Resonance in Large Finite Particle Systems

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Abstract. We consider general linear system of Hamiltonian equations. The corresponding linear operator is assumed to be positive definite for the particles could not escape to infinity. However, there are also external driving forces, that could make the solution unbounded. It is assumed that driving force depends only on time, it can be periodic, almost-periodic and random. Moreover, it acts only on one coordinate. Our main problem here is to understand what restrictions on driving force and/or what dissipative force could be added to escape resonance (unbounded trajectories). Various conditions for existence and non-existence of resonance are obtained, for any number of particles.

Keywords: Linear systems · Hamiltonian dynamics · Resonance · Boundedness

1 The Model

There are extreme models in non-equilibrium statistical physics. First one is the ideal (or almost ideal) gas where the particles are free that is do not interact (or almost do not interact) with each other. The second is when the particles interact but each particle moves inside its own potential well, which also moves due to interaction of particles. The simplest such model is the general linear model with quadratic potential interaction energy. However, besides interaction, there can be also external driving and dissipative forces. There are many different qualitative phenomena concerning such systems. One of the most important is resonance phenomena. That is when the particles start to leave their potential wells, the system becomes unstable and the dynamics becomes unbounded. Here we study the models with large number of particles where the resonance can occur even if the external forces act only on one fixed particle.

We consider general linear system of N_0 point particles in $\mathbb{R}^d$ with $N = dN_0$ coordinates $q_j \in \mathbb{R},\ j = 1,..,N$. Let

$$v_j = \frac{dx_j}{dt}, p_j = m_j v_j,\ j = 1,..,N,$$
$$q = (q_1, ..., q_N)^T, p = (p_1, ..., p_N)^T, \psi(t) = (q_1, ..., q_N, p_1, ..., p_N)^T.$$

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Here m_j is the mass of the particle having q_j as one of its coordinate. Further on we put $m_j = 1$ and thus $p_j = v_j$. Potential and kinetic energies are:

$$U(\psi(t)) = \frac{1}{2} \sum_{1 \le j,l \le N} V_{j,l} q_j q_l = \frac{1}{2}(q, Vq),\ T(\psi(t)) = \sum_{j=1}^{N} \frac{p_j^2}{2} = \frac{1}{2}(p,p),$$

where the matrix $V = (V_{ij})$ is always assumed to be positive definite. Then $q_j(t), v_j(t)$ are bounded for any initial conditions.

If there are also external forces $f_j(t, v_j)$, acting correspondingly on the coordinates j, then we have the following system of equations:

$$\ddot{q}_j = -\sum_l V_{j,l} q_l + f_j(t, v_j),\ j = 1, .., N,$$

or in the first order form:

$$\ddot{q}_j = \dot{v}_j = \sum_l V_{j,l} q_l + f_j(t, v_j).$$

or

$$\dot{\psi} = A_0 \psi + F, \tag{1}$$

where

$$A_0 = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix}, \tag{2}$$

$$F = (0, ..., 0, f_1(t, v_1), ..., f_N(t, v_N))^T \in \mathbb{R}^{2N}.$$

We shall consider the case:

$$f_j(t, v_j) = f(t)\delta_{j,n} - \alpha v_j \delta_{j,k}, \tag{3}$$

where the time dependent external force $f(t)$ acts only on fixed coordinate n, and the dissipative force $-\alpha v_k$ acts only on coordinate k.

For fixed initial conditions and parameters (F, V), we say that resonance takes place if the solutions $x_j(t), v_j(t)$ are not bounded in $t \in [0, \infty)$ at least for one $j = 1, ..., N$.

2 Main Results

As matrix V is positively definite, its eigenvalues are strictly positive and it is convenient to denote them as $a_k = \nu_k^2, k = 1, ..., N$, furthermore, it is convenient to consider all ν_k positive too. Corresponding system of eigenvectors we denote as $\{u_k, k = 1, ..., N\}$ and this system can always be assumed to be orthonormal.

2.1 Periodic Driving Force

Here we consider the case (3) with $f(t) = a \sin \omega t$ and $\alpha = 0$. And for general periodic force one could just (due to linearity of equations) consider its Fourier series.

Denote Ω_N the set of all positive-definite $(N \times N)$-matrices. It is an **open** subset in R^M (the set of all symmetric matrices), where $M = N + \frac{N^2-N}{2} = \frac{N(N+1)}{2}$. It is open because any sufficiently small perturbation does not change positive definiteness. Denote μ the Lebesgue measure on Ω_N, and let (for fixed ω) $\Omega(\omega)$ be the subset of Ω_N such that $\omega^2 = \nu_l^2$ at least for one $l \in \{1, ..., N\}$. It is an algebraic manifold in $\mathbb{R}^M$ and thus $\mu(\Omega(\omega)) = 0$.

Theorem 1. *1). Assume that* $V \notin \Omega(\omega)$ *that is, for all* $j \in \{1, ..., N\}$, $\nu_j^2 \neq \omega^2$. *Then for all* $j \in \{1, ..., N\}$ *and all* $t \geq 0$*:*

$$|q_j(t)| \leq 2d_j\beta, \ |p_j(t)| \leq 2d_j\beta\omega,$$

where

$$\beta = \max_r \frac{1}{|\omega^2 - \nu_r^2|}, d_j = |a| \sum_{k=1}^{N} |(u_k, e_n)(u_k, e_j)|. \tag{4}$$

In other words, there will not be resonance for almost all matrices V*;*

2). Assume $\omega^2 = \nu_l^2$ *at least for one* $l \in \{1, ..., N\}$. *Then* $q_j(t)$, $p_j(t)$ *are bounded uniformly in* $t \geq 0$ *if and only if for this* j *holds:*

$$\sum_{k \in I(\omega)} (u_k, e_n)(u_k, e_j)) = 0, \tag{5}$$

where $I(\omega) = \{k \in \{1, ..., N\} : \omega^2 = \nu_k^2\}$. *Otherwise resonance occurs. Moreover, for all* $j \in \{1, ..., N\}$*:*

$$\liminf_{t\to+\infty} q_j(t) = -\infty, \ \limsup_{t\to+\infty} q_j(t) = +\infty,$$

$$\liminf_{t\to+\infty} p_j(t) = -\infty, \ \limsup_{t\to+\infty} p_j(t) = +\infty,$$

and

$$\limsup_{t\to+\infty} \frac{T(\psi(t))}{t^2} = \limsup_{t\to+\infty} \frac{U(\psi(t))}{t^2} = \lim_{t\to+\infty} \frac{H(\psi(t))}{t^2} = C,$$

where

$$C = \frac{a^2}{8} \sum_{k \in I(\omega)} (u_k, e_n)^2.$$

2.2 Arbitrary Driving Force and Dissipation

Now we consider the force (3), that is the equation:

$$\ddot{q}_j = -\sum_l V_{j,l} q_l + f(t)\delta_{j,n} - \alpha \dot{q}_k \delta_{j,k},\ j = 1, .., N,\ \alpha > 0.$$

In the vector form it can be written as:

$$\begin{aligned} \dot{q}_j &= p_j, \\ \dot{p}_j &= -\sum_l V_{j,l} q_l + f(t)\delta_{j,n} - \alpha p_k \delta_{j,k}, \end{aligned}$$

or

$$\dot{\psi} = A\psi + f(t) g_n, \tag{6}$$

where

$$A = \begin{pmatrix} 0 & E \\ -V & -\alpha D \end{pmatrix} \tag{7}$$

is the $2N \times 2N$-matrix with $N \times N$ blocks, E is the unit $N \times N$-matrix,

$$D = D_k = diag(\delta_{1,k}, .., \delta_{N,k}),$$

$$g_n = (\overline{0}, e_n)^T \in \mathbb{R}^{2N},\ e_n = (\delta_{1,n}, .., \delta_{N,n}),\ \overline{0} = (0, ..., 0) \in \mathbb{R}^N, \tag{8}$$

and again we consider zero initial conditions.

Put

$$S = \begin{pmatrix} 0 & 0 \\ 0 & \alpha D \end{pmatrix}.$$

Then

$$A = A_0 - S,$$

where A_0 was defined in (2). It is known that $Re\, \nu \leq 0$ for all eigenvalues of A [see [2]].

Theorem 2. *1). Assume that the function $f(t)$ (defined by (3)) grows in time t on $[0, \infty)$ not faster than the power function. Then if the spectrum of A does not have pure imaginary eigenvalues, then the solution of the system (6) is bounded on $[0, \infty)$.*

2). Let $\Lambda_N \subset \Omega_N \subset \mathbb{R}^M$ be the set of matrices V such that all eigenvalues of the matrix A lie inside left halfplane. Then the Lebesgue measure $\mu(\Omega_N \backslash \Lambda_N) = 0$, that is for almost all matrices V the spectrum of the corresponding matrices $A = A(V)$ lies inside left halfplane.

2.3 Almost-Periodic Force

Suppose the force $f(t)$ has the form:

$$f(t) = \int_R a(\omega)\cos(\omega t) d\omega,$$

where $a(\omega) \in l_1(\mathbb{R})$ is a sufficiently smooth function. Then the function $f(t)$ on $\mathbb{R}$ is almost periodic.

Theorem 3. *Assume $\alpha = 0$, that is there is no dissipative force. Then for any initial data the solutions $\{x_k(t), v_k(t)\}$ are bounded on the time interval $[0, \infty)$.*

3 Proofs

3.1 Proof of Theorem 1

Note first that the eigenvalues of matrix A_0 are $\pm i\nu_1, ..., \pm i\nu_N$. In fact, if $u = u_k$ - eigenvector of V corresponding to the eigenvalue ν_k^2, $k = 1, ..., N$, (i.e. $Vu = \nu_k^2 u$) then vector $x_\pm = \begin{pmatrix} u \\ \lambda_\pm u \end{pmatrix}$, where $\lambda_\pm = \pm i\nu_k$ is the eigenvector of A_0, corresponding to the eigenvalue $\lambda_\pm = \pm i\nu_k$:

$$A_0 x_\pm = \begin{pmatrix} 0 & E \\ -V & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda_\pm u \end{pmatrix} = \begin{pmatrix} \lambda_\pm u \\ -Vu \end{pmatrix} = \begin{pmatrix} \lambda_\pm u \\ -\nu_k^2 u \end{pmatrix} = \begin{pmatrix} \lambda_\pm u \\ \lambda_\pm^2 u \end{pmatrix} = \lambda_\pm x_\pm.$$

It is well-known that the solution of equation (1) can be written as:

$$\psi(t) = e^{A_0 t}(\int_0^t f(s)e^{-A_0 s} g_n ds + \psi(0)). \tag{9}$$

It is easy to prove that:

$$e^{A_0 t} = \begin{pmatrix} \cos(\sqrt{V}t) & (\sqrt{V})^{-1}\sin(\sqrt{V}t) \\ -\sqrt{V}\sin(\sqrt{V}t) & \cos(\sqrt{V}t) \end{pmatrix},$$

where trigonometric functions of matrices are defined by the corresponding power series. Then we can find $q(t)$, $p(t)$ explicitely:

$$\begin{gathered} q(t) = \int_0^t f(s)(\sqrt{V})^{-1}\sin(\sqrt{V}(t-s))e_n ds + \cos(\sqrt{V}t)q(0) \\ +(\sqrt{V})^{-1}\sin(\sqrt{V}t)p(0), \end{gathered} \tag{10}$$

$$p(t) = \int_0^t f(s)\cos(\sqrt{V}(t-s))e_n ds - \sqrt{V}\sin(\sqrt{V}t)q(0) + \cos(\sqrt{V}t)p(0). \tag{11}$$

Let us expand vectors $e_n, q(0), p(0)$ in the orthonormal basis of eigenvectors of V:

$$e_n = \sum_{k=1}^N (u_k, e_n)u_k,\ q(0) = \sum_{k=1}^N (u_k, q(0))u_k,\ p(0) = \sum_{k=1}^N (u_k, p(0))u_k.$$

Then as

$$(\sqrt{V})^{-1}u_k = \frac{1}{\nu_k}u_k,\ \sin(\sqrt{V}t)u_k = u_k\sin(\nu_k t),$$
$$\cos(\sqrt{V}t)u_k = u_k\cos(\nu_k t),$$

we have:

$$q(t) = \sum_{k=1}^{N}[(u_k, e_n)(\int_0^t f(s)\frac{\sin(\nu_k(t-s))}{\nu_k}ds)$$
$$+ (u_k, q(0))\cos(\nu_k t) + (u_k, p(0))\frac{\sin(\nu_k t)}{\nu_k}]u_k, \qquad (12)$$

$$p(t) = \sum_{k=1}^{N}[(u_k, e_n)(\int_0^t f(s)\cos(\nu_k(t-s))ds)$$
$$- (u_k, q(0))\nu_k\sin(\nu_k t) + (u_k, p(0))\cos(\nu_k t)]u_k. \qquad (13)$$

Thus we reduced the question of boundedness to the question of boundedness of the functions:

$$\widetilde{q}_k(t) = \int_0^t f(s)\sin(\nu_k(t-s))ds,$$
$$\widetilde{p}_k(t) = \int_0^t f(s)\cos(\nu_k(t-s))ds.$$

For those $j \in \{1, ..., N\}$, where $\omega^2 \neq \nu_j^2$:

$$\widetilde{q}_j(t) = \int_0^t \sin(\omega s)\sin(\sqrt{a_j}(t-s))ds = \frac{\sqrt{a_j}}{\omega^2 - a_j}(\sin(\omega t) - \sin(\sqrt{a_j}t)),$$
$$\widetilde{p}_j(t) = \int_0^t \sin(\omega s)\cos(\sqrt{a_j}(t-s))ds = \frac{\omega}{\omega^2 - a_j}(\cos(\sqrt{a_j}t) - \cos(\omega t)).$$

It follows:

$$|q_j(t)| = |(q, e_j)| = |a\sum_{k=1}^{N}(u_k, e_n)\frac{1}{\nu_k}\widetilde{q}_k(t)(u_k, e_j)| \leq 2d_j\beta,$$
$$|p_j(t)| = |(p, e_j)| = |a\sum_{k=1}^{N}(u_k, e_n)\widetilde{p}_k(t)(u_k, e_j)| \leq 2d_j\beta\omega.$$

Now consider $j \in \{1, ..., N\}$, where $\nu_j^2 = \nu_l^2 = \omega^2$:

$$\widetilde{q}_j(t) = \int_0^t \sin(\omega s)\sin(\sqrt{a_j}(t-s))ds = \tfrac{\sin(\omega t)}{2\omega} - \tfrac{t\cos(\omega t)}{2},$$
$$\widetilde{p}_j(t) = \int_0^t \sin(\omega s)\cos(\sqrt{a_j}(t-s))ds = \tfrac{t\sin(\omega t)}{2},$$
$$q_j(t) = (q, e_j) = a\sum_{k=1}^{N}(u_k, e_n)\tfrac{1}{\nu_k}\widetilde{q}_k(t)(u_k, e_j)$$
$$= (a\textstyle\sum_{k\in I(\omega)}(u_k, e_n)(u_k, e_j))(-\tfrac{t\cos(\omega t)}{2\omega}) + \underline{O}(1) = B_j(-\tfrac{t\cos(\omega t)}{2\omega}) + \underline{O}(1),\ t \to +\infty,$$

where $I(\omega) = \{j \in \{1, ..., N\} : \omega^2 = \nu_j^2\}$.

Similarly for $p_j(t)$:

$$p_j(t) = (p, e_j) = a \sum_{k=1}^{N} (u_k, e_n) \widetilde{p}_k(t)(u_k, e_j)$$

$$= (a \sum_{k \in I(\omega)} (u_k, e_n)(u_k, e_j)) \frac{t \sin(\omega t)}{2} + \underline{O}(1) = B_j \frac{t \sin(\omega t)}{2} + \underline{O}(1),\ t \to +\infty.$$

The theorem follows.

3.2 Proof of Theorem 2

In [1–3] the following subspaces of

$$L = \{\psi = (q, p),\ q, p \in \mathbb{R}^N\}$$

were defined (with $H = U + T$):

$$L_- = \{\psi \in L : H(e^{At}\psi) \longrightarrow 0,\ t \longrightarrow +\infty\},$$
$$L_0 = \{\psi \in L : \tfrac{d}{dt} H(e^{At}\psi) = 0\ \forall t\},$$

and was proved that:

1). L_-, L_0 are linear orthogonal subspaces;
2). $L = L_- \oplus L_0$;
3). both L_-, L_0 are invariant with respect to dynamics;
4). $L_0 = \{0\}$ iff A does not have pure imaginary eigenvalues;
5). A does not have pure imaginary eigenvalues iff the vectors $e_n, Ve_n, ..., V^{N-1}e_n$ are linear independent.

Note that resonance is possible for pure imaginary eignvalues as in the integrals, introduced below, secular terms like $t\cos(\omega t)$, $t\sin(\omega t)$ can appear.

The first statement of the theorem follows from Theorem 4.1. (in [4],p.88), where the solution of the system:

$$\dot{\psi} = B\psi + F(t), \tag{14}$$

where B is some linear operator and $F(t)$ is vector function, is considered.

We cite this theorem almost literally.

Theorem 4. *(see [4], Theorem* 4.1, *p.*88)

In order for there to correspond to any bounded-on-the-real-line continuous vector function $F(t)$ one and only one bounded-on-the-real-line solution of (14) it is necessary and sufficient that the spectrum $\sigma(B)$ not intersect the imaginary axis.

The solution is given by formula:

$$x(t) = \int_{\mathbb{R}} G_B(t - s) F(s) ds,$$

where $G_B(t)$ is principal Green function for equation.

In our case $F(t) = f(t)g_n, t \geq 0$, and

$$G_B(t) = e^{Bt}P_-,$$

where P_- is the spectral projection corresponding to the spectrum of B in the left (negative) halfplane.

3.3 Proof of Theorem 3

Using formula (12) we want to prove boundedness in $t \in [0, \infty)$ of the function

$$I(t) = \int_0^t f(s) \sin(\nu_k(t-s))ds.$$

We have:

$$\begin{aligned}\int_0^t \sin(\nu_k(t-s))f(s)ds &= \int_R a(\omega)d\omega \int_0^t \sin(\nu_k(t-s)) \cos(\omega s)ds \\ &= \nu_k \int_R a(\omega) \frac{\cos \omega t - \cos \nu_k t}{\nu_k^2 - \omega^2} d\omega \\ &= 2 \int_R a(\omega) \frac{\sin((\omega+\nu_k)t) \sin((\omega-\nu_k)t)}{(\omega+\nu_k)(\omega-\nu_k)} d\omega.\end{aligned}$$

We see that unboundedness in time can only arise when we integrate in a small neighborhood of ν_k. Denoting $\omega = \nu_k + x$ we get as $\epsilon \to 0$:

$$2\int_{-\epsilon}^{\epsilon} a(\nu_k + x) \frac{\sin((x+2\nu_k)t) \sin(xt)}{(x+2\nu_k)x} dx \sim \frac{a(\nu_k) \sin(2\nu_k t)}{\nu_k} \int_{-\epsilon}^{\epsilon} \frac{\sin xt}{x} dx.$$

At the same time we have that the integral:

$$\int_{-\epsilon}^{\epsilon} \frac{\sin xt}{x} dx = \int_{-t\epsilon}^{t\epsilon} \frac{\sin x}{x} dx$$

is bounded uniformly in t. Indeed, on arbitrary period $(N, N+2\pi)$ put $x = N+y$, then

$$\frac{1}{x} = \frac{1}{N} \frac{1}{1+\frac{y}{N}} = \frac{1}{N} - \frac{y}{N^2} + ...$$

The first term gives 0 in the integrals for such periods, and the rest will give a convergent sum.

4 Conclusion

Note first that the solution boundedness problem with the external force $f(t)$ was in fact reduced to the problem of boundedness of the integral:

$$I(t) = \int_0^t f(s) \sin \omega s ds. \tag{15}$$

Now we want to formulate simple and more difficult problems concerning general situation with resonances.

Let us summarize now how to get rid of resonances without self-isolation from external influence. If the external force is periodic, there are following possibilities:

1). one should choose his own oscillation frequency sufficiently far from the external frequency;
2). use external “smooth” almost-periodic force like in Theorem 3;
3). if all previous is impossible one should be simultaneously be under influence of some external dissipative force;
4). what will be if the external force is neither periodic nor almost-periodic. In particular, what will be if $f(s)$ is a random stationary process.

If it is unbounded then the solution also will be unbounded. If $f(s)$ is bounded, consider the following cases.
5). If $f(s)$ is stationary with fast correlation function decay, then the solution will be unbounded. In fact, let $\tau = \frac{2\pi}{\omega}$ be the period in (15). Consider the sequence of random variables

$$\xi_k = \int_{k\tau}^{(k+1)\tau} f(s) \sin \omega s ds$$

and their sums

$$S_N = \xi_1 + ... + \xi_N.$$

If ξ_k are independent or have sufficient decay of correlations, then just by central limit theorem there cannot be boundedness. Interesting question is to formulate general conditions when, keeping the randomness and wihout dissipation forces, one can have bounded solutions.
6). Complete different situation will be for infinite collection of particles. Namely, in many cases there will not be resonance (unbounded graph) due to phenomenon that energy escaped to infinity.

We consider countable number of point particles (with unit masses) on the real axis $x_k \in \mathbb{R}, k \in \mathbb{Z}$. Intuitively, we would like that each particle $x_k(t)$ were close to $ak \in \mathbb{R}$ for some $a > 0$. That is why we introduce the formal Hamiltonian:

$$H(q,p) = \frac{1}{2} \sum_{k \in \mathbb{Z}} p_k^2 + \frac{1}{2} \sum_{k,j \in \mathbb{Z}} b(k-j) q_k q_j,$$

where $q_k = x_k - ak$, and $p_k(t) = \dot{q}_k(t)$ are momenta of the particles k. The real function $v(k)$ is assumed to satisfy the following conditions:

1. symmetry: $b(k) = b(-k), b(0) > 0$;
2. boundedness of the support, that is there exists $r \in \mathbb{N}$ such that $b(k) = 0$ for any $k, |k| > r$;
3. for any $\lambda \in \mathbb{R}$:

$$\omega^2(\lambda) = \sum_{k \in \mathbb{Z}} b(k) e^{ik\lambda} > 0.$$

It follows that the linear operator V in $l_2(\mathbb{Z})$ with elements $V_{jk} = b(k-j)$ (in the standard orthonormal basis $e_n \in l_2(\mathbb{Z})$, $e_n(j) = \delta_{j,n}$) is a positive definite self-adjoint operator.

The trajectories of the system are defined by the following system of equations:

$$\ddot{q}_j = -\sum_k b(k-j)q_k + f(t)\delta_{j,n},\ j \in \mathbb{Z},$$

where $f(t)$ is some external force which acts only on the particle n, $\delta_{j,n}$ is the Kronecker symbol. We will always assume zero initial conditions:

$$q_k(0) = 0,\ p_k(0) = 0,\ k \in \mathbb{Z}.$$

We can rewrite it in the Hamiltonian form:

$$\dot{q_j} = p_j, \dot{p_j} = -\sum_k b(k-j)q_k + f(t)\delta_{j,n}. \tag{16}$$

In $l_2(\mathbb{Z} \times \mathbb{Z})$ define the (state) vector $\psi(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$ and the linear operator A_0 which was defined in (2). Then the system can be rewritten as follows:

$$\dot{\psi} = A_0\psi + f(t)g_n, \tag{17}$$

where g_n is defined in (8).

Here we assume that $f(t)$ is a real-valued stationary random process (in the wider sense) with zero mean and covariance function $B(s)$, so that:

$$Ef(t) = 0, Ef(t)f(s) = B(t-s).$$

Also assume that there exist random measure $Z(dx)$ and (spectral) measure $\mu(dx)$ such that for any Borel set $D \subset \mathbb{R}$:

$$EZ(D) = 0,\ E|Z|^2(D) = \mu(D),\ EZ(D_1)Z^*(D_2) = 0$$

for nonintersecting D_1 and D_2, and moreover:

$$B(s) = \int_{\mathbb{R}} e^{isx}\mu(dx),\ f(s) = \int_{\mathbb{R}} e^{isx}Z(dx). \tag{18}$$

We assume also that the support of the random measure is "separated" from the spectrum of A_0. Then the following assertion holds.

Theorem 5. *Solution $\psi(t)$ of the system (16) can be presented as the sum of two centered random processes:*

$$\psi(t) = \zeta(t) + \eta(t),$$
$$\eta(t) = -e^{A_0t}\int_{\mathbb{R}} e^{itx}R_{A_0}(ix)Z(dx)g,$$
$$\zeta(t) = e^{A_0t}\int_{\mathbb{R}} R_{A_0}(ix)Z(dx)g = -e^{A_0t}\eta(0),$$

where $R_A(z) = (A_0 - zI)^{-1}$ is the resolvent of the operator A_0 (I is the unit operator in $l_2(\mathbb{Z} \times \mathbb{Z})$). Moreover, components of $\eta(t)$ are stationary (in wider sense) random processes, and each component of $\zeta(t) \to 0$ a.s. as $t \to +\infty$.

Proof of this theorem and the development of this theme will be given elsewhere.

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