On Modification of the Law of Large Numbers and Linear Regression of Fuzzy Random Variables

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Abstract. Extreme properties of the average characteristics of fuzzy random variables are given. A new form of the law of large numbers for fuzzy random variables is established. An optimal linear regression of fuzzy random variables is constructed, in which the coefficients are similar to the Fourier coefficients. It is shown that under certain conditions, the optimal regression has the maximum cosine of the angle with the predicted fuzzy random variable.

Keywords: Fuzzy random variables · Means · Covariance · Variance · Law of large numbers · Linear regression

1 Introduction

Fuzzy random variables originated as a branch of fuzzy mathematics in [1–3]. They are widely used in financial mathematics, forecasting, decision theory, and others. In particular, the mathematical model of a random experiment with fuzzy outcomes is interpreted as a fuzzy random variable. The current state of the theory of fuzzy random variables is reflected in [4–7] and others.

In this paper, a new definition of the quasi-scalar product between fuzzy random variables is introduced, and its relationship with the covariance of fuzzy random variables proposed in [8] is revealed. Extreme properties of expectations and fuzzy expectations of fuzzy random variables are discussed.

The main results of this work are devoted to the law of large numbers (LLN) for fuzzy random variables and linear regression of fuzzy random variables. The difference between our result and those known from the LLN consists in using a special metric associated with the quasi-scalar product introduced by the author. The specificity of our result on linear regression is to derive a formula for optimal linear regression coefficients similar to the Fourier coefficients for an orthonormal system in Hilbert space. This is provided by introduction the definition quasi-scalar product.

In addition, it is shown that under certain conditions, the optimal solution has the maximum cosine of the angle with the predicted fuzzy random variable in the class of linear estimates.

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Below, by the fuzzy set A given on the universal space U, we mean the set of ordered pairs $(u, \mu_A(u))$, where the membership function $\mu_A : U \to [0,1]$, determines the degree to which $\forall u \in U$ belongs to the set A.

We rely on the following definition of a fuzzy number (cf. [9] chap. 2–4). A fuzzy number is called a fuzzy set whose universal set is the set of real numbers R, and which additionally satisfies the following conditions:

1) the support (supp) of a fuzzy number is a closed and bounded (compact) set of real numbers;
2) the membership function of a fuzzy number is convex;
3) the membership function of a fuzzy number is normal, i.e. the supremum of the membership function is equal to one;
4) the membership function of a fuzzy number is semi-continuous from above.

We will use the interval representation of fuzzy numbers. Namely, we will assign a set of α-intervals to each fuzzy number.

The set of fuzzy numbers satisfying the conditions 1)–4) is denoted by J.

As known, the set α-level of a fuzzy number $\tilde{z} \in J$ with the membership function $\mu_{\tilde{z}}(x)$ is defined by the relation

$$Z_\alpha = \{x | \mu_{\tilde{z}}(x) \geq \alpha\} \;\; (\alpha \in (0,1]), \;\; Z_0 = supp(\tilde{z}).$$

According to the above assumptions 1)–4) all α-levels of a fuzzy number are closed and bounded intervals at the real axis. Let's denote the left (lower) border of the interval $z^-(\alpha)$, and the right (upper) - $z^+(\alpha)$, i.e. $Z_\alpha = [z^-(\alpha), z^+(\alpha)]$. Sometimes $z^-(\alpha)$ and $z^+(\alpha)$ they are called the left and right indices of a fuzzy number, respectively.

On a set of fuzzy numbers, you can enter the definition of distances between them in different ways. The interval approach sometimes uses the Hausdorff distance, which for fuzzy numbers $\tilde{z}$, $\tilde{u} \in J$ with α-level sets $Z_\alpha = [z^-(\alpha), z^+(\alpha)]$ and $U_\alpha = [u^-(\alpha), u^+(\alpha)]$ in accordance with [5] is defined by the formula $\rho_H(\tilde{z}, \tilde{u}) = \sup\limits_{0<\alpha\leq 1} d_H(Z_\alpha, U_\alpha)$, where

$$d_H(Z_\alpha, U_\alpha) = \max\left[\sup_{z\in Z_\alpha} \inf_{u\in u_\alpha} |z-u|, \sup_{u\in u_\alpha} \inf_{z\in z_\alpha} |z-u|\right]$$

- Hausdorff metric. Some other distances are also considered (see, for example., [7,8,10]).

Denote by J_0 - the set of fuzzy numbers in the interval representation of which the left and right indices are quadratically summable.

We use the distance $\rho(\tilde{z}, \tilde{u})$ between the fuzzy numbers $\tilde{z}$ and $\tilde{u}$ from J_0 with α-level sets $Z_\alpha = [z^-(\alpha), z^+(\alpha)]$ and $U_\alpha = [u^-(\alpha), u^+(\alpha)]$, which is defined by the formula

$$\rho(\tilde{z}, \tilde{u}) = \left(\int_0^1 (z^-(\alpha) - u^-(\alpha))^2 + (z^+(\alpha) - u^+(\alpha))^2 d\alpha\right)^{\frac{1}{2}}. \quad (1)$$

Here and below, integration is understood by Lebesgue.

This kind of distance was previously used, for example, in [11].

Let the fuzzy number $\tilde{z}$ correspond to α - levels $Z_\alpha = [z^-(\alpha), Z^+(\alpha)]$, with $\alpha \in (0, 1]$. Suppose, as is customary in interval analysis,

$$midZ_\alpha = \frac{1}{2}(z^+(\alpha) + z^-(\alpha)),\ \ radZ_\alpha = \frac{1}{2}(z^+(\alpha) - z^-(\alpha)).$$

For fuzzy numbers $\tilde{z}$ and $\tilde{u}$ from J_0, with sets of α - levels $[z^-(\alpha), z^+(\alpha)]$ and $[u^-(\alpha), u^+(\alpha)]$, we define quasi-scalar product

$$\langle \tilde{z}, \tilde{u} \rangle = \int\limits_0^1 (midZ_\alpha, midU_\alpha + radU_\alpha\, radZ_\alpha) d\alpha$$

$$= 0.5 \int\limits_0^1 (z^+(\alpha)u^+(\alpha) + z^-(\alpha)u^-(\alpha)) d\alpha. \tag{2}$$

The quasinorm $\tilde{z}$ is $\langle \tilde{z}, \tilde{z} \rangle^{1/2}$.

According to (1), (2) the distance between the fuzzy numbers $\tilde{z}$ and $\tilde{u}$ from J_0 with sets of α - levels $[z^-(\alpha), z^+(\alpha)]$ and $[u^-(\alpha), u^+(\alpha)]$ matches the quasinorm of a fuzzy number whose left index is $z^-(\alpha) - u^-(\alpha)$, and the right index $z^+(\alpha) - u^+(\alpha)$.

Under the sum of the fuzzy numbers $\tilde{z}$ and $\tilde{u}$ we will understand a fuzzy number with α - levels $[z^-(\alpha)+u^-(\alpha), z^+(\alpha)+u^+(\alpha)]$. The product of a fuzzy number $\tilde{z}$ by a positive number c is a fuzzy number with α - levels $[cz^-(\alpha), cz^+(\alpha)]$. In the case of $c < 0$ - a fuzzy number with α - levels $[cz^+(\alpha), cz^-(\alpha)]$.

The following properties of the introduced quasi-scalar product are valid:

1) $\langle \tilde{z}, \tilde{u} \rangle = \langle \tilde{u}, \tilde{z} \rangle$ $(\forall \tilde{z}, \tilde{u} \in J_0)$;
2) $\langle c_1\tilde{z}, c_2\tilde{u} \rangle = c_1 c_2 \langle \tilde{z}, \tilde{u} \rangle$ $(\forall \tilde{z}, \tilde{u} \in J_0)$, provided that the product of the numbers $c_1 c_2 > 0$;
3) $\langle \tilde{z}_1 + \tilde{z}_2, \tilde{u} \rangle = \langle \tilde{z}_1, \tilde{u} \rangle + \langle \tilde{z}_2, \tilde{u} \rangle$ $(\forall \tilde{z}_1, \tilde{z}_2, \tilde{u} \in J_0)$;
4) $\langle \tilde{z}, \tilde{z} \rangle \geq 0$ $(\forall \tilde{z} \in J_0)$, and the condition $\langle \tilde{z}, \tilde{z} \rangle = 0$ is equivalent to vanishing the left and right indexes $\tilde{z}$;
5) Cauchy-Bunyakovsky Inequality $|\langle \tilde{z}, \tilde{u} \rangle| \leq \langle \tilde{z}, \tilde{z} \rangle^{1/2} \langle \tilde{u}, \tilde{u} \rangle^{1/2}$ $(\forall \tilde{z}, \tilde{u} \in J_0)$.

The quasi-scalar product of the form $\langle \tilde{z}, \tilde{u} \rangle_1 = \int\limits_0^1 (mid\ Z_\alpha\, mid\ U_\alpha) d\alpha$ is considered in [6]. It is easy to see that in this case, turning the $\langle \tilde{z}, \tilde{z} \rangle_1^{1/2}$ to zero does not guarantee that the left and right indexes of $\tilde{z}$ are equal to zero.

Other definitions of the scalar product of fuzzy numbers are also found in the literature (see, for example, [10]).

Note the following relationship between the quasi-scalar product (2) and the distance (1).

Set the fuzzy number $\tilde{z}$ with indexes $z^{-}(\alpha)$ and $z^{+}(\alpha)$ match the vector function $\bar{z}(\alpha) = (z^{-}(\alpha), z^{+}(\alpha))^{T}$. Scalar product of $\langle \bar{z}, \bar{u} \rangle$ vector functions $\bar{z}$ and $\bar{u}$ define by equality (2). Then $\rho(\tilde{z}, \tilde{u}) = ||\bar{z} - \bar{u}||$.

We introduce the concept of the cosine of the angle between fuzzy numbers. For fuzzy numbers $\tilde{z}$, $\tilde{u} \in J_0$, we put

$$cos(\tilde{z}, \tilde{u}) = \frac{\langle \tilde{z}, \tilde{u} \rangle}{\langle \tilde{z}, \tilde{z} \rangle^{1/2} \langle \tilde{u}, \tilde{u} \rangle^{1/2}}.$$

Note the properties of the cosine.

1. $|cos(\tilde{z}, \tilde{u})| \leq 1 \ \ (forall \tilde{z}, \tilde{u} \in J_0)$.
   This follows from the Cauchy-Bunyakovsky inequality.
2. $cos(\tilde{z}, \tilde{u}) = 0$, if and only if $\tilde{z}$ and $\tilde{u}$ quasi-orthogonal.
3. $cos(\tilde{z}, \tilde{u}) = 1$, if and only if $\tilde{z}$ and $\tilde{u}$ are collinear, i.e. there is a number $\lambda > 0$ such that $\tilde{z} = \lambda \tilde{u}$.

Indeed, $cos(\tilde{z}, \tilde{u})$ matches $cos(\bar{z}, \bar{u})$, where $\bar{z}, \bar{u}$ are vector functions corresponding to $\tilde{z}$ and $\tilde{u}$, respectively. Then the condition $cos(\tilde{z}, \tilde{u}) = cos(\bar{z}, \bar{u}) = 1$ means that $\bar{z} = \lambda \bar{u}$, where $\lambda > 0$. This is equivalent to $\tilde{z} = \lambda \tilde{u}$, i.e. the fuzzy numbers $\tilde{z}$ and $\tilde{u}$ collinear.

2 Fuzzy Random Variables and Their Averages

Let (Ω, Σ, P) be a probability space, where Ω is a set of elementary events, Σ is a σ-algebra consisting of subsets of the set Ω, and P is a probability measure.

A measurable map $\tilde{X} : \Omega \to J_0$ is called a fuzzy random variable if, for any $\omega \in \Omega$, the set $\tilde{X}(\omega)$ is a fuzzy number from J_0.

Consider the intervals of α - levels of a fuzzy random variable $\tilde{X}$ for a fixed ω. Namely, $X_{\alpha}(\omega) = \{t \in R : \mu_{\tilde{X}(\omega)} \geq \alpha\}$, where $\mu_{\tilde{X}(\omega)}$ - membership function of a fuzzy number $\tilde{X}(\omega)$, and $\alpha \in (0, 1]$. The interval $X_{\alpha}(\omega)$ represent as $X_{\alpha}(\omega) = [X^{-}(\omega, \alpha), X^{+}(\omega, \alpha)]$, where the boundaries are $X^{-}(\omega, \alpha)$ and $X^{+}(\omega, \alpha)$ - random variables. They are called, respectively, the left and right index of the fuzzy random variable $\tilde{X}$.

Below, we will consider the class $\mathfrak{X}$ of fuzzy random variables $\tilde{X}$, for which indexes $X^{-}(\omega, \alpha)$ and $X^{+}(\omega, \alpha)$ are functions that are quadratically summable by $\Omega \times [0, 1]$.

Put

$$x^{-}(\alpha) = \int\limits_{\Omega} X^{-}(\omega, \alpha) dP, \quad x^{+}(\alpha) = \int\limits_{\Omega} X^{+}(\omega, \alpha) dP. \tag{3}$$

A fuzzy number $\tilde{x}$ with indexes defined by formula (3) is called the fuzzy expectation of a fuzzy random variable $\tilde{X}$.

Let $X_{\alpha}(\omega) = [X^{-}(\omega, \alpha), X^{+}(\omega, \alpha)]$ - interval α - level of the fuzzy random variable $\tilde{X}$. Put $mid\ X_{\alpha}(\omega) = \frac{1}{2}(X^{+}(\omega, \alpha) + X^{-}(\omega, \alpha))$ and $rad\, X_{\alpha}(\omega) = \frac{1}{2}(X^{+}(\omega, \alpha) - X^{-}(\omega, \alpha))$.

Expectation $E(\tilde{X})$ a fuzzy random variable $\tilde{X}$ is a number defined by the expression

$$E(\tilde{X}) = \int_0^1 \int_\Omega mid\ X_\alpha(\omega) dP d\alpha = 0.5 \int_0^1 \int_\Omega (X^-(\omega,\alpha) + X^+(\omega,\alpha)) dP d\alpha. \quad (4)$$

Note the equality

$$E(\tilde{X}) = \int_0^1 mid\ X_\alpha d\alpha = 0.5 \int_0^1 (X^-(\alpha) + X^+(\alpha)) d\alpha,$$

where $X^-(\alpha)$ and $X^+(\alpha)$ are determined by formulas (3).

We define a quasi-scalar product for fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ with α - level sets $X_\alpha(\omega) = [X^-(\omega,\alpha), X^+(\omega,\alpha)]$ and $Y_\alpha(\omega) = [Y^-(\omega,\alpha), Y^+(\omega,\alpha)]$ formula

$$\left\langle \tilde{X}, \tilde{Y} \right\rangle = \int_0^1 \int_\Omega (mid\ X_\alpha(\omega)\, mid\ Y_\alpha(\omega) + \ rad\ X_\alpha(\omega)\, rad\ Y_\alpha(\omega)) dP d\alpha$$

$$= 0.5 \int_0^1 \int_\Omega (X^+(\omega,\alpha) Y^+(\omega,\alpha) + X^-(\omega,\alpha) Y^-(\omega,\alpha)) dP d\alpha. \quad (5)$$

In this case, the quasinorm of the fuzzy random variable $\tilde{X}$ will be denoted $||\tilde{X}|| = \left\langle \tilde{X}, \tilde{X} \right\rangle^{1/2}$.

Note that the same properties 1)–5) hold for the quasi-scalar product (5) as for the quasi-scalar product of fuzzy numbers.

Some other definitions of the scalar product of fuzzy random variables may be found in [6,10], and others.

Fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ with α-level intervals $[X(\omega,\alpha)^-, X(\omega,\alpha)^+]$ and $[Y(\omega,\alpha)^-, Y(\omega,\alpha)^+]$ are called independent if the random variables $X(\omega,\alpha)^-$ and $Y(\omega,\alpha)^-$, as well as $X(\omega,\alpha)^+$ and $Y(\omega,\alpha)^+$ are pairwise independent for all $\alpha \in (0,1]$.

It is easy to check.

Statement 1. *for independent fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ their quasiscalar product $\left\langle \tilde{X}, \tilde{Y} \right\rangle = \langle \tilde{x}, \tilde{y} \rangle$, where $\tilde{x}, \tilde{y}$-fuzzy expectations $\tilde{x}$ and $\tilde{y}$, respectively.*

Define the distance between fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ of class $\mathfrak{X}$ expression

$$d(\tilde{X}, \tilde{Y}) = (\int_0^1 \int_\Omega (\left[X^-(\omega,\alpha) - Y^-(\omega,\alpha) \right]^2$$

$$+ \left[X^+(\omega,\alpha) - Y^+(\omega,\alpha)\right]^2)dPd\alpha)^{1/2}. \qquad (6)$$

Definition (6) corresponds to the definition of the distance between fuzzy numbers (1). Other definitions of the distance between fuzzy-random variables are used, for example, in the works [3,7,10].

It turns out that the expectation $E(\tilde{X})$ and the fuzzy expectation $\tilde{x}$ of a fuzzy random variable $\tilde{X}$ have certain extreme properties with respect to distances (1) and (6), respectively.

Denote by $\hat{y}$ a singleton corresponding to the number $y \in R$, i.e. a fuzzy number characterized by the membership function $\mu_{\hat{y}}(x)$ equal to 1 for $x = y$ and zero in other cases. By definition, all left and right indexes of $\hat{y}$ are equal to y.

The following statements take place.

Statement 2. *For a given fuzzy random variable $\tilde{X}$ with indexes $X^-(\omega,\alpha)$, $X^+(\omega,\alpha)$ its expectation is $E(\tilde{X})$ is the only solution to the extreme problem*

$$d(\tilde{X},\hat{y}) \to \min \quad (\forall y \in R),$$

where the distance is $d(\tilde{X},\hat{y})$ is defined by the formula (6).

Statement 3. *Expectation $E(\tilde{X})$ is the only solution to the following extreme problem*

$$\rho(\tilde{x},\hat{y}) \to \min \quad (\forall y \in R),$$

where is the distance ρ defined by the formula (2).

These statements are verified by applying an extreme sign for scalar differentiable functions $f(y) = d^2(\tilde{X},y)$ and $g = \rho^2(\tilde{x},y)$, respectively, taking into account the expectation definition $E(\tilde{X})$ and fuzzy expectation $\tilde{x}$ of a fuzzy random variable $\tilde{X}$.

The following theorem is true.

Theorem 1. *The fuzzy expectation $\tilde{x}$ of a fuzzy random variable $\tilde{X}$ is the solution to the following extreme problem*

$$d(\tilde{X},\tilde{y}) \to \min \quad (\forall \tilde{y} \in J_0).$$

The proof is just to check equality

$$d^2(\tilde{X},\tilde{y}) = d^2(\tilde{X},\tilde{x}) + \rho^2(\tilde{x},\tilde{y}) \quad (\forall \tilde{y} \in J_0).$$

We emphasize that the average fuzzy random variables in various aspects are widely discussed in the literature. However, their extreme properties were not previously observed.

3 The Law of Large Numbers

According to [8] we define the covariance between fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ with intervals of α-level $[X^-(\omega,\alpha)X^+(\omega,\alpha)]$ and $[Y^-(\omega,\alpha),Y^+(\omega,\alpha)]$ by formula

$$cov[\tilde{X},\tilde{Y}] = 0.5\int_0^1\int_\Omega((X^-(\omega,\alpha)-x^-(\alpha))(Y^-(\omega,\alpha)-y^-(\alpha))$$

$$+(X^+(\omega,\alpha)-x^+(\alpha))(Y^+(\omega,\alpha)-y^+(\alpha)))dPd\alpha. \quad (7)$$

where $x^-(\alpha)$ and $x^+(\alpha)$ defined by formulas (3) and similarly $y^-(\alpha)$ and $y^+(\alpha)$.

This definition is convenient for us because it is closely related to the quasi-scalar product (5) and distance (6) that we have introduced. Various definitions of covariance of fuzzy random variables are found in the literature. In particular, in [6], the covariance $cov[\tilde{X},\tilde{Y}]$ fuzzy random variables $\tilde{X}$, $\tilde{Y}$ is similar to (7) expression

$$cov[\tilde{X},\tilde{Y}] = 0.25\int_0^1\int_\Omega(X^-(\omega,\alpha)+X^+(\omega,\alpha)-x^-(\alpha)$$

$$-x^+(\alpha))(Y^-(\omega,\alpha)+Y^+(\omega,\alpha)-y^-(\alpha)-y^+(\alpha))dPd\alpha.$$

However, it is easy to see that this formula actually includes covariances of random variables $mid\ X_\alpha(\omega)$ and $mid\ Y_\alpha(\omega)$, but not for $rad\ X_\alpha(\omega)$ and $rad\ Y_\alpha(\omega)$.

In this sense, the formula (7) used below more adequately reflects the structure of fuzzy random variables.

The definition of covariance (7) has a lot of properties, that are such a modification of the case of real random variables (see, [8]).

$$1)\, cov[\tilde{X}+\tilde{Z},\tilde{Y}] = cov[\tilde{X},\tilde{Y}] + cov[\tilde{Z},\tilde{Y}];$$

$$2)\, cov[c_1\tilde{X},c_2\tilde{Y}] = c_1c_2cov[\tilde{X},\tilde{Y}],$$

for any real $c_1, c_2 \in R$ such that $c_1c_2 > 0$.

This definition of sum fuzzy random variables and product of fuzzy random variable with real number understands as the respective definition for fuzzy numbers above.

A specific property of the covariance of fuzzy random variables defined by formula (7) with the quasi-scalar product (5) introduced by us (and not noted in [8]) is the following

$$3.\, cov[\tilde{X},\tilde{Y}] = \left\langle \tilde{X},\tilde{Y} \right\rangle - \langle \tilde{x},\tilde{y} \rangle\,.$$

This property (for other definitions of covariance) was considered, for example, in [6,10].

As usual, fuzzy random variables $\tilde{X}_1, \tilde{X}_2$ are called uncorrelated if $cov[\tilde{X}_1, \tilde{X}_2] = 0$.

Remark 1. If the fuzzy random variables $\tilde{X}_1, \tilde{X}_2$ are independent, they are uncorrelated.

This follows from property 3 of the covariance given statement 1.

Remark 2. If the fuzzy random variables $\tilde{X}_1, \tilde{X}_2$ are uncorrelated, then $\left\langle \tilde{X}_1, \tilde{X}_2 \right\rangle = \langle \tilde{x}_1, \tilde{x}_2 \rangle$, where $\tilde{x}_1, \tilde{x}_2$ - fuzzy expectations $\tilde{x}_1$ and $\tilde{x}_2$. Conversely, if the previous equality is satisfied, the fuzzy random variables $\tilde{X}_1, \tilde{X}_2$ are uncorrelated.

This follows from property 3 of the covariance.

We define the variance of the fuzzy random variable $\tilde{X}$ the equation $D(\tilde{X}) = cov[\tilde{X}, \tilde{X}]$ and note its properties (cf. [8]):

1. $D(c\tilde{X}) = c^2 D(\tilde{X})$ for any real number c.
2. $D(\tilde{X} + \tilde{Y}) = D(\tilde{X}) + D(\tilde{Y}) + 2cov[\tilde{X}, \tilde{Y}]$ for $\forall \tilde{X}, \tilde{Y} \in \mathfrak{X}$.
3. $D(\tilde{z}) = 0$ for any fuzzy number $\tilde{z} \in J_0$.

   Important for us is the following special property of the dispersion

4. $$D(\tilde{X}) = \frac{1}{2} d^2(\tilde{X}, \tilde{x}) \ \ (\forall \tilde{X} \in \mathfrak{X}),$$

where $\tilde{x}$ is the fuzzy expectation of a fuzzy random variable $\tilde{X}$, and $d^2(\tilde{X}, \tilde{x})$ is the distance defined by the formula (6).

It follows from the equality $D(\tilde{X}) = cov[\tilde{X}, \tilde{X}]$ and the definitions (6), (7).

Consider for fuzzy random variables looks Chebyshev's inequality (see, e.g., [12], Chap. 6, Sect. 32 to "normal" random variables).

Lemma 1 (Chebyshev's Inequality). *For a fuzzy random variable $\tilde{X}$ with a fuzzy expectation $\tilde{x}$ and a given $\varepsilon > 0$, the inequality occurs*

$$P(d(\tilde{X}, \tilde{x}) \geq \varepsilon) \leq \frac{2}{\varepsilon^2} D(\tilde{X}). \tag{8}$$

Indeed, by the probability properties

$$P(d(\tilde{X}, \tilde{x}) \geq \varepsilon) = \int\limits_{d(\tilde{X}, \tilde{x}) \geq \varepsilon} dP.$$

Since in the integration domain $\frac{d^2(\tilde{X}, \bar{x})}{\varepsilon^2} \geq 1$, then

$$\int\limits_{d(\tilde{X}, \bar{x}) \geq \varepsilon} dP \leq \frac{1}{\varepsilon^2} \int\limits_{\Omega} d^2(\tilde{X}, \bar{x}) dP = \frac{1}{\varepsilon^2} d^2(\tilde{X}, \bar{x}).$$

Whence, taking into account the property 4 of the variance, follows (8).

Inequality (8) is similar to the corresponding inequality from [8], but it uses a different definition of distance.

Let's look how the law of large numbers turns out in the case of fuzzy random variables. There are a significant number of publications on this subject (see, for example, [7,13–15]). The main difference is in determining the distance between fuzzy numbers (respectively, between fuzzy random variables).

Theorem 2 (Law of large numbers). *Let $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$ be a collection of pairwise uncorrelated fuzzy random variables with fuzzy expectations $\tilde{x}_i$. Let their variances be uniformly bounded, i.e. there is a constant $c > 0$ such that $D(X_i) \leq c$ $(i = 1, ..., n)$. Then the relation is valid*

$$P(d(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i) \geq \varepsilon) \leq \frac{2c}{n\varepsilon^2}. \tag{9}$$

Indeed, putting the Chebyshev's inequality $\tilde{X} = \frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i$, get

$$P(d(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i) \leq \frac{2}{\varepsilon^2}D(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i).$$

Further, under the properties 1, 2 of the variance we have

$$D(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i) = \frac{1}{n^2}D(\sum_{i=1}^{n}\tilde{X}_i) = \frac{1}{n^2}\sum_{i=1}^{n}D(\tilde{X}_i) \leq \frac{c}{n}.$$

Hence the result.

Inequality (9) implies

Corollary 1. *In the conditions of Theorem 2 the relation is valid*

$$P(d(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i) < \varepsilon) \geq 1 - \frac{2c}{n\varepsilon^2}. \tag{10}$$

The law of large numbers means that the probability on the left in (10) tends to 1 for $n \to \infty$.

Let's consider an important special case of the law of large numbers. It is said that fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ with intervals of α - levels $[X^-(\omega,\alpha), X^+(\omega,\alpha)]$ and $[Y^-(\omega,\alpha), Y^+(\omega,\alpha)]$ are equally distributed if $X^-(\omega,\alpha)$ and $Y^-(\omega,\alpha)$, and $X^+(\omega,\alpha)$ and $Y^+(\omega,\alpha)$, are equally distributed for all $\alpha \in [0,1]$.

It is said that $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$ is a fuzzy random sample if $\tilde{X}_i$ are independent and equally distributed. Theorem 2 implies

Corollary 2. *Let $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$ be a fuzzy random sample and $\tilde{x}$ be a fuzzy expectation for each of the fuzzy random variables $\tilde{X}_i$. Then*

$$P(d(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, \tilde{x}) < \varepsilon) \geq 1 - \frac{2c}{n\varepsilon^2},$$

where c is the variance of the fuzzy random variable $\tilde{X}_i$.

Moreover, under conditions of Corollary 2 and properties 1)–4) of the variance the convergence on metric (6) of $\frac{1}{n}\sum\limits_{i=1}^{n}\tilde{X}_i$ to $\tilde{x}$ is valid, when $n \to \infty$.

4 Linear Regression

Let's consider the optimal linear approximation of a (predicted) fuzzy random variable $\tilde{Y}$ using a system of (predictive) fuzzy 1random variables $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$. In a number of works [8,11,15–17] and other tasks of this kind were considered. In this case, the specifics of the problem are determined by the choice of the distance to be minimized. We investigate the question of approximating a fuzzy random variable $\tilde{Y}$ - the linear combinations $\sum\limits_{i=1}^{n}\beta_i\tilde{X}_i$ with real coefficients β_i $(i = 1, ..., n)$ by the criterion of minimizing the distance (6).

Consider first the extreme challenge with nonnegative coefficients $\beta_i \geq 0$ $(i = 1, ..., n)$

$$d(\tilde{Y}, \sum_{i=1}^{n}\beta_i\tilde{X}_i) \to \min \quad (\forall\beta_i \geq 0). \tag{11}$$

Takes place

Lemma 2. *Let the fuzzy random variables $\tilde{X}_i$ be quasi-orthogonal for $i \neq j$, and their quasinorms $\varkappa_j := \left\langle \tilde{X}_j, \tilde{X}_j \right\rangle^{1/2} \neq 0$ $(j = 1, ..., n)$. Let the condition $b_i = \left\langle \tilde{Y}, \tilde{X}_i \right\rangle \geq 0$ $(i = 1, ..., n)$. then problem (11) has a non-negative solution, and the only one. It has the form $\beta_i^* = \frac{b_i}{\varkappa_i^2}$, $(i = 1, ..., n)$.*

Indeed, due to the assumption that the coefficients β_i are non-negative, the left index $\sum\limits_{i=1}^{n}\beta_i\tilde{X}_i$ is equal to $\sum\limits_{i=1}^{n}\beta_i X_i^-(\omega, \alpha)$, and the right one is $\sum\limits_{i=1}^{n}\beta_i X_i^+(\omega, \alpha)$. We will omit the arguments ω, α in the proof below. Put

$$F(\beta_1, ..., \beta_n) = d^2(\tilde{Y}, \sum_{i=1}^{n}\beta_i\tilde{X}_i)$$

$$= \int_0^1 \int_\Omega ((Y^+ - \sum_{i=1}^{n}\beta_i X_i^+)^2 + (Y^- - \sum_{i=1}^{n}\beta_i X_i^-)^2) dP d\alpha. \tag{12}$$

This is a quadratic form in $\beta_1, ..., \beta_n$.

Differentiate with respect to (12) for β_j and equate the derivative to zero

$$\frac{\partial F}{\partial \beta_j} = -2\int_0^1 \int_{\Omega}((Y^+ - \sum_{i=1}^{n}\beta_i X_i^+)X_j^+ + (Y^- - \sum_{i=1}^{n}\beta_i X_i^-)X_j^-)dPd\alpha = 0.$$

Hence, for every $j = 1, 2, ..., n$ we have

$$\sum_{i=1}^{n}\beta_i \int_0^1 \int_{\Omega}(X_i^+ X_j^+ + X_i^- X_j^-)dPd\alpha = \int_0^1 \int_{\Omega}(Y^+ X_j^+ + Y^- X_j^-)dPd\alpha,$$

i.e.

$$\sum_{i=1}^{n}\beta_i \left\langle \tilde{X}_i, \tilde{X}_j \right\rangle = \left\langle \tilde{Y}, \tilde{X}_j \right\rangle \ \ (j = 1, ..., n), \tag{13}$$

We introduce the following notation. Vector B with coefficients $b_i = \left\langle \tilde{Y}, \tilde{X}_i \right\rangle$, matrix A with coefficients $A_{ij} = \left\langle \tilde{X}_i, \tilde{X}_j \right\rangle$, vector β with coefficients β_i. In vector form, system (13) has the form $A\beta = B$. Matrix A due to the quasi-orthogonality of the system $\{\tilde{X}_i\}$ has a diagonal form, with positive numbers on the main diagonal $\varkappa_i^2$ $(i = 1, ..., n)$. then the solution is $\beta^* = A^{-1}B$, i.e. $\beta_i^* = \frac{b_i}{\varkappa_i^2}$ $(i = 1...n)$.

The nonnegativity of the obtained coefficients β_i^* is provided by the condition $b_i = \left\langle \tilde{Y}, \tilde{X}_i \right\rangle \geq 0$ $(i = 1, ..., n)$.

To verify that $\beta^* = A^{-1}B$ is the minimum point, consider the second derivative

$$\frac{\partial^2 F}{\partial \beta_j \partial \beta_s} = 2\int_0^1 \int_{\Omega} 4(X_s^+ X_j^+ + X_s^- X_j^-)dPd\alpha = \left\langle \tilde{X}_s, \tilde{X}_j \right\rangle \ \text{ when } s \neq j.$$

$$\frac{\partial^2 F}{\partial \beta_j^2} = 2\int_0^1 \int_{\Omega}((X_j^+)^2 + (X_j^-)^2)dPd\alpha = 4\left\langle \tilde{X}_j, \tilde{X}_j \right\rangle \ \text{ when } s = j.$$

A sufficient sign of a minimum is the positive definiteness of the Hesse matrix $\{\frac{\partial^2 F}{\partial \beta_j \partial \beta_s}\}$. And this is provided by the quasi-orthogonality of the system $\{\tilde{X}_j\}$.

Remark 3. Condition $\left\langle \tilde{Y}, \tilde{X}_i \right\rangle > 0$ means that there is an acute angle between the fuzzy random variables $\tilde{Y}$ and $\tilde{X}_i$. In other words, the fuzzy-random variables $\tilde{Y}$ and $\tilde{X}_i$ increase (in this sense) or decrease at the same time.

Remark 4. In the conditions of Lemma 2, we can reject the requirement of pairwise quasi-orthogonality of fuzzy random variables $\tilde{X}_1, ..., \tilde{X}_n$. It is sufficient to require positive invertibility of their Gram matrix A with coefficients $a_{ij} = \left\langle \tilde{X}_i, \tilde{X}_j \right\rangle$. In the sense that the inverse matrix A^{-1} exists and converts vectors with non-negative coordinates back to vectors with non-negative coordinates.

Remark 5. If, under Lemma 2, we reject the requirement of pairwise quasi-orthogonality of fuzzy random variables $\tilde{X}_1, ..., \tilde{X}_n$, but additionally assume their pairwise uncorrelability, then it is sufficient to require positive invertibility of the Gram matrix from their fuzzy expectations $\langle \tilde{x}_i, \tilde{x}_j \rangle$.

We emphasize that the coefficients β_i^* are analogous to the Fourier coefficients when decomposing in an orthogonal system in a Hilbert space (see, for example, [17], Chap. II, Sect. 11 for random variables). This is due to the relationship of the metric to be minimized in problem (11) with quasi-scalar product (5).

The proximity of fuzzy random variables (as well as any space with a scalar product) can be characterized by the cosine of the angle between them.

Define the cosine between the fuzzy random variables $\tilde{Y}, \tilde{Z}$ by the equality

$$\cos(\tilde{Y}, \tilde{Z}) = \frac{\left\langle \tilde{Y}, \tilde{Z} \right\rangle}{||\tilde{Y}||||\tilde{Z}||}. \tag{14}$$

According to the definition (14) and the properties of the cosine of the angle between the fuzzy numbers $|cos(\tilde{Y}, \tilde{Z})| \leq 1$. In this case, $cos(\tilde{Y}, \tilde{Z}) = 0$, if and only, if $\tilde{Y}$ and $\tilde{Z}$ are quasi-orthogonal. And $cos(\tilde{Y}, \tilde{Z}) = 1$, if and only, if $\tilde{Z} = \lambda\tilde{Y}$ are collinear ($\lambda > 0$).

Denote, as in Lemma 2, $\beta_i^* = \frac{1}{\varkappa_i^2} \left\langle \tilde{Y}, \tilde{X}_i \right\rangle$ and consider

$$\tilde{Z}_n^* = \sum_{i=1}^{n} \beta_i^* \tilde{X}_i \tag{15}$$

- an optimal estimate of the predicted fuzzy random variable $\tilde{Y}$ from Lemma 2.

Theorem 3. *Let the conditions of Lemma 2. Then the optimal estimate (15) has the maximum cosine with the predicted fuzzy random variable $\tilde{Y}$ in the class of linear estimates of the form $Z_n = \sum\limits_{i=1}^{n} \beta_i \tilde{X}_i$ $(\beta_i \geq 0)$.*

Indeed, we will show that

$$|cos(\tilde{Y}, \tilde{Z}_n)| \leq cos(\tilde{Y}, \tilde{Z}_n^*).$$

Due to the properties of the quasi-scalar product and the non-negativity of the coefficients $\beta_i^* \geq 0$ we have

$$\left\langle \tilde{Y}, \tilde{Z}_n^* \right\rangle = \left\langle \tilde{Y}, \sum_{i=1}^{n} \beta_i^* \tilde{X}_i \right\rangle = \sum_{i=1}^{n} \beta_i^* \left\langle \tilde{Y}, \tilde{X}_i \right\rangle = \sum_{i=1}^{n} (\beta_i^*)^2 \varkappa_i^2.$$

In this case, due to the pairwise quasi-orthogonality of the system $||Z_n^*||^2 = \sum\limits_{i=1}^{n} (\beta_i^*)^2 \varkappa_i^2$. Then

$$cos(\tilde{Y}, \tilde{Z}_n^*) = \frac{\sum\limits_{i=1}^{n} \varkappa_i^2 (\beta_i^*)^2}{||\tilde{Y}||(\sum\limits_{i=1}^{n} \varkappa_i^2 (\beta_i^*)^2)^{1/2}} = \frac{1}{||\tilde{Y}||} (\sum_{i=1}^{n} \varkappa_i^2 (\beta_i^*)^2)^{1/2}.$$

Consider

$$\left\langle \tilde{Y}, \tilde{X}_i \right\rangle = \sum_{i=1}^{n} \beta_i \left\langle \tilde{Y}, \tilde{X}_i \right\rangle = \sum_{i=1}^{n} \beta_i \beta_i^* \varkappa_i^2$$

and $||\tilde{Z}_n||^2 = \sum_{i=1}^{n} \beta_i^2 \varkappa_i^2$.

Then

$$cos(\tilde{Y}, \tilde{Z}_n) = \frac{\sum_{i=1}^{n} \varkappa_i^2 \beta_i \beta_i^*}{||\tilde{Y}||(\sum_{i=1}^{n} \beta_i^2 \varkappa_i^2)^{1/2}}.$$

By the Cauchy-Schwarz inequality

$$|cos(\tilde{Y}, \tilde{Z}_n)| \leq \frac{(\sum_{i=1}^{n} \varkappa_i^2 \beta_i^2)^{1/2} (\sum_{i=1}^{n} \varkappa_i^2 (\beta_i^*)^2)^{1/2}}{||\tilde{Y}||(\sum_{i=1}^{n} \varkappa_i^2 \beta_i^2)^{1/2}}$$

$$= \frac{1}{||\tilde{Y}||} (\sum_{i=1}^{n} \varkappa_i^2 (\beta_i^*)^2)^{1/2} = cos(\tilde{Y}, \tilde{Z}_n^*),$$

which was required to be proved.

Let's consider the optimal regression problem in a situation where all linear approximation coefficients are not assumed to be nonnegative and the condition $\left\langle \tilde{Y}, \tilde{X}_i \right\rangle \geq 0 \ \ (i = 1, ..., n)$ is met.

Note that the explicit form of the formula for the distance $d(\tilde{Y}, \sum_{i=1}^{n} \beta_i \tilde{X}_i)$ in the case of coefficients β_i of an arbitrary sign is inconvenient for research, since in this case the product of the interval α - the level of a fuzzy number $\tilde{z}$ by a clear number β is given by the cumbersome expression

$$\beta[z^-, z^+] = [\min\{\beta z^-, \beta z^+\}, \max\{\beta z^-, \beta z^+\}].$$

However, in the general situation, the following statement is true. Let's say $c_* = \max_{j=1,...,n} \{\frac{||\tilde{Y}||}{||\tilde{X}_j||}\}$.

Theorem 4. *Let the fuzzy-random variables $\tilde{X}_i$ be quasi-orthogonal for $i \neq j$, and all their quasinorms $\varkappa_i \neq 0$ $(i = 1, ..., n)$. then the problem is*

$$d(\tilde{Y}, \sum_{i=1}^{n} \beta_i \tilde{X}_i) \to \min \ \ (\beta_i \in [-c_*, \infty)) \tag{16}$$

has a solution, and the only one. It has the form $\beta_i^* = \frac{b_i}{\varkappa_i^2}$ $(i = 1, ..., n)$.

Note though problem (16) does not assume that the coefficients β_i are positive, the formula for the coefficients β_i has the same form as in Lemma 2. At the same time the coefficients b_i in the condition of Theorem 4 may have different signs.

In the proof of Theorem 4, the following special property of the distance (6) between fuzzy random variables will be used.

Lemma 3. *For any fuzzy random variables* $\tilde{X}$, $\tilde{Y}$ *and* $\tilde{W}$ *in* $\mathfrak{X}$ *the next equality holds*

$$d(\tilde{X} + \tilde{W}, \tilde{Y} + \tilde{W}) = d(\tilde{X}, \tilde{Y}).$$

In fact, this is true because subject to the rules of interval addition on the left are

$$(\tilde{X} + \tilde{W})^- = X^- + W^-,\ (\tilde{Y} + \tilde{W})^- = Y^- + W^-,$$

and similarly for the right indexes.

After substituting the corresponding expressions in (6), we obtain the required equality.

Proof of Theorem 4. Let the condition $\left\langle \tilde{Y}, \tilde{X}_i \right\rangle \geq 0$ not be satisfied for at least one j. Consider the fuzzy random variable $\tilde{Z} = \tilde{Y} + c_* \sum_{i=1}^{n} \tilde{X}_i$. According to the definition $c_* > 0$, $\left\langle \tilde{Z}, \tilde{X}_j \right\rangle \geq 0 \quad (j = 1, ..., n)$. Consider for $\tilde{Z}$ task (11). Let $\gamma_i \geq 0$ be the optimal coefficients of a linear combination $\sum_{i=1}^{n} \gamma_i \tilde{X}_i$ for $\tilde{Z}$, obtained by solving problem (11). The vector γ with coordinates γ_i is defined by the formula $\gamma = A^{-1} f$, for $f_i = \left\langle \tilde{Z}, \tilde{X}_j \right\rangle$.

We show that the coefficients $\gamma_i - c_*$ are optimal for linear approximation of a fuzzy random variable $\tilde{Y}$ by the system $\{\tilde{X}_i\}$.

Consider the distance $d(\tilde{Y}, \sum_{i=1}^{n} (\gamma_i - c_*) X_i)$. By Lemma 3 and taking into account the definition of $\tilde{Z}$, we have

$$d(\tilde{Y}, \sum_{i=1}^{n} (\gamma_i - c_*) X_i) = d(\tilde{Y} + \sum_{i=1}^{n} c_* \tilde{X}_i, \sum_{i=1}^{n} \gamma_i \tilde{X}_i) = d(\tilde{Z}, \sum_{i=1}^{n} \gamma_i \tilde{X}_i). \tag{17}$$

Since $\{\gamma_i\}$ - solution of problem (11) for $\tilde{Z}$, then in accordance with Lemma 2 for any set of numbers $\xi_i \geq 0 \ (i = 1, ..., n)$ can record

$$d(\tilde{Z}, \sum_{i=1}^{n} \gamma_i \tilde{X}_i) \leq d(\tilde{Z}, \sum_{i=1}^{n} \xi_i \tilde{X}_i) = d(\tilde{Y} + \sum_{i=1}^{n} c \tilde{X}_i, \sum_{i=1}^{n} (\xi_i - c_*) X_i + \sum_{i=1}^{n} c_* \tilde{X}_i)$$

Using Lemma 3 again, we get

$$d(\tilde{Z}, \sum_{i=1}^{n} \gamma_i \tilde{X}_i) \leq d(\tilde{Y}, \sum_{i=1}^{n} (\xi_i - c_*) \tilde{X}_i).$$

Then (17) implies the inequality

$$d(\tilde{Y}, \sum_{i=1}^{n} (\gamma_i - c_*) X_i) \leq d(\tilde{Y}, \sum_{i=1}^{n} (\xi_i - c_*) X_i).$$

Since here $(\xi_i - c_*)$ - arbitrary coefficients from a closed interval $[-c_*, \infty)$, then $(\gamma_i - c_*)$ - optimal coefficients.

Note that by definition $\tilde{Z}$ and according to Lemma 2

$$\gamma_j = \frac{1}{\varkappa_j^2} \left\langle \tilde{Z}, \tilde{X}_j \right\rangle = \frac{1}{\varkappa_j^2} \left\langle (\tilde{Y} + \sum_{i=1}^{n} c_* \tilde{X}_i), \tilde{X}_j \right\rangle.$$

Then, taking into account quasiorthogonality system $\{\tilde{X}_i\}$ will receive

$$\gamma_j = \frac{1}{\varkappa_j^2} \left(b_j + c_* \left\langle \tilde{X}_j, \tilde{X}_j \right\rangle \right) = \frac{b_j}{\varkappa_j^2} + c_*.$$

Hence, the optimal coefficients of $\tilde{\beta}_j$ for $\tilde{X}_j$ in the linear approximation $\tilde{Y}$, having the form $(\gamma_j - c_*)$, defined by the equality $\frac{b_j}{\varkappa_j^2}$ $(j = 1, ..., n)$. this is what the statement implies.

Remark 6. Similarly to Remark 4, under the conditions of theorem 4, one can reject the quasi-orthogonality of the system $\{\tilde{X}_i\}$ and instead assume positive invertibility of their Gram matrix. In addition, it is required that the sum of elements of all columns of the Gram matrix be positive.

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