

From Singularities to Graphs



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Abstract In this text I present some problems which led to the introduction of special kinds of graphs as tools for studying singular points of algebraic surfaces. I explain how such graphs were first described using words, and how several classification problems made it necessary to draw them, leading to the elaboration of a special kind of calculus with graphs. This non-technical paper is intended to be readable both by mathematicians and philosophers or historians of mathematics.

Keywords Coxeter diagrams · Dual graphs · Du Val singularities · Graph manifolds · Models of surfaces · Plumbing calculus · Resolution of singularities · Singularity links · Surface singularities

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Introduction

Nowadays, graphs are common tools in singularity theory. They mainly serve to represent morphological aspects of algebraic surfaces in the neighborhoods of their singular points. Three examples of such graphs may be seen in Figs. 1, 2, and 3. They are extracted from the papers Némethi and Szilárd (2000), Neumann and Wahl (2005) and Chung et al. (2009), respectively.

Comparing those figures, one may see that the vertices are diversely depicted by small stars or by little circles, which are either full or empty. These drawing conventions are not important. What matters is that all vertices are decorated with numbers. I will explain their meaning later.

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Applying Step 4, one gets the resolution graph of $(\{y^3 + (x^2 - z^4)^2 + z^{15} = 0\}, 0)$:

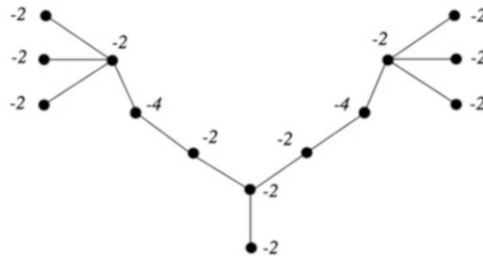
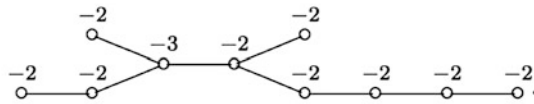


Fig. 1 An example from a 2000 paper of Némethi and Szilárd

Here is another resolution graph with the same splice diagram



It has discriminant 17, so its link has first homology $\mathbb{Z}/17$.

Fig. 2 An example from a 2005 paper of Neumann and Wahl

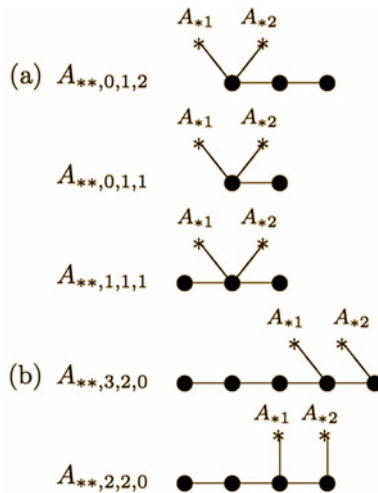


Fig. 3 An example from a 2009 paper of Chung, Xu and Yau

My aim in this paper is to understand which kinds of problems forced mathematicians to associate graphs to surface singularities. I will show how the initial idea, appeared in the 1930s, was only described in words, without any visual representation. Then I will suggest two causes that made the drawing of such

graphs unavoidable, starting from the beginning of the 1960s. One of them was a topological reinterpretation of those graphs. The other one was the growing interest in problems of classification of special types of surface singularities.

Let me describe briefly the structure of the paper. In section “[What Is the Meaning of Such Graphs?](#)”, I explain the meaning of such graphs: they represent configurations of curves which appear by resolving surface singularities. I continue in section “[What Does it Mean to Resolve the Singularities of an Algebraic Surface?](#)” by describing what it means to “resolve” a singularity. In section “[Representations of Surface Singularities Around 1900](#)” I present several models of surface singularities made around 1900 and I discuss one of the oldest configurations of curves, perhaps the most famous of them all: the 27 lines lying on a smooth cubic surface. In section “[Du Val’s Singularities, Coxeter’s Diagrams and the Birth of Dual Graphs](#)”, I present excerpts of Du Val’s 1934 paper in which he described a way of thinking about a special class of surface singularities in terms of graphs. In the same paper, he made an analogy between his configurations of curves and the facets of special spherical simplices analyzed by Coxeter in his 1931 study of finite groups generated by reflections. It is perhaps the fact that Coxeter had described a way to associate a graph to such a simplex which, through this analogy, gave birth to Du Val’s idea of speaking about graphs of curves. In section “[Mumford’s Paper on the Links of Surface Singularities](#)”, I jump to the years 1960s, because until then Du Val’s idea of associating graphs to singularities had almost never been used. I show how things changed with a 1961 paper of Mumford, in which he reinterpreted those graphs in the realm of 3-dimensional topology. Hirzebruch’s 1963 Bourbaki Seminar talk about this work of Mumford seems to be the first place in which graphs representing arbitrary configurations of curves were explicitly defined. I begin section “[Waldhausen’s Graph Manifolds and Neumann’s Calculus with Graphs](#)” by discussing a 1967 paper of Waldhausen, in which he built a subtle theory of the 3-dimensional manifolds associated to graphs as explained in Mumford’s paper. I finish it with a discussion of a 1981 paper of Neumann, which turned Waldhausen’s work into a concrete “calculus” for deciding whether two graphs represent the same 3-dimensional manifold. In section “[Conclusion](#)”, I conclude by mentioning several recent directions of research concerning graphs associated to singularities of algebraic varieties, and by summarizing this paper.

What Is the Meaning of Such Graphs?

Let me begin by explaining the meaning of the graphs associated to singularities of surfaces. In fact, the construction is not specific to singularities, one may perform it whenever is given a configuration of curves on a surface. The rule is very simple:

- each curve of the configuration is represented by a vertex;
- two vertices are joined by an edge whenever the corresponding curves intersect.

A variant of the construction introduces as many edges between two vertices as there are points in common between the corresponding curves.

Note that this construction reverses the dimensions of the input objects. Indeed, the curves, which have dimension one, are represented by vertices of the graph, which have dimension zero. Conversely, the intersection points of two curves, which have dimension zero, are represented by edges of the graph, which have dimension one. Remark also that an intersection point lies on a curve of the configuration if and only if its associated edge of the graph contains the vertex representing the curve. It is customary nowadays in mathematics to speak about “duality” whenever one has such a dimension-reversing and inclusion-reversing correspondence between parts of two geometric configurations. For this reason, one speaks here about the “dual graph” of the curve configuration, a habit which became common at the end of the 1960s.

An example of this construction is represented in Fig. 4, which combines drawings from Michael Artin’s 1962 and 1966 papers (Artin, 1962, 1966). In the upper half, one sees sketches of curve configurations, each curve being depicted as a segment. This representation is schematic, as it does not respect completely the topology of the initial curve configuration, which consists of curves without boundary points. But it represents faithfully the intersections between the curves of the configuration: two of its curves intersect if and only if the associated segments do. The corresponding “dual graphs” are depicted in the lower part of the figure. For instance, the vertex which is joined to three other vertices in the graph of the lower right corner represents the horizontal segment of the curve configuration labeled (v), on the right of the second row.

Note that the representation of the curve configurations as dual graphs emphasizes better visually its overall connectivity pattern than the representation as a

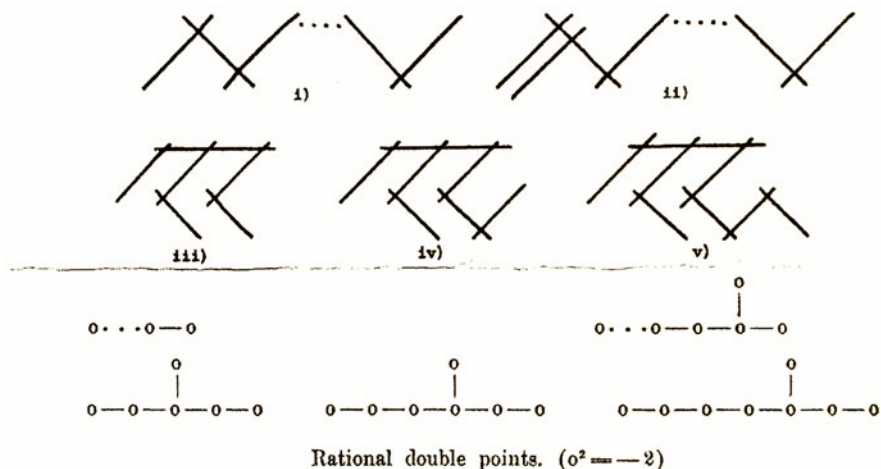


Fig. 4 Artin’s depictions of curve configurations and associated dual graphs

configuration of segments. This is probably one of the reasons which led Artin to pass from drawings of configurations of segments in his 1962 paper to drawings of dual graphs in his 1966 paper.

More generally, dual graphs may be introduced whenever one is interested in the mutual intersections of several subsets of a given set. It is not important that the given sets consist of the points of several curves lying on surfaces, they may for instance be arbitrary subsets of manifolds of any dimension or, less geometrically, the sets of members of various associations of persons. Then, one represents each set by a vertex and one joins two such vertices by an edge if and only if the corresponding sets intersect.

As a general rule, one represents any object of study by a vertex, whenever one is not interested in its internal structure, but in its “sociology”, that is, in its relations or interactions with other objects. A basic way to depict these relations is to join two such vertices whenever the objects represented by them interact in the way under scrutiny. In our context, one considers that two curves interact if and only if they have common points.

Let us come back to the configurations of curves and their associated dual graphs from Fig. 4. Those drawings represent the classification of a special type of surface singularities, namely the “*rational double points*”, in the terminology of Artin’s 1966 paper (Artin, 1966). That list was not new, it had already appeared in the solution of a different classification problem—leading nevertheless to the same objects—in Du Val’s 1934 paper in which he had informally introduced the idea of dual graph. We will further discuss that paper in section “*Du Val’s Singularities, Coxeter’s Diagrams and the Birth of Dual Graphs*”.

Before passing to the next section, let me mention that Artin’s paper (Artin, 1966) contained also a new classification, that of the dual graphs associated to the “*rational triple points*” (see Fig. 5). Being more abundant than those of the lower part of Fig. 4,

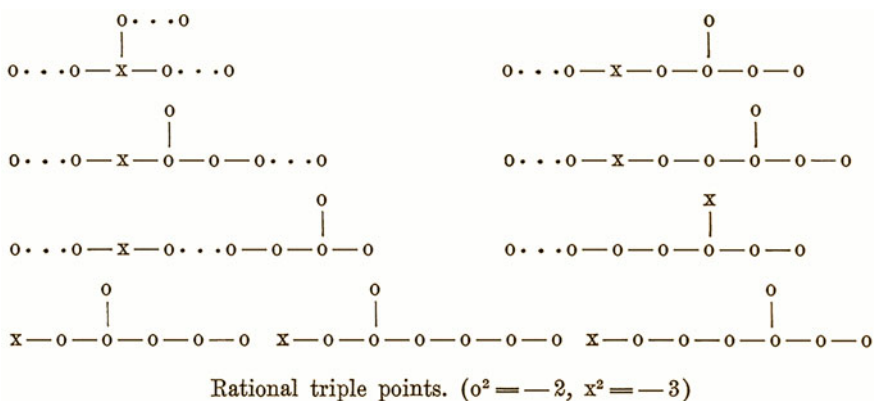


Fig. 5 Artin’s classification of dual graphs of rational triple points

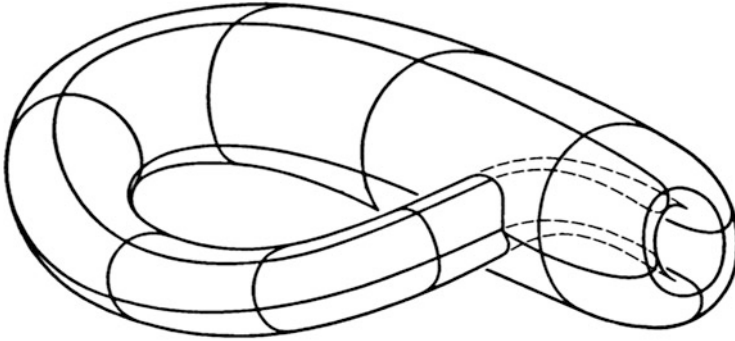


Abb. 297.

Fig. 6 A Klein bottle

it becomes apparent that it is also more economical for printing to draw such graphs rather than configurations of segments.

Now that we understood how configurations of curves lead to dual graphs, let us see in which way singularities of surfaces may lead to configurations of curves. This is the object of the next section.

What Does it Mean to Resolve the Singularities of an Algebraic Surface?

What is a *singularity of an algebraic surface*? It is a special point, at which the surface is not smooth. For instance, a sphere does not have singular points, but a double cone, idealization of the boundary of a nighty region illuminated by a lighthouse, has a singular point at its vertex. In this case, the singular point is isolated, but other surfaces may have whole curves of singularities. Such curves may be either self-intersections of the surface, as shown in Fig. 6,¹ or they may exhibit more complicated behaviour, as shown in Fig. 7.²

In this last figure, a polynomial equation in three variables is written next to each surface. The reason is that each of those surfaces is a portion of the locus of points which satisfy the associated equation in the 3-dimensional cartesian space of coordinates (x, y, z) . As polynomials are algebraic objects, such a locus

¹ This illustration of an immersion of a Klein bottle in three-dimensional cartesian space comes from the wonderful 1932 book (Hilbert and Cohn-Vossen, 1952) of David Hilbert and Stefan Cohn-Vossen. Note that in addition to a circle of self-intersection, this illustration represents also curves of apparent contours and several closed curves drawn on the surface.

² This figure comes from <https://homepage.univie.ac.at/herwig.hauser/gallery.html>, January 2018. Copyright: Herwig Hauser, University of Vienna, www.hh.hauser.cc.

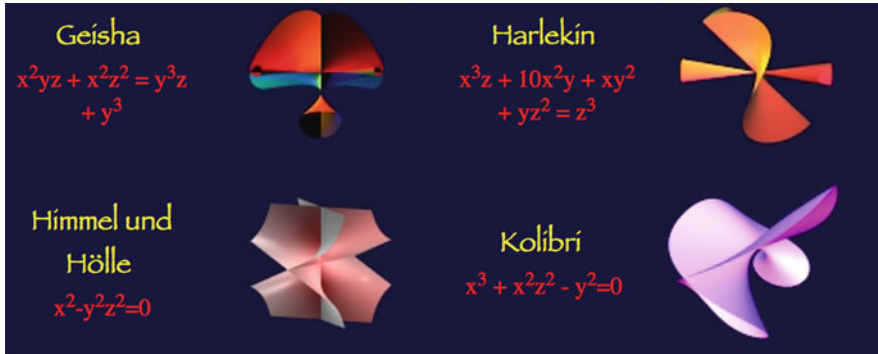


Fig. 7 Several real surface singularities from Hauser's gallery

is called “*algebraic*”. There are also algebraic surfaces in cartesian spaces of higher dimensions, defined by systems of polynomial equations in more than three variables. In fact, all surfaces considered in the papers discussed here are algebraic. One advantage of working with such surfaces is that one may consider not only the real solutions of those equations, but also the complex ones. In this way, one expects in general to make the correspondence between the algebraic properties of the defining equations and the morphological properties of the associated surface easier to understand.

A prototype of this expectation is the fact that a polynomial equation in one variable has as many complex roots as its degree, provided that the roots are counted with suitable “multiplicities” (this is the so-called “fundamental theorem of algebra”, but it is rather a fundamental theorem of the *correspondence* between algebra and topology). If one considers instead only its real roots, then their number is not determined by the degree, but there are several possibilities. In fact, as I will briefly explain at the beginning of section “[Representations of Surface Singularities Around 1900](#)”, whenever one considers families with three parameters of polynomials, these possibilities may be distinguished using “*discriminant surfaces*”, which have in general non-empty singular loci.

Because of this expected relative simplicity of complex algebraic geometry versus real algebraic geometry, it became customary in the nineteenth century to study the sets of *complex* solutions of polynomial equations in three variables. One gets in this way “*complex algebraic surfaces*”. Nevertheless, in order to build an intuition of their properties, it may be useful to practice with concrete *models* of the associated *real* surfaces. Around the end of the nineteenth century, such models were either drawn or manufactured using for instance wood, plaster, cardboard, wires and string. Nowadays they are also built using 3D-printers or, more commonly, simulated using techniques of computer visualization. This is for instance the case of Hauser's images of Fig. 7.

Why is it important to study singular surfaces? Because, in general, surfaces do not appear alone, but rather in families depending on parameters (which, in physical

contexts, may be for instance temperatures or intensities of external fields), and that for some special values of these parameters one gets surfaces with singularities. Understanding the singular members of a family is many times essential for understanding also subtle aspects of its non-singular members. For instance, one may understand part of the structure of a non-singular member by looking at its portions which “vanish” when one converges to a singular member.

The techniques of differential or algebraic geometry used in the study of smooth algebraic surfaces may be extended to singular surfaces using three basic procedures:

- by decomposing a singular surface into smooth “*strata*”, which are either isolated points, smooth portions of curves or smooth pieces of surfaces; this is similar to the decomposition of the surface of a convex polyhedron into vertices, edges and faces;
- by seeing a singular surface as a limit of smooth ones; when this is possible, one says that the surface was “*smoothed*”; such a process is not always possible, and even if it is possible, it can be usually done in various ways;
- by seeing a singular surface as a projection of a smooth one, living in a higher dimensional ambient space; if such a projection leaves the smooth part of the initial singular surface unchanged, then it is called a “*resolution of singularities*”; resolutions of singularities always exist, but are not unique.

Intuitively speaking, resolving the singularities of a surface means to remove its singular locus and to replace it algebraically by another configuration of points and curves, so that the resulting surface is smooth. In the special case of an *isolated* singular point of algebraic surface, one replaces that singular point by a configuration of curves, called the “*exceptional divisor*” of the resolution. For instance, all the graphs appearing in Figs. 1, 2, 3, 4, and 5 are dual graphs of exceptional divisors of resolutions of isolated singularities of complex surfaces.

In simple examples, one may resolve the singularities of an algebraic surface by performing finitely many times the elementary operation called “*blowing up a point*”, which is a mathematical way to look through a microscope at the neighborhood of the chosen point. This operation builds new cartesian spaces starting from the space which contains the initial singular surface. Each of those spaces contains a new surface, which projects onto a part of the initial one. If one of those surfaces is smooth, then one keeps it untouched. Otherwise, one blows up again its isolated singular points. It may happen that finitely many such operations lead to a family of *smooth* surfaces, each one of them projecting onto a portion of the initial singular surface. Those surfaces may be glued, together with their projections, into a global smooth surface which “*resolves the singularities*” of the initial one.

Let us consider for instance the surface with equation $x^2 - y^2 + z^7 = 0$, illustrated in Fig. 8. It has an isolated singularity (called “of type A_6 ”) at the origin.³ If one

³ This figure was taken from the web-page <https://www.krueger-berg.de/anne/auff-bilder/A6.html> of Anne Frühbis-Krüger in September 2020. This is also the case of Figs. 9 and 10.

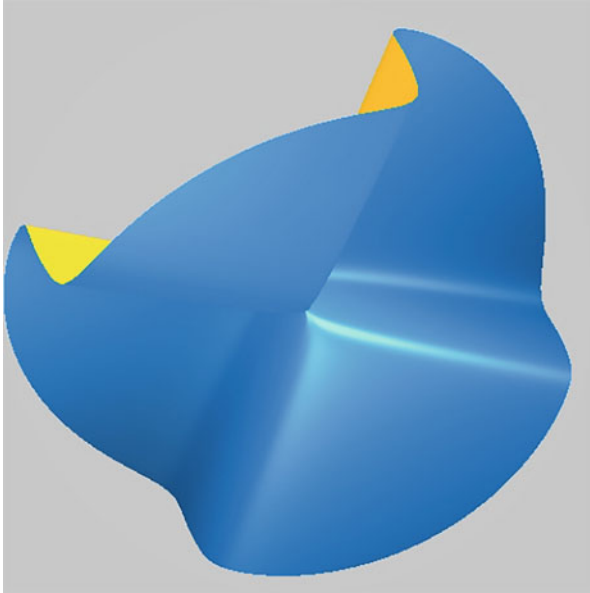


Fig. 8 An isolated A_6 surface singularity

performs the previous iterative process of blowing up the singular points of the intermediate surfaces, one gets a “tree” of surfaces, represented diagrammatically on Fig. 9. The initial surface is indicated in the top-most rectangle, and each edge of the diagram represents a blow-up operation.

The final smooth surfaces produced by the process are represented in Fig. 10. Each of them contains one or more highlighted lines. Those lines glue into a configuration of curves on the total smooth surface which resolves the initial singular one. This configuration is the exceptional divisor of this resolution of the starting isolated singularity. By looking carefully at the way the gluing is performed, one may show that its associated dual graph is a chain of five segments. This means that it is of the type shown on the left of the third row in Fig. 4.

For more complicated singularities, it may not be enough to blow up points, as previous blow-ups may create whole curves of singularities. Other operations which allow to modify the singular locus were introduced in order to deal with this problem. One may learn about them in Kollár’s book (Kollár, 2007), which explains various techniques of resolution of singularities in any dimension. The reader more interested in gaining intuition about resolutions of surfaces may consult Faber and Hauser’s promenade (Faber and Hauser, 2010) through a garden of examples of resolutions or my introduction (Popescu-Pampu, 2011b) to one of the oldest methods of resolution, originating in Jung’s method for parametrizing algebraic surfaces locally.

The Tree of Charts

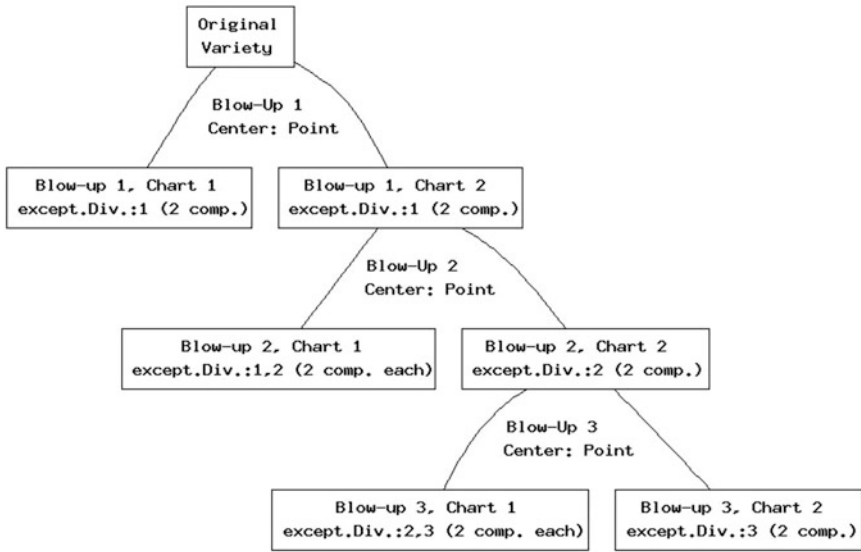


Fig. 9 Frühbis-Krüger’s representation of a resolution process

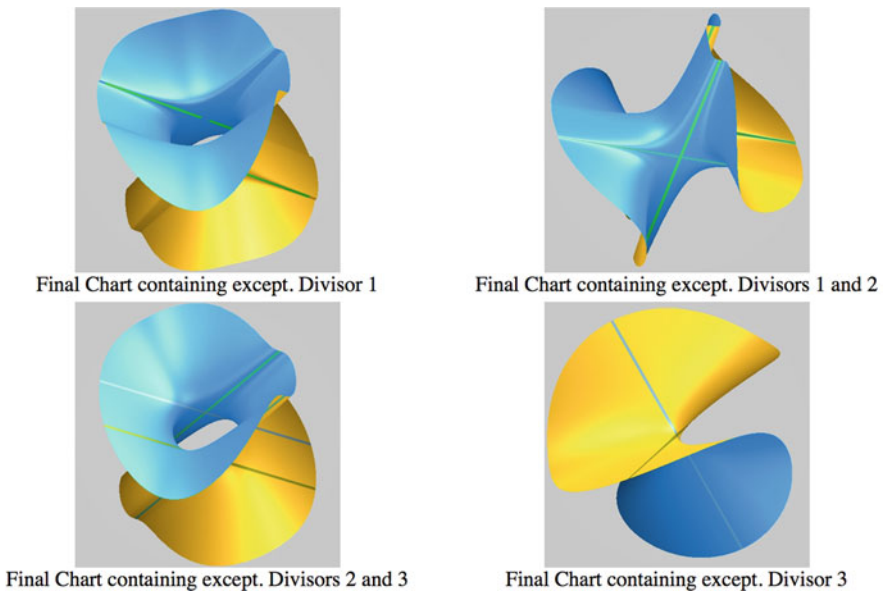


Fig. 10 The final stages of the previous resolution process

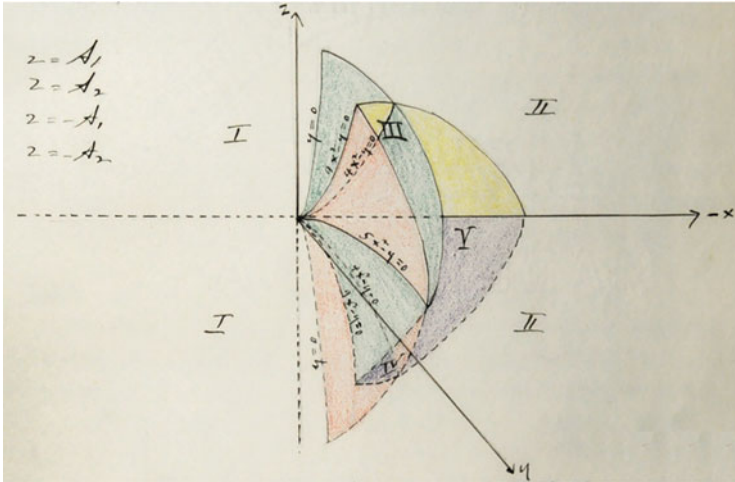


Fig. 11 Sinclair’s representation of a discriminant surface

Representations of Surface Singularities Around 1900

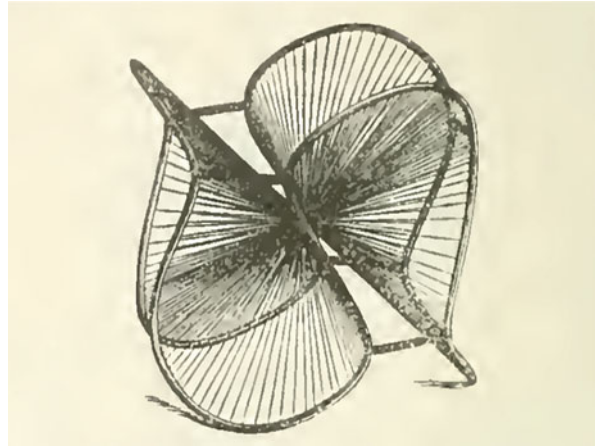
In the previous sections we saw contemporary representations of surface singularities, obtained using computer visualization techniques. Let us turn now to older representations, dating back to the beginning of the twentieth century. This will allow us to present one of the oldest sources of surfaces with singularities—the study of discriminant surfaces—and one of the most famous configurations of curves—consisting of the lines contained in a smooth cubic surface.

Figure 11 shows⁴ a hand-drawn “discriminant surface”, which has whole curves of singular points, as was the case in the examples of Figs. 6 and 7. It reproduces a drawing done by Mary Emily Sinclair in her 1903 thesis. Let me discuss this surface a little bit, as it emphasizes another source of interest on the structure of singular surfaces. As explained in its caption from the paper (Top and Weitenberg, 2011), Sinclair was studying the family with three parameters (x, y, z) of polynomials of the form $t^5 + xt^3 + yt + z$. One may associate to it an algebraic family of sets of points, namely the sets of roots of the polynomial in the variable t obtained for fixed values of the parameters. The “discriminant surface” is the subset of the cartesian space of coordinates (x, y, z) for which the associated polynomial has at least one multiple complex root.

More generally, consider any family of points, curves, surfaces or higher-dimensional algebraic objects, depending algebraically on some parameters. If there are exactly three parameters, then the set of singular objects of the family is usually

⁴This drawing and the text immediately below it were extracted from the paper (Top and Weitenberg, 2011) of Jaap Top and Erik Weitenberg.

Fig. 12 An image from Schilling's catalog of mathematical models



a surface in the space of parameters. All the surfaces obtained in this way are called “*discriminant surfaces*”, because they allow to discriminate the possible aspects of the objects in the family, according to the position of the corresponding point in the space of parameters, relatively to the surface. For instance, by determining in which region of the complement of the surface of Fig. 11 lies the point with coordinates (x, y, z) , one may see if the set of real roots of the polynomial has 1, 3 or 5 elements—those being the only possibilities for a quintic polynomial equation, because the non-real roots come in pairs of complex conjugate numbers. The reader interested in the analogous study of quartic polynomial equations may read Michel Coste’s paper (Coste, 2010).

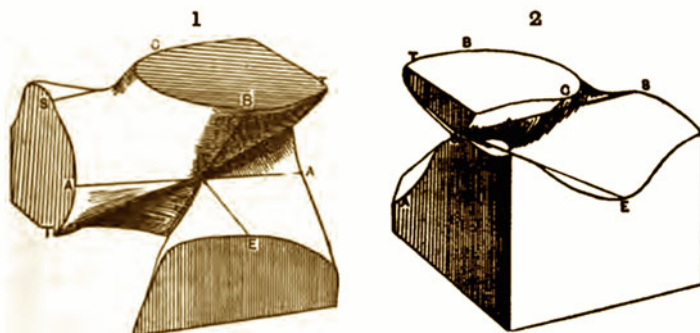
Let us pass now to material models of surfaces with singularities. Figure 12 reproduces an engraving⁵ from the 1911 catalog of mathematical models of Martin Schilling’s enterprise. It depicts a cone over a smooth cubic curve, that is, a smooth curve contained in the projective plane and defined by the vanishing of a homogeneous polynomial in 3 variables. This cone has therefore only one singular point, its vertex. Figure 13 shows a reproduction from the 1905 book (Blythe, 1905) of William Henry Blythe. It depicts two plaster models of cubic surfaces with singularities.

One of the most famous discoveries of the nineteenth century regarding the properties of algebraic surfaces is that all smooth complex algebraic cubic surfaces situated in the projective space of dimension three contain exactly 27 lines. This discovery was done in 1849, during a correspondence between Arthur Cayley and George Salmon, and it triggered a lot of research.⁶ Starting from around 1870,

⁵ It may be found on page 123, part II.3.b of the catalog (Catalog mathematischer Modelle, 1911).

⁶ For instance, in the historical summary of his 1915 thesis (Henderson, 1915), Henderson mentions that “in a bibliography on curves and surfaces compiled by J. E. Hill [in 1897] [...] the section on cubic surfaces contained two hundred and five titles. The Royal Society of London

A model is here given of a surface having a binode B_5 . The lines on it correspond to the table of reference on page 12.



Figures 1 and 2 are two views of the model when placed upon a table, but in figure 3 the under part of the model is shewn.

Fig. 13 An image from Blythe's book on models of cubic surfaces

material models of parts of real cubic surfaces with all 27 lines visible on them started to be built. One may see such a model in Fig. 14. It represents a portion of “Clebsch’s diagonal surface”.⁷ This surface does not contain singular points, but it is interesting in our context because it exhibits a highly sophisticated configuration of curves, composed of its 27 lines.

Much more details about the building of models of algebraic surfaces around 1900 may be found in the books (Mathematical models, 2017; Institut Henri Poincaré, 2017). As illustrated by Fig. 13, plaster models were built not only of smooth cubic surfaces, but also of singular ones. The manufacturing process was based on Rodenberg’s 1878 work (Rodenberg, 1878). The complete classification of the topological types of real cubic surfaces was achieved by Knörrer and Miller in their 1987 paper (Knörrer and Miller, 1987). Other historical details about the study of the configurations of 27 lines lying on smooth cubic surfaces may be found in Polo Blanco’ and Lê’s theses (Lê, 2013; Polo-Blanco, 2007), as well as in Lê’s paper (Lê, 2013) and Labs’ paper (Labs, 2017).

Catalogue of Scientific Papers, 1800–1900, volume for *Pure Mathematics* (1908), contains very many more.”

⁷ Clebsch’s diagonal surface is usually defined by the pair of homogeneous equations $x_0 + \dots + x_4 = 0$ and $x_0^3 + \dots + x_4^3 = 0$ inside the projective space of dimension 4 whose homogeneous coordinates are denoted $[x_0 : \dots : x_4]$. This photograph of a model belonging to the University of Göttingen was taken by Zausig in 2012. It comes from Wikimedia Commons: https://commons.wikimedia.org/wiki/File:Modell_der_Diagonalfleche_von_Clebsch_-_Schilling_VII_1_-44-.jpg.



Fig. 14 A model of Clebsch's diagonal surface and of its 27 lines

Note that at the beginning of the twentieth century, some artists from the Constructivist and Surrealist movements were inspired by material models of possibly singular surfaces, as explained in Vierling-Claassen's article (Vierling-Claassen, 2017). It would be interesting to know in which measure computer models as those of Fig. 7 inspire nowadays other artists.

We are thus led to consider a double point having further double points in its neighbourhoods, all of which are rational. If these are transformed into curves we obtain a “tree” of rational curves, each of which meets enough of the others for the tree to be connected, and each of which (as arising from the neighbourhood of a double point) has grade -2 . Not every such tree however is capable of representing the whole neighbourhood of a multiple point on a surface; since if a system $|f|$ represents the

Fig. 15 Du Val’s introduction of dual graphs

Du Val’s Singularities, Coxeter’s Diagrams and the Birth of Dual Graphs

Let us discuss now the 1934 paper (Du Val, 1935) in which Patrick Du Val considered, seemingly for the first time, the idea of *dual graph* of an exceptional divisor of resolution of surface singularity.

As indicated by the title of his paper, Du Val’s problem was to classify the “*isolated singularities of surfaces which do not affect the conditions of adjunction*”. Given a possibly singular algebraic surface contained in a complex projective space of dimension three, its “*adjoint surfaces*” are other algebraic surfaces contained in the same projective space and defined in terms of double integrals. I will not give here their precise definition, which is rather technical.⁸ Let me only mention that the adjoint surfaces must contain all curves consisting of singularities of the given surface. By contrast, it does not necessarily contain its *isolated* singular points. Those through which the adjoint surfaces are not forced to pass are precisely the singularities “which do not affect the conditions of adjunction”.

Du Val analyzed such singularities by looking at their resolutions. It is in this context that he wrote that for each one of those singularities, there is a resolution whose associated exceptional divisor is a “*tree of rational curves*” with supplementary properties (see Fig. 15). For instance, each curve in this “tree” has necessarily self-intersection -2 in its ambient smooth surface (this is the meaning of the syntagm “*has grade -2* ”). Du Val continued by giving a list of constraints verified by such “trees”, if they were to correspond to singularities which do not affect the conditions of adjunction (see Fig. 16). Using those constraints, he arrived exactly at the list of configurations of curves depicted in Fig. 4. But, in contrast with Artin’s papers (Artin, 1962, 1966) from 1960s, his article does not contain any schematic drawing of a configuration of curves, or of an associated dual graph.

It is not even clear whether Du Val really thought about dual graphs. Perhaps he drew for himself some diagrams resembling those of the upper part of Fig. 4, and he

⁸ The interested reader may find it in Merle and Teissier’s paper (Merle and Teissier, 1977). The whole volume containing that paper is dedicated to a modern study of the singularities analyzed by Du Val.

point. We are thus able to restrict the trees which need to be considered by the elimination of the following :

(i) All trees containing a cycle of curves, each of which meets its two neighbours; since the least sum of positive multiples of the curves of such a cycle which has non-positive intersection number with each of them is just the sum of them all, and this has zero intersection with each (the grade of each being -2 , and each meeting two others).

(ii) All trees containing a curve which meets four others; for twice this curve plus the sum of the other four has zero intersections with each of the five.

(iii) All trees containing more than one curve which meets three others; for if there are two such curves, since the tree is connected there is a chain (or sequence in which each curve meets its predecessor and successor) joining them, and this chain, with the two given curves, forms a total curve of grade -2 meeting four others.

Fig. 16 Du Val's restricted class of graphs

It may be noted that the "trees" of curves which we have had to consider bear a strict formal resemblance to the spherical simplexes whose angles are submultiples of π , considered by Coxeter†. If in fact we let the r curves which form a tree correspond to the bounding primes of a simplex in $[r-1]$, making intersecting curves correspond to primes inclined at an angle $\pi/3$, and non-intersecting curves to mutually perpendicular primes, then comparison of the results obtained above with Coxeter's indicates that to a simplex which can be constructed in spherical space corresponds a tree of curves which can be fundamental to a linear system, i.e. which can actually represent the neighbourhood of a singular point; while a simplex which can be constructed

Fig. 17 Du Val's analogy with Coxeter's spherical simplexes

saw an analogy with some "trees" considered by other mathematicians. Note that it is possible that for Du Val the term "tree" meant what we call "graph". Indeed, one sees him stating in the excerpt of Fig. 16 that the "trees" under scrutiny should not contain "a cycle of curves", a formulation which allows some "trees of curves" to contain such cycles.

At the end of his article, Du Val mentioned an analogy with results of Coxeter⁹ regarding finite groups generated by reflections (see Fig. 17). In order to understand

⁹ Du Val cites the paper (Coxeter, 1932), but Coxeter already considered this problem one year earlier, in Coxeter (1931). Note that Du Val and Coxeter were friends and that they discussed regularly about their research. One may learn a few details about their friendship and discussions in Roberts' book (Roberts, 2007), especially on pages 71–72.

this analogy, we have to know that Coxeter started from a finite set of hyperplanes passing through the origin in a real Euclidean vector space of arbitrary finite dimension. He assumed that they were spanned by the facets of a simplicial cone emanating from the origin, and he looked at the spherical simplex obtained by intersecting the cone with the unit sphere centered at the origin. Coxeter's problem was to classify those spherical simplices for which the group generated by the orthogonal reflections in the given hyperplanes is *finite*.

Du Val realized that his classification of isolated singularities which do not affect the conditions of adjunction corresponds to a part of Coxeter's classification of spherical simplices giving rise to finite groups of reflections. In order to make this correspondence visible, he associated to each curve of a given exceptional divisor a facet of the simplex, two curves being disjoint if and only if the corresponding facets are orthogonal, and having one point of intersection if and only if the facets meet at an angle of $\pi/3$.

Exactly in the same way in which Du Val introduced in 1934 his dual graphs verbally, without drawing them, Coxeter had verbally introduced in 1931 "*diagrams of dots and links*" in order to describe the shapes of his spherical simplices (see Fig. 18). It is only in his 1934 paper (Coxeter, 1934) that he published drawings of such graphs (see Fig. 19), which were to be called later "*Coxeter diagrams*", or "*Coxeter-Dynkin diagrams*", in reference to their reappearance in a slightly different form in Dynkin's 1946 work (Dynkin, 1946) about the structure of Lie groups and Lie algebras.

Much later, Coxeter explained in his 1991 paper (Coxeter, 1991) that analogous diagrams had already been introduced by Rodenberg in 1904, in his description (Rodenberg, 1904) of the plaster models of singular cubic surfaces from Schilling's catalog. Rodenberg's interpretation was different, not related to reflections, but to special subsets of the configuration of 27 lines on a generic smooth cubic surface (see Fig. 20, containing an extract from Coxeter (1991)). More details about Rodenberg's convention, based on his older paper (Rodenberg, 1878), may be found in Barth and Knörrer's text (Barth and Knörrer, 2017).

It is therefore desirable to enumerate all spherical simplexes whose dihedral angles are submultiples of π . Since these angles must not be too small, only a few of them can be acute; the rest are all right angles. It is useful to represent such a spherical simplex by a diagram of dots and links. Every prime is represented by a dot; and if $(rs) = \pi/k$ ($k > 2$), the dots r and s are joined by a link marked " k ". The dots representing perpendicular primes are not joined at all. We can suppose the diagram to consist of a connected chain, since otherwise the corresponding group is merely the direct product of the groups which correspond to the various disconnected portions.

Fig. 18 Coxeter's introduction of his diagrams

GRAPHICAL REPRESENTATION OF THE IRREDUCIBLE GROUPS GENERATED BY REFLECTIONS

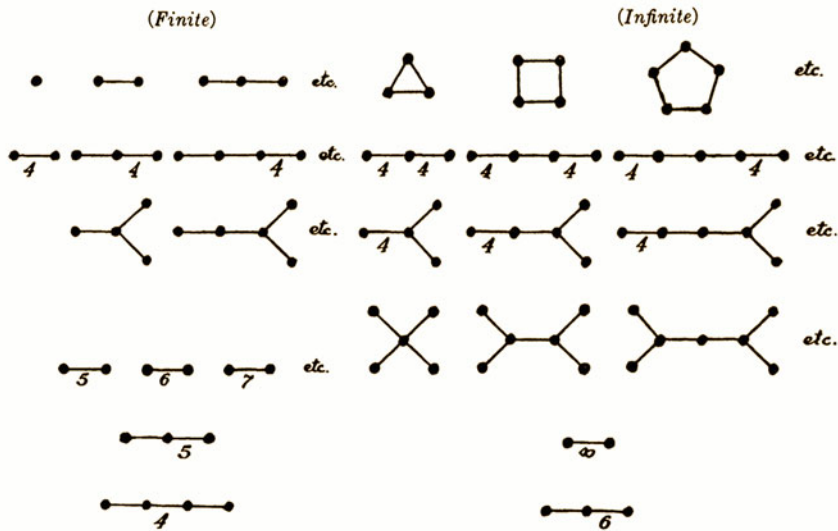


Fig. 19 Coxeter’s pictures of his diagrams

Mumford’s Paper on the Links of Surface Singularities

One could believe that the combination of Du Val’s analogy between his “trees of curves” and Coxeter’s spherical simplices on one side, and Coxeter’s diagrams on another side, would trigger research on the possible dual graphs of isolated surface singularities. Such an interest indeed developed starting from a 1961 paper of David Mumford. In this section I explain the aim of Mumford’s paper and how it led to the first explicit formulation of the notion of dual graph of a configuration of algebraic curves contained in a smooth algebraic surface.

I could find only one article published between 1934 and 1961 which contained a drawing of dual graph of resolution of isolated surface singularity.¹⁰ It is Hirzebruch’s 1953 paper (Hirzebruch, 1953), in which he proved that one could not only resolve the singularities of complex algebraic surfaces, but also of the more general *complex analytic* ones. That article contains a single illustration (see Fig. 21), which depicts the general shape of possible dual graphs of resolutions for a class of singularities which is crucial for his method.¹¹ Unlike the case of Du

¹⁰ One may think, from a rapid glance, that Du Val’s paper (Du Val, 1944) is an exception. But the graphs of that paper, which I rediscovered with a different interpretation in Popescu-Pampu (2011a), are not dual graphs of configurations of curves. They indicate relations between infinitely near points of a given smooth point of an algebraic surface, being variants of Enriques’ diagrams introduced in Enriques and Chisini (1917).

Rodenberg (1904, pp. 5, 32) used the notation $\lambda\mu\nu$ for the double-six corresponding to $N_{\lambda\mu\nu}$, and invented a diagram in which symbols for two double-sixes are linked when they share 6 lines (so that the involutions are braided) but not linked when they share only 4 lines (so that the involutions commute). He was investigating the possible singularities of cubic surfaces. In this project he essentially anticipated the discovery by Du Val (1934) of a connection between double points

$$B_3, B_4, B_5, B_6, U_6, U_7, U_8, U_9$$

(*Biplanar* or *Uniplanar*) and reflection groups

$$A_2, A_3, A_4, A_5, D_4, D_5, E_6, \tilde{E}_6$$

(see also Arnold 1974, pp. 21, 24; Fischer 1986b, p. 13). For instance, Rodenberg described the uniplanar double point U_8 by a diagram

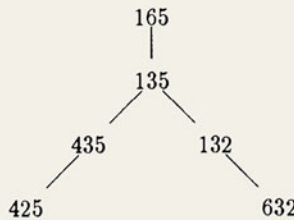


Fig. 20 One of Rodenberg’s diagrams

Dem Sphärenbaum $(t_{P_1} \dots t_{P_k})^{-1} P_1$ ist daher der folgende (Strecken)-Baum zugeordnet: [vgl. I.3 (11)]

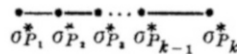


Fig. 21 The dual graph of Hirzebruch’s 1953 resolution paper

Val’s singularities which do not affect the conditions of adjunction, which have resolutions for which all the curves composing the exceptional divisor have self-intersection -2 , here the self-intersections can be arbitrary negative integers.¹² But a common feature of both cases is that all those curves are smooth and

¹¹ Such singularities are variously called nowadays “*Hirzebruch-Jung singularities*”, “*cyclic quotient singularities*” or “*toric surface singularities*”. The first name alludes to their importance for Hirzebruch’s method or resolution of arbitrary complex analytic surfaces (explained in Popescu-Pampu, 2011b), inspired by the ideas of Jung’s method of local parametrization of algebraic surfaces mentioned at the end of section “[What Does it Mean to Resolve the Singularities of an Algebraic Surface?](#)”.

¹² The “normal” surface singularities which admit resolutions with such a dual graph, its composing curves being moreover smooth, rational and pairwise transversal, are exactly the Hirzebruch-Jung singularities alluded to in Footnote 11. Their “*links*”, in the terminology explained later in this section, are the so-called *lens spaces*. One may consult Weber’s recent paper (Weber,

rational. This means that from a topological viewpoint they are 2-dimensional spheres.

Hirzebruch used the expression “*Sphärenbaum*”, that is, “tree of spheres” for the configurations schematically represented in Fig. 21. He explained that this terminology had been introduced by Heinz Hopf in his 1951 paper (Hopf, 1951) for the configurations of 2-dimensional spheres which are created by successive blow-ups of points, starting from a point on a smooth complex algebraic surface. Hopf chose that name because those spheres intersect in the shape of a tree.¹³

Let us look again at the models of algebraic surfaces from Figs. 7, 8 and 12. In each case, one has a representation of only part of the surface, as the whole surface is unbounded. The chosen part is obtained by considering the intersection of the entire surface with a ball centered at the singular point under scrutiny. By this procedure, one obtains a portion of the surface possessing a boundary curve. When the ball’s radius is small enough, one gets a curve whose qualitative shape (number of connected components, number of singular points on each component, etc.) does not depend on the radius. It is called nowadays the “*link*” of the singular point.

One may perform an analogous construction starting from a point of a *complex* algebraic surface. When the point is taken on a “*normal*” complex surface,¹⁴ then its associated link is a 3-dimensional manifold. In the late 1950s, Abhyankar conjectured that it was impossible to obtain a counterexample to the Poincaré conjecture following this procedure. In other words, that it was impossible to find a point on a normal complex surface, whose link is simply connected and different from the 3-dimensional sphere. The aim of Mumford’s paper (Mumford, 1961) was to prove this conjecture, as may be seen in Fig. 22, which reproduces the end of its introduction.

2018) for details about the history of the study of lens spaces and their relation with Hirzebruch-Jung singularities.

¹³ In Section 4 of Hopf (1951), Hopf wrote: “[...] die σ_i lassen sich den Eckpunkten eines Baumes—d. h. eines zusammenhängenden Streckencomplexes, der keinen geschlossenen Streckenzug enthält—so zuordnen, dass zwei σ_i dann und nur dann einen gemeinsamen Punkt haben, wenn die entsprechenden Eckpunkte eine Strecke begrenzen.” This is to be compared with Hirzebruch’s explanation given in section (11) of Hirzebruch (1953): “ K_n wird von H. Hopf als Sphärenbaum bezeichnet, da sich die Sphären σ_i^* eindeutig den Eckpunkten eines Baumes zuordnen lassen: Zwei σ_i^* haben genau dann einen Schnittpunkt, wenn die zugeordneten Eckpunkte im Baum eine Strecke begrenzen.”

¹⁴ Normality is a technical condition which implies that all the singular points of the surface are isolated. One may find details about it in Laufer’s book (Laufer, 1971) or in Mumford’s book (Mumford, 1999).

From the standpoint of the theory of algebraic surfaces, the really interesting case is that of a singular point on a *normal* algebraic surface, and m arbitrary. M is then by no means generally S^3 and consequently its own topology reflects the singularity P ! In this paper, we shall consider this case, first giving a partial construction of $\pi_1(M)$ in terms of a resolution of the singular point P ; secondly we shall sketch the connexion between $H_1(M)$ and the algebraic nature of P . Finally and principally, we shall demonstrate the following theorem, conjectured by Abhyankar:

Theorem. — $\pi_1(M) = (e)$ if and only if P is a simple point of F (a locally normal surface); and F topologically a manifold at P implies $\pi_1(M) = (e)$.

Fig. 22 Mumford’s motivation

are connected ⁽¹⁾. In order not to be lost in a morass of confusion, we shall now restrict ourselves to computing only H_1 in general, and π_1 only if $\pi_1(\cup E_i) = (e)$. Note that this last is equivalent to (a) E_i connected together as a tree (i.e. it never happens $E_1 \cap E_2 \neq \emptyset, E_2 \cap E_3 \neq \emptyset, \dots, E_{k-1} \cap E_k \neq \emptyset, E_k \cap E_1 \neq \emptyset$ and $k > 2$ for some ordering of the E_i ’s), (b) all E_i are rational curves.

Fig. 23 Mumford’s curves “connected together as a tree”

Mumford’s proof started from a resolution of the given point, assuming—which is always possible—that its exceptional divisor has “*normal crossings*”.¹⁵ It proceeded along the following steps:¹⁶

1. Show that the link M of the given point p of a normal complex surface is determined by the exceptional divisor and by the self-intersection numbers of its composing curves.
2. Show that if M is simply connected, then the curves of the exceptional divisor are “*connected together as a tree*” and are all rational (see Fig. 23).
3. Under this hypothesis, write a presentation of the fundamental group $\pi_1(M)$ of M in terms of the configuration of curves of the exceptional divisor and of their self-intersection numbers.
4. Deduce from this presentation that if $\pi_1(M)$ is trivial, then one may contract algebraically one of the curves of the exceptional divisor to a point and obtain again a resolution whose exceptional divisor has normal crossings.
5. Iterating such contractions, show that the given point p is a smooth point of the starting surface, which implies that its link M is the 3-dimensional sphere.

In what concerns step (1), Mumford proved in fact that the link M is determined by the dual graph of the exceptional divisor, decorated by the genera and the

¹⁵ This means that the exceptional divisor is composed of smooth curves intersecting pairwise transversally, and that three such curves do not have common points.

¹⁶ One needs to know a certain amount of algebraic topology in order to understand this proof completely. The reader with a strong taste for visualization may learn the needed notions from the website (Saint Gervais, 2017).

Case 2. — It remains to consider the case where no E_i intersects more than two others. Then the E_i are arranged as follows:

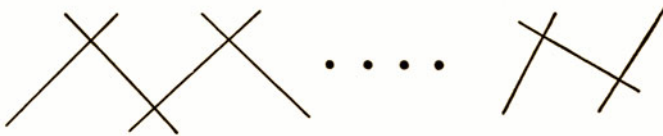


Fig. 3

Fig. 24 Mumford's diagram

self-intersection numbers of the associated curves. This formulation is slightly anachronistic, because he still did not formally introduce this dual graph. He said only that the curves of the exceptional divisor were “*connected together as a tree*”, which is similar to Du Val's terminology of his 1934 paper discussed in section “[Du Val's Singularities, Coxeter's Diagrams and the Birth of Dual Graphs](#)” (see again Fig. 15). Unlike Hirzebruch in his 1953 article, he did not even draw a dual graph, but only a schematic representation of the same type of configuration of curves as that of Hirzebruch's paper (Hirzebruch, 1953) (see Fig. 24). One may notice that the same drawing convention was to be followed by Michael Artin in his 1962 paper (Artin, 1962) (see the upper half of Fig. 4), before his switch to dual graphs in the 1966 paper (Artin, 1966).

It seems that the notion of dual graph of an arbitrary curve configuration, not necessarily formed of smooth rational curves, was formally defined for the first time in Hirzebruch's 1963 Bourbaki Seminar talk (Hirzebruch, 1963) discussing the previous results of Mumford. Hirzebruch associated to any “regular graph of curves” a “graph in the usual sense” (see Fig. 25), without using the terminology “dual graph”, which seems to have appeared later in the 1960s. Then, he stated the result of step (3) formulated above as the fact that the dual graph determines an explicit presentation of the fundamental group of the link M —of course, under Mumford's hypothesis that the exceptional divisor has normal crossings, that all its components are rational curves and that the graph is a tree.

Less than 10 years later, appeared the first books explaining—among other things—algorithms for the computation of dual graphs of resolutions of singularities of normal surfaces: Hirzebruch, Neumann and Koh's book (Hirzebruch et al., 1971) and Laufer's book (Laufer, 1971). All those algorithms followed Hirzebruch's method of his 1953 paper from which Fig. 21 was extracted.

Before passing to the next section, let me quote an e-mail received on 9 January 2018, in which Mumford answered my questions about the evolution of the notion of dual graph:

Perhaps the following is useful. In much of the twentieth century, math papers never had any figures. As a geometer, I always found this absurd and frustrating. In my “red book” intro to AG [Algebraic Geometry, Mumford (1999)], I drew suggestive pictures of various schemes, trying to break through this prejudice. On the other hand, I listened to many lectures by Oscar Zariski and, on rare occasions, we, his students, noticed him making a small drawing

We shall study MUMFORD's results in the complex-analytic case.

1. Regular graphs of curves.

Let X be a complex manifold of complex dimension 2. A regular graph Γ of curves on X is defined as follows.

- i. $\Gamma = \{E_1, E_2, \dots, E_n\}$.
- ii. Each E_i is a compact connected complex submanifold of X of complex dimension 1.
- iii. Each point of X lies on at most two of the E_i .
- iv. If $x \in E_i \cap E_j$ and $i \neq j$, then E_i, E_j intersect regularly in x and $E_i \cap E_j = \{x\}$.

Γ defines a graph Γ' in the usual sense (i. e. a one-dimensional finite simplicial complex) by associating to each E_i a vertex e_i and by joining e_i and e_j by an edge if and only if $E_i \cap E_j$ intersect. Γ' becomes a "weighted graph" by attaching to each e_i the self-intersection number $E_i \cdot E_i$, i. e. the

Fig. 25 Hirzebruch's switch from "graphs of curves" to graphs "in the usual sense"

on the corner of the blackboard. You see, the Italian school had always in mind actual pictures of the real points on varieties. Pictures of real plane curves and plaster casts of surfaces given by the real points were widespread. If you want to go for firsts, check out Isaac Newton's paper classifying plane cubics. So we were trained to "see" the resolution as a set of curves meeting in various ways. The old Italian theory of "infinitely near points" was, I think, always drawn that way. Of course, this worked out well for compactifying moduli space with stable curves. I'm not clear who first changed this to the dual graphs. Maybe it was Fritz [Hirzebruch].

Waldhausen's Graph Manifolds and Neumann's Calculus with Graphs

Mumford's theorem stating that the link of a complex normal surface singularity is determined by the dual graph of any of its resolutions whose exceptional divisor has normal crossings raised the question whether, conversely, it was possible to recover the dual graph from the structure of the link.

Formulated in this way, the problem cannot be solved, because resolutions are not unique. Indeed, given a resolution whose exceptional divisor has normal crossings, one can get another one by blowing up any point of the exceptional divisor. The new resolution has a different dual graph, with one additional vertex. Is there perhaps a minimal resolution, from which all other resolutions are obtained by sequences of

blow ups of points? Such a resolution indeed exists,¹⁷ and one may ask instead whether its dual graph is determined by the corresponding link.

This second question was answered affirmatively in the 1981 paper (Neumann, 1981) of Walter Neumann, building on a 1967 paper (Waldhausen, 1967) of Friedhelm Waldhausen. Let me describe successively the two papers, after a supplementary discussion of Mumford's article (Mumford, 1961).

Mumford looked at the link M of a singular point of a normal complex algebraic surface as the boundary of a suitable "tubular neighborhood" of the exceptional divisor of the chosen resolution. One of his crucial insights was that the assumption that this divisor has normal crossings¹⁸ implies that the 3-dimensional manifold M may be described only in terms of real curves and surfaces, which are objects of smaller dimension. Namely, the link M may be obtained by suitably cutting and pasting continuous families of circles—called *circle bundles*—parametrized by the points of the curves of the exceptional divisor. Let me explain why.

In the simplest case where the exceptional divisor is a single smooth algebraic curve—topologically a *real surface*, because it is a *complex curve*—such a tubular neighborhood has a structure of disk bundle over the curve: one may fill it by discs transversal to the curve. Being its boundary, the link has therefore a structure of circle bundle over this surface.

In general, the exceptional divisor has several components. Then the boundary of one of its tubular neighborhoods has again a structure of circle bundle far from the intersection points of those components, but one has to make a careful analysis near such points. Mumford worked in special neighborhoods of them, which he called "*plumbing fixtures*". These allow to see in which way one passes from the circle bundle over a curve of the exceptional divisor to that over a second such curve, intersecting the first one at the chosen point. In terms of the dual graph, the description is very simple: to every edge of it one can assign a "plumbing fixture". In the link, which is identified with the boundary of the tubular neighborhood, it gives rise to a torus. By moving inside the link M and crossing this torus, one passes from the first circle fibration to the second one. In order to understand precisely in which way the transition is made, one has to look at the relative positions of the circles of both fibrations on the separating torus. These intersect transversally at exactly one point.

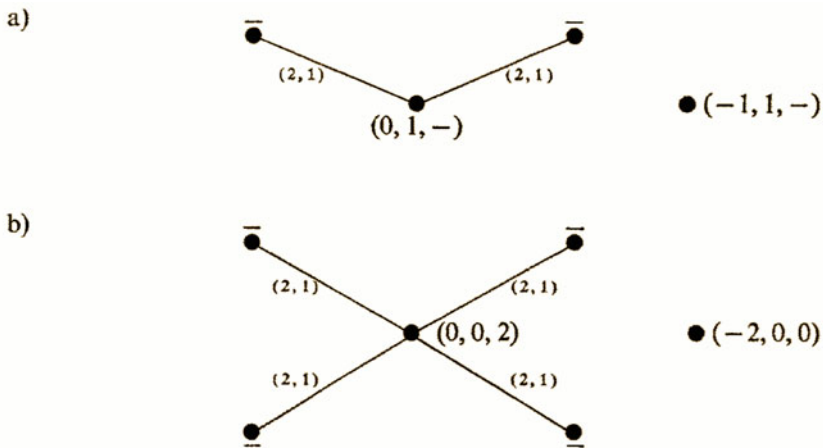
This phenomenon gave rise to the notion of "*plumbed 3-manifold*". It is a 3-dimensional manifold constructed from a decorated graph by performing a "*plumbing*" operation for each edge of the graph, similar to that described by Mumford in his paper. Each vertex comes equipped with two numbers, one representing the genus of a surface and the second its self-intersection number in an associated disk-bundle of dimension four. There is a subtlety related to orientations, which obliges one to decorate the edges with signs.

¹⁷ The figures in sections "Introduction" and "What Is the Meaning of Such Graphs?" present in fact dual graphs of *minimal* resolutions of the corresponding surface singularities.

¹⁸ Recall that this notion was explained in footnote 15.

One witnesses here a metamorphosis of the interpretation of the weighted dual graphs. If they started by representing the configurations of curves obtained as exceptional divisors of resolutions of singularities, they became blueprints for building certain 3-manifolds. It was Waldhausen who developed a subtle theory of those manifolds in Waldhausen (1967). He called them “*Graphenmannigfaltigkeiten*”—that is, “*graph-manifolds*” and not “plumbed manifolds”, in order to emphasize the idea that they are defined by graphs. In fact, he considered slightly more general graphs, whose edges are also decorated with pairs of numbers (see Fig. 26). This convention allowed the transitions from one circle fibration to another one across a torus to be performed by letting the fibers from both sides intersect in any way, not necessarily transversally at a single point. One of his main theorems states that any graph-manifold has a unique minimal graph-presentation, except for an explicit list of ambiguous manifolds.

In his 1981 paper (Neumann, 1981), Neumann turned this theorem into an algorithm, allowing to determine whether two weighted graphs in the original sense of the plumbing operations determined the same oriented 3-dimensional manifold. Roughly speaking, this algorithm consists in applying successively the rules of a “*plumbing calculus*”—some of them being represented in Fig. 27—in the direction which diminishes the number of vertices of the graph. Two graphs determine

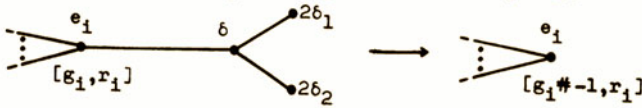


(9.4) Satz. *Eine orientierte reduzierte Graphenmannigfaltigkeit, die nicht eine der in (9.1) genannten ist, bestimmt einen bewerteten Graphen mit den Eigenschaften (9.2). Davon gilt auch die Umkehrung.*

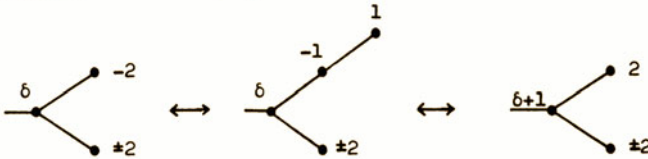
Zwei orientierte Mannigfaltigkeiten sind genau dann orientierungserhaltend homöomorph, wenn die zugeordneten Graphen äquivalent im Sinne von (9.3) sind.

Fig. 26 Two of Waldhausen’s “graph manifolds”

R2 ($\mathbb{R}P^2$ -absorption). Here $\delta_1 = \pm 1, \delta_2 = \pm 1$, and $\delta = (\delta_1 + \delta_2)/2$.



By blowing up (the inverse operation to R1) and blowing down one sees that the three cases of R2 are mutually equivalent. Indeed



R3 (0-chain absorption). Here the edge-signs ϵ'_i are given by $\epsilon'_i = -\bar{\epsilon}\epsilon_i$ if the edge in question is not a loop, and $\epsilon'_i = \epsilon_i$ if it is a loop.

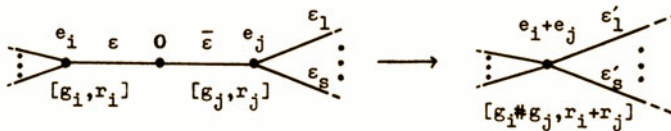


Fig. 27 Part of Neumann’s “plumbing calculus”

the same 3-dimensional manifold if and only if the associated “minimal” graphs coincide—again, up to a little ambiguity related to the signs on the edges.

The algorithm allowed Neumann to prove important topological properties of normal surface singularities and of families of smooth complex curves degenerating to singular ones. For instance, he showed that the decorated dual graph of the minimal resolution with normal crossings is determined by the *oriented* link of the singularity. This unified the two viewpoints on the graphs associated to surface singularities discussed in this paper (as dual graphs of their resolutions, and as blueprints for building their links).

Conclusion

We could continue this presentation of the interaction between graphs and singularities in several directions:

- By examining other classification problems of singularity theory which led to lists of dual graphs, between Mumford’s paper (Mumford, 1961) and Neumann’s paper (Neumann, 1981) (for instance Brieskorn’s paper (Brieskorn, 1968), Wagreich’s paper (Wagreich, 1972) and Laufer’s papers (Laufer, 1973, 1977)). We saw that it was such a classification problem which led Du Val to his consideration of “graphs of curves”, and which led to Artin’s lists of dual graphs

shown in Figs. 4 and 5. Another such problem led to the more recent paper of Chung, Xu and Yau from which Fig. 3 is extracted. Note that in Laufer (1973), Laufer classified all “*taut*” normal surface singularities, that is, those which are determined up to complex analytic isomorphisms by the dual graphs of their minimal resolutions. He showed in particular that all rational double and triple points of Figs. 4 and 5 are taut. This result had already been proved by Brieskorn in Brieskorn (1968) for rational double points. As a consequence, this class of singularities coincides with Du Val’s singularities which do not affect the condition of adjunction. This result is much stronger than the fact that their minimal resolutions have the same dual graphs.

- By examining the applications and developments of Neumann’s “plumbing calculus”. One could analyze its variant developed by Eisenbud and Neumann in Eisenbud and Neumann (1985) for the study of certain links (that is, disjoint unions of knots) in integral homology spheres which are graph-manifolds, its applications initiated by Neumann (1989) to the study of complex plane curves at infinity, or those initiated by Némethi and Szilárd (2012) and continued by Curmi (2020) to the study of boundaries of “*Milnor fibers*” of non-isolated surface singularities.
- By discussing generalizations of dual graphs to higher dimensions. In general, when one has a configuration of algebraic varieties, one may represent them by points, and fill any subset of the total set of such points by a simplex, whenever the corresponding varieties have a non-empty intersection. One gets in this way the so-called “*dual complex*” of the configuration of varieties. In the same way as there was a substantial lapse of time since the idea of dual graph emerged till it became an active object of study, an analogous phenomenon occurred with this more general notion. It seems to have appeared independently in the 1970s, in Danilov’s paper (Danilov, 1975)—whose results were rediscovered with a completely different proof by Stepanov in his 2006 article (Stepanov, 2006)—in Kulikov’s paper (Kulikov, 1977) and in Persson’s book (Persson, 1977). Information about recent works on dual complexes may be found in Payne’s paper (Payne, 2013), in Kollár’s paper (Kollár, 2013) and in the paper (de Fernex et al., 2017) by de Fernex, Kollár and Xu. One may use Nicaise’s paper (Nicaise, 2016) as an introduction to the relations between dual complexes and “*non-Archimedean analytifications in the sense of Berkovich*”.
- By presenting the notion of “*fan*” of the divisor at infinity of a toroidal variety, introduced by Kempf, Knudson, Mumford and Saint-Donat in the 1973 book (Kempf et al., 1973). It is a complex of cones associated to special kinds of configurations of hypersurfaces in complex algebraic varieties. When the configuration has normal crossings, the projectivisation of the fan is in fact the dual complex of the configuration. Fans had been introduced before by Demazure for “*toric varieties*” in the 1970 paper (Demazure, 1970), and since then they were mainly used in “*toric geometry*”. Following this direction, we could arrive at the notion of “*geometric tropicalization*”, which expresses “*tropicalizations*” of subvarieties of algebraic tori in terms of the dual complexes of the divisors at infinity of convenient compactifications (see Hacking et al., 2009 and Maclagan

and Sturmfels, 2015, Theorem 6.5.15). Note that Berkovich’s analytification (alluded to at the end of the previous item) and tropicalization are intimately related, as explained by Payne in Payne (2009, 2015). Note also that the paper Barroso et al. (2019) studies dual graphs of resolutions of normal surface singularities in the same spirit.

- By discussing how Waldhausen’s theory of graph-manifolds led to Jaco-Shalen-Johannson’s theory of canonical decompositions of arbitrary orientable and closed 3-manifolds into elementary pieces, by cutting them along spheres and tori (see Jaco and Shalen’s book (Jaco and Shalen, 1979) and Johannson’s book (Johannson, 1979)). This in turn gave rise to Thurston’s geometrization conjecture of Thurston (1982), proved partially by Thurston, and which was finally completely settled by Perelman’s work (Perelman, 2003). For details on Perelman’s strategy, one may consult the monographs Bessières et al. (2010) and Morgan and Tian (2014).
- By speaking about the second, more recent, main source of graphs in singularity theory: the dual graphs of configurations of “*vanishing cycles*” in Milnor fibers of isolated hypersurface singularities. Such dual graphs, called sometimes “*Dynkin diagrams*”, began to be described and drawn after 1970 for special classes of singularities in A’Campo (1975a,b), Gabrielov (1973, 1974) and Gusein-Zade (1974). One may consult Arnold’s papers (Arnold, 1973, 1975), Gusein-Zade’s survey (Gusein-Zade, 1977) and Brieskorn’s papers (Brieskorn, 1981, 1983) for a description of the context leading to those researches on Dynkin diagrams and of their relations with other invariants of hypersurface singularities. Du Val’s singularities possess configurations of vanishing cycles isomorphic to the dual graphs of their minimal resolutions indicated in Fig. 4. One may consult Brieskorn’s paper (Brieskorn, 2000) for a description of the way he proved this theorem instigated by a question of Hirzebruch. In fact, this property characterizes Du Val’s singularities (see Durfee’s survey (Durfee, 1979) of many other characterizations of those singularities). In general, the relation between the two types of dual graphs is still mysterious. Note that Arnold described in Arnold (1975) a “*strange duality*” inside a set of 14 “*exceptional unimodular singularities*”, relating the two types of dual graphs. This duality was explained in Pinkham (1977) on one side and Dolgachev and Nikulin (1977) on another side (see Dolgachev’s Bourbaki seminar presentation (Dolgachev, 1983)). Later, Dolgachev related it in Dolgachev (1996) to the very recent phenomenon—at the time—of “*mirror symmetry*”, but this seems to be only the tip of an iceberg.

I will not proceed in such directions, because this would be very difficult to do while remaining reasonably non-technical. I made nevertheless the previous list in order to show that dual graphs and their generalizations to higher dimensions are nowadays common tools in singularity theory, in algebraic geometry and in geometric topology. It is for this reason that I found interesting to examine their births and their early uses.

We saw that dual graphs of surface singularities were first used mainly verbally, in expressions like “tree of curves”, “Sphärenbaum”. Drawing them became important for stating results of various problems of classification. This made their verbal description first too cumbersome, then completely inadequate for the description of the wealth of morphologies under scrutiny. Then, their reinterpretation as blueprints for building graph-manifolds led to the development of a “plumbing calculus”, which transformed them into objects of algebra. The necessity to develop an analogous “calculus” appears every time one gets many different encodings of the structure of an object, leading to the problem of deciding which encodings correspond to the same object (another instance of this phenomenon is Kirby’s calculus of Kirby, 1978). In other situations—for instance, that of finite presentations of discrete groups—it is known that the problem is undecidable. But for plumbing graphs it is solvable, as shown by the works of Waldhausen and Neumann alluded to before.

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