated this result of Bollobás into the random setting $G(n, p)$, giving the following bounds.

Theorem 1 [6]. *Let G be a random graph in $G(n, p)$, where $p = \frac{c \log n}{n}$ and $c > 1$. Then a.a.s.,*

$$\frac{\log\left(\frac{c}{11}\right) + \log n}{\log c + \log \log n} \leq \text{diam}(G) \leq \frac{\log\left(\frac{33c^2}{400}\right) + \log \log n + \log n}{\log c + \log \log n} + 2.$$

In other words, Chung and Lu show that the diameter of G is a.a.s. one of at most four consecutive integer values, each within a constant from $\frac{\log n}{\log c + \log \log n}$.

In seeking the diameter of a random graph G , one essentially asks the following question: For which values of s does there a.a.s. exist a path of length at most s between every pair of vertices in G ? In this paper, we ask a similar question in the following *rainbow* setting.

We consider a family $\mathcal{G} = \{G_1, \dots, G_s\}$ of s graphs on a common vertex set $[n]$. We say that a path $P \subseteq \bigcup_{i=1}^s E(G_i)$ is a *rainbow path* if there exists an injection $\phi : E(P) \rightarrow [s]$ such that for each edge $e \in E(P)$, $e \in E(G_{\phi(e)})$. Two vertices $u, v \in [n]$ are *rainbow connected* if there exists a rainbow path with u and v as endpoints. Furthermore, we say that \mathcal{G} is *rainbow connected* if every pair $\{u, v\} \in \binom{[n]}{2}$ is rainbow connected. If we let each graph $G_i \in \mathcal{G}$ have its edges colored with the color i , then we may equivalently define a rainbow path as a path that uses at most one edge of each color. Note that we allow an edge joining u and v to appear in more than one of the graphs in the family and that we consider repetitions as having multiple edges, each of different color. For a given family of random graphs, we will ask, for which values of s does there a.a.s. exist a rainbow path of length at most s between every pair of vertices in $[n]$. Equivalently, we ask, for which values of s is \mathcal{G} a.a.s. rainbow connected.

Rainbow connectivity has been studied by Kamčev, Krivelevich, and Sudakov in [11]. Rainbow paths are also an example of rainbow structures, which are defined as edge-colored subgraphs in which each edge has a distinct color. Depending on the setting, rainbow subgraphs are often referred to either as transversals or partial transversals of a graph family. The study of rainbow graphs dates back as far as the eighteenth century in Euler’s work on Latin squares. Recently, rainbow graph structures have received increasing attention. For instance, Alon, Pokrovskiy, and Sudakov show that every properly colored complete graph contains a long rainbow cycle [2], and Gao et al. prove that a family of sufficiently dense 3-uniform hypergraphs contains a rainbow matching [9]. Additionally, certain classical results have been extended into the rainbow setting. For instance, a famous theorem of Dirac [7] states that a graph on n vertices with minimum degree at least $n/2$ must contain a Hamiltonian cycle. Joos and Kim [10] have generalized Dirac’s result to show that every family $\mathcal{G} = \{G_1, \dots, G_n\}$ of n graphs with $V(G_i) = [n]$ ($1 \leq i \leq n$), each of minimum degree at least $n/2$, contains a rainbow Hamiltonian cycle. Note that if all graphs G_i are the same, this is the same as Dirac’s theorem.

In the same flavor, a classic result of Moon and Moser [14] gives a minimum degree condition for the existence of a Hamiltonian cycle in a bipartite graph.

One of the authors of this paper, generalized this result to the rainbow setting [4]. In addition to rainbow Hamiltonian cycles, certain other rainbow structures have been shown to exist under appropriate conditions. For instance, Aharoni et al. [1] obtained a rainbow version of Mantel’s theorem, proving that a family \mathcal{G} of three graphs on a common set of n vertices contains a rainbow triangle if each graph in \mathcal{G} contains at least $0.2557n^2$ edges.

Our Results

First, we fix some notation that we will use throughout the paper. We let n be a large integer and we consider families of random graphs of order n . We pick an integer $s \geq 1$, depending on n , and we let $c > 1$ be a fixed constant. We let

$$p = \frac{c \log n}{sn}.$$

Then, for $1 \leq i \leq s$, we take a random graph G_i in $G(n, p)$, and let $\mathcal{G} = \{G_1, \dots, G_s\}$. We refer to the values $1, \dots, s$ as *colors*, and we will often use language suggesting that each graph G_i has its edges colored monochromatically with color i . In this setting, it is straightforward to show that an edge $e \in \binom{[n]}{2}$ belongs to at least one graph $G_i \in \mathcal{G}$ with probability $(c - o(1)) \frac{\log n}{n}$, and therefore, by the threshold of Erdős and Rényi [8], $\bigcup_{i=1}^s G_i$ is connected a.a.s. when $c > 1$ (and disconnected when $c < 1$).

In the following two main theorems, we will find a threshold for the number of graphs required in \mathcal{G} to ensure rainbow connectivity.

Theorem 2. *Let $\mathcal{G} = \{G_1, \dots, G_s\}$ be a family of s graphs on a common set of n vertices, each taken randomly from $G(n, p)$, with $p = \frac{c \log n}{sn}$, where $c > 1$ is a constant. If*

$$s \leq \frac{\log n}{\log c - 1 + \log \log n} - \frac{1}{2} + \frac{\log \log \log n}{3 \log \log n},$$

then a.a.s. \mathcal{G} is not rainbow connected.

Theorem 3. *Let $\mathcal{G} = \{G_1, \dots, G_s\}$ be a family of s graphs on a common set of n vertices, each taken randomly from $G(n, p)$, with $p = \frac{c \log n}{sn}$, where $c > 1$ is a constant. If*

$$s \geq \frac{\log n}{\log c - 1 + \log \log n} + \frac{3}{2} + \frac{2\sqrt{\log \log \log n}}{\log \log n},$$

then a.a.s. \mathcal{G} is rainbow connected.

Together, Theorems 2 and 3 show that the minimum value s that guarantees that the s graphs in \mathcal{G} a.a.s. make a rainbow connected family is concentrated on at most three consecutive integer values, each within a constant of $\frac{\log n}{\log c - 1 + \log \log n}$. By comparing this threshold with the result of Theorem 1, we see that the diameter of $\bigcup_{i=1}^s G_i$ is a.a.s. of the form $\frac{\log n}{\log c + o(1) + \log \log n}$, which is slightly smaller (by about $\frac{\log n}{(\log \log n)^2}$) than the minimum value of s that a.a.s. ensures rainbow connectivity in \mathcal{G} .

2 Proof of Theorem 3

Theorem 2 can be proved with a simple application of the First Moment Method (c.f. [13, Chapter 3]). However, proving Theorem 3 takes more work. Therefore we omit the details of the proof of Theorem 2 in this extended abstract and provide more details about the proof of Theorem 3. For this entire section, we set

$$d = \lceil \log \log \log \log n \rceil.$$

2.1 Spheres and Breadth-First Search for Rainbow Paths

We aim to determine the number s of colors needed to make \mathcal{G} a.a.s. rainbow connected. In order to estimate this value s , we will need the following definition.

Definition 1. Let $v \in [n]$, $C \subseteq [s]$, and let $t \geq 0$ be an integer. Then we define $\Gamma_t^C(v)$ to be the set of vertices $u \in [n]$ satisfying the following two conditions:

- u can be reached from v by a rainbow path of length t consisting of edges of graphs G_i for which $i \in C$;
- u cannot be reached from v by a rainbow path of length at most $t - 1$ consisting of edges of graphs G_i for which $i \in C$.

We will refer to these sets $\Gamma_t^C(v)$ as *spheres*. We observe that for each vertex $v \in [n]$, v is rainbow connected with every vertex in $\bigcup_{i=0}^s \Gamma_i^{[s]}(v)$. For a vertex $v \in [n]$, our sets $\Gamma_i^C(v)$ can be computed recursively with a breadth-first search. First, we let $\Gamma_0^C(v) = \{v\}$. Then, for $0 \leq t \leq |C| - 1$, we can compute $\Gamma_{t+1}^C(v)$ from $\Gamma_t^C(v)$ as follows. We consider each vertex $w \in \Gamma_t^C(v)$ individually. There exists a nonempty set \mathcal{P}_w of rainbow paths from v to w of length exactly t . For each path $P \in \mathcal{P}_w$, we denote by $C(P)$ the set of t colors of the edges of P . Then, for each path $P \in \mathcal{P}_w$, we search for vertices u that have not been reached before in some $\Gamma_z^C(v)$ with $z \leq t$ and for which $wu \in E(G_i)$ for some color $i \in C \setminus C(P)$. We add every such u to $\Gamma_{t+1}^C(v)$. By carrying out this process for each vertex $w \in \Gamma_t^C(v)$ and each path $P \in \mathcal{P}_w$, we obtain $\Gamma_{t+1}^C(v)$. When we calculate bounds for s , we will be interested in estimating the sizes of these spheres $\Gamma_t^C(v)$. We have two lemmas to help us.

Lemma 1. It holds a.a.s. that for each graph $G_i \in \mathcal{G}$, $\Delta(G_i) < \frac{2c \log n}{\log \log n}$.

Lemma 2. There exists a value $\varepsilon = \varepsilon(c) > 0$ such that a.a.s., for every vertex $v \in [n]$ and every set $C \subseteq [s]$ of size at most $d + 1$, $|\Gamma_1^{[s] \setminus C}(v)| \geq \varepsilon \log n$.

We will use the value ε from Lemma 2 throughout the rest of the paper.

2.2 The Proof

In this section, we sketch the proof of Theorem 3. Our strategy is to show that for an arbitrary vertex pair $u, v \in [n]$, u and v are rainbow connected with probability $1 - o(\frac{1}{n^2})$, from which it will follow that \mathcal{G} is a.a.s. rainbow connected.

Lemma 3. *It holds a.a.s. that for every vertex $u \in [n]$, there exists a value $t^* \leq d$ for which*

$$|\Gamma_{t^*}^{[s-1]}(u)| \geq \frac{1}{2}\varepsilon(c \log n)^d.$$

Now, we move to the main strategy. We consider a vertex pair $u, v \in [n]$. By Lemma 3, we may assume that $|\Gamma_{t^*}^{[s-1]}(u)| \geq \frac{1}{2}\varepsilon(c \log n)^d$ for some $t^* \leq d$. For a vertex $w \in \Gamma_{t^*}^{[s-1]}$, let P_w be a rainbow path from u to w , and let $C(P_w)$ be the set of colors used in $E(P_w)$. We define $R_w := [s - 1] \setminus C(P_w)$ and $r := s - d - 1 \leq |R_w|$. We will consider the spheres $\Gamma_t^{R_w}(v)$ for $t \leq r$. In order for the growth of our spheres to be independent from observations about P_w , we will not use vertices of $\bigcup_{t=0}^d \Gamma_t^{[s-1]}(u)$ in our search from v , but this does not negatively affect us.

We write $\xi = \frac{n}{(\log n)^{d-1}}$, and we would like to show that $|\Gamma_t^{R_w}(v)| \geq \xi$ for some value $t \leq r$. Then, we can make the following argument.

Claim. For each $w \in \Gamma_{t^*}^{[s-1]}(u)$, let P_w, R_w , and r be defined as above. Suppose that for each $w \in \Gamma_{t^*}^{[s-1]}(u)$, the inequality $|\Gamma_{t_w}^{R_w}(v)| \geq \xi$ holds for some $t_w \leq r$. Then u and v are rainbow connected with probability $1 - o(\frac{1}{n^2})$.

Proof. Consider a vertex $w \in \Gamma_{t^*}^{[s-1]}(u)$. Let E_w be the set of edges with one endpoint at w and the other endpoint in $\Gamma_{t_w}^{R_w}(v)$. Since neither P_w nor $\Gamma_{t_w}^{R_w}(v)$ uses the color s , if some edge in E_w belongs to G_s , then u and v are rainbow connected. Now, if the hypothesis of the claim holds, then E_w must have at least ξ edges. Furthermore, as our spheres $\Gamma_{t_w}^{R_w}(v)$ avoid $\bigcup_{t=0}^d \Gamma_t^{[s-1]}(u)$, for two vertices $w, w' \in \Gamma_{t^*}^{[s-1]}(u)$, E_w and $E_{w'}$ are disjoint. Thus, the union of all sets E_w over all vertices $w \in \Gamma_{t^*}^{[s-1]}(u)$ must have at least $\xi \cdot \frac{1}{2}\varepsilon(\log n)^d > 4 \log n/p$ distinct edges, and if any one of these $4 \log n/p$ edges belongs to G_s , then u and v are rainbow connected. Furthermore, with probability at least $1 - (1 - p)^{4 \log n/p} = 1 - o(n^{-2})$, there exists an edge of G_s in some edge set E_w , giving a rainbow path between u and v .

For $0 \leq t \leq r$, we write $\Gamma_t = \Gamma_t^{R_w}(v)$. By our claim, it suffices to show that for each $w \in \Gamma_{t^*}^{[s-1]}(u)$, we have $|\Gamma_{t_w}(v)| \geq \xi$ for some value $t_w \leq r$, and then Theorem 3 will be proven. We will define values L_t with the goal of showing that if $|\Gamma_{t_w}(v)| \geq \xi$ does not hold for some value $t_w \leq r$, then with high probability, $|\Gamma_t(v)| \geq L_t$ for all values $0 \leq t \leq r$, which will ultimately give us a contradiction. In order to estimate $|\Gamma_t(v)|$ for $0 \leq t \leq r$, we will carry out a breadth first search from v . We define $\delta_1 = \delta_2 = (\log n)^{-1/3}$, and we define

$\delta_t = (\log n)^{-1}$ for $3 \leq t \leq r-1$. We also define $\phi = \frac{1}{\log n}$ and $\alpha(t) = 1 - (r-t)p$. Then, we define $L_0 = 1$ and

$$L_t = \left(\frac{1}{2}\varepsilon \log n\right) \left(\frac{c \log n}{s}\right)^{t-1} (r-1)^{t-1} (1-\phi)^{2t-2} \prod_{i=1}^{t-1} \alpha(i)(1-\delta_i).$$

Using our breadth-first search technique, we can show that with probability $1 - o\left(\frac{1}{n^3}\right)$, for all values $0 \leq t \leq r$, $|L_t| \geq L_t$. By substituting $t = r$ in this inequality and simplifying, we may obtain the following inequality:

$$\begin{aligned} \frac{|L_r|}{n/(\log n)^{d-1}} &\geq \exp(s \log \log n + s(\log c - 1) - \log n - \frac{3}{2} \log \log n + O(d)) \\ &\geq \exp(\sqrt{\log \log \log n} + O(d)) \\ &> 1. \end{aligned}$$

Thus, we conclude that $|L_r| \geq \frac{n}{(\log n)^{d-1}}$. By our claim, the vertex pair u, v is rainbow connected with probability $1 - o\left(\frac{1}{n^2}\right)$, which implies that every vertex pair is rainbow connected with probability $1 - o(1)$. \square

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Robust Connectivity of Graphs on Surfaces

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Abstract. Let $\Lambda(T)$ denote the set of leaves in a tree T . One natural problem is to look for a spanning tree T of a given graph G such that $\Lambda(T)$ is as large as possible. Recently, a similar but stronger notion called the *robust connectivity* of a graph G was introduced, which is defined as the minimum value $\frac{|R \cap \Lambda(T)|}{|R|}$ taken over all nonempty subsets $R \subseteq V(G)$, where $T = T(R)$ is a spanning tree on G chosen to maximize $|R \cap \Lambda(T)|$. We prove a tight asymptotic bound of $\Omega(\gamma^{-\frac{1}{r}})$ for the robust connectivity of r -connected graphs of Euler genus γ . Moreover, we give a surprising connection between the robust connectivity of graphs with an edge-maximal embedding in a surface and the *surface connectivity* of that surface, which describes to what extent large induced subgraphs of embedded graphs can be cut out from the surface without splitting the surface into multiple parts. For planar graphs, this connection provides an equivalent formulation of a long-standing conjecture of Albertson and Berman.

Keywords: Robust connectivity · Graphs on surfaces ·
Albertson-Berman conjecture

1 Introduction

Let $\Lambda(T)$ denote the set of leaves in a tree T . Given a graph G , we denote by τ_G the set of all spanning trees in G . The *maximum leaf number* (or *maxleaf number*) of a graph G is defined as $\ell(G) := \max_{T \in \tau_G} |\Lambda(T)|$.

Questions about maximum leaf number have been thoroughly considered throughout the literature, and MAXIMUM LEAF NUMBER was one of the original NP-complete problems (even when restricted to planar graphs of maximum

T.M. was supported by a postdoctoral fellowship at the Simon Fraser University through NSERC grants R611450 and R611368. J.N. received support for foreign internships for the WDSMCS students in the framework of “Excellence initiative – research university” programme at MIM, University of Warsaw. L.S. was supported by NSERC grant R611368. The full version of the paper can be found on arXiv [4].

degree 4) identified by Garey and Johnson [7]. Storer [10] considered the problem of finding a lower bound for the maximum leaf number of cubic graphs, and he proved that every cubic graph on n vertices has a spanning tree with at least $\lceil \frac{n}{4} + 2 \rceil$ leaves. Kleitman and West [9] gave an algorithm for a connected graph G of minimum degree k that shows $\ell(G) \geq (1 - \frac{2.51 \ln k}{k})n$.

The maximum leaf number problem can be equivalently formulated as a *minimum connected dominating set* problem, which is a problem where the task is to find a smallest connected subset of vertices $D \subseteq V(G)$ of a graph G , such that every vertex of G is in the closed neighborhood of D . Both formulations of the maximum leaf number problem have been studied from the computational point of view in many areas of computer science [6]. In this paper, we will consider a graph invariant related to maximum leaf number known as *robust connectivity*¹. The robust connectivity $\kappa_\rho(G)$ of a graph G is defined as follows.

Definition 1 (Robust connect. [5]).

$$\kappa_\rho(G) := \min_{\substack{R \subseteq V(G) \\ R \neq \emptyset}} \max_{T \in \tau_G} \frac{|R \cap \Lambda(T)|}{|R|}.$$

We often write $\ell(G, R)$ for the maximum value of $\frac{|R \cap \Lambda(T)|}{|R|}$ taken over all spanning trees T of G , in which case $\kappa_\rho(G) = \min_{R \subseteq V(G)} \ell(G, R)$. In [5], it was shown that

a non-regular graph G of maximum degree Δ is $\frac{\kappa_\rho(G)}{2\Delta}$ -flexibly Δ -choosable.

Despite being useful for establishing bounds in certain problems like flexible list coloring, robust connectivity does not appear to be simple to calculate. However, in [5], it was shown that for graphs of bounded degree, 3-connectivity is enough to guarantee some absolute lower bound for robust connectivity.

Theorem 1 (Theorem 22 in [5]). *If $\Delta \geq 3$ is an integer, then there exists a value $\varepsilon = \varepsilon(\Delta) > 0$ such that if G is a 3-connected graph of maximum degree Δ , then $\kappa_\rho(G) \geq \varepsilon$.*

The authors of [5] also showed that 3-connectivity alone is not enough to guarantee a lower bound on a graph’s robust connectivity. To demonstrate this fact, the authors used the Levi graph of the complete 3-uniform hypergraph $K_n^{(3)}$, described in Example 1. They also pointed out that 2-connected cubic planar graphs do not have any guaranteed nonzero lower bound for robust connectivity, as demonstrated by Fig. 1.

Example 1 ([5]). Let G be a graph whose vertex set consists of a set R of at least four vertices and an additional vertex v_A for each triplet $A \in \binom{R}{3}$, and let each vertex of the form v_A be adjacent exactly to those vertices in the triplet A .

It is straightforward to show that the Levi graph G of $K_n^{(3)}$ in Example 1 is 3-connected. However, no more than two vertices of R may be removed from G

¹ This parameter was formerly called “game connectivity” in [5]. However, we believe that the term “robust connectivity” better suits the properties of this parameter.

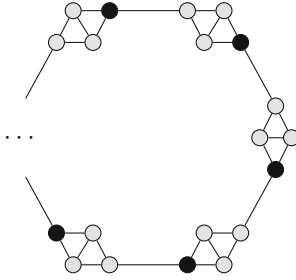


Fig. 1. The graph G in the figure is an arbitrarily large two-connected 3-regular graph. If a set $R \subseteq V(G)$ is chosen as shown by the dark vertices in the figure, then there does not exist a constant $\varepsilon > 0$ such that $\varepsilon|R|$ vertices of R may become leaves of some spanning tree of G . Therefore, the robust connectivity of G is arbitrarily close to zero. For any $k \geq 3$, a similar k -regular graph with robust connectivity arbitrarily close to zero with may be constructed from a cycle C by replacing each vertex of C by a k -clique minus an edge.

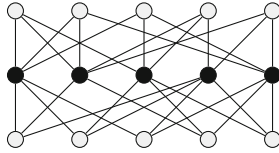


Fig. 2. [Figure 4 in [5]] The figure shows the Levi graph of $K_5^{(3)}$, which is a 3-connected graph constructed based on Example 1 with $|R| = 5$.

without disconnecting G . Therefore, for any spanning tree T on G , the leaves of T include at most two vertices of R . As R becomes arbitrarily large, the proportion of vertices in R that can be included as leaves in a spanning tree on G becomes arbitrarily small. Therefore, Example 1 shows that some 3-connected graphs G do not satisfy $\kappa_\rho(G) \geq \varepsilon$ for any universal $\varepsilon > 0$. Figure 2 shows the Levi graph of $K_n^{(3)}$ from Example 1 when $n = |R| = 5$. Similarly, the Levi graphs of $K_n^{(r)}$ for larger uniformities $r \geq 4$ show that the robust connectivity of r -connected graphs may also be arbitrarily small.

We show a surprising connection between the notion of robust connectivity and the following famous conjecture of Albertson and Berman [1].

Conjecture 1 ([1]). If G is a planar graph on n vertices, then G contains an induced forest of size at least $n/2$.

Conjecture 1 has a long history, and many partial results and theorems of a similar flavor exist; see [2] for a very recent overview of the related results. One of the large motivations for Conjecture 1 was that it would provide a proof that every planar graph on n vertices has an independent set of size $\lceil n/4 \rceil$ without relying on the Four Color Theorem. The currently best known lower bound of $\frac{2}{5}n$ is a consequence of Borodin’s theorem of 5-acyclic colorability [3], which was already published in 1976. Conjecture 1 is proven for only a few subclasses of planar graphs, e.g., outerplanar graphs [8], where the tight lower bound is $\frac{2}{3}n$.

Our Results. We present asymptotically tight lower bounds for the robust connectivity of r -connected graphs in terms of their Euler genus, for $r \geq 3$.

Theorem 2. *If $r \geq 3$ and G is an r -connected graph of Euler genus γ , then $\kappa_\rho(G) \geq \frac{1}{27}\gamma^{-1/r}$.*

The Levi graphs of $K_n^{(r)}$ show that this bound is tight within a constant factor. With more careful calculations, we also derive improved lower bounds for the robust connectivity of 3-connected planar graphs.

Theorem 3. *If G is a 3-connected planar graph, then $\kappa_\rho(G) \geq \frac{21}{256} > \frac{1}{13}$. Moreover, if $\varepsilon > 0$, then there exists a planar 3-connected graph H such that $\kappa_\rho(H) \leq \frac{1}{3} + \varepsilon$.*

In this direction, we may attempt to go even further and exchange the assumption of 3-connectivity with being a planar triangulation. Note that planar triangulations on at least 4 vertices are 3-connected. For planar triangulations, we formulate the following conjecture.

Conjecture 2. If G is a planar triangulation, then $\kappa_\rho(G) \geq \frac{1}{2}$.

Surprisingly, Conjecture 2 turns out to be equivalent to the famous Conjecture 1.

Theorem 4. *Conjecture 1 is equivalent to Conjecture 2.*

Hence, we propose the notion of robust connectivity as another way to attack Conjecture 1. In fact, we will present a further generalization of both conjectures to graphs of arbitrary Euler genus γ . In order to do this, we develop a new notion for an arbitrary surface S that, informally, describes how large of an induced subgraph we can cut out of an edge-maximal graph on S without separating S into multiple pieces. We will often write \tilde{G} to refer to an embedding of graph G . For a surface S , we let \mathcal{G}_S be the family of (simple) embedded graphs on S . A graph G is *edge-maximal* (with respect to a surface S) if G has an embedding $\tilde{G} \in \mathcal{G}_S$, but for each non-edge $e \notin E(G)$, $G + e$ cannot be embedded in S . More often, we will speak about an *edge-maximal embedding* $\tilde{G} \in \mathcal{G}_S$ of graph G if for each $e \notin E(G)$, e cannot be added to the embedding \tilde{G} without creating a crossing on S . Note that every edge-maximal graph G with respect to a surface S has an edge-maximal embedding on S , but not every graph with an edge-maximal embedding on S is edge-maximal with respect to S . For an embedded graph $\tilde{G} \in \mathcal{G}_S$, we write $S \setminus \tilde{G}$ for the surface obtained by cutting S along the edges of \tilde{G} and puncturing S at each isolated vertex of \tilde{G} . Then, we define $m(\tilde{G})$ to be the number of vertices in a largest induced embedded subgraph $\tilde{G}' \subseteq \tilde{G}$ for which $S \setminus \tilde{G}'$ is a connected surface. Now, we define the main parameter of interest.

Definition 2 (Surface connectivity). *The surface connectivity $\kappa_s(S)$ of a surface S is defined as follows: $\kappa_s(S) = \inf \left\{ \frac{m(\tilde{G})}{|\tilde{G}|} : \tilde{G} \in \mathcal{G}_S \right\}$.*

We will show that the minimum robust connectivity over all edge-maximal graphs embedded in a surface S is equal to the surface connectivity of S .

Theorem 5. *Let S be a surface. Every graph G with an edge-maximal embedding on S satisfies $\kappa_\rho(G) \geq k$ if and only if $\kappa_s(S) \geq k$.*

When S is the plane, an edge-maximally embedded graph \tilde{G} is simply a planar triangulation, and $S \not\ll \tilde{G}'$ is connected for an induced embedded subgraph $\tilde{G}' \subseteq \tilde{G}$ if and only if \tilde{G}' is acyclic. Hence, Theorem 5 directly implies Theorem 4.

Proofs of Theorems 2 and 3 are in the full version of this paper [4].

2 Induced Subgraphs on Surfaces

First, we establish an upper bound on the surface connectivity of any surface S . Since every surface S locally resembles the plane, we may embed K_4 on S in such a way that every triangle of K_4 separates S into two connected components. Thus, the largest subgraph of K_4 that does not separate S contains only two out of the four total vertices, and so $\kappa_s(S) \leq \frac{1}{2}$. When S is the plane, Conjecture 1 asserts that $\kappa_s(S) = \frac{1}{2}$.

Lemma 1. *Let G be a graph, and let \tilde{G} be an edge-maximal embedding of G in a surface S . Suppose \tilde{G}' is a proper induced embedded subgraph of \tilde{G} corresponding to an induced subgraph $G' \subseteq G$. If $S \not\ll \tilde{G}'$ is a connected surface, then $G \setminus G'$ is a connected graph.*

The proof of Lemma 1 is in the full version [4]. We note that Lemma 1 immediately implies that an edge-maximal graph on a surface with at least four vertices is 3-connected, since in a simple graph, a pair of vertices can only induce an edge, which cannot separate a surface. We also note that edge-maximality is necessary for Lemma 1. For example, if \tilde{T} is an embedded tree in a surface S , $S \not\ll \tilde{T}'$ is connected for any induced embedded subgraph \tilde{T}' of \tilde{T} , but $T \setminus T'$ is often disconnected. Now, we are ready to prove our main result of this section.

Proof (of Theorem 5). Suppose first that $\kappa_s(S) \geq k$, or in other words, that every embedded graph $\tilde{H} \in \mathcal{G}_S$ on n vertices has an induced embedded subgraph \tilde{H}' of size at least kn for which $S \not\ll \tilde{H}'$ is a connected surface. As shown above using K_4 , $k \leq \frac{1}{2}$. Let G be a graph with an edge-maximal embedding on S . We note that since every edge-maximal graph G with at most three vertices is a clique and hence satisfies $\kappa_\rho(G) > \frac{1}{2} \geq k$, it suffices only to consider edge-maximal graphs G on at least four vertices. In particular, we may assume by Lemma 1 that G is 3-connected.

Now, let $R \subseteq V(G)$. If $|R| \leq 3$, then since G is 3-connected, we may use at least $2 \geq \frac{2}{3}|R| > k|R|$ vertices of R as leaves of some spanning tree on G , and we are done in this case. Now, suppose $|R| \geq 4$. If G has an universal vertex, then we may find a spanning tree on G that uses at least $|R| - 1 \geq \frac{3}{4}|R| > k|R|$ vertices of R as leaves, and we are done. Otherwise, we consider the graph $\tilde{G}[R]$

embedded in S . By our hypothesis, we may find a subset $R' \subsetneq R$ of size at least $k|R|$ for which $S \not\ll G[R']$ is a connected surface. We claim that we may find a spanning tree on G that includes every vertex of R' as a leaf. Indeed, as $S \not\ll G[R']$ is connected, and as G is edge-maximal, it follows from Lemma 1 that $G \setminus R'$ is a connected graph. Furthermore, since G has no universal vertex, $G \setminus N(r)$ is a disconnected graph for each $r \in R'$, so by Lemma 1, $S \not\ll G[N(r)]$ is a disconnected surface. Therefore, for each $r \in R'$, at least one neighbor of r does not belong to R' . Hence, one may take any spanning tree T on $G \setminus R'$, and T will dominate R' ; then one may add each vertex of R' as a leaf of T . As $|R'| \geq k|R|$, and as the choice of R was arbitrary, it follows that $\kappa_\rho(G) \geq k$.

Suppose, on the other hand, that every graph G with an edge-maximal embedding on S satisfies $\kappa_\rho(G) \geq k$. Let \tilde{H} be a graph embedded in S . We seek an induced embedded subgraph $\tilde{H}' \subseteq \tilde{H}$ of size at least $k|\tilde{H}|$ for which $S \not\ll \tilde{H}'$ is a connected surface. It will make our task no easier to add edges to \tilde{H} until \tilde{H} is edge-maximal. Now, let \tilde{G} be a graph embedded in S obtained by adding a vertex v_C to each connected component C of $S \not\ll \tilde{H}$ and making v_C adjacent to all vertices incident to C . We call these vertices v_C *component vertices*. Clearly, \tilde{G} is an edge-maximal embedding, since every possible edge between vertices of \tilde{H} is included, and every possible edge between a component vertex and a vertex of \tilde{H} is included. Now, let $R \subseteq V(\tilde{G})$ denote the vertices that originated in \tilde{H} . Since \tilde{G} has an edge-maximal embedding, we know that $\kappa_\rho(\tilde{G}) \geq k$, and hence we may choose a spanning tree T on \tilde{G} that includes at least $k|R|$ of the vertices of R as leaves. Then $T' = T \setminus (\Lambda(T) \cap R)$ is a connected graph that spans all component vertices of \tilde{G} . We claim that if $H' = H[\Lambda(T) \cap R]$, then $S \not\ll \tilde{H}'$ is a connected surface. Indeed, if $S \not\ll \tilde{H}'$ is disconnected, then there must exist two component vertices of \tilde{G} in distinct connected components of $S \not\ll \tilde{H}'$, and T' cannot contain both of these component vertices, a contradiction. Therefore, the induced subgraph \tilde{H}' is a graph of size at least $k|R| = k|\tilde{H}|$, and $S \not\ll \tilde{H}'$ is a connected surface. □

Conclusions. We have shown tight asymptotic bounds for the robust connectivity of r -connected graphs embedded on surfaces. Moreover, we show a connection between robust connectivity of edge-maximal graphs and the notion surface connectivity. We propose a further study of surface connectivity which, to the best of our knowledge, has not been considered before. For planar graphs, the connection we showed provides another equivalent formulation of the famous Albertson Berman conjecture, and our results may give another direction to attack the conjecture itself. For surfaces of higher genus, this connection gives rise a more general question. The graph K_7 may be embedded on any surface with at least one handle such that no more than three vertices can be removed without disconnected the surface. Hence, we ask widely open Question 1. We do not even know whether the correct bound decreases with increasing Eulerian genus.

Question 1. Is surface connectivity always at least $\frac{3}{7}$ for any surface?

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