Survival Probabilities in Compound Poisson Model with Negative Claims and Investments as Viscosity Solutions of Integro-Differential Equations

Tatiana Belkina(✉)

Central Economics and Mathematics Institute RAS,
Nakhimovsky Prosp. 47, 117418 Moscow, Russia
tbel@cemi.rssi.ru
http://www.cemi.rssi.ru/about/persons/index.php?SECTION_ID=6&ELEMENT_ID=4123

Abstract. This work relates to the problem of the identifying of some solutions to linear integro-differential equations as the probability of survival (non-ruin) in the corresponding collective risk models involving investments. The equations for the probability of non-ruin as a function of the initial reserve are generated by the infinitesimal operators of corresponding dynamic reserve processes. The direct derivation of such equations is usually accompanied by some significant difficulties, such as the need to prove a sufficient smoothness of the survival probability. We propose an approach that does not require a priori proof of the smoothness. It is based on previously proven facts for a certain class of insurance models with investments: firstly, under certain assumptions, the survival probability is at least a viscosity solution to the corresponding integro-differential equation, and secondly, any two viscosity solutions with coinciding boundary conditions are equivalent. We apply this approach, allowing us to justify rigorously the form of the survival probability, to the collective life insurance model with investments.

Keywords: Survival probability · Viscosity solution · Integro-differential equations

1 Introduction

The problem of viscosity solutions of linear integro-differential equations (IDEs) for non-ruin probabilities as a functions of an initial surplus in collective insurance risk models, when the whole surplus is invested into a risky (or risk-free) asset, is considered in [1]. For a rather general model of the resulting surplus process, it is shown that the non-ruin probability always solves corresponding IDE in the viscosity sense. Moreover, for the case when the distributions of claims in the insurance risk process have full support on the half-line, a uniqueness theorem is proved in [1]. In the present paper, we use these results to establish that the solution of some previously formulated and investigated boundary value problem for IDE defines the probability of ruin for the corresponding surplus model.

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Thus, the uniqueness theorem for a viscosity solution plays the role of a verification argument for the solution of the IDE as the probability of non-ruin for the resulting surplus process in the models with investments. The approach proposed here can be considered as an alternative tool along with traditional verification arguments based on the use of the martingale approach (see, e.g., [2,3] and references therein). It can be used when it is possible to determine a priori the value of the probability of non-ruin at an initial surplus which is equal to zero, and its limiting value when the initial surplus tends to infinity.

The mentioned general model, which is studied in [1], considers an insurance risk in the classical actuarial framework but under the assumptions that the price process of the risky asset is a jump-diffusion process defined by the stochastic exponential of the Lévy process. The classical actuarial framework involves two possible versions of the original model (without investment): the classical Cramér-Lundberg model or the so-called dual risk model (also called compound Poisson model with negative claims [4], or life annuity insurance model [5]). To demonstrate the main idea of this paper, we consider the dual risk model and assume that the insurer's reserve is invested to a risky asset with price modelled by the geometric Brownian motion or it is invested to a risk-free asset. We use this particular case of the model considered in [1], because 1) for the case of an exponential distribution of jumps and risky investments, the existence of a twice continuously differentiable solution to the boundary value problem for the corresponding IDE is proved in [6], where its properties also are studied and the numerical calculations are done; for the risk-free investment, a non-smooth, generally speaking, solution is constructed in [7] and 2) the value of the survival probability at zero surplus level is a priory known (unlike, for example, the Cramér-Lundberg model with investment, where it can be determined only numerically; see, e.g. [8]).

The paper is organized as follows. In Sect. 2 the compound Poisson model with negative claims and investments is described. Then the problem of the identifying of some solutions to linear integro-differential equations as the survival probabilities in this model in two cases: risky and risk-free investments is formulated. In Sect. 3 some preliminary results about survival probabilities as viscosity solutions of IDEs are given. In Sect. 4 a general statement concerning the identifying the survival probability in the considered model (Theorem 3) is proved. In this statement, the uniqueness theorem for a viscosity solution as a verification argument for the survival probability is used. Moreover, the results of Theorem 3 with applying to the case of exponential distribution of premiums size (jumps of the compound Poisson process) are given; here risky investments (Sect. 4.1) as well as risk-free investments (Sect. 4.2) are considered. Section 5 deals with proofs. In Sect. 6 some results of numerical calculations from [7] are presented, and Sect. 7 contains the conclusions.

2 The Model Description and Statement of the Problem

The typical insurance contract for the policyholder in the dual risk model is the life annuity with the subsequent transfer of its property to the benefit of the insurance company. Thus, the surplus of a company in a collective risk model is of the form

$$R_t = u - ct + \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0. \tag{1}$$

Here R_t is the surplus of a company at time $t \geq 0$; u is the initial surplus, $c > 0$ is the life annuity rate (or the pension payments per unit of time), assumed to be deterministic and fixed. $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of random revenues up to the time t; Z_k $(k = 1, 2, ...)$ are independent identically distributed random variables (r.v.) with a distribution function $F(z)$ $(F(0) = 0,\ \mathbf{E}Z_1 = m < \infty,$ $m > 0)$ that determine the revenue sizes (premiums) and are assumed to be independent of $N(t)$. These random revenues arise at the final moments of the life annuity contracts realizations.

We assume also that the insurer's reserve is invested to a risky asset with price S_t modelled by the geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \quad t \geq 0,$$

where μ is the stock return rate, σ is the volatility, w_t is a standard Brownian motion independent of $N(t)$ and Z_i's.

Then the resulting surplus process X_t is governed by the equation

$$dX_t = \mu X_t dt + \sigma X_t dw_t + dR_t, \quad t \geq 0, \tag{2}$$

with the initial condition $X_0 = u$, where R_t is defined by (1).

Denote by $\varphi(u)$ the survival probability: $\varphi(u) = \mathbf{P}\left(X_t \geq 0,\ t \geq 0\right)$. Then $\Psi(u) = 1 - \varphi(u)$ is the ruin probability. Then $\tau^u := \inf\{t\colon\ X_t^u \leq 0\}$ is the time of ruin.

Recall at first that the infinitesimal generator $\mathcal{A}$ of the process X_t has the form

$$(\mathcal{A}f)(u) = \frac{1}{2}\sigma^2 u^2 f''(u) + f'(u)(\mu u - c) - \lambda f(u) + \lambda \int_0^\infty f(u+z)\, dF(z), \tag{3}$$

for any function $f(u)$ from a certain subclass of the space $\mathcal{C}^2(\mathbb{R}_+)$ of twice continuously differentiable on $(0, \infty)$ functions (in the case $\sigma > 0$; if $\sigma = 0$ we are dealing with a different class of functions, see [7]).

One of the important questions in this and similar models is the question of whether the survival probability $\varphi(u)$ is a twice continuously differentiable function of the initial capital u on $(0, \infty)$. In the case of a positive answer to this question, we can state that $\varphi(u)$ is a classical solution of the equation

$$(\mathcal{A}f)(u) = 0, \quad u > 0, \tag{4}$$

and the properties of this probability can be investigated as properties of a suitable solution to this equation. In [10], for the case of exponential distribution of Z_k and $\sigma > 0$, such a suitable solution in the set of all solutions of the linear IDE (4) is selected using some results of renewal theory; the regularity (twice continuous differentiability) of $\varphi(u)$ is studied using a method based on integral representations; asymptotic expansions of the survival probability for infinitely large values of the initial capital is obtained.

In contrast to the direct method used in [10], we propose a method based on the assumption of the existence of a classical (or, maybe, viscosity) solution to a boundary value problem for the IDE (4) and verification arguments for the survival probability related to the concept of viscosity. For the case of exponential distribution of the company's random revenues and $\sigma > 0$, the existence theorem for IDE (4) with boundary conditions

$$\lim_{u\to+0} f(u) = 0, \quad \lim_{u\to+\infty} f(u) = 1, \tag{5}$$

is proved in [6]. The uniqueness of the classical solution is also established, as well as its asymptotic behaviour at zero and at infinity. For the case $\sigma = 0$, a non-smooth (generally speaking) solution is presented in [7].

The problem we are solving here: to prove that if there exists a solution f of the problem (4), (5), then it determines the survival probability of the process (2). For the solving this problem, we use the results of [1] on the survival probability as a viscosity solution to equation (4).

3 Survival Probabilities as Viscosity Solutions of IDEs: Preliminary Results [1]

Let denote by $C_b^2(u)$ the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ two times continuously differentiable in the classical sense in a neighbourhood of the point $u \in]0, \infty[$ and equal to zero on $]-\infty, 0]$. For $f \in C_b^2(u)$, the value $(\mathcal{A}f)(u)$ is well-defined.

A function $\Phi :]0, \infty[\to [0, 1]$ is called *a viscosity supersolution* of (4) if for every point $u \in]0, \infty[$ and every function $f \in C_b^2(u)$ such that $\Phi(u) = f(u)$ and $\Phi \geq f$ the inequality $(\mathcal{A}f)(u) \leq 0$ holds.

A function $\Phi :]0, \infty[\to [0, 1]$ is called *a viscosity subsolution* of (4) if for every $u \in]0, \infty[$ and every function $f \in C_b^2(u)$ such that $\Phi(u) = f(u)$ and $\Phi \leq f$ the inequality $(\mathcal{A}f)(u) \geq 0$ holds.

A function $\Phi :]0, \infty[\to [0, 1]$ is a *viscosity solution* of (4) if Φ is simultaneously a viscosity super- and subsolution.

From the results of [1], formulated for the more general model of the surplus process, we have that the following propositions are true:

Theorem 1. *The survival probability φ of the process (2) as a function of an initial surplus u is a viscosity solution of IDE (4) with $\mathcal{A}$ defined by (3).*

Theorem 2. *Suppose that the topological support of the measure $dF(z)$ is $\mathbb{R}_+ \setminus \{0\}$. Let Φ and $\tilde{\Phi}$ be two continuous bounded viscosity solutions of (4) with the boundary conditions $\Phi(+0) = \tilde{\Phi}(+0)$ and $\Phi(\infty) = \tilde{\Phi}(\infty)$. Then $\Phi \equiv \tilde{\Phi}$.*

4 Main Results

Theorem 3. *Let the topological support of the measure $dF(z)$ be $\mathbb{R}_+ \setminus \{0\}$ and the survival probability $\varphi(u)$ of the process (2) be continuous on $[0, \infty[$ and not identically zero. Suppose there is a continuous viscosity solution Φ of IDE (4) with the boundary conditions (5). Then $\varphi \equiv \Phi$.*

Proof. First, we note that, as is easy to see, $\varphi(0) = 0$ (see also [6, Lemma 1]). In addition, if $\varphi(u)$ is not identically zero, then

$$\lim_{u \to +\infty} \varphi(u) = 1. \tag{6}$$

Indeed, by the Markov property for any $t, u \geq 0$ we have the identity $\varphi(u) = \varphi(X_{\tau^u \wedge t})$. Using the Fatou lemma and the monotonicity of φ we get, for $t \to \infty$, that $\varphi(u) = \overline{\lim}_t \mathbf{E}\varphi(X_{\tau^u \wedge t}) \leq \mathbf{E}\, \overline{\lim}_t\, \varphi(X_{\tau^u \wedge t}) \leq \mathbf{E}\, \varphi(X_{\tau^u}) I_{\{\tau^u < \infty\}} + $ $+ \lim_{u \to +\infty} \varphi(u)\, \mathbf{E}\, I_{\{\tau^u = \infty\}}$. In virtue of definitions, the first term in the right-hand side is zero. Then $\varphi(u) \leq \varphi(u) \lim_{u \to +\infty} \varphi(u)$. Since $\varphi(u)$ is monotone, we conclude from this inequality that if it is not identically zero, then equality (6) is true. In view of Theorem 1 the survival probability φ is the viscosity solution of IDE (4). Therefore, from Theorem 2 on the uniqueness of the viscosity solution with fixed boundary conditions, we have $\varphi \equiv \Phi$.

Remark 1. For the case $\sigma = 0$, the equality (6) is also the consequence of the following relation:

$$\varphi(u) \equiv 1, \;\; u \geq c/\mu, \tag{7}$$

(see Lemma 1 and Remark 2 below).

Next, we consider examples of the application of Theorem 3 in the case of exponential distribution of Z_i.

4.1 The Case of Risky Investments ($\sigma > 0$)

In [6] the following proposition is proved.

Theorem 4. *Let $F(z) = 1 - \exp(-z/m)$, all the parameters in (3): c, λ, m, μ, $\sigma > 0$, and $2\mu > \sigma^2$. Then the following assertions hold:*

(I) *there exists a twice continuously differentiable function f satisfying the equation IDE (4) and conditions (5);*

(II) *this solution may be defined by the formula $f(u) = 1 - \int_u^\infty g(s)\, ds$, where $g(u)$ is the unique solution of the following problem for an ordinary differential equation (ODE):*

$$\frac{1}{2}\sigma^2 u^2 g''(u) + \left(\mu u + \sigma^2 u - c - \frac{1}{2m}\sigma^2 u^2\right) g'(u) + \left(\mu - \lambda - \frac{\mu u - c}{m}\right) g(u) = 0, \quad u > 0, \tag{8}$$

$$\lim_{u\to+0}|g(u)|<\infty, \qquad \lim_{u\to+0}[ug'(u)]=0, \tag{9}$$

$$\lim_{u\to\infty}[ug(u)]=0, \qquad \lim_{u\to\infty}[u^2 g'(u)]=0, \tag{10}$$

with the normalizing condition

$$\int_0^\infty g(s)\,ds=1. \tag{11}$$

Moreover, in [6], asymptotic representations of the solution f at zero and at infinity are obtained and examples of its numerical calculations by solving the ODE problem (8)–(11) are given.

Theorem 5. *Let the conditions of Theorem 4 be satisfied. Then the function f defined in this theorem is the survival probability for the process (2), i.e., $\varphi\equiv f$.*

4.2 The Case of Risk-Free Investments ($\sigma = 0$)

For this case, our approach can also be applied to a non-smooth (generally speaking) solution constructed in [7] (see also [9,11]).

We assume here that the insurer's reserve is invested to a risk-free asset with price B_t modelled by the equation

$$dB_t = rB_t dt, \quad t\ge 0,$$

where r is the return rate.

Then the resulting surplus process X_t is governed by the equation

$$dX_t = rX_t dt + dR_t, \quad t\ge 0, \tag{12}$$

with the initial condition $X_0=u$, where R_t is defined by (1).

Recall that, in the case $\sigma=0$, the infinitesimal generator (3) of the corresponding process X_t takes the form

$$(\mathcal{A}f)(u) = f'(u)(ru-c) - \lambda f(u) + \lambda\int_0^\infty f(u+z)\,dF(z) \tag{13}$$

(here we rename the return rate of the risk-free asset from μ to r). From the results of [7] we have the following

Proposition 1. *Let $F(z)=1-\exp(-z/m)$, all the parameters in (13): c, λ, m, $r>0$. Then the following assertions hold:*

(I) *there exists a continuous function Φ, which is twice continuously differentiable on the interval $(0,c/r)$, satisfying the equation IDE (4) (everywhere, except, perhaps, the point c/r) and the conditions*

$$\lim_{u\to+0}\Phi(0)=0, \quad \Phi(u)\equiv 1, \;\; u\ge c/r; \tag{14}$$

(II) *on the interval* $(0, c/r)$, *this solution may be defined by the formula* $\Phi(u) = 1 - \int\limits_u^{c/r} g(s)\,ds$, *where*

$$g(u) = \left[\int\limits_0^{c/r} (c/r - u)^{\lambda/r-1} \exp(u/m)\,du\right]^{-1} (c/r - u)^{\lambda/r-1} \exp(u/m); \quad (15)$$

(III) $\Phi(u)$ *is a viscosity solution of IDE (4);*

(IV) *for* $\lambda > 2r$, $\Phi(u)$ *is a twice continuously differentiable on* $(0, \infty)$ *function, i.e., it is a classical solution of IDE (4); in this case* $\lim_{u\uparrow c/r} \Phi''(u) = \lim_{u\uparrow c/r} \Phi'(u) = 0$*; otherwise,* $\Phi(u)$ *satisfies IDE (4) in the classical sense everywhere except for the point* $u = c/r$*;*

(V) *for* $r < \lambda \le 2r$, $\Phi(u)$ *is smooth but it is not twice continuously differentiable on* $(0, \infty)$, *since* $\lim_{u\uparrow c/r} \Phi''(u) = -\infty$ *for* $\lambda < 2r$, *and*

$$\lim_{u\uparrow c/r} \Phi''(u) = -m^{-2} \left[\exp\big(c/(rm)\big) - 1 - c/(rm)\right]^{-1} \exp\big(c/(rm)\big) < 0,$$

$\lambda = 2r$*;*

(VI) *for* $\lambda \le r$, $\Phi(u)$ *is not smooth, since its derivative is discontinuous at the point* $u = c/r$*:*

$$\lim_{u\uparrow c/r} \Phi'(u) = m^{-1} \left[\exp\big(c/(rm)\big) - 1\right]^{-1} \exp\big(c/(rm)\big) > 0,$$

$\lambda = r$, *and* $\lim_{u\uparrow c/r} \Phi'(u) = \infty$, *wherein* $\Phi'(u)$ *is integrable at the point* $u = c/r$, $\lambda < r$.

Theorem 6. *Let the conditions of Proposition 1 be satisfied. Then the function* Φ *defined in this proposition is the survival probability for the process (12), i.e.,* $\varphi \equiv \Phi$.

5 Proofs

Let us return to the general case of a process of the form (2) and prove auxiliary statements about non-triviality and continuity of its survival probability (Lemma 1 and Lemma 3 below respectively). We also formulate Lemma 2 about zero value of the survival probability at zero surplus level. Then the statement of Theorem 5 is a consequence of Theorems 3, 4 and Lemmas 1–3. The statement of Theorem 6 is a consequence of Theorem 3, Proposition 1 and the same lemmas.

Lemma 1. *Let*

$$2\mu > \sigma^2. \quad (16)$$

Then the survival probability $\varphi(u)$ *of process (2) is not identically zero. Moreover, if* $\sigma = 0$, *then* $\varphi(u) = 1$, $u \ge c/\mu$.

Proof. 1) The case $\sigma > 0$. Let

$$\mu(x) = \mu x - c, \quad \sigma(x) = \sigma x. \tag{17}$$

Let us consider the process $Y_t = Y_t^u$ given by the equation

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dw_t, \tag{18}$$

with initial state $Y_0 = u > 0$ and the same standard Brownian motion as in (2). To understand the qualitative behavior of the process (2), we use the corresponding result for the process (18) at first; this result is given in [12, Chapter 4]. Below we use the following functions:

$$\rho(x) = \exp\left(-\int_a^x \frac{2\mu(y)}{\sigma^2(y)}dy\right), \quad x \in [a, \infty), \tag{19}$$

$$s(x) = -\int_x^\infty \rho(y)dy, \quad x \in [a, \infty). \tag{20}$$

It is easy to check that in the case when the functions $\mu(x)$, $\sigma(x)$ are of the form (17) and the relation (16) is valid, we have

$$\int_a^\infty \rho(x)dx < \infty, \quad \int_a^\infty \frac{|s(x)|}{\rho(x)\sigma^2(x)}dx = \infty.$$

Note that the (strong) solution of equation (18) with coefficients defined in (17) and the initial state $Y_0 = u$ can be represented as

$$Y_t^u = \exp(H_t)\left[u - c\int_0^t \exp(-H_s)\,ds\right], \quad t \geq 0, \tag{21}$$

where

$$H_t = \left(\mu - \sigma^2/2\right)t + \sigma w_t.$$

Let us denote $T_a^u := \inf\{t\colon\ Y_t^u \leq a\}$; for the process Y_t^u, the r.v. T_a^u is the moment of its first hitting the level a. Then, for $a < u$, according to [12, Th. 4.2], we conclude that

$$\mathbf{P}\{T_a^u = \infty\} > 0 \tag{22}$$

and $\lim_{t\to\infty} Y_t = \infty$ $\mathbf{P} - a.s.$ on $\{T_a^u = \infty\}$. For the solution of (2) with the same initial state $X_0 = u$ we can write

$$X_t^u = Y_t^u + \exp(H_t)\left[\sum_{i=1}^{N(t)} Z_i \exp\left(-H_{\theta_i}\right)\right], \quad t \geq 0, \tag{23}$$

where θ_i is the moment of the i-th jump of the process $N(t)$. It is clear that

$$X_t^u \geq Y_t^u \quad \mathbf{P} - a.s.,\ t \geq 0. \tag{24}$$

Hence, taking into account the relation (22) for $a < u$, we have for the time τ^u of ruin of the process X_t^u that

$$\mathbf{P}\{\tau^u = \infty\} > 0,\ u > 0,$$

i.e., $\varphi(u) > 0, \quad u > 0.$

2) The case $\sigma = 0$. It is clear in this case that for $u \geq c/\mu$ the ruin of the process Y_t^u will never occur and relations (23), (24) are true. Hence, for the process X_t^u we have at least that $\varphi(u) = 1,\ u \geq c/\mu$.

Remark 2. For the survival probability $\varphi(u)$ of process (12) we have clearly from Lemma 1 that $\varphi(u) = 1,\ u \geq c/r$.

Lemma 2. *For $\sigma^2 \geq 0$, the survival probability $\varphi(u)$ of process (2) satisfies the condition*

$$\varphi(0) = 0. \tag{25}$$

For the simple proof of this lemma, see ([6]).

Lemma 3. *Let c, λ, m be positive numbers. Then for arbitrary μ, σ the survival probability $\varphi(u)$ of process (2) is continuous on $[0, \infty[$.*

Proof. Let us prove this statement in the case $\sigma^2 > 0$; otherwise the proof is simpler. Let us show first the continuity of $\varphi(u)$ at zero. In other words, we prove the limit equality

$$\lim_{u \to +0} \varphi(u) = 0. \tag{26}$$

Note that, for $u > 0$ and any fixed $t > 0$,

$$\begin{aligned}
\varphi(u) \leq \mathbf{P}(X_t^u > 0) &= \mathbf{P}\left(u - c\int_0^t \exp(-H_s)\,ds + \sum_{i=1}^{N(t)} Z_i \exp\left(-H_{\theta_i}\right) > 0\right) \\
&\leq \mathbf{P}\left(\sum_{i=1}^{N(t)} Z_i \exp\left(-H_{\theta_i}\right) > 0\right) + \mathbf{P}\left(\int_0^t \exp(-H_s)\,ds < u/c\right) \\
&\leq \mathbf{P}(N(t) \geq 1) + \mathbf{P}\left(t \inf_{s \leq t} \exp(-H_s) < u/c\right).
\end{aligned}$$

Denote $M_s = \exp[(\mu - \sigma^2)s - H_s]$. Clear that

$$M_s = \exp(-\frac{\sigma^2}{2}s - \sigma w_s) \tag{27}$$

is a non-negative martingale with $M_0 = 1$. Hence,

$$\varphi(u) \leq \mathbf{P}(N(t) \geq 1) + \mathbf{P}\left(\inf_{s \leq t} M_s < \frac{u}{ctb(t)}\right), \tag{28}$$

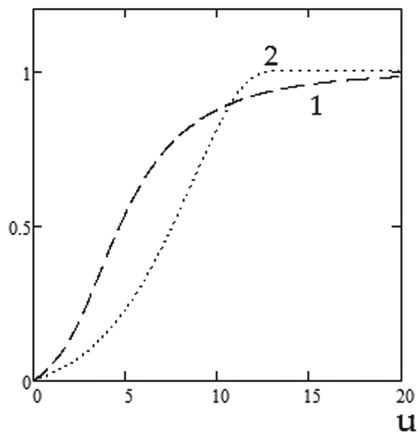

Fig. 1. The case $\lambda > 2r$: r=0.3; $\mu = 0.7$.

where $b(t) = \exp[(\mu - \sigma^2)t]\, I\{\mu > \sigma^2\} + I\{\mu \le \sigma^2\}$ and I is the indicator function of the set. The following inequality is proved in the course of the proof of Lemma 4.2 in [13]. For non-negative supermartingale $M_t, M_0 = 1$, we have

$$\mathbf{P}\left(\inf_{s\le t} M_s < \varepsilon\right) \le 2\mathbf{P}\left(M_t < 2\varepsilon\right), \quad \varepsilon > 0. \tag{29}$$

Setting $\varepsilon = \frac{u}{ctb(t)}$ and applying inequality (29) to the martingale of the form (27), we obtain from (28) that, for any fixed t,

$$\lim_{u\to+0} \varphi(u) \le 1 - \exp\left(-\lambda t\right). \tag{30}$$

Letting $t \to 0$ in (30) and taking into account the non-negativity of φ, we have equality (26).

Let us prove the continuity at any point $u > 0$. Note that the difference between the two processes $X_t(u+\varepsilon) = X_t^{u+\varepsilon}$ and $X_t(u) = X_t^u$ starting at points $u+\varepsilon$ and u respectively, has the form

$$X_t(u+\varepsilon) - X_t(u) = \varepsilon \exp(H_t). \tag{31}$$

For the stopping time $\tau^u \wedge t$, where τ^u is the time of ruin of the process X_t^u, due to the strong Markov property of the process $X_t^{u+\varepsilon}$, we have

$$\begin{aligned}
\varphi(u+\varepsilon) &= \mathbb{E}\varphi(X_{\tau^u\wedge t}(u+\varepsilon)) \\
&= \mathbb{E}[\varphi(X_{\tau^u\wedge t}(u+\varepsilon))I\{\tau^u < \infty\}] + \mathbb{E}[\varphi(X_{\tau^u\wedge t}(u+\varepsilon))I\{\tau^u = \infty\}] \\
&= \mathbb{E}[\varphi(X_t(u+\varepsilon))I\{\tau^u = \infty\}] + \mathbb{E}[\varphi(X_{\tau^u\wedge t}(u+\varepsilon))I\{\tau^u \le t\}] \\
&+\mathbb{E}[\varphi(X_{\tau^u\wedge t}(u+\varepsilon))I\{t < \tau^u < \infty\}] \\
&= \mathbb{E}[\varphi(X_t(u+\varepsilon))I\{\tau^u = \infty\}] + \mathbb{E}[\varphi(X_{\tau^u}(u+\varepsilon))I\{\tau^u \le t\}] \\
&+\mathbb{E}[\varphi(X_t(u+\varepsilon))I\{t < \tau^u < \infty\}].
\end{aligned}$$

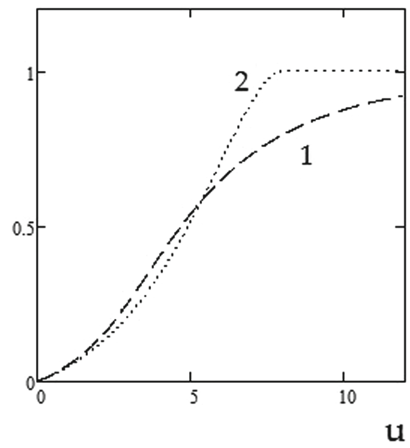

Fig. 2. The case $\lambda = 2r$: r=0.5; $\mu = 0.7$.

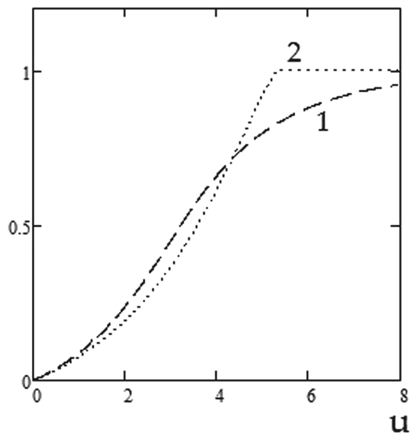

Fig. 3. The case $r < \lambda < 2r$: r=0.75; $\mu = 1$.

For three terms at the end of the last chain of equalities, we have $\mathbb{E}[\varphi(X_t(u+\ \varepsilon))I\{\tau^u = \infty\}] \le \mathbb{P}\{\tau^u = \infty\} = \varphi(u)$,

$$\mathbb{E}[\varphi(X_{\tau^u}(u+\varepsilon))I\{\tau^u \le t\}] = \mathbb{E}[\varphi(\varepsilon \exp(H_{\tau^u})I\{\tau^u \le t\}], \qquad (32)$$

$\mathbb{E}[\varphi(X_t(u+\varepsilon))I\{t < \tau^u < \infty\}] \le \mathbb{P}\{t < \tau^u < \infty\}$ (equality (32) is true due to relation (31) and the fact that $X_{\tau^u} = 0$ for the process with positive jumps). Then

$$\varphi(u+\varepsilon) \le \varphi(u) + \mathbb{E}[\varphi(\varepsilon \exp(H_{\tau^u})I\{\tau^u \le t\}] + \mathbb{P}\{t < \tau^u < \infty\}. \qquad (33)$$

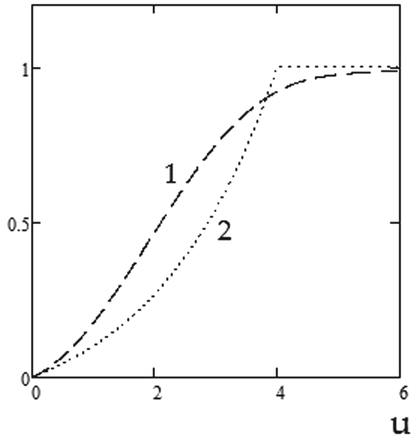

Fig. 4. The case $\lambda = r$: r=1; $\mu = 1.5$.

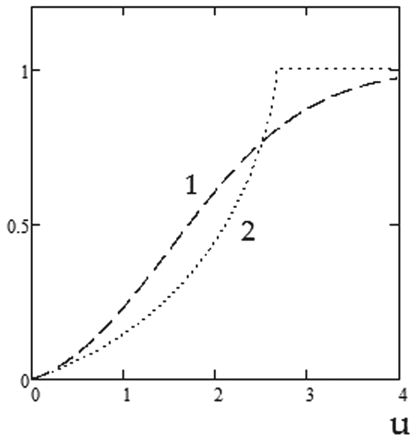

Fig. 5. The case $\lambda < r$: r=1.5; $\mu = 1.75$.

Note that the first term in (33) tends to zero as $\varepsilon \to 0$ due to the proved continuity at zero of $\varphi(u)$, condition (25) and taking into account the dominated convergence theorem. Then, for any t,

$$\lim_{\varepsilon \to +0} (\varphi(u+\varepsilon) - \varphi(u)) \leq \mathbb{P}\{t < \tau^u < \infty\}.$$

Letting $t \to \infty$ in the last inequality and taking into account that the survival probability φ is the non-decreasing on the initial state u, we obtain the right-continuity of this function. The left-continuity may be proved analogously.

6 Numerical Results [7]

For the results of numerical calculations (Figs. 1, 2, 3, 4 and 5), the curves with number 1 (2) correspond to the case of risky investments in shares with parameters μ and σ^2 (risk-free ones with return rate r respectively). The figures are presented in order of decreasing smoothness and increasing discontinuity of derivatives for the curves number 2. For all figures, $c = 4$, $m = 2$, $\lambda = 1$, $\sigma^2 = 0.3$ (the parameter values are relative, they are normalized in such a way that $\lambda = 1$).

7 Conclusions

s A new approach to justifying the survival probabilities in dynamic insurance models with investments as the solutions of corresponding IDE problems is proposed. This approach avoids direct proof of the smoothness of the survival probability by using verification arguments based on the uniqueness of the viscosity solution. It can be applied if it has been previously proved, that the survival probability is continuous, not identically equal to zero function, has a known value at zero initial surplus and is a viscosity solution of some IDE problem. The first two facts can be established quite simply, and the last fact can be proved for a whole class of models, as it is done in [1]. In this case, for specific models from this class, it remains only to prove the existence of a solution (classical or in the sense of viscosity) for the corresponding IDE problem. On the other hand, it remains unclear whether this approach can be applied to models in which the corresponding problem for the IDE is not a boundary problem (see, e.g., [8])

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