On Positive Recurrence of One-Dimensional Diffusions with Independent Switching

In Memory of Svetlana Anulova (19.10.1952–21.11.2020)

Alexander Veretennikov[1,2,3](✉)

[1] University of Leeds, Leeds, UK
ayv@iitp.ru
[2] National Research University Higher School of Economics, Moscow, Russian Federation
[3] Institute for Information Transmission Problems, Moscow, Russian Federation
https://www.hse.ru/en/org/persons/125432921

Abstract. Positive recurrence of one-dimensional diffusion with switching, with an additive Wiener process, and with one recurrent and one transient regime is established under suitable conditions on the drift in both regimes and on the intensities of switching. The approach is based on an embedded Markov chain with alternating jumps: one jump increases the average of the square norm of the process, while the next jump decreases it, and under suitable balance conditions this implies positive recurrence.

Keywords: 1D diffusion · Switching · Positive recurrence

1 Introduction

On a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\geq 0}$ on it, a one-dimensional SDE with switching is considered,

$$dX_t = b(X_t, Z_t)\,dt + dW_t, \quad t \geq 0, \quad X_0 = x,\ Z_0 = z,$$

where Z_t is a continuous-time Markov process on the state space $S = \{0, 1\}$ with (positive) intensities of respective transitions $\lambda_{01} =: \lambda_0$, & $\lambda_{10} =: \lambda_1$; the process Z is assumed to be independent of W and adapted to the filtration $(\mathcal{F}_t)$. We assume that these intensities are constants; this may be relaxed. Under the regime $Z = 0$ the process X is assumed positive recurrent, while under the regime $Z = 1$ its modulus may increase in the square mean with the rate comparable to the decrease rate under the regime $Z = 0$. This vague wording will be specified in the assumptions. Denote

$$b(x, 0) = b_-(x), \quad b(x, 1) = b_+(x).$$

A. N. Shiryaev et al. (Eds.): ICSM-5 2020, PROMS 371, pp. 242–252, 2021.
https://doi.org/10.1007/978-3-030-83266-7_18

The problem addressed in this paper is to find sufficient conditions for the positive recurrence (and, hence, for convergence to the stationary regime) for solutions of stochastic differential equations (SDEs) with switching in the case where not for all values of the modulating process the SDE is recurrent, and where it is recurrent, this property is assumed to be "not very strong". Earlier a similar problem was tackled in [2] in the exponential recurrent case; its method apparently does not work for the weaker polynomial recurrence. A new approach is offered. Other SDEs with switching were considered in [1,4,5,7], see also the references therein. Neither of these works address exactly the problem which is attacked in this paper: some of them tackled an exponential recurrence, some other study the problem of a simple recurrence versus transience.

2 Main Result: Positive Recurrence

The existence and pathwise uniqueness of the solution follows easily from [9], or from [6], or from [8], although, neither of these papers tackles the case with switching. The next theorem is the main result of the paper.

Theorem 1. *Let the drift b be bounded and let there exist $r_-, r_+, M > 0$ such that*

$$xb_-(x) \le -r_-, \quad xb_+(x) \le +r_+, \quad \forall\, |x| \ge M, \tag{1}$$

and

$$2r_- > 1 \quad \& \quad \kappa_1^{-1} := \frac{\lambda_0(2r_+ + 1)}{\lambda_1(2r_- - 1)} < 1. \tag{2}$$

Then the process (X, Z) is positive recurrent; moreover, there exists $C > 0$ such that for all M_1 large enough and all $x \in \mathbb{R}$

$$\mathbb{E}_x \tau_{M_1} \le C(x^2 + 1), \tag{3}$$

where

$$\tau_{M_1} := \inf(t \ge 0 : \, |X_t| \le M_1).$$

Moreover, the process (X_t, Z_t) has a unique invariant measure, and for each nonrandom initial condition x, z there is a convergence to this measure in total variation when $t \to \infty$.

3 Proof

Denote $\|b\| = \sup_x |b(x)|$. Let $M_1 \gg M$ (the value M_1 will be specified later); denote

$$T_0 := \inf(t \ge 0 : Z_t = 0),$$

and

$$0 \le T_0 < T_1 < T_2 < \ldots,$$

where each T_n is defined as the next moment of switch of the component Z; let

$$\tau := \inf(T_n \geq 0 : \, |X_{T_n}| \leq M_1).$$

It suffices to evaluate from above the value $\mathbb{E}_x\tau$ because $\tau \geq \tau_{M_1}$. Let us choose $\epsilon > 0$ such that

$$\lambda_0(2r_+ + 1 + \epsilon) = q\lambda_1(2r_- - 1 - \epsilon) \tag{4}$$

with some $q < 1$ (see (2)). Note that for $|x| \leq M$ there is nothing to prove; so assume $|x| > M$.

Lemma 1. *Under the assumptions of the theorem for any $\delta > 0$ there exists M_1 such that*

$$\max\left[\sup_{|x|>M_1} \mathbb{E}_x\left(\int_0^{T_1} 1(\inf_{0\leq s\leq t}|X_s|\leq M)dt|Z_0=0\right),\right.$$
$$\left.\sup_{|x|>M_1} \mathbb{E}_x\left(\int_0^{T_0} 1(\inf_{0\leq s\leq t}|X_s|\leq M)dt|Z_0=1\right)\right] < \delta. \tag{5}$$

Proof. Let X^i_t, $i = 0, 1$ denote the solution of the equation

$$dX^i_t = b(X^i_t, i)\,dt + dW_t, \quad t \geq 0, \quad X^i_0 = x.$$

Let $Z_0 = 0$; then $T_0 = 0$. The processes X and X^0 coincide a.s. on $[0, T_1]$ due to uniqueness of solution. Therefore, due to the independence of Z and W, and, hence, of Z and X^0, we obtain

$$\mathbb{E}_x\left(\int_0^{T_1} 1(\inf_{0\leq s\leq t}|X_s| \leq M)dt|Z_0 = 0\right) = \mathbb{E}_x\int_0^{T_1} 1(\inf_{0\leq s\leq t}|X^0_s| \leq M)dt$$
$$= \mathbb{E}_x\int_0^\infty 1(t < T_1)1(\inf_{0\leq s\leq t}|X^0_s| \leq M)dt = \int_0^\infty \mathbb{E}_x 1(t < T_1)\mathbb{P}(\inf_{0\leq s\leq t}|X^0_s| \leq M)dt$$
$$= \int_0^\infty \exp(-\lambda_0 t)\mathbb{P}(\inf_{0\leq s\leq t}|X^0_s| \leq M)dt.$$

Let us take t such that

$$\int_t^\infty e^{-\lambda_0 s}ds < \delta/2.$$

Now, by virtue of the boundedness of b, it is possible to choose $M_1 > M$ such that for this value of t we have

$$t\,\mathbb{P}_x(\inf_{0\leq s\leq t}|X^0_s| \leq M) < \delta/2.$$

The bound for the second term in (5) follows by using the process X^1 and the intensity λ_1 in the same way. QED

Lemma 2. *If M_1 is large enough, then under the assumptions of the theorem for any $|x| > M_1$ for any $k = 0, 1, \ldots$*

$$\begin{aligned}\mathbb{E}_x(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) &\le \mathbb{E}_x(X^2_{T_{2k}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) \\ &\quad -1(\tau > T_{2k})\lambda_0^{-1}((2r_- - 1) - \epsilon),\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbb{E}_x(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) &\le \mathbb{E}_x(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) \\ &\quad +1(\tau > T_{2k+1})\lambda_1^{-1}((2r_- + 1) + \epsilon).\end{aligned} \tag{7}$$

Proof. **1.** Recall that $T_0 = 0$ under the condition $Z_0 = 0$. We have,

$$T_{2k+1} = \inf(t > T_{2k} : Z_t = 1).$$

In other words, the moment T_{2k+1} may be treated as "T_1 after T_{2k}". Under $Z_0 = 0$ the process X_t coincides with X_t^0 until the moment T_1. Hence, we have on $[0, T_1]$ by Ito's formula

$$dX_t^2 - 2X_t dW_t = 2X_t b_-(X_t)dt + dt \le (-2r_- + 1)dt,$$

on the set $(|X_t| > M)$ due to the assumptions (1). Further, since $1(|X_t| > M) = 1 - 1(|X_t| \le M)$, we obtain

$$\begin{aligned}&\int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \\ &= \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)1(|X_t| \le M)dt \\ &\le -2r_- \int_0^{T_1\wedge\tau} 1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2M\|b\|1(|X_t| \le M)dt \\ &= -2r_- \int_0^{T_1\wedge\tau} 1dt + \int_0^{T_1\wedge\tau} (2M\|b\| + 2r_-)1(|X_t| \le M)dt \\ &\le -2r_- \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-) \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt.\end{aligned}$$

Thus, always for $|x| > M_1$,

$$\begin{aligned}&\mathbb{E}_x \int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \\ &\le -2r_- E \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)E_x \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt\end{aligned}$$

$$= -2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_x\int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt$$

$$\le -2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\mathbb{E}_x\int_0^{T_1} 1(|X_t| \le M)dt$$

$$\le -2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\delta.$$

For our fixed $\epsilon > 0$ let us choose $\delta = \lambda_0^{-1}\epsilon/(2M\|b\| + 2r_-)$. Then, since $|x| > M_1$ implies $T_1 \wedge \tau = T_1$ on $(Z_0 = 0)$, we get

$$\mathbb{E}_x X^2_{T_1\wedge\tau} - x^2 \le -(2r_- - 1)\mathbb{E}_x\int_0^{T_1} dt + \lambda_0^{-1}\epsilon = -\lambda_0^{-1}((2r_- - 1) - \epsilon).$$

Substituting here $X_{T_{2k}}$ instead of x and writing $\mathbb{E}_x(\cdot|\mathcal{F}_{T_{2k}})$ instead of $\mathbb{E}_x(\cdot)$, and multiplying by $1(\tau > T_{2k})$, we obtain the bound (6), as required.
2. The condition $Z_0 = 1$ implies the inequality $T_0 > 0$. We have,

$$T_{2k+2} = \inf(t > T_{2k+1} : Z_t = 0).$$

In other words, the moment T_{2k+2} may be treated as "T_0 after T_{2k+1}". Under $Z_0 = 1$ the process X_t coincides with X_t^1 until the moment T_0. Hence, we have on $[0, T_0]$ by Ito's formula

$$dX_t^2 - 2X_t dW_t = 2X_t b_+(X_t)dt + dt \le (2r_+ + 1)dt,$$

on the set $(|X_t| > M)$ due to the assumptions (1). Further, since $1(|X_t| > M) = 1 - 1(|X_t| \le M)$, we obtain

$$\int_0^{T_0\wedge\tau} 2X_t b_+(X_t)dt$$

$$= \int_0^{T_0\wedge\tau} 2X_t b_+(X_t)1(|X_t| > M)dt + \int_0^{T_0\wedge\tau} 2X_t b_+(X_t)1(|X_t| \le M)dt$$

$$\le 2r_+\int_0^{T_0\wedge\tau} 1(|X_t| > M)dt + \int_0^{T_0\wedge\tau} 2M\|b\|1(|X_t| \le M)dt$$

$$= 2r_+\int_0^{T_0\wedge\tau} 1dt + \int_0^{T_1\wedge\tau} (2M\|b\| - 2r_+)1(|X_t| \le M)dt$$

$$\le 2r_+\int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\int_0^{T_0\wedge\tau} 1(|X_t| \le M)dt.$$

Thus, always for $|x| > M_1$,

$$\mathbb{E}_x\int_0^{T_0\wedge\tau} 2X_t b_+(X_t)dt$$

$$
\begin{aligned}
&\leq 2r_+ E\int_0^{T_0\wedge\tau} 1dt + 2M\|b\|E_x\int_0^{T_0\wedge\tau} 1(|X_t|\leq M)dt \\
&= 2r_+ \mathbb{E}\int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\mathbb{E}_x\int_0^{T_1\wedge\tau} 1(|X_t|\leq M)dt \\
&\leq 2r_+ \mathbb{E}\int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\mathbb{E}_x\int_0^{T_0} 1(|X_t|\leq M)dt \\
&\leq 2r_+ \mathbb{E}\int_0^{T_0\wedge\tau} 1dt + 2M\|b\|\delta.
\end{aligned}
$$

For our fixed $\epsilon > 0$ let us choose $\delta = \lambda_0^{-1}\epsilon/(2M\|b\|)$. Then, since $|x| > M_1$ implies $T_0 \wedge \tau = T_0$ on $(Z_0 = 1)$, we get

$$
\mathbb{E}_x X^2_{T_1\wedge\tau} - x^2 \leq -(2r_- - 1)\mathbb{E}_x\int_0^{T_1} dt + \lambda_0^{-1}\epsilon = -\lambda_0^{-1}((2r_- - 1) - \epsilon).
$$

Substituting here $X_{T_{2k+1}}$ instead of x and writing $\mathbb{E}_x(\cdot|\mathcal{F}_{T_{2k+1}})$ instead of $\mathbb{E}_x(\cdot)$, and multiplying by $1(\tau > T_{2k+1})$, we obtain the bound (7), as required. QED

Lemma 3. *If M_1 is large enough, then under the assumptions of the theorem for any $k = 0, 1, \ldots$*

$$
\begin{aligned}
\mathbb{E}_x(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) &\leq \mathbb{E}_x(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) \\
&\quad +1(\tau > T_{2k+1})\lambda_1^{-1}((2r_+ + 1) + \epsilon)),
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\mathbb{E}_x(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) &\leq \mathbb{E}_x(X^2_{T_{2k}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) \\
&\quad -1(\tau > T_{2k})\lambda_0^{-1}((2r_+ - 1) - \epsilon)).
\end{aligned} \tag{9}
$$

Proof. Let $Z_0 = 0$; recall that it implies $T_0 = 0$. If $\tau \leq T_{2k+1}$, then (8) is trivial. Let $\tau > T_{2k+1}$. We will substitute x instead of $X_{T_{2k}}$ for a while, and will be using the solution X^1_t of the equation

$$
dX^1_t = b(X^1_t, 1)\, dt + dW_t, \quad t \geq T_1, \quad X^1_{T_1} = X_{T_1}.
$$

For M_1 large enough, since $|x| \wedge |X_{T_1}| > M_1$ implies $T_2 \leq \tau$, and due to the assumptions (1) we guarantee the bound

$$
\begin{aligned}
&1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \\
&\leq 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}}(T_2 - T_1)((2r_+ + 1) + \epsilon))
\end{aligned}
$$

$$= +1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon))$$

in the same way as the bound (7) in the previous lemma. In particular, it follows that for $|x| > M_1$

$$(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau}) \le 1(|X_{T_1}| > M_1)(\mathbb{E}_{X_{T_1}}(T_2 - T_1)((2r_+ + 1) + \epsilon))$$

$$= +1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon)),$$

since $|X_{T_1}| \le M_1$ implies $\tau \le T_1$ and $\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau} = 0$. So, on the set $|x| > M_1$ we have,

$$\mathbb{E}_x(\mathbb{E}_{X_{T_1}} X^2_{T_2\wedge\tau} - X^2_{T_1\wedge\tau})$$

$$\le \mathbb{E}_x 1(|X_{T_1}| > M_1)(\lambda_1^{-1}((2r_+ + 1) + \epsilon)) \le \lambda_1^{-1}((2r_+ + 1) + \epsilon).$$

Now substituting back $X_{T_{2k}}$ in place of x and multiplying by $1(\tau > T_{2k+1})$, we obtain the inequality (8), as required.

For $Z_0 = 1$ we have $T_0 > 0$, and the bound (9) follows in a similar way. QED

Now we can complete the proof of the theorem. Consider the case $Z_0 = 0$ where $T_0 = 0$. Note that the bound (6) of the Lemma 2 together with the bound (8) of the Lemma 3 can be equivalently rewritten as follows:

$$\mathbb{E}_x X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_x X^2_{T_{2k}\wedge\tau} \le -((2r_- - 1) - \epsilon)\mathbb{E}_x(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau), \quad (10)$$

and

$$\mathbb{E}_x X^2_{T_{2k}\wedge\tau} - \mathbb{E}_x X^2_{T_{2k-1}\wedge\tau} \le ((2r_+ + 1) + \epsilon)\mathbb{E}_x(T_{2k}\wedge\tau - T_{2k-1}\wedge\tau). \quad (11)$$

We have the identity

$$\tau\wedge T_n = T_0 + \sum_{m=0}^{n-1}((T_{m+1}\wedge\tau) - (T_m\wedge\tau)).$$

Therefore,

$$\mathbb{E}_x(\tau\wedge T_n) = \mathbb{E}_x T_0 + \mathbb{E}_x\sum_{m=0}^{n-1}((T_{m+1}\wedge\tau) - (T_m\wedge\tau)),$$

Since $T_n \uparrow \infty$, by virtue of the monotonic convergence in both parts and due to Fubini theorem we obtain,

$$\mathbb{E}_x\tau = \mathbb{E}_x T_0 + \sum_{m=0}^{\infty}\mathbb{E}_x((T_{m+1}\wedge\tau) - (T_m\wedge\tau)) \quad (12)$$

$$= \mathbb{E}_x T_0 + \sum_{k=0}^{\infty} \mathbb{E}_x((T_{2k+1} \wedge \tau) - (T_{2k} \wedge \tau))$$

$$+ \sum_{k=0}^{\infty} \mathbb{E}_x((T_{2k+2} \wedge \tau) - (T_{2k+1} \wedge \tau)).$$

Due to (10) and (11) we have,

$$\mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \leq ((2r_- - 1) - \epsilon)^{-1} \left(\mathbb{E}_x X^2_{T_{2k+1} \wedge \tau} - \mathbb{E}_x X^2_{T_{2k} \wedge \tau} \right)$$

$$\mathbb{E}_x X^2_{T_{2m+2} \wedge \tau} - x^2$$

$$\leq ((2r_+ + 1) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_x(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$-((2r_- - 1) - \epsilon) \sum_{k=0}^{m} \mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$= \sum_{k=0}^{m} (-((2r_- - 1) - \epsilon)(\mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$+((2r_+ + 1) + \epsilon)\mathbb{E}_x(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)) \,.$$

By virtue of Fatou's lemma we get

$$\begin{aligned} x^2 \geq ((2r_- - 1) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\ -((2r_+ + 1) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_x(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau). \end{aligned} \tag{13}$$

Note that $1(\tau > T_{2k+1}) \leq 1(\tau > T_{2k})$. So, $\mathbb{P}(\tau > T_{2k}) \geq \mathbb{P}(\tau > T_{2k+1})$. Hence,

$$\lambda_0 \mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) - \lambda_1 \mathbb{E}_x(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$= \lambda_0 \mathbb{E}_x(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) 1(\tau \geq T_{2k})$$

$$-\lambda_1 \mathbb{E}_x (T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) 1(\tau \geq T_{2k+1})$$

$$= \lambda_0 \mathbb{E}_x 1(\tau > T_{2k}) \mathbb{E}_{X_{T_{2k}}} (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$-\lambda_1 \mathbb{E}_x 1(\tau > T_{2k+1}) \mathbb{E}_{X_{T_{2k+1}}} (T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$= \lambda_0 \mathbb{E}_x 1(\tau > T_{2k}) \lambda_0^{-1} - \lambda_1 \mathbb{E}_x 1(\tau > T_{2k+1}) \lambda_1^{-1}$$

$$= \mathbb{E}_x 1(\tau > T_{2k}) - \mathbb{E}_x 1(\tau > T_{2k+1}) \geq 0.$$

Thus,

$$\mathbb{E}_x (T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \leq \frac{\lambda_0}{\lambda_1} \mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).$$

Therefore, we estimate

$$((2r_+ + 1) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_x (T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\leq ((2r_+ + 1) + \epsilon) \frac{\lambda_0}{\lambda_1} \sum_{k=0}^{m} \mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$= q((2r_- - 1) - \epsilon) \sum_{k=0}^{m} \mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).$$

So, (13) implies that

$$x^2 \geq ((2r_- - 1) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$-((2r_+ + 1) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_x (T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\geq (1-q)((2r_- - 1) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$\geq \frac{1-q}{2} ((2r_- - 1) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_x (T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$+\frac{1-q}{2q}\left((2r_{+}+1)+\epsilon\right)\sum_{k=0}^{m}\mathbb{E}_x(T_{2k+2}\wedge\tau-T_{2k+1}\wedge\tau).$$

Denoting $c := \min\left(\frac{1-q}{2q}\left((2r_{+}+1)+\epsilon\right), \frac{1-q}{2}\left((2r_{-}-1)-\epsilon\right)\right)$, we conclude that

$$x^2 \geq c\sum_{k=0}^{2m}\mathbb{E}_x(T_{k+1}\wedge\tau-T_k\wedge\tau).$$

So, as $m\uparrow\infty$, by the monotone convergence theorem we get the inequality

$$\sum_{k=0}^{\infty}\mathbb{E}_x(T_{k+1}\wedge\tau-T_k\wedge\tau)\leq c^{-1}x^2.$$

Due to (12), it implies that (in the case $T_0=0$)

$$\mathbb{E}_x\tau\leq c^{-1}x^2, \tag{14}$$

as required. Recall that this bound is established for $|x|>M_1$, while in the case of $|x|\leq M_1$ the left hand side in this inequality is just zero.

In the case of $Z_0=1$ (and, hence, $T_0>0$), we have to add the value $\mathbb{E}_xT_0=\lambda_1^{-1}$ to the right hand side of (14), which leads to the bound (3), as promised.

In turn, this bound implies existence of the invariant measure, see [3, Section 4.4]. Convergence to it in total variation follows due to the coupling method in a standard way. So, this measure is unique. The details and some extensions of this issue will be provided in another paper. QED

Acknowledgments. The author is deeply indebted to S. Anulova who made useful comments to the first versions of the manuscript and with whom the author started studying the theme of switched diffusions some years ago. This article is devoted to her memory. The article was prepared within the framework of the HSE University Basic Research Program in part which includes Lemmata 1 and 2; the Proof of Lemma 3 and Theorem 1 was funded by Russian Foundation for Basic Research grant 20-01-00575a.

References

1. Anulova, S.V., Veretennikov, A.Y.: Exponential convergence of degenerate hybrid stochastic systems with full dependence. In: Korolyuk, V., Limnios, N., Mishura, Y., Sakhno, L., Shevchenko, G. (eds.) Modern Stochastics and Applications. SOIA, vol. 90, pp. 159–174. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-03512-3_10
2. Cloez, B., Hairer, M.: Exponential ergodicity for Markov processes with random switching. Bernoulli **21**(1), 505–536 (2015)

3. Khasminskii, R.Z.: Stochastic Stability of Differential Equations, 2nd edn. Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-23280-0
4. Khasminskii, R.Z.: Stability of regime-switching stochastic differential equations. Probl. Inf. Transm. **48**, 259–270 (2012). https://doi.org/10.1134/S0032946012030064
5. Mao, X., Yin, G., Yuan, C.: Stabilization and destabilization of hybrid systems of stochastic differential equations. Automatica **43**, 264–273 (2007)
6. Nakao, S.: Comparison theorems for solutions of one-dimensional stochastic differential equations. In: Maruyama, G., Prokhorov, Y.V. (eds.) Proceedings of the Second Japan-USSR Symposium on Probability Theory. LNM, vol. 330, pp. 310–315. Springer, Heidelberg (1973). https://doi.org/10.1007/BFb0061496
7. Shao, J., Yuan, C.: Stability of regime-switching processes under perturbation of transition rate matrices. Nonlinear Anal. Hybrid Syst. **33**, 211–226 (2019)
8. Veretennikov, A.Y.: On strong solutions and explicit formulas for solutions of stochastic integral equations. Math. USSR-Sb. **39**(3), 387–403 (1981). https://doi.org/10.1070/SM1981v039n03ABEH001522
9. Zvonkin, A.K.: A transformation of the phase space of a diffusion process that removes the drift. Math. USSR-Sb. **22**(1), 129–149 (1974)